The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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VOLUME 36 AUGUST 1998 NUMBER 4

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980) and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

A METRIC RESULT CONCERNING THE APPROXIMATION OF REAL NUMBERS BY CONTINUED FRACTIONS

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(*Submitted October 1996*)

1. INTRODUCTION AND STATEMENT OF RESULTS

A basic result in the theory of Diophantine approximations of irrational numbers by rationals, where certain additional congruence relations are satisfied, is given by the following theorem of Uchiyama [8]:

For any irrational number ξ, any $s>1$, and integers a and b, there are infinitely many integers u and $v \neq 0$ satisfying

$$\left|\xi-\frac{u}{v}\right|<\frac{s^2}{4v^2} \tag{1.1}$$

and

$$u \equiv a \bmod s, \quad v \equiv b \bmod s \tag{1.2}$$

provided that a and b are not both divisible by s.

Theorems of the same type with greater constants on the right-hand side of (1.1) were proved by Hartman [3] and Koksma [6]. Recently, the author has shown [1] that the theorem of Uchiyama no longer holds if the constant $s^2/4$ is replaced by any smaller number. Assuming weaker arithmetical restrictions in (1.2) on numerators and denominators of the approximants, the constant in (1.1) can be diminished. For prime moduli p, the author has shown (see [2]):

Let $0<\varepsilon\leq 1$, and let p be a prime with $p>(2/\varepsilon)^2$; h denotes any integer that is not divisible by p. Then, for any real irrational number ξ, there are infinitely many integers u and $v>0$ satisfying

$$\left|\xi-\frac{u}{v}\right| \leq \frac{(1+\varepsilon)p^{3/2}}{\sqrt{5}\,v^2} \tag{1.3}$$

and

$$u \equiv hv \not\equiv 0 \bmod p. \tag{1.4}$$

The object of this paper is to show that almost all irrational numbers (in the sense of the Lebesgue-measure) are better approximated by fractions u/v satisfying (1.4).

Theorem 1.1: Let $\varepsilon>0$, and let p be any prime. Then there is a set $\mathcal{A}\subset(0,1)$ of measure 1 depending at most on ε and p, such that every real number ξ from $\mathcal{A}$ satisfies the following conditions:

If h denotes any integer that is not divisible by p, there are infinitely many integers u and $v>0$ with

$$\left|\xi - \frac{u}{v}\right| < \frac{\varepsilon}{v^2} \tag{1.5}$$

and

$$u \equiv hv \not\equiv 0 \mod p. \tag{1.6}$$

The main tool used in proving this theorem is a certain generalization of the famous average-theorem of Gauss-Kusmin-Lévy concerning the elements of continued fractions (see Satz 35 in [4]), which is stated in Lemma 2.1 below. It follows from [5] or [7].

The set $\mathcal{A}$ given in Theorem 1.1 depends on ε and p. One may ask whether there are irrationals, where (1.5) and (1.6) hold for *every* $\varepsilon > 0$ and for *every* prime p.

Theorem 1.2: There is an uncountable subset $\mathcal{B}$ of (0, 1) such that, for every real number ξ from $\mathcal{B}$, for every $\varepsilon > 0$, every prime p and any integer h that is not divisible by p, the inequality (1.5) and the congruences (1.6) hold for infinitely many integers u and $v > 0$.

2. PROOF OF THE THEOREMS

Lemma 2.1: Let $r_1, r_2, \ldots, r_k$ $(k \geq 1)$ be positive integers. Then the successive elements $r_1, r_2, \ldots, r_k$ occur infinitely often in the sequence $a_1, a_2, a_3, \ldots$ of the continued fraction expansion $\langle 0; a_1, a_2, a_3, \ldots \rangle$ of almost all real numbers from (0, 1).

Proof of Theorem 1.1: Let $\varepsilon > 0$ and p be any prime. Moreover, let $s > 0$ be some integer with

$$\frac{1}{sp} < \varepsilon. \tag{2.1}$$

By Lemma 2.1, there is a subset $\mathcal{A} \subset (0, 1)$ of measure 1 such that the finite sequence

$$\underbrace{sp+1, sp, sp+1, sp, \ldots, sp+1, sp}_{2p} \tag{2.2}$$

occurs infinitely often among the elements of the continued fraction expansion of every number from $\mathcal{A}$. Obviously, $\mathcal{A}$ depends on ε and p.

Now we fix some number ξ from $\mathcal{A}$; by $\xi = \langle 0; a_1, a_2, a_3, \ldots \rangle$ we denote its continued fraction expansion. There are infinitely many integers $j \geq 1$ such that, for all integers n with $1 \leq n \leq 2p$, we have

$$a_{j+n+2} = \begin{cases} sp+1, & \text{if } n \equiv 1 \mod 2, \\ sp, & \text{if } n \equiv 0 \mod 2. \end{cases} \tag{2.3}$$

Let j be such an index, and put

$$w_n := p_{j+n} - hq_{j+n}, \tag{2.4}$$

where $h \not\equiv 0 \mod p$ denotes some integer. The integers p_m and q_m are given by

$$\begin{array}{lll} p_{-1} := 1, & p_0 := 0, & p_{m+2} := a_{m+2}p_{m+1} + p_m \quad (m \geq -1), \\ q_{-1} := 0, & q_0 := 1, & q_{m+2} := a_{m+2}q_{m+1} + q_m \quad (m \geq -1). \end{array} \tag{2.5}$$

It follows easily from (2.5) that w_n, w_{n+1}, and w_{n+2} satisfy

$$w_{n+2} = a_{j+n+2} w_{n+1} + w_n \quad (1 \le n \le 2p). \tag{2.6}$$

We shall show by mathematical induction that the following congruences hold for all odd integers n with $1 \le n \le 2p-1$:

$$w_{n+1} \equiv w_2 \mod p \quad \text{and} \quad w_{n+2} \equiv w_1 + \frac{n+1}{2} w_2 \mod p. \tag{2.7}$$

For $n = 1$ we have, by (2.3) and (2.6),

$$w_3 = a_{j+3} w_2 + w_1 \equiv w_1 + \frac{1+1}{2} w_2 \mod p.$$

Now assume that (2.7) holds for some odd integer n with $1 \le n \le 2p-3$. Since $n+3$ is some even integer and since $3 < n+3 < 2p+2$, we conclude from (2.3), (2.6), and the induction hypothesis that

$$w_{n+3} = a_{j+n+3} w_{n+2} + w_{n+1} \equiv w_{n+1} \equiv w_2 \mod p. \tag{2.8}$$

Similarly, since $n+4$ is some odd integer with $3 < n+4 < 2p+2$,

$$\begin{aligned} w_{n+4} &= a_{j+n+4} w_{n+3} + w_{n+2} \equiv w_{n+3} + w_{n+2} \\ &\overset{(2.7),(2.8)}{\equiv} w_2 + \left(w_1 + \frac{n+1}{2} w_2 \right) = w_1 + \frac{n+3}{2} w_2 \mod p. \end{aligned}$$

Hence, (2.7) holds for all odd integers $1 \le n \le 2p-1$.

In what follows, we distinguish two cases:

Case 1. $w_2 \equiv 0 \mod p$.

This means, by (2.4),

$$p_{j+2} \equiv h q_{j+2} \mod p. \tag{2.9}$$

Case 2. $w_2 \not\equiv 0 \mod p$.

Since p is a prime, we even have $(p, w_2) = 1$. We write down the right-hand congruences in (2.7) for all odd integers $1 \le n \le 2p-1$:

$$\left. \begin{array}{rcl} w_3 & \equiv & w_1 + w_2 \\ w_5 & \equiv & w_1 + 2w_2 \\ w_7 & \equiv & w_1 + 3w_2 \\ w_9 & \equiv & w_1 + 4w_2 \\ & \vdots & \\ w_{2p+1} & \equiv & w_1 + p w_2 \end{array} \right\} \mod p.$$

By $(p, w_2) = 1$, the p integers $w_3, w_5, w_7, \ldots, w_{2p+1}$ represent p distinct residue classes with respect to the modulus p. Hence, there is some odd integer k with $3 \le k \le 2p+1$ and $w_k \equiv 0 \mod p$, which means

$$p_{j+k} \equiv h q_{j+k} \mod p. \tag{2.10}$$

Collecting together from (2.9) and (2.10), we have proved the existence of an integer k with $2 \le k \le 2p+1$ and

$$p_{j+k} \equiv h q_{j+k} \not\equiv 0 \mod p. \tag{2.11}$$

Since h is not divisible by the prime p, the assumption $p_{j+k} \equiv hq_{j+k} \equiv 0 \bmod p$ would imply that $(p_{j+k}, q_{j+k}) > 1$, which is well known to be impossible. Finally we get, by (2.1), (2.3), and (2.5),

$$\left|\xi - \frac{p_{j+k}}{q_{j+k}}\right| < \frac{1}{q_{j+k}q_{j+k+1}} \le \frac{1}{a_{j+k+1}q_{j+k}^2} \le \frac{1}{spq_{j+k}^2} < \frac{\varepsilon}{q_{j+k}^2}; \tag{2.12}$$

note that $3 \le k+1 \le 2p+2$. The integers j from (2.3) are not bounded, hence, by (2.11) and (2.12), there are infinitely many integers $u := p_{j+k}$ and $v := q_{j+k} > 0$ satisfying the conditions (1.5) and (1.6) of the theorem. Thus, the proof is complete.

Examples:

1.) Let $\varepsilon = 1$; (2.1) holds for every prime p with $s = 1$. Hence, the number

$$\xi := \langle 0; \overline{1, p} \rangle = \frac{\sqrt{p^2+4p} - p}{2}$$

belongs to the corresponding set $\mathcal{A}$ in Theorem 1.1.

2.) It is well known that

$$\left|\frac{1+\sqrt{5}}{2} - \frac{F_{n+1}}{F_n}\right| < \frac{1}{F_nF_{n+1}} \quad (n \ge 1).$$

Since F_n and F_{n+1} are coprime integers, and because any prime p divides infinitely many Fibonacci numbers $F_{n-1} = F_{n+1} - F_n$, for every prime p the congruence $F_n \equiv F_{n+1} \not\equiv 0 \bmod p$ is satisfied for infinitely many pairs of Fibonacci numbers F_n and F_{n+1}.

Proof of Theorem 1.2: Let $\mathcal{B} \subset (0, 1)$ be the subset of those numbers $\xi = \langle 0; a_1, a_2, a_3, \ldots \rangle$, where the elements $a_1, a_2, a_3, \ldots$ of the continued fraction expansions are given by the following scheme:

$$\begin{array}{lllllllll} a_2 & a_4 \\ a_6 & a_8 & a_{10} & a_{12} & a_{14} \\ a_{16} & a_{18} & a_{20} & a_{22} & a_{24} & a_{26} & a_{28} & a_{30} & a_{32} \\ & & & & \vdots \end{array} \quad =$$

$$\begin{array}{llllllllllllll} \alpha_1 \cdot 1 \cdot 2 & 1 \cdot 2 \\ \alpha_2 \cdot 2 \cdot 2 & 2 \cdot 2 & 2 \cdot 3 & 2 \cdot 3 & 2 \cdot 3 \\ \alpha_3 \cdot 3 \cdot 2 & 3 \cdot 2 & 3 \cdot 3 & 3 \cdot 3 & 3 \cdot 3 & 3 \cdot 4 & 3 \cdot 4 & 3 \cdot 4 & 3 \cdot 4 \\ \alpha_4 \cdot 4 \cdot 2 & 4 \cdot 2 & 4 \cdot 3 & 4 \cdot 3 & 4 \cdot 3 & 4 \cdot 4 & 4 \cdot 4 & 4 \cdot 4 & 4 \cdot 4 & 4 \cdot 5 & 4 \cdot 5 & 4 \cdot 5 & 4 \cdot 5 & 4 \cdot 5 \\ \alpha_5 \cdot 5 \cdot 2 & 5 \cdot 2 & \cdots \\ \vdots \end{array}$$

In the first row, the integers $\alpha_1, \alpha_2, \alpha_3, \ldots$ are arbitrary numbers from the set $\{1, 2\}$. Let the elements a_m with odd indices m be given by

$$a_m = 1 + a_{m+1} \quad (m \equiv 1 \bmod 2,\ m \ge 1).$$

Obviously, $\mathcal{B}$ is an uncountable set of numbers, since the set of sequences with elements from $\{1, 2\}$ is not countable.

If ξ belongs to $\mathcal{B}$ and if p is some prime, there are infinitely many finite sequences $r_1, r_2, r_3, \ldots, r_{2p}$ among the elements of the continued fraction expansion of ξ, where

$$r_\nu \equiv 1 \mod p \quad (\nu \equiv 1 \bmod 2),$$
$$\text{and} \quad r_\nu \equiv 0 \mod p \quad (\nu \equiv 0 \bmod 2).$$

Moreover, we can choose all the integers $r_1, r_2, r_3, \ldots, r_{2p}$ as large as possible for infinitely many such finite sequences.

Now infinitely many integers u and $v > 0$ satisfying (1.5) and (1.6) can be found in the same way as was shown in the proof of Theorem 1.1.

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AMS Classification Numbers: 11J04, 11J70, 11J83

IDENTITIES FOR A CLASS OF SUMS INVOLVING HORADAM'S GENERALIZED NUMBERS $\{W_n\}$

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(Submitted October 1996-Final Revision August 1997)

1. INTRODUCTION

Following Horadam [8], we consider the sequence $\{W_n = W_n(W_0, W_1; a, b)\}_{n=0}^{\infty}$ generated by the recurrence relation

$$W_{n+2} = aW_{n+1} - bW_n; \tag{1}$$

$W_0 = W_{n=0}$ and $W_1 = W_{n=1}$ are initial values. This sequence can be extended to negative subscripts and W_n has the Binet representation $(n = 0, \pm 1, \pm 2, \ldots)$

$$W_n = A\alpha^n + B\beta^n, \tag{2}$$

where A and B are constants and

$$\alpha, \beta = (a \pm \sqrt{a^2 - 4b}) / 2; \;\; \alpha\beta = b. \tag{3}$$

Expressions for sums involving Fibonacci numbers or Pell numbers have been given by many authors (e.g., [2], [3], [6], [10]-[12], [14]-[18]). The interested reader should consult these references as well as Bicknell's "primer" on Pell numbers [1]for further details and references.

The purpose of this article is to obtain a general identity for the following sum:

$$S(u; m; q, s, n) \equiv \sum_{r=0}^{n-1} r^m u^r W_{qr+s}. \tag{4a}$$

This identity provides a means of evaluating S, where $n \geq 1, m \geq 0$, q and s are integers, and u is an arbitrary parameter. This general identity, which consists of a sum of $(m+1)(m+4)$ terms, regardless of the value of n, collapses into a sum containing only 2 $(m+1)$ terms for certain values of u. This simpler identity applies to the following:

$$S_1(m, p, q, s; n) \equiv \sum_{r=0}^{n-1} r^m (U_p / U_{p+q})^r W_{qr+s}, \tag{4b}$$

for $p \neq 0, q \neq 0, q \neq -p$, as will be shown. Here

$$U_n \equiv (\alpha^n - \beta^n) / (\alpha - \beta); \;\; U_{-n} = -U_n / b^n. \tag{5}$$

The collapsing of the sum, from $(m+1)(m+4)$ terms to 2 $(m+1)$ terms, rests on the following identity:

$$U_p^r W_{qr+s} = \sum_{\ell=0}^{r} (-1)^{r-\ell} \binom{r}{\ell} b^{p(r-\ell)} U_q^\ell \, U_{q-p}^{r-\ell} W_{p\ell+s}, \tag{6}$$

where $r \geq 0, s, p\ (\neq 0)$, and $q\ (\neq 0, p)$ are arbitrary integers. This type of identity is usually referred to as a Fibonacci-binomial identity when it applies specifically to generalized Fibonacci or Lucas numbers. The reader may find an interesting approach to such identities in [4]. Layman

[13] was among the first to consider identities of this type, for Fibonacci numbers, but his results were only partial. Carlitz [3] subsequently gave a generalization in the form of (6), but again for Fibonacci numbers only. Haukkanen [7] recently extended the result of Carlitz by making use of exponential generating functions. His generalization applies to Lucas, Pell, and Pell-Lucas numbers. By contrast, the present approach is more general in some aspects to be clarified later, and it covers the results obtained by Carlitz and by Haukkanen as four special cases.

Several of the identities obtained in the references quoted earlier may be obtained as special cases of (4). Specific cases will be discussed in the closing section of this article.

2. ASSOCIATED GEOMETRIC POLYNOMIALS

Let $x \neq 1$ be a real variable and define the following polynomial for integers $n \geq 1$ and $m \geq 0$:

$$P_{n-1}^m(x) \equiv \sum_{r=0}^{n-1} r^m x^r. \tag{7}$$

$P_{n-1}^m(x)$ can be obtained from the geometric polynomial of degree $n-1$, $P_{n-1}(x) = \sum_{r=0}^{n-1} x^r$, by repeated use of the differential operator $D = xd/dx$:

$$P_{n-1}^m = D^m \sum_{r=0}^{n-1} x^r = \sum_{r=0}^{n-1} r^m x^r. \tag{8}$$

The convention that $r^0 = 1$ for all r, including $r = 0$, will be used throughout. We consider $P_{n-1}^m(x)$ briefly in what follows.

Let

$$f_\nu(x) \equiv \frac{x^\nu}{(1-x)}, \tag{9}$$

where ν is arbitrary, and observe that (7) may be written as

$$\sum_{r=0}^{n-1} r^m x^r = D^m f_0(x) - D^m f_n(x). \tag{10}$$

As a result, a study of the function

$$g_m^\nu(x) = D^m f_\nu(x) \tag{11}$$

will provide the means to calculate $\sum r^m x^r$. In general, one can show that [5]

$$g_m^\nu(x) = \sum_{\ell=0}^{m} a_\ell^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}}, \tag{12}$$

where the set of coefficients $\{a_\ell^m(\nu); \ell = 0, 1, \ldots, m\}$ is simply found by acting m times on $f_\nu(x)$ with the operator D. Here is a short list, for easy reference:

$m=0$: $a_0^0(\nu) = 1$;
$m=1$: $a_0^1(\nu) = \nu, a_1^1(\nu) = 1$;
$m=2$: $a_0^2(\nu) = \nu^2, a_1^2(\nu) = 2\nu+1, a_2^2(\nu) = 2$;
$m=3$: $a_0^3(\nu) = \nu^3, a_1^3(\nu) = 3\nu^2+3\nu+1, a_2^3(\nu) = 6\nu+6, a_3^3(\nu) = 6$;
$m=4$: $a_0^4(\nu) = \nu^4, a_1^4(\nu) = 4\nu^3+6\nu^2+4\nu+1$;
$a_2^4(\nu) = 12\nu^2+24\nu+14, a_3^4(\nu) = 24\nu+36, a_4^4(\nu) = 24$;

... and so on. A hierarchy of equations is easily found for the set of coefficients $\{a_\ell^{m+1}(\nu);\ \ell = 0, 1, 2, \ldots, m+1\}$ in terms of the set $\{a_\ell^m(\nu)\}$. Indeed, from (11) and (12), with m replaced by $m+1$,

$$g_{m+1}^\nu(x) \equiv D^{m+1} f_\nu(x) = \sum_{\ell=0}^{m+1} a_\ell^{m+1}(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}}. \tag{13}$$

But one can also write

$$g_{m+1}^\nu(x) \equiv D g_m^\nu(x) = D \sum_{\ell=0}^{m} a_\ell^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}} \tag{14}$$

when using (12) for $g_m^\nu(x)$. Operating on the sum with D then gives

$$g_{m+1}^\nu(x) = \sum_{\ell=0}^{m} (\nu+\ell) a_\ell^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}} + \sum_{\ell=0}^{m} (\ell+1) a_\ell^m(\nu) \frac{x^{\nu+\ell+1}}{(1-x)^{\ell+2}}. \tag{15}$$

To cast this result in the form of a sum over ℓ from 0 to $m+1$ [see (13)], first define $a_{m+1}^m(\nu) \equiv 0$ and $a_{-1}^m(\nu) \equiv 0$ to obtain, from (15),

$$g_{m+1}^\nu(x) = \sum_{\ell=0}^{m+1} (\nu+\ell) a_\ell^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}} + \sum_{\ell=0}^{m+1} \ell\, a_{\ell-1}^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}}. \tag{16}$$

This is achieved by extending the upper limit of the first sum in (15) to $m+1$ and by shifting ℓ to $\ell-1$ in the second sum; the value $\ell = 0$ can be included in the latter with the help of $a_{-1}^m(\nu) = 0$. A comparison of (13) and (16) then gives the desired recurrence for the unknown coefficients,

$$a_\ell^{m+1}(\nu) = (\nu+\ell) a_\ell^m(\nu) + \ell a_{\ell-1}^m(\nu); \tag{17}$$

here, $m = 0, 1, 2, \ldots$ and $\ell = 0, 1, 2, \ldots, m+1$. This equation was used to generate the list of coefficients $\{a_\ell^m(\nu);\ 0 \le \ell \le m,\ 0 \le m \le 4\}$ presented earlier in this section.

In what follows, the set of coefficients $\{a_\ell^m(\nu)\}$ is assumed to be known and will be used to obtain identities for (4a) and (4b).

3. OBTAINING EXPRESSIONS FOR $S(u; m, q, s, n)$ AND $S_1(m, p, q, s; n)$

To evaluate (4a) and (4b) explicitly, first multiply both sides of (10), (11), and (12) by $A\alpha^s$, where s is an arbitrary integer and get

$$\sum_r A\alpha^s r^m x^r = \sum_{\nu,\ell} \xi_\nu a_\ell^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}} A\alpha^s, \tag{18}$$

where $\nu = 0$ and n, with $\xi_0 = 1$, $\xi_n = -1$. The limits on the sum over ℓ are from 0 to m, as before. To simplify the notation, the limits on all sums will be omitted, as they always remain the same in what follows. Next, replace x by y, and $A\alpha^s$ by $B\beta^s$ in (18), add the resulting expression to (18), and set $x = u\alpha^q$, $y = u\beta^q$ to get

$$\sum_r r^m u^r W_{qr+s} = \sum_{\nu,\ell} \xi_\nu a_\ell^m(\nu) u^{\nu+\ell} \left[A \frac{\alpha^{q(\nu+\ell)+s}}{(1-u\alpha^q)^{\ell+1}} + B \frac{\beta^{q(\nu+\ell)+s}}{(1-u\beta^q)^{\ell+1}} \right]; \tag{19}$$

q is an arbitrary integer and u an arbitrary parameter such that $u\alpha^q \neq 1$ and $u\beta^q \neq 1$. We shall examine two cases, in turn: 1°) the denominators will be inverted and a binomial expansion made; 2°) the denominators will be made proportional to pure powers in α and β.

To invert the denominators without invoking infinite series, consider $(1-u\alpha^q)^{-1}$ first:

$$\begin{aligned}(1-u\alpha^q)^{-(\ell+1)} &= [(1-u\alpha^q)(1-u\beta^q)]^{-(\ell+1)}(1-u\beta^q)^{\ell+1}\\ &= N_q^{\ell+1}(u)(1-ub^q\alpha^{-q})^{\ell+1}\\ &= N_q^{\ell+1}(u)\sum_{j=0}^{\ell+1}(-1)^j\binom{\ell+1}{j}b^{qj}u^j\alpha^{-qj},\end{aligned} \tag{20}$$

where

$$N_q(u) \equiv [1-uV_q+u^2b^q]^{-1};\ V_q \equiv \alpha^q+\beta^q;\ V_{-q}=V_q/b^q. \tag{21}$$

A similar approach for the other denominator finally gives the desired identity for (4a):

$$\begin{aligned}S(u;m,q,s,n) &\equiv \sum_r r^m u^r W_{qr+s}\\ &= \sum_{\nu,\ell,j}(-1)^j\binom{\ell+1}{j}\xi_\nu a_\ell^m(\nu)N_q^{\ell+1}b^{qj}u^{\nu+\ell+j}W_{q(\nu+\ell-j)+s}.\end{aligned} \tag{22}$$

This general identity allows an evaluation of the left-hand sum in a closed form. Consider the case in which $m=0$, for example: $a_\ell^0(\nu)=1,\ \ell=0;\ a_\ell^0(\nu)=0,\ \ell\neq 0$. Then, $j=0,1$ and $\nu=0,n$, so that the right-hand expression reduces to only four terms:

$$\sum_{r=0}^{n-1}u^rW_{qr+s} = N_q[W_s-ub^qW_{-q+s}-u^nW_{qn+s}+u^{n+1}b^qW_{q(n-1)+s}]. \tag{23}$$

Now let us turn to the other situation in which the denominators in (19) are proportional to pure powers of α and β, say, when

$$1-u\alpha^q=\gamma\alpha^p;\ 1-u\beta^q=\gamma\beta^p, \tag{24}$$

with p and q integers; the parameters $u=u(p,q)$, $\gamma=\gamma(p,q)$ are to be determined. To do so, multiply the first equation of (24) by α^{-p}, the second by β^{-p}, and equate the results to get

$$\alpha^{-p}-u\alpha^{q-p}=\gamma=\beta^{-p}-u\beta^{q-p}. \tag{25}$$

For $p\neq 0$, $q\neq 0$, and $q\neq p$, this gives

$$u=U_{-p}/U_{q-p}=-U_p/(b^pU_{q-p}), \tag{26}$$

where U_n is defined in (5). Similarly, multiplying the first equation by α^{-q} and the second by β^{-q}, one finds, for $p\neq 0$, $q\neq 0$, and $q\neq p$:

$$\gamma=U_{-q}/U_{p-q}=U_q/(b^pU_{q-p}). \tag{27}$$

This set of values, (u,γ), satisfies (24), and insertion in (19) gives, for $\gamma\neq 0$:

$$\sum_r r^mu^rW_{qr+s}=\sum_{\nu,\ell}\xi_\nu a_\ell^m(\nu)\frac{u^{\nu+\ell}}{\gamma^{\ell+1}}W_{q(\nu+\ell)-p(\ell+1)+s}, \tag{28}$$

i.e., we get the identity ($p \neq 0, q \neq 0, q \neq -p$) referred to in (4b), namely,

$$\begin{aligned} S_1(m,p,q,s;n) &\equiv \sum_r r^m (U_p / U_{p+q})^r W_{qr+s} \\ &= \sum_{v,\ell} \xi_v a_\ell^m(v) b^{-p(\ell+1)} U_p^{v+\ell} U_q^{-(\ell+1)} U_{p+q}^{1-v} W_{q(v+\ell)+p(\ell+1)+s}, \end{aligned} \tag{29}$$

after replacing p by $-p$ and making use of the second form for γ in (27). As mentioned in the introduction, the right-hand expression in (29) only contains 2 $(m+1)$ terms, whereas that in (22) contains $(m+1)\,(m+4)$ such terms. As a result, (29) represents a very special class of identities. We now use (24) to establish a generalization of the Carlitz theorem [3].

From (24), consider the following,

$$(u\alpha^q)^r = (1-\gamma\alpha^p)^r \tag{30}$$

for $r = 0, 1, 2, \ldots$, where u and γ are as given in (26) and (27). On using the binomial expansion, we readily obtain the identity

$$u^r\alpha^{qr} = \sum_{j=0}^{r}(-1)^j \binom{r}{j} \gamma^j \alpha^{pj}; \tag{31}$$

a similar result also holds for $u^r\beta^{qr}$. Now multiply (31) by $A\alpha^s$, then substitute B for A, β for α, and add the two resulting equations to get

$$u^r W_{qr+s} = \sum_{j=0}^{r}(-1)^j \binom{r}{j} \gamma^j W_{pj+s}. \tag{32}$$

Inserting (u, γ) from (26), (27) then gives (6); this is, as stated earlier, a generalization of the "if" part of a theorem, due to Carlitz [3], for Fibonacci numbers, and recently extended by Haukkanen [7] to Lucas, Pell, and Pell-Lucas numbers. The present proof is simpler and more general. The present approach does not establish the "only if" part of those theorems, however.

The solutions u and γ can also be used to generate interesting associated identities similar to (6). Indeed, from

$$1^r = (u\alpha^q + \gamma\alpha^p)^r, \tag{33}$$

one obtains, for $p \neq 0, q \neq 0, q \neq -p, r = 0, 1, 2, \ldots$, and s an arbitrary integer,

$$b^{pr} U_q^r W_s = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} U_p^\ell U_{q+p}^{r-t} W_{q\ell+pr+s}. \tag{34}$$

The proof is similar to that of (6); thus, we omit the details. Similarly, starting with

$$(u\gamma\beta^{p+q})^{-r} = (\gamma^{-1}\beta^{-p} + u^{-1}\beta^{-q})^r, \tag{35}$$

one finds, again for $p \neq 0, q \neq 0, q \neq -p, r = 0, 1, 2, \ldots$, and s an arbitrary integer,

$$U_{2p+q}^r W_{qr+s} = \sum_{\ell=0}^{r} \binom{r}{\ell} b^{(p+q)(r-\ell)} U_p^{r-\ell} U_{p+q}^\ell W_{(2p+q)\ell-pr+s}. \tag{36}$$

Finally, consider

$$\frac{1}{u\beta^q} = \frac{1}{1-\gamma\beta^p} = N(p,q)(1-\gamma\alpha^p), \tag{37}$$

where

$$N(p,q) \equiv [1-\gamma V_p + \gamma^2 b^p]^{-1}. \tag{38}$$

In the right-hand side of (37), replace $1-\gamma\alpha^p$ by $u\alpha^q$ to obtain the identity

$$1 = b^q u^2(p,q)N(p,q), \tag{39}$$

from which we get, for $p \neq 0, q \neq p$:

$$\begin{aligned} N(p,q) &= b^q\, U_{p-q}^2 / U_p^2 \\ &= b^p\, U_{q-p}^2 / [b^p\, U_{q-p}^2 - V_p U_q U_{q-p} + U_q^2]. \end{aligned} \tag{40}$$

The last line follows from (38).

We now examine special cases of some of the identities that have been obtained.

4. ADDITIONAL PROPERTIES AND SOME APPLICATIONS

In this final section, we apply formulas (4b), (29), and (6) to specific cases, some of which will help support our earlier claim that certain identities contained in [2], [3], [6], [7], [10]-[12], and [14]-[18] are special cases of the present formulas.

We first establish the following, for n, $k = 0, \pm 1, \pm 2, \ldots$,

$$W_{n+k} = f_{k-1}W_{n+1} + f_{k-2}W_n, \tag{41}$$

where the coefficients $\{f_k\}$ are to be determined. (This formula will prove useful in comparing some of the identities presented here to some of the identities given [2], [3], [6], [7], [10]-[12] and [14]-[18], as mentioned in the introduction.) To do so, assume that (41) holds for all n, k and replace k by $k-1$:

$$W_{n+k-1} = f_{k-2}W_{n+1} + f_{k-3}W_n. \tag{42}$$

Multiply (41) by a, (42) by $-b$, and add the results to get

$$aW_{n+k} - bW_{n+k-1} = (af_{k-1} - bf_{k-2})W_{n+1} + (af_{k-2} - bf_{k-3})W_n. \tag{43}$$

Comparing this result to (1) then gives

$$W_{n+k+1} = (af_{k-1} - bf_{k-2})W_{n+1} + (af_{k-2} - bf_{k-3})W_n, \tag{44}$$

so if we let

$$f_k = af_{k-1} - bf_{k-2}, \tag{45}$$

then (41) is satisfied for all n, k. Thus, f_k can be written in the Binet form

$$f_k = A'\alpha^k + B'\beta^k, \tag{46}$$

where A' and B' are constants. The initial conditions on $\{f_k\}$ are obtained by setting $k = 2$ in (41), i.e.,

$$\begin{aligned} W_{n+2} &= f_1 W_{n+1} + f_0 W_n \\ &= aW_{n+1} - bW_n, \end{aligned} \tag{47}$$

where the second line follows from (1). Because n, A, and B are arbitrary, (47) implies

$$f_0 = -b;\ f_1 = a. \tag{48}$$

Use of (48) in (46) gives, after a little algebra,

$$f_k = aU_k + b^2 U_{k-1}. \tag{49}$$

For $a = 1, b = -1, f_k = F_{k+1}$, which is a well-known result.

Now, armed with these preliminaries, we turn to formulas (4b), (29), and (6). $S_1(m, p, q, s; n)$ is as defined in (4b); we also let [see (6)]

$$S_2(p, q, s; r) \equiv U_p^r W_{qr+s}. \tag{50}$$

S_1 and S_2 contain additional parameters, a, b, A, and B, but those are omitted in the notation for the sake of brevity. In this same spirit, the following nomenclature and notation will be used.

1. General Horadam (GH) Case

This is the most general case; previous notation will not be altered except that S_1 and S_2 will be written as S_1^{GH} and S_2^{GH}.

2. General Fibonacci (GF) Case

This is the special case where $a = 1, b = -1$; all previous notation will be used, except for S_1^{GF}, S_2^{GF}, and $W_n = H_n$ (to follow what now appears to be common practice). Also, $U_n = F_n$ and $V_n = L_n$, where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively.

3. Fibonacci (F) and Lucas (L) Cases

Then $a = 1, b = -1$ for both cases, while $A = -B = (\alpha - \beta)^{-1}$ for the F-case and $A = +B = 1$ for the L-case; we also write $S_1^{\text{F}}, S_1^{\text{L}}, S_2^{\text{F}}, S_2^{\text{L}}, W_n = F_n$, and $W_n = L_n$.

4. General Pell (GP) Case

This is the case where $a = 2, b = -1$. The notations $S_1^{\text{GP}}, S_2^{\text{GP}}$, and $W_n = H_n$ will be used (again to follow a very common practice); also, $U_n = P_n, V_n = R_n$, where P_n and R_n are the n^{th} Pell and Pell-Lucas numbers, respectively.

5. Pell (P) and Pell-Lucas (PL) Cases

Then $a = 2, b = -1$ for both cases, while $A = -B = (\alpha - \beta)^{-1}$ for the P-case and $A = +B = 1$ for the PL-case; we also write $S_1^{\text{P}}, S_1^{\text{PL}}, S_2^{\text{P}}, S_2^{\text{PL}}, W_n = H_n$ (for both), and $U_n = P_n, V_n = R_n$.

We now consider S_1 and S_2 in the special cases mentioned above.

I. $S_1(m, p, q, s; n)$

One has

$$S_1(m, -p-q, q, s; n) = \sum_r r^m b^{-qr} (U_{p+q} / U_p)^r W_{qr+s} \tag{51}$$

and

$$S_1(m, -p, -q, s; n) = b^{-s} \sum_r r^m (U_p / U_{p+q})^r \overline{W}_{qr+s} \tag{52}$$

where, for $n = 0, \pm 1, \pm 2, \dots,$

$$\overline{W}_n \equiv b^n W_{-n} = B\alpha^n + A\beta^n \tag{53}$$

will be called the transpose of W_n. The transpose is obtained by interchanging A and B in W_n. In general, $\overline{W}_n \neq W_n$. For Pell and Fibonacci numbers, $\overline{W}_n = -W_n$, while for Lucas and Pell-Lucas numbers, $\overline{W}_n = W_n$.

For $m = 0, p \neq 0, q \neq 0, q \neq -p, b \neq 0$, (29) gives

$$\begin{aligned} S_1(0,p,q,s;n) &\equiv \sum_{r=0}^{n-1} (U_p / U_{p+q})^r W_{qr+s} \\ &= [U_{p+q} W_{p+s} - U_p^n U_{p+q}^{1-n} W_{qn+p+s}] / (b^p U_q). \end{aligned} \tag{54}$$

Specializing further, with $W_{-1} = \overline{W}_1 / b$, $U_{-1} = -b^{-1}$, $U_2 = a$:

$$\begin{aligned} S_1(0,-1,3,0;n) &\equiv \sum_{r=0}^{n-1} (-ab)^r W_{3r} \\ &= ab[a\overline{W}_1 / b - a(-ab)^{-n} W_{3n-1}] / (a^2 - b). \end{aligned} \tag{55}$$

Set $a = 1, b = -1, A = -B = 1/(\alpha - \beta)$ in (55) to obtain

$$\sum_{r=0}^{n-1} F_{3r} = \frac{1}{2}(F_{3n-1} - 1); \tag{56}$$

this corresponds to Iyer's equation (6) in [12], since his result transforms into (56) after use of (41) and (49).

For $m = 3$, (29) gives, with $U_1 = 1$ and $U_2 = a$,

$$\begin{aligned} S_1(3,1,1,0;n) &= \sum_{r=0}^{n-1} r^3 a^{-r} W_r \\ &= \sum_{\nu=0,n} \sum_{\ell=0}^{3} \xi_\nu a_\ell^3(\nu) a^{1-\nu} b^{-(\ell+1)} W_{2\ell+\nu+1}. \end{aligned} \tag{57}$$

From the list of coefficients following (12), we get $a_\ell^3(\nu)$, $\ell = 0, 1, 2, 3$; then

$$\begin{aligned} S_1(3,1,1,0;n) &= a\left[\frac{W_3}{b^2} + 6\frac{W_5}{b^3} + 6\frac{W_7}{b^4}\right] \\ &\quad - a^{1-n}\left[n^3 \frac{W_{n+1}}{b} + (3n^2 + 3n + 1)\frac{W_{n+3}}{b^2} + (6n + 6)\frac{W_{n+5}}{b^3} + 6\frac{W_{n+7}}{b^4}\right]. \end{aligned} \tag{58}$$

Now, specializing to the Fibonacci case ($a = 1, b = -1, A = -B = 1/(\alpha - \beta), W_n = F_n$):

$$S_1^{\mathrm{F}}(3,1,1,0;n) = 50 + n^3 F_{n+1} - (3n^2 + 3n + 1)F_{n+3} + 6(n+1)F_{n+5} - 6F_{n+7}. \tag{59}$$

After using (41) and (49), this gives the identity

$$\sum_{r=0}^{n-1} r^3 F_r = 50 + (n^3 - 3n^2 + 9n - 19)F_{n+1} - (3n^2 - 15n + 31)F_{n+2}. \tag{60}$$

This result is identical to Harris's equation 17 in [6].

We now list a few other sums that can be evaluated using (29):

$$S_1^{\mathrm{GC}}(m,1,1,s;n)=\sum r^m a^{-r}W_{r+s};$$
$$S_1^{\mathrm{GF}}(m,1,1,s;n)=\sum r^m W_{r+s};$$
$$S_1^{\mathrm{GC}}(m,-2,3,s;n)=\sum r^m(-a/b^2)^r W_{3r+s};$$
$$S_1^{\mathrm{GP}}(m,-2,3,s;n)=\sum r^m(-2)^r W_{3r+s}.$$

II. $S_2(p,q,s;r)$

$$S_2^{\mathrm{GP}}(p,q,s;r)=P_p^r H_{qr+s}=\sum_{\ell=0}^{r}(-1)^{(p+1)(r-\ell)}\binom{r}{\ell}P_q^\ell P_{q-p}^{r-\ell}H_{p\ell+s}. \tag{61}$$

If we now set $A=-B=(\alpha-\beta)^{-1}$ (Pell: P) in this result, we obtain

$$S_2^{\mathrm{P}}(p,q,s;r)=P_p^r P_{qr+s}=\sum_{\ell=0}^{r}(-1)^{(p+1)(r-\ell)}\binom{r}{\ell}P_q^\ell P_{q-p}^{r-\ell}P_{p\ell+s}. \tag{62}$$

This is the "if" part of Haukkanen's Theorem 3 in [7]. Now, if $A=B=1$ (Pell-Lucas: PL),

$$S_2^{\mathrm{PL}}(p,q,s;r)=P_p^r P_{qr+s}=\sum_{\ell=0}^{r}(-1)^{(p+1)(r-\ell)}\binom{r}{\ell}P_q^\ell P_{q-p}^{r-\ell}R_{p\ell+s}. \tag{63}$$

This is the "if" part of Haukkanen's Theorem 4 in [7].

$$S_2^{\mathrm{GF}}(p,q,s;r)=F_p^r H_{qr+s}=\sum_{\ell=0}^{r}(-1)^{(p+1)(r-\ell)}\binom{r}{\ell}F_q^\ell F_{q-p}^{r-\ell}H_{p\ell+s}. \tag{64}$$

If $A=-B=(\alpha-\beta)^{-1}$ (F), then

$$S_2^{\mathrm{F}}(p,q,s;r)=F_p^r F_{qr+s}=\sum_{\ell=0}^{r}(-1)^{(p+1)(r-\ell)}\binom{r}{\ell}F_q^\ell F_{q-p}^{r-\ell}F_{p\ell+s}. \tag{65}$$

This is the "if" part of Carlitz's basic theorem [3]. [Note that (4.8) and (4.9) in [3] are in error due to an omission arising after (4.7).]

Finally, if $A=B=1$ (L), then

$$S_2^{\mathrm{L}}=F_p^r L_{qr+s}=\sum_{\ell=0}^{r}(-1)^{(p+1)(r-\ell)}\binom{r}{\ell}F_q^\ell F_{q-p}^{r-\ell}L_{p\ell+s}. \tag{66}$$

This is the "if" part of Haukkanen's Theorems 1 and 2 in [7]. In our treatment, there is no need to consider $s=0$ and $s\neq 0$ separately as Haukkanen does due to the fact that we only prove the "if" part of that theorem.

ACKNOWLEDGMENT

The author is greatly indebted to Dr. J. R. Gosselin for reading and commenting on this manuscript.

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AMS Classification Numbers: 11B37, 11B39

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A NOTE ON INITIAL DIGITS OF RECURRENCE SEQUENCES

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1. INTRODUCTION

The problem set in [3] is: What is the probability that initial digits of n^{th} Lucas and Fibonacci numbers have the same parity? We answer the problem and demonstrate a simple technique that provides answers on similar questions regarding relative frequency ("probability") of initial digits in almost any linear recurrence sequence.

The probability that a random number from the sequence X_n belongs to the set A (which has a certain property) is defined as the value of the limit (if it exists):

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} 1_A(X_i),$$

where 1_A denotes the characteristic function of the set A: $1_A(x)=1$ if $x\in A$, $1_A(x)=0$ if $x\notin A$.

The main tool in the proofs will be the well-known Weyl-Sierpinski equidistribution theorem [1] in its simplest form.

Theorem: Let q be an irrational number, $\widetilde{T}_n = p+nq$ be a sequence and $T_n=\{\widetilde{T}_n\}$ its fractional part. Then the probability that T_n is in the interval $[a,b)$, $0\le a<b\le 1$, is $b-a$. (The fractional part of irrational translation is uniformly distributed on $[0,1)$.)

2. CALCULATION OF PROBABILITIES

The following two lemmas prove that anything that is close enough to irrational translation is uniformly distributed on $[0,1)$. We will apply it to the logarithms of linear recursive sequences.

Lemma 1: Let $\widetilde{T}_n = p+nq$, q irrational, $T_n=\{\widetilde{T}_n\}$ its fractional part, and $\widetilde{X}_n$, $X_n=\{\widetilde{X}_n\}$ another sequence such that $\lim_{n\to\infty}|\widetilde{X}_n-\widetilde{T}_n|=0$. Then the probability that some X_n falls in the interval $A=[a,b)$, $0\le a<b\le 1$ is $b-a$.

Proof: Given $\varepsilon>0$, there exists n_1 such that, for each $m>n_1$, $|\widetilde{X}_m-\widetilde{T}_m|<\frac{\varepsilon}{4}$. If

$$A_\varepsilon=\left[a+\frac{\varepsilon}{4},\, b-\frac{\varepsilon}{4}\right),$$

this means that, for each $m\ge n_1$, $T_m\in A_\varepsilon$ implies $X_m\in A$. Equivalently, for each $m\ge n_1$, $1_A(X_m)\ge 1_{A_\varepsilon}(T_m)$.

There exist $n_0\ge n_1$ such that, for each $n>n_0$, $\frac{1}{n}\sum_{m=0}^{n_1-1}1_{A_\varepsilon}(T_m)\le\frac{\varepsilon}{2}$ (the sum is constant, so we choose n_0 large enough).

For each $n > n_0$, we calculate

$$\begin{aligned}\frac{1}{n}\sum_{m=0}^{n-1} 1_A(X_m) &\geq \frac{1}{n}\sum_{m=n_1}^{n-1} 1_A(X_m) \geq \frac{1}{n}\sum_{m=n_1}^{n-1} 1_{A_\varepsilon}(T_m) \\ &\geq \frac{1}{n}\sum_{m=n_1}^{n-1} 1_{A_\varepsilon}(T_m) + \frac{1}{n}\sum_{m=0}^{n_1-1} 1_{A_\varepsilon}(T_m) - \frac{\varepsilon}{2} \\ &= \frac{1}{n}\sum_{m=0}^{n-1} 1_{A_\varepsilon}(T_m) - \frac{\varepsilon}{2}.\end{aligned} \tag{1}$$

Applying the equidistribution theorem, we get

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} 1_A(X_m) \geq \lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} 1_{A_\varepsilon}(T_m) - \frac{\varepsilon}{2} = b - a - \varepsilon.$$

Since it is valid for each ε, $\liminf_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} 1_A(X_m) \geq b-a$. We apply the same reasoning for intervals $[0, a)$ and $[b, 1)$. Since $1_{[0,a)}(x) + 1_{[a,b)}(x) + 1_{[b,1)}(x) = 1$, we get

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} 1_{[a,b)}(X_m) &\leq 1 + \limsup_{n\to\infty}\left(-\frac{1}{n}\sum_{m=0}^{n-1} 1_{[0,a)}(X_m)\right) + \limsup_{n\to\infty}\left(-\frac{1}{n}\sum_{m=0}^{n-1} 1_{[b,1)}(X_m)\right) \\ &= 1 - \liminf_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} 1_{[0,a)}(X_m) - \liminf_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} 1_{[b,1)}(X_m) \leq b-a.\end{aligned} \tag{2}$$

Now we have $\liminf_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} 1_A(X_m) = \limsup_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} 1_A(X_m) = b-a$, so it converges and the lemma is proved.

The following lemma is a simple generalization that can be proved using the same technique (the proof is omitted).

Lemma 2: Let $\widetilde{T}_n = p + nq$, and let $\widetilde{X}_n^1, \widetilde{X}_n^2, \ldots, \widetilde{X}_n^k$ be k sequences such that, for each i, we have $\lim_{n\to\infty}|\widetilde{X}_n^i - \widetilde{T}_n| = 0$. Let q be irrational, and let $X_n^1, \ldots, X_n^k, T_n$ be the fractional parts of the sequences. Then the probability that, for random n, $X_n^1 \in [a_1, b_1), \ldots, X_n^k \in [a_k, b_k)$ is $b-a$, where

$$\bigcap_{i=1}^{k} [a_i, b_i) = [a, b).$$

Example 1: Probability that the first digit of F_n and that of L_n have the same parity is $\log_{10}\frac{648}{245}$.

Proof: Let $\widetilde{X}_n = \log_{10} F_n - \log_{10} p$, $\widetilde{Y}_n = \log_{10} L_n$, X_n, Y_n their fractional parts, $p = 1/\sqrt{5}$, and $q = (\sqrt{5}+1)/2$. As an example, we calculate the probability that, for given n, F_n begins with 1 and L_n begins with 3.

F_n begins with 1 if and only if, for some $k \in \mathcal{N}$, $F_n \in [10^k, 2\cdot 10^k)$, which is equivalent to

$$\begin{aligned} \log_{10} F_n &\in [k, \log_{10} 2 + k) \\ \Leftrightarrow \widetilde{X}_n = \log_{10} F_n - \log_{10} p &\in [k + \log_{10}\sqrt{5}, k + \log_{10} 2\sqrt{5}) \\ \Leftrightarrow X_n = \{\widetilde{X}_n\} &\in [\log_{10}\sqrt{5}, \log_{10} 2\sqrt{5}). \end{aligned} \tag{3}$$

L_n begins with 3 if and only if

$$Y_n = \{\log_{10} L_n\} \in [\log_{10} 3, \log_{10} 4). \tag{4}$$

Since $\widetilde{X}_n$ and $\widetilde{Y}_n$ asymptotically converge to $\widetilde{T}_n = n \log_{10} q$, $\log_{10} q$ irrational, we can apply Lemma 2. The probability is $\log_{10} 4/3$.

In the following table, we calculated all nonzero probabilities that, for random n, F_n begins with i and L_n begins with j (probability is $\log_{10} x$).

F_n	4	5	6	7	8	8	9	1	1	1	2	2	2	3	3	3	4	4
L_n	1	1	1	1	1	2	2	2	3	4	4	5	6	6	7	8	8	9
x	$\frac{\sqrt{5}}{2}$	$\frac{6}{5}$	$\frac{7}{4}$	$\frac{8}{7}$	$\frac{\sqrt{5}}{2}$	$\frac{9\sqrt{5}}{20}$	$\frac{10}{9}$	$\frac{3\sqrt{5}}{5}$	$\frac{4}{3}$	$\frac{\sqrt{5}}{2}$	$\frac{\sqrt{5}}{2}$	$\frac{6}{5}$	$\frac{\sqrt{5}}{2}$	$\frac{7\sqrt{5}}{15}$	$\frac{8}{7}$	$\frac{\sqrt{5}}{2}$	$\frac{9\sqrt{5}}{20}$	$\frac{10}{9}$

Summing the probabilities from the appropriate columns, we prove the formula. This probability (approximately 0.42241) is in accordance with the numerical test from [3]—4232 out of 10000.

In this example, we can avoid using Lemma 2, noting the fact that the initial digits of F_n and L_n are the same as the initial digits of $p \cdot q^n, q^n$. However, using the described technique, we can answer the same question about, e.g., 5th leftmost digits of F_n and L_n.

It can easily be proved (checking that $[(1-\sqrt{5})/2]^n$ is small enough for large n) that the entries in the table are the only possible ones (and not only with positive probability) [2].

Example 2: We will call a linear recurrence sequence Y_n *random enough* if the root q_1 of the characteristic polynomial that has the largest absolute value is real, positive, not a rational power of 10, unique and has multiplicity 1, and P_1 in equation (5) is positive.

The probability that a random enough recursive sequence begins with the digits 1997 is $\log_{10}(1+\frac{1}{1997})$.

Proof: We can then write the sequence in explicit form [4]:

$$Y_n = P_1 q_1^n + P_2(n) q_2^n + \cdots + P_k(n) q_k^n, \tag{5}$$

where P_1 is a real number and $P_2, \ldots, P_k$ are polynomials. Y_n begins with 1997 if and only if, for some $k \in \mathcal{N}$,

$$Y_n \in [1997 \cdot 10^k, 1998 \cdot 10^k) \Leftrightarrow \tag{6}$$

$$\Leftrightarrow \{\log_{10} Y_n\} \in [\log_{10} 1.997, \log_{10} 1.998). \tag{7}$$

Since $\lim_{n\to\infty} |\log_{10} Y_n - (\log_{10} P_1 + n \cdot \log_{10} q_1)| = 0$, we can apply Lemma 1. The probability is the length of the interval in (7).

We can prove the following formula in the same way.

Example 3: The probability that the i^{th} leftmost digit of a random enough recursive sequence is j obeys the generalized Benford's law (see [3] and [5]):

$$P = \log_{10} \prod_{k=10^{i-2}}^{10^{i-1}-1} \left(1 + \frac{1}{10k+j}\right)$$

for $i \geq 2$, and $P = \log_{10}(1+\frac{1}{j})$ for $i = 1$.

Lemma 1 implies as well that the fractional part of the logarithm of the random enough recursive sequence is uniformly distributed on $[0, 1)$.

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AMS Classification Numbers: 11B39, 11N37

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A CHAOTIC EXTENSION OF THE $3x+1$ FUNCTION TO $\mathbb{Z}_2[i]$

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(Submitted October 1996-Final Revision August 1997)

1. INTRODUCTION

The $3x+1$ problem is most elegantly expressed in terms of iteration of the function $T:\mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$T(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The $3x+1$ conjecture, which is generally attributed to Collatz [3], is that for every positive integer n, $T^{(k)}(n) = 1$ for some k.

The function T can be extended in a natural manner to the 2-adic integers, $\mathbb{Z}_2$, and this extension has proven to be quite fruitful. In this paper, we further extend the domain of T to $\mathbb{Z}_2[i]$, hoping to increase our understanding of the problem.

We construct an extension, $\widetilde{T}$, of the $3x+1$ function, $T:\mathbb{Z}_2 \to \mathbb{Z}_2$ to the metric space $(\mathbb{Z}_2[i], D)$, as follows.

Definition 1: Let $\widetilde{T}:\mathbb{Z}_2[i] \to \mathbb{Z}_2[i]$ by

$$\widetilde{T}(\alpha) = \begin{cases} \frac{\alpha}{2} & \text{if } \alpha \in [0], \\ \frac{3\alpha+1}{2} & \text{if } \alpha \in [1], \\ \frac{3\alpha+i}{2} & \text{if } \alpha \in [i], \\ \frac{3\alpha+1+i}{2} & \text{if } \alpha \in [1+i], \end{cases}$$

where $[x]$ denotes the equivalence class of x in $\mathbb{Z}_2[i]/2\mathbb{Z}_2[i]$.

Our main results separate naturally into three areas.

First, $\widetilde{T}$ is an extension of the original function and is nontrivial in the following sense.

Theorem A:
(a) $\widetilde{T}|\mathbb{Z}_2 = T$;
(b) $\widetilde{T}$ is not conjugate to $T \times T$ via a $\mathbb{Z}_2$-module isomorphism;
(c) $\widetilde{T}$ is, however, topologically conjugate to $T \times T$.

Second, $\widetilde{T}$ preserves the salient qualities of the original function. In particular, there is a "parity vector function," Q_∞, for T which has been extremely important in understanding the nature of the problem, see [1], [2], [5], [6], [7]. We show that Q_∞ can also be extended in an analogous manner. The original parity vector function and the ex-tended parity vector function share several important properties (c.f. [5], Theorem B). We simply state the results here, saving the details for later in the paper.

Theorem B: The extended parity vector function $\widetilde{Q}_k$ is periodic with period 2^k. In addition, $\widetilde{Q}_\infty$ is an isometric homeomorphism.

Finally, T and $T \times T$ are both chaotic functions (in the sense of [4]) and thus it follows from part (c) of Theorem A that

Theorem C: $\widetilde{T}$ is chaotic.

We have constructed $\widetilde{T}$ in the hope that the results above may lead to further insight into the $3x+1$ problem and similar problems.

2. BACKGROUND AND NOTATION

In this section, we establish notation and discuss the relevant background material. Lagarias [5] describes the history of the $3x+1$ problem and gives a survey of the literature on the subject. We follow his notation.

The sequence n, $T(n)$, $T^{(2)}(n)$, $T^{(3)}(n)$, ... is called the *orbit* of n under T. Another way to state the $3x+1$ conjecture is that, for all $n \in \mathbb{Z}^+$, the orbit of n under T enters the cycle $2 \to 1 \to 2 \to 1 \to \dots$. Since T extends naturally to the ring of 2-adic integers, $\mathbb{Z}_2$, the statement of the $3x+1$ problem is also valid on $\mathbb{Z}_2$. For brevity, we shall often refer to a 2-adic integer as simply a "2-adic." Recall that an element, a, of $\mathbb{Z}_2$ is just a formal power series of the form $\sum_{i=0}^{\infty} a_i 2^i$, where $a_i \in \{0, 1\}$. As is common, we will often abbreviate this by writing the sequence of 0s and 1s as $a_0, a_1, a_2, \dots_2$. Note that we add the subscript 2 (to distinguish from base 10) when writing 2-adics and use an overbar to denote a repeating pattern. Note also that both $\mathbb{Z}$ and the set of rationals with odd denominators are subrings of $\mathbb{Z}_2$ and thus, for clarity, we will frequently write an integer or rational number in place of its 2-adic representation. For example, $\overline{10}_2$ denotes the 2-adic $\sum_{i=0}^{\infty} 2^{2i}$ associated with $-\frac{1}{3}$.

Define the *parity vector of length k for T of a* [5] to be the sequence given by the function $Q_k : \mathbb{Z}_2 \to \mathbb{Z}_2 / 2^k \mathbb{Z}_2$ by

$$Q_k(a) = x_0(a), x_1(a), \dots, x_{k-1}(a),$$

where $x_i(n) \equiv T^{(i)}(n) \bmod 2$ and $x_i(n) \in \{0, 1\}$ for all $i \geq 0$. The parity vector, $Q_k(a)$, completely describes the behavior of the first k iterates of a under T. $Q_\infty(a)$ is defined in a similar manner and completely describes all iterates of a under T.

Q_k and Q_∞ have several interesting properties: Q_k is periodic with period 2^k and induces a permutation of $\mathbb{Z}_2 / 2^k \mathbb{Z}_2$, denoted $\overline{Q}_k$; Q_∞ is a continuous bijection. The proofs of these properties of Q_k and Q_∞ may be found in [5]. Both have proven to be extremely useful in the study of the $3x+1$ problem.

In this paper we extend T to the 2-adic integers adjoined with i, $\mathbb{Z}_2[i]$. We choose to extend to $\mathbb{Z}_2[i]$ because many number theoretic problems in $\mathbb{Z}$ have been solved by generalizing to the Gaussian integers $\mathbb{Z}[i]$. In keeping with this theme, we shall refer to $\mathbb{Z}_2[i]$ as the set of *Gaussian 2-adic integers* or, simply, the *Gaussian 2-adics*.

By freely associating $a+bi \in \mathbb{Z}_2[i]$ with $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ we can define the metric D on $\mathbb{Z}_2[i]$ to be the product metric on $\mathbb{Z}_2 \times \mathbb{Z}_2$ induced by the usual metric on $\mathbb{Z}_2$ which is derived from the 2-adic valuation. Addition and multiplication in $\mathbb{Z}_2[i]$ are defined in the usual manner. It is important to note that $\mathbb{Z}_2[i]$ is a commutative ring with identity, but not a field. In addition, $\mathbb{Z}_2$ is a commutative subring of $\mathbb{Z}_2[i]$ with identity and is also not a field.

3. EXTENSION TO $\mathbb{Z}_2[i]$

Since T was piecewise linear depending on equivalence in $\mathbb{Z}/2\mathbb{Z}$, our extension $\widetilde{T}$ is piecewise linear depending on equivalence in $\mathbb{Z}_2[i]/2\mathbb{Z}_2[i]=\{[0],[1],[i],[1+i]\}$.

Definition 1: Let $\widetilde{T}:\mathbb{Z}_2[i]\to\mathbb{Z}_2[i]$ by

$$\widetilde{T}(\alpha)=\begin{cases}\frac{\alpha}{2} & \text{if } \alpha\in[0],\\ \frac{3\alpha+1}{2} & \text{if } \alpha\in[1],\\ \frac{3\alpha+i}{2} & \text{if } \alpha\in[i],\\ \frac{3\alpha+1+i}{2} & \text{if } \alpha\in[1+i].\end{cases}$$

Notice that $\widetilde{T}$ resembles $T\times T$ to a great degree. It is natural to ask how $\widetilde{T}$ is different from T and $T\times T$; after all, we claim that $\widetilde{T}$ is a nontrivial extension of T. It is easily seen that $\widetilde{T}$ is an extension of T, i.e., $\widetilde{T}|\mathbb{Z}_2=T$. It is also clear that $\widetilde{T}$ is not the trivial extension $T\times T:\mathbb{Z}_2\times\mathbb{Z}_2\to\mathbb{Z}_2\times\mathbb{Z}_2$, e.g., $T\times T((1,2))=(2,1)$, while $\widetilde{T}(1+2i)=2+3i$. What is surprising, though, is that $\widetilde{T}$ and $T\times T$ are not conjugate via a $\mathbb{Z}_2$-module isomorphism, as we show below. (However, they are topologically conjugate, as we shall see in Section 6.)

In order to show this, we must formalize our association between elements of $\mathbb{Z}_2[i]$ and $\mathbb{Z}_2\times\mathbb{Z}_2$. Define a continuous bijection $B:\mathbb{Z}_2[i]\to\mathbb{Z}_2\times\mathbb{Z}_2$ by $B(a+bi)=(a,b)$. Let $\hat{T}:\mathbb{Z}_2\times\mathbb{Z}_2\to\mathbb{Z}_2\times\mathbb{Z}_2$ by $\hat{T}=B\circ\widetilde{T}\circ B^{-1}$.

Theorem 1: There is no $\mathbb{Z}_2$-module isomorphism $A:\mathbb{Z}_2\times\mathbb{Z}_2\to\mathbb{Z}_2\times\mathbb{Z}_2$ such that

$$\hat{T}=A^{-1}\circ T\times T\circ A.$$

Proof: Assume that such a $\mathbb{Z}_2$-module isomorphism A exists. Let $e_1=(1,0)$ and $e_2=(0,1)$. Then $Ae_1=(x_1,y_1)$ and $Ae_2=(x_2,y_2)$, where x_1,x_2,y_1, and y_2 are 2-adics and $(a,b)\in\mathbb{Z}_2\times\mathbb{Z}_2$. Then, for all $a,b\in\mathbb{Z}_2\times\mathbb{Z}_2$, $A\circ\hat{T}((a,b))=T\times T\circ A((a,b))$. Let $(a,b)\in(1,0)$.

$$\begin{aligned}T\times T(A((a,b))) &= A(\hat{T}((a,b)))\\ \Rightarrow T\times T\left(\begin{pmatrix}x_1 & x_2\\ y_1 & y_2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\right) &= A\left(\left(\frac{3a+1}{2},\frac{3b}{2}\right)\right)\\ \Rightarrow T\times T((ax_1+bx_2,ay_1+by_2)) &= \begin{pmatrix}x_1 & x_2\\ y_1 & y_2\end{pmatrix}\begin{pmatrix}\frac{3a+1}{2}\\ \frac{3b}{2}\end{pmatrix}\\ \Rightarrow T\times T((ax_1+bx_2,ay_1+by_2)) &= \left(\frac{3ax_1+3bx_2+x_1}{2},\frac{3ay_1+3by_2+y_1}{2}\right).\end{aligned}$$

Thus, we have

$$T\times T((ax_1+bx_2,ay_1+by_2))=\left(\frac{3ax_1+3bx_2+x_1}{2},\frac{3ay_1+3by_2+y_1}{2}\right).$$

In order to evaluate $T\times T((ax_1+bx_2,ay_1+by_2))$, we must determine the parities of ax_1+bx_2 and ay_1+by_2. Because b is even and a is odd, the parities are completely determined by, and equivalent to, the parities of x_1 and y_1. This yields the following four cases:

$$T\times T((ax_1+bx_2, ay_1+by_2)) = \begin{cases} \left(\frac{ax_1+bx_2}{2}, \frac{ay_1+by_2}{2}\right) & \text{if } x_1 \text{ even, } y_1 \text{ even,} \\ \left(\frac{3ax_1+3bx_2+1}{2}, \frac{ay_1+by_2}{2}\right) & \text{if } x_1 \text{ odd, } y_1 \text{ even,} \\ \left(\frac{ax_1+bx_2}{2}, \frac{3ay_1+3by_2+1}{2}\right) & \text{if } x_1 \text{ even, } y_1 \text{ odd,} \\ \left(\frac{3ax_1+3bx_2+1}{2}, \frac{3ay_1+3by_2+1}{2}\right) & \text{if } x_1 \text{ odd, } y_1 \text{ odd.} \end{cases}$$

From this it is easy to check that

$$T\times T((ax_1+bx_2, ay_1+by_2)) = \left(\frac{3ax_1+3bx_2+x_1}{2}, \frac{3ay_1+3by_2+y_1}{2}\right)$$

if and only if $x_1=1$ and $y_1=1$. Thus, A must be of the form $\left(\begin{smallmatrix}1 & x_2\\ 1 & y_2\end{smallmatrix}\right)$. Similarly, choosing $(a,b)\in(0,1)$ implies $x_2=y_2=1$.

This means $A=\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$, but $\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ is not invertible, which contradicts our assumption. □

Since $\hat{T}$ is conjugate to $\widetilde{T}$ via the $\mathbb{Z}_2$-module isomorphism B, we have

Corollary 1: $\widetilde{T}$ is not conjugate to $T\times T$ via a $\mathbb{Z}_2$-module isomorphism.

4. EXTENSION OF Q_k AND Q_∞ TO $\mathbb{Z}_2[i]$

One of our main reasons for extending to $\mathbb{Z}_2[i]$ was to add the tools associated with $\mathbb{Z}_2[i]$ to the current tools for studying the $3x+1$ problem. With this in mind, we redefine Q_k in terms of $\widetilde{T}$. Let $\widetilde{Q}_k:\mathbb{Z}_2[i]\to\mathbb{Z}_2[i]/2^k\mathbb{Z}_2[i]$ by $\widetilde{Q}_k(\alpha)=\widetilde{x}_0(\alpha), \widetilde{x}_1(\alpha), \widetilde{x}_2(\alpha), \ldots, \widetilde{x}_{k-1}(\alpha)$, where $\widetilde{x}_i(\alpha)\equiv\widetilde{T}^{(i)}(\alpha) \bmod 2$ for all $i\geq 0$ and $\widetilde{x}_i(\alpha)\in\{0,1,i,1+i\}$ be the *parity vector of length k for $\widetilde{T}$ of α*.

As with Q_k, $\widetilde{Q}_k$ completely describes the behavior of the first k iterates of α under $\widetilde{T}$.

We also define $\widetilde{Q}_\infty:\mathbb{Z}_2[i]\to\mathbb{Z}_2[i]$ in a similar manner and note that, as one would expect, $\widetilde{Q}_\infty$ completely describes the behavior of all iterates of α under $\widetilde{T}$.

$\widetilde{Q}_k$ and $\widetilde{Q}_\infty$ have properties similar to Q_k and Q_∞ as will be demonstrated in the following theorems which mirror analogous theorems for Q_k and Q_∞ found in [5].

Theorem 2: The function $\widetilde{Q}_k:\mathbb{Z}_2[i]\to\mathbb{Z}_2[i]/2^k\mathbb{Z}_2[i]$ is periodic with period 2^k.

In order to show that $\widetilde{Q}_k$ is periodic, we begin by showing

Lemma 1: $\widetilde{T}^k(\alpha+\omega 2^k)\equiv\widetilde{T}^k(\alpha)+\omega \bmod 2$, for any $\alpha,\omega\in\mathbb{Z}_2[i]$.

Proof: We proceed by induction on k. Let $\alpha,\omega\in\mathbb{Z}_2[i]$.

Base Case (k = 1). In this case,

$$\widetilde{T}(\alpha+\omega 2) = \begin{cases} \frac{\alpha}{2}+\omega\equiv\widetilde{T}(\alpha)+\omega & \bmod 2 & \text{if } \alpha\in[0], \\ \frac{3\alpha+1}{2}+3\omega\equiv\widetilde{T}(\alpha)+\omega & \bmod 2 & \text{if } \alpha\in[1], \\ \frac{3\alpha+i}{2}+3\omega\equiv\widetilde{T}(\alpha)+\omega & \bmod 2 & \text{if } \alpha\in[i], \\ \frac{3\alpha+1+i}{2}+3\omega\equiv\widetilde{T}(\alpha)+\omega & \bmod 2 & \text{if } \alpha\in[1+i]. \end{cases}$$

General Case: Assume $\widetilde{T}^{k-1}(\alpha+\omega 2^{k-1}) \equiv \widetilde{T}^{k-1}(\alpha)+\omega \mod 2$ for all n (inductive hypothesis).

Case 1: $\alpha \in [0]$. Then

$$\begin{aligned}
\widetilde{T}^{k}(\alpha+\omega 2^k) &= \widetilde{T}^{k-1}(\widetilde{T}(\alpha+\omega 2^k)) & \\
&= \widetilde{T}^{k-1}\left(\tfrac{\alpha+\omega 2^k}{2}\right) & \text{(since } \alpha \in [0]) \\
&= \widetilde{T}^{k-1}\left(\tfrac{\alpha}{2}+\omega 2^{k-1}\right) & \\
&\equiv \widetilde{T}^{k-1}\left(\tfrac{\alpha}{2}\right)+\omega \mod 2 & \text{(by ind. hyp.)} \\
&\equiv \widetilde{T}^{k}(\alpha)+\omega \mod 2 & \text{(since } \alpha \in [0]).
\end{aligned}$$

Case 2: $\alpha \in [1]$. Then

$$\begin{aligned}
\widetilde{T}^{k}(\alpha+\omega 2^k) &= \widetilde{T}^{k-1}(\widetilde{T}(\alpha+\omega 2^k)) & \\
&= \widetilde{T}^{k-1}\left(\tfrac{3(\alpha+\omega 2^k)+1}{2}\right) & \text{(since } \alpha \in [1]) \\
&= \widetilde{T}^{k-1}\left(\tfrac{3\alpha+1}{2}+\omega 2^k+\omega 2^{k-1}\right) & \\
&\equiv \widetilde{T}^{k-1}\left(\tfrac{3\alpha+1}{2}+\omega 2^k\right)+\omega \mod 2 & \text{(by ind. hyp.)} \\
&\equiv \widetilde{T}^{k-1}\left(\tfrac{3\alpha+1}{2}+\omega 2^{k-1}\right) \mod 2 & \text{(by ind. hyp.)} \\
&\equiv \widetilde{T}^{k-1}\left(\tfrac{3\alpha+1}{2}\right)+\omega \mod 2 & \text{(by ind. hyp.)} \\
&\equiv \widetilde{T}^{k}(\alpha)+\omega \mod 2 & \text{(since } \alpha \in [1]).
\end{aligned}$$

Case 3 ($\alpha \in [i]$) and Case 4 ($\alpha \in [1+i]$) are very similar to this case. Thus, $\widetilde{T}^{k}(\alpha+\omega 2^k) \equiv \widetilde{T}^{k}(\alpha)+\omega \mod 2$ for all k by induction on k. □

It follows easily that $\widetilde{x}_k$ is also periodic in the same sense.

Corollary 2: For every $\alpha, \omega \in \mathbb{Z}_2[i]$, $\widetilde{x}_j(\alpha+\omega 2^j) \equiv \widetilde{x}_j(\alpha)+\omega \mod 2$ for all $0 \le j \le \infty$.

From this we obtain Theorem 2.

Proof of Theorem 2: We proceed using induction on k.

Base Case (k = 1):

$$\begin{aligned}
\widetilde{Q}_1(\alpha+2\omega) &= \widetilde{x}_0(\alpha+2\omega) & \\
&= \widetilde{x}_0(\alpha) & \text{(by Cor. 2)} \\
&= \widetilde{Q}_1(\alpha). &
\end{aligned}$$

General Case: Assume $\widetilde{Q}_{k-1}(\alpha+\omega 2^{k-1}) = \widetilde{Q}_{k-1}(\alpha)$ (by inductive hypothesis). Then

$$\begin{aligned}
\widetilde{Q}_k(\alpha+\omega 2^k) &= \textstyle\sum_{j=0}^{k-1} \widetilde{x}_j(\alpha+\omega 2^k)2^j & \\
&= \textstyle\sum_{j=0}^{k-2} \widetilde{x}_j(\alpha+\omega 2^k)2^j + \widetilde{x}_{k-1}(\alpha+\omega 2^k)2^{k-1} & \\
&= \widetilde{Q}_{k-1}(\alpha+\omega 2^k) + \widetilde{x}_{k-1}(\alpha)2^{k-1} & \text{(by Cor. 2)} \\
&= \widetilde{Q}_{k-1}(\alpha) + \widetilde{x}_{k-1}(\alpha)2^{k-1} & \text{(by ind. hyp.)} \\
&= \textstyle\sum_{j=0}^{k-2} \widetilde{x}_j(\alpha)2^j + \widetilde{x}_{k-1}(\alpha)2^{k-1} & \\
&= \textstyle\sum_{j=0}^{k-1} \widetilde{x}_j(\alpha)2^j = \widetilde{Q}_k(\alpha), &
\end{aligned}$$

as required. □

Theorem 3: $\widetilde{Q}_\infty$ is an isometric homeomorphism.

Proof: We begin by showing that $\widetilde{Q}_\infty$ is one-to-one.

Let $\alpha, \beta \in \mathbb{Z}_2[i]$, $\alpha \neq \beta$. Then there exists $\omega \in \mathbb{Z}_2[i]$ such that $\alpha = \beta + \omega 2^k$, where $k = \min\{j : \alpha_j \neq \beta_j\}$, $\alpha = \alpha_0, \alpha_1, \ldots$, $\beta = \beta_0, \beta_1, \ldots$, and ω is not equivalent to 0 mod 2. By Corollary 2, $\widetilde{x}_k(\alpha) \equiv \widetilde{x}_k(\beta + \omega 2^k) \equiv \widetilde{x}_k(\beta) + \omega$ mod 2. Consequently, $\widetilde{x}_k(\alpha) - \widetilde{x}_k(\beta) \equiv \omega$ mod 2. Since ω is not equivalent to 0 mod 2, $\widetilde{x}_k(\alpha) \neq \widetilde{x}_k(\beta)$ and, therefore, by definition of $\widetilde{Q}_\infty$, $\widetilde{Q}_\infty(\alpha) \neq \widetilde{Q}_\infty(\beta)$. Thus, $\widetilde{Q}_\infty$ is one-to-one.

Next, we show that $\widetilde{Q}_\infty$ preserves the metric (and is therefore continuous).

Let $\alpha, \beta \in \mathbb{Z}_2[i]$, where $\alpha = \alpha_0, \alpha_1, \ldots, \beta = \beta_0, \beta_1, \ldots$. Choose k so that $D(\alpha, \beta) = 2^{-k}$. Then $\alpha \equiv \beta$ mod 2^k. By Theorem 2, $\widetilde{Q}_k(\alpha) = \widetilde{Q}_k(\beta)$, so $\widetilde{Q}_\infty(\alpha) \equiv \widetilde{Q}_\infty(\beta)$ mod 2^k. Thus, $D(\widetilde{Q}_\infty(\alpha), \widetilde{Q}_\infty(\beta)) \leq 2^{-k}$. However, because α is not equivalent to β mod 2^{k+1}, $\alpha = \beta + \omega 2^k$ for some $\omega \in \mathbb{Z}_2[i]$, where ω is not equivalent to 0 mod 2. Hence, $\widetilde{x}_k(\alpha) = \widetilde{x}_k(\beta + \omega 2^k) \equiv \widetilde{x}_k(\beta) + \omega$ mod 2 by Corollary 2. However, because ω is not equivalent to 0 mod 2, $\widetilde{x}_k(\alpha) \neq \widetilde{x}_k(\beta)$. It follows that $\widetilde{Q}_\infty(\alpha)$ is not equivalent to $\widetilde{Q}_\infty(\beta)$ mod 2^{k+1}. Therefore, $D(\widetilde{Q}_\infty(\alpha), \widetilde{Q}_\infty(\beta)) = 2^{-k}$ and so $\widetilde{Q}_\infty$ preserves the metric.

Finally, we show that $\widetilde{Q}_\infty$ is onto.

Let $\alpha = \alpha_0, \alpha_1, \ldots \in \mathbb{Z}_2[i]$, $\alpha'_k = \alpha_0, \ldots, \alpha_k$, $\overline{0} \in \mathbb{Z}_2[i]$, and let $\hat{\alpha}_k \in \mathbb{Z}_2[i]/2^k\mathbb{Z}_2[i]$ such that $\alpha \in \hat{\alpha}_k$. We first note that $\widetilde{Q}_k$ is onto, as can be seen by induction on k using Corollary 2. There exists a β'_k such that $\widetilde{Q}_k(\beta'_k) = \hat{\alpha}_k$. Let $\beta'_k = \beta_0, \ldots, \beta_k, \overline{0}$. We can see that $\widetilde{Q}_\infty(\beta'_k) = \hat{\alpha}'_k$ mod 2^k. Thus, $\lim_{k\to\infty} D(\widetilde{Q}_\infty(\beta), \alpha'_k)) = 0$. Consequently, $\lim_{k\to\infty} \widetilde{Q}_\infty(\beta'_k) = \lim_{k\to\infty} \alpha'_k = \alpha$. Now, since $\widetilde{Q}_\infty$ is continuous, $\lim_{k\to\infty} \widetilde{Q}_\infty(\beta'_k) = \widetilde{Q}_\infty(\lim_{k\to\infty} \beta'_k) = \alpha$. Therefore, all that remains is to show that $\lim_{k\to\infty} \beta'_k$ exists as a Gaussian 2-adic. Since the sequence $\{\widetilde{Q}_\infty(\beta'_k)\}$ converges to α, it is Cauchy. Because $\widetilde{Q}_\infty$ preserves the metric, the sequence $\{\beta'_k\}$ in $\mathbb{Z}_2[i]$ is also a Cauchy sequence. Now, $\mathbb{Z}_2[i]$ is a compact metric space by the Tychonoff theorem, so every Cauchy sequence in $\mathbb{Z}_2[i]$ has a limit in $\mathbb{Z}_2[i]$. Thus, the sequence $\{\beta'_k\}$ converges to some $\beta \in \mathbb{Z}_2[i]$ and $\widetilde{Q}_\infty(\beta) = \alpha$. Therefore, $\widetilde{Q}_\infty$ is onto.

$\widetilde{Q}_\infty^{-1}$ is continuous because $\widetilde{Q}_\infty$ is an isometry; thus, $\widetilde{Q}_\infty$ is an isometric homeomorphism. □

Now that we have shown $\widetilde{Q}_\infty$ is a continuous bijection, we shall see just how powerful a tool it is in our exploration of the dynamics of $\widetilde{T}$.

5. CHAOS AND THE 3x + 1 PROBLEM

A map $F:(X,d) \to (X,d)$ is defined to be *transitive* if $\forall x, y \in X$, $\forall \varepsilon > 0$, $\exists z \in X$ such that $d(x,z) < \varepsilon$ and $d(y, F^{(k)}(z)) < \varepsilon$ for some $k \geq 0$ [4]. Devaney [4] defines a *chaotic map* to be a transitive map with dense periodic points. Chaoticity in this sense is preserved by topological conjugacy, so we can show that a function is chaotic if it is topologically conjugate to a known chaotic map. Such a map is the shift map on the sequence space, $\sigma : \Sigma \to \Sigma$, where

$$\Sigma = \{s_0, s_1, s_2, \ldots | s_j \in \{0, 1\}\} \quad \text{and} \quad \sigma(s_0, s_1, s_2, \ldots) = s_1, s_2, s_3, \ldots.$$

Q_∞ provides a conjugacy between T and σ. Thus, we have shown

Theorem 4: $T:\mathbb{Z}_2 \to \mathbb{Z}_2$ is chaotic.

It should be noted, however, that the restriction of T to $\mathbb{Z}^+$, an invariant subdomain of $\mathbb{Z}_2$, is not chaotic, according to the $3x+1$ conjecture.

Since the product of chaotic maps is itself a chaotic map, we have

Corollary 3: $T \times T$ is chaotic.

Certainly if T is chaotic, any reasonable extension of T should also be chaotic. We show that $\widetilde{T}$ is chaotic by showing that it is conjugate to some known chaotic system. With this in mind, we define $\sigma_4:(\Sigma_4, d_\delta) \to (\Sigma_4, d_\delta)$ and show that σ_4 and $\sigma \times \sigma$ are conjugate via a homeomorphism, F.

Let $\sigma_4:(\Sigma_4, d_\delta) \to (\Sigma_4, d_\delta)$ be the shift map on the sequence space with four elements $\{0, 1, i, i+1\}$, where

$$d_\delta((s_0, s_1, \ldots), (t_0, t_1, \ldots)) = \sum_{k=0}^{\infty} \frac{\delta_k(s, t)}{4^k}$$

and

$$\delta_k = \begin{cases} 0 & \text{if } s_k = t_k, \\ 1 & \text{otherwise.} \end{cases}$$

It can easily be shown that $\sigma_4:(\Sigma_4, d_\delta) \to (\Sigma_4, d_\delta)$ is chaotic.

Lemma 2: The function $F:(\Sigma_4, d_\delta) \to \Sigma \times \Sigma$ by

$$F(s) = ((a_1(s), a_2(s), a_3(s), \ldots), (b_1(s), b_2(s), b_3(s), \ldots)),$$

where each

$$a_i(s) = \begin{cases} 0 & \text{if } s_i = 0 \text{ or } i, \\ 1 & \text{if } s_i = 1 \text{ or } 1+i, \end{cases}$$

and each

$$b_i(s) = \begin{cases} 0 & \text{if } s_i = 0 \text{ or } 1, \\ 1 & \text{if } s_i = i \text{ or } 1+i, \end{cases}$$

is a homeomorphism.

Proof: It is clear that F is a bijection. We now show that F is continuous. Let $\varepsilon > 0$, $\delta = 4^{-k}$, where k is chosen to make $2^{-k} < \varepsilon$, $s = s_0, s_1, \ldots \in \Sigma_4$, $t = t_0, t_1, \ldots \in \Sigma_4$. If $d_\delta(s, t) < \delta$, then $s_j = t_j$ for all $0 \le j \le k$. Consequently, if we consider that $F(s) = (x, y)$ and $F(t) = (z, w)$ for some $(x, y), (z, w) \in \Sigma \times \Sigma$, where $x = x_0, x_1, \ldots \in \Sigma$, $y = y_0, y_1, \ldots \in \Sigma$, $z = z_0, z_1, \ldots \in \Sigma$, and $w = w_0, w_1, \ldots \in \Sigma$, then $x_j = z_j$ and $y_j = w_j$ for all $0 \le j \le k$ by definition of d_δ. Thus, by definition of F, $d_x(F(s), F(t)) \le 2^{-k} < \varepsilon$, where d_x is the product metric on Σ. So F is continuous.

By letting $\varepsilon > 0$ and choosing $\delta = 2^{-k}$, where k is such that $4^{-k} < \varepsilon$, we can apply a similar argument to show that F^{-1} is continuous. Therefore, F is a homeomorphism. □

It easily follows that

Corollary 4: σ_4 and $\sigma \times \sigma$ are conjugate via F.

Since $\widetilde{T}$ is conjugate to σ_4 via $\widetilde{Q}_\infty$, we have

Theorem 5: $\widetilde{T}$ is chaotic.

It turns out that, in proving the chaoticity of T, $\widetilde{T}$, and $T \times T$, we have defined some very useful conjugacies, as we shall see in the next section.

6. RELATIONSHIP BETWEEN $\widetilde{T}$ AND $T \times T$

Though $\widetilde{T}$ and $T \times T$ are not conjugate via a $\mathbb{Z}_2$-module isomorphism, they are topologically conjugate. Since topological conjugacy is transitive, $\widetilde{T} \cong \sigma_4 \cong \sigma \times \sigma \cong T \times T$, where $\cong$ denotes topological conjugacy and thus

Theorem 6: $\widetilde{T}$ and $T \times T$ are topologically conjugate (via $(Q_\infty \times Q_\infty)^{-1} \circ F \circ \widetilde{Q}_\infty$).

These theorems allow us to work in the system of our choice and then convert the results to any other system using the homeomorphisms defined above.

7. ACKNOWLEDGMENT

This paper was completed as an undergraduate in the Faculty/Student Research Program at the University of Scranton under the direction of Dr. Kenneth Monks. I would like to dedicate this paper to Dr. Monks for all of his time, effort, and assistance.

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AMS Classification Numbers: 11B37, 39B12, 58F08

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WILSON'S THEOREM VIA EULERIAN NUMBERS

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INTRODUCTION

In 1770, Edward Waring, in a work entitled "Meditationes Algebraicae," announced without proof the following result, which he attributed to his student, John Wilson:

$$\text{If } p \text{ is prime, then } (p-1)! \equiv -1 \pmod{p}.$$

This statement, now known as Wilson's Theorem, was first proved by Lagrange in 1771, and may have been known earlier by Leibniz.

In this note, we present a new proof of Wilson's Theorem, based on properties of Eulerian numbers, which are defined below. Consider the following triangular array, which is somewhat reminiscent of Pascal's triangle.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 4 & & 1 & & & \\
 & & 1 & & 11 & & 11 & & 1 & & \\
 & 1 & & 26 & & 66 & & 26 & & 1 & \\
1 & & 57 & & 302 & & 302 & & 57 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

The numbers that appear in this array were first discovered by Euler [1] and are known as *Eulerian numbers*. Following Knuth [2], we denote the k^{th} entry in row n by $\left\langle {n \atop k} \right\rangle$, where $1 \le k \le n$.

Eulerian numbers may be defined recursively via:

$$\left\langle {n \atop 1} \right\rangle = \left\langle {n \atop n} \right\rangle = 1; \quad \left\langle {n \atop k} \right\rangle = k\left\langle {n-1 \atop k} \right\rangle + (n+1-k)\left\langle {n-1 \atop k-1} \right\rangle \quad \text{if } 2 \le k \le n-1. \tag{1}$$

(See [2], p. 35, eq. (2).)

They enjoy a symmetry property:

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n+1-k} \right\rangle \text{ for all } k \text{ such that } 1 \le k \le n. \tag{2}$$

Adding all the Eulerian numbers in a given row, we get

$$\sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle = n! \tag{3}$$

Furthermore,

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j (k-j)^n \binom{n+1}{j}. \tag{4}$$

Remarks: (2) follows easily from (1), (3) follows from (1), using induction on n, and (4) is equation (13) on page 37 in [2].

We will also need

Definition 1: If m and n are integers larger than 1 and k is a nonnegative integer, we say that $O_n(m) = k$ if $n^k \mid m$ but $n^{k+1} \nmid m$.

$$\text{If } p \text{ is prime, } p \nmid a,\ j \le m, \text{ and } 0 < a < p^{m-j}, \text{ then } O_p\left(\binom{p^m}{ap^j}\right) = m - j. \tag{5}$$

Remark: (5) is Theorem 4 in [3].

THE MAIN RESULTS

Lemma 1: If p is prime, $m \ge 1$, and $1 \le k \le p^m - 1$, then $\binom{p^m}{k} \equiv 0 \pmod{p}$.

Proof: This follows from the hypothesis and (5).

Theorem 1: If p is prime, $m \ge 1$, and $1 \le k \le p^m - 1$, then

$$\left\langle {p^m - 1 \atop k} \right\rangle \equiv \begin{cases} 0 \pmod{p} & \text{if } k \equiv 0 \pmod{p}, \\ 1 \pmod{p} & \text{if } k \not\equiv 0 \pmod{p}. \end{cases}$$

Proof: (4) implies

$$\left\langle {p^m - 1 \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j (k-j)^{p^m - 1} \binom{p^m}{j}.$$

Now Lemma 1 implies

$$\left\langle {p^m - 1 \atop k} \right\rangle \equiv k^{p^m - 1} \pmod{p}.$$

If $k \equiv 0 \pmod{p}$, then

$$\left\langle {p^m - 1 \atop k} \right\rangle \equiv 0^{p^m - 1} \equiv 0 \pmod{p}.$$

If $k \not\equiv 0 \pmod{p}$, then, by Fermat's Little Theorem,

$$\left\langle {p^m - 1 \atop k} \right\rangle \equiv (k^{p-1})^{\left(\frac{p^m - 1}{p-1}\right)} \equiv 1^{\left(\frac{p^m - 1}{p-1}\right)} \equiv 1 \pmod{p}.$$

Theorem 2 (Wilson's Theorem): If p is prime, then $(p-1)! \equiv -1 \pmod{p}$.

Proof: (3) implies $(p-1)! = \sum_{k=1}^{p-1} \left\langle {p-1 \atop k} \right\rangle$. Theorem 1 implies $\left\langle {p-1 \atop k} \right\rangle \equiv 1 \pmod{p}$ for $1 \le k \le p-1$. Therefore, $(p-1)! \equiv \sum_{k=1}^{p-1} 1 \equiv p - 1 \equiv -1 \pmod{p}$.

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AMS Classification Number: 11A99

A NOTE ON THE SET OF ALMOST-ISOSCELES RIGHT-ANGLED TRIANGLES

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1. INTRODUCTION

Pythagorean triples have long provided a source of great interest and amusement to mathematicians since antiquity. Not surprisingly, many important properties have and continue to be deduced about these solutions of the diophantine equation

$$x^2+y^2=z^2. \tag{1}$$

The most general solutions of (1) that satisfy the conditions (see [2], p. 190) $x>0$, $y>0$, $z>0$, $(x,y)=1$, $2|x$, are

$$x=2ab,\ y=b^2-a^2,\ z=a^2+b^2, \tag{2}$$

where the integers a, b are of opposite parity and $(a,b)=1$, $b>a>0$.

One important class of Pythagorean triples (x,y,z) that we shall be concerned with are those in which x and y are consecutive integers. These triples, which we shall call *almost-isosceles right-angled* (AIRA) triangles, can be constructed from (2) in the following manner (see [4], p. 13). Take an a and b that generate a triangle whose two shortest sides differ by one, then the next such triangle is constructed by b and $a+2b$. Thus, as $(3,4,5)$ is generated by (2) using $a=1$ and $b=2$, so the next AIRA triangle will be determined via $a=2$ and $b=5$, thereby producing the triple $(20,21,29)$. Clearly, with repeated applications of the rule $(a,b)\mapsto(b,a+2b)$, one can generate an infinite number of these triples. A similar recurrence scheme generating AIRA triangles was also developed in [1] using Pell's equation.

Our aim in this short note is to re-establish the existence of infinitely AIRA triangles via an alternate argument which, unlike the above, does not require the use of (2) or Pell's equation. We shall, as a result of this approach, reveal a surprising connection that exists between these Pythagorean triples and the set of square triangular numbers [see (3)]. This will be employed later to calculate the first six such triples. An additional number fact concerning the primes is also deduced.

2. MAIN CONSTRUCTION

Let us begin by noting the following key observation, a proof is included for completeness.

Lemma 2.1: There are infinitely many perfect squares of the form $n(n+1)/2$.

Proof: If $n\in\mathbb{Z}^+$ is such that $T(n)=n(n+1)/2$ is a perfect square, then so is $T(4n(n+1))$; however, the statement now readily follows because $T(1)=1$ is clearly a perfect square. □

To find all AIRA triangles, we first reduce the problem (as in [1]) to a question of the solubility of a diophantine equation obtained by exploiting an obvious fact, namely, if the sum of two consecutive squares is a perfect square, it must be the square of an odd number. Then, using a series of elementary arguments, one can further reduce this equation to another diophantine equation which is known to be solvable from Lemma 2.1. This approach is contained in the proof of the following theorem.

Theorem 2.1: There are infinitely many nontrivial pairs of consecutive squares whose sum is a perfect square; moreover, all such AIRA triangles are given by

$$\left(\frac{4T_n-1+\sqrt{8T_n^2+1}}{2}, \frac{4T_n+1+\sqrt{8T_n^2+1}}{2}, 2T_n+\sqrt{8T_n^2+1}\right), \tag{3}$$

where T_n denotes the positive square root of the n^{th} square triangular number.

Proof: Suppose $m \in \mathbb{Z}^+$ is such that the sum of m^2 and $(m+1)^2$ is a square, then there must exist an $s \in \mathbb{Z}^+$ such that $m^2+(m+1)^2=(2s+1)^2$. Upon expanding and simplifying, we obtain

$$m(m+1)=2s(s+1). \tag{4}$$

Our argument is therefore reduced to demonstrating the infinitude of solutions (m, s) to the above diophantine equation. To solve (4), first observe that if $m \le s$ then $m(m+1)<2s(s+1)$, while if $m \ge 2s$ then $m(m+1)>2s(s+1)$. Thus, for an arbitrary $s \in \mathbb{Z}^+$, the only integer values m can assume in order that (4) may possibly be satisfied are those in which $2s>m>s$. Hence, if (m, s) is a solution, then there must exist a fixed $r \in \mathbb{Z}^+$ such that $m=s+r$ with

$$(s+r)(s+r+1)=2s(s+1). \tag{5}$$

Expanding and simplifying (5) yields the quadratic, $s^2+s(1-2r)-(r^2+r)=0$, in the variable s, from which it is deduced that

$$s=\frac{2r-1+\sqrt{8r^2+1}}{2}. \tag{6}$$

Note that the positive radical has been taken as $s>0$. Since s is an integer $8r^2+1$ must be an odd perfect square. Consequently, we require that $8r^2+1=(2n+1)^2$ for some $n \in \mathbb{Z}^+$, and so r and n are solutions of the diophantine equation $2r^2=n(n+1)$.

However, by Lemma 2.1, there are infinitely many integer solutions of this equation; hence, we conclude that there are an infinite number of integers s of the form in (6) such that (5) is satisfied. Thus, equation (4) must have infinitely many solutions (m, s) since $m=s+r$.

It is now a simple matter to determine the required expression of the AIRA triples

$$(m, m+1, 2s+1). \tag{7}$$

Let T_n denote the positive square root of the n^{th} square triangular number, then by the above, $r=T_n$ and so, from (6), we have

$$s=\frac{2T_n-1+\sqrt{8T_n^2+1}}{2}.$$

Finally, substituting the corresponding expressions for $m=s+r$ and $m+1$ into (7) produces (3). □

We now prove an interesting fact concerning the prime numbers, which can be deduced by showing that the odd base length in the above AIRA triangles is always a composite number, with the exception of $(3, 4, 5)$.

Corollary 2.1: If p is a prime number greater than 3, then neither $(p-1)^2+p^2$ nor $p^2+(p+1)^2$ is a perfect square.

Proof: Let $m, s \in \mathbb{Z}^+$ be such that $(m, m+1, 2s+1)$ is an AIRA triangle, and assume m is a prime greater than 3. Clearly, the integers satisfy equation (4) of the previous theorem and so $2s > m > s$. But, in view of this inequality and by assumption, we must have $(m, s) = (m, 2) = 1$; this, in turn, implies $(m, 2s) = 1$. Further, note that $m > s+1$, for if $m = s+1 = s+T_1$, then

$$m = \frac{4T_1 - 1 + \sqrt{8T_1^2+1}}{2} = 3,$$

which is contrary to the assumption. Thus, $(m, s+1) = 1$ because m is prime; consequently, $(m, 2s(s+1)) = 1$ and so m is not a divisor of $2s(s+1)$. This is a contradiction, since $m|2s(s+1)$ by equation (4). Hence, m cannot be a prime greater than 3. Assume now that $m+1$ is a prime greater than 3. Clearly, from the above inequality, $m+1 > s+1 > s$, which, via the assumption, implies that $(m+1, s+1) = (m+1, s) = (m+1, 2) = 1$. Thus, $(m+1, 2s(s+1)) = 1$, and so $m+1$ is not a divisor of $2s(s+1)$, again a contradiction because $m+1|2s(s+1)$. Consequently, since both m and $m+1$ are greater than 3 in all Pythagorean triples of the form $(m, m+1, 2s+1)$, with the exception of $(3, 4, 5)$, we deduce that m and $m+1$ must be composite. The result now readily follows. □

In view of the previous result, one may question whether the hypotenuses in the above AIRA triples are similarly composite for large n. This is the motivation behind the following conjecture.

Conjecture 2.1: There are only finitely many primes p, such that p^2 is representable as a sum of two consecutive squares.

At present the author, via an application of Corollary 5.14 in [3], has found that primes of the form $4k+3$ will fail the condition of the conjecture. Whether there exists an infinite subset of primes of the form $4k+1$ satisfying the above, is still an open question.

3. NUMERICAL COMPUTATION

To conclude this note, we shall apply equation (2.1) to calculate the first six triples. Clearly, all that is required is a means of determining T_n; however, we are fortunate in this respect, as one may either make use of the formula (see [4], p. 16)

$$T_n^2 = \frac{1}{32}((17+12\sqrt{2})^n + (17-12\sqrt{2})^n - 2), \tag{8}$$

or the recurrence relation where, for all integers $n \geq 2$,

$$T_{n+1} = 6T_n - T_{n-1}, \tag{9}$$

with $T_1 = 1$, $T_2 = 6$. Curiously, equation (8) coupled with (3) will produce an explicit formula in n for the n^{th} AIRA triangle. However, from a computational viewpoint, it is more efficient to use

(9) because this provides a recurrence scheme for calculating all AIRA triangles entailing fewer arithmetic operations. The results of the first six iterations are tabulated as follows:

TABLE 1. The First Six AIRA Triangles

n	T_n	$(4T_n-1+(8T_n^2+1)^{1/2})/2$	$(4T_n+1+(8T_n^2+1)^{1/2})/2$	$2T_n+(8T_n^2+1)^{1/2}$
1	1	3	4	5
2	6	20	21	29
3	35	119	120	169
4	204	696	697	985
5	1189	4059	4060	5741
6	6930	23660	23661	33461

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AMS Classification Numbers: 11Axx, 11Dxx

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MEAN CROWDS AND PYTHAGOREAN TRIPLES

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1. INTRODUCTION

The opportunity to temporarily avoid computational distractions and thereby concentrate on new concepts is helpful to many students who find themselves struggling in pre-calculus mathematics courses. For this reason, instructors in these classes often present introductory examples that have been carefully designed to require no irrational numbers.

For example, it is likely that anyone who has taught trigonometry appreciates the utility of Pythagorean triples, i.e., nontrivial integer solutions $(x, y; z)$ of $x^2+y^2=z^2$. A well-known characterization theorem for these triples [7, p. 190] enables one to generate an endless supply of nonsimilar right triangles whose angles have rational trigonometric function values.

In the spirit of Pythagorean triples, I present an analogous characterization of those pairs of positive integers whose arithmetic, geometric, harmonic, and certain other means are also integers. In this development, Pythagorean triples will also surprisingly appear in ways that go beyond this motivation by analogy.

2. PYTHAGOREAN MEANS

In his *Commentary on Euclid*, Proclus of Alexandria (ca. 410-485 A.D.) reports that, while traveling in Mesopotamia, Pythagoras learned about the theory of proportionals (means); specifically, the binary operations we call the arithmetic, geometric, and harmonic means. Later, his followers extended this concept to include several other means. The neo-Pythagorean philosopher, Nicomachus of Gerasa (ca. 100 A.D.), and the geometer, Pappus of Alexandria (ca. 300 A.D.), each presented distinct but intersecting lists of ten "means" of the Pythagoreans. Most of these probably would not qualify as means by modern standards, e.g., [11, p. 84], since they fail the (perfectly reasonable) condition that a mean should always return a value between those of its arguments. In this article I will restrict attention to those means of the Pythagoreans that satisfy this intermediacy condition [12], [13].

The first three means common to the lists of Nicomachus and Pappus are the familiar *arithmetic*, *geometric*, and *harmonic means*, defined by

$$a=A(p,y)=\frac{p+y}{2},\ g=G(p,y)=\sqrt{py},\ h=H(p,y)=\frac{2py}{p+y}.$$

Another mean given by both Nicomachus and Pappus is occasionally called the "subcontrary" mean, but that name has also been used for the harmonic mean, so I propose the name *tetra* mean, indicating its fourth position in both lists. It is defined by

$$t=T(p,y)=\frac{p^2+y^2}{p+y}.$$

Results relating the arithmetic, geometric, and harmonic means are abundant (e.g., [1], [3], [4], [5], [6], [10], [11], [13]. [14]). Among the best-known examples, the A-G-H Mean Inequality states that, for any p and y, $h\le g\le a$, with equality holding if and only if $p=y$. An easy exercise will extend this to $h \le g \le a \le t$. The Musical Proportionality and Golden Proportionality state that

$$\frac{y}{a} = \frac{h}{p} \quad \text{and} \quad \frac{a}{g} = \frac{g}{h},$$

respectively. The latter is the basis for the Babylonian method of extraction of square roots [2, p. 56].

To avoid unnecessary computational distractions when introducing these means, we might wish to generate nontrivial examples of positive integers whose arithmetic, geometric, and harmonic means are also integers. (A near-miss is the example $p = 6$ and $y = 12$, which gives $a = 9$, $h = 8$, and $g = 6\sqrt{2}$. The early Pythagorean number-mystic Philolaus of Tarantum saw this 6-8-12 relationship as an indication that the cube, with its 6 faces, 8 vertices, and 12 edges, was particularly harmonious [8, pp. 85-86].)

3. MEAN CROWDS

In the spirit of Pythagorean triples, let us define a *mean crowd* to be an ordered sextuple (p, y, t, h, a, g) of distinct positive integers such that $p < y$ and t, h, a, and g are, respectively, the tetra, harmonic, arithmetic, and geometric means of p and y as defined above.

Lemma 3.1: Let p and y be positive integers. If any two of the means t, h, and a of p and y are integers, then the third of these is also an integer. Moreover, h and t are of the same parity, as are p and y.

Proof: $h + t = \frac{2py}{p+y} + \frac{p^2+y^2}{p+y} = \frac{(p+y)^2}{p+y} = p + y = 2a$. □

With the assistance of this lemma, it takes only a little experimentation to discover the mean crowds $(5, 45, 41, 9, 25, 15)$ and $(10, 40, 34, 16, 25, 20)$. The interested reader might wish to search for other examples before reading on.

Since each of the means A, G, H, and T is homogeneous [i.e., $A(kp, ky) = kA(p, y)$, etc.], it is immediate that any positive integral multiple of a mean crowd is also a mean crowd. Let us borrow a term associated with Pythagorean triples and say that a mean crowd is *primitive* if the greatest common divisor of its members is unity.

Conversely, it is clear (by the well-ordering principle) that any mean crowd must be some integral multiple of a primitive mean crowd; hence, the problem of characterizing mean crowds reduces to that of characterizing primitive mean crowds.

Lemma 3.2: The mean crowd (p, y, t, h, a, g) is primitive if and only if a and h are relatively prime.

Proof: The "if" part is obvious. For the "only if" part, let (p, y, t, h, a, g) be a mean crowd and suppose that q is a prime common divisor of a and h. By Lemma 3.1, $t = 2a - h$; therefore q divides t. By the Golden Proportionality, $ah = g^2$ so that q^2 divides g^2, and (since q is prime), q divides g.

Now $p + y = 2a$ and q divides the right side, so either q divides both p and y, or else q divides neither p nor y. But, by the Musical Proportionality, $ah = py$, so q^2 divides py, implying that q divides p, or q divides y. Hence, q divides both p and y.

Therefore, any prime common divisor of a and h also divides the other four members of the mean crowd (p, y, t, h, a, g), and the desired result follows. □

4. COMPLEMENTS OF MEAN CROWDS

Let us define the *complement* of the mean crowd (p, y, t, h, a, g) by

$$\text{comp}(p, y, t, h, a, g) = (a-g, a+g, a+h, a-h, a, a-p).$$

Lemma 4.1: The "complement" operator is involutory and preserves mean crowds and primitive mean crowds. That is, the complement of the complement of any mean crowd is the original mean crowd, and the complement of a mean crowd is a mean crowd, which is primitive if and only if the original is primitive.

Proof: Let us denote $\text{comp}(p, y, t, h, a, g)$ by (p', y', t', h', a', g'). Then we have

$$\begin{aligned}\text{comp}[\text{comp}(p, y, t, h, a, g)] &= \text{comp}(p', y', t', h', a', g')\\ &= (a'-g', a'+g', a'+h', a'-h', a', a'-p')\\ &= (a-(a-p), a+(a-p), a+(a-h), a-(a-h), a, a-(a-g))\\ &= (p, 2a-p, 2a-h, h, a, g) = (p, y, t, h, a, g).\end{aligned}$$

It is obvious that the members of the complement of a mean crowd are each positive integers. Moreover,

$$\begin{aligned}&A(a-g, a+g) = a = a',\\ &G(a-g, a+g) = \sqrt{(a-g)(a+g)} = \sqrt{a^2-g^2} = a-p = g',\\ &H(a-g, a+g) = \frac{(a-p)^2}{a} = \frac{(g')^2}{a'} = h', \quad \text{and}\\ &T(a-g, a+g) = 2a'-h' = 2a-(a-h) = a+h = t'.\end{aligned}$$

Therefore, the complement of a mean crowd is a mean crowd.

Now (p, y, t, h, a, g) is primitive if and only if $\text{GCD}(a, h) = 1$ (by Lemma 3.2), if and only if $\text{GCD}(a, a-h) = \text{GCD}(a', h') = 1$, if and only if (p', y', t', h', a', g') is primitive (again by Lemma 3.2). □

5. CHARACTERIZATIONS OF MEAN CROWDS

Characterization Theorem (Pythagorean Triple Form): Let p and y be positive numbers with $p < y$ and let a, g, h, and t be the arithmetic, geometric, harmonic, and tetra means, respectively, of p and y. Then the ordered sextuple (p, y, t, h, a, g) is a primitive mean crowd if and only if there exists a primitive Pythagorean triple $(u, v; w)$ such that $p = w^2 - wv$ and $y = w^2 + wv$.

Proof: (If) Let $(u, v; w)$ be a primitive Pythagorean triple with $p = w^2 - wv$ and $y = w^2 + wv$. Then

$$\begin{aligned}&a = w^2, \quad g = \sqrt{(w^2-wv)(w^2+wv)} = \sqrt{w^4 - w^2v^2} = \sqrt{w^2(w^2-v^2)} = wu,\\ &h = \frac{g^2}{a} = \frac{(wu)^2}{w^2} = u^2, \quad \text{and} \quad t = 2a - h = 2w^2 - u^2;\end{aligned}$$

each a positive integer, and therefore, (p, y, t, h, a, g) is a mean crowd. Moreover, $\text{GCD}(a, h) = \text{GCD}(w^2, u^2) = 1$ since $(u, v; w)$ is primitive; hence, (p, y, t, h, a, g) is primitive by Lemma 3.2.

(Only if) Let (p, y, t, h, a, g) be a primitive mean crowd. Then $\mathrm{GCD}(a, h) = 1$ by Lemma 3.2. But $ah = g^2$ by the Golden Proportionality, so a and h are both squares. Also, by Lemma 4.1, $\mathrm{comp}(p, y, t, h, a, g)$ is also a primitive mean crowd, and so, by the same reasoning, h' is also a square. Therefore, $(\sqrt{h}, \sqrt{a-h}; \sqrt{a})$ is a Pythagorean triple. Clearly, any common divisor of $\sqrt{a}$ and $\sqrt{h}$ is also a common divisor of a and h; hence, $(\sqrt{h}, \sqrt{a-h}; \sqrt{a})$ is primitive. □

This form of the Characterization Theorem allows for a striking illustration of the *complement* operator, as follows: The primitive mean crowd $(w^2 - wv, w^2 + wv, 2w^2 - u^2, u^2, w^2, wu)$ is generated by the primitive Pythagorean triple $(u, v; w)$; its complement is $(w^2 - wu, w^2 + wu, 2w^2 - v^2, v^2, w^2, wv)$, which is, of course, generated by the permuted triple $(v, u; w)$—quite literally a "complement."

Characterization Theorem (Sum of Squares Form): The ordered sextuple (p, y, t, h, a, g) is a primitive mean crowd if and only if there exist relatively prime integers α and β of opposite parity and $\beta < \alpha$ such that either $p = (\alpha - \beta)^2(\alpha^2 + \beta^2)$ and $y = (\alpha + \beta)^2(\alpha^2 + \beta^2)$ or $p = 2\beta^2(\alpha^2 + \beta^2)$ and $y = 2\alpha^2(\alpha^2 + \beta^2)$.

Proof: The characterization theorem for primitive Pythagorean triples [7, p. 190] states that $(u, v; w)$ is a primitive Pythagorean triple with v even exactly when there exist relatively prime integers α and β of opposite parity with $\alpha > \beta > 0$ and $u = \alpha^2 - \beta^2$, $v = 2\alpha\beta$, and $w = \alpha^2 + \beta^2$. The desired result now follows immediately from the preceding theorem. □

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AMS Classification Numbers: 11A99, 11D72

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GENERALIZATIONS OF SOME IDENTITIES INVOLVING GENERALIZED SECOND-ORDER INTEGER SEQUENCES

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(*Submitted November 1996*)

In [4], using the method of Carlitz and Ferns [1], some identities involving generalized second-order integer sequences were given. The purpose of this paper is to obtain the more general results.

In the notation of Horadam [2], write $W_n = W_n(a,b;\, p,q)$ so that

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = a, W_1 = b,\ n \geq 2. \tag{1}$$

If α and β, assumed distinct, are the roots of $\lambda^2 - p\lambda + q = 0$, we have the Binet form (see [2])

$$W_n = A\alpha^n + B\beta^n, \tag{2}$$

where $A = \frac{b-a\beta}{\alpha-\beta}$ and $B = \frac{a\alpha-b}{\alpha-\beta}$.

Using this notation, define $U_n = W_n(0,1;\, p,q)$ and $V_n = W_n(2,p;\, p,q)$. The Binet forms for U_n and V_n are given by $U_n = (\alpha^n - \beta^n)/(\alpha-\beta)$ and $V_n = \alpha^n + \beta^n$, where $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences, respectively. They have been studied extensively, particularly by Lucas [3].

Throughout this paper, the symbol $\binom{n}{i,j}$ is defined by $\binom{n}{i,j} = \frac{n!}{i!j!(n-i-j)!}$.

To extend the results of [4], we need the following lemma.

Lemma: Let $u = \alpha$ or β, then

$$-q^{m+1} + pq^m u + u^{2(m+1)} = V_m u^{m+2}. \tag{3}$$

Proof: Since α and β are roots of $\lambda^2 - p\lambda + q = 0$, we have $\alpha^2 = p\alpha - q$ and $\beta^2 = p\beta - q$. Hence,

$$\begin{aligned}-q^{m+1} + pq^m u + u^{2(m+1)} &= q^m(pu-q) + u^{2(m+1)} = q^m u^2 + u^{2(m+1)} \\ &= u^{m+2}(q^m u^{-m} + u^m) = (\alpha^m + \beta^m)u^{m+2} = V_m u^{m+2}.\end{aligned}$$

This completes the proof of the Lemma.

Theorem 1:

$$-q^{m+1}W_k + pq^m W_{k+1} + W_{k+2(m+1)} = V_m W_{k+m+2}. \tag{4}$$

Proof: By the Lemma, we have

$$-q^{m+1} + pq^m\alpha + \alpha^{2(m+1)} = V_m\alpha^{m+2} \quad \text{and} \quad -q^{m+1} + pq^m\beta + \beta^{2(m+1)} = V_m\beta^{m+2}.$$

Theorem 1 follows if we multiply both sides of the previous two identities by α^k and β^k, respectively, and use the Binet form (2).

Theorem 2:

$$W_{n+k} = (pq^m)^{-n} \sum_{i+j+s=n} \binom{n}{i,j} (-1)^j q^{(m+1)s} V_m^i W_{(m+2)i+2(m+1)j+k}. \tag{5}$$

$$W_{(m+2)n+k} = V_m^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^s p^j q^{mj+(m+1)s} W_{2(m+1)i+j+k}. \tag{6}$$

$$W_{2(m+1)n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j p^j q^{(m+1)s+mj} V_m^i W_{(m+2)i+j+k}. \tag{7}$$

Proof: By using the Lemma and the multinomial theorem, we have

$$(pq^m)^n u^n = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j q^{(m+1)s} V_m^i u^{(m+2)i+2(m+1)j},$$

$$V_m^n u^{(m+2)n} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^s p^j q^{mj+(m+1)s} u^{2(m+1)i+j},$$

$$u^{2(m+1)n} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j p^j q^{(m+1)s+mj} V_m^i u^{(m+2)i+j}.$$

If we multiply both sides in the preceding identities by u^k and use the Binet form (2), we obtain (5), (6), and (7), respectively. This completes the proof of Theorem 2.

Theorem 3:

$$p^n q^{mn} W_{n+k} - \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{(m+1)(n-j)} W_{2(m+1)j+k} \equiv 0 \pmod{V_m}. \tag{8}$$

$$W_{2(m+1)n+k} - (-1)^n q^{mn} W_{2n+k} \equiv 0 \pmod{V_m}. \tag{9}$$

Proof: From (5) and (7), by using the decomposition $\sum_{i+j+s=n} = \sum_{i+j+s=n,\, i=0} + \sum_{i+j+s=n,\, i\neq 0}$ and Theorem 2.1 of [4], we get Theorem 3.

Remark: When we take $m = 2, 4$, and 8, the results of this paper become those of [4].

ACKNOWLEDGMENT

The authors would like to thank Professor Peng Ye, who has given us great help during our work.

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AMS Classification Numbers: 11B37, 11B39

SOME IDENTITIES INVOLVING GENERALIZED GENOCCHI POLYNOMIALS AND GENERALIZED FIBONACCI-LUCAS SEQUENCES

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1. INTRODUCTION

In [7], Toscano gave some novel identities between generalized Fibonacci-Lucas sequences and Bernoulli-Euler polynomials. Later, Zhang and Guo [9] and Wang and Zhang [8] discussed the case of Bernoulli-Euler polynomials of higher order and generalized the results of Toscano.

The purpose of this paper is to establish some identities containing generalized Genocchi polynomials that, as one application, yield some results of Toscano [7] and Byrd [1] as special cases, as well as other identities involving Bernoulli-Euler and Fibonacci-Lucas numbers.

2. SOME LEMMAS

It is well known that a general linear sequence $S_n(p,q)$ $(n=0,1,2,\dots)$ of order 2 is defined by the law of recurrence,

$$S_n(p,q)=pS_{n-1}(p,q)-qS_{n-2}(p,q),$$

with S_0, S_1, p, and q arbitrary, provided that $\Delta=p^2-4q>0$.

In particular, if $S_0=0, S_1=1$, or $S_0=2, S_1=p$, we have generalized Fibonacci and Lucas sequences, respectively, in symbols $U_n(p,q), V_n(p,q)$.

If α,β $(\alpha>\beta)$ are the roots of the equation $x^2-px+q=0$, then we have (see [7])

$$U_{n(}(p,q)=\frac{\alpha^n-\beta^n}{\alpha-\beta},\quad V_n(p,q)=\alpha^n+\beta^n, \tag{1}$$

$$S_n(p,q)=\left(S_1-\frac{1}{2}S_0\right)U_n(p,q)+\frac{1}{2}S_0V_n(p,q). \tag{2}$$

We assume

$$S_0=k,\quad S_1=\frac{1}{2}pk+\left(x-\frac{1}{2}k\right)\Delta^{1/2}$$

and, using (1) and (2), we deduce that

$$S_n(x;p,q)=\left(x-\frac{1}{2}k\right)\Delta^{1/2}U_n(p,q)+\frac{1}{2}kV_n(p,q), \tag{3}$$

$$S_n(x;p,q)=x\alpha^n+(k-x)\beta^n. \tag{4}$$

From this point on, we shall use the brief notation U_n, V_n, and $S_n(x)$ to denote $U_n(p,q)$, $V_n(p,q)$, and $S_n(x;p,q)$, respectively.

By adapting the method of [7], [8], and [9] to $S_n(x)$, we have obtained the following results:

$$\begin{aligned}
&S_n^m(x)+(-1)^\nu S_n^m(k-x)\\
&=\sum_{r=0}^{m}\binom{m}{r}\Delta^{r/2}U_n^r\frac{1}{2^{m-r}}k^{m-r}V_n^{m-r}\left(x-\frac{k}{2}\right)^r(1+(-1)^{\nu+r})\\
&=\begin{cases}\dfrac{1}{2^{m-1}}\displaystyle\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^r U_n^{2r}k^{m-2r}V_n^{m-2r}(2x-k)^{2r} & (\nu\text{ even}),\\ \dfrac{1}{2^{m-1}}\displaystyle\sum_{r=0}^{[m/2]}\binom{m}{2r+1}\Delta^{r+(1/2)}U_n^{2r+1}k^{m-2r-1}V_n^{m-2r-1}(2x-k)^{2r+1} & (\nu\text{ odd}),\end{cases}
\end{aligned}\tag{5}$$

$$\begin{aligned}
&S_n^m(x)+(-1)^\nu S_n^m(k-x)\\
&=\sum_{r=0}^{m}\binom{m}{r}q^{nr}[\beta^{n(m-2r)}+(-1)^\nu\alpha^{n(m-2r)}]x^r(k-x)^{m-r}\\
&=\begin{cases}\displaystyle\sum_{r=0}^{m}\binom{m}{r}q^{nr}V_{n(m-2r)}x^r(k-x)^{m-r} & (\nu\text{ even}),\\ -\displaystyle\sum_{r=0}^{m}\binom{m}{r}q^{nr}\Delta^{1/2}U_{n(m-2r)}x^r(k-x)^{m-r} & (\nu\text{ odd}),\end{cases}
\end{aligned}\tag{6}$$

$$\begin{aligned}
&S_n^m(x)+(-1)^\nu S_n^m(k-x)\\
&=\begin{cases}2\dbinom{2m}{n}q^{nm}x^m(k-x)^m+\displaystyle\sum_{r=0}^{m-1}\binom{2m}{r}q^{nr}V_{2n(m-r)}[x^r(k-x)^{2m-r}+x^{2m-r}(k-x)^r] & (\nu\text{ even}),\\ -\displaystyle\sum_{r=0}^{m-1}\binom{2m}{r}q^{nr}\Delta^{1/2}U_{2n(m-r)}[x^r(k-x)^{2m-r}+x^{2m-r}(k-x)^r] & (\nu\text{ odd}),\end{cases}
\end{aligned}\tag{7}$$

$$\begin{aligned}
&S_n^{2m+1}(x)+(-1)^\nu S_n^{2m+1}(k-x)\\
&=\begin{cases}\displaystyle\sum_{r=0}^{m}\binom{2m+1}{r}q^{nr}V_{n(2m-2r+1)}[x^r(k-x)^{2m-r+1}+x^{2m-r+1}(k-x)^r] & (\nu\text{ even}),\\ -\displaystyle\sum_{r=0}^{m}\binom{2m+1}{r}q^{nr}\Delta^{1/2}U_{n(2m-2r+1)}[x^r(k-x)^{2m-r+1}+x^{2m-r+1}(k-x)^r] & (\nu\text{ odd}),\end{cases}
\end{aligned}\tag{8}$$

and the generating functions

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}U_{nr}=\frac{1}{\Delta^{1/2}}[\exp(t\alpha^n)-\exp(t\beta^n)]\tag{9}$$

and

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}V_{nr}=\exp(t\alpha^n)+\exp(t\beta^n).\tag{10}$$

3. THE MAIN RESULTS

The generalized Genocchi polynomial is defined as (see [4])

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}G_r^{(k)}(x)=\left(\frac{2t}{\exp t+1}\right)^k\exp(tx). \tag{11}$$

From this definition, it is easy to deduce the following properties (see [4]):

$$G_n^{(k)}(k-x)=(-1)^{n+k}G_n^{(k)}(x), \tag{12}$$

$$G_n^{(k+1)}(x)=\frac{2n(k-x)}{k}G_{n-1}^{(k)}(x)+\frac{2}{k}(n-k)G_n^{(k)}(x). \tag{13}$$

In particular, $G_n^{(1)}(0)=G_n$ (Genocchi number, see [3]).

From (11), replacing t by $\Delta^{1/2}U_nt$, we have

$$\begin{aligned}\sum_{r=0}^{\infty}G_r^{(k)}(x)\frac{(\Delta^{1/2}U_nt)^r}{r!}&=\frac{2^k\Delta^{k/2}U_n^kt^k}{[\exp(\Delta^{1/2}U_nt)+1]^k}\exp(x\Delta^{1/2}U_nt)\\&=\frac{2^k\Delta^{k/2}U_n^kt^k}{[\exp(\alpha^nt)+\exp(\beta^nt)]^k}\exp[t(x\alpha^n+(k-x)\beta^n)].\end{aligned}$$

Therefore,

$$[\exp(\alpha^nt)+\exp(\beta^nt)]^k\sum_{r=0}^{\infty}G_r^{(k)}(x)\frac{(\Delta^{1/2}U_nt)^r}{r!}=2^k\Delta^{k/2}U_n^kt^k\exp(tS_n(x)).$$

Hence,

$$\left(\sum_{r=0}^{\infty}\frac{t^r}{r!}V_{nr}\right)^k\left(\sum_{r=0}^{\infty}G_r^{(k)}(x)\frac{(\Delta^{1/2}U_nt)^r}{r!}\right)=2^k\Delta^{k/2}U_n^kt^k\exp(tS_n(x)).$$

We now expand the product figuring in the left member into a power series of t, compare with the expansion of the right member, and obtain

$$\begin{aligned}&\sum_{r=0}^{m}\binom{m}{r}\Delta^{r/2}U_n^rG_r^{(k)}(x)(m-r)!\sum_{r_1+\cdots+r_k=m-r}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}\\&=(m)_k2^k\Delta^{k/2}U_n^kS_n^{m-k}(x).\end{aligned} \tag{14}$$

If we replace x by $k-x$ in (14) and use (12), we find

$$\begin{aligned}&\sum_{r=0}^{m}\binom{m}{r}\Delta^{r/2}U_n^r(-1)^rG_r^{(k)}(x)(m-r)!\sum_{r_1+\cdots+r_k=m-r}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}\\&=(m)_k(-1)^k2^k\Delta^{k/2}U_n^kS_n^{m-k}(x).\end{aligned} \tag{15}$$

From (14), (15), (5), (7), and (8), we have

$$\begin{aligned}&\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^rU_n^{2r}G_{2r}^{(k)}(x)(m-2r)!\sum_{r_1+\cdots+r_k=m-2r}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}\\&=(m)_k2^{k-1}\Delta^{k/2}U_n^k[S_n^{m-k}(x)+(-1)^kS_n^{m-k}(x)]\end{aligned}$$

$$=\begin{cases}\dfrac{(m)_k}{2^{m-2k}}\Delta^{k/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k)/2]}\binom{m-k}{2r}\Delta^r U_n^{2r}k^{m-k-2r}V_n^{m-k-2r}(2x-k)^{2r} & (k \text{ even}),\\ \dfrac{(m)_k}{2^{m-2k}}\Delta^{(k+1)/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k)/2]}\binom{m-k}{2r+1}\Delta^r U_n^{2r+1}k^{m-k-2r-1}V_n^{m-k-2r-1}(2x-k)^{2r+1} & (k \text{ odd}),\end{cases} \tag{16}$$

$$=\begin{cases}(m)_k 2^{k-1}U_n^k(1+(-1)^{m-k})q^{n(m-k)/2}\dbinom{m-k}{(m-k)/2}x^{\frac{m-k}{2}}(k-x)^{\frac{m-k}{2}}+(m)_k 2^{k-1}\Delta^{k/2}U_n^k & \\ \quad\times\displaystyle\sum_{r=0}^{m-k-1}\binom{m-k}{r}q^{nr}V_{n(m-k-2r)}[x^r(k-x)^{m-k-r}+x^{m-k-r}(k-x)^r] & (k \text{ even}),\\ -(m)_k 2^{k-1}\Delta^{(k+1)/2}U_n^k\displaystyle\sum_{r=0}^{\left[\frac{m-k-2}{2}\right]}\binom{m-k}{r}q^{nr}U_{n(m-k-2r)}[x^r(k-x)^{m-k-r}+x^{m-k-r}(k-x)^r] & (k \text{ odd}).\end{cases} \tag{17}$$

We also obtain

$$\sum_{r=0}^{[m/2]}\binom{m}{2r+1}\Delta^{r+(1/2)}U_n^{2r+1}G_{2r+1}^{(k)}(x)(m-2r-1)!\sum_{r_1+\cdots r_k=m-2r-1}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}$$
$$=(m)_k 2^{k-1}\Delta^{k/2}U_n^k[S_n^{m-k}(x)-(-1)^k S_n^{m-k}(x)]$$

$$=\begin{cases}\dfrac{(m)_k}{2^{m-2k}}\Delta^{k/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k)/2]}\binom{m-k}{2r+1}\Delta^{r+(1/2)}U_n^{2r+1}k^{m-k-2r-1}V_n^{m-k-2r-1}(2x-k)^{2r+1} & (k \text{ even}),\\ \dfrac{(m)_k}{2^{m-2k}}\Delta^{k/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k)/2]}\binom{m-k}{2r}\Delta^r U_n^{2r}k^{m-k-2r}V_n^{m-k-2r-1}(2x-k)^{2r} & (k \text{ odd}),\end{cases} \tag{18}$$

$$=\begin{cases}(m)_k 2^{k-1}\Delta^{k/2}U_n^k(1+(-1)^{m-k})q^{n(m-k)/2}\dbinom{m-k}{(m-k)/2}x^{(m-k)/2}(k-x)^{(m-k)/2} & \\ \quad+(m)_k 2^{k-1}\Delta^{k/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k-2)/2]}\binom{m-k}{r}q^{nr}V_{n(m-k-2r)}[x^r(k-x)^{m-k-r}+x^{m-k-r}(k-x)^r] & (k \text{ odd}),\\ -(m)_k 2^{k-1}\Delta^{(k+1)/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k-2)/2]}\binom{m-k}{r}q^{nr}U_{n(m-k-2r)}[x^r(k-x)^{m-k-r} & \\ \qquad\qquad+x^{m-k-r}(k-x)^r] & (k \text{ even}).\end{cases} \tag{19}$$

4. SOME CONSEQUENCES

If we take $x=k/2$ in (16) and (18), then

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^r U_n^{2r}G_{2r}^{(k)}(k/2)(m-2r)!\sum_{r_1+\cdots+r_k=m-2r}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}$$
$$=\begin{cases}\dfrac{(m)_k}{2^{(m-2k)}}\Delta^{k/2}U_n^k k^{m-k}V_n^{m-k} & (k \text{ even}),\\ 0 & (k \text{ odd}),\end{cases} \tag{20}$$

$$\sum_{r=0}^{[m/2]}\binom{m}{2r+1}\Delta^r U_n^{2r} G_{2r+1}^{(k)}(k/2)(m-2r-1)! \sum_{r_1+\cdots+r_k=m-2r-1}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}$$

$$=\begin{cases}0 & (k \text{ even}),\\ \frac{(m)_k}{2^{(m-2k)}}\Delta^{(k-1)/2}U_n^{k-1}k^{m-k}V_n^{m-k} & (k \text{ odd}).\end{cases} \tag{21}$$

Taking $k=2$ in (20), again using $G_{2r}(1)=-G_{2r}$ (see [4]) and recurrence relation (13), we get

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^{r-1}U_n^{2(r-1)}(1-2r)\,G_{2r}\sum_{j}^{m-2r}\binom{m-2r}{j}V_{nj}V_{n(m-2r-j)}=2V_n^{m-2}m(m-1). \tag{22}$$

Taking $k=1$ in (21), using $G_{2r+1}(1/2)=\frac{(2r+1)E_{2r}}{2^{2r}}$, where E_{2r} is the Euler number (see [4]), and $m\to m+1$, we get (22) of [8], namely,

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^r U_n^{2r}\frac{E_{2r}}{2^{2r}}V_{n(m-2r)}=\frac{1}{2^{m-1}}V_n^m. \tag{23}$$

In (22), using $G_{2r}=2(1-2^{2r})B_{2r}$, where B_{2r} is the Bernoulli number (see [2], [4]), we obtain

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^{r-1}U_n^{2(r-1)}(1-2r)(1-2^{2r})B_{2r}\sum_{j}^{m-2r}\binom{m-2r}{j}V_{nj}V_{n(m-2r-j)}=V_n^{m-2}m(m-1). \tag{24}$$

If we take $p=1$ and $q=-1$, then $U_n(1,-1)=F_n$ (*Fibonacci number*), $V_n(1,-1)=L_n$ (*Lucas number*), and from (23) and (24) it follows that

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}5^r F_n^{2r}\frac{E_{2r}}{2^{2r}}L_{n(m-2r)}=\frac{1}{2^{m-1}}L_n^m, \tag{25}$$

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}5^{r-1}F_n^{2r}(1-2r)(1-2^{2r})B_{2r}\sum_{j=0}^{m-2r}\binom{m-2r}{j}L_{nj}L_{n(m-2r-j)}=m(m-1)L_n^{m-2}, \tag{26}$$

where (25) is the result of Byrd [1].

If we take $p=2$ and $q=-1$, then $U_n(2,-1)=P_n$ (*Pell number*), $V_n(2,-1)=Q_n$ (*Pell-Lucas number*, see [5]), and from (23) and (24), it follows that

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}2^r P_n^{2r}E_{2r}Q_{n(m-2r)}=\frac{1}{2^{m-1}}Q_n^m, \tag{27}$$

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}8^{r-1}P_n^{2r}(1-2r)(1-2^{2r})B_{2r}\sum_{j=0}^{m-2r}\binom{m-2r}{j}Q_{nj}Q_{n(m-2r-j)}=m(m-1)Q_n^{m-2}. \tag{28}$$

5. RESULTS IN TERMS OF THE POLYNOMIALS $\Xi_n(u)$, $\mathrm{T}_n(u)$, $\Omega_n(u)$, AND $\Psi_n(u)$

The Bernoulli and Euler polynomials allow themselves to be expressed as follows:

$$\begin{aligned}
B_{2n}(x)&=\Xi_n(u),\ u=x^2-x,\ n=0,1,2,\ldots,\\
E_{2n}(x)&=\mathrm{T}_n(u),\ u=x^2-x,\ n=0,1,2,\ldots,\\
B_{2n-1}(x)&=(2x-1)\,\Omega_{n-1}(u),\ u=x^2-x,\ n=1,2,\ldots,\\
E_{2n-1}(x)&=(2x-1)\,\Psi_{n-1}(u),\ u=x^2-x,\ n=1,2,\ldots,
\end{aligned}$$

where Ξ, T, Ω, and Ψ are all polynomials in u (see, e.g., Subramanian and Devanathan [6]). Applying (15), (16), (20), and (21) of [7], we get the following:

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r} \Delta^r U_n^{2r} U_{n(m-2r)} \Xi_r(u) = \frac{m}{2^{m-1}} U_n \sum_{r=0}^{[(m-1)/2]} \binom{m-1}{2r} \Delta^r U_n^{2r} V_n^{m-2r-1} (1+4u)^r, \tag{29}$$

$$\sum_{r=0}^{[(m-2)/2]} \binom{m}{2r+1} \Delta^r U_n^{2r+1} U_{n(m-2r-1)} \Omega_r(u) = \frac{m}{2^{m-1}} U_n \sum_{r=0}^{[(m-2)/2]} \binom{m-1}{2r+1} \Delta^r U_n^{2r+1} V_n^{m-2r-2} (1+4u)^r, \tag{30}$$

$$\sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^r U_n^{2r} V_{n(m-2r)} \mathrm{T}_r(u) = \frac{1}{2^{m-1}} U_n \sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^r U_n^{2r} V_n^{m-2r} (1+4u)^r, \tag{31}$$

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r+1} \Delta^r U_n^{2r+1} V_{n(m-2r-1)} \Psi_r(u) = \frac{1}{2^{m-1}} U_n \sum_{r=0}^{[(m-1)/2]} \binom{m-1}{2r+1} \Delta^r U_n^{2r+1} V_n^{m-2r-1} (1+4u)^r. \tag{32}$$

6. A REMARK

From our main results (16), (17), (18), and (19), according to different choices of k, x, p, and q, using recurrence relation (13), we can obtain many interesting identities.

Our results have been numerically checked and found to be correct for $m \leq 5$.

ACKNOWLEDGMENT

The authors wish to thank the anonymous referee for valuable suggestions and the editor of this journal, Gerald E. Bergum, for posting reference [6], which led to a substantial improvement of the paper.

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AMS Classification Numbers: 11B39, 11B37, 05A15

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THE DIOPHANTINE EQUATIONS $x^2 - k = T_n(a^2 \pm 1)$

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(Submitted November 1996)

It is the object of this note to demonstrate that the two equations of the title have only finitely many solutions in positive integers x and n for any given integers a and k, $k \neq \pm 1$. In these equations, $(T_n)_{n\geq 0}$ is the sequence of Chebyshev polynomials of the first kind.

1. Chebyshev Polynomials of the First Kind $(T_n(x))_{n\geq 0}$.

These polynomials are defined by the recurrence relation

$$T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x), \ (\forall) x \in C, n \in N^*, \tag{1.1}$$

where $T_0(x) = 1$ and $T_1(x) = x$.

We also have the sequence $(\widetilde{T}_n(x))_{n\geq 0}$ of polynomials "associated" with the Chebyshev polynomials $(T_n(x))_{n\geq 0}$:

$$\widetilde{T}_{n+1}(x) = 2x \cdot \widetilde{T}_n(x) + \widetilde{T}_{n-1}(x), \ x \in C, n \in N^*, \tag{1.2}$$

with $\widetilde{T}_0(x) = 1$ and $\widetilde{T}_1(x) = x$.

The connection between the sequence $(\widetilde{T}_n)_{n\geq 0}$ and the sequence $(T_n)_{n\geq 0}$ is given by the simple relations,

$$\begin{cases} \widetilde{T}_k(x) = \frac{T_k(i\cdot x)}{i^k}, \\ T_k(x) = \frac{\widetilde{T}_k(i\cdot x)}{i^k}, \ k \in N, x \in C, \end{cases} \tag{1.3}$$

where $i^2 = -1$.

Two important properties of the polynomials $(T_n)_{n\geq 0}$ are given by the formulas

$$T_n(\cos\varphi) = \cos n\varphi, \ n \in N, \varphi \in C, \tag{1.4}$$

and

$$T_m(T_n(x)) = T_{mn}(x), \ (\forall) m, n \in N, (\forall) x \in C. \tag{1.4'}$$

Also, we observe that

$$\begin{array}{ll}
T_0\left(\frac{x}{\sqrt{2}}\right) = 1 & \widetilde{T}_0\left(\frac{x}{\sqrt{2}}\right) = 1 \\
T_1\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}} \cdot 1 & \widetilde{T}_1\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}} \cdot 1 \\
T_2\left(\frac{x}{\sqrt{2}}\right) = x^2 - 1 & \widetilde{T}_2\left(\frac{x}{\sqrt{2}}\right) = x^2 + 1 \\
T_3\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}} \cdot (2x^2 - 3) & \widetilde{T}_3\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}} \cdot (2x^2 + 3) \\
T_4\left(\frac{x}{\sqrt{2}}\right) = 2x^4 - 4x^2 + 1 & \widetilde{T}_4\left(\frac{x}{\sqrt{2}}\right) = 2x^4 + 4x^2 + 1 \\
T_5\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}} \cdot (4x^4 - 10x^2 + 5) & \widetilde{T}_5\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}} \cdot (4x^2 + 10x^2 + 5) \\
\dots & \dots
\end{array}$$

2. The Equation $x^2 - k = T_n(a^2 - 1)$.

Lemma 1: If $(T_n(x))_{n\geq 0}$ is the sequence of Chebyshev polynomials of the first kind, then one has

$$T_n(a^2-1) = 2\cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right)-1,\ (\forall)n \in N, (\forall)a \in C. \tag{2.1}$$

Proof: Indeed, we have

$$\begin{aligned} T_n(a^2-1) &= T_n\left(2\cdot\left(\tfrac{a}{\sqrt{2}}\right)^2-1\right) = T_n\left(T_2\left(\tfrac{a}{\sqrt{2}}\right)\right) = T_{2n}\left(\tfrac{a}{\sqrt{2}}\right) \\ &= T_2\left(T_n\left(\tfrac{a}{\sqrt{2}}\right)\right) = 2\cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right)-1. \quad \text{Q.E.D.} \end{aligned}$$

Lemma 2: We have

$$2\cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right) = z_m^2,\ z_m \in N^*, \tag{2.2}$$

where $n = 2m+1,\ m \in N$.

Proof: Indeed

$$\begin{aligned} 2\cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right) &= 2\cdot T_{2m+1}^2\left(\tfrac{a}{\sqrt{2}}\right) = 2\cdot\left(\tfrac{a}{\sqrt{2}}\cdot(\cdots)\right)^2 \\ &= a^2\cdot(\cdots)^2 = (a(\cdots))^2 = z_m^2,\ z_m \in N^*. \quad \text{Q.E.D.} \end{aligned}$$

From Lemma 1 and Lemma 2 one obtains, for $n = 2m+1,\ m \in N$, $T_n(a^2-1) = T_{2m+1}(a^2-1) = z_m^2 - 1$, where $z_m \in Z$. Thus, $x^2 - k = z_m^2 - 1$, which can be solved immediately, giving only finitely many possible values of x, if $k \neq \pm 1$ (see [2]); hence, only finitely many possible corresponding values for $n = 2m+1, m \in N$.

For $n = 2m, m \in N$, from Lemma 1, one obtains

$$T_n(a^2-1)+1 = 2\cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right) = 2\cdot z_m^2,\ z_m \in N,$$

where

$$z_m = T_{2m}\left(\tfrac{a}{\sqrt{2}}\right) = T_2\left(T_m\left(\tfrac{a}{\sqrt{2}}\right)\right) = 2\cdot T_m^2\left(\tfrac{a}{\sqrt{2}}\right)-1 = 2\cdot w_m^2 - 1$$

if m is even. If $m = 2\lambda+1$ is odd, we have

$$z_m = T_{2m}\left(\tfrac{a}{\sqrt{2}}\right) = \begin{cases} v_m^2 - 1, & m = 2\lambda+1, \lambda \in N, \\ 2w_m^2 - 1, & m = 2\lambda, \lambda \in N. \end{cases} \tag{2.3}$$

Consequently, one gets

$$\begin{aligned} x^2 - k = T_n(a^2-1) = 2\cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right)-1 &= \begin{cases} 2\cdot(v_m^2-1)^2-1, & m \text{ odd}, \\ 2\cdot(2w_m^2-1)^2-1, & m \text{ even}, \end{cases} \\ &= \begin{cases} 2\cdot v_m^4 - 4v_m^2 + 1, & m \text{ odd}, \\ 8w_m^4 - 8w_m^2 + 1, & m \text{ even}. \end{cases} \end{aligned} \tag{2.4}$$

Thus, we obtain either

$$x^2 = 2v_m^4 - 4v_m^2 + k + 1 = T_4\left(\tfrac{v_m}{\sqrt{2}}\right) + k \tag{2.5}$$

or

$$x^2 = 8w_m^4 - 8w_m^2 + k + 1 = T_4(w_m) + k, \tag{2.6}$$

and each of these equations has but a finite number of solutions in integers for each given $k = \pm 1$ (see [2]). Thus, for each given $k \in Z$, $k \neq \pm 1$, there are but finitely many possible values of x, and hence of corresponding $n = 2m, m \in N$.

3. The Equation $x^2 - k = T_n(a^2 + 1)$.

Lemma 3: If $(\widetilde{T}_n)_{n \geq 0}$ is the sequence of polynomials "associated" with the Chebyshev polynomials $(T_n)_{n \geq 0}$, then one has:

(a) $\widetilde{T}_{2n}\left(\tfrac{a}{\sqrt{2}}\right) = 2 \cdot \widetilde{T}_n^2\left(\tfrac{a}{\sqrt{2}}\right) - (-1)^n$, $n \in N$;

(b) $T_n(a^2+1) = \widetilde{T}_{2n}\left(\tfrac{a}{\sqrt{2}}\right)$, $n \in N$;

(c) $T_n(a^2+1) = 2 \cdot \widetilde{T}_n^2\left(\tfrac{a}{\sqrt{2}}\right) - (-1)^n$, $n \in N$.

Proof:

(a) We have:

$$\begin{aligned}\widetilde{T}_{2n}\left(\tfrac{a}{\sqrt{2}}\right) &= \frac{T_{2n}\left(i \cdot \frac{a}{\sqrt{2}}\right)}{i^{2n}} = (-1)^n \cdot T_{2n}\left(i \cdot \tfrac{a}{\sqrt{2}}\right) = (-1)^n \cdot T_2\left(T_n\left(i \cdot \tfrac{a}{\sqrt{2}}\right)\right) \\ &= (-1)^n \cdot \left[2 \cdot T_n^2\left(i \cdot \tfrac{a}{\sqrt{2}}\right) - 1\right] = (-1)^n \cdot \left[2 \cdot \left(i^n \cdot \widetilde{T}_n\left(\tfrac{a}{\sqrt{2}}\right)\right)^2 - 1\right] \\ &= (-1)^n \cdot \left(2 \cdot (-1)^n \cdot \widetilde{T}_n^2\left(\tfrac{a}{\sqrt{2}}\right) - 1\right) = 2 \cdot \widetilde{T}_n^2\left(\tfrac{a}{\sqrt{2}}\right) - (-1)^n. \quad \text{Q.E.D.}\end{aligned}$$

(b)

$$\begin{aligned}\widetilde{T}_{2n}\left(\tfrac{a}{\sqrt{2}}\right) &= \frac{T_{2n}\left(i \cdot \frac{a}{\sqrt{2}}\right)}{i^{2n}} = (-1)^n \cdot T_{2n}\left(i \cdot \tfrac{a}{\sqrt{2}}\right) = (-1)^n \cdot T_n\left(T_2\left(i \cdot \tfrac{a}{\sqrt{2}}\right)\right) \\ &= (-1)^n \cdot T_n\left(2 \cdot \left(\tfrac{ia}{\sqrt{2}}\right)^2 - 1\right) = (-1)^n \cdot T_n(-a^2 - 1) \\ &= (-1)^n \cdot (-1)^n \cdot T_n(a^2 + 1). \quad \text{Q.E.D.}\end{aligned}$$

(c) For $n = 2m+1, m \in N$, we have

$$\widetilde{T}_{2n}\left(\tfrac{a}{\sqrt{2}}\right) = 2 \cdot \widetilde{T}_{2m+1}^2\left(\tfrac{a}{\sqrt{2}}\right) + 1 = \left(\sqrt{2} \cdot \widetilde{T}_{2m+1}\left(\tfrac{a}{\sqrt{2}}\right)\right)^2 + 1 = z_m^2 + 1,$$

where

$$z_m = \sqrt{2} \cdot \widetilde{T}_{2m+1}\left(\tfrac{a}{\sqrt{2}}\right) \in N^*.$$

Thus, in this case, we obtain $x^2 - k = z_m^2 + 1$, and the result follows as before.

For $n = 2m, m \in N$, we have

$$T_n(a^2+1) = T_{2m}(a^2+1) = 2\cdot \widetilde{T}^2_{2m}\left(\tfrac{a}{\sqrt{2}}\right) - 1 = 2\cdot t_m^2 - 1,$$

where

$$t_m = \widetilde{T}^2_{2m}\left(\tfrac{a}{\sqrt{2}}\right) = 2\cdot \widetilde{T}^2_{m}\left(\tfrac{a}{\sqrt{2}}\right) - (-1)^m = \begin{cases} v_m^2+1, & m \text{ odd}, \\ 2w_m^2 - 1, & m \text{ even}. \end{cases}$$

Consequently, we have

$$T_n(a^2+1) = T_{2m}(a^2+1) = \begin{cases} 2\cdot(v_m^2+1)^2 - 1, & m \text{ odd}, \\ 2\cdot(2w_m^2-1)^2 - 1, & m \text{ even}, \end{cases} = \begin{cases} 2v_m^4 + 4v_m^2 + 1, & m \text{ odd}, \\ 8w_m^4 - 8w_m^2 + 1, & m \text{ even}. \end{cases} \tag{3.1}$$

Thus, we obtain

$$x^2 = 2v_m^4 + 4v_m^2 + k + 1 = \widetilde{T}_4\left(\tfrac{v_m}{\sqrt{2}}\right) + k \tag{3.2}$$

or

$$x^3 = 8w_m^4 - 8w_m^2 + k + 1 = T_4(w_m) + k \tag{3.3}$$

and the result follows. In this case, as before, for each given $k \neq \pm 1$, there are finitely many possible values of x, and hence, only finitely many possible corresponding values for $n = 2m,\ m \in N$.

This concludes the proof of the result of this paper.

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AMS Classification Numbers: 11B37, 11B39, 11D25

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A GENERALIZATION OF THE KUMMER IDENTITY AND ITS APPLICATION TO FIBONACCI-LUCAS SEQUENCES

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(Submitted November 1996-Final Revision April 1997)

1. INTRODUCTION

We assume that the reader is familiar with the basic notations and facts from combinatorial analysis (cf. [1]). In [2] and [4], H. W. Gould and P. Haukkanen discussed the following transformation of the sequence $\{A_k\}_{k=0}^{\infty}$, that is,

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}t^{n-k}s^kA_k \quad \text{for } n=0,1,2,\dots. \tag{1}$$

Let $A(x)$ and $S(x)$ be the formal series determined by $\{A_k\}_{k=0}^{\infty}$ and $\{S(k)\}_{k=0}^{\infty}$. Then

$$S(x)=\frac{1}{1-tx}A\left(\frac{sx}{1-tx}\right). \tag{2}$$

They found that transformation of (1) or (2) is related to Fibonacci numbers. Recently, Shapiro et al. [8] and Sprugnoli [10] introduced the theories of the Riordan array and the Riordan group, respectively, in an effort to answer the following question: What are the conditions under which a combinatorial sum can be evaluated by transforming the generating function? We think that the works of Gould and Haukkanen mentioned above can be extended by using the Riordan group or the Riordan array. We adopt the concept of the Riordan group in this paper because both theories are essentially the same. It is certain that the idea of the Riordan group can be traced back to Mullin and Rota [5], Rota [6], and Roman and Rota [7]. The reader is referred to [5]-[10] for more details. In the present paper we are concerned with the following identity, called the *Kummer identity*:

$$x^n+y^n=\sum_{0\le k\le n/2}(-1)^k\frac{n}{n-k}\binom{n-k}{k}(x+y)^{n-2k}(xy)^k. \tag{3}$$

It is well known that the number $(-1)^k\frac{n}{n-k}\binom{n-k}{k}$ is closely related to the *probléme des ménages* and the Kummer identity is closely related to the Fibonacci-Lucas sequence (cf. [11]). With the help of the Riordan group, we give a generalization of the Kummer identity as follows.

Theorem 1: $\forall\, x, y$ and $z\in R$ with $xy+yz+zx=0$, the following identity holds:

$$x^n+y^n+z^n=\sum_{0\le k\le n/3}\frac{n}{n-2k}\binom{n-2k}{k}(x+y+z)^{n-3k}(xyz)^k. \tag{4}$$

It is reasonable to believe that the above identity can find some application to the second-order Fibonacci-Lucas sequence as the Kummer identity does; this needs to be discussed further.

Remark: According to the referee, it is worth noting that the *Kummer identity* and Theorem 1 can also be obtained by the use of symmetric functions, a large and older literature that dates back to Albert Girard in (1629) and concerns summation formulas for the powers of the roots of algebraic equations. Actually, in view of symmetric functions, these results may be extended in various forms to deal with $t_1^n + t_2^n + \cdots + t_n^n$, where t_i are the distinct roots of certain algebraic equations. However, not completely new as it is, we rederive it by using the Riordan group for the purpose of establishing some combinatorial sums as well as a reciprocal relation satisfied by two famous numbers in the literature.

To make this paper self-contained, we need some elementary results regarding the Riordan group.

Definition 1: Given $d(t)$, $f(t) \in R[t]$ with $f(0) = 0$. Let $d_{n,k}$ be the number given by

$$d_n = [t^n](d(t)f^k(t)), \tag{5}$$

where $[t^n](\cdot)$ denotes the coefficient of t^n in $(\cdot)$. Write $M = (d(t), f(t))$ for matrix $(d_{n,k})$ with entries in R, and M_R for the set $\{M = (d(t), f(t)) \mid f(0) = 0\}$. Then M_R is referred to as the Riordan group endowed with the binary operation $*$ as follows:

$$(d(t), f(t)) * (h(t), g(t)) = (d(t)h(f(t)), g(f(t))), \tag{6}$$

where $h(f(t))$ denotes the composition of $h(t)$ and $f(t)$ just as usual.

Theorem 2: (Cf. [9].) Let $d_{n,k}$ be defined by (5) and $F(t) = \sum_{k\geq 0} a_k t^k$. Then

$$\sum_{n\geq k\geq 0} d_{n,k} a_k = [t^n]d(t)F(f(t)). \tag{7}$$

As far as the Kummer identity (3) and its generalization (4) are concerned, we know from [9] that

$$(d_{n,k}) = \left(\frac{k}{n-k}\binom{n-k}{k}\right) \quad \text{and} \quad \left(\frac{k}{n-2k}\binom{n-2k}{k}\right)$$

are just two elements of M_R. Based on this result, (3) can be verified directly and (4) can be rediscovered.

2. PROOF OF THEOREM 1

Recall the facts mentioned in the last section. We can obtain two preliminary results directly from Theorem 2.

Lemma 1: Let $F(t) = \sum_{k\geq 1} \frac{f_k}{k} t^k$ and $G(t) = \sum_{k\geq 1} \frac{g_k}{k} t^k$. Then

$$\sum_{0\leq k\leq n/2} \frac{n}{n-k}\binom{n-k}{k} f_k = f_0 + n[t^n]F\left(\frac{t^2}{1-t}\right); \tag{8}$$

$$\sum_{0\leq k\leq n/3} \frac{n}{n-2k}\binom{n-2k}{k} g_k = g_0 + n[t^n]G\left(\frac{t^3}{1-t}\right). \tag{9}$$

Proof: Since

$$\binom{n-k}{k} = \frac{n-k}{k}\binom{n-k-1}{k-1}, \quad \binom{n-2k}{k} = \frac{n-2k}{k}\binom{n-2k-1}{k-1},$$

and

$$\binom{n-k-1}{k-1} = [t^n]\left(\frac{t^2}{1-t}\right)^k, \quad \binom{n-2k-1}{k-1} = [t^n]\left(\frac{t^3}{1-t}\right)^k.$$

From Theorem 2, (8) and (9) follow as desired. □

Lemma 2: $\forall\, a, b$ and $c \in R$. Then the following identities hold:

$$a^n + b^n = \sum_{0 \le k \le n/2} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(ab)^k \tag{10}$$

for all a and b satisfying $a+b=1$;

$$a^n + b^n + c^n = \sum_{0 \le k \le n/3} \frac{n}{n-2k}\binom{n-2k}{k}(abc)^k \tag{11}$$

for all $a+b+c=1$ and $ab+bc+ca=0$.

Proof: It suffices to show (10) because a similar argument remains valid for (11). From Lemma 1, it follows that

$$F(t) = \sum_{k \ge 1} \frac{(-ab)^k}{k} t^k = -\ln(1+abt) \tag{12}$$

and

$$F\left(\frac{t^2}{1-t}\right) = \ln(1-t) - \ln(1-t+abt^2). \tag{13}$$

Consider that $a+b=1$ if and only if $1-t+abt^2 = (1-at)(1-bt)$. Thus, it is easy to verify that

$$F\left(\frac{t^2}{1-t}\right) = \ln(1-t) - \ln(1-at) - \ln(1-bt) \tag{14}$$

and

$$n[t^n]f\left(\frac{t^2}{1-t}\right) = -1 + a^n + b^n. \tag{15}$$

Combining the above result with (10) gives the complete proof of the Lemma. □

Proof of Theorem 1 via the Riordan Group: Let

$$a = \frac{x}{x+y+z}, \quad b = \frac{y}{x+y+z}, \quad \text{and} \quad c = \frac{z}{x+y+z}.$$

We see that $ab+bc+ca=0$ equals $xy+yz+zx=0$. Then Theorem 1 follows as desired. □

A similar proof of the Kummer identity (3) was also found by Sprugnoli in [9].

3. APPLICATION

In this section, by setting specified values a and b into Theorem 1, we will find various combinatorial identities.

Example 3.1: Let $a = (x+\sqrt{x^2-4})/2x$ and $b = (x-\sqrt{x^2-4})/2x$. Then (10) gives a generalization of the very old Hardy identity,

$$\sum_{0\le k\le n/2}(-1)^k \frac{n}{n-k}\binom{n-k}{k}x^{n-2k} = \frac{(x+\sqrt{x^2-4})^n+(x-\sqrt{x^2-4})^n}{2^n}. \tag{16}$$

Example 3.2: Let $a=t$ and $b=1-t$. Then

$$t^{2n}+(1-t)^{2n} = \sum_{k=0}^{n}(-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(t(1-t))^k. \tag{17}$$

This implies that

$$\sum_{k=0}^{n}\frac{2n}{2n-k}\binom{2n-k}{k}\binom{k}{m-k}=\binom{2n}{m}, \tag{18}$$

where m is an integer, $0\le m\le n-1$. Thus,

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{2n+1}{2k+1}}{\binom{2n-1}{k}}\equiv 2. \tag{19}$$

It is worth noting that the above two identities are missing from [3].

Proof: Equation (18) can be obtained by comparing the coefficient of t^m and (19) can be obtained by integrating from 0 to 1 on both sides of (17). □

Note that the same method exhibited above is often used when we proceed to set up a combinatorial identity.

Example 3.3: Let F_n be the n^{th} Fibonacci number defined by

$$\begin{cases} F_{n+2}=F_{n+1}+F_n & (n\ge 0); \\ F_1=F_0=1. \end{cases} \tag{20}$$

Then

$$F_n+F_{n-2}=\sum_{k=0}^{n/2}\frac{n}{n-k}\binom{n-k}{k} \quad (n\ge 2). \tag{21}$$

Furthermore, we have an arithmetic identity,

$$F_p+F_{p-2}=1 \pmod p, \tag{22}$$

where p is a prime.

Now let us focus attention on (4). Without loss of generality, we suppose that x, y, and z are all roots of a given equation,

$$X^3-(c-b_1)X^2-(cb_1-b_2)X-cb_2=(X-c)(X^2+b_1X+b_2).$$

Then $xy+yz+zx=0$ is equivalent to condition $cb_1=b_2$. Thus, from the Kummer identity (3) and its generalization (4), it follows that

$$x^n+y^n=\sum_{0\le k\le n/2}(-1)^{n-k}\frac{n}{n-k}\binom{n-k}{k}b_1^{n-2k}b_2^k \tag{23}$$

and

$$x^n+y^n+c^n=\sum_{0\le k\le n/3}\frac{n}{n-2k}\binom{n-2k}{k}(c-b_1)^{n-3k}(cb_2)^k. \tag{24}$$

Under the condition that $cb_1 = b_2$, the above identities lead to the following theorem.

Theorem 3: $\forall\, t_1, t_2 \in R$. Then we have

$$t_1^n + \sum_{0\le k\le n/2} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k} t_1^k t_2^{n-k} = \sum_{0\le k\le n/3} \frac{n}{n-2k}\binom{n-2k}{k}(t_1 - t_2)^{n-3k}(t_1^2 t_2)^k. \tag{25}$$

We are convinced that this result includes a series of combinatorial identities. To justify this claim, we write down some interesting identities below.

Example 3.4: Let $t_1 = 1$, $t_2 = t$. Then

$$1 + \sum_{0\le k\le n/2} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k} t^{n-k} = \sum_{0\le k\le n/3} \frac{n}{n-2k}\binom{n-2k}{k}(1-t)^{n-3k} t^k, \tag{26}$$

which implies that

$$1 + \sum_{0\le k\le n/2} (-1)^{n-k} \frac{n}{(n-k)(n-k+1)}\binom{n-k}{k} = \sum_{0\le k\le n/3} \frac{n}{(n-2k)(n-2k+1)} \tag{27}$$

and

$$\sum_{0\le k\le n/3} (-1)^k \frac{n}{n-2k}\binom{n-2k}{k}\binom{n-3k}{m-k} = \frac{n}{m}\binom{m}{n-m}. \tag{28}$$

Example 3.5:

$$t^n + \sum_{0\le k\le n/2} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k} t^k = \sum_{0\le k\le n/3} \frac{n}{n-2k}\binom{n-2k}{k}(t-1)^{n-3k} t^{2k}, \tag{29}$$

which implies that

$$\frac{1}{n+1} + \sum_{0\le k\le n/2} (-1)^{n-k} \frac{n}{(n-k)(k+1)}\binom{n-k}{k} = \sum_{0\le k\le n/3} (-1)^{n-3k} \frac{n\binom{n}{k}}{(n-2k)(n-k+1)\binom{n}{2k}}, \tag{30}$$

and

$$\sum_{0\le k\le n/3} (-1)^{n-k-m} \frac{n}{n-2k}\binom{n-2k}{m-k}\binom{n-3k}{m-2k} = (-1)^{n-m}\frac{n}{n-m}\binom{n-m}{m}. \tag{31}$$

Exploring further, let $t_1 = 1/\sqrt{1-x}$, $t_2 = \sqrt{1-x}$, and n be replaced by $2n$. Then Theorem 3 suggests the next example.

Example 3.6: $\forall\, 0 \le x \le 1$, the following holds:

$$\begin{aligned}(1-x)^{-n} &+ \sum_{0\le k\le n} (-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(1-x)^{n-k} \\ &= \sum_{0\le k\le 2n/3} \frac{n}{n-k}\binom{2n-2k}{k} x^{2n-3k}(1-x)^{-n+k}.\end{aligned} \tag{32}$$

Using the generating function technique to expand $(1-x)^{-n}$ into a formal series of the power of x, this identity gives

$$\begin{aligned}&\sum_{0\le i\le+\infty}\binom{n-1+i}{n-1}x^1+\sum_{0\le k,\,i\le n}(-1)^{k+i}\frac{2n}{2n-k}\binom{2n-k}{k}\binom{n-k}{i}x^i\\&=\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-k}{k}\binom{n-k-1+i}{n-k-1}x^{2n-3k+i}.\end{aligned}\tag{33}$$

By the same argument as taken in Example 3.2, we can derive

$$\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-2k}{k}\binom{m+2k-n-1}{n-k-1}=\binom{n-1+m}{n-1}\tag{34}$$

for all $m\ge n+1$, and

$$\begin{aligned}&\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-2k}{k}\binom{m+2k-n-1}{n-k-1}\\&=\binom{n-1+m}{n-1}+\sum_{0\le k\le n}(-1)^{m+k}\frac{2n}{2n-k}\binom{2n-k}{k}\binom{n-k}{m}\end{aligned}\tag{35}$$

for all $m\le n$.

As mentioned earlier, the coefficient appearing on the right-hand side of the Kummer identity (3) is closely related to the number $\mu(n)$ of the reduced *ménages* problem with n married couples, for which there exists an explicit computing formula as below:

$$\mu(n)=\sum_{0\le k\le n}(-1)^k\frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!$$

In the meantime, it is also well known that, for the set $[n]$, the number $d(n)$ of derangements of $[n]$ is equal to

$$d(n)=\sum_{k=0}^{n}(-1)^k\frac{n!}{k!}.$$

The most important case related to applications of Theorem 1 is that we find the following reciprocal relation connecting $\mu(n)$ with $d(n)$ be means of (32). Indeed, the following theorem follows from (32).

Theorem 4—The Reciprocal Relation: Given $\mu(n)$ and $d(n)$ as above, we have

$$\mu(n)=\sum_{k=0}^{n}A(n,k)\,d(k)\tag{36}$$

if and only if

$$d(n)=\sum_{k=0}^{n}B(n,k)\,\mu(k),\tag{37}$$

where $A(n,k)$ and $B(n,k)$ are given, respectively, by

$$A(n,k)=\sum_{i=0}^{m}(-1)^i\frac{2n}{2n-i}\binom{2n-i}{i}\binom{n-i}{k}$$

and

$$B(n,k)=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\binom{2i}{k-i}.$$

Proof: To prove (36), we first note that

$$\sum_{0\le i\le n}\binom{n-1+i}{n-1}x^i+\sum_{0\le k,i\le n}(-1)^{k+i}\frac{2n}{2n-k}\binom{2n-k}{k}\binom{n-k}{i}x^i$$
$$=\sum_{n+i\le 3k\le 2n}\frac{n}{n-k}\binom{2n-2k}{k}\binom{n-k-1+i}{n-k-1}x^{2n-3k+i}.$$

Multiply both sides of this identity by e^x and then integrate from $-\infty$ to 1. Consider that

$$\int_{-\infty}^{1}x^ie^x\,dx=(-1)^id(i)e \quad\text{and}\quad \int_{-\infty}^{1}(1-x)^{n-k}e^x\,dx=(n-k)!\,e.$$

Set these into the above identity, to obtain

$$\mu(n)+\sum_{0\le i\le n}\binom{n-1+i}{n-1}(-1)^id(i)$$
$$=\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-k}{k}\binom{n-k-1+i}{n-k-1}(-1)^{2n-3k+i}d(2n-3+i),$$

which leads to

$$\mu(n)=-\sum_{0\le i\le n}\binom{n-1+i}{n-1}(-1)^id(i)$$
$$+\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-k}{k}\binom{n-k-1+i}{n-k-1}(-1)^{2n-3k+i}d(2n-3k+i)$$
$$=\sum_{0\le i\le n}\left\{\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-k}{k}\binom{i+2k-n-1}{n-k-1}-\sum_{0\le i\le n}\binom{n-1-i}{n-1}\right\}(-1)^id(i).$$

Combining the above result with (35) yields the complete proof of the theorem.

Now we proceed to prove (37). Let ϕ_1 and ϕ_2 be the two roots of the equation $X^2-xX+1=0$. For simplicity, we write H_n for $\phi_1^{2n}+\phi_2^{2n}$. Then, from the Hardy identity (16), we have

$$H_n=\sum_{0\le k\le n}(-1)^k\frac{2n}{2n-k}\binom{2n-k}{k}x^{2(n-k)}.$$

On the other hand, observe that $\phi_1+\phi_2=x$ and $\phi_1\phi_2=1$. We can show by induction on n that

$$x^{2n}=\sum_{0\le k\le n}\binom{2n}{n+k}H_k.$$

Thus, relation (37) reads out from the above identities immediately as desired. □

Evidently, $B(n,n)=1$, $B(n,k)=0$ $(n<k)$, which gives a new efficient recurrence relation for $\mu(n)$ as follows:

$$\mu(n)=d(n)-\sum_{0\le k\le n-1}B(n,k)\mu(k). \tag{38}$$

Besides its application to combinatorial identities, we also find that identity (4) is closely related to the second-order Fibonacci-Lucas sequence. Let $\Omega(\lambda_1, \lambda_2)$ denote the set of second-order Fibonacci-Lucas sequences, defined as follows:

$$\begin{cases} L_{n+2} = \lambda_1 L_{n+1} + \lambda_2 L_n \quad (n \geq 0); \\ L_0 = c_0, \ L_1 = c_1. \end{cases}$$

Let $x_1, x_2, x_1 \neq x_2$ be the two roots of the equation $x^2 - \lambda_1 x - \lambda_2 = 0$. Then we have

Lemma 3—De Moivre Formula: (Cf. [11].) Let $L_n^{(i-1)} \in \Omega(\lambda_1, \lambda_2)$ and x_i $(i = 1, 2)$ be given as above. Then

$$x_i^n = L_n^{(1)} x_i + L_n^{(0)}, \tag{39}$$

where $L_n^{(0)}$ and $L_n^{(1)}$ are the second-order Fibonacci-Lucas sequences with $c_0 = 1$, $c_1 = 0$, and $c_0 = 0$, $c_1 = 1$, respectively.

Let $c = t \in R$. Then the conditions that $a + b + c = 1$ and $ab + bc + ca = 0$ amount to the following relations:

$$\begin{cases} a + b = 1 - t; \\ ab = t(t-1). \end{cases}$$

This means that a and b satisfy the equation $X^2 - (1-t)X + t(t-1) = 0$. Therefore, by means of (39), we restate identity (11) in the following form in terms of the second-order Fibonacci-Lucas sequence.

Theorem 5: Let $L_n^{(0)}(t), L_n^{(1)}(t)$ be the second-order Fibonacci-Lucas sequences with $\lambda_1 = 1 - t$, $\lambda_2 = t(1-t)$. Then

$$t^n + (1-t)L_n^{(1)}(t) + 2L_n^{(0)}(t) = \sum_{0 \leq k \leq n/3} \frac{n}{n-2k} \binom{n-2k}{k} (t^2(t-1))^k. \tag{40}$$

We believe that the above result can be useful in finding some arithmetic identities for the second-order Fibonacci-Lucas sequence. For instance, we can obtain

Corollary 5.1:

$$(1-t)L_p^{(1)}(t) + 2L_p^{(0)}(t) \equiv 1 - t \pmod{p}, \tag{41}$$

where $L_p^{(0)}(t)$, $L_p^{(1)}(t)$ are as given by (39) and p is a prime.

ACKNOWLEDGMENT

The author wishes to thank the anonymous referee for supplying useful comments with a professional remark that improved the presentation of this paper.

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AMS Classification Numbers: 05A10, 05A19, 11A25, 11B39, 11B65

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SOME EXTENSIONS OF PROPERTIES OF THE SEQUENCE OF RECIPROCAL FIBONACCI POLYNOMIALS

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(*Submitted November 1996-Final Revision September 1997*)

This paper is, in a sense, dual to the Fibonacci Association paper by J. R. Howell [4]. On the other hand, interest in the *reciprocal* Fibonacci-like polynomials is caused by the very effective propositions 7 and 4 of [3].

It is also the intention of this paper to draw the attention of the Fibonacci Association audience to the vast area of applications of its activities in the domain of computational techniques allowing one to perform quantitative comparisons among various data organizations in the framework defined by the authors of [3].

Let $W_n(x)$ be a polynomial in the variable x; $x \in (c,d) \subset \mathbf{R}$ and $\deg(W_n(x)) = N$. We define the reciprocal polynomial of $W_n(x)$ as follows.

Definition 1:
$$\overline{W}_n(x) = x^N W_n\left(\frac{1}{x}\right). \tag{1}$$

The purpose of this paper is to describe the reciprocal polynomials of Fibonacci-like polynomials that are defined by the recursion formula [4]
$$g_{n+2}(x) = axg_{n+1}(x) + bg_n(x), \tag{2}$$
where a and b are real constants.

It is easy to verify that the reciprocal Fibonacci-like polynomials satisfy
$$\overline{g}_{n+2}(x) = a\overline{g}_{n+1}(x) + bx^2\overline{g}_n(x), \quad n > 2. \tag{3}$$
Indeed, if $\deg g_3(x) = m$, then
$$\deg g_n(x) = n - 3 + m, \quad \text{for } n > 2. \tag{4}$$
From (2),
$$x^{n+m-1}g_{n+2}\left(\frac{1}{x}\right) = ax^{n+m-2}g_{n+1}\left(\frac{1}{x}\right) + bx^{n+m-1}g_n\left(\frac{1}{x}\right).$$
Hence, (3) follows by (1) and (4). If $\deg g_2(x) \geq \deg g_1(x)$, then the recursion formula (3) is true for $n \geq 2$, and if $\deg g_2(x) = \deg g_1(x) + 1$, then (3) holds for each natural number n.

Theorem 1: Suppose that the sequence $\{\overline{g}_n(x)\}$ satisfies (3) for every natural number n. Then the following summation formula holds:

$$\sum_{j=1}^{p} \overline{g}_j(x) = \frac{\overline{g}_{p+1}(x) + bx^2\overline{g}_p(x) + (a-1)\overline{g}_1(x) - \overline{g}_2(x)}{a+bx^2-1}, \tag{5}$$

for each natural number p.

Proof: For $p=1$, formula (5) is trivial. Let (5) hold for $p=k$, then

$$\begin{aligned}\sum_{j=1}^{k+1} \overline{g}_j(x) &= \frac{\overline{g}_{k+1}(x) + bx^2\overline{g}_k(x) + (a-1)\overline{g}_1(x) - \overline{g}_2(x)}{a+bx^2-1} + \overline{g}_{k+1}(x) \\ &= \frac{\overline{g}_{k+1}(x) + bx^2\overline{g}_k(x) + (a-1)\overline{g}_1(x) - \overline{g}_2(x) + a\overline{g}_{k+1}(x) + bx^2\overline{g}_{k+1}(x) - \overline{g}_{k+1}(x)}{a+bx^2-1} \\ &= \frac{\overline{g}_{k+2}(x) + bx^2\overline{g}_{k+1}(x) + (a-1)\overline{g}_1(x) - \overline{g}_2(x)}{a+bx^2-1}\end{aligned}$$

and the result follows by induction on p. □

The inverse of Theorem 1 is also true.

Proof: If the summation formula (5) holds for some sequence of polynomials (for each natural number p), then the identity

$$\overline{g}_{p+1}(x) = \sum_{j=1}^{p+1} \overline{g}_j(x) - \sum_{j=1}^{p} \overline{g}_j(x)$$

may be transformed easily into equation (3). □

For the remainder of this paper, we consider the sequence of Fibonacci-like polynomials $\{w_n(x)\}_{n=1}^{\infty}$ defined by recurrence (2), with initial values

$$w_1(x) = 1, \quad w_2(x) = ax. \tag{6}$$

If $a \neq 0$ and $b \neq 0$, then $w_p(x)$ can be written in the explicit form [4]:

$$w_p(x) = \sum_{j=0}^{[(p-1)/2]} \binom{p-1-j}{j} (ax)^{p-1-2j} b^j. \tag{7}$$

The reciprocal polynomials of $w_p(x)$ are defined by recurrence (3) with the following initial conditions:

$$\overline{w}_1(x) = 1, \quad \overline{w}_2(x) = a. \tag{8}$$

We take $\overline{w}_n(x)$ and $\overline{w}_n$ to mean the same thing.

By a simple transformation of formula (7), we obtain the explicit form of $\overline{w}_p(x)$. From (7),

$$x^{p-1} w_p\left(\frac{1}{x}\right) = \sum_{j=0}^{[(p-1)/2]} \binom{p-1-j}{j} x^{p-1} \left(\frac{a}{x}\right)^{p-1-2j} b^j.$$

Since $\deg w_n(x) = n-1$, we have

$$\overline{w}_p(x) = \sum_{j=0}^{[(p-1)/2]} \binom{p-1-j}{j} x^{2j} a^{p-1-2j} b^j. \tag{9}$$

Thus, $\overline{w}_p(x)$ is a polynomial of degree $2[(p-1)/2]$ with only even powers of x.

Theorem 2:

$$\overline{w}_p(x) = \frac{A^p(x) - B^p(x)}{A(x) - B(x)}, \tag{10}$$

where

$$A(x) = \frac{a + \sqrt{a^2 + 4bx^2}}{2} \quad \text{and} \quad B(x) = \frac{a - \sqrt{a^2 + 4bx^2}}{2}.$$

Proof: It is easy to verify that

$$A^2(x) = aA(x) + bx^2 \quad \text{and} \quad B^2(x) = aB(x) + bx^2.$$

Multiplying both sides of the above identities by A^{p-2} and B^{p-2}, respectively, we see that the sequences $, A, A^2, A^3, \ldots$ and $, B, B^2, B^3, \ldots$ satisfy (3). From these two facts, it follows that the recursion formula (3) holds also for the sequence $A - B, A^2 - B^2, A^3 - B^3, \ldots$ Since

$$\overline{w}_{n+2}(x)(A - B) = a\overline{w}_{n+1}(x)(A - B) + bx^2\overline{w}_n(x)(A - B),$$

the result follows from the identities

$$\overline{w}_1(x)(A - B) = A - B \quad \text{and} \quad \overline{w}_2(x)(A - B) = A^2 - B^2. \ \square$$

Theorem 3: Let $Q = \begin{pmatrix} a & 1 \\ bx^2 & 0 \end{pmatrix}$ and let the sequence $\{\overline{g}_n\}$ be defined by the recursion formula (3). Then, for every natural number p,

$$\begin{pmatrix} \overline{g}_{p+2} & \overline{g}_{p+1} \\ \overline{g}_{p+1} & \overline{g}_p \end{pmatrix} = \begin{pmatrix} \overline{g}_3 & \overline{g}_2 \\ \overline{g}_2 & \overline{g}_1 \end{pmatrix} Q^{p-1}. \tag{11}$$

The proof of this theorem may be realized by a simple induction argument [5]. Theorem 3 provides standard means of obtaining identities for the sequence of reciprocal Fibonacci-like polynomials [5].

For example, computing the determinants in identity (11) leads to

$$\overline{g}_{p+2}\overline{g}_p - \overline{g}_{p+1}^2 = (-b)^{p-1}x^{2p-2}(\overline{g}_3\overline{g}_1 - \overline{g}_2^2). \tag{12}$$

Now if we consider $\{\overline{w}_n\}$ with initial conditions (8), then from identities (11) and (12) we get:

$$\begin{pmatrix} \overline{w}_{p+2} & \overline{w}_{p+1} \\ \overline{w}_{p+1} & \overline{w}_p \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} Q^p; \tag{13}$$

$$\overline{w}_{p+2}\overline{w}_p - \overline{w}_{p+1}^2 = -(-b)^p x^{2p}. \tag{14}$$

Multiplying both sides of (13) on the left by $\begin{pmatrix} 1 & 0 \\ 0 & bx^2 \end{pmatrix}$ yields

$$\begin{pmatrix} \overline{w}_{p+2} & \overline{w}_{p+1} \\ bx^2\overline{w}_{p+1} & bx^2\overline{w}_p \end{pmatrix} = Q^{p+1}. \tag{15}$$

Let p and q denote natural numbers. Using (15) with Q^{p+q}, Q^p, and Q^q, one has

$$\begin{pmatrix} \overline{w}_{p+q+1} & \overline{w}_{p+q} \\ bx^2\overline{w}_{p+q} & bx^2\overline{w}_{p+q-1} \end{pmatrix} = \begin{pmatrix} \overline{w}_{p+1} & \overline{w}_p \\ bx^2\overline{w}_p & bx^2\overline{w}_{p-1} \end{pmatrix} \begin{pmatrix} \overline{w}_{q+1} & \overline{w}_q \\ bx^2\overline{w}_q & bx^2\overline{w}_{q-1} \end{pmatrix}.$$

If we compare the entries on both sides of the above identity, we obtain

$$\overline{w}_{p+q+1} = \overline{w}_{p+1}\overline{w}_{q+1} + bx^2\overline{w}_p\overline{w}_q. \tag{16}$$

Identity (16) is a special case of identity (7) in [5].

We shall now describe some of the divisibility properties of $\{\overline{w}_n\}$. If $a = 0$, then

$$\overline{w}_n(x) = \frac{1-(-1)^n}{2} b^{(n-1)/2} x^{n-1};$$

if $b = 0$, then

$$\overline{w}_n(x) = a^{n-1}.$$

In these cases, the investigation of divisibility properties of $\{\overline{w}_n(x)\}$ is easy. Suppose that a and b are nonzero numbers.

Theorem 4: Let $W(x)$ be a polynomial that divides both $\overline{w}_p$ and $\overline{w}_{p+1}$ for a fixed $p > 1$. Then $W(x)$ divides $\overline{w}_{p-1}$.

Proof: Suppose that $\overline{w}_p = W(x)S(x)$ and $\overline{w}_{p+1} = W(x)T(x)$, where $S(x)$ and $T(x)$ are certain polynomials. From (3), we have

$$x^2\overline{w}_{p-1}(x) = \frac{1}{b}W(x)(T(x) - aS(x)).$$

Thus, $W(x) \mid x^2\overline{w}_{p-1}$. From the fact that x^n does not divide $\overline{w}_n$ for any natural number n [see (9)], we conclude that x^2 and $\overline{w}_{p-1}$ are relatively prime. Finally, since $W(x)$ does not divide x^2, then $W(x) \mid \overline{w}_{p-1}$. □

Theorem 5: For natural numbers p and q, $\overline{w}_p \mid \overline{w}_{pq}$.

Proof: Let p be an arbitrary natural number. The fact that $\overline{w}_p \mid \overline{w}_p$ is trivial. If $\overline{w}_p \mid \overline{w}_{pk}$ for a certain k, then using formula (16) and the fact that $p(k+1) = (pk-1) + p + 1$, we obtain the following identity:

$$\overline{w}_{p(k+1)} = \overline{w}_{pk}\overline{w}_{p+1} + bx^2\overline{w}_{pk-1}\overline{w}_p.$$

Since $\overline{w}_p$ divides the right-hand side of the above identity, we have $\overline{w}_p \mid \overline{w}_{p(k+1)}$. This completes the proof of Theorem 5. □

We now consider some natural corollaries of Theorems 4 and 5.

Corollary 1: Let $W(x)$ be a polynomial that divides both $\overline{w}_p$ and $\overline{w}_{p+1}$ for a fixed $p > 1$. Then $W(x)$ is a constant.

Proof: Corollary 1 follows from Theorem 4 by induction. □

Corollary 2: If n, p, q, and r are natural numbers ($p > 1$) such that $q = np + r$ and if $\overline{w}_p \mid \overline{w}_q$, then $\overline{w}_p \mid \overline{w}_r$.

Proof: From $p > 1$, $np - 1 > 0$, $q = (np-1) + r + 1$ and formula (16), we have

$$\overline{w}_q = \overline{w}_{np}\overline{w}_{r+1} + bx^2\overline{w}_{np-1}\overline{w}_r.$$

Since $\overline{w}_p \mid \overline{w}_q$ and $\overline{w}_p \mid \overline{w}_{np}$, we have

$$\overline{w}_p \,|\, x^2 \overline{w}_{np-1} \overline{w}_r .$$

The greatest common divisor of $\overline{w}_{np}$ and $\overline{w}_{np-1}$ is a constant (Corollary 1), so the greatest common divisor of $\overline{w}_p$ and $\overline{w}_{np-1}$ is a constant. Thus, $\overline{w}_p \,|\, x^2 \overline{w}_r$. Now, reasoning as in the proof of Theorem 4 completes the proof of Corollary 2. □

Corollary 2 implies our final theorem which is analogous to Theorem 10 in [4].

Theorem 6: If p and q are natural numbers and $\overline{w}_p \,|\, \overline{w}_q$, then $p \,|\, q$.

THE MAIN REMARK

If we put $a = 1$ and $b = -1$ in (2) and (6), we obtain

$$w_1(x) = 1, \quad w_2(x) = x, \quad \text{and} \quad w_{n+2}(x) = x w_{n+1}(x) - w_n(x). \tag{17}$$

These are the well-known *Tchebycheff polynomials*. Then the reciprocal Tchebycheff polynomials do satisfy

$$\overline{w}_1(x) = 1, \overline{w}_2(x) = 1, \quad \text{and} \quad \overline{w}_{n+2}(x) = x^2 \overline{w}_n(x) - w_n(x).$$

From [3], it is known that these polynomials are associated with stacks.

Specifically, orthogonal Tchebycheff polynomials are used to calculate the numbers $H_{k,l,n}$ of histories of length n starting at level k and ending at level l, while the reciprocal Tchebycheff polynomials of degree h are used to derive generating functions for histories of height $\leq h$. In this context, the Tchebycheff polynomials are distinguished among the family of Fibonacci-like polynomials defined by (2) and (6), as only for that case (i.e., for $a = 1$ and $b = -1$) the Fibonacci-like polynomials associate with **standard** organizations [3]. This can be seen easily after consulting Theorem 4.1 of [3]. For other admissible values of a and b, the Fibonacci-like polynomials also provide us with an orthogonal polynomial system with respect to the corresponding positive-definite moment functional [3].

The resulting dynamical data organizations are then **nonstandard** ones. (A paper on nonstandard data organizations and Fibonacci-like polynomials is now in preparation.)

FINAL REMARKS

The following two attempts now seem to be natural. First, one may use the number-theoretic properties of Tchebycheff and reciprocal Tchebycheff polynomials developed in [4] and this paper to investigate further stacks in the framework created in [3]. Second, one may look for other data structure organizations relaxing the positive-definiteness of the moment functional. This might be valuable if we knew how to convey contiguous quantum-mechanics-like descriptions of dynamic data structures, which is one of several considerations in [2].

ACKNOWLEDGMENTS

The authors are very indebted to the referee for drawing their attention to Waddill's paper [5] and for many valuable suggestions that led to a more desirable presentation of this paper.

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AMS Classification Number: 11B39

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COMPLETE PARTITIONS

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(Submitted November 1996)

1. INTRODUCTION

MacMahon [4] introduced perfect partitions of a number. He defined a perfect partition of a positive integer n to be a partition such that every number from 1 to n can be represented by the sum of parts of the partition in one and only one way. For instance, (1 1 1 4) is a perfect partition of 7 because we can express each of the numbers 1 through 7 uniquely by using the parts of three 1's and one 4; thus, (1), (1 1), (1 1 1), (4), (1 4), (1 1 4), and (1 1 1 4) are the partitions referred to in this example. MacMahon considered the case of $n = p^{\alpha} - 1$, where p is a prime number, and showed that the enumeration of perfect partitions is identical to the enumeration of compositions of the number α, using the correspondence between the factorizations of $(1-x^{p^{\alpha}})/(1-x)$ and the compositions of α. Further, he considered perfect partitions of the number $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots - 1$, where $p_1, p_2, \ldots$ are primes, and found that the number of perfect partitions of this number is equal to the number of compositions of the multipartite number $(\alpha_1, \alpha_2, \ldots)$. The fact that the number of perfect partitions of n is the same as the number of ordered factorizations of $n+1$ was also shown.

A similar idea of representing numbers as a sum of given numbers was used in later days. It seems that the word "complete" first appeared in a problem suggested by Hoggatt and King in [3], which was solved in Brown's paper [2]. They called an arbitrary sequence $\{f_i\}_{i=1}^{\infty}$ of positive integers "complete" if every positive integer n could be represented in the form $n = \sum_{i=1}^{\infty} \alpha_i f_i$, where each α_i was either 0 or 1. Brown found a simple necessary and sufficient condition for the completeness of such sequences and showed that the Fibonacci numbers are characterized by certain properties involving completeness. His note also considered whether or not completeness was destroyed by the deletion of some terms.

Now, turning our attention to partitions of a positive integer, we apply completeness to partitions. Several properties, recurrence relations, and generating functions for complete partitions will be obtained.

2. COMPLETE PARTITIONS

We begin with a definition of partitions of a positive integer.

Definition 2.1: A partition of a positive integer n is a finite non-decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = n$ and $\lambda_i > 0$ for all $i = 1, \ldots, k$. The λ_i are called the *parts of the partition* and k is called the *length of the partition*.

We sometimes write $\lambda = (1^{m_1} 2^{m_2} \ldots)$, which means there are exactly m_i parts equal to i in the partition λ. For example, there are five partitions of 4: $(1^4), (1^2\,2), (2^2), (1\ 3)$, and (4). We are now ready to define our main topic—complete partitions.

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* This research has been supported in part by YSRF and by Non Directed Research Fund, Korea Research Foundation.

Definition 2.2: A *complete partition* of an integer n is a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, with $\lambda_1 = 1$, such that each integer i, $1 \le i \le n$, can be represented as a sum of elements of $\lambda_1, \ldots, \lambda_k$. In other words, each i can be expressed as $\sum_{j=1}^{k} \alpha_j \lambda_j$, where α_j is either 0 or 1.

Example 2.3: Among the five partitions of 4, (1^4) and $(1^2\,2)$ are complete partitions of 4.

From Definition 2.2, the following is obvious.

Lemma 2.4: Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a complete partition of a positive integer. Then $(\lambda_1, \ldots, \lambda_i)$ is a complete partition of the number $\lambda_1 + \cdots + \lambda_i$ for $i = 1, \ldots, k$. □

Brown [2] found the following three facts on completeness of sequences of positive integers which are also true for partitions.

Proposition 2.5 (Brown [2]): Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a complete partition of a positive integer. Then, for $i = 1, \ldots, k-1$,

$$\lambda_{i+1} \le 1 + \sum_{j=1}^{i} \lambda_j .$$

Proof: Suppose not. Then there exists at least one $r \ge 2$ such that $\lambda_r > 1 + \sum_{i=1}^{r-1} \lambda_i$. Therefore, $\lambda_r > \lambda_r - 1 > \sum_{i=1}^{r-1} \lambda_i$. Thus, the integer $\lambda_r - 1$ cannot be represented as a sum of elements of $\lambda_1, \lambda_2, \ldots, \lambda_k$. □

The converse of Proposition 2.5 is also true, which we shall prove here in a manner different from that of Brown .

Theorem 2.6 (Brown [2]): Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n with $\lambda_1 = 1$ such that

$$\lambda_{i+1} \le 1 + \sum_{j=1}^{i} \lambda_j, \text{ for } i = 1, \ldots, k-1.$$

Then λ is a complete partition of n.

Proof: Suppose not. Then there must be some numbers between 1 and n that cannot be expressed as a sum of elements of $\lambda_1, \ldots, \lambda_k$. Let m be the least such number. Then we have $\lambda_1 + \cdots + \lambda_i < m < \lambda_1 + \cdots + \lambda_i + \lambda_{i+1}$, for some $i \ge 1$. We claim that $m < \lambda_{i+1}$. From our choice of m, we know that $m \ne \lambda_{i+1}$. If $m > \lambda_{i+1}$, then $0 < m - \lambda_{i+1} < m < n$. So $m - \lambda_{i+1}$ must be represented in the form $\sum_{j=1}^{i} \alpha_j \lambda_j$, where $\alpha_i = 0$ or 1, which contradicts our choice of m. Therefore, $m < \lambda_{i+1}$, so $1 + \lambda_1 + \cdots + \lambda_i < a + m \le \lambda_{i+1}$, a contradiction. □

Corollary 2.7 (Brown [2]): Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a complete partition. Then $\lambda_i \le 2^{i-1}$ for each $i = 1, \ldots, k$.

Proof: Obviously it is true for $i = 1, \ldots, k$, since $\lambda_1 = 1 \le 2^0 = 1$. Assuming $\lambda_i \le 2^{i-1}$ for each $i = 1, \ldots, j$, we have $\lambda_{j+1} \le 1 + \sum_{\ell=1}^{j} \lambda_\ell \le 1 + 1 + 2 + 2^2 + \cdots + 2^{j-1} = 2^j$. □

Now let us characterize complete partitions by the length and the size of parts.

Proposition 2.8: Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a complete partition of a positive integer n. Then the minimum possible length k is $\lceil \log_2(n+1) \rceil$, where $\lceil x \rceil$ is the least integer $\ge x$.

Proof: By Corollary 2.7,

$$n=\sum_{i=1}^{k}\lambda_i \le \sum_{i=0}^{k-1}2^i = 2^k-1.$$

Therefore, $n+1\le 2^k$, which gives $k\ge \lceil \log_2(n+1)\rceil$. □

Proposition 2.9 Let $\lambda=(\lambda_1,\ldots,\lambda_k)$ be a complete partition of n. Then the largest possible part is $\lfloor \frac{n+1}{2}\rfloor$, where $\lfloor x\rfloor$ is the greatest integer $\le x$.

Proof: Straightforward from Theorem 2.6. □

3. RECURRENCE RELATIONS AND GENERATING FUNCTIONS

In this section we find some recurrence relations to count complete partitions of a positive integer n. Let $C_{\ell,k}(n)$ be the number of complete partitions of n with length ℓ and largest part k. Then, by Proposition 2.9 and Lemma 2.7, k and ℓ must satisfy

$$1\le k\le \lfloor \tfrac{n+1}{2}\rfloor \text{ and } \lceil \log_2(n+1)\rceil \le \ell \le n.$$

Obviously, $C_{\ell,k}(n)=1$ if $n=k+\ell-1$ and $C_{1,1}(1)=1$ in particular. Since k is the largest part, counting complete partitions of $n-k$ with length $\ell-1$ and largest parts from 1 to k gives the following recurrence relation.

Proposition 3.1: Let $C_{\ell,k}(n)$ be the number of complete partition of n with length ℓ and largest part k. Then

$$C_{\ell,k}(n)=\begin{cases}\sum_{i=1}^{k} C_{\ell-1,i}(n-k) & \text{if } 1\le k\le \lfloor \frac{n+1}{2}\rfloor \text{ and } \lceil \log_2(n+1)\rceil \le \ell\le n,\\ 0 & \text{otherwise,}\end{cases}$$

with the initial condition $C_{1,1}(1)=1$. □

Example 3.2: There are two complete partitions of 8 with length $\ell=4$ and largest part $k=3$: $(12^2\,3)$ and $(1^2\,3^2)$. This is obtained by the recurrence relation

$$\begin{aligned}C_{4,3}(8)&=\sum_{i=1}^{3}C_{3,i}(5)=C_{3,1}(5)+C_{3,2}(5)+C_{3,3}(5)\\&=C_{3,1}(5)+(C_{2,1}(3)+C_{2,2}(3))+C_{2,1}(2)\\&=0+(0+1)+1=2.\end{aligned}$$

Naturally, by adding $C_{\ell,k}(n)$ for all possible ℓ, we obtain the total number of complete partitions of n as follows.

Corollary 3.3: Let $C(n)$ be the number of complete partitions of n. Then

$$C(n)=\sum_{\ell=\lceil \log_2(n+1)\rceil}^{n}\ \sum_{k=1}^{\lfloor (n+1)/2\rfloor} C_{\ell,k}(n).\ \square$$

Now let us take a look at the size of parts to get another recurrence relation. Let $C_k(n)$ be the number of complete partitions of a positive integer n with largest part at most k. We take

$C_0(n) = 0$ for all $n \ge 0$ and $C_k(0) = 0$ for all $k \ge 1$, and $C_1(1) = 1$. So $C_k(n)$ is always positive if $n, k \ge 1$; therefore, k ranges from 1 to $\lfloor \frac{n+1}{2} \rfloor$ by Proposition 2.9. From our definition of $C_k(n)$, for any $k > \lfloor \frac{n+1}{2} \rfloor$,

$$C_k(n) = C_{k-1}(n) = \cdots = C_{\lfloor \frac{n+1}{2} \rfloor}(n).$$

The set of complete partitions of n with largest part at most k can be partitioned into two subsets: one with largest part exactly k; the other with largest part at most $k-1$. It is not difficult to see that the number of complete partitions of n with largest part exactly k is equal to the number of complete partitions of $n-k$ with largest part at most k. Therefore, we have the following theorem.

Theorem 3.4: Let $C_k(n)$ be the number of complete partitions of a positive integer n with largest part at most k $(k \ge 1)$. Then

$$C_k(n) = \begin{cases} C_{k-1}(n) + C_k(n-k) & \text{if } 1 \le k \le \lfloor \frac{n+1}{2} \rfloor, \\ C_{\lfloor \frac{n+1}{2} \rfloor}(n) & \text{if } k > \lfloor \frac{n+1}{2} \rfloor, \end{cases}$$

with the initial conditions $C_0(n) = 0$ for all $n \ge 0$, $C_k(0) = 0$ for all k, and $C_1(1) = 1$.

Note that $C_1(n) = 1$ for all $n \ge 1$. Let us take some examples.

Example 3.5:

1. $C_2(5) = C_1(5) + C_2(3) = C_1(5) + (C_1(3) + C_2(1)) = C_1(5) + (C_1(3) + C_1(1)) = 1 + 1 + 1 = 3.$

2. $C_3(5) = C_2(5) + C_3(2) = C_2(5) + C_1(2) = 3 + 1 = 4$.

Corollary 3.6: If $k = \lfloor (n+1)/2 \rfloor$, then $C_k(n)$ is the number of all complete partitions of n.

Now we count complete partitions by the largest part. Let $D_k(n)$ be the number of complete partitions of a positive integer n with largest part exactly k. Then $D_k(n) = C_k(n) - C_{k-1}(n) = C_k(n-k)$ by the definition of $C_k(n)$ and Theorem 3.4. Obviously, $D_1(n) = 1$ for all $n \ge 1$. Thus, for $k \ge 2$,

$$\begin{aligned} D_k(n) &= C_k(n-k) \\ &= C_k(n-k) - C_{k-1}(n-k) + C_{k-1}(n-k) \\ &= D_k(n-k) + D_{k-1}(n-1). \end{aligned}$$

Since each complete partition of n with largest part exactly k must have at least one k as a part, $D_k(n) = 0$ if $1 \le n \le 2k-2$ and $D_k(n-k) = 0$ if $2k-1 \le n \le 3k-2$. Thus, we obtain

Theorem 3.7: Let $D_k(n)$ be the number of complete partitions of a positive integer n with largest part exactly k. Then $D_1(n) = 1$ for all $n \ge 1$ and for $k \ge 2$,

$$D_k(n) = \begin{cases} D_{k-1}(n-1) + D_k(n-k) & \text{if } n \ge 3k-1, \\ D_{k-1}(n-1) & \text{if } 2k-1 \le n \le 3k-2, \\ 0 & \text{if } 1 \le n \le 2k-2, \end{cases}$$

with the conditions $D_0(n) = 0$ for all n and $D_k(0) = 0$ for all k.

Example 3.8:

1. $D_2(4)=D_1(3)=1$;
2. $D_2(5)=D_1(4)+D_2(3)=D_1(4)+D_1(3)=1+1=2$;
3. $D_3(7)=D_2(6)=D_1(5)+D_2(4)=1+1=2$;
4. $D_2(7)+D_3(5)=(D_1(6)+D_2(5))+D_2(4)=(1+2)+1=4$.

The following table shows the first few values of complete partitions of n with largest part at most k, and $C(n)$ is the total number of complete partitions of n.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	2	2	3	3	4	4	5	5
3					1	2	2	4	5	6	8	10
4							2	2	4	5	8	10
5									2	4	5	8
6											4	5
$C(n)$	1	1	2	2	4	5	8	10	16	20	31	39

Now we find the generating function for the number $D_k(n)$.

Theorem 3.9: Let $f_k(q)=\Sigma_{n=0}^{\infty}D_k(n)q^n$ $(k\geq 2)$. Then we have

$$f_k(q)=\frac{q^{k+1}}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$
$$-\left(\frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k}+\frac{D_{k-2}(2k-5)q^{2k-3}}{(1-q^k)(1-q^{k-1})}+\cdots+\frac{q^{k+1}}{(1-q^k)(1-q^{k-1})\cdots(1-q^3)}\right),$$

with $f_1(q)=\frac{q}{1-q}$.

Proof: Since $D_1(n)=1$ for all n, $f_1(q)=\frac{q}{1-q}$. Let $f_k(q)=\Sigma_{n=0}^{\infty}D_k(n)q^n$. Then

$$\begin{aligned}f_k(q)&=\sum_{n=0}^{\infty}D_k(n)q^n=\sum_{n=2k-1}^{\infty}D_k(n)q^n\\&=\sum_{n=2k-1}^{3k-2}D_{k-1}(n-1)q^n+\sum_{n=3k-1}^{\infty}[D_{k-1}(n-1)+D_k(n-k)]q^n\\&=\sum_{n=2k-1}^{\infty}D_{k-1}(n-1)q^n+\sum_{n=3k-1}^{\infty}D_k(n-k)q^n\end{aligned}$$

$$= q \sum_{n=2k-1}^{\infty} D_{k-1}(n-1)q^{n-1} + q^k \sum_{n=3k-1}^{\infty} D_k(n-k)q^{n-k}$$
$$= q[f_{k-1}(q) - D_{k-1}(2k-3)q^{2k-3}] + q^k f_k(q).$$

Thus, we have

$$f_k(q) = \frac{q}{1-q^k} f_{k-1}(q) - \frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k}.$$

An iteration gives

$$f_k(q) = \frac{q}{1-q^k} f_{k-1}(q) - \frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k}$$
$$= \frac{q}{1-q^k}\left[\frac{q}{1-q^{k-1}} f_{k-2}(q) - \frac{D_{k-2}(2k-5)q^{2k-4}}{1-q^{k-1}}\right] - \frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k}$$
$$= \frac{q^2}{(1-q^k)(1-q^{k-1})} f_{k-2}(q) - \left[\frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k} + \frac{D_{k-2}(2k-5)}{(1-q^k)(1-q^{k-1})}\right].$$

By continuing iteration on $f_k(q)$, we obtain

$$f_k(q) = \frac{q^{k-2}}{(1-q^k)(1-q^{k-1})\cdots(1-q^3)} f_2(q)$$
$$- \left[\frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k} + \frac{D_{k-2}(2k-5)q^{2k-3}}{(1-q^k)(1-q^{k-1})} + \cdots + \frac{D_2(3)q^{k+1}}{(1-q^k)(1-q^{k-1})\cdots(1-q^3}\right].$$

Since $D_2(3) = 1$, $f_1(q) = \frac{q}{1-q}$, and

$$f_2(q) = \frac{q^2}{(1-q^2)(1-q)} - \frac{q^2}{1-q^2} = \frac{q^3}{(1-q^2)(1-q)},$$

the theorem follows. □

Note that the numbers $D_{k-i}(2(k-i)-1)$, for $i = 1, 2, \ldots, k-2$, in $f_k(q)$ can be simplified to

$$D_{\lfloor \frac{k-i+1}{2} \rfloor}\left(\left\lfloor \frac{3(k-i)-1}{2} \right\rfloor\right)$$

by Theorem 3.7.

Example 3.10: The following are generating functions for $k = 3, 4$, and 5.

$$f_3(q) = \frac{q^4}{(1-q^3)(1-q^2)(1-q)} - \frac{q^4}{1-q^3},$$

$$f_4(q) = \frac{q^5}{(1-q^4)(1-q^3)(1-q^2)(1-q)} - \left[\frac{q^6}{1-q^4} + \frac{q^5}{(1-q^4)(1-q^3)}\right],$$

and

$$f_5(q) = \frac{q^6}{(1-q^5)(1-q^4)(1-q^3)(1-q^2)(1-q)} - \left[\frac{2q^8}{1-q^5} + \frac{q^7}{(1-q^5)(1-q^4)} + \frac{q^6}{(1-q^5)(1-q^4)(1-q^3)}\right].$$

By expanding the above, we get the following, which is expected from the above table.

$$f_3(q) = q^5 + 2q^6 + 2q^7 + 4q^8 + 5q^9 + 6q^{10} + 8q^{11} + 10q^{12} + \cdots,$$
$$f_4(q) = 2q^7 + 2q^8 + 4q^9 + 5q^{10} + 8q^{11} + 10q^{12} + \cdots,$$
$$f_5(q) = 2q^9 + 4q^{10} + 5q^{11} + 8q^{12} + \cdots.$$

ACKNOWLEDGMENT

I would like to thank Professor Ira Gessel of Brandeis University for his valuable comments and Professor James Lewis of the University of Rhode Island who brought this topic to my attention.

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AMS Classification Number: 05A17

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PSEUDOPRIMES, PERFECT NUMBERS, AND A PROBLEM OF LEHMER

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(Submitted December 1996-Final Revision April 1997)

1. INTRODUCTION

Two classical problems in elementary number theory appear, at first, to be unrelated. The first, posed by D. H. Lehmer in [7], asks whether there is a composite integer N such that $\phi(N)$ divides $N-1$, where $\phi(N)$ is Euler's totient function. This question has received considerable attention and it has been demonstrated that such an integer, if it exists, must be extraordinary. For example, in [2] G. L. Cohen and P. Hagis, Jr., show that an integer providing an affirmative answer to Lehmer's question must have at least 14 distinct prime factors and exceed 10^{20}.

The second is the ancient question whether there exists an odd perfect number, that is, an odd integer N, such that $\sigma(N) = 2N$, where $\sigma(N)$ is the sum of the divisors of N. More generally, for each integer $k > 1$, one can ask for odd multiperfect numbers, i.e., odd solutions N of the equation $\sigma(N) = kN$. This question has also received much attention and solutions must be extraordinary. For example, in [1] W. E. Beck and R. M. Rudolph show that an odd solution to $\sigma(N) = 3N$ must exceed 10^{50}. Moreover, C. Pomerance [9], and more recently D. R. Heath-Brown [4], have found explicit upper bounds for multiperfect numbers with a bounded number of prime factors.

In recent work [13], L. Somer shows that for fixed d there are at most finitely many composite integers N such that some integer a relatively prime to N has multiplicative order $(N-1)/d$ modulo N. A composite integer N with this property is a Fermat d-pseudoprime. (See [12], p. 117, where Fermat d-pseudoprimes are referred to as Somer d-pseudoprimes.) More recently, Somer [14] showed that under suitable conditions, there are at most finitely many Lucas d-pseudoprimes, i.e., pseudoprimes that arise via tests employing recurrence sequences. (Lucas d-pseudoprimes are discussed on pp. 131-132 of [12] where they are also called Somer-Lucas d-pseudoprimes. For a complete discussion of these and other pseudoprimes that arise from recurrence relations, see [12] or [11].)

The methods used by Somer in his papers motivated the present work. While attempting to simplify and extend the arguments in [13] and [14] we discovered that, in fact, Lehmer's problem, the existence of odd multiperfect numbers, and Somer's theorems about pseudoprimes are intimately related. In this paper we present a unified approach to the study of these four questions.

2. PRELIMINARIES

We adopt the convention that p always represents a prime number. Define the set $\delta(N) = \{p \mid p \text{ divides } N\}$ and for each i such that $1 \le i \le |\delta(N)|$, define $\delta_i(N)$ to the i^{th} largest prime in the decomposition of N. Thus, if N has decomposition

$$N = \prod_{i=1}^{t} p_i^{k_i}, \tag{2.1}$$

with $p_1 < p_2 < \cdots < p_t$, then $\delta_i(N) = p_i$. If Ω is a set of natural numbers, define

$$\delta(\Omega) = \cup_{N \in \Omega} \delta(N)$$

and, similarly, $\delta_i(\Omega) = \{\delta_i(N) \mid N \in \Omega\}$.

In the arguments below we will have need to extract the square-free part of certain integers. If N has decomposition (2.1), we will write

$$N_1 = \prod_{i=1}^{t} p_i \quad \text{and} \quad N_2 = \prod_{i=1}^{t} p_i^{k_i - 1}, \tag{2.2}$$

so that $N = N_1 N_2$ with N_1 square-free.

In the definitions and lemmas below, we will need a semigroup homomorphism from the natural numbers $\mathbf{N}$ to the multiplicative semigroup $\{-1, 0, 1\}$. Such a function will be called a *signature* function, and we will single out the case in which $\varepsilon = 1$, the constant function. Clearly, a signature function is determined by its values on the primes. We say that N is *supported* by ε if $\varepsilon(N) \neq 0$ or, equivalently, if $\varepsilon(p) \neq 0$ for all p that divide N. Similarly, a set Ω of natural numbers is *supported* by ε if $\varepsilon(N) \neq 0$ for all $N \in \Omega$. Note that if D is a fixed integer, the Jacobi symbol $\varepsilon(i) = (\frac{D}{i})$ is a signature function.

If N is any natural number and ε is a signature function, define the number theoretic function $\xi(N)$ as follows:

$$\xi(N) = \xi_\varepsilon(N) = \frac{1}{N} \prod_{p \mid N} (p - \varepsilon(p)). \tag{2.3}$$

Note that if N has decomposition (2.1), we can write $N = N_1 N_2$ as in (2.2) and

$$\xi(N) = \frac{1}{N_2} \prod_{i=1}^{t} \left(\frac{p_i - \varepsilon(p_i)}{p_i} \right) = \frac{1}{N_2} \prod_{i=1}^{t} \left(1 - \frac{\varepsilon(p_i)}{p_i} \right). \tag{2.4}$$

We will be interested in certain limiting values of $\xi(N)$ for N in a set Ω. In particular, if Ω is an infinite set of positive integers, then

$$\lim_{N \in \Omega} \xi(N) = L \tag{2.5}$$

means that for every $\varepsilon > 0$ there is an M such that $|\xi(N) - L| < \varepsilon$ whenever $N > M$ and $N \in \Omega$. Although in most applications the signature ε will be fixed, we also allow ε to vary with N, requiring only that N be supported by its associated signature.

The following elementary lemma is an easy exercise.

Lemma 2.1: Suppose that Ω is a set of positive integers and $f: \Omega \to \mathbf{R}$ a function such that $\lim_{N \in \Omega} f(N) = L$. Suppose as well that there exist functions f_1 and $f_2: \Omega \to \mathbf{R}$ such that

(a) $f(N) = f_1(N) f_2(N)$ for all $N \in \Omega$;

(b) $\{f_2(N) \mid N \in \Omega\}$ has finite cardinality; and

(c) $\lim_{N \in \Omega} f_1(N) = 1$.

Then $f_2(N) = L$ for some $N \in \Omega$.

Lemma 2.2: If $N > 1$ is an integer supported by the signature ε and (c, d) is a pair of integers such that $\xi(N) = c/d$, then $(N, d) \neq 1$.

Proof: If $\xi(N) = c/d$, then

$$d\prod_{p|N}(p-\varepsilon(p)) = cN.$$

Since N is supported by ε, it follows that $\varepsilon(p) \neq 0$ for all p dividing N. Thus, if p is the largest prime divisor of N, then $p|d$. □

Theorem 2.3: Suppose that Ω is an infinite set of positive integers with each $N \in \Omega$ supported by corresponding signature ε and for which $|\delta(N)| = t$ for all $N \in \Omega$. Suppose as well that $\{N_2 \mid N \in \Omega\}$ is bounded. If c and d are integers such that $(N, d) = 1$ for all $N \in \Omega$ and

$$\lim_{N\in\Omega}\xi(N) = c/d, \tag{2.6}$$

then $c = d$.

Proof: If $\delta_t(\Omega)$ is bounded, then $\delta(\Omega)$ is bounded. Since $\{N_2 \mid N \in \Omega\}$ is bounded, it follows from (2.4) that $\xi(N)$ takes on finitely many values as N ranges over Ω. It follows that $\lim_{N\in\Omega}\xi(N) = \xi(N_0)$ for some $N_0 \in \Omega$, and $\xi(N_0) = c/d$, contrary to Lemma 2.2.

Consequently $\delta_t(\Omega)$ is unbounded. Choose s to be minimal such that $\delta_s(\Omega)$ is unbounded. Since $\delta_s(\Omega)$ is unbounded, we can find an infinite subset of Ω such that $\delta_s(N)$ is increasing and, without loss of generality, we may replace Ω with this subset. Now, if

$$f_1(N) = \prod_{i=s}^{t}\frac{\delta_i(N)-\varepsilon(\delta_i(N))}{\delta_i(N)},$$

then

$$\lim_{N\in\Omega} f_1(N) = 1. \tag{2.7}$$

Since $\delta_k(\Omega)$ is bounded for all $k < s$ and $\{N_2 \mid N \in \Omega\}$ is bounded, it follows that

$$f_2(N) = \begin{cases} \dfrac{1}{N_2}\displaystyle\prod_{i=1}^{s-1}\frac{\delta_i(N)-\varepsilon(\delta_i(N))}{\delta_i(N)} & \text{if } s > 1 \\[2ex] \dfrac{1}{N_2} & \text{if } s = 1 \end{cases} \tag{2.8}$$

takes on finitely many values. Since, in both cases, $\xi(N) = f_1(N)f_2(N)$, Lemma 2.1 implies that $f_2(N) = c/d$ for some $N \in \Omega$. If $s > 1$, it follows that

$$d\prod_{i=1}^{s-1}(\delta_i(N)-\varepsilon(\delta_i(N))) = cN_2\prod_{i=1}^{s-1}\delta_i(N). \tag{2.9}$$

But then $\delta_{s-1}(N)$ divides d, contrary to the hypothesis that $(N, d) = 1$. It now follows that $s = 1$. But then Lemma 2.1 implies that $d = cN_2$ for some $N \in \Omega$. Since $(N_2, d) = 1$ for all $N \in \Omega$, this implies that $N_2 = 1$ and $c = d$, as desired. □

Corollary 2.4: Suppose that Ω is an infinite set of positive integers that is supported by the signature ε and for which $\{|\delta(N)|\}_{N\in\Omega}$ is bounded. Suppose as well that $\{N_2 \mid N \in \Omega\}$ is bounded. If c and d are integers such that $(N, d) = 1$ for all $N \in \Omega$ and

$$\lim_{N\in\Omega} \xi(N) = c/d, \tag{2.10}$$

then $c = d$.

Proof: If Ω is infinite and $\{|\delta(N)|\}_{N\in\Omega}$ is bounded, then there is some integer t such that $\hat{\Omega} = \{N \in \Omega \mid t = |\delta(N)|\}$ is infinite. We can now apply Theorem 2.3 to $\hat{\Omega}$. □

3. FERMAT PSEUDOPRIMES

Suppose that N is a composite integer and $a > 1$ is an integer such that $(N, a) = 1$ and $a^{N-1} \equiv 1 \pmod{N}$. Then N is called a *Fermat pseudoprime* to the base a. Moreover, if a has multiplicative order $(N-1)/d$ in $(\mathbf{Z}/N\mathbf{Z})^*$, then N is said to be a *Fermat d-pseudoprime* to the base a. In general, if there exists an integer $a > 1$ such that N is a Fermat d-pseudoprime to the base a, then we call N a Fermat d-pseudoprime.

If N has prime decomposition (2.1), then the structure of the unit group $(\mathbf{Z}/N\mathbf{Z})^*$ is well known. If N is not divisible by 8, then $(\mathbf{Z}/N\mathbf{Z})^*$ is a product of cyclic groups of order $p_i^{k_i-1}(p_i - 1)$, while if N is divisible by 8, then $p_1 = 2$ and $(\mathbf{Z}/N\mathbf{Z})^*$ has an additional factor that is a product of a cyclic group of order 2 and a cyclic group of order 2^{k_1-2}. It follows that the multiplicative orders of integers a relatively prime to N in $(\mathbf{Z}/N\mathbf{Z})^*$ are just the divisors of $\lambda(N) = \mathrm{lcm}\{p_i^{s_i}(p_i - 1)\}$, where $s_i = k_i - 1$ when p_i is odd, $s_1 = k_1 - 1$ if $p_1 = 2$ and $k_1 = 1$ or 2, and $s_1 = k_1 - 2$ if $p_1 = 2$ and $k_1 \geq 3$. Therefore N is a Fermat d-pseudoprime if and only if $(N-1)/d$ divides $\lambda(N)$. Moreover, since $(N, N-1) = 1$, a composite integer N is a Fermat d-pseudoprime if and only if $(N-1)/d$ divides $\lambda'(N) = \mathrm{lcm}\{p_i - 1\}$.

If N has decomposition (2.1), define

$$\psi(N) = \frac{1}{2^s}\prod_{i=1}^{t}(p_i - 1),$$

where $s = t - 2$ when $2 \mid N$ and $t \geq 2$, and $s = t - 1$ otherwise. It is easy to see that if N is composite, then $\psi(N)$ is an integer and $\lambda'(N)$ divides $\psi(N)$. Therefore, if N is a Fermat d-pseudoprime, then $(N-1)/d$ divides $\psi(N)$, and hence, there is an integer c such that

$$\frac{\psi(N)}{N-1} = \frac{c}{d}. \tag{3.1}$$

We will need several lemmas concerning the properties of Fermat d-pseudoprimes and $\psi(N)$. Similar lemmas appear in [13], but the proofs are short and we include them here for completeness.

Lemma 3.1: If N is a Fermat d-pseudoprime with prime decomposition (2.1), then $(N, d) = 1$ and there exists an integer c such that

$$\frac{\psi(N)}{N-1} = \frac{c}{d} < \frac{1}{2^{t-1}}. \tag{3.2}$$

Proof: If $t=1$, then (3.2) follows immediately from the definition of $\psi(N)$ and the fact that N is composite. Assume that $t>1$. By (3.1) and the preceding comments, it suffices to show that $c/d<1/2^{t-1}$. This is immediate from the observation that

$$\frac{\prod_{p|N}(p-1)}{\prod_{p|N}p-1}<1$$

in general, and

$$\frac{\prod_{p|N}(p-1)}{\prod_{p|N}p-1}<\frac{1}{2}$$

when $2|N$. □

Lemma 3.2: If N is a Fermat d-pseudoprime with prime decomposition (2.1), then $t<\log_2(d)+1$.

Proof: By Lemma 3.1,

$$\frac{1}{d}\le\frac{c}{d}<\frac{1}{2^{t-1}},$$

and hence $d>2^{t-1}$. Thus $t-1<\log_2(d)$, and therefore $t<\log_2(d)+1$. □

Lemma 3.3: If N is a Fermat d-pseudoprime with prime decomposition (2.1) and $k_i\ge 2$, then

$$p_i^{k_i-1}<\frac{p_i^{k_i}}{p_i-1}\le d+1. \tag{3.3}$$

Proof: Clearly,

$$\begin{aligned}p_i^{k_i-1}<\prod_{j=1}^{t}\frac{p_j^{k_j}}{p_j-1}&=\frac{1}{2^s}\left(\frac{\Pi p_j^{k_j}}{\frac{1}{2^s}\Pi(p_j-1)}\right)=\frac{1}{2^s}\left(\frac{N}{\psi(N)}\right)\\&=\frac{1}{2^s}\left(\frac{N-1}{\psi(N)}\right)+\frac{1}{2^s\psi(N)}=\frac{1}{2^s}\left(\frac{d}{c}\right)+\frac{1}{2^s\psi(N)}\\&\le\frac{d}{2^s}+\frac{1}{2^s}=\frac{1}{2^s}(d+1)\le d+1.\ \square\end{aligned}$$

The following theorem first appeared in [13].

Theorem 3.4: For fixed positive integer d, there are at most a finite number of Fermat d-pseudoprimes.

Proof: By way of contradiction, suppose that there are an infinite number of Fermat d-pseudoprimes. By Lemma 3.2, there exists an integer t, with $t<\log_2(d)+1$, such that an infinite number of these Fermat d-pseudoprimes have exactly t distinct prime divisors. Moreover, an infinite number of these Fermat d-pseudoprimes have the same parity. Then (3.2) is satisfied by an infinite number of integers N of the same parity. There are, however, only a finite number of possible values for c, and it follows that there is some value of c for which (3.2) has an infinite number of solutions N of the same parity. Fix this value of c and let Ω be an (infinite) set of positive integers N of the same parity that satisfy (3.2) for these fixed values of c and d.

If $\delta(\Omega)$ is bounded, then, by Lemma 3.3, Ω is finite, contrary to our choice of c. Consequently $\delta(\Omega)$ is unbounded. Moreover, by Lemma 3.2, $\{|\delta(N)|\}_{N\in\Omega}$ is bounded, and it follows that

$$\lim_{N\in\Omega}\frac{1}{\psi(N)}=0.$$

Consequently, with constant signature $\varepsilon=1$, and $s=t-2$ if the elements of Ω are even and $t\geq 2$, and $s=t-1$ otherwise, we obtain

$$\begin{aligned}\frac{2^s c}{d}&=2^s\lim_{N\in\Omega}\left(\frac{\psi(N)}{N-1}\right)=2^s\lim_{N\in\Omega}\frac{1}{\left(\frac{N-1}{\psi(N)}\right)}\\&=2^s\lim_{N\in\Omega}\frac{1}{\left(\frac{N}{\psi(N)}-\frac{1}{\psi(N)}\right)}=2^s\lim_{N\in\Omega}\left(\frac{\psi(N)}{N}\right)=\lim_{N\in\Omega}\xi(N).\end{aligned}\tag{3.4}$$

By Lemma 3.3, $\{N_2\mid N\in\Omega\}$ is bounded and, by Lemma 3.1, $(N,d)=1$ for all $N\in\Omega$. Clearly, Ω is supported by the constant signature $\varepsilon=1$. Therefore Theorem 2.3 implies that $2^s c/d=1$.

Finally, by (3.2),

$$1=\frac{2^s c}{d}<\frac{2^s}{2^{t-1}}\leq 1,\tag{3.5}$$

a contradiction. □

4. LUCAS PSEUDOPRIMES

Let $U(P,Q)$ be the recurrence sequence defined by $U_0=0, U_1=1$, and

$$U_{n+2}=PU_{n+1}-QU_n\tag{4.1}$$

for all $n\geq 0$. The sequence $U(P,Q)$ is called a *Lucas sequence* with parameters P and Q. Associated with $U(P,Q)$ is an integer $D=P^2-4Q$ known as the *discriminant* of $U(P,Q)$ and, as noted above, the function $\varepsilon(i)=(\frac{D}{i})$ is a signature function. For the duration of this section, $\varepsilon(N)$ will be the Jacobi symbol.

If N is an integer and $U(P,Q)$ a Lucas sequence, we define $\rho_U(N)$ to be the least positive integer n such that N divides U_n. The number $\rho(N)$ is called the *rank of appearance* (or simply the *rank*) of N in $U(P,Q)$. If $(N,Q)=1$, then it is well known that $U(P,Q)$ is purely periodic modulo N and, since $U_0=0$, $\rho(N)$ exists. Moreover, in this case $U_n\equiv 0 \pmod N$ if and only if $\rho(N)$ divides n. It was proven by Lucas [8] that, if a prime p does not divide $2QD$, then $U_{p-\varepsilon(p)}\equiv 0 \pmod p$ and hence $\rho(p)$ divides $p-\varepsilon(p)$.

Motivated by Lucas' theorem, we say that an odd composite integer N is a *Lucas pseudoprime* if there is a Lucas sequence $U(P,Q)$ with discriminant D such that $(N,QD)=1$ and $U_{N-\varepsilon(N)}\equiv 0 \pmod N$, where $\varepsilon(N)=(\frac{D}{N})$. Moreover, if $\rho(N)=(N-\varepsilon(N))/d$, then N is said to be a *Lucas d-pseudoprime*.

Suppose that ε is any signature function and N an odd integer with decomposition (2.1) that is supported by ε. Analogous to the functions λ, λ', and ψ defined in the previous section, define

$$\lambda(N) = \mathrm{lcm}\{p_i^{k_i-1}(p_i - \varepsilon(p_i))\},$$
$$\lambda'(N) = \mathrm{lcm}\{p_i - \varepsilon(p_i)\}, \text{ and}$$
$$\psi(N) = \frac{1}{2^{t-1}}\prod_{i=1}^{t}(p_i - \varepsilon(p_i)).$$

In [14], L. Somer shows that an integer N is a Fermat d-pseudoprime if and only if it is a Lucas d-pseudoprime with a signature ε satisfying $\varepsilon(p) = 1$ for all primes p dividing N. Since for each d there are only a finite number of Fermat d-pseudoprimes, it may seem reasonable to conjecture that there are also a finite number of Lucas d-pseudoprimes. This conjecture seems highly unlikely, however, since d-pseudoprimes with three prime divisors and d divisible by 4 are easy to construct.

If k is an even integer with the property that $p = 3k-1$, $q = 3k+1$, and $r = 3k^2 - 1$ are prime, set $N = pqr$ and choose D relatively prime to N and congruent to 0 or 1 (mod 4) such that $\varepsilon(p) = 1$ and $\varepsilon(q) = \varepsilon(r) = -1$. Then

$$\begin{aligned} N - \varepsilon(N) = pqr - 1 &= (3k-1)(3k+1)(3k^2-1) - 1 \\ &= 3k^2(9k^2-4) = (3k-2)(3k+2)(3k^2) \\ &= (p-1)(q+1)(r+1). \end{aligned}$$

It is a consequence of elementary properties of Lucas sequences and a theorem of H. C. Williams [15] that for any odd integer N and discriminant D relatively prime to N and satisfying $D \equiv 0$ or 1 (mod 4), there is a Lucas sequence U satisfying $\rho_U(N) = \lambda(N)$. Thus, for

$$d = \frac{(p-1)(q+1)(r+1)}{\mathrm{lcm}(p-1),(q+1),(r+1)} = \frac{N - \varepsilon(N)}{\lambda(N)},$$

Williams' theorem implies that N is a Lucas d-pseudoprime. Since $p-1, q+1$, and $r+1$ are all even, it is clear that d is divisible by 4, and when $\lambda(N)$ is maximal, $d = 4$. For example, taking $k = 4$ yields the Lucas 4-pseudoprime $N = 11\cdot 13\cdot 47 = 6721$ and $k = 60$ yields the 4-pseudoprime $N = 179\cdot 181\cdot 10799 = 349876801$.

More general algorithms for generating Lucas d-pseudoprimes are described in [14] and will be discussed in detail in a future paper. It is worth noting that the computational evidence presented in [14] suggests that there are infinitely many Lucas d-pseudoprimes with exactly three distinct prime divisors when 4 divides d and d is a square, and that there is a relationship between the number of Lucas d-pseudoprimes N, the precise power of 2 that divides d, and the number of prime divisors of N. We prove below that there are at most a finite number of Lucas d-pseudoprimes N such that $2^r \| N$ and $|\delta(N)| \geq r+2$. In light of the computational evidence presented in [14], the requirement that $|\delta(N)| \geq r+2$ appears to be best possible.

As in the previous section, we require a few lemmas that describe properties of Lucas d-pseudoprimes and $\psi(N)$. The following three lemmas can be proved by methods analogous to those used to prove Lemma 3.1, Lemma 3.2, and Lemma 3.3.

Lemma 4.1: If N is a Lucas d-pseudoprime, then $(N, d) = 1$ and there exist integers b and c such that

$$\frac{\lambda'(N)}{N - \varepsilon(N)} = \frac{b}{d} \leq \frac{\psi(N)}{N - \varepsilon(N)} = \frac{c}{d} < 2\left(\frac{2}{3}\right)^t. \tag{4.2}$$

Lemma 4.2: If N is a Lucas d-pseudoprime with prime decomposition (2.1), then $t < \log_{3/2}(2d)$.

Lemma 4.3: If N is a Lucas d-pseudoprime with prime decomposition (2.1) and $k_i \geq 2$, then

$$p_i^{k_i - 1} < 2(2/3)^t (d+1). \tag{4.3}$$

The following theorem is new; it sharpens a result of the third author in [14].

Theorem 4.4: Let d be a fixed positive integer and suppose that 2^r exactly divides d. Then there are at most a finite number of Lucas d-pseudoprimes N such that $|\delta(N)| \geq r+2$.

Proof: Suppose that there are an infinite number of Lucas d-pseudoprimes N with $|\delta(N)| \geq r+2$. By Lemma 4.2, there exists an integer t, with $r+1 < t < \log_{3/2}(2d)$, such that an infinite number of these Lucas d-pseudoprimes have exactly t distinct prime divisors. Thus (4.2) is satisfied by an infinite number of integers N. There are, however, only a finite number of possible values for c, and it follows that there is some value of c for which (4.2) has an infinite number of solutions N. Fix this value of c and let Ω be the (infinite) set of positive integers N that satisfy (4.2) for these fixed values of c and d.

If $\delta(\Omega)$ is bounded, then, by Lemma 4.3, Ω is finite, contrary to our choice of c. Consequently $\delta(\Omega)$ is unbounded. Moreover, by Lemma 4.2, $\{|\delta(N)|\}_{N\in\Omega}$ is bounded and it follows that

$$\lim_{N\in\Omega} \frac{\varepsilon(N)}{\psi(N)} = 0.$$

It then follows that

$$\begin{aligned} \frac{2^{t-1}c}{d} &= 2^{t-1} \lim_{N\in\Omega}\left(\frac{\psi(N)}{N-\varepsilon(N)}\right) = 2^{t-1}\lim_{N\in\Omega}\frac{1}{\left(\frac{N-\varepsilon(N)}{\psi(N)}\right)} \\ &= 2^{t-1}\lim_{N\in\Omega}\frac{1}{\left(\frac{N}{\psi(N)} - \frac{\varepsilon(N)}{\psi(N)}\right)} = 2^{t-1}\lim_{N\in\Omega}\left(\frac{\psi(N)}{N}\right) = \lim_{N\in\Omega}\xi(N). \end{aligned} \tag{4.4}$$

By Lemma 4.3, $\{N_2 \mid N \in \Omega\}$ is bounded and, by Lemma 4.1, $(N, d) = 1$ for all $N \in \Omega$. Moreover, since $\varepsilon(N) = \left(\frac{D}{N}\right)$ and, by definition of Lucas d-pseudoprime, $(D, N) = 1$, it follows that Ω is supported by ε. Therefore Theorem 2.3 implies that $2^{t-1}c/d = 1$. Thus $d = 2^{t-1}c$. Since 2^r exactly divides d, the hypothesis that $t > r+1$ implies that $r \geq t-1 > (r+1)-1 = r$, a contradiction. □

The following two corollaries are stated in [14].

Corollary 4.5: If d is odd, then there are at most finitely many Lucas d-pseudoprimes.

Proof: Theorem 4.4 handles the case in which N has at least 2 distinct prime divisors and Lemma 4.3 handles the case in which N is a prime power. □

Corollary 4.6: If 2 exactly divides d, then there are at most finitely many Lucas d-pseudoprimes.

Proof: Suppose otherwise and fix d such that $d \equiv 2 \pmod 4$ and there are infinitely many d-pseudoprimes N. Then, by Theorem 4.4 and Lemma 4.3, there are infinitely many d-pseudoprimes with $|\delta(N)| = 2$. By Lemma 4.1 and the argument in the proof of Theorem 4.4,

$$\frac{\psi(N)}{N-\varepsilon(N)} = \frac{1}{2}; \tag{4.5}$$

and hence, if N has decomposition (2.1),

$$\frac{(p_1-\varepsilon(p_1))(p_2-\varepsilon(p_2))}{N-\varepsilon(N)}=1. \tag{4.6}$$

If either $k_1>1$ or $k_2>1$, then

$$\begin{aligned}\frac{(p_1-\varepsilon(p_1))(p_2-\varepsilon(p_2))}{N-\varepsilon(N)}&=\frac{(p_1-\varepsilon(p_1))(p_2-\varepsilon(p_2))}{p_1^{k_1}p_2^{k_2}-\varepsilon(N)}\\&\le\frac{(p_1+1)(p_2+1)}{p_1^2p_2-1}\le\frac{(3+1)(5+1)}{9\cdot 5-1}=\frac{24}{44}<1,\end{aligned} \tag{4.7}$$

a contradiction. Therefore $k_1=k_2=1$.

It now follows that

$$\begin{aligned}(p_1-\varepsilon(p_1))(p_2-\varepsilon(p_2))&=p_1p_2-\varepsilon(p_1)\varepsilon(p_2),\quad\text{and}\\ p_1\varepsilon(p_2)+p_2\varepsilon(p_1)&=2\varepsilon(p_1)\varepsilon(p_2).\end{aligned} \tag{4.8}$$

If $\varepsilon(p_1)=\varepsilon(p_2)$, then $p_1+p_2=\pm 2$, which is impossible. Hence, $\varepsilon(p_1)=-\varepsilon(p_2)$.

Since $p_2>p_1$, it now follows that $p_2-p_1=2$, i.e., p_1 and p_2 are twin primes.

Now, by Lemma 4.1,

$$\frac{b}{d}=\frac{\lambda'(N)}{N-\varepsilon(N)}=\frac{\operatorname{lcm}\{(p_1+1),(p_1+2-1)\}}{p_1(p_1+2)+1}=\frac{1}{p_1+1}. \tag{4.9}$$

It follows that $d=b(p_1+1)$. Clearly, there are only finitely many prime twins p_1 and p_1+2 such that p_1+1 divides d. This final contradiction completes the proof of the corollary. □

5. LEHMER'S PROBLEM

In [7], D. H. Lehmer asks whether there exist composite integers N such that $\phi(N)$ divides $N-1$. If N has prime decomposition (2.1), then

$$\phi(N)=N\prod_{p\mid N}\frac{p-1}{p}. \tag{5.1}$$

Consequently, if $d\phi(N)=N-1$, it follows that

$$dN\prod_{p\mid N}(p-1)=(N-1)\prod_{p\mid N}p, \tag{5.2}$$

and therefore

$$dN_2\prod_{p\mid N}(p-1)=(N-1). \tag{5.3}$$

Since $(N, N-1)=1$, this implies that $N_2=1$, i.e., N is square-free.

The following theorem was first proven by C. Pomerance in [10].

Theorem 5.1: For any integers $t>1$ and $d>1$, there are at most a finite number of integers $N>2$ such that $d\phi(N)=N-1$ and $|\delta(N)|\le t$.

Proof: Fix positive integers t and d, and let Ω be the set of all positive integers N such that $d\phi(N) = N-1$ and $|\delta(N)| \le t$. By way of contradiction, assume that Ω has infinite cardinality.

It follows from the hypotheses that $(N, d) = 1$ for all $N \in \Omega$ and, from the remarks above, that N is square-free. Moreover, since $\phi(N)$ is even for N greater than 2, every element of Ω is odd.

It now follows for each $N \in \Omega$ that $\phi(N)/(N-1) = 1/d$. As in the previous sections, replacing Ω with a subset if necessary, we obtain

$$\frac{1}{d} = \frac{\phi(N)}{N-1} = \lim_{N \in \Omega} \frac{\phi(N)}{N-1} = \lim_{N \in \Omega} \frac{N\xi(N)}{N-1} = \lim_{N \in \Omega} \xi(N). \tag{5.4}$$

It now follows from Corollary 2.4 that $d = 1$, a contradiction. □

6. PERFECT NUMBERS

If N is a positive integer, define $\sigma(N)$ to be the sum of the positive divisors of N. A positive integer N is called a *perfect number* if $\sigma(N) = 2N$. It is well known that every even perfect number is a Euclid number, i.e., an integer of the form $2^n(2^{n+1}-1)$, where $2^{n+1}-1$ is a Mersenne prime. Moreover, it is well known that every odd perfect number can be written in the form $N = pM^2$ for some integer $M > 1$. It follows that 6 is the only square-free perfect number.

Recall that if N has decomposition (2.1), then

$$\sigma(N) = \prod_{p|N} \frac{p^{k_i+1}-1}{p-1}. \tag{6.1}$$

If N is square-free, then (6.1) becomes

$$\sigma(N) = \prod_{p|N} \frac{p^2-1}{p-1} = \prod_{p|N} (p+1) = N\xi(N), \tag{6.2}$$

where the signature function ε is given by $\varepsilon(p) = -1$ for all primes p. Thus, for N square-free, N is a perfect number if and only if

$$\xi(N) = 2. \tag{6.3}$$

More generally, we can ask for square-free k-perfect integers N, that is, solutions N of

$$\xi(N) = k. \tag{6.4}$$

L. E. Dickson [3] and I. S. Gradstein [5] have both proven that there are only a finite number of odd perfect numbers N with $|\delta(N)|$ bounded, and Dickson [3] generalized this result to primitive abundant numbers. H.-J. Kanold [6] has studied (6.4) for k rational, and proved that there are only finitely many primitive (and hence only finitely many odd) solutions N with a fixed number of prime factors. As mentioned in the introduction, these results have recently been generalized by Pomerance [9] and D. R. Heath-Brown [4]. Here we apply the methods developed above to prove a similar result for multiperfect numbers.

Theorem 6.1: For fixed k and t, there exist at most finitely many square-free integers N such that $|\delta(N)| \le t$ and

$$\sigma(N) = kN. \tag{6.5}$$

Proof: By the remarks preceding the theorem, the condition $\sigma(N) = kN$ is equivalent to $\xi(N) = k$. Let $\Omega = \{N \mid \xi(N) = k, |\delta(N)| \leq t$, and N is square-free$\}$. By way of contradiction, suppose that Ω has infinite cardinality. Since each $N \in \Omega$ is square-free, $\{N_2 \mid N \in \Omega\}$ is bounded. It is clear that Ω satisfies the hypotheses of Corollary 2.4, and we conclude that $k = 1$. But, clearly, $\sigma(N) \geq N + 1 > kN$, a contradiction. □

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AMS Classification Numbers: 11B39, 11A25, 11A51, 11B36

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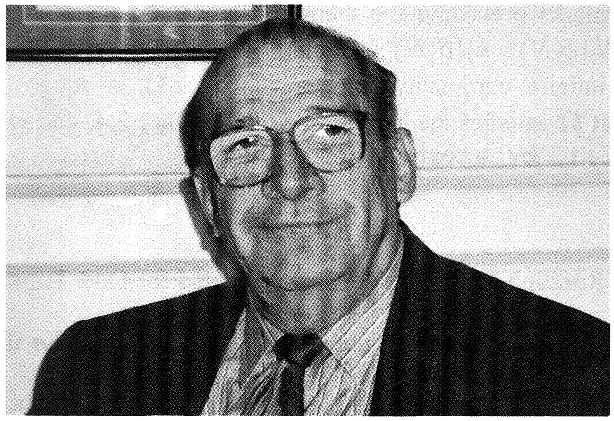
A TOAST TO OUR EDITOR



To Gerald E. Bergum in appreciation of
18 years of dedicated service as Editor of
The Fibonacci Quarterly Journal

So read the inscription on the plaque presented to Jerry Bergum at the banquet of the Eighth International Conference on Fibonacci Numbers and Their Applications held in Rochester, New York, this past June. The attendees of the conference, representing 18 countries, gave Jerry a standing ovation upon his acceptance of the plaque.

But what a colossal understatement of the true feelings of gratitude that the presentation of such a plaque is meant to convey. Dedicated service, indeed! How can words describe the demands of processing over 1000 papers for potential publication in the *Quarterly*; acknowledgment of receipt of these papers from a myriad of authors; arranging for refereeing by a stellar collection of referees; follow-up of authors with articles in the pipeline, deciding whether to accept, reject, or ask for a revision of a paper; providing support and encouragement to inexperienced authors to turn ineptly presented but neat mathematical ideas into well reasoned and publishable papers; coordinating the typing, layout, and printing of each issue of the *Quarterly*; and generating a mountain of ancillary correspondence.

In addition to his *Fibonacci Quarterly* work, Jerry Bergum edited the Proceedings of our biennial international conferences—a task equivalent to producing a full year's volume of the *Quarterly*, with the associated responsibilities of negotiating with the commercial publisher of the Proceedings, admonishing authors to proofread their papers in a timely fashion, and coordinating all activities needed to provide a camera-ready copy to the publisher.

It seems inevitable that the metaphor "tip of the iceberg" comes to mind when describing Jerry's Fibonacci efforts. While handling all these responsibilities, he served as Chairman of the Department of Computer Science at South Dakota State University, taught four courses per term, served as governor for his MAA Section, gave speeches to various academic groups, and assisted in the raising of a family of ten children.

In summation, Jerry is truly a remarkable person, with an infinite capacity for arduous work, a sympathetic and helpful attitude toward authors, and an uncompromising quest for quality of achievement in himself and those with whom he is associated.

Under his leadership, *The Fibonacci Quarterly* has not only prospered but has achieved an international reputation for excellence. The members of The Fibonacci Association, and all others who have read the *Quarterly* over these past 18 years, do indeed owe a considerable debt of gratitude to our Editor for his years of dedicated and selfless effort on behalf of us all.

Last, but certainly far from least, it goes without saying that all of this could not have happened had Jerry not had the strong support and understanding of his family and, particularly, his wife, Shirley. Jerry and Shirley, know that our hats are off to both of you, and that you have the respect, appreciation, and love of the many individuals you have touched during this 18-year saga of service to an idea and an ideal.

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stan@mathpro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-854 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Simplify

$$3\arctan(\alpha^{-1}) - \arctan(\alpha^{-5}).$$

B-855 ***Proposed by the editor***

Let $r_n = F_{n+1}/F_n$ for $n > 0$. Find a recurrence for r_n.

B-856 ***Proposed by Zdravko F. Starc, Vršac. Yugoslavia***

If n is a positive integer, prove that

$$L_1\sqrt{F_1} + L_2\sqrt{F_2} + L_3\sqrt{F_3} + \cdots + L_n\sqrt{F_n} < 8F_n^2 + 4F_n.$$

B-857 ***Proposed by the editor***

Find a sequence of integers $\langle w_n \rangle$ satisfying a recurrence of the form $w_{n+2} = Pw_{n+1} - Qw_n$ for $n \geq 0$, such that for all $n > 0$, w_n has precisely n digits (in base 10).

B-858 ***Proposed by Wolfdieter Lang, Universität Karlsruhe, Germany***

(a) Find an explicit formula for

$$\sum_{k=0}^{n} kF_{n-k}$$

which is the convolution of the sequence $\langle n \rangle$ and the sequence $\langle F_n \rangle$.

(b) Find explicit formulas for other interesting convolutions.

(The convolution of the sequence $\langle a_n\rangle$ and $\langle b_n\rangle$ is the sum $\sum_{k=0}^{n} a_k b_{n-k}$.)

B-859 ***Proposed by Kenneth B. Davenport, Pittsburgh, PA***

Simplify

$$\begin{vmatrix} F_nF_{n+1} & F_{n+1}F_{n+2} & F_{n+2}F_{n+3} \\ F_{n+3}F_{n+4} & F_{n+4}F_{n+5} & F_{n+5}F_{n+6} \\ F_{n+6}F_{n+7} & F_{n+7}F_{n+8} & F_{n+8}F_{n+9} \end{vmatrix}.$$

NOTE: The Elementary Problems Column is in need of more **easy**, yet elegant and nonroutine problems.

SOLUTIONS

Radical Inequality

B-834 ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia (Vol. 35, no. 3, August 1997)***

For x a real number and n an integer larger than 1, prove that

$$(x+1)F_1+(x+2)F_2+\cdots+(x+n)F_n < 2^n\sqrt{\frac{n(n+1)(2n+1+6x)+6nx^2}{6}}.$$

Editorial comment: *In the original statement of the problem in the August issue, the final term "$6nx^2$" was erroneously printed as "nx^2". Nevertheless, Bruckman managed to prove the stronger inequality, as it was actually printed. We now present his solution to this stronger inequality.*

Solution 2 by Paul S. Bruckman, Edmonds, WA

The inequality is false if the radicand is negative, so to simplify matters, we impose the condition $x \ge 0$, which ensures that the right member is well defined.

Let

$$S(x,n)=\sum_{k=1}^{n}(x+k)F_k$$

denote the left member of the putative inequality. Then

$$\begin{aligned} S(x,n) &= \sum_{k=1}^{n}[(x+k)F_{k+2}-F_{k+3}-(x+k-1)F_{k+1}+F_{k+2}] \\ &= (x+n)F_{n+2}-F_{n+3}-xF_2+F_3 \\ &= (x+n)F_{n+2}-F_{n+3}-(x-2) \\ &= x(F_{n+2}-1)+(n-1)F_{n+2}-F_{n+1}+2. \end{aligned}$$

Since $n \ge 2$, we have $n \le 2^{n-1}$, $F_{n+1}\ge 2$, and $F_{n+2}\le 1+2^{n-1}$. Therefore,

$$S(x,n) < x\cdot 2^{n-1}+n(1+2^{n-1}) \le 2^{n-1}(x+n+1).$$

If

$$R(x,n) = 2^n\sqrt{\frac{n(n+1)(2n+1+6x)+nx^2}{6}}$$

represents the right member of the putative inequality, then

$$\begin{aligned} R(x,n) &= 2^n\sqrt{n/6}\cdot\sqrt{x^2+6(n+1)x+(n+1)(2n+1)} \\ &> 2^{n-1}\sqrt{x^2+2(n+1)x+(n+1)^2} \\ &= 2^{n-1}(x+n+1). \end{aligned}$$

Thus, $S(x,n) < R(x,n)$ if $n \ge 2$.

Weighted Binomial Sum

B-839 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY (Vol. 35, no. 4, November 1997)***

Evaluate the sum

$$\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k 2^{-3k}\binom{n-2k}{k}$$

in terms of Fibonacci numbers.

Comment: Seiffert and Prielipp pointed out that the result

$$\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k 2^{-3k}\binom{n-2k}{k} = \frac{F_{n+3}-1}{2^n}$$

was proven by Jaiswal in [1].

Reference

1. D. V. Jaiswal. "On Polynomials Related to the Tchebichef Polynomials of the Second Kind." *The Fibonacci Quarterly* **12.3** (1974):263-65.

Seiffert found several other related identities, such as the remarkable formula

$$\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k 16^k 17^{-3k}\binom{n-2k}{k} = \frac{2^{2n+1}F_{3n+6}+2^{2n+3}F_{3n+3}-1}{31\cdot 17^n}.$$

Using the Binet form for the Fibonacci polynomials, he was able to show that

$$\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k (x^2)^k (x^2+1)^{-3k}\binom{n-2k}{k} = \frac{x^{n+1}F_{n+2}(x)+x^{n+2}F_{n+1}(x)-1}{(x^2+1)^n(2x^2-1)},$$

where $F_{n+2}(x) = xF_{n+1}(x)+F_n(x)$ *with* $F_0(x)=0$ *and* $F_1(x)=1$.

Rabinowitz looked at the sum

$$x_n = 2^n\sum_{k=0}^{\lfloor n/3\rfloor}2^{-3k}\binom{n-2k}{k}$$

and found that it satisfies the recurrence $x_n = 2x_{n-1}+x_{n-3}$, *but he found no further generalization.*

Solutions also received from Paul S. Bruckman, Nenad Cakic, Charles K. Cook, Russell Jay Hendel, H.-J. Seiffert, and the proposer.

An Arcane Formula for a Curious Matrix

B-840 ***Proposed by the editor***
(Vol. 35, no. 4, November 1997)

Let

$$A_n = \begin{pmatrix} F_n & L_n \\ L_n & F_n \end{pmatrix}.$$

Find a formula for A_{2n} in terms of A_n and A_{n+1}.

Solution by Paul S. Bruckman, Edmonds, WA

We require the following identities:

$$F_{n-1}F_n + F_nF_{n+1} = F_{2n}; \quad F_{n-1}L_n + F_nL_{n+1} = L_{2n}.$$

The first identity is obvious, from the relations $F_{n-1} + F_{n+1} = L_n$ and $F_nL_n = F_{2n}$. Since $F_{n-1}L_n = F_{2n-1} + (-1)^n$ and $F_nL_{n+1} = F_{2n+1} - (-1)^n$, we see that the left side of the second identity is equal to $F_{2n-1} + F_{2n+1} = L_{2n}$, as claimed.

From these identities, it follows immediately that

$$A_{2n} = F_{n-1}A_n + F_nA_{n+1}.$$

Comment by the editor: All solvers came up with the formula $A_{2n} = F_{n-1}A_n + F_nA_{n+1}$, but this formula expresses A_{2n} in terms of A_n, A_{n+1}, F_n, and F_{n-1}. What the proposer wanted, and perhaps did not state clearly enough, was a formula for A_{2n} expressed in terms of A_n and A_{n+1} **only**. The proposer's solution was

$$A_{2n} = \frac{1}{4}\left[\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} A_n^2 - \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} A_nA_{n+1} + \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} A_{n+1}^2\right].$$

How the proposer found this arcane formula remains a mystery.

Editorial comment: *In general, we have $H_{2n} = F_{n-1}H_n + F_nH_{n+1}$ for any sequence satisfying the recurrence $H_n = H_{n-1} + H_{n-2}$.*

Solutions also received by Brian D. Beasley, Charles K. Cook, Leonard A. G. Dresel, Russell Jay Hendel, Carl Libis, H.-J. Seiffert, and the proposer.

Integer Quotient

B-841 ***Proposed by David Zeitlin, Minneapolis, MN***
(Vol. 35, no. 4, November 1997)

Let P be an integer. For $n \geq 0$, let $U_{n+2} = PU_{n+1} + U_n$, with $U_0 = 0$ and $U_1 = 1$. Let $V_{n+2} = PV_{n+1} + V_n$, with $V_0 = 2$ and $V_1 = P$. Prove that

$$\frac{V_n^2 + V_{n+a}^2}{U_n^2 + U_{n+a}^2}$$

is always an integer if a is odd.

Solution by Hai-Xing Zhao, Qinghai Normal University, Xining, QingHai, China

From [1], we have

$$U_n = \frac{\alpha^n - \beta^n}{\sqrt{\Delta}} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where $\alpha + \beta = P$, $\alpha\beta = -1$, and $\Delta = P^2 + 4$. Hence,

$$U_n^2 = \frac{1}{\Delta}(V_{2n} - 2(-1)^n) \quad \text{and} \quad V_n^2 = V_{2n} + 2(-1)^n.$$

Therefore,

$$\frac{V_n^2 + V_{n+a}^2}{U_n^2 + U_{n+a}^2} = \Delta \frac{V_{2n} + V_{2n+2a} + 2(-1)^n + 2(-1)^{n+a}}{V_{2n} + V_{2n+2a} - 2(-1)^n - 2(-1)^{n+a}}.$$

When a is odd,

$$\frac{V_n^2 + V_{n+a}^2}{U_n^2 + U_{n+a}^2} = \Delta = P^2 + 4$$

which is an integer.

Reference

1. M. N. S. Swamy. "On a Class of Generalized Polynomials." *The Fibonacci Quarterly* **35.4** (1997):329-40.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Herta T. Freitag, Russell Jay Hendel, H.-J. Seiffert, and the proposer.

Divisibility by $x - 1$

B-842 ***Proposed by the editor***
(Vol. 36, no. 1, February 1998)

The Fibonacci polynomials, $F_n(x)$, and the Lucas polynomials, $L_n(x)$, satisfy

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \quad F_0(x) = 0, \quad F_1(x) = 1;$$
$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x), \quad L_0(x) = 2, \quad L_1(x) = x.$$

Prove that no Lucas polynomial is exactly divisible by $x - 1$.

Solution by Lawrence Somer, The Catholic University of America, Washington, D.C.

We prove, more generally, that if a is a nonzero real number, then no Lucas polynomial is exactly divisible by $x - a$ for $n \ge 0$ and no Fibonacci polynomial is exactly divisible by $x - a$ for $n \ge 1$.

By the Factor Theorem, $x - a$ exactly divides $L_n(x)$ if and only if $L_n(a) = 0$ and $x - a$ exactly divides $F_n(x)$ if and only if $F_n(a) = 0$. Clearly, $L_n(a)$ and $F_n(a)$ both satisfy the second-order recurrence

$$W_{n+2}(a) = aW_{n+1}(a) + W_n(a)$$

with initial terms $L_0(a) = 2$, $L_1(a) = a$, and $F_0(a) = 0$, $F_1(a) = 1$. It is easily proved by induction that $L_n(-a) = (-1)^n L_n(a)$ and $F_n(-a) = (-1)^{n+1} F_n(a)$. Thus,

$$|L_n(-a)| = |L_n(a)| \quad \text{and} \quad |F_n(-a)| = |F_n(a)|.$$

It thus suffices to prove that if $a > 0$, then $L_n(a) > 0$ for $n \geq 0$ and $F_n(a) > 0$ for $n \geq 1$. These assertions easily follow by induction. We are now done.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler & Jawad Sadek (jointly), Russell Jay Hendel, Harris Kwong, Angel Plaza & Miguel A. Pedrón (jointly), H.-J. Seiffert, Indulis Strazdins, and the proposer.

Addenda: The editor wishes to apologize for misplacing some solutions that were sent in on time. We therefore acknowledge solutions from the following solvers.

Peter G. Anderson—B-814
Brian Beasley—B-836, 837, 838
Leonard Dresel—B-784 to 789; 814, 815, 816, 819
Frank Flanigan—B-815
Pentti Haukkanen—B-837
Russell Hendel—B-814, 815, 816, 819, 821, 822, 823
Daina Krigens—B-814
Carl Libis—B-836, 837, 838
Graham Lord—B-815, 816
Bob Prielipp—B-836
Don Redmond—B-795
Lawrence Somer—B-796 to 801, 814

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-542 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the sequence $(c_k)_{k\ge1}$ by

$$c_k = \begin{cases} 1 & \text{if } k \equiv 2 \pmod 5, \\ -1 & \text{if } k \equiv 3 \pmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

Show that, for all positive integers n:

$$\frac{1}{n}\sum_{k=1}^{n} k\binom{2n}{n-k}c_k = F_{2n-2}; \tag{1}$$

$$\frac{1}{2n-1}\sum_{k=1}^{2n-1}(-1)^k k\binom{4n-2}{2n-k-1}c_k = 5^{n-1}F_{2n-2}; \tag{2}$$

$$\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k k\binom{4n}{2n-k}c_k = 5^{n-1}L_{2n-1}. \tag{3}$$

H-543 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY***

Find all positive nonsquare integers d such that, in the continued-fraction expansion

$$\sqrt{d} = [n; \overline{a_1, \ldots, a_{r-1}, 2n}],$$

we have $a_1 = \cdots = a_{r-1} = 1$. (This includes the case $r = 1$ in which there are no a's.)

H-544 ***Proposed by Paul S. Bruckman, Highwood, IL***

Given a prime $p > 5$ such that $Z(p) = p+1$, suppose that $q = \frac{1}{2}(p^2-3)$ and $r = p^2 - p - 1$ are primes with $Z(q) = q+1$, $Z(r) = \frac{1}{2}(r-1)$. Prove that $n = pqr$ is a FPP (see previous proposals for definitions of the Z-function and of FPP's).

SOLUTIONS

Re-enter

H-525 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 35, no. 1, February 1997)

Let p be any prime $\neq 2, 5$. Let

$$q = \frac{1}{2}(p-1), \quad e = \left(\frac{5}{p}\right), \quad r = \frac{1}{2}(p-e).$$

Let $Z(p)$ denote the entry-point of p in the Fibonacci sequence. Given that $2^{p-1} \equiv 1 \pmod p$ and $5^q \equiv e \pmod p$, let

$$A = \frac{1}{p}(2^{p-1}-1), \quad B = \frac{1}{p}(5^q - e), \quad C = \sum_{k=1}^{q} \frac{5^{k-1}}{2k-1}.$$

Prove that $Z(p^2) = Z(p)$ if and only if $eA - B \equiv C \pmod p$.

Solution by the proposer

Unless otherwise indicated, we will assume congruences $(\text{mod } p)$, but will omit the "$(\text{mod } p)$" notation. Note that $(5/p) = (-1/p) = 1$. It follows from [1] that a and q have the same parity and, in fact, are both even. Since $p \equiv 1 \pmod 4$, let $r = q/2$, an integer. Define the function $\delta_p = \delta$ as follows:

$$\delta = \begin{cases} +1 & \text{if } p \equiv 1 \pmod{20}, \\ -1 & \text{if } p \equiv 9 \pmod{20}. \end{cases} \tag{1}$$

We may therefore express the desired result as follows:

$$\delta \cdot 5^r \equiv (-1)^{a/2+r}. \tag{2}$$

The following result was shown in [2]:

$$F_{q+1} \equiv (-1)^{a/2+r}. \tag{3}$$

Also, note that $(\alpha\beta/p) = (-1/p) = 1$, hence $(\alpha/p) = (\beta/p)$; note that since $(5/p) = 1$, $\sqrt{5}$ and, hence, α and β are ordinary residues. Then,

$$F_{q+1} = 5^{-1/2}(\alpha^{q+1} - \beta^{q+1}) = 5^{-1/2}(\alpha^q\alpha - \beta^q\beta) \equiv (\alpha-\beta)^{-1}\{(\alpha/p)\alpha - (\beta/p)\beta\},$$

or

$$F_{q+1} \equiv (\alpha/p). \tag{4}$$

In light of (2), (3), and (4), it suffices to prove that

$$(\alpha/p) \equiv \delta \cdot 5^r. \tag{5}$$

Note that $5^r = (\sqrt{5})^q \equiv (\sqrt{5}/p)$. Therefore, it suffices to prove that

$$(\alpha/p) = \delta(\sqrt{5}/p). \tag{6}$$

However, the last result is an old result attributable to E. Lehmer (see [3]); we have only changed the notation to conform with that employed herein. Thus, the desired result is established.

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1. D. M. Bloom. Problem H-494. *The Fibonacci Quarterly* **33.1** (1995):91. The solution by H.-J. Seiffert appeared in *The Fibonacci Quarterly* **34.2** (1996):190-91.
2. P. S. Bruckman. Problem H-515. *The Fibonacci Quarterly* **34.4** (1996):379.
3. E. Lehmer. "On the Quadratic Character of the Fibonacci Root." *The Fibonacci Quarterly* **4.2** (1966):135-38.

Also solved by H.-J. Seiffert.

Generator Trouble

H-526 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 35, no. 2, May 1997)

Following H-465, let $r_1, r_2,$ and r_3 be natural integers such that

(1) $\sum_{k=1}^{3} kr_k = n$, where n is a given natural integer.

Let

(2) $B_{r_1, r_2, r_3} = \dfrac{1}{r_1+r_2+r_3} \dfrac{(r_1+r_2+r_3)!}{r_1!r_2!r_3!}$.

Also, let

(3) $C_n = \sum B_{r_1, r_2, r_3}$, summed over all possible $r_1, r_2,$ and r_3.

Define the generating function

(4) $$F(x) = \sum_{n=6}^{\infty} C_n x^n :$$

(a) find a closed form for $F(x)$;
(b) obtain an explicit expression for C_n;
(c) show that C_n is a positive integer for all $n \ge 7$, n prime.

Solution by the proposer

Solution of part (a): Note that $2 \le 2r_2 \le n-1-3r_3 \le n-4$ (eliminating $r_1 = n-2r_2-3r_3$). Then

$$F(x) = \sum_{n=6}^{\infty} x^n \sum_{r_3=1}^{[n/3-1]^3} 1/r_3! \sum_{r_2=1}^{\left[\frac{1}{2}(n-1-3r_3)\right]} \frac{(n-2r_3-r_2-1)!}{r_2!(n-3r_3-2r_2)!}$$
$$= \sum_{r_3=1}^{\infty} \frac{1}{r_3!} \sum_{n=3r_3+3}^{\infty} x^n \sum_{r_2=1}^{\left[\frac{1}{2}(n-1-3r_3)\right]} \frac{(n-2r_3-r_2-1)!}{r_2!(n-3r_3-2r_2)!}.$$

Changing variables, we obtain

$$F(x) = \sum_{v=1}^{\infty} \frac{1}{v!} \sum_{m=0}^{\infty} x^{m+3v+3} \sum_{u=1}^{\left[\frac{1}{2}(m+2)\right]} \frac{(m+2+v-u)!}{u!(m+3-2u)!}$$
$$= \sum_{m=0}^{\infty} x^{m+1} \sum_{u,\, v=1}^{\infty} x^{2u+3v} \frac{(m+u+v)!}{(m+1)!\,u!\,v!} = \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{v=1}^{\infty} x^{3v} \binom{n-1+v}{v} \sum_{u=1}^{\infty} x^{2u} \binom{n+v-1+u}{u}$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{v=1}^{\infty} x^{3v} \binom{n-1+v}{v} \cdot [(1-x^2)^{-n-v} - 1]$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{v=1}^{\infty} (-x^3)^v \binom{-n}{v} [(1-x^2)^{-n-v} - 1]$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n} \left\{ (1-x^2)^{-n} \left[\left(1 - \frac{x^3}{1-x^2}\right)^{-n} - 1 \right] - [(1-x^3)^{-n} - 1] \right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left[\left(\frac{x}{1-x^2-x^3}\right)^n - \left(\frac{x}{1-x^2}\right)^n - \left(\frac{x}{1-x^3}\right)^n + x^n \right]$$

$$= -\log\left(1 - \frac{x}{1-x^2-x^3}\right) + \log\left(1 - \frac{x}{1-x^2}\right) + \log\left(1 - \frac{x}{1-x^3}\right) - \log(1-x)$$

$$= -\log(1-x-x^2-x^3) + \log(1-x^2-x^3) + \log(1-x-x^2) - \log(1-x^2)$$
$$+ \log(1-x-x^3) - \log(1-x^3) - \log(1-x),$$

or

$$F(x) = \log\left\{ \frac{(1-x-x^2)(1-x^2-x^3)(1-x-x^3)}{(1-x)(1-x^2)(1-x^3)(1-x-x^2-x^3)} \right\}. \qquad (*)$$

Solution of part (b): Suppose

$$\begin{aligned} 1-x^2-x^3 &= (1-rx)(1-sx)(1-tx), \\ 1-x-x^3 &= (1-ux)(1-vx)(1-wx), \\ 1-x-x^2-x^3 &= (1-fx)(1-gx)(1-hx). \end{aligned} \qquad (**)$$

Then

$$\begin{aligned} F(x) = {} & \log(1-\alpha x) + \log(1-\beta x) + \log(1-rx) + \log(1-sx) + \log(1-tx) \\ & + \log(1-ux) + \log(1-vx) + \log(1-wx) - 3\log(1-x) - \log(1+x) \\ & - \log(1-\omega x) - \log(1-\omega^2 x) - \log(1-fx) - \log(1-gx) - \log(1-hx), \end{aligned}$$

where α and β are the usual Fibonacci constants and $\omega = \exp(2i\pi/3)$. We then obtain

$$\begin{aligned} F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} [& -(\alpha^n + \beta^n) - (r^n + s^n + t^n) - (u^n + v^n + w^n) \\ & + 3 + (-1)^n + \omega^n + \omega^{2n} + (f^n + g^n + h^n)]. \end{aligned}$$

Comparison of coefficients yields the explicit formula:

$$C_n = \frac{1}{n}(J_n + 3 + (-1)^n + \omega^n + \omega^{2n} - L_n - G_n - H_n), \quad n = 1, 2, 3, \ldots, \qquad (***)$$

where

$$L_n = \alpha^n + \beta^n \text{ (Lucas numbers)}, \quad G_n = r^n + s^n + t^n,$$
$$H_n = u^n + v^n + w^n, \quad J_n = f^n + g^n + h^n, \quad n = 1, 2, \ldots. \qquad (****)$$

The initial values and recurrence relations satisfied by the J_n's, G_n's, and H_n's may be obtained from $(**)$, and are as follows:

(i) $J_{n+3} = J_{n+2} + J_{n+1} + J_n,\ n = 1, 2, \ldots;\ J_1 = 1, J_2 = 3, J_3 = 7;$

(ii) $G_{n+3} = G_{n+1} + G_n,\ n = 1, 2, \ldots;\ G_1 = 0, G_2 = 2, G_3 = 3;$

(iii) $H_{n+3} = H_{n+2} + H_n,\ n = 1, 2, \ldots;\ H_1 = H_2 = 1, H_3 = 4.$

If $n \geq 5$ is prime, $\omega^n + \omega^{2n} = -1$; thus, for prime $n \geq 7$, we obtain the slightly simplified formula for C_n:

$$C_n = \frac{1}{n}(J_n + 1 - L_n - G_n - H_n),\ n \geq 7,\ n \text{ prime.} \qquad (*****)$$

To obtain values of J_n, G_n, and H_n without means of the recurrence relations (i)-(iii), we would need to solve for the roots in $(**)$; we shall omit this exercise and assume that these roots are known. Also, it is of interest to note, as can be verified, that C_n given by $(***)$ vanishes for $n = 1, 2, 3, 4, 5$, as we would expect.

Solution of part (c): As was determined in Problem H-465 as a special case, B_{r_1, r_2, r_3} is an integer for prime $n \geq 7$. From (3), it then follows immediately that C_n is an integer if n is prime (even for $n = 2, 3, 5$, since $C_2 = C_3 = C_5 = 0$.)

Note: It may be shown that $L_n \equiv 1 \pmod{n}$ for all prime n; from this result and the expression in $(*****)$, we deduce that

$$J_n \equiv G_n + H_n \pmod{n}, \text{ if } n \text{ is prime.} \qquad (\#)$$

Sum Formula

H-527 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 35, no. 2, May 1997)***

Let q, a, and b be positive integers, with $(a, b) = 1$. Prove or disprove the following:

a) $$\sum_{\substack{r=0 \\ (br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}(-1)^{q(br+as)}L_{2q(br+as)} = \frac{F_{q(a+b-ab)}F_{qab}}{F_{qa}F_{qb}} + (-1)^{q(1-ab)}\frac{F_{q(2ab-1)}}{F_q};$$

b) $$5\sum_{\substack{r=0 \\ (br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}(-1)^{q(br+as)}L_{2q(br+as)} = (-1)^{q(1-ab)}\frac{L_{q(2ab-1)}}{F_q} - \frac{F_{qab}L_{q(a+b-ab)}}{F_{qa}F_{qb}}.$$

Solution by the proposer

Consider

$$S(x; a, b) \equiv \sum_{\substack{r=0 \\ (br+as<ab)}}^{a-1}\sum_{s=0}^{b-1} x^{br+as}, \qquad (1)$$

for a, b positive integers, with $(a, b) = 1$, and $x \neq 1$ an arbitrary variable. L. Carlitz has shown ["Some Restricted Multiple Sums," *The Fibonacci Quarterly* **18.1** (1980):58-65, eqns. (1.1) and (1.2)] that

$$S(x; a, b) = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)} - \frac{x^{ab}}{1 - x}. \qquad (2)$$

Now, for q a positive integer, consider

$$T_{\pm}(q;a,b) \equiv S(\alpha^q/\beta^q;a,b) \pm S(\beta^q/\alpha^q;a,b), \tag{3}$$

where $\alpha = \frac{1}{2}[a+\sqrt{5}]$, $\beta = \frac{1}{2}[1-\sqrt{5}]$, $\alpha\beta = -1$. It is readily seen that (2) in (3) gives

$$\begin{aligned}\sqrt{5}\,T_{\pm} &= -\beta^{q(a+b-ab)}\frac{F_{qab}}{F_{qa}F_{qb}} + \alpha^{qab}\beta^{q(1-ab)}\frac{1}{F_q}\\ &\pm\left[\alpha^{q(a+b-ab)}\frac{F_{qab}}{F_{qa}F_{qb}} - \beta^{qab}\alpha^{q(1-ab)}\frac{1}{F_q}\right],\end{aligned} \tag{4}$$

where $F_n \equiv (\alpha^n - \beta^n)/\sqrt{5}$. Similarly, (1) in (3) gives

$$\begin{aligned}\sqrt{5}\,T_{\pm} &= \sqrt{5}\sum_{\substack{r=0\\(br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}\left[\left(\frac{\alpha^q}{\beta^q}\right)^{br+as} \pm \left(\frac{\beta^q}{\alpha^q}\right)^{br+as}\right]\\ &= \sqrt{5}\sum_{\substack{r=0\\(br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}(-1)^{q(br+as)}\left[\alpha^{2q(br+as)} \pm \beta^{2q(br+as)}\right].\end{aligned} \tag{5}$$

The solution to part (a) follows by choosing T_+ in (4) and (5); equating the results gives

$$\sum_{\substack{r=0\\(br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}(-1)^{q(br+as)}L_{2q(br+as)} = \frac{F_{q(a+b-ab)}F_{qab}}{F_{qa}F_{qb}} + (-1)^{q(1-ab)}\frac{F_{q(2ab-1)}}{F_q}.$$

For the solution to part (b), choose T_- in (4) and (5) to obtain

$$5\sum_{\substack{r=0\\(br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}(-1)^{q(br+as)}L_{2q(br+as)} = (-1)^{q(1-ab)}\frac{L_{q(2ab-1)}}{F_q} - \frac{F_{qab}L_{q(a+b-ab)}}{F_{qa}F_{qb}}.$$

Also solved by P. Bruckman.

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Fibonacci Entry Points and Periods for Primes 100,003 through 415,993 by Daniel C. Fielder and Paul S. Bruckman.

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