The Fibonacci Quarterly

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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

WHEN DOES $m-n$ DIVIDE $f(m)-f(n)$? A LOOK AT COLUMN-FINITE MATRICES

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In this paper we consider the $\mathbf{Z}$-module of integer-valued functions f defined on the nonnegative integers (respectively, on all integers) and characterize the submodule determined by the divisibility relation of the title and also, as a corollary, by the divisibility relation $m+n|f(m)+f(n)$. Our results suggest some rather basic questions about such modules (equivalently, about infinite matrices of integers in which each column has only finitely many nonzero entries). We discuss these questions and pose a conjecture.

The functions f from the nonnegative integers $\mathbf{N}$ to the integers $\mathbf{Z}$ satisfying

$$m-n|f(m)-f(n) \text{ for all } m,n\in\operatorname{dom} f \tag{1}$$

are mentioned in Apostol's textbook [1]. Waterhouse [2] observes that integer-coefficient polynomials f certainly satisfy (1) and asks for a *nonpolynomial* function from $\mathbf{Z}$ to $\mathbf{Z}$ that does so. Myerson [3] supplies one. Problem 4 on the 1995 U.S.A. Mathematical Olympiad asks one to show that nonpolynomial functions from $\mathbf{N}$ to $\mathbf{Z}$ satisfying (1) never exhibit polynomial growth (see [4]). For sharper results on their growth rates, including an open question, see [6].

Our main result is that in both cases, $\mathbf{N}$ to $\mathbf{Z}$ and $\mathbf{Z}$ to $\mathbf{Z}$, there is a simple characterization of *all* functions satisfying (1); in fact, in each case, these functions form a $\mathbf{Z}$-module for which we can give a *basis*. (In the present context, the term *basis*, defined below, has the usual connotations of linearly spanning and being linearly independent, but infinite linear combinations are allowed.)

Let $M_\mathbf{N}$ (resp. $M_\mathbf{Z}$) denote the $\mathbf{Z}$-module of functions $\mathbf{N}$ to $\mathbf{Z}$ (resp. $\mathbf{Z}$ to $\mathbf{Z}$). Let M be any submodule of $M_\mathbf{N}$ or $M_\mathbf{Z}$ and let us define a *basis* for M as a finite or countably infinite set $\{f_0, f_1, f_2, \dots\}$ in M for which each $f\in M$ has a unique expression (up to order of summands) as an integral linear combination $f=\sum_{k\ge0}c_kf_k$. Naturally, $(\sum_{k=0}^\infty c_kf_k)(n)$ means $\sum_{k=0}^\infty c_kf_k(n)$ and, to converge, the series must have only finitely many nonzero terms for any specific value of n (and hence, of course, order of summation does not matter). Equivalently, we may identify $f\in M_\mathbf{N}$ with the infinite sequence (row vector) $(f(j))_{j\ge0}$, identify $M_\mathbf{N}$ with the set of all infinite sequences of integers, and view $\{f_i\}$ as an infinite matrix F with row i the sequence for f_i. Then $\sum_{k\ge0}c_kf_k$ is the vector-matrix product $\mathbf{c}F$, and F must be column-finite (i.e., only finitely many nonzero entries in each column) to ensure $\mathbf{c}F$ is defined for arbitrary $\mathbf{c}$. The conditions for $\{f_i\}$ to be a basis translate into: the rows of F (i) span $M_\mathbf{N}$ and (ii) are linearly independent (both conditions over $\mathbf{Z}$ and in the sense of infinite linear combinations). For example, the identity matrix corresponds to the "natural" basis $\{e_i\}_{i\ge0}$ for $M_\mathbf{N}$ with $e_i(j)=\delta_{ij}$ (Kronecker delta). Pascal's matrix, given by $P=(\binom{j}{i})_{i,j\ge0}$ corresponds to the basis $C_\mathbf{N}:=\{f_i\}_{i\ge0}$ with $f_i(j)=\binom{j}{i}$. The f_i form a basis because, in fact, $f=\sum_{i\ge0}c_if_i$ if and only if $c_j=\sum_{i=0}^j(-1)^{j-i}\binom{j}{i}f(i)$, $j\ge0$.

We digress to give one reason why $C_\mathbf{N}$ is a good basis to work with. Note that $f_k(x)=\binom{x}{k}$ is a *polynomial* in x of degree k with rational coefficients; thus, any *finite* integral linear combination of the f_k is a polynomial in the polynomial ring $\mathbf{Q}[x]$ that assumes integral values at the integers. Conversely, suppose $f(x)\in\mathbf{Q}[x]$ has this property. Let $\deg f=m$. Now $\{1,x,\dots,x^m\}$ is a basis for the $\mathbf{Q}$-vector space $P_m = \{g(x) \in \mathbf{Q}[x] : \deg g \le m\}$, and since $\deg f_k = k$, the set $\{f_k\}_{0\le k\le m}$ is also a basis for P_m. Hence, $f = \sum_{k=0}^{m} c_k f_k$ with $c_k \in \mathbf{Q}$. In fact, $c_k \in \mathbf{Z}$ by the integrality property of f and the last sentence of the preceding paragraph. It follows that the polynomial functions in $M_{\mathbf{N}}$ are precisely the finite integral linear combinations of the f_i. (It also follows that if $f \in \mathbf{Q}[x]$ assumes integral values at the nonnegative integers, then f assumes integral values at all integers and, hence, the polynomial functions in $M_{\mathbf{Z}}$ are the same as the polynomial functions in $M_{\mathbf{N}}$.)

An analogous basis for $M_{\mathbf{Z}}$ is $C_{\mathbf{Z}} := \{g_k\}_{k\ge 0} \cup \{h_k\}_{k\ge 1}$, where

$$g_k(n) = \binom{n+k}{2k} \quad \text{and} \quad h_k(n) = \binom{n+k-1}{2k-1}.$$

In this case, $f \in M_{\mathbf{Z}}$ can be (uniquely) expressed as

$$\sum_{k=0}^{\infty} c_k g_k + \sum_{k=1}^{\infty} d_k h_k,$$

with

$$c_n = \sum_{k=-n}^{n} (-1)^{n-k} \binom{2n}{n-k} f(k) \text{ for } n \ge 0$$

and

$$d_n = \sum_{k=-n}^{n-1} (-1)^{n-k-1} \binom{2n-1}{n-k-1} f(k) \text{ for } n \ge 1.$$

By arranging $C_{\mathbf{Z}}$ in the order $g_0, h_1, g_1, h_2, g_2, \ldots$, and evaluating at the integers in the order 0, -1, 1, -2, 2, ..., verification becomes equivalent to showing that the upper triangular matrices

$$\left(\binom{\lfloor i/2\rfloor + (-1)^j \lceil j/2\rceil}{i}\right)_{i,j\ge 0} \quad \text{and} \quad \left((-1)^{\lceil i/2\rceil + \lfloor j/2\rfloor}\binom{j}{\lceil (i+j)/2\rceil}\right)_{i,j\ge 0}$$

are inverses of one another. This does not particularly facilitate a proof by hand (and we leave the combinatorial identities on which a formal proof rests to the interested reader), but spending a few minutes checking finite sections by computer will convince you that these two matrices are indeed inverse to one another.

Note that h_k is an odd function, that is, $h_k(-n) = -h_k(n)$, $n \in \mathbf{Z}$, and, for $k \ge 1$, $2g_k - h_k$ is an even function, as is g_0. Also, from the above formulas for c_n and d_n, if $f \in M_{\mathbf{Z}}$ is odd, it follows that $c_k = 0$ for $k \ge 0$; thus, the h_k span the *odd* functions in $M_{\mathbf{Z}}$. Similarly, if $f \in M_{\mathbf{Z}}$ is even, it follows that $c_k = -2d_k$ for $k \ge 1$, yielding a spanning set for the *even* functions.

Summarizing, we have the following theorem.

Theorem 1: Let

$$f_k(n) = \binom{n}{k}, \; g_k(n) = \binom{n+k}{2k}, \; h_k(n) = \binom{n+k-1}{2k-1}$$

as above. Then:

(i) $C_{\mathbf{N}} = \{f_k\}_{k\ge 0}$ is a basis for $M_{\mathbf{N}}$;

(ii) $C_{\mathbf{Z}} = \{g_k\}_{k\ge 0} \cup \{h_k\}_{k\ge 1}$ is a basis for $M_{\mathbf{Z}}$;

(iii) $C_{\text{odd}} = \{h_k\}_{k\ge 1}$ is a basis for $\{f \in M_{\mathbf{Z}} : f \text{ is odd}\}$,

$C_{\text{even}} = \{g_0\} \cup \{2g_k - h_k\}_{k\ge 1}$ is a basis for $\{f \in M_{\mathbf{Z}} : f \text{ is even}\}$.

Just as for $C_{\mathbf{N}}$, the finite integral linear combinations in $C_{\mathbf{Z}}$, C_{odd}, C_{even} comprise the polynomial functions in the respective modules. The latter facts—at least for $C_{\mathbf{N}}$, C_{odd}, C_{even}—are noted in Pólya and Szegö [5].

For the sake of clarity, we mostly use the function interpretation for $M_{\mathbf{N}}$ and $M_{\mathbf{Z}}$ in the first part of the paper—Theorems 1 through 4—and then work with sequences and matrices in the second part. We are now ready to state our main result. Let $\operatorname{lcm}[n]$ denote the least common multiple of the first n positive integers (and set $\operatorname{lcm}[0]=1$).

Theorem 2: Let $M'_{\mathbf{N}}$ (resp. $M'_{\mathbf{Z}}$) denote the submodule of functions f in $M_{\mathbf{N}}$ (resp. $M_{\mathbf{Z}}$) that satisfy (1): $m-n|f(m)-f(n)$ for all $m,n\in\operatorname{dom}f$. Then:

(i) $M'_{\mathbf{N}}$ has a basis $C'_{\mathbf{N}}=\{\operatorname{lcm}[k]f_k\}_{k\ge 0}$;

(ii) $M'_{\mathbf{Z}}$ has a basis $C'_{\mathbf{Z}}=\{\operatorname{lcm}[2k]g_k\}_{k\ge 0}\cup\{\operatorname{lcm}[2k-1]h_k\}_{k\ge 1}$.

Proof: First we develop two lemmas. Let $[a,b]$ denote the interval of integers a, $a+1$, ..., b. For prime p and positive integer n, let $v_p(n)$ denote the exponent of the largest power of p that divides n; thus, $p^{v_p(n)}|n$, but $p^{v_p(n)+1}\nmid n$. It is trivial that

$$\left\lfloor\frac{a}{b}\right\rfloor+\left\lfloor\frac{b}{c}\right\rfloor\le\left\lfloor\frac{a+b}{c}\right\rfloor$$

for positive integers a, b, c. Hence, for $n\ge k\ge 1$ and $r\ge 1$,

$$\left\lfloor\frac{k}{p^r}\right\rfloor\le\left\lfloor\frac{n}{p^r}\right\rfloor-\left\lfloor\frac{n-k}{p^r}\right\rfloor.$$

This says that, for each r, the number of integers in $[1,k]$ divisible by p^r is $\le$ the number in $[n-k+1,n]$ so divisible. This fact allows the construction in an obvious way of a bijection $\phi:[1,k]\to[n-k+1,n]$ such that $v_p(i)\le v_p(\phi(i))$ for $1\le i\le k$. (Consider first the integers in $[1,k]$ divisible by the highest power of p that divides k. Let ϕ be any one-to-one map from these integers to the integers in $[n-k+1,n]$ divisible by this power of p and then proceed in turn to the smaller powers of p.) Let $n^{\underline{k}}$ denote the falling factorial $n(n-1)(n-2)\cdots$ to k factors.

Lemma 1:

(i) For $n\ge k\ge 1$ and p prime, $p^{v_p(k!)}|n^{\underline{k}}$.

(ii) If i $(1\le i<k)$ factors are removed from the product $n^{\underline{k}}$, then the resulting product is divisible by $p^{v_p(k!)-iv_p(\operatorname{lcm}[k])}$.

Proof:

(i) Since $\binom{n}{k}=\frac{n^{\underline{k}}}{k!}$ is an integer, $p^{v_p(k!)}|n^{\underline{k}}$.

(ii) This assertion follows from part (i) and the existence of the bijection ϕ which says, so far as divisibility by p goes, the effect of tossing the factor $\phi(i)$ out of the product $n^{\underline{k}}$ is no worse than the effect of tossing the factor i out of $k!$, and $v_p(i)\le v_p(\operatorname{lcm}[k])$.

We also need a result on the divisibility of binomial coefficients.

Lemma 2: If p is a prime that does not divide r, and if $i>j\ge 0$, then $p^{i-j}\left|\binom{p^iq}{p^jr}\right.$.

Proof: An often-quoted result of Kummer (see [7] for a proof) says that the exact power of p that divides a binomial coefficient $\binom{n}{k}$ is the number of "borrows" when k is subtracted from n in base p. For example, in base 5, $(375)_5 = 3000$, $(330)_5 = 2310$, and we have the subtraction

$$\begin{array}{rcl} n &=& 3\ 0\ 0\ 0 \\ k &=& 2_1\ 3_1\ 1\ 0 \\ \hline n-k &=& \quad 1\ 4\ 0 \end{array}$$

with 2 borrows. Note that if (as here) the number of trailing zeros in n exceeds that in k, the number of borrows will always be at least the excess (here, 2). Since $k = p^j r$ has exactly j trailing zeros, this observation translates immediately into Lemma 2.

Now to the proof of the theorem. First, we must show that the elements of C'_{N} actually satisfy (1). Therefore, let $f'_k(n) = \mathrm{lcm}[k]\binom{n}{k}$ denote a typical element of C'_{N} and we must show that $m | f'_k(n+m) - f'_k(n)$, for $m, n, k \ge 1$; that is, $m|\mathrm{lcm}[k]((n+m)^{\underline{k}} - n^{\underline{k}})/k!$ or, equivalently, for each prime divisor p of m,

$$p^{v_p(m)+v_p(k!)-v_p(\mathrm{lcm}[k])} | (n+m)^{\underline{k}} - n^{\underline{k}}. \tag{2}$$

If $v_p(m) \le v_p(\mathrm{lcm}[k])$, (2) is an immediate consequence of Lemma 1(i). So suppose $v_p(m) > v_p(\mathrm{lcm}[k])$ and consider the cases $k > n$ and $k \le n$ separately. If $k > n$, then m is one of the factors in $(n+m)^{\underline{k}}$ and $n^{\underline{k}} = 0$. But $p^{v_p(m)}|m$ by definition, and

$$p^{v_p(k!)-v_p(\mathrm{lcm}[k])} \left| \frac{(n+m)^{\underline{k}}}{m} \right.$$

by Lemma 1(ii), and (2) is obtained by multiplying these divisibility relations. On the other hand, if $k \le n$, consider the powers of m in the expansion

$$(n+m)^{\underline{k}} - n^{\underline{k}} = \sum_{i=1}^{k} [\Sigma \pi(k-i)] m^i,$$

where, in the inner sum, the summand $\pi(k-i)$ runs over all products of $k-i$ factors from $[n-k+1, n]$. We have $p^{v_p(k!)-iv_p(\mathrm{lcm}[k])}|\pi(k-i)$ by Lemma 1(ii) and, trivially, $p^{v_p(m)i}|m^i$. This yields

$$p^{v_p(k!)+i(v_p(m)-v_p(\mathrm{lcm}[k]))}|\pi(k-i)m^i$$

and hence, certainly,

$$p^{v_p(k!)+v_p(m)-v_p(\mathrm{lcm}[k])}|\pi(k-i)m^i$$

since $i \ge 1$, and, by supposition, $v_p(m) > v_p(\mathrm{lcm}[k])$. Summing over i, we obtain (2). Hence, $C'_{\mathrm{N}} \subseteq M'_{\mathrm{N}}$, and the proof that $C'_{\mathrm{Z}} \subseteq M'_{\mathrm{Z}}$ is very similar.

Next, we must show that every $f \in M'_{\mathrm{N}}$ is an integral linear combination of the elements of C'_{N}. We already know there exists a unique sequence of integers $(c_n)_{n\ge 0}$, namely,

$$c_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k),$$

such that

$$f(n) = \sum_{k=0}^{n} c_k \binom{n}{k}.$$

So we must show that $\operatorname{lcm}[k] | c_k$ under the hypothesis that f satisfies (1). To get induction (on n) working, we will prove a little more:

$$\text{For all } a \in \mathbf{N},\ \operatorname{lcm}[n] \text{ divides } c_n(a) := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(a+k) \text{ when } f \text{ satisfies (1).}$$

[Of course, $c_n(0) = c_n$ and, to numerical analysts, $c_n(a)$ is the n^{th} forward difference of f at a.] The base case $n = 1$ is trivial. Since $c_n(a) = c_{n-1}(a+1) - c_{n-1}(a)$ we have, by the induction hypothesis, that $\operatorname{lcm}[n-1] | c_n(a)$ and need only show $n | c_n(a)$. Let p be any prime divisor of n. Write $n = p^i r$ and let $k = p^j s$ with r and s relatively prime to p. Now (1) implies $p^j | f(a+k) - f(a)$. Also, if $i \geq j$, then $p^{i-j} | \binom{n}{k}$ by Lemma 2. In any case, $p^i | \binom{n}{k} [f(a+k) - f(a)]$. Since

$$c_n(a) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} [f(a+k) - f(a)],$$

it follows that $p^i | c_n(a)$, and since p is arbitrary, $n | c_n(a)$ and the induction is complete. The corresponding proof that $C'_{\mathbf{Z}}$ generates $M'_{\mathbf{Z}}$ is analogous: $c_n(a)$ and $d_n(a)$ are defined analogously (except now $a \in \mathbf{Z}$) and the induction is based on the recurrence relations $c_n(a) = d_n(a+1) - d_n(a)$ and $d_n(a) = c_{n-1}(a) - c_{n-1}(a-1)$. This completes the proof of Theorem 2.

Here is one corollary. As noted earlier, the finite linear combinations in $C_{\mathbf{N}}$ (or $C_{\mathbf{Z}}$) yield polynomials, and infinite linear combinations yield nonpolynomials (by uniqueness—the polynomials are already exhausted by the finite linear combinations). The observation (made in the editorial comment on [3]) that there are uncountably many nonpolynomial functions $\mathbf{Z}$ to $\mathbf{Z}$ satisfying (1) follows immediately.

For later use, we remark that the divisibility relation (2) above in fact holds for arbitrary integer n. (This shows that $f'_k(n) = \operatorname{lcm}[k]\binom{n}{k}$, if considered as a function of $n \in \mathbf{Z}$ rather than just $n \in \mathbf{N}$, is in $M'_{\mathbf{Z}}$; thus, each element of the basis $C'_{\mathbf{N}}$ for $M'_{\mathbf{N}}$ extends to an element of $M'_{\mathbf{Z}}$. We will soon see that not every element of $M'_{\mathbf{N}}$ extends in this fashion.) The proof of (2) for $n \leq 0$ is similar to that given above for $n \geq 1$; $n = 0$ is easy, so suppose $n < 0$. In the case $n + m < 0$, the result already established in (2) for $n \geq 1$ applies (with the roles of $n+m$ and n reversed); the case $n + m \geq 0$ should be split into subcases $k > n + m$ and $k \leq n + m$ corresponding to the subcases $k > n$ and $k \leq n$ above. The details are left to the interested reader.

Now we consider an interesting submodule of $M'_{\mathbf{N}}$. Let $\theta : M_{\mathbf{Z}} \to M_{\mathbf{N}}$ be given by restriction of domain, that is, $\theta(f) = f_{|\mathbf{N}}$. (Interpreting the elements of $M_{\mathbf{N}}$ and $M_{\mathbf{Z}}$ as sequences, θ just throws away the left half of a doubly infinite sequence.) Let ψ denote the restriction of θ to $M'_{\mathbf{Z}}$; then it is clear that $\psi : M'_{\mathbf{Z}} \to M'_{\mathbf{N}}$. Note that the range of ψ includes at least all the finite integral linear combinations of $C'_{\mathbf{N}}$, that is, all the polynomial maps in $M'_{\mathbf{N}}$. This is because each $f'_k \in C'_{\mathbf{N}}$ extends, as remarked in the previous paragraph, to an element in $M'_{\mathbf{Z}}$. Of course, the map θ is onto but far from one-to-one. Contrariwise, we have the following result for ψ.

Theorem 3: Let $\psi : M'_{\mathbf{Z}} \to M'_{\mathbf{N}}$ be the map just defined. Then

(i) ψ is one-to-one,

(ii) ψ is not onto.

Proof:

(i) It suffices to show $\ker\psi=(0)$ and, to see this, view an element of $\ker\psi$ as a doubly infinite sequence of integers with a tail of zeros. Then it cannot have any nonzero term [the standing divisibility hypothesis (1) implies any term is divisible by all sufficiently large integers].

(ii) Recall $f_k(n)=\binom{n}{k}$ and, for brevity, set $u_k=\mathrm{lcm}[k]$, $k\ge 0$. Now consider the element of $M'_{\mathbf{N}}$ given by $f=\sum_{k\ge0}u_kf_k$ and let us ask if f can be extended backward, i.e., defined at -1, to yield a (sequence of integers) g that is still in $M'_{\mathbf{N}}$. It suffices to show that this cannot be done. Suppose it can. Then, by Theorem 2(i), $g=\sum_{k\ge0}c_ku_kf_k$ for some sequence of integers (c_k) and $g(n+1)=f(n)$, $n\ge0$. This readily implies that $u_k=c_ku_k+c_{k+1}u_{k+1}$, $k\ge0$. Multiply by $(-1)^k$ and add to obtain

$$\sum_{k=0}^{n-1}(-1)^ku_k=c_0u_0+(-1)^{n-1}c_nu_n$$

and, in particular,

$$\sum_{k=0}^{n}(-1)^ku_k\equiv c_0 \pmod{u_n},\ n\ge1. \tag{3}$$

This infinite set of congruences has no solution for c_0 as follows. Let $s_n=\sum_{k=0}^{n}(-1)^ku_k$ denote the left side of (3). Since $u_n=u_{n-1}$ unless n is a prime power, p^r, in which case $u_n=pu_{n-1}$, it is easy to show by induction that $0<s_n<u_n$ for n even ≥2, and $-u_n<s_n<0$ for n odd ≥3. Thus, if $c_0\ge0$ we have, for all sufficiently large even n, $0\le c_0<u_n$ while $0<s_n<u_n$ and, by (3), $s_n\equiv c_0$ (mod u_n); hence, $s_n=c_0$. It follows that $s_{n+2}=s_n$ and therefore $u_{n+2}=u_{n+1}$, a contradiction since $u_{n+2}=2u_{n+1}$ whenever $n+2$ is a power of 2. A similar contradiction is obtained in case $c_0<0$, completing the proof.

As a curiosity, we can now "analyze" the divisibility relation $m+n|f(m)+f(n)$ with little extra effort. Let $M''_{\mathbf{N}}=\{f\in M_{\mathbf{N}}: m+n|f(m)+f(n),\ m,n\in\mathbf{N}\}$ and analogously for $M''_{\mathbf{Z}}$. Also, let $\rho: M''_{\mathbf{Z}}\to M''_{\mathbf{N}}$ denote the restriction map analogous to $\psi: M'_{\mathbf{Z}}\to M'_{\mathbf{N}}$ above.

Theorem 4: Let

$$h_k(n)=\binom{n+k-1}{2k-1}$$

as in Theorem 1, and let $h_{k|\mathbf{N}}$ denote the restriction of h_k to $\mathbf{N}$. Then:

(i) $\{\mathrm{lcm}[2k-1]h_{k|\mathbf{N}}\}_{k\ge1}$ is a basis for $M''_{\mathbf{N}}$;

(ii) $\{\mathrm{lcm}[2k-1]h_k\}_{k\ge1}$ is a basis for $M''_{\mathbf{Z}}$.

(iii) $\rho: M''_{\mathbf{Z}}\to M''_{\mathbf{N}}$ is both one-to-one and onto (unlike the map ψ of Theorem 3).

Proof: Suppose $f\in M''_{\mathbf{Z}}$. The divisibility hypothesis with $m=-n$ implies that f is odd and, consequently, the divisibility hypothesis also implies that $m-n|f(m)-f(n)$, $m,n\in\mathbf{Z}$. Therefore, $M''_{\mathbf{Z}}=M'_{\mathbf{Z}}\cap\{f\in M_{\mathbf{Z}}: f \text{ is odd}\}$ and part (ii) follows from Theorem 1(iii) and Theorem 2(ii). Now observe that $m+n|f(m)+f(n)$, $m,n\in\mathbf{N}$ implies also $m-n|f(m)-f(n)$, $m,n\in\mathbf{N}$ or, equivalently, $k|f(\ell+k)-f(\ell)$, $k,\ell\in\mathbf{N}$. To see this, write $\ell=kq+r$ with $0\le r<k$ (division algorithm) and apply the hypothesis with $m=kq+r$ and $n=k-r$ to obtain $k|f(kq+r)+f(k-r)$. Replacing q by $q+1$ yields $k|f(kq+r+k)+f(k-r)$. Hence, k divides the difference, that is, $k|f(\ell+k)-f(\ell)$, as desired.

This permits us to extend any $f \in M''_{\mathbf{N}}$ to $f \in M''_{\mathbf{Z}}$ by defining $f(-n) = -f(n)$, $n \geq 1$ (the reader should check this), and this is the *only* way to extend f since $M''_{\mathbf{Z}}$ consists of odd functions. Thus, the restriction map $\rho: M''_{\mathbf{Z}} \to M''_{\mathbf{N}}$ is one-to-one and onto. This is part (iii) and part (i) follows from parts (ii) and (iii).

The preceding results raise the question: To what extent is the above notion of *basis* analogous to a free basis (of a finitely generated module)? To begin with, any finitely generated submodule of $M_{\mathbf{N}}$ is a free $\mathbf{Z}$-module and here the notions *basis* as above and *free basis* coincide. Every submodule M of $M_{\mathbf{N}}$ does possess a basis. To see this, view elements of $M_{\mathbf{N}}$ as sequences, and for $i \geq 0$ let c_i denote the least positive integer occurring in position i among elements of M having zeros in the positions preceding i (but if all these elements have 0 in position i, take $c_i = 0$). If $c_i \neq 0$, let $\mathbf{u}_i$ be any such sequence (with first nonzero entry c_i in position i). Then it is straightforward to verify that these $\mathbf{u}_i$ form a basis for M.

Now we switch perspective from functions and lists of functions to sequences and matrices, and henceforth write $\mathbf{Z}^\infty$ instead of $M_{\mathbf{N}}$ for the infinite sequences of integers (and $\mathbf{Q}^\infty$ for the infinite sequences of rationals). Also $\mathbf{Z}_\infty$ (resp. $\mathbf{Q}_\infty$) will be used to denote the set of infinite matrices of integers (resp. rationals). The terms *span* and *linear independence* will continue to be used in the sense of infinite linear combinations. For $A \in \mathbf{Z}_\infty$, let $R(A)$ denote the set of rows of A. One basic question is: When does $R(A)$ form a basis for $\mathbf{Z}^\infty$? First, as noted above, A must be column-finite. Second, for $R(A)$ to span $\mathbf{Z}^\infty$, it is certainly necessary for $R(A)$ to span the *basic* vectors $\{e_k\}$; equivalently, A must possess a left inverse in $\mathbf{Z}_\infty$, call it B. Third, A must have a trivial left nullspace in $\mathbf{Z}^\infty$ (so A's rows are linearly independent) and this makes A's left inverse B unique. These three conditions, though, do not ensure that $R(A)$ spans all of $M_{\mathbf{N}}$.

Example 1: Let J denote the infinite *Jordan* matrix—all 0's except 1's just below the main diagonal—and set $A = I + 2J$. Then A is column-finite and has a unique left inverse in $\mathbf{Z}_\infty$, but A's rows do not have $\mathbf{Z}$-span $\mathbf{Z}^\infty$.

Proof: The column-finiteness of A is obvious. It is easy to check that J^k has 1's on the k^{th} diagonal below and parallel to the main diagonal and 0's elsewhere, and that a left inverse of A in $\mathbf{Z}_\infty$ is given correctly by the formal expansion $(1+2J)^{-1} = I - 2J + 4J^2 - 8J^3 + \cdots$. The left nullspace of A in $\mathbf{Q}^\infty$ is spanned by $\left(1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \cdots\right)$ and so, clearly, has trivial intersection with $\mathbf{Z}^\infty$, making A's left inverse in $\mathbf{Z}_\infty$ unique. On the other hand, let $\mathbf{e}$ denote the (row) vector of all 1's. Then $\mathbf{e}A = 3\mathbf{e}$, so the general solution of $\mathbf{x}A = \mathbf{e}$ in $\mathbf{Q}^\infty$ is $\mathbf{x} = \frac{1}{3}\mathbf{e} + k\left(1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \cdots\right)$ (arbitrary $k \in \mathbf{Q}$) and, clearly, $\mathbf{x} \notin \mathbf{Z}^\infty$ for any k. Thus, $\mathbf{e}$ is not in the $\mathbf{Z}$-row span of A.

Column-finiteness of A's left inverse (or, rather, the lack of it) plays a role in the preceding example. Note that a product of column-finite matrices is again column-finite, and associativity holds; in fact, for $\mathbf{w} \in \mathbf{Q}^\infty$ and $X, Y \in \mathbf{Q}_\infty$, if $\mathbf{w}X$ is defined and Y is column-finite, then it is easy to check that all four products are defined and $(\mathbf{w}X)Y = \mathbf{w}(XY)$. Three corollaries: (1) if X and Y are column-finite matrices in $\mathbf{Q}_\infty$, then $(WX)Y = W(XY)$ for arbitrary $W \in \mathbf{Q}_\infty$; (2) the column-finite (CF) matrices in $\mathbf{Q}_\infty$ form a ring (with identity). Let us denote this ring by CF($\mathbf{Q}_\infty$) and, analogously, for CF($\mathbf{Z}_\infty$); (3) if $A \in$ CF($\mathbf{Z}_\infty$) has a unique left inverse B in $\mathbf{Z}_\infty$ *that happens to be column-finite*, then we have, for arbitrary $\mathbf{w} \in \mathbf{Z}^\infty$, $\mathbf{w} = \mathbf{w}(BA) = (\mathbf{w}B)A$; thus, $R(A)$ does indeed have $\mathbf{Z}$-span $\mathbf{Z}^\infty$. Of course, this argument breaks down if B is not column-finite because then

$\mathbf{w}B$ might not be defined (and Example 1 shows that the conclusion need not hold). For $A \in \mathrm{CF}(\mathbf{Z}_\infty)$, the assertion that A has a unique left inverse in $\mathbf{Z}_\infty$ that happens to lie in $\mathrm{CF}(\mathbf{Z}_\infty)$ is, on the face of it, stronger than saying that A has a unique left inverse in $\mathrm{CF}(\mathbf{Z}_\infty)$. In fact, the two statements are equivalent since, if A has a left inverse B in $\mathrm{CF}(\mathbf{Z}_\infty)$ and another left inverse C in $\mathbf{Z}_\infty \setminus \mathrm{CF}(\mathbf{Z}_\infty)$, then $B-C$ has a nonzero row, and adding this row to any row of B produces another left inverse for A in $\mathrm{CF}(\mathbf{Z}_\infty)$. Furthermore, from elementary ring theory, for a in any ring with identity R, "a has a unique left inverse in R" is equivalent to "a is invertible (i.e., a unit) in R."

These observations suggest the following conjecture.

Conjecture: Let A be a column-finite infinite matrix of integers, that is, $A \in \mathrm{CF}(\mathbf{Z}_\infty)$. Then the rows of A form a basis for $\mathbf{Z}^\infty$ if and only if A is a unit in $\mathrm{CF}(\mathbf{Z}_\infty)$.

[We have proved the "if" part and for the "only if" part we have shown that A has a unique left inverse B in $\mathbf{Z}_\infty$. The conjecture then is that B must lie in $\mathrm{CF}(\mathbf{Z}_\infty)$.]

Motivated by the preceding observations, let us now consider the subtleties of the concept of inverse for an infinite matrix. In general, we must distinguish between left and right inverses—indeed over the integers, all nine combinations of $0, 1$ or infinitely many left inverses and $0, 1$ or infinitely many right inverses are possible. The following table provides simple examples (J denotes the Jordan matrix with 1's below the diagonal, K is given below, and the superscript t denotes transpose).

# left inverses \ # right inverses	0	1	∞
0	O	K^t	J^t
1	K	I	$I+J^t$
∞	J	$I+J$	$J+J^t$

Here the unique left inverse of K and K itself are given, respectively, by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & \\ 0 & -1 & 0 & 1 & 0 & \\ 0 & -1 & 0 & 0 & 1 & \\ 0 & -1 & 0 & 0 & 0 & \\ \vdots & & & & & \ddots \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & \\ 1 & 0 & 0 & 1 & 0 & \\ \vdots & & & & & \ddots \end{pmatrix}.$$

For $J+J^t$, a right inverse is $J-J^3+J^5-\cdots$ (note that $J^tJ=1$), and a right nullvector is $(1,0,-1,0,1,0,-1,\ldots)^t$. These and the other easy verifications are left to the reader.

We now collect a few simple facts about inverses of column-finite matrices.

Proposition 1: If $A \in \mathrm{CF}(\mathbf{Z}_\infty)$ has a *unique* left inverse B in $\mathbf{Z}_\infty$ and if AB is defined, then $AB = I$.

Proof: By associativity, $(AB-I)A = A(BA)-A = O$. Hence, $AB-I=O$ by uniqueness of A's left inverse.

A diagonal matrix $D \in \mathbf{Q}_\infty$ with nonzero diagonal entries obviously has an unambiguous inverse $D^{-1} \in \mathbf{Q}_\infty$, but already with triangular matrices we must be careful.

Proposition 2: Suppose $U \in \mathrm{CF}(\mathbf{Q}_\infty)$ is block upper triangular with finite square blocks $\{U_{ii}\}$ on the main diagonal.

If the U_{ii} are invertible, then U has a unique left inverse V in $\mathbf{Q}_\infty$ and V is block upper triangular. Moreover, V is a right inverse for U and it is the only *column-finite* right inverse for U in $\mathbf{Q}_\infty$, though U may have other right inverses in $\mathbf{Q}_\infty$ and even in $\mathbf{Z}_\infty$.

Also, if U has integer entries and the U_{ii} are invertible over $\mathbf{Z}$, then V is actually in $\mathbf{Z}_\infty$.

Proof: We can solve $VU = I$ uniquely for the blocks of V in turn: first row, left to right, then second row, left to right, etc., and V has the stated form. In particular, V is column-finite; so UV is defined and, by Proposition 1, $UV = I$.

Similarly, if W is assumed column-finite, we can solve $UW = I$ uniquely for the (block) columns of W left to right, bottom to top. Hence, W must equal V. (Alternatively, invoke the result for rings: if $vu=1$ and $uw=1$, then $w=v$.) However, for the upper triangular matrix $U = I + J^t$ with J the "Jordan" matrix above, $U(1,-1,1,-1,1,\dots)^t = \mathbf{0}$ showing that U has multiple right inverses.

With U and V as in Proposition 2, uniqueness of U's right inverse may be guaranteed by U's zero pattern.

Proposition 3: With U and V as in Proposition 2, suppose U satisfies the following condition:

$$U\mathbf{x} \text{ is defined (i.e., involves no finite sums) only when } \mathbf{x} \text{ is column-finite} \tag{4}$$

[e.g., (4) certainly holds if U's above-diagonal entries are all nonzero]. Then V is the *unique* right inverse for U in $\mathbf{Q}_\infty$.

Proof: Suppose $U\mathbf{x} = \mathbf{0}$. By (4) there exists n such that $x_i = 0$ for $i > n$. Take a large enough square upper left submatrix U_m consisting of whole blocks of U so its size is m by m with $m \ge n$. Then $U_m(x_1, x_2, \dots, x_m)^t = \mathbf{0}$ and, since U_m is a finite invertible matrix, $\mathbf{x} = \mathbf{0}$ and U has trivial right nullspace, making any right inverse for U unique.

Proposition 4: With U and V again as in Proposition 2, condition (4) of Proposition 3 is satisfied if and only if there exists $k \ge 1$ such that the submatrix of U consisting of its first k rows has only finitely many zero columns.

Proof: Exercise.

Corollary: Pascal's matrix of binomial coefficients, $P = \left(\binom{j}{i}\right)_{i,j\ge 0}$, has a unique left inverse and a unique right inverse, and both are given by $P^{-1} = \left((-1)^{j-i}\binom{j}{i}\right)$.

Proof: One verifies that $Q = \left((-1)^{j-i}\binom{j}{i}\right)$ is a left inverse for P, unique by Proposition 2, hence $PQ = I$ by Proposition 1, and Q is P's unique right inverse by Proposition 3.

Referring back to Example 1, note the matrix A given there has multiple left inverses in $\mathbf{Q}_\infty$. This raises the question: Does the phenomenon of Example 1 occur in $\mathbf{Q}_\infty$? Perhaps things are nicer over fields and for a matrix $A \in \mathrm{CF}(\mathbf{Q}_\infty)$ with a unique left inverse B in $\mathbf{Q}_\infty$, perhaps B must lie in $\mathrm{CF}(\mathbf{Q}_\infty)$.

Here are several other questions. Is there an effective method to determine which elements of M_{N}' are in the range of ψ? When can a subset of M_{N} $(=\mathbf{Z}^\infty)$ be enlarged to a basis? Could the matrix K of Example 2 be replaced by a column-finite matrix, that is, in view of Proposition 1, could a matrix $A \in \mathrm{CF}(\mathbf{Z}_\infty)$ have a unique left inverse B in $\mathbf{Z}_\infty$ for which AB is not defined? In the rings $\mathrm{CF}(\mathbf{Z}_\infty)$ and $\mathrm{CF}(\mathbf{Q}_\infty)$, is there a nice characterization (or generating set) for the units? An answer to the latter question for matrices A that are both row- and column-finite and with entries in a *field* was recently given in [7]: A is invertible if and only if both its rows and its columns are (infinitely) linearly independent.

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A FAMILY OF 4-BY-4 FIBONACCI MATRICES

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1. AIM OF THE PAPER

Fibonacci matrices are matrices the entries of the powers of which are related to the Fibonacci numbers F_n and/or the Lucas numbers L_n. The most celebrated among them are the 2-by-2 matrix **Q** (e.g., see [6, p. 65]) first studied by C. H. King in [7, pp. 11-27], and the 3-by-3 matrix **R** (e.g., see [1, p. 26]).

Consider the m-by-m tridiagonal symmetric Toeplitz matrix $\mathbf{S}_m(x,y)$ defined as

$$\mathbf{S}_m(x,y)=\begin{bmatrix} x & y & 0 & 0 & \cdots & 0 & 0 & 0 \\ y & x & y & 0 & \cdots & 0 & 0 & 0 \\ 0 & y & x & y & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & y & x & y \\ 0 & 0 & 0 & 0 & \cdots & 0 & y & x \end{bmatrix}, \tag{1.1}$$

where x and y are arbitrary quantities.

In this paper we show how, for certain (integral) values of x and y, the matrices $\mathbf{S}_4(x,y)$ are Fibonacci matrices. More precisely, after recalling some properties of $\mathbf{S}_m(x,y)$ which are valid for arbitrary couples (x,y), we work out closed-form expressions for the entries $s_{h,k}^{(n)}(x,y)$ of $\mathbf{S}_4^n(x,y)$ and prove that, for special values of the above-mentioned couples, they involve Fibonacci numbers. Further, as application examples, some of these expressions are used, jointly with certain matrix expansions, to obtain some hopefully new Fibonacci identities. It is worth mentioning that the existence of some relations between matrices $\mathbf{S}_m(x,y)$ and the Fibonacci numbers is well known (e.g., see [8] and [9]). For example, the corollary to Theorem 1 of [8] tells us that the permanent of $\mathbf{S}_m(1,1)$ equals F_{m+1}.

2. PRELIMINARY RESULTS

The fundamental properties of $\mathbf{S}_m(x,y)$ have been investigated in [2] where, in particular, the following compact form for the generic entry $f_{h,k}(x,y)$ of any admissible function $f[\mathbf{S}_m(x,y)]$ has been established:

$$f_{h,k}(x,y)=\frac{2}{m+1}\sum_{j=1}^{m} f[\lambda_j(x,y)]\sin\frac{jh\pi}{m+1}\sin\frac{jk\pi}{m+1} \quad (1\le h,k\le m), \tag{2.1}$$

where (see Theorem D1 of [3])

$$\lambda_j(x,y)=x+2y\cos\frac{j\pi}{m+1} \quad (j=1,2,\ldots,m) \tag{2.2}$$

are the eigenvalues of $\mathbf{S}_m(x,y)$.

Remark 1: It is worth noting that formula (2.1) also works if $\mathbf{S}_m(x, y)$ is replaced by any function $g[\mathbf{S}_m(x, y)]$ which, in general, is not a Toeplitz tridiagonal matrix. It is sufficient to replace $f(x+2y\cos\frac{j\pi}{m+1})$ by $f[g(x+2y\cos\frac{j\pi}{m+1})]$.

Specializing f in (2.1) to the n^{th} power ($n \geq 0$) yields the desired expression for the entries $s_{h,k}^{(n)}(x, y)$ of $\mathbf{S}_m^n(x, y)$. Namely, we get the relation

$$s_{h,k}^{(n)}(x, y) = \frac{2}{m+1}\sum_{j=1}^{m}\left(x+2y\cos\frac{j\pi}{m+1}\right)^n \sin\frac{jh\pi}{m+1}\sin\frac{jk\pi}{m+1} \quad (1 \leq h, k \leq m) \tag{2.3}$$

which, in the case $m = 4$ that is of interest to us, becomes

$$s_{h,k}^{(n)}(x, y) = \frac{2}{5}\sum_{j=1}^{4}\left(x+2y\cos\frac{j\pi}{5}\right)^n \sin\frac{jh\pi}{5}\sin\frac{jk\pi}{5} \quad (1 \leq h, k \leq 4). \tag{2.4}$$

Remark 2: For integral values of x and y $(x+y \neq 0)$, the eigenvalues of $\mathbf{S}_4(x, y)$ are nonzero [see (2.2)]. Therefore, under this condition, (2.4) applies for negative values of n as well.

Formula (2.3) [and, in particular, its specialization (2.4)] will play a crucial role throughout the proofs of the results established in this paper. Since it comes from expression (2.1), which has been established in an unpublished paper, the reader might be interested in an alternative proof of (2.3). It can be obtained by induction on n. To do this, we need the following two trigonometrical identities, the proofs of which are omitted for the sake of brevity:

$$\sum_{j=1}^{m}\sin\frac{jh\pi}{m+1}\sin\frac{jk\pi}{m+1} = \frac{m+1}{2}\delta_{h,k}, \tag{2.5}$$

where $\delta_{h,k} = 1\ (0)$ if $h = (\neq)\ k$ is the Kronecker symbol;

$$\sin(p-q)+\sin(p+q) = 2\sin p\cos q. \tag{2.6}$$

Proof of (2.3) (by induction on n): By virtue of (2.5), expression (2.3) clearly holds for $n = 0$. For $n = 1$, from identities (2.5) and (2.6) it is not hard to see that (2.3) holds as well. Suppose it holds for a certain $n > 1$.

Case 1: $k \neq 1, m$. For the inductive step $n \to n+1$, put $\sin[jr\pi/(m+1)] = s(j, r)$ and $\cos[j\pi/(m+1)] = c(j)$ for notational convenience, and use (1.1) and the inductive hypothesis to write

$$\begin{aligned}
s_{h,k}^{(n+1)} &= xs_{h,k}^{(n)} + y(s_{h,k-1}^{(n)} + s_{h,k+1}^{(n)}) \quad \text{(by the usual matrix multiplication rule)} \\
&= \frac{2}{m+1}\left\{x\sum_{j=1}^{m}\lambda_j^n s(j, h)s(j, k) + y\sum_{j=1}^{m}\lambda_j^n s(j, h)[s(j, k-1)+s(j, k+1)]\right\} \\
&= \frac{2}{m+1}\sum_{j=1}^{m}[x\lambda_j^n s(j, h)s(j, k) + 2y\lambda_j^n s(j, h)s(j, k)c(j)] \quad \text{[from (2.6)]} \\
&= \frac{2}{m+1}\sum_{j=1}^{m}\lambda_j^n[x+2yc(j)]s(j, h)s(j, k) \\
&= \frac{2}{m+1}\sum_{j=1}^{m}\lambda_j^{n+1}s(j, h)s(j, k) \quad \text{[from (2.2)]}.
\end{aligned}$$

Case 2: $k = 1$ or m. The proof of this case can be carried out by means of arguments similar to (but much simpler than) those of Case 1, and is omitted for brevity. Q.E.D.

Observe that, since $s_{h,k}^{(1)}(x, y) = x\delta_{h,k} + y\delta_{|h-k|,1}$, letting $n = 1$ in (2.3) and using (2.5) yields the noteworthy trigonometrical identity

$$\sum_{j=1}^{m} \cos\frac{j\pi}{m+1} \sin\frac{jh\pi}{m+1} \sin\frac{jk\pi}{m+1} = \frac{m+1}{4}\delta_{|h-k|,1}. \tag{2.7}$$

3. SOME FIBONACCI MATRICES

In this section, some couples (x, y) for which $\mathbf{S}_4(x, y)$ is a Fibonacci matrix are shown, and closed-form expressions for the entries $s_{h,k}^{(n)}(x, y)$ of $\mathbf{S}_4^n(x, y)$ are established. Of course, since $s_{h,k}^{(n)}(px, py) = p^n s_{h,k}^{(n)}(x, y)$, only *coprime* values of x and y are considered. Further, since it can be proved that the above entries enjoy the symmetry properties

$$\begin{aligned} s_{11} &= s_{44}, \\ s_{12} &= s_{21} = s_{34} = s_{43}, \\ s_{13} &= s_{31} = s_{24} = s_{42}, \\ s_{14} &= s_{41}, \\ s_{22} &= s_{33}, \\ s_{23} &= s_{32}, \end{aligned} \tag{3.1}$$

[here, $s_{h,k}$ stands for $s_{h,k}^{(n)}(x, y)$], expressions will be given only for $s_{1,k}$ $(1 \le k > 4)$, s_{22}, and s_{23}. For the sake of brevity, only a few among these results will be proved in detail by using relation (2.4). On the other hand, once the results have been established, induction on n and some usual Fibonacci identities may provide alternative (even though more tedious) proofs for all of them.

3.1 The Matrix $\mathbf{S}_4(0, 1)$

$$s_{11}^{(n)}(0, 1) = F_{n-1}[1 + (-1)^n]/2, \tag{3.2}$$

$$s_{12}^{(n)}(0, 1) = F_n[1 - (-1)^n]/2, \tag{3.3}$$

$$s_{13}^{(n)}(0, 1) = F_n[1 + (-1)^n]/2, \tag{3.4}$$

$$s_{14}^{(n)}(0, 1) = F_{n-1}[1 - (-1)^n]/2, \tag{3.5}$$

$$s_{22}^{(n)}(0, 1) = F_{n+1}[1 + (-1)^n]/2, \tag{3.6}$$

$$s_{23}^{(n)}(0, 1) = F_{n+1}[1 - (-1)^n]/2. \tag{3.7}$$

From (3.2)-(3.7), it is immediately seen that

$$\begin{cases} s_{11}^{(n)}(0, 1) + s_{14}^{(n)}(0, 1) = F_{n-1}, \\ s_{12}^{(n)}(0, 1) + s_{13}^{(n)}(0, 1) = F_n, \\ s_{22}^{(n)}(0, 1) + s_{23}^{(n)}(0, 1) = F_{n+1}, \end{cases} \tag{3.8}$$

and

$$\mathrm{Tr}[\mathbf{S}_4^n(0, 1)] = L_n[1 + (-1)^n] \quad \text{[from (3.1)]}, \tag{3.9}$$

where the trace Tr(**M**) of any square matrix **M** is the sum of its diagonal entries (or that of its eigenvalues).

Proof of (3.2): From (2.4), write

$$s_{11}^{(n)}(0,1)=\frac{2}{5}\sum_{j=1}^{4}\left(2\cos\frac{j\pi}{5}\right)^{n}\sin^2\frac{j\pi}{5}, \tag{3.10}$$

and observe that $2\cos(j\pi/5)=\alpha,-\beta,\beta$, and $-\alpha$ for $j=1,2,3$, and 4, respectively, where $\alpha=1-\beta=(1+\sqrt{5})/2$. Moreover, observe that $\sin^2(j\pi/5)=(1+\beta^2)/4$ (for $j=1$ and 4) and $(1+\alpha^2)/4$ (for $j=2$ and 3), so that, by using the Binet forms for Fibonacci and Lucas numbers and the property $\alpha\beta=-1$, (3.10) can be rewritten as

$$\begin{aligned} s_{11}^{(n)}(0,1)&=\frac{1}{10}[\alpha^n+\alpha^{n-2}+(-1)^n(\beta^n+\beta^{n-2})+\beta^n+\beta^{n-2}+(-1)^n(\alpha^n+\alpha^{n-2})]\\ &=\begin{cases}(L_n+L_{n-2})/5=F_{n-1} & (n \text{ even}),\\ 0 & (n \text{ odd}).\end{cases}\end{aligned}$$

The proofs of (3.3)-(3.7) can be carried out by means of analogous arguments.

3.2 The Matrix $S_4(1,1)$

$$s_{11}^{(n)}(1,1)=(F_{2n-1}+F_{n+1})/2, \tag{3.11}$$
$$s_{12}^{(n)}(1,1)=(F_{2n}+F_{n})/2, \tag{3.12}$$
$$s_{13}^{(n)}(1,1)=(F_{2n}-F_{n})/2, \tag{3.13}$$
$$s_{14}^{(n)}(1,1)=(F_{2n-1}-F_{n+1})/2, \tag{3.14}$$
$$s_{22}^{(n)}(1,1)=(F_{2n+1}+F_{n-1})/2, \tag{3.15}$$
$$s_{23}^{(n)}(1,1)=(F_{2n+1}-F_{n-1})/2. \tag{3.16}$$

From (3.11)-(3.16), it is immediately seen that

$$\begin{cases} s_{11}^{(n)}(1,1)+s_{14}^{(n)}(1,1)=F_{2n-1},\\ s_{12}^{(n)}(1,1)+s_{13}^{(n)}(1,1)=F_{2n},\\ s_{22}^{(n)}(1,1)+s_{23}^{(n)}(1,1)=F_{2n+1},\end{cases} \tag{3.17}$$

and

$$\mathrm{Tr}[\mathbf{S}_4^n(1,1)]=L_n+L_{2n} \quad [\text{from } (3.1)]. \tag{3.18}$$

Proof of (3.14): From (2.4), write

$$s_{14}^{(n)}(1,1)=\frac{2}{5}\sum_{j=1}^{4}\left(1+2\cos\frac{j\pi}{5}\right)^{n}\sin\frac{j\pi}{5}\sin\frac{4j\pi}{5}, \tag{3.19}$$

and observe that $1+2\cos(j\pi/5)=\alpha^2,\alpha,\beta^2$, and β for $j=1,2,3$, and 4, respectively. Moreover, observe that $\sin(j\pi/5)\sin(4j\pi/5)$ = $(\beta^2+1)/4$, $-(\alpha^2+1)/4$, $(\alpha^2+1)/4$, and $-(\beta^2+1)/4$ for $j=1,2,3$, and 4, respectively. Consequently, by using the Binet form for Lucas numbers and the property $\alpha\beta=-1$, (3.19) can be rewritten as

$$s_{14}^{(n)}(1,1)=\frac{1}{10}(\alpha^{2n-2}+\alpha^{2n}-\alpha^{n+2}-\alpha^n+\beta^{2n-2}+\beta^{2n}-\beta^{n+2}-\beta^n)=$$

$$= \frac{1}{10}(L_{2n-2} + L_{2n} - L_{n+2} - L_n)$$
$$= \frac{1}{10}(5F_{2n-1} + 5F_{n+1}) = (F_{2n-1} + F_{n+1})/2.$$

The proofs of (3.11)-(3.13) and (3.15)-(3.16) can be carried out by means of analogous arguments.

3.3 The Matrix $S_4(1, 2)$

$$s_{11}^{(n)}(1,2) = (F_{3n-1} + 5^{\lfloor n/2 \rfloor})/2, \tag{3.20}$$
$$s_{12}^{(n)}(1,2) = \{F_{3n} + 5^{(n-1)/2}[1-(-1)^n]\}/2, \tag{3.21}$$
$$s_{13}^{(n)}(1,2) = \{F_{3n} - 5^{(n-1)/2}[1-(-1)^n]\}/2, \tag{3.22}$$
$$s_{14}^{(n)}(1,2) = (F_{3n-1} - 5^{\lfloor n/2 \rfloor})/2, \tag{3.23}$$
$$s_{22}^{(n)}(1,2) = (F_{3n+1} + (-1)^n 5^{\lfloor n/2 \rfloor})/2, \tag{3.24}$$
$$s_{23}^{(n)}(1,2) = (F_{3n+1} - (-1)^n 5^{\lfloor n/2 \rfloor})/2, \tag{3.25}$$

where the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function.

From (3.20)-(3.25), it is immediately seen that

$$\begin{cases} s_{11}^{(n)}(1,2) + s_{14}^{(n)}(1,2) = F_{3n-1}, \\ s_{12}^{(n)}(1,2) + s_{13}^{(n)}(1,2) = F_{3n}, \\ s_{22}^{(n)}(1,2) + s_{23}^{(n)}(1,2) = F_{3n+1}, \end{cases} \tag{3.26}$$

and

$$\mathrm{Tr}[\mathbf{S}_4^n(1,2)] = L_{3n} + 5^{n/2}[1+(-1)^n] \quad [\text{from (3.1)}]. \tag{3.27}$$

As an example application of Remark 2, we provide the expression for the entry $s_{11}^{(-n)}(1,2)$ of $\mathbf{S}_4^{-n}(1,2)$. Namely, we have

$$s_{11}^{(-n)}(1,2) = [(-1)^n F_{3n+1} + 5^{-\lfloor (n+1)/2 \rfloor}]/2. \tag{3.28}$$

The proof of (3.28) is left as an exercise for the interested reader.

Proof of (3.21): From (2.4), write

$$s_{12}^{(n)}(1,2) = \frac{2}{5}\sum_{j=1}^{4}\left(1+4\cos\frac{j\pi}{5}\right)^n \sin\frac{j\pi}{5}\sin\frac{2j\pi}{5}, \tag{3.29}$$

and observe that $1+4\cos(j\pi/5) = \alpha^3, \sqrt{5}, \beta^3$, and $-\sqrt{5}$ for $j = 1, 2, 3$, and 4, respectively. Moreover, observe that $\sin(j\pi/5)\sin(2j\pi/5) = \sqrt{5}/4$ (for $j = 1$ and 2) and $-\sqrt{5}/4$ (for $j = 3$ and 4), so that, by using the Binet form for Fibonacci numbers, (3.29) can be rewritten as

$$s_{12}^{(n)}(1,2) = \frac{1}{10}\{\sqrt{5}(\alpha^{3n} - \beta^{3n}) + \sqrt{5}(\sqrt{5})^n[1-(-1)^n]\}$$
$$= \begin{cases} F_{3n}/2 & (n \text{ even}), \\ F_{3n}/2 + 5^{(n-1)/2} & (n \text{ odd}). \end{cases}$$

The proofs of (3.20) and (3.22)-(3.25) can be carried out by means of analogous arguments.

3.4 The Matrix $S_4(F_s, F_{s+1})$

The expressions (3.8), (3.17), and (3.26) can be generalized as follows.

Proposition 1: If s is any integer, then

$$\begin{cases} s_{11}^{(n)}(F_s, F_{s+1}) + s_{14}^{(n)}(F_s, F_{s+1}) = F_{(s+1)n-1}, \\ s_{12}^{(n)}(F_s, F_{s+1}) + s_{13}^{(n)}(F_s, F_{s+1}) = F_{(s+1)n}, \\ s_{22}^{(n)}(F_s, F_{s+1}) + s_{23}^{(n)}(F_s, F_{s+1}) = F_{(s+1)n+1}. \end{cases} \tag{3.30}$$

Proof: To save space, we shall confine ourselves to proving only the second identity of (3.30). The proofs of the first and third identities can be obtained in a similar manner.

First, from (2.4), write

$$s_{1,k}^{(n)}(F_s, F_{s+1}) = \frac{2}{5}\sum_{j=1}^{4}\left(F_s + 2F_{s+1}\cos\frac{j\pi}{5}\right)^n \sin\frac{j\pi}{5}\sin\frac{kj\pi}{5}. \tag{3.31}$$

Then, write down the following chart:

j	$\left(F_s + 2F_{s+1}\cos\frac{j\pi}{5}\right)^n$	$\sin\frac{j\pi}{5}\sin\frac{2j\pi}{5}$	$\sin\frac{j\pi}{5}\sin\frac{3j\pi}{5}$
1	$(F_s + \alpha F_{s+1})^n = \alpha^{(s+1)n}$	$\sqrt{5}/4$	$\sqrt{5}/4$
2	$(F_s - \beta F_{s+1})^n$	$\sqrt{5}/4$	$-\sqrt{5}/4$
3	$(F_s + \beta F_{s+1})^n = \beta^{(s+1)n}$	$-\sqrt{5}/4$	$-\sqrt{5}/4$
4	$(F_s - \alpha F_{s+1})^n$	$-\sqrt{5}/4$	$\sqrt{5}/4$

Finally, put $k = 1$ and 2 in (3.31) and use the chart to obtain

$$\begin{aligned} & s_{12}^{(n)}(F_s, F_{s+1}) + s_{13}^{(n)}(F_s, F_{s+1}) \\ &= \frac{1}{2\sqrt{5}}[\alpha^{(s+1)n} - \beta^{(s+1)n} + (F_s - \beta F_{s+1})^n - (F_s - \alpha F_{s+1})^n] \\ &+ \frac{1}{2\sqrt{5}}[\alpha^{(s+1)n} - \beta^{(s+1)n} - (F_s - \beta F_{s+1})^n + (F_s - \alpha F_{s+1})^n] = F_{(s+1)n}. \end{aligned}$$

3.5 Miscellany

The identities established in Subsections 3.1-3.4 represent only a small sample of the possibilities available to us. As a minor example, we leave the proofs of the following results to the interested reader:

$$s_{22}^{(n)}(2, 1) = (5^{\lfloor n/2 \rfloor} W_{n+1} + F_{2n-1})/2, \tag{3.32}$$

where W stands for F (n even) or L (n odd);

$$s_{h,k}^{(n)}(1, -1) = (-1)^{h+k} s_{h,k}^{(n)}(1, 1); \tag{3.33}$$

$$\mathrm{Tr}[S_4^n(x, 1)] = 2\sum_{j=0}^{\lfloor n/2 \rfloor}\binom{n}{2j}x^{n-2j}L_{2j} \tag{3.34}$$

which, for $x = 0$, reduces to (3.9) by virtue of the usual assumption $0^z = \delta_{z,0}$ $(z \ge 0)$.

4. APPLICATION EXAMPLES

In this section some Fibonacci matrices $\mathbf{S}_4(x,y)$ are used jointly with certain matrix expansions to get Fibonacci and combinatorial relations. Their novelty may be questioned; nevertheless, our aim here is to illustrate some ways of using the results presented in Section 3.

Example 1: Consider the matrix inverse expansion (see [5, p. 113])

$$(\mathbf{I}-\mathbf{A})^{-1}=\sum_{n=0}^{\infty}\mathbf{A}^n \quad (|\lambda|<1;\ \lambda,\text{ any eigenvalue of }\mathbf{A}), \tag{4.1}$$

and put $\mathbf{A}=\frac{1}{2}\mathbf{S}_4(0,1)$ $[$whence $\mathbf{I}_4-\mathbf{A}=\frac{1}{2}\mathbf{S}_4(2,-1)]$ in (4.1), thus getting the relation

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\mathbf{S}_4^n(0,1)=\mathbf{S}_4^{-1}(2,-1). \tag{4.2}$$

Then, from (2.4) and Remark 2, write

$$\begin{aligned} s_{12}^{(-1)}(2,-1)&=\frac{2}{5}\sum_{j=1}^{4}\left(2-2\cos\frac{j\pi}{5}\right)^{-1}\sin\frac{j\pi}{5}\sin\frac{2j\pi}{5}\\ &=\frac{1}{2\sqrt{5}}[\beta^{-2}+(\beta^2+1)^{-1}-\alpha^{-2}-(\alpha^2+1)^{-1}]=\frac{3}{5}. \end{aligned} \tag{4.3}$$

Finally, use (4.2), (3.3), and (4.3) to obtain the relation

$$\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}F_{2n+1}=\frac{6}{5}. \tag{4.4}$$

Example 2: Consider the matrix logarithm expansion (see [5, p. 113])

$$\ln\mathbf{A}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\mathbf{A}-\mathbf{I})^n}{n} \quad (|\lambda-1|<1;\ \lambda,\text{ any eigenvalue of }\mathbf{A}), \tag{4.5}$$

and put $\mathbf{A}=\mathbf{S}_4^2(0,1/\sqrt{2})$ in (4.5), thus getting the relation

$$\ln\mathbf{S}_4^2(0,1/\sqrt{2})=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}[\mathbf{S}_4^2(0,1/\sqrt{2})-\mathbf{I}_4]^n. \tag{4.6}$$

First, find the upper-left entry l_{11} of the matrix on the left-hand side of (4.6). The eigenvalues μ_j of $\mathbf{S}_4(0,1/\sqrt{2})$ are $\mu_1=\mu_4=\alpha/\sqrt{2}$ and $\mu_2=\mu_3=\beta/\sqrt{2}$, so that, from (2.3) and Remark 1, we get

$$\begin{aligned} l_{11}&=\frac{2}{5}\sum_{j=1}^{4}\ln\mu_j^2\sin^2\frac{j\pi}{5}=\frac{2}{5}\left[\frac{1+\beta^2}{2}\ln\frac{\alpha^2}{2}+\frac{1+\alpha^2}{2}\ln\frac{\beta^2}{2}\right]\\ &=\frac{1}{5}[(1+\beta^2)\ln\alpha^2+(1+\alpha^2)\ln\beta^2-5\ln 2]=\frac{1}{5}[\beta^2\ln\alpha^2+\alpha^2\ln\beta^2-5\ln 2]\\ &=\frac{1}{5}[(\beta^2-\alpha^2)\ln\alpha^2-5\ln 2]=-\frac{2}{\sqrt{5}}\ln\alpha-\ln 2. \end{aligned} \tag{4.7}$$

Then, find the upper-left entry $t_{11}^{(n)}$ of $[\mathbf{S}_4^2(0,1/\sqrt{2})-\mathbf{I}_4]^n$. The eigenvalues ξ_j of $\mathbf{S}_4^2(0,1/\sqrt{2})$ $-\,\mathbf{I}_4$ are $\xi_1=\xi_4=\mu_1^2-1$ and $\xi_2=\xi_3=\mu_2^2-1$, so that, from (2.4) and Remark 1, we can write

$$\begin{aligned} t_{11}^{(n)} &= \frac{2}{5}\sum_{j=1}^{4}\xi_j^n \sin^2\frac{j\pi}{5} = \frac{1}{5}\left[\left(\frac{\alpha^2}{2}-1\right)^n(1+\beta^2)+\left(\frac{\beta^2}{2}-1\right)^n(1+\alpha^2)\right] \\ &= \frac{1}{5}\left[\left(-\frac{\beta}{2}\right)^n(1+\beta^2)+\left(-\frac{\alpha}{2}\right)^n(1+\alpha^2)\right] \\ &= \frac{(-1)^n}{2^n 5}(L_n+L_{n-2}) = \frac{(-1)^n}{2^n}F_{n+1}. \end{aligned} \tag{4.8}$$

Finally, use (4.6)-(4.8) to obtain the relation

$$\sum_{n=1}^{\infty}\frac{F_{n+1}}{n2^n} = \frac{2}{\sqrt{5}}\ln\alpha + \ln 2. \tag{4.9}$$

Example 3: Since $\mathbf{S}_4(2,1) = \mathbf{S}_4(0,1)+\mathbf{S}_4(2,0)$, from (1.4) of [4] write

$$\mathbf{S}_4^n(2,1) = \sum_{j=0}^{n}\binom{n}{j}\mathbf{S}_4^j(0,1)\mathbf{S}_4^{n-j}(2,0), \tag{4.10}$$

whence, from (3.6) and (3.32), one obtains the relation

$$\sum_{j=0}^{n}\binom{n}{j}F_{j+1}\frac{1+(-1)^j}{2}2^{n-j} = (5^{\lfloor n/2\rfloor}W_{n+1}+F_{2n-1})/2$$

[where W stands for F (n even) or L (n odd)], which can be rewritten as (cf. identities I_{41} and I_{42} of [6])

$$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}2^{n-2j+1}F_{2j+1} = 5^{\lfloor n/2\rfloor}W_{n+1}+F_{2n-1}. \tag{4.11}$$

Example 4: Put $\mathbf{A}=\mathbf{B}=\mathbf{S}_4(0,1)$ in (1.5) of [4], and write

$$2\mathbf{S}_4^n(0,1) = \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}\mathbf{S}_4^{2j}(0,1)\mathbf{S}_4^{n-2j}(0,2) \quad (n\ge 1). \tag{4.12}$$

First, observe that the upper-left entry of the matrix on the left-hand side of (4.12) is

$$2s_{11}^{(n)}(0,1) = F_{n-1}[1+(-1)^n] \quad \text{[from (3.2)]}. \tag{4.13}$$

Then, use (3.2)-(3.5), (3.1), and the usual matrix multiplication rules to write the upper-left entry u_{11} of $\mathbf{S}_4^{2j}(0,1)\mathbf{S}_4^{n-2j}(0,2)$ as

$$\begin{aligned} u_{11} &= 2^{n-2j-1}[1+(-1)^n](F_{2j-1}F_{n-2j-1}+F_{2j}F_{n-2j}) \\ &= 2^{n-2j-1}F_{n-1}[1+(-1)^n] \quad \text{(from identity } I_{26} \text{ of [6])}. \end{aligned} \tag{4.14}$$

Finally, use (4.12)-(4.14) to obtain the combinatorial identity

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}2^{n-2j-1} = 1 \quad (n\ge 1). \tag{4.15}$$

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COUPLED SEQUENCES OF GENERALIZED FIBONACCI TREES AND UNEQUAL COSTS CODING PROBLEMS

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INTRODUCTION AND BACKGROUND

Fibonacci trees and generalized Fibonacci trees have been defined and studied by Horibe [4], Chang [2], and the author [1]. The k^{th} tree in the sequence of r-ary generalized Fibonacci trees, $T(k)$, has $T(k-c(i))$ as the i^{th} leftmost subtree descending from its root node for $k>r$, and $T(k)$ consists of a single root node for $k=1,\dots,r$. Here $c(i)$, $i=1,\dots,r$, are positive integers with greatest common divisor 1 and are nondecreasing in i. In the case in which $r=2$, $c(1)=1$, $c(2)=2$, the generalized Fibonacci trees are the Fibonacci trees of Horibe [4].

In addition to the construction of generalized Fibonacci trees by the recursive specification of their subtrees, there is an equivalent construction by the method of "types." The two constructions can be seen to be equivalent by induction. In the method of types, each leaf node is assigned one of $c(r)$ "types" denoted by $a_1, a_2, \dots, a_{c(r)}$. Then $T(k+1)$ is constructed from $T(k)$ according to the following set of rules. A leaf node of type a_1 in $T(k)$ will be replaced by r descendant nodes of types $a_{c(1)}$, $a_{c(2)}, \dots, a_{c(r)}$ in left to right order in $T(k+1)$. A leaf node of type a_j in $T(k)$ will be replaced by a node of type a_{j-1} in $T(k+1)$, $j=2,\dots,c(r)$. The sequence of trees begins with $T(1)$ which consists of a single root node of type $a_{c(r)}$.

The construction by leaf node type has an interpretation in connection with Varn's algorithm for the solution of a particular unequal costs coding problem [5]. Thus, the recursive subtree method also generates Varn's code trees. In the coding problem, a path from the root to a leaf describes a codeword, a sequence of r-ary symbols used to represent the source symbol assigned to the leaf. It is assumed that the code trees are exhaustive, that is, every interior node has exactly r descendants. In the case that the i^{th} code symbol, $i=1,\dots,r$, costs $c(i)$, the generalized Fibonacci tree minimizes the average codeword cost for equally likely source symbols when the number of leaves in the generalized Fibonacci tree is the same as the number of source symbols. In Varn's algorithm for the optimal code tree, leaf nodes of least cost, say c, in an optimal tree for a given number of leaves are replaced by r descendant leaf nodes of cost $c+c(i)$, $i=1,\dots,r$, in left to right order in generating the optimal tree for the appropriate larger number of leaves; the sequence begins with a single root node of cost 0. The correspondence between Varn's algorithm and the construction of generalized Fibonacci trees by the method of types is immediate because the method of types is exactly a mechanism for keeping track of each leaf node until it is a node of least cost in Varn's sense. Easy recurrence relations for the resulting number of leaves and average code cost can be obtained through the recursive subtrees perspective.

In this paper, a further generalization of Fibonacci trees is examined: the case of multiple coupled, recursively-generated sequences of trees. These sequences of trees have interesting structure and, under certain conditions, can be interpreted as optimal code trees for a generalization of Varn's unequal costs coding problem. One arbitrarily selected example will be considered in detail. The general case can be treated in an obviously similar fashion; however, this is not done here in order to avoid notational complexities.

EXAMPLE OF COUPLED SEQUENCES OF RECURSIVELY-GENERATED TREES

Consider the particular example of four coupled recursively-generated sequences of binary trees, $T(k)$, $U(k)$, $V(k)$, and $W(k)$, $k=1,2,\dots$, defined as follows. Let $T(k)$ have $U(k-1)$ as its leftmost subtree and $V(k-2)$ as its rightmost subtree, $k>4$, and denote this by $T(k)=U(k-1)^*V(k-2)$. Similarly, let $U(k)=W(k-2)^*V(k-2)$, $V(k)=U(k-1)^*V(k-4)$, and $W(k)=W(k-3)^*V(k-2)$. Initialize by letting $T(k)$, $U(k)$, $V(k)$, and $W(k)$, $k=1,\dots,4$, consist of single root nodes. (More generally, the number of coupled sequences need not be 4 but rather any positive integer, the trees need not be binary but rather r-ary for any integer $r\geq 2$ fixed for all sequences, the positive integer lags can be set arbitrarily, and the assignment of trees from various of the sequences as subtrees in the same or other sequences can be made arbitrarily. The largest lag value is the number of single root node trees used to initialize the sequences. In some cases, some sets of subsequences will not be coupled with others.)

The same sequences of trees can also be constructed by a method based on the notion of node type. Start with $T(1)$ given by a single root node of type a_4, $U(1)$ a single root node of b_4, $V(1)$ a single root node of type c_4, and $W(1)$ a single root node of type d_4. (More generally, the largest index value and the index of the types of nodes used to initialize the sequences is the largest lag value.) A leaf node of type x_j, $x=a,b,c,$ or d, in $X(k)$, $X=T,U,V,$ or W, will be replaced by a node of type x_{j-1} in $X(k+1)$ if $j=2,3,4$. Denote this by $a_2\sim a_1$, and so on. A node of type a_1 in $X(k)$ will be replaced by two descendant nodes of types b_1 and c_2 in left to right order in $X(k+1)$. Denote this by $a_1\sim(b_1+c_2)$. Similarly, let $b_1\sim(d_2+c_2)$, $c_1\sim(b_1+c_4)$, and $d_1\sim(d_3+c_2)$. (In general, the substitution rules correspond to the subtree recursions with one type symbol x identified with each tree sequence X and indices subscripted to x identical to lags in the argument of X. There is one substitution rule involving $\sim$ for each subtree recursion involving *.)

The equivalence of these two sets of sequences of trees can be verified by induction as in the case of a single sequence of trees treated previously in the literature. The set of sequences is given in part in Table 1. The trees are described using the following compact notation. Sibling nodes in left to right order are separated by + signs, and parentheses are used to indicate depth in the tree from the root so that, for example, $(((d_3+c_2)+(b_1+c_4))+((d_2+c_2)+c_3))$ denotes a binary tree with 6 depth 3 leaves and 1 depth 2 leaf from left to right, with the left subtree made up of 4 depth 2 leaves and the right subtree made up of 2 depth 2 leaves and 1 depth 1 leaf. The leaves are labeled by type in left to right order as $d_3, c_2, b_1, c_4, d_2, c_2, c_3$, respectively. Using this notation for the trees, each line of the table comes from the previous line according to the substitution rules given, with $\sim$ used to denote substitution, together with $+\sim+$ and $()\sim()$.

Costs can be assigned to the leaf nodes of these trees in such a way that nodes of types x_1 are of equal cost and of least cost in a tree of index k. Thus, the substitution rules can be interpreted as tracking the leaf nodes by type so that their indices indicate their relative costs until they are of least cost and about to be replaced by their descendants. This is like Varn's algorithm for constructing a code tree, although we do not yet have any coding problem in mind for which the resulting tree is a minimum-average-cost solution. Assign costs, based obviously on the substitution rules as follows: When a node of type a_1, which costs c_a, splits, its descendants cost c_a+1 and c_a+2 in left to right order. When a node of type b_1, which costs c_b, splits, its descendants cost c_b+2 and c_b+2. When a node of type c_1, which costs c_c, splits, its descendants cost c_c+1 and c_c+4. When a node of type d_1, which costs c_d, splits, its descendants cost c_d+3 and c_d+2. The process starts with root nodes of type x_4, which cost 0, and if a node of type x_j, j = 1, 2, or 3, is not created by a split, its cost in $X(k+1)$ is the same as the cost of x_{j+1} in $X(k)$. The trees of the example are given in part in Table 2 with leaf nodes labeled by cost.

TABLE 1. Trees of Example, Leaves Labeled by Type

k	$T(k)$	$U(k)$
1	a_4	b_4
2	a_3	b_3
3	a_2	b_2
4	a_1	b_1
5	(b_1+c_2)	(d_2+c_2)
6	$((d_2+c_2)+c_1)$	(d_1+c_1)
7	$((d_1+c_1)+(b_1+c_4))$	$((d_3+c_2)+(b_1+c_4))$
8	$(((d_3+c_2)+(b_1+c_4))+((d_2+c_2)+c_3))$	$((d_2+c_1)+((d_2+c_2)+c_3))$
9	$(((d_2+c_1)+((d_2+c_2)+c_3))+((d_1+c_1)+c_2))$	$((d_1+(b_1+c_4))+((d_1+c_1)+c_2))$
10	$(((d_1+(b_1+c_4))+((d_1+c_1)+c_2))$ $+(((d_3+c_2)+(b_1+c_4))+c_1))$	$(((d_3+c_2)+((d_2+c_2)+c_3))$ $+(((d_3+c_2)+(b_1+c_4))+c_1))$
...	...	...

We can find expressions for the average costs of the trees from recurrence relations on the number of leaves of each type. First, let $e_{xj}(k)$ denote the number of leaves of type x_j in $T(k)$ for $x=a,b,c,$ or d and $j=1,2,3,$ or 4. Initialize with $e_{a4}=1$ and, except for this case, $e_{xj}(k)=0$ for $k=1,2,3,4$. Then we have, for $k>4$:

$$
\begin{aligned}
&e_{a4}(k)=0\\
&e_{a3}(k)=e_{a4}(k-1)\\
&e_{a2}(k)=e_{a3}(k-1)\\
&e_{a1}(k)=e_{a2}(k-1)\\
&e_{b4}(k)=0\\
&e_{b3}(k)=e_{b4}(k-1)\\
&e_{b2}(k)=e_{b3}(k-1)\\
&e_{b1}(k)=e_{a1}(k-1)+e_{b2}(k-1)+e_{c1}(k-1)\\
&e_{c4}(k)=e_{c1}(k-1)
\end{aligned}
$$

$$
\begin{aligned}
&e_{c3}(k) = e_{c4}(k-1)\\
&e_{c2}(k) = e_{a1}(k-1) + e_{b1}(k-1) + e_{c3}(k-1) + e_{d1}(k-1)\\
&e_{c1}(k) = e_{c2}(k-1)\\
&e_{d4}(k) = 0\\
&e_{d3}(k) = e_{d1}(k-1) + e_{d4}(k-1)\\
&e_{d2}(k) = e_{b1}(k-1) + e_{d3}(k-1)\\
&e_{d1}(k) = e_{d2}(k-1).
\end{aligned}
$$

We can obtain similar expressions for $f_{xj}(k)$, which denotes the corresponding quantities for $U(k)$, as well as $g_{xj}(k)$ for $V(k)$ and $h_{xj}(k)$ for $W(k)$; however, that will not be pursued here.

TABLE 2. Trees of Example, Leaves Labeled by Cost

k	$T(k)$	$U(k)$
1	0	0
2	0	0
3	0	0
4	0	0
5	$(1+2)$	$(2+2)$
6	$((3+3)+2)$	$(2+2)$
7	$((3+3)+(3+6))$	$((5+4)+(3+6))$
8	$(((6+5)+(4+7))+((5+5)+6))$	$((5+4)+((5+5)+6))$
9	$(((6+5)+((6+6)+7))+((5+5)+6))$	$((5+(5+8))+((5+5)+6))$
10	$(((6+(6+9))+((6+6)+7))$ $+(((8+7)+(6+9))+6))$	$(((8+7)+((7+7)+8))$ $+(((8+7)+(6+9))+6))$
...	...	...

By the method of generating functions, with $E_{xj}(z) = \sum e_{xj}(k)z^k$, we have:

$$
\begin{aligned}
&E_{a4}(z) = z^1\\
&E_{a3}(z) = z^2\\
&E_{a2}(z) = z^3\\
&E_{a1}(z) = z^4\\
&E_{b4}(z) = E_{b3}(z) = E_{b2}(z) = 0\\
&E_{b1}(z) = (z^5 + z^7 - z^8 - z^9 - z^{10} + z^{12}) / p(z) = z^5 + z^7 + z^8 + 2z^{10} + \dots\\
&E_{c4}(z) = (z^7 + z^8 - z^{11}) / p(z) = z^7 + z^8 + 2z^{10} + \dots\\
&E_{c3}(z) = (z^8 + z^9 - z^{12}) / p(z) = z^8 + z^9 + \dots\\
&E_{c2}(z) = (z^5 + z^6 - z^9) / p(z) = z^5 + z^6 + 2z^8 + 2z^9 + 2z^{10} + \dots\\
&E_{c1}(z) = (z^6 + z^7 - z^{10}) / p(z) = z^6 + z^7 + 2z^9 + 2z^{10} + \dots
\end{aligned}
$$

$$E_{d4}(z) = 0$$
$$E_{d3}(z) = (z^8 + z^{10} - z^{12}) / p(z) = z^8 + z^{10} + \dots$$
$$E_{d2}(z) = (z^6 + z^8 - z^{10}) / p(z) = z^6 + z^8 + 2z^9 + \dots$$
$$E_{d1}(z) = (z^7 + z^9 - z^{11}) / p(z) = z^7 + z^9 + 2z^{10} + \dots$$

where $p(z) = 1 - 2z^3 - z^4 - z^5 + z^6 + z^7$. The coefficients of z^k obtained from the right-hand sides of these expressions give $e_{xj}(k)$.

It follows from $E(z) = \sum E_{xj}(z)$ that

$$\begin{aligned} E(z) &= (z^1 + z^2 + z^3 - z^4 - z^5 - z^6 + z^7 + 3z^8 + z^9 - z^{11} - z^{12}) / p(z) \\ &= z^1 + z^2 + z^3 + z^4 + 2z^5 + 3z^6 + 4z^7 + 7z^8 + 8z^9 + 11z^{10} + \dots. \end{aligned}$$

Here $E(z)$ is the generating function of $e(k) = \sum e_{xj}(k)$, the number of leaves in $T(k)$. (Similarly, we can find the generating function of $f(k)$, the number of leaves in $U(k)$, to be

$$\begin{aligned} F(z) &= (z^1 + z^2 + z^3 - z^4 - z^5 - 2z^6 + z^7 + z^8 + z^9 - z^{11}) / p(z) \\ &= z^1 + z^2 + z^3 + z^4 + 2z^5 + 2z^6 + 4z^7 + 5z^8 + 6z^9 + 10z^{10} + \dots.) \end{aligned}$$

Inverting the expression for $E(z)$, we have

$$\begin{aligned} e(k) &= 2e(k-3) + e(k-4) + e(k-5) - e(k-6) - e(k-7) \\ &\quad + \delta(k-1) + \delta(k-2) + \delta(k-3) - \delta(k-4) - \delta(k-5) - \delta(k-6) \\ &\quad + \delta(k-7) + 3\delta(k-8) + \delta(k-9) - \delta(k-11) - \delta(k-12), \end{aligned}$$

where $\delta(k) = 1$ for $k = 0$ and 0 otherwise. Equivalently,

$$e(k) = 2e(k-3) + e(k-4) + e(k-5) - e(k-6) - e(k-7)$$

for $k > 12$, where $e(1) = 1$, $e(2) = 1$, $e(3) = 1$, $e(4) = 1$, $e(5) = 2$, $e(6) = 3$, $e(7) = 4$, $e(8) = 7$, $e(9) = 8$, $e(10) = 11$, $e(11) = 17$, $e(12) = 21$. The recurrence for $e(k)$ in terms of its own lagged values does not appear to illuminate the tree structure of $T(k)$.

Then observe that, for the example, in the tree $X(k)$, a leaf node of type x_j costs $k + j - (4+1)$ (and, more generally, this expression is $k + j - (\max\ \text{lag} + 1)$), as can be verified by induction. Thus, the average cost of $T(k)$, $C(T(k))$, is given by

$$C(T(k)) = \sum_{1 \le j \le 4} (k + j - 5)(e_{aj}(k) + e_{bj}(k) + e_{cj}(k) + e_{dj}(k)) / e(k),$$

and similarly for $C(U(k))$ in terms of the corresponding quantities with f replacing e, etc. The generating function for the unnormalized cost of $T(k)$ is then

$$z dE(z)/dz + \sum_{1 \le j \le 4} (j-5)(E_{aj}(z) + E_{bj}(z) + E_{cj}(z) + E_{dj}(z))$$

or

$$\begin{aligned} &(3z^5 + 8z^6 + 15z^7 + 26z^8 + 8z^9 - 6z^{10} - 40z^{11} - 38z^{12} - 18z^{13} \\ &\qquad + 13z^{14} + 28z^{15} + 17z^{16} + 5z^{17} - 9z^{18} - 5z^{19}) / p^2(z) \\ &= 3z^5 + 8z^6 + 15z^7 + 38z^8 + 46z^9 + 76z^{10} + \dots. \end{aligned}$$

Then $C(T(k))$ is given by the ratio of the coefficient of z^k in this expression to the coefficient of z^k in $E(z)$ (see Table 3).

TABLE 3. Costs of Example

k	$T(k)$			$U(k)$		
	$e(k)$	$C(T(k))$	Bound	$f(k)$	$C(U(k))$	Bound
5	2	1.5	10.5	2	2	10.5
6	3	2.7	10.7	2	2	10.5
7	4	3.8	10.8	4	4.5	10.8
8	7	5.4	11.1	5	5	10.9
9	8	5.8	11.1	6	5.7	11.0
10	11	6.9	11.2	10	7.3	11.2
...	...	...	...	...	...	...

CORRESPONDING CODING PROBLEM

The example trees do in fact minimize average cost for particular unequal costs coding problems, although this is not the case for all choices of multiple coupled recursively-generated sequences of trees. First, let us identify the coding problems for which the trees of Table 2 are code trees compatible with the cost structure of the coding problem. Then, let us verify that in this case the substitution rules are such that the code trees are, in addition, the minimum-average-cost code trees for the coding problem.

Each of the $T(k)$ trees is a finite subtree of an infinite tree with the specified cost structure. That infinite tree can be described by the finite state diagram of Figure 1 in which a path through the tree starts with state a (denoting a root node of type a_i for some i), branches left (labeled 0) with cost 1 into state b (denoting a node of type b_i for some i) or right (labeled 1) with cost 2 into state c. Similarly, a path through the diagram from b to c is a right branch in the infinite tree from a node of type b_i for some i to a node of type c_i for some i at a cost of 2, and so on.

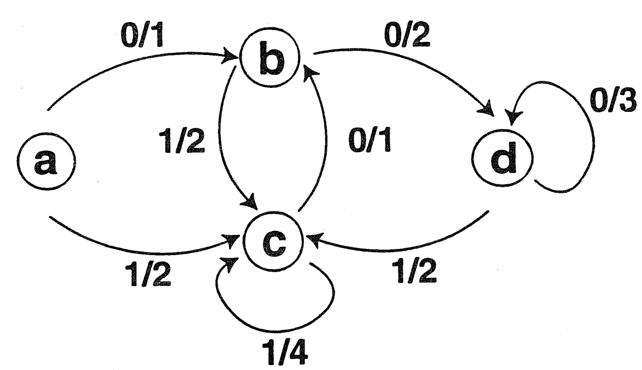

FIGURE 1. Finite State Diagram for Example

Every finite subtree of this infinite tree shares a common cost structure. We can read it off from the tree. In the case in which we start in state a, the first 0 in a row in a codeword costs 1, the second 0 in a row costs 2, the third and subsequent 0's each cost 3, the first 1 in a row in a codeword costs 2, and the second and subsequent 1's each cost 4. Loosely speaking, long runs of 0's or 1's are penalized, and 0's are generally less costly code symbols than 1's. The $T(k)$ trees share this cost structure.

Similarly, the $U(k)$ trees are finite subtrees of the infinite tree described by the finite state diagram of Figure 1 only starting in state b. For the infinite tree starting in b, the first symbol costs 2 whether it is 0 or 1. After that, if the first symbol is 0, subsequent 0's cost 3 until the first 1, which costs 2. From here on, or following the first symbol if it is 1, repeated 1's each cost 4 until the first 0, which costs 1 and restarts the cost rules. This is a slightly different formulation of the notion that long runs of like symbols are penalized in the codewords by the cost structure, with 0's begin generally less costly than 1's. Similarly, for $V(k)$ start in state c, and for $W(k)$ in d.

The coding problem is to find the finite subtrees of particular size that minimize the sum of the costs of the branches along the paths to the leaves. Varn's algorithm for finding the minimum-average-cost tree when the costs of each code symbol are constant throughout the tree, although they may differ from symbol to symbol, does not necessarily carry over to the more general cost structures discussed here. Rather, in general, the sequence of optimal trees can be obtained from the root node by successively creating the least costly substitution of descendant nodes for their parent. For each leaf node y in one tree in the sequence, say its cost is $c(y)$, compute the additional total cost by replacing it by its descendants, $(r-1)c(y)+c(y,1)+c(y,2)+\cdots+c(y,r)$, where $c(y,i)$ is the cost of the i^{th} leftmost branch descending from y in the cost structure, and select the least of these in constructing the next tree in the sequence. The trees constructed in this way will be the same as the trees constructed by splitting the least costly leaf node whenever $(r-1)c(y)+c(y,1)+c(y,2)+\cdots+c(y,r)$ is least whenever $c(y)$ is least.

One sufficient condition on the cost structure for splitting the least costly node to yield the optimal trees is for $c(y,1)+c(y,2)+\cdots+c(y,r)$ to equal either m or $m+1$ or $m+2$ or ... or $m+r-1$ for all nonroot y, where m is a positive integer. In this case $(r-1)c(y)+c(y,1)+c(y,2)+\cdots+c(y,r)$ is not strictly less than $(r-1)c(z)+c(z,1)+c(z,2)+\cdots+c(z,r)$ whenever $c(z)$ is minimum and y is any leaf node other than z. The argument has two parts and is immediate if $c(y)>c(z)$. If $c(y)=c(z)$, consider z to be a leaf node of minimal cost for which $c(z,1)+c(z,2)+\cdots+c(z,r)$ is also minimum. Split z and continue with the now reduced set of leaf nodes y such that $c(y)=c(z)$.

The example treated here satisfies this condition for $T(k), U(k), V(k)$, and $W(k)$, since all nonroot nodes are equivalent to either y_b, y_c, or y_d, where $c(y_b,1)+c(y_b,2)=2+2=4$, $c(y_c,1)+c(y_c,2)=1+4=5$, and $c(y_d,1)+c(y_d,2)=3+2=5$. Therefore, splitting the least costly leaf node at each stage generates the optimal tree, leading to the corresponding recursively-constructed sequence of trees from subtrees.

Clearly, there are many other examples of cost structures or finite state diagrams that also satisfy these conditions and many that do not. One example that does not is to let each 0 cost 1, the first 1 in a codeword cost 1, but each subsequent 1 in a row cost 10, thereby penalizing long runs of 1's only. Here, the splitting algorithm leads to a tree of 4 leaves given by $((2+2)+(2+11))$ while the minimum cost tree is $(((3+3)+2)+1)$.

PERFORMANCE ANALYSIS

An upper bound on the resulting expected cost, in the case that the trees solve a minimum-average-cost coding problem, can be obtained from the work of Csiszar [3] who provides a sub-optimal coding scheme for cost structures represented by finite state diagrams such as the one in Figure 1. Csiszar's method applies to arbitrary probability distributions on the source symbols. Because the coding scheme given here is optimal (for finite state cost structures satisfying the sufficient conditions) for the case of equally likely source symbols, Csiszar's upper bound when specialized to the uniform probability distribution on N leaves applies here as well, but only for binary trees. That is because his code trees are not necessarily exhaustive, and it is possible that the optimal exhaustive code is more costly than the suboptimal nonexhaustive code; however, for binary trees, all codes are exhaustive.

The upper bound on average cost for N equally likely source symbols when optimally binary coded is $(\log P^{-1})(\log N + \log Q) + R$, where P, Q, and R are obtained from the finite state diagram. The notation describing the finite state diagram used below generally follows [3] and should not be confused with the tree or cost notation used in the example. The base of the log is arbitrary and will be taken as 10 in the example for convenience.

For costs described by a finite state diagram, as in Figure 1, let us use the following notation. The set of code symbols is $Y = \{y_j\}$, $j \in J$, and in the example $Y = \{0, 1\}$. The set of states is $S = \{s_i\}$, $i \in I$, and in the example $S = \{a, b, c, d\}$. The function $F(i, j)$ specifies the new state $s_{F(i,j)}$ if the symbol y_j has been used at the state s_i, and in the example $F(a,0) = b$, $F(a,1) = c$, $F(b,0) = d$, $F(b,1) = c$, $F(c,0) = b$, $F(c,1) = c$, $F(d,0) = d$, $F(d,1) = c$. In $F(i, j)$, j takes on values in the set $J(i)$, and in the example $J(i) = J$ for all i. Also define $J_k(i)$ to be the set of symbols in $J(i)$ for which $F(i, j) = k$, and in the example $J_a(a) = J_a(b) = J_a(c) = J_a(d) = J_b(b) = J_b(d) = J_d(a) = J_d(c) = \varnothing$, $J_b(a) = J_b(c) = J_d(b) = J_d(d) = \{0\}$, $J_c(a) = J_c(b) = J_c(c) = J_c(d) = \{1\}$. The quantity t_{ij} is the cost of the symbol y_j if used at the state s_i, and in the example $t_{a0} = 1$, $t_{a1} = 2$, $t_{b0} = 2$, $t_{b1} = 2$, $t_{c0} = 1$, $t_{c1} = 4$, $t_{d0} = 3$, $t_{d1} = 2$.

R is defined as $\max_{I,J(i)} t_{ij}$, and in the example $R = 4$. P and Q are defined in terms of a matrix $A(w)$ whose entries a_{ik}, $i \in I$, $k \in I$, are given by $\sum_{J_k(i)} w^{-t_{ij}} - I_0$, where I_0 is the identity matrix, and in the example

$$A(w) = \begin{bmatrix} -1 & w^{-1} & w^{-2} & 0 \\ 0 & -1 & w^{-2} & w^{-2} \\ 0 & w^{-1} & w^{-4}-1 & 0 \\ 0 & 0 & w^{-2} & w^{-3}-1 \end{bmatrix}.$$

P is defined as the greatest positive root of the equation $\det A(w) = 0$, and in the example this equation is $p(z) = 0$ evaluated at $z = w^{-1}$ and $P = 1/0.73 = 1.37$. For the column vector a with entries a_i and the column vector 0 of all 0 entries, solve the matrix equation $A(P)a = 0$, and in the example $a_1 = c(2 + P^{-2} - P^{-4}) = 2.25c$, $a_2 = c(2 - P^{-4}) / P^{-1} = 2.35c$, $a_3 = c$, $a_4 = cP^{-2} / (2 - P^{-3}) = 0.33c$ for an arbitrary c. Q is defined as $\max a_i / a_k$, and in the example $Q = 2.35 / 0.33 = 7.12$.

Thus, for the example, expected cost is upper bounded by $7.3(\log_{10} N + 0.85) + 4$ as given in Table 3 above.

Also, note that in the general case for finite state costs described in this notation, the corresponding coupled tree sequences for $T_i(k)$, $i \in I$, $k > \max t_{ij}$, are $T_i(k) = *_{j \in J} T_{F(i,y_j)}(k - t_{ij})$ using $*_{j \in J}$ in the obvious sense. The general recurrence relations and generating functions can also be identified using this notation. Again, however, the focus here is on the specific, arbitrary, example in order to avoid the notational complexities of the general case.

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A PASCAL RHOMBUS

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1. INTRODUCTION

A generalization of Pascal's triangle can be defined using the following recurrence scheme. Given two rows of values, compute a new row by adding together the four numbers in the rhombus above the value to be computed. A sample rhombus is given in Figure 1. The value 16 is the sum of the four numbers above it in the rhombus configuration.

$$\begin{array}{c} 3 \\ 4\,5\,4 \\ 16 \end{array}$$

FIGURE 1. Sample Rhombus

In general, we shall start with one 1 in the first row and three 1's in the second row. The recurrence then determines the subsequent rows. The first few rows of the rhombus are given in Figure 2. We assume all blank positions are zero. So, for example, when calculating the second entry in the third row the two zeros are assumed to be up two places and up one and to the left. We call this array of numbers a *Pascal rhombus.*

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 1 & 1 & & & \\ & & 1 & 2 & 4 & 2 & 1 & & \\ & 1 & 3 & 8 & 9 & 8 & 3 & 1 & \\ 1 & 4 & 13 & 22 & 29 & 22 & 13 & 4 & 1 \end{array}$$

FIGURE 2. The First Five Rows of the Rhombus

This pattern generation scheme arose while studying a switch-setting problem [4], [5]. Given an n by m arrangement of switches, some on and some off, the goal is to achieve an all-off configuration of the switches. Many puzzles and computer games, such as "Button Madness" and

"Lights Out" are built using this idea. The operation available involves activating a particular switch, causing it and its rectilinearly adjacent neighbors to change states. Part of our method for solving the switch-setting problem involved the following: begin with an initial (row) vector containing one 1 and a second vector containing the three 1's under the initial vector's 1. We then "grew" new vectors by applying the rhombus rule recursively. Our work on the switches differed in two ways from the Pascal rhombus recurrence presented above. First, the rows in the switch problem are bounded by a certain fixed length and are not allowed to grow outward without bound on either the left or the right. Second, since the switches (in the simplest case) have only two states, all of the arithmetic is done modulo 2. Similar triangles have been studied in a number of papers; a thorough survey can be found in [2]. In particular, the generalized Pascal triangle of order 3 consists of the coefficients $\binom{n}{k}_3$ in the expansion of $(1+x+x^2)^n$ [8]. However, this generalized triangle of order 3 is defined by a recurrence where each value is the sum of three terms, whereas each term in the rhombus is the sum of four terms.

In Section 2 we discuss various properties of the rhombus, show that the rhombus's elements can be given using a family of monic polynomials, and analyze the row sums. In Section 3 we define a modified rhombus by not allowing the rhombus to grow to the left. We exhibit relationships between this *left-bounded* rhombus and Pascal's rhombus and introduce some graphs to help analyze the row sums of the left-bounded rhombus. In Section 4 we discuss an analogy between the left-bounded rhombus and the classic Pascal triangle. Finally, in Section 5 we discuss the coefficients in the rhombus modulo 2 and propose some directions for future work.

2. SOME PROPERTIES OF THE RHOMBUS

In this section we consider some properties of Pascal's rhombus. First, note that each row contains two more entries than the previous row, and each row is symmetric around the center column. Let $[n, k]$ represent the k^{th} value of the n^{th} row. The row numbering begins at 0 and the elements in a row also are numbered beginning at 0. We have $[0, 0] = 1$ and $[n, 0] = [n, 2n] = 1$ for all n. The rhombus then is indexed as follows:

$$\begin{array}{ccccccc} & & & [0,0] & & & \\ & & [1,0] & [1,1] & [1,2] & & \\ & [2,0] & [2,1] & [2,2] & [2,3] & [2,4] & \\ [3,0] & [3,1] & [3,2] & [3,3] & [3,4] & [3,5] & [3,6] \end{array}$$

The rhombus defining recurrence relation can be written as

$$[n+1, k] = [n, k] + [n, k-1] + [n, k-2] + [n-1, k-2]. \tag{1}$$

Letting $k = 1$ in (1) gives the following relationship for the second entry of each row:

$$[n+1, 1] = [n, 1] + [n, 0] = [n, 1] + 1. \tag{2}$$

Two of the terms are missing in (2) because $k-2$ is -1, and the rhombus's values for negative k are taken to be 0. It follows directly from (2) that $[n, 1] = n$ for all n. Writing down the recurrences for subsequent terms and solving them gives rise to the following formulas:

$$[n,2]=(n^2+3n-2)/2!$$
$$[n,3]=(n^3+9n^2-22n+12)/3!$$
$$[n,4]=(n^4+18n^3-49n^2+6n+48)/4!$$
$$[n,5]=(n^5+30n^4-45n^3-570n^2+1904n-1680)/5!$$
$$[n,6]=(n^6+45n^5+55n^4-2865n^3+12184n^2-18780n+8640)/6!$$

We state the general result below.

Theorem 1: $[n,k]$ is a polynomial in n of degree k, such that $k![n,k]$ is monic with integer coefficients.

Proof: First, rewrite (1) as $[n,k]-[n-1,k]=[n-1,k-1]+[n-1,k-2]+[n-2,k-2]$. Treat this as an identity in the variable n and constant k, and sum over n. The least value of n to use is the last nonzero entry in the appropriate diagonal. It can be written as $\lfloor (k+1)/2 \rfloor$ to account for parity of k. Then

$$[n,k]=\sum_{i=\lfloor\frac{k+1}{2}\rfloor}^{n}([i-1,k-1]+[i-1,k-2]+[i-2,k-2])+\left[\left\lfloor\frac{k+1}{2}\right\rfloor,k\right]$$

which, in turn, is equal to

$$\sum_{i=\lfloor\frac{k+1}{2}\rfloor}^{n}([i-1,k-1]+2[i-1,k-2])+\left[\left\lfloor\frac{k+1}{2}\right\rfloor-1,k-2\right]-[n-1,k-2]+\left[\left\lfloor\frac{k+1}{2}\right\rfloor,k\right].$$

The sequence of polynomials thus continues, with the general recurrence establishing by induction that $[n,k]$ is a polynomial in n of degree k, such that $k![n,k]$ is monic with integer coefficients. □

We next analyze the row sums of the rhombus.

Theorem 2: Let T_n be the sum of the elements in row n of the Pascal rhombus. Then

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\frac{(3+\sqrt{13})}{2}.$$

Proof: Using the recurrence in (1), we have that $T_{n+1}=3T_n+T_{n-1}$. Let us solve the recurrence by setting $T_{n+2}-3T_{n+1}-T_n=0$ and using the initial conditions $T_0=1$ and $T_1=3$. (Note: One may find it easier to solve by defining $T_{-1}=0$ and $T_1=1$.) Therefore, the characteristic equation is $r^2-3r-1=0$, which has two solutions: $r_1=(3+\text{sqrt}(13))/2$; $r_2=(3-\text{sqrt}(13))/2$. One can then easily determine that

$$T_n=\frac{1}{\sqrt{13}}\left(\left(\frac{3+\sqrt{13}}{2}\right)^n+\left(\frac{3-\sqrt{13}}{2}\right)^n\right).$$

Taking the limit as n approaches infinity of the ratio T_{n+1}/T_n gives the desired result. Thus, the ratio of sums of consecutive rows of the rhombus approaches $(3+\text{sqrt}(13))/2=3.3027756$. □

The first few values of the row sums are 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, 42837, 141481, and 467280. This sequence has arisen in the literature before, e.g., in [7] and [9]. Theorem 3 shows that the Fibonacci sequence is embedded in the rhombus as alternating sum of row elements.

Theorem 3: $F(n+1)$, the $n+1^{\text{st}}$ Fibonacci number, is equal to

$$F(n+1)=\sum_{i=0}^{2n}(-1)^i[n,i].$$

Proof: By induction on n. The base case is trivial. Assume it is true for the first n Fibonacci numbers. Consider $F(n+1)$, which we claim is the alternating sum of elements on row number n. Now look at $[n-1,i]$. Suppose i is even. By definition, $[n-1,i]$ is used to compute three distinct elements on row n. It is easy to see that two of those elements will have positive coefficients and one a negative coefficient. The net effect is that of adding $[n-1,i]$ once. Likewise, if i is odd, the net effect is that of subtracting $[n-1,i]$ once. By the same token, each term from row $n-2$ is used once (and with the same sign on row n as on row $n-2$) in the computation of the sum of row n. Hence, the alternating sum of the elements on row n is the sum of the alternating sums on rows $n-1$ and $n-2$. □

We shall define a graph based on the rhombus in a straightforward manner as described in Theorem 4. This graph will be used in Section 3 to analyze the left-bounded rhombus.

Theorem 4: Define an infinite directed graph $G=(V,E)$ by using as the vertex set V points corresponding to the nonzero entries $[n,k]$ of the Pascal rhombus, and creating directed edges in E from the vertex $[n,k]$ to the vertices $[n+1,k]$, $[n+1,k+1]$, $[n+1,k+2]$, and $[n+2,k+2]$. Then the number of distinct paths from $[0,0]$ to $[n,k]$ is given by the value of $[n,k]$.

Proof: Again, an easy proof is available by induction. □

3. THE LEFT-BOUNDED RHOMBUS

In the switch-setting problem (see [4], [5]), vectors were built using the rhombus rule modified so as to use leftmost column entries that remained zero. In this case, an array arises that is left-justified: the only new nonzero values in successive rows appear on the right. The result, which we call a *left-bounded rhombus*, is shown in Figure 3.

1					
1	1				
3	2	1			
6	7	3	1		
16	18	12	4	1	
40	53	37	18	5	1

FIGURE 3. Left-Bounded Rhombus

Similar looking triangles, such as Stirling's triangle and Euler's triangles are discussed in, for example, [6]. However, those are generated by different formulas. In our left-bounded rhombus, each row contains one more element than the previous row. Clearly, the last element of each row is 1 and the next to last element is n. For simplicity in what follows, we use the notation (n,k) to index the left-bounded rhombus, as shown below:

$$\begin{array}{lllll} (0,0) \\ (1,0) & (1,1) \\ (2,0) & (2,1) & (2,2) \\ (3,0) & (3,1) & (3,2) & (3,3) \\ (4,0) & (4,1) & (4,2) & (4,3) & (4,4) \end{array}$$

Elements in the left-bounded rhombus and the Pascal rhombus are related by equation (3) below (which can easily be verified inductively):

$$[n, k]-[n, k+2]=(n, k-n), \text{ where } k \geq n. \tag{3}$$

For example, letting $n=3$ and $k=4$, we have $[3,4]-[3,6]=(3,1)$ or $8-1=7$. Equation (3) can be used to extend the left-bounded rhombus leftward beyond the (implicit) column of zeros. Since the Pascal rhombus is symmetrical, what is generated is a mirror image of the left-bounded rhombus, except that all the entries are negative. In fact, one way of obtaining the left-bounded rhombus is to start the Pascal rhombus using the original recurrence, but with the two initial rows 0 and $-1\,0\,1$. Identity (3) also applies to provide an analogous result to Theorem 1, giving (n, k) to be a second family of polynomials in n with integer coefficients. The first few values are listed below:

$$\begin{aligned} &(n, n-2)=(n^2+3n-4)/2! \\ &(n, n-3)=(n^3+9n^2-28n+12)/3! \\ &(n, n-4)=(n^4+18n^3-61n^2-30n+72)/4! \\ &(n, n-5)=(n^5+30n^4-65n^3-750n^2+2344n-1920)/5! \\ &(n, n-6)=(n^6+45n^5+25n^4-3405n^3+13654n^2-18960n+7200)/6! \end{aligned}$$

The row sums for the left-bounded rhombus are denoted by U_n having the first few values: 1, 2, 6, 17, 51, 154, 473, and 1464. These are more difficult to analyze than the row sums in the Pascal rhombus. Nevertheless, the same limiting value of ratios of successive rows exists, as shown in Theorem 5.

Theorem 5: Let U_n be the sum of the elements in row n of the left-bounded rhombus D_4. Then

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{3+\sqrt{13}}{2}.$$

The remainder of this section is devoted to the proof of Theorem 5. First, we define some additional recurrences that will be used in the proof of Theorem 5. Each recurrence defines an array of integers D_i and a graph G_i (using the procedure described in Theorem 4).

D_4: the usual rhombus with left boundary (see Fig. 3 and Fig. 4a). The corresponding graph is denoted G_4.

Define a *double jump* in G_4 to be an edge that goes from row i to row $i+2$.

G_3: take a G_4 graph and remove all double jumps (see Fig. 4b).

G_2: take a G_3 graph and remove all vertical edges (see Fig. 4c).

D_2 and D_3 are defined accordingly and examples shown in Figure 4. As usual, it will be convenient to index the elements by (row, column) beginning with $(0,0)$. We sometimes abuse notation and use (i, j) to refer to a particular vertex in a graph, as well as the value in an array.

To be clear, we sometimes preface the index by a graph or array name, such as $G_4(0, 0)$. Also, call (0, 0) the *root* of each graph and call edges from column i to column $j \neq i$ *diagonal* edges. A connection between D_2 and Pascal's triangle will be discussed in Section 4.

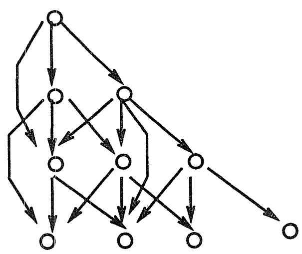
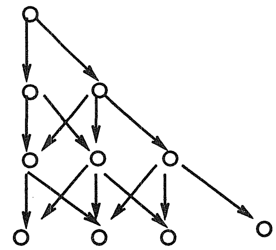
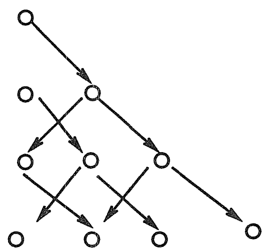


FIGURE 4a. G_4



1			
1	1		
2	2	1	
4	5	3	1

FIGURE 4b. G_3 **and** D_3



1			
0	1		
1	0	1	
0	2	0	1

FIGURE 4c. G_2 **and** D_2

To prove Theorem 5, we know that the row sum recurrence is $U_{n+1} = 3U_n + U_{n-1} - D_4(n, 0)$, so it is enough to show that $D_4(n, 0) = o(U_n)$ as $n \to \infty$ in order to make the argument of Theorem 2 apply. We shall show further that the rows of the left-bounded rhombus are unimodal with a maximum value that moves ever rightward. First, note that the path-counting property of Theorem 4 applies to G_4, G_3, and G_2. The following propositions aid in the proof of Theorem 5 as well as show some interesting properties of the aforementioned recurrences.

Proposition 6: In D_2, for all sufficiently large n and fixed j, $D_2(n, j) < D_2(n, j+2)$.

Proof: Let $f(n)$ denote the column of the maximum value on row n of D_2. If more than one position on row n is equal to the maximum, let $f(n)$ denote the leftmost such column. Our method also shows that, for sufficiently large n, $D_2(n, j) \leq D_2(n, j+2)$. Consider row $2k$ on D_2, for some $k \geq 1$ and column $2p$ for some $p \geq 1$. Represent a path from the root of G_2 to $G_2(2k, 2p)$ as a sequence of -1's and 1's, where -1 indicates an edge from column i to $i-1$ and 1 indicates an edge from i to $i+1$. So, for example, sequence $1, -1$ is a path from $G_2(0, 0)$ to $G_2(2, 0)$. The length of the sequence is $2k$ and its sum is $2p$. Let us first count the total number of sequences from (0, 0)—including "illegal" sequences having prefixes whose sum is negative. There are ($2k$ choose $(k-p)$) such sequences. Now we must subtract the number of illegal sequences. It can be observed that this is equal to the number of sequences of length $2k$ whose sum is $2p+2$. This may be seen by looking at each path in Figure 5 from the root to $G_2(2k, 2p)$ that uses a vertex in column -1. The portion of each of these paths that is below its first visit to column -1 may then be reflected about column -1, leading to paths that terminate at $(2k, -2p-2)$. Thus, the number of illegal sequences is ($2k$ choose $(k-p-1)$), which means that the number of paths in G_2 from the root to $(2k, 2p)$ is ($2k$ choose $(k-p)$) − ($2k$ choose $(k-p-1)$). Comparing (n, j) and $(n, j+2)$ leads us to look to satisfy the following inequality,

$$\binom{2k}{k-p}-\binom{2k}{k-p-1}\leq\binom{2k}{k-p-1}-\binom{2k}{k-p-2},$$

which is equal to

$$\frac{2k!(2p+1)}{(k-p)!(k-p-1)!}\leq\frac{2k!(2p+3)}{(k-p-1)!(k+p+2)!}$$

and simplifying yields $2p^2+4p+1\leq k$. It is easy to see that this inequality is satisfied when $p\sim\text{sqrt}(k)/2$. To be exact, $D_2(n, j) > D_2(n, j-2)$ if and only if $j > \lceil(\text{sqrt}(n+2))\rceil - 2$. The same method works for odd rows/columns; the details are omitted. □

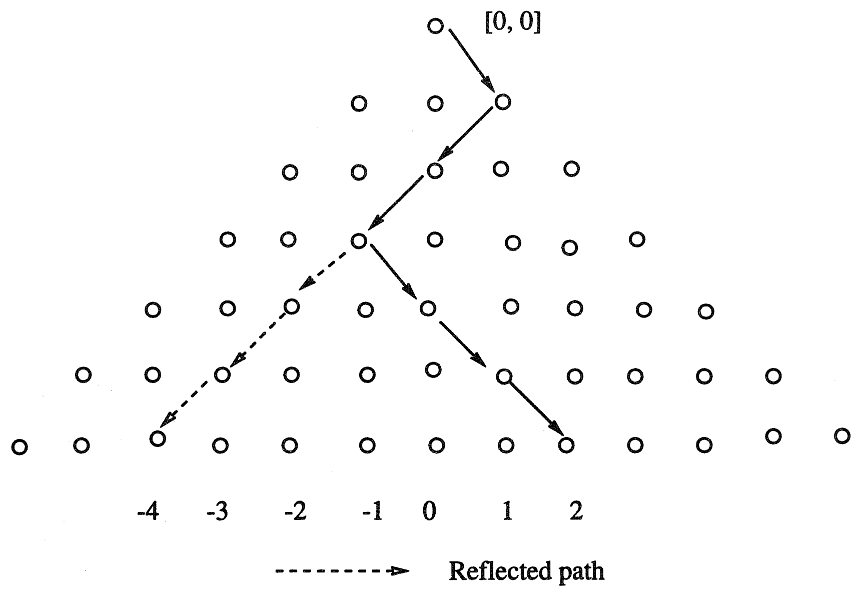


FIGURE 5

Proposition 7: In D_3, for all sufficiently large n and fixed c, $D_3(n, c) < D_3(n, c+1)$.

Proof: In D_3, we want to show that, for any fixed column number $c+1$ and sufficiently large n, $D_3(n, i) < D_3(n, c+1)$ for each $i < c+1$. Consider a G_3 graph with n rows. It is easy to see that the number of paths to $G_3(n, c+1)$ having exactly d diagonal edges is given by

$$D_3(n, c+1)=\binom{n}{n-d}D_2(d, c+1). \tag{4}$$

The careful reader will observe that (4) is often zero, depending on the parity of n and d. Assume without loss of generality both n and $c+1$ are even; otherwise, if $c+1$ is odd, we may choose n

odd and the proof follows in a similar manner. In order to compare $D_3(n, c)$ and $D_3(n, c+1)$, we group paths to $G_3(n, c)$ having d diagonal edges with paths to $G_3(n, c+1)$ having $d+1$ diagonal edges. Based on the parity of n and d, each group has paths to both c and $c+1$ and every path is in some group. Thus, we want to compare the following:

$$D_3(n, c) = \sum_{d=c}^{n-1}\binom{n}{n-d}D_2(d, c) \tag{5}$$

and

$$D_3(n, c+1) = \sum_{d_1=c+1}^{n}\binom{n}{n-d_1}D_2(d_1, c+1). \tag{6}$$

Let us compare the terms of the summations in (5) and (6) one by one: the (n choose x) term of (5) with the (n choose $x-1$) term of (6), and so on. It is easy to see that the D_2 terms in each summation increase as the index—d in the case of (5) and d_1 in the case of (6)—increases. Also, by Proposition 6, we may select n sufficiently large so that, for all $x \ge n/2$, we have that $D_2(x, c+1) \ge 2 * D_2(x-1, c)$. This implies that

$$\binom{n}{n-(d+1)}D_2(d+1, c+1) \ge \binom{n}{n-d}D_2(d, c) + \binom{n}{n-q}D_2(q, c),$$

where $d+1 \le n/2$ and $q = n/2+d+1$. In visual terms, we line up the terms in (5) and (6) in increasing order of the d, d_1 index as shown below.

$$\begin{array}{lccccc|ccc}
\text{Terms from (5):} & \binom{n}{x} & \binom{n}{x-1} & \cdots & \binom{n}{n/2+1} & & \binom{n}{n/2} & \cdots & \binom{n}{1} \\
\text{Terms from (6):} & \binom{n}{x-1} & \binom{n}{x-2} & \cdots & \binom{n}{n/2} & & \binom{n}{n/2-1} & \cdots & \binom{n}{0}
\end{array}$$

"*middle*"

The q term is as far right of the "middle" in (5) as $d+1$ is to the left in (6)—as $d+1$ ranges from $c+1$ to $n/2$. In other words, due to symmetry, these two binomial coefficients yield equal values. In this way, the first $n/2-c$ terms to the right of the middle in (5) may be accounted for using terms to the left of the middle in (6). Selecting n sufficiently large allows the remaining terms to the far right of the middle in (5) to be accounted for by those to the (near) right of the middle in (6). □

We are now ready to complete the proof of Theorem 5.

Proof of Theorem 5: Let m be a row in D_3 such that $D_3(m, j+1) > D_3(m, j)$ for some fixed value of j. Consider a G_4 graph G with $2m$ rows, $0, \ldots, 2m-1$. We show that $D_4(2m-1, j) < D_4(2m-1, j+1)$. Clearly, if we consider all paths in G to row $2m-1$ from the root with no double jumps, the proposition is true from the assumption. Likewise, if we consider all paths in G to row $2m-1$ with a double jump as the first move and no other double jumps. Using this idea, we group paths together as follows: two paths are put in the same group if each have k double jumps and if those double jumps occur in the same positions in the edge sequence that defines the paths—e.g., $k=2$ and the second and fourth edges are double jumps. Note that there are groups having between 1 and m double jumps, and for each k there are approximately ($2m-k$ choose k)

groups. It is easy to see that, for each group of such paths, the claim is true. Since these groups of paths are mutually disjoint, the theorem follows. □

We conjecture that the location of the maximum value in D_4 on row n is at least as large as sqrt(n)$/2$. The proof of Theorem 5 shows an O(sqrt(n)) upper bound on the location of the maximum. An analogy between the left-bounded rhombus and the classic Pascal triangle is explored in the next section.

4. A CONNECTION WITH PASCAL'S TRIANGLE

A seemingly different left-bounded array can be constructed using the recurrence for Pascal's triangle:

$$\begin{array}{ccccccccc}
 & 1 & & & & & & & \\
0 & & 1 & & & & & & \\
 & 1 & & 1 & & & & & \\
0 & & 2 & & 1 & & & & \\
 & 2 & & 3 & & 1 & & & \\
0 & & 5 & & 4 & & 1 & & \\
 & 5 & & 9 & & 5 & & 1 &
\end{array}$$

FIGURE 6. Left-Bounded Pascal Triangle

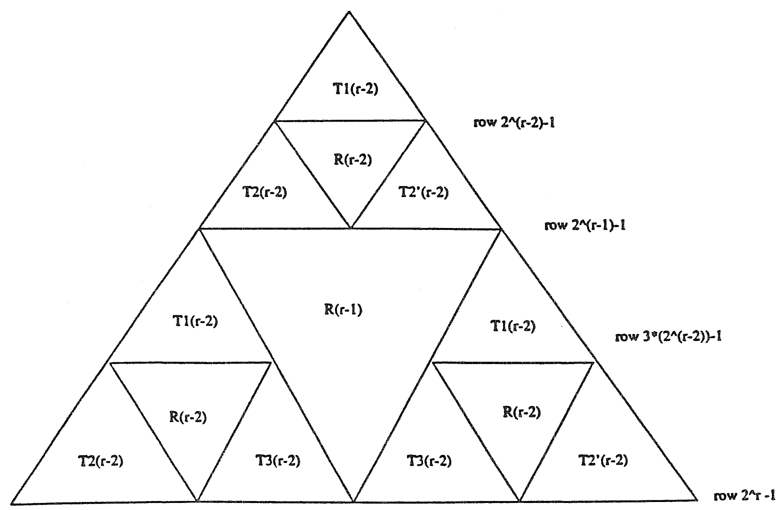
Notice the relationship between this *left-bounded Pascal triangle* and the array D_2 from the previous section. D_2 is identical to the left-bounded Pascal triangle, except that D_2 contains additional 0 elements. In this section, we use a completely different technique than the one used in Section 3 to show that the maximum value moves ever rightward in the left-bounded Pascal triangle. This time, the analog of (3) is easily shown to hold; so these table entries are differences of binomial coefficients. Hence, the maximum value in row n of this array occurs in the column k such that k gives the maximum value of the difference in binomial coefficients in row n of Pascal's triangle. But as n grows, by the classical limit theorem of De Moivre and Laplace [1], [3], the binomial distribution approaches a normal distribution, given that we choose binomial distribution parameters $p=q=1/2$. In this case, the mean is $n/2$ and the standard deviation is sqrt($n/4$). We are interested in where the maximum (absolute) derivative of this function occurs, i.e., the inflection points. Using a well-known result [1] in probability and statistics, we have that the inflection points are given by $x=n/2\pm\sqrt{n}/2$. Thus, the maximum difference on row n of Pascal's triangle occurs in column sqrt(n)$/2$, implying that the maximum value on row n of the left-bounded Pascal triangle is in column sqrt(n)$/2$. For example, if $n=729$, the maximum difference between columns in Pascal's triangle occurs between columns 378 and 379 (note that $n/2=364.5$); computing sqrt(729)$/2$ gives 13.5.

5. THE RHOMBUS MOD 2

In this section we present several conjectures concerning the distribution of the terms in the rhombus when arithmetic is done modulo two. Other problems such as divisibility properties, distribution of coefficients mod p, and the investigation of arithmetic fractal structures have been studied for Pascal's triangle and its generalizations [2] and seem to be rich and interesting in the rhombus (mod 2), though they also appear difficult to analyze formally.

Conjecture 1: For any $n \geq 1$, the sub-rhombus (mod 2) with corner points $[2^n, 2^{n+1}]$, $[2^n + 2^{n-1} - 1, 2*(2^n + 2^{n-1} - 1)]$, and $[2^n + 2^{n-1} - 1, 2^{n+1}]$ is identical to the rhombus (mod 2) with corner points $[0, 0]$, $[2^{n-1} - 1, 2*(2^{n-1} - 1)]$, and $[2^{n-1} - 1, 0]$, and to the sub-rhombus (mod 2) with corner points $[2^n, 0]$, $[2^n + 2^{n-1} - 1, 2*(2^{n-1} - 1)]$, and $[2^n + 2^{n-1} - 1, 0]$.

One can, in fact, make a stronger self-similarity conjecture, which is illustrated in Figure 7.



T2'(r-2) is mirror image of T2(r-2)

FIGURE 7. Self-Similarity in the Rhombus (mod 2)

Conjecture 2: Let $n = 2^s - 1$ be a row number of the rhombus (mod 2) and I_s be the number of ones on that row. Then

$$I_s = \frac{1}{3}[2^{s+2} + (-1)^{1-\delta}], \text{ where } \delta = 2*\text{frac}\left(\frac{s}{2}\right). \tag{7}$$

The "frac" in (7) refers to the fractional part of the term $s/2$. Equation (7) is just a closed form of the recurrence $I_0 = 1$, $I_s = 2*I_{s-1} + 1$ when s is odd, and $I_s = 2*I_{s-1} - 1$ when s is even.

Recurrences similar to that in Conjecture 2 also seem to describe the number of ones on rows whose row number is $2^s - c$ for each constant $c > 1$.

Conjecture 3: The diagonals in the rhombus (mod 2) given by $[n, k]$, for k fixed, are periodic with period length 2^p, where $p = \lceil \log_2 k \rceil + 1$ for $k \geq 1$, and the period of the $[n, 0]$ diagonal is 1.

To illustrate Conjecture 3, observe that diagonal $[n, 6]$ begins 1, 1, 0, 1, 1, 0, 1, 1 and then this sequence of eight values repeats itself.

One can also observe a strong fractal structure to the rhombus, which is characterized by large quadrilateral shaped blocks of zeros, as shown in Figure 8, a depiction of the first 512 rows of the rhombus (mod 2) with odd entries colored black and even entries colored white. This leads to the following conjecture.

Conjecture 4: Let G_n (H_n) be the number of odd (even) coefficients in the first n rows of the rhombus. Then, as n approaches infinity, $\lim G_n / H_n = 0$.

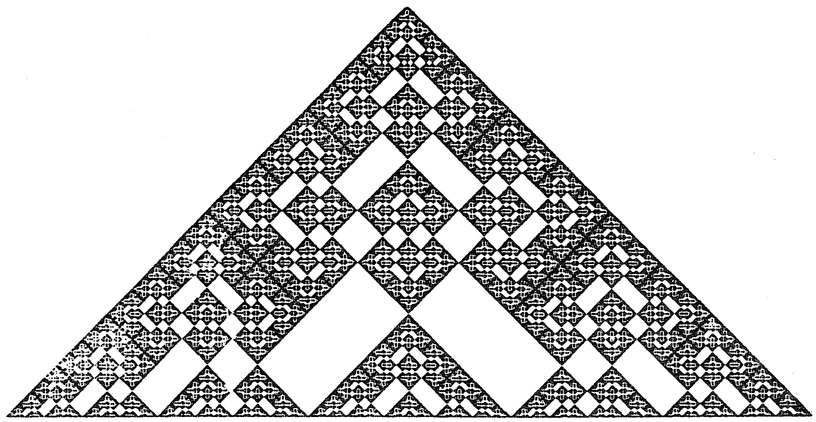


FIGURE 8. Fractal Structure of the Rhombus (mod 2)

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ON A CLASS OF GENERALIZED POLYNOMIALS

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1. INTRODUCTION

In a series of articles [1]-[3], André-Jeannin has recently defined the polynomials $U_n(p,q;x)$ and $V_n(p,q;x)$ by the recurrence relations (1) and (2), and has studied some of the combinatorial properties of the coefficients of U_n and V_n as well as some of the differential properties of these polynomials.

$$U_n = (x+p)U_{n-1} - qU_{n-2} \ (n \geq 2),\ U_0 = 0, U_1 = 1 \tag{1}$$

and

$$V_n = (x+p)V_{n-1} - qV_{n-2} \ (n \geq 2),\ V_0 = 2, V_1 = x+p. \tag{2}$$

The parameters p and q as well as the variable x are real numbers. If α and β are defined by

$$\alpha + \beta = x + p,\ \alpha\beta = q, \tag{3}$$

then it is well known that [5]

$$U_n = \frac{\alpha^n - \beta^n}{\sqrt{\Delta}}, \tag{4a}$$

and

$$V_n = \alpha^n + \beta^n, \tag{4b}$$

where

$$\Delta = (x+p)^2 - 4q. \tag{5}$$

The purpose of this article is to introduce and study some of the properties of the generalized polynomial $W_n(p,q;x)$ defined by

$$W_n = (x+p)W_{n-1} - qW_{n-2} \ (n \geq 2), \tag{6}$$

where W_0 and W_1 are specified, as well as those of two other polynomials $u_n(p,q;x)$ and $v_n(p,q;x)$ that are very closely associated with U_n and V_n. We shall define these polynomials $u_n(p,q;x)$ and $v_n(p,q;x)$ to be

$$u_n = (x+p)u_{n-1} - qu_{n-2} \ (n \geq 2),\ u_0 = 1, u_1 = x + p - \sqrt{q} \tag{7}$$

and

$$v_n = (x+p)v_{n-1} - qv_{n-2} \ (n \geq 2),\ v_0 = 1, v_1 = x + p + \sqrt{q}. \tag{8}$$

2. SOME BASIC RELATIONS AMONG U_n, V_n, u_n AND v_n

Using the well-known properties of $W_n(a,b,p,q)$ introduced by Horadam [5], we may derive a number of relations between U_n and V_n. However, we shall not do so except to list a few of the important ones that will be required for the remainder of this article. It is easy to show that W_n as defined by (6) may be evaluated using the relation [5],

$$W_n = W_1 U_n - qW_0 U_{n-1} \ (n \geq 1). \tag{9}$$

From (9) we can immediately derive the following relations:

$$V_n = U_{n+1} - qU_{n-1}, \tag{10}$$

$$u_n = U_{n+1} - \sqrt{q}\, U_n, \tag{11}$$

$$v_n = U_{n+1} + \sqrt{q}\, U_n, \tag{12}$$

$$V_n = u_n + \sqrt{q}\, u_{n-1} = v_n - \sqrt{q}\, v_{n-1}. \tag{13}$$

From the results in [5], we may also derive the following "Simson" formulas:

$$U_{n+1}U_{n-1} - U_n^2 = -q^{n-1}, \tag{14a}$$

$$V_{n+1}V_{n-1} - V_n^2 = q^{n-1}\Delta, \tag{14b}$$

$$u_{n+1}u_{n-1} - u_n^2 = q^{n-1/2}\Delta_u, \tag{14c}$$

$$v_{n+1}v_{n-1} - v_n^2 = -q^{n-1/2}\Delta_v, \tag{14d}$$

where

$$\Delta_u = x + p - 2\sqrt{q}, \tag{15a}$$

$$\Delta_v = x + p + 2\sqrt{q}, \tag{15b}$$

$$\Delta = \Delta_u \Delta_v. \tag{15c}$$

From (14a-14d), we have the interesting result that

$$q(U_{n+1}U_{n-1} - U_n^2)(V_{n+1}V_{n-1} - V_n^2) = (u_{n+1}u_{n-1} - u_n^2)(v_{n+1}v_{n-1} - v_n^2) = -q^{2n-1}\Delta. \tag{16}$$

3. ZEROS AND ORTHOGONALITY PROPERTY OF U_n, V_n, u_n, AND v_n

André-Jeannin ([1], [2]) has shown that

$$U_n = q^{(n-1)/2} \frac{\sin n\theta}{\sin\theta} \tag{17a}$$

and

$$V_n = 2q^{n/2} \cos n\theta, \tag{17b}$$

where $\cos\theta = (x+p)/2\sqrt{q}$. Hence, from (11) and (17a), we get

$$u_n = q^{n/2} \frac{\cos(2n+1)\theta/2}{\cos\theta/2}. \tag{17c}$$

Similarly, from (12) and (17a), we have

$$v_n = q^{n/2} \frac{\sin(2n+1)\theta/2}{\sin\theta/2}. \tag{17d}$$

Hence, the zeros of U_n, V_n, u_n, and v_n are given by

$$U_n\colon\ x_k = -p + 2\sqrt{q}\cos\left(\frac{k}{n}\cdot\pi\right), \qquad k = 1, 2, \ldots, n-1, \tag{18a}$$

$$V_n\colon\ x_k = -p + 2\sqrt{q}\cos\left(\frac{2k-1}{2n}\cdot\pi\right), \quad k = 1, 2, \ldots, n, \tag{18b}$$

$$u_n\colon\ x_k = -p + 2\sqrt{q}\cos\left(\frac{2k-1}{2n+1}\cdot\pi\right), \quad k = 1, 2, \ldots, n, \tag{18c}$$

$$v_n\colon\ x_k = -p + 2\sqrt{q}\cos\left(\frac{2k}{2n+1}\cdot\pi\right), \quad k = 1, 2, \ldots, n. \tag{18d}$$

Of these, André-Jeannin ([1], [2]) has given the zeros for U_n and V_n. It should be observed that, if $p = 2$ and $q = 1$, then the above results correspond to the already known results for the zeros of $B_n(x)$, $C_n(x)$, $b_n(x)$, and $c_n(x)$ (see [6], [7], [4]).

André-Jeannin ([1], [2]) has shown further that U_n and V_n are orthogonal over the interval $(-p-2\sqrt{q},\ -p+2\sqrt{q})$ with respect to the weight functions $w_U(x) = \sqrt{-\Delta}$ and $w_V(x) = 1/w_U(x)$, respectively. Using expressions (17c) and (17d), we may easily prove that u_n and v_n are also orthogonal over the same interval, but with respect to the weight functions $w_u(x) = \sqrt{-\Delta_u/\Delta_v}$ and $w_v(x) = 1/w_u(x)$, respectively.

4. Q-MATRIX AND FORMULAS FOR W_{nk-1}, W_{nk} AND W_{nk+1}

If we define the generating matrix Q to be

$$Q = \begin{bmatrix} x+p & -q \\ 1 & 0 \end{bmatrix}, \tag{19}$$

then it is straightforward to show by induction that

$$P = Q^k = \begin{bmatrix} U_{k+1} & -qU_k \\ U_k & -qU_{k-1} \end{bmatrix}. \tag{20}$$

The characteristic equation of P is given by

$$\lambda^2 - (U_{k+1} - qU_{k-1})\lambda + q(U_k^2 - U_{k+1}U_{k-1}) = 0.$$

Using relations (10) and (14a), we may reduce the above equation to

$$\lambda^2 - V_k\lambda + q^k = 0.$$

Hence, by the Cayley-Hamilton theorem, we have

$$P^2 = V_kP - q^kI. \tag{21}$$

Starting with (21), we may easily show by induction that

$$P^n(x) = \lambda_n(x)P(x) - q^k\lambda_{n-1}(x)I, \tag{22}$$

where $\lambda_n(x)$ satisfies the recurrence relation

$$\lambda_n(x) = V_k(x)\lambda_{n-1}(x) - q^k\lambda_{n-2}(x) \ \ (n \geq 2), \ \lambda_0 = 0, \lambda_1 = 1. \tag{23}$$

Hence, from (20) and (22), we have

$$Q^{nk}(x) = \lambda_n(x)Q^k(x) - q^k\lambda_{n-1}(x)I. \tag{24}$$

Therefore, we have

$$U_{nk}(x) = \lambda_n(x)U_k(x), \tag{25a}$$

$$U_{nk+1}(x) = \lambda_n(x)U_{k+1}(x) - q^k\lambda_{n-1}(x), \tag{25b}$$

and

$$U_{nk-1}(x) = \lambda_n(x)U_{k-1}(x) + q^{k-1}\lambda_{n-1}(x). \tag{25c}$$

We now derive similar results for the polynomial W, and thus for the polynomials V, u, and v. Consider the matrix

$$R = \begin{bmatrix} W_{nk+1} & -qW_{nk} \\ W_{nk} & -qW_{nk-1} \end{bmatrix}.$$

Using relation (9), we may rewrite R as

$$\begin{aligned} R &= W_1\begin{bmatrix} U_{nk+1} & -qU_{nk} \\ U_{nk} & -qU_{nk-1} \end{bmatrix} - qW_0\begin{bmatrix} U_{nk} & -qU_{nk-1} \\ U_{nk-1} & -qU_{nk-2} \end{bmatrix} \\ &= W_1Q^{nk} - qW_0Q^{nk-1}, \ \text{ using (20),} \\ &= Q^{nk}(W_1I - qW_0Q^{-1}). \end{aligned}$$

Hence,

$$\begin{bmatrix} W_{nk+1} & -qW_{nk} \\ W_{nk} & -qW_{nk-1} \end{bmatrix} = \begin{bmatrix} U_{nk+1} & -qU_{nk} \\ U_{nk} & -qU_{nk-1} \end{bmatrix}\begin{bmatrix} W_1 & -qW_0 \\ W_0 & W_1 - (x+p)W_0 \end{bmatrix}.$$

From the above identity, we may derive the following relations after some manipulations using (9) and (25a-25c):

$$W_{nk} = \lambda_n W_k - q^k W_0 \lambda_{n-1}, \tag{26a}$$

$$W_{nk+1} = \lambda_n W_{k+1} - q^k W_1 \lambda_{n-1}, \tag{26b}$$

$$W_{nk-1} = \lambda_n W_{k-1} + q^{k-1}\lambda_{n-1}\{W_1 - (x+p)W_0\}. \tag{26c}$$

Using appropriate values for W_0 and W_1 in (26a-26c), we may now derive the following relations for the polynomials V, u, and v:

$$V_{nk} = \lambda_n V_k - 2q^k\lambda_{n-1}, \tag{27a}$$

$$V_{nk+1} = \lambda_n V_{k+1} - q^k(x+p)\lambda_{n-1}, \tag{27b}$$

$$V_{nk-1} = \lambda_n V_{k-1} - q^{k-1}(x+p)\lambda_{n-1}; \tag{27c}$$

$$u_{nk} = \lambda_n u_k - q^k\lambda_{n-1}, \tag{28a}$$

$$u_{nk+1} = \lambda_n u_{k+1} - q^k(x+p-\sqrt{q})\lambda_{n-1}, \tag{28b}$$

$$u_{nk-1} = \lambda_n u_{k-1} - q^{k-1/2}\lambda_{n-1}; \tag{28c}$$

$$v_{nk} = \lambda_n v_k - q^k \lambda_{n-1}, \tag{29a}$$

$$v_{nk+1} = \lambda_n v_{k+1} - q^k (x+p+\sqrt{q})\lambda_{n-1}, \tag{29b}$$

$$v_{nk-1} = \lambda_n v_{k-1} + q^{k-1/2}\lambda_{n-1}. \tag{29c}$$

It is clear from (23) that, if $V_k \mid \lambda_{n-2}$, then $V_k \mid \lambda_n$ also. However, $V_k \mid \lambda_2$ since $\lambda_2 = V_k$. Hence, by induction, it follows that $V_k \mid \lambda_n$ when n is even. Thus, we see from (25a) that $V_k \mid U_{kn}$ for even n, while $U_k \mid U_{kn}$ for all n. Further, we see from (27a) that $V_k \mid V_{kn}$ for odd n. Thus, we have the following results:

$$U_k \mid U_{kn} \quad \text{for all } n; \tag{30a}$$

$$V_k \mid U_{kn} \quad \text{for even } n; \tag{30b}$$

$$V_k \mid V_{kn} \quad \text{for odd } n. \tag{30c}$$

It is evident that similar results hold for Fibonacci and Lucas polynomials, Pell and Pell-Lucas polynomials, etc., since these polynomials are special cases of U_n and V_n. In particular, for the Fibonacci, Lucas, Pell, and Pell-Lucas numbers F_n, L_n, P_n, and Q_n, we obtain from (30) the already known results:

$$F_k \mid F_{kn}, \quad P_k \mid P_{kn}, \quad \text{for all } n; \tag{31a}$$

$$L_k \mid F_{kn}, \quad Q_k \mid P_{kn}, \quad \text{for even } n; \tag{31b}$$

$$L_k \mid L_{kn}, \quad Q_k \mid P_{kn}, \quad \text{for odd } n. \tag{31c}$$

5. SPECIAL CASE WHEN $q = 1$

This corresponds to a modified version of the Morgan-Voyce polynomials, where $x+2$ is replaced by $x+p$ in the difference equations. We shall denote the modified Morgan-Voyce polynomials by $\widetilde{B}_n(x)$, $\widetilde{b}_n(x)$, $\widetilde{C}_n(x)$, and $\widetilde{c}_n(x)$, where

$$\widetilde{B}_n(x) = U_{n+1}(p,1;\, x), \tag{32a}$$

$$\widetilde{C}_n(x) = V_n(p,1;\, x), \tag{32b}$$

$$\widetilde{b}_n(x) = u_n(p,1;\, x), \tag{32c}$$

$$\widetilde{c}_n(x) = v_n(p,1;\, x). \tag{32d}$$

Hence, from (14a-14d), we have the "Simson" formulas:

$$\widetilde{B}_{n+1}\widetilde{B}_{n-1} - \widetilde{B}_n^2 = -1; \tag{33a}$$

$$\widetilde{C}_{n+1}\widetilde{C}_{n-1} - \widetilde{C}_n^2 = (x+p)^2 - 4 = \Delta = \Delta_b\Delta_c; \tag{33b}$$

$$\widetilde{b}_{n+1}\widetilde{b}_{n-1} - \widetilde{b}_n^2 = x+p-2 = \Delta_b; \tag{33c}$$

$$\widetilde{c}_{n+1}\widetilde{c}_{n-1} - \widetilde{c}_n^2 = -(x+p+2) = -\Delta_c. \tag{33d}$$

André-Jeannin [3] has shown that $\widetilde{B}_n^{(k)}(x)$ and $\widetilde{C}_n^{(k)}(x)$, $k = 0, 1, 2, \ldots$, where k stands for the k^{th} derivative, satisfy the following second-order differential equations:

$$\widetilde{B}_n^{(k)}(x):\quad \Delta y'' + (2k+3)(x+p)y' + \{(k+1)^2 - (n+1)^2\}y = 0, \tag{34a}$$

$$\widetilde{C}_n^{(k)}(x):\quad \Delta y'' + (2k+1)(x+p)y' + (k^2 - n^2)y = 0, \tag{34b}$$

where

$$\Delta = (x+p)^2 - 4. \tag{34c}$$

We will now derive similar results for $\widetilde{b}_n^{(k)}(x)$ and $\widetilde{c}_n^{(k)}(x)$. It is already known (see [6]) that $b_n(x)$ satisfies the differential equation

$$x(x+4)b_n''(x) + 2(x+1)b_n'(x) - n(n+1)b_n(x) = 0. \tag{35}$$

Changing x to $x+p-2$ and noting that $\widetilde{b}_n(x) = b_n(x+p-2)$, we find that equation (35) reduces to

$$\Delta\widetilde{b}_n''(x) + 2(x+p-1)\widetilde{b}_n'(x) - n(n+1)\widetilde{b}_n(x) = 0, \tag{36}$$

where Δ is given by (34c). Differentiating (36) k times and using the Leibniz rule, we can show that $\widetilde{b}_n^{(k)}(x)$ satisfies the differential equation

$$\widetilde{b}_n^{(k)}(x):\ \Delta y'' + 2\{(k+1)(x+p) - 1\}y' + \{k(k+1) - n(n+1)\}y = 0. \tag{37a}$$

Similarly, we can show that $\widetilde{c}_n^{(k)}(x)$ satisfies the equation

$$\widetilde{c}_n^{(k)}(x):\ \Delta y'' + 2\{(k+1)(x+p) + 1\}y' + \{k(k+1) - n(n+1)\}y = 0. \tag{37b}$$

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❖❖❖

A SEMIGROUP ASSOCIATED WITH THE *k*-BONACCI NUMBERS WITH DYNAMIC INTERPRETATION

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1. INTRODUCTION

In this paper we shall associate a semigroup with the k-bonacci numbers, which describes the self-similar structure of the dynamical system associated with the substitution $1 \to 12, \ldots (k-1) \to 1k, k \to 1$ for $k \geq 3$. The operation that defines the semigroup is used to handle the cylinders of the partitions defined by the self-similar structure of the symbolic system. This collection of cylinders is called the standard partition. The relation between the standard partition and the semigroup is given by Theorem 2.

The dyamical system that arises from this substitution admits geometrical representations as:

- an irrational translation on the $(k-1)$-dimensional torus [7],
- an interval exchange map in the circle [1], and
- a map on a geodesic lamination on the hyperbolic disc [8].

The self-similar structure of the symbolic system is translated to its geometrical realizations. The understanding of the self-similar structure of the symbolic system and its geometric relations on the torus and the circle, using the semigroup, plays an important role in the construction of the geodesic lamination, given in [8], and also in the proofs of other dynamical properties of these systems [8].

2. THE SEMIGROUP

The k-bonacci numbers are obtained by the recurrence relation

$$g_{n+k} = g_{n+k-1} + \cdots + g_{n+1} + g_n \quad \text{for } n \geq 0 \tag{1}$$

with initial conditions $g_j = 2^j$ for $0 \leq j \leq k-1$. We can represent each natural number in a unique way as a sum of the g_i's with no k consecutive g_i's in the present sum. This is a generalization of the Zeckendorf representation of the nonnegative [10] integers using this recurrence relation instead of the Fibonacci relation.

In the rest of this section, we work with the Tribonacci numbers. However, the following constructions and results are valid for all the k-bonacci numbers.

Let n and m be given in the Tribonacci Zeckendorf representation

$$n = \sum_{i=0}^{N} a_i g_i, \quad m = \sum_{j=0}^{M} b_j g_j.$$

Define $n \diamond m$ by

$$n \diamond m = \sum_{i=0}^{N} \sum_{j=0}^{M} a_i b_j g_{i+j}. \tag{2}$$

Unlike the Fibonacci multiplication ([5], [2]), this operation is not associative.

Now we define a new binary operation in **N**: Let $n = g_{i_0} + \cdots + g_{i_\ell}$, with $g_{i_j} < g_{i_q}$ when $j < q$, be the Zeckendorf representation of n. Observe that we can write n in the following way:

$$\begin{aligned} n &= g_{i_0} \diamond (1 + g_{i_1 - i_0} + \cdots + g_{i_\ell - i_0}) \\ &= g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_\ell - i_1}) \\ &\vdots \\ &= g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{\ell-1} - i_{\ell-2}} \diamond (1 + g_{i_\ell - i_{\ell-1}}) \cdots)). \end{aligned}$$

Definition 1: Define the binary operation $*$ by

$$\begin{aligned} &\mathbf{N} \times \mathbf{N} \to \mathbf{N} \\ &n * m = g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{\ell-1} - i_{\ell-2}} \diamond (1 + g_{i_\ell - i_{\ell-1}} \diamond m) \cdots)). \end{aligned}$$

Properties:

- $1 * m = m * 1 = m$.
- If $n = g_q$ then $n * m = g_q \diamond m$.
- $*$ is not commutative: e.g., $9 * 3 = 22$ and $3 * 9 = 18$.
- In general, it is not associative: e.g., $3 * (3 * 2) = 10$ and $(3 * 3) * 2 = 13$.

For this reason, we keep the following convention:

$$m_1 * m_2 * \cdots * m_\ell \overset{\text{def}}{=} m_1 * (m_2 * (\cdots * (m_{\ell-2} * (m_{\ell-1} * m_\ell)) \cdots)).$$

However, this operation is associative in a subset of the natural numbers. Let $n_1 = g_1 = 2$, $n_2 = g_0 + g_2 = 1 + 4 = 5$, $n_3 = g_0 + g_1 + g_3 = 1 + 2 + 7 = 10$, $n_0 = g_0 = 1$, and $\mathcal{P}$ the set generated by n_1, n_2, n_3 under the operation $*$, i.e.,

$$\mathcal{P}_\ell = \{n_{i_1} * \cdots * n_{i_\ell} | i_j = 1, 2, \text{ or } 3 \text{ for all } j\}, \quad \mathcal{P}_0 = \{1\}, \quad \mathcal{P} = \cup_{\ell \geq 0} \mathcal{P}_\ell.$$

Given any three natural numbers n, m, and m' then the associativity in $n * m * m'$ fails when we do the operation $n * m$ and we get an expression with three consecutive g_i's and, therefore, we have to use the relation (1) to express the number according to the Zeckendorf representation.

Easy calculations show that when we compute $n_i * n_j$ for $i, j = 0, 1, 2, 3$ we never get three consecutive g_i's. So the operation $*: \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is associative. Therefore, we have proved

Theorem 1: $(\mathcal{P}, *)$ is a semigroup.

3. THE SUBSTITUTION

A substitution in a finite alphabet $\mathcal{A}$ is a map, Π, from the alphabet to a set of words in this alphabet. This map extends to a map from the set of words in the alphabet $\mathcal{A}$ into itself by juxtaposition, i.e., $\Pi(UV) = \Pi(U)\Pi(V)$, where U and V are words in the alphabet and $\Pi(\emptyset) = \emptyset$. In this way, the substitution is extended to the set of infinite sequences in the alphabet $\mathcal{A}$. See [6] for an introduction to the theory of substitutions. In this paper we consider the substitution

$$\Pi: \{1, 2, \ldots, n\}^{\mathbf{N}^*} \to \{1, 2, \ldots, n\}^{\mathbf{N}^*}$$

$$1 \xrightarrow{\Pi} 12, 2 \xrightarrow{\Pi} 13, \ldots, (k-1) \xrightarrow{\Pi} 1k, k \xrightarrow{\Pi} 1. \tag{3}$$

This substitution is Pisot, since the Perron-Frobenius eigenvalue of the matrix that represents the substitution is a Pisot number. A *Pisot number* is an algebraic integer such that all its Galois conjugates are strictly inside the unit circle [3].

The map Π has a unique fixed point $\underline{\mathbf{u}} = u_0u_1 \ldots$. We consider the closure, in the product topology on $\{1, 2, \ldots, k\}^{\mathbb{N}^*}$, of the orbit under the shift map—$\sigma(u_0u_1u_2\ldots) = u_1u_2\ldots$—of the fixed point. This space is denoted by Ω. The dynamical system (Ω, σ) is minimal. The dynamical and geometrical properties of this substitution have been studied in [7], [1], [4], [8], [9].

Note that the relation of this substitution to the k-bonacci numbers is the following: if $|V|$ denotes the length of the word V, and $g_j = |\Pi^j(1)|$, we have the recurrence relation $g_{n+k} = g_{n+k-1} + \cdots + g_n$, since the substitution satisfies

$$\Pi^{n+k}(1) = \Pi^{n+k-1}(1)\Pi^{n+k-2}(1)\cdots\Pi^{n+1}(1)\Pi^n(1)\ \forall n \geq 0.$$

The space Ω admits a natural self-similar partition $\{\Omega_1, \ldots, \Omega_k\}$, where $\Omega_i = \{\underline{v} \in \Omega | v_0 = i\}$. The self-similarity among the elements of this partition comes from the commutativity of the diagram:

$$\begin{array}{ccc} \Omega & \xrightarrow{\sigma} & \Omega \\ \Pi \downarrow & & \downarrow \Pi \\ \Omega_1 & \xrightarrow{\tilde{\sigma}} & \Omega_1 \end{array}$$

where $\tilde{\sigma}$ denotes the induced map of σ on Ω_1, i.e.,

$$\tilde{\sigma}(\underline{v}) = \sigma^{\min\{\ell | \sigma^\ell(\underline{v}) \in \Omega_1\}}(\underline{v}).$$

In the rest of the paper we will assume that $k = 3$. However, the results are valid for $k \geq 3$.

We are going to show how to express $\sigma^n(\underline{\mathbf{u}})$ as a composition of powers of Π, applied to $\sigma(\underline{\mathbf{u}})$, and σ (without using its powers). In particular, we shall associate with each natural number n an operator $O_{\sigma,\Pi}(n)$ such that $\sigma^n(\underline{\mathbf{u}}) = O_{\sigma,\Pi}(n)(\sigma(\underline{\mathbf{u}}))$. Moreover, we shall prove the property

$$O_{\sigma,\Pi}(m) \circ O_{\sigma,\Pi}(n) = O_{\sigma,\Pi}(m * n).$$

Definition 2: Let $n = g_{i_0} + \cdots + g_{i_\ell}$ be the representation of n according to the recurrence relation (1). Then

$$n = g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{\ell-1} - i_{\ell-2}} \diamond (1 + g_{i_\ell - i_{\ell-1}}) \cdots)).$$

We define

$$O_{\sigma,\Pi}(n) : \Omega \to \Omega;$$
$$O_{\sigma,\Pi}(n) = \Pi^{i_0}\sigma\Pi^{i_1 - i_0}\sigma \cdots \Pi^{i_{\ell-1} - i_{\ell-2}}\sigma\Pi^{i_\ell - i_{\ell-1}}.$$

Lemma 1: The map $O_{\sigma,\Pi}(n)$ satisfies the properties:

(a) $O_{\sigma,\Pi}(n)(\sigma(\underline{\mathbf{u}})) = \sigma^n(\underline{\mathbf{u}})$ for any $n \in \mathbb{N}$.

(b) $O_{\sigma,\Pi}(m) \circ O_{\sigma,\Pi}(n) = O_{\sigma,\Pi}(m * n)$ for $m, n \in \mathcal{P}$.

First, we are going to prove the following proposition.

Proposition 1: Let g_q be the q^{th} Tribonacci number, then

(a) $\sigma^{g_q}(\underline{\mathbf{u}}) = \Pi^q(\sigma(\underline{\mathbf{u}}))$,

(b) $\tilde{\sigma}^{g_q}(\underline{\mathbf{u}}) = \sigma^{g_{q+1}}(\underline{\mathbf{u}})$,

(c) $\sigma^{g_q \diamond n}(\underline{\mathbf{u}}) = \sigma^{n \diamond g_q}(\underline{\mathbf{u}}) = \Pi^q \sigma^n(\underline{\mathbf{u}})$ for all $n \in \mathbf{N}^*$.

Proof of Proposition 1:

(a) This fact is proved by induction on q. In the case $q = 1$:

$$\underline{\mathbf{u}} = u_0\sigma(\underline{\mathbf{u}}) = 1\sigma(\underline{\mathbf{u}}) \text{ so } \underline{\mathbf{u}} = \Pi(\underline{\mathbf{u}}) = \Pi(1)\Pi(\sigma(\underline{\mathbf{u}})) = 12\Pi(\sigma(\underline{\mathbf{u}})).$$

Therefore, $\sigma^2(\underline{\mathbf{u}}) = \Pi\sigma(\underline{\mathbf{u}})$ but $2 = g_1$. Hence, $\sigma^{g_1}(\underline{\mathbf{u}}) = \Pi(\sigma(\underline{\mathbf{u}}))$. Let the expression of $\underline{\mathbf{u}}$ be given as

$$\underline{\mathbf{u}} = u_0 \dots u_{g_q-1}\sigma^{g_q}(\underline{\mathbf{u}}) = \Pi^q(1)\sigma^{g_q}(\underline{\mathbf{u}}).$$

Since $\underline{\mathbf{u}}$ is the fixed point of the substitution, we have $\underline{\mathbf{u}} = \Pi^{q+1}(1)\Pi(\sigma^{g_q}(\underline{\mathbf{u}}))$. Therefore, we have

$$\sigma^{g_{q+1}}(\underline{\mathbf{u}}) = \Pi(\sigma^{g_q}(\underline{\mathbf{u}})) = \Pi(\Pi^q(\sigma(\underline{\mathbf{u}})) = \Pi^{q+1}(\sigma(\underline{\mathbf{u}})).$$

(b) As we showed in part (a) of this proposition, $\sigma^{g_{q+1}}(\underline{\mathbf{u}}) = \Pi(\sigma^{g_q}(\underline{\mathbf{u}}))$. Since $\Pi \circ \sigma = \tilde{\sigma} \circ \Pi$, we have

$$\Pi(\sigma^{g_q}(\underline{\mathbf{u}})) = \tilde{\sigma}^{g_q}(\Pi(\underline{\mathbf{u}}))$$

and, since $\underline{\mathbf{u}}$ is the fixed point of the substitution, we have $\sigma^{g_{q+1}}(\underline{\mathbf{u}}) = \tilde{\sigma}^{g_q}(\underline{\mathbf{u}})$.

(c) Let $n = g_{i_0} + \cdots + g_{i_\ell}$. We can write $\underline{\mathbf{u}} = u_0 \dots u_{n-1}\sigma^n(\underline{\mathbf{u}})$; according to [7]:

$$u_0 \dots u_{n-1} = \Pi^{i_\ell}(1) \cdots \Pi^{i_0}(1),$$

and using the fact that $\underline{\mathbf{u}}$ is a fixed point of the substitution Π, we have

$$\underline{\mathbf{u}} = \Pi^q(\underline{\mathbf{u}}) = \Pi^{i_\ell+q}(1) \cdots \Pi^{i_0+q}(1)\Pi^q\sigma^n(\underline{\mathbf{u}}).$$

Therefore,

$$\Pi^q\sigma^n(\underline{\mathbf{u}}) = \sigma^{g_{i_\ell+q}+\cdots+g_{i_0+q}}(\underline{\mathbf{u}}) = \sigma^{g_q \diamond n}(\underline{\mathbf{u}}). \text{ Q.E.D.}$$

Proof of Lemma 1:

(a) Let

$$\begin{aligned} n &= g_{i_0} + \cdots + g_{i_\ell} \\ &= g_{i_0} \diamond (1 + g_{i_1-i_0} \diamond (1 + \cdots + g_{i_{\ell-1}-i_{\ell-2}} \diamond (1 + g_{i_\ell - i_{\ell-1}}) \cdots)). \end{aligned}$$

By Proposition 1,

$$\begin{aligned} \Pi^{i_\ell - i_{\ell-1}}(\sigma(\underline{\mathbf{u}})) &= \sigma^{g_{i_\ell - i_{\ell-1}}}(\underline{\mathbf{u}}) \\ \sigma\Pi^{i_\ell - i_{\ell-1}}(\sigma(\underline{\mathbf{u}})) &= \sigma^{1+g_{i_\ell - i_{\ell-1}}}(\underline{\mathbf{u}}) \\ \Pi^{i_{\ell-1}-i_{\ell-2}}\sigma\Pi^{i_\ell - i_{\ell-1}}(\sigma(\underline{\mathbf{u}})) &= \sigma^{g_{i_{\ell-1}-i_{\ell-2}} \diamond (1+g_{i_\ell - i_{\ell-1}})}(\underline{\mathbf{u}}) \\ &\vdots \\ \Pi^{i_0}\sigma\Pi^{i_1-i_0}\sigma \cdots \Pi^{i_\ell - i_{\ell-1}}(\sigma(\underline{\mathbf{u}})) &= \sigma^{g_{i_0} \diamond (1+g_{i_1-i_0} \diamond (1+\cdots+g_{i_{\ell-1}-i_{\ell-2}} \diamond (1+g_{i_\ell-i_{\ell-1}})\cdots))}(\underline{\mathbf{u}}). \end{aligned}$$

But the last term is $\sigma^n(\underline{\mathrm{u}})$ by using the expression for n given at the beginning of the proof. But, by Definition 2, $O_{\sigma,\Pi}(n) = \Pi^{i_0}\sigma\Pi^{i_1-i_0}\sigma\cdots\Pi^{i_\ell-i_{\ell-1}}$. Hence, $O_{\sigma,\Pi}(n)(\sigma(\underline{\mathrm{u}})) = \sigma^n(\underline{\mathrm{u}})$.

(b) Let

$$\begin{aligned} m &= g_{j_0} + \cdots + g_{j_q} \quad \text{and } m \in \mathcal{P} \\ &= g_{j_0} \diamond (1 + g_{j_1-j_0} \diamond (1 + \cdots + g_{j_{q-1}-j_{q-2}} \diamond (1 + g_{j_q-j_{q-1}}) \cdots)). \end{aligned}$$

So

$$O_{\sigma,\Pi}(m) = \Pi^{j_0}\sigma\Pi^{j_1-j_0}\sigma\cdots\Pi^{j_q-j_{q-1}}$$

and

$$O_{\sigma,\Pi}(m) \circ O_{\sigma,\Pi}(n) = \underbrace{\Pi^{j_0}\sigma\Pi^{j_1-j_0}\sigma\cdots\Pi^{j_q-j_{q-1}}}_{O_{\sigma,\Pi}(m)}\underbrace{\Pi^{i_0}\sigma\Pi^{i_1-i_0}\sigma\cdots\Pi^{i_\ell-i_{\ell-1}}}_{O_{\sigma,\Pi}(n)}.$$

Since m and $n \in \mathcal{P}$,

$$m * n = g_{j_0} \diamond (1 + \cdots \diamond (1 + g_{j_q-j_{q-1}} \diamond g_{i_0} \diamond (1 + g_{i_1-i_0} \diamond (1 + \cdots \diamond (1 + g_{i_\ell-i_{\ell-1}}) \cdots)))).$$

Therefore,

$$O_{\sigma,\Pi}(m * n) = O_{\sigma,\Pi}(m) \circ O_{\sigma,\Pi}(n). \quad \text{Q.E.D.}$$

Each of the sets Ω_i is similar to Ω:

$$\begin{aligned} \Omega_1 &= \Pi(\Omega); \\ \Omega_2 &= \sigma(\Pi^2(\Omega)) = \sigma(\Pi(\Omega_1)); \\ \Omega_3 &= \sigma(\Pi(\sigma(\Pi^2(\Omega)))) = \sigma(\Pi(\sigma(\Pi(\Omega_1)))) = \sigma(\Pi(\Omega_2)). \end{aligned}$$

This similarity induces a partition in each of the Ω_i's and each of these cylinders can be subdivided into three subcylinders according to the maps Π, $\sigma\Pi^2$, and $\sigma\Pi\sigma\Pi^2$.

Definition 3: The collection of subsets of Ω generated by the system of iterated maps (Π, $\sigma\Pi^2$, $\sigma\Pi\sigma\Pi^2$) is called the *standard partition of* Ω. The elements of this collection are called *cylinders*.

Theorem 2: R is a cylinder of the standard partition if and only if there exists an element n of $\mathcal{P}$ such that $R = O_{\sigma,\Pi}(n)(\Omega)$.

Proof: Let n be an element of $\mathcal{P}$ so $n = n_{i_0} * \cdots * n_{i_\ell}$, where $i_j \in \{1, 2, 3\}$, then $O_{\sigma,\Pi}(n) = O_{\sigma,\Pi}(n_{i_0}) \cdots O_{\sigma,\Pi}(n_{i_\ell})$, since $n_1 = g_1$, $n_2 = g_0 + g_2$, $n_3 = g_0 + g_1 + g_3 = g_0 + g_1 \diamond (g_0 + g_2)$, and we have

$$O_{\sigma,\Pi}(n_1) = \Pi, \quad O_{\sigma,\Pi}(n_2) = \sigma\Pi^2, \quad O_{\sigma,\Pi}(n_3) = \sigma\Pi\sigma\Pi^2.$$

Hence, $O_{\sigma,\Pi}(n)(\Omega)$ is a cylinder of the standard partition.

Reciprocally, given R a cylinder of the standard partition, by construction it is equal to a composition of $\Pi, \sigma\Pi^2$, and $\sigma\Pi\sigma\Pi^2$, i.e., it is of the form $O_{\sigma,\Pi}(n_{i_0})O_{\sigma,\Pi}(n_{i_1})\cdots O_{\sigma,\Pi}(n_{i_k})$. By Lemma 1, we have that $R = O_{\sigma,\Pi}(m)(\Omega)$, where $m = n_{i_0} * \cdots * n_{i_k}$. Q.E.D.

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PROFESSOR LUCAS VISITS THE PUTNAM EXAMINATION

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The following problem appeared on the 1995 William Lowell Putnam Mathematical Competition:

Evaluate

$$\sqrt[8]{2207-\cfrac{1}{2207-\cfrac{1}{2207-\cdots}}}$$

and express the answer in the form $(a+b\sqrt{c})/d$, where a, b, c, and d are integers.

Readers of this journal might recognize that 2207 is the sixteenth Lucas number, L_{16}. Therefore, a more general problem is to evaluate

$$\sqrt[n]{L_{2n}-\cfrac{1}{L_{2n}-\cfrac{1}{L_{2n}-\cdots}}}\,.$$

Let S denote this expression. Then $S^n = L_{2n}-(1/S^n)$, and therefore $S^{2n}-L_{2n}S^n+1=0$. It follows that

$$S^n = \frac{L_{2n}+\sqrt{L_{2n}^2-4}}{2}.$$

Now, using the Binet formula for the Lucas numbers, i.e., $L_n - \alpha^n + \beta^n$, $\alpha = (1+\sqrt{5})/2$, and $\beta = (1+\sqrt{5})/2$, we have

$$S^n = \frac{\alpha^{2n}+\beta^{2n}+\sqrt{(\alpha^{2n}+\beta^{2n})^2-4}}{2} = \frac{\alpha^{2n}+\beta^{2n}+\sqrt{(\alpha^{2n}-\beta^{2n})^2}}{2} = \alpha^{2n}.$$

It follows that $S=\alpha^2=(3+\sqrt{5})/2$ (independent of n).

This technique can be used to simplify a variety of expressions of this general form. A more natural solution to the original problem is to set T, say, equal to the expression and note, as above, that

$$T^{16}-2207T^8+1=0.$$

This can be written in the form $T^{16}+2T^8+1=47^2T^8$. Then, taking square roots (T is positive), we have $T^8+1=47T$. Repeating this gives $T^4+1=7T^2$ and $T^2+1=3T$. Solving the latter equation yields the solution $T=(3+\sqrt{5})/2$.

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THE SUM OF INVERSES OF BINOMIAL COEFFICIENTS REVISITED

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The aim of this note is to generalize the work on finite sums of inverses of binomial coefficients that are part of a paper by Andrew Rockett [1] which was published in this Quarterly in 1981. Our work rests on the following lemma.

Lemma 1: For any positive integers n and p, with $p \le n$:

$$\frac{1}{n+1}\binom{n}{p}^{-1} = \int_0^1 t^p(1-t)^{n-p}dt.$$

Proof: Use the well-known formulas of Euler,

$$\int_0^1 t^{u-1}(1-t)^{v-1}dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \quad \text{and} \quad \Gamma(k+1) = k!,$$

that are valid for any positive real numbers u and v, and any positive integer k.

Now we define

$$q_0 = 1 \quad \text{and} \quad q_n(x, y) = \frac{1}{n+1}\sum_{p=0}^{n}\binom{n}{p}^{-1} x^p y^{n-p}$$

for arbitrary nonzero complex numbers x and y. Let $G(z)$ be the generating function

$$G(z) = \sum_{n=0}^{+\infty} z^n q_n(x, y).$$

Theorem 1: For any *real* numbers x, y, and z such that $|z| < \min(|1/x|, |1/y|)$:

$$G(z) = -\frac{\log(1-xz)+\log(1-yz)}{az-bz^2} = \frac{-\log(1-az+bz^2)}{az-bz^2},$$

where $a = x+y$ and $b = xy$ (that is, x and y are the roots of the equation in S: $S^2 - aS + b = 0$).

Proof: We have

$$q_n(x, y) = \int_0^1 \sum_{p=0}^{n}(xt)^p(y(1-t))^{n-p}dt$$

and, therefore, for any z such that $|z| < \min(|1/x|, |1/y|)$,

$$G(z) = \int_0^1 \sum_{n=0}^{+\infty}\left(\sum_{p=0}^{n} z^n(xt)^p(y(1-t))^{n-p}\right)dt$$

because of the uniform convergence (deduced from its *absolute* convergence) of the series over $t \in [0, 1]$. By the Cauchy rule for the product of powers series, this is equivalent to

$$G(z)=\int_0^1\left(\sum_{n=0}^{+\infty}(zxt)^n\right)\left(\sum_{n=0}^{+\infty}(zy(1-t))^n\right)dt=\int_0^1\frac{1}{1-zxt}\cdot\frac{1}{1-zy(1-t)}\,dt,$$

from which we obtain the stated result by elementary techniques of calculus.

The restriction to real values allows us not to manipulate many-valued complex functions such as the log of complex argument, and will not impair our work since its main objective is the *polynomial* identities that follow; these, once proved for the real values of the variables, are also proved for the complex values because the identity for all real values implies the identity of the coefficients of the like powers of the variables on both sides of the relation.

This leads to Theorem 1 of Rockett [1],

$$\sum_{p=0}^{n}\binom{n}{p}^{-1}=\frac{n+1}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}$$

by taking $x=y=1$, because in that case,

$$G(z)=-\frac{2\log(1-z)}{2z-z^2}=-\frac{\log(1-z)}{z-(z^2/2)}=\left(\sum_{n=0}^{+\infty}\frac{z^n}{n+1}\right)\left(\sum_{n=0}^{+\infty}\frac{z^n}{2^n}\right)$$

and the result follows by applying the Cauchy rule for the product of power series and then equating the coefficients of the like powers of z on both sides of the identity. (This formula previously appeared in the literature in 1947 in a paper by Tor B. Staver [2].)

If we take $x=-1$ and $y=1$, we obtain

$$G(z)=-\frac{\log(1-z^2)}{z^2},$$

from which it is easy to derive the closed formula

$$\sum_{p=0}^{2n}(-1)^p\binom{2n}{p}^{-1}=\frac{2n+1}{n+1},$$

already found independently by Tor B. Staver [2] and T. S. Nanjudiah [3].

Theorem 2: If $(U_n(u,v))$ denotes the generalized Fibonacci sequence of the recurrence $r_{n+2}-ur_{n+1}+vr_n=0$, then

$$q_n(x,y)=\int_0^1 U_{n+1}(at,bt)\,dt. \tag{1}$$

Proof: An equivalent statement of Theorem 1 is, for x, y, and z real:

$$G(z)=\int_0^1\frac{1}{1-atz+btz^2}\,dt.$$

But since

$$\frac{1}{1-uz+vz^2}=\sum_{n=0}^{+\infty}z^nU_{n+1}(u,v),$$

we obtain, by integrating term by term with regard to t,

$$G(z) = \sum_{n=0}^{+\infty} z^n \int_0^1 U_{n+1}(at, bt)\, dt,$$

and the stated theorem follows by equating the coefficients of like powers of z on both sides of the identity. Since both sides of relation (1) are polynomials in x and y, this relation is also valid for all the complex values of these variables.

Given the well-known Lucas identity [4]

$$U_{n+1}(u, v) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} u^{n-2k}(-v)^k,$$

we obtain as a direct consequence of Theorem 2 the explicit expression for $q_n(x, y)$ as a function of a and b,

$$q_n(x, y) = p_n(a, b) = \sum_{k=0}^{[\frac{n}{2}]} \frac{1}{n-k+1} \binom{n-k}{k} a^{n-2k}(-b)^k,$$

from which we can deduce the converse relation

$$\frac{\partial}{\partial t}(tp_n(at, bt)) = \frac{\partial}{\partial t}\left(tq_n\left(\frac{at - \sqrt{(at)^2 - 4bt}}{2}, \frac{at + \sqrt{(at)^2 - 4bt}}{2}\right)\right) = U_{n+1}(at, bt).$$

Another consequence of Theorem 2 is that, for any positive integer m, we can compute recursively the development in powers of z of

$${}^mG(z) = \frac{G(z)}{(a - bz)^m}.$$

In effect, if we define

$${}^1q_n(x, y) = \int_0^1 tU_{n+1}(at, bt)\, dt$$

by the fundamental second-order recurrence satisfied by $(U_n(u, v))$ and Theorem 2, we obtain

$$q_n(x, y) = a\, {}^1q_{n-1}(x, y) - b\, {}^1q_{n-2}(x, y).$$

From this last relation, one can deduce easily that ${}^1G(z)$ is a generating function for $({}^1q_n(x, y))$. By recursion on m, and the same kind of reasoning, it can be shown that ${}^mG(z)$ is a generating function for $({}^mq_n(x, y))$ defined as

$${}^mq_n(x, y) = \int_0^1 t^m U_{n+1}(at, bt)\, dt$$

since

$${}^{m-1}q_n(x, y) = a\, {}^mq_{n-1}(x, y) - b\, {}^mq_{n-2}(x, y).$$

With the previously used Lucas identity, we are able to compute an explicit expression of ${}^mq_n(x, y)$.

Another relation deduced from Theorem 2 is

$$2q_n(x, y) - a\,{}^1q_{n-1}(x, y) = \int_0^1 V_n(at, bt)\, dt$$

if $(V_n(u, v))$ denotes the generalized Lucas sequence of the recursion $r_{n+2} - ur_{n+1} + vr_n = 0$, as a consequence of the well-known relation

$$V_n(u, v) = 2U_{n+1}(u, v) - uU_n(u, v).$$

Theorem 3: The sequence $(q_n(x, y))$ satisfies the following third-order recurrence [by writing for convenience, $q_n(x, y) = q_n$]:

$$a(n+2)\,q_{n+1} - \{a^2(n+1) + b(n+2)\}\,q_n + ab(2n+1)\,q_{n-1} - b^2nq_{n-2} = 0.$$

Proof: From Theorem 1, we deduce that

$$\begin{aligned}(az - bz^2)\sum_{n=0}^{+\infty} z^n q_n(x, y) &= -\log(1-zx) - \log(1-zy) \\ &= \sum_{n=1}^{+\infty} \frac{1}{n} z^n(x^n + y^n) = \sum_{n=1}^{+\infty} \frac{1}{n} z^n V_n(a, b).\end{aligned}$$

By comparing the coefficients of z^{n+2} on both sides, we obtain

$$\frac{V_{n+2}(a, b)}{n+2} = a\,q_{n+1} - b\,q_n.$$

Then the recursion $V_{n+2}(a, b) - aV_{n+1}(a, b) + bV_n(a, b) = 0$ becomes

$$a(n+2)\,q_{n+1} - \{a^2(n+1) + b(n+2)\}\,q_n + ab(2n+1)\,q_{n-1} - b^2nq_{n-2} = 0$$

as stated.

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DIRECTED GRAPHS DEFINED BY ARITHMETIC (MOD n)

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1. INTRODUCTION

Let a and $n>0$ be integers, and define $G(a, n)$ to be the directed graph with vertex set $V=\{0,1,\ldots,n-1\}$ such that there is an arc from x to y if and only if $y \equiv ax \pmod{n}$. Recently, Ehrlich [1] studied these graphs in the special case $a=2$ and n odd. He proved that if n is odd, then the number of cycles in $G(2, n)$ is odd or even according as 2 is or is not a quadratic residue mod n. The aim of this paper is to give the analogous results for all a and all positive n. In particular, we show that if a and n are relatively prime, and n is odd, then the number of cycles in $G(a, n)$ is odd or even according as a is or is not a quadratic residue mod n.

Define $GP(a, n)$ to be the directed graph with vertex set $V=\{0,1,\ldots,n-1\}$ such that there is an arc from x to y if and only if $y \equiv x^a \pmod{n}$. We determine the number of cycles in $GP(a, n)$ for n a prime power.

2. PRELIMINARY RESULTS

We require a few lemmas. In what follows, write $d|n$ to mean that d is a divisor of n and let (x, y) and $[x, y]$ denote the greatest common divisor (GCD) and least common multiple (LCM), respectively, of x and y. If $(a, m)=1$, then (a/m) denotes the familiar Legendre-Jacobi quadratic residue symbol. Finally, let $U_n=\{x: 1\le x\le n \text{ and } (x, n)=1\}$, let $\varphi(n)$ denote the Euler phi-function, and, if $(a, n)=1$, let $\mathrm{ord}_n(a)$ be the least positive integer r such that $a^r \equiv 1 \pmod{n}$.

Lemma 1: Let $(a, n)=1$. If $(x_1, x_2, \ldots, x_r)$ is a cycle in $G(a, n)$, then (n, x_i) is the same for each i, $1\le i\le r$.

Proof: Let $(x_1, x_2, \ldots, x_r)$ be a cycle in $G(a, n)$. Since $(a, n)=1$, it follows that $(n, x_2)=(n, ax_1)=(n, x_1)$; thus, for each i, $(n, x_i)=(n, x_1)$ by induction. [We shall call this common value of (n, x_i) the GCD of the cycle $(x_1, x_2, \ldots, x_r)$.] □

For arbitrary a and n, let $C(a, n)$ denote the number of cycles in $G(a, n)$, and let $c(a, n, d)$ be the number of cycles in $G(a, n)$ with GCD d.

Lemma 2: Let $(a, n)=1$. Then $c(a, n, 1)=\frac{\varphi(n)}{\mathrm{ord}_n(a)}$.

For example, let $a=3$ and $n=65$. Then $\varphi(65)=48$, $\mathrm{ord}_5(3)=4$, and $\mathrm{ord}_{13}(3)=3$; hence, $\mathrm{ord}_{65}(3)=12$. Thus, $c(3, 65, 1)=48/12=4$, and the four relevant cycles are

$$(1, 3, 9, 27, 16, 48, 14, 42, 61, 53, 29, 22),$$
$$(2, 6, 18, 54, 32, 31, 28, 19, 57, 41, 58, 44),$$
$$(4, 12, 36, 43, 64, 62, 56, 38, 49, 17, 51, 23), \text{ and}$$
$$(7, 21, 63, 59, 47, 11, 33, 34, 37, 46, 8, 24).$$

Proof: Let $r = \text{ord}_n(a)$. Then the elements of the cycle $(1, a, \dots, a^{r-1})$ form a subgroup $\langle a \rangle$ of U_n of order r. The claim is that the cosets of $\langle a \rangle$ in U_n and the cycles in $G(a, n)$ with GCD 1 are in one-to-one correspondence. For, writing $x \sim y$ to mean that x and y are in the same coset of $\langle a \rangle$ in U_n, we see that $x \sim y$ if and only if $x^{-1}y \equiv a^i \pmod{n}$ for some integer i. But this is precisely the condition that x and y lie on a cycle in $G(a, n)$. Hence, $c(a, n, 1)$ is equal to the number of cosets of $\langle a \rangle$ in U_n, i.e., the index of $\langle a \rangle$ in U_n. But since the group U_n has order $\varphi(n)$, this index is just $\frac{\varphi(n)}{\text{ord}_n(a)}$. □

Lemma 3: If $(a, n) = 1$ and $d|n$, then $c(a, n, d) = c\left(a, \frac{n}{d}, 1\right)$.

For example, the cycles in $G(2, 45)$ with GCD 3 are (3, 6, 12, 24) and (21, 42, 39, 33); the corresponding cycles in $G(2, 15)$ with GCD 1 are (1, 2, 4, 8) and (7, 14, 13, 11).

Proof: Let $(x_1, x_2, \dots, x_r)$ be a cycle in $G(a, n)$ with GCD d. Then $x_2 \equiv ax_1, \dots, x_r \equiv a^{r-1}x_1$ and $x_1 \equiv a^r x_1 \pmod{n}$ with r positive and minimal. This is true if and only if $(1, a, \dots, a^{r-1})$ is a cycle in $G\left(a, \frac{n}{(n, x_1)}\right) = G\left(a, \frac{n}{d}\right)$ (clearly with GCD 1). Hence, each cycle $G(a, n)$ with GCD d has length $r = \text{ord}_{n/d}(a)$. Furthermore, x and y lie on a cycle in $G(a, n)$ with GCD d if and only if $y \equiv xa^i \pmod{n}$, i.e., $\frac{y}{d} \equiv \frac{x}{d}a^i \left(\text{mod } \frac{n}{d}\right)$—which is precisely the condition that $\frac{x}{d}$ and $\frac{y}{d}$ lie on a cycle in $G\left(a, \frac{n}{d}\right)$. Thus, the number of cycles in $G(a, n)$ with GCD d is the same as the number of cycles in $G\left(a, \frac{n}{d}\right)$ with GCD 1. That is, $c(a, n, d) = c\left(a, \frac{n}{d}, 1\right)$. □

We are now ready for the main result of this section.

Theorem A: If $(a, n) = 1$, then

$$C(a, n) = \sum_{d|n} \frac{\varphi(d)}{\text{ord}_d(a)}.$$

Thus,

$$\begin{aligned} C(5, 77) &= \frac{\varphi(1)}{\text{ord}_1(5)} + \frac{\varphi(7)}{\text{ord}_7(5)} + \frac{\varphi(11)}{\text{ord}_{11}(5)} + \frac{\varphi(77)}{\text{ord}_{77}(5)} \\ &= \frac{1}{1} + \frac{6}{6} + \frac{10}{5} + \frac{60}{30} \\ &= 1 + 1 + 2 + 2 = 6. \end{aligned}$$

Proof: We have

$$\begin{aligned} C(a, n) &= \sum_{d|n} c(a, n, d) \\ &= \sum_{d|n} c\left(a, \frac{n}{d}, 1\right) \quad \text{(by Lemma 3)} \\ &= \sum_{d|n} c(a, d, 1) \quad \text{(by reordering the sum)} \\ &= \sum_{d|n} \frac{\varphi(d)}{\text{ord}_d(a)} \quad \text{(by Lemma 2). } \square \end{aligned}$$

3. THE PARITY OF $C(a, n)$ FOR $(a, n) = 1$

Next, we determine the parity of the number of cycles in $G(a, n)$ with GCD 1; from that, we determine the parity of $C(a, n)$ for $(a, n) = 1$.

Lemma 4: Let p be an odd prime, let r be a positive integer, and let $(a, p) = 1$. Put $p-1 = 2^s q$, where q is odd. ***(a)*** If $(a/p) = 1$, then $\mathrm{ord}_{p^r}(a) | 2^{s-1}qp^{r-1}$. ***(b)*** If $(a/p) = -1$, then $2^s | \mathrm{ord}_{p^r}(a)$.

Proof: Euler's criterion for the Legendre symbol states that $(a/p) \equiv a^{(p-1)/2} \pmod p$. Thus, if $p-1 = 2^s q$, where q is odd, then $(a/p) \equiv a^{2^{s-1}q} \pmod p$. We have two cases:

(a) If $(a/p) = 1$, then $a^{2^{s-1}q} \equiv 1 \pmod p$, so that $\mathrm{ord}_p(a) | 2^{s-1}q$. If the statement is true for some $r \ge 1$, then $a^{2^{s-1}qp^{r-1}} = 1 + kp^r$. Raising both sides to the p^{th} power, we have $a^{2^{s-1}qp^r} = (1+kp^r)^p \equiv 1 \pmod{p^{r+1}}$. Hence, $\mathrm{ord}_{p^r}(a) | 2^{s-1}qp^{r-1}$ by induction.

(b) If $(a/p) = -1$, then $a^{2^{s-1}q} \equiv -1 \pmod p$, so that $2^s | \mathrm{ord}_p(a)$. Since $\mathrm{ord}_p(a)$ is a divisor of $\mathrm{ord}_{p^r}(a)$ for $r \ge 1$, we are done. □

Lemma 5: Let $(a, n) = 1$ with n odd. If $n = p^r$, where p is a prime and if $(a/p) = -1$, then $c(a, n, 1)$ is odd; in all other cases, $c(a, n, 1)$ is even.

Proof: Let $p-1 = 2^s q$, where q is odd. By Lemma 4, if $(a/p) = -1$, then $\mathrm{ord}_{p^r}(a) = 2^s k$ with k odd. Since $\varphi(p^r) = p^{r-1}(p-1) = p^{r-1}2^s q$, it follows from Lemma 2 that

$$c(a, p^r, 1) = \frac{\varphi(p^r)}{\mathrm{ord}_{p^r}(a)} = \frac{p^{r-1}q}{k},$$

which is an odd number. Hence, $c(a, p^r, 1)$ is odd.

We must now show that $c(a, n, 1)$ is even in all other cases.

First, if $n = p^r$ with p as above, and if $(a/p) = 1$, then the highest power of 2 dividing $\mathrm{ord}_{p^r}(a)$ is 2^{s-1}. Since $2^s | \varphi(p^r)$, it follows that the fraction $\frac{\varphi(p^r)}{\mathrm{ord}_{p^r}(a)}$ is even.

Next, if $n = \prod_{i=1}^g p_i^{e_i}$ with $g > 1$ and $p_i - 1 = 2^{s_i} q_i$, then

$$\mathrm{ord}_n(a) \,|\, [p_1^{e_1-1}\cdot 2^{s_1}q_1, \ldots, p_g^{e_g-1}\cdot 2^{s_g}q_g] = \prod_{i=1}^{g} p_i^{e_i-1}[q_1, \ldots, q_g]\cdot 2^M,$$

where $M = \max(s_1, \ldots, s_g)$. Now let $S = \sum_{i=1}^g s_i$. Since n is divisible by at least two distinct odd primes, it follows that $S > M$, so that $c(a, n, 1) = \frac{\varphi(n)}{\mathrm{ord}_n(a)}$ is divisible by 2^{S-M}. Hence, $c(a, n, 1)$ is even. □

A slight modification of the above proof yields the following lemma.

Lemma 6: Let $(a, n) = 1$ with n even.

(a) If n is divisible either by 8 or by more than one odd prime, or if $n = 4p^e$ with p an odd prime, then $c(a, n, 1)$ is even.

(b) If p is an odd prime, then $c(a, p^e, 1) = c(a, 2p^e, 1)$.

(c) $c(a, 1, 1) = c(a, 2, 1) = 1$ and $c(a, 4, 1) = \frac{(-1/a)+3}{2}$.

We may now prove our main results.

Theorem B: Let a and n be relatively prime, and let n be odd. Then the number of cycles in $G(a,n)$ is odd or even according as a is or is not a quadratic residue mod n. That is $C(a,n) \equiv \frac{1+(a/n)}{2}$ (mod 2).

For example, $C(3,1001)$ is even because $(3/1001)=(1001/3)=(2/3)=-1$. A bit of direct calculation reveals that $\mathrm{ord}_7(3)=6$, $\mathrm{ord}_{11}(3)=5$, and $\mathrm{ord}_{13}(3)=3$, so that

$$\begin{aligned} C(3,1001) &= \sum_{d\mid 1001} \frac{\varphi(d)}{\mathrm{ord}_d(a)} \\ &= 1+\frac{6}{6}+\frac{10}{5}+\frac{12}{3}+\frac{60}{30}+\frac{72}{6}+\frac{120}{15}+\frac{720}{30} = 1+1+2+4+2+12+8+24=54, \end{aligned}$$

which is indeed even. Somewhat more tricky is the evaluation of $C(2159,pq)$, where both $p=205909401806482731 2345603$ and $q=53428614127183181483133 3517$ are primes. But, since $pq\equiv 3 \pmod 4$, we see that $(2159/pq)=-(pq/2159)=-(743/2159)=(2159/743)$, which reduces to the product $(2/673)(8/35)$, or -1. Hence, $C(2159,pq)$ is even.

Proof: Let $n=\prod_{i=1}^{g} p_i^{e_i}$ with each p_i odd, and suppose $(a,n)=1$. It follows from Theorem A and Lemma 5 that

$$C(a,n)=\sum_{d\mid n}\frac{\varphi(d)}{\mathrm{ord}_d(a)} \equiv 1+\sum_{i=1}^{g}\sum_{j=1}^{e_i}\frac{\varphi(p_i^j)}{\mathrm{ord}_{p_i^j}(a)} \pmod 2,$$

since all other terms are even. If we order the primes p_i so that for some integer f (which might be 0), $(a/p_i)=1$ if and only if $i>f$, then we see that

$$C(a,n)\equiv 1+\sum_{i\le f}\sum_{j=1}^{e_i}1 \pmod 2 \equiv 1+\sum_{i\le f}e_i \pmod 2.$$

On the other hand, since n is odd and $(a,n)=1$, we use the well-known properties of the Legendre and Jacobi symbols to see that

$$\begin{aligned} (a/n) &= \prod_{i=1}^{g}(a/p_i)^{e_i} = \prod_{i\le f}(-1)^{e_i} \quad [\text{since } (a/p_i)=1 \text{ for } i>f] \\ &= (-1)^{\sum_{i\le f} e_i}, \quad \text{so that} \\ (-1)^{C(a,n)} &\equiv (-1)^{1+\sum_{i\le f} e_i} \equiv -(a/n) \pmod 2. \end{aligned}$$

Hence, $C(a,n)$ is odd if $(a/n)=1$, and $C(a,n)$ is even if $(a/n)=-1$, and we are done. □

Theorem C: Let a and n be relatively prime, let n be even, and write $n=2^e n'$, where n' is odd.

(a) If $e=1$, then $G(a,n)$ has an even number of cycles.

(b) If $e\ge 2$, then the number of cycles in $G(a,n)$ is even or odd according as -1 is or is not a quadratic residue mod n'. That is,

$$C(a,n)\equiv\frac{1-(-1/n')}{2} \pmod 2.$$

Proof: Theorem C follows from Theorem A and Lemma 6 in the same way that Theorem B follows from Theorem A and Lemma 5. □

4. THE PARITY OF $C(a, n)$ FOR ARBITRARY a AND n

We are now ready to extend Theorems B and C to the graphs $G(a, n)$, where a and n are not relatively prime. The principal observation is the correspondence between the cycles in $G(a, qm)$ and the cycles in $G(a, m)$. Specifically, we have the following lemma.

Lemma 7: Suppose that $(m, a) = 1$ and that each prime divisor of q divides a. Then $C(a, qm) = C(a, m)$.

Proof: Let x be an integer mod qm. We may write $x = (x_a, y)$, where $(y, a) = 1$ and each prime divisor of x_a divides a. Thus, $(x_a, q) = 1$. Now let $i \geq 0$ and $r > 0$ be minimal and satisfy $a^{i+r}x \equiv a^i x \pmod{qm}$. This happens if and only if $y(a^r - 1)(a^i x_a) \equiv 0 \pmod{qm}$. But $(a^i x_a, q) = 1$ and $(y(a^r - 1), m) = 1$. Hence, the above congruence holds if and only if $q \mid y(a^r - 1)$ and $m \mid a^i x_a$. Thus, $(a^i x, \ldots, a^{i+r-1}x)$ is a cycle in $G(a, qm)$ if and only if i is the least nonnegative integer such that $m \mid a^i x$ and $(y, ay, \ldots, a^{r-1}y)$ is a cycle in $G(a, q)$, where y is the largest divisor of x relatively prime to m. But this means that the cycles of $G(a, qm)$ and the cycles of $G(a, q)$ are in one-to-one correspondence, i.e., $C(a, qm) = C(a, m)$. □

As a direct consequence of Lemma 7, we have the following result.

Theorem D: If a and n are positive integers, then the parity of $C(a, n)$ is equal to the parity of $C(a, n')$, where n' is the largest divisor of n that is relatively prime to a.

5. THE CYCLE STRUCTURE OF THE GRAPHS $GP(a, n)$ FOR n A PRIME

Let $GP(a, n)$ be the directed graph with vertex set $V = \{0, 1, \ldots, n-1\}$ such that there is an arc from x to y if and only if $y \equiv x^a \pmod{n}$. Let $CP(a, n)$ denote the number of cycles in the graph $GP(a, n)$.

There are some interesting differences between the graphs $GP(a, n)$ and $G(a, n)$. For example, if $(a, n) = 1$, then every vertex of $G(a, n)$ lies on a cycle. This is not the case for the vertices of $GP(a, n)$. If p^n is a prime power, then $GP(a, p^n)$ looks like a union of charm bracelets, with each charm a tree that corresponds to a coset of a certain subgroup U of roots of unity mod p^n. In particular, if we write $\varphi(p^n) = qr$, where $(q, a) = 1$, every prime divisor of r divides a, and m is the least positive integer such that $r \mid a^m$, then U consists of the a^mth roots of unity mod $\varphi(p^n)$.

Our principal result of this section is the following theorem.

Theorem P: If p^n is an odd prime, then there is a one-to-one correspondence between the cycles of $GP(a, p^n)$ and the cycles of $G(a, q)$, where q is the largest divisor of $\varphi(p^n)$ that is relatively prime to a. Furthermore,

$$CP(a, p^n) = 1 + \sum_{d \mid \varphi(p^n), (d, a)=1} \frac{\varphi(d)}{\mathrm{ord}_d(a)}.$$

The following lemma leads us to the proof of Theorem P.

Lemma 8: Let p^n be a prime power, let g be a primitive root (mod p), let $(a, p) = 1$, and write $\varphi(p^n) = qr$, where $(q, a) = 1$ and every prime divisor of r divides a. Then x and y lie on a cycle in $GP(a, p^n)$ if and only if either ***(a)*** there exist integers j and k such that $x \equiv g^{rj} \pmod{p^n}$, $y \equiv g^{rk}$ (mod p), and j and k lie on a cycle of $G(a, q)$, or ***(b)*** $x = y = 0$.

Proof: If $p|x$, then for some positive integer s, $x^{a^s} \equiv 0 \pmod{p^n}$. Thus, if $p|x$, then x lies on a cycle in $GP(a, p^n)$ if and only if $x \equiv 0 \pmod{p^n}$. From here on, we assume that x and y are relatively prime to p.

If x is a vertex of $GP(a, p^n)$, then we may write $x \equiv g^t \pmod{p^n}$ for some integer t with $0 \le t < \varphi(p^n)$. Let us first show that x lies on a cycle of $GP(a, p^n)$ if and only if $r|t$. We have the following sequence of equivalent statements:

$$\begin{aligned} & x \text{ lies on a cycle of } GP(a, p^n) \\ \text{if and only if} \quad & x^{a^s} \equiv x \pmod{p^n} \text{ for some positive integer } s, \\ \text{if and only if} \quad & g^{t(a^s-1)} \equiv 1 \pmod{p^n} \text{ for some positive integer } s, \\ \text{if and only if} \quad & \varphi(p^n) | t(a^s-1). \end{aligned}$$

Hence, if x lies on a cycle of $GP(a, p^n)$, then $rq|t(a^s-1)$. Now each prime divisor of r divides a, so it follows that $(r, a^s-1) = 1$. We conclude that $r|t$.

Conversely, suppose that $r|t$, so that $x \equiv g^{rj} \pmod{p^n}$ for some integer j. If $j = 0$, then $x = 1$, which is clearly on its own cycle; since $g^{\varphi(p^n)} \equiv 1 \pmod{p^n}$, we may assume that $1 \le j \le q-1$. The above argument shows that x is on a cycle if and only if $rq|rj(a^s-1)$ for some integer s. Since $1 \le j \le q-1$, it follows that $q|(a^s-1)$. In particular, if $s = \mathrm{ord}_q(a)$, then we may conclude that x lies on a cycle of length s.

Next, x and y will lie on a common cycle if and only if $x \equiv g^{rj} \pmod{p^n}$ and $y \equiv g^{rk}$ (mod p^n) lie on a common cycle of $GP(a, p^n)$. It is straightforward to verify that this happens if and only if there exists an integer m such that $ja^m \equiv k \pmod q$—i.e., that j and k lie on a cycle of $G(a, q)$.

Finally, if $(j, ja, \ldots, k \equiv ja^m, \ldots, ja^{s-1})$ is a cycle in $G(a, q)$, then it follows that $s = \mathrm{ord}_q(a)$, which means that $(g^{rj}, g^{rja}, \ldots, g^{rja^m}, \ldots, g^{rja^{s-1}})$ is a cycle in $PG(a, p^n)$, and we are done. □

Theorem P now follows from Lemma 8 and Theorem A, and from the fact that there is one extra cycle in $PG(a, p^n)$—the cycle consisting of the directed loop from the vertex 0 to itself.

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DERIVATIVE SEQUENCES OF JACOBSTHAL AND JACOBSTHAL-LUCAS POLYNOMIALS

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1. AIM OF THE PAPER

The *Jacobsthal polynomials* $J_n(x)$ and the *Jacobsthal-Lucas polynomials* $j_n(x)$, whose properties have been investigated in [4], are a natural extension of the *Jacobsthal numbers* J_n and the *Jacobsthal-Lucas numbers* j_n which, in turn, have been investigated in [3]. These polynomials are defined by the second-order recurrence relations

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x), \quad [J_0(x) = 0, \ J_1(x) = 1] \tag{1.1}$$

and

$$j_{n+2}(x) = j_{n+1}(x) + 2xj_n(x), \ [j_0(x) = 2, \ j_1(x) = 1], \tag{1.2}$$

respectively, where x is an indeterminate.

Since throughout this paper we shall make use of the notation and the formulas found in [3] and [4], the reader is assumed to be aware of the contents of these papers.

Definitions: Following the idea exploited in [1], let us define the polynomials $J_n^{(1)}(x)$ and $j_n^{(1)}(x)$ {see (3.9) and (3.10) of [4] for the combinatorial representations of $J_n(x)$ and $j_n(x)$} as

$$J_n^{(1)}(x) = \frac{d}{dx} J_n(x) = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} 2^r r \binom{n-1-r}{r} x^{r-1} \quad (n \ge 0), \tag{1.3}$$

$$j_n^{(1)}(x) = \frac{d}{dx} j_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{2^r nr}{n-r} \binom{n-r}{r} x^{r-1} \quad (n \ge 1), \tag{1.4}$$

and

$$J_0^{(1)}(x) = j_0^{(1)}(x) = 0, \tag{1.5}$$

where the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function, and the bracketed superscript symbolizes the first derivative with respect to x.

The aim of this paper is to study some properties of the above sequences just as was done in [1] for the Fibonacci and Lucas polynomials. Here, we shall also confine ourselves to considering the case $x = 1$. Since letting $x = 1$ in (1.1) and (1.2) will yield the Jacobsthal numbers and the Jacobsthal-Lucas numbers {cf. (2.3) and (2.4) of [3]}

$$J_n = \frac{2^n - (-1)^n}{3} \quad \text{and} \quad j_n = 2^n + (-1)^n, \tag{1.6}$$

the sequences $\{J_n^{(1)}(1)\}$ and $\{j_n^{(1)}(1)\}$ will be referred to as *Jacobsthal* and *Jacobsthal-Lucas derivative sequences*. For notational convenience, their terms $J_n^{(1)}(1)$ and $j_n^{(1)}(1)$ will be denoted by H_n and K_n, respectively. From (1.3)-(1.5), the numbers H_n and K_n can be obtained readily for the first few values of n. They are shown in Table 1.

TABLE 1. The Numbers H_n and K_n for $0 \le n \le 8$

n	0	1	2	3	4	5	6	7	8
$J_n^{(1)}(1) = H_n$	0	0	0	2	4	14	32	82	188
$j_n^{(1)}(1) = K_n$	0	0	4	6	24	50	132	294	688

2. CLOSED-FORM EXPRESSIONS FOR H_n AND K_n

Closed-form expressions for H_n and K_n are, quite obviously, useful tools for discovering their properties. They are established in this section, where some equivalent expressions for these numbers are also found.

By using formulas (1.4), (1.5), (3.3), and (3.4) of [4], we easily see that

$$\Delta^{(1)}(x) = \frac{d}{dx}\Delta(x) = 4/\Delta(x),$$

$$\alpha^{(1)}(x) = \frac{d}{dx}\alpha(x) = \Delta^{(1)}(x)/2 = 2/\Delta(x),$$

$$\beta^{(1)}(x) = \frac{d}{dx}\beta(x) = -\Delta^{(1)}(x)/2 = -2/\Delta(x),$$

$$[\alpha^n(x)]^{(1)} = \frac{d}{dx}\alpha^n(x) = n\alpha^{n-1}(x)\alpha^{(1)}(x) = 2n\alpha^{n-1}(x)/\Delta(x),$$

and

$$[\beta^n(x)]^{(1)} = \frac{d}{dx}\beta^n(x) = n\beta^{n-1}(x)\beta^{(1)}(x) = -2n\beta^{n-1}(x)/\Delta(x).$$

Hence, we have

$$J_n^{(1)}(x) = \frac{d}{dx}\left[\frac{\alpha^n(x)-\beta^n(x)}{\Delta(x)}\right] = 2\frac{nj_{n-1}(x)-2J_n(x)}{\Delta^2(x)}, \tag{2.1}$$

and

$$j_n^{(1)}(x) = 2nJ_{n-1}(x). \tag{2.2}$$

Letting $x = 1$ in (2.1) and (2.2) leads to the relations

$$H_n = J_n^{(1)}(1) = \frac{2(nj_{n-1}-2J_n)}{9} \tag{2.3}$$

and

$$K_n = j_n^{(1)}(1) = 2nJ_{n-1} \tag{2.4}$$

which express H_n and K_n in terms of J_n and j_n.

By (2.3) and (2.4) above, and (1.6), the following relations can be obtained readily:

$$H_n = \frac{2^n(3n-4)-(6n-4)(-1)^n}{27} \tag{2.5}$$

and

$$K_n = \frac{n[2^n + 2(-1)^n]}{3}, \tag{2.6}$$

which express H_n and K_n in terms of their subscripts.

Observe that using (2.5) and (2.6) above, along with (1.6), we obtain the relations

$$H_n = \frac{(3n-4)J_n - n(-1)^n}{9} \tag{2.7}$$

and

$$K_n = \frac{n[j_n + (-1)^n]}{3}, \tag{2.8}$$

which express H_n in terms of J_n and K_n in terms of j_n, respectively.

3. BASIC PROPERTIES OF H_n AND K_n

Some relations involving H_n and K_n are established in this section, most of which are the analogs of those found by Horadam in [3] for J_n and j_n. Some simple but sometimes tedious manipulations involving the use of (2.3)-(2.8) provide the required proofs. To save space, only the proofs of Theorems 1-3 will be given in detail in Subsection 3.2.

3.1. Results

Generating functions

$$\sum_{n=0}^{\infty} H_n y^n = \frac{2y^3}{(2y^2+y-1)^2}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} K_n y^n = \frac{2y^2(2-y)}{(2y^2+y-1)^2}. \tag{3.2}$$

These functions can be obtained readily from (3.1) and (3.2) of [4].

Recurrence relations

$$H_{n+2} = H_{n+1} + 2H_n + 2J_n, \tag{3.3}$$

$$K_{n+2} = K_{n+1} + 2K_n + 2j_n. \tag{3.4}$$

These relations can be obtained readily by calculating at $x = 1$ the first derivative with respect to x of both sides of (1.1) and (1.2).

Some identities

$$H_n K_n = \frac{n}{9}[K_{2n-1} - 2J_{n-1}(4J_n - j_{n-1})] \tag{3.5}$$

$$= \frac{n}{3}H_{2n} - \frac{n}{81}[(-2)^{n+2} + 3n4^n - 4] \tag{3.5'}$$

$$H_{n+1} + 2H_{n-1} = K_n - 2J_{n-1} \tag{3.6}$$

$$= 2(n-1)J_{n-1} \quad \text{[by (2.4)]}, \tag{3.6'}$$

$$K_{n+1} + 2K_{n-1} = 9H_n + 2J_n + 2^n, \tag{3.7}$$

$$H_n + K_n = 2H_{n+1} \quad \text{[from (2.5) and (2.6)]}, \tag{3.8}$$

$$K_n - H_n = 4(H_{n-1} + J_{n-1}), \tag{3.9}$$

$$K_n = 3H_n + \frac{1}{9}[4(-1)^n(3n-1) + 2^{n+2}]. \tag{3.10}$$

Observe that identity (3.8) is an important feature of H_n and K_n, being analogous to $J_n + j_n = 2J_{n+1}$ for Jacobsthal and Jacobsthal-Lucas numbers.

Simson formula analogs

$$H_{n+1}H_{n-1} - H_n^2 = \frac{1}{81}[(-2)^n(9n^2 - 18n + 5) - 4^n - 4], \tag{3.11}$$

$$K_{n+1}K_{n-1} - K_n^2 = -\frac{1}{9}[(-2)^n(9n^2 - 5) + 4^n + 4]. \tag{3.12}$$

Limits

$$\lim_{n\to\infty} H_{n+1} / H_n = \lim_{n\to\infty} K_{n+1} / K_n = 2, \tag{3.13}$$

$$\lim_{n\to\infty} K_n / H_n = 3. \tag{3.14}$$

Evaluation of some finite sums

$$S_n \stackrel{\text{def}}{=} \sum_{k=0}^{n} H_k = 2H_n - \frac{1}{18}[2^{n+2} - (-1)^n(6n-5) - 9], \tag{3.15}$$

$$T_n \stackrel{\text{def}}{=} \sum_{k=0}^{n} K_k = 2K_n - \frac{1}{6}[(-1)^n(6n-1) + 2^{n+2} - 3]. \tag{3.16}$$

Alternative, but perhaps less elegant, expressions for S_n and T_n can be obtained after several tedious manipulations involving the use of (2.4) and (2.7). They are

$$S_n = \frac{1}{108}[2^n(27n - 56) - 9K_n - (-1)^n(12\lceil n/2 \rceil - 5) + 51], \tag{3.15$'$}$$

where the symbol $\lceil x \rceil$ denotes the least integer not less than x, and

$$T_n = \frac{1}{2}[K_n + J_{n+1} + 2^n(n-2) + 1]. \tag{3.16$'$}$$

$$\sum_{k=0}^{n} \binom{n}{k} H_k = 2(n-2)3^{n-3} + \frac{1}{9}\left[2\delta_{1,n} + \frac{4}{3}\delta_{0,n}\right], \tag{3.17}$$

where $\delta_{a,b} = 1$ (0) if $a = (\neq) b$ is the Kronecker symbol,

$$\sum_{k=0}^{n} \binom{n}{k} K_k = 2n3^{n-2} - \frac{2}{3}\delta_{1,n}. \tag{3.18}$$

Convolution properties

$$H_n = \sum_{k=0}^{n} J_k J_{n-k} - \frac{1}{9}[2^n + (-1)^n(3n-1)], \tag{3.19}$$

$$K_n = \frac{1}{3}\sum_{k=0}^{n} j_k j_{n-k} - \frac{1}{9}[2^n 7 - (-1)^n(3n-5)], \tag{3.20}$$

$$\sum_{k=0}^{n} H_k H_{n-k} = 3^{-7}[2^{n-1}(9n^3-72n^2+159n-80)+(-1)^n(18n^3-72n^2+30n+40)], \quad (3.21)$$

$$\sum_{k=0}^{n} K_k K_{n-k} = 3^{-5}[2^{n-1}(9n^3-57n+16)+(-1)^n(18n^3-42n-8)]. \quad (3.22)$$

***Remarks*:**

(i) The geometric series formula has to be used along with (2.3)-(2.8) to prove (3.15)-(3.22).

(ii) The identities (3.19) and (3.20) can be checked easily by using (2.5), (2.6), and the identities

$$\sum_{k=0}^{n} J_k J_{n-k} = \frac{1}{9}[(n+1)j_n - 2J_{n+1}] \quad (3.23)$$

and

$$\sum_{k=0}^{n} j_k j_{n-k} = (n+1)j_n + 2J_{n+1}, \quad (3.24)$$

which are obtainable by using (1.6) and the geometric series formula.

Congruence properties

Congruence properties of H_n and K_n deserve a thorough investigation. Nevertheless, in this paper we shall confine ourselves to considering the residue of these numbers modulo their subscripts. That K_n is divisible by n for all $n>0$ is patent by (2.4). A brief computer experiment showed that the values of $n \le 1000$ for which H_n is divisible by n are 1, 2, 4, 20, 100, 220, 500, 620, and 820.

Theorem 1: There exist infinitely many values of n for which $H_n \equiv 0 \pmod{n}$.

Theorem 2: If $p \neq 3$ is a prime, then

$$H_p \equiv -12\left[\frac{1+p(-1)^{p\,(\mathrm{mod}\,3)}}{3}\right]^3 \pmod{p}.$$

Theorem 3: If $p \neq 3$ is a prime, then $K_p \equiv 0 \pmod{p^2}$.

3.2. Proofs of Special Results

Proof of Theorem 1: We shall prove that, if $n = n(k) = 5^k 4$ $(k = 0, 1, 2, \ldots)$, then $H_n \equiv 0$ (mod n). Let B_n denote the numerator of the fraction on the right-hand side of (2.5). Since $n(k)$ and 27 are coprime, it suffices to prove that $B_{n(k)} \equiv 0 \pmod{n(k)}$. After some simple manipulation, it is apparent that this is equivalent to proving that $4[2^{n(k)} - (-1)^{n(k)}] \equiv 0 \pmod{n(k)}$, that is, to proving the validity of the congruence

$$2^{n(k)} \equiv 1 \pmod{5^k}. \quad (3.25)$$

By Euler's theorem, it is known that $2^{n(k)} = 2^{5^k 4} = 2^{\phi(5^{k+1})} \equiv 1 \pmod{5^{k+1}}$, whence (3.25) is satisfied *a fortiori*.

By Table 1, it is immediately seen that the congruence $H_n \equiv 0 \pmod{n}$ holds for n = 1, 2, and 4. We now state a proposition that gives the general solution to the problem of finding *all* $n > 4$

for which this congruence is satisfied. Of course, this general solution encompasses the case $n = 5^k 4$ considered in the proof of Theorem 1.

Proposition 1: For $n > 4$, $H_n \equiv 0 \pmod{n}$ if and only if

$$n = 4p_2^{a_2} p_3^{a_3} \cdots (p_2 = 5, p_3 = 7, p_4 = 11, \ldots;\ a_2 \geq 1, a_i \geq 0 \text{ for } i > 3),$$

and $\operatorname{ord}(2, p_i^{a_i})$ divides n for all i such that $a_i \geq 1$, where (see [2], p. 71) the symbol $\operatorname{ord}(a, b)$ [defined for g.c.d.$(a, b) = 1$] denotes the least exponent x for which $a^x \equiv 1 \pmod{b}$.

The proof of Proposition 1 is extremely long and cumbersome; it is omitted for the sake of brevity, but it is available on request.

Proof of Theorem 2: By (2.5), we get the congruence

$$H_p \equiv \frac{-2^{p+2} - 4}{27} \equiv -\frac{12}{27} \pmod{p} \quad \text{(by Fermat's Little Theorem).}$$

The desired result is obtained readily by observing that the multiplicative inverse of 27 modulo a prime $p \neq 3$ is $\{[1 + p(-1)^{p \pmod{3}}]/3\}^3$.

Proof of Theorem 3: First, by Table 1, we observe that $K_2 \equiv 0 \pmod{4}$ and $K_3 \equiv 6 \pmod{9}$. Then, for $p > 5$, let us define $M_p = K_p / p$ and prove that $M_p \equiv 0 \pmod{p}$. By (2.4) and (1.6), we can write

$$M_p = 2J_{p-1} = \frac{2^p - 2}{3} \equiv \frac{2-2}{3} \equiv 0 \pmod{p} \quad \text{(by Fermat's Little Theorem).}$$

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THE $(2, T)$ GENERALIZED FIBONACCI SEQUENCES

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Recently the $(2, F)$ and $(3, F)$ generalized Fibonacci sequences were considered and the generating functions for these sequences were derived (see [1] through [8]). The purpose of this note is to derive generating functions for the $(2, T)$ generalized Tribonacci sequences.

Let $S = (a, b)$ and S_b be the group of permutations on S. Let i be the identity and $\alpha = (a, b)$. Let τ_i be a permutation of S_b for $0 \le i \le 2$ and $Y_i = \{a_i, b_i\}$ for $i \ge 0$. Finally, let a_1, a_2, a_3, b_1, b_2, and b_3 be six distinct real numbers. Then

$$Y_{n+3} = \sum_{i=0}^{2} \tau_i Y_{n+i}, \quad n \ge 0, \tag{1}$$

with initial conditions $Y_i = \{a_i, b_i\}$ for $0 \le i \le 2$, are the eight systems of third-order difference equations defining the $(2, T)$ generalized Tribonacci sequences.

Define

$$\delta_i = \begin{cases} 0 & \text{if } \tau_i = i, \\ 1 & \text{if } \tau_i = (a, b), \end{cases}$$

and $S = \sum_{i=0}^{2} \delta_i 2^i$. Then each of the eight systems (1) corresponds to an integer S where $0 \le s \le 7$. When S is expressed as a binary number and the right-hand member of (1) is arranged in descending order of subscripts, then the 1's in the binary number indicate the position(s) of the elements b_i in the equation for a_n and the position(s) of the elements a_i in the equation for b_n. If $s = 0 = 000_2$ the system is

$$a_{n+3} = a_{n+2} + a_{n+1} + a_n,$$
$$b_{n+3} = b_{n+2} + b_{n+1} + b_n.$$

In this case the $(2, T)$ generalized Fibonacci sequences are a pair of generalized Tribonacci sequences. This case is excluded from further consideration.

Consider the seven difference systems

$$iY_{n+3}^s = \tau_2 Y_{n+2}^s + \tau_1 Y_{n+1}^s + \tau_0 Y_n^s, \quad 1 \le s \le 7, \tag{2}$$

with initial conditions

$$Y_i^s = \{a_i, b_i\}, \quad 0 \le i \le 2.$$

Atanassov [3] proved that these systems are equivalent to seven sixth-order systems

$$\sum_{i=0}^{6} k_i^s a_{n+6-i}^s = 0, \quad \sum_{i=0}^{6} k_i^s b_{n+6-i}^s = 0, \quad n \ge 0, \tag{3}$$

with initial conditions $\langle a_i \rangle_0^5$ and $\langle b_i \rangle_0^5$, respectively. The values for k_i^s for $1 \le s \le 7$ and $0 \le i \le 6$ are given in Table 1.

TABLE 1. Values of k_i^s

s	0	1	2	3	4	5	6
1	1	−2	−1	2	1	0	−1
2	1	−2	1	−2	1	0	1
3	1	−2	1	0	−1	−2	−1
4	1	0	−3	−2	1	2	1
5	1	0	−3	0	−1	0	−1
6	1	0	−1	−4	−1	0	1
7	1	0	−1	−2	−3	−2	−1

Let $p^s(x)=\sum_{i=0}^{6}k_i^s x^i$ and let $\{P_j^s\}_{j=0}^{\infty}$ be the recursive numbers (of order six) determined by $1/p^s(x)$. Then the seven recursion relations and first terms of the sequences are given in Table 2.

TABLE 2

S	Recursive Relations	First 7 Terms						
1	$P_{n+6}=2P_{n+5}+P_{n+4}-2P_{n+3}-P_{n+2}+P_n$	1	2	5	10	20	38	72
2	$P_{n+6}=2P_{n+5}-P_{n+4}+2P_{n+3}-P_{n+2}-P_n$	1	2	3	6	12	22	40
3	$P_{n+6}=2P_{n+5}+P_{n+4}+P_{n+2}+2P_{n+1}+P_n$	1	2	3	4	6	12	26
4	$P_{n+6}=3P_{n+4}+2P_{n+3}-P_{n+2}-2P_{n+1}-P_n$	1	0	3	2	8	10	24
5	$P_{n+6}=3P_{n+4}+P_{n+2}+P_n$	1	0	3	0	10	0	34
6	$P_{n+6}=P_{n+4}+4P_{n+3}+P_{n+2}-P_n$	1	0	1	4	2	8	18
7	$P_{n+6}=P_{n+4}+2P_{n+3}+3P_{n+2}+2P_{n+1}+P_n$	1	0	1	2	4	6	12

Let $f^s(x)$ and $g^s(x)$ be the solutions to the seven systems and let

$$f^s(x)=\sum_{j=0}^{\infty}a_j^s x^j \quad \text{and} \quad g^s(x)=\sum_{j=0}^{\infty}b_j^s x^j.$$

Substituting $f^s(x)$ into the difference systems (3) yields

$$f^s(x)=\left(\sum_{i=0}^{5}q_i^s x^i\right)\left(\sum_{j=0}^{\infty}P_j^s x^j\right),$$

where P_j^s are from the sequences in Table 2 and $q_i^s=\sum_{m=0}^{i}k_m^s a_{i-m}^s$, $0\le i\le 5$.

Expanding and collecting terms gives

$$f^s(x)=\sum_{j=0}^{4}\left(\sum_{i=0}^{j}q_i^s P_{j-i}^s\right)x^j+\sum_{j=5}^{\infty}\sum_{i=0}^{5}(q_i^s P_{j-i}^s)\,x^j$$

for the generating function of $\{a_i^s\}_0^{\infty}$. The terms of the sequence are given by

$$a_j^s=\sum_{i=0}^{j}q_i^s P_{j-i}^s=\sum_{i=0}^{j}\left[\sum_{m=0}^{i}k_m^s a_{i-m}^s\right]P_{j-i}^s \quad \text{for } j<5,$$

and

$$a_j^s = \sum_{i=0}^{5} q_i^s P_{j-i}^s = \sum_{i=0}^{5} \left[\sum_{m=0}^{i} k_m^s a_{i-m}^s \right] P_{j-i}^s, \quad \text{for } j \geq 5.$$

The values of a_i^s, $3 \leq i \leq 5$, are computed in terms of a_0^s, a_1^s, a_2^s, b_0^s, b_1^s, and b_2^s by use of equations (2). The sequences $\{b_i^s\}_0^\infty$ have the same form for each s.

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RODRIGUES' FORMULAS FOR JACOBSTHAL-TYPE POLYNOMIALS

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1. INTRODUCTION

Motivation

Recently [2], some second-order differential properties of generalized Fibonacci polynomials and generalized Lucas polynomials were exhibited.

Here, we intend to

(i) obtain similar differential equations from a slightly different viewpoint in the more general context of the polynomials $W_n(x)$ and $\mathcal{W}_n(x)$ [3], and

(ii) discover analogous equations for Jacobsthal polynomials $J_n(x)$ and Jacobsthal-Lucas polynomials $j_n(x)$ [4], i.e., non-Fibonacci and non-Lucas polynomials.

Central to the process is the question:

Can we determine Rodrigues' formulas for $J_n(x)$ and $j_n(x)$ corresponding to those (in a somewhat different notation) for $U_n(x)$ and $V_n(x)$ in [2]?

Background

Essentially, the following basic material [3] is needed:

$$W_{n+2}(x) = p(x)W_{n+1}(x) + q(x)W_n(x), \quad W_0(x) = 0,\ W_1(x) = 1, \tag{1.1}$$

$$\mathcal{W}_{n+2}(x) = p(x)\mathcal{W}_{n+1}(x) + q(x)\mathcal{W}_n(x), \quad \mathcal{W}_0(x) = 2,\ \mathcal{W}_1(x) = p(x), \tag{1.2}$$

leading to (if we drop the functional notation)

$$W_n = \frac{\alpha^n - \beta^n}{\Delta}, \tag{1.3}$$

$$\mathcal{W}_n = \alpha^n + \beta^n, \tag{1.4}$$

where

$$\left.\begin{aligned} \alpha &= \tfrac{1}{2}\{p+\Delta\}, \\ \beta &= \tfrac{1}{2}\{p-\Delta\}, \\ \Delta &= \sqrt{p^2+4q} = \alpha - \beta. \end{aligned}\right\} \tag{1.5}$$

Differentiating once w.r.t. x gives

$$\Delta' = \frac{pp' + 2q'}{\Delta}. \tag{1.6}$$

Specialized cases of (1.1) and (1.2) are generalized the Fibonacci and Lucas polynomials $F_n = W_n$ and $L_n = \mathcal{W}_n$, for which $p = x, q = 1$, and the Jacobsthal and Jacobsthal-Lucas polynomials J_n and j_n, for which $p = 1, q = 2x$. (See [3] for other examples of "Fibonacci-type" polynomials, e.g., Pell, Chebyshev, and Fermat.)

Two dichotomous situations thus arise:

A. $q' = 0$ for "Fibonacci-type" polynomials like F_n and L_n;
B. $p' = 0$ for J_n and j_n.

Immediately from (1.6) we have

$$\Delta' = \begin{cases} \dfrac{pp'}{\Delta}, & (1.6A) \\ \dfrac{2q'}{\Delta}. & (1.6B) \end{cases}$$

Crucial to the theory are the derivatives [3]

$$\mathcal{W}_n' = \begin{cases} np'W_n & (q'=0), \\ nq'W_{n-1} & (p'=0), \end{cases} \tag{1.7}$$

so, in particular,

$$j_n' = 2nJ_{n-1}. \tag{1.8}$$

Finally, we record for later use the notation [2]

$$c_{n,0} = 2\frac{n!}{(2n)!} \quad (n \geq 0), \tag{1.9}$$

and

$$c_{n,r} = 2\frac{n!n(n+r)!}{(2n)!(n+r)(n-r)!} \quad (n \geq r \geq 1), \tag{1.10}$$

whence

$$c_{n,r+1} = (n^2 - r^2)c_{n,r} \quad (n \geq r+1 \geq 1). \tag{1.11}$$

Notation for Theorems: Letters F and $J(j)$ will be appended as subscripts to the Theorem number of theorems relating to Fibonacci-type polynomials and Jacobsthal-type polynomials, respectively. In this symbolism, we will have Theorem $1_F, \ldots,$ Theorem 3_J.

2. SOME BASIC DIFFERENTIAL EQUATIONS FOR RECURRENCES

A. Fibonacci-type Polynomials ($q' = 0$)

From (1.3)-(1.7), double differentiation of $\mathcal{W}_n$ leads to

$$\Delta^2 \mathcal{W}_n'' = n^2(p')^2 \mathcal{W}_n - np(p')^2 W_n$$

whence, with $\mathcal{W}_n = y$,

$$\Delta^2 y'' + pp'y' - (np')^2 y = 0. \tag{2.1}$$

Alternatively, if we follow the procedure in [2], while using our notation, then we arrive at (2.1) also, a process left to the reader.

Differentiating (2.1) r times in conjunction with Leibniz' rule, we deduce that $z = y^{(r)} = \mathcal{W}_n^{(r)}$ satisfies the differential equation

$$\Delta^2 z'' + (2r+1)pp'z' + (p')^2(r^2 - n^2)z = 0, \tag{2.2}$$

of which (2.1) is the special case when $r = 0$.

Illustrations of (2.1) are:

(i) the associated Morgan-Voyce polynomial $C_n = y$, for which $p = 2 + x, q = -1$, leading to [2]

$$x(x+4)y'' + (x+2)y' - n^2 y = 0;$$

(ii) the Chebyshev polynomial $T_n = y$, in which $p = 2x, q = -1\,(x = \cos\theta)$, yielding

$$(1-x^2)y'' - xy' + n^2 y = 0,$$

in conformity with [6, p. 260].

Starting now with the double differentiation of W_n in (1.3), we eventually arrive at the differential equation

$$\Delta^2 W_n'' + 3pp'W_n' - (p')^2(n^2-1)W_n = 0. \tag{2.3}$$

Compare this with (2.1). A quick check confirms that $r = 1$ in (2.2) does indeed give us (2.3), where we invoke (1.7) for $q' = 0$. Particular instances of (2.3) are

(a) the Morgan-Voyce polynomial B_n, for which $p = 2 + x, q = -1$, giving

$$x(x+4)B_n'' - 3(x+2)B_n' - (n^2-1)B_n = 0,$$

in conformity with [2, p. 455] on making the transformation $n \to n-1$ for our B_n;

(b) The Chebyshev polynomial S_n (in the notation of [2, p. 453]), where $p = 2x$, $q = -1$ $(x = \cos\theta)$, for which

$$(1-x^2)S_n'' - 3xS_n' + (n^2-1)S_n = 0$$

as in [6, p. 260], n being replaced by $n-1$ for our S_n.

Now (1.7), where $q' = 0$, immediately shows that $\mathcal{W}_n^{(r)} = np'W_n^{(r-1)}$ $(r \geq 1)$, i.e.,

$$W_n^{(r-1)} = \frac{1}{np'}\mathcal{W}_n^{(r)}. \tag{2.4}$$

Hence, $W_n^{(r-1)}$ satisfies (2.2). Combining this with (2.2), we deduce that

Theorem 1$_F$: $W_n^{(r-1)}$ and $\mathcal{W}_n^{(r)}$ both satisfy (2.2).

Example ($r = 2, n = 4; p = 2x, q = 1$, Pell-type polynomials [3]): $P_4^{(1)} = (8x^3 + 4x)'$ and $Q_4^{(2)} = (16x^4 + 16x^2 + 2)''$ both satisfy

$$(x^2+1)z'' + 5xz' - 12z = 0.$$

Observe that (2.2) can be cast in the more general form (cf. [2]):

$$\left[\Delta^{2r+1}z'\right]' = (p')^2(n^2 - r^2)\Delta^{2r-1}z. \tag{2.5}$$

Following the technique in [2] and using (2.5), we may establish the results corresponding to equations (2.9)-(2.11) in [2], namely $\left(\text{with } D^{(r)} \equiv \frac{d^r}{dx^r}\right)$:

$$D\left[\Delta^{2r+1}D^{(n+r)}\Delta^{2n-1}\right] = (p')^2(n^2-r^2)\Delta^{2r-1}D^{(n+r-1)}\Delta^{2n-1}, \tag{2.6}$$

$$D\left[\Delta^{-2r-1}D^{(n-r-1)}\Delta^{2n-1}\right] = (p')^2(n^2-(r+1)^2)\Delta^{-2r-3}D^{(n-r-2)}\Delta^{2n-1}, \tag{2.7}$$

$$D\left[\Delta D^{(n+1)}\Delta^{2n+1}\right] = (p')^2(n+1)^2\Delta^{-1}D^{(n)}\Delta^{2n+1}. \tag{2.8}$$

B. Jacobsthal (≡ non-Fibonacci)-type Polynomials ($p' = 0$)

Trying to apply the method used in [2], or variations of it, to J_n and j_n is likely to lead to frustration.

Therefore, we abandon this approach and start afresh.

Differentiate twice in the pivotal relation (1.7) for $p' = 0$. Then

$$\Delta^2 \mathcal{W}_n'' + (q')^2 \mathcal{W}_n' - n(n-1)(q')^2 \mathcal{W}_{n-2} = 0, \tag{2.9}$$

wherein the diminished subscript in the undifferentiated polynomial is particularly to be noted. [Check (2.9) when, for example, $j_4 = 8x^2+8x+1$, $j_6 = 16x^3+36x^2+12x+1$, for which $p=1$, $q=2x$, $\Delta^2 = 1+8x$.]

Continued differentiation with recourse to Leibniz' rule, as in [2], reveals the generalized form of (2.9) to be ($z_n = \mathcal{W}_n^{(r)}$)

$$\Delta^2 z_n'' + (4r+q')q'z_n' - n(n-1)(q')^2 z_{n-2} = 0. \tag{2.10}$$

Putting $r=0$ in (2.10) obviously leads us back to (2.9).

Repeated differentiation in (1.3) next yields, with little difficulty,

$$\Delta^2 W_n'' + 3(q')^2 W_n' - n(n-1)(q')^2 W_{n-2} = 0. \tag{2.11}$$

Contrast this with (2.3). One may readily verify (2.11) for, say, $J_5 = 4x^2+6x+1$, $J_7 = 8x^3+24x^2+10x+1$.

Proceeding for the sake of interest to differentiate (2.11) may times, we eventually arrive at the generalization ($z_n = W_n^{(r-1)}$)

$$\Delta^2 z_n'' + (4r+q')q'z_n' - n(n-1)(q')^2 z_{n-2} = 0. \tag{2.12}$$

Substituting $r=1$ clearly reproduces (2.11), since $q'=2$.

Bearing in mind (1.7) with $p'=0$ and (2.12), we conclude that

Theorem $\mathbf{1}_J$: $J_n^{(r-1)}$ and $j_n^{(r)}$ both satisfy (2.10).

Analogously to (2.5), we see that (2.10) may be reformulated as

$$\left[\Delta^{2r+1}z_n'\right]' = (q')^2 n(n-1)\Delta^{2r-1}z_{n-2}.$$

Corresponding to (2.6)-(2.8), we derive

$$D\left[\Delta^{2r+1}D^{(n+r)}\Delta^{2n-1}\right] = (q')^2 n(n-1)\Delta^{2r-1}D^{(n+r-3)}\Delta^{2n-1}, \tag{2.13}$$

$$D\left[\Delta^{-2r-1}D^{(n-r-1)}\Delta^{2n-1}\right] = (q')^2 n(n-1)\Delta^{-2r-3}D^{(n-r-4)}\Delta^{2n-1}, \tag{2.14}$$

$$D\left[\Delta D^{(n+1)}\Delta^{2n+1}\right] = (q')^2 n(n+1)\Delta^{-1}D^{(n-2)}\Delta^{2n+1}. \tag{2.15}$$

3. RODRIGUES' FORMULAS

Rodrigues' formulas for W_n, $\mathscr{W}_n$ (when $q' = 0$) and for J_n, j_n (when $p' = 0$) are now determined.

A. Case $q' = 0$.

Procedures followed in [2] using (1.9) will largely be applied here.

Theorem 2_F:

(i) $W_n = \dfrac{nc_{n,0}}{(p')^{n-1}} \Delta^{-1} D^{(n-1)} \Delta^{2n-1}$;

(ii) $\mathscr{W}_n = \dfrac{c_{n,0}}{(p')^n} \Delta D^{(n)} \Delta^{2n-1}$.

Proof: Definitions (1.3) and (1.4) disclose that

$$\mathscr{W}_{n+1} = \frac{1}{2}\left[p\mathscr{W}_n + \Delta^2 W_n\right]. \tag{3.1}$$

Assuming (i), (ii) in Theorem 2_F, we then have, on simplifying,

$$\mathscr{W}_{n+1} = \frac{n!\,\Delta}{(2n)!(p')^n}\left[pD^{(n)}\Delta^{2n-1} + np'D^{(n-1)}\Delta^{2n-1}\right]. \tag{3.2}$$

But, by Leibniz' rule,

$$\begin{aligned} D^{(n+1)}\Delta^{2n+1} &= D^{(n)}\{(2n+1)pp'\Delta^{2n-1}\} \\ &= (2n+1)p'\{pD^{(n)}\Delta^{2n-1} + np'D^{(n-1)}\Delta^{2n-1}\}, \end{aligned} \tag{3.3}$$

since $p'' = 0$. Accordingly, (3.2), (3.3) yield

$$\mathscr{W}_{n+1} = \frac{2(n+1)!\,\Delta}{(2n+2)!(p')^{n+1}} D^{(n+1)}\Delta^{2n+1}$$

in conformity with Theorem 2_F(ii) and (1.9).

Furthermore, from (1.7),

$$\begin{aligned} W_{n+1} &= \frac{1}{(n+1)p'}\mathscr{W}'_{n+1} \\ &= \frac{1}{(n+1)p'}\frac{c_{n+1,0}}{(p')^{n+1}} D\left(\Delta D^{(n+1)}\Delta^{2n+1}\right) \quad \text{by Theorem } 2_F\text{(ii)} \\ &= \frac{2(n+1)}{(p')^n} c_{n+1,0}\Delta^{-1}D^{(n)}\Delta^{2n+1} \quad \text{by (2.8)} \end{aligned}$$

in agreement with Theorem 2_F(i). Consequently, Theorem 2_F is completely proved.

***Example* (Chebyshev polynomials [3], $p = 2x$, $q = -1$):**

$$W_5 = 16x^4 - 12x^2 + 1 \quad (= U_4 \text{ [5, p. 256]}),$$
$$\mathscr{W}_5 = 2(16x^5 - 20x^3 + 5x) \quad (= 2T_5 \text{ [5, p. 256]}).$$

See also [7, p. 755]. Be it noted that in [6] the Rodrigues formulas for Chebyshev polynomials are given in terms of Gamma functions.

More generally,

Theorem 3_F:

(i) $W_n^{(r)} = \dfrac{c_{n,r+1}}{n(p')^{n-2r-1}} \Delta^{-2r-1} D^{(n-r-1)} \Delta^{2n-1}$;

(ii) $\mathscr{W}_n^{(r)} = \dfrac{c_{n,r}}{(p')^{n-2r}} \Delta^{-2r+1} D^{(n-r)} \Delta^{2n-1}$.

Proof:

(i) Induction on r is employed. The Theorem is true for $r = 0$ [Theorem 2_F(i)] and may be verified for $r = 1, 2$. Assume it is true for $r = k$. Then

$$\begin{aligned} W_n^{(k+1)} &= \frac{c_{n,k+1}}{n(p')^{n-2k-1}} D\left[\Delta^{-2k-1} D^{(n-k-1)} \Delta^{2n-1}\right] \quad \text{by Theorem } 3_F\text{(i)} \\ &= \frac{c_{n,k+2}}{n(p')^{n-2(k+1)-1}} \left[\Delta^{-2(k+1)-1} D^{(n-(k+1)-1)} \Delta^{2n-1}\right] \quad \text{by (2.7)} \end{aligned}$$

as expected. Thus, the Theorem is true for $r = k+1$. Hence, it is true for all r.

(ii)

$$\begin{aligned} \mathscr{W}_n^{(r)} &= np' W_n^{(r-1)} \quad \text{by (1.7)} \\ &= np' \frac{c_{n,r}}{n(p')^{n-2r+1}} \Delta^{-2r+1} D^{(n-r)} \Delta^{2n-1} \quad \text{by Theorem } 3_F\text{(i)} \\ &= \frac{c_{n,r}}{(p')^{n-2r}} \Delta^{-2r+1} D^{(n-r)} \Delta^{2n-1} \end{aligned}$$

as desired. Thus, Theorem 3_F is completely established.

Examples:

Chebyshev: $W_5^{(1)} = 8x(8x^2 - 3)$;

Fermat: $\mathscr{W}_4^{(2)} = 36(27x^2 - 4)$. (Here, $p = 3x, q = -2$.)

B. Case $p' = 0$.

Efforts to exploit the techniques of the theory when $q' = 0$ to the related situation when $p' = 0$ are doomed to disappointment, due mainly to the differing natures of Δ' in (1.6A) and (1.6B). A fresh approach is therefore necessary.

Computations rapidly show that, since $\Delta' = 2q'/\Delta$ (1.6B),

$$\begin{aligned} D^{(1)}\Delta^{2n-1} &= (2n-1)(2q')\Delta^{2n-3}, \\ D^{(2)}\Delta^{2n-1} &= (2n-1)(2n-3)(2q')^2\Delta^{2n-5}, \\ &\cdots \\ D^{(n-1)}\Delta^{2n-1} &= (2n-1)(2n-3)(2n-5)\cdot\cdots\cdot 3(2q')^{n-1}\Delta, \end{aligned} \tag{3.4}$$

whence

$$
\begin{aligned}
&\binom{n}{1}\frac{c_{n,0}}{(q')^{n-1}}\Delta^{-1}D^{(n-1)}\Delta^{2n-1}=\binom{n}{1},\\
&\binom{n}{3}\frac{c_{n-2,0}}{(q')^{n-3}}\Delta^{-1}D^{(n-2)}\Delta^{2n-1}=\binom{n}{3}\Delta^{2},\\
&\qquad\cdots\\
&\begin{cases}\binom{n}{n-1}\frac{c_{2,0}}{(q')^{1}}\Delta^{-1}D^{(1)}\Delta^{2n-1}=\binom{n}{n-1}\Delta^{n-2}, & n \text{ even},\\ \binom{n}{n}\frac{c_{1,0}}{(q')^{1}}\Delta^{-1}D^{(1)}\Delta^{2n-1}=\binom{n}{n}\Delta^{n-1}, & n \text{ odd}.\end{cases}
\end{aligned}
\tag{3.5}
$$

Differentiating once more in (3.4) gives rise to

$$D^{(n)}\Delta^{2n-1}=(2n-1)(2n-3)(2n-5)\cdot\cdots\cdot 3\cdot 1(2q')^{n}\Delta^{-1}. \tag{3.6}$$

Initially

$$D^{(0)}\Delta^{2n-1}=\Delta^{2n-1}. \tag{3.7}$$

Reassembling the ideas in (3.4), (3.5), and (3.6), we arrive at

$$
\begin{aligned}
&\binom{n}{0}\frac{c_{n,0}}{(q')^{n}}\Delta D^{(n)}\Delta^{2n-1}=\binom{n}{0},\\
&\binom{n}{2}\frac{c_{n-2,0}}{(q')^{n-2}}\Delta D^{(n-1)}\Delta^{2n-1}=\binom{n}{2}\Delta^{2},\\
&\qquad\cdots\\
&\begin{cases}\binom{n}{n}\frac{c_{2,0}}{(q')^{0}}\Delta D^{(1)}\Delta^{2n-1}=\binom{n}{n}\Delta^{n}, & n \text{ even},\\ \binom{n}{n-1}\frac{c_{1,0}}{(q')^{0}}\Delta D^{(1)}\Delta^{2n-1}=\binom{n}{n-1}\Delta^{n-1}, & n \text{ odd}.\end{cases}
\end{aligned}
\tag{3.8}
$$

Because the left-hand forms in (3.5) and (3.8) resemble the Rodrigues formulas in Theorem 2_F, we feel justified to appropriate to them the name of *Rodrigues-type* expressions.

Now, $p=1$ and $q=2x$ in (1.1), (1.3), and (1.5) indicate that

$$
\begin{aligned}
J_n&=\frac{(1+\Delta)^n-(1-\Delta)^n}{\Delta}\qquad(\Delta^2=1+8x)\\
&=\frac{1}{2^{n-1}}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}\Delta^{2k}=\frac{1}{2^{n-1}}\left[\binom{n}{1}+\binom{n}{3}\Delta^2+\binom{n}{5}\Delta^4+\cdots+\begin{Bmatrix}\binom{n}{n-1}\Delta^{n-2}\\ \binom{n}{n}\Delta^{n-1}\end{Bmatrix}\right]\begin{cases}n \text{ even},\\ n \text{ odd},\end{cases}\\
&=\text{a sum of expressions of Rodrigues-type (3.5).}
\end{aligned}
\tag{3.9}
$$

Similarly, use of (1.2), (1.4), and (1.5) gives rise to

$$
\begin{aligned}
j_n&=(1+\Delta)^n+(1-\Delta)^n=\frac{1}{2^{n-1}}\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{2k}\Delta^{2k}\\
&=\frac{1}{2^{n-1}}\left[\binom{n}{0}+\binom{n}{2}\Delta^2+\binom{n}{4}\Delta^4+\cdots+\begin{Bmatrix}\binom{n}{n}\Delta^{n}\\ \binom{n}{n-1}\Delta^{n-1}\end{Bmatrix}\right]\begin{cases}n \text{ even},\\ n \text{ odd},\end{cases}\\
&=\text{a sum of expressions of Rodrigues-type (3.8).}
\end{aligned}
\tag{3.10}
$$

Combining (3.9) and (3.10), we then conclude that

***Theorem* 2_J:** The Rodrigues formula analogs for J_n and j_n are given by (3.9) and (3.10).

Examples:

$$J_6 = \frac{1}{32}\left[\binom{6}{1} + \binom{6}{3}(1+8x) + \binom{6}{5}(1+8x)^2\right] = 1 + 8x + 12x^2;$$

$$j_6 = \frac{1}{32}\left[\binom{6}{0} + \binom{6}{2}(1+8x) + \binom{6}{4}(1+8x)^2 + \binom{6}{6}(1+8x)^3\right] = 1 + 12x + 36x^2 + 16x^3.$$

Our last major program is to generalize Theorem 2_J. Recall, first, that $j_n^{(1)} = 2nJ_{n-1}$ (1.8). Elementary calculations involving (1.3) and (1.4) for J_n and j_n quickly tell us that

$$J_n^{(1)} = \frac{4}{\Delta^2}\left(\frac{n}{2}j_{n-1} - J_n\right). \tag{3.11}$$

Subsequent differentiation reveals that

$$J_n^{(2)} = \frac{4}{\Delta^2}[n(n-1)J_{n-2} - 3J_n^{(1)}],$$

$$J_n^{(3)} = \frac{4}{\Delta^2}[n(n-1)J_{n-2}^{(1)} - 5J_n^{(2)}],$$

$$J_n^{(4)} = \frac{4}{\Delta^2}[n(n-1)J_{n-2}^{(2)} - 7J_n^{(3)}],$$

and so on, suggesting the proposition that

***Theorem* 3_J:** $J_n^{(r)} = \dfrac{4}{\Delta^2}\{n(n-1)J_{n-2}^{(r-2)} - (2r-1)J_n^{(r-1)}\},\ r \ge 2.$

Proof: Induction on r demonstrates the validity of this assertion.

Successive differentiations in (1.8) then establish that

***Theorem* 3_j:** $j_n^{(r)} = 2nJ_{n-1}^{(r-1)} = \dfrac{8n}{\Delta^2}[(n-1)(n-2)J_{n-3}^{(r-3)} - (2r-3)J_{n-1}^{(r-2)}],\ r \ge 3.$

***Example of Theorems* $3_J, 3_j$ ($r = 2, n = 9$):**

$$J_9^{(2)} = \frac{12}{8x+1}[24J_7 - J_9^{(1)}] = 24(8x^2 + 20x + 5) = \frac{1}{20}j_{10}^{(3)}.$$

Observations

(i) Summation procedures beginning with the definitions (1.3) and (1.4), and ending with (3.9) and (3.10), cannot be applied to the Fibonacci-type polynomials. This is because (3.9) and (3.10) are tied irrevocably to (3.5) and (3.8), both of which depend on $p' = 0$.

(ii) Corresponding to (3.1) for Fibonacci-type polynomials, for J_n and j_n we may derive

$$^{c}W_{n+1} = \Delta^2 W_n - q\,{}^{c}W_{n-1}. \tag{3.12}$$

Use of the Leibniz rule nexus in Theorem 2_F is impossible in the case of Jacobsthal-type polynomials J_n and j_n because of the diminished subscript for ${}^{c}W$ on the right-hand side.

(iii) In (3.11), where $r=1$, the appearance of $\frac{n}{2}j_{n-1}$, which seems to break the pattern of the theorem, requires explanation. From (1.8),

$$\frac{n}{2}j_{n-1}=\frac{n}{2}\int_0^x \frac{d}{dx}j_{n-1}\,dx=\frac{n}{2}\cdot 2(n-1)\int_0^x J_{n-2}\,dx=n(n-1)J_{n-2}^{(-1)},$$

where integration is represented by the negative unit superscript. With this symbolism, the pattern in Theorem 3_J is valid for $r\ge 1$, and hence that in Theorem 3_j for $r\ge 2$.

4. ILLUSTRATION OF THEORY WHEN $n=5$ (i.e., $2n-1=9$)

Now

$$\begin{aligned}
&D^{(1)}\Delta^9=9(pp'+2q')\Delta^7,\quad \text{where } \Delta^2=p^2+4q \ (1.5),\\
&D^{(2)}\Delta^9=9\{7(pp'+2q')^2\Delta^5+(p')^2\Delta^7\},\\
&D^{(3)}\Delta^9=9\{7[5(pp'+2q')^3\Delta^3+3(p')^2(pp'+2q')\Delta^5]\},\\
&D^{(4)}\Delta^9=9\cdot 7\cdot 3[5(pp'+2q')^4\Delta+10(p')^2(pp'+2q')^2\Delta^3+(p')^4\Delta^5].
\end{aligned}\tag{I}$$

Therefore,

$$\frac{5\Delta^{-1}D^{(4)}\Delta^9}{9\cdot 7\cdot 5\cdot 3\cdot (p')^4}=\frac{1}{(p')^4}\left\{\binom{5}{4}(pp'+2q')^4+\binom{5}{2}(p')^2(pp'+2q')^2\Delta^2+\binom{5}{0}(p')^4\Delta^4\right\}.\tag{I'}$$

So, for $q'=0$, on simplifying,

$$\frac{1}{2^4}(\text{R.H.S.})=p^4+3p^2q+q^2=W_5,$$
$$=\begin{cases}16x^4+12x^2+1 & \text{for the Pell polynomial } P_5\,[3]\text{: } p=2x,\ q=1,\\ (1+6x+4x^2 & \text{for the Jacobsthal polynomial } J_5\text{: } p=1,\ q=2x).\end{cases}$$

Differentiate (I) again to get

$$D^{(5)}\Delta^9=9\cdot 7\cdot 5\cdot 3\cdot\left[\frac{(pp'+2q')^5}{\Delta}+10(p')^2(pp'+2q')^3\Delta^2+5(p')^4(pp'+2q')\Delta^3\right].\tag{II}$$

Then

$$\frac{\Delta D^{(5)}\Delta^9}{9\cdot 7\cdot 5\cdot 3\cdot 1\cdot (p')^5}=\frac{1}{(p')^5}\left\{\binom{5}{5}(pp'+2q')^5+\binom{5}{3}(p')^2(pp'+2q')^3\Delta^2+\binom{5}{1}(p')^4(pp'+2q')\Delta^4\right\},\tag{II'}$$

whence, for $q'=0$,

$$\frac{1}{2^4}(\text{R.H.S.})\ =p^5+5p^3q+5pq^2=\mathcal{W}_5,$$
$$=\begin{cases}32x^5+40x^3+10x & \text{for Pell-Lucas polynomials [3]},\\ (1+10x+20x^2 & \text{for Jacobsthal-Lucas polynomials}).\end{cases}$$

On the other hand, when $p'=0$, we obtain the results (3.4)-(3.8) and hence (3.9) and (3.10).

Notice, particularly, that the general expressions for W_5 and $\mathcal{W}_5$ above are valid for both Fibonacci-type and Jacobsthal-type polynomials, even though $q'=0$.

This is because the binomial coefficients associated with the powers of Δ *in (I') and (II') are the same as those in (3.9) and (3.10), since* $\binom{n}{m} = \binom{n}{n-m}$.

Expressions for W_n and $\mathcal{W}_n$ may be sighted in [5] in a notation slightly varied from that used here.

5. CONCLUSION

While the author of [2] evidently did not consider this theory as applying to Jacobsthal-type polynomials, one observes that if his numerical parameter q in [2, eqns. (1.1), (1.2)] is allowed to be functional $q(x) = -2x$ with accompanying change in his x and p, then J_n and j_n can be incorporated into his system. For example, his U_5 ([2, eqn. (1.12)] reduces to $1 + 6x + 4x^2 = J_5$.

So we come to our rest, having achieved the objectives (i) and (ii) in Section 1 which motivated our undertaking. Many facets of the work were revealed with others to be investigated. The unexpected complications in the patterns of behavior of J_n and j_n (and W_n and $\mathcal{W}_n$) have added zest to the hunt.

Questions: Does there exist a general formula for the coefficients of the Jacobsthal-type polynomials in terms of the Gamma function in the sense of [1, Table 22.3]? If so, is it attainable using the techniques of this paper? Can, further, the theory be extended to the situation when both $p(x)$ and $q(x)$ are linear polynomials?

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2. R. André-Jeannin. "Differential Properties of a General Class of Polynomials." *The Fibonacci Quarterly* **32.5** (1994):445-54.
3. A. F. Horadam. "A Synthesis of Certain Polynomial Sequences." In *Applications of Fibonacci Numbers* **6**:215-29. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1996.
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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Proposers should inform us of the history of the problem if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-836 ***Proposed by Al Dorp, Edgemere, NY***

Replace each of "W", "X", "Y", and "Z" by either "F" or "L" to make the following an identity:

$$W_n^2 - 6X_{n+1}^2 + 2Y_{n+2}^2 - 3Z_{n+3}^2 = 0.$$

B-837 ***Proposed by Joseph J. Koštál, Chicago, IL***

Let

$$P(x) = x^{1997} + x^{1996} + x^{1995} + \cdots + x^2 + x + 1$$

and let $R(x)$ be the remainder when $P(x)$ is divided by $x^2 - x - 1$. Show that $R(x)$ is divisible by F_{999}.

B-838 ***Proposed by Peter G. Anderson, Rochester Institute of Technology, Rochester, NY***

Define a sequence of linear polynomials, $f_n(x) = m_n x + b_n$, by the recurrence

$$f_n(x) = f_{n-1}(f_{n-2}(x)),\quad n \geq 3,$$

with initial conditions

$$f_1(x) = \frac{1}{2}x$$

and

$$f_2(x) = \frac{1}{2}x + \frac{1}{2}.$$

Find a formula for m_n.

Extra credit: Find a formula for b_n.

B-839 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY***

Evaluate the sum

$$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 2^{-3k} \binom{n-2k}{k}$$

in terms of Fibonacci numbers.

B-840 ***Proposed by the editor***

Let

$$A_n = \begin{pmatrix} F_n & L_n \\ L_n & F_n \end{pmatrix}.$$

Find a formula for A_{2n} in terms of A_n and A_{n+1}.

B-841 ***Proposed by David Zeitlin, Minneapolis, MN***

Let P be an integer. For $n \geq 0$, let $U_{n+2} = PU_{n+1} + U_n$, with $U_0 = 0$ and $U_1 = 1$. Also let $V_{n+2} = PV_{n+1} + V_n$, with $V_0 = 2$ and $V_1 = P$. Prove that

$$\frac{V_n^2 + V_{n+a}^2}{U_n^2 + U_{n+a}^2}$$

is always an integer if a is odd.

NOTE: The Elementary Problems Column is in need of more **easy**, yet elegant and non-routine problems.

SOLUTIONS

Nonstandard Recurrence

B-820 ***Proposed by the editor; Dedicated to Herta Freitag***
(Vol. 34, no. 5, November 1996)

Find a recurrence (other than the usual one) that generates the Fibonacci sequence.

[The usual recurrence is a second-order linear recurrence with constant coefficients. Can you find a first-order recurrence that generates the Fibonacci sequence? Can you find a third-order linear recurrence? a nonlinear recurrence? one with nonconstant coefficients? etc.]

There were so many fine formulas sent in that we will only list a selection of them. We omit the obvious initial conditions.

First-Order Recurrences

$F_{n+1} = \alpha F_n + \beta^n$. Dresel

$F_{n+1} = \beta F_n + \alpha^n$. Bruckman

$F_{n+1} = \lfloor \alpha F_n + 0.4 \rfloor,\ n \geq 2.$ Dresel

$F_{n+1} = \frac{1}{2}(F_n + \sqrt{5F_n^2 + 4(-1)^n}).$ Dresel

Second-Order Nonlinear Recurrences

$F_{n+1} = ((-1)^n + F_n^2) / F_{n-1}.$ Anderson/Bruckman

Third-Order Recurrences

$F_{n+3} = 2F_{n+1} + F_n.$ Hendel

$F_{n+3} = 2F_{n+2} - F_n.$ Bruckman

$F_{n+3} = [pF_{n+2} + (p+2q)F_{n+1} + qF_n] / (p+q).$ Dresel

Fourth-Order Recurrences

$F_{n+4} = (1+p+q)F_{n+3} + (1-p-q-pq)F_{n+2} + (pq-p-q)F_{n+1} + pqF_n.$ Taylor

k^{th}-Order Linear Recurrences with Constant Coefficients

$F_n = F_{n-k} + \sum_{j=1}^{k} F_{n-j-1}.$ Anderson

$F_n = F_{n-2k+2} + \sum_{j=1}^{k-1} F_{n+2j-2k+1}.$ Freitag

Other Nonlinear Recurrences

$F_n = \sqrt[3]{F_{n-1}^3 + 3F_{n-1}^2 F_{n-2} + F_{n-2}^3 + 4F_{n-1}F_{n-2}^2}.$ Anderson/Seiffert

$F_n = \frac{1}{2}\left(L_k F_{n-k} + F_k \sqrt{5F_{n-k}^2 + 4(-1)^{n-k}}\right).$ Seiffert

$F_n = (F_{n-k}^2 - (-1)^n F_k^2) / F_{n-2k}.$ Seiffert

$F_n = F_{\lfloor (n+2)/2 \rfloor}^2 - (-1)^n F_{\lfloor (n-1)/2 \rfloor}^2.$ Seiffert

Taylor sent an 11-page tome of formulas, too many to list them all.

Also solved by Peter G. Anderson, Paul S. Bruckman, Leonard A. G. Dresel, Herta T. Freitag, Russell Jay Hendel, H.-J. Seiffert, and Joan Marie Taylor.

Fibonacci Rectangle

B-821 ***Proposed by L. A. G. Dresel, Reading, England (Vol. 35, no. 1, February 1997)***

Consider the rectangle with sides of lengths F_{n-1} and F_{n+1}. Let A_n be its area, and let d_n be the length of its diagonal. Prove that $d_n^2 = 3A_n \pm 1$.

Solution by Steve Scarborough, Loyola Marymount University, Los Angeles, CA

$$\begin{aligned} d_n^2 - 3A_n &= F_{n-1}^2 + F_{n+1}^2 - 3F_{n-1}F_{n+1} \\ &= F_{n-1}^2 + (F_n + F_{n-1})^2 - 3F_{n-1}(F_n + F_{n-1}) \\ &= F_n^2 - F_{n-1}^2 - F_{n-1}F_n = F_n^2 - F_{n-1}^2 - F_{n-1}(F_{n+1} - F_{n-1}) \\ &= F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}. \end{aligned}$$

The last step is Hoggatt's identity (I_{13}) from [1].

Reference

1. Verner E. Hoggatt, Jr.. *Fibonacci and Lucas Numbers.* Santa Clara, Calif.: The Fibonacci Association, 1979.

Also solved by Peter G. Anderson, Michel A. Ballieu, Brian D. Beasley, Scott H. Brown, Paul S. Bruckman, Charles K. Cook, Steve Edwards, Russell Euler & Jawad Sadek, Herta T. Freitag, Hans Kappus, Daina A. Krigens, Harris Kwong, Carl Libis, Bob Prielipp, Don Redmond, Maitland A. Rose, H.-J. Seiffert, Sahib Singh, Lawrence Somer, I. Strazdins, and the proposer.

A Tricky n^{th} Root

B-822 ***Proposed by Anthony Sofo, Victoria University of Technology, Australia (Vol. 35, no. 1, February 1997)***

For $n > 0$, simplify

$$\sqrt[n]{\alpha F_n + F_{n-1}} + (-1)^{n+1}\sqrt[n]{F_{n-1} - \alpha F_n}.$$

Solution by Hans Kappus, Rodersdorf, Switzerland

It is well known (see, for example, page 34 of [1]) that $\alpha F_n + F_{n-1} = \alpha^n$ and $F_{n+1} - \alpha F_n - \beta^n$. Ignoring complex n^{th} roots, we presume that the symbol $\sqrt[n]{x}$ denotes the principal root of the real quantity x. Since $\alpha > 0$ and $\beta < 0$, we have

$$\sqrt[n]{\alpha F_n + F_{n-1}} = \alpha$$

and

$$\sqrt[n]{F_{n+1} - \alpha F_n} = \sqrt[n]{\beta^n} = (-1)^n|\beta| = (-1)^{n+1}\beta.$$

Therefore, the expression given in the problem has value $\alpha + \beta = 1$.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, Calif: The Fibonacci Association, 1979.

Several readers incorrectly assumed that $\sqrt[n]{\beta^n} = \beta$, which is not true when n is even. Haukkanen found several analogs, such as $\sqrt[n]{F_{n+1} - \beta F_n} + (-1)^{n+1}\sqrt[n]{\beta F_n + F_{n-1}} = 1$.

Also solved by Brian D. Beasley, Paul S. Bruckman, Leonard A. G. Dresel, Herta T. Freitag, Pentti Haukkanen, Bob Prielipp, Steve Scarborough, H.-J. Seiffert, Lawrence Somer, and the proposer.

Solving a Simple Recurrence

B-823 ***Proposed by Pentti Haukkanen, University of Tampere, Finland (Vol. 35, no. 1, February 1997)***

It is easy to see that the solution of the recurrence relation

$$A_{n+2} = -A_{n+1} + A_n,\ A_0 = 0,\ A_1 = 1,$$

can be written as $A_n = (-1)^{n+1}F_n$.

Find a solution to the recurrence

$$A_{n+2} = -A_{n+1} + A_n,\ A_0 = 1,\ A_1 = 1,$$

in terms of F_n and L_n.

Solution by Hans Kappus, Rodersdorf, Switzerland

Let $A_n = (-1)^n B_n$. Then, for the B_n, we have the recurrence

$$B_{n+2} = B_{n+1} + B_n,\ \ B_0 = 1,\ \ B_1 = -1.$$

Hence, $B_n = aF_n + bL_n$ with constants a and b determined by the initial conditions, i.e., $2b = 1$ and $a + b = -1$. The result is

$$A_n = (-1)^n (L_n - 3F_n)/2.$$

An equivalent form of the answer is $A_n = (-1)^{n+1} F_{n-2}$.

Also solved by Michel A. Ballieu, Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Aloysius Dorp, Leonard A. G. Dresel, Russell Euler & Jawad Sadek, Daina A. Krigens, Harris Kwong, Carl Libis, Bob Prielipp, Maitland A. Rose, H.-J. Seiffert, Sahib Singh, Lawrence Somer, I. Strazdins, and the proposer.

Solving a Harder Recurrence

B-824 ***Proposed by Brian D. Beasley, Presbyterian College, Clinton, SC (Vol. 35, no. 1, February 1997)***

Fix a nonnegative integer m. Solve the recurrence $A_{n+2} = L_{2m+1}A_{n+1} + A_n$, for $n \geq 0$, with initial conditions $A_0 = 1$ and $A_1 = L_{2m+1}$, expressing your answer in terms of Fibonacci and/or Lucas numbers.

Solution by Paul S. Bruckman, Highwood, IL

The characteristic polynomial of the given recurrence is given by

$$p(z) = z^2 - L_{2m+1}z - 1 = (z - \alpha^{2m+1})(z - \beta^{2m+1}).$$

Therefore, there exist constants A and B, dependent solely on the initial conditions, such that

$$A_n = A\alpha^{(2m+1)n} + B\beta^{(2m+1)n}.$$

Setting $n = 0$ and $n = 1$, we obtain $A + B = 1$ and $A\alpha^{2m+1} + B\beta^{2m+1} = L_{2m+1}$. Solving this pair of equations, we obtain $A = D\alpha^{2m-1}$ and $B = -D\beta^{2m+1}$, where $D = (\alpha^{2m+1} - \beta^{2m+1})^{-1}$.

Thus,

$$A_n = D[\alpha^{(2m+1)(n+1)} - \beta^{(2m+1)(n+1)}],$$

which simplifies to

$$A_n = F_{(2m+1)(n+1)} / F_{2m+1}.$$

Several readers pointed out that the result follows from Problem B-748.

Also solved by Charles K. Cook, Leonard A. G. Dresel, Steve Edwards, Russell Euler & Jawad Sadek, Herta T. Freitag, Hans Kappus, Daina A. Krigens, Harris Kwong, Carl Libis, Don Redmond, H.-J. Seiffert, Lawrence Somer, and the proposer.

Divisors of Lucas Sequences

B-825 ***Proposed by Lawrence Somer, University of America, Washington, D.C. (Vol. 35, no. 1, February 1997)***

Let $\langle V_n \rangle$ be a sequence defined by the recurrence $V_{n+2} = PV_{n+1} - QV_n$, where P and Q are integers and $V_0 = 2$, $V_1 = P$. The integer d is said to be a divisor of $\langle V_n \rangle$ if $d | V_n$ for some $n \geq 1$.

(a) If P and Q are both even, show that 2^m is a divisor of $\langle V_n \rangle$ for any $m \geq 1$.

(b) If P or Q is odd, show that there exists a fixed nonnegative integer k such that 2^k is a divisor of $\langle V_n \rangle$ but 2^{k+1} is not a divisor of $\langle V_n \rangle$. If exactly one of P or Q is even, show that $2^k | V_1$; if P and Q are both odd, show that $2^k | V_3$.

Solution by the proposer

First, suppose that P and Q are both even. Using the recursion relation defining $\langle V_n \rangle$, it follows by induction that $2^{i+1} | V_{2i}$ and $2^{i+1} | V_{2i+1}$ for $i \geq 0$. Thus, 2^m is a divisor of $\langle V_n \rangle$ for all $m \geq 1$.

Now, suppose that $2 | Q$ but $2 \nmid P$. One sees by induction that V_n is odd for all $n \geq 1$. Hence, $2^0 = 1$ is a divisor of $\langle V_n \rangle$ but $2^1 = 2$ is not a divisor of $\langle V_n \rangle$. Clearly, $2^0 | V_1$.

We now assume that $2^k \| P$ but $2 \nmid Q$, where $k \geq 1$. It follows by induction that $2^k \| V_{2n-1}$ and $2 \| V_{2n}$ for $n \geq 1$. Then 2^k is a divisor of $\langle V_n \rangle$ but 2^{k+1} is not a divisor of $\langle V_n \rangle$. Clearly, $2^k | V_1$.

Finally, assume that P and Q are both odd. By inspection, one sees that $2 | V_n$ if and only if $3 | n$. Suppose that $2^k \| V_3$, where $k \geq 1$. By the Binet formula, $V_n = \gamma^n + \delta^n$, where γ and δ are roots of the equation $x^2 - Px + Q = 0$. Consider the sequence $\langle V_n' \rangle$ defined by $V_n' = V_{3n}$. Then

$$V_n' = \gamma^{3n} + \delta^{3n} = (\gamma^3)^n + (\delta^3)^n,$$

where γ^3 and δ^3 are roots of the equation $x^2 - V_3 x + Q^3 = 0$. Thus, $\langle V_n' \rangle$ is a Lucas sequence of the second kind satisfying the second-order linear recursion relation

$$V_{n+2}' = P' V_{n+1}' - Q' V_n',$$

where $P' = V_3$ is even, $Q' = Q^3$ is odd, $V_0' = 2$, and $V_1' = P' = V_3$. Hence, $2^k \| V_1'$. By our previous argument in the case in which P is even and Q is odd, we see that

$$2^k \| V_{2n-1}' = V_{3(2n-1)} \quad \text{and} \quad 2 \| V_{2n}' = V_{3(2n)}$$

for all $n \geq 1$. Thus, $2^k \| V_1$ and 2^k is a divisor of $\langle V_n \rangle$ but 2^{k+1} is not a divisor of $\langle V_n \rangle$. The result now follows.

Also solved by Paul S. Bruckman and Leonard A. G. Dresel.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-532 *Proposed by Paul S. Bruckman, Highwood, IL*

Let $V_n = V_n(x)$ denote the generalized Lucas polynomials defined as follows: $V_0 = 2$; $V_1 = x$; $V_{n+2} = xV_{n+1} + V_n$, $n = 0, 1, 2, \ldots$. If n is an odd positive integer and y is any real number, find all (exact) solutions of the equation: $V_n(x) = y$.

H-533 *Proposed by Andrej Dujella, University of Zagreb, Croatia*

Let $Z(n)$ be the entry point for positive integers n. Prove that $Z(n) \le 2n$ for any positive integer n. Find all positive integers n such that $Z(n) = 2n$.

H-534 *Proposed by Piero Filipponi, Rome, Italy*

An interesting question posed to me by Evelyn Hart (Colgate University, Hamilton, NY) led me to pose, in turn, the following two problems to the readers of *The Fibonacci Quarterly*.

Problem A: For k a fixed positive integer, let n_k be *any* integer representable as

$$n_k = \sum_{j=1}^{k} v_j F_j, \tag{1}$$

where v_j equals either j or zero.

Remarks:

(i) Clearly, we have that $0 \le n_k \le f(k) = (k+1)F_{k+2} - F_{k+4} + 2$ (see Hoggatt's identity I_{40}).

(ii) In general, the representation (1) is not unique, as shown by the following example: $91 = 7F_7 = 6F_6 + 5F_5 + 4F_4 + 3F_3$.

(iii) Not all integers can be represented as (1), 4, 5, 10, 11, 16, 17, 22, 23, and 24 being the smallest among such integers.

Let $S(k)$ be the number of all n_k. Is it possible to evaluate $\lim_{k\to\infty} \frac{S(k)}{f(k)}$?

Problem B: Is it possible to characterize the set of all positive integers k for which kF_k is representable as

$$kF_k = \sum_{j=1}^{k-1} v_j F_j,$$

where v_j is as in Problem A?

Remarks:

(*i*) Since $kF_k > \Sigma_{j=1}^{k-1} jF_j$ for $k \leq 6$, we must have $k \geq 7$. In fact, $7F_7 = 91$ can be represented in this form [see Remark (ii) in Problem A].

(*ii*) The numerical inspection of earliest cases shows that other values of k are 10, 11, 12, 13, 15, and 16. As an example, we have: $16F_{16} = 15F_{15} + 14F_{14} + 11F_{11} + 9F_9 + 6F_6 + 5F_5 + 3F_3$.

H-535 ***Proposed by Piero Filipponi & Adina Di Porto, Rome, Italy***

For given positive integers n and m, find a closed form expression for $\Sigma_{k=1}^{n} k^m F_k$.

Conjecture by the proposers:

$$\Sigma_{m,n} = \sum_{k=1}^{n} k^m F_k = p_1^{(m)}(n)F_{n+1} + p_2^{(m)}(n)F_n + C_m, \tag{1}$$

where $p_1^{(m)}(n)$ and $p_2^{(m)}(n)$ are polynomials in n of degree m,

$$p_1^{(m)}(n) = \sum_{i=0}^{m} (-1)^i a_{m-i}^{(m)} n^{m-i}, \quad p_2^{(m)}(n) = \sum_{i=0}^{m} (-1)^i b_{m-i}^{(m)} n^{m-i}, \tag{2}$$

the coefficients $a_k^{(m)}$ and $b_k^{(m)}$ $(k = 0, 1, \ldots, m)$ are positive integers, and C_m is an integer.

On the basis of the well-known identity

$$\Sigma_{1,n} = (n-2)F_{n+1} + (n-1)F_n + 2, \tag{3}$$

which is an alternate form of Hoggatt's identity I_{40}, the above quantities can be found recursively by means of the following algorithm:

1. $p_1^{(m+1)}(n) = (m+1)\int p_1^{(m)}(n)\,dn + (-1)^{m+1}a_0^{(m+1)}$, $p_2^{(m+1)}(n) = (m+1)\int p_2^{(m)}(n)\,dn + (-1)^{m+1}b_0^{(m+1)}$.

2. $a_0^{(m+1)} = \sum_{i=1}^{m+1} (a_i^{(m+1)} + b_i^{(m+1)})$.

3. $b_0^{(m+1)} = \sum_{i=1}^{m+1} a_i^{(m+1)}$.

4. $C_{m+1} = (-1)^m a_0^{(m+1)}$.

Example: The following results were obtained using the above algorithm:

$$\Sigma_{2,n} = (n^2 - 4n + 8)F_{n+1} + (n^2 - 2n + 5)F_n - 8;$$

$$\Sigma_{3,n} = (n^3 - 6n^2 + 24n - 50)F_{n+1} + (n^3 - 3n^2 + 15n - 31)F_n + 50;$$

$$\Sigma_{4,n} = (n^4 - 8n^3 + 48n^2 - 200n + 416)F_{n+1} + (n^4 - 4n^3 + 30n^2 - 124n + 257)F_n - 416;$$

$$\begin{aligned}\Sigma_{5,n} = {} & (n^5 - 10n^4 + 80n^3 - 500n^2 + 2080n - 4322)F_{n+1} \\ & + (n^5 - 5n^4 + 50n^3 - 310n^2 + 1285n - 2671)F_n + 4322.\end{aligned}$$

Remarks:

(*i*) These results can obviously be proved by induction on n.

(*ii*) It can be noted that, using the same algorithm, $\Sigma_{1,n}$ can be obtained by the identity $\Sigma_{0,n} = F_{n+1} + F_n - 1$.

(*iii*) It appears that $a_k^{(m+k)} / b_k^{(m+k)} = \text{const.} = a_0^{(m)} / b_0^{(m)}$, $(k = 1, 2, \ldots)$ and $\lim_{m\to\infty} a_0^{(m)} / b_0^{(m)} = \alpha$.

SOLUTIONS

Limits

H-514 ***Proposed by Juan Pla, Paris, France***
(Vol. 34, no. 4, August 1996)

I) Let (L_n) be the generalized Lucas sequence of the recursion $U_{n+2}-2aU_{n+1}+U_n=0$ with a real such that $a>1$. Prove that

$$\lim_{n\to+\infty}\frac{L_2L_{2^2}L_{2^3}\dots L_{2^n}}{L_{2^{n+1}}}=\frac{1}{4}\frac{1}{a\sqrt{a^2-1}}.$$

II) Show that the above expression has a limit when (L_n) is the *classical* Lucas sequence.

Solution by H.-J. Seiffert, Berlin, Germany

Let (L_n) be the generalized Lucas sequence of the recursion $U_{n+2}-2aU_{n+1}+bU_n=0$ with a and b real such that $a>0$ and $a^2>b$. Then L_n has the Binet form $L_n=\alpha^n+\beta^n$, $n\in N_0$, where $\alpha=a+\sqrt{a^2-b}$ and $\beta=a-\sqrt{a^2-b}$. Let $F_n=(\alpha^n-\beta^n)/(\alpha-\beta)$, $n\in N_0$. Since $\alpha>|\beta|$ by $a>0$ and $a^2>b$, we have

$$\lim_{n\to+\infty}\frac{F_n}{L_n}=\lim_{n\to+\infty}\frac{\alpha^n-\beta^n}{(\alpha-\beta)(\alpha^n+\beta^n)}=\frac{1}{\alpha-\beta}\lim_{n\to+\infty}\frac{1-(\beta/\alpha)^n}{1+(\beta/\alpha)^n}=\frac{1}{\alpha-\beta}$$

or

$$\lim_{n\to+\infty}\frac{F_n}{L_n}=\frac{1}{2\sqrt{a^2-b}}. \tag{1}$$

It is easily verified that $F_{2n}-F_nL_n$, $n\in N_0$. Now, a simple induction argument yields

$$L_{2k}L_{2^2k}L_{2^3k}\dots L_{2^nk}=\frac{F_{2^{n+1}k}}{F_{2k}},\quad k\in N,\ n\in N_0.$$

Hence, by (1),

$$\lim_{n\to+\infty}\frac{L_{2k}L_{2^2k}L_{2^3k}\dots L_{2^nk}}{L_{2^{n+1}k}}=\frac{1}{2F_{2k}\sqrt{a^2-b}} \tag{2}$$

for all $k\in N$. In the special case $k=1$, this limit is $1/(4a\sqrt{a^2-b})$. The more special case $b=1$ and $(a>1)$ solves the first part of the proposal. Taking $a=1/2$, $b=-1$, and $k=1$, (2) gives the value $1/\sqrt{5}$ for the limit considered in the second part of the proposal.

Also solved by P. Bruckman, C. Georgiou, J. Koštál, and the proposer.

Some Entry

H-515 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 34, no. 4, August 1996)

For all primes $p\neq 2,5$, let $Z(p)$ denote the entry-point of p in the Fibonacci sequence. It is known that $Z(p)\,|\,\left(p-\left(\frac{5}{p}\right)\right)$. Let $a(p)=\left(p-\left(\frac{5}{p}\right)\right)/Z(p)$, $q=\frac{1}{2}\left(p-\left(\frac{5}{p}\right)\right)$. Prove that if $p\equiv 1$ or 9 (mod 20) then

$$F_{q+1}\equiv(-1)^{\frac{1}{2}(q+a(p))}\pmod{p}. \tag{*}$$

Solution by H.-J. Seiffert, Berlin, Germany

We will use the easily verifiable equations

$$F_{2n+1} = F_n L_{n+1} + (-1)^n \quad \text{and} \quad F_{2n+1} = F_{n+1} L_n - (-1)^n, \tag{1}$$

where n is any integer, and the following known results:

$$\left(\tfrac{5}{p}\right) = 1 \text{ if } p \equiv 1 \text{ or } 9 \pmod{10}, \tag{2}$$

$$p \mid F_q \text{ if and only if } p \equiv 1 \pmod 4, \tag{3}$$

$$Z(p) \mid m \text{ if and only if } p \mid F_m, \tag{4}$$

where $p \neq 5$ denotes an odd prime and m a positive integer.

Let p be an odd prime such that $p \equiv 1$ or $9 \pmod{20}$. From (2), we have $\left(\frac{5}{p}\right) = 1$, so that $q = \frac{1}{2}(p-1)$.

First, suppose that $p \mid F_{q/2}$. Then $Z(p) \mid q/2$ by (4), which yields $a(p) \equiv 0 \pmod 4$. Using the left equation of (1) with $n = q/2$, it follows that

$$F_{q+1} = F_{q/2} L_{\frac{1}{2}(q+2)} + (-1)^{q/2} \equiv (-1)^{q/2} \pmod p,$$

which proves $(*)$ in this case.

If $p \nmid F_{q/2}$, then $p \mid L_{q/2}$, since p divides $F_q = F_{q/2} L_{q/2}$, by (3). Since $Z(p) \mid q$ and $Z(p) \nmid q/2$, by (4), we have $a(p) \equiv 2 \pmod 4$. Using the right equation of (1) with $n = q/2$, we obtain

$$F_{q+1} = F_{\frac{1}{2}(q+2)} L_{q/2} - (-1)^{q/2} \equiv (-1)^{\frac{1}{2}(q+2)} \pmod p,$$

proving $(*)$ in such case.

Also solved by the proposer.

Mod Squad

H-516 ***Proposed by Paul S. Bruckman, Highwood, IL (Vol. 34, no. 4, August 1996)***

Given p an odd prime, let $\bar{k}(p)$ denote the *Lucas period (mod p)*, that is, $\bar{k}(p)$ is the smallest positive integer $m = m(p)$ such that $L_{m+n} \equiv L_n \pmod p$ for all integers n. Prove the following:

(a) Let $u = u(p)$ denote the smallest positive integer such that $\alpha^u \equiv \beta^u \equiv 1 \pmod p$. Then $u = m = \bar{k}(p)$.

(b) $\bar{k}(p)$ is even for all (odd) p.

(c) $p \equiv 1 \pmod{\bar{k}(p)}$ iff $p = 5$ or $p \equiv \pm 1 \pmod{10}$.

(d) $p \equiv -1 + \frac{1}{2}\bar{k}(p) \pmod{\bar{k}(p)}$ iff $p = 5$ or $p \equiv \pm 3 \pmod{10}$.

Solution by the proposer

We will use the following fairly well-known result that $\alpha^p \equiv \alpha$, $\beta^p = \beta \pmod p$ iff $p = 5$ or $p \equiv \pm 1 \pmod{10}$, while $\alpha^p \equiv \beta$, $\beta^p \equiv \alpha \pmod p$ iff $p = 5$ or $p \equiv \pm 3 \pmod{10}$. Also, we shall use the easily demonstrable result that $\bar{k}(p) = 4$ iff $p = 5$. The first result implies that u always exists.

Proof of (a): If $p=5$, then $\alpha \equiv \beta \equiv 2^{-1} \equiv -2 \pmod 5$; we see readily that $u=m=\bar{k}(5)=4$.

If $p \neq 5$, suppose the congruence in the statement of the problem. Then, for all integers n, we have $\alpha^{u+n} \equiv \alpha^n$, $\beta^{u+n} \equiv \beta^n \pmod p$, which implies (by addition) $L_{u+n} \equiv L_n \pmod p$. This, in turn, implies that $m|u$. On the other hand, $L_{m+n} \equiv L_n \pmod p$ for all integers n, and in particular for $n=-1$, 0, and 1; hence, $L_{m-1} \equiv L_{-1} \equiv -1$, $L_m \equiv L_0 \equiv 2$, $L_{m+1} \equiv L_1 \equiv 1 \pmod p$. Then $L_{m-1} + L_{m+1} = 5F_m = 5^{1/2}(\alpha^m - \beta^m) \equiv 0 \pmod p$, so $\alpha^m \equiv \beta^m \pmod p$. Since $L_m = \alpha^m + \beta^m \equiv 2 \pmod p$, we have $\alpha^m \equiv \beta^m \equiv 1 \pmod p$. From this, it follows that $u|m$. Hence, $u=m$. Q.E.D.

Proof of (b): Since $\alpha^m \equiv \beta^m \equiv 1 \pmod p$, we have that $(\alpha\beta)^m = (-1)^m \equiv 1 \pmod p$, which implies that $m=\bar{k}(p)$ must be even.

Proof of (c): Since $\bar{k}(p)=4$ iff $p=5$, we see that the first congruence in the statement of (c) is satisfied by $p=5$. Suppose $p \neq 5$ and $p \equiv 1 \pmod{\bar{k}(p)}$. Then $\alpha^p \equiv \alpha$, $\beta^p \equiv \beta \pmod p$, which implies $p \equiv \pm 1 \pmod{10}$.

Conversely, if $p \equiv \pm 1 \pmod{10}$, then $\alpha^p \equiv \alpha$, $\beta^p \equiv \beta \pmod p$. so $\alpha^{p-1} \equiv \beta^{p-1} \equiv 1 \pmod p$. Then $\bar{k}(p)|(p-1)$ or $p \equiv 1 \pmod{\bar{k}(p)}$.

Proof of (d): We see that the first congruence in the statement of (d) is satisfied by $p=5$. Suppose that it is satisfied by $p \neq 5$. Then $\bar{k}(p)|(2p+2)$, $\bar{k}(p)\nmid(p+1)$, so $\alpha^{p+1} \equiv \beta^{p+1} \equiv -1 \pmod p$; for if $\alpha^{p+1} \equiv -\beta^{p+1} \equiv \pm 1 \pmod p$, then $(\alpha\beta)^{p+1} \equiv -1$, which is absurd, since $(-1)^{p+1}=1$ (for odd p). Then $\alpha^p \equiv \beta$, $\beta^p \equiv \alpha \pmod p$, which implies $p \equiv \pm 3 \pmod{10}$.

Conversely, if $p \equiv \pm 3 \pmod{10}$, then $\alpha^p \equiv \beta$, $\beta^p \equiv \alpha \pmod p$, which implies $\alpha^{p+1} \equiv \beta^{p+1} \equiv -1$, $\alpha^{2p+2} \equiv \beta^{2p+2} \equiv 1 \pmod p$. Therefore, $\bar{k}(p)|(2p+2)$, $\bar{k}(p)\nmid(p+1)$, which implies $p \equiv -1 + \frac{1}{2}\bar{k}(p) \pmod{\bar{k}(p)}$.

Also solved by L. A. G. Dresel.

Divide and Conquer

H-517 ***Proposed by Paul S. Bruckman, Highwood, IL (Vol. 34, no. 5, November 1996)***

Given a positive integer n, define the sums $P(n)$ and $Q(n)$ as follows:

$$P(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) L_d, \quad Q(n) = \sum_{d|n} \Phi\left(\frac{n}{d}\right) L_d, \tag{1}$$

where μ and Φ are the Möbius and Euler functions, respectively. Show that $n|P(n)$ and $n|Q(n)$.

Solution by H.-J. Seiffert, Berlin, Germany

It is well known that

$$L_{kp^r} \equiv L_{kp^{r-1}} \pmod{p^r} \text{ if } p \text{ is a prime and } k, r \in N. \tag{1}$$

Let $n \in N$ be divisible by the prime p. Then there exist $m, e \in N$ such that $p \nmid m$ and $n = mp^e$.

Using $\mu(d)=0$ if $d \in N$ and $p^2|d$, $\mu(jp) = -\mu(j)$ if $j \in N$ and $p \nmid j$, and (1), modulo p^e we obtain

$$P(n) = P(mp^e) = \sum_{d \mid mp^e} \mu(d) L_{\frac{m}{d}p^e} = \sum_{d \mid m} \mu(d) L_{\frac{m}{d}p^e} + \sum_{j \mid m} \mu(jp) L_{\frac{m}{j}p^{e-1}}$$
$$\equiv \sum_{d \mid m} \mu(d) L_{\frac{m}{d}p^{e-1}} - \sum_{j \mid m} \mu(j) L_{\frac{m}{j}p^{e-1}} \equiv 0 \pmod{p^e}.$$

Clearly, this proves the desired relation $P(n) \equiv 0 \pmod{n}$.

Modulo p^e we have

$$Q(n) = Q(mp^e) = \sum_{d \mid mp^e} \Phi(d) L_{\frac{m}{d}p^e} = \sum_{d \mid m} \Phi(d) L_{\frac{m}{d}p^e} + \sum_{\substack{d \mid mp^e \\ p \mid d}} \Phi(d) L_{\frac{m}{d}p^e}$$
$$\equiv \sum_{d \mid m} \Phi(d) L_{\frac{m}{d}p^{e-1}} + \sum_{s=1}^{e} \sum_{j \mid m} \Phi(jp^s) L_{\frac{m}{j}p^{e-s}} \pmod{p^e},$$

where we have used (1). Since $\Phi(jp^s) = (p^s - p^{s-1})\Phi(j)$ if $j, s \in N$ and $p \nmid j$, we obtain

$$\sum_{s=1}^{e} \sum_{j \mid m} \Phi(jp^s) L_{\frac{m}{j}p^{e-s}} = \sum_{s=1}^{e} (p^s - p^{s-1}) \sum_{j \mid m} \Phi(j) L_{\frac{m}{j}p^{e-s}}$$
$$= \sum_{s=1}^{e} p^s \sum_{j \mid m} \Phi(j) L_{\frac{m}{j}p^{e-s}} - \sum_{t=0}^{e-1} p^t \sum_{j \mid m} \Phi(j) L_{\frac{m}{j}p^{e-t-1}}$$
$$\equiv \sum_{s=1}^{e-1} p^s \sum_{j \mid m} \Phi(j) L_{\frac{m}{j}p^{e-s}} - \sum_{t=0}^{e-1} p^t \sum_{j \mid m} \Phi(j) L_{\frac{m}{j}p^{e-t-1}}$$
$$\equiv \sum_{s=1}^{e-1} p^s \sum_{j \mid m} \Phi(j) L_{\frac{m}{j}p^{e-s-1}} - \sum_{t=0}^{e-1} p^t \sum_{j \mid m} \Phi(j) L_{\frac{m}{j}p^{e-t-1}}$$
$$= -\sum_{j \mid m} \Phi(j) L_{\frac{m}{j}p^{e-1}} \pmod{p^e},$$

where we have used (1) again. It follows that $Q(n) \equiv 0 \pmod{p^e}$. Of course, this proves the desired result $Q(n) \equiv 0 \pmod{n}$.

Also solved by P. Haukkanen and the proposer.

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