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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

HARMONIC SEEDS

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1. INTRODUCTION

Harmonic numbers were introduced by Ore [6] in 1948, though not under that name. A natural number n is harmonic if the harmonic mean of its positive divisors is an integer. Equivalently, n is harmonic if $H(n)$ is an integer, where

$$H(n)=\frac{n\tau(n)}{\sigma(n)},$$

and $\tau(n)$, $\sigma(n)$, respectively, are the number of and sum of the positive divisors of n.

Ore listed all harmonic numbers up to 10^5, and this list was extended by Garcia [3] to 10^7 and by Cohen [2] to $2\cdot 10^9$. The second author of this paper has continued the list up to 10^{10}. In all of these cases, straightforward direct searches were used. No odd harmonic numbers have been found, giving the main interest to the topic since all perfect numbers are easily shown to be harmonic. If it could be proved that there are no odd harmonic numbers, then it would follow that there are no odd perfect numbers.

A number might be labeled also as arithmetic or geometric if the arithmetic mean, or geometric mean, respectively, of its positive divisors were an integer. Most harmonic numbers, but not all, appear to be also arithmetic (see Cohen [2]). It is easy to see that the set of geometric numbers is in fact simply the set of perfect squares, and it is of interest, according to Guy [4], that no harmonic numbers are known that are also geometric.

Although it is impractical to extend the direct search for harmonic numbers, we shall show, through the introduction of harmonic seeds, that no harmonic number less than 10^{12} is powerful. (We say that n is powerful if $p\mid n$ implies $p^2\mid n$, where p is prime.) In particular, then, no harmonic number less than 10^{12} is also geometric. We have also used harmonic seeds to show that there are no odd harmonic numbers less than 10^{12}.

To define harmonic seeds, we first recall that d is a unitary divisor of n (and n is a unitary multiple of d) if $d\mid n$ and $\gcd(d, n/d)=1$; we call the unitary divisor d proper if $d>1$. Then:

Definition: A harmonic number (other than 1) is a harmonic seed if it does not have a smaller proper unitary divisor which is harmonic (and 1 is deemed to be the harmonic seed only of 1).

Then any harmonic number is either itself a harmonic seed or is a unitary multiple of a harmonic seed. For example, $n=2^3 3^3 5^2 31$ is harmonic (with $H(n)=27$); the proper unitary divisors of n are the various products of $2^3, 3^3, 5^2$, and 31. Since $2^3 5^2 31$ is harmonic and does not itself have a proper unitary harmonic divisor, it is a harmonic seed of n. (We are unable to prove that a harmonic number's harmonic seed is always unique, but conjecture that this is so.)

It is not as difficult to generate harmonic seeds only, and our two results on harmonic numbers less than 10^{12}, that (except for 1) none are powerful and none are odd, will clearly follow when the corresponding properties are seen to be true of the harmonic seeds less than 10^{12}.

2. COMPUTATION AND USE OF HARMONIC SEEDS

We shall need the following lemmas. Always, p and q will denote primes. We write $p^a \| n$ to mean $p^a | n$ and $p^{a+1} \nmid n$, and we then call p^a a component of n.

Lemma 1: Besides 1, the only squarefree harmonic number is 6 (Ore [6]).

Lemma 2: There are no harmonic numbers of the form p^a (Ore [6]). The only harmonic numbers of the form $p^a q^b$, $p \neq q$, are perfect numbers (Callan [1], Pomerance [7]).

Lemma 3: If n is an odd harmonic number and $p^a \| n$, then $p^a \equiv 1 \pmod 4$ (Garcia [3], Mills [5]).

Lemma 4: If n is an odd harmonic number greater than 1, then n has a component exceeding 10^7 (Mills [5]).

We first illustrate the algorithm for determining all harmonic seeds less than 10^{12}.

By Lemma 2, even perfect numbers are harmonic seeds and all other harmonic seeds, besides 1, have at least three distinct prime factors. Then, in seeking harmonic seeds n with $2^a \| n$, we must have $a \leq 35$, since $2^{36} 3 \cdot 5 > 10^{12}$.

We build even harmonic seeds n, based on specific components 2^a, $1 \leq a \leq 35$, by calculating $H(n)$ simultaneously with n until $H(n)$ is an integer, using the denominators in the values of $H(n)$ to determine further prime factors of n. This uses the fact that H, like σ and τ, is multiplicative. For example, taking $a = 13$,

$$H(2^{13}) = \frac{2^{13}\tau(2^{13})}{\sigma(2^{13})} = \frac{2^{14}7}{3 \cdot 43 \cdot 127}.$$

Choosing the largest prime in the denominator, either $127^b \| n$ for $1 \leq b \leq 3$ (since $2^{13}3 \cdot 127^4 > 10^{12}$), or $p^{126} | n$ for some prime p so that $127 | \tau(n)$. In the latter case, clearly $n > 10^{12}$. With $b = 1$, we have

$$H(2^{13}127) = \frac{2^8 7}{3 \cdot 43},$$

so that $43^c \| n$ for $1 \leq c \leq 3$ (since $2^{13}43^4 127 > 10^{12}$) or $p^{42} | n$. In similar fashion, we then take, in particular,

$$H(2^{13}127 \cdot 43) = \frac{2^7 7}{3 \cdot 11}, \quad H(2^{13}127 \cdot 43 \cdot 11) = \frac{2^6 7}{3^2}.$$

At this stage we must have either $3^d \| n$ for $1 \leq d \leq 6$, or $p^2 | n$ for *two* primes p, or $p^8 | n$ for some prime p. All possibilities must be considered. We find

$$H(2^{13}127 \cdot 43 \cdot 11 \cdot 3^3) = \frac{2^5 3 \cdot 7}{5}, \quad H(2^{13}127 \cdot 43 \cdot 11 \cdot 3^3 5) = 2^5 7,$$

and so $2^{13}3^3 5 \cdot 11 \cdot 43 \cdot 127$ must be a harmonic seed.

Odd harmonic seeds up to 10^{12} were sought in the same way. Each odd prime was considered in turn as the smallest possible prime factor of an odd harmonic number. By Lemma 4, only the primes less than 317 needed to be considered since $317 \cdot 331 \cdot 10^7 > 10^{12}$. Lemmas 1 and 3 were also taken into account.

The list of all harmonic seeds less than 10^{12} is given in Table 1. Inspection of this list allows us to conclude the following.

Theorem 1: There are no powerful harmonic numbers less than 10^{12}.

Theorem 2: There are no odd harmonic numbers less than 10^{12}.

We had hoped originally that we would be able to generate easily all harmonic numbers, up to some bound, with a given harmonic seed. This turns out to be the case for those harmonic numbers which are squarefree multiples of their harmonic seed. In fact, we have the following result.

Theorem 3: Suppose n and $nq_1 \ldots q_t$ are harmonic numbers, where $q_1 < \cdots < q_t$ are primes not dividing n. Then nq_1 is harmonic, except when $t \geq 2$ and $q_1 q_2 = 6$, in which case $nq_1 q_2$ is harmonic.

Proof: We may assume $t \geq 2$. Suppose first that $q_1 \geq 3$. Since $nq_1 \ldots q_t$ is harmonic and H is multiplicative,

$$H(nq_1 \ldots q_t) = H(n)H(q_1) \ldots H(q_t) = H(n)\frac{2q_1}{q_1+1} \cdots \frac{2q_t}{q_t+1} = h,$$

say, where h is an integer. Then

$$H(n)q_1 \ldots q_t = h\frac{q_1+1}{2} \cdots \frac{q_t+1}{2}.$$

Since $(q_1+1)/2 < \cdots < (q_t+1)/2 < q_t$, we have $q_t \mid h$, and then

$$H(nq_1 \ldots q_{t-1}) = H(n)\frac{2q_1}{q_1+1} \cdots \frac{2q_{t-1}}{q_{t-1}+1} = \frac{h}{q_t}\frac{q_t+1}{2},$$

an integer. Applying the same argument to the harmonic number $nq_1 \ldots q_{t-1}$, and repeating it as necessary, leads to our result in this case. In the less interesting case when $q_1 = 2$ (since then n must be an odd harmonic number), we again find that nq_1 is harmonic except perhaps if $q_2 = 3$, in which case $nq_1 q_2$ is harmonic. These details are omitted.

The point of Theorem 3 is that harmonic squarefree multiples of harmonic seeds may be built up a prime at a time. Furthermore, when n and nq_1 are harmonic numbers, with $q_1 > 2, q_1 \nmid n$, we have

$$H(nq_1)\frac{q_1+1}{2} = H(n)q_1,$$

so that $(q_1+1)/2 \mid H(n)$. Thus, $q_1 \leq 2H(n)-1$, implying a relatively short search for all possible q_1, and then for q_2, and so on.

There does not seem to be a corresponding result for non-squarefree multiples of harmonic seeds. For example, $2^6 3^2 5 \cdot 13^3 17 \cdot 127$ is harmonic, but no unitary divisors of this number other than its seed $2^6 127$ and 1 are harmonic.

As an example of the application of Theorem 3, in Table 2 we give a list of all harmonic numbers n that are squarefree multiples of the seed 2457000. It is not difficult to see that the list is complete, and in fact it seems clear that there are only finitely many harmonic squarefree multiples of any harmonic number, all obtainable by the algorithm described above. However, a proof of this statement appears to be difficult.

TABLE 1

Harmonic seeds n less than 10^{12}			
n	$H(n)$	n	$H(n)$
1	1	$2876211000 = 2^3 3^2 5^3 13^2 31 \cdot 61$	150
$6 = 2 \cdot 3$	2	$8410907232 = 2^5 3^2 7^2 13 \cdot 19^2 127$	171
$28 = 2^2 7$	3	$8589869056 = 2^{16} 131071$	17
$270 = 2 \cdot 3^3 5$	6	$8628633000 = 2^3 3^3 5^3 13^2 31 \cdot 61$	195
$496 = 2^4 31$	5	$8698459616 = 2^5 7^2 11^2 19^2 127$	121
$672 = 2^5 3 \cdot 7$	8	$10200236032 = 2^{14} 7 \cdot 19 \cdot 31 \cdot 151$	96
$1638 = 2 \cdot 3^2 7 \cdot 13$	9	$14182439040 = 2^7 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19$	384
$6200 = 2^3 5^2 31$	10	$19017782784 = 2^9 3^2 7^2 11 \cdot 13 \cdot 19 \cdot 31$	336
$8128 = 2^6 127$	7	$19209881600 = 2^{11} 5^2 7^2 13 \cdot 19 \cdot 31$	256
$18620 = 2^2 5 \cdot 7^2 19$	14	$35032757760 = 2^9 3^2 5 \cdot 7^3 11 \cdot 13 \cdot 31$	392
$30240 = 2^5 3^3 5 \cdot 7$	24	$43861478400 = 2^{10} 3^3 5^2 23 \cdot 31 \cdot 89$	264
$32760 = 2^3 3^2 5 \cdot 7 \cdot 13$	24	$57575890944 = 2^{13} 3^2 11 \cdot 13 \cdot 43 \cdot 127$	192
$173600 = 2^5 5^2 7 \cdot 31$	25	$57648181500 = 2^2 3^2 5^3 7^3 13^3 17$	273
$1089270 = 2 \cdot 3^2 5 \cdot 7^2 13 \cdot 19$	42	$66433720320 = 2^{13} 3^3 5 \cdot 11 \cdot 43 \cdot 127$	224
$2229500 = 2^2 5^3 7^3 13$	35	$71271827200 = 2^8 5^2 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73$	270
$2457000 = 2^3 3^3 5^3 7 \cdot 13$	60	$73924348400 = 2^4 5^2 7 \cdot 31^2 83 \cdot 331$	125
$4713984 = 2^9 3^3 11 \cdot 31$	48	$77924700000 = 2^5 3^3 5^5 7^2 19 \cdot 31$	375
$6051500 = 2^2 5^3 7^2 13 \cdot 19$	50	$81417705600 = 2^7 3 \cdot 5^2 7 \cdot 11^2 17 \cdot 19 \cdot 31$	484
$8506400 = 2^5 5^2 7^3 31$	49	$84418425000 = 2^3 3^2 5^5 7^2 13 \cdot 19 \cdot 31$	375
$17428320 = 2^5 3^2 5 \cdot 7^2 13 \cdot 19$	96	$109585986048 = 2^9 3^7 7 \cdot 11 \cdot 31 \cdot 41$	324
$23088800 = 2^5 5^2 7^2 19 \cdot 31$	70	$110886522600 = 2^3 3 \cdot 5^2 7 \cdot 31^2 83 \cdot 331$	155
$29410290 = 2 \cdot 3^5 5 \cdot 7^2 13 \cdot 19$	81	$124406100000 = 2^5 3^2 5^5 7^3 13 \cdot 31$	375
$33550336 = 2^{12} 8191$	13	$137438691328 = 2^{18} 524287$	19
$45532800 = 2^7 3^3 5^2 17 \cdot 31$	96	$156473635500 = 2^2 3^2 5^3 7^2 13^3 17 \cdot 19$	390
$52141320 = 2^3 3^4 5 \cdot 7 \cdot 11^2 19$	108	$183694492800 = 2^7 3^2 5^2 7^2 13 \cdot 17 \cdot 19 \cdot 31$	672
$81695250 = 2 \cdot 3^3 5^3 7^2 13 \cdot 19$	105	$206166804480 = 2^{11} 3^2 5 \cdot 7 \cdot 13^2 31 \cdot 61$	384
$115048440 = 2^3 3^2 5 \cdot 13^2 31 \cdot 61$	78	$221908282624 = 2^8 7 \cdot 19^2 37 \cdot 73 \cdot 127$	171
$142990848 = 2^9 3^2 7 \cdot 11 \cdot 13 \cdot 31$	120	$271309925250 = 2 \cdot 3^7 5^3 7^2 13 \cdot 19 \cdot 41$	405
$255428096 = 2^9 7 \cdot 11^2 19 \cdot 31$	88	$428440390560 = 2^5 3^2 5 \cdot 7^2 13^2 19 \cdot 31 \cdot 61$	546
$326781000 = 2^3 3^3 5^3 7^2 13 \cdot 19$	168	$443622427776 = 2^7 3^4 11^3 17 \cdot 31 \cdot 61$	352
$459818240 = 2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	96	$469420906500 = 2^2 3^3 5^3 7^2 13^3 17 \cdot 19$	507
$481572000 = 2^5 3^3 5^3 7^3 13$	168	$513480135168 = 2^9 3^5 7^2 11 \cdot 13 \cdot 19 \cdot 31$	648
$644271264 = 2^5 3^2 7 \cdot 13^2 31 \cdot 61$	117	$677701763200 = 2^7 5^2 7 \cdot 11 \cdot 17^2 31 \cdot 307$	340
$1307124000 = 2^5 3^3 5^3 7^2 13 \cdot 19$	240	$830350521000 = 2^3 3^4 5^3 7^3 11^2 13 \cdot 19$	756
$1381161600 = 2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 31$	240	$945884459520 = 2^9 3^5 5 \cdot 7^3 11 \cdot 13 \cdot 31$	756
$1630964808 = 2^3 3^4 11^3 31 \cdot 61$	99	$997978703400 = 2^3 3^3 5^2 7 \cdot 31^2 83 \cdot 331$	279
$1867650048 = 2^{10} 3^4 11 \cdot 23 \cdot 89$	128		

Of the harmonic seeds in Table 1, the most prolific in producing harmonic squarefree multiples is 513480135168, with 216 such multiples. The largest is the 32-digit number

$$N_1 = 29388663214285910932405215567360$$
$$= 2^9 3^5 5 \cdot 7^2 11 \cdot 13 \cdot 19 \cdot 23 \cdot 31 \cdot 137 \cdot 821 \cdot 8209 \cdot 16417 \cdot 32833,$$

with $H(N_1) = 65666$. Much larger harmonic numbers were given by Garcia [3] and the algorithm above can be applied to give a great many harmonic squarefree multiples of those which are of truly gigantic size. Furthermore, most known multiperfect numbers (those n for which $\sigma(n) = kn$, for some integer $k \geq 2$) are also harmonic, for these are known only for $k \leq 10$ and nearly always it is the case that $k \mid \tau(n)$. For example, the largest known 4-perfect number (i.e., having $k = 4$) is

$$N_2 = 2^{37}3^{10}7 \cdot 11 \cdot 23 \cdot 83 \cdot 107 \cdot 331 \cdot 3851 \cdot 43691 \cdot 174763 \cdot 524287;$$

this has 169 harmonic squarefree multiples, the largest of which is

$$N_3 = N_2 \cdot 31 \cdot 61 \cdot 487 \cdot 3343 \cdot 3256081 \cdot 6512161 \cdot 13024321 \approx 5.53 \cdot 10^{73}$$

with $H(N_3) = 13024321$.

TABLE 2

All squarefree harmonic multiples n of 2457000	
n	$H(n)$
$27027000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13$	110
$513513000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 19$	209
$18999981000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37$	407
$1386998613000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 73$	803
$1162161000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 43$	215
$2945943000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 109$	218
$2457000 = 2^3 3^3 5^3 7 \cdot 13$	60
$46683000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19$	114
$1727271000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 37$	222
$126090783000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73$	438
$765181053000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 37 \cdot 443$	443
$5275179000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 113$	226
$10597041000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 227$	227
$56511000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 23$	115
$12941019000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 23 \cdot 229$	229
$5914045683000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 23 \cdot 229 \cdot 457$	457
$71253000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 29$	116
$144963000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 59$	118

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MORGAN-VOYCE TYPE GENERALIZED POLYNOMIALS WITH NEGATIVE SUBSCRIPTS

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1. PURPOSE OF THE PAPER

Previous papers ([1], [2], [3], and [4]) have investigated aspects of the Morgan-Voyce polynomials $B_n(x)$, $b_n(x)$ and certain polynomials $C_n(x)$, $c_n(x)$ associated with them, together with generalizations of them, $P_n^{(r)}(x)$, $Q_n^{(r)}(x)$, namely,

$$P_{n-1}^{(0)}(x) = b_n(x), \tag{1.1}$$

$$P_{n-1}^{(1)}(x) = B_n(x), \tag{1.2}$$

$$P_{n-1}^{(2)}(x) = c_n(x), \tag{1.3}$$

$$Q_n^{(0)}(x) = C_n(x). \tag{1.4}$$

Both generalizations are absorbed into a *composite* polynomial $R_n^{(r,u)}(x)$ such that [4]

$$R_n^{(r,1)}(x) = P_n^{(r)}(x), \tag{1.5}$$

$$R_n^{(r,2)}(x) = Q_n^{(r)}(x). \tag{1.6}$$

Here we consider the implications for the theory in the case $R_{-n}^{(r,u)}(x)$, where $n > 0$.

Because of the detailed information in the previous papers, only the algebraic skeletal structure of the new system of polynomials will be outlined.

For the record, we list the following equalities involving negative subscripts which are readily obtainable from the Binet forms in [2]:

$$B_{-n}(x) = -B_n(x), \tag{1.7}$$

$$b_{-n}(x) = b_{n+1}(x), \tag{1.8}$$

$$C_{-n}(x) = C_n(x), \tag{1.9}$$

$$c_{-n}(x) = -c_{n+1}(x). \tag{1.10}$$

Additionally, we require

$$P_n^{(r)}(1) = F_{2n+1} + rF_{2n} \qquad [1], \tag{1.11}$$

$$Q_n^{(r)}(1) = L_{2n} + rF_{2n} \qquad [3], \tag{1.12}$$

$$Q_n^{(2r+1)}(1) = 2P_n^{(r)}(1) \qquad [3], \tag{1.13}$$

$$Q_n^{(r)}(x) = P_n^{(r)}(x) + P_{n-1}^{(0)}(x) \quad (n \geq 1) \quad [3], \tag{1.14}$$

whence

$$Q_n^{(r)}(1) - P_n^{(r)}(1) = F_{2n-1}. \tag{1.15}$$

Worth recording finally is ([2], (1.7), (1.9)) the differential equation

$$\frac{dC_{-n}(x)}{dx} = -nB_{-n}(x). \tag{1.16}$$

2. THE POLYNOMIALS $R_{-n}^{(r,u)}(x)$

Define the polynomials $\{R_{-n}^{(r,u)}(x)\}$ by means of a Morgan-Voyce type recurrence

$$R_{-n}^{(r,u)}(x) = (x+2)R_{-n-1}^{(r,u)}(x) - R_{-n-2}^{(r,u)}(x) \quad (n>0) \tag{2.1}$$

with

$$R_0^{(r,u)}(x) = u, \quad R_{-1}^{(r,u)}(x) = (u-1)x+u-r. \tag{2.2}$$

Paralleling the data in [4], we postulate the existence of a sequence of integers $\{c_{-n,k}^{(r,u)}\}$, $n \geq 0$, for which

$$R_{-n}^{(r,u)}(x) = \sum_{k=0}^{n} c_{-n,k}^{(r,u)} x^k, \tag{2.3}$$

in which

$$c_{-n,n}^{(r,u)} = \begin{cases} u, & n=0, \\ u-1, & n>0, \end{cases} \tag{2.4}$$

and

$$c_{-n,0}^{(r,u)} = u - nr. \tag{2.5}$$

Moreover, for $x=0$ in (2.1) and (2.3),

$$c_{-n,0}^{(r,u)} = 2c_{-n-1,0}^{(r,u)} - c_{-n-2,0}^{(r,u)}. \tag{2.6}$$

Furthermore, (2.1) leads to ($k \geq 1$)

$$c_{-n,k}^{(r,u)} = 2c_{-n-1,k}^{(r,u)} - c_{-n-2,k}^{(r,u)} + c_{-n-1,k-1}^{(r,u)}. \tag{2.7}$$

The Coefficients $c_{-n,k}^{(r,u)}$

Repeated use of (2.1) and (2.2) allows us to construct a table of the coefficients $c_{-n,k}^{(r,u)}$ as follows.

TABLE 1. The Coefficients $c_{-n,k}^{(r,u)}$ $(n \geq 0)$

$n \backslash k$	0	1	2	3	4	5	6
0	u						
−1	$u-r$	$-1+u$					
−2	$u-2r$	$-2+3u-r$	$-1+u$				
−3	$u-3r$	$-3+6u-4r$	$-4+5u-r$	$-1+u$			
−4	$u-4r$	$-4+10u-10r$	$-10+15u-6r$	$-6+7u-r$	$-1+u$		
−5	$u-5r$	$-5+15u-20r$	$-20+35u-21r$	$-21+28u-8r$	$-8+9u-r$	$-1+u$	
−6	$u-6r$	$-6+21u-35r$	$-35+70u-56r$	$-56+84u-36r$	$-36+45u-10r$	$-10+11u-r$	$-1+u$

Comparison of this table with the corresponding table for $c_{n,k}^{(r,u)}$ in [4] reveals that the sign of the constants and the sign of the coefficients for r have both changed from + to −. On the other hand, the sign of the coefficients of u remains unchanged (+), but n has been replaced by $-n+1$. That is, from [4], we have the key formula

$$c_{-n,k}^{(r,u)} = -\binom{n+k-1}{2k-1} - r\binom{n+k}{2k+1} + u\binom{n+k}{2k} \tag{2.8}$$

$$= \binom{n+k-1}{2k} - r\binom{n+k}{2k+1} + (u-1)\binom{n+k}{2k}, \tag{2.9}$$

by Pascal's Theorem.

Suitable specializations $u = 1, 2$ in (1.5) and (1.6) reduce this to

$$a_{-n,k}^{(r)} = \binom{n+k-1}{2k} - r\binom{n+k}{2k+1} \tag{2.10}$$

and

$$b_{-n,k}^{(r)} = \binom{n+k}{2k} + \binom{n+k-1}{2k} - r\binom{n+k}{2k+1} \tag{2.11}$$

for $P_{-n}^{(r)}(x)$ and $Q_{-n}^{(r)}(x)$, respectively, tables for which the reader may care to construct.

Further specializations are obvious, e.g., $a_{-n,k}^{(0)} = \binom{n+k-1}{2k}$.

Next, multiply (2.9) throughout by x^k and sum. Then, by (1.5) $(n \to -n)$, (1.8), and (2.3), we deduce that

Theorem 1: $R_{-n}^{(r,u)}(x) = P_{-n}^{(r)}(x) + (u-1)b_{n+1}(x)$.

Numerical Specializations

Using (1.1)-(1.14) variously, we deduce that

$$R_{-n}^{(0,1)}(1) = P_{-n}^{(0)}(1) = b_n(1) = F_{2n-1}, \tag{2.12}$$

$$R_{-n}^{(1,1)}(1) = P_{-n}^{(1)}(1) = -B_{n-1}(1) = -F_{2n-2}, \tag{2.13}$$

$$R_{-n}^{(2,1)}(1) = P_{-n}^{(2)}(1) = -c_n(1) = -L_{2n-1}, \tag{2.14}$$

$$R_{-n}^{(0,2)}(1) = Q_{-n}^{(0)}(1) = C_n(1) = L_{2n}, \tag{2.15}$$

$$R_{-n}^{(1,2)}(1) = Q_{-n}^{(1)}(1) = 2P_{-n}^{(0)}(1) = 2b_n(1) = 2F_{2n-1}, \tag{2.16}$$

$$R_{-n}^{(2,2)}(1) = Q_{-n}^{(2)}(1) = F_{2n+3}. \tag{2.17}$$

Also [cf. (2.13)],

$$R_{-n}^{(0,0)}(1) = B_{-n}(1) = F_{2n}. \tag{2.18}$$

Moreover, we have from (2.1) that

$$R_{-n-1}^{(r,u)}(-1) = R_{-n}^{(r,u)}(-1) + R_{-n-2}^{(r,u)}(-1), \tag{2.19}$$

$$R_{-n-1}^{(r,u)}(-3) = -(R_{-n}^{(r,u)}(-3) + R_{-n-2}^{(r,u)}(-3)), \tag{2.20}$$

$$R_{n}^{(r,u)}(-2) = -R_{-n}^{(r,u)}(-2) \tag{2.21}$$

[e.g., $R_{-1}^{(r,u)}(-2) = -u-r+2 = -R_{-3}^{(r,u)}(-2)$].

3. MISCELLANEOUS RESULTS

Chebyshev Polynomials

Employing the notation in [4] for the Chebyshev polynomials $U_n(x)$ and $T_n(x)$, we discover that, with (1.7)-(1.10),

$$B_{-n}(x) = -U_n\left(\frac{x+2}{2}\right), \tag{3.1}$$

$$C_{-n}(x) = 2T_n\left(\frac{x+2}{2}\right), \tag{3.2}$$

$$b_{-n}(x) = U_{n+1}\left(\frac{x+2}{2}\right) - U_n\left(\frac{x+2}{2}\right), \tag{3.3}$$

$$c_{-n}(x) = -U_{n+1}\left(\frac{x+2}{2}\right) + U_n\left(\frac{x+2}{2}\right). \tag{3.4}$$

[Ordinarily, $U_{-n}(x) = -U_{n-2}(x)$, but this is not true when x is replaced by $\frac{x+2}{2}$.]

As in [4], we have

Theorem 2: $R_{-n}^{(r,u)}(x) = -B_{-n-1}(x) - (r+u-2)B_{-n}(x) + (u-1)B_{-n+1}(x)$.

Theorem 3: $R_{-n}^{(r,u)}(x) = ((u-1)x - r + u)B_{-n}(x) - uB_{-n-1}$.

Both these theorems can, by (3.1), be cast in terms of $U_n(\frac{x+2}{2})$. Theorem 3 is, in fact, an equivalent of the Binet form for $R_{-n}(x)$. A Simson formula analog for $R_{-n}(x)$ corresponding to that in [4] for $R_n(x)$ is left to the reader's interest, and likewise for a generating function analog.

Zeros and Orthogonality

These properties for $B_{-n}(x), \ldots, c_{-n}(x)$ may be approached as for those of $B_n(x), \ldots, c_n(x)$ in [2], by referring to (1.7)-(1.10).

Rising Diagonals

Rising diagonal polynomials (functions) are obtained from Table 1 by considering a set of upward-slanting parallel diagonal lines (cf. [2]). Designate these polynomials by $\mathfrak{R}_{-n}^{(r,u)}(x)$ or just $\mathfrak{R}_{-n}(x)$ for brevity. Then $\mathfrak{R}_0(x) = u - r$, $\mathfrak{R}_{-1}(x) = u - 2r + (u-1)x$.

A little tricky exploration enables us to affirm that [see (2.9)]

$$\mathfrak{R}_{-n}(x) = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} c_{-n-1+k,\,k}\, x^k. \tag{3.5}$$

Comparison with [2, (7.1)] is worthwhile at this point. The contrast in the two forms demonstrates that, in passing from $\mathfrak{R}_n(x)$ in [2] to $\mathfrak{R}_{-n}(x)$ here, we cannot with impunity always merely replace n by its negative. Asymmetry in the two patterns of rising diagonals explains this dilemma. [Indeed, $\mathfrak{R}_0(x)$ is chosen to be different in [2] and here.]

Adopting [2] as our model, we are able to establish the following corresponding results (no proofs offered.)

Theorem 4 (Recurrence): $\mathfrak{R}_{-n}(x) = 2\mathfrak{R}_{-n+1}(x) + (x-1)\mathfrak{R}_{-n+2}(x)$.

Corollary 1: $\mathfrak{R}_{-n}(1) = 2^{n-1}\{2u - 2r - 1\}$.

Theorem 5 (Generating function):

$$\sum_{i=0}^{\infty} \mathfrak{R}_{-i}(x)y^i = \{u - r + [-u + x(u-1)]y\}\{1 - (2y + (x-1)y^2)\}^{-1}.$$

Analogously to the procedures in [2], we may derive a Binet form and a Simson formula for $\mathfrak{R}_{-n}(x)$.

4. CONCLUSION

The development outlined above complements that in [4] and thus rounds out the general theory for integer n (about which more could be written).

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ACKNOWLEDGMENT OF PRIORITY

It has been brought to my attention by Dr. John Holte by way of Dr. Bergum that there was a failure to give a "complete list of references" in my article "The Fibonacci Triangle Modulo *p*" (June-July 1998 issue of *The Fibonacci Quarterly*). My research was performed in spring and summer of 1995. The paper did not appear until 1998 because it references an unpublished paper of Dr. William Webb and Dr. Diana Wells that Dr. Bergum asked me to get permission to cite. My research therefore post-dates Dr. Holte's article "A Lucas-Type Theorem for Fibonomial Coefficient Residues" (February 1994 issue of *The Fibonacci Quarterly*) of which I was unaware until after the publication of my article. While the results were obtained independently and without knowledge of Dr. Holte's work, Dr. Holte has asked that I give an acknowledgment of priority. I acknowledge that Dr. Holte has priority for any results common to the two papers. As a final note, I would like to add that the starting point for my research was a paper by Dr. Diana Wells, "The Fibonacci and Lucas Triangle Modulo 2" (April 1994 issue of *The Fibonacci Quarterly*) which also failed to reference Dr. Holte's paper and contains results that Holte claims priority for in his letter to Dr. Bergum.

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ON USING PATTERNS IN BETA-EXPANSIONS TO STUDY FIBONACCI-LUCAS PRODUCTS

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1. INTRODUCTION

The Zeckendorf decomposition of a natural number n is the unique expression of n as a sum of Fibonacci numbers with nonconsecutive indices and with each index greater than 1, where $F_0 = 0$, $F_1 = 1$, and $F_{i+2} = F_i + F_{i+1}$ form the Fibonacci numbers for $i \geq 0$ (see [13] and [17], or see [16, pp. 108-09]). The Zeckendorf decompositions of products of the forms kF_m and kL_m with $k, m \in \mathbb{N}$ (where $L_m = F_{m-1} + F_{m+1}$ is the m^{th} Lucas number) have occurred in questions in cryptography [3] and in the study of periodic points in algebraic topology [11]. They are also the subject of study in [5]. We describe here a simple method for finding results concerning the Zeckendorf decomposition of such a product. We let $\beta = (1+\sqrt{5})/2$ throughout the paper, and we make use of the connection between the β-expansion and the Zeckendorf decomposition as developed by Grabner et al. in [8] and [9].

The β-expansion of $n \in \mathbb{N}$ is the unique finite sum of integral powers of β that equals n and contains no consecutive powers of β. Grabner et al., in [8] and [9], prove that for m sufficiently large the Zeckendorf decomposition of kF_m can be produced by replacing β^i in the β-expansion of k with F_{m+i}. For example, the β-expansion of 5 is $\beta^3 + \beta^{-1} + \beta^{-4}$, and the Zeckendorf decomposition of $5F_{10}$ is $F_{13} + F_9 + F_6$. See [1], [2], [6], [10], [14], and Section 2 for background on the β-expansion.

We have found that by studying short lists of β-expansions of small positive integers we can easily observe patterns that represent new results. In Section 4 we improve upon the results of [5] involving the number of addends in the Zeckendorf decomposition of mF_m and we include a proof of Conjecture 3 from the same paper. This conjecture states that, for certain values of m and k, the Zeckendorf decomposition of $(mL_{2k}+1)(F_{mL_{2k}+1})$ contains $F_{mL_{2k}+1}$ as one of its terms. This is equivalent to saying that β^0 occurs in the β-expansion of $mL_{2k}+1$. Most of the identities in [5] can be discovered easily using the techniques given here, as we demonstrate in Section 3. While a computer can be used to form lists of β-expansions, we were able to discover all the results in Sections 3 and 4 easily by hand. All proofs are provided in Section 6.

The developments presented here provide the background necessary for [12], joint work with L. Sanchis, in which we prove Conjecture 1 from [5]. The conjecture involves the ratio of natural numbers k that do not have F_k in the Zeckendorf decomposition of kF_k to those natural numbers that do. The list of β-expansions of k for $1 \leq k \leq 500$, produced easily by a computer, was sufficient to allow us to discover the recursive patterns in the β-expansions and then to prove that the conjecture is correct. This result also answers an equivalent question posed by Bergman in [1] concerning the frequency of positive integers n with β^0 appearing in the β-expansion of n.

We present an algorithm for finding the β-expansion of a positive integer that can be used to efficiently produce a list of β-expansions. The beginning of this list is given in Section 2. The algorithm actually applies more generally. Given a sum $n = \sum_{i=m}^{M} \lambda_1 F_i$ with $m, M \in \mathbb{Z}$ and $\lambda_i \in \mathbb{N}$

for all i, the algorithm produces a representation of n as a sum of nonconsecutive Fibonacci numbers, some of which may have negative indices. If the smallest index in the resulting sum is at least 2, then the algorithm has produced the Zeckendorf decomposition of n without requiring the calculation of the value of n. This algorithm runs in time that is linear in $M-m+\sum_{i=m}^{M}\lambda_i$. For another algorithm that produces the Zeckendorf decomposition of n with the same input (but does not give the β-expansion of a number) see, for example, [7].

2. PRELIMINARIES

Remark 2.1: Note that in [8] and [9] the indices for Fibonacci and Lucas numbers are different from the standard used here. We use $F_0=0$, $F_1=1$, $L_0=2$, and $L_1=1$. For $x<0$, let F_x be equal to $(-1)^{-x+1}F_{-x}$.

Definition 2.2: Let $n\in\mathbb{N}$. The Zeckendorf decomposition of n is the unique expression of n as a sum of Fibonacci numbers of the form $\sum_{i=2}^{r}\mu_iF_i$, with $r\in\mathbb{N}$, $\mu_i\in\{0,1\}$, and with $\mu_i\mu_{i+1}=0$.

Definition 2.3: Let β be the golden ratio $(1+\sqrt{5})/2$. For any $n\in\mathbb{N}$, the β-expansion of n is the unique expression of n as a finite sum of integral powers of β with no consecutive powers occurring. That is, $n=\sum_{i=-\infty}^{\infty}e_i\beta^i$ with $e_i\in\{0,1\}$, $e_ie_{i+1}=0$, and with at most finitely many e_i equal to one.

For this value of β, the β-expansion was first defined by Bergman in 1957 in [1]. For generalizations using other values of β, see, for example, [2], [6], [14], and [15].

Definition 2.4: For $k\in\mathbb{N}$, the lower width of k, $\ell(k)$ [resp. the upper width of k, $u(k)$] is defined to be the absolute value of the smallest (resp. largest) exponent that appears in the β-expansion of k.

For example, the β-expansion of 12 is $\beta^{-6}+\beta^{-3}+\beta^{-1}+\beta^5$, so $\ell(12)=6$ and $u(12)=5$.

The following is a restatement of Lemma 1 and Theorem 1 in [9] for the special case of Fibonacci numbers. See also Theorem 1 in [8].

Theorem 2.5 (Grabner et al. [8]): For $k\in\mathbb{N}$ and for $n\geq\ell(k)+2$, if the β-expansion of k is $\sum_{i=-\ell(k)}^{u(k)}e_i\beta^i$, then the Zeckendorf decomposition of kF_n is $\sum_{i=-\ell(k)}^{u(k)}e_iF_{i+n}$. For $k\in\mathbb{N}$, we have that $\ell(k)$ is the even number defined by $L_{\ell(k)-1}<k\leq L_{\ell(k)+1}$. If $2<k<L_{\ell(k)}$, then $u(k)=\ell(k)-1$. If $k\geq L_{\ell(k)}$, then $u(k)=\ell(k)$. We also have that $u(1)=0$ and $u(2)=1$.

For example, the β-expansion of 10 is $\beta^{-4}+\beta^{-2}+\beta^2+\beta^4$, as can be determined quickly by the algorithm of Section 5 (see 5.7), and the Zeckendorf decomposition of $10F_{5000}$ is $F_{4996}+F_{4998}+F_{5002}+F_{5004}$. The power of Theorem 2.10 in [8] is clear here. Using the greedy algorithm, we would have needed to calculate the value of $10F_{5000}$, which is daunting.

As usual, a sum of Fibonacci numbers will be represented by a vector of zeros and ones. A one occurs in coordinate s if F_s appears in the sum. We allow negative indices.

Definition 2.6: We define V to be the infinite dimensional vector space over $\mathbb{Z}$ given by

$$V:=\{(\ldots,v_{-1},v_0,v_1,\underline{v_2},v_3,\ldots):v_i\in\mathbb{Z}\,\forall_i,\text{ with at most finitely many } v_i \text{ nonzero}\}.$$

For convenience, we underline the second coordinate. We define V^+ to be the subset of V that consists of all vectors of V with all entries nonnegative.

All vectors in V are infinite dimensional, but we will abuse notation and omit the entries before the first possibly nonzero entry and after the last possibly nonzero entry. If the entries are all single digits, we may omit commas and parentheses.

Definition 2.7: Let $n \in \mathbb{N}$. Let $\vec{z}(n)$ be the vector in V^+ corresponding to the Zeckendorf decomposition of n that has 0 in the first coordinate.

In Definition 2.7, we must require that the vector have zero in the first coordinate in order to have $\vec{z}$ well defined. For example, the Zeckendorf decomposition of 4 is $1+3$, which can be represented by either F_1+F_4 or F_2+F_4. Whenever 1 occurs in the Zeckendorf decomposition, we always represent it as F_2 in the image of $\vec{z}$. Thus, $\vec{z}(4) = (0, \underline{1}, 0, 1)$.

Definition 2.8: We define the function $\vec{\beta}: \mathbb{N} \to V^+$ so that $\vec{\beta}(n)$ is the vector in V^+ with $v_i = e_{i-2}$ when the β-expansion of n is $\sum_{i=-\infty}^{\infty} e_i \beta^i$. Thus, the coefficient of β^0 is underlined.

For example, $\vec{\beta}(12)$ is represented by $100101\underline{0}00001$. Here the exponents of β increase from left to right, which does not match the usual notation for a β-expansion. We must choose between the usual notation for $\vec{z}(n)$ and for $\vec{\beta}(n)$. Because this paper concerns Zeckendorf decompositions, we have chosen the former.

The β-expansion of k is as follows for $1 \le k \le 20$, with the exponents of β increasing from left to right.

k	$\vec{\beta}(k)$												
1							$\underline{1}$						
2					1	0	$\underline{0}$	1					
3					1	0	$\underline{0}$	0	1				
4					1	0	$\underline{1}$	0	1				
5			1	0	0	1	$\underline{0}$	0	0	1			
6			1	0	0	0	$\underline{0}$	1	0	1			
7			1	0	0	0	$\underline{0}$	0	0	0	1		
8			1	0	0	0	$\underline{1}$	0	0	0	1		
9			1	0	1	0	$\underline{0}$	1	0	0	1		
10			1	0	1	0	$\underline{0}$	0	1	0	1		
11			1	0	1	0	$\underline{1}$	0	1	0	1		
12	1	0	0	1	0	1	$\underline{0}$	0	0	0	0	1	
13	1	0	0	1	0	0	$\underline{0}$	1	0	0	0	1	
14	1	0	0	1	0	0	$\underline{0}$	0	1	0	0	1	
15	1	0	0	1	0	0	$\underline{1}$	0	1	0	0	1	
16	1	0	0	0	0	1	$\underline{0}$	0	0	1	0	1	
17	1	0	0	0	0	0	$\underline{0}$	1	0	1	0	1	
18	1	0	0	0	0	0	$\underline{0}$	0	0	0	0	0	1
19	1	0	0	0	0	0	$\underline{1}$	0	0	0	0	0	1
20	1	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	1

It is possible to generate the k^{th} row in this list by applying the algorithm developed in Section 5 to the vector $(0, \underline{k}, 0)$ (see 5.5). We will see in Remark 5.9 that we may instead move from one row to the next by adding one to the underlined entry and applying $\mathcal{A}$ to the result. This second method is much more efficient.

Definition 2.9: We define a linear transformation that shifts the entries of a vector. For $t \in \mathbb{Z}$, let $s_t : V \to V$ be given by the following. If $\vec{v} \in V$ has coordinates v_i for $i \in \mathbb{Z}$, then $s_t(\vec{v})$ is the vector with coordinates $w_i := v_{i-t}$.

Next, we restate part of Theorem 2.5 using the notation of this section.

Theorem 2.10 (Grabner et al. [8]): For $k \in \mathbb{N}$ and for $n \in \mathbb{Z}$, kF_n is represented by the vector $s_{n-2}(\vec{\beta}(k))$. For $n \geq \ell(k)+2$, this vector is $\vec{z}(kF_n)$.

3. PATTERNS IN β-EXPANSIONS

Most of the identities in [5] can be found using lists of β-expansions. For example, (2.4) of [5] gives the Zeckendorf decomposition for $4F_kF_{n+k}$ whenever $k, n \geq 3$. This could be determined by adding the formulas given in [5] for F_kF_{n+k} and for $3F_kF_{n+k}$ and reducing the result so that no two consecutive Fibonacci numbers occur. On the other hand, we are able to determine the pattern for the β-expansions of $4F_k$ from scratch very quickly by considering the list below. We then arrive at the Zeckendorf decomposition of $4F_kF_{n+k}$ by simply shifting the vector $\vec{\beta}(4F_k)$ to the right by $n+k-2$ spaces.

The following list provides the β-expansion for $4F_k$ as k increases from 3 to 10. Note that we can add two consecutive rows in this list and then apply the algorithm from Section 5 to the sum. The result will be the next row in the list. This is easy to do by hand. The diagonal lines of ones appear in a predictable pattern that will continue, as can be proven by induction.

$k=3:$									1	0	0	0	$\underline{1}$	0	0	0	1							
$k=4:$							1	0	0	1	0	1	$\underline{0}$	0	0	0	0	1						
$k=5:$							1	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	1					
$k=6:$					1	0	0	1	0	1	0	0	$\underline{0}$	0	1	0	0	0	0	1				
$k=7:$					1	0	0	0	1	0	0	1	$\underline{0}$	0	0	1	0	0	0	0	1			
$k=8:$			1	0	0	1	0	1	0	0	0	0	$\underline{1}$	0	0	0	1	0	0	0	0	1		
$k=9:$			1	0	0	0	1	0	0	1	0	0	$\underline{0}$	1	0	0	0	1	0	0	0	0	1	
$k=10:$	1	0	0	1	0	1	0	0	0	0	1	0	$\underline{0}$	0	1	0	0	0	1	0	0	0	0	1

For $n \geq 3$ and $k \geq 3$ we have, as in (2.4) of [5],

$$4F_kF_{n+k} = \begin{cases} F_{n-2} + F_{n+1} + F_{n+3} + F_{2k+n+1} + \sum_{j=1}^{(k-4)/2} F_{4j+n+4} & (k \text{ even}), \\ F_{n-1} + F_{n+3} + F_{2k+n+1} + \sum_{j=1}^{(k-3)/2} F_{4j+n+2} & (k \text{ odd}). \end{cases}$$

Note that a similar method can be used for finding the β-expansion of mL_k and hence for finding the Zeckendorf decomposition of products of the form mL_kF_{n+k}.

4. NEW RESULTS

We summarize here the new results we have found using the β-expansion. Proofs are provided in Section 6. We begin with a technical definition and then state precisely in 4.2 the useful fact that, if two Zeckendorf decompositions have indices that do not overlap significantly, then the two Zeckendorf decompositions can meld into the Zeckendorf decomposition of the sum.

Definition 4.1: For $x \in \mathbb{Z}$, we say that a $\vec{v} \in V^+$ is reduced to index x if every entry with index (i.e., coordinate) $\geq x$ is either zero or one and $\vec{v}$ has at most finitely many ones with index $\geq x$ and $\vec{v}$ has no consecutive ones with indices $\geq x$. If $\vec{v}$ is reduced for all $x \in \mathbb{Z}$, then $\vec{v}$ is totally reduced.

Note that, for all $n \in \mathbb{N}$, the vectors $\vec{z}(n)$ and $\vec{\beta}(n)$ are totally reduced.

Fact 4.2: If $n, m \in \mathbb{N}$ and if $\vec{z}(n)+\vec{z}(m)$ is totally reduced, then $\vec{z}(n+m)=\vec{z}(n)+\vec{z}(m)$. The same is true for β-expansions by 2.10. One way to determine whether $\vec{z}(n)+\vec{z}(m)$ is totally reduced is to consider the following two sets. Let $I_n = \{i \in \mathbb{Z}: F_i$ is in the Zeckendorf decomposition of $n\}$, and let I_m be the corresponding set for m. Let $d = \min\{|i-j|: i \in I_n, j \in I_m\}$. Then $\vec{z}(n)+\vec{z}(m)$ is totally reduced if $d \geq 2$.

Let $Q(n) = nF_n$, and note that we will find it useful to use exponents in vectors. For example, the β-expansion of L_8 is given by $10000000\underline{0}00000001$, which we may write as $10^7\underline{0}0^71$. Similarly, the β-expansion of L_9 is $10101010\underline{1}01010101$, which we may write as $(10)^4\underline{1}(01)^4$.

Proposition 4.3: For $k \geq 2$, we have $\vec{\beta}(2L_{2k}) = 110010^{2k-2}\underline{0}0^{2k-3}1001$. Thus, the Zeckendorf decomposition of $Q(2L_{2k}) = F_{2k+1+2L_{2k}} + F_{2k-2+2L_{2k}} + F_{-2k-2+2L_{2k}} + F_{-2k+1+2L_{2k}}$.

The preceding proposition is proven in detail in Section 6, but to give an idea of the flavor of such proofs, we provide a sketch here. We have $\vec{\beta}(L_{2k}) = 10^{2k-1}\underline{0}0^{2k-1}1$ (see (1.5) of [4] and apply 2.10). We think of $2L_{2k}$ as $20^{2k-1}\underline{0}0^{2k-1}2$ and we prove that $\vec{\beta}(2L_{2k})$ is given by

$$\overbrace{1001}0^{2k-2}\underline{0}0^{2k-3}\overbrace{1001},$$

where the braces mark the vectors $s_{-2k}(\vec{\beta}(2))$ and $s_{2k}(\vec{\beta}(2))$. Because the two braces do not touch, the entire vector is totally reduced.

In the following propositions, let $f[n]$ denote the number of addends in the Zeckendorf decomposition of n as in [5]. Note that $f[Q(m)]$ is equal to the number of ones in the vector $\vec{\beta}(m)$ by 2.10. The next two propositions are generalizations of (3.3) and (3.4) in [5].

Proposition 4.4: If $k \geq 2$, and if $1 \leq m \leq L_{2k-1}$, then $f[Q(L_{2k}+m)] = 2 + f[Q(m)] \leq 2k+1$. Moreover, $\vec{z}(Q(L_{2k}+m)) = \vec{z}(L_{2k}F_{L_{2k}+m}) + \vec{z}(mF_{L_{2k}+m})$.

Proposition 4.5: If $k \geq 2$, and if $1 \leq m \leq L_{2k-3}$, then $f[Q(2L_{2k}+m)] = 4 + f[Q(m)]$. Moreover, $\vec{z}(Q(2L_{2k}+m)) = \vec{z}(2L_{2k}F_{2L_{2k}+m}) + \vec{z}(mF_{2L_{2k}+m})$.

In [5] a positive integer n is said to have Property $\mathcal{P}$ if F_n occurs in the Zeckendorf decomposition of nF_n. This is equivalent to stating that a one occurs in the underlined coordinate of $\vec{\beta}(n)$. We prove Conjecture 3 of [5] in the following proposition.

Proposition 4.6: If $m, k \in \mathbb{N}$ with $1 \leq k$ and $1 \leq m \leq L_{2k-1}$, then mL_{2k} does not have Property $\mathcal{P}$, and $mL_{2k}+1$ does have Property $\mathcal{P}$.

Proposition 4.7: For $k \geq 2$, we have $\vec{\beta}(L_{2k+1}+L_{2k-1}) = 100100(10)^{k-2}\underline{1}(01)^{k-1}001$. Thus, we see that $Q(L_{2k+1}+L_{2k-1})$ has Property $\mathcal{P}$.

See Section 6 for proofs of the propositions in this section.

5. THE ALGORITHM

The algorithm begins with a positive integer n expressed as $n = \sum_{i=m}^{M} \lambda_i F_i$, with λ_i any non-negative integer for $i \in \mathbb{Z}$, and with $m, M \in \mathbb{Z}$. It ends with an expression for n as a sum of Fibonacci numbers with nonconsecutive (possibly negative) indices. This sum is the Zeckendorf decomposition for n under certain conditions. There are other algorithms that produce Zeckendorf decompositions (normal forms) in this setting (see [7]). The advantage of the algorithm given here is that it allows us to find the β-expansion of $k \in \mathbb{N}$ by applying the algorithm to $(0, \underline{k}, 0)$ (see 5.5).

Definition 5.1: Let $\vec{v} \in V$ be a vector with coordinates v_i for $i \in \mathbb{Z}$. Let $\sigma: V \to \mathbb{Z}$ be the function given by $\sigma(\vec{v}) := \sum_{i=-\infty}^{\infty} v_i F_i$. Note that σ is a linear function and that it is not injective.

Verbose Description of the Algorithm: We begin with n represented by the vector $\vec{v} := (\ldots, \lambda_0, \lambda_1, \underline{\lambda_2}, \lambda_3, \ldots)$, where $\lambda_i = 0$ for $i > M$ and for $i < m$ as above. Thus, the initial values for the entries in $\vec{v}$ are $v_i = \lambda_i$ for $i \in \mathbb{Z}$. First, we search for the smallest integer x for which the vector $\vec{v}$ is reduced to index x. If there is no such integer, then we are done. Details of the search are below in the second description of the algorithm. We assign $t := x-1$ if $v_x = 0$ and $t := x$ if $v_x = 1$. Note that this implies that $v_{t+1} = 0$ and $v_t \geq 1$.

Case 1. $v_{t-1} \neq 0$. We have $(\ldots, v_{t-1}, v_t, 0, \ldots)$. We replace $v_{t-1}, v_t, 0$ with $v_{t-1}-1,\ v_t - 1,\ 1$. This does not change the value of $\sigma(\vec{v})$, because $F_t + F_{t-1} = F_{t+1}$. We return to the beginning of the algorithm and search for a new value of x.

Case 2. We have $(\ldots, v_{t-2}, 0, v_t, 0, \ldots)$, and, because the vector is not reduced to index $t-1$, $v_t > 1$. We replace $v_{t-2}, 0, v_t, 0$ with $v_{t-2}+1, 0, v_t - 2, 1$. This does not change the value of $\sigma(\vec{v})$. To see this, consider two smaller steps. We can replace $v_{t-2}, 0, v_t, 0$ with $v_{t-2}+1, 1, v_t - 1, 0$ because $F_t = F_{t-1} + F_{t-2}$. Now we have two consecutive nonzero entries, so we can do as in the first case. This results in $v_{t-2}+1, 0, v_t - 2, 1$. Note that the sum of all the entries in the vector $\vec{v}$ has not changed. We return to the beginning of the algorithm.

As stated above, the algorithm terminates when there is no minimal value x.

Definition 5.2: Let $\mathcal{A}: V^+ \to V^+$ be the function that assigns to a vector $\vec{v} \in V^+$ the result of applying this algorithm to $\vec{v}$.

Precise Description of the Algorithm: As above, $n = \sum_{i=m}^{M} \lambda_i F_i$.

$\max := M$, $\min := m$;
$t := \max$;
while $(t \geq \min)$ do {
    if $(v_t = 0)$ then $t := t-1$;
    else if $(v_{t-1} = 0$ and $v_t = 1)$ then $t := t-2$;
    else if $(v_{t-1} \neq 0)$ then {
        $v_{t+1} := 1$; $v_t := v_t - 1$; $v_{t-1} := v_{t-1} - 1$;
        if $(v_{t+2} = 0)$ then $t := t+1$;
        else $t := t+2$;
    }

```
    else {
        v_{t+1} := 1; v_t := v_t - 2; v_{t-2} := v_{t-2} + 1;
        if (t - 2 < min) then min := t - 2;
        if (v_{t+2} = 0) then t := t + 1;
        else t := t + 2;
    }
}
```

Remark 5.3: The algorithm $\mathscr{A}$ is designed so that, for all $\vec{v} \in V^+, \sigma(\vec{v}) = \sigma(\mathscr{A}(\vec{v}))$, and $s_t(\mathscr{A}(\vec{v})) = \mathscr{A}(s_t(\vec{v}))$ for all $t \in \mathbb{Z}$. The second equality follows from the fact that the algorithm is independent of the numbering of the coordinates of the vector $\vec{v}$.

Proofs of the results from this section are postponed until Section 6.

Proposition 5.4: The algorithm terminates in a finite number of steps for any vector $\vec{v} \in V^+$. The result $\mathscr{A}(\vec{v})$ is totally reduced.

Proposition 5.5: For all $k \in \mathbb{N}$, $\vec{\beta}(k) = \mathscr{A}(0, \underline{k}, 0)$.

Remark 5.6: If $\vec{v} \in V^+$, and if $\mathscr{A}(\vec{v})$ has no nonzero entries for all coordinates with index less than 2, then $\mathscr{A}(\vec{v}) = \vec{z}(\sigma(\vec{v}))$. For $k \in \mathbb{N}$ and $n \geq \ell(k)+2$, we have $\vec{z}(kF_n) = s_{n-2}(\vec{\beta}(k))$ as in Theorem 2.10.

Example 5.7: We apply the algorithm to $10F_2$ to find the β-expansion of 10.

$$
\begin{array}{ccccccccccc}
 & & & & \underline{10} & & & & & \\
 & & 1 & 0 & \underline{8} & 1 & & & & \\
 & & 1 & 0 & \underline{7} & 0 & 1 & & & \\
 & & 2 & 0 & \underline{5} & 1 & 1 & & & \\
 & & 2 & 0 & \underline{5} & 0 & 0 & 1 & & \\
 & & 3 & 0 & \underline{3} & 1 & 0 & 1 & & \\
 & & 3 & 0 & \underline{2} & 0 & 1 & 1 & & \\
 & & 3 & 0 & \underline{2} & 0 & 0 & 0 & 1 & \\
 & & 4 & 0 & \underline{0} & 1 & 0 & 0 & 1 & \\
1 & 0 & 2 & 1 & \underline{0} & 1 & 0 & 0 & 1 & \\
1 & 0 & 1 & 0 & \underline{1} & 1 & 0 & 0 & 1 & \\
1 & 0 & 1 & 0 & \underline{0} & 0 & 1 & 0 & 1 & = \vec{\beta}(10)
\end{array}
$$

Note that in the 9th row we have in coordinates 2 through 6 the Zeckendorf decomposition of 10, with a 4 in the 0th coordinate. A similar pattern occurs whenever this method is used to find the β-expansion of any positive integer.

Having determined the β-expansion of 10, we can apply Theorem 2.10 and see that $\vec{z}(10F_{5000}) = s_{4998}(1010\underline{0}0101)$. This is much easier than calculating the value of $10F_{5000}$ and applying the greedy algorithm.

Theorem 5.8: For $\vec{v}, \vec{w} \in V^+$ and for $k \in \mathbb{N}$, we have $\mathscr{A}(\mathscr{A}(\vec{v}) + \vec{w}) = \mathscr{A}(\vec{v} + \vec{w})$ and $\mathscr{A}(k\vec{v}) = \mathscr{A}(k\mathscr{A}(\vec{v}))$. In addition, for all $n, m \in \mathbb{N}$, we have $\vec{\beta}(nm) = \mathscr{A}(n\vec{\beta}(m))$ and $\vec{\beta}(n+m) = \mathscr{A}(\vec{\beta}(n) + \vec{\beta}(m))$.

Remark 5.9: The list of β-expansions in Section 2 above can be generated by applying the algorithm to $(0, \underline{k}, 0)$ for each k (see 5.5). Theorem 5.8 provides a more efficient method for deriving the list. Once we have found that $\vec{\beta}(2) = 10\underline{0}1$, we note that

$$\vec{\beta}(3) = \mathcal{A}(0, \underline{3}, 0) = \mathcal{A}(\mathcal{A}(0, \underline{2}, 0) + (0, \underline{1}, 0)) = \mathcal{A}((\vec{\beta}(2)) + (0, \underline{1}, 0)).$$

To move from the β-expansion of $k-1$ to that of k, we need only add one to the underlined entry (which corresponds to β^0) and then apply the algorithm.

6. PROOFS

Lemma 6.1: If a vector $\vec{v} \in V^+$ is reduced to index $s+1$, and if $v_s = 1$, then, when the algorithm is applied, none of the entries with index less than s will be changed until after the algorithm has changed $\vec{v}$ into a vector that is reduced to index s.

Proof of 6.1: We induct upon n using the following induction hypothesis:

If $\vec{v}$ is reduced to index $t+1$ for some t with exactly n nonzero entries (ones) with index greater than t, and if $v_t = 1$, then none of the entries with index less than t will be changed until after the algorithm has changed $\vec{v}$ into a vector that is reduced to index t.

Suppose $n=1$. Then either $v_s = 1$ and $v_{s+1} = 0$, which means that $\vec{v}$ is already reduced to index s, or $v_s = 1 = v_{s+1}$ and $v_{s+i} = 0$ for $i \geq 2$, which means that the algorithm will change the vector so that $v_s = 0 = v_{s+1}$ and $v_{s+2} = 1$ without changing any other entries. The new vector is reduced to index s. Thus, the statement is true for $n=1$.

Now induct on n. Consider the triple $1, v_{s+1}, v_{s+2}$. If this triple is $1, 0, 0$ or $1, 0, 1$, then $\vec{v}$ is already reduced to index s. If the triple is $1, 1, 0$, the algorithm first replaces the triple with $0, 0, 1$, and we can use the inductive hypothesis. We now have a vector that is reduced to index $s+3$ that has $v_{s+2} = 1$. The number of ones with index greater than $s+2$ is one smaller than the number of ones we had originally with index greater than s. Thus, the algorithm does not change the values of entries with index less than $s+2$ until the vector has been changed to a new vector that is reduced to index $s+2$. This means that we will have the triple $0, 0, 1$ either unchanged or replaced with $0, 0, 0$. In either case, the resulting vector is reduced to index s.

Proof of 5.4: If $\vec{v} \in V^+$ is reduced to index s for all s, then the algorithm does not ever change the vector. We have $\mathcal{A}(\vec{v}) = \vec{v}$, and the proposition is proven for that case.

Otherwise, there is a unique $x(\vec{v}) \in \mathbb{Z}$ with $\vec{v}$ reduced to index $x(\vec{v})$ and with $\vec{v}$ not reduced to index $x(\vec{v})-1$. In this case, we define $r(\vec{v})$ to be the sum of all entries of $\vec{v}$ with index less than $x(\vec{v})$. We will see that $r(\vec{v})$ will reach zero in a finite number of steps. This means that the algorithm stops in a finite number of steps and that the vector $\mathcal{A}(\vec{v})$ is in the desired form.

We refer now to the cases given in the *Verbose Description of the Algorithm* in Section 5. The algorithm first assigns $t := x(\vec{v}) - 1$ if $v_x = 0$ and assigns $t := x(\vec{v})$ if $v_x = 1$.

In *Case 1*, the triple $(v_{t-1}, v_t, 0)$ is replaced by $(v_{t-1} - 1, v_t - 1, 1)$ and the new vector is reduced to index $t+2$ with $v_{t+1} = 1$. By Lemma 6.1, we know that the algorithm will next change the vector so that it is reduced to index $t+1$ without changing the values of entries with index less than $t+1$. At this stage, the new vector $\vec{v}$ has a new $x(\vec{v})$-value that is less than or equal to $t+1$. Thus, the new value of $r(\vec{v})$ is at most 2 less than the old value of $r(\vec{v})$.

In *Case 2*, we see that $(v_{t-2}, 0, v_t, 0)$ is replaced by $(v_{t-2}+1, 0, v_t-2, 1)$ and the new vector is reduced to index $t+2$ with $v_{t+1}=1$. By Lemma 6.1, we know that the algorithm will next change the vector so that it is reduced to index $t+1$ without changing the values of entries with index less than $t+1$. At this stage, the new vector $\vec{v}$ has a new $x(\vec{v})$-value that is less than or equal to $t+1$. Thus, the new value of $r(\vec{v})$ is at most 1 less than the old value of $r(\vec{v})$.

In both cases, the value of $r(\vec{v})$ decreases. Thus, the algorithm terminates in a finite number of steps. The vector that results will be reduced to index s for all $s \in \mathbb{Z}$.

Proof of 5.5: This result follows from the work of Grabner et al., but a direct proof is as follows. Because $\beta^i + \beta^{i+1} = \beta^{i+2}$ for all $i \in \mathbb{Z}$, we can replace each F_i in the description of the algorithm with β^{i-2} and replace each σ with σ', where $\sigma'(\vec{v}) = \sum_{i=-\infty}^{\infty} v_i\beta^{i-2}$. Because $kF_2 = k\beta^0$, for the vector $(0, \underline{k}, 0)$ the algorithm will produce the same result either way. Thus, $\mathcal{A}(0, \underline{k}, 0)$ is $\vec{\beta}(k)$.

Proof of 5.8: Let $\vec{x} = \mathcal{A}(\vec{v}) + \vec{w}$, and let $\vec{y} = \vec{v} + \vec{w}$. We first prove that, for all $t \in \mathbb{Z}$, we have $\sigma(s_t(\vec{x})) = \sigma(s_t(\vec{y}))$. We have, using Remark 5.3 and the fact that s_t and σ are linear, $\sigma(s_t(\vec{x})) = \sigma(s_t\mathcal{A}(\vec{v})) + \sigma(s_t(\vec{w})) = \sigma(s_t\mathcal{A}(\vec{v})) + \sigma(s_t(\vec{w})) = \sigma(s_t(\vec{v})) + \sigma(s_t(\vec{w})) = \sigma(s_t(\vec{y}))$.

Next we prove that, for all $t \in \mathbb{Z}$ and for all $k \in \mathbb{N}$, we have $\sigma(s_t(k\vec{v})) = \sigma(s_t(k\mathcal{A}\vec{v}))$. We have $\sigma(s_t(k\vec{v})) = k\sigma(s_t(\vec{v})) = k\sigma(\mathcal{A}(s_t(\vec{v}))) = k\sigma(s_t(\mathcal{A}\vec{v})) = \sigma(s_t(k\mathcal{A}\vec{v}))$.

Theorem 2.10 implies the following. There exist $t_1, t_2, t_3, t_4 \in \mathbb{N}$ such that, for all $i \geq 0$, we have $\mathcal{A}(s_{t_1+i}(\vec{x})) = \vec{z}(\sigma(s_{t_1+i}(\vec{x})))$, $\mathcal{A}(s_{t_2+i}(\vec{y})) = \vec{z}(\sigma(s_{t_2+i}(\vec{y})))$, $\mathcal{A}(s_{t_3+i}(k\vec{v})) = \vec{z}(\sigma(s_{t_3+i}(k\vec{v})))$, and $\mathcal{A}(s_{t_4+i}(k\mathcal{A}\vec{v})) = \vec{z}(\sigma(s_{t_4+i}(k\mathcal{A}\vec{v})))$. Let $t = \max\{t_1, t_2, t_3, t_4\}$.

Using 5.3 again, we see that $s_t(\mathcal{A}(k\vec{v})) = \mathcal{A}(s_t(k\vec{v})) = \vec{z}(\sigma(s_t(k\mathcal{A}\vec{v})))$, and $\vec{z}(\sigma(s_t(k\mathcal{A}\vec{v}))) = \mathcal{A}(s_t(k\mathcal{A}\vec{v})) = s_t(\mathcal{A}(k\mathcal{A}\vec{v}))$. But s_t is one-to-one. Thus, $\mathcal{A}(k\vec{v}) = \mathcal{A}(k\mathcal{A}(\vec{v}))$.

Similarly, $s_t(\mathcal{A}(\vec{x})) = \mathcal{A}(s_t(\vec{x})) = \vec{z}(\sigma(s_t(\vec{x}))) = \vec{z}(\sigma(s_t(\vec{y}))) = \mathcal{A}(s_t(\vec{y})) = s_t(\mathcal{A}(\vec{y}))$. Again because s_t is one-to-one, $\mathcal{A}(\vec{x}) = \mathcal{A}(\vec{y})$. Thus, $\mathcal{A}(\mathcal{A}(\vec{v}) + \vec{w}) = \mathcal{A}(\vec{v} + \vec{w})$.

Next, let $m, n \in \mathbb{N}$. We have that $\vec{\beta}(nm) = \mathcal{A}(0, \underline{nm}, 0) = \mathcal{A}(n(0, \underline{m}, 0)) = \mathcal{A}(n\mathcal{A}(0, \underline{m}, 0)) = \mathcal{A}(n\vec{\beta}(m))$, and also, $\vec{\beta}(nm) = \mathcal{A}(0, \underline{n+m}, 0) = \mathcal{A}((0, \underline{n}, 0) + (0, \underline{m}, 0)) = \mathcal{A}(\mathcal{A}(0, \underline{n}, 0) + \mathcal{A}(0, \underline{m}, 0)) = \mathcal{A}(\vec{\beta}(n) + \vec{\beta}(m))$. Thus, Theorem 5.8 is proven.

Proof of 4.3: We have, for all $k \in \mathbb{N}$, that $\vec{\beta}(L_{2k}) = 10^{2k-1}\underline{0}0^{2k-1}1$ (see Proposition 10 of [5] and apply 2.10). Thus, using Theorem 5.8, we have $\vec{\beta}(2L_{2k}) = \mathcal{A}(2\vec{\beta}(L_{2k})) = \mathcal{A}(20^{2k-1}\underline{0}0^{2k-1}2)$ $= \mathcal{A}(s_{-2k}(0, \underline{2}, 0) + s_{2k}(0, \underline{2}, 0)) = \mathcal{A}(\mathcal{A}s_{-2k}(0, \underline{2}, 0) + \mathcal{A}s_{2k}(0, \underline{2}, 0)) = \mathcal{A}(s_{-2k}\mathcal{A}(0, \underline{2}, 0) + s_{2k}\mathcal{A}(0, \underline{2},))$ $= \mathcal{A}(s_{-2k}\vec{\beta}(2) + s_{2k}\vec{\beta}(2)) = \mathcal{A}((10010^{2k-2}\underline{0}) + (\underline{0}0^{2k-3}1001)) = 10010^{2k-2}\underline{0}0^{2k-3}1001$. Finally, we apply 2.10 to complete the proof.

Proof of 4.4: We have that $\vec{\beta}(L_{2k}) = 10^{2k-1}\underline{0}0^{2k-1}1$, as in the proof of Theorem 4.3. Because $m \leq L_{2k-1}$, we have by 2.5 that $\ell(m), u(m) \leq 2k-2$. Thus, using 4.2, we see that $\vec{\beta}(L_{2k}+m) = \vec{\beta}(L_{2k}) + \vec{\beta}(m)$. Thus, $f[Q(L_{2k}+m)]$ is the number of ones in $\vec{\beta}(L_{2k})$ plus the number of ones in $\vec{\beta}(m)$, and $f[Q(L_{2k}+m)] = 2 + f[Q(m)]$. Because $\ell(L_{2k}) = u(L_{2k}) = 2k$, there can be at most $2k+1$ addends in $Q(m)$. This proves the last inequality.

Proof of 4.5: Using 4.3, we have that $\vec{\beta}(2L_{2k}) = 10010^{2k-2}\underline{0}0^{2k-3}1001$. Because $m \le L_{2k-3}$, we have by 2.5 that $\ell(m), u(m) \le 2k-4$. Using 4.2, we see that $\vec{\beta}(2L_{2k}+m) = \vec{\beta}(2L_{2k}) + \vec{\beta}(m)$. Thus, $f[Q(2L_{2k}+m)]$ is the number of ones in $\vec{\beta}(2L_{2k})$ plus the number of ones in $\vec{\beta}(m)$, and $f[Q(2L_{2k}+m)] = 4 + f[Q(m)]$.

Proof of 4.6: We have $\vec{\beta}(L_{2k}) = 10^{2k-1}\underline{0}0^{2k-1}1$, as in the proof of 4.3. Let $\vec{v} = s_{-2k}(0\underline{1}0)$, and let $s_{2k}(0\underline{1}0)$. Then $\vec{\beta}(L_{2k}) = \vec{v} + \vec{w}$, and so by Theorem 5.8 we have $\vec{\beta}(mL_{2k}) = \mathscr{A}(m\vec{v} + m\vec{w}) = \mathscr{A}(\mathscr{A}(m\vec{v}) + \mathscr{A}(m\vec{w})) = \mathscr{A}(\mathscr{A}(s_{-2k}(0, \underline{m}, 0)) + \mathscr{A}(s_{2k}(0, \underline{m}, 0))) = \mathscr{A}(s_{-2k}(\mathscr{A}(0, \underline{m}, 0)) + s_{2k}(\mathscr{A}(0, \underline{m}, 0))) = \mathscr{A}(s_{-2k}(\vec{\beta}(m)) + s_{2k}(\vec{\beta}(m)))$. By Fact 4.2 and Theorem 2.5, $s_{-2k}(\vec{\beta}(m)) + s_{2k}(\vec{\beta}(m))$ is totally reduced whenever $m \le L_{2k-1}$. Thus, $\vec{\beta}(mL_{2k}) = s_{-2k}(\vec{\beta}(m)) + s_{2k}(\vec{\beta}(m))$ for $1 \le m \le L_{2k-1}$. Whenever $m \le L_{2k-1}$, the two shifted β-expansions of m will not overlap, and in fact there will be zeros in the coordinates corresponding to β^{-1}, β^0, and β^1. Thus, mL_{2k} does not have Property $\mathscr{P}$. When a one is inserted in the coordinate corresponding to β^0, the resulting vector is totally reduced and equals $\vec{\beta}(mL_{2k}+1)$ (see 5.9). Thus, $mL_{2k}+1$ does have Property $\mathscr{P}$.

Proof of 4.7: We have, for all $k \in \mathbb{N}$, that $\vec{\beta}(L_{2k+1}) = (10)^k\underline{1}(01)^k$ (see (3.1) of [5]). Thus, using 5.8, we have $\vec{\beta}(L_5 + L_3) = \mathscr{A}(\vec{\beta}(L_5) + \vec{\beta}(L_3)) = \mathscr{A}(10202\underline{0}201) = 100100\underline{1}01001$, so, the result holds for $k = 2$. We induct on k. We assume that $\vec{\beta}(L_{2k-1} + L_{2k-3}) = 100100(10)^{k-3}\underline{1}(01)^{k-2}001$. Fact 4.2 implies that $\vec{\beta}(L_{2k} + L_{2k-2}) = 1010^{2k-3}\underline{0}0^{2k-3}101$. Therefore, $\vec{\beta}(L_{2k+1} + L_{2k-1}) = \vec{\beta}(L_{2k-1} + L_{2k} + L_{2k-3} + L_{2k-2}) = \mathscr{A}(\vec{\beta}(L_{2k-1} + L_{2k-3}) + \vec{\beta}(L_{2k} + L_{2k-2})) = \mathscr{A}(201100(10)^{k-3}\underline{1}(01)^{k-2}0111) = \mathscr{A}(201100(10)^{k-3}\underline{1}(01)^{k-2}01001)$. Note that this last vector is reduced to index $-2k+5$. The algorithm will not change any of the entries except that the 20110 that occurs on the left changes to 1001001. Thus, $\vec{\beta}(L_{2k+1} + L_{2k-1}) = 10010010(10)^{k-3}\underline{1}(01)^{k-2}01001$, and the induction is completed.

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ON CERTAIN SUMS OF FUNCTIONS OF BASE B EXPANSIONS

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0. INTRODUCTION

Let $s_b(i)$ denote the base 10 sum of the digits in the base b representation of the nonnegative integer i and $L_b(i)$ denote the number of large digits ($\lceil b/2 \rceil$ or more) in the base b representation of the nonnegative integer i. For example, $s_{10}(4567)=22$, $s_7(7079)=17$ since $7079=26432_7$, and $s_2(19)=3$ since $19=10011_2$. In addition, $L_{10}(4567)=3$, $L_7(7079)=2$, and $L_2(19)=3$. The mathematical literature has many instances of sums involving s_b and L_b. Bush [1] showed that

$$\frac{1}{x}\sum_{n<x} s_b(n) \sim \frac{b-1}{2\log b}\log x.$$

Here, $\log x$ denotes the natural logarithm of x. Mirsky [7], and later Cheo and Yien [2], proved that

$$\frac{1}{x}\sum_{n<x} s_b(n) = \frac{b-1}{2\log b}\log x + O(1).$$

Trollope [9] discovered the following result. Let $g(x)$ be periodic of period one and defined on [0, 1] by

$$g(x)=\begin{cases}\frac{1}{2}x, & 0\le x\le \frac{1}{2},\\ \frac{1}{2}(1-x), & \frac{1}{2}<x\le 1,\end{cases}$$

and let

$$f(x)=\sum_{i=0}^{\infty}\frac{1}{2^i}g(2^i x).$$

Now, if $n=2^m(1+x)$, $0\le x<1$, then

$$\sum_{i<n} s_2(i) = \frac{1}{2\log 2}n\log n - E_2(n),$$

where

$$E_2(n)=2^{m-1}\left\{2f(x)+(1+x)\frac{\log(1+x)}{\log 2}-2x\right\}.$$

In addition, it was shown in [6] that

$$\sum_{i=1}^{\infty}\frac{L_{10}(2^i)}{2^i}=\frac{2}{9}.$$

We will discuss some other sums involving s_b and L_b. In particular, we will give formulas for

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^m \quad\text{and}\quad \frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^m,$$

where m and n are positive integers. Then, we will find a formula for

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1} s_b(i)\cdot L_b(i).$$

We define $C_b(x; y)$ to be the sum of the carries when the positive integer x is multiplied by y, using the normal multiplication algorithm in base b arithmetic. That is, we convert x and y to base b and then multiply in base b. In this algorithm, we consider the carries above the numbers as well as in the columns. We will prove that

$$\sum_{i=1}^{\infty}\frac{C_b(a; a^i)}{(s_b(a))^i}=\frac{s_b(a)}{b-1}.$$

We will conclude the paper with some open questions.

1. FIRST SUM

To compute

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^m,$$

we begin with the function

$$f(x)=(\underbrace{1+\cdots+1}_{\lceil b/2\rceil \text{ times}}+\underbrace{e^x+\cdots+e^x}_{\lfloor b/2\rfloor \text{ times}})^n=(\lceil b/2\rceil+\lfloor b/2\rfloor e^x)^n.$$

The motivation for this function comes from the fact that in the base b representation of $i=i_n\ldots i_2i_1$, the j^{th} digit of i, i_j, is either small or large and thus contributes 0 or 1 to the number of large digits in i. Expanding the product, we see that there is a 1-1 correspondence between the numbers $0\le i\le b^n-1$ and the b^n terms $1\cdot e^{L_b(i)x}$. Therefore,

$$f(x)=(\lceil b/2\rceil+\lfloor b/2\rfloor e^x)^n=\sum_{i=0}^{b^n-1}1\cdot e^{L_b(i)x}.$$

Thus,

$$f^{(m)}(x)=\sum_{i=0}^{b^n-1}(L_b(i))^m e^{L_b(i)x},$$

and so we have that

$$f^{(m)}(0)=\sum_{i=0}^{b^n-1}(L_b(i))^m.$$

To continue our discussion, we need the idea of Stirling numbers of the first and second kinds. A discourse on this subject can be found in [3]. A Stirling number of the second kind, denoted by $\left\{ {n \atop k} \right\}$, symbolizes the number of ways to partition a set of n things into k nonempty subsets. A Stirling number of the first kind, denoted by $\left[{n \atop k} \right]$, counts the number of ways to arrange n objects into k cycles. These cycles are cyclic arrangements of the objects. We will use the notation $[A, B, C, D]$ to denote a clockwise arrangement of the four objects A, B, C, and D in a circle. For example, there are eleven different ways to make two cycles from four elements:

$$\begin{array}{llll} [1,2,3][4], & [1,2,4][3], & [1,3,4][2] & [2,3,4][1], \\ [1,3,2][4], & [1,4,2][3], & [1,4,3][2], & [2,4,3][1], \\ [1,2][3,4] & [1,3][2,4], & [1,4][2,3]. & \end{array}$$

Hence, $\left[{4 \atop 2}\right] = 11$. Now it can be shown, by induction on m, that

$$f^{(m)}(x) = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} n^{\underline{j}} (\lfloor b/2 \rfloor e^x)^j (\lceil b/2 \rceil + \lfloor b/2 \rfloor e^x)^{n-j},$$

where $n^{\underline{j}} = n(n-1)\cdots(n-j+1)$. The last quantity is known as the j^{th} falling factorial of n. A discussion of this idea can be found in [3]. Thus,

$$\sum_{i=0}^{b^n-1} (L_b(i))^m = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} n^{\underline{j}} \lfloor b/2 \rfloor^j \cdot b^{n-j} = b^n \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \left(\frac{\lfloor b/2 \rfloor}{b} \right)^j n^{\underline{j}}.$$

Since $n^{\underline{j}} = j!\binom{n}{j}$, we have proved the following theorem.

Theorem 1: Let m and n be nonnegative integers. Then

$$\frac{1}{b^n} \sum_{i=0}^{b^n-1} (L_b(i))^m = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \left(\frac{\lfloor b/2 \rfloor}{b} \right)^j \cdot j! \binom{n}{j}.$$

To illustrate this theorem, if $b = 5$, $m = 3$, and n is a nonnegative integer, then

$$\frac{1}{5^n} \sum_{i=0}^{5^n-1} (L_5(i))^3 = \frac{8}{125} n^3 + \frac{36}{125} n^2 + \frac{6}{125} n.$$

2. SECOND SUM

Let m and n be positive integers. The determination of the sum

$$\frac{1}{10^n} \sum_{i=0}^{10^n-1} (s_{10}(i))^m$$

was an open question in [4]. In [10], David Zeitlin presented the following answer to the problem in base 10. He stated that if $B_i^{(n)}$ denotes Bernoulli numbers of order n, where

$$\binom{n-1}{i} \cdot B_i^{(n)} = \left[{n \atop n-i} \right],$$

then

$$\frac{1}{10^n} \sum_{i=0}^{10^n-1} (s_{10}(i))^m = \binom{n+m}{m}^{-1} \sum_{i=0}^{m} 10^i \cdot \binom{n+m}{m-i} \left\{ {n+i \atop n} \right\} \cdot B_{m-i}^{(n)}.$$

To compute

$$\frac{1}{b^n} \sum_{i=0}^{b^n-1} (s_b(i))^m,$$

we make use of the function $(g(x))^n$, where $g(x) = 1 + e^x + e^{2x} + \cdots + e^{(b-1)x}$. The motivation for this function comes from the fact that in the base b representation of $i = i_n \ldots i_2 i_1$, the j^{th} digit of i, i_j, contributes i_j to the digital sum of i. Expanding the product, we see that there is a 1-1 correspondence between the numbers $0 \le i \le b^n - 1$ and the b^n terms $1 \cdot e^{s_b(i)x}$. Therefore,

$$(g(x))^n = \sum_{i=0}^{b^n-1} 1 \cdot e^{s_b(i)x}.$$

Thus, for $m > 1$, we have

$$\frac{d^m}{dx^m}(g(x))^n = \sum_{i=0}^{b^n-1}(s_b(i))^m e^{s_b(i)x},$$

and so we have that

$$\frac{d^m}{dx^m}(g(0))^n = \sum_{i=0}^{b^n-1}(s_b(i))^m.$$

Now we need Faá di Bruno's formula [8]. This formula states that if $f(x)$ and $g(x)$ are functions for which all the necessary derivatives are defined and m is a positive integer, then

$$\frac{d^m}{dx^m}f(g(x)) = \sum_{n_1+2n_2+\cdots+mn_m=m} \frac{m!}{n_1!\ldots n_m!}\left(\frac{d^{n_1+\cdots+n_m}}{dx^{n_1+\cdots+n_m}}f\right)(g(x)) \cdot\left(\frac{\frac{d}{dx}g(x)}{1!}\right)^{n_1}\cdots\left(\frac{\frac{d^m}{dx^m}g(x)}{m!}\right)^{n_m},$$

where $n_1, n_2, \ldots, n_m$ are nonnegative integers.

It follows that

$$\frac{d^m}{dx^m}(g(x))^n = \sum_{n_1+2n_2+\cdots+mn_m=m} n^{\underline{n_1+n_2+\cdots+n_m}} g(x)^{n-n_1-n_2-\cdots-n_m} \cdot\frac{m!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(m!)^{n_m}n_m!}(g^{(1)}(x))^{n_1}(g^{(2)}(x))^{n_2}\cdots(g^{(m)}(x))^{n_m},$$

where m is a positive integer and $n_1, n_2, \ldots, n_m$ are nonnegative integers. Thus,

$$\frac{d^m}{dx^m}(g(0))^n = \sum_{n_1+2n_2+\cdots+mn_m=m} n^{\underline{n_1+n_2+\cdots+n_m}} g(0)^{n-n_1-n_2-\cdots-n_m} \cdot\frac{m!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(m!)^{n_m}n_m!}(g^{(1)}(0))^{n_1}(g^{(2)}(0))^{n_2}\cdots(g^{(m)}(0))^{n_m}.$$

Equating the two expressions for $\frac{d^m}{dx^m}(g(0))^n$ and simplifying gives the following theorem.

Theorem 2: Let n and m be positive integers and $n_1, n_2, \ldots, n_m$ be nonnegative integers. Then

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^m = \sum_{n_1+2n_2+\cdots+mn_m=m} \frac{m!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(m!)^{n_m}n_m!} \cdot(g^{(1)}(0)/b)^{n_1}(g^{(2)}(0)/b)^{n_2}\cdots(g^{(m)}(0)/b)^{n_m}n^{\underline{n_1+n_2+\cdots+n_m}},$$

where $g^{(i)}(0) = 0^i + 1^i + \cdots + (b-1)^i$.

It might be noted that, in [4], formulas for the sums

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1}(s_{10}(i))^m$$

were given for $m = 0, 1, \ldots, 8$. Using the formulas we just derived, we have the new formula for $m = 9$, that is,

$$\begin{aligned}\frac{1}{10^n}\sum_{i=0}^{10^n-1}(s_{10}(i))^9 &= \frac{387420489}{512}\cdot n^9 + \frac{1420541793}{128}\cdot n^8\\ &+\frac{12153524229}{256}\cdot n^7 + \frac{7215728751}{160}\cdot n^6\\ &-\frac{30325460319}{512}\cdot n^5 - \frac{2286016425}{128}\cdot n^4\\ &+\frac{30058716303}{640}\cdot n^3 - \frac{2699999973}{160}\cdot n^2.\end{aligned}$$

3. THIRD SUM

We next try to tackle the sum

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1} s_b(i)\cdot L_b(i).$$

The base 10 result is

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s_{10}(i)\cdot L_{10}(i) = \frac{9}{4}n^2 + \frac{5}{4}n.$$

From the previous two sections, we have established the formulas

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^2 = \frac{b^2-2b+1}{4}n^2 + \frac{b^2-1}{12}n$$

and

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^2 = \left(\frac{\lfloor b/2\rfloor}{b}\right)^2 n^2 + \left(\left(\frac{\lfloor b/2\rfloor}{b}\right) - \left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)n.$$

Now, consider the function

$$h(x) = (1+e^x+e^{2x}+\cdots+e^{(\lceil b/2\rceil-1)x}+e^{(\lceil b/2\rceil+1)x}+\cdots+e^{bx})^n.$$

The motivation for this function comes from the fact that, in the base b representation of $i = i_n \ldots i_2 i_1$, the j^{th} digit of i, i_j, contributes either i_j or i_j+1, depending upon whether or not the i_j^{th} digit is small or large, respectively. That is, the $h(x)$ function considers both the digital sum and the number of large digits, compared to the $g(x)$ function, where we were only concerned with the digital sum. Expanding the product, we see that there is a 1-1 correspondence between the numbers $0 \le i \le b^n-1$ and the b^n terms $1\cdot e^{(s_b(i)+L_b(i))x}$. Therefore,

$$\begin{aligned}h(x) &= (1+e^x+e^{2x}+\cdots+e^{(\lceil b/2\rceil-1)x}+e^{(\lceil b/2\rceil+1)x}+\cdots+e^{bx})^n\\ &= \sum_{i=0}^{b^n-1} 1\cdot e^{(s_b(i)+L_b(i))x}.\end{aligned}$$

Thus,

$$h''(x) = \sum_{i=0}^{b^n-1}(s_b(i)+L_b(i))^2 e^{(s_b(i)+L_b(i))x},$$

and so we have that

$$h''(0)=\sum_{i=0}^{b^n-1}(s_b(i)+L_b(i))^2.$$

Computing $h''(0)$ and dividing by b^n, we obtain

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i)+L_b(i))^2=n(n-1)b^{-2}\cdot\left(\frac{b(b+1)}{2}-\left\lceil\frac{b}{2}\right\rceil\right)^2+nb^{-1}\cdot\left(\frac{b(b+1)(2b+1)}{6}-\left\lceil\frac{b}{2}\right\rceil^2\right)$$

$$=\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2n^2+\left(\left(\frac{2b^3+3b^2+b-6\lceil b/2\rceil^2}{6b}\right)-\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2\right)n.$$

But,

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}s_b(i)\cdot L_b(i)=\frac{1}{b^n}\sum_{i=0}^{b^n-1}\frac{(s_b(i)+L_b(i))^2-(s_b(i))^2-(L_b(i))^2}{2}$$

$$=\frac{1}{2}\left(\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i)+L_b(i))^2-\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^2-\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^2\right).$$

Substituting our three formulas in the above expression, we have

$$\begin{aligned}\frac{1}{b^n}\sum_{i=0}^{b^n-1}s_b(i)\cdot L_b(i)=&\frac{1}{2}\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2n^2\\&+\frac{1}{2}\left(\left(\frac{2b^3+3b^2+b-6\lceil b/2\rceil^2}{6b}\right)-\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2\right)n\\&-\frac{1}{2}\left(\frac{b^2-2b+1}{4}n^2+\frac{b^2-1}{12}n\right)\\&-\frac{1}{2}\left(\left(\frac{\lfloor b/2\rfloor}{b}\right)^2n^2+\left(\left(\frac{\lfloor b/2\rfloor}{b}\right)-\left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)n\right).\end{aligned}$$

Collecting like terms, we have the following theorem.

Theorem 3: Let n be a positive integer. Then

$$\begin{aligned}\frac{1}{b^n}\sum_{i=0}^{b^n-1}s_b(i)\cdot L_b(i)=&\frac{1}{2}\left(\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2-\frac{b^2-2b+1}{4}-\left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)n^2\\&+\frac{1}{2}\left(\left(\frac{2b^3+3b^2+b-6\lceil b/2\rceil^2}{6b}\right)-\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2\right.\\&\left.-\frac{b^2-1}{12}-\left(\left(\frac{\lfloor b/2\rfloor}{b}\right)-\left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)\right)n.\end{aligned}$$

Furthermore, we have the following corollary.

Corollary: Let n be a positive integer and b be a positive even integer. Then

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1} s_b(i)\cdot L_b(i) = \frac{b-1}{4}n^2 + \frac{b}{8}n.$$

4. FOURTH SUM

We next determine the sum

$$\sum_{i=1}^{\infty}\frac{C_b(a;a^i)}{(s_b(a))^i}$$

where $C_b(x;y)$ denotes the sum of the carries when the positive integer x is multiplied by y, using the normal multiplication algorithm in base b arithmetic.

Noting that $L_{10}(2^i) = C_{10}(2;2^i)$, this sum is a generalization of the sum

$$\sum_{i=1}^{\infty}\frac{L_{10}(2^i)}{2^i}$$

which was a problem considered in [6].

To compute this sum, we need the following lemma.

Lemma 1: Let d be a digit in base b and y be any positive integer. Then

$$C_b(d;y) = \frac{1}{b-1}(d\cdot s_b(y) - s_b(dy)).$$

Proof: The proof of Lemma 1 relies on Legendre's theorem,

$$s_b(n) = n - (b-1)\sum_{t\ge1}\left\lfloor\frac{n}{b^t}\right\rfloor,$$

where n is a positive integer. Legendre's theorem and its proof can be found in [5].

To prove Lemma 1, we note that

$$s_b(y) = y - (b-1)\sum_{t\ge1}\left\lfloor\frac{y}{b^t}\right\rfloor \quad\text{and}\quad s_b(dy) = dy - (b-1)\sum_{t\ge1}\left\lfloor\frac{dy}{b^t}\right\rfloor.$$

Multiplying the first equality by d and subtracting the second equality from the first yields

$$d\cdot s_b(y) - s_b(dy) = (b-1)\sum_{t\ge1}\left(\left\lfloor\frac{dy}{b^t}\right\rfloor - d\left\lfloor\frac{y}{b^t}\right\rfloor\right).$$

Dividing by $b-1$ and observing that the sum is $C(d;y)$ gives us the result.

Armed with Lemma 1, we have the next lemma.

Lemma 2: Let $s_b(n)$ denote the base b digital sum of the positive integer n and $C_b(a;a^i)$ denote the base b carries in the normal multiplication algorithm of multiplying a and a^i. Let x and y be positive integers. Then $s_b(x\cdot y) = s_b(x)\cdot s_b(y) - (b-1)C_b(x;y)$.

Proof: Consider $x = \sum_{i=0}^{n} x_i b^i$, the base b representation of x. Then, counting the top carries from the multiplication using Lemma 1 and counting the bottom carries from the addition, we have

$$\begin{aligned}
C_b(x;y) &= \frac{1}{b-1}\sum_{i=0}^{n}(x_i s_b(y) - s_b(x_i y)) + \sum_{t\geq 1}\left(\left\lfloor \frac{\sum_{i=0}^{n} x_i b^i y}{b^t}\right\rfloor - \sum_{i=0}^{n}\left\lfloor \frac{x_i b^i y}{b^t}\right\rfloor\right) \\
&= \frac{1}{b-1}s_b(x)s_b(y) - \frac{1}{b-1}\sum_{i=0}^{n} s_b(x_i y) + \sum_{t\geq 1}\left\lfloor \frac{xy}{b^t}\right\rfloor - \sum_{i=0}^{n}\sum_{t\geq 1}\left\lfloor \frac{x_i b^i y}{b^t}\right\rfloor \\
&= \frac{1}{b-1}s_b(x)s_b(y) - \frac{1}{b-1}\sum_{i=0}^{n} s_b(x_i y) + \frac{1}{b-1}(xy - s_b(xy)) \\
&\quad - \sum_{i=0}^{n}\frac{1}{b-1}(x_i b^i y - s_b(x_i b^i y)) \\
&= \frac{1}{b-1}(s_b(x)s_b(y) - s_b(xy)).
\end{aligned}$$

Next, applying Lemma 2, we obtain $s_b(a^{i+1}) = s_b(a)\cdot s_b(a^i) - (b-1)C_b(a;a^i)$. Thus, if n is a positive integer,

$$\begin{aligned}
\sum_{i=1}^{n}\frac{C_b(a;a^i)}{s_b(a)^i} &= \frac{1}{b-1}\sum_{i=1}^{n}\left(\frac{s_b(a^i)}{(s_b(a))^{i-1}} - \frac{s_b(a^{i+1})}{(s_b(a))^i}\right) \\
&= \frac{1}{b-1}s_b(a) - \frac{1}{b-1}\frac{s_b(a^{n+1})}{(s_b(a))^n}.
\end{aligned}$$

Therefore, we have the following theorem.

Theorem 4: Let $s_b(n)$ denote the base b digital sum of the positive integer n and $C_b(a;a^i)$ denote the base b carries in the normal multiplication algorithm of multiplying a and a^i. Then

$$\sum_{i=1}^{\infty}\frac{C_b(a;a^i)}{(s_b(a))^i} = \frac{s_b(a)}{b-1}.$$

To illustrate this theorem, if $b=3$ and $a=14$, then

$$\sum_{i=1}^{\infty}\frac{C_3(14;14^i)}{4^i} = 2.$$

That is, if we count the carries in multiplying $14 = 112_3$ by powers of 14, using the usual base 3 multiplication algorithm, and divide by the appropriate power of 4, the result is 2. In fact, the infinite series begins with

$$\frac{5}{4} + \frac{7}{16} + \frac{14}{64} + \frac{18}{256} + \cdots.$$

5. QUESTIONS

Some open questions remain. Can a formula be found for

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^{n_1}\cdot(L_b(i))^{n_2},$$

where n, n_1, and n_2 are positive integers? Can a formula be found for

$$\frac{1}{b^n}\sum_{i=1}^{b^n-1}\frac{1}{s_b(i)}?$$

Also, can a formula be found for

$$\frac{1}{b_1^n}\sum_{i=0}^{b_1^n-1}s_{b_1}(i)\cdot s_{b_2}(i),$$

where $b_1 = b_2^m$? What about a formula for

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}s_b(s_b(i))?$$

Finally, find the sum

$$\sum_{i=1}^{\infty}\frac{s_b(a^i)}{a^i}.$$

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Among key words associated with Fibonacci numbers are *Lucas* numbers, *Binet* form, *Pell* numbers, *Wythoff* pairs, and *Zeckendorf* sums. The term *Zeckendorf sum* entered the language some time after 1939—a surprisingly late date, as it would appear that most named mathematical entities as simple as these sums were named centuries ago!

Actually, it was not until the early 1950s that Zeckendorf sums were first discussed in a publication, and not until 1972 that the chronology was clarified, by Zeckendorf himself. In an introductory Summary [7], he writes:

> Every natural number can be represented as a sum of distinct and non consecutive Fibonacci numbers or of non consecutive Lucas numbers. Using Fibonacci numbers, such a representation is always unique.

It is the unique representations that are now known as Zeckendorf sums, and their existence and uniqueness, as Zeckendorf's theorem. Shortly after the above-mentioned Summary, Zeckendorf indicates that these sums date from the year 1939.

In 1952, C. G. Lekkerkerker published an account [5] of Zeckendorf's theorem. This article, in Dutch, led to a longer work in 1960, in the prestigious *Journal of the London Mathematical Society*; there, D. E. Daykin [2] proves that the Fibonacci numbers form the *only* sequence of natural numbers for which Zeckendorf's theorem holds. Daykin's paper is cited by many later papers on Zeckendorf sums and their generalizations.

In view of the widespread currency of the terms "Zeckendorf sum" and "Zeckendorf representation," it is surprising how little is known about the life of Zeckendorf. Fortunately, Jean Godeaux [3] was able to obtain the reminiscences of P. R. Charlier, a retired engineer who knew Zeckendorf when both were prisoners of war. In the material that follows, Mr. Charlier's account is supplemented with information provided by Centre de Documentation Historique, Forces Armees Belges [6].

At the end of the nineteenth century, Dr. Abraham Zeckendorf, a dentist, and his wife, Henriette van Gelder, set up his practice in Liège, Belgium. Dr. Zeckendorf was a Dutch citizen and an active Jew. In May 1940, because of the Nazi invasion of Belgium, the Zeckendorf family fled to Nice, France.

The son, Edouard, born in Liège on May 2, 1901, was recognized early as a brilliant student. Speaking fluently both Dutch, the language of the Zeckendorf family, and French, the official language of Liège, Edouard attended the Royal Athenaeum of Liège from 1912 to 1919. There he studied Greek, Latin, English, German, mathematics, and drawing.

Soon after the end of World War I, Edouard enrolled in the University of Liège, where, in 1925, he became a medical doctor, specializing in surgery and delivery. In the same year, he became an officer in the Belgian army. Between 1927 and 1931 he obtained a License for Dental Surgery, and in 1929 he married Elsa Schwers, a nurse and member of the Reformed Church, born July 2, 1889 in Liège.

Elsa, like Edouard, was an artist. Before the marriage, Elsa had produced many fine drawings of Paris, both in pencil and charcoal. She continued to paint many oils; during his free time, Edouard continued his own drawing and mathematical investigations. The two often attended art exhibits and were friends of the best local artists. They had no children, and Elsa's sudden death in 1944 was extremely painful for Edouard.

According to Mr. Charlier, from 1930 to 1940 Dr. Zeckendorf was in charge of the military Hôpital Saint Laurent in Liège. On May 28, 1940, with the surrender of the Belgian army, Dr. Zeckendorf was taken prisoner by the Germans. As a POW, he stayed in several oflags until 1945. (An *Oflag* was an *Officierenlager*, a camp for imprisoned officers, as contrasted to a *Stalag*.)

During his captivity Dr. Zeckendorf provided medical care to allied POWs. He also sketched soldiers representing the many various peoples of the Soviet Union. Mr. Charlier wrote that Dr. Zeckendorf escaped from a camp, and afterwards, his status as a nonpracticing Jew was ignored by the Germans. Records described in [6] confirm that Dr. Zeckendorf did attempt an escape, but no details are given in [6]. Both Mr. Charlier's account and the military records indicate that Dr. Zeckendorf chose to continue his care of POWs in Germany despite opportunities to return to his home.

After his liberation, Dr. Zeckendorf returned to Liège, where he found the family house occupied by the army. At first, the house had been deemed "abandoned" by the Germans, who occupied it and stole or destroyed the furnishings and other possessions. Later, the house had been occupied by Americans. Dr. Zeckendorf decided to go to Nice to care for his aging mother, his father having died only a few months after Elsa had died.

From March 16, 1949, to March 23, 1950, Dr. Zeckendorf headed the Belgian mission near the United Nations Commission for India and Pakistan. He was in charge of the inspection of the 500-mile long cease-fire line. When he returned from India, Dr. Zeckendorf brought with him many original sketches and photographs of the Himalayan foothills.

During his military career, Dr. Zeckendorf was honored with the following awards: Officer of the Order of the Crown (1946); Prisoner of War Medal (1946); Officer of the Order of Leopold (1949); Officer of the Order of Leopold II (1950).

Dr. Zeckendorf married Marie Jeanne Lempereur in Brussels, Belgium, on July 27, 1959. Miss Lempereur's family was Belgian but had lived in Manitoba, Canada, at the time of her birth in 1908. When she was a young girl, the family had returned to Belgium. During the eighteen years of their marriage, Dr. and Mrs. Zeckendorf enjoyed an active life, visiting exhibits and museums, traveling and visiting cities of artistic interest, and reading. After his second wife's death in July 1977, Dr. Zeckendorf continued his activities, even after the discovery of cancer. Near the end of his life, he often visited friends in Liège, and he regularly attended the monthly meetings of the Société Royale des Sciences de Liège. He died in Liège on May 16, 1983.

It appears that [8] was Dr. Zeckendorf's only publication in English, whereas some thirty others in French were published in *Mathesis* and *Bulletin de la Société Royale des Sciences de Liège*. These include several articles on each of the following subjects: Fibonacci and Lucas numbers, primes, quadratic equations, and combinatorial arrangements of letters. As citations of these publications can be downloaded easily from the MathSci database, they are not listed here.

Dr. Zeckendorf published one paper [8] in *The Fibonacci Quarterly*. In the same issue, the founding editor, V. E. Hoggatt, Jr., also published a paper [4] dedicated to Dr. E. Zeckendorf. A few of Dr. Zeckendorf's letters to Dr. Hoggatt, dating from July 1971 to February 1973, survive. They reveal a warm friendship and enthusiasm for recurrence sequences. Their tone is, of course, much less formal than other materials unearthed for this sketch. Of particular note is the distinctive signature found on all the letters and reproduced here:

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The author thanks Professor emeritus of zoology Jean E. A. Godeaux of the Société Royale des Sciences de Liège for acquiring and passing along information from Mr. Charlier. It was Professor Godeaux's father, Lucien Godeaux, who formally introduced Edouard Zeckendorf to the Société Royale des Sciences de Liège as an associate member on June 20, 1957. After several diligent searchers for a photograph had given up, Dr. Joseph Lahr discovered, upon visiting a military archive in person, the passport photograph of Dr. Zeckendorf seen above. Finally, a most cordial thank you to Herta Hoggatt and Marjorie Bicknell-Johnson for providing copies of Dr. Zeckendorf's letters to Vern Hoggatt.

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DUCCI-PROCESSES OF 5-TUPLES

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Ducci-sequences are successive iterations of the function

$$f(x_1, x_2, \dots, x_n) = (|x_1 - x_2|, |x_2 - x_3|, \dots, |x_n - x_1|).$$

Note that $f: Z^n \to Z^n$, where Z^n is the set of n-tuples with integer entries. Since they were introduced in 1937, Ducci-sequences, also known as the n-number game, have been studied extensively (e.g., [1], [3], [5], [6], [7], [8]). In 1982, Wong suggested a generalization which he called Ducci-processes [12]. Ducci-processes are successive iterations of a function $g: Z^n \to Z^n$ which satisfies the following three conditions:

(i) there exists a function $h: Z^2 \to Z$;
(ii) $g(x_1, x_2, \dots, x_n) = (h(x_1, x_2), h(x_2, x_3), \dots, h(x_n, x_1))$
(iii) the n entries of $g^k(x_1, x_2, \dots, x_n)$ are bounded for all k.

Note that Ducci-sequences are an example of a Ducci-process with $h(x, y) = |x - y|$.

In [4], Engel introduced the Ducci-process D_m, where $h(x, y) = (x + y) \pmod{m}$:

$$D_m(x_1, x_2, \dots, x_n) = (x_1 + x_2 \pmod{m}, x_2 + x_3 \pmod{m}, \dots, x_n + x_1 \pmod{m}).$$

Since numbers are reduced modulo m, we can view the domain and range of D_m as Z_m^n, the set of n-tuples with entries from Z_m. Because Z_m^n is a finite set, the iterations $\{D_m^j(X)\}$ will eventually repeat, resulting in a cycle. As with Ducci-sequences, the goal is to characterize cycles in terms of n and m. This is done in [9] for $n = 4$.

We will begin with some general observations about D_m. Then we will focus on 5-tuples, where the Fibonacci numbers play a prominent role.

GENERAL OBSERVATIONS

To simplify notation, we define two functions on Z^n. For $X = (x_1, x_2, \dots, x_n) \in Z^n$,

$$D(X) = D(x_1, x_2, \dots, x_n) = (x_1 + x_2, x_2 + x_3, \dots, x_n + x_1),$$
$$H(X) = H(x_1, x_2, \dots, x_n) = (x_2, x_3, \dots, x_n, x_1).$$

We write $D(X) \equiv (x_1 + x_2, x_2 + x_3, \dots, x_n + x_1) \pmod{m}$ in lieu of $D_m(X)$. Note that D and H are commutative, linear operators; moreover, $D(X) = X + H(X)$. Iterations of D and H are defined as $D^j(X) = D(D^{j-1}(X))$ and $H^j(X) = H(H^{j-1}(X))$, respectively. Thus, $H^n(X) = X$ and $H^j(X) = H^{j \pmod{n}}(X)$.

A further simplification occurs with the introduction of the special n-tuple $A = (1, 0, \dots, 0)$. Using the function H, we can write $X = (x_1, x_2, \dots, x_n)$ in terms of A:

$$\begin{aligned} X &= x_1 \cdot (1, 0, \dots, 0) + x_2 \cdot (0, 1, \dots, 0) + \cdots + x_n \cdot (0, 0, \dots, 1) \\ &= x_1 \cdot A + x_2 \cdot H^{n-1}(A) + x_3 \cdot H^{n-2}(A) + \cdots + x_{n-1} \cdot H^2(A) + x_n \cdot H(A) \\ &= \textstyle\sum_{1 \le i \le n} x_i H^{n+1-i}(A). \end{aligned}$$

Hence,

$$D(X) = \Sigma_{1\le i\le n} x_i H^{n+1-i}(D(A)).$$

Similarly, $D^j(X)$ can be written in terms of $D^j(A)$.

As we noted above, the iterations of X will eventually lead to a cycle. That is, there exist nonnegative integers l and s for which $D^{l+s}(X) \equiv D^s(X) \pmod{m}$. If l and s are as small as possible, then we will write $\mathfrak{l}_m(X) = l$ and $\mathfrak{s}_m(X) = s$. When the context is clear, we will omit the subscript m. Thus, $\mathfrak{l}(X)$ is the length of the cycle generated by X, while $\mathfrak{s}(X)$ is the number of iterations necessary to reach that cycle. Considering all members of Z_m^n, let Y and W be the tuples for which $\mathfrak{l}(Y)$ and $\mathfrak{s}(W)$ are maximum, respectively. We denote these maximum lengths by $\mathfrak{l}(m)$ and $\mathfrak{s}(m)$. Our goal is to characterize $\mathfrak{l}(m)$ and $\mathfrak{s}(m)$.

Theorem 1: For all n, $\mathfrak{l}(m) = \mathfrak{l}(A)$ and $\mathfrak{s}(m) = \mathfrak{s}(A)$. Further, if $m = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_j^{k_j}$, where the p_i's are distinct primes, then $\mathfrak{l}(m) = \operatorname{lcm}\{\mathfrak{l}(p_1^{k_1}), \dots, \mathfrak{l}(p_j^{k_j})\}$ and $\mathfrak{s}(m) = \max\{\mathfrak{s}(p_1^{k_1}), \dots, \mathfrak{s}(p_j^{k_j})\}$.

Proof: Let $X = (x_1, x_2, \dots, x_n) \in Z_m^n$. As we noted above, $D^j(X) = \Sigma_{1\le i\le n} x_i H^{n+1-i}(D^j(A))$. Thus,

$$\begin{aligned} D^{\mathfrak{l}(A)+\mathfrak{s}(A)}(X) &\equiv \Sigma_{1\le i\le n} x_i H^{n+1-i}(D^{\mathfrak{l}(A)+\mathfrak{s}(A)}(A)) \pmod{m} \\ &\equiv \Sigma_{1\le i\le n} x_i H^{n+1-i}(D^{\mathfrak{s}(A)}(A)) \pmod{m} \\ &\equiv D^{\mathfrak{s}(A)}(X). \end{aligned}$$

Hence, for all X, $\mathfrak{s}(X) \le \mathfrak{s}(A)$ and $l(X) | l(A)$. We conclude that $\mathfrak{l}(m) = \mathfrak{l}(A)$ and $\mathfrak{s}(m) = \mathfrak{s}(A)$.

Using the prime decomposition of m, we know that

$$Z_m \cong Z_{p_1^{k_1}} \oplus Z_{p_2^{k_2}} \oplus \cdots \oplus Z_{p_j^{k_j}},$$

where $\oplus$ denotes the direct sum. For an n-tuple

$$\underset{\in Z_m^n}{(x_1, x_2, \dots, x_n)} \cong (\underset{\in Z^n_{p_1^{k_1}}}{(x_1, x_2, \dots, x_n)}, \dots, \underset{\in Z^n_{p_j^{k_j}}}{(x_1, x_2, \dots, x_n)}).$$

Thus, $D^{l+s}(X) \equiv D^s(X) \pmod{m}$ if and only if $D^{l+s}(X) \equiv D^s(X) \pmod{p_i^{k_i}}$ for $1 \le i \le j$. Consequently, $\mathfrak{l}(m) = \operatorname{lcm}\{\mathfrak{l}(p_1^{k_1}), \dots, \mathfrak{l}(p_j^{k_j})\}$ and $\mathfrak{s}(m) = \max\{\mathfrak{s}(p_1^{k_1}), \dots, \mathfrak{s}(p_j^{k_j})\}$. $\square$

Theorem 1 greatly simplifies our work. To determine $\mathfrak{l}(m)$ and $\mathfrak{s}(m)$, it suffices to calculate $\mathfrak{l}_u(A)$ and $\mathfrak{s}_u(A)$ for $u = p^k$ with p a prime. Since our ultimate goal is to characterize $\mathfrak{l}(m)$ and $\mathfrak{s}(m)$ for 5-tuples, we narrow our focus to n-tuples with n odd.

Lemma 1: Let n be odd. If m is odd, then for each n-tuple X there exists a unique n-tuple Y such that $D(Y) \equiv X \pmod{m}$.

Proof: Let $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ be n-tuples. In order for $D(Y) \equiv X \pmod{m}$, we must have

$$(y_1 + y_2, y_2 + y_3, \dots, y_n + y_1) \equiv (x_1, x_2, \dots, x_n) \pmod{m}. \tag{1}$$

Hence,

$$(y_1 + y_2) - (y_2 + y_3) + \cdots + (-1)^{i+1}(y_i + y_{i+1}) + \cdots + (y_n + y_1) \equiv \Sigma_{1\le i\le n}(-1)^{i+1} x_i \pmod{m},$$

which simplifies to

$$2y_1 \equiv \Sigma_{1\le i\le n}(-1)^{i+1}x_i \pmod{m}. \tag{2}$$

Since m is odd, 2 has an inverse in Z_m, so (2) has a solution for y_1. We solve, in turn, for the other entries of Y using (1):

$$y_2 \equiv x_1 - y_1 \pmod{m},\ y_3 \equiv x_2 - y_2 \pmod{m}, \ldots, y_n \equiv x_{n-1} - y_{n-1} \pmod{m}. \tag{3}$$

Since the solutions in (2) and (3) are unique, Y is unique. □

Theorem 2: Let n be odd. Then $\mathfrak{S}(m) = 0$ if and only if m is odd.

Proof: We begin with the case in which m is even. Suppose there exists an n-tuple $Y = (y_1, y_2, \ldots, y_n)$ such that $D(Y) \equiv A \pmod{m}$. Then

$$(y_1 + y_2, y_2 + y_3, \ldots, y_n + y_1) \equiv (1, 0, \ldots, 0) \pmod{m}. \tag{4}$$

As in Lemma 1, (4) implies $2y_1 \equiv 1 \pmod{m}$. But this is impossible since m is even. Thus A is not in a cycle and $\mathfrak{S}(A) > 0$. Hence, when m is even, $\mathfrak{S}(m) \neq 0$.

When m is odd, we know from Lemma 1 that every n-tuple has a predecessor. For $X \in Z_m^n$, we can find a sequence of n-tuples such that

$$D(Y_1) \equiv X,\ D(Y_2) \equiv Y_1,\ D(Y_3) \equiv Y_2,\ D(Y_4) \equiv Y_3,\ D(Y_5) \equiv Y_4, \ldots \pmod{m} \tag{5}$$

or, equivalently,

$$D(Y_1) \equiv X,\ D^2(Y_2) \equiv X,\ D^3(Y_3) \equiv X,\ D^4(Y_4) \equiv X,\ D^5(Y_5) \equiv X, \ldots \pmod{m}.$$

Since there are only a finite number of n-tuples, eventually the sequence in (5) must repeat. That is, $Y_i \equiv Y_j \pmod{m}$ for some $i > j$. This implies $D^j(Y_i) \equiv D^j(Y_j) \equiv X \pmod{m}$. Hence, X is in a cycle and $\mathfrak{S}(X) = 0$. We conclude that $\mathfrak{S}(m) = 0$. □

Using Theorems 1 and 2, we see that, when n is odd,

$$\mathfrak{S}(m) = \max\{\mathfrak{S}(2^k), \mathfrak{S}(p_2^{k_2}), \ldots, \mathfrak{S}(p_j^{k_j})\} = s(2^k),$$

where the p_i's are distinct primes and $m = 2^k \cdot p_2^{k_2} \cdot \cdots \cdot p_j^{k_j}$. Thus, finding $\mathfrak{S}(m)$ requires only calculating $\mathfrak{S}(2^k)$.

As for $\mathfrak{l}(m)$, since A is in a cycle if and only if m is odd, there are two cases: $\mathfrak{l}(p^k)$, where p is an odd prime and $\mathfrak{l}(2^k)$, In much of what follows, we will consider the first case, leaving the second, special case for the end.

Theorem 3: Let n be odd and p be an odd prime. Suppose that $D^t(A) \equiv A \pmod{p^k}$. Then $D^{pt}(A) \equiv A \pmod{p^{k+1}}$. Thus, $\mathfrak{l}(p^{k+1})$ equals either $\mathfrak{l}(p^k)$ or $p \cdot \mathfrak{l}(p^k)$.

Proof: We begin by noting that Theorem 2 guarantees the existence of $t > 0$ for which $D^t(A) \equiv A \pmod{p^k}$. Rewriting the congruence as an equation gives

$$D^t(A) = A + (b_1p^k, b_2p^k, \ldots, b_np^k) = A + \Sigma_{1\le i\le n} b_ip^kH^{n+1-i}(A).$$

Thus

$$\begin{aligned}D^{2t}(A) &= D^t\left(A+\sum_{1\le i\le n} b_i p^k H^{n+1-i}(A)\right)\\ &= D^t(A)+\sum_{1\le i\le n} b_i p^k H^{n+1-i}(D^t(A))\\ &= A+\sum_{1\le i\le n} b_i p^k H^{n+1-i}(A)+\sum_{1\le i\le n} b_i p^k H^{n+1-i}\left(A+\sum_{1\le j\le n} b_j p^k H^{n+1-j}(A)\right)\\ &= A+\sum_{1\le i\le n} 2b_i p^k H^{n+1-i}(A)+p^{2k}\sum_{1\le i\le n}\sum_{1\le j\le n} b_i b_j H^{2n+2-i-j}(A)\\ &= A+\sum_{1\le i\le n} 2b_i p^k H^{n+1-i}(A)+p^{2k}X_2,\end{aligned}$$

where $X_2=\sum_{1\le i\le n}\sum_{1\le j\le n} b_i b_j H^{2n+2-i-j}(A)$. By induction,

$$D^{ht}(A)=A+\sum_{1\le i\le n} hb_i p^k H^{n+1-i}(A)+p^{hk}X_h$$

for some n-tuple X_h. Hence,

$$\begin{aligned}D^{pt}(A) &= A+\sum_{1\le i\le n} pb_i p^k H^{n+1-i}(A)+p^{pk}X_p\\ &= A+\sum_{1\le i\le n} b_i p^{k+1} H^{n+1-i}(A)+p^{pk}X_p\\ &\equiv A \pmod{p^{k+1}}.\end{aligned}$$

Now let $t=\text{Ɩ}(p^k)$. If $p|b_i$ for all i, then $D^t(A)\equiv A \pmod{p^{k+1}}$. In this case, $\text{Ɩ}(p^{k+1})=\text{Ɩ}(p^k)$. On the other hand, if $p\nmid b_i$ for some i, then $\text{Ɩ}(p^{k+1})=p\cdot\text{Ɩ}(p^k)$. □

Corollary 1: Let n be odd and p be an odd prime.

(i) If $\text{Ɩ}(p^2)\ne\text{Ɩ}(p)$, then $\text{Ɩ}(p^k)=p^{k-1}\cdot\text{Ɩ}(p)$ for all $k\ge 2$.

(ii) If $\text{Ɩ}(p^2)=\text{Ɩ}(p)$, then there exists $u\ge 2$ such that $\text{Ɩ}(p^k)=\text{Ɩ}(p)$ for all $k\le u$ and $\text{Ɩ}(p^k)=p^{k-u}\cdot\text{Ɩ}(p)$ for all $k>u$.

Proof: The proof of Theorem 3 shows that, if $D^t(A)\equiv A \pmod{p^k}$ and $D^t(A)\not\equiv A \pmod{p^{k+1}}$, then $D^{pt}(A)\equiv A \pmod{p^{k+1}}$ and $D^{pt}(A)\not\equiv A \pmod{p^{k+2}}$. Hence, if $\text{Ɩ}(p^{k+1})=p\cdot\text{Ɩ}(p^k)$, then $\text{Ɩ}(p^{k+2})=p^2\cdot\text{Ɩ}(p^k)$. Results (i) and (ii) follow immediately from this observation. □

Corollary 1 greatly reduces our work since $\text{Ɩ}(p^k)=p^s\cdot\text{Ɩ}(p)$ for some $s\le k-1$. This allows us to focus on $\text{Ɩ}(p)$

5-TUPLES AND FIBONACCI NUMBERS

We now restrict our attention to 5-tuples. We begin by considering $D^j(A)$. Surprisingly, $D^j(A)$ can be expressed in terms of the Fibonacci numbers. We will use the standard notation: $F_0=0$, $F_1=1$, $F_2=1$, and $F_{j+1}=F_{j-1}+F_j$.

Theorem 4: For $i\ge 1$,

$$\begin{aligned}D^{2i}(A) &= (2^{2i-4}F_2+2^{2i-6}F_4+\cdots+2^2F_{2i-4}+F_{2i-2})\cdot(1,1,1,1,1)\\ &\quad + H^i(F_{2i+1},F_{2i},0,0,F_{2i}).\end{aligned} \tag{6}$$

Proof: We proceed by induction. First, note that

$A=(1,0,0,0,0)$,

$D(A)=(1,0,0,0,1)$,

$D^2(A)=(1,0,0,1,2)=H^1(F_3,F_2,0,0,F_2)$.

Thus (6) holds for $i = 1$. Now assume (6) holds for i. Then

$$\begin{aligned} D^{2i+1}(A) &= (2^{2i-4}F_2 + 2^{2i-6}F_4 + \cdots + 2^2F_{2i-4} + F_{2i-2}) \cdot (2, 2, 2, 2, 2) \\ &\quad + H^i(F_{2k+2}, F_{2i}, 0, F_{2i}, F_{2k+2}) \end{aligned} \tag{7}$$

and

$$\begin{aligned} D^{2i+2}(A) &= (2^{2i-4}F_2 + 2^{2i-6}F_4 + \cdots + 2^2F_{2i-4} + F_{2i-2}) \cdot (4, 4, 4, 4, 4) \\ &\quad + H^i(F_{2i+2} + F_{2i}, F_{2i}, F_{2i}, F_{2i} + F_{2i+2}, F_{2i+2} + F_{2i+1} + F_{2i}) \\ &= (2^{2i-2}F_2 + 2^{2i-4}F_4 + \cdots + 2^4F_{2i-4} + 2^2F_{2i-2}) \cdot (1, 1, 1, 1, 1) \\ &\quad + (F_{2i}, F_{2i}, F_{2i}, F_{2i}, F_{2i}) + H^i(F_{2i+2}, 0, 0, F_{2i+2}, F_{2i+3}) \\ &= (2^{2i-2}F_2 + 2^{2i-4}F_4 + \cdots + 2^4F_{2i-4} + 2^2F_{2i-2} + F_{2i}) \cdot (1, 1, 1, 1, 1) \\ &\quad + H^{i+1}(F_{2i+3}, F_{2i+2}, 0, 0, F_{2i+2}). \quad \square \end{aligned}$$

Since the sum in (6) will occur frequently, we will adopt the following notation:

$$\mathrm{SUM}(2i) = 2^{2i-4}F_2 + 2^{2i-6}F_4 + \cdots + 2^2F_{2k-4} + F_{2i-2}.$$

Note that SUM is defined only for even integers. We use this notation to rewrite (6) and (7) for $i \geq 1$:

$$D^{2i}(A) = \mathrm{SUM}(2i) \cdot (1, 1, 1, 1, 1) + H^i(F_{2i+1}, F_{2i}, 0, 0, F_{2i}); \tag{8}$$

$$D^{2i+1}(A) = 2 \cdot \mathrm{SUM}(2i) \cdot (1, 1, 1, 1, 1) + H^i(F_{2i+2}, F_{2i}, 0, F_{2i}, F_{2i+2}). \tag{9}$$

Theorem 5: Let m be odd. Suppose $D^{\ell}(A) \equiv A \pmod{m}$.

If ℓ is even, then $F_{\ell} \equiv 0 \pmod{m}$, $F_{\ell+1} \equiv 1 \pmod{m}$, $\mathrm{SUM}(\ell) \equiv 0 \pmod{m}$, and $5 \mid \ell$.

If ℓ is odd, then $F_{\ell} \equiv 0 \pmod{m}$, $F_{\ell+1} \equiv -1 \pmod{m}$.

Proof: If ℓ is even, then (8) applies with $2i = \ell$. To simplify notation, let $s = \mathrm{SUM}(\ell)$. Then

$$\begin{aligned} D^{\ell}(A) &= (s, s, s, s, s) + H^{\ell/2}(F_{\ell+1}, F_{\ell}, 0, 0, F_{\ell}) \\ &= H^{\ell/2}(s + F_{\ell+1}, s + F_{\ell}, s, s, s + F_{\ell}) \\ &\equiv (1, 0, 0, 0, 0) \pmod{m}. \end{aligned}$$

Hence, $5 \mid \ell$, $s \equiv 0 \pmod{m}$, $F_{\ell} \equiv 0 \pmod{m}$, and $F_{\ell+1} \equiv 1 \pmod{m}$.

If ℓ is odd, then (9) applies with $2i + 1 = \ell$. Let $s = \mathrm{SUM}(\ell - 1)$. Then

$$\begin{aligned} D^{\ell}(A) &= s \cdot (2, 2, 2, 2, 2) + H^{(\ell-1)/2}(F_{\ell+1}, F_{\ell-1}, 0, F_{\ell-1}, F_{\ell+1}) \\ &= H^{(\ell-1)/2}(2 \cdot s + F_{\ell+1}, 2 \cdot s + F_{\ell-1}, 2 \cdot s, 2 \cdot s + F_{\ell-1}, 2 \cdot s + F_{\ell+1}) \\ &= H^{(\ell-1)/2+2}(2 \cdot s, 2 \cdot s + F_{\ell-1}, 2 \cdot s + F_{\ell+1}, 2 \cdot s + F_{\ell+1}, 2 \cdot s + F_{\ell-1}) \\ &\equiv (1, 0, 0, 0, 0) \pmod{m}. \end{aligned}$$

Hence, $2 \cdot s \equiv 1 \pmod{m}$, $F_{\ell-1} \equiv -1 \pmod{m}$, and $F_{\ell+1} \equiv -1 \pmod{m}$. The last two congruences imply that $F_{\ell} \equiv 0 \pmod{m}$. $\square$

PROPERTIES OF $F_K \equiv 0$ (mod m)

For m odd, $\mathfrak{l}(m)$ equals the smallest positive integer $\mathfrak{l}$ for which $D^{\mathfrak{l}}(A) \equiv A \pmod{m}$. From Theorem 5, we know that $F_{\mathfrak{l}} \equiv 0 \pmod{m}$ and either $F_{\mathfrak{l}+1} \equiv 1 \pmod{m}$ or $F_{\mathfrak{l}+1} \equiv -1 \pmod{m}$, depending on whether $\mathfrak{l}$ is even or odd, respectively. Thus, we now consider numbers K for which $F_K \equiv 0 \pmod{m}$. We begin by observing that there does exist a $K > 0$ such that $F_K \equiv 0 \pmod{m}$. Since Z_m is finite, there exist $i > j$ such that $F_i \equiv F_j \pmod{m}$ and $F_{i+1} \equiv F_{j+1} \pmod{m}$. These congruences together imply that $F_{i-1} \equiv F_{j-1} \pmod{m}$ which, in turn, implies $F_{i-2} \equiv F_{j-2} \pmod{m}$. Continuing, we see that $F_{i-j} \equiv F_0 \equiv 0 \pmod{m}$.

Numbers K for which $F_K \equiv 0 \pmod{m}$ have been studied in [2], [10], and [11]. The lemmas that follow, as well as the observations in the previous paragraph, are well known. Their proofs are included because they involve techniques that we will use when we derive results about $\mathfrak{l}(m)$.

Lemma 2: Suppose $F_K \equiv 0 \pmod{m}$ and $F_{K+1} \equiv a \pmod{m}$ with $K > 0$. Then

$$F_{K-j} \equiv (-1)^{j+1} a \cdot F_j \pmod{m} \tag{10}$$

and

$$F_{iK+j} \equiv a^i \cdot F_j \pmod{m} \tag{11}$$

for all $i \geq 1$ and $j = 0, 1, \ldots, K-1$.

Proof: To prove (10), we first note that $F_K \equiv 0 \equiv -a \cdot F_0 \pmod{m}$ and $F_{K-1} = F_{K+1} - F_K \equiv a - 0 \equiv a \cdot F_1 \pmod{m}$. Thus, (10) holds for $j = 0$ and $j = 1$. Now assume (10) holds for $j-1$ and j; then

$$\begin{aligned} F_{K-(j+1)} &= F_{K-(j-1)} - F_{K-j} \\ &\equiv (-1)^j a \cdot F_{j-1} - (-1)^{j+1} a \cdot F_j \pmod{m} \\ &\equiv (-1)^j a \cdot (F_{j-1} + F_j) \pmod{m} \\ &\equiv (-1)^{j+2} a \cdot F_{j+1} \pmod{m}. \end{aligned}$$

To prove (11), we make use of the well-known identity: $F_{i+j} = F_{i-1}F_j + F_iF_{j+1}$. Now

$$F_{K+j} = F_{j+1+K-1} = F_jF_{K-1} + F_{j+1}F_K \equiv F_j \cdot a + F_{j+1} \cdot 0 \equiv a \cdot F_j \pmod{m}.$$

Thus (11) holds for $i = 1$. Now assume (11) holds for i. Then

$$F_{(i+1)K+j} = F_{iK+j+1+K-1} = F_{iK+j}F_{K-1} + F_{iK+j+1}F_K \equiv a^i \cdot F_j \cdot a \equiv a^{i+1} \cdot F_j \pmod{m}. \ \square$$

Lemma 3: Suppose $F_K \equiv 0 \pmod{m}$ and $F_{K+1} \equiv a \pmod{m}$ with $K > 0$. Then $a^2 \equiv (-1)^K \pmod{m}$. Thus, when $m > 2$, $a^2 \equiv 1 \pmod{m}$ if and only if K is even.

Proof: By (10),

$$1 = F_1 = F_{K-(K-1)} \equiv (-1)^K \cdot a \cdot F_{K-1} \equiv (-1)^K \cdot a \cdot a \equiv (-1)^K \cdot a^2 \pmod{m}.$$

Thus $a^2 \equiv (-1)^K \pmod{m}$. As for the second statement, when $m > 2$, $(-1)^K \equiv 1 \pmod{m}$ if and only if K is even. $\square$

In Theorem 5 we consider $\mathfrak{l}$ for which $D^{\mathfrak{l}}(A) \equiv A \pmod{m}$ where, of course, m is odd. We showed that $F_{\mathfrak{l}} \equiv 0 \pmod{m}$ and either $F_{\mathfrak{l}+1} \equiv 1 \pmod{m}$ or $F_{\mathfrak{l}+1} \equiv -1 \pmod{m}$ depending on whether I is even or odd, respectively. Lemma 3 shows that the second case with I odd is impossible. Thus, when $D^I(A) \equiv A \pmod{m}$, I is even with $F_I \equiv 0 \pmod{m}$ and $F_{I+1} \equiv 1 \pmod{m}$. We now show that there is always a $K > 0$ for which $F_K \equiv 0 \pmod{m}$ and $F_{K+1} \equiv 1 \pmod{m}$.

Lemma 4: Suppose $F_K \equiv 0 \pmod{m}$ and $F_{K+1} \equiv a \pmod{m}$ with $K > 0$. Then:

$$F_{2K} \equiv 0 \pmod{m} \text{ and } F_{2K+1} \equiv (-1)^K \pmod{m};$$
$$F_{4K} \equiv 0 \pmod{m} \text{ and } F_{4K+1} \equiv 1 \pmod{m}.$$

Proof: Using Lemmas 2 and 3, we find:

$$F_{2K} \equiv a^2 F_0 \equiv 0 \pmod{m} \text{ and } F_{2K+1} \equiv a^2 F_1 \equiv (-1)^K \pmod{m};$$
$$F_{4k} \equiv a^4 F_0 \equiv 0 \pmod{m} \text{ and } F_{4k+1} \equiv a^4 F_1 \equiv (-1)^{2K} \equiv 1 \pmod{m}. \quad \square$$

Thus that there is always a $K > 0$ for which $F_K \equiv 0 \pmod{m}$ and $F_{K+1} \equiv 1 \pmod{m}$. We denote the smallest such integer by $K(m)$. That is,

$$K(m) = \min\{K > 0 \mid F_K \equiv 0 \pmod{m} \text{ and } F_{K+1} \equiv 1 \pmod{m}\}.$$

By Lemma 3, $K(m)$ is even when $m > 2$. We note that $K(2) = 3$. The next lemma contains a useful property of $K(m)$.

Lemma 5: Let $K > 0$. Then $F_K \equiv 0 \pmod{m}$ and $F_{K+1} \equiv 1 \pmod{m}$ if and only if $K(m) \mid K$.

Proof: Suppose $F_K \equiv 0 \pmod{m}$ and $F_{K+1} \equiv 1 \pmod{m}$. By definition, $K(m)$ is the smallest number satisfying these conditions. Thus $K(m) \le K$. Let $K = q \cdot K(m) + r$, where $0 \le r < K(m)$. Then by Lemma 2, $F_K = F_{qK(m)+r} \equiv F_r \pmod{m}$. Since $F_K \equiv 0 \pmod{m}$, $F_r \equiv 0 \pmod{m}$. Hence $r = 0$. The converse follows immediately from Lemma 2. $\square$

Corollary 2: Let m be odd. Then $I(m) = \operatorname{lcm}\{5, j \cdot K(m)\}$, where j is the smallest integer for which $\mathrm{SUM}(j \cdot K(m)) \equiv 0 \pmod{m}$.

Proof: We know that $I(m)$ is the smallest I for which $D^I(A) \equiv A \pmod{m}$. As we observed above, I is even, $F_I \equiv 0 \pmod{m}$ and $F_{I+1} \equiv 1 \pmod{m}$. By Lemma 5, I is a multiple of $K(m)$. The conclusion now follows immediately from Theorem 5. $\square$

By Corollary 2, when p is an odd prime, $I(p^k)$ is a multiple $K(p^k)$. The following lemma connects $K(p^k)$ and $K(p)$. This relationship will greatly aid in the calculation of $I(p^k)$.

Lemma 6: Let p be a prime. Then

(i) For $k \ge 1$, $K(p^{k+1})$ equals either $K(p^k)$ or $p \cdot K(p^k)$.
(ii) If $K(p^2) \ne K(p)$, then $K(p^k) = p^{k-1} \cdot K(p)$ for $k \ge 2$.
(iii) If $K(p^2) = K(p)$, then there exists $u \ge 2$ such that $K(p^k) = K(p)$ for $k \le u$ and $K(p^k) = p^{k-u} \cdot K(p)$ for $k > u$.

Proof: This is a well-known result; its proof is given in [1]. Note the similarities between the properties of K in this lemma and the properties of I in Theorem 3 and Corollary 1. $\square$

We know that $I(p^k)$ is a multiple of $K(p^k)$, while the latter is a multiple of $K(p)$. Thus, $I(p^k)$ is a multiple of $K(p)$. We conclude this section with a lemma that gives bounds on $K(p)$.

Although the result is well known [10], the proof is included because it shows the way in which the different cases arise. As we will see, $K(p)$ depends on the value of $5^{(p-1)/2}$ modulo p. When $p=5$, $5^{(p-1)/2}$ is, of course, congruent to 0. Hence $p=5$ is a special case. For all other odd primes, $5^{(p-1)/2}$ is congruent to 1 or -1 depending on whether 5 is a quadratic residue or non-residue of p, respectively. If p is congruent to 1 or 9 modulo 10, then

$$(5/p)=(p/5)=([10q\pm 1]/5)=(\pm 1/5)=1,$$

where $(\cdot)$ is the Legendre symbol. Hence 5 is a quadratic residue and $5^{(p-1)/2}\equiv 1 \pmod{p}$. On the other hand, if p is congruent to 3 or 7 modulo 10, then

$$(5/p)=(p/5)=([10q\pm 3]/5)=(\pm 3/5)=-1.$$

In this case, 5 is a nonresidue and $5^{(p-1)/2}\equiv -1 \pmod{p}$.

Lemma 7: Let p be an odd prime. Then

$$\begin{array}{ll} K(p)\,|\,(p-1) & p\equiv 1 \text{ or } 9 \pmod{10},\\ K(p)\,|\,(2p+2) & p\equiv 3 \text{ or } 7 \pmod{10},\\ K(5)=20. & \end{array}$$

Proof: By Binet's formula,

$$\begin{aligned} F_p &= \frac{(1+\sqrt{5})^p-(1-\sqrt{5})^p}{\sqrt{5}\,2^p}\\ &= \frac{1}{2^{p-1}}\left[\binom{p}{1}+\binom{p}{3}5+\cdots+\binom{p}{p-2}5^{(p-3)/2}+\binom{p}{p}5^{(p-1)/2}\right]\\ &\equiv 5^{(p-1)/2} \pmod{p} \end{aligned}$$

and

$$\begin{aligned} F_{p+1} &= \frac{(1+\sqrt{5})^{p+1}-(1-\sqrt{5})^{p+1}}{\sqrt{5}\,2^{p+1}}\\ &= \frac{1}{2^p}\left[\binom{p+1}{1}+\binom{p+1}{3}5+\cdots+\binom{p+1}{p-2}5^{(p-3)/2}+\binom{p+1}{p}5^{(p-1)/2}\right]\\ &\equiv 2^{-1}[1+5^{(p-1)/2}] \pmod{p}. \end{aligned}$$

When $p\equiv 1$ or 9 (mod 10), $F_p\equiv 1$ (mod p) and $F_{p+1}\equiv 1$ (mod p). These imply $F_{p-1}\equiv 0$ (mod p). Hence by Lemma 5, $K(p)\,|\,(p-1)$.

When $p\equiv 3$ or 7 (mod 10), $F_p\equiv -1$ (mod p) and $F_{p+1}\equiv 0$ (mod p). These imply $F_{p+2}\equiv -1$ (mod p). By Lemma 4, $F_{2p+2}\equiv 0$ (mod p) and $F_{2p+3}\equiv 1$ (mod p). Hence $K(p)\,|\,(2p+2)$.

By direct calculation we find that $K(5)=20$. □

PROPERTIES OF SUM (mod m)

By Corollary 2, for odd m, $\mathfrak{l}(m)=\operatorname{lcm}\{5, j\cdot K(m)\}$, where j is the smallest integer for which $\operatorname{SUM}(j\cdot K(m))\equiv 0$ (mod m). We now consider such sums.

Lemma 8: Suppose $F_K \equiv 0 \pmod{m}$ and $F_{K+1} \equiv 1 \pmod{m}$, where K is an even positive integer. Then

$$\mathrm{SUM}(jK) \equiv (2^{(j-1)K} + \cdots + 2^K + 1) \cdot \mathrm{SUM}(K) \pmod{m}.$$

Proof: The congruence certainly holds for $j = 1$. Assume it holds for j and consider $j+1$:

$$\begin{aligned}\mathrm{SUM}((j+1)\cdot K) &= 2^{(j+1)K-4}F_2 + 2^{(j+1)K-6}F_4 + \cdots + 2^{jK}F_{K-2} + 2^{jK-2}F_K \\ &\quad + 2^{jK-4}F_{K+2} + \cdots + 2^2 F_{(j+1)K-4} + F_{(j+1)K-2} \\ &= 2^{jK}\cdot(2^{K-4}F_2 + 2^{K-6}F_4 + \cdots + F_{K-2}) + 2^{jK-2}F_K \\ &\quad + 2^{jK-4}F_{K+2} + \cdots + 2^2 F_{(j+1)K-4} + F_{(j+1)K-2}.\end{aligned}$$

Now by Lemma 2, $F_{K+2} \equiv F_2 \pmod{m}, \ldots, F_{(j+1)K-2} \equiv F_{K-2} \equiv F_{jK-2} \pmod{m}$. Thus

$$\begin{aligned}\mathrm{SUM}((j+1)\cdot K) &\equiv 2^{jK}\cdot(2^{K-4}F_2 + 2^{K-6}F_4 + \cdots + F_{K-2}) + 2^{jK-2}F_K \\ &\quad + 2^{jK-4}F_2 + \cdots + 2^2 F_{jK-4} + F_{jK-2} \pmod{m} \\ &\equiv 2^{jK}\cdot \mathrm{SUM}(K) + 0 + \mathrm{SUM}(jK) \pmod{m} \\ &\equiv 2^{jK}\cdot \mathrm{SUM}(K) + (2^{(j-1)K} + \cdots + 2^K + 1)\cdot \mathrm{SUM}(K) \pmod{m} \\ &\equiv (2^{jK} + 2^{(j-1)K} + \cdots + 2^K + 1)\cdot \mathrm{SUM}(K) \pmod{m}. \ \square\end{aligned}$$

For odd m, $\mathfrak{l}(m)$ is a multiple of $j\cdot K(m)$. Lemma 8 tells us how to find j. First, we calculate $\mathrm{SUM}(K(m))$. If $\mathrm{SUM}(K(m)) \equiv 0 \pmod{m}$, $\mathfrak{l}(m) = \mathrm{lcm}\{5, K(m)\}$. On the other hand, if $\mathrm{SUM}(K(m)) \not\equiv 0 \pmod{m}$, then we must select j so that

$$(2^{(j-1)K(m)} + \cdots + 2^{K(m)} + 1)\cdot \mathrm{SUM}(K(m)) \equiv 0 \pmod{m}.$$

The next lemma will aid in calculating $\mathrm{SUM}(K(m))$ modulo m.

Lemma 9: Suppose $F_K \equiv 0 \pmod{m}$ and $F_{K+1} \equiv 1 \pmod{m}$, where K is an even positive integer. Then

$$\mathrm{SUM}(K) \equiv \sum_{j=1}^{K/2}\binom{K}{2j}5^{j-1} \pmod{m}.$$

Proof: By Lemma 2, $F_{K-j} \equiv (-1)^{j+1}F_j \pmod{m}$. Thus

$$\begin{aligned}\mathrm{SUM}(K) &= 2^{K-4}F_2 + 2^{K-6}F_4 + \cdots + 2^2F_{K-4} + F_{K-2} \\ &\equiv -2^{K-4}F_{K-2} - 2^{K-6}F_{K-4} - \cdots - 2^2F_4 - F_2 \pmod{m}.\end{aligned} \tag{12}$$

In preparation for using Binet's formula, let $a = (1+\sqrt{5})$ and $b = (1-\sqrt{5})$. Note that

$$a^2 - 1 = 5 + 2\sqrt{5},\ b^2 - 1 = 5 - 2\sqrt{5}, \text{ and } (a^2-1)\cdot(b^2-1) = 5.$$

Now, by Binet's formula, $F_j = [a^j - b^j]/(2^j\sqrt{5})$. Thus $2^{j-2}F_j = [a^j - b^j]/(2^2\sqrt{5})$. Hence

$$\begin{aligned}&2^{K-4}F_{K-2} + 2^{K-6}F_{K-4} + \cdots + 2^2F_4 + F_2 \\ &= [a^{K-2} - b^{K-2} + a^{K-4} - b^{K-4} + \cdots + a^2 - b^2]/(2^2\sqrt{5}) \\ &= [(a^{K-2} + a^{K-4} + \cdots + a^2 + 1) - (b^{K-2} + b^{K-4} + \cdots + b^2 + 1)]/(2^2\sqrt{5}) \\ &= [(a^K - 1)/(a^2-1) - (b^K-1)/(b^2-1)]/(2^2\sqrt{5})\end{aligned}$$

$$
\begin{aligned}
&= [(a^K-1)\cdot(5-2\sqrt{5})-(b^K-1)\cdot(5+2\sqrt{5})]/(2^2\cdot 5\sqrt{5})\\
&= [5a^K-2\sqrt{5}\,a^K-5+2\sqrt{5}-5b^K-2\sqrt{5}\,b^K+5+2\sqrt{5}]/(2^2\cdot 5\sqrt{5})\\
&= (a^K-b^K)/(2^2\sqrt{5})-(a^K+b^K)/(2\cdot 5)+1/5\\
&= 2^{K-2}F_K-5^{-1}[(a^K+b^K)/2-1].
\end{aligned}
\tag{13}
$$

We use the Binomial Theorem to rewrite $5^{-1}[(a^K+b^K)/2-1]$ as

$$
5^{-1}\left[\frac{(1+\sqrt{5})^K+(1-\sqrt{5})^K}{2}-1\right]=\sum_{j=1}^{K/2}\binom{K}{2j}5^{j-1}.
\tag{14}
$$

We now combine (12), (13), and (14) and reduce modulo m:

$$
\begin{aligned}
\mathrm{SUM}(K) &\equiv -2^{K-4}F_{K-2}-2^{K-6}F_{K-4}-\cdots-2^2F_4-F_2 \pmod{m}\\
&\equiv -2^{K-2}F_K+5^{-1}[(a^K+b^K)/2-1] \pmod{m}\\
&\equiv \sum_{j=1}^{K/2}\binom{K}{2j}5^{j-1} \pmod{m}. \quad \square
\end{aligned}
\tag{15}
$$

Note that Lemmas 8 and 9 hold for all m so long as K is an even positive integer.

DETERMINING $I(p)$ FOR ODD PRIMES

We are now going to determine $I(p)$ for odd primes. We will consider four cases: $p=3$, $p=5$, $p\equiv 1$ or 9 (mod 10), and $p\equiv 3$ or 7 (mod 10). Although the derivations will be different, the final result will be the same. In order to state the result, we need some additional notation. For $a\in Z_m$ with $\gcd(a,m)=1$, we will denote the order of a in Z_m by $o_m(a)$. Thus, if $s>0$ is the smallest positive integer for which $a^s\equiv 1$ (mod m), we will write $o_m(a)=s$. Of course, if $a\equiv 1$ (mod m), $o_m(a)=1$. What we will show is that for odd p,

$$
I(p)=\mathrm{lcm}\{5,\, o_p(2^{K(p)})\cdot K(p)\}.
\tag{16}
$$

We showed in Corollary 2 that $I(p)$ is the least common multiple of 5 and $j\cdot K(p)$, where j is the smallest integer for which $\mathrm{SUM}(j\cdot K(p))\equiv 0$ (mod p). As we observed above, to find j we first calculate $\mathrm{SUM}(K(m))$. If $\mathrm{SUM}(K(m))\equiv 0$ (mod m), $I(m)=\mathrm{lcm}\{5,K(m)\}$. On the other hand, if $\mathrm{SUM}(K(m))\not\equiv 0$ (mod m), then we must select j so that

$$
(2^{(j-1)K(m)}+\cdots+2^{K(m)}+1)\cdot\mathrm{SUM}(K(m))\equiv 0 \pmod{m}.
$$

We begin with the two special cases, $p=3$ and $p=5$.

Theorem 6: $I(3)=40$ and $I(5)=20$.

Proof: By direct calculation, it is easy to verify that $K(3)=8$. Now, by Lemma 9,

$$
\begin{aligned}
\mathrm{SUM}(8) &\equiv \sum_{j=1}^{4}\binom{8}{2j}5^{j-1} \pmod{3} \equiv \binom{8}{2}+\binom{8}{4}5+\binom{8}{6}5^2+\binom{8}{8}5^3 \pmod{3}\\
&\equiv 1+2+1+2 \pmod{3} \equiv 0 \pmod{3}.
\end{aligned}
$$

Hence $I(3)=\mathrm{lcm}\{5,8\}=40$.

It is also easy to verify that $K(5)=20$. Now, by Lemma 9,

$$\text{SUM}(20) \equiv \sum_{j=1}^{10} \binom{20}{2j} 5^{j-1} \pmod 5 \equiv \binom{20}{2} \pmod 5 \equiv 0 \pmod 5.$$

Hence $\text{I}(5) = \text{lcm}\{5, 20\} = 20$. □

We note that (16) holds for $p = 3$ and $p = 5$. In both cases, $K(p)$ is a multiple of $\phi(p) = p-1$; hence, $2^{K(p)} \equiv 1 \pmod p$. Therefore, for $p = 3$ and $p = 5$, $o_p(2^{K(p)}) = 1$ and $\text{I}(p) = \text{lcm}\{5, o_p(2^{K(p)}) \cdot K(p)\}$.

Next we consider primes for which $p \equiv 1$ or $9 \pmod{10}$. We begin with a lemma which deals with $\text{SUM}(K(p^j))$ modulo p^j for $j \geq 1$. At the moment we are concerned only when $j = 1$. However, we state and prove the more general case since we will need it later.

Lemma 10: Let p be a prime such that $p \equiv 1$ or $9 \pmod{10}$. Let $q = p^j$ for $j \geq 1$. Then

$$\text{SUM}(K(q)) \equiv 5^{-1}[2^{K(q)} - 1] \pmod q.$$

Proof: To simplify notation, let $K = K(q)$. Since 5 is quadratic residue, the congruence $x^2 \equiv 5 \pmod q$ has a solution in Z_q. Let r be such a solution. Then Binet's formula holds in Z_p:

$$F_K \equiv [(1+r)^K - (1-r)^K] / (2^K r) \equiv 0 \pmod q \tag{17}$$

and

$$F_{K+1} \equiv [(1+r)^{K+1} - (1-r)^{K+1}] / (2^{K+1} r) \equiv 1 \pmod q. \tag{18}$$

From (17), we see that $(1+r)^K \equiv (1-r)^K \pmod p$. Thus, we can rewrite (18) as

$$1 \equiv (1+r)^K \cdot [(1+r) - (1-r)] / (2^{K+1} r) \equiv (1+r)^K / 2^K \pmod q.$$

Hence $(1+r)^K \equiv 2^K \pmod p$. Now, by (15) of Lemma 9,

$$\begin{aligned} \text{SUM}(K) &\equiv 5^{-1}[((1+r)^K + (1-r)^K)/2 - 1] \pmod q \\ &\equiv 5^{-1}[2^K - 1] \pmod q. \quad \square \end{aligned}$$

Theorem 7: Let p be a prime such that $p \equiv 1$ or $9 \pmod{10}$. Then

$$\text{I}(p) = \text{lcm}\{5, o_p(2^{K(p)}) \cdot K(p)\}.$$

Proof: Again, to simplify notation, we let $K = K(p)$. Using Lemmas 8 and 10, we have

$$\begin{aligned} \text{SUM}(j \cdot K) &\equiv (2^{(j-1)K} + \cdots + 2^K + 1) \cdot \text{SUM}(K) \pmod p \\ &\equiv (2^{(j-1)K} + \cdots + 2^K + 1) \cdot 5^{-1}[2^K - 1] \pmod p \\ &\equiv \{[2^{jK} - 1] / [2^K - 1]\} \cdot 5^{-1}[2^K - 1] \pmod p \\ &\equiv 5^{-1}[2^{jK} - 1] \pmod p. \end{aligned}$$

We want the smallest j for which $\text{SUM}(j \cdot K) \equiv 0 \pmod p$. Clearly, $j = o_p(2^K)$ and hence $\text{I}(p) = \text{lcm}\{5, o_p(2^K) \cdot K\}$. □

We now consider the case in which $p \equiv 3$ or $7 \pmod{10}$. Since 5 is a nonresidue in this case, Binet's theorem cannot be used as above.

Lemma 11: Let p be a prime such that $p \equiv 3$ or $7 \pmod{10}$ with $p > 3$. Then $\mathrm{SUM}(2p+2) \equiv 3 \pmod{p}$ and $\mathrm{SUM}(K(p)) \not\equiv 0 \pmod{p}$.

Proof: By Lemma 9,

$$\mathrm{SUM}(2p+2) \equiv \sum_{j=1}^{p+1} \binom{2p+2}{2j} 5^{j-1} \pmod{p}.$$

For $1 < j < (p+1)/2$,

$$\binom{2p+2}{2j} \equiv 0 \pmod{p} \quad \text{and} \quad \binom{2p+2}{2p+2-2j} \equiv 0 \pmod{p}.$$

Also

$$\binom{2p+2}{2} \equiv 1 \pmod{p}, \quad \binom{2p+2}{2p} \equiv 1 \pmod{p},$$

and

$$\binom{2p+2}{p+1} = \frac{(2p+2)(2p+1)2p(p+p-1)\cdots(p+2)}{(p+1)p(p-1)\cdots 2} \equiv \frac{2\cdot 1\cdot 2\cdot (p-1)!}{1\cdot (p-1)!} \equiv 4 \pmod{p}.$$

Since 5 is nonresidue, $5^{(p-1)/2} \equiv -1 \pmod{p}$. Hence

$$\begin{aligned}\mathrm{SUM}(2p+2) &\equiv \binom{2p+2}{2} + \binom{2p+2}{p+1} 5^{(p-1)/2} + \binom{2p+2}{2p} 5^{p-1} + \binom{2p+2}{2p+2} 5^{p} \pmod{p} \\ &\equiv 1 + 4\cdot(-1) + 1 + 5 \pmod{p} \equiv 3 \pmod{p}.\end{aligned}$$

By Lemma 7, $K(p) \mid (2p+2)$. If $K(p) \neq (2p+2)$, let $j = (2p+2)/K(p)$. Then by Lemma 8,

$$\mathrm{SUM}(2p+2) \equiv \mathrm{SUM}(j\cdot K(p)) \equiv (2^{(j-1)K(p)} + \cdots + 2^{K(p)} + 1)\cdot \mathrm{SUM}(K(p)) \pmod{p}.$$

Since $\mathrm{SUM}(2p+2) \not\equiv 0 \pmod{p}$ when $p > 3$, $\mathrm{SUM}(K(p)) \not\equiv 0 \pmod{p}$. □

Lemma 12: Let p be a prime such that $p \equiv 3$ or $7 \pmod{10}$ with $p > 3$. Then $2^{K(p)} \not\equiv 1 \pmod{p}$.

Proof: Assume to the contrary that $2^{K(p)} \equiv 1 \pmod{p}$. This means $o_p(2) \mid K(p)$ which, in turn, implies that $o_p(2) \mid (2p+2)$. But we know $o_p(2) \mid (p-1)$. Since $\gcd(p-1, p+1) = 2$, $o_p(2)$ must equal 2 or 4. For $p > 3$, $2^2 \not\equiv 1 \pmod{p}$ and so $o_p(2) \neq 2$. Now $2^4 \equiv 2 \pmod{7}$, $2^4 \equiv 3 \pmod{13}$ and, for all other p, $2^4 < p$. Thus, for $p > 3$, $2^4 \not\equiv 1 \pmod{p}$, so $o_p(2) \neq 4$. We conclude that $2^{K(p)} \not\equiv 1 \pmod{p}$. □

Theorem 8: Let p be a prime such that $p \equiv 3$ or $7 \pmod{10}$ with $p > 3$. Then we have $l(p) = \operatorname{lcm}\{5, o_p(2^K)\cdot K\}$

Proof: To simplify notation, we let $K = K(p)$. Using Lemma 8, we have $\mathrm{SUM}(j\cdot K) \equiv (2^{(j-1)K} + \cdots + 2^K + 1)\cdot \mathrm{SUM}(K) \pmod{p}$. By Lemma 11, $\mathrm{SUM}(K) \not\equiv 0 \pmod{p}$, so we want the smallest j for which

$$2^{(j-1)K} + \cdots + 2^K + 1 \equiv 0 \pmod{p}. \tag{19}$$

Now $2^K \not\equiv 1 \pmod{p^k}$ by Lemma 12. Thus, the smallest j for which (19) holds is $o_p(2^K)$. □

DETERMINING $\ell(p^k)$ FOR ODD PRIMES

We know that $\ell(p^k) = p^s \cdot \ell(p)$ for some $s \le k-1$. We now show that for most, if not all, primes, $\ell(p^k) = p^{k-1} \cdot \ell(p)$. There are several cases.

Corollary 3: Let p be an odd prime with $p \ne 5$. If $K(p^2) \ne K(p)$, then $\ell(p^k) = p^{k-1} \cdot \ell(p)$.

Proof: By the theorems above, we know that $\ell(p) = \text{lcm}\{5, o_p(2^{K(p)}) \cdot K(p)\}$. Of course, $o_p(2^{K(p)})$ is relatively prime to p. Further, for $p \ne 5$, by Lemma 7, $K(p)$ is also relatively prime to p. Hence, $\gcd(\ell(p), p) = 1$.

By Lemma 6(ii), $K(p^k) = p^{k-1} \cdot K(p)$ for $k \ge 2$. We know from Corollary 2 that $\ell(p^2)$ is a multiple of $K(p^2) = p \cdot K(p)$; hence, $p \mid \ell(p^2)$. On the other hand, Corollary 1 tells us that $\ell(p^2)$ equals either $\ell(p)$ or $p \cdot \ell(p)$. Since $\gcd(\ell(p), p) = 1$, $\ell(p^2) = p \cdot \ell(p)$. This, in turn, implies by Corollary 1 that $\ell(p^k) = p^{k-1} \cdot \ell(p)$ for $k \ge 2$. $\square$

When $p = 5$, the proof of Corollary 3 does not apply since $K(5) = 20$, hence $\gcd(\ell(5), 5) \ne 1$. However, direct calculation shows that $\ell(5^2) \ne \ell(5)$. Thus $\ell(5^k) = 5^{k-1} \cdot \ell(5)$ for $k \ge 2$.

Even if $K(p^2) = K(p)$, it may still be the case that $\ell(p^2) = p \cdot \ell(p)$. We now consider this possibility.

Corollary 4: Let p be an odd prime with $p \ne 5$. Suppose that $K(p^2) = K(p)$. If $o_{p^2}(2^{K(p)}) \ne o_p(2^{K(p)})$, then $\ell(p^k) = p^{k-1} \cdot \ell(p)$.

Proof: Let $K = K(p)$. By Corollary 2 and Lemma 8, $\ell(p^2) = \text{lcm}\{5, j \cdot K\}$, where j is the smallest integer for which

$$\text{SUM}(j \cdot K) \equiv (2^{(j-1)K} + \cdots + 2^K + 1) \cdot \text{SUM}(K) \equiv 0 \pmod{p^2}. \tag{20}$$

First, suppose that $\text{SUM}(K) \equiv 0 \pmod{p^2}$; this implies $\text{SUM}(K) \equiv 0 \pmod{p}$. By Lemmas 10 and 11, this can occur only when $p \equiv 1$ or $9 \pmod{10}$, $o_p(2^K) = 1$ and $o_{p^2}(2^K) = 1$. But this contradicts the hypothesis. Thus, $\text{SUM}(K) \not\equiv 0 \pmod{p^2}$ and $2^K \not\equiv 1 \pmod{p}$. Hence, the smallest j for which (20) holds is $o_{p^2}(2^K) = p \cdot o_p(2^K)$. The proof now proceeds in the same manner as the proof of Corollary 3. We conclude that $\ell(p^2) \ne \ell(p)$ and hence $\ell(p^k) = p^{k-1} \cdot \ell(p)$. $\square$

As Wall points out, it is not known whether there exists a prime p for which $K(p^2) = K(p)$ [11]. It has been verified that $K(p^2) \ne K(p)$ for $p < 10{,}000$. Even if there is a prime for which $K(p^2) \ne K(p)$, it may still be the case that $\ell(p^2) = p \cdot \ell(p)$. In order for $\ell(p^2) \ne p \cdot \ell(p)$, two conditions must hold: $K(p^2) = K(p)$ and $o_{p^2}(2^K) = o_p(2^K)$. Of course, although rare, it is possible for an element to have the same order modulo p and p^2.

BOUNDS ON $\ell(p)$

We now use the results from the previous sections to find bounds on $\ell(p)$. First, we note an alternate way to calculate $\ell(p)$.

Corollary 5: Let p be prime with $p > 5$. Then $\ell(p) = \text{lcm}\{5, o_p(2), K(p)\}$.

Proof: By Theorems 7 and 8, it suffices to show that $o_p(2^{K(p)})\cdot K(p)=\operatorname{lcm}\{o_p(2), K(p)\}$. Now $o_p(2^{K(p)})=o_p(2)/\gcd(K(p), o_p(2))$. Hence

$$o_p(2^{K(p)})\cdot K(p)=o_p(2)/\gcd(K(p), o_p(2))\cdot K(p)=\operatorname{lcm}\{o_p(2), K(p)\}.\ \square$$

Corollary 6: Let p be prime with $p>5$. Define $B(p)$ as follows:

$$\begin{array}{ll} B(p)=(p-1) & p\equiv 1 \pmod{10}; \\ B(p)=5\cdot(p-1) & p\equiv 9 \pmod{10}; \\ B(p)=5\cdot(p^2-1)/2 & p\equiv 3 \text{ or } 7 \pmod{10} \text{ and } p\equiv 1 \pmod 4; \\ B(p)=5\cdot(p^2-1) & p\equiv 3 \text{ or } 7 \pmod{10} \text{ and } p\equiv 3 \pmod 4) \end{array}$$

Then $l(p)\,|\,B(p)$.

Proof: We know by Lemma 7 that $o_p(2)\,|\,(p-1)$. Now, for $p\equiv 1$ or $9 \pmod{10}$, we have $K(p)\,|\,(p-1)$. Hence, for these primes, $\operatorname{lcm}\{o_p(2), K(p)\}\,|\,(p-1)$. Thus $l(p)\,|\,B(p)$.

For $p\equiv 3$ or $7 \pmod{10}$, $K(p)\,|\,2\cdot(p+1)$. Therefore, for these primes,

$$\operatorname{lcm}\{o_p(2), K(p)\}\,|\operatorname{lcm}\{p-1, 2\cdot(p+1)\}.$$

Since

$$\begin{array}{ll} \operatorname{lcm}\{p-1, 2\cdot(p+1)\}=(p^2-1)/2 & p\equiv 1 \pmod 4, \\ \operatorname{lcm}\{p-1, 2\cdot(p+1)\}=(p^2-1) & p\equiv 3 \pmod 4, \end{array}$$

$l(p)=B(p)$. $\square$

For the bounds given in Corollary 6, the most common situation is that $l(p)=B(p)$. This is certainly the case when $o_p(2)$ equals $p-1$ and $K(p)$ equals $p-1$ or $2p+2$, depending on whether p is congruent to ± 1 or ± 3 modulo 10, respectively.

For $p\equiv 1$ or $9 \pmod{10}$, $l(p)$ can equal $B(p)$ even when $K(p)<(p-1)$. The smallest examples are

$$\begin{array}{ll} p=101: & K(101)=50, o_{101}(2)=100, \text{ so } l(101)=100=B(101), \\ p=29: & K(29)=14, o_{29}(2)=28, \text{ so } l(29)=5\cdot 28=B(29). \end{array}$$

However, $l(p)<B(p)$ if and only if both $K(p)$ and $o_p(2)$ are less than $p-1$. The smallest examples are

$$\begin{array}{ll} p=401: & K(401)=200, o_{401}(2)=200, \text{ so } l(401)=200<400=B(401), \\ p=89: & K(89)=44, o_{89}(2)=11, \text{ so } l(89)=5\cdot 44<5\cdot 88=B(89). \end{array}$$

On the other hand, for $p\equiv 3$ or $7 \pmod{10}$, $l(p)\neq B(p)$ if $K(p)<(2p+2)$. The proof of Lemma 7 shows that $K(p)\neq p+1$. Hence, if $K(p)\neq(2p+2)$, then $K(p)<(p+1)$. However, $l(p)$ can be less than $B(p)$ in a variety of ways. As we have already noted, this is the case when $K(p)<(2p+2)$. It can also occur even when $K(p)=(2p+2)$. There are 8 possibilities: $p\equiv 3$ or $7 \pmod{10}$, $p\equiv 1$ or $3 \pmod 4$, $K(p)$ less than or equal to $(2p+2)$. Here are examples of each:

$$\begin{array}{ll} p=113: & K(113)=76, o_{113}(2)=28, \text{ so } l(113)=5\cdot 532<5\cdot 6384=B(113), \\ p=73: & K(73)=148, o_{73}(2)=8, \text{ so } l(73)=5\cdot 296<5\cdot 2664=B(73), \\ p=43: & K(43)=88, o_{43}(2)=14, \text{ so } l(43)=5\cdot 616<5\cdot 1848=B(43), \\ p=263: & K(263)=176, o_{263}(2)=131, \text{ so } l(263)=5\cdot 23056<5\cdot 69168=B(263), \end{array}$$

$p=557$: $K(557)=124, o_{557}(2)=556$, so $\mathfrak{L}(557)=5\cdot 17236<5\cdot 155124=B(557)$,
$p=17$: $K(17)=36, o_{17}(2)=8$, so $\mathfrak{L}(17)=5\cdot 72<5\cdot 144=B(17)$,
$p=47$: $K(47)=32, o_{47}(2)=23$, so $\mathfrak{L}(47)=5\cdot 736<5\cdot 2208=B(47)$,
$p=127$: $K(127)=256, o_{127}(2)=7$, so $\mathfrak{L}(127)=5\cdot 1792<5\cdot 16128=B(127)$.

DETERMINING $\mathfrak{S}(2^k)$ AND $\mathfrak{L}(2^k)$

Finally, we consider powers of 2.

Lemma 13: For $k>1$, $K(2^k)=3\cdot 2^{k-1}$. Further, $\gcd(\mathrm{SUM}(5\cdot \mathrm{K}(2^k)), 2)=1$.

Proof: It is easy to verify that $K(2)=3$ and $K(2^2)\neq 3$. Thus, for $k>1$, $K(2^k)=3\cdot 2^{k-1}$.

To simplify notation, let $K=K(2^k)$, where $k>1$. Since $\phi(2^k)=2^{k-1}$, $K=3\cdot\phi(2^k)$. Thus $2^K\equiv 1 \pmod{2^k}$. Combining this observation with Lemma 8 gives us

$$\begin{aligned}\mathrm{SUM}(5K)&\equiv(2^{4K}+2^{3K}+2^{2K}+2^K+1)\cdot\mathrm{SUM}(K) \pmod{2^k}\\&\equiv 5\cdot\mathrm{SUM}(K) \pmod{2^k}.\end{aligned}$$

Thus, to show $\gcd(\mathrm{SUM}(5K),2)=1$, it suffices to show that $\gcd(\mathrm{SUM}(K),2)=1$. By Lemma 9,

$$\begin{aligned}\mathrm{SUM}(K)&\equiv\sum_{j=1}^{3\cdot 2^{k-2}}\binom{3\cdot 2^{k-1}}{2j}5^{j-1} \pmod{2^k}\\&\equiv\sum_{j=1}^{3\cdot 2^{k-2}-1}\binom{3\cdot 2^{k-1}}{2j}5^{j-1}+5^{3\cdot 2^{k-2}-1} \pmod{2^k}\\&\equiv 0+1 \pmod{2}.\end{aligned}$$

Hence $\gcd(\mathrm{SUM}(K),2)=1$. □

Theorem 9: $\mathfrak{S}(2^k)=k$ and $\mathfrak{L}(2^k)=15\cdot 2^{k-1}$.

Proof: As can easily be verified, $D^{16}(A)\equiv D(A) \pmod 2$. Thus $\mathfrak{S}(2)=1$ and $\mathfrak{L}(2)=15$.

For $k>1$, set $K=K(2^k)=3\cdot 2^{k-1}$. Note that K is even and $\gcd(K,5)=1$. Now, by Theorem 5,

$$\begin{aligned}D^{k-1+5K}(A)&=D^{k-1}(D^{5K}(A))\\&\equiv D^{k-1}(\mathrm{SUM}(5K)\cdot(1,1,1,1,1)+H^{5K/2}(A)) \pmod{2^k}\\&\equiv 2^{k-1}\cdot\mathrm{SUM}(5K)\cdot(1,1,1,1,1)+D^{k-1}(A) \pmod{2^k}.\end{aligned}$$

Since $\gcd(\mathrm{SUM}(5K),2)=1$, $2^{k-1}\cdot\mathrm{SUM}(5K)\not\equiv 0 \pmod{2^k}$. Hence, $D^{k-1+5K}(A)\not\equiv D^{k-1}(A) \pmod{2^k}$. On the other hand,

$$\begin{aligned}D^{k+5K}(A)&=D^k(D^{5k}(A))\\&\equiv D^k(\mathrm{SUM}(5K)\cdot(1,1,1,1,1)+H^{5K/2}(A)) \pmod{2^k}\\&\equiv 2^k\cdot\mathrm{SUM}(5K)\cdot(1,1,1,1,1)+D^k(A) \pmod{2^k}\\&\equiv D^k(A) \pmod{2^k}.\end{aligned}$$

Thus $\mathfrak{S}(2^k)=k$ and $\mathfrak{L}(2^k)=15\cdot 2^{k-1}$. □

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CORRIGENDUM TO THE PAPER "ON MULTIPLICITY SEQUENCES"
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It was pointed out by Professor Harvey L. Abbott that the statement in the Theorem from the paper is not true. The counterexample given by Professor Abbot is as follows:

If $g(1)=1$ *and* $g(n)=2n$ *for* $n>1$, *then* $L.C.M.(g(m),g(n))=g(L.C.M.(m,n))$ *for any* m, n *and* $G.C.D.(g(m),g(n))\neq g(G.C.D.(m,n))$ *for some* m, n.

The Theorem is true in a weaker form:

If g *is a multiplicity sequence and* g *is also quasi-multiplicative which means that* $g(m)g(n)=cg(mn)$ *for any relatively prime* m,n, *then* g *is a strong divisibility sequence.*

LACUNARY RECURRENCES FOR SUMS OF POWERS OF INTEGERS

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1. INTRODUCTION

Let k and n be nonnegative integers with $n > 0$ and let

$$S_n(k) = 1^k + 2^k + \cdots + n^k. \tag{1.1}$$

Thus, $S_n(0) = n$, $S_n(1) = n(n+1)/2$, $S_n(2) = n(n+1)(2n+1)/6$, and so forth. A well-known recurrence is

$$\sum_{j=0}^{k-1} \binom{k}{j} S_n(j) = (1+n)^k - 1 \quad (k \ge 1).$$

It is also known (and easy to prove) that

$$2\sum_{j=0}^{k-1} \binom{2k}{2j} S_n(2j) = (1+n)^{2k} - n^{2k} - 1 \quad (k \ge 1), \tag{1.2}$$

$$2\sum_{j=0}^{k-1} \binom{2k+1}{2j+1} S_n(2j+1) = (1+n)^{2k+1} - n^{2k+1} - 1 \quad (k \ge 1) \tag{1.3}$$

(see, e.g., [8, p. 160]). Howard [4] proved the following formula. For $r = 0, 1, \ldots, 5$ and $n > 0$, $k \ge 1$:

$$6\sum_{j=0}^{k-1} \binom{6k+r-3}{6j+r} S_n(6j+r) = \sum_{s=1}^{6k+r-5} \binom{6k+r-3}{s} w_{r-s} n^s, \tag{1.4}$$

where $w_j = w_{6+j}$ for $j = 0, \pm 1, \pm 2, \ldots$, and the values of w_j for $j = 0, 1, \ldots, 5$ are given by 3, 2, 0, -1, 0, and 2, respectively.

These formulas suggest there may be other simple recurrences involving only $S_n(mj+r)$, where m, n, and r are fixed and $0 \le r \le m-1$. We call such formulas "lacunary," meaning they have lacunae, or gaps. That is, the value of $S_n(mk+r)$ does not depend on *all* the previous $S_n(j)$ $(0 \le j \le mk+r)$, but only on the terms $S_n(mj+r)$ $(0 \le j \le k)$.

In the present paper the main result is Theorem 3.1, which is a general lacunary recurrence for the sums $S_n(mj+r)$. After proving Theorem 3.1 in Section 3, we illustrate it by proving the following theorem for $m = 4$.

Theorem 1.1: For $k \ge 1$ and $r = 0, 1, 2, 3$,

$$4\sum_{j=0}^{k-1} \binom{4k+r}{4j+r} [(-4)^{k-j} - 2] S_n(4j+r) = \sum_{s=1}^{4k+r-3} \binom{4k+r}{s} c_{4k+r-s} n^s,$$

where the numbers c_j are determined by the following formulas: for $j = 0, 1, 2, \ldots$,

$$c_{4j} = 2(-4)^j - 4, \quad c_{4j+1} = 2(-4)^j - 2, \quad c_{4j+2} = 0, \quad c_{4j+3} = -4(-4)^j - 2.$$

After proving Theorem 1.1, we use it to compute $S_n(5)$. One of the key ideas in all of these results is the generating function $(e^x-1)(e^{\theta x}-1)\cdots(e^{\theta^{m-1}x}-1)$, where θ is any primitive m^{th} root of unity. This generating function, an interesting topic in its own right, is discussed in Section 2.

We also prove similar formulas for the alternating sums

$$T_n(k) = 1^k - 2^k + 3^k - \cdots + (-1)^{n-1}n^k, \tag{1.5}$$

and, finally, we show how the results of this paper can be applied to the Bernoulli and Genocchi numbers.

2. GENERATING FUNCTIONS

Let θ be a primitive m^{th} root of unity so that $\theta^m = 1$ and $\theta^h \neq 1$ for $0 < h < m$. For example, we could let $\theta = e^{2\pi i/m}$.

Define the numbers b_j and c_j by means of the generating functions

$$\prod_{u=0}^{m-1}(e^{\theta^u x}-1) = (e^x-1)(e^{\theta x}-1)\cdots(e^{\theta^{m-1}x}-1) = \sum_{j=0}^{\infty} b_j \frac{x^j}{j!}, \tag{2.1}$$

and

$$e^x \prod_{u=1}^{m-1}(e^{\theta^u x}-1) = \sum_{j=0}^{\infty} c_j \frac{x^j}{j!}. \tag{2.2}$$

Note that any primitive m^{th} root of unity can be used in (2.1) and (2.2). The numbers b_j and c_j depend on m, but the value of m will always be clear when we use this notation. Note also that $b_0 = 0$ and for $m = 1$, we have $c_j = 1$.

If we replace x by $-x$ in (2.2), we have

$$\begin{aligned}\sum_{j=0}^{\infty}(-1)^j c_j \frac{x^j}{j!} &= e^{-x}\prod_{u=1}^{m-1}(e^{-\theta^u x}-1) = \prod_{u=1}^{m-1}(e^{\theta^u x}-1)/[(-1)^{m-1}e^{x(1+\theta+\cdots+\theta^{m-1})}] \\ &= (-1)^{m-1}\prod_{u=1}^{m-1}(e^{\theta^u x}-1).\end{aligned}$$

This gives us another useful generating function for c_j:

$$\prod_{u=1}^{m-1}(e^{\theta^u x}-1) = \sum_{j=0}^{\infty}(-1)^{m+j-1}c_j \frac{x^j}{j!}. \tag{2.3}$$

From (2.1), (2.2), and (2.3), we have

$$\begin{aligned}\sum_{j=0}^{\infty} c_j \frac{x^j}{j!} &= e^x\prod_{u=1}^{m-1}(e^{\theta^u x}-1) = \prod_{u=0}^{m-1}(e^{\theta^u x}-1) + \prod_{u=1}^{m-1}(e^{\theta^u x}-1) \\ &= \sum_{j=0}^{\infty} b_j \frac{x^j}{j!} + \sum_{j=0}^{\infty}(-1)^{m+j-1}c_j \frac{x^j}{j!}.\end{aligned} \tag{2.4}$$

Thus, we have $b_j = (1+(-1)^{m+j})c_j$; that is,

$$b_j = \begin{cases} 2c_j & \text{if } (m+j) \text{ is even,} \\ 0 & \text{if } (m+j) \text{ is odd.}\end{cases} \tag{2.5}$$

To prove the main result of this section, Theorem 2.1, we need the following lemma.

Lemma 2.1 (multisection of series): Let θ be any primitive m^{th} root of unity (such as $e^{2\pi i/m}$) and let $F(x)=\sum_{k=0}^{\infty}a_k x^k$ for complex numbers a_k. Then, for $r=0,1,\ldots,m-1$,

$$\sum_{j=0}^{\infty}a_{mj+r}x^{mj+r}=\frac{1}{m}\sum_{j=0}^{m-1}\theta^{(m-j)r}F(\theta^j x). \tag{2.6}$$

If z_0 is a complex number in the circle of convergence of $F(x)$, we can replace x by z_0 in (2.6). Multisection is discussed in [8, p. 131], and a proof of Lemma 2.1 is given in [3].

Theorem 2.1: Let θ be a primitive m^{th} root of unity, and let b_j be defined by (2.1). Then $b_j=0$ unless j is a multiple of m. Furthermore, if m is odd then $b_j=0$ unless j is an *odd* multiple of m.

Proof: We take the logarithm of both sides of (2.1) to obtain

$$\log(e^x-1)+\log(e^{\theta x}-1)+\cdots+\log(e^{\theta^{m-1}x}-1)=\log\sum_{j=0}^{\infty}b_j\frac{x^j}{j!}. \tag{2.7}$$

In (2.6), let $F(x)=\log(e^x-1)$ and $r=0$, and compare the left side of (2.7) with the right side of (2.6) to obtain

$$m\sum_{j=0}^{\infty}a_{mj}x^{mj}=\log\sum_{j=0}^{\infty}b_j\frac{x^j}{j!}. \tag{2.8}$$

Applying the exponential function to (2.8), we have

$$\exp\left(m\sum_{j=0}^{\infty}a_{mj}x^{mj}\right)=\sum_{j=0}^{\infty}b_j\frac{x^j}{j!}. \tag{2.9}$$

We now compare coefficients of x^j on both sides of (2.9) and see that $b_j=0$ unless j is a multiple of m. Now suppose m is odd. Replacing x by $-x$ in (2.1), we have

$$(e^{-x}-1)(e^{-\theta x}-1)\cdots(e^{-\theta^{m-1}x}-1)=\sum_{j=0}^{\infty}(-1)^{mj}b_{mj}\frac{x^{mj}}{(mj)!}.$$

Thus,

$$\prod_{u=0}^{m-1}(e^{-ux}-1)+\prod_{u=0}^{m-1}(e^{ux}-1)=2\sum_{j=0}^{\infty}b_{2mj}\frac{x^{2mj}}{(2mj)!}. \tag{2.10}$$

Now we observe that

$$\prod_{u=0}^{m-1}(e^{-\theta^u x}-1)=\prod_{u=0}^{m-1}(e^{\theta^u x}-1)/[(-1)^m e^{x(1+\theta+\cdots+\theta^{m-1})}]=-\prod_{u=0}^{m-1}(e^{\theta^u x}-1).$$

Thus, the left side of (2.10) is equal to 0 and, therefore, $b_{2mj}=0$ for $j\geq 0$. This completes the proof. □

Theorem 2.1 tells us that the generating function (2.1) could be written as:

$$\prod_{u=0}^{m-1}(e^{\theta^u x}-1)=\sum_{j=0}^{\infty}b_{mj}\frac{x^{mj}}{(mj)!}\quad (m\text{ even}); \tag{2.11}$$

$$\prod_{u=0}^{m-1}(e^{\theta^u x}-1)=\sum_{j=0}^{\infty} b_{m(2j+1)}\frac{x^{m(2j+1)}}{(m(2j+1))!} \quad (m \text{ odd}). \tag{2.12}$$

3. A GENERAL FORMULA

We are now ready to prove our main result, a general lacunary recurrence for the sums $S_n(mj+r)$. We will need the following generating function:

$$F_1(x)=\frac{e^{(1+n)x}-e^x}{e^x-1}=e^x+e^{2x}+\cdots+e^{nx}=\sum_{j=0}^{\infty}S_n(j)\frac{x^j}{j!}. \tag{3.1}$$

Theorem 3.1: Let $S_n(j)$, b_j, and c_j be defined by (1.1), (2.1), and (2.2), respectively. If m is a positive integer, then, for $r=0,1,\ldots,m-1$:

$$\sum_{j=0}^{k-1}\binom{mk+r}{mj+r}b_{(k-j)m}S_n(mj+r)=\sum_{s=1}^{(k-1)m+r+1}\binom{mk+r}{s}c_{mk+r-s}n^s. \tag{3.2}$$

If m is odd and $k\ge 1$, then, for $r=0,1,\ldots,2m-1$:

$$\sum_{j=0}^{k-1}\binom{(2k-1)m+r}{2mj+r}b_{(2k-1-2j)m}S_n(2mj+r)=\sum_{s=1}^{2m(k-1)+r+1}\binom{(2k-1)m+r}{s}c_{(2k-1)m+r-s}n^s. \tag{3.3}$$

Proof: Let $F_1(x)$ be defined by (3.1). We multiply both sides of (2.1) by $F_1(x)$ to obtain

$$(e^{nx}-1)e^x\prod_{u=1}^{m-1}(e^{\theta^u x}-1)=F_1(x)\sum_{j=0}^{\infty}b_j\frac{x^j}{j!}. \tag{3.4}$$

Recalling (2.2) and (2.11), we compare coefficients of x^j on both sides of (3.4) to derive (3.2) for m even or odd.

If m is odd, then by (2.12) we can let $k-j$ be odd in (3.2). We now consider the cases of k even and k odd to obtain (3.3).

Case 1: k is even. In (3.2), since $k-j$ is odd, replace k by $2k$ and j by $2j+1$ to obtain

$$\begin{aligned}&\sum_{j=0}^{k-1}\binom{(2k-1)m+(m+r)}{2mj+(m+r)}b_{(2k-1-2j)m}S_n(2mj+m+r)\\&=\sum_{s=1}^{2m(k-1)+m+r+1}\binom{(2k-1)m+(m+r)}{s}c_{(2k-1)m+(m+r)-s}n^s.\end{aligned} \tag{3.5}$$

If we let $r'=(m+r)$;in (3.5), we get (3.3) with r replaced by r' and $m\le r'<2m$.

Case 2: k is odd. In (3.2), replace k by $2k-1$ and replace j by $2j$ to obtain (3.3) with $0\le r<m$.

Combining the two cases gives us (3.3) with $0\le r<2m$. This completes the proof. □

We illustrate Theorem 3.1 by proving formulas (1.2) and (1.3) and Theorem 1.1.

Let $m=2$. From definitions (2.1) and (2.2), we see that $b_{2j+1}=0$ and, for $j>0$, $b_{2j}=-2$ and $c_j=-1$. Thus,

$$2\sum_{j=0}^{k-1}\binom{2k+r}{2j+r}S_n(2j+r)=\sum_{s=1}^{2k+r-1}\binom{2k+r}{s}n^s,$$

which is equivalent to (1.2) and (1.3).

Formula (1.4) can be deduced from Theorem 3.1 by letting $m=3$.

To prove Theorem 1.1, we let $m=4$ and $\theta=i$. The left side of

$$e^x(e^{ix}-1)(e^{i^2x}-1)(e^{i^3x}-1)=\sum_{j=0}^{\infty}c_j\frac{x^j}{j!}$$

can be written

$$2-2e^x-e^{ix}-e^{-ix}-e^{(1+i)x}-e^{(1-i)x},$$

so for $j>0$,

$$c_j=-2-i^j-(-i)^j+(1+i)^j+(1-i)^j.$$

This gives us

$$c_{4j}=-4+2(-4)^j,\quad c_{4j+1}=-2+2(-4)^j,\quad c_{4j+2}=0,\quad c_{4j+3}=-2-4(-4)^j. \tag{3.6}$$

By (2.5) we have, for $j\geq 1$,

$$b_{4j}=2c_{4j}=-8+4(-4)^j. \tag{3.7}$$

For $m=4$ and the values of b_j and c_j given by (3.6) and (3.7), equation (3.2) gives Theorem 1.1. This completes the proof. □

To illustrate Theorem 1.1, we compute $S_n(5)$. In Theorem 1.1, let $r=1$ and $k=2$ to obtain

$$-24\binom{9}{5}S_n(5)=-56\binom{9}{1}S_n(1)+\sum_{s=1}^{6}\binom{9}{s}c_{9-s}n^s.$$

Using (3.6) and the formula $S_n(1)=n(n+1)/2$, we have

$$S_n(5)=-\frac{1}{12}n^2+\frac{5}{12}n^4+\frac{1}{2}n^5+\frac{1}{6}n^6.$$

We could easily keep going here and compute $S_n(9)$, $S_n(13)$, and so on.

4. ALTERNATING SUMS

The methods of Sections 2 and 3 can be used just as easily on the alternating sums $T_n(k)$ defined by (1.5). Let θ be any primitive m^{th} root of unity, and define the numbers g_j and h_j by means of the generating functions

$$\prod_{u=0}^{m-1}(e^{\theta^u x}+1)=(e^x+1)(e^{\theta x}+1)\cdots(e^{\theta^{m-1}x}+1)=\sum_{j=0}^{\infty}g_j\frac{x^j}{j!}, \tag{4.1}$$

and

$$e^x\prod_{u=1}^{m-1}(e^{\theta^u x}+1)=\sum_{j=0}^{\infty}h_j\frac{x^j}{j!}. \tag{4.2}$$

Note that g_j and h_j are functions of m.

Analogous to (2.3), and proved in the same way, is another generating function for h_j:

$$\prod_{u=1}^{m-1}(e^{\theta^u x}+1)=\sum_{j=0}^{\infty}(-1)^j h_j \frac{x^j}{j!}. \tag{4.3}$$

Equations (4.1), (4.2), and (4.3) give us the relationship $h_j=\frac{1}{2}g_j$ if j is even.

Theorems 4.1 and 4.2 are analogous to Theorems 2.1 and 3.1, and they are proved in exactly the same say. The following generating function is used in the proof of Theorem 4.2:

$$F_2(x)=\frac{(-1)^{1+n}e^{(1+n)x}+e^x}{e^x+1}=e^x-e^{2x}+\cdots+(-1)^{1+n}e^{nx}=\sum_{j=0}^{\infty}T_n(j)\frac{x^j}{j!}. \tag{4.4}$$

Theorem 4.1: Let θ be a primitive m^{th} root of unity and let g_j be defined by (4.1). Then $g_j=0$ unless j is a multiple of m. Furthermore, if m is odd, then $g_j=0$ unless j is an *even* multiple of m.

Theorem 4.2: Let $T_n(j)$ be defined by (1.5) and let g_j and h_j be defined by (4.1) and (4.2), respectively. If m is a positive integer, then, for $r=0,1,\ldots,m-1$,

$$\sum_{j=0}^{k}\binom{mk+r}{mj+r}g_{(k-j)m}T_n(mj+r)=(-1)^{n+1}\sum_{s=0}^{mk+r}\binom{mk+r}{s}h_{mk+r-s}n^s+h_{mk+r}. \tag{4.5}$$

If m is odd, then, for $r=0,1,\ldots,2m-1$,

$$\sum_{j=0}^{k}\binom{2mk+r}{2mj+r}g_{(k-j)2m}T_n(2mj+r)=(-1)^{n+1}\sum_{s=0}^{2mk+r}\binom{2mk+r}{s}h_{2mk+r-s}n^s+h_{2mk+r}. \tag{4.6}$$

We note that, by (4.2), $h_0=2^{m-1}$, so the right sides of (4.5) and (4.6) are polynomials in n of degrees $mk+r$ and $2mk+r$, respectively. We also note that $g_0=2^m$.

For example, let $m=2$ and $\theta=-1$. Then we have $g_0=4$, $g_{2j}=2$ if $j>0$, $h_0=2$, and $h_j=1$ if $j>0$. Theorem 4.2 gives us

$$\begin{aligned}4T_n(2k+r)=&-2\sum_{j=0}^{k-1}\binom{2k+r}{2j+r}T_n(2j+r)+1+(-1)^{n+1}\\&+(-1)^{n+1}\left\{\sum_{s=1}^{2k+r-1}\binom{2k+r}{s}n^s+2n^{2k+r}\right\}.\end{aligned}$$

The following formula for $m=3$, which is analogous to (1.4), was given in [4]. Let $m=3$, let θ be a primitive third root of unity, and let w_j be defined as in Section 1. Then, for $n>0$ and $k\geq 0$, with r and k not both 0,

$$\begin{aligned}8T_n(6k+r)=&-6\sum_{j=0}^{k-1}\binom{6k+r}{6j+r}T_n(6j+r)+[1+(-1)^{n+1}]w_r\\&+(-1)^{n+1}\left\{\sum_{s=1}^{6k+r-1}\binom{6k+r}{s}w_{r-s}n^s+4n^{6k+r}\right\}.\end{aligned}$$

If $m=4$, we use Theorem 4.2 to prove the following new result.

Theorem 4.3: Let $n>0$ and $k\ge 0$. Then, for $r=0,1,2,3,$ and r and k not both 0,

$$16T_n(4k+r)=-4\sum_{j=0}^{k-1}\binom{4k+r}{4j+r}[(-4)^{k-j}+2]T_n(4j+r)+h_{4k+r}$$
$$+(-1)^{n+1}\sum_{s=0}^{4k+r}\binom{4k+r}{s}h_{4k+r-s}n^s,$$

where $h_0=8$ and the numbers h_j, except for h_0, are determined by the following formulas: for $j=0,1,2,\dots,$

$$h_{4j}=2(-4)^{k-h}+4,\quad h_{4j+1}=2(-4)^{k-h}+2,\quad h_{4j+2}=0,\quad h_{4j+3}=-4(-4)^{k-h}+2.$$

5. BERNOULLI AND GENOCCHI NUMBERS

The methods of this paper can be applied to other special number sequences. For example, consider the Bernoulli numbers B_n defined by the generating function

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}. \tag{5.1}$$

These numbers are well known and have been studied extensively (see, e.g., [7, ch. 2]). It is well known that $B_0=1$, $B_1=-\frac{1}{2}$, and $B_{2k+1}=0$ for $k>0$.

We can use the methods of this paper to derive the following general lacunary recurrence for the Bernoulli numbers.

Theorem 5.1: Let B_n be defined by (5.1) and let b_j and c_j be defined by (2.1) and (2.2), respectively. If m is a positive integer and $k>0$, then, for r even, $0\le r<m$,

$$\sum_{j=0}^{k-1}\binom{mk+r}{mj+r}b_{(k-j)m}B_{mj+r}=(mk+r)c_{mk+r-1}.$$

If m is odd and $k>0$, then, for r even, $0\le r<2m$,

$$\sum_{j=0}^{k-1}\binom{(2k-1)m+r}{2mj+r}b_{(2k-1-2j)m}B_{2mj+r}=[(2k-1)m+r]c_{(2k-1)m+r-1}.$$

Proof: Multiply both sides of (2.1) by $x/(e^x-1)$ to obtain

$$x\prod_{u=1}^{m-1}(e^{\theta^u x}-1)=\frac{x}{e^x-1}\sum_{j=0}^{\infty}b_j\frac{x^j}{j!}. \tag{5.2}$$

By (2.3) and (5.2) we have, for $n>0$,

$$(-1)^{m+n}nc_{n-1}=\sum_{j=0}^{n}\binom{n}{j}B_jb_{n-j}.$$

The remainder of the proof is similar to the proof of Theorem 3.1. □

Several writers, like Chellali [1], Lehmer [5], Ramanujan [7], and Riordan [8, pp. 136-40] have developed lacunary formulas for the Bernoulli numbers (see [2] for references).

Obviously, the methods of this paper can also be used on the Genocchi numbers G_n, which are defined by

$$\frac{2x}{e^x+1}=\sum_{n=1}^{\infty}G_n\frac{x^n}{n!}.$$

Lacunary recurrences for the Genocchi numbers can be found in [2]. Incidentally, it is well known that the Genocchi numbers are integers and that $G_{2j}=2(1-2^{2j})B_{2j}$.

As a final comment, we note that the numbers b_j and g_j of this paper are special cases of the generalized Bernoulli and Euler numbers of Nörlund [6, pp. 142-43], which are defined by

$$(e^{\omega_1 x}-1)(e^{\omega_2 x}-1)\cdots(e^{\omega_m x}-1)=(\omega_1\omega_2\ldots\omega_m)\sum_{j=0}^{\infty}B_j^{(-m)}(\omega_1,\ldots,\omega_m)\frac{x^{j+m}}{j!}$$

and

$$(e^{\omega_1 x}+1)(e^{\omega_2 x}+1)\cdots(e^{\omega_m x}+1)=\sum_{j=0}^{\infty}2^{m-j}C_j^{(-m)}(\omega_1,\ldots,\omega_m)\frac{x^j}{j!},$$

where $\omega_1,\ldots,\omega_m$ are arbitrary complex numbers. To the writer's knowledge, none of the properties of b_n and g_n developed in this paper were proved by Nörlund.

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RECURRENCE RELATIONS FOR POWERS OF RECURSION SEQUENCES

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1. INTRODUCTION

In this article I present some partial answers to the open questions raised by Cooper and Kennedy in [1]. In that article the authors asked whether there exists a recurrence relation among powers x_n^l, where x_n represents the solution to a given recurrence relation. I answer this in the affirmative below, include a few details about the corresponding order, then indicate a way to calculate any such relation the reader might seek, and, finally, state a few results from such calculations.

An informal sketch of the proof and procedure runs as follows. Every solution to a recurrence relation can be expressed as a linear combination of powers of roots to the characteristic polynomial. The coefficients of the original recurrence relation are the elementary symmetric polynomials in these roots. Every power of a solution can be expressed as a linear combination of products of powers of these roots by using the general multinomial theorem. These products can be used as roots to form a new characteristic polynomial. On inspection, the coefficients of this new characteristic polynomial are symmetric in the roots of the old characteristic polynomial, and, therefore, can be expressed as polynomials in the elementary symmetric polynomials of the roots; that is, the coefficients of the new recurrence relation can be expressed in terms of the coefficients of the original characteristic polynomial. There is a method for obtaining the expression, amounting to a multivariate version of the Euclidean algorithm.

2. EXISTENCE

Let $x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}$ be a linear homogeneous recurrence relation with constant coefficients $\{a_i \mid i = 1, \ldots, k\}$ and of order k. Let $p(x) = x^k - a_1x^{k-1} - \cdots - a_k$ be the characteristic polynomial for this relation. Let $p(x)$ factor as $p(x) = (x - r_1)(x - r_2)\cdots(x - r_k)$ over the field of complex numbers and suppose that the roots are distinct. We can write the Binet closed form for x_n as $x_n = A_1r_1^n + A_2r_2^n + \cdots + A_kr_k^n$. The constants $\{A_i \mid i = 1, \ldots, k\}$ are determined by the initial conditions specified in a particular solution to the recurrence relation.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ be two k-tuples. Define the symbol α^β to be the product of all terms α_i raised to the β_i power:

$$\alpha^\beta = \prod_{i=1}^{k} (\alpha_i^{\beta_i}).$$

Writing $X_n = (A_1r_1^n, A_2r_2^n, \ldots, A_kr_k^n)$, $A = (A_1, A_2, \ldots, A_k)$, and $R = (r_1, r_2, \ldots, r_k)$, we see that $X^\alpha = A^\alpha(R^\alpha)^n$ for each k-tuple α.

Recall the definition of the multinomial coefficient $c(\alpha) = (\alpha_1 + \cdots + \alpha_k)!/(\alpha_1! \ldots \alpha_k!)$.

Introduce the indexing set $B_l = \{(i_1, \ldots, i_k) \mid \text{each } i_j \text{ is a nonnegative integer and } i_1 + \cdots + i_k = l\}$.

Theorem 1: If $x_n = \sum_{i=1}^{k} A_i r_i^n$, then $x_n^l = \sum_{\alpha \in B_l} c(\alpha) A^\alpha (R^\alpha)^n$.

Proof:

$$x_n^l = \left(\sum_{i=1}^{k} A_i r_i^n \right)^l = \sum_{\alpha \in B_l} c(\alpha) X^\alpha = \sum_{\alpha \in B_l} c(\alpha) A^\alpha (R^\alpha)^n$$

by the general multinomial Theorem (see Hungerford [3, Th. 1.6, p. 118]). □

Therefore, $y_n = x_n^l$ is a linear combination of terms that are products of roots from the original characteristic polynomial of total degree l, raised to the n^{th} power. Thus, y_n is a solution to a recurrence relation. The next theorem tells us more.

Theorem 2: The characteristic polynomial for the sequence y_n can be written in terms of the coefficients of the characteristic polynomial $p(x)$ for x_n.

Proof: The characteristic polynomial for y_n is

$$q(x) = \prod_{\alpha \in B_l} (x - R^\alpha) \in \mathbb{C}[x] \text{ by Theorem 1.}$$

Consider some permutation $\sigma : \{r_1, \ldots, r_k\} \to \{r_1, \ldots, r_k\}$ of the roots of $p(x)$. σ can be decomposed into a product of transpositions [3, p. 48]. Each transposition interchanges two roots, say r_m and r_n. The effect of this transposition is to interchange the exponents from $\alpha = (i_1, \ldots, i_m, \ldots, i_n, \ldots, i_k)$ to $\alpha' = (i_1, \ldots, i_n, \ldots, i_m, \ldots, i_k)$ within the indexing set B_l. Thus, each transposition represents a transposition of the elements of B_l because the conditions defining k-tuples in B_l are unchanged by switching values positionally. The composition of transpositions that give σ also describe a composition of transpositions in B_l. Thus, σ gives rise to a permutation of B_l. Since the product for $q(x)$ is formed over the entire set B_l, this permutation leaves $q(x)$ fixed.

If we were to expand $q(x)$ into its standard form, the coefficients would be polynomial expressions in the roots $\{r_1, \ldots, r_k\}$. These coefficients are invariant under permutation of the roots and so are symmetric polynomials in the roots. Any such symmetric polynomial can be expressed as a polynomial in the elementary symmetric functions [2, p. 307].

Since these elementary symmetric polynomials are exactly the coefficients of the characteristic polynomial $p(x)$, the coefficients of $q(x)$ can be written as expressions in the coefficients of $p(x)$. □

3. ORDER

What is the order of the recurrence relation $y_n = x_n^l$?

It should be the degree of the characteristic polynomial $q(x)$. This degree is counted by the number of elements in B_l. Given a value of k, define $S(k, 1) = |B_l|$ where, recall, $B_l = \{(i, \ldots, i_k) |$ each i_j is a nonnegative integer and $i_1 + \cdots + i_k = l\}$.

Theorem 3: $S(k, l)$ obeys the relations: $S(k, 1) = k$ for all k, $S(1, l) = l$ for all l, and $S(k, l) = S(k-1, l) + S(k, l-1)$ for every k and l.

Proof: Proceed inductively.

$S(k, 1)$ represents the number of ways to define a k-tuple of nonnegative integers that add to 1. There are k ways to do this, corresponding to placing a 1 in any one of the k places in the k-tuple and 0 everywhere else.

$S(1, l)$ represents the number of ways to have a 1-tuple of nonnegative integers that add to l. There is only one such way.

Let $\alpha = (i_1, \ldots, i_k)$ be a generic element of B_l. Either $i_1 = 0$ or $i_1 > 0$. If $i_1 = 0$, then $i_2 + \cdots + i_k = l$, and the number of possible ways to select such i_j's is $S(k-1, l)$; in other words, a sum of l obtained with $k-1$ variables. If $i_1 > 0$, then we can subtract one from i_1 to get $(i_1 - 1, \ldots, i_k)$ as an element of B_{l-1} that has $S(k, l-1)$ elements. Therefore, the total number of possibilities is $S(k-1, l) + S(k, l-1)$. □

Theorem 4: $S(k, l) = {}_{l+k-1}C_{k-1}$, where ${}_iC_j$ stands for the binomial coefficient $i!/[j!(i-j)!]$.

Proof: Substituting $l = 1$ gives ${}_{l+k-1}C_{k-1} = {}_kC_{k-1} = k$ while substituting $k = 1$ gives ${}_{l+k-1}C_{k-1} = {}_lC_0 = 1$. To check that the given binomial coefficient satisfies the recurrence relation, simplify

$$ {}_{l+(k-1)-1}C_{(k-1)-1} + {}_{(l-1)+k-1}C_{k-1} = {}_{l+k-2}C_{k-2} + {}_{l+k-2}C_{k-1} = {}_{l+k-1}C_{k-1}. $$

The binomial coefficient ${}_{l+k-1}C_{k-1}$ satisfies the recurrence relation and the initial conditions. Therefore, it is the solution to this recurrence relation and, by Theorem 3, $S(k, l) = {}_{l+k-1}C_{k-1}$. □

This answer, an order of ${}_{l+k-1}C_{k-1}$ for y_n, represents the largest order sufficient to express y_n as a recurrence relation. It is not the least order necessary. The reason for the discrepancy is that the various values for the products of powers of roots might not be distinguishable arithmetically, while in the above proof the various terms were distinguished symbolically. As an example, suppose $x_n = 1^n + 2^n + 3^n + 6^n$ with characteristic equation

$$ p(x) = (x-1)(x-2)(x-3)(x-6). $$

The process above indicates that a characteristic polynomial for $y_n = x_n^2$ would be

$$ q(x) = (x-1)(x-2)(x-3)(x-6)(x-4)(x-6)(x-12)(x-9)(x-18)(x-36); $$

However, we do not require a double root of 6, obtained on the one hand by $r_1 r_4 = 1 * 6 = 6$ and on the other hand by $r_2 r_3 = 2 * 3 = 6$.

We can obtain a sharp result if we assume that all the elements of $\alpha \in B_l$ gives rise to a unique value for R^α. We would wish for some general criteria for determining whether all such values are distinct, without arithmetically checking all the possibilities. One such criterion would be the assumption that each root is an integer and each root is divisible by a different prime. Then, given an arithmetic value for a product of roots, we could identify the factors by determining the power of the corresponding prime unique to each root. This would determine the power of the root that comprises the overall product.

4. GENERATING A RELATIONSHIP

One starts with the order k of the recurrence relation for x_n and one decides upon the power l in $y_n = x_n^l$. Next, construct $q(x)$ symbolically, and expand the expression algebraically to obtain the coefficients for $q(x)$ as explicit symmetric polynomials. Finally, write these coefficients in terms of the elementary symmetric polynomials (see Cox [2, pp. 307-09] for more details). The algorithm amounts to successively subtracting appropriate powers of the elementary symmetric polynomials. The powers are obtained by identifying the leading term of the symmetric expression, and using the powers of this monomial to determine powers for products of the elementary symmetric polynomials.

Example: Let $x_n = A_1x_{n-1} + A_2x_{n-2}$, which gives rise to the characteristic polynomial

$$p(x) = (x - r_1)(x - r_2).$$

Notice that $A_1 = r_1 + r_2$ and $A_2 = -r_1r_2$. Let $y_n = x_n^2$, which has derived characteristic polynomial

$$q(x) = (x - r_1^2)(x - r_1r_2)(x - r_2^2).$$

This expands to

$$q(x) = x^3 - (r_1^2 + r_1r_2 + r_2^2)x^2 + (r_1^3r_2 + r_1^2r_2^2 + r_1r_2^3)x - r_1^3r_2^3,$$

which leads to the recurrence relation

$$y_n = (r_1^2 + r_1r_2 + r_2^2)y_{n-1} - (r_1^3r_2 + r_1^2r_2^2 + r_1r_2^3)y_{n-2} + r_1^3r_2^3y_{n-3}.$$

Performing the algorithm indicated in Cox, Little, and O'Shea [2] gives

$$(r_1^2 + r_1r_2 + r_2^2) = (r_1 + r_2)^2 - r_1r_2 = A_1^2 + A_2,$$
$$(r_1^3r_2 + r_1^2r_2^2 + r_1r_2^3) = (r_1 + r_2)^2(r_1r_2) - r_1^2r_2^2 = -A_1^2A_2 - A_2^2,$$
$$r_1^3r_2^3 = (r_1r_2)^3 = -A_2^3.$$

Consequently,

$$y_n = (A_1^2 + A_2)y_{n-1} + (A_1^2A_2 + A_2^2)y_{n-2} - (A_2^3)y_{n-3}. \quad \square$$

5. RESULTS

Using the computer algebra system Maple® and the easier algorithm indicated in [2, pp. 309-10] yielded the following results: If $x_n = ax_{n-1} + bx_{n-2} + cx_{n-3}$ and $y_n = x_n^3$, then

$$y_n = \sum_{i=1}^{10} a_i y_{n-i}$$

with

$$\begin{aligned}
a_1 &= a^3 + 2ba + c,\\
a_2 &= 2b^3 + ba^4 + 3b^2a^2 + ca^3 + 2cba,\\
a_3 &= 3c^3 + ca^6 - 2b^4a - b^3a^3 + 11c^2ba + 7cba^4 + 5c^2a^3 + 10cb^2a^2,\\
a_4 &= -3cb^3a^3 + 4cb^4a + 2c^3a^3 + 2c^2b^3 - cb^2a^5 + c^2a^6 - 13c^3ba + c^2ba^4 - 13c^2b^2a^2 - 3c^4 - b^6,\\
a_5 &= c^2b^2a^5 - 4c^4ba - 7c^3ba^4 - 5c^4a^3 + 5c^3b^3 - c^3a^6 - cb^6 + 7c^2b^4a - cb^5a^2 + 8c^2b^3a^3,\\
a_6 &= -2c^5a^3 - c^4a^6 + c^2b^6 - 2c^4b^3 - 4c^4ba^4 - 13c^4b^2a^2 - c^3b^4a + c^2b^5a^2 - 3c^3b^3a^3 - 3c^6 - 13c^5ba,\\
a_7 &= -c^4b^3a^3 + 2c^5ba^4 + c^3b^6 - 7c^4b^4a + 10c^5b^2a^2 - 5c^5b^3 + 11c^6ba + 3c^7,\\
a_8 &= -2c^7a^3 + 2c^7ba - b^3c^6 - c^5b^4a + 3a^2c^6b^2,\\
a_9 &= 2c^8ba + c^9 - b^3c^7, \quad \text{and}\\
a_{10} &= -c^{10}.
\end{aligned}$$

If $x_n = ax_{n-1} + bx_{n-2} + cx_{n-3} + dx_{n-4}$ and $y_n = x_n^2$, then

$$y_n = \sum_{i=1}^{10} a_i y_{n-i}$$

with

$$\begin{aligned}
a_1 &= a^2 + b, \\
a_2 &= b^2 + ba^2 + ca + d, \\
a_3 &= -b^3 + ca^3 + 2c^2 + 4cba, \\
a_4 &= db^2 + 4dba^2 + 5dca + 2d^2 + da^4 - c^2b - cb^2a + c^2a^2, \\
a_5 &= dca^3 - db^2a^2 - 2d^2b + d^2a^2 - dc^2 - 2db^3 - c^3a + c^2b^2, \\
a_6 &= -c^4 + dcb^2a - 2d^3 - 5d^2ca - d^2ba^2 + 4dc^2b - dc^2a^2 - d^2b^2, \\
a_7 &= 4d^2cba - dac^3 - 2a^2d^3 - d^2b^3, \\
a_8 &= -d^3b^2 - cd^3a - d^4 + c^2d^2b, \\
a_9 &= bd^4 - d^3c^2, \quad \text{and} \\
a_{10} &= d^5.
\end{aligned}$$

6. FURTHER RESEARCH

One of the assumptions throughout this article is that the roots $\{r_i \mid i = 1, \ldots, k\}$ are all distinct. The next step in investigating this problem would be to allow for several repeated roots. Each of the roots r_i could have a multiplicity k_i, which would lead to a polynomial of degree $(k_i - 1)$ in n as the coefficient of r_i^n in the Binet form. These different polynomials would then combine in the multinomial theorem to form repeated roots of high degree. Tracing this argument through carefully might yield precise estimates for the order of the recurrence relation $y_n = x_n^l$.

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A CLOSED FORM OF THE (2, F) GENERALIZATIONS OF THE FIBONACCI SEQUENCE

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1. INTRODUCTION

In this paper we consider the generalized (2, F) sequences. They are introduced in [1] and [2], and some of their properties are studied in [1], [2], [5], [7], [8], and [9]. The generalized (2, F) sequences $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ are defined by their first two elements and two linear equalities:

$$x_0 = a,\ x_1 = b,\ y_0 = c,\ y_1 = d,$$
$$x_{n+2} = \alpha x_{n+1} + \beta y_n,\ y_{n+2} = \gamma y_{n+1} + \delta x_n,$$

for $n \geq 0$. In [1] the following open problem is given: Find a closed form of x_n and y_n for arbitrary n, i.e., represent them as functions of n, a, b, c, d, α, β, γ, and δ. In [5] such functions are obtained. They have one of the following five forms:

$$\begin{array}{lll}
x_n = C_1\rho_1^n + C_2\rho_2^n + C_3\rho_3^n + C_4\rho_4^n, & y_n = C_5\rho_1^n + C_6\rho_2^n + C_7\rho_3^n + C_8\rho_4^n, & \text{or} \\
x_n = C_1\rho_1^n + C_2\rho_2^n + (C_3 + nC_4)\rho_3^n, & y_n = C_5\rho_1^n + C_6\rho_2^n + (C_7 + nC_8)\rho_3^n, & \text{or} \\
x_n = C_1\rho_1^n + (C_2 + C_3 n + C_4 n^2)\rho_2^n, & y_n = C_5\rho_1^n + (C_6 + C_7 n + C_8 n^2)\rho_2^n, & \text{or} \\
x_n = (C_1 + C_2 n + C_3 n^2 + C_4 n^3)\rho_1^n, & y_n = (C_5 + C_6 n + C_7 n^2 + C_8 n^3)\rho_1^n, & \text{or} \\
x_n = (C_1 + C_2 n)\rho_1^n + (C_3 + C_4 n)\rho_2^n, & y_n = (C_5 + C_6 n)\rho_1^n + (C_7 + C_8 n)\rho_2^n, &
\end{array}$$

where ρ_1, ρ_2, ρ_3, and ρ_4 are the roots (complex in the general case) of the equation

$$\rho^4 - (\alpha + \gamma)\rho^2 + \alpha\gamma\rho - \beta\delta = 0$$

(the above five cases correspond to four simple roots, two simple roots and one double root, ..., two double roots, respectively) and C_i, $1 \leq i \leq 8$ are (complex) constants depending on a, b, c, d, and ρ_i, $1 \leq i \leq 4$.

We shall give an alternative closed form for x_n and y_n. Our approach is fully combinatorial (it is based on an enumeration of weighted paths in an infinite graph) whereas the Georgieu-Atanassov method is from linear algebra (it uses Jordan's factorization form of some matrix). More concretely, we shall prove the following.

Theorem 1 (Main result): The equalities

$$x_n = a \sum_{4p+q+r=n-4} \binom{p+q}{q}\binom{p+r}{r} \alpha^q \beta^{p+1} \gamma^r \delta^{p+1} + b \sum_{4p+q+r=n-1} \binom{p+q}{q}\binom{p+r-1}{r} \alpha^q \beta^p \gamma^r \delta^p$$
$$+ c \sum_{4p+q+r=n-2} \binom{p+q}{q}\binom{p+r-1}{r} \alpha^q \beta^{p+1} \gamma^r \delta^p + d \sum_{4p+q+r=n-3} \binom{p+q}{q}\binom{p+r}{r} \alpha^q \beta^{p+1} \gamma^r \delta^p$$

and

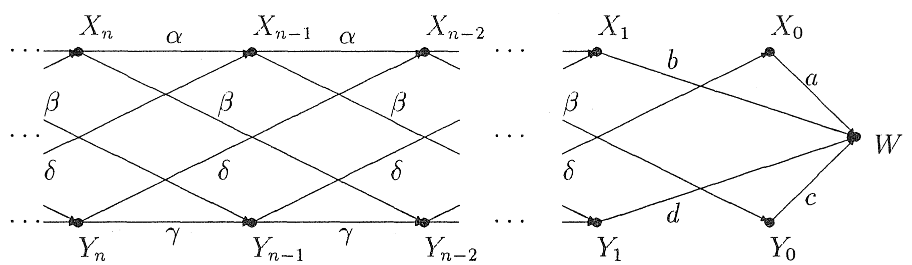
$$y_n = a \sum_{4p+q+r=n-2} \binom{p+q-1}{q}\binom{p+r}{r}\alpha^q\beta^p\gamma^r\delta^{p+1} + b \sum_{4p+q+r=n-3} \binom{p+q}{q}\binom{p+r}{r}\alpha^q\beta^p\gamma^r\delta^{p+1}$$
$$+ c \sum_{4p+q+r=n-4} \binom{p+q-1}{q}\binom{p+r}{r}\alpha^q\beta^{p+1}\gamma^r\delta^{p+1} + d \sum_{4p+q+r=n-1} \binom{p+q-1}{q}\binom{p+r}{r}\alpha^q\beta^p\gamma^r\delta^p$$

hold for every $n \geq 2$, where all sums are taken for nonnegative integer values of p, q, and r.

2. PROOF OF THE MAIN RESULT

Our basic construction is an infinite directed graph $G = (V, E)$ with weighted edges:

The set of vertices is $V = \{W\} \cup \{X_i \mid i \in Z_{\geq 0}\} \cup \{Y_i \mid i \in Z_{\geq 0}\}$ (here $Z_{\geq 0}$ denotes the set of nonnegative integers). The set of edges is $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_0$, where $E_1 = \{(X_i, X_{i-1}) \mid i \geq 2\}$, all edges from E_1 have weight α and we shall call them edges of type A. Analogously, the set of edges of type B with weight β is $E_2 = \{(X_i, Y_{i-2}) \mid i \geq 2\}$, the set of edges of type C with weight γ is $E_3 = \{(Y_i, Y_{i-1}) \mid i \geq 2\}$, and the set of edges of type D with weight δ is $E_4 = \{(Y_i, X_{i-2}) \mid i \geq 2\}$. The last set E_0 consists of the following four edges: (X_1, W) with weight a, (X_0, W) with weight b, (Y_1, W) with weight c, and (Y_0, W) with weight d. A graphical representation of G is given in the figure below.



We define the weight of a path in G as the product of weights of its edges. For two arbitrary vertices $v_1, v_2 \in V$, $v_1 \neq v_2$, we define the function $\omega(v_1, v_2)$ as the sum of the weights of all paths from v_1 to v_2 in G; for $v_1 = v_2$, we set $\omega(v_1, v_2) = 1$. The following lemma shows the connection between function ω and sequences $\{x_i\}_{i=0}^{\infty}$, $\{y_i\}_{i=0}^{\infty}$.

Lemma 1: $\omega(X_i, W) = x_i$ and $\omega(Y_i, W) = y_i$ hold for every $i \in Z_{\geq 0}$.

Proof: The proof is straightforward by induction on i. For $i \in \{0, 1\}$, we have $\omega(X_0, W) = a$, $\omega(x_1, W) = b$, $\omega(Y_0, W) = c$, and $\omega(Y_1, W) = d$. For $i \geq 2$, we observe that every path from X_i to W starts with the edge (X_i, X_{i-1}) or with the edge (X_i, Y_{i-2}). Thus, $\omega(X_i, W) = \alpha\omega(X_{i-1}, W) + \beta\omega(Y_{i-2}, W) = \alpha x_{i-1} + \beta y_{i-2} = x_i$. The proof for $\omega(Y_i, W)$ is similar. □

We shall compute some values of the function ω that we shall use further.

Lemma 2: The following equalities hold for every $i, j \in \mathbf{Z}$, $i \geq j \geq 1$ (all sums are taken for nonnegative integer values of p, q, and r):

1. $\omega(X_i, X_j) = \sum_{4p+q+r=i-j} \binom{p+q}{q}\binom{p+r-1}{r}\alpha^q\beta^p\gamma^r\delta^p,$

2. $$\omega(Y_i, Y_j) = \sum_{4p+q+r=i-j} \binom{p+q-1}{q}\binom{p+r}{r}\alpha^q\beta^p\gamma^r\delta^p,$$

3. $$\omega(X_i, Y_j) = \sum_{4p+q+r=i-j-2} \binom{p+q}{q}\binom{p+r}{r}\alpha^q\beta^{p+1}\gamma^r\delta^p,$$

4. $$\omega(Y_i, X_j) = \sum_{4p+q+r=i-j-2} \binom{p+q}{q}\binom{p+r}{r}\alpha^q\beta^p\gamma^r\delta^{p+1}.$$

Proof: We shall prove case 1 only; the proofs of 2, 3, and 4 are similar.

Let us consider the structure of an arbitrary path from X_i to X_j. Edges of type B and D alternate, starting with an edge of type B and ending with an edge of type D. It is clear also that there are edges of type C only between neighboring pairs (B, D) and there are edges of type A only between neighboring pairs (D, B) at the beginning and at the end. Therefore, the considered path has the form

$$\underbrace{A\dots A}_{q_1}\,B\,\underbrace{C\dots C}_{r_1}\,D\,\underbrace{A\dots A}_{q_2}\,B\,\underbrace{C\dots C}_{r_2}\,D\dots\underbrace{A\dots A}_{q_p}\,B\,\underbrace{C\dots C}_{r_p}\,D\,\underbrace{A\dots A}_{q_{p+1}},$$

where the number of edges of types B and D is p, the number of edges of type A is $q = \sum_{k=1}^{p+1} q_k$, and the number of edges of type C is $r = \sum_{k=1}^{p} r_k$. It is known that the number of all nonnegative ordered $p+1$-tuples with sum q is $\binom{p+q}{q}$ and the number of all nonnegative ordered p-tuples with sum r is $\binom{p+r-1}{r}$. Since the tuples $(q_1, q_2, \dots, q_{p+1})$ and $(r_1, r_2, \dots, r_p)$ are independent, we obtain that the total number of paths from X_i to X_j with q edges of type A, p edges of type B, r edges of type C, and p edges of type D is $\binom{p+q}{q}\binom{p+r-1}{r}$. Their weight is $\alpha^q\beta^p\gamma^r\delta^p$. Thus, we need all admissible values of p, q, and r to compute $\omega(X_i, X_j)$. Since the difference between indices of the vertices adjacent to the edge of type B or D is 2 and the difference for the edges of type A or C is 1, we have that $i-j = 4p+q+r$. That is why we obtain

$$\omega(X_i, X_j) = \sum_{4p+q+r=i-j} \binom{p+q}{q}\binom{p+r-1}{r}\alpha^q\beta^p\gamma^r\delta^p,$$

where the sum is taken for nonnegative integer values of p, q, and r. □

Now we are able to prove our main result (Theorem 1).

Proof of the Main Result: Let us observe that the last edge of an arbitrary path from X_n to W is (X_0, W) or (X_1, W) or (Y_0, W) or (Y_1, W). Thus,

$$x_n = \omega(X_n, W) = a\omega(X_n, X_0) + b\omega(X_n, X_1) + c\omega(X_n, Y_0) + d\omega(X_n, Y_1).$$

Let us observe also that every path from X_n to X_0 ends with edge (Y_2, X_0) and every path from X_n to Y_0 ends with edge (X_2, Y_0). That is why

$$x_n = a\delta\omega(X_n, Y_2) + b\omega(X_n, X_1) + c\beta\omega(X_n, X_2) + d\omega(X_n, Y_1).$$

The proof for y_n is similar. □

Finally, we mention that some other problems from [1]-[11] can also be solved using the described method.

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GENERALIZED TRIPLE PRODUCTS

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1. INTRODUCTION

For arbitrary integers a and b, Horadam [2] and [3] established the notation

$$W_n = W_n(a, b; p, q), \tag{1.1}$$

meaning that

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = a,\ W_1 = b,\quad n \geq 2. \tag{1.2}$$

The sequence $\{W_n\}_{n=0}^{\infty}$ thus defined can be extended to negative integer subscripts by the use of (1.2), and with this understanding we write simply $\{W_n\}$. In this paper we assume that a, b, p, and q are arbitrary real numbers.

By using the generating functions of $\{F_{n+m}\}_{n=0}^{\infty}$ and $\{L_{n+m}\}_{n=0}^{\infty}$ Hansen [1] obtained expansions for $F_jF_kF_l$, $F_jF_kL_l$, $F_jL_kL_l$, and $L_jL_kL_l$. By following the same techniques, Serkland [5] produced similar expansions for the Pell and Pell-Lucas numbers defined by

$$\begin{cases} P_n = W_n(0, 1; 2, -1), \\ Q_n = W_n(2, 2; 2, -1). \end{cases} \tag{1.3}$$

Later Horadam [4] generalized the results of both these writers to the sequences

$$\begin{cases} U_n = W_n(0, 1; p, -1), \\ V_n = W_n(2, p; p, -1). \end{cases} \tag{1.4}$$

Define the sequences $\{W_n\}$ and $\{X_n\}$ by

$$\begin{cases} W_n = W_n(a, b; p, -1), \\ X_n = W_{n+1} + W_{n-1}. \end{cases} \tag{1.5}$$

Here we emphasize that W_n is as in (1.2) but with $q = -1$, and this is the case for the remainder of the paper. Since $\{W_n\}$ generalizes $\{U_n\}$, then $\{X_n\}$ generalizes $\{V_n\}$ by virtue of the fact that $V_n = U_{n+1} + U_{n-1}$. The object of this paper is to generalize the results of Horadam, and so also of Serkland and of Hansen, by incorporating terms from the sequences $\{W_n\}$ and $\{X_n\}$ into the products.

Since $\Delta = p^2 + 4 \neq 0$, the roots α and β of $x^2 - px - 1 = 0$ are distinct. Hence, the Binet form (see [2] and [3]) for W_n is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $A = b - a\beta$ and $B = b - a\alpha$. It can also be shown that

$$X_n = A\alpha^n + B\beta^n.$$

2. SOME PRELIMINARY RESULTS

We shall need the following, each of which can be proved with the use of Binet forms:

$$(-1)^n U_{-n} = -U_n, \tag{2.1}$$

$$(-1)^n V_{-n} = V_n, \tag{2.2}$$

$$\Delta W_n = X_{n+1} + X_{n-1}, \tag{2.3}$$

$$\Delta U_{m+d} W_{n-d} - V_m X_n = (-1)^{m+1} V_d X_{n-m-d}, \tag{2.4}$$

$$W_{m+d} V_{n-d} - U_m X_n = (-1)^m W_d V_{n-m-d}, \tag{2.5}$$

$$W_n U_m + W_{n-1} U_{m-1} = W_{n+m-1}, \tag{2.6}$$

$$W_n V_m + W_{n-1} V_{m-1} = X_{n+m-1}, \tag{2.7}$$

$$U_n X_m + U_{n-1} X_{m-1} = X_{n+m-1}, \tag{2.8}$$

$$X_n V_m + X_{n-1} V_{m-1} = X_{n+m} + X_{n+m-2} = \Delta W_{n+m-1}. \tag{2.9}$$

3. THE MAIN RESULTS

Using the Binet form for W_n we have, for m an integer and $|x|$ small,

$$\sum_{n=0}^{\infty} W_{n+m} x^n = \sum_{n=0}^{\infty} \frac{(A\alpha^{n+m} - B\beta^{n+m})x^n}{\alpha - \beta} = \frac{1}{\alpha - \beta}\left(\alpha^m \sum_{n=0}^{\infty} A\alpha^n x^n - \beta^m \sum_{n=0}^{\infty} B\beta^n x^n\right)$$

$$= \frac{1}{\alpha - \beta}\left(\frac{A\alpha^m}{1-\alpha x} - \frac{B\beta^m}{1-\beta x}\right) = \frac{1}{\alpha - \beta}\left(\frac{(A\alpha^m - B\beta^m) - \alpha\beta(A\alpha^{m-1} - B\beta^{m-1})x}{(1-\alpha x)(1-\beta x)}\right).$$

Then, putting $D = 1 - px - x^2$, we have

$$\sum_{n=0}^{\infty} W_{n+m} x^n = \frac{W_m + W_{m-1} x}{D}. \tag{3.1}$$

Of course, in (3.1), we can replace $\{W_n\}$ by any of the sequences in this paper. In particular, with $m = 1$ and $\{W_n\} = \{U_n\}$, (3.1) becomes

$$\sum_{n=0}^{\infty} U_{n+1} x^n = \frac{1}{D}. \tag{3.2}$$

The following result, which is essential for what follows, can be proved with partial fractions techniques:

$$\begin{aligned} \frac{(j+kx)}{D} \cdot \frac{(l+tx)}{D} &= \frac{jl + (jt + kl)x + ktx^2}{D^2} \\ &= \frac{-kt}{D} + \frac{(jl + kt) + (jt + kl - pkt)x}{D^2}. \end{aligned} \tag{3.3}$$

Now

$$\frac{U_m + U_{m-1}x}{D} \cdot \frac{X_s + X_{s-1}x}{D} = \sum_{n=0}^{\infty} U_{n+m} x^n \cdot \sum_{n=0}^{\infty} X_{n+s} x^n = \sum_{n=0}^{\infty} \sum_{i=0}^{n} U_{i+m} X_{n-i+s} x^n. \tag{3.4}$$

Alternatively, using (3.3), we have

$$\frac{U_m+U_{m-1}x}{D}\cdot\frac{X_s+X_{s-1}x}{D}$$
$$=\frac{-U_{m-1}X_{s-1}}{D}+\frac{(U_mX_s+U_{m-1}X_{s-1})+(U_mX_{s-1}+U_{m-1}X_s-pU_{m-1}X_{s-1})x}{D^2}.$$

Then, by using (2.8) and the recurrence relation (1.2), this becomes

$$\frac{-U_{m-1}X_{s-1}}{D}+\frac{X_{m+s-1}+(U_{m-1}X_s+U_{m-2}X_{s-1})x}{D^2}$$
$$=\frac{-U_{m-1}X_{s-1}}{D}+\frac{X_{m+s-1}+X_{m+s-2}x}{D^2}=-(U_{m-1}X_{s-1})\cdot\frac{1}{D}+\frac{X_{m+s-1}+X_{m+s-2}x}{D}\cdot\frac{1}{D}.$$

Now, by using (3.1) and (3.2), this in turn becomes

$$-U_{m-1}X_{s-1}\sum_{n=0}^{\infty}U_{n+1}x^n+\sum_{n=0}^{\infty}X_{n+m+s-1}x^n\cdot\sum_{n=0}^{\infty}U_{n+1}x^n$$
$$=\sum_{n=0}^{\infty}(-U_{n+1}U_{m-1}X_{s-1})x^n+\sum_{n=0}^{\infty}\sum_{i=0}^{n}U_{i+1}X_{n-i+m+s-1}x^n$$
$$=\sum_{n=0}^{\infty}\left(-U_{n+1}U_{m-1}X_{s-1}+\sum_{i=0}^{n}U_{i+1}X_{n-i+m+s-1}\right)x^n.$$

By equating the coefficients of x^n in the last line and the right side of (3.4), we obtain

$$\sum_{i=0}^{n}U_{i+m}X_{n-i+s}=-U_{n+1}U_{m-1}X_{s-1}+\sum_{i=0}^{n}U_{i+1}X_{n-i+m+s-1}.$$

Finally, putting $j=m-1$, $k=n+1$, and $l=s-1$, we get

$$U_jU_kX_l=\sum_{i=0}^{k-1}(U_{i+1}X_{j+k+l-i}-U_{j+i+1}X_{k+l-i}). \tag{3.5}$$

If we replace X by V, we see that this generalizes Horadam's Theorem 4, which contains a typographical error in one of the subscripts.

In exactly the same manner, taking the product of

$$\frac{U_m+U_{m-1}x}{D}\quad\text{and}\quad\frac{W_s+W_{s-1}x}{D}$$

and using (2.6), we obtain

$$W_jU_kU_l=\sum_{i=0}^{l-1}(W_{j+k+l-i}U_{i+1}-W_{j+i+1}U_{k+l-i}). \tag{3.6}$$

This generalizes Horadam's Theorem 5.

Again, taking the product of

$$\frac{V_m+V_{m-1}x}{D}\quad\text{and}\quad\frac{X_s+X_{s-1}x}{D}$$

and using (2.9) yields

$$U_jV_kX_l = \sum_{i=0}^{j-1}(\Delta U_{j-i}W_{k+l+i+1} - V_{k+i+1}X_{j+l-i}). \tag{3.7}$$

This generalizes Horadam's Theorem 6.

Further, taking the product of

$$\frac{W_m + W_{m-1}x}{D} \quad \text{and} \quad \frac{V_s + V_{s-1}x}{D}$$

and using (2.7) leads to

$$W_jU_kV_l = \sum_{i=0}^{k-1}(U_{i+1}X_{j+k+l-i} - W_{j+i+1}V_{k+l-i}). \tag{3.8}$$

Making use of (3.7), we have

$$\begin{aligned} V_jV_kX_l &= (U_{j+1} + U_{j-1})V_kX_l = U_{j+1}V_kX_l + U_{j-1}V_kX_l \\ &= \sum_{i=0}^{j}(\Delta U_{j-i+1}W_{k+l+i+1} - V_{k+i+1}X_{j+l-i+1}) + \sum_{i=0}^{j-2}(\Delta U_{j-i-1}W_{k+l+i+1} - V_{k+i+1}X_{j+l-i-1}) \\ &= \left(\sum_{i=0}^{j-2}(\Delta W_{k+l+i+1}(U_{j-i+1} + U_{j-i-1}) - V_{k+i+1}(X_{j+l-i+1} + X_{j+l-i-1}))\right) \\ &\quad + (\Delta U_2W_{k+l+j} - V_{k+j}X_{l+2}) + (\Delta U_1W_{k+l+j+1} - V_{k+j+1}X_{l+1}). \end{aligned}$$

We now use (2.4) and (2.2) to simplify the last two terms on the right side. Finally, recalling that $U_{n+1} + U_{n-1} = V_n$ and using (2.3), we obtain

$$V_jV_kX_l = \left(\Delta\sum_{i=0}^{j-2}(W_{k+l+i+1}V_{j-i} - W_{j+l-i}V_{k+i+1})\right) + pX_lV_{j+k-1}. \tag{3.9}$$

This generalizes Horadam's Theorem 7, and is more concisely written.

To obtain our final product, we write

$$W_jV_kV_l = W_j(U_{k+1} + U_{k-1})V_l.$$

Then proceeding in the same manner we use (3.8) and (2.5) to obtain

$$W_jV_kV_l = \left(\Delta\sum_{i=0}^{k-2}(W_{j+k+l-i}U_{i+1} - W_{j+i+1}U_{k+l-i})\right) + (-1)^{k+1}pW_jV_{l+1-k}. \tag{3.10}$$

Of course, in each summation identity, the parameter contained in the upper limit of summation must be chosen so that the sum is well defined. For example, in (3.10), we assume $k \ge 2$.

4. THE MAIN RESULTS SIMPLIFIED

We have chosen to present the results (3.5)-(3.10) in the given manner in order to facilitate comparison with the results of Horadam, Serkland, and Hansen. We now demonstrate that they can be simplified considerably.

By using Binet forms, it can be shown that

$$U_{i+1}X_{j+k+l-i} - U_{j+i+1}X_{k+l-i} = (-1)^i U_j X_{k+l-2i-1}, \tag{4.1}$$

$$W_{j+k+l-i}U_{i+1} - W_{j+i+1}U_{k+l-i} = (-1)^i W_j U_{k+l-2i-1}, \tag{4.2}$$

$$\Delta U_{j-i}W_{k+l+i+1} - V_{k+i+1}X_{j+l-i} = (-1)^{i+j+1} X_l V_{k+2i+1-j}, \tag{4.3}$$

$$U_{i+1}X_{j+k+l-i} - W_{j+i+1}V_{k+l-i} = (-1)^i W_j V_{k+l-2i-1}, \tag{4.4}$$

$$W_{k+l+i+1}V_{j-i} - W_{j+l-i}V_{k+i+1} = (-1)^{i+j} X_l U_{k+2i+1-j}, \tag{4.5}$$

$$W_{j+k+l-i}U_{i+1} - W_{j+i+1}U_{k+l-i} = (-1)^i W_j U_{k+l-2i-1}. \tag{4.6}$$

Now, if we substitute the left side of (4.1) into (3.5) and replace k by j and l by k, we obtain

$$U_j X_k = \sum_{i=0}^{j-1} (-1)^i X_{j+k-2i-1}. \tag{4.7}$$

In the same manner, we use (4.2)-(4.6) to simplify (3.6)-(3.10), which become, respectively,

$$U_j U_k = \sum_{i=0}^{k-1} (-1)^i U_{j+k-2i-1}, \tag{4.8}$$

$$U_j V_k = \sum_{i=0}^{j-1} (-1)^{i+j+1} V_{k+2i+1-j}, \tag{4.9}$$

$$U_j V_k = \sum_{i=0}^{j-1} (-1)^i V_{j+k-2i-1}, \tag{4.10}$$

$$V_j V_k = \left(\Delta \sum_{i=0}^{j-2} (-1)^{i+j} U_{k+2i+1-j} \right) + pV_{j+k-1}, \tag{4.11}$$

$$V_j V_k = \left(\Delta \sum_{i=0}^{j-2} (-1)^i U_{j+k-2i-1} \right) + (-1)^{j+1} pV_{k+1-j}. \tag{4.12}$$

By noting that $\Sigma_{i=0}^n f(i) = \Sigma_{i=0}^n f(n-i)$, we see that the right sides of (4.9) and (4.10) are identical. However, the right sides of (4.11) and (4.12) are different expressions which reduce to $V_j V_k$.

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REPRESENTING GENERALIZED LUCAS NUMBERS IN TERMS OF THEIR α-VALUES

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1. INTRODUCTION

The elements of the sequences $\{A_k\}$ obeying the second-order recurrence relation

$$A_0,\ A_1 \text{ arbitrary real numbers, and } A_k = PA_{k-1} - QA_{k-2} \text{ for } k \geq 2 \tag{1.1}$$

are commonly referred to as *generalized Lucas numbers* with generating parameters P and Q (e.g., see [7], p. 41 ff.), and the equation

$$x^2 - Px + Q = 0 \tag{1.2}$$

is usually called the characteristic equation of $\{A_k\}$. In what follows, the root

$$\alpha_{P,Q} = (P + \sqrt{P^2 - 4Q})/2 \tag{1.3}$$

of (1.2) will be referred to as the *α-value* of $\{A_k\}$. Here, we shall confine ourselves to considering the well-known sequences $\{A_k\}$ that have $(A_0, A_1) = (2, P)$ or $(0, 1)$ as initial conditions, and

$$(P, Q) = (m, -1) \text{ or } (1, -m) \quad (m \text{ a natural number}) \tag{1.4}$$

as generating parameters (e.g., see [2] and [3]).

The aim of this note is to find the representation of the elements of these sequences in terms of their α-values. More precisely, we shall express A_k as

$$A_k = \sum_{r=-\infty}^{\infty} c_r \alpha_{P,Q}^r \quad [c_r = 0 \text{ or } m^{s(r)},\ s(r) \text{ nonnegative integers}] \tag{1.5}$$

with $c_r c_{r+1} = 0$, and with at most finitely many nonzero c_r.

The special case $m = 1$, for which, depending on the initial conditions, A_k equals the k^{th} Lucas number L_k or the k^{th} Fibonacci number F_k, is perhaps the most interesting (see Section 4).

2. REPRESENTING $V_k(m)$ AND $U_k(m)$

If we let $(P, Q) = (m, -1)$ in (1.1)-(1.3), then we get the numbers $V_k(m)$ and $U_k(m)$ (e.g., see [2]). They are defined by the second-order recurrence relation

$$A_k(m) = mA_{k-1}(m) + A_{k-2}(m) \tag{2.1}$$

(here A stands for either V or U) with initial conditions $V_0(m) = 2$, $V_1(m) = m$, $U_0(m) = 0$, and $U_1(m) = 1$. Their Binet forms are

$$V_k(m) = \alpha_m^k + \beta_m^k \quad \text{and} \quad U_k(m) = (\alpha_m^k - \beta_m^k)/\sqrt{m^2+4}, \tag{2.2}$$

where

$$\alpha_m := \alpha_{m,-1} = (m+\sqrt{m^2+4})/2 \tag{2.3}$$

is the α-value of (2.1) and

$$\beta_m = -1/\alpha_m = m - \alpha_m. \tag{2.3'}$$

Observe that $V_k(1) = L_k$ and $U_k(1) = F_k$, whereas $V_k(2) = Q_k$ (the k^{th} Pell-Lucas number) and $U_k(2) = P_k$ (the k^{th} Pell number) (e.g., see [5]).

The α-representations of $V_k(m)$ and $U_k(m)$ are presented in Subsection 3.1 below and then proved in detail in Subsection 3.2.

2.1 Results

$$V_{2k}(m) = \alpha_m^{-2k} + \alpha_m^{2k} \quad (k = 0, 1, 2, \dots). \tag{2.4}$$

Remark 1: For $k = 0$ the r.h.s. of (2.4) is correct but it is not the α-representation of $V_0(m)$.

$$V_{2k+1}(m) = \sum_{r=1}^{2k+1} m\alpha_m^{2r-2(k+1)} \quad (k = 0, 1, 2, \dots), \tag{2.5}$$

$$U_{2k}(m) = \sum_{r=1}^{k} m\alpha_m^{4r-2(k+1)} \quad (k = 1, 2, 3, \dots), \tag{2.6}$$

$$U_{2k+1}(m) = \alpha_m^{-2k} + \sum_{r=1}^{k} m\alpha_m^{4r-2k-1} \quad (k = 0, 1, 2, \dots). \tag{2.7}$$

Remark 2: Under the usual assumption that a sum vanishes whenever the upper range indicator is less than the lower one, (2.7) applies also for $k = 0$.

2.2 Proofs

The proof of (2.4) can be obtained trivially by using (2.2) and (2.3′).

Proof of (2.5): Use the geometric series formula (g.s.f.) and (2.2)-(2.3′) to rewrite the r.h.s. of (2.5) as

$$\frac{m\alpha_m^{-2k}(\alpha_m^{4k+2}-1)}{\alpha_m^2-1} = \frac{m\alpha_m}{\alpha_m^2-1}[\alpha_m^{2k+1} - \alpha_m^{-(2k+1)}] = \alpha_m^{2k+1} - \alpha_m^{-(2k+1)} = V_{2k+1}(m). \quad \square$$

Proof of (2.6): Use the g.s.f. and (2.2)-(2.3′) to rewrite the r.h.s. of (2.6) as

$$\frac{m\alpha_m^{2-2k}(\alpha_m^{4k}-1)}{\alpha_m^4-1} = \frac{m\alpha_m(\alpha_m^{2k+1}-\alpha_m^{1-2k})}{(\alpha_m^2+1)(\alpha_m^2-1)} = \frac{\alpha_m^{2k+1}-\alpha_m^{1-2k}}{\alpha_m^2+1}$$

$$= \frac{\alpha_m(\alpha_m^{2k}-\alpha_m^{-2k})}{\alpha_m\sqrt{m^2+4}} = U_{2k}(m). \quad \square$$

To prove (2.7), we need the identity

$$\alpha_m U_n(m) + (-1)^n \alpha_m^{-n} = U_{n+1}(m). \tag{2.8}$$

Proof of (2.8): Use (2.2), (2.3′), and the relation $\sqrt{m^2+4} = \alpha_m - \beta_m$ to rewrite the l.h.s. of (2.8) as

$$\frac{\alpha_m(\alpha_m^n - \beta_m^n)}{\sqrt{m^2+4}} + \beta_m^n = \frac{\alpha_m^{n+1} + \beta_m^{n-1}}{\sqrt{m^2+4}} + \beta_m^n = \frac{\alpha_m^{n+1} + \beta_m^{n-1}(\beta_m\sqrt{m^2+4}+1)}{\sqrt{m^2+4}}$$

$$= \frac{\alpha_m^{n+1} + \beta_m^{n-1}(-\beta_m^2)}{\sqrt{m^2+4}} = U_{n+1}(m). \quad \square$$

Proof of (2.7): Use the g.s.f., (2.2)-(2.3′), and (2.8) to rewrite the r.h.s. of (2.7) as

$$\alpha_m^{-2k} + \frac{m\alpha_m^{3-2k}(\alpha_m^{4k}-1)}{\alpha_m^4 - 1} = \alpha_m^{-2k} + \frac{m\alpha_m(\alpha_m^{2k+2} - \alpha_m^{2-2k})}{(\alpha_m^2+1)(\alpha_m^2-1)} = \alpha_m^{-2k} + \frac{\alpha_m^{2k+2} - \alpha_m^{2-2k}}{\alpha_m^2+1}$$

$$= \alpha_m^{-2k} + \frac{\alpha_m^2(\alpha_m^{2k} - \alpha_m^{-2k})}{\alpha_m\sqrt{m^2+4}} = \alpha_m^{-2k} + \alpha_m U_{2k}(m) = U_{2k+1}(m). \quad \square$$

3. REPRESENTING $H_k(m)$ AND $G_k(m)$

If we let $(P, Q) = (1, -m)$ in (1.1)-(1.3), then we get the numbers $H_k(m)$ and $G_k(m)$ (e.g., see [3]). They are defined by the second-order recurrence relation

$$A_k(m) = A_{k-1}(m) + mA_{k-2}(m) \tag{3.1}$$

(here A stands for H or G) with initial conditions $H_0(m) = 2$, $H_1(m) = G_1(m) = 1$, and $G_0(m) = 0$. Their Binet forms are

$$H_k(m) = \gamma_m^k + \delta_m^k \quad \text{and} \quad G_k(m) = (\gamma_m^k - \delta_m^k)/\sqrt{4m+1}, \tag{3.2}$$

where

$$\gamma_m := \alpha_{1,-m} = (1+\sqrt{4m+1})/2 \tag{3.3}$$

is the α-value of (3.1), and

$$\delta_m = -m/\gamma_m = 1 - \gamma_m. \tag{3.3′}$$

Observe that $H_k(1) = V_k(1) = L_k$ and $G_k(1) = U_k(1) = F_k$, whereas $H_k(2) = j_k$ (the k^{th} Jacobsthal-Lucas number) and $G_k(2) = J_k$ (the k^{th} Jacobsthal number) (see [6]).

The α-representations of $H_k(m)$ and $G_k(m)$ are shown in Subsection 3.1 below. To save space, only identity (3.7) will be proved in detail in Subsection 3.2.

3.1 Results

$$H_{2k}(m) = m^{2k}\gamma_m^{-2k} + \gamma_m^{2k} \quad (k = 0, 1, 2, \ldots; \text{ see Remark 1}), \tag{3.4}$$

$$H_{2k+1}(m) = \sum_{r=1}^{2k+1} m^{2k+1-r}\gamma_m^{2r-2(k+1)} \quad (k = 0, 1, 2, \ldots), \tag{3.5}$$

$$G_{2k}(m) = \sum_{r=1}^{k} m^{2(k-r)}\gamma_m^{4r-2(k+1)} \quad (k = 1, 2, 3, \ldots), \tag{3.6}$$

$$G_{2k+1}(m) = m^{2k}\gamma_m^{-2k} + \sum_{r=1}^{k} m^{2(k-r)}\gamma_m^{4r-2k-1} \quad (k = 0, 1, 2, \ldots; \text{ see Remark 2}). \tag{3.7}$$

A Special Case (cf. (2.3) and (2.4) of [6]): For $m=2$ $(=\gamma_2)$, identities (3.4)-(3.7) reduce to

$$H_{2k}(2)=4^k+1=j_{2k}, \tag{3.4$'$}$$

$$H_{2k+1}(2)=\sum_{r=1}^{2k+1}2^{r-1}=2^{2k+1}-1=j_{2k+1} \tag{3.5$'$}$$

$$G_{2k}(2)=\sum_{r=1}^{k}2^{2(r-1)}=(4^k-1)/3=J_{2k}, \tag{3.6$'$}$$

$$G_{2k+1}(2)=1+\sum_{r=1}^{k}2^{2r-1}=(2^{2k+1}+1)/3=J_{2k+1}. \tag{3.7$'$}$$

3.2 A Proof

To prove (3.7), we need the identity

$$\gamma_m G_n(m)+\delta_m^n=G_{n+1}(m). \tag{3.8}$$

Proof of (3.8): Use (3.2) and the relation $\sqrt{4m+1}=\gamma_m-\delta_m$ to rewrite the l.h.s. of (3.8) as

$$\frac{\gamma_m(\gamma_m^n-\delta_m^n)}{\sqrt{4m+1}}+\delta_m^n=\frac{\gamma_m^{n+1}-\delta_m^n(\gamma_m-\sqrt{4m+1})}{\sqrt{4m+1}}=\frac{\gamma_m^{n+1}-\delta_m^n\delta_m}{\sqrt{4m+1}}=G_{n+1}(m).\ \square$$

Proof of (3.7): Use the g.s.f., (3.2)-(3.3$'$), and (3.8) to rewrite the r.h.s. of (3.7) as

$$\begin{aligned}m^{2k}\gamma_m^{-2k}+m^{2(k-1)}\gamma_m^{3-2k}\sum_{s=0}^{k-1}(\gamma_m^4/m^2)^s&=m^{2k}\gamma_m^{-2k}+m^{2(k-1)}\gamma_m^{3-2k}\frac{(\gamma_m^4/m^2)^k-1}{\gamma_m^4/m^2-1}\\
&=m^{2k}\gamma_m^{-2k}+\frac{\gamma_m^{2k+3}-m^{2k}\gamma_m^{3-2k}}{\gamma_m^4-m^2}=m^{2k}\gamma_m^{-2k}+\frac{\gamma_m^{2k+3}-m^{2k}\gamma_m^{3-2k}}{\gamma_m^2\sqrt{4m+1}}\\
&=m^{2k}\gamma_m^{-2k}+\frac{\gamma_m^{2k+1}-m^{2k}\gamma_m^{1-2k}}{\sqrt{4m+1}}=\delta_m^{2k}+\frac{\gamma_m^{2k+1}-\delta_m^{2k}\gamma_m}{\sqrt{4m+1}}\\
&=\delta_m^{2k}+\gamma_m G_{2k}(m)=G_{2k+1}(m).\ \square\end{aligned}$$

4. A REMARKABLE CASE ($m = 1$)

The *β-expansion* of any natural number N is the *unique* finite sum of distinct integral powers of the golden section α that equals N and contains no consecutive powers of α. This expansion was first studied by Bergman in [1] where the author used the symbol β instead of α to denote the golden section.

As already mentioned in the previous sections, if we let $m=1$ in (2.1)-(2.3) [or in (3.1)-(3.3)], then we get either the Lucas numbers L_k or the Fibonacci numbers F_k, depending on the initial conditions of the recurrence relation (2.1) [or (3.1)] whose α-value

$$\alpha:=\alpha_1=\gamma_1=(1+\sqrt{5})/2 \tag{4.1}$$

is the golden section. Consequently, if we let $m=1$ in (1.5), then it is evident that $c_r\in\{0,1\}$ so that letting $m=1$ in (2.4)-(2.7) [or in (3.4)-(3.7)] yields the β-expansions of L_k and F_k. As an illustration, from (2.5) [or (3.5)], we see that the β-expansion of L_{2k+1} is

$$L_{2k+1}=\alpha^{-2k}+\alpha^{-2k+2}+\cdots+\alpha^0+\cdots+\alpha^{2k-2}+\alpha^{2k}. \tag{4.2}$$

Remark 3: By letting $m = 1$ in (2.4), one gets $L_{2k} = \alpha^{-2k} + \alpha^{2k}$. This expression works correctly also for $k = 0$ but, in this case, it is not the β-expansion of L_0, as stated in Remark 1. In fact, this expansion is $L_0 = 2 = 2\alpha^{-1}\alpha = \alpha^{-1}(1+\sqrt{5}) = \alpha^{-1}(\alpha + \alpha^{-1} + 1) = \alpha^{-1}(\alpha^2 + \alpha^{-1}) = \alpha^{-2} + \alpha$.

5. CONCLUDING COMMENTS

The representations established in this note for certain generalized Lucas numbers, besides being of some interest *per se*, allow us to derive some cute identities involving them. For example, by using (2.5), (2.6), and (2.4), we get

$$\frac{V_{2k+1}(m)}{m} = 1 + \sum_{r=1}^{k} V_{2r}(m), \tag{5.1}$$

$$\frac{U_{2k}(m)}{m} = \begin{cases} \sum_{r=1}^{k/2} V_{4r-2}(m) & (k \text{ even}), \\ 1 + \sum_{r=1}^{(k-1)/2} V_{4r}(m) & (k \text{ odd}), \end{cases} \tag{5.2}$$

$$\frac{V_{2k+1}^2(m)}{m^2} = 2k + 1 + \sum_{r=1}^{2k} r V_{4k+2-2r}(m), \tag{5.3}$$

whereas, from (3.5) and (3.4), we obtain

$$H_{2k+1}(m) = m^k + \sum_{r=1}^{k} m^{k-r} H_{2r}. \tag{5.4}$$

The most interesting among the above identities is, perhaps, the identity (5.3) which, for $m = 1$, gives a rather unusual expression for the squares of odd-subscripted Lucas numbers. Let us give a sketch of its proof.

Proof of (5.3) (a sketch): Use (2.5) and (2.4) to write

$$\begin{aligned} V_{2k+1}^2(m) &= m^2[\alpha_m^{-4k} + 2\alpha_m^{-4k+2} + 3\alpha_m^{-4k+4} + \cdots + (2k+1)\alpha_m^0 + \cdots + 3\alpha_m^{4k-4} + 2\alpha_m^{4k-2} + \alpha_m^{4k}] \\ &= m^2[V_{4k}(m) + 2V_{4k-2}(m) + \cdots + 2kV_2(m) + 2k + 1]. \ \square \end{aligned}$$

Using the same technique led us to discover a quite amazing expression for the cubes of odd-subscripted Lucas numbers. Namely, we get

$$L_{2k+1}^3 = \sum_{r=1}^{2k+1} T_r L_{6k+2-2r} + \sum_{r=1}^{k-1} (S_k - r^2) L_{2r} + S_k \quad (k \geq 1), \tag{5.5}$$

where T_k denotes the k^{th} triangular number and $S_k = 3k(k+1) + 1$. A direct proof of (5.5) can be carried out by using expressions for $\sum_r L_{a+hr}$, $\sum_r rL_{a+hr}$, and $\sum_r r^2 L_{a+hr}$, the last two of which can be obtained from the Binet form for Lucas numbers and (3.1)-(3.2) of [4].

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A NOTE ON A PAPER BY GLASER AND SCHÖFFL

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1. INTRODUCTION

Ducci sequences are sequences of integer n-tuples $\mathbf{u}_0, \mathbf{u}_1, \ldots$ generated by the relation $\mathbf{u}_{k+1} = D\mathbf{u}_k$, where $D(x_1, x_2, \ldots, x_n) = (|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_n - x_1|)$. We say that $\mathbf{u}_0$ generates the Ducci sequence $\mathbf{u}_0, \mathbf{u}_1, \ldots$. It has been shown (e.g., in [3]) that every Ducci sequence reduces to a sequence of binary tuples $\mathbf{u}_k = (x_1, \ldots, x_n)$, where $x_i \in \{0, c\}$ for all i and some constant c. As $D(\lambda\mathbf{u}) = \lambda D\mathbf{u}$ for all $\lambda \geq 1$, it is customary to assume $c = 1$. At this point it is obvious that every Ducci sequence eventually forms a cycle, called a Ducci cycle.

Many aspects of Ducci sequences have been studied, such as the smallest k such that $\mathbf{u}_k$ is part of the Ducci cycle (this is also known as the "n-Number Game," see [4]). In this note we will only concern ourselves with the Ducci cycles themselves. We list some of the known results and conventions used in this note:

1. For binary tuples, D becomes the linear operator $D(x_1, x_2, \ldots, x_n) = (x_1 + x_2, x_2 + x_3, \ldots, x_n + x_1) \pmod 2$.
2. If $\mathbf{a}_0 = (0, 0, \ldots, 0, 1)$, then the Ducci sequence $\mathbf{a}_0, \mathbf{a}_1, \ldots$ is called the *basic Ducci sequence* and the resulting cycle the *basic Ducci cycle*. From now on, $\mathbf{a}_k$ will always denote a tuple in the basic Ducci sequence.
3. The *period* (also referred to as the *length*) of the basic Ducci cycle is denoted by $P(n)$ and many properties of $P(n)$ are given in [1] and [2]. When we speak of the period of a Ducci sequence, we actually mean the period of the Ducci cycle produced by that Ducci sequence.
4. If $n > 1$ is odd and there exists an M such that $2^M \equiv -1 \pmod n$, then n is said to be "with a -1"; otherwise, n is said to be "without a -1". This useful convention was introduced in [1] and used extensively in [2].

In this paper we generalize two of the results in [2]: first, we show how Pascal's triangle can be used to construct any tuple in the basic Ducci sequence; second, we determine, in general, the first tuple in the basic Ducci sequence that is part of the basic Ducci cycle. We also provide counter-examples of two other remarks in [2] concerning the number of Ducci cycles of maximum length and determine the cause for these errors (which do not, however, affect any other results in [2]).

The author would like to thank Dr. A. B. van der Merwe for many stimulating discussions.

2. USING PASCAL'S TRIANGLE TO CONSTRUCT DUCCI SEQUENCES

Glaser and Schöffl described how Pascal's triangle can be used to find the first n-tuples of the basic Ducci sequence. If we assume the convention that

$$\binom{k}{r} = 0 \text{ if } r > k \text{ or } r < 0,$$

then this method can be expressed as the following theorem (Theorem 1 in [2]).

Theorem 1: The r^{th} entry of $\mathbf{a}_k$ is

$$x_r = \binom{k}{r+k-n} \pmod 2 \text{ if } k < n.$$

We shall prove a more general result.

Theorem 2: For all $k \geq 0$ the r^{th} entry of $\mathbf{a}_k$ is

$$x_r = \sum_{i \equiv r \pmod n} \binom{k}{i+k-n} \pmod 2.$$

Proof: From Theorem 1 we know that this is true for $k < n$. Assume it is also true for some $k = p$, so

$$x_r = \sum_{i \equiv r \pmod n} \binom{p}{i+p-n} \pmod 2.$$

Let us denote the r^{th} entry of $\mathbf{a}_p$ by x_r, and the r^{th} entry of $\mathbf{a}_{p+1}$ by x_r'. Now, if $k = p+1$, then we have two cases: If $r < n$, then we have

$$\begin{aligned} x_r' = x_r + x_{r+1} &= \sum_{i \equiv r} \binom{p}{i+p-n} + \sum_{i \equiv r+1} \binom{p}{i+p-n} \pmod 2 \\ &= \sum_{i \equiv r} \left[\binom{p}{i+p-n} + \binom{p}{i+1+p-n} \right] \pmod 2 \\ &= \sum_{i \equiv r} \binom{p+1}{i+p+1-n} \pmod 2; \end{aligned}$$

if $r = n$, then we have

$$\begin{aligned} x_n' = x_n + x_1 &= \sum_{i \equiv 0} \binom{p}{i+p-n} + \sum_{i \equiv 1} \binom{p}{i+p-n} \pmod 2 \\ &= \sum_{i \equiv 0} \left[\binom{p}{i+p-n} + \binom{p}{i+1+p-n} \right] \pmod 2 \\ &= \sum_{i \equiv 0} \binom{p+1}{i+p+1-n} \pmod 2. \end{aligned}$$

So, by induction, the theorem is true for all natural numbers k. □

This result suggests that we can construct the basic Ducci sequence by wrapping Pascal's triangle (mod 2) around a cylinder of circumference n and adding (mod 2) those entries that overlap. We demonstrate this below for the first seven tuples of the basic Ducci sequence for $n = 3$.

$$\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

giving

$$\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

as the first seven 3-tuples in the basic Ducci sequence.

3. TWO COUNTER-EXAMPLES

Just below their proof of Theorem 2 [2, p. 316], Glaser and Schöffl mention that for every n with a -1 there exists only one cycle of maximum length. This conclusion is false, as the counter-example (1, 0, 0, 1, 0, 0, 0, 0, 0) demonstrates; this tuple is not part of the basic Ducci sequence (which has maximum length) but is part of another Ducci sequence that also has maximum length. (This can be checked by computer: $n = 9$ is with a -1 and $P(9) = 63$. For values of $P(n)$ for $n \leq 165$, see [1].)

In Corollary 2 [2, p. 319], Glaser and Schöffl remark that for $n = 2^r - 1$ ($r \geq 2$) there are exactly n different cycles of maximum length, namely, the n cyclic permutations of the basic Ducci cycle. Again, we offer a counter-example: (0, 1, 1, 1, 0, 0, 1) forms an 8th cycle of maximum length (shown below).

$$\begin{array}{ccccccc} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}$$

The cycle given above is of maximum length as $P(7) = 7$.

Both of these errors arise from the misconception that the cycles of maximum length are precisely the basic cycles, an assumption that may have been implicit in earlier papers. The errors did not influence any other results in [2].

4. THE FIRST TUPLE OF A CYCLE

Glaser and Schöffl mentioned that Ehrlich [1] was able to describe the first n-tuple in the basic Ducci cycle if n is odd, but that nothing was known about the case in which n is even. They solved this problem for the cases $n = 2^r + 2^s$ and $n = 2^r - 2^s$, $r > s \geq 0$.

We give the general solution here, but first we must recall a result of Ludington Furno [3].

Let $n = 2^r k$, where k is odd.

Definition 1: We say the n-tuple $(x_1, \ldots, x_n)$ is r-even if

$$\sum_{i=0}^{k-1} x_{2^r i+j} \equiv 0 \pmod 2, \ \forall j = 1, \ldots, 2^r.$$

Let us count the number of n-tuples that are r-even. We can choose the first $n - 2^r$ entries arbitrarily, but the last 2^r entries are then uniquely determined by previous entries in order for the tuple to be r-even, so we have a total of 2^{n-2^r} r-even n-tuples. For example, if $n = 6 = 2^1 \cdot 3$, then the first four entries can be arbitrary, but the last two entries must have specific values in order for the tuple to be 1-even. So we have a total of $2^4 = 16$ 1-even 6-tuples.

Theorem 3 (Ludington Furno): An n-tuple is in a cycle if and only if it is r-even.

From this, follows

Theorem 4: $\mathbf{a}_{2^r} = (0, \ldots, 0, 1, \underbrace{0, \ldots, 0}_{2^r-1 \text{ zeros}}, 1)$ is the first n-tuple in the basic Ducci cycle.

Proof: Obviously, $\mathbf{a}_{2^r}$ is r-even, so it is in a cycle. As it is the $2^{r\text{th}}$ row in Pascal's triangle, it is in the basic Ducci cycle. To show that $\mathbf{a}_t = (x_1, \ldots, x_n)$ is not in a cycle for $t < 2^r$, we need only note that $x_1 = x_2 = \cdots = x_{n-2^r} = 0$; thus,

$$S = \sum_{i=0}^{k-1} x_{2^r i+j} \leq 1 \text{ for all } j = 1, \ldots, 2^r.$$

But $S \neq 0$ for at least one j; therefore, $\mathbf{a}_t$ is not r-even (therefore, not in a cycle) for $t < 2^r$. □

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stan@mathpro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

The Fibonacci polynomials $F_n(x)$ and the Lucas polynomials $L_n(x)$ satisfy

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x),\quad F_0(x) = 0,\quad F_1(x) = 1;$$
$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x),\quad L_0(x) = 2,\quad L_1(x) = x.$$

Also,

$$F_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)} \quad \text{and} \quad L_n(x) = \alpha(x)^n + \beta(x)^n,$$

where $\alpha(x) = (x + \sqrt{x^2+4})/2$ and $\beta(x) = (x - \sqrt{x^2+4})/2$.

PROBLEMS PROPOSED IN THIS ISSUE

B-860 ***Proposed by Herta T. Freitag, Roanoke, VA***

Let k be a positive integer. The sequence $\langle A_n \rangle$ is defined by the recurrence $A_{n+2} = 2kA_{n+1} - A_n$ for $n \geq 0$ with initial conditions $A_0 = 0$ and $A_1 = 1$. Prove that $(k^2-1)A_n^2 + 1$ is a perfect square for all $n \geq 0$.

B-861 ***Proposed by the editor***

The sequence $w_0, w_1, w_2, w_3, w_4, \ldots$ satisfies the recurrence $w_n = Pw_{n-1} - Qw_{n-2}$ for $n > 1$. If every term of the sequence is an integer, must P and Q both be integers?

B-862 ***Proposed by Charles K. Cook, University of South Carolina, Sumter, SC***

Find a Fibonacci number and a Lucas number whose sum is 114,628 and whose least common multiple is 567,451,586.

B-863 ***Proposed by Stanley Rabinowitz, Westford, MA***

Let

$$A=\begin{pmatrix}-9 & 1\\ -89 & 10\end{pmatrix},\quad B=\begin{pmatrix}-10 & 1\\ -109 & 11\end{pmatrix},\quad C=\begin{pmatrix}-7 & 5\\ -11 & 8\end{pmatrix},\quad \text{and}\quad D=\begin{pmatrix}-4 & 19\\ -1 & 5\end{pmatrix},$$

and let n be a positive integer. Simplify $30A^n - 24B^n - 5C^n + D^n$.

B-864 ***Proposed by Stanley Rabinowitz, Westford, MA***

The sequence $\langle Q_n \rangle$ is defined by $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n > 1$ with initial conditions $Q_0 = 2$ and $Q_1 = 2$.

(a) Show that $Q_{7n} \equiv L_n \pmod{159}$ for all n.
(b) Find an integer $m > 1$ such that $Q_{11n} \equiv L_n \pmod{m}$ for all n.
(c) Find an integer a such that $Q_{an} \equiv L_n \pmod{31}$ for all n.
(d) Show that there is no integer a such that $Q_{an} \equiv L_n \pmod{7}$ for all n.
(e) Extra credit: Find an integer $m > 1$ such that $Q_{19n} \equiv L_n \pmod{m}$ for all n.

B-865 ***Proposed by Alexandru Lupas, University Lucian Blaga, Sibiu, Romania***

Let $f(x) = (x^2+4)^{n-1/2}$ where n is a positive integer. Let

$$g(x) = \frac{d^n f(x)}{dx^n}.$$

Express $g(1)$ in terms of Fibonacci and/or Lucas numbers.

Note: The Elementary Problems Column is in need of more **easy**, yet elegant and non-routine problems.

SOLUTIONS

See "Basic Formulas" at the beginning of this column for notation about Fibonacci and Lucas polynomials.

It Repeats!

B-843 ***Proposed by R. Horace McNutt, Montreal, Canada***
(Vol. 36, no. 1, February 1998)

Find the last three digits of $L_{1998}(114)$.

Solution by L. A. G. Dresel, Reading, England

We note that when $x = 114$, $x^2+4 = 13000$. From the recurrence for $L_n(x)$, we have $L_0(x) = 2$, $L_1(x) = x$, $L_2(x) = x^2+2 \equiv -2 \pmod{x^2+4}$ so that $L_3(x) \equiv -x$, $L_4(x) \equiv -x^2-2 \equiv 2$, and $L_5(x) \equiv x \pmod{x^2+4}$.

Hence, modulo 13000, the sequence $L_n(114)$ is periodic with period 4, so that

$$L_{1998}(114) \equiv L_2(114) \equiv -2 \equiv 12998 \pmod{13000}.$$

Therefore, the last three digits of $L_{1998}(114)$ are 998.

Solutions also received by Brian D. Beasley, Paul S. Bruckman, Mario DeNobili, Aloysius Dorp, Russell Jay Hendel, Harris Kwong, H.-J. Seiffert, Indulis Strazdins, and the proposer. One incorrect solution was received.

A Polynomial Identity

B-844 ***Proposed by Mario DeNobili, Vaduz, Lichtenstein (Vol. 36, no. 1, February 1998)***

If $a+b$ is even and $a>b$, show that

$$[F_a(x)+F_b(x)][F_a(x)-F_b(x)] = F_{a+b}(x)F_{a-b}(x).$$

Solution 1 by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Since $a+b$ is even, a and b have the same parity, and $a-b$ is also even. For brevity, we shall write $\alpha(x)$, $\beta(x)$, $F_a(x)$, and $F_b(x)$ as α, β, F_a, and F_b, respectively. It follows from $\alpha\beta=-1$ that

$$\begin{aligned}(\alpha-\beta)^2(F_a^2-F_b^2) &= (\alpha^a-\beta^a)^2-(\alpha^b-\beta^b)^2\\ &= \alpha^{2a}-2(\alpha\beta)^a+\beta^{2a}-[\alpha^{2b}-2(\alpha\beta)^b+\beta^{2b}]\\ &= \alpha^{2a}+\beta^{2a}-\alpha^{2b}-\beta^{2b}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\alpha-\beta)^2F_{a+b}F_{a-b} &= (\alpha^{a+b}-\beta^{a+b})(\alpha^{a-b}-\beta^{a-b})\\ &= \alpha^{2a}-(\alpha\beta)^{a-b}(\beta^{2b}+\alpha^{2b})+\beta^{2a}\\ &= \alpha^{2a}+\beta^{2a}-\alpha^{2b}-\beta^{2b}.\end{aligned}$$

Therefore, $F_{a+b}F_{a-b}=F_a^2-F_b^2=[F_a+F_b][F_a-F_b]$.

Solution 2 by H.-J. Seiffert, Berlin, Germany

It is known ([1], p. 12, formula 3.26) that

$$F_{n+m}^2(x)-F_{n-m}^2(x)=F_{2n}(x)F_{2m}(x)$$

for all integers m and n. If $a+b$ is even, then $a-b$ is also even. With $n=(a+b)/2$ and $m=(a-b)/2$, the above identity gives the desired one.

Reference

1. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.

Comment by the editor: No reader sent in any generalizations related to Lucas polynomials. If $\langle v_n\rangle$ is the generalized Lucas sequence defined by the recurrence $v_n=Pv_{n-1}-Qv_{n-2}$ with initial conditions $v_0=2$ and $v_1=P$, then one can investigate the expression

$$(v_a+v_b)(v_a-v_b)-v_{a+b}v_{a-b}.$$

Applying Algorithm `LucasSimplify` (from [1]), shows that this expression simplifies to

$$Q^{-b}(4Q^{a+b}-Q^a v_b^2-Q^b v_b^2).$$

Thus, if $a+b$ is odd, $a>b$, and $Q=-1$, then we have

$$(v_a+v_b)(v_a-v_b)-v_{a+b}v_{a-b}=4(-1)^a.$$

In particular, for the Lucas polynomials, we have $P=x$ and $Q=-1$. This shows that if $a+b$ is odd with $a>b$, then

$$[L_a(x)+L_b(x)][L_a(x)-L_b(x)]=L_{a+b}(x)L_{a-b}(x)+4(-1)^a.$$

Reference

1. Stanley Rabinowitz. "Algorithmic Manipulation of Fibonacci Identities." In *Applications of Fibonacci Numbers* **6**:389-408. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.

Solutions also received by Brian D. Beasley, Paul S. Bruckman, Leonard A. G. Dresel, Steve Edwards, Russell Euler & Jawad Sadek, Russell Jay Hendel, Harris Kwong, Gene Ward Smith, Lawrence Somer, Indulis Strazdins, and the proposer.

Curious Commuting Composition

B-845 ***Proposed by Gene Ward Smith, Brunswick, ME***
(Vol. 36, no. 1, February 1998)

Show that, if m and n are odd positive integers, then $L_n(L_m(x)) = L_m(L_n(x))$.

Solution 1 by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Given x, choose $\theta = \theta(x)$ such that $x = 2i\sin\theta$. Then $\sqrt{x^2+4} = 2\cos\theta$, hence

$$\alpha(x) = \cos\theta + i\sin\theta \quad\text{and}\quad \beta(x) = -(\cos\theta - i\sin\theta).$$

Consequently, for any odd positive integer n, we have

$$L_n(x) = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta) = 2i\sin n\theta.$$

In other words, $\theta(L_n(x)) = n\theta(x)$. It follows immediately that if m and n are odd positive integers, then $L_m(L_n(x)) = 2i\sin mn\theta(x) = L_n(L_m(x))$.

Solution 2 by Indulis Strazdins, Riga, Latvia

Put $y = L_m(x)$ in the basic equality

$$L_n(y) = yL_{n-1}(y) + L_{n-2}(y)$$

to get

$$L_n(L_m(x)) = L_m(x)L_{n-1}(L_m(x)) + L_{n-2}(L_m(x)).$$

It is easily proved by induction that

$$L_{nm}(x) = L_m(x)L_{(n-1)m}(x) + (-1)^{m-1}L_{(n-2)m}(x)$$

so, if m is odd, we have

$$L_n(L_m(x)) = L_{nm}(x).$$

From this it follows that if m and n are both odd, then

$$L_n(L_m(x)) = L_m(L_n(x)) = L_{nm}(x).$$

Comment by L. A. G. Dresel, Reading, England: Di Porto and Filipponi (see [1], p. 221) have proven the following Lemma:

If m and n are integers with m odd, then $L_n(L_m(x)) = L_{nm}(x)$.

If m and n are both odd, then the desired result follows.

Reference

1. A. Di Porto & P. Filipponi. "A Probabilistic Primality Test Based on the Properties of Certain Generalized Lucas Numbers." *Lecture Notes in Computer Science* **330** (1988):211-23.

No reader submitted any related results for compositions of Fibonacci polynomials.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler & Jawad Sadek, R. Horace McNutt, H.-J. Seiffert, and the proposer.

Integer Sum

B-846 ***Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy (Vol. 36, no. 1, February 1998)***

Show that

$$\sum_{n=1}^{5} \frac{F_n(40k+1)}{n!}$$

is an integer for all integral k. Generalize.

Solution by Gene Ward Smith, Brunswick, ME

The sum in question,

$$\sum_{n=1}^{5} \frac{F_n(40k+1)}{n!},$$

is a polynomial of degree 4, namely,

$$(64000k^4+14400k^3+1760k^2+135k+6)/3.$$

Writing this in terms of binomial coefficients gives us

$$2+26765\binom{k}{1}+328640\binom{k}{2}+796800\binom{k}{3}+512000\binom{k}{4};$$

which has integer coefficients. The polynomial therefore is integer-valued for integer values of k. Alternatively, we may factor

$$64000k^4+14400k^3+1760k^2+135k+6,$$

modulo 3, and obtain $k^2(k+1)(k+2)$, from which it follows that 3 is a divisor for any integer k, so that the initial polynomial is integer-valued.

We may generalize in various ways, most obviously by considering instead

$$\sum_{n=1}^{r} \frac{F_n(ak+b)}{n!}$$

for various values of a, b, and r. In this way we may, for instance, similarly prove that

$$\sum_{n=1}^{5} \frac{F_n(40k+9)}{n!}$$

is integral.

Bruckman and Seiffert showed that $\sum_{n=1}^{5} \frac{F_n(x)}{n!}$ is an integer if and only if $x \equiv 1$ or $9 \pmod{40}$. Bruckman showed that $\sum_{i=1}^{n} \frac{F_i(x)}{i!}$ is never an integer if $n = 3$ or 4. The proposed stated that if k is a positive odd integer, then $\sum_{n=1}^{5} \frac{F_n^k(x)}{n!}$ is an integer if and only if $x \equiv 1$ or $9 \pmod{40}$; but he did not include a proof.

Solutions also received by Paul S. Bruckman and H.-J. Seiffert.

Polynomial GCD

B-847 ***Proposed by Gene Ward Smith, Brunswick, ME***
(Vol. 36, no. 1, February 1998)

Find the greatest common polynomial divisor of $F_{n+4k}(x)+F_n(x)$ and $F_{n+4k-1}(x)+F_{n-1}(x)$.

Solution by Paul S. Bruckman, Highland, IL

Let $d=d(x)=\sqrt{x^2+4}$. For brevity, write $\alpha(x)$ as α and $\beta(x)$ as β. Note that $\alpha\beta=-1$. Then

$$\begin{aligned}F_{n+4k}(x)+F_n(x)&=\frac{\alpha^{n+4k}-\beta^{n+4k}}{d}+\frac{\alpha^n-\beta^n}{d}\\&=\alpha^n\frac{\alpha^{4k}+1}{d}-\beta^n\frac{\beta^{4k}+1}{d}\\&=\alpha^{n+2k}\frac{\alpha^{2k}+\beta^{2k}}{d}-\beta^{n+2k}\frac{\alpha^{2k}+\beta^{2k}}{d}\\&=\frac{\alpha^{n+2k}-\beta^{n+2k}}{d}\cdot(\alpha^{2k}+\beta^{2k})\\&=F_{n+2k}(x)L_{2k}(x).\end{aligned}$$

Replacing n by $n-1$, we find that

$$F_{n-1+4k}(x)+F_{n-1}(x)=F_{n-1+2k}(x)L_{2k}(x).$$

If g is the desired gcd of the two given expressions, then $g=L_{2k}(x)\cdot\gcd(F_{n+2k}(x),F_{n-1+2k}(x))$.

Given any integer u, if d is the gcd of $F_u(x)$ and $F_{u-1}(x)$, then by the recurrence relation, d is a common divisor of $F_1(x)=1$ and $F_2(x)=x$. Thus, $d=\gcd(F_{n+2k}(x),F_{n-1+2k}(x))=1$ and $g=L_{2k}(x)$.

Solutions also received by Leonard A. G. Dresel, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Addenda. We wish to belatedly acknowledge solutions from the following solvers:

Brian Beasley—B-842
Glenn A. Bookhout—B-784
Andrej Dujella—B-772 through B-777
Steve Edwards—B-837, B-840, B-842
Russell Euler—B-788
Herta Freitag—B-791, B-793
Hans Kappus—B-784 through B-786
Carl Libis—B-784, B-785
Graham Lord—B-784, B-785

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-545 ***Proposed by Paul S. Bruckman, Highwood, IL***

Prove that for all odd primes p,

(a) $\sum_{k=1}^{p-1} L_k \cdot k^{-1} \equiv \frac{-2}{p}(L_p - 1) \pmod p$; *(b)* $\sum_{k=1}^{p-1} F_k \cdot k^{-1} \equiv 0 \pmod p$.

H-546 ***Proposed by R. André-Jeannin, Longwy, France***

Find the triangular Mersenne numbers (the sequence of Mersenne numbers is defined by $M_n = 2^n - 1$).

SOLUTIONS

A Prime Problem

H-528 ***Proposed by Paul S. Bruckman, Highwood, IL (Vol. 35, no. 2, May 1997)***

Let $\Omega(n) = \sum_{p^e \| n} e$, given the prime decomposition of a natural number $n = \prod p^e$. Prove the following:

$$\sum_{d \mid n} (-1)^{\Omega(d)} F_{\Omega(n/d)-\Omega(d)} = 0; \tag{A}$$

$$\sum_{d \mid n} (-1)^{\Omega(d)} L_{\Omega(n/d)-\Omega(d)} = 2U_n, \text{ where } U_n = \prod_{p^e \| n} F_{e+1}. \tag{B}$$

Solution by H.-J. Seiffert, Berlin, Germany

Define the Fibonacci and Lucas polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_{n+1}(x) = xF_n(x) + F_{n-1}(x),\ n \in Z,$$
$$L_0(x) = 2,\ L_1(x) = x,\ L_{n+1}(x) = xL_n(x) + L_{n-1}(x),\ n \in Z,$$

respectively. We shall prove that for all complex numbers x and all positive integers n,

$$\sum_{d \mid n} (-1)^{\Omega(d)} F_{\Omega(n/d)-\Omega(d)}(x) = 0; \tag{A'}$$

$$\sum_{d \mid n} (-1)^{\Omega(d)} L_{\Omega(n/d)-\Omega(d)}(x) = 2\prod_{p^e \| n} F_{e+1}(x); \tag{B'}$$

$$\sum_{d\mid n} L_{2\Omega(n/d)-2\Omega(d)}(x) = 2x^{-\omega(n)} \prod_{p^e \| n} F_{2e+2}(x), \tag{C'}$$

where $\omega(n)$ denotes the number of distinct prime factors of n.

The desired identities (A) and (B) are obtained from (A′) and (B′), respectively, by taking $x = 1$.

We need the following known equations [see A. F. Horadam & Bro. J. M. Mahon, "Pell and Pell-Lucas Polynomials, *The Fibonacci Quarterly* **23.1** (1985):7-20, equations (2.1), (3.23), and (3.25)],

$$L_j(x) = F_{j-1}(x) + F_{j+1}(x), \ j \in Z, \tag{1}$$

$$L_{j+k}(x) + (-1)^k L_{j-k}(x) = L_j(x) L_k(x), \ j, k \in Z, \tag{2}$$

and the easily verified relations,

$$L_{-j}(x) = (-1)^j L_j(x) \quad \text{and} \quad F_{-j}(x) = (-1)^{j-1} F_j(x), \ j \in Z.$$

Proposition: For all nonnegative integers m and e, we have

$$\sum_{j=0}^{m-1} (-1)^j L_{e-2j}(x) = F_{e+1}(x) - (-1)^{m-e} F_{2m-e-1}(x).$$

Proof: This is true for $m = 0$ (empty sums have the value zero). Suppose that the equation holds for $m, m \in N_0$ (whole numbers). Then

$$\begin{aligned}\sum_{j=0}^{m} (-1)^j L_{e-2j}(x) &= \sum_{j=0}^{m-1} (-1)^j L_{e-2j}(x) + (-1)^m L_{e-2m}(x) \\ &= F_{e+1}(x) - (-1)^{m-e} F_{2m-e-1}(x) + (-1)^m L_{e-2m}(x) \\ &= F_{e+1}(x) - (-1)^{m+1-e} (L_{2m-e}(x) - F_{2m-e-1}(x)) \\ &= F_{e+1}(x) - (-1)^{m+1-e} F_{2m-e+1}(x),\end{aligned}$$

where we have used (1). This completes the induction proof. Q.E.D.

Corollary: For all nonnegative integers e, we have

$$\sum_{j=0}^{e} (-1)^j L_{e-2j}(x) = 2F_{e+1}(x).$$

Proof: Take $m = e + 1$ in the equation of the Proposition. Q.E.D.

Now we are able to prove the desired identities. We note that if d runs through all positive divisors of n, so does n/d. Hence, if $S(n)$ denotes the left side of (A′), then

$$S(n) = \sum_{d\mid n} (-1)^{\Omega(n/d)} F_{\Omega(d)-\Omega(n/d)}(x) = -\sum_{d\mid n} (-1)^{\Omega(d)} F_{\Omega(n/d)-\Omega(d)}(x) = -S(n),$$

or $S(n) = 0$. This proves (A′).

The proof of (B′) is more interesting. Let $T(n)$ denote the left side of (B′). If $n = p^e$ is a prime power, then by the identity of the above Corollary,

$$T(n)=T(p^e)=\sum_{j=0}^{e}(-1)^j L_{e-2j}(x)=2F_{e+1}(x).$$

Thus, (B′) holds for all prime powers n. The proof of (B′) is completed by showing that the function $f:N\to C$ defined by $f(n)=T(n)/2$, $n\in N$, is multiplicative. Let m and n be coprime natural numbers. If $c\,|\,m$ and $d\,|\,n$, then

$$\Omega\left(\frac{mn}{cd}\right)-\Omega(cd)=\Omega\left(\frac{m}{c}\right)-\Omega(c)+\Omega\left(\frac{n}{d}\right)-\Omega(d)$$

and

$$\Omega\left(\frac{m}{c}d\right)-\Omega\left(c\frac{n}{d}\right)=\Omega\left(\frac{m}{c}\right)-\Omega(c)+\Omega(d)-\Omega\left(\frac{n}{d}\right),$$

so that by (2),

$$(-1)^{\Omega(cd)}L_{\Omega(\frac{mn}{cd})-\Omega(cd)}(x)+(-1)^{\Omega(\frac{cn}{d})}L_{\Omega(\frac{m}{c}d)-\Omega(c\frac{n}{d})}(x)=(-1)^{\Omega(c)}(-1)^{\Omega(d)}L_{\Omega(\frac{m}{c})-\Omega(c)}(x)L_{\Omega(\frac{n}{d})-\Omega(d)}(x).$$

Summing over all positive divisors c of m and d of n, we obtain the claimed equation:

$$f(mn)=f(m)f(n).$$

This completes the proof of (B′).

The desired identity (C′) easily follows from (B′) when we replace x by $i(x^2+2)$, where $i=\sqrt{(-1)}$, and use the known relations

$$L_j(i(x^2+2))=i^j L_{2j}(x)$$

and

$$F_j(i(x^2+2))=i^{j-1}F_{2j}(x)/x,\ j\in Z.$$

Let us look at what we get from (B′) if we set $x=2i$. Now, since $L_j(2i)=2i^j$ and $F_j(2i)=ji^{j-1}$, $j\in Z$, (B′) gives, after some simplification,

$$\tau(n)=\sum_{d\,|\,n}1=\prod_{p^e\|n}(e+1),$$

where $\tau(n)$ denotes the number of positive divisors of n. This is a well-known identity from Analytic Number Theory.

Also solved by the proposer.

Triple Play

H-529 ***Proposed by Paul S. Bruckman, Highwood, IL (Vol. 35, no. 3, August 1997)***

Let ρ denote the set of Pythagorean triples (a,b,c) such that $a^2+b^2=c^2$. Find all pairs of integers $m,n>0$ such that $(a,b,c)=(F_mF_n,F_{m+1}F_{n+2},F_{m+2}F_{n+1})\in\rho$.

Solution by L. A. G. Dresel, Reading, England

Let $a=F_mF_n$, $b=F_{m+1}F_{n+2}$, $c=F_{m+2}F_{n+1}$. We shall prove that there is only one such Pythagorean triple with $m,n>0$, namely $m=3,n=6$, giving $a=16,b=63,c=65$. We use the identity

$5F_mF_n = L_{n+m} - (-1)^m L_{n-m}$, so that $5c = L_{n+m+3} + (-1)^{m+1} L_{n-m-1}$ and $5b = L_{n+m+3} - (-1)^{m+1} L_{n-m+1}$. Hence, $5(c+b) = 2L_{n+m+3} - (-1)^{m+1} L_{n-m}$ and $(c-b) = (-1)^{m+1} F_{n-m}$. Since F_t and F_{t+1} have no common factor, it follows that a, b, and c have no common factor, and the Pythagorean triples must take the form $2uv$, $u^2 - v^2$, $u^2 + v^2$, where $u > v > 0$, have no common factor; hence, c is odd, while just one of a and b is even. We now consider these two cases in turn.

Case A. Let $a = 2uv$, then b and c are odd, and we have $3 \mid m$ and $3 \mid n$, while $c - b = 2v^2$ gives $(-1)^{m+1} F_{n-m} = 2v^2$. Using a result proved by J. H. E. Cohn in [1], this implies that $|n-m| = 0$ or 6. We can reject $n = m$, since this gives $b = c$ and $a = 0$. Taking $|n-m| = 3$, we have $F_{\pm 3} = 2$ and $v = 1$, so that m must be odd. Furthermore, we have $5(c+b) = 10u^2$. Hence, if $n = m+3$, then $10u^2 = 2(L_{2(m+3)} - 2) = 10(F_{m+3})^2$ gives $u = F_{m+3}$; if $n = m-3$, then $10u^2 = 2(L_{2m} + 2) = 10(F_m)^2$ gives $u = F_m$, since m is odd. Also, $a = 2uv = 2u = 2F_{m+3}$ or $2F_{n+3}$. But we also have $a = F_mF_n$; therefore, the smaller factor must be $F_3 = 2$, and this must be F_m, since m is odd. Hence, $m = 3$ and $n = 6$ is the only solution when $|n-m| = 3$.

Next, take $|n-m| = 6$, so that $2v^2 = (-1)^{m+1} F_{n-m} = 8$. If $n - m = 6$, m must be odd, and we obtain $10u^2 = 2(L_{2m+9} - 9)$; then, since $3 \mid m$, $2m+9$ is an odd multiple of 3, and $4 \mid L_{2m+9}$. Therefore, $5u^2 \equiv u^2 \equiv -1 \pmod 4$, which shows that there are no solutions in this case.

Finally, if $n - m = -6$, m must be even, and we have $6 \mid m$ and $6 \mid n$, so that $F_6 \mid F_m$ and $F_6 \mid F_n$, making F_mF_n divisible by 64. But we have $2v^2 = 8$, giving $v = 2$, so that $a = 2uv = 4u$, where u is odd, since $(u, v) = 1$. Hence, it is not possible to satisfy $a = F_mF_n$ if $n - m = -6$.

Case B. Now, if $b = 2uv$, then $c - b = u^2 + v^2 - 2uv = (u-v)^2$, so that $(-1)^{m+1} F_{n-m} = (u-v)^2$. It was also proved by J. H. E. Cohn in [1] that this implies $|n-m| = 0, 1, 2$, or 12. But since a and c are odd, we must have both $3 \mid (m+1)$ and $3 \mid (n+2)$. This implies $3 \mid (n-m+1)$, which rules out $|n-m| = 0$ and 12, and we are left with $(-1)^{m+1} F_{n-m} = 1$. We then find that m must be odd, of the form $m = 6t - 1$ (with $t \geq 1$), while the corresponding n can be either $n = 6t+1$ or $n = 6t-2$. But $c - b = 1$, so that $a^2 = c^2 - b^2 = c + b$. Since $a = F_mF_n$, this gives

$$(L_{2m}+2)(L_{2n} \pm 2) = 5\{2L_{n+m+3} - (-1)^{m+1} L_{n-m}\}.$$

Approximating by putting $L_r = \alpha^r$ and ignoring terms that are small compared to L_r, we obtain $\alpha^{2(m+n)} = 10\alpha^{n+m+3}$ approximately, and since $\alpha^5 > 11$, our equation gives $\alpha^{m+n} < 11\alpha^3 < \alpha^8$. But the smallest pair of values for m and n is given above as $m = 5$ and $n = 4$, giving $m + n = 9$. This gives a contradiction, and proves that there are no acceptable solutions in Case B.

Reference

1. J.H.E. Cohn. "Square Fibonacci Numbers, etc." *The Fibonacci Quarterly* **2.2** (1964):109-13.

Also solved by H.-J. Seiffert, I. Strazdins, and the proposer.

Some Period

H-530 ***Proposed by Andrej Dujella, University of Zagreb, Croatia (Vol. 35, no. 3, August 1997)***

Let $k(n)$ be the period of a sequence of Fibonacci numbers $\{F_i\}$ modulo n. Prove that $k(n) \leq 6n$ for any positive integer n. Find all positive integers n such that $k(n) = 6n$.

Solution by Paul S. Bruckman, Highwood, IL

For the first part of the problem, it suffices to prove the following lemma.

Lemma 1: For all odd n, $k(n) \le 4n$.

Of course, $k(1) = 1$, hence the result is trivially true for $n = 1$. If $n > 1$ is odd, let K_e denote $k(2^e n)$, $N_e = 2^e n$, $k = k(n)$, and $R_e = K_e / N_e$. Assuming the result of Lemma 1, $K_1 = \text{LCM}(3, k) \le 3k$, hence $R_1 \le 3k/2n \le 6$, Next, $K_2 = \text{LCM}(6, k) \le 6k$, hence $R_2 \le 6k/4n \le 6$. Next, $K_3 = \text{LCM}(6, k) \le 6k$, hence $R_3 \le 6k/8n \le 3$. Finally, if $e \ge 4$, $K_e = \text{LCM}(3 \cdot 2^{e-1}, k) \le 3k \cdot 2^{e-1}$, hence $R_e \le 3k/2n \le 6$. Thus, the result of Lemma 1 implies that $k(n) \le 6n$ for all $n > 1$; it therefore suffices to prove Lemma 1.

Proof of Lemma 1: We first assume that $\gcd(n, 10) = 1$. The following results are well known for all primes $p \ne 2, 5$: $k(n)$ is even for all $n > 2$; $k(p) \mid (p-1)$ if $(5/p) = 1$, $k(p) \mid (2p+2)$ if $(5/p) = -1$. Also, $k(p^e) = p^{e-t} k(p)$ for some t with $1 \le t \le e$. Therefore, if $(5/p) = 1$, $k(p^e) = 2p^{e-t}(p-1)/2a$ for some integer a, while if $(5/p) = -1$, $k(p^e) = 4p^{e-t}(p+1)/2a$ for some integer a. If $n = \prod p^e$, $k(n) = \text{LCM}\{k(p^e)\}$. We then see that $k(n) \le 4\prod_{p^e \| n} p^{e-1}(p+1)/2$. Then $k(n)/n \le 4\prod_{p|n}(p+1)/2p < 4$, since $(p+1)/2p < 1$ for all p.

On the other hand, if we assume that $n = 5^e$, then $Z(n) = n$ and $k(n) = 4n$. If $n = 5^e m$, where $\gcd(m, 10) = 1$, then $k(n) = \text{LCM}(k(5^e), k(m)) = \text{LCM}(4 \cdot 5^e, k(m)) < 4n$. This proves Lemma 1. In conjunction with our earlier discussion, it follows that $k(n) \le 6n$ for all n.

From Lemma 1 and the earlier discussion, it is seen that the upper bound of $6n$ is possibly reached only if $n = 2^a 5^b$ for some integers a and b. Note that

$$k(2 \cdot 5^b) = \text{LCM}(3, 4 \cdot 5^b) = 12 \cdot 5^b = 6n.$$

Next,

$$k(4 \cdot 5^b) = k(8 \cdot 5^b) = \text{LCM}(6, 4 \cdot 5^b) = 12 \cdot 5^b = 3n \text{ or } 3n/2 < 6n.$$

Finally, if $a \ge 4$,

$$k(n) = \text{LCM}(3 \cdot 2^{a-1}, 4 \cdot 5^b) = 3 \cdot 2^{a-1} \cdot 5^b = 3n/2 < 6n.$$

Thus, $k(n) = 6n$ if and only if $n = 2 \cdot 5^b$, $b = 1, 2, \ldots$.

Also solved by D. Bloom, L. Dresel, and the proposer.

A Rational Decision

H-531 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 35, no. 3, August 1997)

Consider the sum $S = \sum_{n=1}^{\infty} t(n)/n^2$, where $t(1) = 1$ and $t(n) = \prod_{p|n}(1-p^{-2})^{-1}$, $n > 1$, the product taken over all prime p dividing n. Evaluate S and show that it is rational.

Solution by H.-J. Seiffert, Berlin, Germany

We need the following results.

Theorem 1: If $f: N \to C$ is a multiplicative function such that $\sum_{n=1}^{\infty} f(n)/n^s$ converges absolutely for $\sigma = \text{Re}(s) > \sigma_0$, then

$$\sum_{n=1}^{\infty} f(n)/n^s = \prod_p \left(1+\sum_{j=1}^{\infty} f(p^j)/p^{js}\right) \text{ for } \sigma > \sigma_0,$$

where the product is over all primes p.

Proof: See ([1], pp. 230-31).

Theorem 2: For $\sigma > 1$, we have

$$\prod_p (1-p^{-s}) = 1/\zeta(s) \text{ and } \prod_p (1+p^{-s}) = \zeta(s)/\zeta(2s),$$

where ζ denotes the Riemann Zeta function.

Proof: See ([1], p. 231).

Let $S_k = \sum_{n=1}^{\infty} t_k(n)/n^k$, $k \in C$, $\text{Re}(k) > 1$, where $t_k(1) = 1$ and $t_k(n) = \prod_{p|n}(1-p^{-k})^{-1}$ for $n > 1$. Clearly, t_k is a multiplicative function. Since $t_k(p^j) = (1-p^{-k})^{-1}$ for all $j \in N$ and all primes p, we have

$$\sum_{j=1}^{\infty} t_k(p^j)/p^{jk} = p^{-k}(1-p^{-k})^{-2} \text{ for all primes } p,$$

where we have used the closed form expression for infinite geometric sums. Using

$$1+p^{-k}(1-p^{-k})^{-2} = (1-p^{-k})^{-1}(1-p^{-2k})^{-1}(1+p^{-3k}),$$

it follows from Theorems 1 and 2 that

$$S_k = \zeta(k)\zeta(2k)\zeta(3k)/\zeta(6k),\ k \in C,\ \text{Re}(k) > 1. \tag{1}$$

Since ([1], p. 266)

$$\zeta(2j) = (-1)^{j+1}\frac{(2\pi)^{2j}}{2(2j)!}B_{2j},\ j \in N,$$

where the B's are the Bernoulli numbers defined by ([1], p. 265, or [2], p. 9)

$$B = 1 \text{ and } B_n = \sum_{r=0}^{n}\binom{n}{r}B_r,\ n \in N,\ n \ge 2,$$

from (1) we obtain

$$S_{2j} = \frac{(12j)!}{4(2j)!(4j)!(6j)!}\frac{B_{2j}B_{4j}B_{6j}}{B_{12j}},\ j \in N, \tag{2}$$

showing that S_{2j}, $j \in N$, is a rational number. Using the values ([2], p. 10) $B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}$, $B_6 = \frac{1}{42}$, and $B_{12} = \frac{-691}{2730}$, from (2) it is easily calculated that $S = S_2 = \frac{5005}{2764}$. This solves the present proposal.

References

1. T.M. Apostol. *Introduction to Analytic Number Theory*. New York: Springer-Verlag, 1976.
2. H. Rademacher. *Topics in Analytic Number Theory*. New York: Springer-Verlag, 1973.

Also solved by K. Lau, and the proposer.

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