The Fibonacci Quarterly

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980) and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

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ON A GENERALIZATION OF THE BINOMIAL THEOREM

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1. INTRODUCTION

The elementary binomial theorem is arguably one of the oldest and perhaps most well-known result in mathematics. This famous theorem, which was known to Chinese mathematicians from as early as the thirteenth century, has been subject since that time to a number of generalizations, one of which is attributable to Newton. In this result, commonly referred to today as the General Binomial Theorem, Newton asserted that the expansion of $(1+x)^n$ for negative and fractional exponents consisted of the following series

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)\cdots(n-p+1)}{p!}x^p+\cdots, \tag{1}$$

where the variable x was assumed "small." This binomial series was applied to great effect by Newton in such diverse problems as the quadrature of the hyperbola, root extraction, and the approximation of π. In contrast, the second and perhaps more obvious extension to the binomial theorem can be found in the so-called multinomial theorem of Leibniz, where the expansion of a general multinomial

$$(x_1+x_2+\cdots+x_m)^n \tag{2}$$

into a polynomial of m variables was considered (see [1], p. 340). This particular result, which has found numerous applications in the area of combinatorics, is somewhat more "algebraic" in character when compared with the former generalization, which is essentially a statement concerning the power series representation of a function. In keeping with the "algebraic" spirit of (2), we present in this paper an additional extension to the binomial theorem via the development of an expansion theorem for the following class of polynomial functions, denoted

$$(x)_{a_n} = \prod_{i=1}^{n}(x+a_i), \tag{3}$$

in which the sequence $\{a_n\}$ of complex numbers is assumed in arithmetic progression. It should be noted that the construction of this expansion theorem can be viewed as a "connection constant" problem of the Umbral Calculus (see [4], p. 120) in which real numbers c_{nk} are sought so that a given polynomial sequence $p_n(x)$ can be expanded in terms of another, as follows:

$$p_n(x) = \sum_{k>0} c_{nk}q_k(x).$$

In this article we shall not make use of the Umbral Calculus to derive the desired expansion theorem; rather, we shall be content with applying more elementary methods to effect the said result. The outline of this paper is as follows. To facilitate the main result, it will first be necessary to formulate an expression for the coefficients within the polynomial expansion of (3) in terms of the elements of an arbitrary sequence. This is achieved in Section 2, where the coefficient of x^{n-p} for $p=1,2,\ldots,n$, denoted $\phi_p(n)$, will be shown to consist of a p-fold summation of a p-fold product. When $\{a_n\}$ is substituted with an arithmetic progression, these summands then reduce, as demonstrated in Section 3, to a linear combination of binomial coefficients as follows:

$$\phi_p(n) = \sum_{m=1}^{p+1} \theta_m^{(p)} \binom{n+p+1-m}{2p+1-m}. \tag{4}$$

Moreover, the scalars $\theta_m^{(p)}$, which vary in accordance with the particular arithmetic progression chosen, will be calculated via an accompanying algorithm, thereby determining completely the equation for the coefficient of x^{n-p}. This use of an algorithm in the formulation of $\phi_p(n)$ highlights one major difficulty when attempting to construct a general expansion theorem for (3), namely that, in most instances, no simple closed-form expression exists for $\theta_m^{(p)}$ in terms of the parameters m and p. However, all such apparent difficulties diminish when dealing with a constant sequence (say $a_n = a$), as the corresponding scalars will assume the following simple form,

$$\theta_m^{(p)} = \begin{cases} 0 & \text{for } m = 1, 2, \ldots, p, \\ a^p & \text{for } m = p+1, \end{cases}$$

which, when combined with equations (3) and (4), will yield the binomial theorem. An alternate expansion theorem is also derived when $\{a_n\}$ is in geometric progression. Finally, in Section 4, we will explore an application of the above expansion theorem to the Pochhammer family of polynomial functions that result when $a_n = n-1$. Of particular interest will be the derivation of closed-form expressions for the Stirling numbers of first order, which shall mirror existing formulas for the Stirling numbers of second order (see [6], p. 233).

2. PRELIMINARIES

In this section we shall be concerned with the expansion of a class of polynomial functions which result from the n-fold binomial product $(x+a_1)(x+a_2)\cdots(x+a_n)$ for a given sequence $\{a_n\}$. Our aim is to derive a closed-form expression for the coefficients within these polynomial expansions in terms of the elements of $\{a_n\}$. We begin with a formal definition.

Definition 2.1: Let $\{a_n\}$ be an arbitrary sequence of complex numbers. Then the following n-fold binomial product $(x+a_1)(x+a_2)\cdots(x+a_n)$ shall be denoted by $(x)_{a_n}$. In addition, the coefficient of x^{n-p} for $p = 1, 2, \ldots, n$ within the polynomial expansion of $(x)_{a_n}$ will be written as $\phi_p(n)$.

Remark 2.1: The notation $(x)_{a_n}$ has been improvised from the Pochhammer symbol $(x)_n$, which denotes the rising factorial polynomial of degree n given by $x(x+1)\cdots(x+n-1)$.

It is clear from the definition that each coefficient $\phi_p(n)$ in $(x)_{a_n}$ is an elementary symmetric function in $a_1, a_2, \ldots, a_n$. Although it is well known (see [2], p. 252) that these functions can be expressed in terms of a multiple summation of a p-fold product, the formulation provided is somewhat incomplete for our purposes here. This is the motivation behind the following discussion, which will lead to a more satisfactory representation of $\phi_p(n)$ in Proposition 2.1. We return now to the expansion of $(x)_{a_n}$.

To determine how the coefficients within $(x)_{a_n}$ are formed by the terms of an arbitrary sequence, let us examine $\phi_p(n)$ for $p = 1, 2, 3$ in the cases $n = 2, \ldots, 5$. Beginning with $\phi_1(n)$, it is evident upon expanding that the coefficient of x^{n-1} is equal to the n^{th} partial sum of the sequence $\{a_n\}$. Next, by grouping lower-order terms in each expansion, we observe the following for increasing n:

$$\begin{aligned}
&\phi_2(2) = a_2a_1,\\
&\phi_2(3) = a_2a_1 + a_3(a_1 + a_2),\\
&\phi_2(4) = a_2a_1 + a_3(a_1 + a_2) + a_4(a_1 + a_2 + a_3).
\end{aligned}$$

Thus, it would appear, at least empirically, that $\phi_2(n)$ consists of a summation of $n-1$ terms, each of which is the sum of a 2-fold product. Therefore, if the outer and inner terms of each product in the above summands were indexed by i_1 and i_2, respectively, one may then infer that

$$\phi_2(n) = \sum_{i_1=1}^{n-1} a_{i_1+1}\left\{\sum_{i_2=1}^{i_1} a_{i_2}\right\} = \sum_{i_1=1}^{n-1}\sum_{i_2=1}^{i_1} a_{i_1+1}a_{i_2}. \tag{5}$$

Finally, for simplicity, set $\tilde{\phi}_2(n) = \phi_2(n+1)$. Then a similar arrangement of lower-order terms reveals

$$\begin{aligned}
&\phi_3(3) = a_3\tilde{\phi}_2(1),\\
&\phi_3(4) = a_3\tilde{\phi}_2(1) + a_4\tilde{\phi}_2(2),\\
&\phi_3(5) = a_3\tilde{\phi}_2(1) + a_4\tilde{\phi}_2(2) + a_5\tilde{\phi}_2(3).
\end{aligned}$$

Once again we are presented with a clear pattern in which $\phi_3(n)$ appears to consist of a summation of $n-2$ terms each of the form $a_{i_1+2}\tilde{\phi}_2(i_1)$. Thus, after relabeling index variables from i_m to i_{m+1} in (5), we propose

$$\phi_3(n) = \sum_{i_1=1}^{n-2} a_{i_1+2}\{\tilde{\phi}_2(i_1)\} = \sum_{i_1=1}^{n-2}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1+2}a_{i_2+1}a_{i_3}. \tag{6}$$

Consequently, with the aid of equations (5) and (6), one may conjecture that $\phi_p(n)$ is formed from a p-fold summation of the product $a_{i_1+p-1}a_{i_2+p-2}\cdots a_{i_p}$ in which the index of the outer summand assumes the values $i_1 = 1, 2, \ldots, n-p+1$ with all subsequent indexes i_m ranging over $i_m = 1$, 2, ..., i_{m-1} for $m = 2, 3, \ldots, p$. By continuing as above and using (5) and (6), one may construct similar expressions for the coefficients of lower-order powers in $(x)_{a_n}$ that are in agreement with the previously suggested rule of formation. Hence, we now consider the following result, which is stated in terms of elementary symmetric functions.

Proposition 2.1: Suppose $\{a_n\}$ is an arbitrary sequence, then for $n = 2, 3, \ldots$ the elementary symmetric function $\phi_p(n)$ in $a_1, a_2, \ldots, a_n$ is given by

$$\phi_p(n) = \begin{cases} \displaystyle\sum_{i_1=1}^{n} a_{i_1} & \text{for } p = 1,\\[2ex] \displaystyle\sum_{i_1=1}^{n-p+1}\sum_{i_2=1}^{i_1}\cdots\sum_{i_p=1}^{i_{p-1}} a_{i_1+p-1}a_{i_2+p-2}\cdots a_{i_p} & \text{for } p = 2, 3, \ldots, n. \end{cases} \tag{7}$$

Proof: Fix the sequence $\{a_n\}$ in question and set $n = 2$ as the base for the following inductive argument. Clearly, (7) holds for the case $n = 2$ since

$$\phi_1(2) = \sum_{i_1=1}^{2} a_{i_1} = a_1 + a_2 \quad \text{and} \quad \phi_2(2) = \sum_{i_1=1}^{1} \sum_{i_2=1}^{i_1} a_{i_1+1} a_{i_2} = a_1 a_2,$$

which are in agreement with the coefficients found in the expansion

$$(x+a_1)(x+a_2) = x^2 + (a_1+a_2)x + a_1 a_2.$$

Assume the result holds for $n = k$ where $k \geq 2$. Thus,

$$(x)_{a_k} = x^k + \phi_1(k)x^{k-1} + \phi_2(k)x^{k-2} + \cdots + \phi_k(k), \tag{8}$$

where the coefficients $\phi_p(k)$ are of the form as stated above. Multiplying (8) by the term $(x + a_{k+1})$ and collecting like powers of x yields a polynomial of degree $k+1$ with coefficients defined as follows:

$$\phi_1(k+1) = a_{k+1} + \phi_1(k), \tag{9}$$

$$\phi_p(k+1) = a_{k+1}\phi_{p-1}(k) + \phi_p(k) \quad \text{for } p = 2, 3, \ldots, k, \tag{10}$$

$$\phi_{k+1}(k+1) = a_{k+1}\phi_k(k). \tag{11}$$

From this set of equations we now generate via the inductive hypothesis corresponding expressions for $\phi_p(k+1)$. Beginning with (9), it is immediately apparent that

$$\phi_1(k+1) = \sum_{i_1=1}^{k+1} a_{i_1}.$$

Now from (10) we have, for $p = 2, 3, \ldots, k$,

$$\begin{aligned}\phi_p(k+1) = a_{k+1} &\sum_{i_1=1}^{k-p+2} \sum_{i_2=1}^{i_1} \cdots \sum_{i_{p-1}=1}^{i_{p-2}} a_{i_1+p-2} a_{i_2+p-3} \cdots a_{i_{p-1}} \\ &+ \sum_{i_1=1}^{k-p+1} \sum_{i_2=1}^{i_1} \cdots \sum_{i_p=1}^{i_{p-1}} a_{i_1+p-1} a_{i_2+p-2} \cdots a_{i_p}.\end{aligned} \tag{12}$$

Relabeling index variables from i_m to i_{m+1} in the expression for $\phi_{p-1}(k)$, observe that $a_{k+1}\phi_{p-1}(k)$ is equal to

$$a_{i_1+p-1} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \cdots \sum_{i_p=1}^{i_{p-1}} a_{i_2+p-2} a_{i_3+p-3} \cdots a_{i_p}$$

when $i_1 = k - p + 2$. Consequently, by factoring a_{i_1+p-1} in the above $(p-1)$-fold summation and adding the result to the second summand of (12) yields

$$\phi_p(k+1) = \sum_{i_1=1}^{k-p+2} \sum_{i_2=1}^{i_1} \cdots \sum_{i_p=1}^{i_{p-1}} a_{i_1+p-1} a_{i_2+p-2} \cdots a_{i_p}.$$

Finally, from (11), we deduce $\phi_{k+1}(k+1) = a_{k+1}a_k \cdots a_1$ which, clearly, is in agreement with the hypothesized expression for the coefficient of x^0 in $(x)_{a_{k+1}}$. Thus, the result holds for $n = k+1$. Hence, by induction, (7) is valid for all $n = 2, 3, \ldots$. □

3. MAIN RESULTS

With the formulation in Section 2 of a precise relationship between the coefficients of $(x)_{a_n}$ and the elements of $\{a_n\}$, it is now possible to determine, for suitable classes of sequences, explicit algebraic expressions for $\phi_p(n)$ in terms of the parameters n and p. Clearly, those sequences of interest must possess a closed-form expression for their respective partial sums. However, in general, this will not guarantee the existence of explicit formulas for subsequent $\phi_p(n)$, as the following simple example indicates. Let $a_n = \frac{1}{n(n+1)}$, then an elementary calculation establishes $\phi_1(n) = \frac{n}{n+1}$. This, in turn, implies that

$$\phi_2(n) = \sum_{i_1=1}^{n-1} a_{i_1+1}\left\{\sum_{i_2=1}^{i_1} a_{i_2}\right\} = \sum_{i_1=1}^{n-1} \frac{i_1}{(i_1+1)^2(i_1+2)},$$

which cannot be expressed as a rational function in n due to the presence of the factor $\frac{1}{(i_1+1)^2}$.

Remark 3.1: We note that the function $\phi_2(n)$ in the previous example can be written as the sum of a rational function in n and the di-gamma function $\psi'(z)$. Indeed, by decomposing into partial fractions, observe that

$$\begin{aligned}\phi_2(n) &= \sum_{i_1=1}^{n-1}\left\{\frac{2}{i_1+1} - \frac{2}{i_1+2}\right\} - \sum_{i_1=1}^{n-1}\frac{1}{(i_1+1)^2} = 1 - \frac{2}{n+1} - \left(\frac{\pi^2}{6} - 1 - \sum_{i_1=n}^{\infty}\frac{1}{(i_1+1)^2}\right)\\ &= 1 - \frac{\pi^2}{6} + \frac{n^2}{(n+1)^2} + \sum_{i_1=n+1}^{\infty}\frac{1}{(i_1+1)^2} = 1 - \frac{\pi^2}{6} + \frac{n^2}{(n+1)^2} + \psi'(n+2),\end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$.

Thus, in addition to the previous condition, those sequences under consideration should also admit for each $p = 2, 3, \ldots$ a closed-form expression for the n^{th} partial sum of

$$a_{i_1+p-1}\left\{\sum_{i_2=1}^{i_1} \cdots \sum_{i_p=1}^{i_{p-1}} a_{i_2+p-2} \cdots a_{i_p}\right\}.$$

Recalling that the partial sum of an arithmetic progression can be expressed as a linear combination of at most two binomial coefficients, we observe from the following result (see [3]) that $a_n = a_1 + (n-1)d$ (where a_1, $d \in \mathbb{C}$) is one such sequence that satisfies the required properties.

Lemma 3.1: Let $r \in \mathbb{N}^+$, then

$$\sum_{i=1}^{n}\binom{i+r}{r+1} = \binom{n+r+1}{r+2},$$

$$\sum_{i=1}^{n} i\binom{i+r}{r+1} = (r+2)\binom{n+r+2}{r+3} - (r+1)\binom{n+r+1}{r+2}.$$

Therefore, with the aid of Lemma 3.1, we can now state and prove the desired expansion theorem.

Theorem 3.1: Suppose $\{a_n\}$ is an arithmetic progression where $a_n = a_1 + (n-1)d$ for a given a_1, $d \in \mathbb{C}$. Then the equation for the coefficient of x^{n-p} for $n = 2, 3, \ldots$ and $p = 1, 2, \ldots, n$ in the resulting polynomial expansion of $(x)_{a_n}$ consists of a linear combination of binomial coefficients as follows,

$$\phi_p(n) = \sum_{m=1}^{p+1} \theta_m^{(p)} \binom{n+p+1-m}{2p+1-m}, \tag{13}$$

in which the corresponding scalars $\theta_m^{(p)}$ are determined via the accompanying algorithm.

Algorithm 3.1: Set $\theta_1^{(1)} = d$ and $\theta_2^{(1)} = a_1 - d$, then calculate remaining scalars $\theta_m^{(p)}$ iteratively as follows:

for $i = 2, 3, \ldots, n$,

$\quad \theta_1^{(i)} = \theta_1^{(i-1)} d(2i-1);$

for $j = 2, 3, \ldots, i$,

$\quad \theta_j^{(i)} = \theta_{j-1}^{(i-1)}(d(j-i-2)+a_1) + \theta_j^{(i-1)}(2i-j)d;$

$\quad \theta_{i+1}^{(i)} = \theta_i^{(i-1)}(a_1 - d).$

Proof: In Proposition 2.1, let $a_n = a_1 + (n-1)d$ and set $\tilde{\phi}_p(n) = \phi_p(n+p-1)$, noting that in the resulting p-fold summation for $\tilde{\phi}_p(n)$ we no longer need require $p \le n$. Thus, it suffices to demonstrate via the following inductive argument on the parameter p, that there exists $\theta_m^{(p)} \in \mathbb{C}$ such that

$$\tilde{\phi}_p(n) = \sum_{m=1}^{p+1} \theta_m^{(p)} \binom{n+2p-m}{2p+1-m}. \tag{14}$$

Beginning with $p = 1$, we have

$$\tilde{\phi}_1(n) = \sum_{i_1=1}^{n} (a_1 - d) + i_1 d = \sum_{m=1}^{2} \theta_m^{(1)} \binom{n+2-m}{3-m},$$

where $\theta_1^{(1)} = d$ and $\theta_2^{(1)} = a_1 - d$; consequently, (14) is valid for $p = 1$. Assume now that the result holds for $p = k$ where $k \ge 1$. To facilitate the inductive step, consider from (7) the expression for $\tilde{\phi}_{k+1}(n)$ as follows:

$$\tilde{\phi}_{k+1}(n) = \sum_{i_1=1}^{n} a_{i_1+k} \left\{ \sum_{i_2=1}^{i_1} \cdots \sum_{i_{k+1}=1}^{i_k} a_{i_2+k-1} \cdots a_{i_{k+1}} \right\}.$$

If necessary, by relabeling index variables, observe that the k-fold summation within the parentheses of the above equation, is equal to $\tilde{\phi}_k(i_1)$. Therefore, by assumption, we have

$$\tilde{\phi}_{k+1}(n) = \sum_{i_1=1}^{n} a_{i_1+k} \left\{ \sum_{m=1}^{k+1} \theta_m^{(k)} \binom{i_1+2k-m}{2k+1-m} \right\} = \sum_{m=1}^{k+1} \theta_m^{(k)} \left\{ \sum_{i_1=1}^{n} a_{i_1+k} \binom{i_1+2k-m}{2k+1-m} \right\}. \tag{15}$$

Now, for each m, an application of Lemma 3.1 yields

$$\begin{aligned} \sum_{i_1=1}^{n} a_{i_1+k} \binom{i_1+2k-m}{2k+1-m} &= a_k \sum_{i_1=1}^{n} \binom{i_1+2k-m}{2k+1-m} + d \sum_{i_1=1}^{n} i_1 \binom{i_1+2k-m}{2k+1-m} \\ &= \alpha(k,m) \binom{n+2k+1-m}{2k+2-m} + \beta(k,m) \binom{n+2k+2-m}{2k+3-m}, \end{aligned}$$

where $\alpha(k,m) = d(m-k-2) + a_1$ and $\beta(k,m) = d(2k-m+2)$. As a result, (15) reduces to

$$\tilde{\phi}_{k+1}(n)=\sum_{m=1}^{k+1}\theta_m^{(k)}\beta(k,m)\binom{n+2k+2-m}{2k+3-m}+\sum_{m=1}^{k+1}\theta_m^{(k)}\alpha(k,m)\binom{n+2k+1-m}{2k+2-m}.$$

Finally, since

$$\sum_{m=1}^{k+1}\theta_m^{(k)}\alpha(k,m)\binom{n+2k+1-m}{2k+2-m}=\sum_{m=2}^{k+2}\theta_{m-1}^{(k)}\alpha(k,m-1)\binom{n+2k+2-m}{2k+3-m},$$

we deduce that

$$\tilde{\phi}_{k+1}(n)=\sum_{m=1}^{k+2}\theta_m^{(k+1)}\binom{n+2k+2-m}{2k+3-m},$$

where

$$\theta_1^{(k+1)}=\theta_1^{(k)}\beta(k,1), \tag{16}$$

$$\theta_m^{(k+1)}=\theta_m^{(k)}\beta(k,m)+\theta_{m-1}^{(k)}\alpha(k,m-1)\ \text{ for }\ m=2,3,\dots,k+1, \tag{17}$$

$$\theta_{k+2}^{(k+1)}=\theta_{k+1}^{(k)}\alpha(k,k+1). \tag{18}$$

Hence, the result holds for $p=k+1$. and so, by induction, (14) is valid for all $p=1,2,\dots$. Having established an explicit equation for $\phi_p(n)$, we note that it may be extended to encompass the case $p=0$ by defining $\theta_1^{(0)}\equiv 1$. It is now a simple matter to construct the accompanying algorithm. We begin by arranging those scalars involved in the first $n+1$ coefficients into a lower-triangular matrix as follows:

$$A_n=\begin{bmatrix}\theta_1^{(0)} & 0 & 0 & \cdots & 0\\ \theta_1^{(1)} & \theta_2^{(1)} & 0 & \cdots & 0\\ \theta_1^{(2)} & \theta_2^{(2)} & \theta_3^{(2)} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \theta_1^{(n)} & \theta_2^{(n)} & \theta_3^{(n)} & \cdots & \theta_{n+1}^{(n)}\end{bmatrix}.$$

Suppose it is required that the n rows of A_n are to be determined. Clearly, from above, the entries of row two are given by $\theta_1^{(1)}=d$ and $\theta_2^{(1)}=a_1-d$. Now let us assume for argument's sake that the $(i-1)^{\text{th}}$ row has been calculated where $i\geq 2$. Then the following row of values can be obtained from the former by setting $k=i-1$ in equations (16), (17), and (18). Consequently, we deduce from (16) that

$$\theta_1^{(i)}=\theta_1^{(i-1)}d(2i-1). \tag{19}$$

While (17) implies, for $j=2,3,\dots,i$,

$$\theta_j^{(i)}=\theta_j^{(i-1)}\beta(i-1,j)+\theta_{j-1}^{(i-1)}\alpha(i-1,j-1), \tag{20}$$

where $\alpha(i-1,j-1)=d(j-i-2)+a_1$ and $\beta(i-1,j)=d(2i-j)$. Similarly, from (18),

$$\theta_{i+1}^{(i)}=\theta_i^{(i-1)}(a_1-d). \tag{21}$$

Then, clearly, as the initial two rows of values are known, we may calculate all remaining $n-1$ rows by applying equations (19), (20), and (21) in succession for each $i=2,3,\dots,n$, thus completing A_n. The algorithm now readily follows. □

The binomial theorem will now follow from Theorem 3.1 by demonstrating that, for a constant sequence (say $a_n = a$), the matrix A_n is rendered diagonal with $\theta_p^{(p+1)} = a^p$ for $p = 0, 1, \ldots, n$.

Corollary 3.1: Let $a, b \in \mathbb{C}$. Then, for all integers $n \geq 1$,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}. \tag{22}$$

Proof: In what follows, assume $n \geq 2$, as (22) holds trivially for $n = 1$. Consider an arithmetic progression defined by $a_1 = a$ and $d = 0$. Then, by Theorem 3.1, the coefficient of x^{n-p} for $p = 0, 1, \ldots, n$ in the polynomial expansion of $(x+a)^n$ is equal to

$$\phi_p(n) = \sum_{m=1}^{p+1} \theta_m^{(p)} \binom{n+p+1-m}{2p+1-m}.$$

We assert that $\theta_j^{(i)} = 0$ for every $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, i$. From Algorithm 3.1, it is clear that $\theta_1^{(1)} = 0$. Now, if the result is assumed to hold for the $(i-1)^{\text{th}}$ row where $2 \leq i \leq n$, then $\theta_1^{(i)} = \theta_1^{(i-1)} d(2i-1) = 0$, while $\theta_j^{(i)} = \theta_{j-1}^{(i-1)} a = 0$ for $j = 2, 3, \ldots, i$. Hence, via the principle of finite mathematical induction the assertion is valid, and so

$$\phi_p(n) = \theta_{p+1}^{(p)} \binom{n}{p}$$

for $p = 0, 1, \ldots, n$. Now, as $\theta_2^{(1)} = a$ and $\theta_{i+1}^{(i)} = a\theta_i^{(i-1)}$ for $i = 2, 3, \ldots, n$, a similar inductive argument establishes $\theta_{p+1}^{(p)} = a^p$ for $p = 1, 2, \ldots, n$. Consequently, by recalling that $\theta_1^{(0)} \equiv 1$, we deduce that the coefficient of x^{n-p} for $p = 0, 1, \ldots, n$ in the above expansion is equal to

$$a^p \binom{n}{p}.$$

Setting $x = b$ yields the statement of the binomial theorem for the given n; however, the result now follows as this was arbitrarily chosen. □

To contrast the previous result, we shall consider now an alternate expansion theorem for $(x)_{a_n}$ where a_n is a geometric progression (i.e., $a_n = az^n$ for $a, z \in \mathbb{C}$); however, unlike Theorem 3.1, no algorithm will be required to complete the formulation of $\phi_p(n)$. It should be noted that setting $z = 1$ in this result will not produce the binomial theorem, as the expressions for $\phi_p(n)$ in this case reduce to an indeterminate form.

Theorem 3.2: For a given integer $n = 2, 3, \ldots$, set $I_n = \{z \in \mathbb{C} \mid z = \sqrt[r]{1},\ r = 1, 2, \ldots, n\}$. If $a_n = az^n$ with $a, z \in \mathbb{C}$ and $z \notin I_n$, then the coefficient of x^{n-p} for $p = 1, 2, \ldots, n$ in the polynomial expansion of $(x)_{a_n}$ is given by

$$\phi_p(n) = a^p z^{\frac{1}{2}p(p+1)} \prod_{j=1}^{p} \frac{1 - z^{n-j+1}}{1 - z^j}. \tag{23}$$

Proof: The result clearly holds for $p = 1$ as $\phi_1(n)$ is equal to the n^{th} partial sum of $\{a_n\}$. To demonstrate (23) for $p = 2, 3, \ldots, n$, consider for a fixed $z \notin I_n$ the polynomial function in x,

$$p_n(x) = (1+zx)(1+z^2x)\cdots(1+z^nx) = \sum_{m=0}^{n} c_m x^m.$$

Clearly, $p_n(x)$ satisfies the following functional identity:

$$(1+z^{n+1}x)p_n(x) = (1+zx)p_n(zx). \tag{24}$$

Substituting the above partial sum for $p_n(x)$ in (24) and equating coefficients of x^m yields

$$c_m + c_{m-1}z^{n+1} = (c_m + c_{m-1})z^m,$$

where $m = 1, 2, \ldots, n$. Thus, after some rearrangement of terms, we obtain the recurrence relation

$$c_m = \frac{z^m(1-z^{n-m+1})}{1-z^m}c_{m-1},$$

from which one easily deduces, as $c_0 = 1$, the formula

$$c_p = z^{\frac{1}{2}p(p+1)}\prod_{j=1}^{p}\frac{1-z^{n-j+1}}{1-z^j}, \tag{25}$$

where $p = 2, 3, \ldots, n$, noting here that the expression in (25) is well-defined due to the restriction $z \notin I_n$. Now let $x = y^{-1}$ for $y \neq 0$ and observe that $y^n p_n(y^{-1}) = (y)_{a_n}$, where $a_n = z^n$; hence, $\phi_p(n) = c_p$. Therefore, by Proposition 2.1, we find that

$$c_p = z^{\frac{1}{2}p(p-1)}\sum_{i_1=1}^{n-p+1}\sum_{i_2=1}^{i_1}\cdots\sum_{i_p=1}^{i_{p-1}} z^{i_1+i_2+\cdots+i_p} \tag{26}$$

for $p = 2, 3, \ldots, n$. As the coefficient of x^{n-p} for $p = 2, 3, \ldots, n$ in $(x)_{a_n}$, where $a_n = az^n$ is equal to $a^p c_p$, we deduce from (25) the desired expression. □

It is possible to retrieve a binomial coefficient from the expression in (23) by taking the limit as $z \to w$, where w is a root of unity. The result that follows may be obtained by an application of L'Hôpital's rule for indeterminant forms; however, the argument used below is probably more direct. We will require the following technical lemma.

Lemma 3.2: If w is a primitive m^{th} root of unity, where m is a positive even integer, then

$$(x+w)(x+w^2)\cdots(x+w^m) = x^m - 1.$$

Proof: Let $a_n = w^n$ and consider the polynomial $(x)_{a_m}$. By making the substitution $x = -y$, observe that

$$(-y)_{a_m} = (-1)^m\prod_{j=1}^{m}(y-w^j) = \prod_{j=1}^{m}(y-w^j).$$

Since w is a primitive root of unity, the set $\{w, w^2, \ldots, w^m\}$ contains all the m^{th} roots of unity without repetition. Hence, the product on the right of the above is equal to $y^m - 1 = x^m - 1$. □

Corollary 3.2: Suppose n, m, and p are positive integers with m even and $1 \leq p \leq mn$. If w is a primitive m^{th} root of unity, then the following limit holds:

$$\lim_{z\to w}\prod_{j=1}^{p}\frac{1-z^{mn-j+1}}{1-z^j}=\begin{cases}\frac{(-1)^s}{w^{\frac{ms}{2}}}\binom{n}{s} & \text{for } p=ms,\\ 0 & \text{for } p\neq ms.\end{cases}$$

Proof: Set $a_n=z^n$ and consider for a fixed x the polynomial in z and x given by

$$f(z)=(x)_{a_{mn}}=(x+z)(x+z^2)\cdots(x+z^{mn}).$$

As $f(z)$ is a continuous function of the complex variable z, we have

$$\lim_{z\to w}f(z)=f(w)=(x^m-1)^n=\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}x^{mr},\tag{27}$$

noting here that the right-hand side follows from Lemma 3.2 and the periodicity of the sequence $\{w^n\}$. Now when $z\notin I_{mn}$ one can expand $f(z)$ in a polynomial in x as follows,

$$f(z)=\sum_{p=0}^{mn}\phi_p(mn)x^{mn-p},\tag{28}$$

where the complex coefficients $\phi_p(mn)$ are of the form as stated in Theorem 3.2. As the set I_{mn} contains only finitely many complex numbers, there must exist, for $\delta>0$ sufficiently small, an open neighborhood about w of the form $B_\delta(w):=\{z\in\mathbb{C}:|z-w|<\delta\}$ such that $B_\delta(w)\cap I_{mn}=\{w\}$. Hence, the expression for $f(z)$ in (28) is valid in the deleted neighborhood $B_\delta(w)\setminus\{w\}$ and so, by (27)

$$\lim_{z\to w}\sum_{p=0}^{mn}\phi_p(mn)x^{mn-p}=\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}x^{mr}.$$

Clearly, as the right-hand side of the above consists of a polynomial in x^m, we have

$$\lim_{z\to w}\frac{\phi_p(mn)}{z^{\frac{1}{2}p(p+1)}}=0\quad\text{when}\quad p\neq ms,$$

while, if $p=ms$, then by setting $r=n-s$ one deduces again from (27) that

$$\lim_{z\to w}\frac{\phi_p(mn)}{z^{\frac{1}{2}p(p+1)}}=\frac{(-1)^s}{w^{\frac{1}{2}ms(ms+1)}}\binom{n}{n-s}=\frac{(-1)^s}{w^{\frac{ms}{2}}}\binom{n}{s}.\ \square$$

Remark 3.2: In the case in which $m=4$, we have for $w=\pm i$ the limit

$$\lim_{z\to w}\prod_{j=1}^{4s}\frac{1-z^{4n-j+1}}{1-z^j}=\binom{n}{s}$$

where $1\le s\le n$.

4. APPLICATION

We now turn our attention to the Pochhammer class of polynomial functions which result from (3) by setting $a_n=n-1$. This family of polynomials was first studied by Stirling in 1730 and later by Appell; however, the name Pochhammer is used in recognition for the invention of the symbol $(x)_n$. These polynomials feature in many areas of analysis, including the study of special functions, where they occur in the coefficients of hypergeometric series (see [5], p. 149). When expanded into a polynomial, $(x)_n$ can be written as

$$(x)_n = \sum_{r=0}^{n} |S_r^{(n)}| x^r,$$

where the integers $S_r^{(n)}$ are the Stirling numbers of first order. This group of numbers are normally calculated by first defining $S_0^{(n)} = 0$, $S_n^{(n)} = 1$, and then applying the recursion formula $S_n^{(r)} = S_{n-1}^{(r-1)} - rS_{n-1}^{(r)}$ for each $n = 1, 2, \ldots$ and $r = 1, 2, \ldots, n$ in succession. However, the Stirling numbers $S_{n-p}^{(n)}$ for $p = 1, 2, \ldots$ also appear as the coefficients of x^{n-p}, in the falling factorial polynomial of degree n which results from (3) by setting $a_n = 1 - n$ (see [5], p. 20). Thus, by applying Theorem 3.1 with the parameters $a_1 = 0$ and $d = -1$, we can now derive algebraic expressions for $S_{n-p}^{(n)}$. To illustrate, suppose three iterations of Algorithm 3.1 are performed, thereby producing the matrix

$$A_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ -15 & 25 & -11 & 1 \end{bmatrix}.$$

Then, by reading directly from this matrix we deduce, using (13), the following formulas:

$$S_{n-1}^{(n)} = \binom{n}{1} - \binom{n+1}{2},$$

$$S_{n-2}^{(n)} = \binom{n}{2} - 4\binom{n+1}{3} + 3\binom{n+2}{4},$$

$$S_{n-3}^{(n)} = \binom{n}{3} - 11\binom{n+1}{4} + 25\binom{n+2}{5} - 15\binom{n+3}{6}.$$

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FIBONACCI MATRICES

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1. INTRODUCTION

In this paper we discuss the Fibonacci matrices which are matrices whose elements are the classical Fibonacci numbers. Some properties are given for these matrices. The relations between these matrices and the units of the field $\mathcal{Q}(\theta)$, $(\theta^4+\theta^3+\theta^2+\theta+1=0)$ is also discussed. As an application, we deduce an interesting relation which includes the Fibonacci and Lucas numbers by using the properties of these Fibonacci matrices.

2. FIBONACCI NUMBERS AND FIBONACCI MATRICES

It is well known that if $\mathcal{Q}=\binom{11}{10}$ then $\mathcal{Q}^n=\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1}\end{pmatrix}$. Many Fibonacci and Lucas identities have been developed using $\mathcal{Q}$ (see [1]).

We are interested in finding other matrices like $\mathcal{Q}$ so that the n^{th} power of the matrix has only the Fibonacci numbers as its elements. If such matrices exist, then we want to know their properties and what relations exist between the matrices and the Fibonacci numbers.

For matrices of order 2, we examine the set

$$F=\left\{\begin{pmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{pmatrix} \,\middle|\, \ell_i = 0 \text{ or } 1\right\}.$$

One can easily see that the only matrices that work are

$$\mathcal{Q}_1=\begin{pmatrix} 01 \\ 11 \end{pmatrix},\ \mathcal{Q}_2=\begin{pmatrix} 11 \\ 10 \end{pmatrix},\ \pm\mathcal{Q}_1,\ \pm\mathcal{Q}_2,\ \pm\mathcal{Q}_1^{-1},\ \pm\mathcal{Q}_2^{-1}.$$

For matrices of order 4, we let

$$H_1=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},\quad H_2=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Note that H_1 and H_2 behave like $\mathcal{Q}$ and are made up of submatrices that are Fibonacci matrices of order 2, the null matrix or matrices that have properties similar to the Fibonacci matrices of order 2. However, H_1 and H_2 are not irreducible, so we ask whether there exists any irreducible matrix of order 4 that behaves like $\mathcal{Q}$.

Definition: A square matrix $\mathcal{A}$ of order r with integer elements is called a Fibonacci matrix if and only if:

(a) $\mathcal{A}^n$, $n=1,2,\ldots$ has only Fibonacci numbers as its elements. The elements may be positive, zero, or negative.

(b) $\mathcal{A}^n$ is irreducible (a matrix B is called irreducible if the matrix B cannot be reduced to a block diagonal matrix by permuting some rows or some columns). Our definition of irreducible is different from the common definition in order to avoid combining two Fibonacci matrices of order 2 to obtain a Fibonacci matrix of order 4.

(c) $\{\mathscr{A}^n \mid n \geq 1\}$ has $F_i \in \mathscr{A}^n$ for all i and some n.

$\mathscr{A}$ is called a basic Fibonacci matrix if $\mathscr{A}$ has only 1, -1, and 0 as its elements.

3. FIBONACCI MATRICES OF ORDER 4 AND THEIR PROPERTIES

Proposition 1: The matrices

$$\begin{pmatrix} -F_1 & -F_2 & -F_1 & 0 \\ 0 & -F_1 & -F_2 & -F_1 \\ F_1 & F_1 & 0 & -F_0 \\ F_0 & F_2 & F_2 & F_0 \end{pmatrix}, \begin{pmatrix} -F_1 & -F_1 & 0 & -F_2 \\ F_2 & F_0 & F_0 & F_2 \\ -F_2 & 0 & -F_1 & -F_1 \\ F_1 & -F_0 & F_1 & 0 \end{pmatrix}, \begin{pmatrix} -F_1 & -F_1 & 0 & F_0 \\ -F_0 & -F_2 & -F_2 & -F_0 \\ F_0 & 0 & -F_1 & -F_1 \\ F_1 & F_2 & F_1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -F_2 & 0 & -F_1 & -F_1 \\ F_1 & -F_0 & F_1 & 0 \\ 0 & F_1 & -F_0 & F_1 \\ -F_1 & -F_1 & 0 & -F_2 \end{pmatrix}, \begin{pmatrix} -F_1 & F_0 & -F_1 & 0 \\ 0 & -F_1 & F_0 & -F_1 \\ F_1 & F_1 & 0 & F_2 \\ -F_2 & -F_0 & -F_0 & -F_2 \end{pmatrix}, \begin{pmatrix} -F_2 & -F_0 & -F_0 & -F_2 \\ F_2 & 0 & F_1 & F_1 \\ -F_1 & F_0 & -F_1 & 0 \\ 0 & -F_1 & F_0 & -F_1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -F_1 & -F_2 & -F_1 \\ F_1 & F_1 & 0 & -F_0 \\ F_0 & F_2 & F_2 & F_0 \\ -F_0 & 0 & F_1 & F_1 \end{pmatrix}, \begin{pmatrix} -F_0 & -F_2 & -F_2 & -F_0 \\ F_0 & 0 & -F_1 & -F_1 \\ F_1 & F_2 & F_1 & 0 \\ 0 & F_1 & F_2 & F_1 \end{pmatrix}, \begin{pmatrix} 0 & -F_1 & F_0 & -F_1 \\ F_1 & F_1 & 0 & F_2 \\ -F_2 & -F_0 & -F_0 & -F_2 \\ F_2 & 0 & F_1 & F_1 \end{pmatrix},$$

and

$$\begin{pmatrix} F_0 & 0 & -F_1 & -F_1 \\ F_1 & F_2 & F_1 & 0 \\ 0 & F_1 & F_2 & F_1 \\ -F_1 & -F_1 & 0 & F_0 \end{pmatrix}$$

are all basic Fibonacci matrices. We denote these matrices, respectively, by $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_{10}$.

Proof: For $\overline{F}_1$, we can easily calculate $\overline{F}_1^2, \overline{F}_1^3, \ldots, \overline{F}_1^{10}$. For example,

$$\overline{F}_1^{10} = \begin{pmatrix} F_9 & 0 & -F_{10} & -F_{10} \\ F_{10} & F_{11} & F_{10} & 0 \\ 0 & F_{10} & F_{11} & F_{10} \\ -F_{10} & -F_{10} & 0 & F_9 \end{pmatrix}.$$

By using the basic definition and the well-known properties of the Fibonacci numbers, one can easily prove by induction that

$$\overline{F}_1^{10k} = \begin{pmatrix} F_{10k-1} & 0 & -F_{10k} & -F_{10k} \\ F_{10k} & F_{10k+1} & F_{10k} & 0 \\ 0 & F_{10k} & F_{10k+1} & F_{10k} \\ -F_{10k} & -F_{10k} & 0 & F_{10k-1} \end{pmatrix}, \quad k = 1, 2, \ldots.$$

If we multiply $\overline{F}_1^i$, $i = 1, 2, \ldots, 9$, by $\overline{F}_1^{10k}$, $k = 1, 2, \ldots$, and use the basic properties of the Fibonacci numbers, we have the 10 patterns $\overline{F}_1^{10k+i}$. For example,

$$\overline{F}_1^{10k+1} = \begin{pmatrix} -F_{10k+1} & -F_{10k+2} & -F_{10k+1} & 0 \\ 0 & -F_{10k+1} & -F_{10k+2} & -F_{10k+1} \\ F_{10k+1} & F_{10k+1} & 0 & -F_{10k} \\ F_{10k} & F_{10k+2} & F_{10k+2} & F_{10k} \end{pmatrix},$$

$$\overline{F}_1^{10k+2} = \begin{pmatrix} 0 & F_{10k+2} & F_{10k+3} & F_{10k+2} \\ -F_{10k+1} & -F_{10k+2} & 0 & F_{10k+1} \\ -F_{10k+1} & -F_{10k+3} & -F_{10k+3} & -F_{10k+1} \\ F_{10k+1} & 0 & -F_{10k+2} & -F_{10k+2} \end{pmatrix}.$$

This completes the proof of part (a) of the Definition for $\overline{F}_1$. Part (b) of the Definition can be proved by the exhaustive method for all permutations of rows and columns. Part (c) of the Definition is obvious. Similar proof exists for $\overline{F}_2, \ldots, \overline{F}_{10}$. We would like to observe that the proofs for $\overline{F}_4$ and $\overline{F}_{10}$ can be simpler.

Proposition 2: If $\overline{F}_k$ is a Fibonacci matrix, then so is $-\overline{F}_k$, $k = 1, \ldots, 10$.

Proof: This is obvious since $(-\overline{F})^n = \pm\overline{F}^n$, $n = 1, 2, \ldots$.

We now let $\overline{F}_{21-k} = -\overline{F}_k$, $k = 1, \ldots, 10$.

Proposition 3: If $\overline{F}_k$ is a Fibonacci matrix, then so is $\overline{F}_k^T$, $k = 1, \ldots, 20$, where $\mathscr{A}^T$ denotes the transpose of the matrix $\mathscr{A}$.

Proof: This is obvious since $(\overline{F}_k^T)^n = (\overline{F}_k^n)^T$.

Thus, we obtain 40 Fibonacci matrices of order 4. However, it is sufficient to discuss only $\overline{F}_1, \ldots, \overline{F}_{20}$. We let $\mathfrak{F} = \{\overline{F}_k \mid k = 1, \ldots, 20\}$.

Proposition 4: If $\overline{F}_k \in \mathfrak{F}$, then $\overline{F}_k^{-1} \in \mathfrak{F}$ for $k = 1, \ldots, 20$.

Proof: It is not difficult to verify that $\overline{F}_1^{-1} = \overline{F}_6$, $\overline{F}_2^{-1} = \overline{F}_{14}$, $\overline{F}_3^{-1} = \overline{F}_{12}$, $\overline{F}_4^{-1} = \overline{F}_{10}$, and $\overline{F}_5^{-1} = \overline{F}_{13}$. The rest can be proved by using the relations $\overline{F}_{21-k} = -\overline{F}_k$, $k = 1, \ldots, 10$. Another interesting result is the following.

Proposition 5: If $\overline{F}_k \in \mathfrak{F}$, then $\det(\overline{F}_k) = 1$, where the $\det(A)$ denotes the determinant of matrix A.

It is well known that, in general, the multiplication of matrices is noncommutative. However, for these Fibonacci matrices, we have the following.

Proposition 6: If $\overline{F}_k$, $\overline{F}_h \in \mathfrak{F}$, then $\overline{F}_k\overline{F}_h = \overline{F}_h\overline{F}_k$.

Proof: One can easily verify that this is true. In order to investigate the properties of multiplication for the matrices in $\mathfrak{F}$, we start by studying the following 10 matrices. Let

$$A_1 = A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad A_i = A^i,\ i = 1, 2, \ldots.$$

It is easy to verify that $A_5 = -E$ and $A_{10} = E$, where the E is the identify a matrix of order 4. Let $\mathscr{A} = \{A_k \mid k = 1, 2, \ldots, 10\}$. Obviously, the multiplication group of $\mathscr{A}$ is isomorphic to the group $(\gamma^k \mid \gamma = \exp(2\pi i/10),\ k = 0, 1, 2, \ldots, 9)$. It is also easy to verify that $A_k\overline{F}_h = \overline{F}_h A_k$ or that the multiplication of the A's and $\overline{F}$'s matrices is commutative. Furthermore, one can easily show that $\mathscr{A}\mathfrak{F} \subset \mathfrak{F}$. In fact, we have the following multiplication table, where the product array is $\overline{F}_n = A_k\overline{F}_h$. For example, $\overline{F}_{19} = A_4\overline{F}_9$. From the table and the properties of A_k, it is easy to see the results in Proposition 7.

$k \backslash h$	1	2	3	4	5	6	7	8	9	10
1	10	9	7	6	4	2	20	18	16	13
2	13	16	20	2	6	9	11	14	17	3
3	3	17	11	9	2	16	8	1	15	7
4	7	15	8	16	9	17	18	10	19	20
5	20	19	18	17	16	15	14	13	12	11
6	11	12	14	15	17	19	1	3	5	8
7	8	5	1	19	15	12	10	7	4	18
8	18	4	10	12	19	5	13	20	6	14
9	14	6	13	5	12	4	3	11	2	1
10	1	2	3	4	5	6	7	8	9	10

Proposition 7: Let $\mathfrak{F}_1 = \{\overline{F}_k \mid k = 1, 3, 7, 8, 10, 11, 13, 14, 18, 20\}$ and $\mathfrak{F}_2 = \mathfrak{F} \setminus \mathfrak{F}_1$. Then

(a) For $\overline{F}_k\overline{F}_h \in \mathfrak{F}_i$, there exists $\overline{F}_n \in \mathfrak{F}_i$ such that $\overline{F}_k\overline{F}_h = \pm\overline{F}_n^2$, $i = 1, 2$.

(b) For $\overline{F}_k \in \mathfrak{F}_1$, $\overline{F}_h \in \mathfrak{F}_2$, there exist $A_n \in \mathscr{A}$ such that $\overline{F}_k\overline{F}_h = A_n$.

(c) For any $\overline{F}_k, \overline{F}_h \in \mathfrak{F}_i$, $\overline{F}_k^{10n} = \overline{F}_h^{10n}$, where $n = 0, 1, 2, \ldots$, and $i = 1, 2$.

The proof is omitted since it is very straightforward.

4. THE CHARACTERISTIC POLYNOMIAL, CHARACTERISTIC VALUE, AND CHARACTERISTIC VECTOR OF A FIBONACCI MATRIX

It is not difficult to compute the characteristic polynomial for $\overline{F}_k \in \mathfrak{F}$.

Proposition 8: The characteristic polynomials of $\overline{F}_1$ (or $\overline{F}_2$), $\overline{F}_3$ (or $\overline{F}_5$), and $\overline{F}_4$ are, respectively, $\lambda^4 + 2\lambda^3 + 4\lambda^2 + 3\lambda + 1$, $\lambda^4 + 3\lambda^3 + 4\lambda^2 + 2\lambda + 1$, and $\lambda^4 + 2\lambda^3 - \lambda^2 - 2\lambda + 1$. The other characteristic polynomials can easily be reduced by using Proposition 4 and the fact that $\overline{F}_{21-k} = -\overline{F}_k$. The proofs are omitted.

There is a very nice property for $\overline{F}_k$ if $k \equiv h \pmod{10}$. We can verify the following.

Proposition 9: Let $G_k = \overline{F}_k^{10}$, $k = 1, \ldots, 20$. Then:

(a) G_k only takes one of two forms, i.e., either

$$G_k = G_1 = \begin{pmatrix} F_9 & 0 & -F_{10} & -F_{10} \\ F_{10} & F_{11} & F_{10} & 0 \\ 0 & F_{10} & F_{11} & F_{10} \\ -F_{10} & -F_{10} & 0 & F_9 \end{pmatrix} \quad \text{or} \quad G_k = G_2 = \begin{pmatrix} F_{11} & 0 & F_{10} & F_{10} \\ -F_{10} & F_9 & -F_{10} & 0 \\ 0 & -F_{10} & F_9 & -F_{10} \\ F_{10} & F_{10} & 0 & F_{11} \end{pmatrix};$$

(b) $G_1 = G_2^{-1}$;

(c) G_k, $k = 1, \ldots, 20$, all satisfy the same characteristic equation,

$$G_k^4 - 246G_k^3 + 15131G_k^2 - 246G_k + E = 0.$$

We conclude this section by giving some properties of the characteristic roots of the Fibonacci matrices and looking at the characteristic vectors of $\overline{F}_n$.

Theorem 1:

(A) Each characteristic root of $\overline{F}_h \in \mathfrak{F}$ is a linear combination of $\exp(2\pi ik/5)$, $k = 0, 1, 2, 3$, with integer coefficients.

(B) Each characteristic root of $\overline{F}_k^n$, $n = 1, 2, \ldots$, is a linear combination of $\exp(2\pi ik/5)$, $k = 0, 1, 2, 3$, with integer coefficients.

Proof:

(A) One method of proof is the following. Let the elements of the first row of $\overline{F}_h$ be f_{h11}, f_{h12}, f_{h13}, f_{h14}, $h = 1, \ldots, 20$, and let $\theta_k = \exp(2\pi ik/5)$, $k = 0, 1, 2, 3, 4$. It is easy to verify that the $\sum_{j=1}^4 f_{h1j}\theta_k^{j-1}$ are the roots of the characteristic equation of $\overline{F}_h$. Noticing that $\sum_{k=0}^4 \theta_k = 0$, we see that θ_k, $k = 4, 5, \ldots$, can be written as a linear combination of θ_k, $k = 0, 1, 2, 3$, with integer coefficients. Hence, the conclusion of (A) is true.

(B) We notice that $|\lambda E - A| = 0$ implies that $|\lambda^n E - A^n| = |\lambda E - A| \cdot |\lambda^{n-1}E + \lambda^{n-2}A + \cdots + A^{n-1}| = 0$. Hence, it follows that the characteristic root of $\overline{F}_h^n$ is λ^n, where λ is the characteristic root of $\overline{F}_h$. Looking at the proof of (A), the proof of (B) is now obvious.

Concerning the characteristic vector of $\overline{F}_k^n$, we have the following theorem.

Theorem 2: Let $\theta_k = (1, \theta_k, \theta_k^2, \theta_k^3)^T$, $\theta_k = \exp(2\pi i/5)$, $k = 1, 2, 3, 4$, and let the f_{h1j}'s have the same meaning as in the proof of Theorem 1(A). Then:

(A) θ_k is the characteristic vector of $\overline{F}_h$ corresponding to the characteristic value of $\sum_{j=1}^4 f_{h1j}\theta_k^{j-1}$, $k = 1, 2, 3, 4$;

(B) θ_k is the characteristic vector of $\overline{F}_h^n$ corresponding to the characteristic value of $(\sum_{j=1}^4 f_{h1j}\theta_k^{j-1})^n$, $k = 1, 2, 3, 4$, $n = 1, 2, \ldots$.

Proof:

(A) The proof of this is trivial.

(B) First, we notice that $(\lambda E - A) = 0$ implies

$$\lambda^n E - A^n = (\lambda^{n-1}E + \lambda^{n-2}A + \cdots + A^{n-1})(\lambda E - A) = 0.$$

The conclusion is now obtained directly from (A).

5. APPLICATIONS AND REMARKS

When we proved Proposition 1, we saw that the proofs for $\overline{F}_4$ and $\overline{F}_{10}$ could be simpler. That is so because the patterns of the signs for their powers has relatively small numbers. The matrix $(\operatorname{sgn} a_{ij})$ is called the pattern of signs for the matrix (a_{ij}), where we have

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

One can easily verify that the pattern of signs for $\overline{F}_h^n, n = 1, 2, \ldots$, is a periodic function of n that the period is never more than ten. In fact, one can easily compute the following:

$$\text{The period of sign's pattern for } \overline{F}_k = \begin{cases} 10, & \text{when } k = 1,2,7,12,13,15,16,18, \\ 5, & \text{when } k = 3,5,6,8,9,14,19,20, \\ 2, & \text{when } k = 4,11, \\ 1, & \text{when } k = 10,17. \end{cases}$$

It is worth mentioning that in the sign's pattern of $\overline{F}_k$, even $F_0 = 0$, we understand that the F_0 has a positive or negative sign.

As an application, we look at $\overline{F}_{17}$ and deduce some wonderful relations between the Fibonacci and Lucas numbers.

By a tedious and careful investigation, one can obtain many relations like the following.

Theorem 3: For n odd, we have

$$F_{4n+2} - 2L_nF_{3n+2} + (L_n^2 - 2)F_{2n+2} + 2L_nF_{n+2} + 1 = 0, \tag{1}$$
$$F_{4n+1} - 2L_nF_{3n+1} + (L_n^2 - 2)F_{2n+1} + 2L_nF_{n+1} + 1 = 0, \tag{2}$$
$$F_{4n-1} - 2L_nF_{3n-1} + (L_n^2 - 2)F_{2n-1} + 2L_nF_{n-1} + 1 = 0, \tag{3}$$
$$F_{4n} - 2L_nF_{3n} + (L_n^2 - 2)F_{2n} + 2L_nF_n = 0. \tag{4}$$

For n even, we have

$$F_{4n+2} - 2L_nF_{3n+2} + (L_n^2 + 2)F_{2n+2} - 2L_nF_{n+2} + 1 = 0, \tag{5}$$
$$F_{4n+1} - 2L_nF_{3n+1} + (L_n^2 + 2)F_{2n+1} - 2L_nF_{n+1} + 1 = 0, \tag{6}$$
$$F_{4n-1} - 2L_nF_{3n-1} + (L_n^2 + 2)F_{2n-1} - 2L_nF_{n-1} + 1 = 0, \tag{7}$$
$$F_{4n} - 2L_nF_{3n} + (L_n^2 + 2)F_{2n} - 2L_nF_n = 0. \tag{8}$$

In order to prove Theorem 3, we need the following proposition.

Proposition 10: Let S_n denote the sum of all principal 2 minors of $\overline{F}_{17}^n$. Then

$$S_n = \begin{cases} L_n^2 + 2 & \text{when } n \text{ is even,} \\ L_n^2 - 2 & \text{when } n \text{ is odd.} \end{cases}$$

A careful examination of $\overline{F}_{17}$ will show that Proposition 10 is equivalent to the following.

Proposition 11: $F_{n-1}^2 + 4F_{n-1}F_{n+1} + F_{n+1}^2 - 2F_n^2 = \begin{cases} L_n^2 + 2 & \text{when } n \text{ is even,} \\ L_n^2 - 2 & \text{when } n \text{ is odd.} \end{cases}$

Proof: Obviously, the left side of this equation is equal to $(F_{n-1} + F_{n+1})^2 + 2(F_{n-1}F_{n+1} - F_n^2)$. However, this is equal to the right side since we have $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ and $L_n = F_{n-1} + F_{n+1}$.

We now give the proof of Theorem 3. Using the relation between the coefficients of the characteristic polynomial and the principal minors of a matrix, applying Proposition 10, and doing some proper computation, we can see that the characteristic equation for the matrix $G_n = \overline{F}_{17}^n$ is $\lambda_n^4 - 2L_n\lambda_n^3 + (L_n^2 - 2)\lambda_n^2 + 2L_n\lambda_n + 1 = 0$ when n is odd. Hence, we have

$$\overline{F}_{17}^{4n} - 2L_n\overline{F}_{17}^{3n} + (L_n^2 - 2)\overline{F}_{17}^{2n} + 2L_n\overline{F}_{17}^n + E = 0$$

by the Hamilton-Cayley theorem. Substituting $\overline{F}_{17}$'s expression by F_n into the last equality and comparing the coefficients of the (1, 1)-, (2, 2)-, and (2, 3)-elements of the resulting matrices, we obtain (2), (3), and (4) and $(1) = (2) + (4)$.

In a similar manner, we can prove the results when n is even.

Remark 1: One can find a more uniform pattern than is given in Theorem 3 using the following proposition as an example.

Proposition 12: The sum of all principal 2-minors of $\overline{F}_{18}^n$ is equal to $L_n^2 \pm 3$.

Remark 2: In this paper, it is shown that the Fibonacci matrices play an important role in the connection between the ancient Fibonacci numbers and some properties of the field $\mathfrak{Q}(\theta)$, where the θ is a zero of the polynomial $x^4 + x^3 + x^2 + x + 1 = 0$.

Remark 3: Research problems.

(a) Are there other Fibonacci matrices of order 4 besides the 40 matrices dealt with in this paper?

(b) Are there any Fibonacci matrices of order higher than 4?

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ON THE OCCURRENCE OF F_n IN THE ZECKENDORF DECOMPOSITION OF nF_n

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1. INTRODUCTION

The Zeckendorf decomposition of a natural number n is the unique expression of n as a sum of Fibonacci numbers with nonconsecutive indices and with each index greater than 1, where $F_0 = 0$, $F_1 = 1$, and $F_{i+1} = F_i + F_{i-1}$ form the Fibonacci numbers for $i \geq 0$ (see [6] and [8], or see pages 108-09 in [7]). The Zeckendorf decomposition of products of the form kF_m for $k, m \in \mathbb{N}$ is studied in [2] and [5]. For each positive integer n, let $Ratio(n)$ be the ratio of the number of $k \in \mathbb{N}$ with $k \leq n$ that do have F_k in the Zeckendorf decomposition of kF_k to those that do not. In this paper we prove Conjecture 1 from [2], which essentially states that as $n \to \infty$ we have $Ratio(n) \to \beta^{-2}$, where $\beta = (1+\sqrt{5})/2$. This result, Theorem 4.9, is proved using methods introduced in [5] by Hart.

The β-expansion of a natural number n, first introduced in [1], is the unique finite sum of integral, nonconsecutive powers of β that equals n. Grabner et al., in [3] and [4], prove that for $m \geq \log_\beta k$ the Zeckendorf decomposition of kF_m can be produced by replacing each β^i in the β-expansion of k with F_{m+i}. Thus, our result also answers a question posed by Bergman in [1] that asks for the frequency of the occurrence of β^0 in the β-expansions of the natural numbers. For simplicity, all results and proofs in the rest of the paper will be stated in terms of β-expansions.

Our proof entails first finding formulas for $Ratio(L_{2k})$ and $Ratio(L_{2k+1})$, where $L_0 = 2$, $L_1 = 1$, and $L_{i+1} = L_i + L_{i-1}$ form the Lucas sequence for $i \geq 0$. We prove that as $k \to \infty$ the two sequences of ratios for odd- and even-indexed Lucas numbers both decrease to β^{-2}. We then prove that for values of n between two Lucas numbers we have $Ratio(n)$ trapped between the two sequences.

The recursive pattern we have discovered in the β-expansions, and upon which our proof is based, can be used to find the frequency of the occurrence of other powers of β as well. This extension of the current problem will be addressed in a future paper.

2. DEFINITIONS AND PRELIMINARIES

We use definitions and notation similar to those in [5]. In particular, $\ell(n)$ denotes the absolute value of the smallest power of β in the β-expansion of n, and $u(n)$ denotes the largest such power.

The following is a restatement of Theorem 1 from [4] in terms of the β-expansion.

Theorem 2.1 (Grabner et al.): For $k \geq 1$, we have $\ell(n) = u(n) = 2k$ whenever $L_{2k} \leq n \leq L_{2k+1}$, and we have $\ell(n) = 2k+2$ and $u(n) = 2k+1$ whenever $L_{2k+1} < n < L_{2k+2}$.

Definition 2.2: We define V to be the infinite dimensional vector space over $\mathbb{Z}$ given by $V := \{(\ldots, v_{-1}, \underline{v_0}, v_1, v_2, \ldots) : v_i \in \mathbb{Z}\,\forall i$, with at most finitely many v_i nonzero$\}$. For convenience, we underline the zeroth coordinate.

Definition 2.3: Define $\hat{V}$ to be the subset of V consisting of all vectors whose entries are in the set $\{0, 1\}$ and which have no two consecutive ones. We will call the elements of $\hat{V}$ totally reduced vectors.

As in [5], we represent β-expansions by vectors of ones and zeros, where a one in the j^{th} coordinate represents β^j. The powers of β increase from left to right in the vector.

Definition 2.4: We define the function $\beta : \mathbb{N} \to \hat{V}$ so that, when the β-expansion of n is $\sum_{i=-\infty}^{\infty} e_i \beta^i$, $\beta(n)$ is the vector in $\hat{V}$ with $v_i = e_i$.

Definition 2.5: The function $\sigma : V \to \mathbb{N}$ is defined as follows: $\sigma((\ldots, v_{-1}, v_0, v_1, \ldots)) = \sum_{i=-\infty}^{\infty} v_i \beta^i$.

Thus, $\sigma(\beta(n)) = n$ for all natural numbers n. (Note that the definition of σ in [5] is in terms of Fibonacci numbers and is not equivalent to the one given here. Specifically, the two functions are only guaranteed to be equal when applied to $\beta(n)$ where $n \in \mathbb{N}$.)

Figure 1 shows the vector representations of the β-expansion of the first 30 natural numbers. Note that the coefficient of β^0 is underlined.

n									$\beta(n)$								
1									$\underline{1}$								
2							1	0	$\underline{0}$	1							
3							1	0	$\underline{0}$	0	1						
4							1	0	$\underline{1}$	0	1						$4 = L_3$
5					1	0	0	1	$\underline{0}$	0	0	1					
6					1	0	0	0	$\underline{0}$	1	0	1					
7					1	0	0	0	$\underline{0}$	0	0	0	1				$7 = L_4$
8					1	0	0	0	$\underline{1}$	0	0	0	1				
9					1	0	1	0	$\underline{0}$	1	0	0	1				
10					1	0	1	0	$\underline{0}$	0	1	0	1				
11					1	0	1	0	$\underline{1}$	0	1	0	1				$11 = L_5$
12			1	0	0	1	0	1	$\underline{0}$	0	0	0	0	1			
13			1	0	0	1	0	0	$\underline{0}$	1	0	0	0	1			
14			1	0	0	1	0	0	$\underline{0}$	0	1	0	0	1			
15			1	0	0	1	0	0	$\underline{1}$	0	1	0	0	1			
16			1	0	0	0	0	1	$\underline{0}$	0	0	1	0	1			
17			1	0	0	0	0	0	$\underline{0}$	1	0	1	0	1			
18			1	0	0	0	0	0	$\underline{0}$	0	0	0	0	0	1		$18 = L_6$
19			1	0	0	0	0	0	$\underline{1}$	0	0	0	0	0	1		
20			1	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	1		
21			1	0	0	0	1	0	$\underline{0}$	0	1	0	0	0	1		
22			1	0	0	0	1	0	$\underline{1}$	0	1	0	0	0	1		$22 = 2L_5$
23			1	0	1	0	0	1	$\underline{0}$	0	0	1	0	0	1		
24			1	0	1	0	0	0	$\underline{0}$	1	0	1	0	0	1		
25			1	0	1	0	0	0	$\underline{0}$	0	0	0	1	0	1		$25 = L_6 + L_4$
26			1	0	1	0	0	0	$\underline{1}$	0	0	0	1	0	1		
27			1	0	1	0	1	0	$\underline{0}$	1	0	0	1	0	1		
28			1	0	1	0	1	0	$\underline{0}$	0	1	0	1	0	1		
29			1	0	1	0	1	0	$\underline{1}$	0	1	0	1	0	1		$29 = L_7$
30	1	0	0	1	0	1	0	1	$\underline{0}$	0	0	0	0	0	0	1	

FIGURE 1. β-Expansions

Following [2], we say that n has property $\mathcal{P}$ if β^0 appears in the β-expansion of n.

Definition 2.6: For natural numbers n, m, we let

$$Ones(n,m) = |\{k \in \mathbb{N} : n < k \le m, k \text{ has property } \mathcal{P}\}|,$$

$$Zeros(n,m) = |\{k \in \mathbb{N} : n < k \le m, k \text{ does not have property } \mathcal{P}\}|.$$

We also define, for $n > 1$, $Ratio(n)$ to be $Ones(0,n)/Zeros(0,n)$.

We will be using the following known facts about Fibonacci and Lucas numbers:

$$\lim_{x\to\infty} (F_x / F_{x-1}) = \beta; \tag{1}$$

$$\lim_{x\to\infty} (F_x / F_{x-2}) = 1 + \beta = \beta^2; \tag{2}$$

$$F_{n+h}F_{n+k} - F_n F_{n+h+k} = (-1)^n F_h F_k. \tag{3}$$

Formula (3) is from [7], page 177, (20a).

Note that in [3] and [4] the indices for Fibonacci and Lucas numbers are different from the indices used here. We use $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$.

3. THE RATIO FOR LUCAS NUMBERS

Our first goal is to prove the following proposition.

Proposition 3.1: For $k \ge 1$,

$$Ratio(L_{2k}) = \frac{F_{2k-1}}{F_{2k+1}} \quad \text{and} \quad Ratio(L_{2k+1}) = \frac{F_{2k}+1}{F_{2k+2}-1}.$$

Thus, both ratios decrease to β^{-2} as k increases.

We shall devote this section to developing the facts needed for the proof of Proposition 3.1. Recall that we express β-expansions of integers with powers of β increasing from left to right.

Lemma 3.2:

(1) $\beta(L_{2k}) = (10^{2k-1}\underline{0}0^{2k-1}1)$ for $k \ge 1$.

(2) $\beta(2L_{2k-1}) = (1000(10)^{k-2}\underline{1}(01)^{k-2}0001)$ for $k \ge 2$.

(3) $\beta(L_{2k} + L_{2k-2}) = (1010^{2k-3}\underline{0}0^{2k-3}101)$ for $k \ge 2$.

(4) $\beta(L_{2k+1}) = ((10)^k\underline{1}(01)^k)$ for $k \ge 1$.

(5) $\beta(2L_{2k}) = (10010^{2k-2}\underline{0}0^{2k-3}1001)$ for $k \ge 2$.

(6) $\beta(L_{2k+1} + L_{2k-1}) = (100100(10)^{k-2}\underline{1}(01)^{k-1}001)$ for $k \ge 2$.

Proof: Parts (1), (2), and (4) follow from results in [2] and from the relationship between the Zeckendorf expansion of nF_n and the β-expansion of n as developed in [3] and [4] (see the Introduction to this paper). Part (3) follows from part (1) when we apply Proposition 4.4 from [5]. Parts (5) and (6) are proved in [5]. □

Lemma 3.3:

(1) $\beta(L_{2k}-1)=(10^{2k-1}\underline{0}(10)^{k-1}1)$ for $k\geq 1$.

(2) $\beta(L_{2k+1}+1)=(10(01)^k\underline{0}0^{2k}1)$ for $k\geq 1$.

(3) $\beta(2L_{2k}-1)=(10010^{2k-2}\underline{0}(10)^{k-1}001)$ for $k\geq 2$.

(4) $\beta(L_{2k+1}+L_{2k-1}+1)=(1000(01)^{k-1}\underline{0}0^{2k-2}101)$ for $k\geq 2$.

Proof: We make repeated applications of Theorem 5.8 from [5]. If $v\in V$, $w=\mathscr{A}(v)$ is the vector in $\hat{V}$ obtained by applying the algorithm in [5]; by the properties of this algorithm, $\sigma(v)=\sigma(w)$.

Part (1): Define $v(k)=(10^{2k-1}\underline{0}(10)^{k-1}1)$ and $w(k)=(\underline{0}(10)^{k-1}1)$. We show that $\mathscr{A}(w(k)+(\underline{1}))=(\underline{0}0^{2k-1}1)$ by induction on k. For $k=1$, $\mathscr{A}(w(1)+(\underline{1}))=\mathscr{A}((\underline{1}1))=(\underline{0}01)$. Assume that $k\geq 2$ and that the formula is correct for smaller k. Then

$$\begin{aligned}\mathscr{A}(w(k)+(\underline{1}))&=\mathscr{A}(w(k-1)+(\underline{0}0^{2k-2}1)+(\underline{1}))=\mathscr{A}(\mathscr{A}(w(k-1)+(\underline{1}))+(\underline{0}0^{2k-2}1))\\&=\mathscr{A}((\underline{0}0^{2k-3}1)+(\underline{0}0^{2k-2}1))=\mathscr{A}((\underline{0}0^{2k-3}11))=(\underline{0}0^{2k-1}1).\end{aligned}$$

Since $v(k)=w(k)+(10^{2k-1}\underline{0})$, we have

$$\begin{aligned}\mathscr{A}(v(k)+(\underline{1}))&=\mathscr{A}(\mathscr{A}(w(k)+(\underline{1}))+(10^{2k-1}\underline{0}))\\&=\mathscr{A}((\underline{0}0^{2k-1}1)+(10^{2k-1}\underline{0}))=(10^{2k-1}\underline{0}0^{2k-1}1)=\beta(L_{2k}).\end{aligned}$$

By uniqueness of the β-representation, this implies that $\beta(L_{2k}-1)=v(k)$.

Part (2): This is proved by induction on k. For $k=1$, we have

$$\beta(L_3+1)=\mathscr{A}((10\underline{1}01)+(\underline{1}))=\mathscr{A}((10\underline{2}01))=\mathscr{A}((20\underline{0}11))=(1001\underline{0}001).$$

For the inductive step, we assume that $k\geq 2$. We have

$$\begin{aligned}\beta(L_{2k+1}+1)&=\beta(L_{2k-1}+L_{2k}+1)=\mathscr{A}(\beta(L_{2k-1}+1)+\beta(L_{2k}))\\&=\mathscr{A}((10(01)^{k-1}\underline{0}0^{2k-2}1)+(10^{2k-1}\underline{0}0^{2k-1}1))=\mathscr{A}((20(01)^{k-1}\underline{0}0^{2k-2}11))\\&=\mathscr{A}((1001(01)^{k-1}\underline{0}0^{2k-2}001))=(10(01)^k\underline{0}0^{2k}1).\end{aligned}$$

Part (3): Using part (1), we have

$$\begin{aligned}\beta(2L_{2k}-1)&=\beta(L_{2k}-1+L_{2k})=\mathscr{A}(\beta(L_{2k}-1)+\beta(L_{2k}))\\&=\mathscr{A}((10^{2k-1}\underline{0}(10)^{k-1}1)+(10^{2k-1}\underline{0}0^{2k-1}1))=\mathscr{A}((20^{2k-1}\underline{0}(10)^{k-1}11))\\&=(10010^{2k-2}\underline{0}(10)^{k-1}001).\end{aligned}$$

Part (4): We use induction on k. For $k=2$, we have

$$\begin{aligned}\beta(L_5+L_3+1)&=\mathscr{A}(\beta(L_5+1)+\beta(L_3))=\mathscr{A}((100101\underline{0}00001)+(10\underline{1}01))\\&=\mathscr{A}((100111\underline{1}01001))=\mathscr{A}((100110\underline{0}11001))=(100001\underline{0}00101).\end{aligned}$$

For the inductive step, we assume that $k\geq 3$. We have

$$\begin{aligned}\beta(L_{2k+1}+L_{2k-1}+1) &= \beta(L_{2k-1}+L_{2k}+L_{2k-3}+L_{2k-2}+1)\\ &= \mathcal{A}(\beta(L_{2k-1}+L_{2k-3}+1)+\beta(L_{2k}+L_{2k-2}))\\ &= \mathcal{A}((1000(01)^{k-2}\underline{0}0^{2k-4}101)+(1010^{2k-3}\underline{0}0^{2k-3}101))\\ &= \mathcal{A}((2010(01)^{k-2}\underline{0}0^{2k-4}1111)).\end{aligned}$$

We apply the algorithm and have

$$\beta(L_{2k+1}+L_{2k-1}+1) = \mathcal{A}((100110(01)^{k-2}\underline{0}0^{2k-4}00101)) = (1000(01)^{k-1}\underline{0}0^{2k-2}101). \quad \square$$

Note that in Figure 1, for $L_4 < n \le L_5$, $\beta(n)$ and $\beta(L_5+n)$ are identical in the coordinate positions within 3 positions of the center. The same relationship can be observed for $L_3 < n \le L_4$ between $\beta(n)$ and $\beta(L_6+n)$. These types of relationships are described in Lemma 3.8. We now define a transformation that will allow us to discuss these relationships in a precise way.

We define $\mathcal{S}(v, T)$ to be the vector obtained by switching (from 0 to 1 or vice versa) the values of all entries of v whose coordinates are in a finite set $T \subset \mathbb{Z}$. This is equivalent to adding and subtracting powers of β. [In our applications, when applying the transformation to $\beta(n)$ we will switch only those entries with coordinate positions close to $u(n)$ or $-\ell(n)$, and will leave the central entries unchanged.]

Definition 3.4: Let $v \in \hat{V}$. Let T be a finite set of coordinates. Define $w = \mathcal{S}(v, T) \in V$ to be the vector with $w_i \in \{0, 1\}$ for all $i \in \mathbb{Z}$ and with $w_i \ne v_i$ if $i \in T$ and $w_i = v_i$ if $i \notin T$.

Definition 3.5: If $v, w \in \hat{V}$ and T is a finite set of coordinates, then we say $v \equiv_T w$ (v is congruent to w mod T) if $v_i = w_i$ $\forall i \in T$.

Lemma 3.6: Let $n, m, x \in \mathbb{N}$ and let T be a finite set of coordinates. Suppose that $\mathcal{S}(\beta(n), T) = \beta(m)$, $\beta(n) \equiv_T \beta(n+x)$ and $\mathcal{S}(\beta(n+x), T) \in \hat{V}$. Then $\mathcal{S}(\beta(n+x), T) = \beta(m+x)$.

Proof: We are given β-expansions for n and x as follows: $n = \sum_{i=-\infty}^{\infty} \varepsilon_i \beta^i$ and $x = \sum_{i=-\infty}^{\infty} \delta_i \beta^i$. Let $T(0) = T \cap \{i : \varepsilon_i = 0\}$ and let $T(1) = T \cap \{i : \varepsilon_i = 1\}$.

Let $d = \sum_{i \in T(0)} \beta^i - \sum_{i \in T(1)} \beta^i$. The fact that $\mathcal{S}(\beta(n), T) = \beta(m)$ means that $n + d = m$. Because $n, m \in \mathbb{N}$, we have $d \in \mathbb{Z}$.

We know that $\mathcal{S}(\beta(n+x), T) \in \hat{V}$, which means that $\mathcal{S}(\beta(n+x), T)$ is the β-expansion of some real number. Since $\beta(n) \equiv_T \beta(n+x)$, we have $\sigma(\mathcal{S}(\beta(n+x), T)) = n + x + d = m + x \in \mathbb{N}$. The β-expansion of a natural number is unique, so $\mathcal{S}(\beta(n+x), T) = \beta(m+x)$. $\square$

Corollary 3.7: Let n_1, n_2, m_1, m_2 be natural numbers where $n_1 \le n_2, m_1 \le m_2$. Let T be a finite set of coordinates such that $0 \notin T$ and, for $0 \le x \le n_2 - n_1$, $\beta(n_1+x) \equiv_T \beta(n_1)$, $\mathcal{S}(\beta(n_1+x), T) \in \hat{V}$, $\mathcal{S}(\beta(n_1), T) = \beta(m_1)$, and $\mathcal{S}(\beta(n_2), T) = \beta(m_2)$. Then $\mathcal{S}(\beta(n_1+x), T) = \beta(m_1+x)$ and, thus, $Ones(n_1, n_1+x) = Ones(m_1, m_1+x)$ and $Zeros(n_1, n_1+x) = Zeros(m_1, m_1+x)$.

Figure 1 above and Figure 2 below illustrate the relationships in parts (6) and (10) of Lemma 3.8. The formulas from Lemmas 3.2 and 3.3 are used in the proof.

n									$\beta(n)$									
1									$\underline{1}$									
2							1	0	$\underline{0}$	1								
3							1	0	$\underline{0}$	0	1							
4							1	0	$\underline{1}$	0	1							
5					1	0	0	1	$\underline{0}$	0	0	1						
6					1	0	0	0	$\underline{0}$	1	0	1						
7					1	0	0	0	$\underline{0}$	0	0	0	1					$7 = L_4$
8					1	0	0	0	$\underline{1}$	0	0	0	1					
9					1	0	1	0	$\underline{0}$	1	0	0	1					
10					1	0	1	0	$\underline{0}$	0	1	0	1					
11					1	0	1	0	$\underline{1}$	0	1	0	1					$11 = L_5$
12			1	0	0	1	0	1	$\underline{0}$	0	0	0	0	1				
13			1	0	0	1	0	0	$\underline{0}$	1	0	0	0	1				
14			1	0	0	1	0	0	$\underline{0}$	0	1	0	0	1				
15			1	0	0	1	0	0	$\underline{1}$	0	1	0	0	1				
16			1	0	0	0	0	1	$\underline{0}$	0	0	1	0	1				
17			1	0	0	0	0	0	$\underline{0}$	1	0	1	0	1				
18			1	0	0	0	0	0	$\underline{0}$	0	0	0	0	0	1			$18 = L_6$
19			1	0	0	0	0	0	$\underline{1}$	0	0	0	0	0	1			
20			1	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	1			
21			1	0	0	0	1	0	$\underline{0}$	0	1	0	0	0	1			
22			1	0	0	0	1	0	$\underline{1}$	0	1	0	0	0	1			
23			1	0	1	0	0	1	$\underline{0}$	0	0	1	0	0	1			
24			1	0	1	0	0	0	$\underline{0}$	1	0	1	0	0	1			
25			1	0	1	0	0	0	$\underline{0}$	0	0	0	1	0	1			
26			1	0	1	0	0	0	$\underline{1}$	0	0	0	1	0	1			
27			1	0	1	0	1	0	$\underline{0}$	1	0	0	1	0	1			
28			1	0	1	0	1	0	$\underline{0}$	0	1	0	1	0	1			
29			1	0	1	0	1	0	$\underline{1}$	0	1	0	1	0	1			$29 = L_7$
30	1	0	0	1	0	1	0	1	$\underline{0}$	0	0	0	0	0	0	1		
31	1	0	0	1	0	1	0	0	$\underline{0}$	1	0	0	0	0	0	1		
32	1	0	0	1	0	1	0	0	$\underline{0}$	0	1	0	0	0	0	1		
33	1	0	0	1	0	1	0	0	$\underline{1}$	0	1	0	0	0	0	1		
34	1	0	0	1	0	0	0	1	$\underline{0}$	0	0	1	0	0	0	1		
35	1	0	0	1	0	0	0	0	$\underline{0}$	1	0	1	0	0	0	1		
36	1	0	0	1	0	0	0	0	$\underline{0}$	0	0	0	1	0	0	1		$36 = 2L_6$
37	1	0	0	1	0	0	0	0	$\underline{1}$	0	0	0	1	0	0	1		
38	1	0	0	1	0	0	1	0	$\underline{0}$	1	0	0	1	0	0	1		
39	1	0	0	1	0	0	1	0	$\underline{0}$	0	1	0	1	0	0	1		
40	1	0	0	1	0	0	1	0	$\underline{1}$	0	1	0	1	0	0	1		$40 = L_7 + L_5$
41	1	0	0	0	0	1	0	1	$\underline{0}$	0	0	0	0	1	0	1		
42	1	0	0	0	0	1	0	0	$\underline{0}$	1	0	0	0	1	0	1		
43	1	0	0	0	0	1	0	0	$\underline{0}$	0	1	0	0	1	0	1		
44	1	0	0	0	0	1	0	0	$\underline{1}$	0	1	0	0	1	0	1		
45	1	0	0	0	0	0	0	1	$\underline{0}$	0	0	1	0	1	0	1		
46	1	0	0	0	0	0	0	0	$\underline{0}$	1	0	1	0	1	0	1		
47	1	0	0	0	0	0	0	0	$\underline{0}$	0	0	0	0	0	0	0	1	$47 = L_8$
48	1	0	0	0	0	0	0	0	$\underline{1}$	0	0	0	0	0	0	0	1	
49	1	0	0	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	0	0	1	
50	1	0	0	0	0	0	1	0	$\underline{0}$	0	1	0	0	0	0	0	1	

FIGURE 2. More β-Expansions

Lemma 3.8: For $k \geq 2$ with *Digits* = *Ones* throughout or *Digits* = *Zeros* throughout:

(1) If $0 \leq x \leq L_{2k-3}$, then

$$\begin{aligned} Digits(L_{2k-2}, L_{2k-2}+x) &= Digits(L_{2k}, L_{2k}+x) \\ &= Digits(L_{2k}+L_{2k-2}, L_{2k}+L_{2k-2}+x) \\ &= Digits(2L_{2k}, 2L_{2k}+x). \end{aligned}$$

(2) If $0 \le x \le L_{2k-2}$, then
$$\begin{aligned} Digits(L_{2k-1}, L_{2k-1}+x) &= Digits(2L_{2k+1}, 2L_{2k+1}+x) \\ &= Digits(L_{2k+1}, L_{2k+1}+x) \\ &= Digits(L_{2k+1}+L_{2k-1}, L_{2k+1}+L_{2k-1}+x). \end{aligned}$$

(3) $Digits(L_{2k-2}, L_{2k-1}) = Digits(L_{2k}, 2L_{2k-1})$.

(4) $Digits(L_{2k-3}, L_{2k-2}) = Digits(2L_{2k-1}, L_{2k}+L_{2k-2})$.

(5) $Digits(L_{2k-2}, L_{2k-1}) = Digits(L_{2k}+L_{2k-2}, L_{2k+1})$.

(6) $Digits(L_{2k}, L_{2k+1}) = 2Digits(L_{2k-2}, L_{2k-1}) + Digits(L_{2k-3}, L_{2k-2})$.

(7) $Digits(L_{2k-1}, L_{2k}) = Digits(L_{2k+1}, 2L_{2k})$.

(8) $Digits(L_{2k-2}, L_{2k-1}) = Digits(2L_{2k}, L_{2k+1}+L_{2k-1})$.

(9) $Digits(L_{2k-1}, L_{2k}) = Digits(L_{2k+1}+L_{2k-1}, L_{2k+2})$.

(10) $Digits(L_{2k+1}, L_{2k+2}) = 2Digits(L_{2k-1}, l_{2k}) + Digits(L_{2k-2}, L_{2k-1})$.

(11) For $L_{2k+1} < n \le L_{2k+2}$, $\beta(n)$ starts with 100 [i.e., the values at coordinate positions $-\ell(n)$, $-\ell(n)+1$, and $-\ell(n)+2$ of $\beta(n)$ are 1, 0, and 0, respectively].

Proof: Let $T_1 = \{-2k, -2k+2, 2k-2, 2k\}$. It can be checked that $\mathcal{S}(\beta(L_{2k-2}), T_1) = \beta(L_{2k})$ and that $\mathcal{S}(\beta(L_{2k-1}), T_1) = \beta(2L_{2k-1})$. It follows from 2.1 that, for all n with $L_{2k-2} \le n \le L_{2k-1}$, $\beta(n) \equiv_T \beta(L_{2k-2})$ and $\mathcal{S}(\beta(n), T_1) \in \hat{V}$. Let $x = n - L_{2k-2}$ so that $0 \le x \le L_{2k-3}$ and note that, by 3.7, $Digits(L_{2k-2}, L_{2k-2}+x) = Digits(L_{2k}, L_{2k}+x)$.

Let $T_2 = \{-2k-2, 2k+2\}$. It can be checked that $\mathcal{S}(\beta(L_{2k-1}), T_2) = \beta(2L_{2k+1})$ and that $\mathcal{S}(\beta(L_{2k}), T_2) = \beta(L_{2k+2}+L_{2k})$. By 2.1, for all n satisfying $L_{2k-1} \le n \le L_{2k}$, $\beta(n) \equiv_{T_2} \beta(L_{2k-1})$ and $\mathcal{S}(\beta(n), T_2) \in \hat{V}$. Let $x = n - L_{2k-1}$ so that $0 \le x \le L_{2k-2}$ and note that, by 3.7, $Digits(L_{2k-1}, L_{2k-1}+x) = Digits(2L_{2k+1}, 2L_{2k+1}+x)$.

Let $T_3 = \{-2k, 2k\}$. Then $\mathcal{S}(\beta(L_{2k-2}), T_3) = \beta(L_{2k}+L_{2k-2})$, $\mathcal{S}(\beta(L_{2k-1}), T_3) = \beta(L_{2k+1})$ and, for all n satisfying $L_{2k-2} \le n \le L_{2k-1}$, $\beta(n) \equiv_{T_3} \beta(L_{2k-2})$, and $\mathcal{S}(\beta(n), T_3) \in \hat{V}$. Let $x = n - L_{2k-2}$ so that $0 \le x \le L_{2k-3}$ and note that, by 3.7, $Digits(L_{2k-2}, L_{2k-2}+x) = Digits(L_{2k}+L_{2k-2}, L_{2k}+L_{2k-2}+x)$.

Using the largest values of x possible in the above arguments, we have formulas (3) and (5) of the lemma proven as well as formula (4) for $k \ge 3$. To complete the proof of formula (4), we check by hand that it holds for $k = 2$ as well. Thus,
$$\begin{aligned} &Digits(L_{2k}, L_{2k+1}) \\ &= Digits(L_{2k}, 2L_{2k-1}) + Digits(2L_{2k-1}, L_{2k}+L_{2k-2}) + Digits(L_{2k}+L_{2k-2}, L_{2k+1}) \\ &= Digits(L_{2k-2}, L_{2k-1}) + Digits(L_{2k-3}, L_{2k-2}) + Digits(L_{2k-2}, L_{2k-1}), \end{aligned}$$
which proves (6).

Formulas (7)-(11) remain to be proven as well as the third equality from (1) and the second and third equalities from (2). The proof for these remaining formulas is by induction on k. For $k = 2$, the formulas can be checked directly. Assume $k \ge 3$ and all the remaining formulas hold for smaller k.

Let $T_4 = \{-2k-2, -2k, -2k+1, 2k-1, 2k+1\}$. Note that $\mathcal{S}(\beta(L_{2k-1}+1), T_4) = \beta(L_{2k+1}+1)$ and $\mathcal{S}(\beta(L_{2k}-1), T_4) = \beta(2L_{2k}-1)$. By 2.1 and part (11) of the induction hypothesis, we see that for all n where $L_{2k-1} < n < L_{2k}$, $\beta(n) \equiv_{T_4} \beta(L_{2k-1}+1)$ and $\mathcal{S}(\beta(n), T_4) \in \hat{V}$. Let $x = n - L_{2k-1}$ so that $0 < x \le L_{2k-2}-1$. We have $Digits(L_{2k-1}+1, L_{2k-1}+x) = Digits(L_{2k+1}+1, L_{2k+1}+x)$. Since $(\beta(L_{2k-1}+1))_0 = (\beta(L_{2k+1}+1))_0$, we have $Digits(L_{2k-1}, L_{2k-1}+x) = Digits(L_{2k+1}, L_{2k+1}+x)$ for $0 < x \le L_{2k-2}-1$. When $x=0$ the equality is trivially true. We note that $(\beta(L_{2k}))_0 = (\beta(2L_{2k}))_0$ so, for $x = L_{2k-2}$, we have $Digits(L_{2k-1}, L_{2k}) = Digits(L_{2k+1}, 2L_{2k})$. Note also that the above shows that, for $L_{2k-1} < n \le L_{2k}$, $\mathcal{S}(\beta(n), T_4)$ starts with 100 and, hence, $\beta(\ell)$ starts with 100 for $L_{2k+1} < \ell \le 2L_{2k}$. We have proven (7), some of (11), and the second equality of (2).

Let $T_5 = \{-2k-2, -2k+1, -2k+2, 2k+1\}$. We can check that $\mathcal{S}(\beta(L_{2k-2}), T_5) = \beta(2L_{2k})$, $\mathcal{S}(\beta(L_{2k-1}), T_5) = \beta(L_{2k+1}+L_{2k-1})$ and, for all n satisfying $L_{2k-2} \le n \le L_{2k-1}$, $\beta(n) \equiv_{T_5} \beta(L_{2k-2})$ and $\mathcal{S}(\beta(n), T_5) \in \hat{V}$. Let $x = n - L_{2k-2}$ so that $0 \le x \le L_{2k-3}$ and note that, by 3.7, we have $Digits(L_{2k-2}, L_{2k-2}+x) = Digits(2L_{2k}, 2L_{2k}+x)$. Therefore, $Digits(L_{2k-2}, L_{2k-1}) = Digits(2L_{2k}, L_{2k+1}+L_{2k-1})$. Using 2.1, we have, for $L_{2k-2} \le n \le L_{2k-1}$, that $\mathcal{S}(\beta(n), T_5)$ starts with 100; thus, $\beta(\ell)$ starts with 100 for $2L_{2k} \le \ell \le L_{2k+1}+L_{2k-1}$. We have proven (8), some more of (11), and the last equality from (1).

Let $T_6 = \{-2k-2, -2k, 2k+1\}$. Then we have $\mathcal{S}(\beta(L_{2k-1}+1), T_6) = \beta(L_{2k+1}+L_{2k-1}+1)$ and $\mathcal{S}(\beta(L_{2k}-1), T_6) = \beta(L_{2k+2}-1)$. For all n satisfying $L_{2k-1} < n < L_{2k}$, we have $\beta(n) \equiv_{T_6} \beta(L_{2k-1}+1)$ and $\mathcal{S}(\beta(n), T_6) \in \hat{V}$. Let $x = n - L_{2k-1}$ so that $0 < x \le L_{2k-2}-1$. Then $Digits(L_{2k-1}+1, L_{2k-1}+x) = Digits(L_{2k+1}+L_{2k-1}+1, L_{2k+1}+L_{2k-1}+x)$. We note that $(\beta(L_{2k-1}+1))_0 = (\beta(L_{2k+1}+L_{2k-1}+1))_0$, so we actually have $Digits(L_{2k-1}, L_{2k-1}+x) = Digits(L_{2k+1}+L_{2k-1}, L_{2k+1}+L_{2k-1}+x)$ for $0 \le x \le L_{2k-2}-1$. Since also $(\beta(L_{2k}))_0 = (\beta(L_{2k+2}))_0$ we have, for $x = L_{2k-2}$, that $Digits(L_{2k-1}, L_{2k}) = Digits(L_{2k+1}+L_{2k-1}, L_{2k+2})$. We have proven (9) and the last equality in (2).

The above also shows that, for $\ell_{2k-1} < n < L_{2k}$, we know that $\mathcal{S}(\beta(n), T_6)$ starts with 100. Thus, for $L_{2k+1}+L_{2k-1}+1 \le \ell \le L_{2k+2}-1$, $\beta(\ell)$ starts with 100. After we check that $\beta(\ell)$ starts with 100 for $\ell = L_{2k+2}$, we have completed the proof of (11).

Hence,

$$\begin{aligned} Digits(L_{2k+1}, L_{2k+2}) &= Digits(L_{2k+1}, 2L_{2k}) + Digits(2L_{2k}, L_{2k+1}+L_{2k-1}) \\ &\quad + Digits(L_{2k+1}+L_{2k-1}, L_{2k+2}) \\ &= 2Digits(L_{2k-1}, L_{2k}) + Digits(L_{2k-2}, L_{2k-1}), \end{aligned}$$

and (10) is proven. □

Proposition 3.9: For $k \ge 1$:

(1) $Ones(L_{2k}, L_{2k+1}) = F_{2k-2}+1$ and $Zeros(L_{2k}, L_{2k+1}) = F_{2k}-1$.

(2) $Ones(L_{2k+1}, L_{2k+2}) = F_{2k-1}-1$ and $Zeros(L_{2k+1}, L_{2k+2}) = F_{2k+1}+1$.

(3) $Ones(0, L_{2k}) = F_{2k-1}$ and $Zeros(0, L_{2k}) = F_{2k+1}$.

(4) $Ones(0, L_{2k+1}) = F_{2k}+1$ and $Zeros(0, L_{2k+1}) = F_{2k+2}-1$.

Proof: The first two results are proved by induction on k. It may be checked by inspection that the formulas hold for $k=1$ and $k=2$. Let $k \geq 3$ and assume that the formulas hold for smaller k. Then, by 3.8,

$$Ones(L_{2k}, L_{2k+1}) = 2Ones(L_{2k-2}, L_{2k-1}) + Ones(L_{2k-3}, L_{2k-2}) = 2F_{2k-4} + 2 + F_{2k-5} - 1 = F_{2k-2} + 1$$

by the induction hypothesis. Similarly, using Lemma 3.8 and the induction hypothesis

$$Ones(L_{2k+1}, L_{2k+2}) + Ones(L_{2k-2}, L_{2k-1}) = 2F_{2k-3} - 2 + F_{2k-4} + 1 = F_{2k-1} - 1.$$

To prove the last two results, we note that

$$Ones(L_{2k}, L_{2k+2}) = Ones(L_{2k}, L_{2k+1}) + Ones(L_{2k+1}, L_{2k+2}) = F_{2k}.$$

So

$$\begin{aligned} Ones(0, L_{2k}) &= 1 + \sum_{i=1}^{k-1} Ones(L_{2i}, L_{2i+2}) = 1 + \sum_{i=1}^{k-1} F_{2i} \\ &= F_1 + F_2 + \cdots + F_{2k-2} = F_{2k-1}. \end{aligned}$$

And

$$Ones(0, L_{2k+1}) = Ones(0, L_{2k}) + Ones(L_{2k}, L_{2k+1}) = F_{2k-1} + F_{2k-2} + 1 = F_{2k} + 1. \quad \square$$

Proof of 3.1: The formulas for $Ratio(L_{2k})$ and $Ratio(L_{2k+1})$ follow from 3.9. The limit follows from equation (2). To see that both sequences are decreasing, we use equation (3). We have $(F_{2k+1})^2 - F_{2k-1}F_{2k+3} = -1 < 0$, which implies that $Ratio(L_{2k}) > Ratio(L_{2k+2})$. We also have $(F_{2k})^2 - F_{2k-2}F_{2k+2} = 1 < F_{2k+2} - F_{2k-2}$. This implies that $Ratio(L_{2k+1}) < Ratio(L_{2k-1})$. $\square$

4. THE RATIO FOR NON-LUCAS NUMBERS

In this section we prove that the sequence $Ratio(n)$ for $n \geq 2$ is trapped between the two decreasing sequences of Proposition 3.1, which both approach β^{-2}.

Proposition 4.1: Let $k \geq 1$. Then, for $n \in \mathbb{N}$ with $L_{2k} < n \leq L_{2k+2}$, $Ratio(L_{2k}) \leq Ratio(n) \leq Ratio(L_{2k-1})$.

We devote the rest of the paper to developing the lemmas which, when combined with some of the results of the previous section, will allow us to prove Proposition 4.1.

The following lemma will be used repeatedly.

Lemma 4.2: Let $a, b, c, d \in \mathbb{N}$ and $x, y \in \mathbb{R}$. If $\frac{a}{b} \leq x$ and $\frac{c}{d} \leq y$, then $\frac{a+c}{b+d} \leq \max\{x, y\}$. When each $\leq$ is replaced by $\geq$, the result holds with *max* replaced by *min*.

Lemma 4.3: For $k \geq 1$:

(1) $Ones(0, 2L_{2k-1}) = 2F_{2k-2} + 1$, $Zeros(0, 2L_{2k-1}) = 2F_{2k} - 1$.

(2) $Ones(0, L_{2k} + L_{2k-2}) = F_{2k-1} + F_{2k-3}$, $Zeros(0, L_{2k} + L_{2k-2}) = F_{2k+1} + F_{2k-1}$.

(3) $Ones(0, 2L_{2k}) = 2F_{2k-1}$, $Zeros(0, 2L_{2k}) = 2F_{2k+1}$.

(4) $Ones(0, L_{2k+1} + L_{2k-1}) = F_{2k} + F_{2k-2} + 1$, $Zeros(0, L_{2k+1} + L_{2k-1}) = F_{2k+2} + F_{2k} - 1$.

Proof: We use Lemma 3.8 and Proposition 3.9. For example, for $k \geq 2$, $Ones(0, 2L_{2k-1}) = Ones(0, L_{2k}) + Ones(L_{2k}, 2L_{2k-1}) = Ones(0, L_{2k}) + Ones(L_{2k-2}, L_{2k-1})$, using part (1) of Lemma

3.8. Therefore, $Ones(0, 2L_{2k-1}) = F_{2k-1} + F_{2k-4} + 1 = 2F_{2k-2} + 1$. The case $k = 1$ can be checked directly. The rest of the proofs are similar. □

Lemma 4.4 $\{F_{2k-2} / F_{2k}\}_k$ is an increasing sequence that approaches β^{-2} as $k \to \infty$.

Proof: Apply equations (2) and (3). □

Lemma 4.5: Let $k \geq 1$. If $0 < x < L_{2k+2}$, then $Ones(L_{2k+1}, L_{2k+1} + x) / Zeros(L_{2k+1}, L_{2k+1} + x) \leq \beta^{-2}$.

Proof: In the proof of this lemma, *Digits* stands for either *Ones* or *Zeros*. The proof is by induction on *k*. The cases for $k = 1$ and $k = 2$ can be checked directly. Assume $k \geq 3$, and the result is true for smaller *k*.

Case 1. $1 \leq x \leq L_{2k-2}$. By Lemma 3.8 and the induction hypothesis,

$$\frac{Ones(L_{2k+1}, L_{2k+1} + x)}{Zeros(L_{2k+1}, L_{2k+1} + x)} = \frac{Ones(L_{2k-1}, L_{2k-1} + x)}{Zeros(L_{2k-1}, L_{2k-1} + x)} \leq \beta^{-2}.$$

Case 2. $L_{2k-2} < x < L_{2k-1}$. Let $y = x - L_{2k-2}$ and $z = y + L_{2k-4}$. Then $0 < y < L_{2k-3}$ and $L_{2k-4} < z < L_{2k-2}$. This implies that $L_{2k+1} + x = 2L_{2k} + y$. We have

$$\frac{Ones(L_{2k+1}, L_{2k+1} + x)}{Zeros(L_{2k+1}, L_{2k+1} + x)} = \frac{Ones(L_{2k+1}, 2L_{2k}) + Ones(2L_{2k}, 2L_{2k} + y)}{Zeros(L_{2k+1}, 2L_{2k}) + Zeros(2L_{2k}, 2L_{2k} + y)}.$$

Note that using Lemma 3.8,

$$\begin{aligned} Digits(2L_{2k}, 2L_{2k} + y) &= Digits(L_{2k-2}, L_{2k-2} + y) \\ &= Digits(L_{2k-3}, L_{2k-2} + y) - Digits(L_{2k-3}, L_{2k-2}) \\ &= Digits(L_{2k-3}, L_{2k-3} + z) - Digits(L_{2k-3}, L_{2k-2}). \end{aligned}$$

Hence, using Lemma 4.3 and Proposition 3.9, we see that

$$\begin{aligned} \frac{Ones(L_{2k+1}, L_{2k+1} + x)}{Zeros(L_{2k+1}, L_{2k+1} + x)} &= \frac{2F_{2k-1} - (F_{2k} + 1) - F_{2k-3} + (F_{2k-4} + 1) + Ones(L_{2k-3}, L_{2k-3} + z)}{2F_{2k+1} - (F_{2k+2} - 1) - F_{2k-1} + (F_{2k-2} - 1) + Zeros(L_{2k-3}, L_{2k-3} + z)} \\ &= \frac{F_{2k-4} + Ones(L_{2k-3}, L_{2k-3} + z)}{F_{2k-2} + Zeros(L_{2k-3}, L_{2k-3} + z)} \leq \beta^{-2}, \end{aligned}$$

since $F_{2k-4} / F_{2k-2} \leq \beta^{-2}$ by Lemma 4.4 and $Ones(L_{2k-3}, L_{2k-3} + z) / Zeros(L_{2k-3}, L_{2k-3} + x) \leq \beta^{-2}$ by the induction hypothesis.

Case 3. $x = L_{2k-1}$. Then

$$\frac{Ones(L_{2k+1}, L_{2k+1} + L_{2k-1})}{Zeros(L_{2k+1}, L_{2k+1} + L_{2k-1})} = \frac{F_{2k} + F_{2k-2} + 1 - (F_{2k} + 1)}{F_{2k+2} + F_{2k} - 1 - (F_{2k+2} - 1)} = \frac{F_{2k-2}}{F_{2k}} \leq \beta^{-2}$$

using Proposition 3.9, Lemma 4.3, and Lemma 4.4.

Case 4. $L_{2k-1} < x \leq L_{2k}$. Let $y = x - L_{2k-1}$. So $0 < y < L_{2k-2}$. We have

$$\frac{Ones(L_{2k+1}, L_{2k+1} + x)}{Zeros(L_{2k+1}, L_{2k+1} + x)} = \frac{Ones(L_{2k+1}, L_{2k+1} + L_{2k-1}) + Ones(L_{2k-1}, L_{2k-1} + y)}{Zeros(L_{2k+1}, L_{2k+1} + L_{2k-1}) + Zeros(L_{2k-1}, L_{2k-1} + y)}$$

$$=\frac{F_{2k}+F_{2k-2}+1-(F_{2k}+1)+Ones(L_{2k-1},L_{2k-1}+y)}{F_{2k+2}+F_{2k}-1-(F_{2k+2}-1)+Zeros(L_{2k-1},L_{2k-1}+y)}$$

$$=\frac{F_{2k-2}+Ones(L_{2k-1},L_{2k-1}+y)}{F_{2k}+Zeros(L_{2k-1},L_{2k-1}+y)}\leq\beta^{-2}$$

using Lemma 3.8, Proposition 3.9, Lemma 4.3, Lemma 4.4, and the induction hypothesis.

Case 5. $L_{2k}<x<L_{2k+1}$. Let $y=x-L_{2k}$ and $z=y+L_{2k-2}$. Then $0<y<L_{2k-1}$ and $L_{2k-2}<z<L_{2k}$. As before, we have

$$Digits(L_{2k+1},L_{2k+1}+x)=Digits(L_{2k+1},L_{2k+2}+y)$$
$$=Digits(L_{2k+1},L_{2k+2})+Digits(L_{2k},L_{2k}+y)$$
$$=Digits(L_{2k+1},L_{2k+2})+Digits(L_{2k-1},L_{2k-1}+z)-Digits(L_{2k-1},L_{2k}).$$

Thus,

$$\frac{Ones(L_{2k+1},L_{2k+1}+x)}{Zeros(L_{2k+1},L_{2k+1}+x)}=\frac{F_{2k-1}-1-(F_{2k-3}-1)+Ones(L_{2k-1},L_{2k-1}+z)}{F_{2k+1}+1-(F_{2k-1}+1)+Zeros(L_{2k-1},L_{2k-1}+z)}$$

$$=\frac{F_{2k-2}+Ones(L_{2k-1},L_{2k-1}+z)}{F_{2k}+Zeros(L_{2k-1},L_{2k-1}+z)}\leq\beta^{-2}$$

using the induction hypothesis.

Case 6. $x=L_{2k+1}$. We have

$$\frac{Ones(L_{2k+1},2L_{2k+1})}{Zeros(L_{2k+1},2L_{2k+1})}=\frac{(2F_{2k}+1)-(F_{2k}+1)}{(2F_{2k+2}-1)-(F_{2k+2}-1)}$$

$$=\frac{F_{2k}}{F_{2k+2}}\leq\beta^{-2}.$$

Case 7. $L_{2k+1}<x\leq 2L_{2k}$. Let $y=x-L_{2k+1}$. So $0<y\leq L_{2k-2}$. We have, by the induction hypothesis and using Lemma 4.3,

$$\frac{Ones(L_{2k+1},L_{2k+1}+x)}{Zeros(L_{2k+1},L_{2k+1}+x)}=\frac{Ones(L_{2k+1},2L_{2k+1})+Ones(L_{2k-1},L_{2k-1}+y)}{Zeros(L_{2k+1},2L_{2k+1})+Zeros(L_{2k-1},L_{2k-1}+y)}$$

$$=\frac{(2F_{2k}+1)-(F_{2k}+1)+Ones(L_{2k-1},L_{2k-1}+y)}{(2F_{2k+2}-1)-(F_{2k+2}-1)+Zeros(L_{2k-1},L_{2k-1}+y)}$$

$$=\frac{F_{2k}+Ones(L_{2k-1},L_{2k-1}+y)}{F_{2k+2}+Zeros(L_{2k-1},L_{2k-1}+y)}\leq\beta^{-2}.$$

Case 8. $2L_{2k}<x<L_{2k+2}$. Let $y=x-2L_{2k}$ and let $z=y+L_{2k-2}$. Then $0<y<L_{2k-1}$ and $L_{2k-2}<z<L_{2k}$. We have

$$Digits(L_{2k+1},L_{2k+1}+x)=Digits(L_{2k+1},L_{2k+2}+L_{2k})+Digits(L_{2k+2}+L_{2k},L_{2k+2}+L_{2k}+y)$$
$$=Digits(L_{2k+1},L_{2k+2}+L_{2k})+Digits(L_{2k},L_{2k}+y)$$
$$=Digits(L_{2k+1},L_{2k+2}+L_{2k})+Digits(L_{2k-1},L_{2k-1}+z)-Digits(L_{2k-1},L_{2k}).$$

Thus,

$$\frac{Ones(L_{2k+1}, L_{2k+1}+x)}{Zeros(L_{2k+1}, L_{2k+1}+x)} = \frac{F_{2k+1}+F_{2k-1}-(F_{2k}+1)-(F_{2k-3}-1)+Ones(L_{2k-1}, L_{2k-1}+z)}{F_{2k+3}+F_{2k+1}-(F_{2k+2}-1)-(F_{2k-1}+1)+Zeros(L_{2k-1}, L_{2k-1}+z)}$$
$$= \frac{F_{2k}+Ones(L_{2k-1}, L_{2k-1}+z)}{F_{2k+2}+Zeros(L_{2k-1}, L_{2k-1}+z)} \le \beta^{-2}$$

by the induction hypothesis. □

Proposition 4.6: Let $k \ge 1$. If $0 < x < L_{2k+2}$, then $Ratio(L_{2k+1}+x) \le Ratio(L_{2k+1})$.

Proof:

$$Ratio(L_{2k+1}+x) = \frac{Ones(0, L_{2k+1})+Ones(L_{2k+1}, L_{2k+1}+x)}{Zeros(0, L_{2k+1})+Zeros(L_{2k+1}, L_{2k+1}+x)} \le Ratio(L_{2k+1})$$

since

$$\frac{Ones(L_{2k+1}, L_{2k+1}+x)}{Zeros(L_{2k+1}, L_{2k+1}+x)} \le \beta^{-2} \le Ratio(L_{2k+1})$$

by Lemma 4.5 and Proposition 3.1. □

Lemma 4.7: Let $k \ge 1$. If $0 < x < L_{2k+1}$, then

$$\frac{Ones(L_{2k}, L_{2k}+x)}{Zeros(L_{2k}, L_{2k}+x)} \ge Ratio(L_{2k}).$$

Proof: The proof of this lemma is similar to that of Lemma 4.5 and is omitted.

Proposition 4.8: Let $k \ge 1$. If $0 < x < L_{2k+1}$, then $Ratio(L_{2k}+x) \ge Ratio(L_{2k})$.

Proof: For $x > 1$, we have

$$Ratio(L_{2k}+x) = \frac{Ones(0, L_{2k})+Ones(L_{2k}, L_{2k}+x)}{Zeros(0, L_{2k})+Zeros(L_{2k}, L_{2k}+x)} \ge Ratio(L_{2k})$$

by Proposition 4.7 and Lemma 4.2. For $x = 1$, $Ones(L_{2k}, L_{2k}+x) = 1$, $Zeros(L_{2k}, L_{2k}+x) = 0$, and the result follows. □

Proof of Proposition 4.1: By Propositions 4.6 and 4.8, it follows for $k \ge 1$ that, if $L_{2k+1} \le n \le L_{2k+2}$, then $Ratio(L_{2k}) \le Ratio(n) \le Ratio(L_{2k+1}) \le Ratio(L_{2k-1})$. Also, for $k \ge 1$, if $L_{2k} < n < L_{2k+1}$, then $Ratio(L_{2k}) \le Ratio(n) \le Ratio(L_{2k-1})$. □

The following theorem has now been proven.

Theorem 4.9: The limit of the ratio of natural numbers having Property $\mathcal{P}$ to those not having Property $\mathcal{P}$ is $\lim_{n\to\infty} Ratio(n) = \beta^{-2}$.

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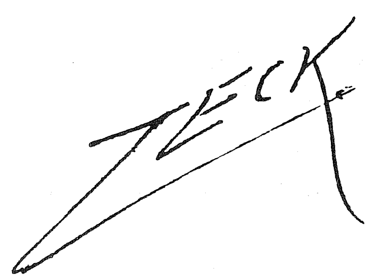
CORRIGENDUM

In the November 1998 issue of *The Fibonacci Quarterly* (Vol. 36, no. 5), Clark Kimberling's article entitled "Edouard Zeckendorf" appeared on pages 416-418. Due to an unfortunate printing error, the signature which was to accompany the article was inadvertently omitted. We apologize to the author, and are pleased to present the missing signature below:



APPLICATION OF MARKOV CHAINS PROPERTIES TO r-GENERALIZED FIBONACCI SEQUENCES

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1. INTRODUCTION

Let $a_0, \ldots, a_{r-1}$ $(r \geq 2, a_{r-1} \neq 0)$ be some fixed real numbers. An r-generalized Fibonacci sequence $\{V_n\}_{n=0}^{+\infty}$ is defined by the linear recurrence relation of order r,

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_{r-1} V_{n-r+1}, \text{ for } n \geq r-1, \tag{1}$$

where $V_0, \ldots, V_{r-1}$ are specified by the initial conditions. A first connection between Markov chains and sequence (1), whose coefficients a_i $(0 \leq i \leq r-1)$ are nonnegative, is considered in [6]. And we established that the limit of the ratio V_n / q^n exists if and only if $\mathrm{CGD}\{i+1; a_i > 0\} = 1$, where CGD means the common great divisor and q is the unique positive root of the characteristic polynomial $P(x) = x^r - a_0 x^{r-1} - \cdots - a_{r-2} x - a_{r-1}$ (cf. [6] and [7]).

Our purpose in this paper is to give a second connection between Markov chains and sequence (1) when the a_i are nonnegative. This allows us to express the general term V_n $(n \geq r)$ in a combinatoric form. Note that the combinatoric form of V_n has been studied by various methods and techniques (cf. [2], [4], [5], [8], [9], and [10], for example). However, our method is different from those above, and it allows us to study the asymptotic behavior of the ratio V_n / q^n, from which we derive a new approximation of the number q.

This paper is organized as follows. In Section 2 we study the connection between Markov chains and sequence (1) when the coefficients a_j are nonnegative and $a_0 + \cdots + a_{r-1} = 1$, and we establish the combinatoric form of V_n for $n \geq r$. In Section 3 we are interested in the asymptotic behavior of V_n when the coefficients a_j are arbitrary nonnegative real numbers.

2. COMBINATORIC FORM OF SEQUENCE (1) WITH NONNEGATIVE COEFFICIENTS OF SUM 1

2.1 Sequence (1) and Markov Chains

Let $\{V_n\}_{n=0}^{+\infty}$ be a sequence (1) whose coefficients $a_0, \ldots, a_{r-1}$ $(a_{r-1} \neq 0)$ are nonnegative with $a_0 + \cdots + a_{r-1} = 1$. Set

$$X = \begin{pmatrix} V_0 \\ V_1 \\ \vdots \\ V_n \\ \vdots \end{pmatrix} \quad \text{and} \quad P = \left(\begin{array}{cccccccccc} & & I_r & & \vert & & 0 & & & \\ a_{r-1} & a_{r-2} & \cdots & a_0 & 0 & \cdots & & & & \\ 0 & a_{r-1} & a_{r-2} & \cdots & a_0 & 0 & \cdots & & & \\ 0 & 0 & a_{r-1} & a_{r-2} & \cdots & a_0 & 0 & \cdots & & \\ \vdots & & & & & & & & & \end{array}\right), \tag{2}$$

where I_r is the identity $r \times r$ matrix. The condition $\sum_{i=0}^{r-1} a_i = 1$ implies that $P = (P(n,m))_{n \geq 0, m \geq 0}$ is a stochastic matrix. Consider the following general theorem on the convergence of the matrix sequence $\{P^{(k)}\}_{k=0}^{+\infty}$, where $P^{(k)} = P \cdot P \cdot \cdots \cdot P$ (k times).

Theorem 2.1 (e.g., cf. [3], [11]): Let $P = (P(n, m))_{n\geq 0,\, m\geq 0}$ be the transition matrix of a Markov chain. Then, the sequence $\{P^{(k)}\}_{k=0}^{+\infty}$ converges in the Cesaro mean. More precisely, the sequence $\{Q_k\}_{k=1}^{+\infty}$ defined by

$$Q_k = \frac{P + P^{(2)} + \cdots + P^{(k)}}{k}$$

converges to the matrix $Q = \{q(n, m)\}_{n\geq 0,\, m\geq 0}$ with

$$q(n, m) = \frac{\rho(n, m)}{\mu_m},$$

where $\rho(n, m)$ is the probability that starting from the state n the system will ever pass through m, and μ_m is the mean of the real variable which gives the time of return for the first time to the state m, starting from m.

It is obvious that the particular matrix given by (2) is the transition matrix of a Markov chain whose state space is $\mathbb{N} = \{0, 1, 2, \ldots\}$. We observe that $0, 1, \ldots, r-1$ are absorbing states and the other states $r, r+1, \ldots$ are transient, because starting from a state $n \geq r$ the process will be absorbed with probability 1 by one of the states $0, 1, \ldots, r-1$ after $n-r+1$ transitions. If m is a transient state, we have $\mu_m = +\infty$ (cf. [3]), hence $q(n, m) = 0$ for $m = r, r+1, \ldots$. If n and m are absorbing states, we have $\mu_m = 1$ and $\rho(n, m) = \delta_{n,m}$. Therefore, the limit matrix Q of Theorem 2.1 has the following form:

$$Q = \begin{pmatrix} & I_r & & \mid & 0 & \\ \rho(r, 0) & \cdots & \rho(r, r-1) & & 0 & \cdots \\ \rho(r+1, 0) & \cdots & \rho(r+1, r-1) & & 0 & \cdots \\ \vdots & & \vdots & & \vdots & \\ \rho(n, 0) & \cdots & \rho(n, r-1) & & 0 & \cdots \\ \vdots & & \vdots & & \vdots & \end{pmatrix}. \tag{3}$$

The sequence defined by (1) may be written in the following form,

$$X = PX. \tag{4}$$

From expression (4) we derive easily that $X = P^{(n)}X$ for $n \geq 1$, which is equivalent to $X = Q_n X$ for $n \geq 1$, where $Q_n = \frac{P + P^{(2)} + \cdots + p^{(n)}}{n}$. Thus, we have $X = QX$, where Q is given by (3). We then derive the following result.

Theorem 2.2: Let $\{V_n\}_{n=0}^{+\infty}$ be a sequence (1) such that the real numbers $a_0, \ldots, a_{r-1}$ are non-negative with $\sum_{i=1}^{r-1} a_i = 1$. Then, for any $n \geq r$, we have

$$V_n = \rho(n, 0)V_0 + \rho(n, 1)V_1 + \cdots + \rho(n, r-1)V_{r-1}. \tag{5}$$

Note that the number $\rho(n, j)$ $(0 \leq j \leq r-1)$ is the probability of absorption of the process by the state j, starting from the state n. Theorem 2.2 gives the expression of the general term V_n for $n \geq r$ as a function of the initial conditions $V_0, \ldots, V_{r-1}$ and the absorption probabilities $\rho(n, j)$.

2.2 Combinatoric Expression of $\rho(n, m)$

For $n > m \geq r$, the number $\rho(n, m)$ is the probability to reach the state m starting from the state n, because m is a transient state. To reach the state m starting from the state n, the process makes k_j jumps of $j+1$ units with the probability a_j $(0 \le j \le r-1)$. The total number of jumps is $k_0+k_1+\cdots+k_{r-1}$ and the number of units is $k_0+2k_1+\cdots+rk_{r-1}=n-m$. The number of ways to choose the k_j $(0 \le j \le r-1)$ is

$$\frac{(k_0+k_1+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!}$$

and the probability for each choice is $a_0^{k_0}a_1^{k_1}\cdots a_{r-1}^{k_{r-1}}$. Hence, we have the following result.

Theorem 2.3: For any two states n, m $(n > m \ge r)$, the probability $\rho(n, m)$ to reach m starting from n is given by

$$\rho(n,m) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-m} \frac{(k_0+k_1+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} a_0^{k_0}a_1^{k_1}\cdots a_{r-1}^{k_{r-1}}. \tag{6}$$

Note that, for $n > m \ge r$, $\rho(n,m) = H_{n-m+1}^{(r)}(a_0, \dots, a_{r-1})$, where $\{H_{n-m+1}^{(r)}(a_0, \dots, a_{r-1})\}_{n\ge 0}$ is the sequence of multivariate Fibonacci polynomials of order r of Philippou (cf. [1]).

Let n and j be two states such that $0 \le j < r \le n$. Then n is a transient state, j is an absorbing one, and $\rho(n, j)$ is the probability of absorption of the process by j starting from n. First, we suppose that $n \ge 2r$ and $j = 0$. To reach 0 starting from n, the state r is the last transient state visited by the process. And a_{r-1} is the probability of the jump from r to 0, which implies that we have $\rho(n, 0) = a_{r-1}\rho(n, r)$. More precisely, to reach j $(0 \le j \le r-1)$ starting from n $(n \ge 2r)$, the process must visit one of the following states $r, r+1, \dots, r+j$, because they are the only states from which the process can reach j in one jump. As $a_{r+k-j-1}$ $(0 \le k \le j)$ is the probability to go from $r+k$ to j and $\rho(n, r+k)$ is the probability to go from n to $r+k$, we obtain

$$\rho(n, j) = a_{r-j-1}\rho(n,r)+a_{r-j}\rho(n,r+1)+\cdots+a_{r-1}\rho(n,r+j). \tag{7}$$

From expression (6), we deduce that $\rho(n,r+l) = \rho(n-l,r)$ for any $n > r+l$. Thus, for any $n \ge 2r$ and j $(0 \le j < r)$, we have

$$\rho(n, j) = a_{r-j-1}\rho(n,r)+a_{r-j}\rho(n-1,r)+\cdots+a_{r-1}\rho(n-j,r).$$

Now suppose that $n < 2r$. Then we have two cases. If $r+j \le n$, expression (7) is still verified. For the second case, $r \le n < r+j$, we have

$$\rho(n, j) = a_{r-j-1}\rho(n,r)+a_{r-j}\rho(n-1,r)+\cdots+a_{n-j-1}\rho(r,r).$$

Hence, the expression of the absorption probabilities is given by Theorem 2.4.

Theorem 2.4: Let n and j be two states such that $0 \le j < r \le n$. Then, if we set $\rho(i, i) = 1$ and $\rho(i, k) = 0$ if $i < k$, the probability of absorption $\rho(n, j)$ is given by

$$\rho(n, j) = a_{r-j-1}\rho(n,r)+a_{r-j}\rho(n-1,r)+\cdots+a_{r-1}\rho(n-j,r), \tag{8}$$

where $\rho(n,0) = a_{r-1}\rho(n,r)$.

2.3 Combinatoric Expression of V_n

By substituting expression (8) in (5), we obtain the following result.

Theorem 2.5: Let $\{V_n\}_{n=0}^{+\infty}$ be a sequence (1). Suppose that the coefficients $a_0, \dots, a_{r-1}$ are nonnegative with $\sum_{i=0}^{r-1} a_i = 1$. Then, for any $n \ge r$, we have

$$V_n = A_0\rho(n,r) + A_1\rho(n-1,r) + \cdots + A_{r-1}\rho(n-r+1,r), \tag{9}$$

where $A_m = a_{r-1}V_m + \cdots + a_m V_{r-1}$; $m = 0, 1, \ldots, r-1$ and the $\rho(k,r)$ are given by (6) with $\rho(r,r) = 1$ and $\rho(k,r) = 0$ if $k < r$.

If we take $V_0 = 1$ and $V_1 = \cdots = V_{r-1} = 0$, we get $V_n = a_{r-1}\rho(n,r)$. Therefore, the sequence $\{(\rho(n,r)\}_{n=0}^{+\infty}$ satisfies the following relation:

$$\rho(n+1,r) = a_0\rho(n,r) + a_1\rho(n-1,r) + \cdots + a_{r-1}\rho(n-r+1,r). \tag{10}$$

Relation (10) may be proved otherwise by considering the jumps of the process from the state $n+1$ to the state r.

2.4 General Case and Levesque Result

Now suppose that the coefficients $a_0, \ldots, a_{r-1}$ are arbitrary real numbers and define the number $\rho(n,r)$ by (6). Then we can prove by induction on n that expression (10) is satisfied. Hence, Theorem 2.5 is still valid in this general case. Such a result was established by Levesque in [5].

3. ASYMPTOTIC BEHAVIOR OF $\rho(n, r)$

Let $\{V_n\}_{n=0}^{+\infty}$ be a sequence (1). Suppose that $a_0, \ldots, a_{r-1}$ are nonnegative real numbers with $\sum_{i=0}^{r-1} a_i = 1$. We have established, using some Markov chains properties, that sequence (1) converges for any $V_0, \ldots, V_{r-1}$ if and only if $\mathrm{CGD}\{i+1; a_i > 0\} = 1$ (cf. [6], Theorem 2.2). When this condition is satisfied, we obtain

$$\lim_{n\to+\infty} V_n = \Pi(0)V_{r-1} + \Pi(1)V_{r-2} + \cdots + \Pi(r-1)V_0, \tag{11}$$

where

$$\Pi(m) = \frac{\sum_{j=m}^{r-1} a_j}{\sum_{i=0}^{r-1}(i+1)a_i} \quad \text{(cf. [6], Theorem 2.4).}$$

By using expressions (9) and (11), we derive the following result.

Theorem 3.1: Suppose that the real numbers $a_0, \ldots, a_{r-1}$ are nonnegative with $\sum_{i=0}^{r-1} a_i = 1$. Then, if $\mathrm{CGD}\{i+1; a_i > 0\} = 1$, we have

$$\lim_{n\to+\infty} \rho(n,r) = \frac{1}{\sum_{j=0}^{r-1}(j+1)a_j},$$

where $\rho(n,r)$ is given by (6).

Now suppose $\sum_{i=0}^{r-1} a_i \neq 1$. It was shown in [7] that under the condition $\mathrm{CGD}\{i+1; a_i > 0\} = 1$, the characteristic equation $x^r = a_0 x^{r-1} + \cdots + a_{r-2}x + a_{r-1}$ of sequence (1) has a unique simple nonnegative root q, and the moduli of all other roots is less than q. If we set $b_i = a_i / q^{i+1}$, we have $b_i \geq 0$, $\sum_{i=0}^{r-1} b_i = 1$, and $\mathrm{CGD}\{i+1; a_i > 0\} = \mathrm{CGD}\{i+1; b_i > 0\}$. Thus, the sequence $\{W_n\}_{n=0}^{+\infty}$ defined by $W_n = V_n / q^{n+1}$ is a sequence (1) whose initial conditions are $W_i = V_i / q^{i+1}$ for $i = 0, 1, \ldots, r-1$ and $W_{n+1} = \sum_{i=0}^{r-1} b_i W_{n-i}$ for $n \geq r-1$ (cf. [6]). Therefore, we derive from Theorem 3.1 the following result.

Theorem 3.2: Suppose that $a_0, \ldots, a_{r-1}$ are nonnegative real numbers and $\text{CGD}\{i+1; a_i > 0\} = 1$. Then we have

$$\lim_{n\to+\infty} \frac{1}{q^n} \rho(n,r) = \frac{q^{-r}}{\sum_{j=0}^{r-1}(j+1)\frac{a_j}{q^{j+1}}},$$

where $\rho(n,r)$ is given by (6).

From Theorem 3.2, we can derive a new approximation of the number q. More precisely, we have the following corollary.

Corollary 3.3: Suppose that $a_0, \ldots, a_{r-1}$ are nonnegative real numbers and $\text{CGD}\{i+1; a_i > 0\} = 1$. Then the unique simple nonnegative root q of the characteristic equation of (1) is given by

$$q = \lim_{n\to+\infty} \sqrt[n]{\rho(n,r)},$$

where $\rho(n,r)$ is given by (6).

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A GENERAL CONCLUSION ON LUCAS NUMBERS OF THE FORM px^2 WHERE p IS PRIME

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1. INTRODUCTION

Let L_n be the n^{th} Lucas number, that is, $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$ for $n \geq 2$. Let p be prime. Consider the equation

$$L_n = px^2 \quad (n, x > 0). \tag{1.1}$$

In [1], Cohn solved (1.1) for $p = 2$. In [3], Goldman solved (1.1) for $p = 3, 7, 47$, and 2207. In [5], Robbins solved (1.1) for $p < 1000$. He proved that, for $2 < p < 1000$, (1.1) holds iff

$$\begin{aligned}(p, n, x) = {} & (3, 2, 1), (7, 4, 1), (11, 5, 1), (19, 9, 2), \\ & (29, 7, 1), (47, 8, 1)\ (199, 11, 1),\ (521, 13, 1).\end{aligned} \tag{1.2}$$

Besides, he proved that, for $p = 14503$, (1.1) holds iff

$$(n, x) = (28, 7). \tag{1.3}$$

Following Robbins, denote $z(n) = \min\{m : n \mid F_m,\ m > 0\}$, where F_m is the m^{th} Fibonacci number, that is, $F_1 = F_2 = 1$, $F_{m+1} = F_m + F_{m-1}$ for $m \geq 2$. If p is odd and $2 \mid z(p)$, denote $y(p) = \frac{1}{2}z(p)$. Then we observe that every (n, x) in (1.2) and (1.3) satisfies $n = y(p)$. Furthermore, if $2 \mid n$, then either $n = 2^r$ or $n = 2^r q$, where q is an odd prime and $l_{2^r} = q\square$; if $2 \nmid n$, then n is a prime except $n = 9$ for $p = 19$. The question is: Does the above conclusion holds for arbitrary p? Our answer is affirmative. In this paper, we state and prove this general conclusion in Section 3. Some preliminaries are given in Section 2. In Section 4, we give an algorithm which we can use to solve (1.1) for given p. For example, we have given the solutions of (1.1) for $1000 < p < 60000$. A conjecture is also given in Section 4.

2. PRELIMINARIES

Let (n / m) be the Jacobi symbol. (For odd prime m, (n / m) is the Legendre symbol; see [9].) Denote $O_p(n) = k$ if $p^k \| n$.

(1) If $m \geq 2$, then $m \mid F_n$ iff $z(m) \mid n$.

(2) If m is odd and $m \geq 3$, then $m \mid L_n$ iff $n / y(m)$ is an odd integer.

(3) $F_{2n} = L_n F_n$.

(4) $L_{2n} = L_n^2 - 2(-1)^n = 5F_n^2 + 2(-1)^n$.

(5) $L_{-n} = (-1)^n L_n$.

(6) If p is an odd prime, then $z(p) \mid (p - e)$, where $e = (5 / p) = 1, -1, 0$ for $p = \pm 1, \pm 2, 0$ (mod 5), respectively.

(7) $L_n \mid L_{kn}$ iff k is odd or $n = 1$.

(8) If k is odd, then $(L_n,\ L_{kn} / L_n) \mid k$.

(9) $F_n = (\alpha^n - \beta^n) / (\alpha - \beta)$ and $L_n = \alpha^n + \beta^n$, where $\alpha = (1 + \sqrt{5}) / 2$, $\beta = (1 - \sqrt{5}) / 2$.

(10) If p is an odd prime, $p \mid F_m$ and $p \nmid a$, then $O_p(F_{p^k am} / F_m) = k$.

(11) If p is an odd prime, $p \mid L_m$, a is an odd integer, and $p \nmid a$, then $O_p(L_{p^k am} / L_m) = k$.

(12) $O_2(L_n) = \begin{cases} 1 & \text{if } n = 0 \pmod 6, \\ 2 & \text{if } n = 3 \pmod 6, \\ 0 & \text{otherwise.} \end{cases}$

(13) $L_{12m+n} \equiv L_n \pmod 8$; furthermore, $L_n \equiv 1, -1, 3, -3 \pmod 8$ for $n = 1, -1$ or $\pm 4, \pm 2$ or $5, -5 \pmod{12}$, respectively.

(14) $L_n = x^2$ iff $n = 1$ or 3.

(15) $L_{n+k} + (-1)^k L_{n-k} = L_n L_k$.

(16) If $m > 0$, then $L_{2mk+t} \equiv (-1)^{m(k-1)} L_t \pmod{L_k}$.

Remarks: (1) through (10), (12), (14), and (15) can be found in [4], [8], or [6]; (13) follows from the observation of the sequence $\{L_n \pmod 8\}$. We give the proofs of (11) and (16) below.

Proof of (11): From (9), it is easy to see that $\sqrt{5}\alpha^m = L_m \alpha + L_{m-1}$. Then

$$(\sqrt{5})^t \alpha^{tm} = \sum_{i=0}^{t} \binom{t}{i} L_{m-1}^{t-i} L_m^i \alpha^i.$$

For the same reason, we have

$$(-\sqrt{5})^t \beta^{tm} = \sum_{i=0}^{t} \binom{t}{i} L_{m-1}^{t-i} L_m^i \beta^i.$$

If $2 \nmid t$, then, by using (9), we get

$$5^{(t-1)/2} L_{tm} / L_m = \sum_{i=1}^{t} \binom{t}{i} L_{m-1}^{t-i} L_m^{i-1} F_i = \sum_{i=1}^{t} h_i. \tag{2.1}$$

Let $t = p^k a$. If $i \geq p^{k+1}$, then $p^{k+1} \mid L_m^{i-1}$ since $p \mid L_m$, whence $p^{k+1} \mid h_i$. If $2 \leq i \leq p^{k+1}$, let $i = rp^s$ $(p \nmid r,\ s \leq k)$, then

$$p^{k-s} \left| \binom{p^k a}{p^s r} = \binom{t}{i} \right. \text{ (see [7], Th. 2.1),}$$

whence $p^{k-s+i-1} \mid h_i$. Since $p \geq 3$, we have $i \geq s+2$, so $k - s + i - 1 \geq k+1$. Hence, $p^{k+1} \mid h_i$ for $i \geq 2$. Now $h_1 = tL_{m-1}^{t-1}$. Suppose that $p \mid L_{m-1}$, then $p \mid L_m$ and the recurrence $L_{n+1} = L_n + L_{n-1}$ implies $p \mid L_1 = 1$. This is impossible. Hence $p \nmid L_{m-1}$, whence $O_p(h_1) = O_p(t) = k$. Summarizing the above, we have that $p^k \parallel \sum_{i=1}^{t} h_i$. From $\{L_n \pmod 5\}_0^{+\infty} = \{2, 1, 3, 4, 2, 1, \ldots\}$, we observe that $5 \nmid L_m$, thus $p \neq 5$. Then, (11) follows from (2.1). □

Proof of (16): In (15), take $n = k + t$. Then we get $L_{2k+t} \equiv (-1)^{k-1} L_t \pmod{L_k}$. This means that (16) holds for $m = 1$. Assume that (16) holds for m. In (15), taking $n = (2m+1)k + t$, we get $L_{2(m+1)k+t} \equiv (-1)^{k-1} L_{2mk+t} \pmod{L_k}$. By the induction hypothesis, we have

$$L_{(2m+1)k+t} \equiv (-1)^{k-1}(-1)^{m(k-1)} L_t = (-1)^{(m+1)(k-1)} L_t \pmod{L_k},$$

thus (16) is proved. □

Note: (1) through (16) can also be found in [10] which was published in Chinese.

3. THE MAIN RESULT AND ITS PROOF

In the following discussion, we always assume $n, x > 0$.

Theorem: Let p be an odd prime, and $L_n = px^2$, then $n = y(p)$. Furthermore, let $2^r \| y(p)$.

(a) If $r = 0$, then $p = \pm 1 \pmod 5$ and $y(p)$ is prime except $y(p) = 9$ for $p = 19$.

(b) If $r = 1$, then $(p, n, x) = (3, 2, 1)$.

(c) If $r \geq 2$, then $p \equiv 7$ or $23 \pmod{40}$ and either $y(p) = 2^r$ or $y(p) = 2^r q$, where q is an odd prime satisfying $L_2^r = q\square$.

Clearly, the theorem is a considerable improvement of both Theorem 9 and Theorem 11 in [5]. To prove the theorem we need the following lemmas.

Lemma 1: Let p be an odd prime and let $L_n = px^2$. Then $3 \nmid n$ except $n = 9$ for $p = 19$, and so $2 \nmid x$ for $p \neq 19$ (see [5], Th. 3 and Th. 4).

Lemma 2: Let p be prime, $t \equiv \pm 1 \pmod 6$, and $p \equiv \pm L_{5t} \pmod 8$. Then $p \equiv \mp L_t \pmod 4$ and $(2/p)(2/L_t) = -1$.

Proof: If $t \equiv \pm 1 \pmod{12}$, then $5t \equiv \pm 5 \pmod{12}$, whence (13) implies $L_t \equiv \pm 1 \pmod 8$ and $L_{5t} \equiv \pm 3 \pmod 8$. Hence, the lemma holds. If $t \equiv \pm 5 \pmod{12}$, the lemma is proved in the same way. $\square$

Lemma 3: Let p be prime, $n = (12s \pm 1)t$, $s > 0$, $t \equiv \pm 1 \pmod 6$, and $p \mid L_t$. Then $L_n \neq px^2$.

Proof: Suppose $L_n = px^2$. Then, from (13) and (5), we have $L_n \equiv L_{\pm t} \equiv \pm L_t \pmod 8$. (12) implies $2 \nmid L_n$, $2 \nmid L_t$, so $2 \nmid x$. Thus,

$$p \equiv L_n \equiv \pm L_t \pmod 8. \tag{3.1}$$

Rewrite $n = 2 \cdot 3^a \cdot k \pm t$, where $k \equiv \pm 2 \pmod 6$. From (16), it follows that

$$px^2 = L_{2 \cdot 3^a \cdot k \pm t} \equiv -L_{\pm t} = \mp L_t \pmod{L_k} \tag{3.2}$$

It is easy to see that $k = 2ht$. (16) implies $L_k = L_{2ht+0} \equiv L_0 = 2 \pmod{L_t}$. This and $2 \nmid L_t$ imply $(L_k, L_t) = 1$. Since $p \mid L_t$, we have $L_k \equiv 2 \pmod p$ and $(L_k, p) = 1$. (13) implies $L_k \equiv -1 \pmod 4$. From (3.1), we have

$$\begin{aligned}(p(\mp L_t)/L_k) &= \mp(p/L_k)(L_t/L_k) = (\mp)(\pm)(L_k/p)(L_k/L_t)\\ &= -(L_k/p)(L_k/L_t) = -(2/p)(2/L_t) = -1.\end{aligned}$$

This contradicts (3.2). Hence, $L_n \neq px^2$. $\square$

Lemma 4: Let p be prime, $n = (12s \pm 5)t$, $t \equiv \pm 1 \pmod 6$, and $p \mid L_t$. Then $L_n \neq px^2$.

Proof: Suppose $L_n = px^2$. For the same reason as in the proof of Lemma 3, we have

$$p \equiv L_n \equiv \pm L_{5t} \pmod 8. \tag{3.3}$$

Rewrite $n = 2(6s \pm 2)t \pm t = 2k \pm t$. Then (16) implies

$$px^2 = L_{2k \pm t} \equiv -L_{\pm t} = \mp L_t \pmod{L_k}. \tag{3.4}$$

For the same reason as above, $L_k \equiv 2 \pmod{L_t}$ and $L_k \equiv 2 \pmod{p}$, $(L_k, L_t) = (L_k, p) = 1$, and $L_k \equiv -1 \pmod{4}$. Thus, from Lemma 2, we have

$$(p(\mp L_t)/L_k) = \mp(p/L_k)(L_t/L_k) = (\mp)(\mp)(L_k/p)(L_k/L_t) = (2/p)(2/L_t) = -1.$$

This contradicts (3.4). Hence, $L_n \neq px^2$. □

Lemma 5: Let p be prime, $n = (12s \pm 5)t$, $t = 2^r d$, $r \geq 2$, $d \equiv \pm 1 \pmod{6}$, and $p \mid L_t$. Then $L_n \neq px^2$.

Proof: Suppose $L_n = px^2$. Since $2^r = 4 \cdot 2^{r-2} \equiv 4(-1)^{r-2} = \pm 4 \pmod{12}$ for $r \geq 2$, we have $n \equiv \pm 5t \equiv \mp t \equiv \pm 4$ or $\mp 4 \pmod{12}$. (13) implies $L_n \equiv L_t \equiv -1 \pmod{8}$, and so $L_n = px^2$ implies $p \equiv -1 \pmod{8}$. Let $3s \pm 1 = 2^a m$, $2 \nmid m$. Then $n = 2m \cdot 2^{a+1} t \pm t = 2mk \pm t$. (16) implies

$$px^2 = L_n \equiv -L_{\pm t} = -L_t \pmod{L_k}. \tag{3.5}$$

Again, (16) implies $L_k = L_{2 \cdot 2^a t + 0} \equiv (-1)^{2^a (t-1)} L_0 = \pm 2 \pmod{L_t}$, and so $L_k \equiv \pm 2 \pmod{p}$. For the same reason as given above, $L_k \equiv -1 \pmod{8}$ and $(L_k, L_t) = (L_k, p) = 1$. Thus,

$$(p(-L_t)/L_k) = -(p/L_k)(L_t/L_k) = -(-1)(L_k/p)(-1)(L_k/L_t) = -(\pm 2/p)(\pm 2/L_t) = -1$$

This contradicts (3.5). Hence, $L_n \neq px^2$. □

Lemma 6: Let p be prime, $n = (12s \pm 1)t$, $t = 2^r d$, $s > 0$, $r \geq 2$, $d \equiv \pm 1 \pmod{6}$, and $p \mid L_t$. Then $L_n \neq px^2$.

Proof: Suppose $L_n = px^2$. Let $3s = 2^a m$, $2 \nmid m$. Then $n = 2 \cdot m \cdot 2^{a+1} t \pm t = 2mk \pm t$. The proof is completed in the same way as the proof of Lemma 5. □

Lemma 7: Let p be an odd prime, and $L_n = px^2$. Then $n = y(p)$.

Proof: From (1.2), we know that the lemma holds for $p = 19$. Now we assume that $p \neq 19$. Then Lemma 1 implies $3 \nmid n$ and (2) implies $n = mt$, where $t = y(p)$ and $2 \nmid m$. Therefore, $m \equiv \pm 1 \pmod{6}$. If $m > 1$, then $m = 12s \pm 1$ or $m = 12s \pm 5$. Let $t = 2^r d$, $r \geq 0$, $d \equiv \pm 1 \pmod{6}$. When $r = 0$, the conditions of Lemma 3 and Lemma 4 are fulfilled. When $r \geq 2$, the conditions of Lemma 5 and Lemma 6 are fulfilled. These all lead to $L_n \neq px^2$. Hence, $m = 1$, and so $n = y(p)$. When $r = 1$, (12) implies $3 \| L_n$, whence $L_n = px^2$ iff $(p, n, x) = (3, 2, 1)$. Obviously, $2 = y(3)$, and we are done. □

Lemma 8: Let p be prime, $p > 3$, and $t = y(p) \equiv \pm 1 \pmod{6}$. If $L_t = px^2$, then $p \equiv \pm 1 \pmod{5}$ and t is prime.

Proof: $L_t = px^2$, $2 \nmid t$, and (4) imply $5F_t^2 \equiv 4 \pmod{p}$. This implies $(5/p) = 1$, and so $p \equiv \pm 1 \pmod{5}$. Suppose that t is a composite. Then $t = kq$, where q is a prime greater than 3, and $k > 1$. (14) implies $L_q \neq \square$. Since $2 \nmid L_q$, there exists an odd prime r such that $r \mid L_q$ and $2 \nmid O_r(L_q)$. From (2), it is clear that

$$y(r) = q. \tag{3.6}$$

If $r = q$, then $z(q) = 2 \cdot y(q) = 2 \cdot y(r) = 2q$. (6) implies $2q \mid (q - (5/q))$. This is impossible. Hence, $r \neq q$. If $r \nmid k$, then (11) implies $O_r(L_{kq}) = O_r(L_q)$. Therefore, $2 \nmid O_r(L_t)$. This means that

$L_t = px^2$ implies $r = p$. Thus, from (3.6), we get $y(p) = q < kq = y(p)$. This is a contradiction! Hence, $r \mid k$. Let $k = rh$, then $L_{qh} \cdot L_{qrh} / L_{qh} = px^2$. Let $(L_{qh}, L_{qrh} / L_{qh}) = d$. (8) implies $d \mid r$, (2) implies $r \mid L_{qh}$, and so (11) implies $O_r(L_{qrh} / L_{qh}) = 1$. Thus, $r \mid d$; hence, $d = r$. Then we have either (i) $L_{qh} = ru^2$ or (ii) $L_{qh} = rpu^2$. (ii) contradicts the fact that $y(p) = qrh$, since $qh < y(p)$. If (i) holds, then, from Lemma 7, we have $y(r) = qh$. Comparing it with (3.6), we get $h = 1$ and $t = qr$.

For the same reason, there exists an odd prime s such that $s \mid L_r$ and $2 \nmid O_s(L_r)$. And we also have

$$y(s) = r \tag{3.7}$$

and $s \neq r$. Again, for the same reason as $r \mid k$, we have $s \mid q$, whence $s = q$. Thus, (3.7) becomes

$$y(q) = r. \tag{3.8}$$

Equations (3.6) and (3.8) imply that $z(r) = 2q$ and $z(q) = 2r$. Thus, (6) implies $2q \mid (r - (5/r))$ and $2r \mid (q - (5/q))$. Clearly, this is impossible. Hence, t is prime. □

Lemma 9: Let p be prime, $p > 3$, $2^r \| t = y(p)$, and $r \geq 2$. If $L_t = px^2$, then $p \equiv 7$ or 23 (mod 40) and either $t = 2^r$ or $t = 2^r q$, where q is a prime satisfying $l_{2^r} = q\square$.

Proof: From the proof of Lemma 7, we know that $t = 2^r d$, $d \equiv \pm 1$ (mod 6). From the proof of Lemma 5, we know that $p \equiv -1$ (mod 8). $L_t = px^2$, $2 \mid t$, and (4) imply $5F_t^2 \equiv -4$ (mod p), and so $(-5/p) = -(5/p) = 1$. This leads us to $p \equiv \pm 2$ (mod 5). Summarizing the above, we obtain $p = 7$ or 23 (mod 40).

From the proof of Lemma 8, we know that there exists an odd prime q such that $q \mid l_{2^r}$ and $2 \nmid O_q(l_{2^r})$. From (2), it is clear that $y(q) = 2^r$. If $d \neq 1$, then, for the same reason as in the proof of Lemma 8, we have $q \mid d$. Let $d = qh$, then $l_{2^r h} \cdot l_{2^r qh} / l_{2^r h} = px^2$. Now (8), (2), and (11) imply $(l_{2^r h}, l_{2^r qh} / l_{2^r h}) = q$, so we get either (i) $l_{2^r h} = qu^2$ or (ii) $l_{2^r h} = qpu^2$. (ii) contradicts the fact that $y(p) = 2^r qh$. If (i) holds, then Lemma 7 implies $y(q) = 2^r h$. Comparing this with $y(q) = 2^r$, we get $h = 1$ and $l_{2^r} = qu^2$. Thus, the lemma is proved. □

Proof of the Theorem: The Theorem follows from Lemmas 7 through 9. □

4. AN ALGORITHM AND EXAMPLES

From the Theorem in Section 3 and using (1) and (6), we can give the following algorithm.

Algorithm: Let p be a given odd prime, $p \neq 3$, 19.

I. If $p \not\equiv \pm 1$ (mod 5) and $p \not\equiv 7, 23$ (mod 40), then (1.1) has no solution.

II. For $p \equiv \pm 1$ (mod 5), let $A = \{q_1, \ldots, q_k\}$ be the set of distinct prime factors greater than 3 of $p - 1$.

- ***(a)*** If A is empty, then (1.1) has no solution.
- ***(b)*** For $i = 1, \ldots, k$, calculate L_{q_i} (mod p).
- ***(c)*** If there exists an $i = j$ such that $L_{q_j} \equiv 0$ (mod p), then calculate L_{q_j}. If $L_{q_j} = pu^2$ $(u > 0)$, then $(n, x) = (q_j, u)$ is the solution of (1.1), otherwise (1.1) has no solution.
- ***(d)*** If, for all $i = 1, \ldots, k$, $L_{q_i} \not\equiv 0$ (mod p), then (1.1) has no solution.

III. For $p \equiv 7$ or 23 (mod 40), let $2^a \| (p+1)$ and $A = \{q_1, \ldots, q_k\}$ be the set of distinct prime factors greater than 3 of $p+1$.

- ***(a)*** For $s = 2, 3, \ldots, a-1$, calculate l_{2^s} (mod p).
- ***(b)*** If there exists an $s = r$ such that $l_{2^r} \equiv 0$ (mod p), then calculate l_{2^r}. If $l_{2^r} = pu^2$ $(u > 0)$, then $(n, x) = (2^r, u)$ is the solution of (1.1), otherwise (1.1) has no solution.
- ***(c)*** If, for all $s = 2, 3, \ldots, a-1$, $l_{2^r} \not\equiv 0$ (mod p), then $s = 2, 3, \ldots, a-1$ and, for every q_i in A such that $q_i \equiv 7$ or 23 (mod 40), calculate l_{2^s} (mod q_i). Let B be the set of such (s, i)'s that $l_{2^s} = q_i \square$.
- ***(d)*** If B is empty, then (1.1) has no solution.
- ***(e)*** For each (s, i) in B, calculate $L_{2^s q_i}$ (mod p).
- ***(f)*** If there exists an $(s, i) = (r, j)$ in B such that $L_{2^r q_j} \equiv 0$ (mod p), then calculate $L_{2^r q_j}$. If $L_{2^r q_j} \equiv pu^2$ $(u > 0)$, then $(n, x) = (2^r q_j, u)$ is a solution of (1.1), otherwise (1.1) has no solution.
- ***(g)*** If, for all (s, i) in B, $L_{2^s q_i} \not\equiv 0$ (mod p), then (1.1) has no solution.

Remark: For calculating L_m (mod p) and L_m, there is an algorithm that determines the result after $[\log_2 m]$ recursive calculations (see [2]).

Example 1: $p = 63443 \not\equiv \pm 1$ (mod 5) and $p \not\equiv 7, 23$ (mod 40). Hence, (1.1) has no solution.

Example 2: $p = 19489 \equiv -1$ (mod 5), $p-1 = 2^5 \times 3 \times 7 \times 29$, $A = \{7, 29\}$. By calculating, we get $L_{29} \equiv 0$ (mod p). But $L_{29} = 59p \neq px^2$, so (1.1) has no solution.

Example 3: $p = 4481 \equiv 1$ (mod 5), $p-1 = 2^9 \times 5 \times 7$, $A = \{5, 7\}$. Since $L_5, L_7 \not\equiv 0$ (mod p), (1.1) has no solution.

Example 4: $p = 9349 \equiv -1$ (mod 5), $p-1 = 2^2 \times 3 \times 19 \times 41$, $A = \{19, 41\}$. By calculating, we get $L_{19} \equiv 0$ (mod p) and $L_{19} = p$. Hence, $(n, x) = (19, 1)$ is the solution of (1.1).

Example 5: $p = 1103 \equiv 23$ (mod 40), $p+1 = 2^4 \times 3 \times 23$, $A = \{23\}$. Since $l_{2^2}, l_{2^3} \not\equiv 0$ (mod p) and $l_{2^2}, l_{2^3} \not\equiv 0$ (mod 23), (1.1) has no solution.

Example 6: $p = 1097 \equiv 7$ (mod 40), $p+1 = 2^6 \times 17$, $A = \{17\}$. Since $l_{2^5} \equiv 0$ (mod p) but $l_{2^5} = 1087 \times 4481 \neq px^2$, (1.1) has no solution.

Example 7: $p = 3607 \equiv 7$ (mod 40), $p+1 = 2^3 \times 11 \times 41$, $A = \{11, 41\}$. Since $l_{2^2}, l_{2^3} \not\equiv 0$ (mod p) and 11 and $41 \not\equiv 7, 23$ (mod 40), (1.1) has no solution.

Example 8: $p = 14503 \equiv 23$ (mod 40), $p+1 = 2^3 \times 7^2 \times 37$, $A = 7, 37\}$. By the Algorithm, we get $l_{2^2} = 7$ and $l_{2^2 \cdot 7} = p \cdot 7^2$. Hence, $(n, x) = (28, 7)$ is a solution of (1.1).

Remark: In II(c), III(b), and III(d) of the Algorithm, it is unnecessary to calculate L_t, where $t = q_j$, 2^r, or $2^r q_j$ for most of the t's. The reason is that, if pL_t is a quadratic nonresidue (mod m), where m is some prime, then $L_t \neq px^2$. For example, by using the Algorithm and making m

run through the first 20 odd primes, and by means of a computer, we have verified the following proposition.

Proposition: Let p be prime, $10^3 < p < 6\times 10^4$. Then (1.1) holds iff

$$(p, n, x) = (2207, 16, 1),\ (3571, 17, 1),\ (9349, 19, 1),\ (14503, 28, 7). \tag{4.1}$$

Extensive numeric results inspire the following conjecture.

Conjecture: Let p be an odd prime and $p \neq 3$, 19. Then $L_n = px^2$ iff one of the following conditions holds:

(a) $p \equiv \pm 1 \pmod 5$, $y(p)$ is prime, and $L_{y(p)} = p$, so $(n, x) = (y(p), 1)$;

(b) $p \equiv 7$ or 23 (mod 40), $y(p) = 2^r$, and $L_{y(p)} = p$, so $(n, x) = (y(p), 1)$;

(c) $p \equiv 7$ or 23 (mod 40), $y(p) = 2^r q$, where q is a prime greater than 3 satisfying $l_{2^r} = q$ and $L_{y(p)} = pq^2$, so $(n, x) = (y(p), q)$.

We point out that the conjecture would hold if we could show $p^2 \nmid L_{y(p)}$ for all odd prime p. At this time, it remains unknown whether there exists an odd prime p such that $p^2 \mid L_{y(p)}$.

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THE EIGHTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Herta T. Freitag

Rochester Institute of Technology, which has been internationally respected as a world leader in career-oriented and professional education since 1829, was the inspired choice of setting for our Eighth Conference. We are indeed grateful to RIT, as well as to the Fibonacci Association, for sponsoring our conference. A special word of thanks goes to Dr. Albert Simone, President of RIT, Wiley R. McKinzie, Dean of Applied Science and Technology, and Dr. Walter A. Wolf, Chair of the Computer Science Department.

The participants came from eighteen different countries: 28 from the USA, three each from Australia, Canada, and Japan, two each from Germany and Italy, and one participant from each of Austria, Belarus, Cyprus, Denmark, England, France, Greece, Iceland, New Zealand, Poland, Romania, and Scotland. Five of the presenters were women. There were four mathematicians who have attended all of the eight conferences, many who have attended several, most happily, several who have attended for the first time. The magnetism of Fibonacci-type mathematics drew even some who did not present a paper. The ages ranged from a few who were in their twenties to one who will soon earn the title of nonagenarian.

There have been two major changes since our last conference, both involving Jerry Bergum. After eighteen years as editor of *The Fibonacci Quarterly*, he has handed over the baton to Curtis Cooper. We wish Curtis all success and fulfillment in his new role. At the same time, Jerry has been succeeded as conference organizer by Fred Howard, who also has our best wishes. Fred is already widely respected for his wisdom and kindliness. We hope that Jerry will attend many more of our conferences. We deeply appreciate all he has done. He has been in a very real sense the heart and soul of our Association. We would also like to renew our thanks to Calvin Long for his continuing work as our President. Our discussions have been illuminated by his fine mathematical insights.

This is a conference where all of the talks are attended by almost all of the participants, who appreciated the diversity of topics covered and the remarkable level of quality of the papers. All of the presentations displayed the high enthusiasm of the speakers for their studies. And they all showed enjoyment over the opportunity of sharing their ideas with each others.

As well as working hard (51 talks in five days!), the group also enjoyed some delightful social events, the highlight being a "cook-out" at the Anderson's' home with Peter and Jane our gracious hosts. Through his wit and warmth, Peter immediately set the stage for a conference where we not only saw a fellow mathematician in each other (which would already be enjoyable) but, moreover, a friend. We are deeply grateful to Peter and his helpers for all their hard work in preparing those delightful outings for us and the extra care they took in looking after us in Rochester.

The friendships created by this sequence of Fibonacci conferences has produced many worthwhile results in the area of mathematics. At this conference we enjoyed renewing old friendships and beginning new ones. The "Goddess Mathesis" (to use Howard Eves' term) looks favorably on these friendships.

Finally, we had to part. But now, we greatly look forward to meeting again in two years in Luxembourg in 2000.

THE NUMBER OF REPRESENTATIONS OF N USING DISTINCT FIBONACCI NUMBERS, COUNTED BY RECURSIVE FORMULAS

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0. INTRODUCTION

Let $R(N)$ be the number of representations of the nonnegative integer N as a sum of distinct Fibonacci numbers. For $N=F_n-1$, $n\geq 1$, the Zeckendorf representation, in which no two consecutive Fibonacci numbers appear in the sum, is the only possible representation, and $R(F_n-1)=1$, as proved by Carlitz [3] and Klarner [4]. The sequences $\{b_n-1\}$, $b_{n+1}=b_n+b_{n-1}$, arise as a generalization, having the property that $R(b_n-1)=R(b_{n+1}-1)=k$ for all sufficiently large n (see [1] and [4]). The generation of the specialized and related sequence 1, 3, 8, 16, 24, ..., A_n, whose n^{th} term is the least N such that $n=R(N)$, spurred efforts to find recursive relationships for the values $R(N)$ and ways to compute $R(N)$ for large values of N. Some authors have used $T(N)$ and some $R(N)$ in counting representations; we will use $R(N)$ for the number of ways to represent N as a sum of distinct Fibonacci numbers (without F_1) and $T(N)$ for the number of representations if both F_1 and F_2 are used. In our notation, Carlitz and Klarner both give $R(F_n)=[n/2]$, $n\geq 2$, where $[x]$ is the greatest integer in x. Since $T(N)=R(N)+R(N-1)$, we have concentrated on formulas for $R(N)$.

Earlier authors have used generating functions and combinatorics to develop and prove representation theorems. In this paper we concentrate on properties of the integers whose representations are being counted. We prove Conjectures 1, 2, and 3 from [1] as well as writing formulas for $R(MF_k)$ and $R(ML_k)$, $M\geq 1$, and solving $R(N)=mR(N-1)-q$ for integers M, m, and q.

1. THE SYMMETRIC PROPERTY AND A BASIC RECURSION

The most obvious property in a table of $R(N)$ is the palindromic subsequences it contains, beginning and ending with 1, for N in the interval $F_n-1\leq N\leq F_{n+1}-1$; i.e., when $0\leq M\leq F_{n-1}$, $n\geq 3$,

$$R(F_{n+1}-1-M)=R(F_n-1+M). \tag{1}$$

Since these values $R(N)$ are symmetric about the center of each palindromic segment, we only have to compute the values of the first half of the interval. Symmetric property (1) is a variation of Theorem 1, whose results appear in Klarner [5], as specialized for the Fibonacci sequence $\{F_{n+1}\}$.

Theorem 1:

$$R(F_{n+1}-2-M)=R(F_n+M),\ 0\leq M\leq F_{n-1},\ n\geq 3.$$

Values for $R(N)$ for $0 \le N \le 60$

N	$R(N)$	N	$R(N)$	N	$R(N)$	N	$R(N)$
0	1						
1	1	16	4	31	3	46	2
2	1	17	2	32	4	47	5
3	2	18	3	33	1	48	5
4	1	19	3	34	4	49	3
5	2	20	1	35	4	50	6
6	2	21	4	36	3	51	3
7	1	22	3	37	6	52	4
8	3	23	3	38	3	53	4
9	2	24	5	39	5	54	1
10	2	25	2	40	5	55	5
11	3	26	4	41	2	56	4
12	1	27	4	42	6	57	4
13	3	28	2	43	4	58	7
14	3	29	5	44	4	59	3
15	2	30	3	45	6	60	6

It is a simple matter to compute a table for $R(N)$ from generating functions for small N, but as N gets larger, the computer's memory will eventually be exceeded. We have calculated $R(N)$ for $1 \le N \le 257{,}115$ and have capabilities of calculating individual values for $R(N)$ for very large N; for example, $R(3{,}000{,}000{,}000) = 6165$. We have listed $\{A_n\}$ for $1 \le n \le 330$. But to study the mysteries of $\{A_n\}$ or to compute $R(N)$ for large N by hand, we need some recursive relationships. Klarner [5] proved Theorem 2 for generalized Fibonacci numbers.

Theorem 2 (Basic Recursion Formula): If $F_n \le M \le F_{n+1} - 2$, then

$$R(M) = R(F_{n+1} - 2 - M) + R(M - F_n),\ n \ge 4. \tag{2}$$

Lemma 1: If $F_n \le M \le F_{n+1} - 2$, then $R(M - F_n)$ is the number of representations of M using F_n, while the number of representations of M using F_{n-1} is $R(F_{n+1} - 2 - M)$.

Proof: The largest Fibonacci number in M is F_n. $R(M)$ is the sum of the number of representations of M that use F_n and the number of those that use F_{n-1}. Since $M \le F_{n+1} - 2$, no representations of M use both F_n and F_{n-1}; else $M > F_{n+1}$. There are no representations of M that use neither F_n nor F_{n-1}, since $F_n - 2 = F_{n-2} + F_{n-3} + \cdots + F_3 + F_2 < M$. Note that $M = F_n + M_1$, where the largest possible Fibonacci number in M_1 is F_{n-2}; else M could contain F_{n+1}. The number of representations of M that use F_n is $R(M_1) = R(M - F_n)$ since F_n is added to each possible representation of M_1 to make a representation of M using F_n. To list representations of M using F_{n-1}, if we write $M = F_{n-1} + F_{n-2} + M_1$ and then list representations of M_1, there can be a repetition of terms, such as F_{n-2} appearing twice, so we need sums using disjoint sets of Fibonacci numbers. Representations of $(F_{n+1} - 2 - M) = (F_{n-1} + F_{n-2} + \cdots + F_3 + F_2) - M$ will use a set of Fibonacci numbers disjoint from those selected to represent M. Thus, $R(F_{n+1} - 2 - M)$ must give the number of representations of M that use F_{n-1} by examining Theorem 2. □

In counting by hand, $R(M) = R(M - F_n) + R(M - F_{n-1})$ if $M - F_{n-1} < F_{n-1}$. For example, $23 = 21 + 2 = 13 + 10$, and $R(23) = R(2) + R(10)$. If $M - F_{n-1} > F_{n-1}$, an adjustment must be made;

$30 = 21+9 = 13+17 = 13+(13+4)$, and $R(30) = R(9)+R(17)-R(4)$. Lemma 2 makes this counting correction. We take $R(0)=1$ and $R(K)=0$ when $K<0$ in Lemmas 2 through 6, and $[x]$ denotes the greatest integer in x.

Lemma 2: If $F_n \le M \le F_{n+1}-2$, then

$$\begin{aligned} R(M) &= R(M-F_n)+R(M-F_{n-1})-R(M-2F_{n-1}); \\ R(F_{n+1}-2-M) &= R(M-F_{n-1})-R(M-2F_{n-1}). \end{aligned} \tag{3}$$

Proof: $R(M)$ is the number of representations of M using F_n plus the number of representations of M using F_{n-1} corrected for the number of representations of $(M-F_{n-1})$ using F_{n-1}, is any exist. A second way to write the representations of M that use F_{n-1} is to write $M = F_{n-1} + (M-F_{n-1})$ and observe that the number of representations that use F_{n-1} is $R(M-F_{n-1})$ if F_{n-1} is not used in representing $(M-F_{n-1})$. If $M > 2F_{n-1}$, $R(M-2F_{n-1})$ is the number of representations of $(M-F_{n-1})$ using F_{n-1}, since $M-F_{n-1} = F_{n-1}+((M-F_{n-1})-F_{n-1})$. Thus, the representations of M using F_{n-1} are counted by $[R(M-F_{n-1})-R(M-2F_{n-1})]$, which count appeared in Lemma 1 as $R(F_{n+1}-2-M)$. □

Lemma 3:

$$R(F_n+K) = R(F_{n-1}-2-K)+R(K),\ \ 0 \le K \le F_{n-1}-2. \tag{4}$$

Lemma 2 is another form of Theorem 2, while Lemma 3 results when $M = K+F_n$ in (2), and is useful in computation. For example, let $K=24$, $R(K)=5$; since $0 \le K \le F_{n-1}-2$, take $n \ge 10$.

$$\begin{aligned} n=12:&\quad R(24+144) = R(87-24)+R(24) = 8+5; \quad R(168)=13, \\ n=13:&\quad R(24+233) = R(142-24)+R(24) = 10+5; \quad R(257)=15, \\ n=14:&\quad R(24+377) = R(231-24)+R(24) = 13+5; \quad R(401)=18, \\ n=16:&\quad R(24+987) = R(608-24)+R(24) = 18+5; \quad R(1011)=23, \end{aligned}$$

where we recognize 24, 168, 257, 401, and 1011 as members of our specialized sequence $\{A_n\}$.

Lemma 4:

$$R(M) = R(M-F_n)+R(M-F_{n-1}),\ \ F_n \le M \le F_n+F_{n-3}-1.$$

Proof: Because $2F_{n-1} = F_n + F_{n-3}$, $R(M-2F_{n-1}) = 0$ in Lemma 2 throughout the interval chosen. □

Lemma 5: $R(N)$ for the interval $F_n \le N \le F_{n+1}-1$ is given by:

$$\begin{aligned} R(F_n+K) &= R(F_{n-2}+K)+R(K), && 0 \le K \le F_{n-3}-1; \\ R(F_n+K) &= 2R(K), && F_{n-3} \le K \le F_{n-2}-1; \\ R(F_n+K) &= R(F_{n+1}-2-K), && F_{n-2} \le K \le F_{n-1}-1. \end{aligned} \tag{5}$$

Proof: Let $M = F_n + K$ in Lemma 4 and use Theorem 1 to write the first and last F_{n-3} values of $R(N)$. Let $F_{n-3}+p = K$ in Lemma 3, followed by application of Theorem 1 since $0 \le p \le F_{n-4}$:

$$\begin{aligned} R(F_n+F_{n-3}+p) &= R(F_{n-1}-2-(F_{n-3}+p))+R(F_{n-3}+p) \\ &= R(F_{n-2}-2-p)+R(F_{n-3}+p) \\ &= R(F_{n-3}+p)+R(F_{n-3}+p). \end{aligned}$$

Thus, $R(F_n+K) = 2R(K)$ when $F_{n-3} \le K \le F_{n-2}-1$. □

Lemma 6:

$$R(F_n+K)=R(F_{n-2}+K)+R(K)-R(K-F_{n-3}),\ 0\le K\le F_{n-1}. \tag{6}$$

Proof: For $0\le K\le F_{n-1}-2$, take $M=F_n+K$ in Lemma 2, so that $M-2F_{n-1}=(M-F_n)+(F_n-2F_{n-1})=K-F_{n-3}$. Then let $K=F_{n-1}-1$ in the expression above, using $R(F_n-1)=1$. Finally, take $K=F_{n-1}$, using $R(F_{n+2})=[(n+2)/2]=R(F_n)+1$ from [3] and [4]. □

2. SPECIAL VALUES FOR $R(b_n-1)$ AND $R(b_n)$

Recursive sequences $\{b_n-1\}$, $b_{n+1}=b_n+b_{n-1}$, have $R(b_n-1)=R(b_{n+1}-1)=k$ for n sufficiently large (see [1] and [4]). We can write sequences for which $R(N-1)=k$, a given constant, as indicated in the following example. Say $k=5$ is given. Find a particular value, i.e., $R(24)=5$. Write $24+1=25=21+3+1$ in Zeckendorf form, or

$$R(24)=R(F_8+F_4+F_1-1)=R(F_8+F_4+F_2-1)=5.$$

These are the first terms, when $F_n=1$, in sequences we seek. Thus,

$$R(F_{n+7}+F_{n+3}+F_n-1)=5=R(F_{n+7}+F_{n+3}+F_{n+1}-1),\ n\ge 1.$$

The symmetric property gives $R(F_{n+7}-1+M)=R(F_{n+8}-1-M)=5$ for $M=F_{n+3}+F_n$, so that we can write

$$R(F_{n+8}-1-(F_{n+3}+F_n))=R(F_{n+7}+F_{n+5}+F_{n+1}-1)=5,\ n\ge 1.$$

Since $R(F_{10})=R(F_{10}+1-1)=5$, again using the symmetric property,

$$R(F_{n+9}+F_n-1)=R(F_{n+9}+F_{n+1}-1)=5,\quad n\ge 1,$$
$$R(F_{n+10}-F_n-1)=R(F_{n+10}-F_{n+1}-1)=5,\quad n\ge 1.$$

Since $R(F_{2k})=R(F_{2k+1})=k$, we can derive in a similar way, for $n\ge 1$:

$$\begin{aligned}&R(F_{2k-1+n}+F_n-1)=k=R(F_{2k-1+n}+F_{n+1}-1);\\ &R(F_{2k+n}-F_n-1)=k=R(F_{2k+n}-F_{n+1}-1),\ \text{for } n\ge 1.\end{aligned} \tag{7}$$

For a given value of k, there are many infinite sequences such that $R(b_n-1)=k$. All ways of writing infinite sequences such that $R(b_n-1)=k$, for $k=1,2,3$, were given by Klarner [4] as

$$\begin{aligned}R(F_n-1)&=R(F_{n+1}-1)&&=1;\\ R(F_{n+3}+F_n-1)&=R(F_{n+3}+F_{n+1}-1)&&=2;\\ R(F_{n+5}+F_n-1)&=R(F_{n+5}+F_{n+1}-1)&&=3;\\ R(F_{n+6}-F_n-1)&=R(F_{n+6}-F_{n+1}-1)&&=3.\end{aligned}$$

Some useful equivalent statements are

$$\begin{aligned}R(2F_{n+2}-1)&=R(L_{n+2}-1)&&=2;\\ R(3F_{n+3}-1)&=R(4F_{n+3}-1)&&=3;\\ R(L_{n+1}+F_n-1)&=R(L_n+F_{n+1}-1)&&=3.\end{aligned}$$

Lemma 7: Let $\{b_n\}$ be a sequence of natural numbers such that $b_{n+2}=b_{n+1}+b_n$. Then $\{b_n\}$ has the following properties:

(i) $R(b_n-1)=R(b_k-1)$ for all $n\geq k$ if F_k is the smallest Fibonacci number used in the Zeckendorf representation of b_k, $k\geq 2$, or if $\{b_n\}$ has $b_2\geq 2b_1$ and $F_{k-1}<b_2-b_1\leq F_k$.

(ii) $R(b_n-1)=R(b_n-1)R(F_m)-q$, q a constant, $0\leq q\leq R(b_n-1)$, where F_m is the smallest Fibonacci number used in the Zeckendorf representation of b_n, $m\geq 2$;

(iii) $R(b_{n+2})=R(b_n)+R(b_n-1)=T(b_n)$, $n\geq k$, as in (i), where $T(N)$ is the number of representations of N as sums of Fibonacci numbers, where both F_1 and F_2 can be used;

(iv) $R(b_{n+2c})=R(b_n)+cR(b_n-1)=R(b_{n+2c-2})+R(b_n-1)$, $n\geq k$.

Proof: Klarner [4] used the Zeckendorf representation of b_n to prove (i) for n sufficiently large; $n\geq k$ as in the second statement appears in [1]. The proof of (ii) relies on Lemma 5 and mathematical induction. Take $F_n\leq b_n\leq F_{n+1}-1$. Let $b_n=F_n+K$, $0\leq K\leq F_{n-1}-1$. Assume part (ii) holds for all integers $K=F_{n-1}$. If $0\leq K\leq F_{n-3}-1$, Lemma 5 and the inductive hypothesis give

$$\begin{aligned}R(b_n)&=R(K)+R(F_{n-2}+K)\\&=[R(K-1)R(F_m)-q_1]+[R(F_{n-2}+K-1)R(F_m)-q_2]\\&=[R(K-1)+R(F_{n-2}+K-1]R(F_m)-(q_1+q_2)\\&=R(F_n+K-1)R(F_m)-q_3\\&=R(b_n-1)R(F_m)-q_3,\quad 0\leq q_3<R(b_n-1),\end{aligned}$$

since $0\leq q_1+q_2\leq R(K-1)+R(F_{n-2}+K-1)=R(F_n+K-1)=R(b_n-1)$, again using the inductive hypothesis. A proof by induction can be made from each of the other two parts of Lemma 5, extending K to the intervals $F_{n-3}\leq K\leq F_{n-2}-1$, and $F_{n-2}\leq K\leq F_{n-1}-1$, but is omitted here in the interest of brevity.

To prove (iii), using (i) and (ii),

$$\begin{aligned}R(b_{n+2})&=R(b_{n+2}-1)R(F_{m+2})-q=R(b_n-1)(R(F_m)+1)-q\\&=(R(b_n-1)R(F_m)-q)+R(b_n-1)=R(b_n)+R(b_n-1).\end{aligned}$$

Next, take $N=b_n$ and use $T(N)=R(N)+R(N-1)$ as in [4]. Note: The notation is not standardized; the meanings of $R(N)$ and $T(N)$ are reversed in [4] from those used in this paper. Part (iv) follows from $R(F_{n+2c})=R(F_n)+c$, using (ii) to write

$$\begin{aligned}R(b_{n+2c})&=R(b_{n+2c}-1)R(F_{m+2c})-q=R(b_n-1)(R(F_m)+c)-q\\&=(R(b_n-1)R(F_m)-q)+cR(b_n-1)=R(b_n)+cR(b_n-1),\end{aligned}$$

where, also from (iii) and (i),

$$R(b_{n+2c})=R(b_{n+2c-2})+R(b_{n+2c-2}-1)=R(b_{n+2c-2})+R(b_n-1).\ \square$$

3. FORMULAS FOR $R(N)$ BASED ON ZECKENDORF REPRESENTATION

A formula for $R(N)$ for whole sequences $\{b_n\}$, $b_{n+2}=b_{n+1}+b_n$, can be written, or $R(N)$ for large integers N based on the Zeckendorf representation of N, by repeatedly using Theorem 2, Lemmas 2 and 6, and formulas for $R(F_{n+p}+N)$ as developed next. Let the largest Fibonacci number contained in N be F_n; equivalently, F_n is the largest term in the Zeckendorf representation of N, and $F_n\leq N\leq F_{n+1}-2$. To count the number of ways to represent N as sums of distinct

Fibonacci numbers, first find the largest two Fibonacci numbers in N and then apply formulas of the form $R(F_{n+p}+N)$.

Lemma 8: Let $F_n \le N \le F_{n+1}-2$. Then

$$\begin{aligned}R(F_{n+1}+N) &= R(N)+R(N-F_n);\\ R(F_{n+2}+N) &= R(N)+R(F_{n+1}-2-N);\\ R(F_{n+3}+N) &= 2R(N).\end{aligned}$$

Proof: Let $M = N+F_{n+1}$ in Lemma 2, where $F_{n+2} \le M < F_{n+3}-2$. Then

$$\begin{aligned}R(F_{n+1}+N) &= R(F_{n+1}+N-F_{n+2})+R(F_{n+1}+N-F_{n+1})-R(F_{n+1}+N-2F_{n+1})\\ &= R(N-F_n)+R(N)-R(N-F_{n+1}) = R(N-F_n)+R(N),\end{aligned}$$

where $R(N-F_{n+1})=0$ because $N<F_{n+1}$.

Let $M = N+F_{n+3}$ in Lemma 2, where $F_{n+3} \le M < F_{n+4}-2$;

$$\begin{aligned}R(F_{n+3}+N) &= R(F_{n+3}+N-F_{n+3})+R(F_{n+3}+N-F_{n+2})-R(F_{n+3}+N-2F_{n+2})\\ &= R(N)+R(N+F_{n+1})-R(N-F_n)\\ &= R(N)+[R(N-F_n)+R(N)]-R(N-F_n) = 2R(N).\end{aligned}$$

Let $M = N+F_{n+2}$ in Theorem 2, where $F_{n+2} \le M < F_{n+3}-2$;

$$\begin{aligned}R(F_{n+2}+N) &= R(F_{n+3}-2-(F_{n+2}+N))+R((F_{n+2}+N)-F_{n+2}))\\ &= R(F_{n+1}-2-N)+R(N). \ \square\end{aligned}$$

Theorem 3: Let $F_n \le N \le F_{n+1}-2$. Then

$$R(F_{n+2k+1}+N) = (k+1)R(N),\ k \ge 1; \tag{9}$$

$$R(F_{n+2k}+N) = kR(N)+R(F_{n+1}-2-N),\ k \ge 1. \tag{10}$$

Proof: Assume that $R(F_{n+2j+1}+N) = (j+1)R(N)$ holds for $j \le k$; the case $k=1$ was established in Lemma 8. Consider

$$R(F_{n+2(k+1)+1}+N) = R(F_{(n+2k+1)+2}+N),\ n < F_{n+1} < F_{(n+2k+3)-3}.$$

By the first part of Lemma 5,

$$\begin{aligned}R(F_{n+2k+3}+N) &= R(F_{n+2k+1}+N)+R(N)\\ &= (k+1)R(N)+R(N) \ = \ [(k+1)+1]R(N),\end{aligned}$$

establishing the formula for $R(F_{n+2k+1}+N)$ by induction.

The proof of the even case is similar, again taking the case $k=1$ from Lemma 8, and using Lemma 5; therefore, it is omitted here. $\square$

Theorem 3 can be used as a reduction formula to write $R(N)$ for large N. For example,

$$R(1694) = R(F_{17}+97) = 3R(97)+R(144-2-97) = 3(9)+6 = 33,$$

so $R(1694)=33$ since $R(97)=9$ and $R(45)=6$ are known from data. However, Theorem 3 can be written in another form that is even more useful for computation, as given in Corollary 3.1.

Corollary 3.1: Let $F_n \le N \le F_{n+1} - 2$. Then

$$R(F_m + N) = R(F_{m-n+1})R(N) + r, \quad m - n \ge 2, \tag{11}$$

where $r = 0$ if $m-n$ is odd, and $r = R(F_{n+1} - 2 - N)$ if $m-n$ is even.

Proof: The result follows from $R(F_n) = [n/2]$ for $[x]$ the greatest integer in x from [3] and [4]. Let $m-n = 2k+1$, then $m = n+2k+1$ and $[(m-n+1)/2] = k+1$; therefore, $R(F_m + N) = [(m-n+1)/2]R(N)$ by (9). Similarly, let $m-n = 2k$ in (10). □

4. SPECIAL VALUES FOR $R(F_n \pm K)$

We write some special formulas useful in breaking down expressions for $R(N)$ by putting special values into equation (1) and Corollary 3.1. Expressions for $k = 0, 1$, and 2 in Lemma 9 appear in [4]. We also find integers m and q such that $R(M) = mR(M-1) - q$.

Lemma 9–Special values for $R(F_n - 1 \pm k)$: Let $[x]$ be the greatest integer contained in x, and let $0 \le k \le F_{n-1}$. Then $R(F_n - 1 + k) = R(F_{n+1} - 1 - k)$ has the following values, $0 \le k \le 8$.

$$\begin{array}{llllll}
k=0: & R(F_n - 1) & = R(F_{n+1} - 1) & & = 1, & n \ge 2; \\
k=1: & R(F_n) & = R(F_{n+1} - 2) & = R(F_n) & = [n/2], & n \ge 3; \\
k=2: & R(F_n + 1) & = R(F_{n+1} - 3) & = R(F_{n-1}) & = [(n-1)/2], & n \ge 4; \\
k=3: & R(F_n + 2) & = R(F_{n+1} - 4) & = R(F_{n-2}) & = [(n-2)/2], & n \ge 5; \\
k=4: & R(F_n + 3) & = R(F_{n+1} - 5) & & = n-3, & n \ge 6; \\
k=5: & R(F_n + 4) & = R(F_{n+1} - 6) & = R(F_{n-3}) & = [(n-3)/2], & n \ge 6; \\
k=6: & R(F_n + 5) & = R(F_{n+1} - 7) & & = n-4, & n \ge 7; \\
k=7: & R(F_n + 6) & = R(F_{n+1} - 8) & & = n-4, & n \ge 7; \\
k=8: & R(F_n + 7) & = R(F_{n+1} - 9) & = R(F_{n-4}) & = [(n-4)/2], & n \ge 7.
\end{array}$$

Lemma 10–Special values for $R(F_{2c} \pm K)$ and $R(F_{2c+1} \pm K)$: Considering n even and n odd, $R(F_n \pm K)$ has the following values:

$$\begin{array}{lll}
R(F_{2c}) & = R(F_{2c+1}) & = R(F_{2c-2}) + 1; \\
R(F_{2c} + 1) & = R(F_{2c-1}) & = R(F_{2c-2}); \\
R(F_{2c+1} + 1) & = R(F_{2c}) & = R(F_{2c+1}); \\
R(F_{2c+1} + 2) & = R(F_{2c-1}) & = R(F_{2c} + 2); \\
R(F_{2c+1} - 1) & = R(F_{2c} - 1) & = 1.
\end{array}$$

Lemma 11: Let K be an integer whose Zeckendorf representation has $F_m + F_k$ for its smallest two terms.

If $k = 2$ so that K ends with $F_m + 1$, $m \ge 4$, then

$$R(K) = R(K-1),\ m \text{ odd};\ R(K) = R(K-1) - R(K-2),\ m \text{ even}; \tag{12}$$

If $k = 3$ so that K ends in $F_m + 2$, $m \ge 5$, then

$$R(K) = R(K-1) - R(K-3),\ m \text{ odd};\ R(K) = R(K-1),\ m \text{ even}; \tag{13}$$

If K ends in $F_m + F_{2c}$, $2c \ge 4$, then $R(K) = R(K-1) + R(K+1)$; (14)

If K ends in $F_m + F_{2c+1}$, $2c \ge 4$, then $R(K) = R(K-1) + R(K+2)$. (15)

Proof: A proof can be written by induction following this outline. Calculate (12) for $K = F_{2c}+1$ and $K = F_{2c+1}+1$. Equation (12) can also be calculated for $K = F_m + F_{2c}+1$ and $K = F_m + F_{2c+1}+1$. Then assume that (12) holds for all K such that $K \le F_{n-1}-1$ and use (5) from Lemma 5, $R(F_n + K) = R(F_{n-2}+K) + R(K)$, $0 \le K \le F_{n-3}-1$, calculating each part of (12). Repeat for the other two parts of Lemma 5. (14) and (15) can be proved by substitution into (12) and (13). When K ends in $F_m + F_{2c}$, $K+1$ ends in $F_m + F_{2c}+1$, so replacing K by $K+1$ in (12) in the even case yields (14). When K ends in $F_m + F_{2c+1}$, then $K+1$ ends in $F_m + F_{2c+1}+1$, which means that $R(K+1) = R(K)$ for the odd case of (12). Also, $K+2$ ends in $F_m + F_{2c+1}+2$, which means that $R(K+2) = R(K+1) - R(K-1)$ from the odd case of (13). Putting these together gives (15). □

Theorem 4: Let $F_m + F_k$ be the smallest Fibonacci numbers in the Zeckendorf representation of M. Then

$$R(M) = R(M-1)R(F_k) - q, \quad 0 \le q < R(M-1). \tag{16}$$

If the Zeckendorf representation of M ends in $F_{2c+1}+1$ or $F_{2c}+2$, where $2c \ge 4$, then $q = 0$. If M ends in $F_{2c+1}+2$, $q = R(M-3)$; $F_{2c}+1$, $q = R(M-2)$. If $m-k$ is odd, $q = 0$. If M ends in $F_{2w}+F_{2c}$, $2c \ge 4$, then $q = (c-1)R(M-1) - R(M+1)$; if M ends in $F_{2w+1}+F_{2c+1}$, $2c \ge 4$, then $q = (c-1)R(M-1) - R(M+2)$.

Proof: Apply Lemma 7(ii) and Lemma 11. When $m-k$ is odd, $q = 0$ by Theorem 3. □

Corollary 4.1: Let K be an integer whose Zeckendorf representation has smallest two terms $F_m + F_k$. Then $R(K) = cR(K-1)$ when $k = 2c$ and m is odd, and when $k = 2c+1$ and m is even.

5. $R(MF_k)$ AND $R(ML_k)$

Below, $R(MF_k)$ can be obtained by putting MF_k into Zeckendorf form and then applying Theorem 3 repeatedly. We list Zeckendorf representations of MF_k for $M \le 18$, taking smallest entry $F_{k-2c} \ge F_2$ and write $R(MF_k)$ for $M \le 29 = L_7$.

$$\begin{array}{rrll}
 & 2F_k & = F_{k+1}+F_{k-2} & \\
L_2F_k = & 3F_k & = F_{k+2}+F_{k-2} & \\
L_3F_k = & 4F_k & = F_{k+2}+F_k+F_{k-2} & = F_{k+3}-F_{k-3} \\
 & 5F_k & = F_{k+3}+F_{k-1}+F_{k-4} & = F_{k+3}+F_k-F_{k-3} \\
 & 6F_k & = F_{k+3}+F_{k+1}+F_{k-4} & \\
L_4F_k = & 7F_k & = F_{k+4}+F_{k-4} & \\
 & 8F_k & = F_{k+4}+F_k+F_{k-4} & = F_{k+4}+F_k+F_{k-4} \\
 & 9F_k & = F_{k+4}+F_{k+1}+F_{k-2}+F_{k-4} & = F_{k+4}+2F_k+F_{k-4} \\
 & 10F_k & = F_{k+4}+F_{k+2}+F_{k-2}+F_{k-4} & = F_{k+4}+3F_k+F_{k-4} \\
L_5F_k = & 11F_k & = F_{k+4}+F_{k+2}+F_k+F_{k-2}+F_{k-4} & = F_{k+4}+4F_k+F_{k-4} = F_{k+5}-F_{k-5} \\
 & 12F_k & = F_{k+5}+F_{k-1}+F_{k-3}+F_{k-6} & = F_{k+5}+F_k-F_{k-5} \\
 & 13F_k & = F_{k+5}+F_{k+1}+F_{k-3}+F_{k-6} & = F_{k+5}+2F_k-F_{k-5} \\
 & 14F_k & = F_{k+5}+F_{k+2}+F_{k-3}+F_{k-6} & = F_{k+5}+3F_k-F_{k-5} \\
 & 15F_k & = F_{k+5}+F_{k+2}+F_k+F_{k-3}+F_{k-6} & = F_{k+5}+4F_k-F_{k-5} \\
 & 16F_k & = F_{k+5}+F_{k+3}+F_{k-1}+F_{k-6} & = F_{k+5}+5F_k-F_{k-5} \\
 & 17F_k & = F_{k+5}+F_{k+3}+F_{k+1}+F_{k-6} & = F_{k+5}+6F_k-F_{k-5} \\
L_6F_k = & 18F_k & = F_{k+6}+F_{k-6} &
\end{array}$$

Lemma 12: For MF_k such that $L_{2c-1} < M \le L_{2c+1}$, $k \ge 2c+2$, the smallest Fibonacci number in the Zeckendorf representation of MF_k is F_{k-2c}, and the largest is F_{k+2c-1} or F_{k+2c}, depending upon the interval, where

$$F_{k+2c-1} \le MF_k < F_{k+2c}, \qquad L_{2c-1} < M < L_{2c};$$
$$F_{k+2c} \le MF_k < F_{k+2c+1}, \qquad L_{2c} \le M \le L_{2c+1}.$$

Proof: Lemma 12 is illustrated for $M \le 18$. Assume it holds for all integers $0 \le Q \le L_{2c-1}$; i.e., the largest term in QF_k is F_{k+2c-2} and the smallest is F_{k-2c-2} when $L_{2c-2} \le Q \le L_{2c-1}$. Since $L_{2c}F_k = F_{k+2c} + F_{k-2c}$ (see [6]), $MF_k = QF_k + L_{2c}F_k = F_{k+2c} + QF_k + F_{k-2c}$ has largest term F_{k+2c} and smallest term F_{k-2c} for $L_{2c} \le M = L_{2c} + Q \le L_{2c+1}$. The subscript difference between F_{k-2c-2} and the next smallest Fibonacci number used in the Zeckendorf representation of MF_k is even. For $L_{2c-1} < M < L_{2c}$, since $L_{2c-1}F_k = F_{k+2c-1} - F_{k-2c+1}$ (see [6]), $MF_k = L_{2c-1}F_k + QF_k = F_{k+2c-1} - F_{k-2c+1} + QF_k$, $0 < Q < L_{2c-2}$.

Assume the largest possible term in the Zeckendorf representation of QF_k is F_{k+2c-3} and the smallest term is F_{k-2i} for $L_{2c-3} < Q < L_{2c-2}$. There is no modification of terms for the Zeckendorf representation in adding F_{k+2c-1}, but the smallest term in the Zeckendorf representation of MF_k becomes F_{k-2c} for $L_{2c-1} < M < L_{2c}$ since

$$\begin{aligned} F_{k-2i} - F_{k-2c+1} &= (F_{k-2i} - F_{k-2c+2}) + F_{k-2c} \\ &= (F_{k-2i-1} + F_{k-2i-3} + \cdots + F_{k-2c+3}) + F_{k-2c}. \end{aligned}$$

Thus, the largest term is $F_{2k+2c-1}$ and the smallest is F_{k-2c} for MF_k, when $L_{2c-1} < M < L_{2c}$. Note that the subscript difference between F_{k-2c} and the next smallest Fibonacci number used in the Zeckendorf representation is odd. □

$R(MF_k)$, $1 \le M \le 29 = L_7$, $k \ge 2c+2$ for Smallest Term F_{k-2c}

$R(F_k)$	$= R(F_{k-0})$		
$R(2F_k)$	$= 2R(F_{k-2})$		
$R(3F_k)$	$= 3R(F_{k-2}) - 1$		
$R(4F_k)$	$= 3R(F_{k-2}) - 2$	$= 3R(F_{k-4}) + 1$	$4 = L_3$
$R(5F_k)$	$= 5R(F_{k-4})$		
$R(6F_k)$	$= 5R(F_{k-4})$		
$R(7F_k)$	$= 5R(F_{k-4}) - 1$		$7 = L_4$
$R(8F_k)$	$= 8R(F_{k-4}) - 3$		
$R(9F_k)$	$= 8R(F_{k-4}) - 4$		
$R(10F_k)$	$= 8R(F_{k-4}) - 5$		
$R(11F_k)$	$= 5R(F_{k-4}) - 4$	$= 5R(F_{k-6}) + 1$	$11 = L_5$
$R(12F_k)$	$= 10R(F_{k-6})$		
$R(13F_k)$	$= 13R(F_{k-6})$		
$R(14F_k)$	$= 12R(F_{k-6})$		
$R(15F_k)$	$= 12R(F_{k-6})$		
$R(16F_k)$	$= 13R(F_{k-6})$		
$R(17F_k)$	$= 10R(F_{k-6})$		
$R(18F_k)$	$= 7R(F_{k-6}) - 1$		$18 = L_6$
$R(19F_k)$	$= 15R(F_{k-6}) - 4$		
$R(20F_k)$	$= 18R(F_{k-6}) - 6$		
$R(21F_k)$	$= 21R(F_{k-6}) - 8$		
$R(22F_k)$	$= 16R(F_{k-6}) - 7$		
$R(23F_k)$	$= 20R(F_{k-6}) - 10$		
$R(24F_k)$	$= 20R(F_{k-6}) - 10$		
$R(25F_k)$	$= 16R(F_{k-6}) - 9$		
$R(26F_k)$	$= 21R(F_{k-6}) - 13$		
$R(27F_k)$	$= 18R(F_{k-6}) - 12$		
$R(28F_k)$	$= 15R(F_{k-6}) - 11$		
$R(29F_k)$	$= 7R(F_{k-6}) - 6$	$= 7R(F_{k-8}) + 1$	$29 = L_7$

Theorem 5: When $L_{2c-1} < M \le L_{2c+1}$, $k \ge 2c+2$,

$$R(MF_k) = R(MF_k - 1)R(F_{k-2c}) - q, \tag{17}$$

where $R(MF_k - 1) = R(MF_{2c+2} - 1)$. Further, $q = 0$ for $L_{2c-1} < M < L_{2c}$ while $q = R(MF_{2c+2} - 2)$ for $L_{2c} \le M \le L_{2c+1}$.

Proof: The assertions follow from Theorem 4 by taking $k = 2c+2$ in (17), since we have F_{k-2c} as the smallest term of the Zeckendorf representation of MF_k by Lemma 12. When $L_{2c-1} < M < L_{2c}$, the last two terms in the Zeckendorf representation are $F_m + F_{k-2c}$, where $(m-k+2c)$ is odd; thus, in using Theorem 3 repeatedly to evaluate $R(MF_k)$ from its Zeckendorf representation, we will have $q = 0$ by Corollary 3.1. When $L_{2c} \le M \le L_{2c+1}$, the subscripts of the last two terms will have an even difference, so a remainder term will be involved. Taking $k = 2c+2$ to give the smallest $F_m = F_2$ gives $q = R(MF_{2c+2} - 2)$ by Theorem 4 in the interval where $q \ne 0$. □

Next, we note that the values $R(MF_{2c+2} - 1)$ form palindromic subsequences such that:

$$R((L_{2c-1} + K)F_k - 1) = R((L_{2c} - K)F_k - 1), \quad 1 \le K \le [L_{2c-2}/2];$$
$$R((L_{2c} + K)F_k - 1)) = R((L_{2c+1} - K)F_k - 1), \quad 0 \le K \le [L_{2c-1}/2].$$

Also of interest, we have

$$R(L_{2c}F_k - 1) = R(L_{2c+1}F_k - 1);$$
$$R(L_{2c-1}F_k - 1) + 2 = R(L_{2c}F_k - 1).$$

Corollary 5.1: $R(L_nL_p - 1) = 4(p-1)$, $n \ge p+3$, $p \ge 2$.

Proof: Vajda [6] gives equation (17a), equivalent to

$$\begin{cases} L_{n+p} + L_{n-p} = L_nL_p, & p \text{ even}, \\ L_{n+p} - L_{n-p} = L_nL_p, & p \text{ odd}. \end{cases}$$

Since $L_{n+p} + L_{n-p} = F_{n+p+1} + F_{n+p-1} + F_{n-p+1} + F_{n-p-1}$, the smallest Fibonacci number used in the Zeckendorf representation is F_{n-p-1}. Theorem 4 gives

$$\begin{aligned} R(L_{n+p} + L_{n-p}) &= R(L_{n+p} + L_{n-p} - 1)R(F_{n-p-1}) - q \\ &= R(L_nL_p - 1)R(F_{n-p-1}) - q. \end{aligned}$$

Since we only want $R(L_nL_p - 1)$, we calculate $R(L_{n+p} + L_{n-p} - 1)$ when $F_{n-p-1} = F_2$ or, for $n-p=3$, $n = p+3$, so that $R(L_{n+p} + L_{n-p} - 1)$ has a constant value for $n \ge p+3$.

$$\begin{aligned} R(L_{n+p} + L_{n-p} - 1) &= R(L_{2p+3} + L_3 - 1) \\ &= R(F_{2p+4} + F_{2p+2} + 3) \\ &= R(F_{2p+2} + 3) + R(F_{2p+1} - 5) \\ &= (2p-1) + (2p-3) = 4(p-1), \end{aligned}$$

where we have applied earlier formulas from Theorem 3 and special values for $R(F_{n+1} - 1 - K)$. Thus, $R(L_nL_p - 1) = 4(p-1)$ for p even.

Similarly, for p odd, $L_{n+p} - L_{n-p}$ has F_{n-p-2} as the smallest Fibonacci number in its Zeckendorf representation. Again calculate $R(L_{n+p} - L_{n-p} - 1)$ for the smallest value for $F_{n-p-2} = F_2$, which occurs for $n-p=4$, $n=p+4$. Then

$$\begin{aligned} R(L_{2p+4} - L_4 - 1) &= R(F_{2p+5} + F_{2p+3} - 8) \\ &= 2R(F_{2p+3} - 8) = 2(2p-2) = 4(p-1). \end{aligned}$$

Thus, $R(L_n L_p - 1) = 4(p-1)$ for p odd, establishing Corollary 5.1 and proving Conjecture 2 of [1]. □

Corollary 5.2: $R(F_p F_n - 1) = F_p$, $n \geq p$, $p \geq 3$.

Proof: $F_{2c+1}F_k$ and $F_{2c+2}F_k$ both have F_{k-2c} as the smallest term in the Zeckendorf representation. Thus,

$$\begin{aligned} R(F_{2c+1}F_k) &= R(F_{2c+1}F_k - 1)R(F_{k-2c}) - q; \\ R(F_{2c+2}F_k) &= R(F_{2c+2}F_k - 1)R(F_{k-2c}) - q. \end{aligned}$$

When $k \geq 2c+2$, $R(MF_k - 1)$ has a constant value. When $k = 2c+2$,

$$R(F_{2c+1}F_k - 1) = R(F_{2c+1}F_{2c+2} - 1) = F_{2c+1}$$

while

$$R(F_{2c+2}F_k - 1) = R(F_{2c+2}F_{2c+2} - 1) = 2c+2,$$

applying two identities from Carlitz [3]. Thus, $R(F_p F_n - 1) = F_p$, establishing Corollary 5.2 and making a second proof of Theorem 3 in [1]. □

The Lucas case $R(ML_k)$ is very similar, relying on [6] for $F_p L_k = F_{k+p} + F_{k-p}$, p odd, and $F_p L_k = F_{k+p} - F_{k-p}$, p even. When $F_{2c-2} < M \leq F_{2c}$, the smallest term in the Zeckendorf representation of ML_k is F_{k-2c+1}, while the largest is F_{k+2c-2}, $F_{2c-2} < M < F_{2c-1}$, or F_{k+2c-1}, $F_{2c-1} \leq M \leq F_{2c}$, $k \geq 2c+1$.

Zeckendorf Representations for ML_k, $1 \leq M \leq 13$

$$\begin{array}{rrll}
F_2L_k = & L_k &= F_{k+1} + F_{k-1} & = F_{k+2} - F_{k-2} \\
F_3L_k = & 2L_k &= F_{k+3} + F_{k-3} & \\
F_4L_k = & 3L_k &= F_{k+3} + F_{k+1} + F_{k-1} + F_{k-3} & = F_{k+4} - F_{k-4} \\
 & 4L_k &= F_{k+4} + F_{k+1} + F_{k-2} + F_{k-5} & \\
F_5L_k = & 5L_k &= F_{k+5} + F_{k-5} & \\
 & 6L_k &= F_{k+5} + F_{k+1} + F_{k-1} + F_{k-5} & \\
 & 7L_k &= F_{k+5} + F_{k+3} + F_{k-3} + F_{k-5} & \\
F_6L_k = & 8L_k &= F_{k+5} + F_{k+3} + F_{k+1} + F_{k-1} + F_{k-3} + F_{k-5} & = F_{k+6} - F_{k-6} \\
 & 9L_k &= F_{k+6} + F_{k+1} + F_{k-2} + F_{k-4} + F_{k-7} & \\
 & 10L_k &= F_{k+6} + F_{k+3} + F_{k-4} + F_{k-7} & \\
 & 11L_k &= F_{k+6} + F_{k+3} + F_{k+1} + F_{k-1} + F_{k-4} + F_{k-7} & \\
 & 12L_k &= F_{k+6} + F_{k+4} + F_{k+1} + F_{k-2} + F_{k-7} & \\
F_7L_k = & 13F_k &= F_{k+7} + F_{k-7} &
\end{array}$$

$R(ML_k)$, $1 \le M \le 21 = F_8$, $k \ge 2c+1$ for Smallest Term F_{k-2c-1}

$R(L_k) = 2R(F_{k-1}) - 1$	$R(12L_k) = 18R(F_k - 7)$
$R(2L_k) = 4R(F_{k-3}) - 1$	$R(13L_k) = 8R(F_{k-7}) - 1$
$R(3L_k) = 4R(F_{k-3}) - 3$	$R(14L_k) = 24R(F_{k-7}) - 7$
$R(4L_k) = 8R(F_{k-5})$	$R(15L_k) = 30R(F_{k-7}) - 11$
$R(5L_k) = 6R(F_{k-5}) - 1$	$R(16L_k) = 20R(F_{k-7}) - 9$
$R(6L_k) = 12R(F_{k-5}) - 5$	$R(17L_k) = 32R(F_{k-7}) - 16$
$R(7L_k) = 12R(F_{k-5}) - 7$	$R(18L_k) = 20R(F_{k-7}) - 11$
$R(8L_k) = 6R(F_{k-5}) - 5$	$R(19L_k) = 30R(F_{k-7}) - 19$
$R(9L_k) = 18R(F_{k-7})$	$R(20L_k) = 24R(F_{k-7}) - 17$
$R(10L_k) = 16R(F_{k-7})$	$R(21L_k) = 8R(F_{k-7}) - 7$
$R(11L_k) = 16R(F_{k-7})$	

Theorem 6: When $F_{2c-2} < M \le F_{2c}$, $k \ge 2c+1$,

$$R(ML_k) = R(ML_k - 1)R(F_{k-2c+1}) - q, \tag{18}$$

where $R(ML_k - 1) = R(ML_{2c+1} - 1)$; further, $q = 0$ for $F_{2c-2} < M < F_{2c-1}$, and $q = R(ML_{2c+1} - 2)$ when $F_{2c-1} \le M \le F_{2c}$.

The proof of Theorem 6 depends on Theorem 4, and being similar to the proof of Theorem 5 is omitted here. We note that the values $R(ML_{2c+1} - 1)$ form palindromic subsequences such that

$$R((F_{2c-1} + K)L_k - 1) = R((F_{2c} - K)L_k - 1), \quad 0 \le K \le [F_{2c-2}/2];$$
$$R((F_{2c} + K)L_k - 1) = R((F_{2c} - K)L_k - 1), \quad 1 \le K \le [F_{2c-1}/2].$$

6. $R(F_m \pm F_k)$

Theorem 7: $R(F_m \pm F_k)$ and $R(F_m \pm F_k - 1)$ have the following values:

$$\begin{aligned}
R(F_m + F_k) &= R(F_{m-k+2})R(F_k), && (m-k) \text{ odd;}\\
R(F_m + F_k) &= R(F_{m-k+2})R(F_k) - 1, && (m-k) \text{ even;}\\
R(F_m - F_k) &= R(F_{m-k+1})R(F_{k-1}) + 1, && (m-k) \text{ even;}\\
R(F_m - F_k) &= R(F_{m-k+1})R(F_{k-1}), && (m-k) \text{ odd;}\\
R(F_m + F_k - 1) &= R(F_{m-k+2});\\
R(F_m - F_k - 1) &= R(F_{m-k+1}).
\end{aligned}$$

Proof: By Corollary 3.1,

$$R(F_m + F_k) = R(F_{m-k+1})R(F_k) + r.$$

If $m-k$ is odd, $r = 0$, and $R(F_{m-k+1}) = R(F_{m-k+2})$, making $R(F_m + F_k) = R(F_{m-k+2})R(F_k)$. If $m-k$ is even, $r = R(F_{k+1} - 2 - F_k) = R(F_{k-2}) = R(F_k) - 1$, and $R(F_{m-k+1}) + 1 = R(F_{m-k+2})$, making $R(F_m + F_k) = R(F_{m-k+2})R(F_k) - 1$.

Equations (7) give $R(F_m \pm F_k - 1)$ by examining the difference of the subscripts; note that the results for $R(F_m + F_k)$ agree with Theorem 4. Using Theorem 1, followed by Corollary 3.1, while noting that the greatest Fibonacci number in $F_k - 2$ is F_{k-1},

$$R(F_m - F_k) = R(F_{m-1} + (F_k - 2)) = R(F_{m-k+1})R(F_k - 2) + r.$$

Note that $R(F_k - 2) = R(F_{k-1})$. If $m-k$ is odd, $r = 0$, while if $m-k$ is even,

$$r = R(F_k - 2 - (F_k - 2)) = R(0) = 1. \ \square$$

Corollary 7.1: $R(F_r L_t - 1)$ can be written as

(i) $R(F_n L_p - 1) = 2R(F_p) + 1, \ n \ge p+2, \ p \ge 1;$

(ii) $R(L_n F_p - 1) = 2R(F_{p+1}), \ n \ge p+1, \ p \ge 2.$

Proof: Vajda [6] gives equation (15a), equivalent to

$$\begin{cases} F_{n+p} + F_{n-p} = F_n L_p, & p \text{ even}, \\ F_{n+p} - F_{n-p} = F_n L_p, & p \text{ odd}, \end{cases}$$

By Theorem 7, $R(F_{n+p} + F_{n-p} - 1) = R(F_{2p+2}) = p+1$, while $R(F_{n+p} - F_{n-p} - 1) = R(F_{2p}) = p$. So $R(F_n L_p - 1) = p+1$, p even, and $R(F_n L_p - 1) = p$, p odd, which makes $R(F_n L_p - 1) = 2[p/2]+1$, proving part (i) as well as Conjecture 3 of [1]. Since [6] also gives

$$\begin{cases} F_{n+p} + F_{n-p} = L_n F_p, & p \text{ odd}, \\ F_{n+p} - F_{n-p} = L_n F_p, & p \text{ even}. \end{cases}$$

in the same way, we can show that $R(L_n F_p - 1) = p+1$, p odd, and $R(L_n F_p - 1) = p$, p even, which can be rewritten in the form of (ii). Thus, we have proved part (ii) as well as Conjecture 1 of [1]. $\square$

Corollary 7.2: Let $F_n \le N < F_{n+1} - 2$.

(i) $R(L_{p+1}) = 2R(F_p) - 1 = R(L_{p-1}) + 2, \ p \ge 4;$

(ii) $R(L_{n+p} + N) = R(F_{n+p-1} + N) + R(F_{n+p-3} + N) = R(L_{p+1})R(N) + 2r,$
where $r = 0$ if p is odd, and $r = R(F_{n+1} - 2 - N)$ if p is even;

(iii) $R(L_{n+p} - K) = 2R(F_{n+p-2} + (K-2)), \ 2 \le K \le F_{n+p-3}.$

Proof: Since $L_{p+1} = F_{p+2} + F_p$, let $m = p+2$ and $k = p$ in Theorem 7 to write (i). Apply equation (10) to $R(F_{n+p+1} + F_{n+p-1} + N)$ followed by Theorem 1 to write the first part of (ii). Then use Corollary 3.1 and (i) to simplify, finally obtaining (ii).

When $2 \le K \le F_{n+p-3}$, the largest term in the Zeckendorf representation of $F_{n+p-1} - K$ is F_{n+p-2}. Then

$$\begin{aligned} R(L_{n+p} - K) &= R(F_{n+p+1} + (F_{n+p-1} - K)) \\ &= 2R(F_{n+p-1} - K) = 2R(F_{n+p-2} - 2 + K). \ \square \end{aligned}$$

Corollary 7.3:

$$R(L_{n+p} + L_{n-p}) = (2p-2)R(L_{n-p}) - 1 = 4(p-1)R(F_{n-p-1}) - (2p-1);$$

$$R(L_{n+p} - L_{n-p}) = 4(p-1)R(F_{n-p-2}), \ n-p \ge 3.$$

Proof: Let $N = L_{n-p} = F_{n-p+1} + F_{n-p-1}$ in Corollary 7.2. Then

$$\begin{aligned} &R(F_{n+p-1} + N) + R(F_{n+p-3} + N) \\ &= (p-1)R(L_{n-p}) + (p-2)R(L_{n-p}) + 2R(F_{n-p+2} - 2 - F_{n-p+1} - F_{n-p-1}) \end{aligned}$$

$$
\begin{aligned}
&= (2p-3)R(L_{n-p}) + 2R(F_{n-p-3}) = (2p-3)R(L_{n-p}) + R(L_{n-p}) - 1 \\
&= (2p-2)R(L_{n-p}) - 1 = (2p-2)[2R(F_{n-p-1}) - 1] - 1 \\
&= 4(p-1)R(F_{n-p-1}) - (2p-1).
\end{aligned}
$$

Now let $K = L_{n-p}$ in Corollary 7.2. Then

$$
\begin{aligned}
R(L_{n+p} - L_{n-p}) &= 2R(F_{n+p-2} + F_{n-p+1} + F_{n-p-1} - 2) \\
&= 2(p-1)R(F_{n-p+1} + F_{n-p-1} - 2) \\
&= 2(p-1)(2R(F_{n-p-1} - 2)) = 4(p-1)R(F_{n-p-2}),
\end{aligned}
$$

finishing Corollary 7.3. □

Corollary 7.4: $R(L_nL_p - 1) = 4(p-1),\ n \ge p+3,\ p \ge 2$.

Proof: Vajda [6] gives $L_{n+p} + L_{n-p} = L_nL_p$ when p is even, and $L_{n+p} - L_{n-p} = L_nL_p$ when p is odd. The smallest Fibonacci numbers in the Zeckendorf representations are F_{n-p-1} and F_{n-p-2}, respectively. Since also $R(L_{n+p} \pm L_{n-p} - 1) = R(L_nL_p - 1)$, apply Theorem 4 to Corollary 7.3. This also proves Conjecture 2 in [1]. □

Corollary 7.5: $R(5F_nF_p - 1) = 4(p-1),\ n \ge p+3,\ p \ge 2$.

Proof: $L_{n+p} + L_{n-p} = 5F_nF_p$, p odd; $L_{n+p} - L_{n-p} = 5F_nF_p$, p even, also appear in [6], giving an easy identity as in Corollary 7.4. □

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POLYNOMIALS RELATED TO MORGAN-VOYCE POLYNOMIALS

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1. INTRODUCTION

In this note we shall study two classes of polynomials, $\{P_{n,m}^{(r)}(x)\}$ and $\{Q_{n,m}^{(r)}(x)\}$, where r is integer. For $m=1$, these polynomials are the known polynomials $P_n^{(r)}(x)$ (see [1]) and $Q_n^{(r)}(x)$ (see [4]). Particularly, $P_n^{(r)}(x)$ and $Q_n^{(r)}(x)$ are the well-known classical Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$ (see [1], [2], [3], [4]). In Section 2 we shall study the class of polynomials $P_{n,m}^{(r)}(x)$. The polynomials $Q_{n,m}^{(r)}(x)$ are given in Section 3. The main results in this paper relate to the determination of coefficients of the polynomials $P_{n,m}^{(r)}(x)$ and $Q_{n,m}^{(r)}(x)$. Also, we give some interesting relations between the polynomials $P_{n,m}^{(r)}(x)$ and $Q_{n,m}^{(r)}(x)$.

2. POLYNOMIALS $P_{n,m}^{(r)}(x)$

We shall introduce the polynomials $P_{n,m}^{(r)}(x)$ by

$$P_{n,m}^{(r)}(x)=2P_{n-1,m}^{(r)}(x)-P_{n-2,m}^{(r)}(x)+xP_{n-m,m}^{(r)}(x),\ n>m \tag{2.1}$$

with

$$P_{n,m}^{(r)}(x)=1+nr \text{ for } n=0,1,\dots,m-1,\quad P_{m,m}^{(r)}(x)=1+mr+x. \tag{2.2}$$

So, by (2.1) and (2.2), we find the first $(m+2)$-members of the sequence $\{P_{n,m}^{(r)}(x)\}$:

$$\begin{aligned}&P_{0,m}^{(r)}(x)=1,\quad P_{1,m}^{(r)}(x)=1+r,\dots,P_{m,m}^{(r)}(x)=1+mr+x,\\&P_{m+1,m}^{(r)}(x)=1+(m+1)r+(3+r)x.\end{aligned} \tag{2.3}$$

From (2.3), by induction on n, we see that there exists a sequence $\{b_{n,k}^{(r)}\}$ ($n\ge 0$ and $k\ge 0$) of numbers such that

$$P_{n,m}^{(r)}(x)=\sum_{k=0}^{[n/m]}b_{n,k}^{(r)}x^k, \tag{2.4}$$

with $b_{n,k}^{(r)}=0$ for $k>[n/m]$.

By (2.4), we get

$$b_{n,0}^{(r)}=P_{n,m}^{(r)}(0). \tag{2.5}$$

Let us take $x=0$ in (2.1). Now, using (2.5), we obtain the following difference equation:

$$b_{n,0}^{(r)}=2b_{n-1,0}^{(r)}-b_{n-2,0}^{(r)},\ n\ge 2, m\ge 1, \tag{2.6}$$

with initial values $b_{0,0}^{(r)}=1$ and $b_{1,0}^{(r)}=1+r$.

Solving (2.6), we get

$$b_{n,0}^{(r)}=1+nr,\ n\ge 0. \tag{2.7}$$

From (2.1), we obtain the following recurrence relation:

$$b_{n,k}^{(r)} = 2b_{n-1,k}^{(r)} - b_{n-2,k}^{(r)} + b_{n-m,k-1}^{(r)},\ n \geq m, k \geq 1. \tag{2.8}$$

Next, we can write the sequence $\{b_{n,k}^{(r)}\}$ into the form of the general triangle:

TABLE 1

n/k	0	1	2	3	$\cdots$
1	1	$\cdots$	$\cdots$	$\cdots$	$\cdots$
2	$1+r$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$m-1$	$1+(m-1)r$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
m	$1+mr$	1	$\cdots$	$\cdots$	$\cdots$
$m+1$	$1+(m+1)r$	$3+r$	$\cdots$	$\cdots$	$\cdots$
$m+2$	$1+(m+2)r$	$6+4r$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Remark 1: For $m=1$, $r=0$ and $r=1$, Table 1 is exactly the *DFF* and the DFF_x triangle, respectively (see [2], [3]).

Theorem 2.1: The coefficients $b_{n,k}^{(r)}$ satisfy the relation

$$b_{n,k}^{(r)} = b_{n-1,k}^{(r)} + \sum_{s=0}^{n-m} b_{s,k-1}^{(r)},\ n \geq m, k \geq 1. \tag{2.9}$$

Proof: We shall use induction on n. By direct computation, we see that (2.9) holds for every $n = 0, 1, \ldots, m-1$. If we suppose that (2.9) is true for n $(n \geq m)$, then, from (2.8) for $n+1$, we have

$$\begin{aligned} b_{n+1,k}^{(r)} &= 2b_{n,k}^{(r)} - b_{n-1,k}^{(r)} + b_{n+1-m,k-1}^{(r)} \\ &= b_{n,k}^{(r)} + b_{n-1,k}^{(r)} + \sum_{s=0}^{n-m} b_{s,k-1}^{(r)} + b_{n+1-m,k-1}^{(r)} - b_{n-1,k}^{(r)} \\ &= b_{n,k}^{(r)} + \sum_{s=0}^{n+1-m} b_{s,k-1}^{(r)}. \end{aligned}$$

Thus, statement (2.9) follows from the last equalities. □

One of the main results is given by the following theorem.

Theorem 2.2: For any $n \geq 0$ and any $k \geq 0$ such that $0 \leq k \leq [n/m]$, we get

$$b_{n,k}^{(r)} = \binom{n-(m-2)k}{2k} + r\binom{n-(m-2)k}{2k+1}, \tag{2.10}$$

where $\binom{p}{s} = 0$ for $s > p$.

Proof: We use induction on n. First, from (2.7), we see that (2.10) is true for $k=0$. Also, if $n=0,1,\ldots,m-1$, then $k=0$, so (2.10) is true. Assume that (2.10) holds for $n-1$ $(n>m)$. Then, by (2.8) for n, we get

$$b_{n,k}^{(r)} = 2b_{n-1,k}^{(r)} - b_{n-2,k}^{(r)} + b_{n-m,k-1}^{(r)} = x_{n,k} + ry_{n,k},$$

where

$$x_{n,k} = 2\binom{n-1-(m-2)k}{2k} - \binom{n-2-(m-2)k}{2k} + \binom{n-m-(m-2)(k-1)}{2k-2}$$

and

$$y_{n,k} = 2\binom{n-1-(m-2)k}{2k+1} - \binom{n-2-(m-2)k}{2k+1} + \binom{n-m-(m-2)(k-1)}{2k-1}.$$

Next, from the well-known relation

$$\binom{p}{s} = \binom{p-1}{s} + \binom{p-1}{s-1},$$

we find that

$$x_{n,k} = \binom{n-(m-2)k}{2k} \quad\text{and}\quad y_{n,k} = \binom{n-(m-2)k}{2k+1}. \quad \square$$

Particular Cases

For $m=1$ and $r=0$, and for $m=1$ and $r=1$, by (2.10), we get

$$b_{n,k}^{(0)} = \binom{n+k}{2k} \quad\text{and}\quad b_{n,k}^{(1)} = \binom{n+k}{2k} + \binom{n+k}{2k+1} = \binom{n+1+k}{2k+1}.$$

These are the coefficients of the classical Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$, respectively (see [3], [4]). Namely, we have

$$b_{n+1}(x) = \sum_{k=0}^{n} \binom{n+k}{2k} x^k \quad\text{and}\quad B_{n+1}(x) = \sum_{k=0}^{n} \binom{n+1+k}{2k+1} x^k.$$

We shall now prove the following lemma.

Lemma 2.1:

$$b_{n,k}^{(1)} - b_{n-2,k}^{(1)} = b_{n,k}^{(0)} + b_{n-1,k}^{(0)}, \quad n \ge 2. \tag{2.11}$$

Proof: From (2.10), for $r=1$, we get

$$b_{n,k}^{(1)} - b_{n-2,k}^{(1)} = \binom{n-(m-2)k}{2k} + \binom{n-(m-2)k}{2k+1} - \binom{n-2-(m-2)k}{2k} - \binom{n-2-(m-2)k}{2k+1}$$

$$= \binom{n-(m-2)k}{2k} + \binom{n-1-(m-2)k}{2k} = b_{n,k}^{(0)} + b_{n-1,k}^{(0)}.$$

From the last equalities, we get (2.11). $\square$

Remark 2: For $m=1$, from (2.11), we obtain (see [5])

$$B_n(x)-B_{n-2}(x)=b_n(x)+b_{n-1}(x),$$

where $B_n(x)$ and $b_n(x)$ are the classical Morgan-Voyce polynomials.

3. POLYNOMIALS $Q_{n,m}^{(r)}(x)$

First, we are going to define the polynomials $Q_{n,m}^{(r)}(x)$, which are the generalization of the polynomials $Q_n^{(r)}(x)$ (see [4]). The polynomials $Q_{n,m}^{(r)}(x)$ are given by

$$Q_{n,m}^{(r)}(x)=2Q_{n-1,m}^{(r)}(x)-Q_{n-2,m}^{(r)}(x)+xQ_{n-m,m}^{(r)}(x),\ n\geq m, \tag{3.1}$$

with the initial values

$$Q_{n,m}^{(r)}(x)=2+nr \text{ for } n=0,1,\dots,m-1,\quad Q_{m,m}^{(r)}(x)=2+mr+x. \tag{3.2}$$

From (3.2) and (3.1), by induction on n, we see that there exists a sequence $\{d_{n,k}^{(r)}\}$ ($n\geq 0$ and $k\geq 0$) of integers such that

$$Q_{n,m}^{(r)}(x)=\sum_{k=0}^{[n/m]}d_{n,k}^{(r)}x^k, \tag{3.3}$$

where

$$d_{n,n}^{(r)}=\begin{cases}1, & n\geq 1,\\ 2, & n=0.\end{cases} \tag{3.4}$$

From (3.3), we get

$$Q_{n,m}^{(r)}(0)=d_{n,0}^{(r)}.$$

Thus, by (3.1) and (3.2), we have

$$d_{n,0}^{(r)}=2d_{n-1,0}^{(r)}-d_{n-2,0}^{(r)}\quad (n\geq 2), \tag{3.5}$$

with

$$d_{0,0}^{(r)}=2 \quad\text{and}\quad d_{1,0}^{(r)}=2+r. \tag{3.6}$$

Solving (3.5), by (3.6), we obtain

$$d_{n,0}^{(r)}=2+nr,\ n\geq 0. \tag{3.7}$$

Furthermore, from (3.1), we get

$$d_{n,k}^{(r)}=2d_{n-1,k}^{(r)}-d_{n-2,k}^{(r)}+d_{n-m,k-1}^{(r)}\quad (n\geq m, m\geq 1, k\geq 1). \tag{3.8}$$

In Table 2, we write the coefficients $d_{n,k}^{(r)}$. Thus, from Tables 1 and 2, we see that

$$d_{n,k}^{(r)}=b_{n,k}^{(r)}+b_{n-1,k}^{(0)},\quad n=0,1,\dots,m-1.$$

TABLE 2

n/k	0	1	2	$\cdots$
0	2	$\cdots$	$\cdots$	$\cdots$
1	$2+r$	$\cdots$	$\cdots$	$\cdots$
2	$2+r$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$m-1$	$2+(m-1)r$	$\cdots$	$\cdots$	$\cdots$
m	$2+mr$	1	$\cdots$	$\cdots$
$m+1$	$2+(m+1)r$	$4+r$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Now we shall prove the following theorem.

Theorem 3.1: For $n \geq 1$, the following equalities hold:

$$\begin{aligned} d_{n,k}^{(r)} &= b_{n,k}^{(r)} + b_{n-1,k}^{(0)} \\ &= \binom{n-(m-2)k}{2k} + \binom{n-1-(m-2)k}{2k} + r\binom{n-(m-2)k}{2k+1}. \end{aligned} \tag{3.9}$$

Proof: In the proof, we use induction on n. For $n=1$, by direct computation, we conclude that (3.9) is true. We assume that (3.9) is true for n $(n \geq 1)$. Then, for $n+1$, we get

$$\begin{aligned} b_{n+1,k}^{(r)} + b_{n,k}^{(0)} &= 2b_{n,k}^{(r)} - b_{n-1,k}^{(r)} + b_{n+1-m,k-1}^{(r)} + 2b_{n-1,k}^{(0)} - b_{n-2,k}^{(0)} + b_{n-m,k-1}^{(0)} \quad \text{[by (2.8)]} \\ &= 2(b_{n,k}^{(r)} + b_{n-1,k}^{(0)}) - (b_{n-1,k}^{(r)} + b_{n-2,k}^{(0)}) + b_{n+1-m,k-1}^{(r)} + b_{n-m,k-1}^{(0)} \\ &= 2d_{n,k}^{(r)} - d_{n-1,k}^{(r)} + d_{n+1-m,k-1}^{(r)} = d_{n+1,k}^{(r)} \quad \text{[by (3.8)]}. \end{aligned}$$

Now, from (2.10), we obtain (3.9). This completes the proof. □

Corollary 1:

$$d_{n,k}^{(r)} = \frac{n-(m-1)k}{k}\binom{n-1-(m-2)k}{2k-1} + r\binom{n-(m-2)k}{2k+1}.$$

Hence, for $m=1$ and $k>0$, we get (see [4])

$$d_{n,k}^{(r)} = \frac{n}{k}\binom{n-1+k}{2k-1} + r\binom{n+k}{2k+1}.$$

Corollary 2:

$$Q_{n,1}^{(r)}(1) = L_{2n} + rF_{2n} \quad \text{(see [4])}.$$

Corollary 3:

$$Q_{n,1}^{(2u+1)}(1) = 2P_{n,1}^{(u)} \quad \text{(see [4])}.$$

Theorem 3.2: The polynomials $P_{n,m}^{(r)}(x)$ and $Q_{n,m}^{(r)}(x)$ satisfy the relation

$$Q_{n,m}^{(r)}(x) = P_{n,m}^{(r)}(x) + P_{n-1,m}^{(0)}(x), \quad n \geq 1. \tag{3.10}$$

Proof: Multiply both sides of (3.9) by x^k and sum. Immediately, from (2.4) and (3.3), we obtain (3.10). □

Remark 3: For $m = 1$, (3.10) becomes (see [4])

$$Q_n^{(r)}(x) = P_n^{(r)}(x) + P_{n-1}^{(0)}(x), \quad n \geq 1.$$

Theorem 3.3:

$$Q_{n,m}^{(0)}(x) = P_{n,m}^{(1)}(x) - P_{n-2,m}^{(1)}(x).$$

Proof:

$$\begin{aligned} Q_{n,m}^{(0)}(x) &= \sum_{k=0}^{[n/m]} d_{n,k}^{(0)} x^k && \text{[by (3.3)]} \\ &= \sum_{k=0}^{[n/m]} (b_{n,k}^{(0)} + b_{n-1,k}^{(0)}) x^k && \text{[by (3.9)]} \\ &= \sum_{k=0}^{[n/m]} (b_{n,k}^{(1)} + b_{n-2,k}^{(1)}) x^k && \text{[by (2.11)]} \\ &= P_{n,m}^{(1)}(x) - P_{n-2,m}^{(1)}(x) && \text{[by (2.4)].} \ \square \end{aligned}$$

Corollary 4: For $m = 1$, we get (see [4])

$$Q_n^{(0)}(x) = P_n^{(1)}(x) - P_{n-2}^{(1)}(x) = B_{n+1}(x) - B_{n-1}(x).$$

Thus, we obtain

$$Q_n^{(0)}(x) = \sum_{k=1}^{n} \frac{n}{k} \binom{n-1+k}{2k-1} x^k + 2.$$

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A GENERALIZATION OF THE EULER AND JORDAN TOTIENT FUNCTIONS

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1. THE FUNCTION $S_m^k(n)$ AND RELATED RESULTS

This article was motivated by a question posed to me by Professor H. W. Gould [2], specifically: What can be said about the number theoretic function

$$G_m(n) = \sum_{\substack{1\le a_1, \dots, a_m \le n \\ (a_1, \dots, a_m)=1}} 1 \quad , \quad \text{where } m \ge 2, n \ge 1? \tag{1}$$

The Jordan totient function $J_m(n)$ generalizes Euler's totient function $\phi(n)$. In this paper we investigate the function $S_m^k(n)$, which generalizes both Jordan and Euler's totient functions. Thus, with $m \ge 1$, $n \ge 1$, and $k \ge 1$, let

$$S_m^k(n) = \sum_{\substack{1\le a_1, \dots, a_m \le n \\ (a_1, \dots, a_m, k)=1}} 1 \quad . \tag{2}$$

The case $k = n$ retrieves $J_m(n)$, while $S_1^n(n)$ is Euler's totient function. Also, it is clear that $S_m^1(n) = n^m = I_m(n)$. In fact, $\sigma_m(n) = \Sigma_{d|n} S_m^1(d)$, from which we obtain by Möbius inversion that

$$S_m^1(n) = \sum_{d|n} \mu(d)\sigma_m\left(\frac{n}{d}\right).$$

Also, since $\Sigma_{d|n} J_m(d) S_m^1(n)$, it follows that

$$J_m(n) = \sum_{d|n} \mu(d) S_m^1\left(\frac{n}{d}\right) = n^m \sum_{d|n} \frac{\mu(d)}{d^m} \quad \text{and} \quad \sigma_m(n) = \sum_{d|n} \sum_{t|d} J_m(t).$$

We shall make use of the following known result.

Theorem 1: Let $f(n)$ and $F(n)$ be number theoretic functions such that $F(n) = \Sigma_{d|n} f(d)$. Then, for any integer N,

$$\sum_{n=1}^{N} F(n) = \sum_{n=1}^{N} \sum_{d|n} f(d) = \sum_{j=1}^{N} f(j)\left[\frac{N}{j}\right].$$

We may use this theorem to obtain the result that

$$\sum_{j=1}^{n} S_m^1(j) = \sum_{j=1}^{n} j^m = \sum_{j=1}^{n} \sum_{d|n} J_m(d) = \sum_{j=1}^{n} \left[\frac{n}{j}\right] J_m(j). \tag{3}$$

We now prove our next result.

Theorem 2: Let $k = \prod_{i=1}^{s} p_i^{e_i}$ be the prime decomposition of k, where $e_i \ge 1$, then

$$S_m^k(n) = \sum_{d|k} \mu(d) \left[\frac{n}{d}\right]^m.$$

Proof: It follows by the inclusion-exclusion theorem that

$$S_m^k(n) = n^m - \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{\substack{i_m=1\\ p_i|(i_1,\dots,i_m,k)\\ 1\le i\le s}}^{n}1 + \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{\substack{i_1=1\\ p_ip_j|(i_1,\dots,i_m,k)\\ 1\le i<j\le s}}^{n}1 + \cdots + (-1)^s\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{\substack{i_m=1\\ p_1p_2\cdots p_s|(i_1,\dots,i_m,k)}}^{n}1$$

$$= n^m - \sum_{1\le i_m\le\left[\frac{n}{p_i}\right]}\sum_{1\le i_m\le\left[\frac{n}{p_i}\right]}\cdots\sum_{1\le i_m\le\left[\frac{n}{p_i}\right]}1 + \sum_{1\le i_1\le\left[\frac{n}{p_ip_j}\right]}\sum_{1\le i_2\le\left[\frac{n}{p_ip_j}\right]}\cdots\sum_{1\le i_m\le\left[\frac{n}{p_ip_j}\right]}1 + \cdots + (-1)^s\sum_{1\le i_1\le\left[\frac{n}{p_1\cdots p_s}\right]}\cdots\sum_{1\le i_m\le\left[\frac{n}{p_1\cdots p_s}\right]}1$$

$$= n^m - \left[\frac{n}{p_i}\right]^m + \left[\frac{n}{p_ip_j}\right]^m + \cdots + (-1)^s\left[\frac{n}{p_1p_2\cdots p_s}\right]^m = \sum_{d|k}\mu(d)\left[\frac{n}{d}\right]^m,$$

where the subindices are as defined in the first line.

For the special case of $k=n$ and $m=1$, it follows that

$$\phi(n) = n - \frac{n}{p_i} + \frac{n}{p_ip_j} + \cdots + (-1)^s\frac{n}{p_1\cdots p_s} = n\prod_{p|n}\left(1-\frac{1}{p}\right) = n\sum_{d|n}\frac{\mu(d)}{d},$$

as expected. Also

$$J_m(n) = n^m - \left(\frac{n}{p_i}\right)^m + \left(\frac{n}{p_ip_j}\right)^m + \cdots + (-1)^s\left(\frac{n}{p_1\cdots p_s}\right)^m = n^m\prod_{p|n}\left(1-\frac{1}{p^m}\right) = n^m\sum_{d|n}\frac{\mu(d)}{d^m}$$

again as expected.

Similarly, it may be shown that

$$S_m^n(n^\alpha) = n^{m\alpha}\prod_{p|n}\left(1-\frac{1}{p^m}\right) = n^{m\alpha}\sum_{d|n}\frac{\mu(d)}{d^m}. \tag{4}$$

Further, by setting , we obtain the result,

$$S_1^k(n) = \sum_{\substack{1\le i\le n\\ (i,k)=1}}1 = \sum_{d|k}\mu(d)\left[\frac{n}{d}\right].$$

On the other hand, by defining

$$S_\tau^k(n) = \sum_{\substack{d|n\\ (d,k)=1}}1,$$

we obtain the following result.

Theorem 3: $S_\tau^k(n) = \sum_{\substack{d|n\\ (d,k)=1}}1 = \sum_{d|(k,n)}\mu(d)\tau\left(\frac{n}{d}\right).$

We may generalize the function $S_m^k(n)$ by setting

$$S_m^k(n,a) = \sum_{\substack{1\le a_1,\dots,a_m\le n\\ (a_1,\dots,a_m,k)=a}}1 = \sum_{\substack{1\le b_1,\dots,b_m\le\left[\frac{n}{a}\right]\\ \left(b_1,\dots,b_m,\frac{k}{a}\right)=1}}1 = \begin{cases} S_m^{a/k}\left(\left[\frac{n}{a}\right]\right), & \text{if } a/k,\\ 0, & \text{otherwise.}\end{cases}$$

We now let $S_1^k(x)$ denote the generating function for $S_1^k(n)$, then

$$S_1^k(x) = \sum_{n=1}^{\infty} S_1^k(n)\,x^n = \sum_{n=1}^{\infty} nx^n - \sum_{n=1}^{\infty}\left[\frac{n}{p_i}\right]x^n + \sum_{n=1}^{\infty}\left[\frac{n}{p_ip_j}\right]x^n + \cdots + (-1)^s\sum_{n=1}^{\infty}\left[\frac{n}{p_1p_2\cdots p_s}\right]x^n$$

$$= \sum_{n=1}^{\infty} nx^n - x^{p_i}\sum_{n=p_i}^{\infty}\left[\frac{n}{p_i}\right]x^{n-p_i} + \cdots + (-1)^s x^{p_1\cdots p_s}\sum_{n=p_1\cdots p_s}^{\infty}\left[\frac{n}{p_1p_2\cdots p_s}\right]x^{n-p_1\cdots p_s}$$

$$= \frac{x}{(1-x)^2} - \frac{x^{p_i}}{(1-x)(1-x^{p_i})} + \frac{x^{p_ip_j}}{(1-x)(1-x^{p_ip_j})} + \cdots + (-1)^s\frac{x^{p_i\cdots p_s}}{(1-x)(1-x^{p_1\cdots p_s})},$$

where we have used the result,

$$\sum_{n=k}^{\infty}\left[\frac{n}{k}\right]x^{n-k} = \frac{1}{(1-x)(1-x^k)},\quad |x|<1.$$

We now use Theorem 2 to partially answer Gould's question, as follows.

Theorem 4: Let $G_m(n) = \sum_{\substack{1\le a_1,\ldots,a_m\le n\\(a_1,\ldots,a_m)=1}} 1$, where $m\ge 2$, $n\ge 1$. Then

$$G_m(n) = \sum_{k=1}^{n}\sum_{\substack{1\le a_1,\ldots,a_{m-1}\le n\\(a_1,\ldots,a_{m-1},k)=1}} 1 = \sum_{k=1}^{n} S_{m-1}^k(n) = \sum_{k=1}^{n}\sum_{d\mid k}\mu(d)\left[\frac{n}{d}\right]^{m-1},\text{ by Theorem 2.}$$

We now restrict the function $S_m^k(n)$ somewhat and define a new function thus:

$$L_m^k(n) = \sum_{\substack{1\le a_1\le a_2\le\cdots\le a_m\le n\\(a_1,\,\ldots,\,a_m,\,k)=1}} 1,\quad\text{where } m\ge 1,\ n\ge 1,\ k\ge 1. \tag{5}$$

The case $k=1$ gives the following result.

Theorem 5: $L_m^1(n) = \binom{n+m-1}{m}$.

Proof: We prove the result by induction on m. First of all, the case $m=2$ gives

$$L_2^1(n) = \sum_{\substack{1\le a\le b\le n\\(a,b,1)=1}} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} 1 = \frac{n^2}{2}+\frac{n}{2} = \binom{n+2-1}{2}.$$

We now assume the result true for 1, 2, 3, ..., m and consider

$$L_{m+1}^1(n) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\cdots\sum_{i_{m+1}=1}^{i_m} 1 = \sum_{i_1=1}^{n} L_m^1(i_1) = \sum_{j=1}^{n}\binom{j+m-1}{m}.$$

Now let $j' = j+m-1$. After reverting back to the original variable, we obtain

$$L_{m+1}^1(n) = \sum_{j=m}^{j+n-1}\binom{j}{m} = \binom{m+n}{m+1},$$

see Gould [3, (1.52), p. 7], and hence, the induction goes through.

Alternatively, we may show that

$$L_m^1(n) = \frac{\sum_{i=0}^{m-1} S_{m-i}^1(n) S_1(m-1,i)}{\sum_{i=0}^{m-1} S_1(m-1,i)},$$

where $S_1(m,i)$ represents Stirling numbers of the first kind in Gould's notation [4]. We note that $s(n,m) = (-1)^{n-m} S_1(n-1, n-m)$, where $s(n,m)$ represents Stirling numbers of the first kind in Riordan's notation [6]. The equivalence follows from the fact that

$$\sum_{i=0}^{m-1} S_{m-i}^1(n) S_1(m-1,i) = \sum_{i=1}^{m} S_1(m-1, m-i) n^i$$
$$= \sum_{i=1}^{m} (-1)^{i-m} s(m,i) n^i = (-1)^m m! \binom{-n}{m} = m! \binom{n+m-1}{m}.$$

And, of course,

$$\sum_{i=0}^{m-1} S_1(m-1,i) = (-1)^i m! \binom{-1}{m} = m!.$$

From Theorem 5 and the standard result,

$$F_{n+1} = \sum_{i=0}^{[\frac{n}{2}]} \binom{n-j}{j},$$

where the F_n are Fibonacci numbers, we may deduce that

$$F_n = \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-1-j}{j} = \sum_{j=1}^{[\frac{n-1}{2}]+1} \binom{n-j}{j-1} = \sum_{j=1}^{[\frac{n+1}{2}]} \binom{n-j}{n-2j+1} = \sum_{j=1}^{[\frac{n+1}{2}]} L_{(n-2j+1)}^1(j),$$

where $L_0^1(n) = 1 \,\forall\, n$.

We now let $k = \prod_{i=1}^{s} p_i^{e_i}$, where $e_i \geq 1$, and prove our next result.

Theorem 6: $L_m^k(n) = \sum_{d|k} \mu(d) L_m^1\left(\left[\frac{n}{d}\right]\right)$.

Proof:

$$L_m^k(n) = L_m^1(n) - \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \cdots \sum_{\substack{i_m=1 \\ p_i|(i_1,\ldots,i_m,k)}}^{i_{m-1}} 1 + \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} + \cdots + \sum_{\substack{i_m=1 \\ p_i p_j|(i_1,\ldots,i_m,k)}}^{i_{m-1}} 1$$
$$+ \cdots + (-1)^s \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \cdots \sum_{\substack{i_m=1 \\ p_1 p_2 \cdots p_s|(i_1,\ldots,i_m,k)}}^{i_{m-1}} 1$$
$$= L_m^1(n) - L_m^1\left(\left[\frac{n}{p_i}\right]\right) + L_m^1\left(\left[\frac{n}{p_i p_j}\right]\right) + \cdots + (-1)^s L_m^1\left(\left[\frac{n}{p_1 p_2 \cdots p_s}\right]\right) = \sum_{d|k} \mu(d) L_m^1\left(\left[\frac{n}{d}\right]\right).$$

The special case $k = n$ gives the result

$$L_m^n(n) = \sum_{d|n} \mu(d) L_m^1\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \binom{d+m-1}{m},$$

which implies that $L_m^1(n) = \sum_{d|n} L_m^d(d)$. Equivalently,

$$\sum_{n=1}^{\infty} L_m^n(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} L_m^1(n) x^n = \frac{x}{(1-x)^{m+1}} \tag{6}$$

or

$$\zeta(s) \sum_{n=1}^{\infty} \frac{L_m^n(n)}{n^s} = \sum_{n=1}^{\infty} \frac{L_m^1(n)}{n^s}.$$

It follows from Theorem 1 that

$$\sum_{j=1}^{n} L_m^1(j) = \sum_{j=1}^{n} \sum_{d|j} L_m^d(d) = \sum_{j=1}^{n} \left[\frac{n}{j}\right] L_m^j(j),$$

that is,

$$\sum_{j=1}^{n} \binom{m+j-1}{m} = \sum_{j=1}^{n} \left[\frac{n}{j}\right] L_m^j(j)$$

which, on letting $m+j-1 = j'$ and reverting back to the original variable, gives

$$\sum_{j=m}^{m+n-1} \binom{j}{m} = \binom{m+n}{m+1} = \sum_{j=1}^{n} \left[\frac{n}{j}\right] L_m^j(j). \tag{7}$$

The case $m = 1$ gives the result $L_1^n(n) = \phi(n)$. Following are the tables of the values of the $L_m^n(n)$ and $L_m^1(n)$ arrays.

TABLE 1. Values of the $L_m^n(n)$ Array

$m \backslash n$	1	2	3	4	5	6	7	8
1	1	1	2	2	4	2	6	4
2	1	2	5	7	14	13	27	26
3	1	3	9	16	34	43	83	100
4	1	4	14	30	69	107	209	295
5	1	5	20	50	125	226	461	736
6	1	6	27	77	209	428	923	1632
7	1	7	35	112	329	749	1715	3312
8	1	8	44	156	494	1234	3002	6270

TABLE 2. Values of the $L_m^1(n)$ Array

$m \backslash n$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	1	3	6	10	15	21	28	36
3	1	4	10	20	35	56	84	120
4	1	5	15	35	70	126	210	330
5	1	6	21	56	126	252	462	792
6	1	7	28	84	252	462	924	1716
7	1	8	36	120	462	924	1716	3432
8	1	9	45	165	792	1716	3003	6435

We obtain a recurrence relation for $L_m^1(n)$ as follows:

$$L_m^1(n+1) = \binom{m+n}{m} = \binom{m+n-1}{m} + \binom{n+m-1}{m-1} = L_m^1(n) + L_{m-1}^1(n+1).$$

With the exception of boundary conditions, we note that this relation is the same as equation (1.1) of Carlitz and Riordan [1]. Its generating function $L_m^1(x)$ is

$$\begin{aligned} L_m^1(x) &= \sum_{n=1}^{\infty} L_m^1(n)x^n = \sum_{n=1}^{\infty} L_m^1(n+1)x^n - \sum_{n=1}^{\infty} L_{m-1}^1(n+1)x^n \\ &= \sum_{n=2}^{\infty} L_m^1(n)x^{n-1} - \sum_{n=2}^{\infty} L_{m-1}^1(n)x^{n-1}, \end{aligned}$$

which implies that

$$xL_m^1(x) = \sum_{n=1}^{\infty} L_m^1(n)x^n - xL_m^1(1) - \sum_{n=1}^{\infty} L_{m-1}^1(n)x^n + xL_{m-1}^1(1),$$

that is,

$$L_m^1(x) = \frac{L_{m-1}^1(x)}{1-x} = \frac{L_1^1(x)}{(1-x)^{m-1}}.$$

But

$$L_1^1(x) = \sum_{n=1}^{\infty} L_1^1(n)x^n = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

and, therefore,

$$L_m^1(x) = \frac{x}{(1-x)^{m+1}}, \quad |x| < 1.$$

We may also let

$$S_n = \sum_{m=1}^{n} L_m^1(n) = \sum_{m=1}^{n} \binom{n+m-1}{m} = \sum_{m=0}^{n} \binom{n+m-1}{m} - \binom{n-1}{0} = \binom{2n}{n} - 1.$$

Similarly, we may define and show that

$$T_m = \sum_{n=1}^{m} \binom{n+m-1}{m} = \sum_{j=m}^{2m-1} \binom{j}{m} = \binom{2m}{m+1}.$$

We now seek the generating function of S_n. And so, with $S_0 = 0$, let

$$S(x) = \sum_{n=0}^{\infty} S_n x^n = \sum_{n=0}^{\infty} \left\{ \binom{2n}{n} - 1 \right\}.$$

We now use the result

$$\binom{-\frac{1}{2}}{n} = (-1)^n \binom{2n}{n} 2^{-2n}, \quad n \geq 0,$$

to obtain

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{-1}{2}}{n} 2^{2n} x^n = (1-4x)^{\frac{-1}{2}};$$

hence,

$$S(X) = \frac{1}{\sqrt{1-4x}} - \frac{1}{1-x}$$

see Gould [4, p. 16].

Finally, we may consider the function

$$T_m^k(n) = \sum_{\substack{1 \le a_1 < a_2 < \cdots < a_m \le n \\ (a_1, \ldots, a_m, k) = 1}} 1 \quad , \quad n \ge m. \tag{8}$$

The case $k = 1$ gives

$$\begin{aligned} T_m^1(n) &= \sum_{i_1=1}^{n-(m-1)} \sum_{i_2=i_1+1}^{n-(m-2)} \cdots \sum_{i_m=i_{m-1}+1}^{n} 1 = \frac{\sum_{i=1}^m s(m,i)n^i}{\sum_{i=1}^m |s(m,i)|} \\ &= \frac{\sum_{i=0}^m s(m,i)n^i}{\sum_{i=0}^{m-1} S_1(m-1,i)} = \frac{m!\binom{n}{m}}{m!} = \binom{n}{m}. \end{aligned} \tag{9}$$

This is a known result.

2. INVERSE AND ORTHOGONAL RELATIONS

Using Theorem 2, we may now prove our next result.

Theorem 7: $T_m^k(n) = \sum_{d|k} \mu(d) T_m^1\left(\left[\frac{n}{d}\right]\right)$.

The case $k = n$ gives

$$T_m^n(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\binom{d}{m},$$

from which it follows that $T_m^n(n) = \phi(n)$ and $T_m^m(m) = 1$. Möbius inversion then gives

$$\binom{n}{m} = \sum_{d|n} T_m^d(d);$$

hence,

$$\sum_{n=1}^{\infty} \frac{T_m^n(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} T_m^1(n)x^n = x^m \sum_{n=m}^{\infty} \binom{n}{m} x^{n-m} = \frac{x^m}{(1-x)^{m+1}} \tag{10}$$

or

$$\zeta(s) \sum_{n=1}^{\infty} \frac{T_m^n(n)}{n^s} = \sum_{n=1}^{\infty} \frac{T_m^1(n)}{n^s}.$$

It follows from Theorem 1 that

$$\sum_{j=m}^{n} \binom{j}{m} = \binom{n+1}{m+1} = \sum_{j=m}^{n} \sum_{d|j} T_m^d(d) = \sum_{j=m}^{n} \left[\frac{n}{j}\right] T_m^j(j). \tag{11}$$

Following a technique of Gould [5, p. 252], we may set

$$\left[\frac{j}{m}\right] = \sum_{i=m}^{j} \binom{j+1}{i+1} K_m(i);$$

hence,

$$\sum_{j=m}^{n}(-1)^{n-j-1}\binom{n}{j+1}\left[\frac{j}{m}\right]=\sum_{j=m}^{n}\sum_{i=m}^{j}(-1)^{n-j-1}\binom{n}{j+1}\binom{j+1}{i+1}K_m(i)$$

$$=\sum_{i=m}^{n}\sum_{j=i}^{n}(-1)^{n-j-1}\binom{n}{j+1}\binom{j+1}{i+1}K_m(i)=\sum_{i=m}^{n}K_m(i)\sum_{j=i+1}^{n+1}(-1)^{n-j}\binom{n}{j}\binom{j}{i+1}=K_m(n-1).$$

From this, we may obtain the inverse to $T_m^n(n)$ as

$$K_m(n)=\sum_{j=m}^{n}(-1)^{n-j}\binom{n+1}{j+1}\left[\frac{j}{m}\right]. \tag{12}$$

Also, since

$$\sum_{n=1}^{\infty}\frac{T_m^n(n)x^n}{1-x^n}=\frac{x^m}{(1-x)^{m+1}},$$

we may consider

$$\sum_{j=1}^{\infty}K_m(j)\frac{x^j}{(1-x)^{j+1}}=\sum_{j=m}^{\infty}\sum_{i=m}^{j}(-1)^{j-i}\binom{j+1}{i+1}\left[\frac{i}{m}\right]\frac{x^j}{(1-x)^{j+1}}$$

$$=\frac{1}{1-x}\sum_{i=m}^{\infty}(-1)^i\left[\frac{i}{m}\right]\sum_{j=i}^{\infty}\binom{j+1}{i+1}\left(\frac{x}{x-1}\right)^j.$$

But

$$\sum_{n=1}^{\infty}\binom{n+1}{m+1}x^n=\sum_{n=1}^{\infty}\sum_{j=m}^{n}\binom{j}{m}x^n=\sum_{j=1}^{\infty}\binom{j}{m}\sum_{n=j}^{\infty}x^n=\frac{x^m}{1-x}\sum_{j=m}^{\infty}\binom{j}{m}x^{j-m}=\frac{x^m}{(1-x)^{m+2}}.$$

Therefore,

$$\sum_{j=i}^{\infty}\binom{j+i}{i+1}\left(\frac{x}{x-1}\right)^j=\left(\frac{x}{x-1}\right)^i(1-x)^{i+2}$$

and so

$$\sum_{j=1}^{\infty}K_m(j)\frac{x^j}{(1-x)^{j+1}}=\frac{(x-1)^2x^m}{1-x}\sum_{i=m}^{\infty}\left[\frac{i}{m}\right]x^{i-m}=\frac{x^m}{1-x^m}. \tag{13}$$

From equations (10) and (13), we obtain the following result.

Theorem 8: The functions $K_m(n)$ and $T_m^n(n)$ satisfy the orthogonality relations

$$\sum_{j=m}^{n}T_m^j(j)K_j(n)=\delta_m^n \quad\text{and}\quad \sum_{j=m}^{n}K_m(j)T_j^n(n)=\delta_m^n.$$

Therefore, we have the following general inversion result.

Theorem 9: For any ordered function sequence pair, $\langle f(n,m), g(n,m)\rangle$,

$$f(n,m)=\sum_{j=m}^{n}g(n,j)T_m^j(j)\ \text{if and only if}\ g(n,m)=\sum_{j=m}^{n}f(n,j)K_m(j).$$

The ordered function pair

$$\left\langle \binom{n+1}{m+1}, \left[\frac{n}{m}\right] \right\rangle$$

is a particular case of this theorem.

We also note the following concerning $T_m^n(n)$:

$$\sum_{j=1}^{n} T_j^n(n) = \sum_{j=1}^{n} \sum_{d|n} \mu(d) \binom{\frac{n}{d}}{j} = n \sum_{j=1}^{n} j \sum_{d|n} \frac{\mu(d)}{d} \binom{\frac{n}{d}-1}{j-1},$$

that is, $\sum_{j=1}^{n} T_j^n(n)$ is divisible by n. Furthermore,

$$\sum_{j=1}^{n} T_j^n(n) x^n = \sum_{j=1}^{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \binom{d}{j} x^n = \sum_{d|n} \mu\left(\frac{n}{d}\right) (x+1)^d,$$

from which we obtain

$$\sum_{j=1}^{n} T_j^n(n) = \sum_{j=1}^{n} \sum_{\substack{1 \le a_1 < a_2 < \cdots < a_j \le n \\ (a_1, \ldots, a_j, n) = 1}} 1 = \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d.$$

Similarly,

$$\sum_{j=1}^{n} K_j(n) x^j = \sum_{j=1}^{n} \sum_{i=j}^{n} (-1)^{n-i} \binom{n+1}{i+1} \left[\frac{i}{m}\right] x^j = \sum_{i=1}^{n} (-1)^{n-i} \binom{n+1}{i+1} \sum_{j=1}^{i} \left[\frac{i}{m}\right] x^j,$$

From which we obtain

$$\sum_{j=1}^{n} K_j(n) = \sum_{i=1}^{n} (-1)^{n-i} \binom{n+1}{i+1} \sum_{j=1}^{i} \left[\frac{i}{m}\right] = \sum_{j=1}^{n} \tau(j) \sum_{i=j}^{n} (-1)^{n-i} \binom{n+1}{i+1} = \sum_{j=1}^{n} \tau(j)(-1)^{n+j} \binom{n}{j}$$

Inversely, it may be shown that

$$\tau(n) = \sum_{j=1}^{n} \binom{n}{j} \sum_{i=1}^{j} K_i(j),$$

a result similar to one obtained by Gould [5, p. 255]. Following are tables of the arrays of the two functions $T_m^n(n)$ and $K_m(n)$.

TABLE 3. The $T_m^n(n)$ Array

$m \backslash n$	1	2	3	4	5	6	7	8
1	1	1	2	2	4	2	6	4
2	0	1	3	5	10	11	21	22
3	0	0	1	4	10	19	35	52
4	0	0	0	1	5	15	35	69
5	0	0	0	0	1	6	21	56
6	0	0	0	0	0	1	7	28
7	0	0	0	0	0	0	1	8
8	0	0	0	0	0	0	0	1

TABLE 4. The $K_m(n)$ Array

$m \backslash n$	1	2	3	4	5	6	7	8
1	1	−1	1	−1	1	−1	1	−1
2	0	1	−3	7	−15	−4	7	127
3	0	0	1	−4	10	−19	28	−28
4	0	0	0	1	−5	15	−34	71
5	0	0	0	0	1	−6	21	−56
6	0	0	0	0	0	1	−7	28
7	0	0	0	0	0	0	1	−8
8	0	0	0	0	0	0	0	1

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FIBONACCI PRIMITIVE ROOTS AND WALL'S QUESTION

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1. INTRODUCTION

Let F_n denote the n^{th} member of the Fibonacci sequence. Fix a positive integer m. We reduce $\{F_n\}_{n=0}^{\infty}$ modulo m, taking least positive residues. If $x = g$ satisfies the congruence

$$f(x) = x^2 - x - 1 \equiv 0 \pmod{m},$$

then, by setting $u_0 = 1$, $u_1 = g$, and $u_n = u_{n-1} + u_{n-2}$, we have that $u_n \equiv g^n \pmod{m}$. We have given particular attention to those cases having the longest possible cycles, i.e., the number g being a primitive root modulo m. We call g a Fibonacci primitive root modulo m if g is a root of $x^2 - x - 1 \equiv 0 \pmod{m}$ and g is a primitive root modulo m. For a fixed prime p, Fibonacci primitive roots modulo p have an extensive literature (see, e.g., [1], [3], [4], [5], [6], and [7]).

Consider the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ modulo m. The positive integer $z(m)$ is called the rank of apparition of m in the Fibonacci sequence if it is the smallest positive integer such that $F_{z(m)} \equiv 0 \pmod{m}$; furthermore, $k(m)$ is called the period of the Fibonacci sequence modulo m if it is the smallest positive integer for which $F_{k(m)} \equiv 0 \pmod{m}$ and $F_{k(m)+1} \equiv 1 \pmod{m}$. For a fixed prime p, Wall [10] has proved that, if $k(p) = k(p^e) \neq k(p^{e+1})$, then $k(p^l) = p^{l-e}k(p)$ for $l \geq e$. Wall asked whether $k(p) = k(p^2)$ is always impossible; up to now, this is still an open question. According to Williams [2], $k(p) \neq k(p^2)$ for every odd prime p less than 10^9. Sun and Sun [8] proved that the affirmative answer to Wall's question implies the first case of Fermat's last theorem.

In this paper we reproduce and improve upon some results for the Fibonacci primitive roots mentioned above. Especially, we give connections among the existence of the Fibonacci primitive roots modulo p^n and Wall's question. Our main theorem says that the affirmative answer to Wall's question [i.e., $k(p) \neq k(p^2)$] and the existence of Fibonacci primitive roots modulo p implies the existence of Fibonacci primitive roots modulo p^n for all positive integers n. This theorem overlaps in part with theorems proved by Phong [5], but our point of view and our methods are different from those of Phong, so that we obtain an effective method to decide whether $k(p) = k(p^2)$ or not.

2. PRELIMINARY RESULTS

In this section we briefly review some elementary results concerning primitive roots and some well-known results concerning the rank of apparition and the period of the Fibonacci sequence.

By Euler's theorem, if m is a positive integer and if a is an integer relatively prime to m, then $a^{\phi(m)} \equiv 1 \pmod{m}$, where $\phi(m)$ is defined to be the number of positive integers not exceeding m which are relatively prime to m. Denote by $\operatorname{ord}_m(a)$ the least positive integer x such that $a^x \equiv 1 \pmod{m}$. If $\operatorname{ord}_m(a) = \phi(m)$, then a is called a primitive root modulo m.

First, we observe that, if $f(x)$ is a polynomial in x with integer coefficients and x_k is a solution to $f(x) \equiv 0 \pmod{p^k}$, then $x_k + p^k y$ is a solution to $f(x) \equiv 0 \pmod{p^{k+1}}$ exactly when

$$f(x_k)+f'(x_k)p^k y \equiv 0 \pmod{p^{k+1}}.$$

This congruence is equivalent to

$$\frac{f(x_k)}{p^k}+f'(x_k)y \equiv 0 \pmod{p}.$$

In particular, if $p \nmid f'(x_k)$, then $(f'(x_k))^{-1}$ exists modulo p. Therefore,

$$y \equiv \frac{-f(x_k)}{p^k}(f'(x_k))^{-1} \pmod{p}$$

is the unique solution modulo p. On the other hand, if $p \mid f'(x_k)$, then y has p solutions modulo p or no solution depends on $f(x_k) \equiv 0 \pmod{p^{k+1}}$ or not. We now have the following lemma.

Lemma 2.1: Suppose that x_k is a solution to $f(x) \equiv 0 \pmod{p^k}$ and $p \nmid f'(x_k)$. Then there exists a unique x_{k+1} modulo p^{k+1} such that $x_{k+1} \equiv x_k \pmod{p^k}$ and $f(x_{k+1}) \equiv 0 \pmod{p^{k+1}}$. On the other hand, suppose that $p \mid f'(x_k)$ and $f(x_k) \not\equiv 0 \pmod{p^{k+1}}$. Then there exists no solution to $f(x) \equiv 0 \pmod{p^{k+1}}$.

A simple application of Lemma 2.1 is the following: suppose that

$$d \mid p-1 \quad \text{and} \quad a^d \equiv 1 \pmod{p}.$$

Since a is a solution to $f(x)=x^d-1 \equiv 0 \pmod{p}$ and $f'(a)=da^{d-1} \not\equiv 0 \pmod{p}$ [note that $(d,p)=(a,p)=1$], we have that there exists exactly one solution b modulo p^2 such that $b \equiv a \pmod{p}$ and $b^d \equiv 1 \pmod{p^2}$.

Lemma 2.2: Suppose that g is a primitive root modulo p. Then there exists a unique g' modulo p^2 such that $g' \equiv g \pmod{p}$ but g' is not a primitive root modulo p^2.

Proof: Suppose that $g' \equiv g \pmod{p}$ and $\mathrm{ord}_{p^2}(g')=m$. We have that

$$p-1 \mid m \quad \text{and} \quad m \mid p(p-1),$$

so $m=p(p-1)$ if and only if $(g')^{p-1} \not\equiv 1 \pmod{p^2}$. By the remark above, our claim follows. □

Let p be an odd prime. Suppose that g is a primitive root modulo p^2. Then we have that $g^{p-1} \not\equiv 1 \pmod{p^2}$. Thus, $g^{p-1}=1+\lambda p$ for some λ such that $p \nmid \lambda$. Hence,

$$g^{p(p-1)}=(1+\lambda p)^p \equiv 1+\lambda p^2 \pmod{p^3}.$$

By induction, we have that

$$g^{p^k(p-1)} \equiv 1+\lambda p^{k+1} \pmod{p^{k+2}}.$$

Lemma 2.3: Let p be an odd prime and let g be a primitive root modulo p^2. Then g is also a primitive root modulo p^n for all positive integers n.

Proof: Suppose that $\mathrm{ord}_{p^3(g)}=m$. Since g is a primitive root modulo p^2, we have that $p(p-1) \mid m \mid p^2(p-1)$. By the argument above, we have that $g^{p(p-1)} \not\equiv 1 \pmod{p^3}$. This implies that $m=p^2(p-1)$, i.e., g is a primitive root modulo p^3. Again, by the argument above and by induction, our claim follows. □

Let α and β be two distinct solutions to $x^2 - x - 1 \equiv 0 \pmod{m}$. Then we have the Binet form

$$F_n \equiv \frac{\alpha^n - \beta^n}{\alpha - \beta} \pmod{m}.$$

Since $\alpha^n \equiv \alpha^{n-1} + \alpha^{n-2} \pmod{m}$ and $\beta^n \equiv \beta^{n-1} + \beta^{n-2} \pmod{m}$, we also have that

$$\alpha^n \equiv \alpha F_{n-2} + \alpha^2 F_{n-1} \pmod{m} \quad \text{and} \quad \beta^n \equiv \beta F_{n-2} + \beta^2 F_{n-1} \pmod{m}.$$

This tells us that, if $k(m)$ is the period of the Fibonacci sequence modulo m, then $\mathrm{ord}_m(\alpha) \mid k(m)$ and $\mathrm{ord}_m(\beta) \mid k(m)$.

Lemma 2.4: Let α and β be two distinct solutions to $x^2 - x - 1 \equiv 0 \pmod{m}$ and let $k(m)$ be the period of the Fibonacci sequence modulo m. Then $k(m) = [\mathrm{ord}_m(\alpha), \mathrm{ord}_m(\beta)]$, where $[a, b]$ denotes the least common multiple of a and b.

Proof: Let $l = [\mathrm{ord}_m(\alpha), \mathrm{ord}_m(\beta)]$. By the argument above, we have that $l \mid k(m)$. On the other hand, $\alpha^l - \beta^l \equiv 0 \pmod{m}$ and $\alpha^{l+1} - \beta^{l+1} \equiv \alpha - \beta \pmod{m}$. This implies that $F_l \equiv 0$ (mod m) and $F_{l+1} \equiv 1 \pmod{m}$. Thus, $k(m) \mid l = [\mathrm{ord}_p(\alpha), \mathrm{ord}_p(\beta)]$, and our proof is complete. □

Let $\mathrm{ord}_m(\alpha) = n_1$ and $\mathrm{ord}_m(\beta) = n_2$. Suppose that $n_1 \geq n_2$. Since $\alpha\beta \equiv -1 \pmod{m}$, we have that $(\alpha)^{n_2} \equiv (\alpha\beta)^{n_2} \equiv (-1)^{n_2} \pmod{m}$. If n_2 is even, then $\alpha^{n_2} \equiv 1 \pmod{m}$. Thus, $n_1 \mid n_2$; hence, $n_1 = n_2$ by assumption. If n_2 is odd, then we have that $\alpha^{n_2} \equiv -1 \pmod{m}$ and so $n_1 \mid 2n_2$. This implies that $n_1 = n_2$ if n_1 is also odd and $n_1 = 2n_2$ if n_1 is even. However, it is impossible that $n_1 = n_2 \equiv 1 \pmod{2}$; otherwise, we will have that $1 \equiv (\alpha\beta)^{n_1} \equiv (-1)^{n_1} \equiv -1 \pmod{m}$. Hence, we have that n_1 is always even. Moreover, suppose that n_2 is odd. Then $n_1 = 2n_2$. Therefore, if $n_1 \equiv 0 \pmod{4}$, then $n_1 = n_2$. On the other hand, suppose that m is an odd prime power and suppose that $n_1 = 2r \equiv 2 \pmod{4}$, where r is odd. Then $\alpha^r \equiv -1 \pmod{m}$ and, hence, $-1 \equiv (\alpha\beta)^r \equiv -\beta^r \pmod{m}$. This implies that $\beta^r \equiv 1 \pmod{m}$. Thus, $n_2 = r$, and we have the following lemma.

Lemma 2.5: Let m be an odd prime power and let α and β be distinct roots of $x^2 - x - 1 \equiv 0$ (mod m). Suppose that $\mathrm{ord}_m(\alpha) \geq \mathrm{ord}_m(\beta)$. Then we have either $\mathrm{ord}_m(\alpha) = \mathrm{ord}_m(\beta) \equiv 0 \pmod{4}$ or $\mathrm{ord}_m(\alpha) = 2\mathrm{ord}_m(\beta) \equiv 2 \pmod{4}$.

Let $z(m)$ be the rank of apparition of m and let $k(m)$ be the period modulo m in the Fibonacci sequence. Wall [10] has shown that $z(m) \mid k(m)$. Vinson [9] gave criteria for the evaluation of $k(m) / z(m)$.

Lemma 2.6: Let p be an odd prime and let e be any positive integer. Then:

(1) $k(p^e) = 4z(p^e)$ if $z(p^e) \not\equiv 0 \pmod{2}$;

(2) $k(p^e) = z(p^e)$ if $z(p^e) \equiv 2 \pmod{4}$;

(3) $k(p^e) = 2z(p^e)$ if $z(p^e) \equiv 0 \pmod{4}$.

Proof: Please see Vinson [9, Theorem 2]. □

3. FIBONACCI PRIMITIVE ROOTS MODULO p

We begin with an easy observation that $x^2 - x - 1 \equiv 0 \pmod{p}$ is solvable if and only if $y^2 \equiv 5 \pmod{p}$ has solutions. If $p = 5$, then $x^2 - x - 1 \equiv 0 \pmod{5}$ has a double root $x \equiv 3 \pmod{5}$. Therefore, 3 is the unique Fibonacci primitive root modulo 5. $x^2 - x - 1 \equiv 0 \pmod{p}$ has two distinct solutions modulo p if p is an odd prime with $(5/p) = 1$, where $(5/p)$ is the Legendre symbol.

For the remainder of this section, we assume that p is an odd prime with $(5/p) = 1$.

The relation of the rank of apparition to the period modulo p in the Fibonacci sequence has been studied extensively by Wall [10] and Vinson [9]. We state their results in the next lemma without proof.

Lemma 3.1: Let $z(p)$ and $k(p)$ be the rank of apparition of p and the period modulo p in the Fibonacci sequence, respectively.

(1) Suppose that $p \equiv 11$ and $p \equiv 19 \pmod{20}$ [i.e., $(5/p) = 1$ and $(-1/p) = -1$]. Then we have $z(p) \mid p-1$, but $z(p) \nmid \frac{p-1}{2}$. Furthermore, $k(p) = z(p)$.

(2) Suppose that $p \equiv 1$ and $p \equiv 9 \pmod{20}$ [i.e., $(5/p) = 1$ and $(-1/p) = 1$]. Then we have $z(p) \mid \frac{p-1}{2}$. Furthermore, $k(p) = z(p)$, $2z(p)$, or $4z(p)$ depending on whether $z(p) \equiv 2$, 0, or $\pm 1 \pmod{4}$, respectively.

The conditions for the existence of Fibonacci primitive roots modulo p and their properties were studied by several authors. Our next theorem overlaps in part with theorems proved by Phong [5].

Theorem 3.2: Let $z(p)$ be the rank of apparition of p in the Fibonacci sequence.

(1) There is exactly one Fibonacci primitive root modulo p if and only if $p \equiv 11$ or $19 \pmod{20}$ and $z(p) = p-1$.

(2) There are two Fibonacci primitive roots modulo p if and only if $p \equiv 1$ or $9 \pmod{40}$ and $z(p) = \frac{p-1}{2}$ or $p \equiv 21$ or $29 \pmod{40}$ and $z(p) = \frac{p-1}{4}$.

Proof: We know that $(5/p) = 1$ if and only if $p \equiv \pm 1 \pmod{10}$. Let α and β be two distinct roots of $x^2 - x - 1 \equiv 0 \pmod{p}$ with $\mathrm{ord}_p(\alpha) \geq \mathrm{ord}_p(\beta)$.

(1) Suppose that $p \equiv 11$ or $19 \pmod{20}$ and $z(p) = p-1$. Then, since $p - 1 \equiv 2 \pmod{4}$, by Lemma 2.4, Lemma 2.5, and Lemma 3.1, $z(p) = k(p) = \mathrm{ord}_p(\alpha) = 2\mathrm{ord}_p(\beta) = p-1$. Conversely, suppose that there exists exactly one Fibonacci primitive root modulo p. Then, by Lemma 2.5, $\mathrm{ord}_p(\alpha) = 2\mathrm{ord}_p(\beta) \equiv 2 \pmod{4}$. Therefore, by Lemma 2.4, $k(p) = \mathrm{ord}_p(\alpha) = p-1$. Hence, $p \equiv 11$ or $19 \pmod{20}$ and $z(p) = k(p) = p-1$ by Lemma 3.1.

(2) Suppose that $p \equiv 1$ or $9 \pmod{40}$ and $z(p) = \frac{p-1}{2}$. Then, since $\frac{p-1}{2} \equiv 0 \pmod{4}$, by the lemmas mentioned in (1), $2z(p) = k(p) = \mathrm{ord}_p(\alpha) = \mathrm{ord}_p(\beta) = p-1$. Suppose that $p \equiv 21$ or $29 \pmod{40}$ and $z(p) = \frac{p-1}{2}$. Then, since $\frac{p-1}{4} \equiv 1 \pmod{2}$, again by the lemmas mentioned in (1), $4z(p) = k(p) = \mathrm{ord}_p(\alpha) = \mathrm{ord}_p(\beta) = p-1$. Conversely, suppose that there exist two Fibonacci primitive roots modulo p. Then, by Lemma 2.5, $\mathrm{ord}_p(\alpha) = \mathrm{ord}_p(\beta) \equiv 0 \pmod{4}$. Therefore, by Lemma 2.4, $k(p) = p-1 \equiv 0 \pmod{4}$. Hence, by Lemma 3.1, our claim follows. □

Theorem 3.2 reproduces results for Fibonacci primitive roots modulo p in [1], [3], [4], [6], and [7]. For example, Mays [4] showed that, if both $p = 60k - 1$ and $q = 30k - 1$ are primes, then there is exactly one Fibonacci primitive root modulo p. In fact, since $p \equiv 19 \pmod{20}$ and $2q = p - 1$, by Lemma 3.1, we have either $z(p) = p - 1$ or $z(p) = \frac{p-1}{q} = 2$ (by the assumption that q is a prime). We obtain $z(p) \neq 2$, because $F_2 = 1$. Therefore, $z(p) = p - 1$. By the theorem above, we conclude that there exists exactly one Fibonacci primitive root modulo p. By a similar method, we have the following proposition.

Proposition 3.3: Let p be a prime such that $p \equiv 11$ or $19 \pmod{20}$ and $p - 1 = 2q$, where q is a prime. Then there exists exactly one Fibonacci primitive root modulo p.

Example 1: In the case , $p - 1 = 2 \cdot 5$. There is exactly one Fibonacci primitive root modulo 11, which is 8. When $p = 59$, $p - 1 = 2 \cdot 29$. There is exactly one Fibonacci primitive root modulo 59, which is 34.

When $p \equiv 1$ or $9 \pmod{20}$, the situation is more complicated, because it is possible that $4z(p) \mid p - 1$. There are many articles discussed for which p, $4z(p) \mid p - 1$ (see, e.g., [2], [8], and [11]). Here, we quote the result in [8].

Lemma 3.4: Let p be a prime such that $p \equiv 1$ or $9 \pmod{20}$ and, hence, $p = x^2 + 5y^2$ for some integers x and y. Then $4z(p) \mid p - 1$ if and only if $4 \mid xy$.

Suppose that $p \equiv 1$ or $9 \pmod{40}$ [resp. $p \equiv 21$ or $29 \pmod{40}$]. By Theorem 3.2, there exist Fibonacci primitive roots modulo p only if $4z(p) \nmid p - 1$ [resp. $4z(p) \mid p - 1$].

Proposition 3.5: Let p be a prime such that $p \equiv 1$ or $9 \pmod{20}$ and, hence, $p = x^2 + 5y^2$ for some integers x and y.

(1) Suppose that $p \equiv 1$ or $9 \pmod{40}$. Then there is no Fibonacci primitive root modulo p if $4 \mid xy$. Suppose that $4 \nmid xy$ and $p - 1 = 8q$, where q is a prime. Then there exist two Fibonacci primitive roots modulo p.

(2) Suppose that $p \equiv 21$ or $29 \pmod{40}$. Then there is no Fibonacci primitive root modulo p if $4 \nmid xy$. Suppose that $4 \mid xy$ and $p - 1 = 4q$, where q is a prime. Then there exist two Fibonacci primitive roots modulo p.

Proof:

(1) Suppose that $4 \mid xy$. By Lemma 3.4, $4z(p) \mid p - 1$. We have that $k(p) \leq \frac{p-1}{2}$, by Lemma 2.6. Hence, there is no Fibonacci primitive root modulo p. Suppose that $4 \nmid xy$ and $p - 1 = 8q$, where q is a prime. Then we have either $z(p) = \frac{p-1}{2}$ or $z(p) = \frac{p-1}{2q} = 4$. However, $z(p) \neq 4$, because $F_4 = 3$. By Theorem 3.2, our claim follows.

(2) Suppose that $4 \nmid xy$. By Lemma 3.4, $4z(p) \nmid p - 1$. Since $2z(p) \mid p - 1$, this implies that $k(p) = z(p) \leq \frac{p-1}{2}$, by Lemma 2.6. Hence, there is no Fibonacci primitive root modulo p. Suppose that $4 \mid xy$ and $p - 1 = 4q$, where q is a prime. Then we have either $z(p) = \frac{p-1}{4}$ or $z(p) = \frac{p-1}{4q} = 1$. However, $z(p) \neq 1$, because $F_1 = 1$. By Theorem 3.2, our claim follows.

Example 2: Since $29 = 3^2 + 5(2^2)$ and $4 \nmid 3 \cdot 2$, there is no Fibonacci primitive root modulo 29. Since $41 = 6^2 + 5$, $4 \nmid 6$, and $41 - 1 = 8 \cdot 5$, there are two Fibonacci primitive roots modulo 41 (namely, 35 and 7). There are two Fibonacci primitive roots modulo 149 (namely, 41 and 109), because $149 = 12^2 + 5$, $4 \mid 12$, and $149 - 1 = 4 \cdot 37$.

Remark 1: Since $F_8 = 3 \cdot 7$, $F_{16} = 3 \cdot 7 \cdot 47$, and $F_{32} = 3 \cdot 7 \cdot 47 \cdot 2207$, we have that, for $p \equiv 1$ or 9 (mod 40), $z(p) \neq 8$, 16, or 32. Therefore, part (1) of Proposition 3.5 is also true, if $p - 1 = 16q$, $32q$, or $64q$ for some odd prime q.

4. FIBONACCI PRIMITIVE ROOTS MODULO p^n

It is well known that the positive integer m possesses a primitive root if and only if $m = 2, 4, p^n$, or $2p^n$, where p is an odd prime. Since there is no solution to $x^2 - x - 1 \equiv 0 \pmod 2$, we only have to consider the case $m = p^n$.

First, we consider the case $p = 5$. Let $f(x) = x^2 - x - 1$. We have that $f(3) = 5 \equiv 0 \pmod 5$. However, since $f'(3) = 5 \equiv 0 \pmod 5$, by Lemma 2.1, there is no solution to $f(x) = x^2 - x - 1 \equiv 0 \pmod{5^2}$. Hence, there is no Fibonacci primitive root modulo 5^n for $n \geq 2$. On the other hand, suppose that $p \neq 5$ and $(5/p) = 1$. There exist two distinct roots, α and β such that $f(\alpha) \equiv f(\beta) \equiv 0 \pmod p$. We have that $f'(\alpha) = 2\alpha - 1 \not\equiv 0 \pmod p$; otherwise, $0 \equiv 4\alpha^2 - 4\alpha - 4 \equiv 1 - 2 - 4 \equiv -5 \pmod p$ contradicts our assumption. Using the same reasoning, we have that $f'(\beta) \not\equiv 0 \pmod p$. Therefore, by Lemma 2.1, we conclude that there exist two distinct roots to $x^2 - x - 1 \equiv 0 \pmod{p^2}$. By induction, we have the following lemma.

Lemma 4.1: Let p be an odd prime such that $p \equiv \pm 1 \pmod{20}$. Then there exist two distinct roots to $x^2 - x - 1 \equiv 0 \pmod{p^n}$ for every positive integer n. Furthermore, suppose that α is a root to $x^2 - x - 1 \equiv 0 \pmod p$. Then there exists a unique α_n modulo p^n such that $\alpha_n^2 - \alpha_n - 1 \equiv 0 \pmod{p^n}$ and $\alpha_n \equiv \alpha \pmod p$.

Suppose that α is a Fibonacci primitive root modulo p. By the argument above, there exists exactly one α_2 modulo p^2 such that $\alpha_2^2 - \alpha_2 - 1 \equiv 0 \pmod{p^2}$ and $\alpha_2 \equiv \alpha \pmod p$. Suppose that α_2 is a primitive root modulo p^2. Then α_2 is a Fibonacci primitive root modulo p^2. In this case, by Lemma 2.4, $k(p^2)$, the period of the Fibonacci sequence modulo p^2, is equal to $\mathrm{ord}_{p^2}(\alpha_2) = p(p-1) = pk(p)$, and since p is odd, by Lemma 2.6, this is equivalent to $z(p^2) = pz(p)$, i.e., $p^2 \nmid F_{z(p)}$. On the other hand, suppose that $p^2 \nmid F_{z(p)}$. Then $k(p^2) = pk(p) = p(p-1)$. By Lemma 2.4 and Lemma 2.5, this implies that $\mathrm{ord}_{p^2}(\alpha_2) = \frac{p(p-1)}{2}$ or $\mathrm{ord}_{p^2}(\alpha_2) = p(p-1)$. By assumption, α_2 is a primitive root modulo p and, hence, $\mathrm{ord}_{p^2}(\alpha_2)$ is either $(p-1)$ or $p(p-1)$. This implies that α_2 is a primitive root modulo p^2.

Theorem 4.2: Let p be an odd prime such that $p \equiv \pm 1 \pmod{20}$. Suppose that there is a Fibonacci primitive root modulo p. Then there is a Fibonacci primitive root modulo p^n for every positive integer n if and only if $p^2 \nmid F_{z(p)}$, where $z(p)$ is the least positive integer such that $p \mid F_{z(p)}$.

Proof: We only have to claim that the existence of a Fibonacci primitive root modulo p^2 implies the existence of a Fibonacci primitive root modulo p^n. Suppose that α_2 is a Fibonacci primitive root modulo p^2. By a similar argument as in Lemma 4.1, there exists α_n such that $\alpha_n^2 - \alpha_n - 1 \equiv 0 \pmod{p^n}$ and $\alpha_n \equiv \alpha_2 \pmod{p^2}$. However, Lemma 2.3 says that α_2 is a primitive root modulo p^n for every positive integer n. $\alpha_n \equiv \alpha_2 \pmod{p^2}$ implies that α_n is also a primitive root modulo p^n. Hence, α_n is a Fibonacci primitive root modulo p^n.

Remark 2: According to Williams [12], $p^2 \nmid F_{p-(5/p)}$ [this is equivalent to $p^2 \nmid F_{z(p)}$] for every odd prime p less than 10^9. Therefore, for $p < 10^9$, suppose that there exists a Fibonacci primitive root modulo p. Then there exists a Fibonacci primitive root modulo p^n. Furthermore, since p is odd, by Lemma 2.5, the number of distinct Fibonacci primitive roots modulo p^n is the same as the number of distinct Fibonacci primitive roots modulo p.

Suppose that α is a root to $x^2 - x - 1 \equiv 0 \pmod{p}$. Then there exists a unique α_2 modulo p^2 such that $\alpha_2 \equiv \alpha \pmod{p}$ and $\alpha_2^2 - \alpha_2 - 1 \equiv 0 \pmod{p^2}$. On the other hand, suppose that α is a primitive root modulo p. By Lemma 2.2, there exists a unique α' modulo p^2 such that $\alpha' \equiv \alpha \pmod{p}$ and α' is not a primitive root modulo p^2. Therefore, $\alpha' \equiv \alpha_2 \pmod{p^2}$ if and only if $p^2 \mid F_{z(p)}$ [or, equivalently, $k(p) = k(p^2)$].

Theorem 4.3: Let p be an odd prime such that $(5/p) = 1$ and let α be a Fibonacci primitive root modulo p. Then there exists a Fibonacci primitive root modulo p^n for every positive integer n if and only if $2\alpha^{p+1} - \alpha^p - \alpha^2 - 1 \not\equiv 0 \pmod{p^2}$.

Proof: By Theorem 4.2, the existence of a Fibonacci primitive root modulo p^2 implies the existence of a Fibonacci primitive root modulo p^n for every positive integer n. By the argument above, there is no Fibonacci primitive root modulo p^2 if and only if there exists λ such that $(\alpha + \lambda p)^2 - (\alpha + \lambda p) - 1 \equiv 0 \pmod{p^2}$ and $(\alpha + \lambda p)^{p-1} - 1 \equiv 0 \pmod{p^2}$. Expand both congruence equations and eliminate λ. This implies that α must satisfy $2\alpha^{p+1} - \alpha^p - \alpha^2 - 1 \equiv 0 \pmod{p^2}$. Conversely, suppose that $\alpha_2 \equiv \alpha + \lambda p \pmod{p^2}$ is a solution to $x^2 - x - 1 \equiv 0 \pmod{p^2}$ and suppose that $2\alpha^{p+1} - \alpha^p - \alpha^2 - 1 \equiv 0 \pmod{p^2}$. We have that

$$\begin{aligned} 2\alpha_2^{p+1} - \alpha_2^p - \alpha_2^2 - 1 &= 2\alpha_2^{p+1} - 2\alpha_2^p - 2\alpha_2^{p-1} + \alpha_2^p + 2\alpha_2^{p-1} - \alpha_2 - 2 \\ &\equiv (\alpha_2 + 2)(\alpha_2^{p-1} - 1) \pmod{p^2}. \end{aligned}$$

Since $2\alpha^{p+1} - \alpha^p - \alpha^2 - 1 \equiv 2\alpha_2^{p+1} - \alpha_2^p - \alpha_2^2 - 1 \pmod{p^2}$, this implies that $(\alpha_2 + 2)(\alpha_2^{p-1} - 1) \equiv 0 \pmod{p^2}$. Suppose that $\alpha_2 + 2 \equiv 0 \pmod{p}$. Then, since $\alpha_2^2 - \alpha_2 - 1 \equiv 0 \pmod{p}$, this implies that $5 \equiv 0 \pmod{p}$, which contradicts our assumption that $p \neq 5$. Hence, $\alpha_2^{p-1} \equiv 1 \pmod{p^2}$. This implies that α_2 is not a primitive root modulo p^2, and our proof is complete. □

Remark 3: From our proof, we have a more general result concerning Wall's question. We have the following result: suppose that α is a solution to $x^2 - x - 1 \equiv 0 \pmod{p}$ (we do not need the assumption that α is a primitive root modulo p). Then $k(p) = k(p^2)$ if and only if

$$2\alpha^{p+1} - \alpha^p - \alpha^2 - 1 \equiv 0 \pmod{p^2}.$$

For the case $(5/p) = -1$, we have a similar result. We should consider everything in the ring $Z[\frac{1+\sqrt{5}}{2}]$ modulo p. We have the following result: suppose $\alpha \in Z[\frac{1+\sqrt{5}}{2}]$ is a solution to $x^2 - x - 1 \equiv 0 \pmod{p}$. Then $k(p) = k(p^2)$ if and only if

$$2\alpha^{p^2+1} - \alpha^{p^2} - \alpha^2 - 1 \equiv 0 \pmod{p^2}.$$

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stan@wwa.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

B-866 ***Proposed by the editor***

For n an integer, show that $L_{8n+4} + L_{12n+6}$ is always divisible by 25.

B-867 ***Proposed by the editor***

Find small positive integers a and b so that 1999 is a member of the sequence $\langle u_n \rangle$, defined by $u_0 = 0$, $u_1 = 1$, $u_n = au_{n-1} + bu_{n-2}$ for $n > 1$.

B-868 ***Based on a proposal by Richard André-Jeannin, Longwy, France***

Find an integer $a > 1$ such that, for all integers n, $F_{an} \equiv aF_n \pmod{25}$.

B-869 ***Based on a communication by Larry Taylor, Rego Park, NY***

Find a polynomial $f(x)$ such that, for all integers n, $2^n F_n \equiv f(n) \pmod 5$.

B-870 ***Proposed by Richard André-Jeannin, Longwy, France***

Solve the equation

$$\tan^{-1} y - \tan^{-1} x = \tan^{-1} \frac{1}{x+y}$$

in nonnegative integers x and y, expressing your answer in terms of Fibonacci and/or Lucas numbers.

B-871 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Prove that

$$\sum_{k=0}^{2n} \binom{2n}{k} |n-k|^3 = n^2 \binom{2n}{n}.$$

Notice to proposers: All problems submitted prior to 1999 for consideration for the Elementary Problems Column that have not yet been used are hereby released back to their authors.

SOLUTIONS

Class Identity

B-848 ***Proposed by Russell Euler's Fall 1997 Number Theory Class, Northwest Missouri State University, Maryville, MO (Vol. 36, no. 2, May 1998)***

Prove that $F_nF_{n+1} - F_{n+6}F_{n-5} = 40(-1)^{n+1}$ for all integers n.

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

We shall prove a generalization. For all integers n and k, since $\alpha\beta = -1$, we have

$$\begin{aligned}(\alpha^n - \beta^n)(\alpha^{n+1} - \beta^{n+1}) &- (\alpha^{n+k} - \beta^{n+k})(\alpha^{n-k+1} - \beta^{n-k+1}) \\ &= -\alpha^n\beta^{n+1} - \beta^n\alpha^{n+1} + \alpha^{n+k}\beta^{n-k+1} + \beta^{n+k}\alpha^{n-k+1} \\ &= (\alpha\beta)^{n+1}(\alpha+\beta) + (\alpha\beta)^{n-k+1}(\alpha^{2k-1} + \beta^{2k-1}) \\ &= (\alpha\beta)^{n-k+1}[(\alpha\beta)^k(\alpha+\beta) + \alpha^{2k-1} + \beta^{2k-1}] \\ &= (-1)^{n-k+1}[-\alpha^k\beta^{k-1} - \alpha^{k-1}\beta^k + \alpha^{2k-1} + \beta^{2k-1}] \\ &= (-1)^{n-k+1}(\alpha^k - \beta^k)(\alpha^{k-1} - \beta^{k-1}).\end{aligned}$$

It follows from the Binet Formula that $F_nF_{n+1} - F_{n+k}F_{n-k+1} = (-1)^{n-k+1}F_kF_{k-1}$. In particular,

$$F_nF_{n+1} - F_{n+6}F_{n-5} = (-1)^{n-5}F_6F_5 = 40(-1)^{n+1}$$

for all integers n.

Several readers found the generalization

$$F_{n+a}F_{n+b} - F_nF_{n+a+b} = (-1)^nF_aF_b,$$

which comes from formula (20a) of [1].

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Solutions also received by Brian Beasley, David M. Bloom, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Herta T. Freitag, Pentti Haukkanen, Russell Jay Hendel, Hans Kappus, Carl Libis, Bob Prielipp, Maitland A. Rose, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Fibonacci Arithmetic Progression

B-849 ***Proposed by Larry Zimmerman & Gilbert Kessler, New York, NY (Vol. 36, no. 2, May 1998)***

If F_a, F_b, F_c, x forms an increasing arithmetic progression, show that x must be a Lucas number.

Solution by H.-J. Seiffert, Berlin, Germany

In view of the counterexample, F_{-4}, F_0, F_4, 6, we must suppose that $a \geq 0$.

Let a, b, and c be nonnegative integers such that $F_a < F_b < F_c < x$ and $F_b - F_a = F_c - F_b = x - F_c$. If $c-2 \geq b$, then $F_{c-2} \geq F_b$, and from

$$2F_b = F_a + F_c \geq F_c = F_{c-1} + F_{c-2} \geq 2F_b,$$

it follows that we get equality; hence, $F_a = 0$ and $F_{c-1} = F_{c-2} = F_b$. Thus, $a = 0$, $b = 1$, and $c = 3$, so that $x = 2F_c - F_b = 2F_3 - F_1 = 3 = L_2$ is a Lucas number.

Now suppose that $c-2 < b$ or $c-1 \leq b$. Since $0 \leq F_b < F_c$, we must have $b < c$. Thus, $c-1 \leq b < c$ and we must have $b = c-1$. In this case, $x = 2F_c - F_{c-1} = L_{c-1}$ is also a Lucas number.

Solutions also received by Paul S. Bruckman, Aloysius Dorp, Leonard A. G. Dresel, Russell Euler, Russell Jay Hendel, N. J. Kuenzi & Bob Prielipp, Bob Prielipp, Indulis Strazdins, and the proposers.

Unknown Subscripts

B-850 ***Proposed by Al Dorp, Edgemere, NY***
(Vol. 36, no. 2, May 1998)

Find distinct positive integers a, b, and c so that $F_n = 17F_{n-a} + cF_{n-b}$ is an identity.

Solution by Leonard A. G. Dresel, Reading, England

We shall find two solutions, namely $a = 6, b = 9, c = 4$, and $a = 12, b = 9, c = 72$, and show that these are the only solutions.

Putting $n = b$ in the given identity, we have $F_b = 17F_{b-a}$, so that 17 divides F_b, giving $b = 9$, 18, 27, ... In fact, $b = 9$ is the only solution since, for $b > 9$, we have $17_{b-6} < F_b < 17_{b-5}$. Letting $b = 9$, we have $17F_{9-a} = F_9 = 34 = 17F_3$, which gives $9 - a = \pm 3$, so that we have $a = 6$ or $a = 12$. To determine c, we put $n = 10$ in the given identity. Then, with $a = 6$, we have $F_{10} = 17F_4 + cF_1$, giving $c = 55 - 51 = 4$; whereas, with $a = 12$, we have $F_{10} = 17F_{-2} + cF_1$, giving $c = 55 + 17 = 72$. We therefore obtain the two identities $F_n = 17F_{n-6} + 4F_{n-9}$ and $F_n = 17F_{n-12} + 72F_{n-9}$. Each identity is true for $n = 9$ and $n = 10$, and can therefore be shown to be true for all n by induction.

Most solvers only found one solution.

Solutions also received by Brian Beasley, Paul S. Bruckman, Russell Jay Hendel, Daina A. Krigens, H.-J. Seiffert, Indulis Strazdins, and the proposer. Partial solution by A. Plaza & M. A. Padrón.

Repeating Series

B-851 ***Proposed by Pentti Haukkanen, University of Tampere, Tampere, Finland***
(Vol. 36, no. 2, May 1998)

Consider the repeating sequence $\langle A_n \rangle_{n=0}^{\infty} = 0, 1, -1, 0, 1, -1, 0, 1, -1, \ldots$.

(a) Find a recurrence formula for A_n.

(b) Find an explicit formula for A_n of the form $(a^n - b^n)/(a-b)$.

Solution by H.-J. Seiffert, Berlin, Germany

Since the sum of any three consecutive terms of the sequence is seen to be 0, we have the recurrence $A_{n+2} = -A_{n+1} - A_n$, for $n \geq 0$.

Based on the equation

$$\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2},$$

a simple induction argument shows that

$$A_n=\frac{2}{\sqrt{3}}\sin\frac{2n\pi}{3},\quad \text{for } n\geq 0.$$

Using Euler's Relation

$$\sin x=\frac{e^{ix}-e^{-ix}}{2i},$$

from above we get

$$A_n=\frac{e^{2n\pi i/3}-e^{-2n\pi i/3}}{e^{2\pi i/3}-e^{-2\pi i/3}},\quad \text{for } n\geq 0,$$

i.e., $a=e^{2\pi i/3}$ and $b=e^{-2\pi i/3}$ work. Equivalently,

$$a=\frac{-1+i\sqrt{3}}{2}\quad\text{and}\quad b=\frac{-1-i\sqrt{3}}{2}.$$

Cook found the recurrence $A_{n+3}=A_n$. Libis found the amazing recurrence

$$A_n=(-1)^{1+A_{n-2}}A_{n-1}+(-1)^{1+A_{n-1}}A_{n-2}.$$

Solutions also received by Brian Beasely, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Russell Jay Hendel, Hans Kappus, Harris Kwong, Carl Libis, A. Plaza & M. A. Padrón, Indulis Strazdins, and the proposer.

The Determinant Vanishes

B-852 ***Proposed by Stanley Rabinowitz, Westford, MA***
(Vol. 36, no. 2, May 1998)

Evaluate

$$\begin{vmatrix} F_0 & F_1 & F_2 & F_3 & F_4 \\ F_9 & F_8 & F_7 & F_6 & F_5 \\ F_{10} & F_{11} & F_{12} & F_{13} & F_{14} \\ F_{19} & F_{18} & F_{17} & F_{16} & F_{15} \\ F_{20} & F_{21} & F_{22} & F_{23} & F_{24} \end{vmatrix}.$$

Solution by Indulis Strazdins, Riga Tech. University, Riga, Latvia

Adding the 1st row and the 5th row, we obtain $F_{n+20}+F_n$ in the n^{th} column, $n=0,1,2,3,4$. This expression is equal to $L_{10}F_{n+10}$ by using identity (15a) of [1], which says that

$$F_{n+m}+(-1)^m F_{n-m}=L_m F_n$$

for all integers m and n. The corresponding element of the 3rd row is F_{n+10}, $n=0,1,2,3,4$. These rows are proportional, and hence the value of the determinant is 0.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Generalization by Pentti Haukkanen, University of Tampere, Tampere, Finland

Let $\langle w_n \rangle$ be a recurrence sequence defined by $w_{n+2} = aw_{n+1} + bw_n$, $n \geq 0$. Define the $m \times m$-determinant as

$$D_m = \begin{vmatrix} w_0 & w_1 & \cdots & w_{m-1} \\ w_{2m-1} & w_{2m-2} & \cdots & w_m \\ w_{2m} & w_{2m+1} & \cdots & w_{3m-1} \\ \vdots & \vdots & \ddots & \vdots \end{vmatrix}.$$

The m^{th} row is

$$w_{(m-1)m} \quad w_{(m-1)m+1} \quad \cdots \quad w_{m^2-1}$$

if m is odd, and

$$w_{m^2-1} \quad w_{m^2-2} \quad \cdots \quad w_{(m-1)m}$$

if m is even.

We show that $D_m = 0$ whenever $m \geq 5$.

We add the $(m-1)^{\text{th}}$ column multiplied with a and the $(m-2)^{\text{th}}$ column multiplied with b to the m^{th} column. Then D_m reduces to the form

$$D_m = \begin{vmatrix} w_0 & w_1 & \cdots & w_{m-2} & 0 \\ w_{2m-1} & w_{2m-2} & \cdots & w_{m+1} & * \\ w_{2m} & w_{2m+1} & \cdots & w_{3m-2} & 0 \\ w_{4m-1} & w_{4m-2} & \cdots & w_{3m+1} & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{vmatrix}.$$

Proceeding in a similar way with respect to the $(m-1)^{\text{th}}$, the $(m-2)^{\text{th}}$, ..., the 3^{th} column, the determinant D_m reduces to the form

$$D_m = \begin{vmatrix} w_0 & w_1 & 0 & 0 & \cdots & 0 \\ w_{2m-1} & w_{2m-2} & * & * & \cdots & * \\ w_{2m} & w_{2m+1} & 0 & 0 & \cdots & 0 \\ w_{4m-1} & w_{4m-2} & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix}.$$

Now, it is easy to see that $D_m = 0$ whenever $m \geq 5$, since we have a square matrix of zeros involving more than half of the rows of the whole matrix.

Comment by the proposer: Let $\langle w_n \rangle$ be any second-order linear recurrence defined by the recurrence $w_n = Pw_{n-1} - Qw_{n-2}$. Consider the determinant

$$\begin{vmatrix} w_0 & w_1 & w_2 & w_3 & w_4 & \cdots \\ * & * & * & * & * & \cdots \\ w_a & w_{a+1} & w_{a+2} & w_{a+3} & w_{a+4} & \cdots \\ * & * & * & * & * & \cdots \\ w_{2a} & w_{2a+1} & w_{2a+2} & w_{2a+3} & w_{2a+4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix},$$

where the asterisks can be any values whatsoever. All the rows after the fifth one can have any values as well. Then the value of this determinant is 0 because the 1^{st}, 3^{rd}, and 5^{th} rows are linearly dependent. This follows from the identity $w_{an+b} = v_a w_{a(n-1)+b} - Q^a w_{a(n-2)+b}$, which is straightforward to verify by using algorithm `LucasSimplify` from [1].

Reference

1. Stanley Rabinowitz. "Algorithmic Manipulation of Fibonacci Identities." In *Applications of Fibonacci Numbers* **6**:389-408. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.

Solutions also received by Charles Ashbacher, Brian Beasley, David M. Bloom, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Pentti Haukkanen, Russell Jay Hendel, Hans Kappus, H.-J. Seiffert, and the proposer. One incorrect solution was received.

A Deranged Sequence

B-853 ***Proposed by Gene Ward Smith, Brunswick, ME (Vol. 36, no. 2, May 1998)***

Consider the recurrence $f(n+1)=n(f(n)+f(n-1))$ with initial conditions $f(0)=1$ and $f(1)=0$. Find a closed form for the sum

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}f(k).$$

Solution by Hans Kappus, Rodersdorf, Switzerland

We claim that $S(n)=n!$.

Proof:

$$\begin{aligned}
S(n+1)-S(n) &= \sum_{k=0}^{n}\left[\binom{n+1}{k}-\binom{n}{k}\right]f(k)+f(n+1)\\
&= \sum_{k=1}^{n}\binom{n}{k-1}f(k)+f(n+1)\\
&= \sum_{k=0}^{n}\binom{n}{k}f(k+1) \qquad (*)\\
&= n\sum_{k=1}^{n}\binom{n-1}{k-1}[f(k)+f(k-1)]\\
&= n\left[\sum_{k=0}^{n-1}\binom{n-1}{k}f(k+1)+\sum_{k=0}^{n-1}\binom{n-1}{k}f(k)\right]\\
&= n[S(n)-S(n-1)+S(n-1)] \quad \text{because of } (*)\\
&= nS(n).
\end{aligned}$$

Hence, $S(n+1)=(n+1)S(n)$. Since $S(0)=1$, the proof is complete.

Cook observes that S satisfies the same recurrence as f with different initial conditions. Many readers pointed out that the recurrence is well known for the number $f(n)$ of derangements (permutations with no fixed points) of the set $\{1,2,3,\ldots,n\}$. See J. Riordan's Introduction to Combinatorial Analysis *(New York: Wiley, 1958) for more information about the derangement number.*

Solutions also received by David M. Bloom, Paul S. Bruckman, Charles K. Cook, Carl Libis, H.-J. Seiffert, Indulis Strazdins, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-547 ***Proposed by T. V. Padmakumar, Thycaud, India***

Prove: If p is a prime number, then

$$\left[\sum_{n=1}^{p}\frac{1}{(2n-1)}\right]^2-\left[\sum_{n=1}^{p}\frac{1}{(2n-1)^2}\right]\equiv 0 \pmod p.$$

H-548 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the sequence of Pell numbers by $P_0=0$, $P_1=1$, and $P_{n+2}=2P_{n+1}+P_n$ for $n\geq 0$. Show that if q is a prime such that $q\equiv 1 \pmod 8$ then

$$q\,|\,P_{(q-1)/4} \text{ if and only if } 2^{(q-1)/4}\equiv(-1)^{(q-1)/8} \pmod q.$$

H-549 ***Proposed by Paul S. Bruckman, Highwood, IL***

Evaluate the expression:

$$\sum_{n\geq 1}(-1)^{n-1}\tan^{-1}(1/F_{2n}). \tag{1}$$

SOLUTIONS

Exactly Right

H-532 ***Proposed by Paul S. Bruckman, Highwood, IL (Vol. 35, no. 4, November 1997)***

Let $V_n=V_n(x)$ denote the generalized Lucas polynomials defined as follows: $V_0=2$; $V_1=x$; $V_{n+2}=xV_{n+1}+V_n$, $n=0,1,2,\ldots$. If n is an odd positive integer and y is any real number, find all (exact) solutions of the equation: $V_n(x)=y$.

Solution by H.-J. Seiffert, Berlin, Germany

It is well known that $V_n(x)$ is a polynomial of degree n and that, for all complex numbers x, $V_n(x)=\alpha(x)^n+\beta(x)^n$, where

$$\alpha(x)=(x+\sqrt{x^2+4})/2 \quad\text{and}\quad \beta(x)=(x-\sqrt{x^2+4})/2.$$

Here, $\sqrt{x^2+4}$ can be any of the at most two possible roots of x^2+4.

Let $n \in N$ be odd and $y \in R$. We show that the solutions of the equation $V_n(x) = y$ are

$$x = x_k = \sqrt[n]{\alpha(y)}\exp\left(\frac{2k\pi i}{n}\right) + \sqrt[n]{\beta(y)}\exp\left(-\frac{2k\pi i}{n}\right)$$
$$= \left(\sqrt[n]{\alpha(y)} + \sqrt[n]{\beta(y)}\right)\cos\left(\frac{2k\pi}{n}\right) + i\left(\sqrt[n]{\alpha(y)} - \sqrt[n]{\beta(y)}\right)\sin\left(\frac{2k\pi}{n}\right),\ k = 0, \ldots, n-1.$$

Here, we consider the main branch of the n^{th} root.

Since $\alpha(y) + \beta(y) = y$ is real and n is odd, it is easily seen that $x_0, \ldots, x_{n-1}$ are n distinct complex numbers. However, the equation $V_n(x) = y$ cannot have more than n distinct solutions, so that we are done if we prove that $V_n(x_k) = y$ for $k = 0, \ldots, n-1$.

Since n is odd and $\alpha(y)\beta(y) = -1$, we find

$$x_k^2 + 4 = \sqrt[n]{\alpha(y)^2}\exp\left(\frac{4k\pi i}{n}\right) + \sqrt[n]{\beta(y)^2}\exp\left(-\frac{4k\pi i}{n}\right) + 2,$$

which implies

$$\sqrt{x_k^2 + 4} = \pm\left(\sqrt[n]{\alpha(y)}\exp\left(\frac{2k\pi i}{n}\right) - \sqrt[n]{\beta(y)}\exp\left(-\frac{2k\pi i}{n}\right)\right).$$

It follows that

$$x_k \pm \sqrt{x_k^2 + 4} = 2\sqrt[n]{\alpha(y)}\exp\left(\frac{2k\pi i}{n}\right) \quad \text{and} \quad x_k \mp \sqrt{x_k^2 + 4} = 2\sqrt[n]{\beta(y)}\exp\left(-\frac{2k\pi i}{n}\right).$$

In each case, we have $V_n(x_k) = \alpha(x_k)^n + \beta(x_k)^n = \alpha(y) + \beta(y) = y$.

Also solved by G. Smith and the proposer.

Enter at Your Own Risk

H-533 ***Proposed by Andrej Dujella, University of Zagreb, Croatia (Vol. 35, no. 4, November 1997)***

Let $Z(n)$ be the entry point for positive integers n. Prove that $Z(n) \le 2n$ for any positive integer n. Find all positive integers n such that $Z(n) = 2n$.

Solution by Paul S. Bruckman, Highwood, IL

We first assume that $\gcd(n, 10) = 1$. The following results are well known for all primes $p \ne 2, 5$: $Z(p) \mid (p - (5/p))$; also, $Z(p^e) = p^{e-t}Z(p)$ for some t with $1 \le t \le e$. Then $Z(p^e) = p^{e-t}(p - (5/p))/a$ for some integer $a = a(p)$. If $n = \Pi p^e$, let $n = PQ$, where P consists of those prime powers p^e exactly dividing n and with $a(p) = 1$, and Q is the corresponding product with $a(p) \ge 2$. Note that

$$Z(P) \le 2\prod_{p^e \| P} p^{e-1}\{(p+1)/2\},$$

since $2 \mid ((p - (5/p))$, while

$$Z(Q) \le \prod_{p^e \| Q} p^{e-1}\{(p+1)/2\};$$

therefore,

$$Z(n) = \text{ LCM}\{Z(p^e) : p^e \| n\} \le 2\prod_{p^e \| n} p^{e-1}\{(p+1)/2\}.$$

Then

$$Z(n)/n \le 2\prod_{p|n}\{(p+1)/2p\} \le 4/3,$$

since $(p+1)/2p \le 2/3$ for all $p > 2$, with equality iff $p = 3$.

If $n = 5^e m$, where $\gcd(m, 10) = 1$, then

$$Z(n) = \text{LCM}(Z(5^e), Z(m)) = \text{LCM}(5^e, Z(m)) \le 5^e \cdot (4m/3) = 4n/3.$$

Therefore, $Z(n) \le 4n/3$ for all odd n.

If $n = 2m$, where m is odd, then

$$Z(n) = \text{LCM}(3, Z(m)) \le 3Z(m) \le 3(4m/3) = 4m = 2n.$$

If $n = 4m$, where m is odd, then

$$Z(n) = \text{LCM}(6, Z(m)) \le 6Z(m) \le 6(4m/3) = 8m = 2n.$$

If $n = 2^e m$, where $e \ge 3$ and $\gcd(m, 10) = 1$, then

$$Z(n) = \text{LCM}(Z(2^e), Z(m)) \le \text{LCM}(3\cdot 2^{e-2}, Z(m)) \le 3\cdot 2^{e-2}\cdot 4m/3 = n.$$

In all cases, $Z(n) \le 2n$ for all $n \ge 1$. □

If we examine the various parts of the foregoing proof, we see that $Z(n)$ has a chance of being exactly equal to $2n$ only if 2^1 or 2^2 is the highest power of 2 exactly dividing n. Moreover, if $\gcd(m, 30) = 1$, if $m > 1$, and if $n = 2m$ or $4m$, we see that $Z(m) < 4m/3$; in this case, $Z(n) < 2n$. Note that the factor 5^f of n does not affect the ratio $Z(n)/n$, since $Z(5^f) = 5^f$.

Thus, any n with $Z(n) = 2n$ must be of the form $2^d \cdot 3^e \cdot 5^f$, where $d = 1$ or 2, $e \ge 1$, $f \ge 0$. We observe that $Z(2\cdot 3^e \cdot 5^f) = \text{LCM}(3, 4\cdot 3^{e-1}, 5^f) = 12\cdot 5^f$ if $e = 1$, or $4\cdot 3^{e-1}\cdot 5^f$ if $e \ge 2$. Thus, $Z(n) = 2n$ if $e = 1$, $Z(n) < 2n$ if $e > 1$.

Therefore, if $2^1 \| n$, $Z(n) = 2n$ iff $e = 1$, i.e., iff $n = 6\cdot 5^f$. On the other hand, we find that if $n = 4\cdot 3^e \cdot 5^f$ then $Z(n) = n$ or $n/3 < 2n$ in either case.

In conclusion, $Z(n) = 2n$ iff $n = 6\cdot 5^f$, $f = 0, 1, 2, \ldots$. □

Also solved by L. A. G. Dresel and the proposer.

Representation

H-534 ***Proposed by Piero Filipponi, Rome, Italy***
(Vol. 35, no. 4, November 1997)

An interesting question posed to me by Evelyn Hart (Colgate University, Hamilton, NY) led me to pose, in turn, the following two problems to the readers of *The Fibonacci Quarterly*. (Please see the above volume of the *Quarterly* for a complete statement of Problem H-534.)

Problem A: For k a fixed positive integer, let n_k be *any* integer representable as

$$n_k = \sum_{j=1}^{k} v_j F_j, \tag{1}$$

where v_j equals either j or zero.

Problem B: Is it possible to characterize the set of all positive integers k for which kF_k is representable as

$$kF_k = \sum_{j=1}^{k-1} v_j F_j$$

where v_j is as in Problem A?

Solution by Paul S. Bruckman, Highwood, IL

Solution to Problem A: We first make some notational changes, for convenience. Let $\theta_j = jF_j$. The set of positive integers that may be represented as a sum $\sum_{j=1}^{k} \varepsilon_j \theta j$ with $\varepsilon_j = 0$ or 1, $\varepsilon_k = 1$, is denoted by τ_k. Let $\tau = \bigcup_{k=1}^{\infty} \tau k$. If a positive integer n *cannot* be represented as such a sum for any value of k, we write $n \notin \tau$. Also, define $S(0) = 1$.

We note that we have the following generating function:

$$\prod_{j=1}^{\infty} (1 + x^{\theta_j}) = \sum_{k=0}^{\infty} S(k) x^k . \tag{1}$$

We use a comparison test to determine the following result:

$$\lim_{k \to \infty} S(k) / f(k) = 0. \tag{2}$$

The comparison is made with the more well-known generating function:

$$\prod_{j=1}^{\infty} (1 + x^j) = \sum_{k=0}^{\infty} q(k) x^k , \tag{3}$$

where $q(k)$ is the number of decompositions of k into distinct positive integer summands without regard to order; for example, $q(7) = 5$, since $7 = 1+6 = 2+5 = 3+4 = 1+2+4$. Since the θ_j's are natural numbers, it is clear that

$$0 \le S(k) \le q(k), \;\; k = 0, 1, 2, \ldots . \tag{4}$$

Indeed, all of the $q(k)$'s are > 0. The following asymptotic formula (paraphrased to conform with our notation) is given in [1]:

$$q(k) \sim \frac{1}{4} (3k^3)^{-1/4} \exp(\pi \sqrt{k/3}), \text{ as } k \to \infty. \tag{5}$$

Thus, $\log q(k) \sim \pi\sqrt{k/3}$. On the other hand, $\log f(k) \sim k \log \alpha$. Hence, $\log\{q(k)/f(k)\} \to -\infty$, which implies $\lim_{k\to\infty} q(k)/f(k) = 0$. This, together with (4), implies (3). □

Partial Solution to Problem B: We see that if $\theta_k = \sum_{j=1}^{k-1} \varepsilon_j \theta_j$ then $\varepsilon_{k-1} = 1$ and $k \ge 7$, by the proposer's comments. For, otherwise, $\theta_k \le f(k-2) = \theta_k - L_{k+1} + 2$, which is clearly impossible. Therefore, either $\theta_k \in \tau_{k-1}$ or $\theta_k \notin \tau$. For brevity, we let U denote the set of $k \ge 7$ such that $\theta_k \in \tau_{k-1}$. Note that $S(\theta_k) \ge 1$ for all $k \ge 1$. One way to characterize U, albeit not a very satisfactory way from a theoretical standpoint, is to observe that U is precisely the set of k such that $S(\theta_k) \ge 2$; this, however, is little more than a restatement of the definition of the $S(k)$'s.

Some other observations may be made, which may or may not be useful. For example, we can determine the characteristic polynomial of the θ_k's. The following relation is easily found:

$$\theta_k - \theta_{k-1} - \theta_{k-2} = L_{k-1}. \tag{6}$$

Thus, the characteristic, or "annihilating," polynomial of the θ_k's is $(z^2-z-1)^2 = z^4-2z^3-z^2+2z+1$; that is, we have the pure recurrence

$$\theta_k - 2\theta_{k-1} - \theta_{k-2} + 2\theta_{k-3} + \theta_{k-4} = 0. \tag{7}$$

We may define the following quantity:

$$u_k \equiv 2\theta_k + \theta_{k-1},\ k = 1, 2, \ldots \ (\text{with } \theta_0 \equiv 0). \tag{8}$$

Then we may recast (7) as follows:

$$u_{k-1} - u_{k-3} = \theta_k,\ k = 4, 5, \ldots. \tag{9}$$

A consequence of these relations is the following:

$$u_{2k} + 2 = \sum_{i=0}^{k} \theta_{2i+1}, \quad u_{2k-1} = \sum_{i=1}^{k} \theta_{2i}, \quad k = 1, 2, \ldots. \tag{10}$$

This shows that u_{2k-1} and $(u_{2k}+2)$ are elements of τ_{2k} and τ_{2k+1}, respectively. We also see that

$$u_k + u_{k-1} + 2 = f(k+1),\ k = 2, 3, \ldots. \tag{11}$$

It is not clear at this point how these relations may be useful in determining which values of k are "acceptable," in the sense that $k \in U$. We observe from (6), however, that if $L_{k-1} \in \tau_m$ for some $m \le k-3$ then $k \in U$.

One practical approach is simply to expand the generating function to any desired number of terms and pick out the values of k for which $S(k) \ge 2$. To ensure that we are not omitting some values of k that *eventually* generate $S(k) \ge 2$, we need to take enough terms in the product. If the partial products $\prod_{j=1}^{n}(1+x^{\theta_j})$ have the expansion $\sum_{k=0}^{f(n)} S(k,n)x^k$, and if the integer $\mu = \mu(k)$ is determined from $\theta_\mu \le k < \theta_{\mu+1}$ then $S(k,n) = S(k)$ for all $n \ge \mu$. In particular, $S(\theta_k, n) = S(\theta_k)$ for all $n \ge \theta_k$.

We conclude with a table indicating the first 25 values of θ_k, $S(k)$, and $f(k)$, also indicating all acceptable representations of θ_k as an element of τ_{k-1} for $k \ge 7$, if such representations exist. We denote such representations in an abbreviated form, where the indicated m-tuple gives the subscripts r of the θ_r's entering in the representation, shown in descending order.

The table was *not* generated by expansion, as might be suggested by the previous comments. Rather, we used a constructive algorithm for generating the representations (if any) in τ_{k-1} of θ_k. Following is a brief description of the algorithm.

We begin by assuming that $\theta_k \in \tau_{k-1}$ and compute the difference $N_1 \equiv \theta_k - \theta_{k-1}$. There exists an index r such that $\theta_r \le N_1 < \theta_{r+1}$. The next term is either θ_r or θ_{r-1}. If $N_1 > f(r-1)$, such next term *must* be θ_r. If $N_1 \le f(r-1)$, such next term is either θ_r or θ_{r-1}; both cases are possible *a priori* and must be examined separately. Let $N_2 = N_1 - \theta_s$, where θ_s is the next term selected (i.e., $s = r$ or $r-1$) and repeat the process with N_2. The algorithm continues until a final difference N_ω, say, is either determined to be representable as a sum of the θ_j's or recognized as impossible to be thus represented. Note: If $N_j = f(m)$ for some m and j, we may either stop at the term θ_m or replace $f(m)$ by $\theta_1 + \theta_2 + \cdots + \theta_m$. Keeping track of all "forks in the road" (where two choices were possible *a priori*), we thereby generate all possible representations, if any.

It is tempting on the basis of the data, to make the conjecture that $k \in U$ for *all* values except 1, 2, 3, 4, 5, 6, 8, 8, and 14. It would seem unlikely that $S(\theta_k) = 1$ for any value of $k > 25$, but these methods did not resolve this question.

TABLE

k	θ_k	$S(k)$	τ_{k-1} Representation(s)	$f(k)$
1	1	1	–	1
2	2	1	–	3
3	6	1	–	9
4	12	1	–	21
5	25	1	–	46
6	48	1	–	94
7	91	2	{6,5,4,3}	185
8	168	1	–	353
9	306	1	–	659
10	550	2	{9,8,6,5,2,1}	1209
11	979	2	{10,9,7,5,3,1}	2188
12	1728	2	{11,10,8,5,3}	3916
13	3029	2	{12,11,8,7,6,4,2,1}	6945
14	5278	1	–	12223
15	9150	3	{14,13,10,8,7,5,3,2,1}, {14,12,11,10,9,8,7,6,2}	21373
16	15792	3	{15,14,11,9,6,5,3}, {15,13,12,11,10,9,6,2}	37165
17	27149	3	{16,15,12,9,7,6,5,3,2,1}, {16,14,13,12,11,9,5,4}	64314
18	46512	4	{17,16,13,9,8,6,4,3,2}, {17,16,12,11,10,8,7,6,3,1}, {17,15,14,13,12,7,6,5,4,2}	110826
19	79439	5	{18,17,14,9,8,5,1}, {18,17,13,12,10,9,7,6,5,1}, {18,16,15,14,12,11}, {18,16,15,14,12,10,9,7,5,3,1}	190265
20	135300	5	{19,18,15,8,5,3}, {19,18,14,6,4,2,1}, {19,18,14,13,10,9,8,4,3}, {19,17,16,15,12,11,10,9,8,5,4,2}	325565
21	229866	4	{20,19,15,13,12,11,8,6,5}, {20,18,17,16,13,12,9,6,2}, {20,18,17,16,13,11,10,9,8,6,5,3,2}	555431
22	389642	2	{21,19,18,16,15,14,13,10,5,1}	945073
23	659111	4	{22,21,17,15,12,11,9,8,7,5,3,1}, {22,20,19,17,16,15,12,10,9,6,3,1}, {22,20,19,17,16,14;13,12,11,10,8,6,3,2,1}	1604184
24	1112832	8	{23,22,18,15,14,13,7,4,3,1}, {23,22,17,16,15,14,13,12,11,10,9,7,5,2}, {23,21,20,19,14,13,10,8,6,5,4,3}, {23,21,20,18,17,15,14,9,7,6,4,3,2,1}, {23,21,20,19,14,13,10,8,7}, {23,21,20,1118,17,15,13,12,10,9,7,5,4,2,1}	2717016
25	1875625	8	{24,23,19,16,14,12,11,9,7,6,4,3,2,1}, {24,23,18,17,16,15,13,12,8,7,6,4,2,1}, {24,22,21,20,14,12,10,9,7,5,3,1}, {24,22,21,20,14,12,11}, {24,22,21,19,18,16,11,10,4,1}, {24,22,21,19,18,15,14,12,11,8,5,3}, {24,22,21,19,18,15,14,12,10,9,8,7,6,4,2,1}	4592641

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