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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

SOME PROPERTIES OF GENERALIZED PASCAL SQUARES AND TRIANGLES*

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1. INTRODUCTION

Four properties, related to columns, column sums, diagonal sums, and determinants, will be considered for

(a) the "Pascal square" recurrence relation and its variations,
(b) the "Pascal triangle" recurrence relation and its variations, and
(c) more general recurrence relations which admit these properties.

Associated basic linear recursive sequences are also outlined. Other research may be found in Bollinger [2], Philippou & Georghiou [9], and Carlitz & Riordan [4] who discuss the recurrence relation (2.1) in depth, but with different boundary conditions. In the following, $\left\{ {n \atop p} \right\}$ represents the entry in the n^{th} row, p^{th} column of a square array.

2. GENERALIZED PASCAL SQUARES

Bondarenko [3] presents an extremely useful collation of the myriad results concerning Pascal triangles and their generalizations. We attempt to provide additional insights and unification of some of these by considering properties of square arrays in which the entries are governed by linear partial recurrence relations of a particular form. A number of illustrative cases are given followed by more general results.

2.1 Case 1: The Pascal Square and Variations

The Pascal array in Table 1 is formed by the use of the recurrence relation

$$\left\{ {n \atop p} \right\} = \left\{ {n-1 \atop p} \right\} + \left\{ {n \atop p-1} \right\} \quad (n \geq 1, p \geq 0) \tag{2.1}$$

with

$$\left\{ {n \atop -1} \right\} = 0 \;\; (n \geq 0), \quad \left\{ {0 \atop p} \right\} = 1 \;\; (p \geq 1), \quad \text{and} \quad \left\{ {0 \atop 0} \right\} = 1.$$

This is clearly just a rotation of the usual Pascal triangle. We highlight four properties, some well known, which will be generalized.

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* This paper, presented at the Seventh International Research Conference held in Graz, Austria, in July 1996, was scheduled to appear in the Conference Proceedings. However, due to limitations placed by the publisher on the number of pages allowed for the Proceedings, we are publishing the article in this issue of *The Fibonacci Quarterly* to assure its presentation to the widest possible number of readers in the mathematics community.

TABLE 1. Pascal's Square

$n \backslash p$	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9
2	1	3	6	10	15	21	28	36	45
3	1	4	10	20	35	56	84	120	165
4	1	5	15	35	70	126	210	330	495
5	1	6	21	56	126	252	462	792	1287
6	1	7	28	84	210	462	924	1716	3003
7	1	8	36	120	330	792	1716	3432	6435
8	1	9	45	165	495	1287	3003	6435	12870

Property 1 (Columns): The form of the recurrence relation implies that first differences by column give entries in the previous column. As the 0^{th} column is constant, entries in the p^{th} column are given by the p^{th} order polynomial in n which interpolates the first (or any consecutive) $p+1$ entries in that column

In this case, since $\left\{{n \atop p}\right\} = \binom{n+p}{p}$, the polynomial is $(n+p)(n+p-1)\dots(n+1)/p!$.

Property 2 (Column Sums): For $n \geq 1$ and $p \geq 0$,

$$\begin{aligned}\sum_{i=0}^{n}\left\{{i \atop p}\right\} &= \sum_{i=1}^{n}\left(\left\{{i \atop p+1}\right\}-\left\{{i-1 \atop p+1}\right\}\right)+\left\{{0 \atop p}\right\}\\ &= \left\{{n \atop p+1}\right\}-\left\{{0 \atop p+1}\right\}+\left\{{0 \atop p}\right\}.\end{aligned}$$

For the Pascal square,

$$\sum_{i=0}^{n}\left\{{i \atop p}\right\} = \left\{{n \atop p+1}\right\},$$

which is better known as the combinatorial identity

$$\sum_{i=0}^{n}\binom{i+p}{p} = \binom{n+p+1}{p+1}.$$

Property 3 (Diagonal Sums): Let the n^{th} diagonal sum be

$$d_n = \sum_{i=0}^{n}\left\{{n-i \atop i}\right\},$$

then, for $n \geq 1$,

$$\begin{aligned}d_n - d_{n-1} &= \sum_{i=0}^{n-1}\left(\left\{{n-1 \atop i}\right\}-\left\{{n-i-1 \atop i}\right\}\right)+\left\{{0 \atop n}\right\}\\ &= \sum_{i=0}^{n-1}\left\{{n-i \atop i-1}\right\}+\left\{{0 \atop n}\right\}\end{aligned}$$

$$= \sum_{i=1}^{n-1} \left\{ {n-i \atop i-1} \right\} + \left\{ {n \atop -1} \right\} + \left\{ {0 \atop n} \right\}$$

$$= \sum_{i=0}^{n-1} \left\{ {n-i-1 \atop i} \right\} - \left\{ {0 \atop n-1} \right\} + \left\{ {0 \atop n} \right\}$$

$$= d_{n-1} - \left\{ {0 \atop n-1} \right\} + \left\{ {0 \atop n} \right\},$$

and so

$$d_n = 2d_{n-1} - \left\{ {0 \atop n-1} \right\} + \left\{ {0 \atop n} \right\}. \tag{2.2}$$

In this case, $d_n = 2d_{n-1} = 2^n$ (as $d_0 = 1$) as expected.

Property 4 (Determinants): Let

$$\left[\left\{ {n \atop p} \right\} \right]_{(a,b)}$$

denote the square array of given entries with $a \le (n, p) \le b$, then, taking determinants and using elementary determinantal row operations:

$$\left| \left[\left\{ {n \atop p} \right\} \right]_{(0,m)} \right| = \begin{vmatrix} \left\{ {0 \atop 0} \right\} & \left\{ {0 \atop 1} \right\} & \cdots & \left\{ {0 \atop m} \right\} \\ \left\{ {1 \atop 0} \right\} & \left[\left\{ {n \atop p} \right\} \right]_{(1,m)} & & \\ \cdots & & & \\ \left\{ {m \atop 0} \right\} & & & \end{vmatrix}$$

$$= \begin{vmatrix} \left\{ {0 \atop 0} \right\} & \left\{ {0 \atop 1} \right\} & \cdots & \left\{ {0 \atop m} \right\} \\ \left\{ {1 \atop 0} \right\} - \left\{ {0 \atop 0} \right\} & \left[\left\{ {n \atop p} \right\} - \left\{ {n-1 \atop p} \right\} \right]_{(1,m)} & & \\ \cdots & & & \\ \left\{ {m \atop 0} \right\} - \left\{ {m-1 \atop 0} \right\} & & & \end{vmatrix}$$

$$= \begin{vmatrix} \left\{ {0 \atop 0} \right\} & \left\{ {0 \atop 1} \right\} & \cdots & \left\{ {0 \atop m} \right\} \\ 0 & \left[\left\{ {n \atop p-1} \right\} \right]_{(1,m)} & & \\ \cdots & & & \\ 0 & & & \end{vmatrix} \quad \text{(by use of the recurrence relation)}$$

$$= \left\{ {0 \atop 0} \right\} \left| \left[\left\{ {n \atop p-1} \right\} \right]_{(1,m)} \right| = \left\{ {0 \atop 0} \right\} \left\{ {1 \atop 0} \right\} \left| \left[\left\{ {n \atop p-2} \right\} \right]_{(2,m)} \right|$$

$$= \cdots = \prod_{i=0}^{m} \left\{ {i \atop 0} \right\} = \left\{ {0 \atop 0} \right\}^{m+1}.$$

Thus, all square sub-arrays of the Pascal array with top-left corner $\{{0 \atop 0}\} = 1$ are unimodular.

Property 5 (Generalizations): The derivations of Properties 1-4 rely (if at all) only on the left-hand ($p = -1$) zero boundary conditions. They thus apply to the Pascal array generalized by arbitrary top-row entries and hence to left-justified sub-arrays of the Pascal square. In particular, all square sub-arrays of the Pascal array with left side in the $p = 0$ column (or, by symmetry, top row in the $n = 0$ row) are unimodular, as noted by Bicknell & Hoggatt [1] for the simple Pascal array.

Two examples, formed by varying the top row ($n = 0$) boundary conditions, follow.

Case 1, Example 1 (Vieta's Array): Using (2.1) with $\{{0 \atop p}\} = 2$ $(p \geq 1)$ gives the array in Table 2. Applying Properties 1-5, one need only interpolate a p^{th}-order polynomial to $p+1$ (consecutive) entries of the p^{th} column to determine the column; column sums are as indicated for the Pascal array $(p > 1)$; diagonal sums obey $d_n = 2d_{n-1} = 3 \cdot 2^{n-1}$ (as $d_1 = 3$) and all (left-justified) square sub-arrays are unimodular. This is known as Vieta's array [10].

TABLE 2. Vieta's Array

1	2	2	2	2	2	2	2	2
1	3	5	7	9	11	13	15	17
1	4	9	16	25	36	49	64	81
1	5	14	30	55	91	140	204	285
1	6	20	50	105	196	336	540	825
1	7	27	77	182	378	714	1254	2079
1	8	35	112	294	672	1386	2640	4719
1	9	44	156	450	1122	2508	5148	9867
1	10	54	210	660	1782	4290	9438	19305

Case 1, Example 2 (A Fibonacci Array): Similar results hold for the array in Table 3, which is (2.1) with $\{{0 \atop p}\} = F_{p+1}$ $(p \geq 0)$, except now the p^{th} column sum is $\{{n \atop p+1}\} = F_p$, while diagonal sums obey $d_n = 2d_{n-1} + F_{n-1} = 2^n + \sum_{i=1}^{n-1} 2^{i-1} F_{n-1}$.

Again, by Property 5, all (left-justified) square sub-arrays are unimodular.

TABLE 3. Fibonacci Array of Case 1, Example 2

1	1	2	3	5	8	13	21	34
1	2	4	7	12	20	33	54	88
1	3	7	14	26	46	79	133	221
1	4	11	25	51	97	176	309	530
1	5	16	41	92	189	365	674	1204
1	6	22	63	155	344	709	1383	2587
1	7	29	92	247	591	1300	2683	5270
1	8	37	129	376	967	2267	4950	10220
1	9	46	175	551	1518	3785	8735	18955

2.2 Case 2: The Pascal Triangle and Variations

The Pascal triangle array in Table 4 is formed by use of the recurrence relation

$$\begin{Bmatrix} n \\ p \end{Bmatrix} = \begin{Bmatrix} n-1 \\ p \end{Bmatrix} + \begin{Bmatrix} n-1 \\ p-1 \end{Bmatrix} \quad (n \geq 1, p \geq 0) \tag{2.3}$$

with

$$\left\{ \begin{matrix} n \\ -1 \end{matrix} \right\} = 0 \ (n \geq 0), \quad \left\{ \begin{matrix} 0 \\ p \end{matrix} \right\} = 0 \ (p \geq 1), \quad \text{and} \quad \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = 1.$$

Precisely the same methods apply to this case as presented for Case 1. Corresponding results are given.

TABLE 4. Pascal Triangle

1	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0
1	3	3	1	0	0	0	0	0
1	4	6	4	1	0	0	0	0
1	5	10	10	5	1	0	0	0
1	6	15	20	15	6	1	0	0
1	7	21	35	35	21	7	1	0
1	8	28	56	70	56	28	8	1

Property 6 (Columns): It is interesting to note that the p^{th} column of the Pascal triangle is thus determined by the polynomial in n which interpolates p zeros followed by 1.

Property 7 (Column Sums):

$$\sum_{i=0}^{n} \left\{ \begin{matrix} i \\ p \end{matrix} \right\} = \left\{ \begin{matrix} n+1 \\ p+1 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ p+1 \end{matrix} \right\} + \left\{ \begin{matrix} 0 \\ p \end{matrix} \right\}. \tag{2.4}$$

For the Pascal triangle,

$$\sum_{i=0}^{n} \left\{ \begin{matrix} i \\ p \end{matrix} \right\} = \left\{ \begin{matrix} n+1 \\ p+1 \end{matrix} \right\} \quad (n, p \geq 0)$$

as expected (since here $\{ \substack{n \\ p} \}$ is just the binomial coefficient).

Property 8 (Diagonal Sums):

$$d_n - d_{n-1} = d_{n-2} + \left\{ \begin{matrix} 0 \\ n \end{matrix} \right\} = d_{n-2},$$

$d_0 = d_1 = 1$ in this case (very well known).

Property 9 (Determinants):

$$\left| \left[\left\{ \begin{matrix} n \\ p \end{matrix} \right\} \right]_{(0,m)} \right| = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} \left| \left[\left\{ \begin{matrix} n-1 \\ p-1 \end{matrix} \right\} \right]_{(1,m)} \right| = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}^{m+1}$$

$$= 1 \quad \text{in this case.}$$

Similarly (see Property 5), all (left-justified) square sub-arrays are unimodular (as noted by Bicknell & Hoggatt [1]). Also, as before, Properties 6-9 apply to the array formed with arbitrary initial row. Three examples follow.

Case 2, Example 1 (Division of p-Space by n (p – 1)-Spaces): The array in Table 5 is (2.3) with $\{ \substack{0 \\ 0} \} = 1$, $\{ \substack{0 \\ p} \} = 1$ $(p \geq 1)$. This relation is a generalization of the recurrence relations governing the maximum number of parts into which p-space can be divided by n $(p-1)$-spaces for p = 1, 2, 3. (Shannon [12] discusses these three instances in the context of the pedagogy of problem-solving.)

Properties 6-9 reduce to: entries in the p^{th} column are given by the p^{th}-order polynomial which interpolates $\{1, 2, 4, \ldots, 2^p\}$, the p^{th} column sum is given by

$$\begin{Bmatrix} n+1 \\ p+1 \end{Bmatrix} - 1, \quad d_n - d_{n-1} = d_{n-2} + \begin{Bmatrix} 0 \\ n \end{Bmatrix} = d_{n-2} + 1 \ (d_0 = 1, d_1 = 2)$$

(thus, the diagonal sums are the partial sums of the Fibonacci sequence), and all left-justified square sub-arrays are unimodular.

TABLE 5. Array in Case 2, Example 1

1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2
1	3	4	4	4	4	4	4	4
1	4	7	8	8	8	8	8	8
1	5	11	15	16	16	16	16	16
1	6	16	26	31	32	32	32	32
1	7	22	42	57	63	64	64	64
1	8	29	64	99	120	127	128	128
1	9	37	93	163	219	247	255	256

Case 2, Example 2 (Another Fibonacci Array): Using (2.3) with $\{{0 \atop p}\} = F_{p+1}$ $(p \geq 0)$ gives the array in Table 6. In this case, the p^{th} column is determined by the interpolating polynomial for the sequence $\{F_{p+1}, F_{p+2}, \ldots, F_{2p+1}\}$, the p^{th} column sum is given by

$$\begin{Bmatrix} n+1 \\ p+1 \end{Bmatrix} - F_{p+2},$$

diagonal sums obey $d_n = d_{n-1} + d_{n-2} + F_{n+1}$ $(d_0 = 1, d_1 = 2)$ and all left-justified square sub-arrays are unimodular.

TABLE 6. Array in Case 2, Example 2

1	1	2	3	5	8	13	21	34
1	2	3	5	8	13	21	34	55
1	3	5	8	13	21	34	55	89
1	4	8	13	21	34	55	89	144
1	5	12	21	34	55	89	144	233
1	6	17	33	55	89	144	233	377
1	7	23	50	88	144	233	377	610
1	8	30	73	138	232	377	610	987
1	9	38	103	211	370	609	987	1597

It is of interest to note that Lavers [8] found the corresponding "Fibonacci triangle" in his investigation of certain idempotent transformations.

Case 2, Example 3 (The Lucas Triangle): Setting $\{{0 \atop 0}\} = 1$, $\{{0 \atop 1}\} = 2$, and $\{{0 \atop p}\} = 0$ $(p \geq 2)$ in (2.3) gives the array in Table 7. This is the so-called Lucas triangle ([3], p. 26). Here the p^{th} column is determined by the interpolating polynomial for the sequence $\{0, \ldots, 0, 2, 2p+1\}$, the p^{th} column sum is given by

$$\begin{Bmatrix} n+1 \\ p+1 \end{Bmatrix} \ (p \geq 1),$$

diagonal sums obey $d_n = d_{n-1} + d_{n-2}$ $(d_0 = 1, d_1 = 3)$ (the Lucas numbers) and, once again, all left-justified square sub-arrays are unimodular.

TABLE 7. Array in Case 2, Example 3

1	2	0	0	0	0	0	0	0
1	3	2	0	0	0	0	0	0
1	4	5	2	0	0	0	0	0
1	5	9	7	2	0	0	0	0
1	6	14	16	9	2	0	0	0
1	7	20	30	25	11	2	0	0
1	8	27	50	55	36	13	2	0
1	9	35	77	105	91	49	15	2
1	10	44	112	182	196	140	64	17

2.3 Generalizations

We now generalize the foregoing to recursive relations of the form

$$\left\{ {n \atop p} \right\} = b\left\{ {n-1 \atop p} \right\} + \sum_{i=r}^{s} a_i \left\{ {n+i \atop p-1} \right\}, \tag{2.5}$$

with $\{{n \atop -1}\} = 0 \;\forall n$ and other boundary conditions for $\{{n \atop p}\}$ $(n \le 0,\, p \ge 0)$ given as necessary.

The above covers each of the earlier cases and others including:

(a) the complementary binomial coefficients of Puritz ([3], p. 33) (Table 8) where

$$\left\{ {n \atop p} \right\} = \left\{ {n-1 \atop p} \right\} - \left\{ {n \atop p-1} \right\} \quad (n \ge 1,\, p \ge 0),$$

with

$$\left\{ {n \atop -1} \right\} = 0 \;\forall n, \quad \left\{ {0 \atop p} \right\} = 0 \;(p \ge 1), \quad \text{and} \quad \left\{ {0 \atop 0} \right\} = 1;$$

TABLE 8: Array in 2.3(a)

1	0	0	0	0	0	0	0	0
1	−1	1	−1	1	−1	1	−1	1
1	−2	3	−4	5	−6	7	−8	9
1	−3	6	−10	15	−21	28	−36	45
1	−4	10	−20	35	−56	84	−120	165
1	−5	15	−35	70	−126	210	−330	495
1	−6	21	−56	126	−252	462	−792	1287
1	−7	28	−84	210	−462	924	−1716	3003
1	−8	36	−120	330	−792	1716	−3432	6435

(b) an analog of the Pascal triangle (Table 9) studied by Wong & Maddocks ([3], p. 36), where

$$\left\{ {n \atop p} \right\} = \left\{ {n-1 \atop p} \right\} + \left\{ {n-1 \atop p-1} \right\} + \left\{ {n-2 \atop p-1} \right\} \quad (n \ge 1,\, p \ge 0),$$

with

$$\left\{ {n \atop -1} \right\} = 0 \;\forall n, \quad \left\{ {0 \atop p} \right\} = 0 \;(p \ge 1),$$

$$\left\{ {-1 \atop p} \right\} = 0 \;(p \ge 0), \quad \text{and} \quad \left\{ {0 \atop 0} \right\} = 1,$$

for which row sums are Pell numbers and diagonal sums are "Tribonacci" numbers;

TABLE 9. Array in 2.3(b)

1	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	3	1	0	0	0	0	0	0
1	5	5	1	0	0	0	0	0
1	7	13	7	1	0	0	0	0
1	9	25	25	9	1	0	0	0
1	11	41	63	41	11	1	0	0
1	13	61	129	129	61	13	1	0
1	15	85	231	321	231	85	15	1

and

(c) the recurrence relation

$$\left\{ {n \atop p} \right\} = b \left\{ {n-1 \atop p} \right\} + a_{-1} \left\{ {n-1 \atop p-1} \right\}$$

studied by Cadogan ([3], p. 30).

Property 10 (Column Sums): The most straightforward generalization is for the special case of (2.5),

$$\left\{ {n \atop p} \right\} = \left\{ {n-1 \atop p} \right\} + a_{-k} \left\{ {n-k \atop p-1} \right\} \quad (k \geq 0), \tag{2.6}$$

with $\{{n \atop -1}\} = 0 \ \forall n$ and other boundary conditions for $\{{n \atop p}\}$ $(n \leq 0,\ p \geq 0)$ given as necessary.

For $n \geq 1$ and $p \geq 0$,

$$\begin{aligned} a_{-k} \sum_{i=0}^{n} \left\{ {i \atop p} \right\} &= \sum_{i=1}^{n} \left(\left\{ {i+k \atop p+1} \right\} - \left\{ {i+k-1 \atop p+1} \right\} \right) + \left\{ {0 \atop p} \right\} \\ &= \left\{ {n+k \atop p+1} \right\} - \left\{ {k \atop p+1} \right\} + \left\{ {0 \atop p} \right\}. \end{aligned}$$

Property 11 (Diagonal Sums): For $n \geq 1$,

$$\begin{aligned} d_n - b d_{n-1} &= \sum_{i=0}^{n-1} \left(\left\{ {n-i \atop i} \right\} - b \left\{ {n-i-1 \atop i} \right\} \right) + \left\{ {0 \atop n} \right\} \\ &= \sum_{i=0}^{n-1} \sum_{j=r}^{s} a_j \left\{ {n-i+j \atop i-1} \right\} + \left\{ {0 \atop n} \right\} \\ &= \sum_{j=r}^{s} a_j d_{n+j-1} - \sum_{j=r}^{s} a_j \sum_{i=0}^{j} \left\{ {j-i \atop n+i-1} \right\} + \left\{ {0 \atop n} \right\} \quad (n+r-1 \geq 0). \end{aligned}$$

Property 12 (Determinants): For $b \neq 0$, and noting that $\{{n \atop 0}\} = ab^n$ $(a = \{{0 \atop 0}\},\ n \geq 0)$,

$$\left| \left[\left\{ {n \atop p} \right\} \right]_{(0,m)} \right| = \left| \begin{array}{ccc} \left\{ {0 \atop 0} \right\} & \left\{ {0 \atop 1} \right\} \quad \cdots & \left\{ {0 \atop m} \right\} \\ \left\{ {1 \atop 0} \right\} - b \left\{ {0 \atop 0} \right\} & & \\ \cdots & \left[\left\{ {n \atop p} \right\} - b \left\{ {n-1 \atop p} \right\} \right]_{(1,m)} & \\ \left\{ {m \atop 0} \right\} - b \left\{ {m-1 \atop 0} \right\} & & \end{array} \right| = \left\{ {0 \atop 0} \right\} \left| \left[\sum_{i=r}^{s} a_i \left\{ {n+i \atop p-1} \right\} \right]_{(1,m)} \right|$$

$$
= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \sum_i a_i \begin{Bmatrix} 1+i \\ 0 \end{Bmatrix} \sum_{i,j} a_i a_j \begin{Bmatrix} 2+i+j \\ 0 \end{Bmatrix} \cdots \sum_{i,j,\dots k} a_i a_j \dots a_k \begin{Bmatrix} m+i+j+\cdots+k \\ 0 \end{Bmatrix}
$$

$$
= a^{m+1} \left(b \sum_{i=r}^{s} a_i b^i \right)^{m(m+1)/2}.
$$

Sufficient conditions for this derivation are that each element below and including $\{{}^{m+(m-1)r}_{\;\;\;p}\}$ $(m \geq 2, 0 \leq p < m-1)$ has been formed by the given recurrence relation. (Thus, if $m+(m-1)r \leq 0$, the result will only apply to sub-arrays beginning at row $1-m-(m-1)r$. This is not a restriction if $r \geq -1$.)

When $b = 0$, we need only restrict the previous formula to $r = s = -1$ (though it will apply to $s \geq -1$), hence,

$$
\left| \left[\begin{Bmatrix} n \\ p \end{Bmatrix} \right]_{(0,m)} \right| = a^{m+1} a_{-1}^{m(m+1)/2}.
$$

Thus, other unimodular arrays can be formed by setting, for example, $a = b = \sum_i a_i = 1$.

3. GENERALIZED PASCAL TRIANGLES

Consider the square with the rule of formation,

$$
\begin{Bmatrix} n \\ p \end{Bmatrix} = \begin{Bmatrix} n \\ p-2 \end{Bmatrix} + \begin{Bmatrix} n-1 \\ p-1 \end{Bmatrix} + \begin{Bmatrix} n-2 \\ p-1 \end{Bmatrix}, \tag{3.1}
$$

with

$$
\begin{Bmatrix} n \\ p \end{Bmatrix} = 0 \ (p < 0), \quad \begin{Bmatrix} -1 \\ p \end{Bmatrix} = 0 \ (p > 0), \quad \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ p \end{Bmatrix} = 1 \ (p > 0).
$$

TABLE 10. Generalized Pascal Triangle

1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	2	2	3	3	4	4	5	5	6	6	7	7
1	2	3	5	6	9	10	14	15	20	21	27		
1	2	5	7	13	16	26	30	45	50	71			
1	3	6	13	19	35	45	75	90	140				
1	3	9	16	35	51	96	126	216					
1	4	10	26	45	96	141	267						
1	4	14	30	75	126	267							
1	5	15	45	90	216								
1	5	20	50	140	266								
1	6	21	71	161									
1	6	27	77										
1	7	28	105										
1	7	35											

If we add along the diagonals in Table 10, we get the sequence $\{1, 2, 3, 6, 18, 27, 54, 81, 162, \dots\}$, which is generated by the recurrence relation $W_n = W_{n-1} + W_{n-2} + \delta(2, n) W_{n-3}$, $n > 3$, where

$$
\delta(m, n) = \begin{cases} 1 & \text{if } m | n, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{(Shannon [11])}
$$

and the initial terms are $W_i = i$, $i = 1, 2, 3$. We can bifurcate this sequence into $W_{1n} = \{1, 3, 9, 27, 81, \ldots\}$ and $W_{2n} = (2, 6, 18, 54, 162, \ldots\}$, which are generated by the recurrence relation

$$W_{jn} = 2W_{j,n-1} + 3W_{j,n-2}, \quad n > 2. \tag{3.3}$$

This bifurcation enables us to distinguish two triangles within the square array, as in Tables 11 and 12.

TABLE 11. Triangle Corresponding to $\{W_{1n}\}$ [1]

1												
1	1	1										
1	2	3	2	1								
1	3	6	7	6	3	1						
1	4	10	16	19	16	10	4	1				
1	5	15	30	45	51	45	30	15	5	1		
1	6	21	50	90	126	141	126	90	50	21	6	1

TABLE 12. Triangle Corresponding to $\{W_{2n}\}$

1	1										
1	2	2	1								
1	3	5	5	3	1						
1	4	9	13	13	9	4	1				
1	5	14	26	35	35	26	14	5	1		
1	6	20	45	75	96	96	75	45	20	6	1

Notice that the triangle in Table 12 has the feature that

$$\sum_{p=0}^{2n+1} \left\{ {n \atop p} \right\} = 2 \cdot 3^n, \quad n = 0, 1, \ldots.$$

Obviously we get the ordinary Pascal triangle if we take the diagonals of the Pascal square (see Table 1). Similarly, if we consider

$$\left\{ {n \atop p} \right\} = \left\{ {n-1 \atop p} \right\} + \left\{ {n-1 \atop p-1} \right\} + \left\{ {n \atop p-1} \right\} \tag{3.4}$$

with $\{{n \atop 0}\} = 1$, $\{{0 \atop p}\} = 1$, which is also of the form (2.5), we get the square in Table 13(a) and the triangle in Table 13(b). In addition to the properties of Section 2, the numbers in Table 11(a), $D(n, m)$, are Delanoy numbers [14] and are linked to minimal paths in lattices. We observe here that the row sums yield the Pell sequence $\{P_n\} = \{1, 2, 5, 12, 29, 70, \ldots\}$ defined by the initial terms $P_1 = 1$, $P_2 = 2$, and the second-order recurrence relation (Horadam [7]) $P_n = 2P_{n-1} + P_{n-2}$, $n > 2$.

TABLE 13. Arrays Corresponding to (3.4)

(a)

1	1	1	1	1
1	3	5	7	9
1	5	13	25	41
1	7	25	63	129
1	9	41	129	321

(b)

1					
1	1				
1	3	1			
1	5	5	1		
1	7	13	7	1	
1	9	25	25	9	1

This is a particularly rich triangle because, when we add along the diagonals, we obtain the third-order sequence $\{0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots\}$.

Again, if we take the triangle from the diagonals of the array formed from equation (2.3),

$$\left\{ n \atop p \right\} = \left\{ n-1 \atop p \right\} + \left\{ n-1 \atop p-1 \right\} \tag{3.5}$$

with $\left\{ n \atop 0 \right\} = 1$, $\left\{ 0 \atop p \right\} = 1$, we get the square and triangle of Table 14(a) and (b).

TABLE 14. Arrays Associated with (3.5)

(a)

1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2
1	3	4	4	4	4	4	4	4
1	4	7	8	8	8	8	8	8
1	5	11	15	16	16	16	16	16
1	6	16	26	31	32	32	32	32
1	7	22	42	57	63	64	64	64
1	8	29	64	99	120	127	128	128
1	9	37	93	163	219	247	255	256

(b)

1							
1	1						
1	2	1					
1	3	2	1				
1	4	4	2	1			
1	5	7	4	2	1		
1	6	11	8	4	2	1	
1	7	16	15	8	4	2	1

Here, too, we find two sequences. The row sums yield $\{x_n\} = \{1, 2, 4, 7, 12, 20, 33, \ldots\}$, with the nonhomogenous second-order recurrence relation $x_n = x_{n-1} + x_{n-2} + 1$, while the diagonal sums yield $\{y_n\} = \{1, 1, 2, 3, 5, 7, 11, 16, 24, 35, 52, \ldots\}$, which is formed from the fifth-order recurrence relation $y_n = y_{n-1} + y_{n-2} - y_{n-5}$, $n > 5$, which is a particular case of equation (1) found in Dubeau & Shannon [5].

4. CONCLUSION

We have demonstrated that a number of well-known properties of Pascal-type arrays are consequences of a more general partial recurrence relation. Further investigations could include relating the various sequences to standard sequences identified by Sloane [13]. Algebraic structural properties can be studied along the lines of Korec [6] who has, in effect, generalized some of the work of Wells [15].

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❖❖❖

ON A GENERALIZATION OF A CLASS OF POLYNOMIALS

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1. INTRODUCTION

In [1], R. André-Jeannin considered a class of polynomials $U_n(p,q;x)$ defined by

$$U_n(p,q;x)=(x+p)U_{n-1}(p,q;x)-qU_{n-2}(p,q;x),\quad n>1,$$

with initial values $U_0(p,q;x)=0$ and $U_1(p,q;x)=1$.

Particular cases of $U_n(p,q;x)$ are: the well-known Fibonacci polynomials $F_n(x)$; the Pell polynomials $P_n(x)$ (see [4]); the Fermat polynomials of the first kind $\phi(x)$ (see [5], [3]); and the Morgan-Voyce polynomials of the second kind $B_n(x)$ (see [2]).

In this paper we shall consider the polynomials $\phi_n(p,q;x)$ defined by

$$\phi_n(p,q;x)=(x+p)\phi_{n-1}(p,q;x)-q\phi_{n-3}(p,q;x),\tag{1.0}$$

with initial values $\phi_{-1}(p,q;x)=\phi_0(p,q;x)=0$ and $\phi_1(p,q;x)=1$. The parameters p and q are arbitrary real numbers, $q\neq 0$.

Let us denote by α, β, and γ the complex numbers, so that they satisfy

$$\alpha+\beta+\gamma=p,\quad \alpha\beta+\alpha\gamma+\beta\gamma=0,\quad \alpha\beta\gamma=-q.\tag{1.1}$$

The first few members of the sequence $\{\phi_n(p,q;x)\}$ are:

$$\phi_2(p,q;x)=p+x;\quad \phi_3(p,q;x)=p^2+2px+x^2;\quad \phi_4(p,q;x)=p^3-q+3p^2x+3px^2+x^3.$$

By induction on n, we can say that there is a sequence $\{c_{n,k}(p,q)\}_{n\geq 0,\,k\geq 0}$ of numbers, so that it holds

$$\phi_{n+1}(p,q;x)=\sum_{k\geq 0}c_{n,k}(p,q)x^k,\tag{1.2}$$

where $c_{n,k}(p,q)=0$ for $k>n$ and $c_{n,n}(p,q)=1$. Therefore, if we set $c_{-1,k}(p,q)=c_{-2,k}(p,q)=0$, $k\geq 0$, then we have

$$\phi_{-1}(p,q;x)=\sum_{k\geq 0}c_{-2,k}(p,q)x^k\quad\text{and}\quad \phi_0(p,q;x)=\sum_{k\geq 0}c_{-1,k}(p,q)x^k.$$

Later on, we consider some other interesting sequences of numbers, define the polynomials $\phi_n^1(p,q;x)$ and $\phi_n^2(p,q;x)$, which are rising diagonal polynomials of $\phi_n(p,q;x)$ and $\phi_n^1(p,q;x)$, respectively, and finally, consider the generalized polynomials $\phi_n^m(x)$.

2. DETERMINATION OF THE COEFFICIENTS $c_{n,k}(p,q)$

The main purpose of this section is to determine the coefficients $c_{n,k}(p,q)$. First, for $n\geq 1$, $k\geq 1$, from (1.0), (1.1), and (1.2), we obtain

$$\begin{aligned}c_{n,k}(p,q)&=c_{n-1,k-1}(p,q)+pc_{n-1,k}(p,q)-qc_{n-3,k}(p,q)\\&=c_{n-1,k-1}(p,q)+(\alpha+\beta)c_{n-1,k}(p,q)+\gamma(c_{n-1,k}(p,q)-\gamma(\alpha+\beta)c_{n-3,k}(p,q)).\end{aligned}\tag{2.0}$$

Therefore, we shall prove the following lemma.

Lemma 2.1: For every $k \geq 0$, we have

$$(1-pt+qt^3)^{-(k+1)} = \sum_{n\geq 0} d_{n,k} t^n, \tag{2.1}$$

where

$$d_{n,k} = \sum_{i+j+s=n} \binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k} \alpha^i \beta^j \gamma^s. \tag{2.2}$$

Proof: From (2.1), using (1.1), we get

$$\begin{aligned}(1-pt+qt^3)^{-(k+1)} &= (1-\alpha t)^{-(k+1)}(1-\beta t)^{-(k+1)}(1-\gamma t)^{-(k+1)} \\ &= \sum_{n\geq 0}\sum_{i+j+s=n} \binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k} \alpha^i \beta^j \gamma^s t^n.\end{aligned}$$

Statement (2.2) follows immediately from the last equality. □

Now we shall prove the following theorem.

Theorem 2.1: The coefficients $c_{n,k}(p,q)$ are given by

$$c_{n,k} = \sum_{i+j+s=n-k} \binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k} \alpha^i \beta^j \gamma^s. \tag{2.3}$$

Proof: First, let us define the generating function of the sequence $\phi_n(p,q;x)$ by

$$F(x,t) = \sum_{n\geq 0} \phi_{n+1}(p,q;x) t^n. \tag{2.4}$$

Then, using (1.0), we find

$$F(x,t) = (1-(p+x)t+qt^3)^{-1}. \tag{2.5}$$

Now, from (2.5) and (2.4), we deduce that

$$\frac{\partial^k F(x,t)}{\partial x^k} = \frac{k!\,t^k}{(1-(x+p)t+qt^3)^{k+1}} = \sum_{n\geq 0} \phi^{(k)}_{n+1+k}(p,q;x) t^{n+k},$$

since $\phi_{n+1}(p,q;x)$ is a polynomial of degree n. If we take $x=0$ in the last formula and recall that

$$c_{n+k,k}(p,q) = \frac{1}{k!}\phi^{(k)}_{n+1+k}(p,q;0),$$

then from (3), and by Taylor's formula, we get

$$(1-pt+qt^3)^{-(k+1)} = \sum_{n\geq 0} c_{n+k,k}(p,q) t^n. \tag{2.6}$$

Comparing (2.6) to (2.1) and (2.2), we see that

$$\begin{aligned}c_{n+k,k}(p,q) &= \frac{1}{k!}\phi^{(k)}_{n+1+k}(p,q;0) = d_{n,k} \\ &= \sum_{i+j+s=n} \binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k} \alpha^i \beta^j \gamma^s.\end{aligned} \tag{2.7}$$

By (2.7), we see that

$$c_{n,k}(p,q)=d_{n-k,k}=\sum_{i+j+s=n-k}\binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k}\alpha^i\beta^j\gamma^s.$$

This completes the proof of Theorem 2.1. □

Remarks:

(i) If $k=0$, then (2.3) becomes

$$c_{n,0}(p,q)=\sum_{i+j+s=n}\alpha^i\beta^j\gamma^s=\phi_{n+1}(p,q;0).$$

(ii) If $p=0$, then (2.1) becomes

$$(1+qt^3)^{-(k+1)}=\sum_{n\geq 0}(-1)^n\binom{k+n}{n}q^n t^{3n}.$$

Thus, we get

$$c_{n,n-3k}(0,q)=(-1)^k\binom{n-2k}{k}q^k,\ c_{n,n-3k-1}(0,q)=0,\ c_{n,n-3k-2}(0,q)=0,$$

for $k\leq[n/3]$. Now, from (1.2), we find that

$$\phi_{n+1}(0,q;x)=\sum_{k=0}^{[n/3]}c_{n,n-3k}(0,q)x^{n-3k}=\sum_{k=0}^{[n/3]}(-1)^k\binom{n-2k}{k}q^k x^{n-3k}. \tag{2.8}$$

We shall prove the following theorem.

Theorem 2.2: The coefficients $c_{n,k}(p,q)$ have the following form:

$$c_{n,k}(p,q)=\sum_{r=0}^{[(n-k(/3]}(-1)^r\binom{n-2r}{r}\binom{n-3r}{k}q^r p^{n-3r-k},\quad n\geq k. \tag{2.9}$$

Proof: Using (1.0), we see that $\phi_{n+1}(p,q;x)=\phi_{n+1}(0,q;x+p)$. Thus,

$$c_{n,k}(p,q)=\frac{1}{k!}\phi_{n+1}^{(k)}(p,q;0)=\frac{1}{k!}\phi_{n+1}^{(k)}(0,q;p).$$

Now, by (2.8), it follows that

$$\frac{1}{k!}\phi_{n+1}(0,q;p)=\sum_{r=0}^{[(n-k(/3]}(-1)^r\binom{n-2r}{r}\binom{n-3r}{k}q^r p^{n-3r-k}.$$

This is the desired equality (2.9). □

Corollary 2.1: From (2.9) or (2.3), we find that:

$$-\alpha-\beta-\gamma=-p;$$
$$(-\alpha)(-\beta)+(-\beta)(-\gamma)+(-\alpha)(-\gamma)=0;$$
$$(-\alpha)(-\beta)(-\gamma)=q.$$

Hence,

$$c_{n,k}(-p,-q)=(-1)^{n-k}c_{n,k}(p,q).$$

3. A PARTICULAR CASE

In this section we shall consider a particular case of the polynomials $\phi_n(p,q;x)$.

If $\alpha=\beta\neq\gamma$, then $\alpha=\beta=2p/3$, $\gamma=-p/3$, and $27q=4p^3$. In this case, by (2.1), we get

$$(1-pt+qt^3)^{-(k+1)}=(1-\alpha t)^{-2(k+1)}(1-\gamma t)^{-(k+1)}$$
$$=\sum_{n\geq 0}\left(\sum_{i+j=n}\binom{2k+1+i}{i}\binom{k+j}{j}\alpha^i\gamma^j\right)t^n.$$

Therefore, we have

$$c_{n,k}(p,q)=(p/3)^{n-k}\sum_{i+j=n-k}(-1)^j2^i\binom{2k+1+i}{i}\binom{k+j}{j}.$$

4. SOME INTERESTING SEQUENCES OF NUMBERS

Here we shall consider the following sequences of numbers.

(a) If we take $x=-p$, we get the sequence $\phi_n(p,q;-p)=0$. This sequence has the following properties: $\phi_{3n}(p,q;-p)=\phi_{3n+2}(p,q;-p)=0$ and $\phi_{3n+1}(p,q;-p)=(-1)^nq^n$. From relation (1.2), it follows that

$$\sum_{k=0}^{3n+l}(-1)^kp^kc_{3n+l,k}(p,q)=0,$$

for $l=1$, and

$$\sum_{k=0}^{3n}(-1)^kp^kc_{3n,k}(p,q)=(-1)^nq^n,$$

for $l=2$.

(b) Using (1.0), for $x=0$, we have the sequence $\{\phi_n(p,q;0)\}$, which is defined by

$$\phi_n(p,q;0)=p\phi_{n-1}(p,q;0)-q\phi_{n-3}(p,q;0),$$

for $n\geq 2$, with initial values $\phi_{-1}(p,q;0)=\phi_0(p,q;0)=0$ and $\phi_1(p,q;0)=1$.

5. RISING DIAGONAL POLYNOMIALS

Now, we define the polynomials $\phi_n^1(p,q;x)$ and $\phi_n^2(p,q;x)$. Also, we define the polynomials $\phi_n^m(x)$. First, we shall write the polynomials $\phi_n(p,q;x)$ in tabular form (see Table 1). We define the polynomials $\phi_n^1(p,q;x)$ by

$$\phi_{n+1}^1(p,q;x)=\sum_{k=0}^{[n/2]}c_{n,k}^1(p,q)x^k=\sum_{k=0}^{[n/2]}c_{n-k,k}(p,q)x^k, \tag{5.1}$$

where $\phi_0^1(p,q;x)=0$ and $c_{n,k}^1(p,q)=0$ for $k>[n/2]$. Also, from Table 1, we get

$$\begin{aligned}&\phi_1^1(p,q;x)=1,\quad \phi_2^1(p,q;x)=p,\quad \phi_3^1(p,q;x)=p^2+x,\\&\phi_4^1(p,q;x)=p^3-q+2px,\quad \phi_5^1(p,q;x)=p^4-2pq+3p^2x+x^2.\end{aligned} \tag{5.2}$$

TABLE 1

n							
0	0	.	.	.	.	.	...
1	1	.	.	.	.	.	...
2	p	$+x$	.	.	.	.	...
3	p^2	$+2px$	$+x^2$	.	.	.	...
4	p^3-q	$+3p^2x$	$+3px^2$	$+x^3$	.	.	...
5	p^4-2pq	$+(4p^3-q)x$	$+6p^2x^2$	$+4px^3$	$+x^4$	.	...
6	p^5-3p^2q	$+(5p^4-6pq)x$	$+(10p^3-3q)x^2$	$+10p^2x^3$	$+5px^4$	$+x^5$	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	...

In fact, we will prove the following theorem.

Theorem 5.1: The polynomials $\phi_n^1(p, q; x)$ satisfy the following recurrence relation:

$$\phi_n^1(p, q; x) = p\phi_{n-1}^1(p, q; x) + x\phi_{n-2}^1(p, q; x) - q\phi_{n-3}^1(p, q; x), \quad n \geq 3. \tag{5.3}$$

Proof: To prove (5.3), we will use the notations $\phi_n^1(x)$ and $c_{n,k}$ instead of $\phi_n^1(p, q; x)$ and $c_{n,k}(p, q)$, respectively, and proceed by induction on n. From (5.2), we see that statement (5.3) holds for $n = 3$. Suppose statement (5.3) is true for $n \geq 3$. Using (5.1), and by (2.0), we obtain

$$\begin{aligned}
\phi_{n+1}^1(x) &= c_{n,0} + \sum_{k=1}^{[n/2]} c_{n-k,k}x^k \\
&= pc_{n-1,0} - qc_{n-3,0} + \sum_{k=1}^{[n/2]} (c_{n-1-k,k-1} + pc_{n-1-k,k} - qc_{n-3-k,k})x^k \\
&= p\sum_{k=0}^{[(n-1)/2]} c_{n-1-k,k}x^k - q\sum_{k=0}^{[(n-3)/2]} c_{n-3-k,k}x^k + x\sum_{k=0}^{[(n-2)/2]} c_{n-2-k,k}x^k,
\end{aligned}$$

since the relation $c_{n,0} = pc_{n-1,0} - qc_{n-3,0}$ is valid for $n \geq 1$. Thus, statement (5.3) follows by the last equality. This completes the proof. □

Similarly, let $\phi_n^2(p, q; x)$ be the rising diagonal polynomial of $\phi_n^1(p, q; x)$, i.e.,

$$\phi_{n+1}^2(p, q; x) = \sum_{k=0}^{[n/3]} c_{n-k,k}^1(p, q)x^k.$$

Furthermore, if we denote the process

$$\phi_n^0(x) \mapsto \phi_n^1(x) \mapsto \phi_n^2(x) \mapsto \cdots \mapsto \phi_n^m(x)$$

by $\phi_n^0(x) \equiv \phi_n(p, q; x)$, then we have

$$c_{n,k}^0 = c_{n,k} \quad \text{and} \quad c_{n,k}^{m+1} = c_{n-k,k}^m. \tag{5.4}$$

From relations (5.4), we get

$$c_{n,k}^m = c_{n-k,k}^{m-1} = \cdots = c_{n-mk,k}^0.$$

Hence, for $k = 0$, we have

$$c_{n,0}^m = c_{n,0}^0 = c_{n,0}.$$

If $n = 0, 1, \ldots, m$, then $[n/(m+1)] = 0$, so we have

$$\phi_{n+1}^m(x) = c_{n,0}^m = c_{n,0}, \quad n = 0, 1, \ldots, m.$$

Also, we get

$$\phi_{n+1}^m(x) = \sum_{k=0}^{[n/(m+1)]} c_{n-mk,k} x^k, \tag{5.5}$$

where $c_{n,k}^m = 0$ for $k > [n/(m+1)]$. Therefore, we are going to prove the following theorem.

Theorem 5.2: The polynomials $\phi_n^m(x)$ satisfy the recurrence relation

$$\phi_{n+1}^m(x) = p\phi_n^m(x) - q\phi_{n-2}^m(x) + x\phi_{n-m}^m(x), \quad n \ge m \ge 2, \tag{5.6}$$

where $\phi_{-1}^m(x) = \phi_0^m(x) = 0$ and $\phi_{n+1}^m(x) = c_{n,0}^m$, $n = 0, 1, \ldots, m$.

Proof: We prove that (5.6) holds for $n \ge m \ge 2$. If $n = m$, then

$$\begin{aligned} \phi_{m+1}^m(x) &= c_{m,0} = pc_{m-1,0} - qc_{m-3,0} \\ &= p\phi_m^m(x) - q\phi_{m-2}^m(x) + x\phi_0^m(x) \quad (\phi_0(p, q; x) = 0). \end{aligned}$$

Assume now that $n \ge m+1$, then, by (2.0), we have

$$\begin{aligned} \phi_{n+1}^m(x) &= \sum_{k=0}^{[n/(m+1)]} c_{n-mk,k} x^k = c_{n,0} + \sum_{k=1}^{[n/(m+1)]} c_{n-mk,k} x^k \\ &= pc_{n-1,0} - qc_{n-3,0} + \sum_{k=1}^{[n/(m+1)]} (pc_{n-1-mk,k} - qc_{n-mk-3,k} + c_{n-mk-1,k-1}) x^k \quad (n - mk \ge 1) \\ &= p\sum_{k=0}^{[n/(m+1)]} c_{n-1-mk,k} x^k - q\sum_{k=0}^{[n/(m+1)]} c_{n-3-mk,k} x^k + x\sum_{k=0}^{[n/(m+1)]} c_{n-1-mk,k-1} x^{k-1} \\ &= p\sum_{k=0}^{[(n-1)/(m+1)]} c_{n-1-mk,k} x^k - q\sum_{k=0}^{[(n-3)/(m+1)]} c_{n-3-mk,k} x^k + x\sum_{k=0}^{[(n-1-m)/(m+1)]} c_{n-m-mk-1,k} x^k \\ &= p\phi_n^m(x) - q\phi_{n-2}^m(x) + x\phi_{n-m}^m(x). \quad \square \end{aligned}$$

Corollary 5.1: The coefficients $c_{n,k}^m$ satisfy the following relation,

$$c_{n,k}^m = pc_{n-1,k}^m - qc_{n-3,k}^m + c_{n-1-m,k-1}^m, \quad m \ge 0, n \ge 2, n \ge m, k \ge 1,$$

where $c_{n,k}^m = c_{n,k}^m(p, q)$.

Corollary 5.2: For $m = 2$, from (5.6), we have

$$\phi_n^2(x) = p\phi_{n-1}^2(x) + (x - q)\phi_{n-2}^2(x), \quad n \ge 2, \tag{5.7}$$

with $\phi_0^2(x) = 0$, $\phi_{n+1}^2(x) = c_{n,0}^1 = c_{n,0}$, $n = 0, 1$.

Remark: For every $n \ge 1$, we have

$$\phi_n^2(p, q; x) = \phi_n(p, x - q; 0). \tag{5.8}$$

Proof: By (1.0), the sequence $\{\phi_n(p, x-q; 0)\}$ satisfies relation (5.7) with $\phi_0(p, q-x; 0) = 0$, $\phi_1(p, q-x; 0) = 1$, $\phi_2(p, q-x; 0) = p$. From this and (5.7), we see that (5.8) holds for $n = 1$ and $n = 2$. If (5.8) holds for $n \le m$, then for $n = m+1$ we get

$$\begin{aligned}\phi^2_{m+1}(p, q; x) &= p\phi^2_m(p, q; x) - (q-x)\phi^2_{m-2}(p, q; x)\\ &= p\phi_m(p, q-x; 0) - (q-x)\phi_{m-2}(p, q-x; 0) = \phi_{m+1}(p, q-x; 0).\end{aligned}$$

Using induction on n, we conclude that relation (5.8) holds for every $n \ge 1$. By (5.8), and from (2.9) with $k = 0$, we get

$$\phi^2_{n+1}(p, q; x) = \sum_{r=0}^{[n/3]} \binom{n-2r}{r} (x-q)^r p^{n-3r}. \tag{5.9}$$

Special Cases

For $x = q$, by (5.9), we have

$$\sum_{k=0}^{[n/3]} q^k c^1_{n-k,k}(p, q) = p^n.$$

For $p = 2$ and $q = 1$, the last equality becomes

$$\sum_{k=0}^{[n/3]} c^1_{n-k,k}(2, 1) = 2^n.$$

For $p = 0$, the polynomials $\phi^2_{n+1}(p, q; x)$ have the following representations:

$$\phi^2_{n+1}(0, q; x) = (x-q)^s$$

for $n = 3s$, and

$$\phi^2_{n+1}(0, q; x) = 0$$

for $n = 3s+1$ and for $n = 3s+2$.

6. GENERALIZATION

If we consider the general recurrence relation

$$U_n(x) = (x+p)U_{n-1}(x) - qU_{n-2}(x) + rU_{n-3}(x), \quad n \ge 3,$$

we find that

$$U_{n+1}(x) = \sum_{k=0}^{n} c_{n,k}(p, q, r)x^k,$$

where

$$\sum_{n\ge 0} c_{n+k,k}(p, q, r)t^n = (1 - pt + qt^2 - rt^3)^{-(k+1)}.$$

In this case, we have $\alpha + \beta + \gamma = p$, $\alpha\beta + \alpha\gamma + \beta\gamma = q$, and $\alpha\beta\gamma = r$. Particularly, if $\alpha = \beta = \gamma = p/3$, then $q = p^2/3$ and $r = p^3/27$. So we get

$$\sum_{n\ge 0} c_{n+k,k}(p, q, r)t^n = (1-\alpha t)^{-3(k+1)} = \sum_{n\ge 0} \binom{3k+2+n}{3k+2} (p/3)^n t^n;$$

hence,

$$c_{n,k}(p,q,r) = \binom{2k+2+n}{3k+2}(p/3)^{n-k}.$$

Thus, we can define $B_n^1(x)$, i.e., a generalization of Morgan-Voyce polynomials, by setting $\alpha = \beta = \gamma = 1$ (i.e., $p = 3, q = 3, r = 1$),

$$B_n^1(x) = \sum_{k=0}^{n} \binom{n+2k+2}{3k+2} x^k.$$

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OBTAINING DIVIDING FORMULAS $n|Q(n)$ FROM ITERATED MAPS

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1. INTRODUCTION

In this work we show that we may use iterated maps to understand and generate a dividing formula $n|Q(n)$, where n is any positive integer. A well-known example of a dividing formula concerning Fibonacci numbers, for instance, is

$$n|(F_{n+1}+F_{n-1}-1), \tag{1.1}$$

where n is a prime and F_n is the n^{th} Fibonacci number.

We will show that from iterated maps we have a systematic way to construct functions $Q(n)$ such that $n|Q(n)$. In this paper, we show how to derive the above dividing formula from an iterated map. We also generalize the result to the case in which n is any positive integer and to the case of Fibonacci numbers of degree m. We begin with Theorem 2.1 below.

2. THE FUNDAMENTAL THEOREM: $n|N(n)$

Theorem 2.1: For an iterated map,

$$n|N(n), \tag{2.1}$$

where $N(n)$ is the number of period-n points for the map.

Proof: If $N(n)=0$, formula (2.1) is obvious. If $N(n)\neq 0$, then the orbit of a period-n point is an n-cycle containing n distinct period-n points. Since there are no common elements in any two distinct n-cycles, $N(n)$ must be a multiple of n, i.e., $n|N(n)$, and $N(n)/n$ is an integer representing the number of n-cycles for the map.

As a consequence of this fundamental theorem, *each iterated map*, in principle, offers a desired $Q(n)$ function such that $n|Q(n)$, where $Q(n)=N(n)$, the number of period-n points of an iterated map. Therefore, we have an additional way to understand the dividing formula $n|Q(n)$ from the point of view of iterated maps.

3. THE $N(n)$ OF AN ITERATED MAP

For a general discussion, we consider a map $f(x)$ in some interval. The fixed points of f are determined from the formula

$$f(x)=x. \tag{3.1}$$

The number of fixed points for f can be determined from the number of intersections of the curve $y=f(x)$ with the diagonal line $y=x$ in the interval. We define $f^{[n]}(x)$ for the n^{th} iterate of x for f, then $f^{[n]}(x)\equiv f(f^{[n-1]}(x))$. We should distinguish a fixed point of $f^{[n]}$, and a period-n point of f. The fixed points of $f^{[n]}$ are determined from the formula $f^{[n]}(x)=x$; however, the period-n points of f are determined from the following two equations:

$$f^{[n]}(x) = x, \tag{3.2}$$

$$f^{[i]}(x) \neq x \quad \text{for } i = 1, 2, \ldots, n-1. \tag{3.3}$$

We need (3.3) because an x satisfying only (3.2) is not necessarily a period-n point of f, since it could be a fixed point of f or, in general, a period-m point of f, where $m < n$ and $m|n$. Formulas (3.2) and (3.3) together ensure that x is a period-n point of f. Then $N(n)$ represents the number of points satisfying both (3.2) and (3.3). We let $N_\Sigma(n)$ represent the number of points satisfying only (3.2); hence, $N_\Sigma(n)$ represents the number of fixed points for $f^{[n]}$. Accordingly, we have

$$N_\Sigma(n) = \sum_{d|n} N(d),$$

where the sum is over all the divisors of n (including 1 and n). $N_\Sigma(n)$ is simply determined from intersections of the curve $y = f^{[n]}(x)$ with the diagonal line. Therefore, what we obtain directly from an iterated map is not $N(n)$ but $N_\Sigma(n)$. We need a reverse formula expressing $N(n)$ in terms of $N_\Sigma(d)$. This has already been done, because we know the following two formulas from [2] and [7]:

$$N_\Sigma(n) = \sum_{d|n} N(d), \tag{3.4}$$

and

$$N(n) = \sum_{d|n} \mu(n/d) N_\Sigma(d) = \sum_{d|n} \mu(d) N_\Sigma(n/d), \tag{3.5}$$

where $\mu(d)$ is the Möbius function. $N_\Sigma(n)$ is called the *Möbius transform* of $N(n)$, and $N(n)$ is the *inverse Möbius transform* of $N_\Sigma(n)$. Hence, after calculating the $N_\Sigma(n)$ of an iterated map, we obtain a dividing formula $n|N(n)$ from (3.5). There are examples of iterated maps for which $N_\Sigma(n)$ are calculated (see [1], [5]). We summarize these in the following theorem.

Theorem 3.1: For an iterated map,

$$n|N(n), \quad \text{with } N(n) = \sum_{d|n} \mu(n/d) N_\Sigma(d) \tag{3.6}$$

and, especially,

$$n|(N_\Sigma(n) - N_\Sigma(1)) \quad \text{for } n \text{ a prime}. \tag{3.7}$$

4. APPLICATIONS

We consider the $B(\mu; x)$ map defined by

$$B(\mu; x) = \begin{cases} \mu x & \text{for } 0 \leq x \leq 1/2, \\ \mu(x - 1/2) & \text{for } 1/2 < x \leq 1, \end{cases} \tag{4.1}$$

where μ is the parameter whose value is restricted to the range $0 \leq \mu \leq 2$ so that an x in the interval [0, 1] is mapped to the same interval. We now have the following theorem.

Theorem 4.1: Line segments in $B^{[n]}(\mu; x)$ are all parallel with slope μ^n.

Proof: In the beginning, for a given μ, $B(\mu; x)$ contains two parallel line segments with slope μ. From (4.1), we see that each line segment will multiply its slope by a factor μ after one iteration. Q.E.D.

We consider the following cases.

4.1 The Case in Which $x = 1/2$ Is a Period-2 Point

If $x = 1/2$ is a period-2 point of the $B(\mu)$ map, it requires that $B^2(\mu, 1/2) = 1/2$. This then requires that $\mu > 1$ and $\mu^2 - \mu - 1 = 0$. Solving this, we have $\mu = (1+\sqrt{5})/2 \approx 1.618$, the well-known golden mean. We denote this μ by Σ_2, indicating that for this parameter value $x = 1/2$ is a period-2 point. To obtain $N_\Sigma(n)$, we need to count the number of line segments in $B^{[n]}(\mu)$ that intersect the diagonal line. Detailed discussions of this map can be seen in [3] and [4]. Briefly, we see that starting from $x_0 = 1/2$, we have a 2-cycle, $\{x_0, x_1\}$, where $x_1 = \mu/2$. It follows that there are two types of line segments. We denote by L the type of line segments connecting points $(x_a, 0)$ and $(x_b, \mu/2)$ with $0 \le x_a < x_b \le 1$, and denote by S the type of line segments connecting points $(x_c, 0)$ and $(x_d, 1/2)$ with $0 \le x_c < x_d \le 1$. Since $x_0 \to x_1$ and $x_1 \to x_0$ under an iteration, it follows that the behavior of line segments of these two types under iteration is

$$L \to L + S \quad \text{and} \quad S \to L. \tag{4.2}$$

Using the symbols L and S, we see that the graph of $B(\mu)$ contains two L. (4.2) shows how the number of line segments increases under the action of iteration. Let $L(n)$ and $S(n)$ be the number of line segments of type L and S in $B^{[n]}(\mu)$, respectively. (4.2) shows simply that each L is from previous L and S, so $L(n) = L(n-1) + S(n-1)$, and each S is from previous L, so $S(n) = L(n-1)$. From these, we conclude that $L(n) = L(n-1) + L(n-2)$ and $S(n) = S(n-1) + S(n-2)$. That is, $L(n)$ and $S(n)$ are both the type of sequences of which each element is the sum of its previous two elements. Starting with an L, according to (4.2), the orbit of which is

$$L \to LS \to 2LS \to 3L2S \to 5L3S \to 8L5S \to \cdots,$$

we easily see that $L(n) = F_n$ and $S(n) = F_{n-1}$. In conclusion, starting from an L, under the action of iteration there are F_n (L-type) and $S(n)$ (S-type) parallel line segments generated in $B^{[n]}(\mu)$. We will use this result. $N_\Sigma(n)$ is then determined from the number of intersections of these line segments with the diagonal line.

Consider first the S-type line segments. We note that all S-type line segments in $B^{[n]}(\mu)$ are parallel and can be divided into two parts, one part in the range $0 \le x \le 1/2$ and the other in the range $1/2 < x \le 1$. We easily see that each line segment in the range $0 \le x \le 1/2$ intersects the diagonal line once, and others in the range $1/2 < x \le 1$ cannot intersect the diagonal line. The original line segment in the range $0 \le x \le 1/2$ is an L, so the number of S-type line segments in $B^{[n]}(\mu)$ in this range is $S(n) = F_{n-1}$.

Consider next the L-type line segment. Similarly, line segments of this type in $B^{[n]}(\mu)$ are parallel, and each of those that is in the range $0 \le x \le \mu/2$ intersects the diagonal line once. Others in the range $\mu/2 \le x \le 1$ cannot intersect the diagonal line. We divide the range $0 \le x \le \mu/2$ into $0 \le x \le 1/2$ and $1/2 < x \le \mu/2$. The original line segment in the range $0 \le x \le 1/2$ is an L, so the number of L-type line segments in $B^{[n]}(\mu)$ in this range is $L(n) = F_n$. Next, the original line segment in the range $1/2 < x \le \mu/2$ is an S. After one iteration, $S \to L$; the L in the right-hand side, after $n-1$ iterations, generates all the L-type line segments in $B^{[n]}(\mu)$ in this range, the number of which is therefore $L(n-1) = F_{n-1}$.

In all, the total number of intersections of line segments of types L and S with the diagonal line is thus $F_{n-1} + F_n + F_{n-1} = F_{n+1} + F_{n-1}$. We conclude that

$$N_\Sigma(n) = F_{n+1} + F_{n-1} = F_n + 2F_{n-1}. \tag{4.3}$$

$N_\Sigma(n)$ is, in fact, the Lucas number L_n. From (3.6) and (3.7), we have

$$n|N(n), \quad \text{with } N(n) = \sum_{d|n} \mu(n/d)(F_{n+1} + F_{n-1}) \tag{4.4}$$

and

$$n|(F_{n+1} + F_{n-1} - 1), \quad \text{for } n \text{ a prime.} \tag{4.5}$$

Formula (4.5) is also a known result [6]; however, we see that it is traceable from the point of view of iterated maps.

For n a composite number, (4.4) offers additional relations for Fibonacci numbers or Lucas numbers. This result seems to be a new one. Consider, as a simple example, taking $n = 12$; since $N(12) = 300$, we can easily check that $12|300$.

4.2 The Case in Which $x = 1/2$ Is a Period-m Point

We now consider the general case in which $x = 1/2$ is a period-m point. In general, there are many values of μ in the range $0 \le \mu \le 2$ such that $x = 1/2$ is a period-m point. If there are k such parameter values, we denote these by $\mu_1 < \mu_2 < \mu_3 < \cdots < \mu_k \equiv \Sigma_m$. We will choose the largest one as the parameter value, i.e., $\mu = \Sigma_m$. It follows that Σ_m satisfies the equation:

$$\mu^m - \sum_{i=0}^{m-1} \mu^i = 0. \tag{4.6}$$

For instance, we have $\Sigma_3 \cong 1.839$, $\Sigma_4 \cong 1.9276$. We refer the reader to [4] for details. Briefly, we see that, starting from $x_0 = 1/2$, we have an m-cycle, $\{x_0, x_1, \ldots, x_{m-1}\}$, where $x_i = f^{[i]}(x_0)$ is the i^{th} iterate of x_0. It is convenient also to define $x_m = x_0$. We have, for instance, $x_1 = (1/2)\mu$, and from (4.6) we have $x_1 = (1/2)(1 + 1/\mu + 1/\mu^2 + \cdots + 1/\mu^{m-1})$. We list all the values of x_i in this way:

$$\begin{aligned}
&x_1 = (1/2)(1 + 1/\mu + 1/\mu^2 + \cdots + 1/\mu^{m-1}),\\
&x_2 = (1/2)(1 + 1/\mu + 1/\mu^2 + \cdots + 1/\mu^{m-2}),\\
&\ldots,\\
&x_{m-2} = (1/2)(1 + 1/\mu + 1/\mu^2),\\
&x_{m-1} = (1/2)(1 + 1/\mu),\\
&x_m = x_0 = 1/2.
\end{aligned} \tag{4.7}$$

We see that $x_1 > x_2 > x_3 > \cdots > x_{m-1} > x_m$. It follows that there are m types of line segments. We denote by L_i the type of line segments connecting points $(x_a, 0)$ and (x_b, x_i), where $0 \le x_a < x_b \le 1$ and $1 \le i \le m$. The behavior of line segments of these types after one iteration is

$$\begin{aligned}
&L_1 \to L_1 + L_2,\\
&L_2 \to L_1 + L_3,\\
&\ldots,\\
&L_{m-1} \to L_1 + L_m,
\end{aligned} \tag{4.8}$$

and

$$L_m \to L_1. \tag{4.9}$$

We note that L_m (or L_0) is the only type of line segment that does not break into two line segments after one iteration. The original graph of $B(\mu)$ contains two L_1. Equations (4.8) and (4.9) show how the number of line segments increases under the action of iteration. Let $L_i(n)$ be the number of line segments of type L_i in $B^{[n]}(\mu)$, then (4.8) and (4.9) show that

$$\begin{aligned}
&L_2(n) = L_1(n-1),\\
&L_3(n) = L_2(n-1) = L_1(n-2),\\
&L_4(n) = L_3(n-1) = L_1(n-3),\\
&\dots,\\
&L_m(n) = L_{m-1}(n-1) = L_1(n-m+1),
\end{aligned}$$

or

$$L_j(n) = L_{j-1}(n-1) = L_1(n-j+1), \quad \text{for } 2 \le j \le m, \tag{4.10}$$

and especially,

$$L_1(n) = \sum_{j=1}^{m} L_j(n-1) = \sum_{j=1}^{m} L_1(n-j). \tag{4.11}$$

It follows that all these $L_i(n)$ are sequences of the following type:

$$L_i(n) = \sum_{j=1}^{m} L_i(n-j), \quad 1 \le i \le m, \tag{4.12}$$

i.e., each element of which is the sum of its previous m elements. Starting with an L_1, according to (4.8) and (4.9), the orbit of L_1 is $L_1 \to L_1L_2 \to 2L_1L_2L_3 \to \cdots$. We see that $L_1(n) = F_n^{(m)}$, the Fibonacci numbers of degree m, whose definition is

$$F_n^{(m)} = \sum_{i=1}^{m} F_{n-i}^{(m)},$$

with the first m elements defined by

$$F_1^{(m)} = 1 \quad \text{and} \quad F_i^{(m)} = 2^{i-2} \quad \text{for } 2 \le i \le m. \tag{4.13}$$

Conventionally, we define $F_j^{(m)} = 0$ for $j \le 0$.

We conclude that, starting from an L_1, we have $L_1(n) = F_n^{(m)}$ and, in general, $L_i(n) = F_{n-i+1}^{(m)}$, where $1 \le i \le m$. In order to discuss the number of intersections of these line segments with the diagonal line, we need to know, starting from an L_a-type line segment, how many L_b-type line segments there are in $B^{[n]}(\mu)$. We let $L_b(n, L_a)$ represent the number of L_b-type line segments generated after n iterations of a starting line segment L_a. Using this notation, we have

$$L_b(n, L_1) = F_{n-b+1}^{(m)}, \tag{4.14}$$

and we have the following theorem.

Theorem 4.2: $L_b(n, L_a) = \sum_{s=0}^{m-a} F_{n-b-s}^{(m)}$, where $1 \le a,\ b \le m$.

Proof: We will prove this theorem by induction. We start from a line segment L_a and calculate the number of L_b-type line segments generated after n iterations of L_a. Consider first $L_a = L_m$. To calculate $L_b(n, L_m)$, we note that after one iteration we have $L_m \to L_1$. The L_1 in the right-hand side, after $n-1$ iterations, generates all the L_b-type line segments in $B^{[n]}(\mu)$ in this range, the number of which is $L_b(n-1, L_1)$. Using (4.14), we conclude that

$$L_b(n, L_m) = L_b(n-1, L_1) = F_{n-b}^{(m)} = \sum_{s=0}^{m-a} F_{n-b-s}^{(m)}, \quad \text{where } a = m. \tag{4.15}$$

Consider next $L_a = L_{m-1}$. After one iteration, $L_{m-1} \to L_1 + L_m$. The L_1 and L_m in the right-hand side, after $n-1$ iterations, generate all the L_b-type line segments in $B^{[n]}(\mu)$ in this range, the number of which is $L_b(n-1, L_1) + L_b(n-1, L_m)$. Using (4.14) and (4.15), we conclude that

$$\begin{aligned} L_b(n, L_a) &= L_b(n-1, L_1) + L_b(n-1, L_m) \\ &= F_{n-b}^{(m)} + F_{n-b-1}^{(m)} = \sum_{s=0}^{m-a} F_{n-b-s}^{(m)}, \quad \text{where } a = m-1. \end{aligned} \tag{4.16}$$

We can now prove the general case by induction. Consider $L_a = L_{m-i}$, and suppose that

$$L_b(n, L_a) = \sum_{s=0}^{m-a} F_{n-b-s}^{(m)}, \quad \text{where } a = m-i. \tag{4.17}$$

Now consider $L_a = L_{m-i-1}$. After one iteration, $L_{m-i-1} \to L_1 + L_{m-i}$. The L_1 and L_{m-i} in the right-hand side, after $n-1$ iterations, generates all the L_b-type line segments in $B^{[n]}(\mu)$ in this range, the number of which is $L_b(n-1, L_1) + L_b(n-1, L_{m-i})$. Using (4.14) and (4.16), we conclude that

$$\begin{aligned} L_b(n, L_a) &= L_b(n-1, L_1) + L_b(n-1, L_{m-i}) \\ &= F_{n-b}^{(m)} + \sum_{s=0}^{i} F_{n-b-s-1}^{(m)} = \sum_{s=0}^{m-a} F_{n-b-s}^{(m)}, \quad \text{where } a = m-i-1. \text{ Q.E.D.} \end{aligned} \tag{4.18}$$

With Theorem 4.2 established, we can now discuss the intersections of line segments of these m-types in $B^{[n]}(\mu)$ with the diagonal line.

We consider line segments of type L_b in $B^{[n]}(\mu)$. All these line segments are parallel and can be divided into two parts; one part is in the range $0 \le x \le x_b$, the other is in the range $x_b < x \le 1$. We easily see that each line segment of those in the range $0 \le x \le x_b$ intersects the diagonal line once and others in the range $x_b < x \le 1$ cannot intersect the diagonal line. We divide the range $0 \le x \le x_b$ into $0 \le x \le 1/2$ and $1/2 < x \le x_b$.

Consider first the range $0 \le x \le 1/2$. The original line segment in this range is an L_1. The number of line segments of type L_b in $B^{[n]}(\mu; x)$ in this range is $L_b(n, L_1) = F_{n-b+1}^{(m)}$.

Consider next the range $1/2 < x \le x_b$. The original line segment in the range $1/2 < x \le x_b$ is L_{b+1}. The number of L_b-type line segments in $B^{[n]}(\mu)$ in this range is $L_b(n, L_{b+1}) = \sum_{s=0}^{m-b-1} F_{n-b-s}^{(m)}$.

Therefore, the total number of intersections of these m-type line segments with the diagonal line is

$$\begin{aligned} \sum_{b=1}^{m} \left[F_{n-b+1}^{(m)} + \sum_{s=0}^{m-b-1} F_{n-b-s}^{(m)} \right] &= \sum_{b=1}^{m} F_{n-b+1}^{(m)} + \sum_{b=1}^{m} \sum_{s=0}^{m-b-1} F_{n-b-s}^{(m)} \\ &= \sum_{b=0}^{m-1} F_{n-b}^{(m)} + \sum_{k=1}^{m-1} k F_{n-k}^{(m)} = \sum_{k=0}^{m-1} (k+1) F_{n-k}^{(m)} \end{aligned} \tag{4.19}$$

$$\equiv L_n^{(m)}. \tag{4.20}$$

$L_n^{(m)}$, defined above, may be called the Lucas numbers of degree m, whose definition is

$$L_n^{(m)} = \sum_{j=1}^{m} L_{n-j}^{(m)}, \text{ with the first } m \text{ elements defined by}$$

$$L_i^{(m)} = 2^i - 1, \quad 1 \le i \le m. \tag{4.21}$$

We then have the final results:

$$N_\Sigma(n) = L_n(m), \tag{4.22}$$

$$n|N(n), \text{ with } N(n) = \sum_{d|n} \mu(n/d) L_n^{(m)}, \tag{4.23}$$

and

$$n|(L_n^{(m)} - 1), \text{ for } n \text{ a prime.} \tag{4.24}$$

5. CONCLUSIONS

Many $N(n)$ such that $n|N(n)$ can be obtained in this way for other iterated maps. In principle, infinite $N(n)$ can be obtained, since each iterated map contributes an $N(n)$. It seems that the existence of dividing formulas is not so rare and not so mysterious.

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BRAHMAGUPTA'S THEOREMS AND RECURRENCE RELATIONS

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1. INTRODUCTION

In this paper, we relate the positive integer solutions of the Diophantine equation of the type $x^2 - Dy^2 = \lambda$ with the generalized sequence of numbers $W_n(a,b;p,q)$ defined by Horadam [3]. We do this by utilizing the *principle of composition*, or *Bhavana*, first enunciated in the sixth century by the Indian astronomer and mathematician Brahmagupta while dealing with the integer solutions of the indeterminate equation $x^2 - Dy^2 = \lambda$, D being a positive integer that is not a perfect square and λ a positive or negative integer [1], [2]. Further, we show that all the integer solutions of the equation $x^2 - Dy^2 = \pm 1$ are related to the Chebyshev polynomials of the first and second kinds if the positive sign is taken in the equation and to the Pell and Pell-Lucas polynomials if the negative sign is taken in the equation. It may be of interest to note that Bhaskara II, another Indian mathematician of the twelfth century dealt extensively with the positive integer solutions of the equation $x^2 - Dy^2 = \lambda$, and gave an elegant method for finding a positive integer solution for an equation of the type $x^2 - Dy^2 = 1$. His technique is known as the *chakravala*, or cyclic method. Of course, this solution, in conjunction with Brahmagupta's method of composition, may be used to generate an infinite number of solutions to the equation $x^2 - Dy^2 = 1$ (see [1] and [2]). Among the many examples that Bhaskara II considered and solved are the equations $x^2 - 61y^2 = 1$ and $x^2 - 67y^2 = 1$. It is interesting to note that the equation $x^2 - 61y^2 = 1$ was proposed by Fermat to Frenicle in 1657 and that it was solved by Euler in 1732.

2. BRAHMAGUPTA'S THEOREMS

We first enunciate the following two theorems originally proposed by Brahmagupta.

Theorem 1 (Bhavana, or the Principle of Composition): If (x_1, y_1) is a solution of the equation $x^2 - Dy^2 = \lambda_1$ and (x_2, y_2) is a solution of the equation $x^2 - Dy^2 = \lambda_2$, then $(x_1x_2 \pm Dy_1y_2, x_1y_2 \pm x_2y_1)$ is a solution of the equation $x^2 - Dy^2 = \lambda_1\lambda_2$.

Theorem 2: If (x_1, y_1) is a solution of the equation $x^2 - Dy^2 = \pm d\lambda^2$ such that $\lambda | x_1$ and $\lambda | y_1$, then $(x_1/\lambda,\ y_1/\lambda)$ is a solution of $x^2 - Dy^2 = \pm d$.

As a consequence of Theorem 1, we can see that if (α, β) is a solution of the equation $x^2 - Dy^2 = -1$ then $(\alpha^2 - D\beta^2, 2\alpha\beta)$ is a solution of $x^2 - Dy^2 = 1$. It is well known that the equation $x^2 - Dy^2 = 1$ is always solvable in integers, while $x^2 - Dy^2 = -1$ may have no integer solutions [4], [5]. Bhaskara has shown that $x^2 - Dy^2 = -1$ has no integer solutions unless D is expressible as the sum of two squares [2].

Let us consider the different positive integer solutions of the equation

$$x^2 - Dy^2 = \lambda. \tag{1}$$

Let (a, b) be the "smallest" solution of (1), which is also referred to as the "fundamental" solution. Then, by repeated application of Theorem 1 (using the positive sign only), we can readily see that (x_n, y_n) is a solution of the equation

$$x_n^2 - Dy_n^2 = \lambda^n, \tag{2}$$

where (x_n, y_n) satisfy the recurrence relations

$$\begin{aligned} x_n &= ax_{n-1} + Dby_{n-1}, \\ y_n &= bx_{n-1} + ay_{n-1}. \end{aligned} \tag{3}$$

From (3), we see that (x_n, y_n) satisfy the difference equations

$$\begin{aligned} x_n &= 2ax_{n-1} - \lambda x_{n-2}, \quad & x_0 = 1,\ x_1 = a, \\ y_n &= 2ay_{n-1} - \lambda y_{n-2}, \quad & y_0 = 0,\ y_1 = b. \end{aligned} \tag{4}$$

Hence, (x_n, y_n) may be expressed in terms of the generalized sequence $W_n(a, b; p, q)$ defined by Horadam [3] in the form

$$x_n = W_n(1, a; 2a, \lambda), \qquad y_n = W_n(0, b; 2a, \lambda), \tag{5}$$

where

$$W_n = pW_{n-1} - qW_{n-2} \ (n \ge 2), \qquad W_0 = a,\ W_1 = b. \tag{6}$$

The difference equations given by (4) have been established recently by Suryanarayan [6], who has very appropriately called x_n and y_n "Brahmagupta polynomials." In the same context, it is appropriate to call equation (1) "Bhaskara's equation" (rather than a Pellian equation), since Pell has made no contribution to this topic, while Bhaskara (in the twelfth century) was the first to present a method for finding a positive integer solution of (1) when $\lambda = 1$.

Using the properties of the sequence $W_n(a, b; p, q)$, it is easy to show that

$$x_n = \frac{1}{2}v_n(2a, \lambda), \qquad y_n = bu_n(2a, \lambda), \tag{7}$$

where $u_n(x, y)$ and $v_n(x, y)$ are generalized polynomials in two variables defined by

$$u_n(x, y) = xu_{n-1}(x, y) - yu_{n-2}(x, y), \quad u_0(x, y) = 0,\ u_1(x, y) = 1, \tag{8}$$

and

$$v_n(x, y) = xv_{n-1}(x, y) - yv_{n-2}(x, y), \quad v_0(x, y) = 2,\ v_1(x, y) = x. \tag{9}$$

A number of properties of the polynomials $u_n(x, y)$ and $v_n(x, y)$ have been derived recently [7]. In particular, we have

$$\begin{aligned} u_n(x, y) &= \sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-r-1}{r} x^{n-2r-1} y^r, \\ v_n(x, y) &= \sum_{r=0}^{[n/2]} (-1)^r \frac{n}{n-r} \binom{n-r}{r} x^{n-2r} y^r. \end{aligned} \tag{10}$$

It has been shown by Suryanarayan [6] that the Brahmagupta polynomials x_n and y_n have the property that

$$Y(s) = sbe^{2X(s)}, \tag{11}$$

where

$$X(s)=\sum_1^\infty \frac{x_n}{n}s^n, \qquad Y(s)=\sum_1^\infty y_n s^n. \tag{12}$$

Since x_n and y_n are related to the polynomials u_n and v_n by (7), it follows that

$$U_n(s)=e^{V_n(s)}, \tag{13}$$

where

$$U_n(s)=\sum_1^\infty u_n s^{n-1}, \qquad V_n(s)=\sum_1^\infty v_n s^n. \tag{14}$$

Relations similar to (13) hold good between Fibonacci and Lucas polynomials $F_n(x)$ and $L_n(x)$, Pell and Pell-Lucas polynomials $P_n(x)$ and $Q_n(x)$, Morgan-Voyce polynomials $B_n(x)$ and $C_n(x)$, Chebyshev polynomials of the first and second kind $T_n(x)$ and $S_n(x)$, etc., since

$$\begin{aligned} F_n(x)&=u_n(x,-1), & L_n(x)&=v_n(x,-1);\\ P_n(x)&=u_n(2x,-1), & Q_n(x)&=v_n(2x,-1);\\ B_n(x)&=u_{n+1}(x+2,1), & C_n(x)&=v_n(x+2,1);\\ S_n(x)&=u_n(2x,1), & 2T_n(x)&=v_n(2x,1). \end{aligned} \tag{15}$$

3. BHASKARA'S EQUATION WITH $\lambda=1$ ($x^2-Dy^2=1$)

Letting $\lambda=1$ in (7), we see that the positive integer solutions of the equation

$$x^2-Dy^2=1 \tag{16}$$

are given by

$$x_n=\frac{1}{2}v_n(2a,1), \qquad y_n=bu_n(2a,1), \tag{17}$$

$(n=1,2,3,\ldots)$, where (a,b) is the fundamental solution of equation (16). Since $u_n(2x,1)=S_n(x)$ and $v_n(2x,1)=2T_n(x)$, the Chebyshev polynomials of the first and second kind, we see that the positive integer solutions of equation (16) are given by

$$x_n=T_n(a), \qquad y_n=bS_n(a). \tag{18}$$

4. BHASKARA'S EQUATION WITH $\lambda=-1$ ($x^2-Dy^2=-1$)

It is well known that it may not always be possible to obtain positive integer solutions to the equation

$$x^2-Dy^2=-1. \tag{19}$$

In fact, it is not solvable unless the length of the period in the continued fraction expansion of $\sqrt{D}$ is odd [1]. Let us assume so, and let the fundamental solution of (19) be (a,b). Then, from (2) and (4), we have that (x_n,y_n) is a solution of the equation

$$x^2-Dy^2=(-1)^n, \tag{20}$$

where

$$\begin{aligned} x_n &= 2ax_{n-1} + x_{n-2}, \quad x_0 = 1,\ x_1 = a, \\ y_n &= 2ay_{n-1} + y_{n-2}, \quad y_0 = 0,\ y_1 = b. \end{aligned} \tag{21}$$

Hence,

$$x_n = \frac{1}{2}v_n(2a, -1), \qquad y_n = bu_n(2a, -1). \tag{22}$$

Thus, (22) gives the various solutions for the equations $x^2 - Dy^2 = -1$ and $x^2 - Dy^2 = 1$, respectively, depending on whether n is odd or even. Since $u_n(2x, -1) = P_n(x)$ and $v_n(2x, -1) = Q_n(x)$, where $P_n(x)$ and $Q_n(x)$ are the Pell and Pell-Lucas polynomials, respectively, we may rewrite (22) as

$$x_n = \frac{1}{2}Q_n(a), \qquad y_n = bP_n(a). \tag{23}$$

Now we see that

$$x_n = \frac{1}{2}Q_{2n-1}(a), \qquad y_n = bP_{2n-1}(a) \tag{24}$$

are the various integer solutions of $x^2 - Dy^2 = -1$, while

$$x_n = \frac{1}{2}Q_{2n}(a), \qquad y_n = bP_{2n}(a) \tag{25}$$

are those of $x^2 - Dy^2 = 1$, where (a, b) is the fundamental solution of $x^2 - Dy^2 = -1$.

Hence, we see that all the integer solutions of $x^2 - Dy^2 = 1$ are expressible in terms of the Chebyshev polynomials of the first and second kinds, while those of $x^2 - Dy^2 = -1$ are expressible in terms of the Pell and Pell-Lucas polynomials.

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SOLVING GENERALIZED FIBONACCI RECURRENCES

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1. GENERALIZATIONS

We consider finding a Binet formula for the continuous function $f:\Re \to \Im$ which has the property

$$f(x) = \sum_{1 \le l \le k} f(x-l), \tag{1}$$

where

- k is a given integer and $k > 1$,
- either $f(0), \ldots, f(k-1)$ are given initial values,
- or $f:[0, k) \to \Im$ is a given continuous *initial function* where

$$\lim_{x \to k} f(x) = \sum_{0 \le x \le k-1} f(x).$$

This generalizes the Fibonacci sequence in new ways. Instead of viewing the sequence as an automorphism on the integers, its domain becomes the reals. The Binet formula also allows the initial values to be arbitrary values, possibly complex ones. Instead of having k initial values for the function of order k, we also allow an *initial function* which is defined on the interval $[0, k)$.

When only k initial values are given, there can be many possible functions f. However, the following can be shown by induction.

Lemma 1.1: Given an initial function, f is uniquely defined on $\Re$, and if

$$\lim_{x \to k} f(x) = \sum_{0 \le x \le k-1} f(x),$$

then f is continuous.

2. RELATED WORK

In 1961, Horadam wrote that generalizations of Fibonacci's sequence either involved changes to the Fibonacci recurrence or allowed its initial values to be changed or, possibly, a combination of these [10].

Since then, the main contributions to a general theory seem to involve generalizations of the Fibonacci recurrence [17], [20]:

$$f(x) = \sum_{1 \le l \le k} f(x-l). \tag{2}$$

When $k = 3$ and $f(0), f(1)$, and $f(2)$ are arbitrary constants, this is the recurrence of the *generalized Tribonacci sequence*. The *Tetranacci* or *Quadranacci sequence* is similarly defined when $k = 4$.

Direct evaluation of equation (2) can have exponential complexity. Burstall and Darlington [2] gave a linear algorithm for computing Fibonacci numbers as an example of their program transformation methods:

$$\begin{aligned}
&\mathrm{f}(0) \Leftarrow 1\\
&\mathrm{f}(1) \Leftarrow 1\\
&\mathrm{f}(\mathrm{x}+2) \Leftarrow \mathrm{u}+\mathrm{v}, \text{where } \langle \mathrm{u}, \mathrm{v}\rangle = \mathrm{g}(\mathrm{x})\\
&\mathrm{g}(0) \Leftarrow \langle 1,\ 1\rangle\\
&\mathrm{g}(\mathrm{x}+1) \Leftarrow \langle \mathrm{u}+\mathrm{v}, \mathrm{u}\rangle, \text{where } \langle \mathrm{u}, \mathrm{v}\rangle = \mathrm{g}(\mathrm{x})
\end{aligned}$$

This approach can, of course, be generalized by allowing different initial values and letting $k>2$. For example, if `f(0)` is defined to be 0 instead, we have that `f(n)` is the n^{th} Fibonacci number. Given an efficient implementation of exponentiation, using the Binet formula

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

for the same task can have lower complexity.

A similar formula, where $k=2$ and the initial values are arbitrary, was given by Horadam [10]. A Binet formula for the recursive sequence of order k was given by Miles [13] for the special case $f(x)=0$, where $0 \le x \le k-2$ and $f(k-1)=1$. Spickerman and Joyner [18] gave another solution for the special case $f(0)=1$, and $f(x)=2^{x-1}$ for $1 \le x \le k-1$. Our approach subsumes these results as special cases.

We have also derived a solution to equation (1), where an arbitrary initial function is specified. This does not seem to have been considered before.

In the next section, we discuss properties of the characteristic equation, the coefficients of generalized Binet formulas, and solutions that use the initial values, and the initial function. When the initial values are given, we present two methods of solution: one uses Binet formulas and the other uses an exponential generating function and the Laplace Transform. We use the latter method to find solutions when $2 \le k \le 4$. They are equivalent to those found with Binet formulas, but they are more complicated and do not involve complex roots.

3. THE CHARACTERISTIC EQUATION

We consider properties of the characteristic equation associated with Fibonacci recurrences of equation (1). These properties include its discriminant, location of roots, reducibility, and solvability in radicals. Several of the results here are used in later sections.

Equation (1) is a homogeneous linear difference equation. Its characteristic equation* is given below:

$$y^k - \sum_{0 \le l < k} y^l = 0. \tag{3}$$

The form of the general solution of such difference equations depends on whether the roots of its characteristic equation are simple [12]. We define the *characteristic function* of order k to be $c_k(y) = y^k - \sum_{0 \le l < k} y^l$.

* See Liu [12], §3-2, for example.

Lemma 3.1: For every solution r of c_k,

$$c_j(r)+\frac{c_{k-j}(r)}{r^{k-j}}=1,$$

where $0\le j\le k$.

Proof: This follows from the definition of c_k. □

Corollary 3.2: For every solution r of c_k,

$$c_j(r)=\sum_{1\le i\le k-j} r^{-i}.$$

Theorem 3.3: The discriminant of the characteristic equation is

$$(-1)^{\frac{k(k+1)}{2}}\left[\frac{(k+1)^{k+1}-2(2k)^k}{(k-1)^2}\right]$$

when $k>1$.

Proof: Let the resultant of $c_k(y)$ and $\frac{dc_k}{dy}$ be $R(c,c')$. The discriminant of the characteristic equation is $(-1)^{\frac{k(k-1)}{2}}R(c,c')$ [11]. The resultant when $k=3$ is

$$R(c,c')=\begin{vmatrix}1&-1&-1&-1&0\\0&1&-1&-1&-1\\3&-2&-1&0&0\\0&3&-2&-1&0\\0&0&3&-2&-1\end{vmatrix}.$$

This can be simplified by partial Gaussian elimination. First, we interchange elements by moving element $a_{i,j}$ to element $a_{2k-i,\,2k-j}$, where $1\le i,\ j\le 2k-1$. This does not change the sign of the determinant. In the example above, we obtain

$$R(c,c')=\begin{vmatrix}-1&-2&3&0&0\\0&-1&-2&3&0\\0&0&-1&-2&3\\-1&-1&-1&1&0\\0&-1&-1&-1&1\end{vmatrix}.$$

Subtracting row 1 from row 4 and row 2 from row 5 yields

$$R(c,c')=\begin{vmatrix}-1&-2&3&0&0\\0&-1&-2&3&0\\0&0&-1&-2&3\\0&1&-4&1&0\\0&0&-1&-4&1\end{vmatrix}.$$

If we then add row 2 to row 4 and row 3 to row 5, we obtain

$$R(c,c')=\begin{vmatrix}-1&-2&3&0&0\\0&-1&-2&3&0\\0&0&-1&-2&3\\0&0&-6&4&0\\0&0&0&-6&4\end{vmatrix}.$$

In general, the last two row operations above can be defined as the replacement of element $a_{k+i,j}$ by $a_{k+i,j}-a_{i,j}+a_{i+1,j}$, where $1 \le i \le k-1$ and $1 \le j \le 2k-1$. Rows $k+1$ to $2k-1$ in these determinants have the form

$$0 \;\cdots\; 0 \;\; -2k \;\; k+1 \;\; 0 \;\cdots\; 0,$$

where row $l: k+1 \le l \le 2k-1$ has 0 in columns 1 to $l-2$. By induction on k, we can show that, for all $k \ge 1$,

$$R(c,c') = (-1)^k\left[-\frac{(2k)^k}{2} + \sum_{0 \le i \le k-2}(i+1)(2k)^i(k+1)^{k-1-i}\right].$$

The following identity can be used to simplify the summation in this expression:

$$\sum_{0 \le l \le n-1}(a+ld)x^l = \frac{a-(a+(n-1)d)x^n}{1-x} + \frac{dx(1-x^{n-1})}{(1-x)^2}.$$

With $x = \frac{2k}{k+1}$ and $n = k-1$, we obtain

$$R(c,c') = (-1)^k\left[\frac{(k+1)^{k+1}-2(2k)^k}{(k-1)^2}\right]$$

when $k > 1$, and the result follows. □

Miles [13] and Miller [14] have shown that the characteristic equation has simple roots. Corollary 3.4 below shows this by different means. Its proof will be used in the proof of Theorem 3.9 below.

Corollary 3.4: The characteristic equation has simple roots.

Proof: It suffices to show that $R(c,c') \ne 0$. The resultant equals -5 when $k=2$, and 44 when $k=3$. It could only be zero if $(k+1)^{k+1} = 2(2k)^k$, which occurs if $(k+1)\log_2(k+1)-1-k-k\log_2 k = 0$. Now $\log_2(k+1)-\log_2 k < 0.6$ when $k > 2$, so that

$$(k+1)\log_2(k+1)-1-k-k\log_2 k < \log_2 k - 0.4(k+1).$$

When $1 \le k \le 4$, $\log_2 k - 0.4(k+1) \le 0$. The derivative of $\log_2 k - 0.4(k+1)$ is negative when $k \ge 4$. Thus, $(k+1)\log_2(k+1)-1-k-k\log_2 k < 0$ and $R(c,c') \ne 0$ when $k > 1$, as required. □

We call the k roots of equation (3), $r_1, \ldots, r_k$.

Corollary 3.5 The general solution to equation (1) when x is an integer has the form

$$f(x) = \sum_{1 \le i \le k} C_i r_i^x, \tag{4}$$

where the C_i are constant coefficients.

To find a solution to equation (1), we need to find a version of this summation where x can be a real number.

The following lemma identifies the locations and limits of the roots of the characteristic equation more precisely than previous results by Miles [13] and Miller [14].

Lemma 3.6: The characteristic equation $y^k - \sum_{0\le l<k} y^l = 0$ has one positive real root in the interval (1, 2). This root approaches 2 as k approaches infinity, and it is greater than $2(1-2^{-k})$.

It has one negative real root in the interval (−1, 0) when k is even. This root and each complex root r has modulus $3^{-k} < |r| < 1$.

Proof: By Descartes' Rule of Signs [6], the characteristic equation has one variation, and so has at most one positive real root. There must be just one positive root because the characteristic function is $-k+1$ when $y=1$, and 1 when $y=2$. Another proof of this follows immediately from Pólya and Szegö ([15], Vol. I, Pt. III, Prob. 16). The characteristic function c_{k+1} is

$$y\left(y^k - \frac{y^k-1}{y-1}\right) - 1.$$

At $y=r$, this equals −1. By Lemma 3.6, there is only one positive root, so that positive root of c_{k+1} is greater than r. Hence, r is always greater than $\frac{1}{2}$ when $k \ge 2$, so that

$$r^k - \frac{\frac{1}{2}^k - 1}{-\frac{1}{2}}$$

is always positive. The root r must be less than 2 because the characteristic equation always equals 1 when $x = 2$. Therefore, r lies between $2(1-2^{-k})$ and 2. As k approaches infinity, r approaches 2.

If we replace y by $-x$, the characteristic function is $-x^k - \frac{x^k+1}{x+1}$ when k is odd. This is negative when $x \ge 0$, and so the characteristic equation does not have any negative real roots.

When k is even, replacing y by $-x$ in the characteristic function gives $x^k + \frac{x^k-1}{x+1}$. This is positive when $x \ge 1$, so there is at least one negative root of the characteristic equation between −1 and 0. The derivative of the preceding function is

$$\frac{kx^{k+1} + (3k-1)x^k + 2kx^{k-1} + 1}{(x+1)^2},$$

which is positive when $x \ge 0$. Therefore, $x^k + \frac{x^k-1}{x+1}$ is strictly increasing when $x \ge 0$. It follows that there is only one negative root of the characteristic equation.

We now consider the complex roots of the characteristic equation. From Miles [13] and Miller [14], each of them has modulus less than one. For every root r, $|r| = |r-2|^{-1/k}$. In the region where $|z|<1$, we have $1<|z-2|<3$, and so each of the complex roots and the negative real root satisfies $3^{-1/k} < |r| < 1$. □

Corollary 3.7: $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = r_1$ and $\lim_{x\to\infty} \frac{f(x)}{r_1^x} = C_1$.

Proof: This follows immediately from Corollary 3.5, and that $|r_i|<1$ for $i: 2 \le i \le k$. □

Corollary 3.8: c_k is irreducible over the rationals where $k>1$.

Proof: By Gauss's lemma, the irreducibility of c_k over the rationals is equivalent to its irreducibility over the integers [11]. If c_k were reducible, the roots of one of its factors would all have moduli that are strictly less than 1. The product of these roots cannot be an integer. This leads to a contradiction because the modulus of the product of these roots must equal the modulus of the constant term of this factor.* □

We now give a series that can be used to evaluate the positive real root of the characteristic equation.

Theorem 3.9: Let $2(1-\varepsilon_k)$ be the positive root of the characteristic equation. Then

$$\varepsilon_k = \sum_{i\geq 1} \binom{(k+1)i-2}{i-1} \frac{1}{i2^{(k+1)i}}.$$

Proof: We have

$$\varepsilon_k = \sum_{i\geq 0} \binom{(k+1)i+(k-1)}{i} \frac{1}{(i+1)2^{(k+1)(i+1)}}.$$

Define

$$\varepsilon_k(z) = \sum_{i\geq 0} \binom{(k+1)i+(k-1)}{i} \frac{z^{i+1}}{i+1}.$$

This is equivalent to the previous expression when $z = \left(\frac{1}{2}\right)^{k+1}$. We have

$$\frac{d\varepsilon_k(z)}{dz} = \sum_{i\geq 0} \binom{(k+1)i+(k-1)}{i} z^i.$$

Identity 29 on page 713 of Prudnikov, Brychkov, and Marichev [16] states**

$$\sum_{k=0}^{\infty} \frac{\Gamma(k\nu+\mu)}{k!\,\Gamma(k\nu-k+\mu)} x^k = \frac{y^\mu}{(1-\nu)y+\nu}, \tag{5}$$

where

$$x = \frac{y-1}{y^\nu} \quad \text{and} \quad |x| = \left|\frac{(\nu-1)^{\nu-1}}{\nu^\nu}\right|.$$

If we rename k by i, and then let $\nu = k+1$, $\mu = k$, $x = z$, and $y = x$, we have

$$\frac{d\varepsilon_k(z)}{dz} = \frac{x^k}{k+1-kx},$$

where

$$z = \frac{x-1}{x^{k+1}}, \text{ provided that } |z| < \left|\frac{k^k}{(k+1)^{k+1}}\right|. \tag{6}$$

When $z = \left(\frac{1}{2}\right)^{k+1}$, this simplifies to $2(2k)^k - (k+1)^{k+1} > 0$. It is remarkable that this is the same condition as in the proof of Corollary 3.4, so that it always holds when $k > 1$.

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* David Boyd told me of this proof. It is known from the theory of Pisot numbers [1].

** Prudnikov, Brychkov, and Marichev [16] seem to refer to Gould [9] for this result. Gould gives a more restricted form where combinations rather than Gamma functions are used (Identity 1.120 on p. 15). Gould, in turn, apparently refers to the 1925 German edition of Pólya and Szegö [15]. The identity appears as a solution to problem **216** of Part III of Volume I of the 1972 English translation of that work [15]. The convergence condition (6) is discussed by Gould [8].

Now

$$\frac{dz}{dx} = \frac{k+1-kx}{x^{k+2}}$$

so that

$$\frac{d\varepsilon_k(z)}{dz}\frac{dz}{dx} = \frac{1}{x^2}.$$

We have

$$\int_0^{\left(\frac{1}{2}\right)^{k+1}} \frac{d\varepsilon_k(z)}{dz} dz = \int_1^{x_0} \frac{1}{x^2} dx,$$

where

$$\frac{x_0 - 1}{x_0^{k+1}} = \left(\tfrac{1}{2}\right)^{k+1}.$$

Since $\frac{dz}{dx} = \frac{k+1-kx}{x^{k+2}}$, the value of z as a function of x is increasing when $1 \le x \le \frac{k+1}{k}$. In this interval, z increases from 0 to $\frac{k^k}{(k+1)^{k+1}}$. We have shown that condition (6) holds when $z = \left(\frac{1}{2}\right)^{k+1}$. This implies that $< x_0 < \frac{k+1}{k}$, and for all $x : 1 \le x \le x_0$, condition (6) is satisfied.

Therefore, $\varepsilon_k = 1 - \frac{1}{x_0}$ and the positive root of the characteristic equation is $\frac{2}{x_0}$. To check this, we can write the characteristic equation as

$$\frac{y^k(y-2)+1}{y-1} = 0,$$

which holds when $y^k(y-2)+1=0$ and $y \ne 1$. On substitution of $y = \frac{2}{x_0}$, we obtain

$$\left(\frac{2}{x_0}\right)^{k+1} - 2\left(\frac{2}{x_0}\right)^k + 1 = 0.$$

This is equivalent to $\frac{x_0-1}{x_0^{k+1}} = \left(\frac{1}{2}\right)^{k+1}$, as required. □

Remark 3.10: Condition (6) and the one for equation (5) [16] should be strengthened. We have used a value $x_0 : 1 < x_0 < \frac{k+1}{k}$ such that $\frac{x_0-1}{x_0^{k+1}} = \left(\frac{1}{2}\right)^{k+1}$, but we could have chosen $x_0 = 2$ instead. This value also satisfies the condition, but in general,

$$\frac{x_0^k}{k+1-kx_0} \ne \frac{2^k}{1-k}.$$

3.1 Solvability in Radicals

We now consider the roots of specific characteristic equations. When $k = 2$, we have $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$, or $-r_1^{-1}$. Approximate values of r_1 and r_2 are 1.618033988749895 and −0.618033988749895, respectively.

When $k = 3$, we let the real root of equation (3) be

$$r_1 = \frac{1 + (19 - 3\sqrt{33})^{1/3} + (19 + 3\sqrt{33})^{1/3}}{3}.$$

This was found by using "Cardan's Method" of 1545, due to Ferro and Tartaglia, for the solution of the general cubic equation (see [6], [7]). This constant r_1 will be used to find a solution to equation (1) above. The complex solutions are $-\omega p-\omega^2 q+\frac{1}{3}$ and $-\omega^2 p-\omega q+\frac{1}{3}$, where

$$\omega=\frac{-1+\sqrt{3}i}{2},\ \omega^2=\frac{-1-\sqrt{3}i}{2},\ p=\frac{-(19-3\sqrt{33})^{1/3}}{3},\ q=\frac{-(19+3\sqrt{33})^{1/3}}{3}.$$

An approximate value of r_1 is 1.83928675521416. The approximate values of the complex roots are $-0.419643377607081 \pm 0.606290729207199i$.

Similarly, when $k=4$, the two real solutions of (3) are given by

$$\left(p_1+\frac{1}{4}\right)\pm\sqrt{\left(p_1+\frac{1}{4}\right)^2-\frac{2\lambda_1}{p_1}\left(p_1+\frac{1}{4}\right)+\frac{7}{24p_1}+\frac{1}{6}},$$

where

$$\lambda_1=\frac{(3\sqrt{1689}-65)^{1/3}-(3\sqrt{1689}+65)^{1/3}}{12\cdot 2^{1/3}} \quad \text{and} \quad p_1=\sqrt{\lambda_1+\frac{11}{48}}.$$

The complex roots are given by

$$\left(\frac{1}{4}-p_1\right)\pm\sqrt{\left(\frac{1}{4}-p_1\right)^2+\frac{2\lambda_1}{p_1}\left(\frac{1}{4}-p_1\right)-\frac{7}{24p_1}+\frac{1}{6}}.$$

These were found by Ferrari's Solution to the general quartic (see [6], [7]). Approximations of the real solutions that we call r_1 and r_2 are 1.92756197548293 and -0.77480411321543, respectively. The approximate values of the complex solutions are $-0.0763789311337454 \pm 0.814703647170387i$.

Lemma 3.11: There are no solutions in radicals to the characteristic equation when $5 \le k \le 11$.

Proof: The Galois group of the characteristic equation is S_k when $1 \le k \le 11$. These groups were found by using Magma* [3], and they are not soluble [11]. □

We conjecture that the Galois group of the characteristic equation is also S_k when $k>11$. In general, computing the Galois group of a polynomial currently seems to be intractable when $k \ge 12$ (see [19]).

4. THE COEFFICIENTS

We consider the problem of finding the coefficients C_i in the equation

$$f(x)=\sum_{1\le i\le k} C_i r_i^x, \tag{7}$$

where $f(0), \ldots, f(k-1)$ are given. This is the problem of finding a general solution of a homogeneous linear difference equation whose characteristic equation has simple roots.

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* These computations were done by John Cannon. Robert Low also told me independently that Maple [4] gave the same answers where $5 \le k \le 8$. Values of k outside this range were not used.

We use the elementary symmetric polynomials ([7], [11]) σ_i^{k-1} defined over $\{y_1, \ldots, y_{k-1}\}$, where $1 \le i \le k-1$, and define $\sigma_0^{k-1} = 1$. The coefficient C_i is then given by the function

$$h(y_1, \ldots, y_k) = \frac{\sum\limits_{0 \le j < k} (-1)^j f(k-1-j)\sigma_j^{k-1}}{\prod\limits_{1 \le j < k} (y_k - y_j)}, \tag{8}$$

where $y_k = r_i$ and $y_1, \ldots, y_{k-1}$ are assigned, respectively, to the other $k-1$ roots of equation (3) in any order. Equation (8) can be verified by induction on k. The formula was derived by Gaussian elimination and back substitution on systems such as

$$\begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$$

when $k = 3$. More generally, the leftmost $k \times k$ matrix has elements $a_{i,j} = r_j^{i-1}$. The determinant of this matrix is the Vandermonde determinant. Lang [11] in Exercise 33(c) of Chapter V discusses how this determinant can be used to find the coefficients, but no explicit formula is given.

Example 4.1: When $k = 4$, we have

$$f(x) = C_1 r_1^x + C_2 r_2^x + C_3 r_3^x + C_4 r_4^x,$$

where the function $h(y_1, y_2, y_3, y_4)$ is

$$\frac{f(3) - (y_1 + y_2 + y_3) f(2) + (y_1 y_2 + y_2 y_3 + y_3 y_1) f(1) - y_1 y_2 y_3 f(0)}{(y_4 - y_1)(y_4 - y_2)(y_4 - y_3)}.$$

Thus, $f(x)$ is

$$h(r_2, r_3, r_4, r_1) r_1^x + h(r_1, r_3, r_4, r_2) r_2^x + h(r_1, r_2, r_4, r_3) r_3^x + h(r_1, r_2, r_3, r_4) r_4^x.$$

Some special cases have appeared in the literature. Horadam [10] presented a Binet formula called equation (δ), which is equivalent to the one below, where $k = 2$, $f(0) = q$, and $f(1) = p$:

$$f(x) = \frac{1}{2\sqrt{5}}(2(p - qr_2)r_1^x - 2(p - qr_1)r_2^x).$$

From equation (8), we find

$$h(y_1, y_2) = \frac{f(1) - y_1 f(0)}{y_2 - y_1},$$

so that

$$C_1 = \frac{p - r_2 q}{\sqrt{5}} \quad \text{and} \quad C_2 = \frac{p - r_1 q}{-\sqrt{5}}$$

in agreement with Horadam's result.

Miles [13] discussed the special case in which $f(x) = 0$, where $0 \le x \le k-2$ and $f(k-1) = 1$. Equation (7) becomes

$$h(y_1,\dots,y_k)=\frac{1}{\prod\limits_{1\le j<k}(y_k-y_j)}$$

in agreement with his equation (2″).

Spickerman and Joyner [18] considered the special case in which $f(0)=1$ and $f(x)=2^{x-1}$, for $1\le x\le k-1$. Their solution is

$$C_i=\frac{r_i^{k+1}-r_i^k}{2r_i^k-(k+1)}. \tag{9}$$

This again is equivalent to a particular solution using equation (8).

Theorem 4.2: When $f(0)=1$ and $f(x)=2^{x-1}$, for $1\le x\le k-1$, equation (9) is equivalent to equation (8).

Proof: The numerator of equation (8) with Spickerman and Joyner's initial values is equal to $\frac{1}{2}\left(\frac{1}{2-r_i}+\frac{1}{r_i}\right)$. This follows from the observation that $\prod_{1\le i\le k}r_i=(-1)^{k-1}$, $c_k(2)=1$, and

$$\frac{c_k(2)}{2-r_i}=\frac{-1}{r_i}+2\sum_{0\le j<k}(-1)^j f(k-1-j)\sigma_j^{k-1}.$$

The expression for the numerator simplifies to y_k^{k-1} by use of the identity $r^k(2-r)=1$, where r is any root of the characteristic equation.

The denominator of equation (8) can be expressed in terms of r_i by using the property that, for this problem, $\sigma_j^k=(-1)^{j-1}$, where $1\le j\le k$ [11]. By induction on k, we can show that

$$\prod_{1\le j<k}(y_k-y_j)=(k-1)y_k^{k-1}+\frac{1}{y_k}-\sum_{1\le l\le k-2}ly_k^l.$$

The summation can be removed by use of the identity

$$\sum_{0\le l\le n-1}(a+ld)x^l=\frac{a-(a+(n-1)d)x^n}{1-x}+\frac{dx(1-x^{n-1})}{(1-x)^2}$$

to give

$$\prod_{1\le j<k}(y_k-y_j)=(k-1)y_k^{k-1}+\frac{1}{y_k}-\frac{(k-2)y_k^{k-1}}{1-y_k}-\frac{y_k(1-y_k^{k-2})}{(1-y_k)^2}.$$

After some algebraic simplifications involving uses of the identity $r^{k+1}-r^k=r^k-1$, the expression for the denominator can be simplified to

$$\frac{y_k^{k-1}(y_k^{k+1}-k)}{y_k^k-1}.$$

Hence, in this case, equation (8) is equivalent to

$$\frac{y_k^k-1}{(y_k^{k+1}-k)}.$$

Equation (9) can be derived from this by using the above identity, as required. □

The formula for the coefficient C_i of a generalized Fibonacci recurrence can be expressed merely in terms of the initial values and the root r_i of the characteristic equation.

Corollary 4.3: In the case of the recurrence in equation (1), equation (8) is equivalent to

$$h(y_1, \ldots, y_k) = \frac{(y_k^k - 1) \sum_{0 \le j < k} f(k-1-j) c_j(y_k)}{y_k^{k-1}(y_k^{k+1} - k)}.$$

Proof: This follows from the proof of Theorem 4.2 and by induction on k to show that

$$\sigma_j^{k-1} = (-1)^j \left(y_k^j - \sum_{0 \le i < j} y_k^i \right),$$

where $0 \le j < k$. □

Since the coefficient C_i only depends on the root r_i and the k initial values, we write $h(y)$ instead of $h(y_1, \ldots, y_k)$. From Corollaries 3.2 and 4.3, we obtain

$$\sigma_j^{k-1} = (-1)^j \sum_{1 \le i \le k-j} y_k^{-i} \tag{10}$$

and

$$h(y) = \frac{(y^k - 1) \sum_{0 \le j < k} f(k-1-j) \sum_{1 \le i \le k-j} y^{-i}}{y^{k-1}(y^{k+1} - k)}. \tag{11}$$

Lemma 4.4: Suppose that the k initial values of f are real numbers. When k is even,

$$f(x) = h(r_1)r_1^x + h(r_2)r_2^x + \sum_{1 \le i \le \frac{k}{2}-1} 2v_i w_i^x \cos(\theta_i + \gamma_i x),$$

where v_i and w_i are real constants, and $-\pi < \theta_i, \gamma_i \le \pi$. When k is odd,

$$f(x) = h(r_1)r_1^x + \sum_{1 \le i \le \frac{k-1}{2}} 2v_i w_i^x \cos(\theta_i + \gamma_i x).$$

Proof: When k is even, from Lemma 3.6, let r_1 be the positive real root of the characteristic equation and r_2 be the negative real root. The $k-2$ complex roots can be paired as conjugates. From Corollary 4.3, if r and $\bar{r}$ are such a pair, then $h(r)$ and $h(\bar{r})$ are also conjugates. From equation (4), it follows that $f(x)$ can be expressed using the terms $h(r_1)r_1^x$ and $h(r_2)r_2^x$, and $(k-2)/2$ terms of the form $h(r)r^x + h(\bar{r})\bar{r}^x$.

Suppose that $h(r) = l_1 + il_2$ and $r = l_3 + il_4$. We can show that

$$h(r)r^x + h(\bar{r})\bar{r}^x = 2vw^x \cos(\theta + \gamma x),$$

where

$$v = \sqrt{l_1^2 + l_2^2} \quad \text{and} \quad w = \sqrt{l_3^2 + l_4^2}.$$

Let $\operatorname{sgn}(x)$ equal 1 if $x \ge 0$, and equal -1 if $x < 0$. We have $\theta = \operatorname{sgn}(l_2)\arccos(l_1 / v)$ and $\gamma = \operatorname{sgn}(l_4)\arccos(l_3 / w)$. We assume that, for all $x: -1 \le x \le 1$, $0 \le \arccos x \le \pi$. The case when k is odd is similar. □

5. SOLUTIONS

Substituting expressions for the coefficients C_i from equation (8) into equation (4) gives the unique general solution to equation (1) when the domain of f is restricted to the integers and the k initial values are known. This has been the usual application of Binet formulas.

Our solution seems to be more general than those considered previously (see [10], [13], [18], and [20]). It can also be used to find the general solutions of all homogeneous linear difference equations whose characteristic equations have simple roots.

The new generalization of the Fibonacci sequence we present defines the domain of f to be the reals. This introduces some additional questions. We first consider the solution of equation (1) when k initial values are given, and then where there is a given initial function.

6. USING THE INITIAL VALUES

6.1 Direct Solutions

Direct solutions use equation (7), $f(x) = \sum_{1 \le i \le k} C_i r_i^x$. It is not difficult to show that, if $x \in \Re$ rather than the integers, then f is a solution to equation (1), as required.

The coefficients C_i can be computed following Corollary 4.3 or equation (11) by using the k initial values and k roots r_i of the characteristic equation.

From Lemma 4.4 we have that, when k is odd, and the initial values are real, then f is a real-valued function for all $x \in \Re$. When k is even and the initial values are real, the image of f can be complex when x is not an integer. This arises from the term r_2^x because $r_2 < 0$. This term can be written $(\cos(\pi x) + i\sin(\pi x))(-r_2)^x$. We can show that the real part of f is

$$h(r_1)r_1^x + h(r_2)\cos(\pi x)(-r_2)^x + \sum_{1 \le i \le \frac{k}{2}-1} 2v_i w_i^x \cos(\theta_i + \gamma_i x).$$

The imaginary part of f is $h(r_2)\sin(\pi x)(-r_2)^x$. The real and imaginary parts of f individually satisfy equation (1). The real part has the same initial values as f, but the imaginary part is zero when x is an integer.

More generally, when k is even, we can replace $h(r_2)r_2^x$ with $h(r_2)m(x)(-r_2)^x$, where m is any continuous function that satisfies $m(x+1) = -m(x)$ for all $x \in \Re$, and $m(x) = (-1)^x$ when x is an integer. This family of solutions satisfies equation (1).

6.2 Laplace Transform Method

Another approach we use for finding solutions to equation (1) is based on the exponential generating function

$$G(x) = \sum_{0 \le l} \frac{f(l)x^l}{l!}, \tag{12}$$

where the function f is a solution to equation (1) for a given k, where $k > 1$. First, we solve the differential equation

$$G^{(k)}(x) = \sum_{0 \le i < k} G^{(i)}(x), \tag{13}$$

where $G(0) = f(0),\ G^{(1)}(0) = f(1), \ldots, G^{(k-1)}(0) = f(k-1)$. This is done by means of the Laplace Transform [5].

We then use this solution to find an expression for $G^{(n)}(0)$, where n is a nonnegative integer. Finally, we replace the variable n by a variable $x \in \Re$, and find $f(x) = G^{(x)}(0)$.

6.2.1 Fibonacci Function

We apply the method of Section 6.2 in the case $k = 2$. The Laplace Transform of $G^{(2)}(x) = G^{(1)}(x) + G(x)$ yields

$$\bar{x} = \frac{f(0)s + f(1) - f(0)}{s^2 - s - 1} = \frac{K_1}{s - r_1} + \frac{K_2}{s - r_2}.$$

The constants K_1 and K_2 can be found by solving the following system:

$$\begin{bmatrix} 1 & 1 \\ -r_2 & -r_1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) - f(0) \end{bmatrix}.$$

This system is equivalent to that discussed in Section 4 above when $k = 2$. The solution is

$$K_1 = f(0) - K_2,$$

$$K_2 = \frac{f(1) - f(0) + r_2 f(0)}{r_2 - r_1}.$$

Applying the inverse Laplace transform yields the same result as the direct method:

$$f(x) = \frac{(f(1) - f(0)r_2)r_1^x - (f(1) - f(0)r_1)r_2^x}{\sqrt{5}}.$$

A special case occurs when $f(0) = 2$ and $f(1) = 1$. The x^{th} Lucas number L_x equals $f(x)$ when x is an integer. We call this function $L(x)$: $L(x) = r_1^x + r_2^x$. If we call $F(x)$ the solution to equation (1), where $f(0) = 0$ and $f(1) = 1$, it is not difficult to show that $L(x) = F(x-1) + F(x+1)$, $r_1^x = \frac{L(x) + \sqrt{5}F(x)}{2}$, and $(-1)^x = \frac{L^2(x) - 5F^2(x)}{4}$ for all $x \in \Re$.

6.2.2 Tribonacci Function

The Laplace Transform of equation (13) when $k = 3$ is

$$\begin{aligned}\bar{x} &= \frac{s^2 f(0) + (f(1) - f(0))s + f(2) - f(1) - f(0)}{s^3 - s^2 - s - 1} \\ &= \frac{K_1}{s - r_1} + \frac{K_2 s + K_3}{s^2 + s(r_1 - 1) + \frac{1}{r_1}}.\end{aligned}$$

The constants K_1, K_2, and K_3 can be found by solving the following system:

$$\begin{bmatrix} 1 & 1 & 0 \\ r_1 - 1 & -r_1 & 1 \\ 1/r_1 & 0 & -r_1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) - f(0) \\ f(2) - f(1) - f(0) \end{bmatrix}.$$

We have

$$K_1 = \frac{f(0)}{r_1(r_1-1)(3r_1+1)} + \frac{f(1)}{3r_1+1} + \frac{f(2)}{(r_1-1)(3r_1+1)},$$
$$K_2 = f(0) - K_1,$$
$$K_3 = \frac{K_1 - r_1(f(2)-f(1)-f(0))}{r_1^2}.$$

Hence

$$\bar{x} = \frac{K_1}{s-r_1} + \frac{K_2\left(s+\left(\frac{r_1-1}{2}\right)\right)}{\left(s+\left(\frac{r_1-1}{2}\right)\right)^2\left(\frac{1}{r_1}-\left(\frac{r_1-1}{2}\right)^2\right)} + \frac{K_3 - K_2\left(\frac{r_1-1}{2}\right)}{\left(s+\left(\frac{r_1-1}{2}\right)\right)^2 + \left(\frac{1}{r_1}-\left(\frac{r_1-1}{2}\right)^2\right)}.$$

Applying the inverse Laplace Transform gives

$$G(x) = K_1e^{r_1x} + K_2e^{\left(\frac{1-r_1}{2}\right)x}\cos\sqrt{\tfrac{1}{r_1}-\left(\tfrac{r_1-1}{2}\right)^2}\,x + \frac{K_2\left(\frac{1-r_1}{2}\right)+K_3}{\sqrt{\frac{1}{r_1}-\left(\frac{r_1-1}{2}\right)^2}}\,e^{\left(\frac{1-r_1}{2}\right)x}\sin\sqrt{\tfrac{1}{r_1}-\left(\tfrac{r_1-1}{2}\right)^2}\,x.$$

We now use the observation that the n^{th} derivative of $e^{k_1x}\cos(k_2x)$ at $x=0$, where k_1 and k_2 are constants, is $l^n\cos(\theta n)$, where $l=\sqrt{k_1^2+k_2^2}$ and $\theta=\operatorname{sgn}(k_2)\arccos(k_1/l)$. Similarly, the n^{th} derivative of $e^{k_1x}\sin(k_2x)$ at $x=0$ is $l^n\sin(\theta n)$. We obtain

$$f(x) = K_1r_1^x + K_2r_1^{-x/2}\cos(\theta x) + r_1^{-x/2}\left(\frac{K_2\left(\frac{1-r_1}{2}\right)+K_3}{\sqrt{\frac{1}{r_1}-\left(\frac{r_1-1}{2}\right)^2}}\right)\sin(\theta x). \tag{14}$$

The angle θ is $\arccos\left(\frac{1-r_1}{2}\sqrt{r_1}\right)$. We can verify by induction on x that equation (14) is a solution when $k=3$, that $C_1=K_1$, and that this is the same function as that found by the direct method. It is interesting to note that, unlike the direct one, this solution does not use the complex roots.

6.2.3 Tetranacci Function

We shall use the method of solution of Section 6.2 when $k=4$. For brevity, we define $V_0=f(0)$, $V_1=f(1)-f(0)$, $V_2=f(2)-f(1)-f(0)$, and $V_3=f(3)-f(2)-f(1)-f(0)$. The Laplace Transform of (13) in this case is

$$\bar{x} = \frac{V_0s^3+V_1s^2+V_2s+V_3}{s^4-s^3-s^2-s-1}.$$

This is equivalent to

$$\bar{x} = \frac{V_0s^3+V_1s^2+V_2s+V_3}{(s-r_1)(s-r_2)(s^2+(r_1+r_2-1)s+r_1^2+r_2^2-r_1-r_2+r_1r_2-1)}.$$

We have

$$\bar{x} = \frac{K_1}{s-r_1} + \frac{K_2}{s-r_2} + \frac{K_3s+K_4}{s^2+(r_1+r_2-1)s+r_1^2+r_2^2-r_1-r_2+r_1r_2-1},$$

where $K_1, K_2, K_3,$ and K_4 are constants. They can be found by solving the following system:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ r_1-1 & r_2-1 & -r_1-r_2 & 1 \\ r_1^2-r_1-1 & r_2^2-r_2-1 & r_1r_2 & -r_1-r_2 \\ 1/r_1 & 1/r_2 & 0 & r_1r_2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

After finding the constants, and continuing with the other steps of the method of solution, we obtain the following solution which can be verified by induction:

$$f(x) = K_1r_1^x + K_2r_2^x + y^{x/2}(K_3\cos(\theta x) + K_5\sin(\theta x)),$$

where

$$y = r_1^2 - r_1 + r_2^2 - r_2 + r_1r_2 - 1,$$

$$\theta = \arccos\frac{1-r_1-r_2}{2\sqrt{y}},$$

$$K_1 = f(0) - K_2 - K_3,$$

$$K_2 = \frac{f(1) - r_1f(0) - K_4 - (1-2r_1-r_2)K_3}{r_2 - r_1},$$

$$K_3 = \frac{f(2) + r_1r_2f(0) - (r_1+r_2)f(1) - (1-2r_1-2r_2)K_4}{r_1^2 - 2r_1 + r_2^2 - 2r_2 + 4r_1r_2 + 2},$$

$$K_4 = \frac{f(3) - \left(1 - \frac{1}{r_1r_2} - (r_1+r_2)K_7\right)f(1) - (1+K_7)f(2) - K_6}{r_1r_2 + \frac{1}{r_1r_2} - (1-2r_1-2r_2)K_7},$$

$$K_5 = \frac{K_4 - \left(\frac{r_1+r_2-1}{2}\right)K_3}{\sqrt{y - \left(\frac{r_1+r_2-1}{2}\right)^2}},$$

$$K_6 = \left(1 + \frac{1}{r_1} + \frac{1}{r_2} + r_1r_2K_7\right)f(0),$$

$$K_7 = \frac{1-2r_1-2r_2}{(r_1^2 + r_2^2 + 4r_1r_2 - 2r_1 - 2r_2 + 2)r_1r_2}.$$

Lemma 6.1: $f(x)$ is symmetric in r_1 and r_2.

Proof: It is easy to check that $y^{x/2}(K_3\cos(\theta x) + K_5\sin(\theta x))$ is symmetric in r_1 and r_2 because y, θ, and K_3 to K_7 are symmetric. Now K_1 is equal to

$$\frac{f(1) - r_2f(0) - K_4 - (1-2r_2-r_1)K_3}{r_1 - r_2}.$$

This is K_2 with r_1 and r_2 interchanged. Hence, $K_1r_1^x + K_2r_2^x$ is also symmetric. □

This solution is also extensionally equivalent to the one found by the direct method, and $C_1 = K_1$, and $C_2 = K_2$. Again we see that it is not necessary to find the complex roots.

Solutions similar to this one, and the ones in Sections 6.2.2 and 6.2.1 above have appeared previously (see [21], [22], [23]) but without the preceding derivations. The method of solution described in Section 6.2 above can also be applied when the roots are expressed numerically.

7. USING THE INITIAL FUNCTION

If $k \geq 2$, then given an initial function f whose domain is the interval $[0, k)$, we can compute every value of the k-step function $f(x)$ where $x \in \Re$. To do this, we define a function F_i. This is a k^{th}-order function on the integers that satisfies equation (1) and whose initial values are

$$F_i(x) = \begin{cases} 0 & \text{if } x \neq i, \\ 1 & \text{if } x = i, \end{cases}$$

where $0 \leq i,\ x \leq k-1$. In general,

$$f(l+\varepsilon) = \sum_{0 \leq i < k} f(i+\varepsilon) F_i(l), \tag{15}$$

where l is an integer, $x = l + \varepsilon$, and $\varepsilon \in [0, 1)$. We can show by induction that

$$F_i(l) = \sum_{0 \leq j \leq i} F_0(l-j). \tag{16}$$

Equation (15) can thus be written as

$$f(l+\varepsilon) = \sum_{0 \leq i < k} f(i+\varepsilon) \sum_{0 \leq j \leq i} F_0(l-j). \tag{17}$$

Equation (17) shows that f can be defined on the real numbers in terms of the initial function and the k-step function F_0 whose domain is the integers. It is not unique. For example, from equation (16) we have, for a fixed k, that

$$F_{k-1}(l-1) = \sum_{0 \leq j \leq k-1} F_0(l-j-1),$$

i.e., $F_{k-1}(l-1) = F_0(l)$. It follows that

$$f(l+\varepsilon) = \sum_{0 \leq i < k} f(i+\varepsilon) \sum_{0 \leq j \leq i} F_{k-1}(l-j-1). \tag{18}$$

Now, from equation (11), the coefficients of equation (7), for F_{k-1}, are given by

$$h(y) = \frac{y^k - 1}{y^{k-1}(y^{k+1} - 1)}.$$

On substitution into equation (18), we have

$$f(l+\varepsilon) = \sum_{0 \leq i < k} f(i+\varepsilon) \sum_{0 \leq j \leq i} \sum_{1 \leq v \leq k} \frac{(r_v^k - 1) r_v^{l-k-j}}{r_v^{k+1} - 1},$$

where the r_v are the roots of the characteristic equation.

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APPROXIMATION OF QUADRATIC IRRATIONALS AND THEIR PIERCE EXPANSIONS

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1. INTRODUCTION

In the year 1937, E. B. Escott published his paper "Rapid Method for Extracting a Square Root" [4], where he presented an algorithm to find rational approximations for the square root of any real number. Escott's algorithm is based upon the algebraic identity

$$\sqrt{\frac{x_1+2}{x_1-2}}=\frac{x_1+1}{x_1-1}\cdot\frac{x_2+1}{x_2-1}\cdot\frac{x_3+1}{x_3-1}\cdot\dots,$$

where the x_i are obtained through the following recurrence:

$$x_n=x_{n-1}(x_{n-1}^2-3).$$

It is obvious that, in order to calculate $\sqrt{N}$, Escott's algorithm must use rational x_i and thus the actual computation is considerably retarded.

More recently, in 1993, Y. Lacroix [6] referred to Escott's algorithm in the context of the representation of real numbers by generalized Cantor products and their metrical study.

In Section 2 of this paper, we present an algorithm similar to Escott's but improved in the sense that we only use positive integers in the recurrence leading to the computation of $\sqrt{N}$. Moreover, the approximating fractions obtained by our algorithm constitute best approximations (of the second kind).

In 1984, J. O. Shallit [15] published the recurrence relations followed by the coefficients in the Pierce series development of irrational quadratics of the form $(c-\sqrt{c^2-4})/2$. His method is based on Pierce's algorithm [12] applied to the polynomial x^2-cx+1.

In Section 3, we use the infinite product expansion provided by our square root algorithm to find the Pierce expansions corresponding to irrationals of the form $p-\sqrt{p^2-1}$, as an alternative way to the one used by Shallit [15]. The same method can also be used in the case of irrationals of the form $2(p-1)(p-\sqrt{p^2-1})$, as we show in Section 4.

2. THE EXPANSION OF AN IRRATIONAL QUADRATIC AS AN INFINITE PRODUCT

It is well known (see [9], [10]) that the convergents p_n / q_n of the *regular continued fraction* development of $\sqrt{r}$, with r a positive integer, verify, alternatively, Pell's equation $p_n^2 - rq_n^2 = \pm 1$, and we get the recurrence relationships,

$$p_n = 2p_1 p_{n-1} - p_{n-2}, \quad q_n = 2p_1 q_{n-1} - q_{n-2}, \tag{1}$$

that allow us to find all the solutions of Pell's equation from the first one (p_1, q_1); (we take $p_0 = 1$, $q_0 = 0$).

Lemma 1: Let (p_1, q_1) be a positive solution $(p_1 > 0,\ q_1 > 0)$ of Pell's equation $x^2 - ry^2 = 1$, where r is a positive integer free of squares. The sequence $\{(\overline{p}_n,\ \overline{q}_n)\}$, obtained recurrently in the following way,

$$\begin{cases} \overline{p}_n = \overline{p}_{n-1}(4\overline{p}_{n-1}^2 - 3), & \overline{p}_1 = p_1, \\ \overline{q}_n = \overline{q}_{n-1}(4\overline{p}_{n-1}^2 - 1), & \overline{q}_1 = q_1, \end{cases} \tag{2}$$

is a subsequence of the sequence $\{(p_n,\ q_n)\}$ of all solutions of the given Pell equation, with the peculiarity that each solution is an integer multiple of the preceding one.

Proof: We shall proceed by induction on n. Let us suppose that $\overline{p}_{n-1}^2 = r\overline{q}_{n-1}^2 + 1$ is verified. We must ascertain that

$$\overline{p}_n^2 = r\overline{q}_n^2 + 1. \tag{3}$$

We replace $\overline{p}_n$ and $\overline{q}_n$ using the recurrence (2):

$$\overline{p}_{n-1}^2(4\overline{p}_{n-1}^2 - 3)^2 = r\overline{q}_{n-1}^2(4\overline{p}_{n-1}^2 - 1)^2 + 1. \tag{4}$$

To simplify, let us denote by α the expression $\alpha = 4\overline{p}_{n-1}^2 - 2$. Equality (4) becomes

$$\overline{p}_{n-1}^2(\alpha - 1)^2 = r\overline{q}_{n-1}^2(\alpha + 1)^2 + 1,$$

which can be written as $\overline{p}_{n-1}^2(\alpha^2 - 2\alpha + 1) = r\overline{q}_{n-1}^2(\alpha^2 + 2\alpha + 1) + 1$. Grouping together the terms corresponding to $\alpha^2 + 1$, we obtain the equality

$$(\alpha^2 + 1)(\overline{p}_{n-1}^2 - r\overline{q}_{n-1}^2) - 2\alpha(\overline{p}_{n-1}^2 + r\overline{q}_{n-1}^2) = 1, \tag{5}$$

and, by the induction hypothesis, $\overline{p}_{n-1}^2 - r\overline{q}_{n-1}^2 = 1$, and also $\overline{p}_{n-1}^2 + r\overline{q}_{n-1}^2 = 2\overline{p}_{n-1}^2 - 1$. Therefore, (5) becomes

$$\alpha^2 - 2\alpha(2\overline{p}_{n-1}^2 - 1) = 0, \tag{6}$$

and, as we have $\alpha = 2(2\overline{p}_{n-1}^2 - 1)$, we deduce that (6) is, in point of fact, an algebraic identity. □

Theorem 1: $\sqrt{r}$ expands in an infinite product of the form

$$\sqrt{r} = \frac{p_1}{q_1} \prod_{n=1}^{\infty} \frac{\alpha_n^2 - 3}{\alpha_n^2 - 1} = \frac{p_1}{q_1} \prod_{n=1}^{\infty} \left(1 - \frac{2}{\alpha_n^2 - 1}\right),$$

where $(p_1,\ q_1)$ is a positive solution of Pell's equation, $x^2 - ry^2 = 1$; $\alpha_1 = 2p_1$, $\alpha_n = \alpha_{n-1}(\alpha_{n-1}^2 - 3)$.

Proof: With the same notation as in Theorem 1, we have, on the one hand,

$$\sqrt{r} = \lim_{n\to\infty} \frac{\overline{p}_n}{\overline{q}_n}, \tag{7}$$

and, on the other, we have the recurrence

$$\frac{\overline{p}_n}{\overline{q}_n} = \frac{\overline{p}_{n-1}}{\overline{q}_{n-1}} \cdot \frac{4\overline{p}_{n-1}^2 - 3}{4\overline{p}_{n-1}^2 - 1}. \tag{8}$$

Iterating, we obtain the expansion

$$\frac{\overline{p}_n}{\overline{q}_n} = \frac{\overline{p}_1}{\overline{q}_1} \cdot \frac{4\overline{p}_1^2 - 3}{4\overline{p}_1^2 - 1} \cdot \frac{4\overline{p}_2^2 - 3}{4\overline{p}_2^2 - 1} \cdot \cdots \cdot \frac{4\overline{p}_{n-1}^2 - 3}{4\overline{p}_{n-1}^2 - 1}, \tag{9}$$

or, if we prefer, we can simplify expression (9) by defining the new recurrence

$$\alpha_n = \alpha_{n-1}(\alpha_{n-1}^2 - 3), \qquad \alpha_1 = 2p_1, \tag{10}$$

which allows us to write

$$\frac{\overline{p}_n}{\overline{q}_n} = \frac{p_1}{q_1} \cdot \frac{\alpha_1^2 - 3}{\alpha_1^2 - 1} \cdot \cdots \cdot \frac{\alpha_{n-1}^2 - 3}{\alpha_{n-1}^2 - 1}. \tag{11}$$

Finally, taking limits as $n \to \infty$, the expansion of $\sqrt{r}$ in an infinite product is

$$\sqrt{r} = \frac{p_1}{q_1} \prod_{n=1}^{\infty} \frac{\alpha_n^2 - 3}{\alpha_n^2 - 1} = \frac{p_1}{q_1} \prod_{n=1}^{\infty} \left(1 - \frac{2}{\alpha_n^2 - 1}\right). \quad \square \tag{12}$$

Using the recurrence (10) in (11), we obtain

$$\frac{\overline{p}_n}{\overline{q}_n} = \frac{\alpha_n}{2q_1(\alpha_1^2 - 1) \cdots (\alpha_{n-1}^2 - 1)}. \tag{13}$$

The recurrence (10) is a fast way to compute the fractions of (13) which constitute best approximations of the second kind of any irrational quadratic of the form $\sqrt{r}$, where r is a positive integer; to start, we need only a positive solution of Pell's equation $x^2 - ry^2 = 1$. With ten iterations of the algorithm, we obtain a fraction whose approximation to the irrational is of the order $10^{-30,000}$. With 14 iterations, the approximation gives us a million correct decimal figures.

Expansion (12), among others, is the one considered by Y. Lacroix [6] in connection with Cantor's representation of real numbers by infinite products (see [1]).

3. THE PIERCE EXPANSION OF $p - (p^2 - 1)^{1/2}$

Any real number $\alpha \in (0, 1]$ has a unique Pierce expansion of the form

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{n+1}}{a_1 a_2 \cdots a_n} + \cdots, \tag{14}$$

where $\{a_n\}$ is a strictly increasing sequence of positive integers. These a_i will be called *coefficients* or *partial quotients* of the development.

Following Erdös and Shallit [3], we will denote the right-hand side of (14) by the special symbol $\langle a_1, a_2, \ldots, a_n, \ldots \rangle$. If expansion (14) is infinite, α is irrational. Otherwise, α is rational.

One of the first mathematicians to consider these developments was Lambert in [8]. Later, Lagrange refers to them in [7], but their numerical properties were not discussed until the independent studies of Sierpinski [18] and Ostyrogadsky [13] were published. Perron mentions them in [11] among other unusual series representations of real numbers. T. A. Pierce [12] used these series to approximate roots of algebraic equations and, in 1986, Shallit [16] studied their metrical properties using methods developed by Rényi [14] to study the metrical properties of Engel's series, i.e., series of type (14) but with all its signs positive (see [2], [11], [14]). In 1984, using Pierce's algorithm, Shallit [15] obtained the Pierce expansion of all irrational quadratics of the form

$$\frac{c-\sqrt{c^2-4}}{2} \text{ with integer } c,\ c \ge 3. \tag{15}$$

Quite recently, in 1994, Shallit [17] used Pierce expansions to propose a very nice method for determining leap years which generalizes those existent and, in 1995, Knopfmacher and Mays [5] related the expansions obtained in (15) to the Pierce expansions of some particular quotients of consecutive Fibonacci numbers.

If $c = 2k$, the irrational in (15) is directly of the form $k-\sqrt{k^2-1}$. If $c = 2k+1$, it can be seen that the irrational in (15) can be written as

$$\frac{1}{2k}-\frac{1}{2k(2k+2)}+\frac{1}{2k(2k+2)}\cdot(p-\sqrt{p^2-1})$$

with $p=(2k+1)(2k^2+2k-1)$. Thus, the Pierce expansion of irrationals of the form studied by Shallit are directly related to the irrationals of the form $p-\sqrt{p^2-1}$. The aim of this section is to find the Pierce expansion of all irrationals of this particular form using different methods than those in [15].

Now, if $\sqrt{p^2-1}=q\sqrt{r}$ with r free of squares, (p, q) is a solution of Pell's equation

$$x^2-ry^2=1.$$

Theorem 2: Given r, a positive integer free of squares, let (p, q) be a positive solution of Pell's equation $x^2-ry^2=1$. The Pierce expansion of the irrational $p-q\sqrt{r}$ is exactly

$$p-q\sqrt{r}=\langle \alpha_1-1,\ \alpha_1+1,\ \alpha_2-1,\ \alpha_2+1, \ldots\rangle, \tag{16}$$

where $\alpha_1=2p$ and $\alpha_{n+1}=\alpha_n(\alpha_n^2-3)$.

Proof: Using expression (13),

$$\frac{\overline{p}_n}{\overline{q}_n}=\frac{\alpha_n}{2q_1(\alpha_1^2-1)\cdots(\alpha_{n-1}^2-1)}.$$

We can write its right-hand side as

$$\frac{\alpha_{n-1}}{2q_1(\alpha_1^2-1)\cdots(\alpha_{n-2}^2-1)}\cdot\frac{\alpha_{n-1}^2-3}{\alpha_{n-1}^2-1}=\frac{\alpha_{n-1}}{2q_1(\alpha_1^2-1)\cdots(\alpha_{n-2}^2-1)}\left(1-\frac{2}{\alpha_{n-1}^2-1}\right).$$

Now, as we have the algebraic identity

$$\frac{\alpha_{n-1}}{b(\alpha_{n-1}^2-1)}=\frac{1}{b(\alpha_{n-1}-1)}-\frac{1}{b(\alpha_{n-1}-1)(\alpha_{n-1}+1)},$$

iterating the former process we eventually reach the expansion,

$$\begin{aligned}\frac{\overline{p}_n}{\overline{q}_n}=\frac{p_1}{q_1}-\frac{1}{q_1(\alpha_1-1)}+\frac{1}{q_1(\alpha_1-1)(\alpha_1+1)}+\cdots\\ +\frac{1}{q_1(\alpha_1^2-1)\cdots(\alpha_{n-2}^2-1)(\alpha_{n-1}-1)}-\frac{1}{q_1(\alpha_1^2-1)\cdots(\alpha_{n-1}^2-1)}.\end{aligned}$$

In our case, $p_1=p$ and $q_1=q$; thus, we can write

$$\begin{aligned}\frac{p}{q}-\frac{p_n}{q_n}=\frac{1}{q(\alpha_1-1)}-\frac{1}{q(\alpha_1-1)(\alpha_1+1)}+\cdots\\ +\frac{1}{q(\alpha_1^2-1)\cdots(\alpha_{n-2}^2-1)(\alpha_{n-1}-1)}-\frac{1}{q(\alpha_1^2-1)\cdots(\alpha_{n-1}^2-1)}.\end{aligned}\tag{17}$$

As $n\to\infty$ we obtain the infinite Pierce expansion,

$$\frac{p}{q}-\sqrt{r}=\sum_{i=1}^{\infty}\left(\frac{1}{q\prod_{k=1}^{i-1}(\alpha_k^2-1)\cdot(\alpha_i-1)}-\frac{1}{q\prod_{k=1}^{i}(\alpha_k^2-1)}\right),\tag{18}$$

which is equivalent to (16). □

4. THE PIERCE EXPANSION OF $2(p-1)[p-(p^2-1)^{1/2}]$

In this section we are going to see how the method we have just used can be extended to find the Pierce expansion of irrational quadratics of the form $2(p-1)(p-\sqrt{p^2-1})$.

As above, our starting point will be Pell's equation $x^2-ry^2=1$, and we will choose a subsequence of the sequence of its solutions. We will need the following result.

Lemma 2: Given a positive solution (p,q) of Pell's equation $x^2-ry^2=1$, with r free of squares, the recurrent sequence $\{(\overline{p}_n,\overline{q}_n)\}$, obtained in the form

$$\begin{cases}\overline{p}_n=2\overline{p}_{n-1}^2-1, & \overline{p}_1=p,\\ \overline{q}_n=2\overline{p}_{n-1}\overline{q}_{n-1}, & \overline{q}_1=q,\end{cases}$$

is a subsequence of the sequence $\{(p_n,q_n)\}$ of all the equation solutions.

Proof: The result is easily proved by induction. Let us suppose that $\overline{p}_{n-1}$ and $\overline{q}_{n-1}$ verify $\overline{p}_{n-1}^2-r\overline{q}_{n-1}^2=1$. For the next index we will have

$$\begin{aligned}\overline{p}_n^2&=(2\overline{p}_{n-1}^2-1)^2=4\overline{p}_{n-1}^4-4\overline{p}_{n-1}^2+1,\\ r\overline{q}_n^2&=r4\overline{p}_{n-1}^2\overline{q}_{n-1}^2,\end{aligned}$$

and subtracting gives $\overline{p}_n^2-r\overline{q}_n^2=4\overline{p}_{n-1}^2(\overline{p}_{n-1}^2-r\overline{q}_{n-1}^2)-4\overline{p}_{n-1}^2+1=1$. □

Having proved that all pairs $(\overline{p}_n,\overline{q}_n)$ are solutions of the given Pell equation and using the fact that $\sqrt{r}=\lim_{n\to\infty}(\overline{p}_n/\overline{q}_n)$, we will try, as before, to expand the fraction $\overline{p}_n/\overline{q}_n$ as a finite

Pierce expansion and then, taking limits, obtain the infinite Pierce expansion corresponding to the irrational $\sqrt{r}$ or an equivalent one.

Let us start with the fraction $\overline{p}_n / \overline{q}_n$, and let us express its numerator and denominator in terms of the preceding pair of solutions, i.e.,

$$\frac{\overline{p}_n}{\overline{q}_n} = \frac{2\overline{p}_{n-1}^2 - 1}{2\overline{p}_{n-1}\overline{q}_{n-1}} = \frac{\overline{p}_{n-1}}{\overline{q}_{n-1}} - \frac{1}{2\overline{p}_{n-1}\overline{q}_{n-1}}. \tag{19}$$

Proceeding with the expansion of the equation above, we will eventually reach the first one, $\overline{p}_1 / \overline{q}_1$, and the chain of equalities

$$\begin{aligned}\frac{\overline{p}_n}{\overline{q}_n} &= \frac{\overline{p}_1}{\overline{q}_1} - \frac{1}{2\overline{p}_1\overline{q}_1} - \frac{1}{2\overline{p}_2\overline{q}_2} - \cdots - \frac{1}{2\overline{p}_{n-1}\overline{q}_{n-1}} \\ &= \frac{\overline{p}_1}{\overline{q}_1} - \frac{1}{\overline{q}_1 2\overline{p}_1} - \frac{1}{\overline{q}_1 2\overline{p}_1 2\overline{p}_2} - \cdots - \frac{1}{\overline{q}_1 2\overline{p}_1 2\overline{p}_2 \ldots 2\overline{p}_{n-1}} \\ &= \frac{\overline{p}_1}{\overline{q}_1} - \frac{1}{\overline{q}_1}\left(\frac{1}{2\overline{p}_1} + \frac{1}{2\overline{p}_1 2\overline{p}_2} + \cdots + \frac{1}{2\overline{p}_1 2\overline{p}_2 \ldots 2\overline{p}_{n-1}}\right).\end{aligned}$$

Taking limits in this last expression and remembering that $\overline{p}_1 = p$ and $\overline{q}_1 = q$, we obtain

$$\sqrt{r} = \frac{p}{q} - \frac{1}{q}\left(\frac{1}{2p} + \frac{1}{2p2\overline{p}_2} + \cdots + \frac{1}{2p2\overline{p}_2 \ldots 2\overline{p}_{n-1}} + \cdots\right), \tag{20}$$

where the $\overline{p}_i$ follow the recurrence $\overline{p}_n = 2\overline{p}_{n-1}^2 - 1$, $\overline{p}_1 = p$. The series within parentheses on the right-hand side of (20) is an Engel series.

Equality (20) can also be expressed in the form

$$p - q\sqrt{r} = \frac{1}{2p} + \frac{1}{2p\overline{p}_2} + \cdots + \frac{1}{2p2\overline{p}_2 \ldots 2\overline{p}_{n-1}} + \cdots, \tag{21}$$

or, if we prefer, we can state the following result.

Lemma 3: For all positive integers p, we have

$$p - \sqrt{p^2 - 1} = \frac{1}{2\overline{p}_1} + \frac{1}{2\overline{p}_1\overline{p}_2} + \cdots + \frac{1}{2\overline{p}_1 2\overline{p}_2 \ldots 2\overline{p}_{n-1}} + \cdots, \tag{22}$$

with $\overline{p}_i = 2\overline{p}_{i-1}^2 - 1$, $\overline{p}_1 = p$.

Expression (22) is known as *Stratemeyer's formula*, and can be obtained algebraically by the method described in Perron (see [11], Ch. IV). We mention in passing that the recurrence (22) verified by the $\overline{p}_i$ is exactly the recurrence verified by the denominators in the infinite product expansion presented by Cantor in [1]. We are now ready for the following result.

Theorem 3: If p is a positive integer greater than one, then $2(p-1)(p - \sqrt{p^2 - 1}) = \langle 1, p_1, p_2, p_3, \ldots \rangle$, where $p_1 = p$ and the p_i verify

$$\begin{cases} p_{2n} = 4(p_{2n-1} + 1), \\ p_{2n+1} = 2p_{2n-1}^2 - 1. \end{cases}$$

Proof: To prove Theorem 3, we just have to change the Engel series in (22) into a Pierce expansion. In order to do that, let us consider the Pierce expansion of Theorem 3,

$$\langle 1, p_1, p_2, p_3, \dots \rangle, \tag{23}$$

with the recurrence $p_{2n} = 4(p_{2n-1}+1)$, $p_{2n+1} = 2p_{2n-1}^2 - 1$, $p_1 = p$. If we denote by S the irrational number represented by (23), we have the following expansion:

$$S = 1 - \frac{1}{p_1} + \frac{1}{p_1 p_2} - \dots = \frac{p_1 - 1}{p_1} + \frac{p_3 - 1}{p_1 p_2 p_3} + \dots + \frac{p_{2n+1} - 1}{p_1 p_2 \cdots p_{2n+1}}.$$

We want to see that each fraction in the sum above is of the form

$$\frac{p_{2n+1} - 1}{p_1 p_2 \cdots p_{2n+1}} = \frac{p_1 - 1}{p_1 2p_3 \dots 2p_{2n+1}}. \tag{24}$$

We will proceed by induction on n. For $n = 0$, it is trivially true. Let us expand the left-hand side of (24) in the following way:

$$\frac{p_{2n+1} - 1}{p_1 \cdots p_{2n-1} p_{2n} p_{2n+1}} = \underbrace{\frac{p_{2n-1} - 1}{p_1 \cdots p_{2n-1}}}_{(*)} \cdot \underbrace{\left(\frac{p_{2n+1} - 1}{p_{2n+1} p_{2n}}\right) \cdot \frac{1}{p_{2n-1} - 1}}_{(**)}. \tag{25}$$

The term $(**)$ can be written as follows:

$$\frac{2p_{2n-1}^2 - 1 - 1}{p_{2n+1} 4(p_{2n-1} + 1)} \cdot \frac{1}{p_{2n-1} - 1} = \frac{2(p_{2n-1}^2 - 1)}{p_{2n+1} 4(p_{2n-1}^2 - 1)} = \frac{1}{2p_{2n+1}}.$$

Finally, by the induction hypothesis applied to factor $(*)$ in (25), we obtain

$$\frac{p_{2n+1} - 1}{p_1 \cdots p_{2n-1} p_{2n} p_{2n+1}} = \frac{p_1 - 1}{p_1 2p_3 \dots 2p_{2n-1}} \cdot \frac{1}{2p_{2n+1}}. \tag{26}$$

Thus, S can be written as

$$\begin{aligned} S = \langle 1, p_1, p_2, \dots \rangle &= \frac{p_1 - 1}{p_1} + \frac{p_1 - 1}{p_1 2p_3} + \dots + \frac{p_1 - 1}{p_1 2p_3 \dots 2p_{2n+1}} + \dots \\ &= \frac{p_1 - 1}{p_1}\left(1 + \frac{1}{2p_3} + \frac{1}{2p_3 2p_5} + \dots + \frac{1}{2p_3 2p_5 \dots 2p_{2n+1}} + \dots\right) \\ &= 2(p_1 - 1)\underbrace{\left(\frac{1}{2p_1} + \frac{1}{2p_1 2p_3} + \dots + \frac{1}{2p_1 2p_3 \dots 2p_{2n+1}}\right)}_{(***)}. \end{aligned}$$

But, by Stratemeyer's formula (22), the term $(***)$ is precisely $p_1 - \sqrt{p_1^2 - 1}$. $\square$

5. CONCLUSIONS

The algorithm presented in this article provides fast best approximations to any irrational of the form $\sqrt{r}$, where r is a positive integer. At the same time, the algorithm provides the necessary background to obtain the Pierce expansion of some quadratic irrationals whose partial quotients, a_i, grow as x^3. The procedure used proves also that the convergents in the Pierce expansions of these irrationals are best approximations of the second kind.

We also present the Pierce series development of irrationals of the form

$$2(p-1)(p-\sqrt{p^2-1}),$$

whose partial quotients grow as x^2.

However, there exist quadratic irrationals that escape the above laws, whose partial quotients obey the metrical behavior, $\lim_{n\to\infty}\sqrt[n]{a_n}=e$, found by Shallit in [16].

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SOME IDENTITIES INVOLVING THE EULER AND THE CENTRAL FACTORIAL NUMBERS

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1. INTRODUCTION AND RESULTS

Let x be a real number with $|x| < \pi/2$. The Euler sequence $E = (E_{2n})$, $n = 1, 2, \ldots$, is defined by the coefficients in the expansion of

$$\sec x = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n}.$$

That is, $E_0 = 1$, $E_2 = 1$, $E_4 = 5$, $E_6 = 61$, $E_8 = 1385$, $E_{10} = 50521, \ldots$. These numbers arose in some combinatorial contexts, and were investigated by many authors. For example, see Lehmer [7] and Powell [8]. The main purpose of this paper is to study the calculating problem of the summation involving the Euler numbers, i.e.,

$$\sum_{a_1+a_2+\cdots+a_k=n} \frac{E_{2a_1}E_{2a_2}\ldots E_{2a_k}}{(2a_1)!(2a_2)!\ldots(2a_k)!}, \tag{1}$$

where the summation is over all k-dimension nonnegative integer coordinates $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$, and k is any odd number with $k > 1$.

This problem is interesting because it can help us to find some new recurrence properties for (E_{2n}). In this paper we use the differential equation of the generating function of the sequence (E_{2n}) to study the calculating problems of (1), and give an interesting identity for (1) for any fixed odd number $k > 1$. That is, we shall prove the following main conclusion.

Theorem: Let n and m be nonnegative integers and $k = 2m+1$. Then we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n} \frac{E_{2a_1}E_{2a_2}\ldots E_{2a_k}}{(2a_1)!(2a_2)!\ldots(2a_k)!}$$
$$= \frac{1}{(k-1)!(2n)!}\sum_{i=0}^{m}(-1)^i 4^i t(2m+1, 2m-2i+1)E_{2n+2m-2i},$$

where $t(n, k)$ are central factorial numbers.

From the above theorem, we may immediately deduce the following.

Corollary 1: For any odd prime p, we have the congruence

$$E_{p-1} \equiv \begin{cases} 0 \pmod{p}, & \text{if } p \equiv 1 \pmod{4}, \\ -2 \pmod{p}, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Corollary 2: For any integer $n > 0$, we have the congruences

(a) $E_{2n+2} + E_{2n} \equiv 0 \pmod{6}$,

(b) $E_{2n+4} + 10E_{2n+2} + 9E_{2n} \equiv 0 \pmod{24}$,

(c) $E_{2n+6} + E_{2n} \equiv 0 \pmod{42}$.

2. PROOF OF THE THEOREM

In this section, we shall complete the proof of the theorem. First, we give an elementary lemma which is described as follows.

Lemma: Let $F(x) = 1/\cos x$. Then, for any odd number $k = 2m+1 > 1$, $F(x)$ satisfies the differential equation

$$(2m)!\,F^k(x) = \sum_{i=0}^{m} c_i(m) F^{(2m-2i)}(x),$$

where $F^{(r)}(x)$ denotes the r^{th} derivative of $F(x)$, and the constants $c_i(m)$, $i = 0, 1, 2, \ldots, m$, are defined by the coefficients of the polynomial

$$G_m(x) = (x+1^2)(x+3^2)(x+5^2)\cdots(x+(2m-1)^2) = \sum_{i=0}^{m} c_i(m) x^{m-i}.$$

Note: The constants $c_i(m)$ in the Lemma are special cases of the generalized Stirling numbers of the first kind, $s_\xi(n, k)$, introduced by Comtet [2], i.e.,

$$(x-\xi_0)(x-\xi_1)\cdots(x-\xi_{n-1}) = \sum_{i=0}^{n} s_\xi(n, i) x^i.$$

Moreover, the constants $c_i(m)$ are, in fact, the central factorial numbers $t(n, k)$ (see Riordan [9]). The inverse and similar numbers are treated in many important papers by Carlitz [3] and [4], and by Carlitz and Riordan [5]. For some generalizations, see Charalambides [6].

Now we prove the Lemma by induction. From the definition of $F(x)$, and differentiating it, we may obtain

$$F'(x) = \frac{\sin x}{\cos^2 x}, \quad F''(x) = \frac{\cos^3 x + 2\sin^2 x \cos x}{\cos^4 x} = \frac{2}{\cos^3 x} - \frac{1}{\cos x},$$

i.e.,

$$2F^3(x) = F''(x) + F(x). \tag{2}$$

This proves that the Lemma is true for $m = 1$. Assume, then, that it is true for a positive integer $m = u$. That is,

$$(2u)!\,F^{2u+1}(x) = \sum_{i=0}^{u} c_i(u) F^{(2u-2i)}(x). \tag{3}$$

We shall prove it is also true for $m = u+1$. Differentiating (3), we have

$$(2u+1)!\,F^{2u}(x)F'(x) = \sum_{i=0}^{u} c_i(u) F^{(2u-2i+1)}(x),$$

$$2u(2u+1)!\,F^{2u-1}(x)(F'(x))^2 + (2u+1)!\,F^{2u}(x)F''(x) = \sum_{i=0}^{u} c_i(u) F^{(2u-2i+2)}(x). \tag{4}$$

From the equality

$$4^{-n}(4x^2-1^2)(4x^2-3^2)\cdots(4x^2-(2n-1)^2) = \sum_{k=0}^{n} t(2n+1, 2k+1) x^{2k},$$

we get

$$c_k(n) = (-1)^k 4^k t(2n+1, 2n-2k+1). \tag{5}$$

These numbers are tabulated in Riordan [9]. Using this expression and the recursive relation $t(n, k) = t(n-2, k-2) - \frac{1}{4}(n-2)^2 t(n-2, k)$, we have the recurrence relation

$$c_k(n+1) = c_k(n) + (2n+1)^2 c_{k-1}(n), \tag{6}$$

with initial conditions $c_0(n) = 1$, $c_n(n) = 1^2 3^2 \ldots (2n-1)^2$. Substituting $(F'(x))^2$ by $F^4(x) - F^2(x)$ and $F''(x)$ by $2F^3(x) - F(x)$ in (4) and applying (3) and (6), we have

$$\begin{aligned}
(2u+2)!\,F^{2u+3}(x) &= (2u)!(2u+1)^2 F^{2u+1}(x) + \sum_{i=0}^{u} c_i(u) F^{(2u+2-2i)}(x) \\
&= (2u+1)^2 \sum_{i=0}^{u} c_i(u) F^{(2u-2i)}(x) + \sum_{i=0}^{u} c_i(u) F^{(2u+2-2i)}(x) \\
&= c_0(u) F^{(2u+2)}(x) + (2u+1)^2 c_u(u) F(x) + \sum_{i=0}^{u-1} (c_{i+1}(u) + (2u+1)^2 c_i(u)) F^{(2u-2i)}(x) \\
&= c_0(u+1) F^{(2u+2)}(x) + c_{u+1}(u+1) F(x) + \sum_{i=1}^{u} c_i(u+1) F^{(2u+2-2i)}(x) \\
&= \sum_{i=0}^{u+1} c_i(u+1) F^{(2u+2-2i)}(x).
\end{aligned}$$

That is, the Lemma is also true for $m = u+1$. This proves the Lemma.

Now we complete the proof of the Theorem. Note that

$$F^{(2i)}(x) = \sum_{n=0}^{\infty} \frac{E_{2n+2i}}{(2n)!} x^{2n}, \quad i = 0, 1, 2, \ldots.$$

Comparing the coefficient of x^{2n} on both sides of the Lemma and applying (5), we immediately obtain

$$\begin{aligned}
(2m)! \sum_{a_1+a_2+\cdots+a_k=n} \frac{E_{2a_1} E_{2a_2} \ldots E_{2a_k}}{(2a_1)!(2a_2)! \ldots (2a_k)!} &= \frac{1}{(2n)!} \sum_{i=0}^{m} c_i(m) E_{2n+2m-2i} \\
&= \frac{1}{(2n)!} \sum_{i=0}^{m} (-1)^i 4^i t(2m+1, 2m-2i+1) E_{2n+2m-2i},
\end{aligned}$$

where the constants $c_i(m), i = 0, 1, 2, \ldots, m$ are the coefficients of the polynomial

$$G_m(x) = (x+1^2)(x+3^2)(x+5^2) \cdots (x+(2m-1)^2) = \sum_{i=0}^{m} c_i(m) x^{m-i}.$$

This completes the proof of the Theorem.

Proof of the Corollaries: Taking $n = 0$ and $k = p$ in the Theorem, and noting that $E_0 = 1$, $(p-1)! \equiv -1 \pmod p$ (Wilson's theorem, see Apostol [1]), we can get

$$\begin{aligned}
-1 \equiv (p-1)! &= \sum_{i=0}^{\frac{p-1}{2}} c_i\left(\frac{p-1}{2}\right) E_{p-1-2i} \equiv E_{p-1} + c_{\frac{p-1}{2}}\left(\frac{p-1}{2}\right) E_0 \\
&\equiv E_{p-1} + 1^2 3^2 5^2 7^2 \ldots (p-2)^2 \equiv E_{p-1} + (-1)^{\frac{p-1}{2}} (p-1)! \equiv E_{p-1} - (-1)^{\frac{p-1}{2}} \pmod p,
\end{aligned}$$

where we have used the congruences

$$c_i\left(\frac{p-1}{2}\right) \equiv 0 \pmod p, \quad i = 1, 2, \ldots, \frac{p-3}{2}.$$

Therefore,

$$E_{p-1} \equiv \begin{cases} 0 \pmod p, & \text{if } p \equiv 1 \pmod 4, \\ -2 \pmod p, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

This completes the proof of Corollary 1.

Taking $m = 1$ and 2 in the Theorem, respectively, we can get

$$E_{2n+4} + E_{2n+2} \equiv E_{2n+2} + E_{2n} \equiv 0 \pmod 2,$$

$$E_{2n+4} + 10E_{2n+2} + 9E_{2n} \equiv 0 \pmod{24}.$$

Thus, $0 \equiv E_{2n+4} + 10E_{2n+2} + 9E_{2n} \equiv E_{2n+4} + E_{2n+2} \equiv 0 \pmod 3$. Since $(2, 3) = 1$, $E_{2n+4} + E_{2n+2} \equiv 0 \pmod 2$, we have $E_{2n+4} + E_{2n+2} \equiv 0 \pmod 6$, that is, $E_{2n+2} + E_{2n} \equiv 0 \pmod 6$, $n = 1, 2, 3, \ldots$.

Similarly, taking $m = 4$ in the Theorem, we can obtain the congruent equation

$$E_{2n+8} + 84E_{2n+6} + 1974E_{2n+4} + 12916E_{2n+2} + 11025E_{2n} \equiv 0 \pmod{40320}.$$

Thus, $0 \equiv E_{2n+8} + 84E_{2n+6} + 1974E_{2n+4} + 12916E_{2n+2} + 11025E_{2n} \equiv E_{2n+8} + E_{2n+2} \pmod{21}$, that is, $E_{2n+6} + E_{2n} \equiv 0 \pmod{21}$, $n = 1, 2, 3, \ldots$. Noting that $E_{2n+6} + E_{2n} \equiv 0 \pmod 2$ and $(2, 21) = 1$, we get $E_{2n+6} + E_{2n} \equiv 0 \pmod{42}$. This proves Corollary 2.

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THE FACTORIZATION OF $x^5 \pm x^a + n$

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1. INTRODUCTION

Rabinowitz [5] has determined all integers n for which $x^5 \pm x + n$ factors as a product of an irreducible quadratic and an irreducible cubic with integral coefficients. Using the properties of Fibonacci numbers, he showed that, in fact, there are only ten such integers n.

Theorem (Rabinowitz [5]): The only integral n for which $x^5 + x + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = \pm 1$ and $n = \pm 6$. The factorizations are

$$x^5 + x \pm 1 = (x^2 \pm x \pm 1)(x^3 \mp x^2 \pm 1),$$
$$x^5 + x \pm 6 = (x^2 \pm x + 2)(x^3 \mp x^2 - x \pm 3).$$

The only integral n for which $x^5 - x + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = \pm 15$, $n = \pm 22{,}440$, and $n = \pm 2{,}759{,}640$. The factorizations are

$$x^5 - x \pm 15 = (x^2 \pm x + 3)(x^3 \mp x^2 - 2x \pm 5),$$
$$x^5 - x \pm 22440 = (x^2 \mp 12x + 55)(x^3 \pm 12x^2 + 89x \pm 408),$$
$$x^5 - x \pm 2759640 = (x^2 \pm 12x + 377)(x^3 \mp 12x^2 - 233x \pm 7320).$$

In this paper we investigate the corresponding question for the quintics $x^5 \pm x^a + n$, where $a = 2, 3$, and 4. We show that for $a = 2, 3$ there are only finitely many n for which $x^5 \pm x^a + n$ factors as a product of an irreducible quadratic and an irreducible cubic, whereas, for $a = 4$, rather surprisingly we show that there are infinitely many such n, which can be parameterized using the Fibonacci numbers. Our treatment of the polynomials $x^5 \pm x^a + n$ makes use of the following three results about Fibonacci numbers.

Theorem (Cohn [1], [2]): The only Fibonacci numbers F_k $(k \geq 0)$ that are perfect squares are $F_0 = 0^2$, $F_1 = F_2 = 1^2$, and $F_{12} = 12^2$.

Theorem (London and Finkelstein [3]): The only Fibonacci numbers F_k $(k \geq 0)$ that are perfect cubes are $F_0 = 0^3$, $F_1 = F_2 = 1^3$, and $F_6 = 2^3$.

Theorem (Wasteels [7], May [4]): If x and y are nonzero integers such that $x^2 - xy - y^2 = \varepsilon$, where $\varepsilon = \pm 1$, then there exists a positive integer k such that

$$x = F_{k+1}, \quad y = F_k, \quad \varepsilon = (-1)^k, \quad \text{if } x > 0, y > 0,$$
$$x = F_k, \quad y = -F_{k+1}, \quad \varepsilon = (-1)^{k+1}, \quad \text{if } x > 0, y < 0,$$

$$x = -F_k, \quad y = F_{k+1}, \quad \varepsilon = (-1)^{k+1}, \quad \text{if } x < 0, y > 0,$$
$$x = -F_{k+1}, \quad y = -F_k, \quad \varepsilon = (-1)^k, \quad \text{if } x < 0, y < 0.$$

We remark that the above formulation corrects, and makes more precise, May's extension of Wasteels' theorem. To see that May's result is not correct, take $x = 13$ and $y = -8$ in part (3) of her theorem. Clearly

$$y^2 - xy - x^2 + 1 = (-8)^2 - 13(-8) - 13^2 + 1 = 64 + 104 - 169 + 1 = 0,$$

but there does not exist an integer n such that $13 = F_{n-1}$, $-8 = -F_n$, *or* $13 = -F_{n-1}$, $-8 = F_n$, since $F_{-7} = 13$, $F_{-6} = -8$, $F_6 = 8$, and $F_7 = 13$.

We prove the following results.

Theorem 1: The only integers n for which $x^5 + x^2 + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = -90, -4, 18$, and 11466. The factorizations are

$$x^5 + x^2 - 90 = (x^2 + 4x + 6)(x^3 - 4x^2 + 10x - 15),$$
$$x^5 + x^2 - 4 = (x^2 - x + 2)(x^3 + x^2 - x - 2),$$
$$x^5 + x^2 + 18 = (x^2 + x + 3)(x^3 - x^2 - 2x + 6),$$
$$x^5 + x^2 + 11466 = (x^2 + 4x + 42)(x^3 - 4x^2 - 26x + 273).$$

The only integers n for which $x^5 - x^2 + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = -11466, -18, 4$, and 90. The factorizations are

$$x^5 - x^2 - 11466 = (x^2 - 4x + 42)(x^3 + 4x^2 - 26x - 273),$$
$$x^5 - x^2 - 18 = (x^2 - x + 3)(x^3 + x^2 - 2x - 6),$$
$$x^5 - x^2 + 4 = (x^2 + x + 2)(x^3 - x^2 - x + 2),$$
$$x^5 - x^2 + 90 = (x^2 - 4x + 6)(x^3 + 4x^2 + 10x + 15).$$

Theorem 2: The only integers n for which $x^5 - x^3 + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = \pm 8$. The factorizations are

$$x^5 - x^3 \pm 8 = (x^2 \pm x + 2)(x^3 \mp x^2 - 2x \pm 4).$$

There are no integers n for which $x^5 + x^3 + n$ factors into the product of an irreducible quadratic and an irreducible cubic.

Theorem 3: Apart from the factorizations

$$x^5 + x^4 + 1 = (x^2 + x + 1)(x^3 - x + 1),$$
$$x^5 - x^4 - 1 = (x^2 - x + 1)(x^3 - x - 1),$$

all factorizations of $x^5 \pm x^4 + n$ as a product of an irreducible quadratic and an irreducible cubic with n integral are given by

$$\begin{aligned} x^5 + \theta(-1)^k x^4 + \theta F_{k-1}^2 F_{k+1}^4 F_{k+2}^4 &= (x^2 + \theta F_{k-1} F_{k+2} x + F_{k-1} F_{k+1} F_{k+2}^2) \\ &\quad \times (x^3 - \theta F_k F_{k+1} x^2 - F_{k-1} F_{k+1}^2 F_{k+2} x + \theta F_{k-1} F_{k+1}^2 F_{k+2}^2) \end{aligned} \tag{1.1}$$

and

$$x^5+\theta(-1)^k x^4-\theta F_{k-1}^4 F_k^4 F_{k+2}^2=(x^2+\theta F_{k-1}F_{k+2}x+F_{k-1}^2F_kF_{k+2}) \times(x^3-\theta F_kF_{k+1}x^2+F_{k-1}F_k^2F_{k+2}x-\theta F_{k-1}^2F_k^3F_{k+2}), \tag{1.2}$$

where $\theta=\pm 1$ and k is an integer with $k\geq 2$ and F_k denotes the k^{th} Fibonacci number.

Taking $k=2$ and 3 in Theorem 3, we obtain the factorizations

$$\begin{aligned} x^5\pm x^4\pm 1296&=(x^2\pm 3x+18)(x^3\mp 2x^2-12x\pm 72),\\ x^5\pm x^4\mp 9&=(x^2\pm 3x+3)(x^3\mp 2x^2+3x\mp 3),\\ x^5\pm x^4\mp 50625&=(x^2\mp 5x+75)(x^3\pm 6x^2-45x\mp 675),\\ x^5\pm x^4\pm 400&=(x^2\mp 5x+10)(x^3\pm 6x^2+20x\pm 40).\end{aligned}$$

2. FACTORIZATION OF $x^5 \pm x^2 + n$

Let m and n be integers with $n\neq 0$. Suppose that

$$x^5+mx^2+n=(x^2+ax+b)(x^3+cx^2+dx+e), \tag{2.1}$$

where a, b, c, d, and e are integers. Then, equating coefficients in (2.1), we obtain

$$be=n, \tag{2.2}$$

$$ae+bd=0, \tag{2.3}$$

$$ad+bc+e=m, \tag{2.4}$$

$$b+ac+d=0, \tag{2.5}$$

$$a+c=0. \tag{2.6}$$

From (2.2), as $n\neq 0$, we deduce that

$$b\neq 0, \quad e\neq 0. \tag{2.7}$$

We show next that $a\neq 0$. Suppose, on the contrary, that $a=0$. From (2.3) we see that $bd=0$. Hence, from (2.7), we have $d=0$. Then, from (2.5), we deduce that $b=0$, contradicting (2.7). Hence, we must have

$$a\neq 0. \tag{2.8}$$

Next, we show that $a^2-2b\neq 0$. For, if $a^2-2b=0$, then, from (2.3), (2.5), and (2.6), we deduce that

$$b=a^2/2, \quad c=-a, \quad d=a^2/2, \quad e=-a^3/4. \tag{2.9}$$

Then, from (2.2), (2.4), and (2.9), we have

$$m=-a^3/4, \quad n=-a^5/8. \tag{2.10}$$

From (2.1), (2.9), and (2.10), we obtain the factorization

$$x^5-\frac{a^3}{4}x^2-\frac{a^5}{8}=\left(x^2+ax+\frac{a^2}{2}\right)\left(x^3-ax^2+\frac{a^2}{2}x-\frac{a^3}{4}\right). \tag{2.11}$$

As $-a^3/4\neq\pm 1$, this factorization is not of the required type. Hence, we may suppose that

$$a^2-2b\neq 0. \tag{2.12}$$

Equations (2.3), (2.4), and (2.5) can be written as three linear equations in the three unknowns c, d, and e:

$$\begin{cases} bc + ad + e = m, \\ ac + d \quad\quad = -b, \\ bd + ae \quad\ = 0. \end{cases} \tag{2.13}$$

Solving the system (2.13) for c, d, and e, we have

$$c = \frac{-am + b^2 - a^2 b}{a^3 - 2ab}, \quad d = \frac{a^2 m + ab^2}{a^3 - 2ab}, \quad e = \frac{-abm - b^3}{a^3 - 2ab}. \tag{2.14}$$

Putting these values into (2.6), we obtain

$$a^4 - 3a^2 b + b^2 = am. \tag{2.15}$$

Now let

$$a = a_1 a_2^2, \tag{2.16}$$

where a_1 is a squarefree integer and a_2 is a positive integer. Then (2.15) becomes

$$a_1^4 a_2^8 - 3a_1^2 a_2^4 b + b^2 = a_1 a_2^2 m. \tag{2.17}$$

From (2.17), we see that $a_1 a_2^2 \mid b^2$, so that $a_1 a_2 \mid b$, say,

$$b = a_1 a_2 r, \tag{2.18}$$

where r is a nonzero integer. From (2.17) and (2.18), we deduce that

$$a_1^3 a_2^6 - 3a_1^2 a_2^3 r + a_1 r^2 = m. \tag{2.19}$$

We now suppose that $m = \pm 1$. From (2.19), we see that $a_1 = \pm 1$. Hence, $a_1^2 = 1$ and (2.19) gives

$$a_2^6 - 3a_1 a_2^3 r + r^2 = a_1 m. \tag{2.20}$$

We define integers s (> 0) and t by

$$s = a_2^3, \quad t = r - \tfrac{1}{2}(3a_1 - 1)s. \tag{2.21}$$

From (2.20) and (2.21), we obtain

$$t^2 - st - s^2 = a_1 m. \tag{2.22}$$

First, we deal with the possibility $t = 0$. If $a_1 = 1$, then, from (2.21), we deduce that $r = s$ and, from (2.22), that $-s^2 = m$. Hence, $m = -1$ and $r = s = \pm 1$. Then, by (2.21), we have $a_2 = s = \pm 1$. Hence, by (2.16) and (2.18), we have $a = 1$ and $b = 1$. Then, from (2.14), we get $e = 0$, contradicting (2.7). If $a_1 = -1$, then, from (2.21), we deduce that $r = -2s$ and, from (2.22), that $s^2 = m$. Hence, $m = 1$, $s = \pm 1$, and $r = \mp 2$. Then, by (2.21), we have $a_2 = s = \pm 1$. Next, by (2.16) and (2.18), we have $a = -1$ and $b = 2$. Then, from (2.2) and (2.14), we get $c = 1$, $d = -1$, $e = -2$, $n = -4$, and (2.1) becomes

$$x^5 + x^2 - 4 = (x^2 - x + 2)(x^3 + x^2 - x - 2),$$

which is one of the factorizations listed in Theorem 1.

Now we turn to the case $t \neq 0$. As $t \neq 0$ and $s > 0$, by the theorem of Wasteels and May, there is a positive integer k such that $s = F_k$. Thus, by (2.21), we have $F_k = a_2^3$. Appealing to the theorem of London and Finkelstein, we deduce that $s = F_k = a_2^3 = 1^3$ *or* 2^3, so that $a_2 = 1$ *or* 2. We have eight cases to consider according as $a_1 = 1$ *or* -1, $a_2 = 1$ *or* 2, $m = 1$ *or* -1. In each case, we determine a from (2.16). Then we determine the possible values of r (if any) from the quadratic equation (2.20). Next, we determine b from (2.18). Then the values of c, d, and e are determined from $c = -a$, $d = -b - ac$, and $e = -bd / a$. Finally, n is determined using $n = be$. We obtain the following table:

a_1	a_2	m	a	r	b	c	d	e	n
1	2	1	4	3	6	−4	10	−15	−90
				21	42	−4	−26	273	11466
1	2	−1	4	(none)					
1	1	1	1	0	0	(inadmissible	as	$b \neq 0$)	
				3	3	−1	−2	6	18
1	1	−1	1	1	1	−1	0	0	(inadmissible as $e \neq 0$)
				2	2	−1	−1	2	4
−1	2	1	-4	(none)					
−1	2	−1	−4	−3	6	4	10	15	90
				−21	42	4	−26	−273	−11466
−1	1	1	−1	−1	1	1	0	0	(inadmissible as $e \neq 0$)
				−2	2	1	−1	−2	−4
−1	1	−1	−1	0	0	(inadmissible	as	$b \neq 0$)	
				−3	3	1	−2	−6	−18

These give the eight factorizations listed in the statement of Theorem 1. It is easy to check in each case that the quadratic and cubic factors are irreducible.

3. FACTORIZATION OF $x^5 \pm x^3 + n$

Let m and n be integers with $n \neq 0$. Suppose that

$$x^5 + mx^3 + n = (x^2 + ax + b)(x^3 + cx^2 + dx + e), \tag{3.1}$$

where a, b, c, d, and e are integers. Equating coefficients in (3.1), we obtain

$$be = n, \tag{3.2}$$
$$ae + bd = 0, \tag{3.3}$$
$$ad + bc + e = 0, \tag{3.4}$$
$$b + ac + d = m, \tag{3.5}$$
$$a + c = 0. \tag{3.6}$$

From (3.6), we obtain

$$c = -a. \tag{3.7}$$

As $n \neq 0$, we see from (3.2) that

$$b \neq 0, \quad e \neq 0. \tag{3.8}$$

Suppose that $a = 0$. From (3.7), we have $c = 0$. Then, from (3.4), we deduce that $e = 0$, contradicting (3.8). Hence, we have

$$a \neq 0. \tag{3.9}$$

Suppose next that $b = a^2$. Then, from (3.5) and (3.7), we deduce that $d = m$. Then, from (3.4), we obtain $e = a^3 - am$. Next, (3.3) gives $a = 0$, contradicting (3.9). Thus, we have

$$b \neq a^2. \tag{3.10}$$

Using (3.7) in (3.4), we obtain

$$ad + e = ab. \tag{3.11}$$

Solving (3.3) and (3.11) for d and e, we find that

$$d = \frac{-a^2 b}{b - a^2}, \quad e = \frac{ab^2}{b - a^2}. \tag{3.12}$$

From (3.2) and (3.5), we deduce that

$$m = \frac{a^4 - 3a^2 b + b^2}{b - a^2}, \quad n = \frac{ab^3}{b - a^2}. \tag{3.13}$$

We define the nonzero integer h by

$$h = b - a^2. \tag{3.14}$$

Then, from (3.7), (3.12), (3.13), and (3.14), we obtain

$$\begin{aligned} b &= a^2 + h, & e &= \frac{a^5}{h} + 2a^3 + ah, \\ c &= -a, & m &= -\frac{a^4}{h} - a^2 + h, \\ d &= -\frac{a^4}{h} - a^2, & n &= \frac{a^7}{h} + 3a^5 + 3a^3 h + ah^2. \end{aligned}$$

These values of b, c, d, e, m, and n satisfy the equations (3.2)-(3.6). The equation for m can be rewritten as $h^2 - (a^2 + m)h - a^4 = 0$. Solving this quadratic equation for h, we obtain

$$h = \tfrac{1}{2}\left(a^2 + m + \varepsilon\sqrt{(a^2 + m)^2 + 4a^4}\right), \tag{3.15}$$

where $\varepsilon = \pm 1$. Relation (3.15) shows that $\sqrt{(a^2 + m)^2 + 4a^4}$ is an integer, namely, $\varepsilon(2h - a^2 - m)$. Hence, there is an integer w such that

$$(a^2 + m)^2 + (2a^2)^2 = w^2. \tag{3.16}$$

From (3.9) and (3.16), we see that $w \neq 0$. As $\{a^2 + m, 2a^2, w\}$ is a Pythagorean triple, there exist integers r, s, and t with $\gcd(r, s) = 1$ such that

$$a^2 + m = 2rst, \; 2a^2 = (r^2 - s^2)t, \; w = (r^2 + s^2)t \tag{3.17}$$

or

$$a^2 + m = (r^2 - s^2)t, \; 2a^2 = 2rst, \; w = (r^2 + s^2)t. \tag{3.18}$$

We assume now that $m = \pm 1$.

If (3.17) holds, then $(r^2 - 4rs - s^2)t = 2a^2 - 2(a^2 + m) = -2m = \pm 2$, so that $t = \pm 1$ or $t = \pm 2$. If $t = \pm 1$, then $r^2 - 4rs - s^2 = \pm 2$, so that $r^2 - s^2 \equiv 2 \pmod 4$, which is impossible, since $r^2 - s^2 \equiv$ 0, 1, or 3 (mod 4). Hence, $t = \pm 2$ and

$$r^2 - 4rs - s^2 = \pm 1. \tag{3.19}$$

From (3.19), we see that $r + s$ and $r - s$ are both odd integers so that, in particular, we have $r + s \neq 0$ and $r - s \neq 0$. Moreover, from (3.19), we have $(r + s)^2 - (r + s)(r - s) - (r - s)^2 = \pm 1$.

Therefore, by the theorem of Wasteels and May, there are positive integers k and l such that $|r+s| = F_k$ and $|r-s| = F_l$. Now, by (3.17), we have

$$a^2 = |r+s||r-s|. \tag{3.20}$$

As $|r+s|$ and $|r-s|$ are both odd, and $\gcd(r, s) = 1$, we have

$$\gcd(|r+s|, |r-s|) = 1. \tag{3.21}$$

From (3.20) and (3.21), we deduce that each of $|r+s|$ and $|r-s|$ is a perfect square. Hence, F_k and F_l are both perfect squares so, by Cohn's theorem, we have $|r+s| = F_k = 1$ or 144, $|r-s| = F_l = 1$ or 144. However, $|r+s|$ and $|r-s|$ are both odd, so $|r+s| = 1$ and $|r-s| = 1$. Therefore, $(r, s) = (\pm 1, 0)$ or $(0, \pm 1)$. Hence, by (3.17), we have $a^2 + m = 2rst = 0$, and, as $m = \pm 1$, we have $m = -1$, $a = \theta$, where $\theta = \pm 1$. From (3.15), we deduce that $h = \varepsilon$; thus, $a = \theta$, $b = 1 + \varepsilon$, $c = -\theta$, $d = -(1+\varepsilon)$, $e = 2\theta(1+\varepsilon)$, $m = -1$, and $n = 4\theta(1+\varepsilon)$. Since $b \neq 0$, we must have $\varepsilon = 1$. Thus, $a = \theta, b = 2, c = -\theta, d = -2, e = 4\theta, m = -1, n = 8\theta$, which gives the factorization

$$x^5 - x^3 + 8\theta = (x^2 + \theta x + 2)(x^3 - \theta x^2 - 2x + 4\theta),\ \theta = \pm 1.$$

If (3.18) holds, then

$$(r^2 - rs - s^2)t = (r^2 - s^2)t - rst = (a^2 + m) - a^2 = m,$$

so that $t = \pm 1$, $r^2 - rs - s^2 = mt$. If r or $s = 0$, then, by (3.18), we have $a = 0$, contradicting (3.9). Hence, $r \neq 0$ and $s \neq 0$. Then, by the theorem of Wasteels and May, we have $|r| = F_k$, $|s| = F_l$, for positive integers k and l. Now

$$a^2 = rst = |r||s|,\ \gcd(|r|, |s|) = 1,$$

so each of $|r|$ and $|s|$ is a perfect square. Thus, both F_k and F_l are perfect squares. Hence, by Cohn's theorem, we have $|r| = F_k = 1$ or 144 and $|s| = F_l = 1$ or 144. Therefore, $r = \pm 1, \pm 144$ and $s = \pm 1, \pm 144$.

From $r^2 - rs - s^2 = mt$, we deduce that

$$\begin{array}{ll} (\alpha) & r = 1,\ s = 1,\ mt = -1, \text{ or} \\ (\beta) & r = 1,\ s = -1,\ mt = 1, \text{ or} \\ (\gamma) & r = -1,\ s = 1,\ mt = 1, \text{ or} \\ (\delta) & r = -1,\ s = -1,\ mt = -1. \end{array}$$

Then, from $a^2 = rst$ we deduce that

$$\begin{array}{ll} (\alpha) & t = 1,\ m = -1,\ a = \theta, \\ (\beta) & t = -1,\ m = -1,\ a = \theta, \\ (\gamma) & t = -1,\ m = -1,\ a = \theta, \\ (\delta) & t = 1,\ m = -1,\ a = \theta, \end{array}$$

where $\theta = \pm 1$. In all four cases, $a^2 + m = 0$ so that, by (3.15), $h = \varepsilon$. Thus, $b = a^2 + h = 1 + \varepsilon$. But $b \neq 0$, so $\varepsilon \neq -1$, that is, $\varepsilon = 1$. Hence, $a = \theta$, $b = 2$, $c = -\theta$, $d = -2$, $e = 4\theta$, $m = -1$, $n = 8\theta$, which gives the same factorization as before. Since $x^2 + \theta x + 2$ and $x^3 - \theta x^2 - 2x + 4\theta$ are both irreducible, this completes the proof of Theorem 2.

4. FACTORIZATION OF $x^5 \pm x^4 + n$

Let m and n be integers with $n \neq 0$. Suppose that

$$x^5 + mx^4 + n = (x^2 + ax + b)(x^3 + cx^2 + dx + e), \tag{4.1}$$

where a, b, c, d, and e are integers. Equating coefficients in (4.1), we obtain

$$be = n, \tag{4.2}$$
$$ae + bd = 0, \tag{4.3}$$
$$ad + bc + e = 0, \tag{4.4}$$
$$b + ac + d = 0, \tag{4.5}$$
$$a + c = m. \tag{4.6}$$

As $n \neq 0$ we have, from (4.2),

$$b \neq 0, \quad e \neq 0. \tag{4.7}$$

We show next that $a \neq 0$. Suppose $a = 0$. Then, by (4.3) and (4.7), we have $d = 0$. From (4.5), we deduce that $b = 0$, contradicting (4.7). Hence,

$$a \neq 0. \tag{4.8}$$

Suppose next that $b = a^2/2$. Then, from (4.3) and (4.8) we obtain $e = -ad/2$. Next, from (4.4) and (4.8), we deduce that $d = -ac$. Then (4.5) gives $b = 0$, contradicting (4.7). Hence,

$$b \neq a^2/2. \tag{4.9}$$

If $a = m$ then, from (4.6), we have $c = 0$. Then (4.5) gives $d = -b$. Next, (4.4) gives $e = bm$. Now (4.3) and (4.7) give $b = m^2$, so that $e = m^3$ and $d = -m^2$. Finally, from (4.2), we obtain $n = m^5$. Thus, (4.1) becomes

$$x^5 + mx^4 + m^5 = (x^2 + mx + m^2)(x^3 - m^2x + m^3).$$

With $m = \pm 1$ we have

$$x^5 + mx^4 + m = (x^2 + mx + 1)(x^3 - x + m).$$

It is easy to check that $x^2 + mx + 1$ and $x^3 - x + m$ are irreducible for $m = \pm 1$.

Thus, we may suppose from now on that $a \neq m$. Replacing x by $-x$ in (4.1), we obtain the factorization

$$x^5 - mx^4 - n = (x^2 - ax + b)(x^3 - cx^2 + dx - e).$$

Thus, in view of (4.8), we may suppose without loss of generality that $a > 0$. Solving (4.3), (4.4), and (4.5) for c, d, and e, we obtain

$$c = \frac{-b(a^2 - b)}{a(a^2 - 2b)}, \tag{4.10}$$

$$d = \frac{b^2}{a^2 - 2b}, \tag{4.11}$$

$$e = \frac{-b^3}{a(a^2 - 2b)}. \tag{4.12}$$

Then, from (4.6) and (4.2), we deduce that

$$m = \frac{a^4 - 3a^2b + b^2}{a(a^2 - 2b)}, \quad n = \frac{-b^4}{a(a^2 - 2b)}. \tag{4.13}$$

Assume now that $m = \pm 1$. Writing the equation for m in (4.13) as a quadratic equation in b, we have

$$b^2 + a(2m - 3a)b + a^3(a - m) = 0.$$

Solving for b, we find that

$$b = \frac{a}{2}(3a - 2m + \varepsilon\sqrt{5a^2 - 8ma + 4}), \tag{4.14}$$

where $\varepsilon = \pm 1$. The equation (4.14) shows that $z = +\sqrt{5a^2 - 8ma + 4}$ is a nonnegative rational number. As a and m are integers, z must be a nonnegative integer such that $5a^2 - 8ma + 4 = z^2$, that is,

$$a^2 + (2a - 2m)^2 = z^2. \tag{4.15}$$

As $a \neq 0$ and $a \neq m$, we have $z \geq 2$, and there exist nonzero integers r, s, and t with $\gcd(r, s) = 1$ such that

$$a = (r^2 - s^2)t, \; 2a - 2m = 2rst, \; z = (r^2 + s^2)t \tag{4.16}$$

or

$$a = 2rst, \; 2a - 2m = (r^2 - s^2)t, \; z = (r^2 + s^2)t. \tag{4.17}$$

Clearly, as $z > 0$, we have $t > 0$. Replacing (r, s) by $(-r, -s)$, if necessary, we may suppose that $r > 0$.

We suppose first that (4.16) holds. Then

$$(r^2 - rs - s^2)t = (r^2 - s^2)t - rst = a - (a - m) = m.$$

Now $m = \pm 1$, so $t = 1$ and $r^2 - rs - s^2 = m$. Appealing to the theorem of Wasteels and May, we have

$$\begin{aligned} r &= F_{k+1}, \; s = F_k, \quad && m = (-1)^k, && \text{if } s > 0, \\ r &= F_k, \quad s = -F_{k+1}, && m = (-1)^{k+1}, && \text{if } s < 0, \end{aligned}$$

for some positive integer k. Then, from (4.16), we obtain

$$a = r^2 - s^2 = \begin{cases} F_{k+1}^2 - F_k^2 = F_{k-1}F_{k+2}, & \text{if } s > 0, \\ F_k^2 - F_{k+1}^2 = -F_{k-1}F_{k+2}, & \text{if } s < 0. \end{cases}$$

As $a > 0$, we must have $s > 0$ so that $r = F_{k+1}$, $s = F_k$, $m = (-1)^k$ and

$$a = F_{k-1}F_{k+2}. \tag{4.18}$$

Further, from (4.16), we have

$$z = r^2 + s^2 = F_k^2 + F_{k+1}^2 = F_kF_{k+2} + F_{k-1}F_{k+1} \tag{4.19}$$

and

$$a - m = rs = F_kF_{k+1}. \tag{4.20}$$

Also, as $a > 0$, we have $F_{k-1} \neq 0$ so $k \neq 1$ and thus $k \geq 2$. From (4.14), we have

$$\begin{aligned} b &= \frac{a}{2}(a+2(a-m)+\varepsilon z) \\ &= \begin{cases} (1/2)F_{k-1}F_{k+2}(F_{k-1}F_{k+2}+2F_kF_{k+1}+F_kF_{k+2}+F_{k-1}F_{k+1}), & \text{if } \varepsilon = 1, \\ (1/2)F_{k-1}F_{k+2}(F_{k-1}F_{k+2}+2F_kF_{k+1}-F_kF_{k+2}-F_{k-1}F_{k+1}), & \text{if } \varepsilon = -1, \end{cases} \\ &= \begin{cases} F_{k-1}F_{k+1}F_{k+2}^2, & \text{if } \varepsilon = 1, \\ F_{k-1}^2F_kF_{k+2}, & \text{if } \varepsilon = -1. \end{cases} \end{aligned}$$

Thus,

$$a^2-2b = \begin{cases} F_{k-1}^2F_{k+2}^2 - 2F_{k-1}F_{k+1}F_{k+2}^2 = -F_{k-1}F_{k+2}^3, & \text{if } \varepsilon = 1, \\ F_{k-1}^2F_{k+2}^2 - 2F_{k-1}^2F_kF_{k+2} = F_{k-1}^3F_{k+2}, & \text{if } \varepsilon = -1, \end{cases}$$

and

$$a^2-b = \begin{cases} F_{k-1}^2F_{k+2}^2 - F_{k-1}F_{k+1}F_{k+2}^2 = -F_{k-1}F_kF_{k+2}^2, & \text{if } \varepsilon = 1, \\ F_{k-1}^2F_{k+2}^2 - F_{k-1}^2F_kF_{k+2} = F_{k-1}^2F_{k+1}F_{k+2}, & \text{if } \varepsilon = -1. \end{cases}$$

Then, from (4.10), (4.11), and (4.12), we obtain

$$c = -F_kF_{k+1}, \quad \text{if } \varepsilon = \pm 1,$$

$$d = \begin{cases} -F_{k-1}F_{k+1}^2F_{k+2}, & \text{if } \varepsilon = 1, \\ F_{k-1}F_k^2F_{k+2}, & \text{if } \varepsilon = -1, \end{cases}$$

$$e = \begin{cases} F_{k-1}F_{k+1}^3F_{k+2}^2, & \text{if } \varepsilon = 1, \\ -F_{k-1}^2F_k^3F_{k+2}, & \text{if } \varepsilon = -1. \end{cases}$$

From (4.13), we get

$$n = \begin{cases} F_{k-1}^2F_{k+1}^4F_{k+2}^4, & \text{if } \varepsilon = 1, \\ -F_{k-1}^4F_k^4F_{k+2}^2, & \text{if } \varepsilon = -1. \end{cases}$$

Then (4.1) gives the factorizations

$$\begin{aligned} x^5+(-1)^kx^4+F_{k-1}^2F_{k+1}^4F_{k+2}^4 &= (x^2+F_{k-1}F_{k+2}x+F_{k-1}F_{k+1}F_{k+2}^2) \\ &\quad \times (x^3-F_kF_{k+1}x^2-F_{k-1}F_{k+1}^2F_{k+2}x+F_{k-1}F_{k+1}^3F_{k+2}^2), \\ x^5+(-1)^kx^4-F_{k-1}^4F_k^4F_{k+2}^2 &= (x^2+F_{k-1}F_{k+2}x+F_{k-1}^2F_kF_{k+2}) \\ &\quad \times (x^3-F_kF_{k+1}x^2+F_{k-1}F_k^2F_{k+2}x-F_{k-1}^2F_k^3F_{k+2}), \end{aligned}$$

and two more obtained by changing x to $-x$. These are the factorizations given in the statement of Theorem 3.

We now suppose that (4.17) holds. Then

$$(r^2-4rs-s^2)t = (r^2-s^2)t - 4rst = (2a-2m) - 2a = -2m.$$

As $m = \pm 1$ and $t > 0$, we have $t = 1$ or $t = 2$. If $t = 1$, then

$$r^2 - 4rs - s^2 = -2m. \tag{4.21}$$

Hence, $r \equiv r^2 \equiv s^2 \equiv s \pmod 2$. But $\gcd(r, s) = 1$, so $r \equiv s \equiv 1 \pmod 2$. Then $r^2 - s^2 \equiv 0$ (mod 4), which contradicts (4.21). Therefore, we must have $t = 2$, in which case $r^2 - 4rs - s^2 = -m$, so that

$$(2r)^2 - (2r)(r+s) - (r+s)^2 = -m.$$

As $a > 0$, $r > 0$, and $t > 0$, we see from (4.17) that $s > 0$. Thus, $2r$ and $r+s$ are positive integers, and so, by the theorem of Wasteels and May, we have

$$2r = F_{k+1}, \quad r+s = F_k, \quad -m = (-1)^k,$$

for some positive integer k. Thus,

$$r = \tfrac{1}{2}F_{k+1}, \quad s = \tfrac{1}{2}F_{k-2}, \quad m = (-1)^{k+1}.$$

As $s \neq 0$, we see that $k \neq 2$. Now $2|F_h \Leftrightarrow 3|h$ (see [6], p. 32), so as r and s are integers, we have

$$r = \tfrac{1}{2}F_{3l+3}, \quad s = \tfrac{1}{2}F_{3l}, \quad m = (-1)^{l+1},$$

for some integer $l \geq 1$. Hence, by (4.17), we have

$$a = F_{3l}F_{3l+3}, \tag{4.22}$$

$$z = \tfrac{1}{2}(F_{3l}^2 + F_{3l+3}^2) = F_{3l+1}F_{3l+3} + F_{3l}F_{3l+2}, \tag{4.23}$$

and

$$a - m = \tfrac{1}{4}(F_{3l+3}^2 - F_{3l}^2) = F_{3l+1}F_{3l+2}. \tag{4.24}$$

Comparing (4.22), (4.23), and (4.24) to (4.18), (4.19), and (4.20), respectively, we see that the possibility (4.17) just leads to a special case $k = 3l+1$ $(l \geq 1)$ of the previous case and, therefore, does not lead to any new factorizations.

The discriminant of $x^2 + \theta F_{k-1}F_{k+2}x + F_{k-1}F_{k+1}F_{k+2}^2$ is

$$\begin{aligned} F_{k-1}^2F_{k+2}^2 - 4F_{k-1}F_{k+1}F_{k+2}^2 &= F_{k-1}F_{k+2}^2(F_{k-1} - 4F_{k+1}) \\ &= -F_{k-1}F_{k+2}^2(3F_{k-1} + 4F_k), \end{aligned}$$

which is negative for $k \geq 2$. Hence, $x^2 + \theta F_{k-1}F_{k+2}x + F_{k-1}F_{k+1}F_{k+2}^2$ is irreducible. Similarly, the discriminant of $x^2 + \theta F_{k-1}F_{k+2}x + F_{k-1}^2F_kF_{k+2}$ is

$$\begin{aligned} F_{k-1}^2F_{k+2}^2 - 4F_{k-1}^2F_kF_{k+2} &= F_{k-1}^2F_{k+2}(F_{k+2} - 4F_k) \\ &= F_{k-1}^2F_{k+2}(F_{k+1} - 3F_k) \\ &= F_{k-1}^2F_{k+2}(F_{k-1} - 2F_k) \\ &= -F_{k-1}^2F_{k+2}(F_{k-1} + 2F_{k-2}), \end{aligned}$$

which is negative for $k \geq 2$. Thus, $x^2 + \theta F_{k-1}F_{k+2}x + F_{k-1}^2F_kF_{k+2}$ is irreducible. To complete the proof of Theorem 3, it remains to show that the cubic polynomials

$$x^3 - \theta F_kF_{k+1}x^2 - F_{k-1}F_{k+1}^2F_{k+2}x + \theta F_{k-1}F_{k+1}^3F_{k+2}^2$$

and

$$x^3 - \theta F_kF_{k+1}x^2 + F_{k-1}F_k^2F_{k+2}x - \theta F_{k-1}^2F_k^3F_{k+2}$$

are irreducible over the rational field Q for $k \geq 2$ and $\theta = \pm 1$. This is done in the next section. It clearly suffices to treat only the case $\theta = 1$.

5. IRREDUCIBILITY OF TWO CUBIC POLYNOMIALS

In this section we prove that the two cubic polynomials

$$f(x) = x^3 - F_k F_{k+1} x^2 - F_{k-1} F_{k+1}^2 F_{k+2} x + F_{k-1} F_{k+1}^3 F_{k+2}^2 \tag{5.1}$$

and

$$g(x) = x^3 - F_k F_{k+1} x^2 + F_{k-1} F_k^2 F_{k+2} x - F_{k-1}^2 F_k^3 F_{k+2} \tag{5.2}$$

are irreducible over the rationals for $k \geq 2$. Before proving this (see Theorem 4 below), we prove three lemmas.

Lemma 1: If N is a nonzero integer, then the quintic equation $x^5 + x^4 + N = 0$ has exactly one real root.

Proof: The function $F(x) = x^5 + x^4 + N$ has a local maximum at $x = -4/5$ and a local minimum at $x = 0$. There are no other local maxima or local minima. Clearly, $F(-4/5) = N + 4^4/5^5$ and $F(0) = N$. As N is a nonzero integer, we cannot have $N \leq 0 \leq N + 4^4/5^5$. Hence, either $N > 0$ or $N + 4^4/5^5 < 0$. If $N > 0$, the curve $y = F(x)$ meets the x-axis at exactly one point x_0 $(x_0 < -4/5)$. If $N + 4^4/5^5 < 0$, the curve $y = F(x)$ meets the x-axis at exactly one point x_1 $(x_1 > 0)$. Hence, $F(x) = 0$ has exactly one real root.

Lemma 2: For $k \geq 2$, each of the quintic polynomials

$$A(x) = x^5 + (-1)^k x^4 + F_{k-1}^2 F_{k+1}^4 F_{k+2}^4 \tag{5.3}$$

and

$$B(x) = x^5 + (-1)^k x^4 - F_{k-1}^4 F_k^4 F_{k+2}^2 \tag{5.4}$$

has exactly one real root.

Proof: As $k \geq 2$, $(-1)^k F_{k-1}^2 F_{k+1}^4 F_{k+2}^4$ is a nonzero integer. Hence, by Lemma 1, the quintic polynomial $Q(y) = y^5 + y^4 + (-1)^k F_{k-1}^2 F_{k+1}^4 F_{k+2}^4$ has exactly one real root. Thus, the quintic polynomial $A(x) = (-1)^k Q((-1)^k x)$ has exactly one real root. The quintic polynomial $B(x)$ can be treated similarly.

Lemma 3: For $k \geq 2$, each of the cubic polynomials $f(x)$ and $g(x)$ has exactly one real root.

Proof: From (1.1), (1.2), (5.1), (5.2), (5.3), and (5.4), we have

$$A(x) = (x^2 + F_{k-1} F_{k+2} x + F_{k-1} F_{k+1} F_{k+2}^2) f(x) \quad \text{and} \quad B(x) = (x^2 + F_{k-1} F_{k+2} x + F_{k-1}^2 F_k F_{k+2}) g(x).$$

Since the two quadratics have no real roots, the result follows from Lemma 2.

Theorem 4: For $k \geq 2$ the cubic polynomials $f(x)$ and $g(x)$ are irreducible over the rationals.

Proof: Suppose $f(x)$ is reducible over the rationals. Then, by Lemma 3, $f(x)$ has exactly one real root, which must be rational and, in fact, an integer. Thus,

$$f_1(x) = \frac{1}{F_{k+1}^3} f(F_{k+1} x) = x^3 - F_k x^2 - F_{k-1} F_{k+2} x + F_{k-1} F_{k+2}^2$$

has exactly one real root, which must be an integer. Hence,

$$f_2(x) = f_1(x - F_{k+1}) = x^3 - (3F_{k+1} + F_k)x^2 + F_{2k+3}x + (-1)^k F_{k+2}$$

has exactly one real root r, which must be an integer. If k is even, then $f_2(0) = F_{k+2} > 0$ and

$$f_2(-1) = -1 - 3F_{k+1} - F_k - F_{2k+3} + F_{k+2} < F_{k+2} - F_{2k+3} < 0,$$

so $-1 < r < 0$, which is impossible. If k is odd, then $f_2(0) = -F_{k+2} < 0$ and

$$\begin{aligned} f_2(1) &= 1 - 3F_{k+1} - F_k + F_{2k+3} - F_{k+2} \\ &= 1 + (F_{2k+1} - F_{k+3}) + (F_{2k+2} - F_{k+3}) \geq 1 > 0, \end{aligned}$$

so $0 < r < 1$, which is impossible. Hence, $f(x)$ is irreducible over Q.

We now turn to $g(x)$. Suppose $g(x)$ is reducible over Q. Then, by Lemma 3, $g(x)$ has exactly one real root, which must be rational and, in fact, integral. Thus,

$$g_1(x) = \frac{1}{F_k^3} g(F_k x) = x^3 - F_{k+1}x^2 + F_{k-1}F_{k+2}x - F_{k-1}^2 F_{k+2}$$

has exactly one real root, which must be an integer. Therefore,

$$g_2(x) = g_1(x + F_k) = x^3 + (F_k + F_{k-2})x^2 + F_{2k-1}x + (-1)^{k-1} F_{k-1}$$

has exactly one real root s, which must be an integer. If k is even, then $g_2(0) = -F_{k-1} < 0$ and

$$\begin{aligned} g_2(1) &= 1 + F_k + F_{k-2} + F_{2k-1} - F_{k-1} \\ &\geq 1 + F_{k-2} + F_{2k-1} > 0, \end{aligned}$$

so that $0 < s < 1$, which is impossible. If k is odd, then $g_2(0) = F_{k-1} > 0$ and

$$\begin{aligned} g_2(-1) &= -1 + F_k + F_{k-2} - F_{2k-1} + F_{k-1} \\ &= -1 + 2F_k - 2F_{2k-3} - F_{2k-4} \\ &\leq -1 - 2(F_{2k-3} - F_k) \leq -1 < 0, \end{aligned}$$

so that $-1 < s < 0$, which is impossible. Hence, $g(x)$ is irreducible over Q.

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PALINDROMIC SEQUENCES FROM IRRATIONAL NUMBERS

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In this paper, a *palindrome* is a finite sequence $(x(1), x(2), \ldots, x(n))$ of numbers satisfying $(x(1), x(2), \ldots, x(n)) = (x(n), x(n-1), \ldots, x(1))$. Of course, an infinite sequence cannot be a palindrome—however, we shall call an infinite sequence $x = (x(1), x(2), \ldots)$ a *palindromic sequence* if for every N there exists $n > N$ such that the finite sequence $(x(1), x(2), \ldots, x(n))$ is a palindrome. If α is an irrational number, then the sequence Δ defined by $\Delta(n) = \lfloor n\alpha \rfloor - \lfloor n\alpha - \alpha \rfloor$ is, we shall show, palindromic.

Lemma: Suppose $\sigma = (\sigma(0), \sigma(1), \sigma(2), \ldots)$ is a sequence of numbers, and $\sigma(0) = 0$. Let Δ be the sequence defined by $\Delta(n) = \sigma(n) - \sigma(n-1)$ for $n = 1, 2, 3, \ldots$. Then Δ is a palindromic sequence if and only if there are infinitely many n for which the equations

$$F_{n,k}: \ \sigma(k) + \sigma(n-k) = \sigma(n) \tag{1}$$

hold for $k = 1, 2, \ldots, n$.

Proof: Equations $F_{n,k}$ and $F_{n,k-1}$ yield $\sigma(k) + \sigma(n-k) = \sigma(n) = \sigma(k-1) + \sigma(n-k+1)$, so that the equations

$$E_{n,k}: \ \sigma(k) - \sigma(k-1) = \sigma(n-k+1) - \sigma(n-k) \tag{2}$$

or

$$\Delta(k) = \Delta(n-k+1)$$

follow, for $k = 1, 2, \ldots, n$. Thus, if the n equations (1) hold for infinitely many n, then Δ is a palindromic sequence.

For the converse, suppose n is a positive integer for which the equations $E_{n,k}$ in (2) hold. The equations $E_{n,1}$, $E_{n,1} + E_{n,2}$, $E_{n,1} + E_{n,2} + E_{n,2}$, $\ldots$, $E_{n,1} + E_{n,2} + \cdots + E_{n,n}$ readily reduce to the equations $F_{n,k}$. Thus, if Δ is palindromic, then the equations $F_{n,k}$, for $k = 1, 2, \ldots, n$, hold for infinitely many n. □

To see how a positive irrational number α can be used to generate palindromic sequences, we recall certain customary notations from the theory of continued fractions. Suppose α has continued fraction $[\![a_0, a_1, a_2, \ldots]\!]$, and let $p_{-2} = 0, p_{-1} = 1, p_i = a_i p_{i-1} + p_{i-2}$ and $q_{-2} = 1, q_{-1} = 0$, $q_i = a_i q_{i-1} + q_{i-2}$ for $i \geq 0$. The *principal convergents of* α are the rational numbers p_i / q_i for $i \geq 0$. Now, for all nonnegative integers i and j, define $p_{i,j} = jp_{i+1} + p_i$ and $q_{i,j} = jq_{i+1} + q_i$. The fractions

$$\frac{p_{i,j}}{q_{i,j}} = \frac{jp_{i+1} + p_i}{jq_{i+1} + q_i}, \qquad 1 \leq j \leq a_{i+2} - 1, \tag{3}$$

are the i^{th} *intermediate convergents of* α. As proved in [2, p. 16],

$$\cdots < \frac{p_i}{q_i} < \cdots < \frac{p_{i,j}}{q_{i,j}} < \frac{p_{i,j+1}}{q_{i,j+1}} < \cdots < \frac{p_{i+2}}{q_{i+2}} < \cdots \text{ if } i \text{ is even,} \tag{4}$$

$$\cdots > \frac{p_i}{q_i} > \cdots > \frac{p_{i,j}}{q_{i,j}} > \frac{p_{i,j+1}}{q_{i,j+1}} > \cdots > \frac{p_{i+2}}{q_{i+2}} > \cdots \text{ if } i \text{ is odd,} \tag{5}$$

and $p_{i,j-1}q_{ij} - p_{ij}q_{i,j-1} = (-1)^j$ for $i = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, a_{i+2} - 1$. If the range of j in (3) is extended to $0 \le j \le a_{i+2} - 1$, then the principal convergents are included among the intermediate convergents. We shall refer to both kinds simply as *convergents*—those in (4) as *even-indexed convergents* and those in (5) as *odd-indexed convergents*.

We shall use the notation (( )) for the fractional-part function, defined by $((x)) = x - \lfloor x \rfloor$.

Theorem 1: Suppose p/q is a convergent to a positive irrational number α. Then for $k = 1, 2, \ldots, q-1$, the sum $((k\alpha)) + (((q-k)\alpha))$ is invariant of k; in fact,

$$((k\alpha)) + (((q-k)\alpha)) = \begin{cases} ((q\alpha)) + 1 & \text{if } p/q \text{ is an even-indexed convergent,} \\ ((q\alpha)) & \text{if } p/q \text{ is an odd-indexed convergent.} \end{cases}$$

Proof: Suppose p/q is an even-indexed convergent and $1 \le k \le q-1$. Then $p/q < \alpha$, so that

$$kp/q < k\alpha. \tag{6}$$

Suppose there is an integer h such that $kp/q \le h < k\alpha$. Then

$$p/q < h/k < \alpha. \tag{7}$$

However, as an even-indexed convergent to α, the rational number p/q is the best lower approximate (as defined in [1]), which means that $k \ge q$ in (7). This contradiction to the hypothesis, together with (6), shows that

$$((kp/q)) < ((k\alpha)). \tag{8}$$

Since $1 \le q-k \le q-1$, we also have $1 = ((kp/q)) + (((q-k)p/q)) < ((k\alpha)) + (((q-k)\alpha))$. Since $((k\alpha)) + (((q-k)\alpha))$ has the same fractional part as $q\alpha$, we conclude that

$$((k\alpha)) + (((q-k)\alpha)) = ((q\alpha)) + 1.$$

The proof for odd-indexed convergents p/q is similar and omitted. □

Theorem 2: Suppose $\Delta(n) = \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor$ for some positive irrational number, for $n = 1, 2, 3, \ldots$. Then Δ is a palindromic sequence.

Proof: By Theorem 1, if p/q is an odd-indexed convergent to α, then

$$((k\alpha)) + (((q-k)\alpha)) = ((q\alpha)) \text{ for } k = 1, 2, \ldots, q-1,$$

and clearly this holds for $k = q$, also. Consequently,

$$\lfloor k\alpha \rfloor + \lfloor (q-k)\alpha \rfloor = \lfloor q\alpha \rfloor,$$
$$\sigma(k) + \sigma(q-k) = \sigma(q),$$

for $k = 1, 2, \ldots, q$. By the lemma, Δ is a palindromic sequence. □

Example 1: There is only one positive irrational number for which all the convergents are principal convergents, shown here along with its continued fraction:

$$\alpha = (1+\sqrt{5})/2 = [\![1, 1, 1, \ldots]\!].$$

The convergents are quotients of consecutive Fibonacci numbers, and the sequence σ given by $\sigma(n) = \lfloor n\alpha \rfloor$ begins with 0, 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, 27, 29, 30, 32, 33, 35, 37, 38, 40, so that the difference sequence Δ begins with 1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2. The sequence Δ is palindromic, since $(\Delta(1), \ldots, \Delta(q))$ is a palindrome for

$$q \in \{1, 3, 8, 21, 55, 144, 377, 987, \ldots\}.$$

Moreover, $(\Delta(2), \ldots, \Delta(q-1))$ is a palindrome for

$$q \in \{2, 5, 13, 34, 89, 233, 610, \ldots\}.$$

In both cases, Fibonacci numbers abound.

Example 2: For $\alpha = e$, approximately 2.718281746, the continued fraction is

$$[\![2,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,1,1,\ldots]\!],$$

and the first twenty convergents (both principal and intermediate) are:

$$\begin{array}{rr}
2/1 = p_{00}/q_{00} & 106/39 = p_{60}/q_{60} \\
3/1 = p_{10}/q_{10} & 193/71 = p_{70}/q_{70} \\
5/2 = p_{01}/q_{01} & 299/110 = p_{61}/q_{61} \\
8/3 = p_{20}/q_{20} & 492/181 = p_{62}/q_{62} \\
11/4 = p_{30}/q_{30} & 685/252 = p_{63}/q_{63} \\
19/7 = p_{40}/q_{40} & 878/323 = p_{64}/q_{64} \\
30/11 = p_{31}/q_{31} & 1071/394 = p_{65}/q_{65} \\
49/18 = p_{32}/q_{32} & 1264/465 = p_{80}/q_{80} \\
68/25 = p_{33}/q_{33} & 1457/536 = p_{90}/q_{90} \\
87/32 = p_{50}/q_{50} & 2721/1001 = p_{81}/q_{81}
\end{array}$$

Here, Δ begins with 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, and $(\Delta(1), \ldots, \Delta(q))$ is a palindrome for

$$q \in \{1, 4, 11, 18, 25, 32, 71, 536, \ldots\},$$

and $(\Delta(2), \ldots, \Delta(q-1))$ is a palindrome for

$$q \in \{2, 3, 7, 39, 110, 181, 252, 323, 394, 465, 1001, \ldots\}.$$

Opportunities: The foregoing theorems and examples suggest the problem of describing *all* the palindromes within the difference sequence Δ given by $\Delta(n) = \lfloor n\alpha \rfloor - \lfloor n\alpha - \alpha \rfloor$ for irrational α. One might then investigate what happens when $n\alpha$ is replaced by $n\alpha + \beta$, where β is a real number.

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ON THE INTEGRITY OF CERTAIN INFINITE SERIES

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1. INTRODUCTION

We consider the second-order recurring relation

$$W_0 = a,\ W_1 = b,\ W_n = PW_{n-1} - QW_{n-2} \quad (n \ge 2), \tag{1.1}$$

where a, b, P, and Q are integers, with $P > 0$, $Q \ne 0$, and $\Delta = P^2 - 4Q > 0$. Particular cases of $\{W_n\}$ are the sequences $\{U_n\} = \{U_n(P, Q)\}$ of Fibonacci and $\{V_n\} = \{V_n(P, Q)\}$ of Lucas defined by $U_0 = 0, U_1 = 1$ and $V_0 = 2, V_1 = P$, respectively. It is well known that

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \tag{1.2}$$

where

$$\alpha = \frac{P + \sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{\Delta}}{2}, \tag{1.3}$$

so that $\alpha + \beta = P$ and $\alpha\beta = Q$.

Since $P > 0$, notice that $\alpha > 1$, $\alpha > |\beta|$, so that $U_n > 0$ $(n \ge 1)$, $V_n > 0$ $(n \ge 0)$.

Recently, several papers ([2], [3], and [6]) have been devoted to the study of the infinite sum

$$S_U(x) = S_U(x; P, Q) = \sum_{n=0}^{\infty} \frac{U_n}{x^n}. \tag{1.4}$$

The main known results can be summarized as follows.

Theorem 1:

(i) If $Q = -1$, the rational values of $x = r/s$ for which $S_U(x)$ is an integer are given by

$$x = \frac{U_{2n+1}}{U_{2n}} \quad (n = 1, 2, \ldots), \tag{1.5}$$

and the corresponding value of S_U is given by

$$S_U(x) = U_{2n}U_{2n+1}. \tag{1.6}$$

(ii) If $Q = 1$ and $P \ge 3$, the rational values of $x = r/s$ for which $S_U(x)$ is an integer are given by

$$x = \frac{U_{n+1}}{U_n} \quad (n = 1, 2, \ldots), \tag{1.7}$$

and the corresponding value of S_U is given by

$$S_U(x) = U_nU_{n+1}. \tag{1.8}$$

The aim of this paper is to extend the above result to the infinite sum

$$S_V(x) = S_V(x; P, Q) = \sum_{n=0}^{\infty} \frac{V_n}{x^n}, \quad \text{where } Q = \pm 1. \tag{1.9}$$

Using the Binet forms (1.2) and the geometric series formula, we get the closed-form expression

$$S_V(x) = \frac{x(2x-P)}{x^2-Px+Q}, \quad |x|>\alpha. \tag{1.10}$$

Remark 1.1: We have assumed that $P>0$. Actually, it is well known that

$$U_n(-P,Q) = (-1)^{n-1}U_n(P,Q) \quad \text{and} \quad V_n(-P,Q) = (-1)^n V_n(P,Q).$$

From this, we get

$$S_U(x;\,-P,Q) = -S_U(-x;\,P,Q) \quad \text{and} \quad S_V(x;\,-P,Q) = S_V(-x;\,P,Q).$$

Thus, the case $P<0$ cannot give really new results.

Remark 1.2: It is clear by (1.9) that $S_V(x) > V_0 = 2$ for $x>\alpha$, since $V_n > 0$ for every $n \geq 0$.

In what follows, we shall make use of the well-known identities:

$$V_n + PU_n = 2U_{n+1}; \tag{1.11}$$
$$\Delta U_n + PV_n = 2V_{n+1}; \tag{1.12}$$
$$U_{2n} = U_n V_n; \tag{1.13}$$
$$V_{2n} + 2Q^n = V_n^2; \tag{1.14}$$
$$V_{2n} - 2Q^n = \Delta U_n^2; \tag{1.15}$$
$$V_n^2 - \Delta U_n^2 = 4Q^n; \tag{1.16}$$
$$U_{2n+1} = U_{n+1}^2 - QU_n^2. \tag{1.17}$$

All of these identities can be proved by using the Binet forms (1.2).

2. MAIN RESULTS

Theorem 2: If $Q=\pm 1$, there do not exist negative rational values of x such that $S_V(x)$ is an integer, except when $Q=-1$ and $P=1$. In this case, the only solution is given by $x=-2$, with $S_V(-2)=2$.

Remark 2.1: Since $V_n(1,-1) = L_n$ (the n^{th} Lucas number), we see by Theorem 2 that

$$\sum_{n=0}^{\infty} \frac{(-1)^n L_n}{2^n} = 2. \tag{2.1}$$

Theorem 3: If $Q=-1$, the positive rational values of x for which $S_V(x)$ is integral are given by

$$x = \frac{U_{2n+1}}{U_{2n}} \quad (n=1,2,\ldots) \tag{2.2}$$

and

$$x = \frac{V_{2n+2}}{V_{2n+1}} \quad (n=0,1,\ldots). \tag{2.3}$$

The corresponding values of $S_V(x)$ are given by

$$S_V(U_{2n+1}/U_{2n}) = U_{2n+1}V_{2n} \tag{2.4}$$

and

$$S_V(V_{2n+2}/V_{2n+1}) = U_{2n+1}V_{2n+2}. \tag{2.5}$$

Theorem 4: If $Q=1$ and $P \geq 3$, the positive rational values of x for which $S_V(x)$ is integral are given by

$$x = \frac{U_{n+1}}{U_n} \quad (n = 1, 2, \ldots) \tag{2.6}$$

and

$$x = \frac{X_{n+1}}{X_n} \quad (n = 0, 1, \ldots), \tag{2.7}$$

where $X_n = U_{n+1} + U_n$.

The corresponding values of $S_V(x)$ are given by

$$S_V(U_{n+1}/U_n) = U_{n+1}V_n \tag{2.8}$$

and

$$S_V(X_{n+1}/X_n) = X_{n+1}(U_{n+1} - U_n). \tag{2.9}$$

3. PROOF OF THEOREM 2

Consider the function ϕ defined by

$$\phi(x) = \frac{x(2x-P)}{x^2 - Px + Q}, \quad x \neq \alpha \text{ and } x \neq \beta. \tag{3.1}$$

From (1.10) it is clear that $\phi(x) = S_V(x)$ when $|x| > \alpha$, and one can see immediately that

$$\lim_{x \to -\infty} \phi(x) = 2 \quad \text{and} \quad \phi(-\alpha) = 1 + \frac{\sqrt{\Delta}}{2P} > 1. \tag{3.2}$$

Assuming first that $Q=1$, we see that ϕ is decreasing on $]-\infty, \beta]$ and thus on $]-\infty, -\alpha]$ (recall that $-\alpha < \beta$, since $P > 0$). By (3.2), it is clear that there does not exist a number $x < -\alpha$ with $\phi(x)$ an integer.

Assuming now that $Q = -1$, we see that ϕ is decreasing on $]-\infty, \gamma]$ with $\gamma = \frac{-2-\sqrt{\Delta}}{P}$, and it is not hard to prove that $\phi(\gamma) = 1 + \frac{2}{\sqrt{\Delta}} > 1$. If $P \geq 2$, one verifies that $-\alpha < \gamma$, and the same conclusion follows. On the other hand, if $P = 1$, we have $\gamma = -2 - \sqrt{5} = -4.2\ldots$, $\phi(\gamma) = 1 + \frac{2}{\sqrt{5}} = 1.8\ldots$, $-\alpha = -1.6\ldots$, $\phi(-\alpha) = 1 + \frac{\sqrt{5}}{2} = 2.1\ldots$, and that ϕ is increasing on $[\gamma, -\alpha]$. Thus, 2 is the only integer value of ϕ within this interval, and it is immediate that $\phi(-1) = 2$ This completes the proof.

To prove Theorems 3 and 4, we need some further mathematical tools. These will be discussed in Sections 4 and 6.

4. A PELL EQUATION

In this section we shall suppose that $Q = \pm 1$. Let $x = r/s > \alpha$, where r and s are positive integers with $\gcd(r, s) = 1$. We see by (1.10) that

$$S_V(r/s) = rk, \tag{4.1}$$

where

$$k = \frac{(2r - Ps)}{r^2 - Prs + Qs^2}. \tag{4.2}$$

It is clear that $k > 0$, since $S_V(r/s) > 0$ by Remark 1.2. We also see that $\gcd(r, r^2 - Prs + Qs^2) = 1$, since $Q = \pm 1$ and $\gcd(r, s) = 1$. From this fact, we see that $S_V(r/s)$ is an integer if and only if

$$k = \frac{2r - Ps}{r^2 - Prs + Qs^2} = \frac{4(2r - Ps)}{(2r - Ps)^2 - \Delta s^2}$$

is an integer. Putting $z = 2r - Ps$ for notational convenience, we get the second-degree equation in the unknown z

$$\frac{4z}{z^2 - \Delta s^2} = k, \tag{4.3}$$

which can be written as

$$kz^2 - 4z - k\Delta s^2 = 0. \tag{4.4}$$

Notice that $z > s\sqrt{\Delta} > 0$, since $r/s > \alpha = \frac{P}{2} + \frac{\sqrt{\Delta}}{2}$. The only positive root of (4.4) is given by

$$z = \frac{2 + \sqrt{d}}{k}, \tag{4.5}$$

where $d = 4 + \Delta(ks)^2$. For z to be an integer, the inequality

$$4 + \Delta(ks)^2 = y^2 \quad (y = 0, 1, \dots) \tag{4.6}$$

must hold. Observing that $\Delta = P^2 \pm 4$ is never a square, it follows by (1.16) and the theory of Pell equation (see, e.g., [5] and [7]) that the solutions of (4.6) in the unknown y and ks are given by

$$y = V_{2n},\ ks = U_{2n}\ \ (n \geq 0),\ \text{if } Q = -1, \tag{4.7}$$

and by

$$y = V_n,\ ks = U_n\ \ (n \geq 0),\ \text{if } Q = 1. \tag{4.8}$$

In our problem we can suppose that $n \geq 1$, since $ks > 0$, and we have to consider the two cases ($Q = 1$ and $Q = -1$), separately.

5. PROOF OF THEOREM 3

In this section we suppose that $Q = -1$. Assuming that $S_V(r/s)$ is an integer, we see by (4.7) that $ks = U_{2n}$ and $\sqrt{d} = y = V_{2n}$ for $n \geq 1$. It follows by (4.5), (1.13), (1.14), and (1.15) that

$$z = s\frac{2 + \sqrt{d}}{ks} = s\frac{2 + V_{2n}}{U_{2n}} \tag{5.1}$$

$$= \begin{cases} s\dfrac{V_n^2}{U_{2n}} = s\dfrac{V_n}{U_n}, & n \geq 2 \text{ even}, \\[2ex] s\dfrac{\Delta U_n^2}{U_{2n}} = s\dfrac{\Delta U_n}{V_n}, & n \text{ odd}. \end{cases} \tag{5.2}$$

On the other hand, recalling that $z = 2r - Ps$ and using (1.11) and (1.12), we see that

$$r = \frac{z+Ps}{2} = \begin{cases} s\dfrac{V_n + PU_n}{2U_n} = s\dfrac{U_{n+1}}{U_n}, & n \ge 2 \text{ even,} \\ s\dfrac{\Delta U_n + PV_n}{2V_n} = s\dfrac{V_{n+1}}{V_n}, & n \text{ odd.} \end{cases} \tag{5.3}$$

Finally, we get

$$x = r/s = \begin{cases} \dfrac{U_{n+1}}{U_n}, & n \text{ even and positive,} \\ \dfrac{V_{n+1}}{V_n}, & n \text{ odd,} \end{cases} \qquad \text{[cf. (2.2) and (2.3)].}$$

To prove the second part of the theorem, notice first that $\frac{U_{n+1}}{U_n} > \alpha$ ($n \ge 2$ even) and $\frac{V_{n+1}}{V_n} > \alpha$ (n odd), since $Q = -1$. From (4.1), we see that

$$S_V(r/s) = rk = \frac{r}{s}ks. \tag{5.4}$$

Putting $r/s = U_{n+1}/U_n$ ($n \ge 2$ even) in (5.4) and using (4.7) and (1.13), we get

$$S_V(U_{n+1}/U_n) = \frac{U_{n+1}}{U_n}U_{2n} = U_{n+1}V_n \quad (n \ge 2 \text{ even}) \quad \text{[cf. (2.4)].}$$

Now, putting $r/s = V_{n+1}/V_n$ (n odd) in (5.4), we obtain

$$S_V(V_{n+1}/V_n) = \frac{V_{n+1}}{V_n}U_{2n} = V_{n+1}U_n \quad (n \text{ odd}) \quad \text{[cf. (2.5)].}$$

This completes the proof of Theorem 3. For the proof of Theorem 4, we need some results on the Fibonacci and Lucas numbers with real subscripts. These will be discussed in Section 6.

6. FIBONACCI AND LUCAS FUNCTIONS

Several definitions of Fibonacci and Lucas numbers with real subscripts are available in the literature (see, e.g., [1] and [4]) for the case in which $P = -Q = 1$.

Let us suppose here that $Q = 1$ and $P \ge 3$. Thus, α and β as defined by (1.3) are positive quantities and we can define, for every real number x, the real quantities

$$U_x = \frac{\alpha^x - \beta^x}{\alpha - \beta} \quad \text{and} \quad V_x = \alpha^x + \beta^x. \tag{6.1}$$

Using (6.1), the following identities can readily be found:

$$U_x = PU_{x-1} - U_{x-2} \quad \text{and} \quad V_x = PV_{x-1} - V_{x-2}, \tag{6.2}$$

$$U_x = U_{x/2}V_{x/2}, \tag{6.3}$$

$$V_x + 2 = V_{x/2}^2, \tag{6.4}$$

$$V_x + PU_x = 2U_{x+1}, \tag{6.5}$$

$$U_{x+y} + U_{x-y} = U_xV_y, \text{ for every } x \text{ and every } y. \tag{6.6}$$

The sequences $Y_n = U_{n+1/2}$ will be of particular interest for our purposes. Putting $x = n+1/2$ and $y = 1/2$ in (6.6), we get

$$Y_n = V_{1/2}^{-1} X_n, \tag{6.7}$$

where $X_n = U_{n+1} + U_n$, as specified in Theorem 4.

7. PROOF OF THEOREM 4

In this section we suppose that $Q = 1$ and $P \geq 3$. Assuming by (4.8) that $ks = U_n$ and $\sqrt{d} = y = V_n$ $(n \geq 1)$, it follows by (6.3) and (6.4) that

$$z = s\frac{2+\sqrt{d}}{ks} = s\frac{2+V_n}{U_n} = s\frac{V_{n/2}^2}{U_n} = s\frac{V_{n/2}}{U_{n/2}}. \tag{6.8}$$

Now by (6.5) we get

$$r = \frac{z+Ps}{2} = s\frac{V_{n/2}+PU_{n/2}}{2U_{n/2}} = s\frac{U_{n/2+1}}{U_{n/2}}. \tag{6.9}$$

Hence, letting $n = 2m$ and $n = 2m+1$ in (6.9) and using (6.7) in the latter case, one obtains

$$x = r/s = \begin{cases} \dfrac{U_{m+1}}{U_m}, & m > 0, \\ \dfrac{Y_{m+1}}{Y_m} = \dfrac{X_{m+1}}{X_m}, & m \geq 0, \end{cases} \quad \text{[cf. (2.6) and (2.7)]}.$$

As for the second part of the theorem, we see that $U_{m+1}/U_m > \alpha$ $(m > 0)$ and $X_{m+1}/X_m > \alpha$ $(m \geq 0)$, since $Q = 1$. Putting $r/s = U_{m+1}/U_m$ in (5.4) and using (4.8) (with $n = 2m$), we get

$$S_V(U_{m+1}/U_m) = \frac{U_{m+1}}{U_m}U_{2m} = U_{m+1}V_m \quad \text{[cf. (2.8)]}.$$

Finally, from (4.8) (with $n = 2m+1$) and (1.17), we get

$$\begin{aligned} S_v(X_{m+1}/X_m) &= \frac{X_{m+1}}{X_m}U_{2m+1} = \frac{X_{m+1}}{X_m}(U_{m+1}^2 - U_m^2) \\ &= X_{m+1}(U_{m+1} - U_m) \quad \text{[cf. (2.9)]}, \end{aligned}$$

and the proof is complete.

Concluding Remark: From Theorems 3 and 4, one can study the integrity of the infinite sum

$$T_k(x) = \sum_{n=0}^{\infty} \frac{V_{kn}}{x^n}, \quad k > 0.$$

This investigation might be the aim of a future work.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stan@wwa.com *on the Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-848 ***Proposed by Russell Euler's Fall 1997 Number Theory Class, Northwest Missouri State University, Maryville, MO***

Prove that $F_nF_{n+1} - F_{n+6}F_{n-5} = 40(-1)^{n+1}$ for all integers n.

B-849 ***Proposed by Larry Zimmerman & Gilbert Kessler, New York, NY***

If F_a, F_b, F_c, x forms an increasing arithmetic progression, show that x must be a Lucas number.

B-850 ***Proposed by Al Dorp, Edgemere, NY***

Find distinct positive integers a, b, and c so that $F_n = 17F_{n-a} + cF_{n-b}$ is an identity.

B-851 ***Proposed by Pentti Haukkanen, University of Tampere, Tampere, Finland***

Consider the repeating sequence $\langle A_n \rangle_{n=0}^{\infty} = 0, 1, -1, 0, 1, -1, 0, 1, -1, \ldots$.

(a) Find a recurrence formula for A_n.

(b) Find an explicit formula for A_n of the form $(a^n - b^n)/(a-b)$.

B-852 ***Proposed by Stanley Rabinowitz, Westford, MA***

Evaluate

$$\begin{vmatrix} F_0 & F_1 & F_2 & F_3 & F_4 \\ F_9 & F_8 & F_7 & F_6 & F_5 \\ F_{10} & F_{11} & F_{12} & F_{13} & F_{14} \\ F_{19} & F_{18} & F_{17} & F_{16} & F_{15} \\ F_{20} & F_{21} & F_{22} & F_{23} & F_{24} \end{vmatrix}.$$

B-853 ***Proposed by Gene Ward Smith, Brunswick, ME***

Consider the recurrence $f(n+1)=n(f(n)+f(n-1))$ with initial conditions $f(0)=1$ and $f(1)=0$. Find a closed form for the sum

$$\sum_{k=0}^{n}\binom{n}{k}f(k).$$

NOTE: The Elementary Problems Column is in need of more **easy**, yet elegant and nonroutine problems.

SOLUTIONS

Pattern Detective

B-832 ***Proposed by Andrew Cusumano, Great Neck, NY***
(Vol. 35, no. 3, August 1997)

Find a pattern in the following numerical identities and create a formula expressing a more general result.

$$\begin{aligned}
3^5+2^5+1^5+1^5 &= 5\cdot 3^4-128\\
5^5+3^5+2^5+1^5+1^5 &= 8\cdot 5^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
8^5+5^5+3^5+2^5+1^5+1^5 &= 13\cdot 8^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
&\quad -3\cdot5\cdot8(19+2\cdot3\cdot5+2\cdot5\cdot8)\\
13^5+8^5+5^5+3^5+2^5+1^5+1^5 &= 21\cdot13^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
&\quad -3\cdot5\cdot8(19+2\cdot3\cdot5+2\cdot5\cdot8)\\
&\quad -5\cdot8\cdot13(19+2\cdot3\cdot5+2\cdot5\cdot8+2\cdot8\cdot13)\\
21^5+13^5+8^5+5^5+3^5+2^5+1^5+1^5 &= 34\cdot21^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
&\quad -3\cdot5\cdot8(19+2\cdot3\cdot5+2\cdot5\cdot8)\\
&\quad -5\cdot8\cdot13(19+2\cdot3\cdot5+2\cdot5\cdot8+2\cdot8\cdot13)\\
&\quad -8\cdot13\cdot21(19+2\cdot3\cdot5+2\cdot5\cdot8+2\cdot8\cdot13+2\cdot13\cdot21)
\end{aligned}$$

Solution by H.-J. Seiffert, Berlin, Germany

Let k be an integer. Then $F_{k-2}(F_k^2+F_{k-1}F_{k+1})=F_{k-2}(F_k^2+F_{k-1}F_k+F_{k-1}^2)=(F_k-F_{k-1})(F_k^2+F_{k-1}F_k+F_{k-1}^2)=F_k^3-F_{k-1}^3$. Using $F_{k-1}F_{k+1}-F_k^2=(-1)^k$, which is identity ($I_{13}$) of [1], we have

$$F_{k-2}(2F_k^2+(-1)^k)=F_k^3-F_{k-1}^3.$$

Multiplying this equation by $F_{k-1}F_k$, adding F_k^5 on both sides of the resulting equation, and using $F_k^5+F_{k-1}F_k^4=F_{k+1}F_k^4$, we obtain

$$F_k^5+F_{k-2}F_{k-1}F_k(2F_k^2+(-1)^k)=F_{k+1}F_k^4-F_kF_{k-1}^4.$$

Summing as k ranges from 1 to n yields

$$\sum_{k=1}^{n}F_k^5=F_{n+1}F_n^4-\sum_{k=1}^{n}F_{k-2}F_{k-1}F_k(2F_k^2+(-1)^k). \tag{1}$$

From identity (I_{13}), again, we find

$$F_j^2-F_{j-1}^2+(-1)^j=F_{j-1}F_{j+1}-F_{j-1}^2=F_{j-1}F_j.$$

Summing as j ranges from 1 to k gives

$$F_k^2+\frac{1}{2}((-1)^k-1)=\sum_{j=1}^{k}F_{j-1}F_j \quad \text{or} \quad 2F_k^2+(-1)^k=1+2\sum_{j=1}^{k}F_{j-1}F_j.$$

Now equation (1) may be rewritten as

$$\sum_{k=1}^{n}F_k^5=F_{n+1}F_n^4-\sum_{k=1}^{n}F_{k-2}F_{k-1}F_k\left(1+2\sum_{j=1}^{k}F_{j-1}F_j\right).$$

Let $n\geq 4$. Using

$$\sum_{k=1}^{4}F_{k-2}F_{k-1}F_k\left(1+2\sum_{j=1}^{k}F_{j-1}F_j\right)=128 \quad \text{and} \quad 1+2\sum_{j=1}^{4}F_{j-1}F_j=19,$$

we find

$$\sum_{k=1}^{n}F_k^5=F_{n+1}F_n^4-128-\sum_{k=5}^{n}F_{k-2}F_{k-1}F_k\left(19+2\sum_{j=5}^{k}F_{j-1}F_j\right),$$

valid for all $n\geq 4$. For $n=4,5,6,7,$ and 8, this produces the numerical identities given in the proposal.

The proposer sent along many related identities.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Santa Clara, Calif.: The Fibonacci Association, 1979.

Also solved by Paul S. Bruckman, L. A. G. Dresel, and the proposer.

Newton Meets Lucas

B-833 ***Proposed by Al Dorp, Edgemere, NY***
(Vol. 35, no. 3, August 1997)

For n a positive integer, let $f(x)$ be the polynomial of degree $n-1$ such that $f(k)=L_k$ for $k=1,2,3,\dots,n$. Find $f(n+1)$.

Solution by Paul S. Bruckman, Highwood, IL

We employ the Pochhammer notation $x^{(m)}=x(x-1)(x-2)\cdots(x-m+1)$ and use Newton's Forward Difference Formula ([1], p. 29), which says that if f_k is a polynomial of degree n, then

$$f_k=f_0+\frac{\Delta f_0}{1!}k^{(1)}+\frac{\Delta^2 f_0}{2!}k^{(2)}+\cdots+\frac{\Delta^n f_0}{n!}k^{(n)},$$

where the operator Δ is defined by $\Delta f(x)=f(x+1)-f(x)$.

In our example, $\Delta f(k)=f(k+1)-f(k)=L_{k+1}-L_k=L_{k-1}$ for $k=1,2,\dots,n-1$. Similarly, $\Delta^2 f(k)=\Delta(\Delta f(k))=\Delta L_{k-1}=L_{k-2}$ for $k=1,2,\dots,n-2$. Continuing, we find $\Delta^3 f(k)=L_{k-3}$ for $k=1,2,\dots,n-3$, etc., until $\Delta^{n-1}f(1)=L_{2-n}$.

Applying Newton's formula, we obtain

$$f(x+1)=\sum_{k=0}^{n-1}\frac{L_{1-k}}{k!}x^{(k)}=\sum_{k=0}^{n-1}(-1)^{k-1}L_{k-1}x^{(k)}/k!.$$

Then

$$\begin{aligned}f(n+1) &= \sum_{k=0}^{n-1}(-1)^{k-1}\binom{n}{k}L_{k-1}\\ &= \sum_{k=1}^{n-1}\binom{n}{k}[(-\alpha)^{k-1}+(-\beta)^{k-1}]\\ &= \beta(1-\alpha)^n+\alpha(1-\beta)^n-(-\alpha)^{n-1}-(-\beta)^{n-1}\\ &= \beta^{n+1}+\alpha^{n+1}+(-1)^n[\alpha^{n-1}+\beta^{n-1}]\\ &= L_{n+1}+(-1)^n L_{n-1}.\end{aligned}$$

This is equivalent to L_n if n is odd and $5F_n$ if n is even.

Reference

1. Ronald E. Mickens. *Difference Equations*. New York: Van Nostrand Reinhold, 1990.

Also solved by Charles K. Cook, L. A. G. Dresel, Hans Kappus, Harris Kwong, R. Horace McNutt, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Radical Inequality

B-834 ***Proposed by Zdravko F. Starc, Vrsac, Yugoslavia (Vol. 35, no. 3, August 1997)***

For x a real number and n an integer larger than 1, prove that

$$(x+1)F_1+(x+2)F_2+\cdots+(x+n)F_n < 2^n\sqrt{\frac{n(n+1)(2n+1+6x)+6nx^2}{6}}.$$

Solution by the proposer

From the identity $\alpha^n = \alpha F_n + F_{n-1}$, we see that for $n \geq 2$,

$$F_n = \alpha^{n-1} - \frac{F_{n-1}}{\alpha} < \alpha^{n-1} < 2^{n-1}.$$

Thus,

$$\sqrt{F_n F_{n+1}} < 2^n. \qquad (*)$$

Cauchy's Inequality ([1], p. 20) says that for all real numbers a_i and b_i,

$$(a_1b_1+a_2b_2+\cdots+a_nb_n)^2 \leq (a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2).$$

Let $a_i = x+i$ and $b_i = F_i$. Then, using the facts

$$F_1^2+F_2^2+\cdots+F_n^2 = F_nF_{n+1},$$

$$(x+1)^2+(x+2)^2+\cdots+(x+n)^2 = \frac{n(n+1)(2n+1+6x)+6nx^2}{6},$$

and the inequality $(*)$, we obtain the desired inequality for all x for which the radicand is non-negative.

Reference

1. D. S. Mitrinovic. *Analytic Inequalities*. Berlin: Springer Verlag, 1970.

Also solved by Paul S. Bruckman.

Cryptarithmic Identity

B-836 ***Proposed by Al Dorp, Edgemere, NY***
(Vol. 35, no. 4, November 1997)

Replace each of W, X, Y, and Z by either F or L to make the following an identity:

$$W_n^2 - 6X_{n+1}^2 + 2Y_{n+2}^2 - 3Z_{n+3}^2 = 0.$$

Solution by L. A. G. Dresel, Reading, England

Putting $n=-1$, we find

$$W_1^2 + 2Y_1^2 = 3(Z_2^2 + 2X_0^2). \tag{a}$$

Putting $n=0$, we find

$$W_0^2 + 2Y_2^2 = 3(Z_3^2 + 2X_1^2). \tag{b}$$

Since $F_1 = L_1 = 1$, we have $W_1 = Y_1 = 1$, so that (a) implies $3 = 3(Z_2^2 + 2X_0^2)$, giving $X_0 = F_0 = 0$ and $Z_2 = F_2 = 1$. Substituting in (b) gives $W_0^2 + 2Y_2^2 = 3(F_3^2 + 2F_1^2) = 18$, which requires $W_0 = F_0 = 0$ and $Y_2 = L_2 = 3$. Therefore, the required identity is

$$F_n^2 - 6F_{n+1}^2 + 2L_{n+2}^2 - 3F_{n+3}^2 = 0.$$

This is satisfied for $n=-1$, $n=0$, and also for $n=1$. Therefore, by the Verification Theorem of [1], this is an identity for all values of n.

Reference

1. L. A. G. Dresel. "Transformations of Fibonacci-Lucas Identities." In *Applications of Fibonacci Numbers* **5**:169-84. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1993.

Also solved by Paul S. Bruckman, Charles K. Cook, Russell Jay Hendel, Daina A. Krigens, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Polynomial Remainder

B-837 ***Proposed by Joseph J. Koštál, Chicago, IL***
(Vol. 35, no. 4, November 1997)

Let $P(x) = x^{1997} + x^{1996} + x^{1995} + \cdots + x^2 + x + 1$ and let $R(x)$ be the remainder when $P(x)$ is divided by $x^2 - x - 1$. Show that $R(x)$ is divisible by L_{999}.

Solution by L. A. G. Dresel, Reading, England

Consider, more generally, the polynomial

$$P_n(x) = x^{2n-1} + x^{2n-2} + x^{2n-3} + \cdots + x^2 + x + 1 = \frac{x^{2n}-1}{x-1}.$$

Then, if $R(x)$ is the remainder on dividing by $x^2 - x - 1$, we have the identity

$$P_n(x) = (x^2 - x - 1)Q(x) + R(x),$$

where $Q(x)$ is a polynomial in x. Putting $x = \alpha$, and using the fact that $\alpha^2 - \alpha - 1 = 0$, we find that $P_n(\alpha) = R(\alpha)$ and $P_n(\beta) = R(\beta)$. Hence, $\alpha^{2n} - 1 = (\alpha - 1)R(\alpha)$ and $\beta^{2n} - 1 = (\beta - 1)R(\beta)$. Now $\alpha - 1 = -\beta$, $\beta - 1 = -\alpha$, and $R(x) = Ax + B$, where A and B are constants. Hence, $\alpha^{2n} - 1 =$

$A-B\beta$ and $\beta^{2n}-1=A-B\alpha$. Subtracting gives $(\alpha-\beta)F_{2n}=B(\alpha-\beta)$, so that $B=F_{2n}=F_nL_n$, while adding gives $L_{2n}-2=2A-B$. When n is odd, we have $L_n^2=(\alpha^n+\beta^n)^2=L_{2n}-2=2A-B$, whereas when n is even, we have $5F_n^2=(\alpha^n-\beta^n)^2=L_{2n}-2=2A-B$. It follows that when n is odd, $R(x)$ is divisible by L_n, whereas when n is even, $R(x)$ is divisible by F_n.

Also solved by Charles Ashbacher, David M. Bloom, Paul S. Bruckman, Al Dorp, Russell Euler & Jawad Sadek, Russell Jay Hendel, Hans Kappus, Harris Kwong, R. Horace McNutt, Bob Prielipp, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Composite Linear Recurrence

B-838 ***Proposed by Peter G. Anderson, Rochester Institute of Technology, Rochester, NY (Vol. 35, no. 4, November 1997)***

Define a sequence of linear polynomials, $f_n(x)=m_nx+b_n$, by the recurrence $f_n(x)=f_{n-1}(f_{n-2}(x))$, $n\ge 3$, with initial conditions $f_1(x)=\frac{1}{2}x$ and $f_2(x)=\frac{1}{2}x+\frac{1}{2}$. Find a formula for m_n.

Extra credit: Find a formula for b_n.

Solution by Charles Ashbacher, Hiawatha, IA

We claim that $m_n=1/2^{F_n}$ for $n\ge 1$. The proof is by induction on n. For the basis step, we have the given initial conditions, showing that the result is true for $n=1$ and $n=2$. Now assume

$$f_{k-1}(x)=\frac{1}{2^{F_{k-1}}}x+b_1 \quad\text{and}\quad f_k(x)=\frac{1}{2^{F_k}}x+b_2.$$

Then

$$\begin{aligned}f_{k+1}(x)&=\frac{1}{2^{F_k}}\left(\frac{1}{2^{F_{k-1}}}x+b_1\right)+b_2=\frac{1}{2^{F_k}2^{F_{k-1}}}x+\frac{1}{2^{F_k}}b_1+b_2\\&=\frac{1}{2^{F_k+F_{k-1}}}x+\frac{1}{2^{F_k}}b_1+b_2=\frac{1}{2^{F_{k+1}}}x+\frac{1}{2^{F_k}}b_1+b_2\end{aligned}$$

and the result follows for all n.

Bruckman receives extra credit for finding that

$$b_n=\sum_{k=1}^{F_{n-1}}\frac{1}{2^{\lfloor k\alpha\rfloor}}.$$

Strazdins reports that this sequence is studied in [1].

Reference

1. H. W. Gould, J. B. Kim, & V. E. Hoggatt, Jr. "Sequences Associated with t-ary Coding of Fibonacci's Rabbits." *The Fibonacci Quarterly* **15.4** (1977):311-18.

Also solved by Paul S. Bruckman, Charles K. Cook, L. A. G. Dresel, Russell Jay Hendel, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Late solutions to problems B-821 through B-824 were received from David Stone.

Errata: In the solution to problem B-830 (Feb. 1998, p. 89), in the second line of the proof of part (b), the subscript $19\cdot 109$ should be $19\cdot 108$ in three places. On the first line of page 90, "$n+1$" should read "$n+a$".

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-539 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Let

$$H_m(p)=\sum_{j=1}^{m}B\left(\frac{j}{2},p\right), m\in N,\ p>0,$$

where

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}dt$$

denotes the Betafunction. Show that for all positive reals p and all positive integers n,

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{2k}(p)=4^{n+p-1}B(n+p,n+p-1)+\frac{1}{n+p-1}. \tag{1}$$

From (1), deduce the identities

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{4^k}\binom{n}{k}\binom{2k}{k}=\frac{2}{4^n}\binom{2n-2}{n-1} \tag{2}$$

and

$$\sum_{k=1}^{n}(-1)^{k-1}4^k\binom{n}{k}\Big/\binom{2k}{k}=\frac{2n}{2n-1}. \tag{3}$$

H-540 ***Proposed by Paul S. Bruckman, Highwood, IL***

Consider the sequence $U=\{u(n)\}_{n=1}^{\infty}$, where $u(n)=[n\alpha]$, its characteristic function $\delta_U(n)$, and its counting function $\pi_U(n)\equiv\sum_{k=1}^{n}\delta_U(k)$, representing the number of elements of U that are $\leq n$. Prove the following relationships:

(a) $\delta_U(n)=u(n+1)-u(n)-1,\ n\geq 1;$

(b) $\pi_U(F_n)=F_{n-1},\ n>1.$

H-541 ***Proposed by Stanley Rabinowitz, Westford, MA***

The simple continued fraction expansion for F_{13}^5/F_{12}^5 is

$$11+\cfrac{1}{11+\cfrac{1}{375131+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{9+\cfrac{1}{11}}}}}}}}}}}}}}$$

This can be written more compactly using the notation $[11,11,375131,1,1,1,1,1,1,1,1,1,2,9,11]$. To be even more concise, we can write this as $[11^2,375131,1^9,2,9,11]$, where the superscript denotes the number of consecutive occurrences of the associated number in the list.

If $n>0$, prove that the simple continued fraction expansion for $(F_{10n+3}/F_{10n+2})^5$ is

$$[11^{2n}, x, 1^{10n-1}, 2, 9, 11^{2n-1}],$$

where x is an integer and find x.

SOLUTIONS

A Fibo Matrix?

H-522 ***Proposed by N. Gauthier, Royal Military College, Kingston, Ontario, Canada (Vol. 35, no. 1, February 1997)***

Let A and B be the following 2×2 matrices:

$$A=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \quad\text{and}\quad B=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Show that, for $m\ge 1$,

$$\sum_{n=0}^{m-1} 2^n A^{2^n}(A^{2^n}+B^{2^n})^{-1} = c_{2^m}C_{2^m}-(A+B),$$

where

$$c_m = m/(F_{m+1}+F_{m-1}-2) \quad\text{and}\quad C_m=\begin{pmatrix}F_{m+1}-1 & F_m\\ F_m & F_{m-1}-1\end{pmatrix};$$

F_m is the m^{th} Fibonacci number.

Solution by Paul S. Bruckman, Highwood, IL

We begin by noting that the matrix B is the identity matrix I (as is any power of B). Let S_m denote the sum in the left member of the statement of the problem; let $W(n)=nA^n(A^n+I)^{-1}$. Note that

$$c_m=m(L_m-2)^{-1},\ c_2=2,\ C_2=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix},\ A+B=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}.$$

Now

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

a well-known result. Then $|A^n + I| = F_{n+1}F_{n-1} + F_{n+1} + F_{n-1} + 1 - (F_n)^2 = L_n + 1 + (-1)^n = L_n + 2e_n$, where e_n is the characteristic function of the even integers. Then

$$(A^n + I)^{-1} = (L_n + 2e_n)^{-1}\begin{pmatrix} F_{n-1}+1 & -F_n \\ -F_n & F_{n+1}+1 \end{pmatrix},$$

and

$$W(n) = n(L_n + 2e_n)^{-1}\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}\begin{pmatrix} F_{n-1}+1 & -F_n \\ -F_n & F_{n+1}+1 \end{pmatrix}$$

$$= n(L_n + 2e_n)^{-1}\begin{pmatrix} F_{n+1}+(-1)^n & F_n \\ F_n & F_{n-1}+(-1)^n \end{pmatrix},$$

after simplification. In particular,

$$W(1) = S_1 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Note that

$$c_2C_2 - (A+I) = 2\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = S_1;$$

thus, the statement of the problem is valid for $m = 1$.

Let N denote the set of positive integers m for which the statement of the problem is valid. As we have just shown, $1 \in N$. Suppose that $m \in N$. Then, letting $u = 2^m$ and using the inductive hypothesis,

$$S_{m+1} = S_m + W(u) = c_uC_u - (A+I) + u(L_u+2)^{-1}\begin{pmatrix} F_{u+1}+1 & F_u \\ F_u & F_{u-1}+1 \end{pmatrix}$$

$$= u(L_u-2)^{-1}\begin{pmatrix} F_{u+1}-1 & F_u \\ F_u & F_{u-1}-1 \end{pmatrix} + u(L_u+2)^{-1}\begin{pmatrix} F_{u+1}+1 & F_u \\ F_u & F_{u-1}+1 \end{pmatrix} - (A+I)$$

$$= u\{(L_u)^2-4\}^{-1}\begin{pmatrix} (L_u+2)\{F_{u+1}-1\}+(L_u-2)\{F_{u+1}+1\} & 2L_uF_u \\ 2L_uF_u & (L_u+2)\{F_{u-1}-1\}+(L_u-2)\{F_{u-1}+1\} \end{pmatrix}$$

$$-(A+I)$$

$$= 2u(L_{2u}-2)^{-1}\begin{pmatrix} L_uF_{u+1}-2 & F_{2u} \\ F_{2u} & L_uF_{u-1}-2 \end{pmatrix} - (A+I)$$

$$= c_{2u}\begin{pmatrix} F_{2u+1}-1 & F_{2u} \\ F_{2u} & F_{2u-1}-1 \end{pmatrix} - (A+I) = c_{2u}C_{2u} - (A+I).$$

Comparison with the expression given in the statement of the problem shows, therefore, that $m \in N$ implies $(m+1) \in N$. This is the required inductive step, and the desired result is proven.

Also solved by H. Kappus, H.-J. Seiffert, and the proposer.

Enter!

H-523 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 35, no. 1, February 1997)

Let $Z(n)$ denote the "Fibonacci entry-point" of n, i.e., $Z(n)$ is the smallest positive integer m such that $n|F_m$. Given any odd prime p, let $q=\frac{1}{2}(p-1)$; for any integer s, define $g_p(s)$ as follows:

$$g_p(s)=\sum_{k=1}^{q}\frac{s^k}{k}.$$

Prove the following assertion:

$$Z(p^2)=Z(p) \text{ iff } g_p(1)\equiv g_p(5) \pmod{p}. \tag{$*$}$$

Solution by H.-J. Seiffert, Berlin, Germany

We need the following results.

Proposition 1: For all positive integers n, it holds that:

(a) $2^{n-1}=\sum_{k=0}^{[n/2]}\binom{n}{2k}$;

(b) $2^{n-1}L_n=\sum_{k=0}^{[n/2]}\binom{n}{2k}5^k$.

Proof: The first equation can be found on page 4 in [1] and the second on page 69 in [3].

Proposition 2: If p is any odd prime, then $Z(p^2)=Z(p)$ if and only if $L_p\equiv 1 \pmod{p^2}$.

Proof: Since $Z(25)=25\neq 5=Z(5)$ and $L_5=11\not\equiv 1 \pmod{25}$, we do suppose that $p\neq 5$. Then (see [2], p. 386, Lemma 5), $Z(p^2)=Z(p)$ if and only if $F_{p-e}\equiv 0 \pmod{p^2}$, where $e=(5|p)$ denotes Legendre's symbol, and (see [4], p. 367, eq. (2.10)) $F_{p-e}\equiv 2e(F_p-e) \pmod{p^2}$. Our claim now easily follows from $p\neq 5$, $e\in\{-1,+1\}$, and the equations $L_p=2F_{p+1}-F_p=F_p+2F_{p-1}$. Q.E.D.

Lemma: If p is a prime, then

$$\binom{p}{j}\equiv(-1)^{j+1}\frac{p}{j} \pmod{p^2},\ j=1,2,\ldots,p-1.$$

Proof: For $j=1,2,\ldots,p-1$, we have

$$(-1)^j\binom{p}{j}=\frac{(-p)(1-p)\cdots(j-1-p)}{1\cdot 2\cdot\cdots\cdot(j-1)\cdot j}\equiv-\frac{p}{j} \pmod{p^2}.$$

This proves this well-known congruence. Q.E.D.

Let p be an odd prime. From Proposition 1(a) and the lemma, modulo p^2 we obtain

$$2^{p-1}=1+\sum_{k=1}^{q}\binom{p}{2k}\equiv 1-\frac{p}{2}g_p(1) \pmod{p^2}$$

or, equivalently,

$$pg_p(1) \equiv 2 - 2^p \pmod{p^2}. \tag{1}$$

Similarly, using Proposition 1(b) and the above lemma, modulo p^2 we find

$$2^{p-1}L_p = 1 + \sum_{k=1}^{q} \binom{p}{2k} 5^k \equiv 1 - \frac{p}{2} g_p(5) \pmod{p^2},$$

giving

$$pg_p(5) \equiv 2 - 2^p L_p \pmod{p^2}. \tag{2}$$

Hence, by (1) and (2), we have $g_p(1) \equiv g_p(5) \pmod{p}$ if and only if $L_p \equiv 1 \pmod{p^2}$. The desired equivalence relation now follows from Proposition 2.

Remark: In 1960, D. D. Wall posed the problem of whether there exists a prime p such that $p^2 | F_{p-e}$. It is still not known whether such a prime exists although it is known that it must exceed 10^9 (see [4], p. 366). In [2] (p. 384, Theorem 4), it was proved that if p is an odd prime such that Fermat's last theorem fails for the exponent p in the first case, then $p^2 | F_{p-e}$. Conversely, it seems that Andrew Wiles' proof of Fermat's last theorem does not imply that such primes cannot exist.

References

1. I. S. Gradsteyn & I. M. Ryzhik. *Table of Integrals, Series, and Products*. 5th ed. New York:: Academic Press, 1994.
2. Z. H. Sun & Z.-W. Sun. "Fibonacci Numbers and Fermat's Last Theorem." *Acta Arith.* **60** (1992):371-88.
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4. H. C. Williams. "A Note on the Fibonacci Quotient F_{p-e}/p." *Can. Math. Bull.* **25** (1982): 366-70.

Also solved by the proposer.

$Z(p)$ ed di do da

H-524 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 35, no. 1, February 1997)***

Let p be a prime with $p \equiv 1$ or $9 \pmod{20}$. It is known that $a := (p-1)/Z(p)$ is an even integer, where $Z(p)$ denotes the entry-point in the Fibonacci sequence [1]. Let $q := (p-1)/2$. Show that

(1) $(-1)^{a/2} \equiv (-5)^{q/2} \pmod{p}$ if $p \equiv 1 \pmod{20}$,

(2) $(-1)^{a/2} \equiv -(-5)^{q/2} \pmod{p}$ if $p \equiv 9 \pmod{20}$.

Reference

1. P. S. Bruckman. Problem H-515. *The Fibonacci Quarterly* **34.4** (1996):379.

Solution by Paul S. Bruckman, Highwood, IL

We will make use of the following easily verified (or well-known) results:

(a) $p | F_r$ and $F_p - e = F_r L_{r+e}$ iff $\left(\frac{-1}{p}\right) = 1$;

(b) $p | L_r$ and $F_p - e = F_{r+e} L_r$ iff $\left(\frac{-1}{p}\right) = -1$;

(c) $e = (-1)^r \left(\frac{-1}{p}\right)$;

(d) $5F_r^2 - L_{r+e}L_{r-e} = 5F_{r+e}F_{r-e} - L_r^2 = (-1)^r$;
(e) for all positive integers m and $n > 1$, $Z(m)|n$ iff $m|F_n$;
(f) $Z(p)|(p-e)$;
(g) $Z(p^2) = pZ(p)$ or $Z(p)$.

(A) Suppose $eA - B \equiv C \pmod{p}$. Then $eAp - Bp \equiv Cp \pmod{p^2}$

$$\Rightarrow e(2^{p-1}-1) - (5^q - e) \equiv p\sum_{k=1}^{q} \frac{5^{k-1}}{2k-1} \equiv \sum_{k=1}^{p-1} \frac{p}{k}\cdot\frac{1}{2}(1-(-1)^k)\cdot 5^{\frac{1}{2}(k-1)} \pmod{p^2}$$

$$\Rightarrow e\cdot 2^{p-1} \equiv \sum_{k=1}^{p} \frac{p}{k}\cdot\frac{1}{2}(1-(-1)^k)\cdot 5^{\frac{1}{2}(k-1)} \pmod{p^2}.$$

Now, if $1 \le k \le p$,

$$\binom{p}{k} = \frac{p}{k}\cdot\binom{p-1}{k-1} \equiv \frac{p}{k}\cdot\binom{-1}{k-1} \equiv \frac{p}{k}(-1)^{k-1} \pmod{p^2}.$$

Thus,

$$e\cdot 2^{p-1} \equiv \sum_{k=1}^{p}(-1)^{k-1}\cdot\frac{1}{2}(1-(-1)^k)\cdot\binom{p}{k}\cdot 5^{\frac{1}{2}(k-1)}$$

$$\equiv 5^{-\frac{1}{2}}\sum_{k=0}^{p}\binom{p}{k}\cdot\frac{1}{2}(1-(-1)^k)\cdot 5^{\frac{1}{2}k} \pmod{p^2}$$

$$\Rightarrow e\cdot 2^p \equiv 5^{-\frac{1}{2}}[(1+\sqrt{5})^p - (1-\sqrt{5})^p] \pmod{p^2} \Rightarrow F_p \equiv e \pmod{p^2}.$$

From (a) and (b), we see that $p|F_r$ and $p^2|F_rL_{r+e}$ if $\left(\frac{-1}{p}\right) = 1$, or $p|L_r$ and $p^2|F_{r+e}L_r$ if $\left(\frac{-1}{p}\right) = -1$. From (d) and (e), $\gcd(F_r, L_{r+e}) = \gcd(F_{r+e}, L_r) = 1$. Then $p^2|F_r$ if $\left(\frac{-1}{p}\right) = 1$, or $p^2|L_r$ if $\left(\frac{-1}{p}\right) = -1$. In any event, $p^2|F_{2r} = F_rL_r$. Then, from (e), $Z(p^2)|2r = p-e$. Since $p\nmid(p-e)$, it follows from (f) and (g) that $Z(p^2) = Z(p)$.

(B) The steps in (A) are reversible. Thus,

$$Z(p^2) = Z(p) \Rightarrow p^2|F_{2r} \Rightarrow p^2|(F_p - e) \Rightarrow eAp - Bp$$
$$\equiv C_p \pmod{p^2} \Rightarrow eA - B \equiv C \pmod{p}. \text{ Q.E.D.}$$

Also solved by the proposer.

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Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book, Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972

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Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965

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A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Applications of Fibonacci Numbers, Volumes 1-7. Edited by G.E. Bergum, A.F. Horadam and A.N. Philippou

Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger. FA, 1993.

Fibonacci Entry Points and Periods for Primes 100,003 through 415,993 by Daniel C. Fielder and Paul S. Bruckman.

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