The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

TABLE OF CONTENTS

VOLUME 37 AUGUST 1999 NUMBER 3

PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

EDITORIAL POLICY

THE FIBONACCI QUARTERLY seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

SUBMITTING AN ARTICLE

Articles should be submitted using the format of articles in any current issues of **THE FIBONACCI QUARTERLY.** They should be typewritten or reproduced typewritten copies, that are clearly readable, double spaced with wide margins and on only one side of the paper. The full name and address of the author must appear at the beginning of the paper directly under the title. Illustrations should be carefully drawn in India ink on separate sheets of bond paper or vellum, approximately twice the size they are to appear in print. Since the Fibonacci Association has adopted $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \geq 2$ and $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$, $n \geq 2$ as the standard definitions for The Fibonacci and Lucas sequences, these definitions ***should not*** be a part of future papers. However, the notations ***must*** be used. One to three ***complete*** A.M.S. classification numbers ***must*** be given directly after references or on the bottom of the last page. **Papers not satisfying all of these criteria will be returned.** See the new worldwide web page at:

http://www.sdstate.edu/~wcsc/http/fibhome.html

for additional instructions.

Two copies of the manuscript should be submitted to: **GERALD E. BERGUM, EDITOR, THE FIBONACCI QUARTERLY, DEPARTMENT OF COMPUTER SCIENCE, SOUTH DAKOTA STATE UNIVERSITY, BOX 2201, BROOKINGS, SD 57007-1596,** until March 1, 1998. ***After March 1, 1998,*** send articles to **CURTIS COOPER, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, CENTRAL MISSOURI STATE UNIVERSITY, WARRENSBURG, MO 64093-5045.**

Authors are encouraged to keep a copy of their manuscripts for their own files as protection against loss. The editor will give immediate acknowledgment of all manuscripts received.

The journal will now accept articles via electronic services. However, electronic manuscripts must be submitted using the typesetting mathematical wordprocessor AMS-TeX. Submitting manuscripts using AMS-TeX will speed up the refereeing process. AMS-TeX can be downloaded from the internet via the homepage of the American Mathematical Society.

SUBSCRIPTIONS, ADDRESS CHANGE, AND REPRINT INFORMATION

Address all subscription correspondence, including notification of address change, to: **PATTY SOLSAA, SUBSCRIPTIONS MANAGER, THE FIBONACCI ASSOCIATION, P.O. BOX 320, AURORA, SD 57002-0320.**

Requests for reprint permission should be directed to the editor. However, general permission is granted to members of The Fibonacci Association for noncommercial reproduction of a limited quantity of individual articles (in whole or in part) provided complete reference is made to the source.

Annual domestic Fibonacci Association membership dues, which include a subscription to **THE FIBONACCI QUARTERLY**, are \$37 for Regular Membership, \$42 for Library, \$47 for Sustaining Membership, and \$74 for Institutional Membership; foreign rates, which are based on international mailing rates, are somewhat higher than domestic rates; please write for details. **THE FIBONACCI QUARTERLY** is published each February, May, August and November.

All back issues of **THE FIBONACCI QUARTERLY** are available in microfilm or hard copy format from **BELL & HOWELL INFORMATION & LEARNING, 300 NORTH ZEEB ROAD, P.O. BOX 1346, ANN ARBOR, MI 48106-1346.** Reprints can also be purchased from **BELL & HOWELL** at the same address.

The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

ALMOST SQUARE TRIANGULAR NUMBERS

K. B. Subramaniam
Dept. of Education in Science & Math., Regional Institute of Education (NCERT), Bhopal-462 013, India
(Submitted July 1997-Final Revision April 1998)

1. INTRODUCTION

While undergoing the study of Square Triangular Numbers (STN), it was observed that there are certain triangular numbers (TN) which, although not squares, are "very close" to squares. If we restrict this closeness to just unity, we obtain what we shall call "Almost Square Triangular Numbers" (ASTN). More precisely, an ASTN is a TN that differs from a perfect square exactly by unity.

The very description of ASTN leads to their two types: first, those TN that exceed a perfect square by one; second, those that fall short of a perfect square by one.

The purpose of this paper is to account for all the ASTN of both types by linking them with STN.

2. SOME PRELIMINARIES

2.1 (Def.) α-ASTN

A TN x will be called an ASTN of the type α (α-ASTN) iff $x-1$ is a perfect square.

The first ten α-ASTN are:

10, 325, 11026, 374545, 12723490, 432224101, 14682895930, 498786237505, 16944049179226, and 575598885856165.

2.2 (Def.) β-ASTN

A TN y will be called an ASTN of the type β (β-ASTN) iff $y+1$ is a perfect square.

The first ten β-ASTN are:

3, 15, 120, 528, 4095, 17955, 139128, 609960, 4726276, and 20720703.

We will need the following notations:

$$\alpha_n = \text{ the } n^{\text{th}}\ \alpha\text{-ASTN}, \quad \beta_n = \text{ the } n^{\text{th}}\ \beta\text{-ASTN}, \quad t_n = \text{ the } n^{\text{th}} \text{ STN},$$
$$U_n = \sqrt{t_n}, \quad a_n = (\alpha_n - 1)^{1/2}, \quad b_n = (\beta_n + 1)^{1/2}.$$

We will also need the results (in addition to the well-known fact that x is a triangular number iff $8x+1$ is a perfect square) from our earlier works:

$$U_n = 6U_{n-1} - U_{n-2} \qquad \text{(from [1]);} \tag{2.1}$$

$$U_{n-1}U_{n+1} + 1 = U_n^2 \qquad \text{(from [2]).} \tag{2.2}$$

3. THE α-ASTN

Our first result paves the way for constructing an α-ASTN using a given STN, thus guaranteeing the infinitude of the set of all α-ASTN.

Lemma 3.1: If x is an STN, then $9x+1$ is an α-ASTN.

Proof: Note that $8(9x+1)+1=(3\sqrt{8x+1})^2$. Since x is a TN, $8x+1$ must be a perfect square, thus making $9x+1$ a TN. Moreover, x itself is a perfect square, say z^2, so $9x+1=(3z)^2+1$, which means that $9x+1$ is an α-ASTN. □

That this construction indeed exhausts all the α-ASTN is confirmed by the following result.

Lemma 3.2: If x is an α-ASTN, then $(x-1)/9$ must be an STN.

Proof: In order that the lemma may make any sense, we must ensure that $x-1$ is indeed a multiple of 9. For this, we note that whenever x is an α-ASTN, $x-1$ is a perfect square. As a result, $8x \equiv 8, 7, 4, 1 \pmod 9$. On the other hand, whenever x is a TN, $8x+1$ is a perfect square. Thus, $8x \equiv 8, 0, 3, 6 \pmod 9$. Therefore, $x \equiv 1 \pmod 9$. Let $(x-1)/9=z$. Clearly, z is a perfect square. Also, $8z+1=(8x+1)/9$ is a perfect square. This means that z is a TN and, hence, an STN. □

Our next result establishes a direct link between α_n and t_n. In what follows, n will always denote an arbitrary natural number.

Theorem 3.1: $\alpha_n = 9t_n+1$.

Proof: First, note that $\alpha_1 = 10 = 9t_1+1$. Assume the assertion is true for $n=k$, so that $\alpha_k = 9t_k+1$. If possible, let $\alpha_{k+1} \neq 9t_{k+1}+1$. But $(\alpha_{k+1}-1)/9$ is an STN (by Lemma 3.2), so let $(\alpha_{k+1}-1)/9 = t_m$ for some m. We have $\alpha_{k+1} > \alpha_n$ so that $t_m > t_k$. This means $m > k$. But m cannot be equal to $k+1$ (by our assumption), so $m > k+1$. Also, $9t_{k+1}+1$ is an α-ASTN (by Lemma 3.1), so let $9t_{k+1}+1=\alpha_p$ for some p. We have $t_k < t_{k+1} < t_m$. This leads to $\alpha_k < \alpha_p < \alpha_{k+1}$, an absurdity. Hence, by mathematical induction, $\alpha_n = 9t_n+1$. □

4. THE β-ASTN

As in the case of the α-ASTN, our first attempt would be toward constructing a β-ASTN from STN. But here, unlike the case of α-ASTN, we need two consecutive STN. First, we will need the following auxiliary results.

Lemma 4.1: $4U_nU_{n+1}+1=(U_{n+1}-U_n)^2$.

Proof: We have

$$\begin{aligned} U_n^2 &= U_{n-1}U_{n+1}+1 && \text{[by (2.2)]} \\ &= 6U_nU_{n+1}-U_{n+1}^2+1 && \text{[by (2.1)]}. \end{aligned}$$

Hence, $4U_nU_{n+1}+1=(U_{n+1}-U_n)^2$. □

Lemma 4.2: $8U_nU_{n+1}+1=(U_{n+1}+U_n)^2$.

Proof: Proceed as in Lemma 4.1. □

Lemma 4.3: $U_{n+1}=3U_n+\sqrt{8U_n^2+1}$.

Proof: While proving Lemma 4.1, we found that $U_n^2 = 6U_{n+1}U_n - U_{n+1}^2 + 1$. This yields $(U_{n+1} - 3U_n)^2 = 8U_n^2 + 1$. But $U_{n+1} - 3U_n = 3U_n - U_{n-1} > 0$. Hence,

$$U_{n+1} - 3U_n = \sqrt{8U_n^2 + 1}. \ \square$$

Theorem 4.1: $(U_{n+1} - 2U_n)^2 - 1$ is a β-ASTN.

Proof: Let

$$x = (U_{n+1} - 2U_n)^2 - 1 = \left\{U_n + \sqrt{8U_n^2 + 1}\right\}^2 - 1 \quad \text{(by Lemma 4.3)}.$$

Thus, $8x + 1 = \left\{8U_n + \sqrt{8U_n^2 + 1}\right\}^2$, a perfect square. As a result, x is a TN and, consequently, $(U_{n+1} - 2U_n)^2 - 1$ is a β-ASTN. $\square$

Theorem 4.1 guarantees the infinitude of the set of all β-ASTN, but it does not guarantee that this construction accounts for all the β-ASTN. In fact, it cannot do so because there do exist β-ASTN that cannot be obtained by the application of this theorem, e.g., the very first β-ASTN viz. 3 cannot be expressed as $(U_{n+1} - 2U_n)^2 - 1$ for any n.

In fact, there are infinitely many such exceptions viz. 3, 120, 4095, ... (i.e., all the odd-indexed β-ASTN). Of course, all the even-indexed β-ASTN are taken care of by the above theorem.

Theorem 4.2: $(U_{n+1} - 4U_n)^2 - 1$ is a β-ASTN.

Proof: Let

$$y = (U_{n+1} - 4U_n)^2 - 1 = \left\{\sqrt{8U_n^2 + 1} - U_n\right\}^2 - 1 \quad \text{(by Lemma 4.3)}.$$

Hence, $8y + 1 = \left\{8U_n - \sqrt{8U_n^2 + 1}\right\}^2$, so that y is a TN. This means that $(U_{n+1} - 4U_n)^2 - 1$ is a β-ASTN. $\square$

It appears that Theorems 4.1 and 4.2 jointly account for all the β-ASTN. The same is confirmed by the following theorem.

Theorem 4.3: $b_{2n} = U_{n+1} - 2U_n$ and $b_{2n-1} = U_{n+1} - 4U_n$.

Before attacking the proof of Theorem 4.3 (our main theorem), we must prove the following three lemmas.

Lemma 4.4: If $b^2 - 1$ is a β-ASTN, then either $\{(R+b)/7\}^2$ or $\{(R-b)/7\}^2$ must be an STN, where $R = (8b^2 - 7)^{1/2}$.

Proof: For this lemma to make any sense, we have to ensure that either $(R+b)/7$ or $(R-b)/7$ must be an integer. To this end, we argue that whenever $b^2 - 1$ is a β-ASTN, $b^2 - 1$ is a TN, so $8(b^2 - 1) + 1 = R^2$ must be a perfect square. Thus, R is an integer. Also $(R-b)(R+b) = 7(b^2 - 1)$. This ensures that $(R-b)/7$ or $(R+b)/7$ is an integer.

<u>**Case 1**</u>. Let $(R+b)/7$ be an integer, say x. Then $8x^2 + 1 = \{(8b+R)/7\}^2$, a perfect square. Hence, x^2 must be a TN. This means that $\{(R+b)/7\}^2$ is an STN.

Case 2. Let $(R-b)/7$ be an integer, say y. Then $8y^2+1=\{(8b-R)/7\}^2$, a perfect square. Hence, y^2 must be a TN. This means that $\{(R-b)/7\}^2$ is an STN. Now, we claim that $(R+b)/7$ and $(R-b)/7$ cannot both be integers at the same time. For, if the contrary is true, then $\{(R+b)/7\}\{(R-b)/7\}=(b^2-1)/7$, which means that b^2-1 is a multiple of 7. Also, $\{(R+b)/7\}-\{(R-b)/7\}=2b/7$, which would mean b is a multiple of 7. This leads to a contradiction. □

Lemma 4.5: If b^2-1 is a β-ASTN and $R-b$ is a multiple of 7, then $b=U_{m+1}-2U_m$ for some m.

Proof: By Lemma 4.4, $\{(R-b)/7\}^2$ is an STN. Hence, $(R-b)/7=U_m$ for some m, so that $(b-U_m)^2=8U_m^2+1$. We claim that $b>U_m$, otherwise b will become $U_m-(8U_m^2+1)^{1/2}$ which is negative, an absurdity. Thus, $b-U_m=(8U_m^2+1)^{1/2}$, i.e., $b=U_{m+1}-2U_m$ (by Lemma 4.3). □

Lemma 4.6: If b^2-1 is a β-ASTN and $R+b$ is a multiple of 7, then $b=U_{k+1}-4U_k$ for some k.

Proof: As before, $(R+b)/7=U_k$ for some k, so that $b=-U_k+(8U_k^2+1)^{1/2}$. □

Proof of Theorem 4.3: Define the sequences $\langle x_r\rangle$ and $\langle y_r\rangle$, respectively, by $x_r=U_{r+1}-4U_r$ and $y_r=U_{r+1}-2U_r$. Clearly, for each r, $x_r<y_r$. Also,

$$x_{r+1}=U_{r+2}-4U_{r+1}=2U_{r+1}-U_r=y_r+(U_{r+1}+U_r)>y_r.$$

Thus, $x_r<y_r<x_{r+1}<y_{r+1}$. Hence, the sequence $\langle z_r\rangle$, defined by $z_{2r-1}=x_r$ and $z_{2r}=y_r$, is monotonically increasing. We claim that the sequence $\langle b_n\rangle$ is a subsequence of the sequence $\langle z_n\rangle$ because, for any n, either $(R_n+b)/7$ or $(R_n-b)/7$ is equal to U_k for some k [where $R_n=(8b_n^2-7)^{1/2}$]. Thus, $b_n=U_{k+1}-2U_k$ or $b_n=U_{k+1}-4U_k$. Also, by Theorems 4.1 and 4.2, for each r, $y_r=b_m$ and $x_r=b_k$ for some m and k. Hence, $\langle z_n\rangle$ and $\langle b_n\rangle$ are identical. □

We conclude by rewriting the statement of Theorem 4.3 in a more useful form, as follows.

Corollary: $\beta_{2n}=(U_{n+1}-2U_n)^2-1$ and $\beta_{2n-1}=(U_{n+1}-4U_n)^2-1$.

ACKNOWLEDGMENT

The author is grateful to an anonymous referee for pointing out many grammatical and typographical lapses in the original manuscript and offering many useful comments and suggestions.

REFERENCES

1. K. B. Subramaniam. "A Simple Computation of Square Triangular Numbers." *International Journal of mathematical Education in Science and Technology* **23.5** (1992):790-93.
2. K. B. Subramaniam. "A Divisibility Property of Square Triangular Numbers." *International Journal of mathematical Education in Science and Technology* **26.2** (1995):284-86.

AMS Classification Numbers: 11B37, 11A99

❖❖❖

NOTE ON THE PIERCE EXPANSION OF A LOGARITHM

Pelegrí Viader

Applied Math., Dept. d'Economia i Empresa,Univ. Pompeu Fabra
Ramon Trias Fargas 25-27, 08005 Barcelona, Spain
pelegri.viader@econ.upf.es

Jaume Paradís

Applied Math., Dept. d'Economia i Empresa,Univ. Pompeu Fabra
Ramon Trias Fargas 25-27, 08005 Barcelona, Spain
jaume.paradis@econ.upf.es

Lluís Bibiloni

Facultat de Ciències de l'Educació, Univ. Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
l.bibiloni@cc.uab.es

(*Submitted July 1997-Final Revision November 1997*)

1. INTRODUCTION

Daniel Shanks, in [10], introduced a very interesting algorithm for obtaining the regular continued fraction expansion of the logarithm of a number. A concise presentation can be found in [5]; however, it is difficult to find in the literature (e.g., texts such as [1], [2], [4], and [11] do not mention it). Shanks' algorithm is the inspiration for the one we present here using Pierce expansions, but it can be adapted easily to other well-known expansions such as Engel's or Sylvester's (see [3] and [6] for details).

A very brief description of Pierce expansions is given below. A more complete account can be found in [7], [8], [9], and [12].

Definition 1: The Pierce expansion of a real number $\alpha \in (0, 1]$ is an expression of the form

$$\frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{n-1}}{a_1 a_2 \ldots a_{n-1} a_n} + \cdots, \tag{1}$$

where $a_1, a_2, \ldots, a_n, \ldots$ constitute a strictly increasing sequence of positive integers. In the case that the sum above is finite, we call it a *terminating* (or *finite*) expansion and then we add the condition that the last two terms, a_{n-1} and a_n, are not consecutive: $a_{n-1} < a_n - 1$.

We denote (1) by $\langle a_1, a_2, \ldots, a_n, \ldots \rangle$. The requirement that $a_{n-1} < a_n - 1$ is to ensure uniqueness in the case of terminating expansions as $\langle a, \ldots, k \rangle = \langle a, \ldots, k, k+1 \rangle$.

Pierce expansions constitute a system of representation of real numbers in (0, 1], as the following theorem proves.

Theorem 1: Any real number α, $0 < \alpha \le 1$, has a unique representation as a Pierce expansion: rationals as finite expansions and irrationals as infinite expansions.

We include a sketch of the proof as a guide for the algorithm of the next section.

Proof: Uniqueness is the result of observing that a Pierce expansion verifies

$$\frac{1}{a_1 + 1} < \langle a_1, a_2, \ldots, a_n, \ldots \rangle \le \frac{1}{a_1}.$$

The existence is easily justified by the following algorithm. If $x \in (0, 1]$, its Pierce expansion, $\langle a_1, a_2, \dots, a_n, \dots\rangle$, is obtained as follows:

Step 1. $x_0 \leftarrow x$; $i \leftarrow 1$.

Step 2. $a_i = \lfloor 1/x_{i-1} \rfloor$; $x_i \leftarrow 1 - a_i x_{i-1}$.

Step 3. If $x_i = 0$, then stop; else $i \leftarrow i+1$ and go to Step 2.

If x is a rational number p/q, the algorithm will eventually terminate as Step 2 requires, on the first iteration, that we perform the division of q by p and after that the division of q by the successive remainders which are obviously decreasing and eventually must become 0. In that case, the expansion will be finite. If x is not rational, the algorithm will never terminate but will provide a series that is easily seen to converge to x. □

2. THE PIERCE EXPANSION OF A LOGARITHM

The following algorithm will provide us with the Pierce expansion of the logarithm of a number in base $b > 1$. It is easily extended to bases < 1. Let x be a real number, $1 < x < b$. Our aim is to find $\log_b x = \langle a_1, \dots, a_n, \dots\rangle$, where the Pierce expansion can be terminating or not.

Let $x_1 = x$. If we have $1 < x_i < b$, let a_i be the sole positive integer verifying

$$x_i^{a_i} \le b < x_i^{a_i+1}. \tag{2}$$

The integer a_i is well defined as $1 < x_i < b$ and the sequence $\{x_i^n\}_{n \in \mathbb{N}}$ is strictly increasing. Now we define

$$x_{i+1} = \frac{b}{x_i^{a_i}}. \tag{3}$$

From (2), we immediately have $1 \le x_{i+1} < x_i < b$. If $x_{i+1} > 1$, we can continue. Let us suppose we have reached an x_{n+1} such that $1 \le x_{n+1} < x_n < \cdots < x_1 < b$.

Lemma 1: For all i $(1 \le i \le n)$,

$$x = b^{\left(\frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{i-1}}{a_1 a_2 \dots a_i}\right)} \cdot x_{i+1}^{\frac{(-1)^i}{a_1 \dots a_i}}. \tag{4}$$

Proof: We shall proceed by induction on i. For $i = 1$ we have, from (3),

$$x_1^{a_1} = b \cdot x_2^{-1} \Rightarrow x = x_1 = b^{\frac{1}{a_1}} \cdot x_2^{-\frac{1}{a_1}},$$

therefore (4) is verified. If we assume it is verified until $i = k-1$, from the definition of x_{k+1} we have

$$x_{k+1} = \frac{b}{x_k^{a_k}} \Rightarrow x_k = (b \cdot x_{k+1}^{-1})^{\frac{1}{a_k}} = b^{\frac{1}{a_k}} \cdot x_{k+1}^{-\frac{1}{a_k}}.$$

By the induction hypothesis,

$$x = b^{\left(\frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{k-2}}{a_1 a_2 \dots a_{k-1}}\right)} \cdot x_k^{\frac{(-1)^{k-1}}{a_1 \dots a_{k-1}}},$$

and, replacing x_k by its value, we have

$$x = b^{\left(\frac{1}{a_1}-\frac{1}{a_1a_2}+\cdots+\frac{(-1)^{k-2}}{a_1a_2\ldots a_{k-1}}\right)} \cdot b^{\frac{(-1)^{k-1}}{a_1\ldots a_{k-1}a_k}} \cdot x_k^{\frac{(-1)^{k-1}}{a_1\ldots a_{k-1}}}$$

$$= b^{\left(\frac{1}{a_1}-\frac{1}{a_1a_2}+\cdots+\frac{(-1)^{k-1}}{a_1a_2\ldots a_k}\right)} \cdot x_{k+1}^{\frac{(-1)^k}{a_1\ldots a_k}}.$$

This completes the proof of Lemma 1. □

Now, if $x_{n+1}=1$, by the former lemma,

$$x = b^{\frac{1}{a_1}-\frac{1}{a_1a_2}+\cdots+\frac{(-1)^{n-1}}{a_1a_2\ldots a_n}},$$

and we shall be done as soon as we prove $a_1 < a_2 < \cdots < a_{n-1} < a_n - 1$, to which we turn at once.

In general, if $1 \le k < n-1$, from the definition of a_k,

$$x_k^{a_k} < b < x_k^{a_k+1},$$

there exists a real number σ, $a_k < \sigma < a_k+1$, such that $b = x_k^{\sigma}$. Consequently, we have

$$\frac{1}{a_k+1} < \frac{1}{\sigma} < \frac{1}{a_k} \quad \text{and} \quad x_k = b^{\frac{1}{\sigma}}. \tag{5}$$

Now, the first inequality in (5) implies the existence of a real number α, $\alpha > a_k+1$, such that

$$\frac{1}{\sigma} = \frac{1}{a_k} - \frac{1}{a_k\alpha}. \tag{6}$$

We can write

$$x_k = b^{\frac{1}{\sigma}} = b^{\frac{1}{a_k}-\frac{1}{a_k\alpha}} \Rightarrow x_k^{a_k} \cdot b^{\frac{1}{\alpha}} = b;$$

therefore, from the definition of x_{k+1}, we can also write

$$x_{k+1} = \frac{b}{x_k^{a_k}} = b^{\frac{1}{\alpha}}. \tag{7}$$

Now, since $b < x_{k+1}^{a_{k+1}+1}$, if we replace x_{k+1} by the value we have just obtained, we have

$$b < \left(b^{\frac{1}{\alpha}}\right)^{a_{k+1}+1} = b^{\frac{a_{k+1}+1}{\alpha}},$$

which implies that

$$1 < \frac{a_{k+1}+1}{\alpha} \Leftrightarrow \alpha < a_{k+1}+1.$$

Finally, since $\alpha > a_k+1$, we conclude that $a_k < a_{k+1}$.

If in the former reasoning we set $k = n-1$, we can find out what happens with the last two terms when $x_{n+1}=1$. In that case, we have $x_n^{a_n} = b$ and from (7), which tells us that $x_n = b^{1/\alpha}$, we can say $(b^{1/\alpha})^{a_n} = x^n a_n = b \Rightarrow a_n = \alpha$. On the other hand, since from the definition of α, (6), we have $\alpha > a_{n-1}+1$, we can conclude that $a_{n-1} < a_n - 1$.

Thus, we have proved that the expression in the exponent of b given by (4) is a true Pierce expansion: a terminating one in the case $x_{n+1}=1$ or a nonterminating one in the case in which, for all $n \in \mathbb{N}$, $x_{n+1} > 1$. In the latter instance, we have to prove that, for $n \to \infty$,

$$b^{\left(\frac{1}{a_1}-\frac{1}{a_1a_2}+\cdots+\frac{(-1)^{n-1}}{a_1a_2\ldots a_n}\right)}\cdot x_{n+1}^{\frac{(-1)^n}{a_1\ldots a_n}} \to b^{\langle a_1,\ldots,a_n,\ldots\rangle}$$

or, equivalently

$$\lim_{n\to\infty} x_{n+1}^{\frac{(-1)^n}{a_1\ldots a_n}} = 1. \tag{8}$$

It is clear that the sequence $\{x_{n+1}^{(-1)^n/(a_1\ldots a_n)}\}_n$ can be split into two subsequences,

$$x_3^{\frac{1}{a_1a_2}},\ x_5^{\frac{1}{a_1a_2a_3a_4}},\ldots \quad\text{and}\quad x_2^{-\frac{1}{a_1}},\ x_4^{-\frac{1}{a_1a_2a_3}},\ldots, \tag{9}$$

and since $\forall n$, $1 < x_n < b$, and $a_1a_2\ldots a_n \to \infty$, both subsequences in (9) have limit 1, thus proving (8) along with

$$x = b^{\langle a_1,\cdots,a_n,\cdots\rangle} \Leftrightarrow \log_b x = \langle a_1,\ldots,a_n,\ldots\rangle. \ \square$$

3. PRACTICAL USE OF THE ALGORITHM

The present algorithm is purely theoretical and of little practical interest. The difficulties of carrying out the calculations involved are quire important due to the size of the integers appearing in them. In that sense, Shanks' algorithm is much easier to use, thanks mainly to the actual distribution of partial quotients in a continued fraction in which a given integer k occurs with the approximate probability,

$$\log_2\frac{(1+k)^2}{(1+k)^2-1}$$

(see [4], pp. 351-52), thus making small integers much more abundant and, consequently, calculations much simpler. Let us consider the following example.

Example: Pierce expansion of $\log_{10}2 = 0.30102999\ldots$. We have:

$$x_1 = 2; \qquad a_1 = 3; \qquad x_2 = \frac{5}{4}; \qquad a_2 = 10;$$

$$x_3 = \frac{2097152}{1953125}; \quad a_3 = 32; \quad x_4 = \frac{10\cdot 1953125^{32}}{2097152^{32}}; \quad a_4 = 89;$$

and

$$\langle 3, 10, 32\rangle = \frac{289}{960} = 0.30104\ldots; \quad \langle 3, 10, 32, 89\rangle = \frac{643}{2136} = 0.30102996\ldots .$$

Using Shanks' algorithm, we would obtain

$$\log_{10}2 = [0; 3, 3, 9, 2, 2, \ldots] = \cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}}}.$$

In this case, the sixth convergent is $146/485 = 0.30103092783\ldots$. As Olds mentions (see [5], p. 87), each convergent approximates log 2 to one more correct decimal place than the previous one.

REFERENCES

1. C. Brezinski. *History of Continued Fractions and Padé Approximants.* Springer Series on Computational Mathematics (12). Berlin: Springer-Verlag, 1991.
2. G. Dahlquist & A. Björck. *Numerical Methods.* Tr. N. Anderson. New Jersey: Prentice-Hall, 1974.
3. P. Erdös, A. Rényi, & P. Szüsz. "On Engel's and Sylvester's Series." *Ann. Univ. Sci. Budapest.* Sect. Math. **1** (1958):7-32.
4. D. E. Knuth. *The Art of Computer Programming.* 2nd ed. Vol. 2. Reading, Mass.: Addison Wesley, 1973.
5. C. D. Olds. *Continued Fractions.* Washington, D.C.: The Mathematical Association of America, 1963.
6. O. Perron. *Irrazionalzahlen.* 2nd ed. Berlin & Leipzig: De Gruyter, 1939.
7. T. A. Pierce. "On an Algorithm and Its Use in Approximating Roots of Algebraic Equations." *Amer. Math. Monthly* **36** (1929):532-35.
8. E. Ya. Remez. "On Series with Alternating Signs, Which May Be Related to Two Algorithms of M. V. Ostrogradski for the Approximation of Irrational Numbers." *Uspekhi Matem. Nauk.* (N.S.) 6, 5, **45** (1951):33-42.
9. J. O. Shallit. "Metric Theory of Pierce Expansions." *The Fibonacci Quarterly* **24.1** (1986): 22-40.
10. Daniel Shanks. "A Logarithm Algorithm." *Mathematical Tables and Other Aids to Computation* **8.45** (1954):60-64.
11. J. Stoer & R. Bulirsch. *Introduction to Numerical Analysis.* 2nd ed. Berlin: Springer-Verlag, 1993.
12. K. G. Valeyev & E. D. Zlebov. "The Metric Theory of an Algorithm of M. V. Ostrogradski." *Ukrain Mat. Z.* **27** (1975):64-69.

AMS Classification Numbers: 11J70, 65D99

GENERALIZED FIBONACCI SEQUENCES AND A GENERALIZATION OF THE Q-MATRIX*

Zhizheng Zhang
Institute of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P.R. China
(*Submitted August 1997-Final Revision April 1998*)

1. INTRODUCTION

In the notation of Horadam [7], let $W_n = W_n(a, b; p, q)$, where

$$\begin{aligned} &W_n = pW_{n-1} - qW_{n-2} \quad (n \geq 2) \\ &W_0 = a, \ W_1 = b. \end{aligned} \tag{1}$$

If α and β, assumed distinct, are the roots of

$$\lambda^2 - p\lambda + q = 0, \tag{2}$$

we have the Binet form

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{3}$$

in which $A = b - a\beta$ and $B = b - a\alpha$.

The n^{th} terms of the well-known Fibonacci and Lucas sequences are then $F_n = W_n(0, 1; 1, -1)$ and $L_n = W_n(2, 1; 1, -1)$.

We also write

$$U_n = W_n(0, 1; p, q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = W_n(2, p; p, q) = \alpha^n + \beta^n.$$

Throughout this paper, d is a natural number.

Define the Aitken transformation (see [1]) by

$$A(x, x', x'') = \frac{xx'' - x'^2}{x - 2x' + x''}. \tag{4}$$

In 1984, Phillips discovered the following relation between Fibonacci numbers and the Aitken transformation: $A(r_{n-t}, r_n, r_{n+t}) = r_{2n}$, where $r_n = F_{n+1} / F_n$ and $t < n$ is a positive integer, and an account of this work is also given by Vajda in [16]. Later, some articles discussed and extended Phillips' results. For example, McCabe and Phillips [11], Muskat [14], Jamieson [10]. More recently, Zhang [17] defined a generalized Fibonacci sequence as

$$W_{n,d}^{(k)} = W_{n,d}^{(k)}(a, b; p, q) = \frac{A^k\alpha^{nk+d} - B^k\beta^{nk+d}}{\alpha - \beta} \tag{5}$$

and obtained

$$A(R_{n-t}^{(k)}, R_n^{(k)}, R_{n+t}^{(k)}) = R_n^{(2k)}, \tag{6}$$

where $R_n^{(k)} = W_{n,d}^{(k)} / W_{n,0}^{(k)}$. This work generalizes the results of [11], [14], and [10].

* This research was supported by the Natural Science Foundation of Education Committee of Henan Province, P. R. China.

Applying the definition of $W_{n,d}^{(k)}$, we can easily prove that $W_{n,d}^{(k)}$ satisfies the following recurrence relation:

$$W_{n+1,d}^{(k)} = (\alpha^k + \beta^k) W_{n,d}^{(k)} - \alpha^k \beta^k W_{n-1,d}^{(k)}, \tag{7}$$

which has the characteristic equation with roots α^k and β^k.

In this article, Section 2 contains the relation between ratios of $W_{n,d}^{(k)}$ and other transformations and Section 3 gives a generalization of the Q-matrix.

2. THE SECANT, NEWTON-RAPHSON, AND HALLEY TRANSFORMATIONS

If the roots of (2) are real when k tends to infinity, then the sequences of ratios

$$\left\{ R_n^{(k)} = \frac{W_{n,d}^{(k)}}{W_{n,0}^{(k)}} \right\}$$

converges to the d^{th} power of a root of (2). In other words, the sequences of ratios $\{R_n^{(k)}\}$ converges to a root of

$$x^2 - (\alpha^d + \beta^d)x + \alpha^d \beta^d = x^2 - V_d x + q^d = 0, \tag{8}$$

namely, $R_n^{(k)} \to \alpha^d$ or β^d as $k \to \infty$.

Define the Secant transformation $S(x, x')$ (see [14]) for equation (8) by

$$S(x, x') = \frac{x(x'^2 - V_d x' + q^d) - x'(x^2 - V_d x + q^d)}{(x'^2 - V_d x' + q^d) - (x^2 - V_d x + q^d)} = \frac{xx' - q^d}{x + x' - V_d}. \tag{9}$$

Define the Newton-Raphson transformation $N(x)$ (see [14]) for equation (8) by

$$N(x) = x - \frac{x^2 - V_d x + q^d}{2x - V_d} = \frac{x^2 - q^d}{2x - V_d}, \tag{10}$$

and the Halley transformation $H(x)$ (see [4]) for equation (8) by

$$H(x) = x - \frac{x^2 - V_d x + q^d}{(2x - V_d) - \frac{x^2 - V_d x + q^d}{2x - V_d}} = \frac{x^3 - 3q^d x + V_d q^d}{3x^2 - 3V_d x + V_d^2 - q^d}. \tag{11}$$

Then we have the following result.

Theorem 1: Let n and m be integers such that $m+n$ is even, and assume that division by zero does not occur. Then:

(i) $S(R_n^{(k)}, R_m^{(k)}) = R_{(m+n)/2}^{(2k)}$, where

$$R_{(m+n)/2}^{(2k)} = \frac{W_{(m+n)/2,d}^{(2k)}}{W_{(m+n)/2,0}^{(2k)}} = \frac{A^{2k}\alpha^{(m+n)k+d} - B^{2k}\beta^{(m+n)k+d}}{A^{2k}\alpha^{(m+n)k} - B^{2k}\beta^{(m+n)k}}; \tag{12}$$

(ii) $N(R_n^{(k)}) = R_n^{(2k)}$; (13)

(iii) $H(R_n^{(k)}) = R_n^{(3k)}$. (14)

Proof: We prove only part (i). The proofs of (ii) and (iii) are similar. Applying the definition and properties—see (3.1)-(3.5) of [17]—of $W_{n,d}^{(k)}$, we have

$$S(R_n^{(k)}, R_m^{(k)}) = \frac{R_n^{(k)}R_m^{(k)} - q^d}{R_n^{(k)} + R_m^{(k)} - V_d} = \frac{\left(W_{n,d}^{(k)} / W_{n,0}^{(k)}\right)\left(W_{m,d}^{(k)} / W_{m,0}^{(k)}\right) - q^d}{\left(W_{n,d}^{(k)} / W_{n,0}^{(k)}\right) + \left(W_{m,d}^{(k)} / W_{m,0}^{(k)}\right) - V_d}$$
$$= \frac{W_{n,d}^{(k)} W_{m,d}^{(k)} - q^d W_{n,0}^{(k)} W_{m,0}^{(k)}}{W_{n,d}^{(k)} W_{m,0}^{(k)} + W_{m,d}^{(k)} W_{n,0}^{(k)} - V_d W_{n,0}^{(k)} W_{m,0}^{(k)}} = \frac{W_{n,d}^{(k)} W_{m,d}^{(k)} - q^d W_{n,0}^{(k)} W_{m,0}^{(k)}}{W_{n,d}^{(k)} W_{m,0}^{(k)} + W_{n,0}^{(k)}\left[W_{m,d}^{(k)} - V_d W_{n,0}^{(k)}\right]}$$
$$= \frac{(\alpha^d - \beta^d)(A^{2k}\alpha^{(m+n)k+d} - B^{2k}\beta^{(m+n)k+d}}{(\alpha^d - \beta^d)(A^{2k}\alpha^{(m+n)k} - B^{2k}\beta^{(m+n)k})}.$$

This completes the proof of Theorem 1.

We define $\{\Psi_{n,d}^{(k)}\}$, the conjugate sequence of $\{W_{n,d}^{(k)}\}$, by

$$\Psi_{n,d}^{(k)} = \Psi_{n,d}^{(k)}(a, b; p, q) = A^k\alpha^{nk+d} + B^k\beta^{nk+d}. \tag{15}$$

Using (15), it is easy to prove that $\{\Psi_{n,d}^{(k)}\}$ also satisfies the recurrence relation (7). If $\Psi_{n,0}^{(k)} \neq 0$, we use $R_n^{(k)}$ again to denote $\Psi_{n,d}^{(k)} / \Psi_{n,0}^{(k)}$, then this $R_n^{(k)}$ also satisfies the same four relations: (6), (12), (13), and (14).

3. A GENERALIZATION OF THE Q-MATRIX

Before proceeding, we state some results that will be used subsequently. These results can be proved using definitions (5) and (15):

$$\Psi_{n,0}^{(2k)} - 2A^kB^kq^{nk} = \Delta(W_{n,0}^{(k)})^2, \tag{16}$$

$$(W_{n,0}^{(k)})^2 - q^d(W_{n,-d}^{(k)})^2 = U_d W_{n,-d}^{(2k)}, \tag{17}$$

$$(W_{n,d}^{(k)})^2 - q^d(W_{n,0}^{(k)})^2 = U_d W_{n,d}^{(2k)}, \tag{18}$$

$$W_{n,d}^{(k)} W_{n,0}^{(k)} - q^d W_{n,0}^{(k)} W_{n,-d}^{(k)} = U_d W_{n,0}^{(2k)}, \tag{19}$$

$$W_{n,d}^{(m+k)} - A^kB^kq^{nk}W_{n,d}^{(m-k)} = W_{n,0}^{(k)}W_{n,d}^{(m)}, \tag{20}$$

$$W_{n,d}^{(k)} - q^d W_{n,-d}^{(k)} = U_d\Psi_{n,0}^{(k)}, \tag{21}$$

$$(W_{n,0}^{(k)})^2 - W_{n,d}^{(k)} W_{n,-d}^{(k)} = A^kB^kq^{nk-d}U_d^2, \tag{22}$$

where $\Delta = p^2 - 4q$.

Following Hoggatt (see [6]), the Q-matrix is defined by

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Generalizations of the Q-;matrix are to be found in Ivie [9], Filipponi and Horadam [3], Filipponi [2], and Horadam and Filipponi [8]. For a comprehensive history, see Gould [5]. Recently, Melham and Shannon [12], [13], gave the following generalization of the Q-matrix:

$$M=\begin{pmatrix} U_{k+m} & -q^m U_k \\ U_k & -q^m U_{k-m} \end{pmatrix}.$$

We now give a generalization of the matrix M. Associated with the recurrence relation (7) and with $\{W_{n,d}^{(k)}\}$ and $\{\Psi_{n,d}^{(k)}\}$ as in (5) and (15), respectively, define

$$M_{n,d}^{(k)}=\begin{pmatrix} W_{n,d}^{(k)} & -q^d W_{n,0}^{(k)} \\ W_{n,0}^{(k)} & -q^d W_{n,-d}^{(k)} \end{pmatrix},$$

where k, n, and d are integers.

By induction and making use of (17) and (18), it can be shown that, for all integral n,

$$(M_{n,d}^{(k)})^m = U_d^{m-1}\begin{pmatrix} W_{n,d}^{(mk)} & -q^d W_{n,0}^{(mk)} \\ W_{n,0}^{(mk)} & -q^d W_{n,-d}^{(mk)} \end{pmatrix}.$$

Applying (16)-(20), we obtain the following theorem.

Theorem 2:

$$(M_{n,d}^{(k_1)})^{m_1}(M_{n,d}^{(k_2)})^{m_2} = U_d^{m_1+m_2-1}\begin{pmatrix} W_{n,d}^{(m_1k_1+m_2k_2)} & -q^d W_{n,0}^{(m_1k_1+m_2k_2)} \\ W_{n,0}^{(m_1k_1+m_2k_2)} & -q^d W_{n,-d}^{(m_1k_1+m_2k_2)} \end{pmatrix}. \tag{23}$$

4. A REMARK

In fact, the sequences $W_{n,d}^{(k)}$ and $\Psi_{n,d}^{(k)}$ may be regarded as two double sequences (in n and k, d being a parameter). The interesting properties of the sequences $W_{n,d}^{(k)}$ and $\Psi_{n,d}^{(k)}$ still need further research.

ACKNOWLEDGMENT

The author wishes to thank the anonymous referees for their patience and for suggestions that led to a substantial improvement of this paper. He also wishes to thank his instructor, Professor Jun Wang, for his help and instruction.

REFERENCES

1. C. Brezinski. *Accélération de la convergence en analyse numérique*. New York: Springer, 1977.
2. P. Filipponi. "Waring's Formula, the Binomial Formula, and Generalized Fibonacci Matrices." *The Fibonacci Quarterly* **30.3** (1992):225-31.
3. P. Filipponi & A. F. Horadam. "A Matrix Approach to Certain Identities." *The Fibonacci Quarterly* **26.2** (1988):115-26.
4. W. Gander. "On Halley's Iteration Method." *Amer. Math. Monthly* **92** (1985):131-34.
5. H. W. Gould. "A History of the Fibonacci Q-Matrix and a Higher-Dimensional Problem." *The Fibonacci Quarterly* **19.3** (1981):250-57.
6. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton-Mifflin, 1969; rpt. Santa Clara, Calif.: The Fibonacci Association, 1979.
7. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.2** (1965):161-76.

8. A. F. Horadam & P. Filipponi. "Cholesky Algorithm Matrices of Fibonacci Type and Properties of Generalized Sequences." *The Fibonacci Quarterly* **29.2** (1991):164-73.
9. J. Ivie. "A General Q-Matrix." *The Fibonacci Quarterly* **10.3** (1972):255-61, 264.
10. M. J. Jamieson. "Fibonacci Numbers and Aitken Sequences Revisited." *Amer. Math. Monthly* **97** (1990):829-31.
11. J. H. McCabe & G. M. Phillips. "Aitken Sequences and Generalized Fibonacci Numbers." *Math. Comp.* **45** (1985):553-58.
12. R. S. Melham & A. G. Shannon. "Some Infinite Series Summations Using Power Series Evaluated at a Matrix." *The Fibonacci Quarterly* **33.1** (1995):13-20.
13. R. S. Melham & A. G. Shannon. "Some Summation Identities Using Generalized Q-Matrices." *The Fibonacci Quarterly* **33.1** (1995):64-73.
14. B. Muskat. "Generalized Fibonacci and Lucas Sequences and Rootfinding Methods." *Math. Comp.* **61** (1993):365-72.
15. G. M. Phillips. "Aitken Sequences and Fibonacci Numbers." *Amer. Math. Monthly* **91** (1984):354-57.
16. S. Vajda. *Fibonacci & Lucas Numbers and the Golden Section: Theory and Applications*, pp. 103-04. New York: Ellis Horwood, 1989.
17. Zhizheng Zhang. "A Class of Sequences and the Aitken Transformation." *The Fibonacci Quarterly* **36.1** (1998):68-71.

AMS Classification Numbers: 11B37, 11B39, 65H05

❖❖❖

NEW PROBLEM WEB SITE

Readers of *The Fibonacci Quarterly* will be pleased to know that many of its problems can now be searched electronically (at no charge) on the World Wide Web at

http://problems.math.umr.edu

Over 23,000 problems from 42 journals and 22 contests are referenced by the site, which was developed by Stanley Robinowitz's MathPro Press. Ample hosting space for the site was generously provided by the Department of Mathematics and Statistics at the University of Missouri-Rolla, through Leon M. Hall, Chair.

Problem statements are included in most cases, along with proposers, solvers (whose solutions were published), and other relevant bibliographic information. Difficulty and subject matter vary widely; almost any mathematical topic can be found.

The site is being operated on a volunteer basis. Anyone who can donate journal issues or their time is encouraged to do so. For further information, write to:

Mr. Mark Bowron
Director of Operations, MathPro Press
1220 East West Highway #1010A
Silver Spring, MD 290910
(301) 587-0618 (Voice mail)
bowron@compuserve.com (e-mail)

LAMBERT SERIES AND ELLIPTIC FUNCTIONS AND CERTAIN RECIPROCAL SUMS

R. S. Melham
School of Mathematical Sciences, University of Technology, Sydney
PO Box 123, Broadway, NSW 2007, Australia
(*Submitted August 1997-Final Revision February 1998*)

1. INTRODUCTION

For p a strictly positive real number define, for all integers n, the sequences

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0, \quad U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2, \quad V_1 = p. \end{cases} \tag{1.1}$$

Then U_n and V_n generalize F_n and L_n, respectively. Their Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{p + \sqrt{p^2+4}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2+4}}{2}.$$

We see that $\alpha\beta = -1$, $\alpha > 1$, and $-1 < \beta < 0$.

It is known that the infinite sums

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{L_{2n}}$$

can be found by using certain constants associated with Jacobian elliptic functions, while the sums

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{L_{2n+1}}$$

involve the Lambert series. For an introduction to these matters we recommend Horadam [6], which contains a wealth of references to original sources. Further excellent references are Bruckman [5], Almkvist [1], and Borwein and Borwein [3]. Other types of reciprocal sums which involve Lambert series can be found in André-Jeannin [2].

In the above four sums, the task of summation is shared equally between the Lambert series and the Jacobian elliptic functions. The purpose of this paper is to give further reciprocal sums in which the task of summation is similarly shared, thus exhibiting a pleasing symmetry of method.

While the results in Section 3 are believed to be new, they are variations and extensions of known results, and so their proofs contain nothing truly innovative. For this reason we simply state each result and indicate where in the literature a similar proof can be found. In Section 4 we obtain results the like of which we have not seen, and which involve Lambert series. Interestingly, certain special cases of these results have known "dual" results which involve the Jacobian elliptic functions, further highlighting our comments above.

2. NOTATION AND PRELIMINARY RESULTS

In the theory of Jacobian elliptic functions we have, in standard notation,

$$K=\int_0^{\pi/2}\frac{dt}{\sqrt{1-k^2\sin^2 t}} \quad\text{and}\quad K'=\int_0^{\pi/2}\frac{dt}{\sqrt{1+k'^2\sin^2 t}},$$

where $0<k,\ k'<1$, and $k^2+k'^2=1$. See, for example, [5] and [7]. Write $q=e^{-K'\pi/K}$ $(0<q<1)$. Then (see [7])

$$\frac{2K}{\pi}=1+\frac{4q}{1+q^2}+\frac{4q^2}{1+q^4}+\frac{4q^3}{1+q^6}+\cdots, \tag{2.1}$$

$$\frac{2kK}{\pi}=\frac{4\sqrt{q}}{1+q}+\frac{4\sqrt{q^3}}{1+q^3}+\frac{4\sqrt{q^5}}{1+q^5}+\cdots. \tag{2.2}$$

Thus, for a given q $(0<q<1)$, we are able to find the unique values of K, k, K', and k'.

The Lambert series is defined as

$$L(x)=\sum_{n=1}^{\infty}\frac{x^n}{1-x^n},\quad |x|<1.$$

For $|x|<1$ we require the following three results, which occur as Lemma 1 in [2]:

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1-x^{2n+1}}=L(x)-L(x^2); \tag{2.3}$$

$$\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}=L(x)-2L(x^2); \tag{2.4}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1+x^{2n+1}}=L(x)-3L(x^2)+2L(x^4). \tag{2.5}$$

Finally, we require the following lemma.

Lemma 1: Let m be a positive integer. Then

$$\frac{1}{(\alpha-\beta)}\left[\frac{1}{\alpha^{(2n+1)m}V_{(2n+1)m}}-\frac{1}{\alpha^{(2n+1)2m}V_{(2n+1)2m}}\right]=\frac{U_{(2n+1)m}}{V_{(2n+1)m}V_{(2n+1)2m}},\ m\text{ even}; \tag{2.6}$$

$$\frac{1}{(\alpha-\beta)}\left[\frac{1}{\alpha^{(2n+1)m}V_{(2n+1)m}}+\frac{1}{\alpha^{(2n+1)2m}V_{(2n+1)2m}}\right]=\frac{U_{(2n+1)m}}{V_{(2n+1)m}V_{(2n+1)2m}},\ m\text{ odd}; \tag{2.7}$$

$$\frac{1}{(\alpha-\beta)}\left[\frac{1}{\alpha^{nm}V_{nm}}-\frac{1}{\alpha^{2nm}V_{2nm}}\right]=\frac{U_{nm}}{V_{nm}V_{2nm}},\ m\text{ even}. \tag{2.8}$$

Proof: We prove only (2.8) since the proofs of (2.6) and (2.7) are similar.

$$\frac{1}{(\alpha-\beta)}\left[\frac{1}{\alpha^{nm}V_{nm}}-\frac{1}{\alpha^{2nm}V_{2nm}}\right]$$

$$=\frac{1}{(\alpha-\beta)}\left[\frac{\alpha^{nm}V_{2nm}-V_{nm}}{\alpha^{2nm}V_{nm}V_{2nm}}\right]$$

$$=\frac{1}{(\alpha-\beta)}\left[\frac{\alpha^{nm}(\alpha^{2nm}+\beta^{2nm})-(\alpha^{nm}+\beta^{nm})}{\alpha^{2nm}V_{nm}V_{2nm}}\right]$$

$$=\frac{1}{(\alpha-\beta)}\left[\frac{\alpha^{3nm}-\alpha^{nm}}{\alpha^{2nm}V_{nm}V_{2nm}}\right] \text{ (since } m \text{ is even and } \alpha\beta=-1)$$

$$=\frac{1}{(\alpha-\beta)}\left[\frac{\alpha^{nm}-\beta^{nm}}{V_{nm}V_{2nm}}\right]=\frac{U_{nm}}{V_{nm}V_{2nm}}. \quad \square$$

3. RECIPROCAL SUMS I

Using the notation in Section 2, we now state the results of this section in the following theorem.

Theorem 1: Let m be a positive integer. Then

$$\sum_{n=1}^{\infty}\frac{1}{U_{2nm}}=(\alpha-\beta)[L(\beta^{2m})-L(\beta^{4m})], \tag{3.1}$$

$$\sum_{n=0}^{\infty}\frac{1}{V_{2nm}}=\frac{1}{4}\left[\frac{2K(\beta^{2m})}{\pi}+1\right], \tag{3.2}$$

$$\sum_{n=0}^{\infty}\frac{1}{U_{(2n+1)m}}=\begin{cases}(\alpha-\beta)[L(\beta^{m})-2L(\beta^{2m})+L(\beta^{4m})], & m \text{ even},\\ \dfrac{(\alpha-\beta)k(\beta^{2m})K(\beta^{2m})}{2\pi}, & m \text{ odd},\end{cases} \tag{3.3}$$

$$\sum_{n=0}^{\infty}\frac{1}{V_{(2n+1)m}}=\begin{cases}\dfrac{k(\beta^{2m})K(\beta^{2m})}{2\pi}, & m \text{ even},\\ -L(\beta^{m})+2L(\beta^{2m})-L(\beta^{4m}), & m \text{ odd}.\end{cases} \tag{3.4}$$

For a special case of (3.1) concerning the Fibonacci numbers, see the paper of Brady [4], where there is an obvious misprint (for $2m\beta$ and $4m\beta$ read β^{2m} and β^{4m}, respectively). Also of interest is (2.1) in Shannon and Horadam [8]. The proof of (3.2) proceeds along the same lines as the proof of (3.12) in Horadam [6]. The proofs of the first part of (3.3) and the second part of (3.4) are similar to the proof of (4.12) in Horadam [6]. For the proofs, one uses the identity

$$\frac{x^{2n+1}}{1-x^{4n+2}}=\frac{x^{2n+1}}{1-x^{2n+1}}-\frac{x^{4n+2}}{1-x^{4n+2}}$$

together with (2.3). Finally, the proofs of the second part of (3.3) and the first part of (3.4) are similar to the proof on page 103 of the above-mentioned paper of Horadam.

4. RECIPROCAL SUMS II

The results of this section are contained in the following theorem.

Theorem 2: Let m be a positive integer. Then

$$\sum_{n=1}^{\infty}\frac{U_{nm}}{V_{nm}V_{2nm}}=\begin{cases}\dfrac{1}{(\alpha-\beta)}[L(\beta^{2m})-3L(\beta^{4m})+2L(\beta^{8m})], & m \text{ even},\\ \dfrac{1}{(\alpha-\beta)}[L(\beta^{2m})+L(\beta^{4m})-6L(\beta^{8m})+4L(\beta^{16m})], & m \text{ odd},\end{cases} \tag{4.1}$$

$$\sum_{n=0}^{\infty}\frac{U_{(2n+1)m}}{V_{(2n+1)m}V_{(2n+1)2m}}=\begin{cases}\dfrac{1}{(\alpha-\beta)}[L(\beta^{2m})-4L(\beta^{4m})+5L(\beta^{8m})-2L(\beta^{16m})], & m \text{ even},\\ \dfrac{1}{(\alpha-\beta)}[L(\beta^{2m})-3L(\beta^{8m})+2L(\beta^{16m})], & m \text{ odd}.\end{cases} \tag{4.2}$$

Proof: If m is even we have, from (2.8),

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{U_{nm}}{V_{nm}V_{2nm}} &= \frac{1}{(\alpha-\beta)}\left[\sum_{n=1}^{\infty}\frac{1}{\alpha^{nm}V_{nm}}-\sum_{n=1}^{\infty}\frac{1}{\alpha^{2nm}V_{2nm}}\right]\\ &= \frac{1}{(\alpha-\beta)}\left[\sum_{n=1}^{\infty}\frac{1}{\alpha^{2nm}+1}-\sum_{n=1}^{\infty}\frac{1}{\alpha^{4nm}+1}\right]\\ &= \frac{1}{(\alpha-\beta)}\left[\sum_{n=1}^{\infty}\frac{(\beta^{2m})^n}{1+(\beta^{2m})^n}-\sum_{n=1}^{\infty}\frac{(\beta^{4m})^n}{1+(\beta^{4m})^n}\right] \text{ (since } \alpha\beta=-1),\end{aligned}$$

and the first part of (4.1) follows from (2.4). To prove (4.2) we begin with (2.6) and (2.7) and proceed in the same manner, making use of (2.3) and (2.5). Now to the second part of (4.1). In the first part of (4.1), we replace m by $2m$ to obtain

$$\sum_{n=1}^{\infty}\frac{U_{2nm}}{V_{2nm}V_{4nm}}=\frac{1}{(\alpha-\beta)}[L(\beta^{4m})-3L(\beta^{8m})+2L(\beta^{16m})],$$

which is valid for all positive integers m. When we add this sum to the second sum in (4.2), we obtain the second sum in (4.1). This completes the proof. □

5. THE DUAL RESULTS

In the introduction we referred to known dual results of special cases of (4.1) and (4.2). To obtain these, we replace $U(V)$ by $V(U)$ in (4.1) and (4.2). Then, with the identity $U_{2n}=U_nV_n$, the summands become $1/U_{nm}^2$ and $1/U_{(2n+1)m}^2$. Now if we take $U_n=F_n$, then the sums

$$\sum_{n=1}^{\infty}\frac{1}{F_n^2},\quad \sum_{n=1}^{\infty}\frac{1}{F_{2n}^2},\quad \text{and}\quad \sum_{n=0}^{\infty}\frac{1}{F_{2n+1}^2}$$

are known. See, for example (44), (48), and (55) of Bruckman [5], where elliptic functions are used. See also (f) and (h) on page 320 of Almkvist [1], where theta functions are used.

We have not found the more general sums

$$\sum_{n=1}^{\infty}\frac{1}{U_{nm}^2} \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{1}{U_{(2n+1)m}^2} \quad \text{(for the two parities of } m\text{)}$$

in the literature available to us, and we suspect that their determination is much more difficult.

REFERENCES

1. G. Almkvist. "A Solution to a Tantalizing Problem." *The Fibonacci Quarterly* **24.4** (1986): 316-22.
2. R. André-Jeannin. "Lambert Series and the Summation of Reciprocals in Certain Fibonacci-Lucas-Type Sequences." *The Fibonacci Quarterly* **28.3** (1990):223-26.
3. J. M. Borwein & P. B. Borwein. *Pi and the AGM.* New York: Wiley, 1987.
4. W. G. Brady. "The Lambert Function." *The Fibonacci Quarterly* **10.2** (1972):199-200.
5. P. S. Bruckman. "On the Evaluation of Certain Infinite Series by Elliptic Functions." *The Fibonacci Quarterly* **15.4** (1977):293-310.
6. A. F. Horadam. "Elliptic Functions and Lambert Series in the Summation of Reciprocals in Certain Recurrence-Generated Sequences." *The Fibonacci Quarterly* **26.2** (1988):98-114.
7. C. G. J. Jacobi. "Fundamenta Nova Theoriae Functionum Ellipticarum." *Gesammelte Werke* **1** (1881):159.
8. A. G. Shannon & A. F. Horadam. "Reciprocals of Generalized Fibonacci Numbers." *The Fibonacci Quarterly* **9.3** (1971):299-306, 312.

AMS Classification Numbers: 11B37, 11B39

❖❖❖

GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS AND THEIR ASSOCIATED DIAGONAL POLYNOMIALS

M. N. S. Swamy
Concordia University, Montreal, Quebec H3G 1M8, Canada
(*Submitted August 1997-Final Revision November 1997*)

1. INTRODUCTION

Horadam [7], in a recent article, defined two sequences of polynomials $J_n(x)$ and $j_n(x)$, the Jacobsthal and Jacobsthal-Lucas polynomials, respectively, and studied their properties. In the same article, he also defined and studied the properties of the rising and descending polynomials $R_n(x)$, $r_n(x)$, $D_n(x)$, and $d_n(x)$, which are fashioned in a manner similar to those for Chebyshev, Fermat, and other polynomials (see [2], [3], [4], [5], and [6]).

The purpose of this article is to extend these results to the generalized Fibonacci and Lucas polynomials defined by

$$U_n(x, y) = xU_{n-1}(x, y) + yU_{n-2}(x, y) \quad (n \geq 2), \tag{1.1a}$$

with

$$U_0(x, y) = 0, \quad U_1(x, y) = 1, \tag{1.1b}$$

and

$$V_n(x, y) = xV_{n-1}(x, y) + yV_{n-2}(x, y) \quad (n \geq 2), \tag{1.2a}$$

with

$$V_0(x, y) = 2, \quad V_1(x, y) = x. \tag{1.2b}$$

In Section 2, we will give some basic properties of the polynomials $U_n(x, y)$ and $V_n(x, y)$, most of which are generalizations of those given in [7] for $J_n(x)$ and $j_n(x)$. In Section 3, we will derive some new properties of $U_n(x, y)$ and $V_n(x, y)$ concerning their derivatives, as well as the differential equations they satisfy. In the remaining sections, we will define and study the properties of the rising and descending diagonal polynomials associated with $U_n(x, y)$ and $V_n(x, y)$, thus generalizing the results already known for Fibonacci, Lucas, Chebyshev, Fermat, and Jacobsthal polynomials.

2. BASIC PROPERTIES OF $U_n(x, y)$ AND $V_n(x, y)$

Binet Forms:

$$U_n(x, y) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2.1}$$

$$V_n(x, y) = \alpha^n + \beta^n, \tag{2.2}$$

where

$$\alpha + \beta = x, \quad \alpha\beta = -y, \tag{2.3a}$$

$$\alpha - \beta = \sqrt{\Delta}, \quad \Delta = x^2 + 4y, \tag{2.3b}$$

$$2\alpha = x + \sqrt{\Delta}, \quad 2\beta = x - \sqrt{\Delta}. \tag{2.3c}$$

Simson Formulas:

$$U_{n+1}(x, y)U_{n-1}(x, y) - U_n^2(x, y) = (-1)^n y^{n-1}, \tag{2.4}$$

$$V_{n+1}(x, y)V_{n-1}(x, y) - V_n^2(x, y) = (-1)^n y^{n-1}\Delta. \tag{2.5}$$

Summation Formulas:

$$\sum_{0}^{n} U_i(x,y) = \frac{1}{x+y-1}[U_{n+1}(x,y) + yU_n(x,y) - 1], \tag{2.6}$$

$$\sum_{0}^{n} V_i(x,y) = \frac{1}{x+y-1}[V_{n+1}(x,y) + yV_n(x,y) + (x-2)]. \tag{2.7}$$

Important Interrelations:

$$V_n(x,y) = U_{n+1}(x,y) + yU_{n-1}(x,y), \tag{2.8}$$
$$V_n(x,y) + xU_n(x,y) = 2U_{n+1}(x,y), \tag{2.9}$$
$$V_n(x,y) - xU_n(x,y) = 2yU_{n-1}(x,y), \tag{2.10}$$
$$\Delta U_n(x,y) = V_{n+1}(x,y) + yV_{n-1}(x,y), \tag{2.11}$$
$$\Delta U_n(x,y) = 2V_{n+1}(x,y) - xV_n(x,y), \tag{2.12}$$
$$U_{2n}(x,y) = U_n(x,y)V_n(x,y), \tag{2.13}$$
$$V_{2n}(x,y) = V_n^2(x,y) - 2(-y)^n, \tag{2.14}$$
$$V_{2n}(x,y) = \Delta U_n^2(x,y) + 2(-y)^n, \tag{2.15}$$
$$\Delta U_n^2(x,y) + V_n^2(x,y) = 2V_{2n}(x,y), \tag{2.16}$$
$$2U_{m+n}(x,y) = U_m(x,y)V_n(x,y) + V_m(x,y)U_n(x,y), \tag{2.17}$$
$$2V_{m+n}(x,y) = V_m(x,y)V_n(x,y) + \Delta U_m(x,y)U_n(x,y). \tag{2.18}$$

All the above results from (2.4)-(2.18) may be derived using the Binet forms (2.1) and (2.2) or, alternately, using the earlier results of Horadam [8]. Most of these results are to be found in Lucas ([10], Ch. 18). Now we let $X = \alpha$ and $Y = \beta$ in the following identities, where α and β are given by (2.3), X and Y arbitrary:

$$\frac{X^n - Y^n}{X - Y} = \sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-r-1}{r} (XY)^r (X+Y)^{n-2r-1} \quad (n \ge 0), \tag{2.19}$$

$$X^n + Y^n = \sum_{r=0}^{[n/2]} (-1)^r \frac{n}{n-r} \binom{n-r}{r} (XY)^r (X+Y)^{n-2r} \quad (n > 0). \tag{2.20}$$

We can then easily establish the following expressions for $U_n(x,y)$ and $V_n(x,y)$.

Closed Form Expressions:

$$U_n(x,y) = \sum_{r=0}^{[(n-1)/2]} \binom{n-r-1}{r} x^{n-2r-1} y^r, \tag{2.21}$$

$$V_n(x,y) = \sum_{r=0}^{[n/2]} \frac{n}{n-r} \binom{n-r}{r} x^{n-2r} y^r \quad (n > 0). \tag{2.22}$$

It is seen from (2.21) and (2.22) that $U_{2n}(x,y)$ and $V_{2n-1}(x,y)$ are odd polynomials in x of degree $(2n-1)$ and polynomials in y of degree $(n-1)$, while $U_{2n+1}(x,y)$ and $V_{2n}(x,y)$ are even polynomials in x of degree $2n$ and polynomials in y of degree n. It may be mentioned that expression (2.21) for $U_n(x,y)$ has already been established by Hoggatt and Long [3]; however, the expression for $V_n(x,y)$ is new. By letting $x = 1$ and $y = 2x$, we obtain the results of Horadam [7] for the polynomials $J_n(x)$ and $j_n(x)$.

Hoggatt and Long [3] have shown that

$$U_n(x, y) = \prod_{k=1}^{n-1}\left\{x - 2\sqrt{-y}\cos\left(\frac{k}{n}\pi\right)\right\} \quad (n \geq 2). \tag{2.23}$$

Using a similar procedure, or by using the technique used by Swamy [11] in obtaining the zeros of Morgan-Voyce polynomials, we can show that

$$V_n(x, y) = \prod_{k=1}^{n}\left\{x - 2\sqrt{-y}\cos\left(\frac{2k-1}{2n}\pi\right)\right\} \quad (n \geq 2). \tag{2.24}$$

We may now rewrite expressions (2.23) and (2.24) to express the polynomials $U_n(x, y)$ and $V_n(x, y)$ in the product form.

Product Form:

$$U_n(x, y) = x^{\delta_n}\prod_{k=1}^{[(n-1)/2]}\left\{x^2 + 4y\cos^2\left(\frac{k}{n}\pi\right)\right\} \quad (n > 2), \tag{2.25}$$

$$V_n(x, y) = x^{1-\delta_n}\prod_{k=1}^{[n/2]}\left\{x^2 + 4y\cos^2\left(\frac{2k-1}{2n}\pi\right)\right\} \quad (n \geq 2), \tag{2.26}$$

where

$$\delta_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{2.27}$$

By letting $x = 1$ and $y = 2x$ in (2.26) and (2.27), we get the zeros of the Jacobsthal polynomials $J_n(x)$ and $j_n(x)$ to be, respectively,

$$x = -\frac{1}{8}\sec^2\left(\frac{k}{n}\pi\right), \quad k = 1, 2, \ldots, (n-1), \tag{2.28}$$

and

$$x = -\frac{1}{8}\sec^2\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, 2, \ldots, n. \tag{2.29}$$

The generating functions for $U_n(x, y)$ and $V_n(x, y)$ are given below.

Generating Functions:

$$U(x, y, t) = \sum_{i=1}^{\infty} U_i(x, y)t^{i-1} = \{1 - t(x + yt)\}^{-1}, \tag{2.30}$$

$$V(x, y, t) = \sum_{i=1}^{\infty} V_i(x, y)t^{i} = (2 - xt)\{1 - t(x + yt)\}^{-1} \tag{2.31}$$

$$= 1 + (1 + yt^2)\{1 - t(x + yt)\}^{-1} \tag{2.32}$$

3. DERIVATIVE PROPERTIES

From (2.30), (2.31), and (2.32), a number of relations involving the derivatives of $U_n(x, y)$ and $V_n(x, y)$ may be derived. However, only the following derivative relations are listed here. Throughout this section, where not explicitly mentioned, U, V, U_n, and V_n stand for $U(x, y, t)$, $V(x, y, t)$, $U_n(x, y)$, and $V_n(x, y)$, respectively. We can prove that

$$\frac{\partial V}{\partial x}=t\frac{\partial}{\partial t}(tU), \tag{3.1}$$

$$\frac{\partial V}{\partial y}=t\frac{\partial}{\partial t}(t^2U), \tag{3.2}$$

$$\frac{\partial U}{\partial y}=t\frac{\partial U}{\partial x}, \tag{3.3}$$

$$\frac{\partial V}{\partial y}=t\frac{\partial V}{\partial x}+t^2U, \tag{3.4}$$

$$x\frac{\partial U}{\partial x}+2y\frac{\partial U}{\partial y}=t\frac{\partial U}{\partial t}, \tag{3.5}$$

$$x\frac{\partial V}{\partial x}+2y\frac{\partial V}{\partial y}=t\frac{\partial V}{\partial t}. \tag{3.6}$$

The above results are now established. From the generating function (2.30), we have

$$\frac{1}{t}\frac{\partial U}{\partial x}=\frac{1}{t^2}\frac{\partial U}{\partial y}=\frac{1}{x+2yt}\frac{\partial U}{\partial t}=U^2. \tag{3.7}$$

We see that (3.3) and (3.5) follow directly from (3.7). Now, from (2.32) and (2.31), we have

$$\frac{\partial V}{\partial x}=t(1+yt^2)U^2, \tag{3.8a}$$

$$\frac{\partial V}{\partial y}=t^2(2-xt)U^2. \tag{3.8b}$$

However,

$$\frac{\partial}{\partial t}(tU)=(1+yt^2)U^2, \tag{3.9a}$$

$$\frac{\partial}{\partial t}(t^2U)=t(2-xt)U^2. \tag{3.9b}$$

Relation (3.1) follows directly from (3.8a) and (3.9a), while (3.2) follows directly from (3.8b) and (3.9b). Also, from (3.8a) and (3.8b), we have

$$x\frac{\partial V}{\partial x}+2y\frac{\partial V}{\partial y}=t(x-xyt^2+4yt)U^2=t\frac{\partial V}{\partial t},$$

thus establishing (3.6). Finally, we have, from (3.8a),

$$t^2U+t\frac{\partial V}{\partial x}=t^2(1-xt-yt^2)U^2+t^2(1+yt^2)U^2$$

$$=t^2(2-xt)U^2=\frac{\partial V}{\partial y},\ \text{using (3.8b)}.$$

Thus, relation (3.4) is established. Using the above relations (3.1) to (3.6) and the generating functions for $U(x,y,t)$ and $V(x,y,t)$ given by (2.30) to (2.32), we can obtain the following relationships, where the primes indicate partial derivatives with respect to x and dots those with respect to y:

$$V_n'(x,y)=nU_n(x,y),\ \text{using (3.1), (2.30), and (2.31)}, \tag{3.10}$$

$$\dot{V}_n(x,y)=nU_{n-1}(x,y),\ \text{using (3.2), (2.3), and (2.31)}, \tag{3.11}$$

$$\dot{U}_{n+1}(x,y)=U_n'(x,y),\ \text{using (3.3) and (2.30), or from (3.10) and (3.11),} \tag{3.12}$$

$$n\dot{V}_{n+1}(x,y)=(n+1)V_n'(x,y),\ \text{using (3.10) and (3.11),} \tag{3.13}$$

$$\dot{V}_{n+1}(x,y)=V_n'(x,y)+U_n(x,y),\ \text{using (3.4), (2.30), and (2.31), or from (3.10) and (3.13),} \tag{3.14}$$

$$xU_n'(x,y)+2y\dot{U}_n(x,y)=(n-1)U_n(x,y),\ \text{using (3.5) and (2.30),} \tag{3.15}$$

$$xV_n'(x,y)+2y\dot{V}_n(x,y)=nV_n(x,y),\ \text{using (3.6) and (2.31).} \tag{3.16}$$

We shall illustrate the procedure for proving the above results by establishing (3.12) and (3.16); the other results may be established in a similar manner. Substituting (2.30) and (2.31) in (3.3) and (3.6), respectively, we get:

$$\sum_1^\infty U_i'(x,y)t^{i-1}=t\sum_1^\infty \dot{U}_i(x,y)t^{i-1};$$

$$x\sum_0^\infty V_i'(x,y)t^i+2y\sum_0^\infty \dot{V}_i(x,y)t^i=t\sum_1^3 iV_i(x,y)t^{i-1}.$$

Comparing the coefficients of like powers of t on both sides of the above equations, we obtain (3.12) and (3.16), respectively.

Using the results of (3.12) and (3.13), we may now derive the following relations for the higher-order derivatives of $U_n(x,y)$ and $V_n(x,y)$, where $D_x^{(r)}$ and $D_y^{(r)}$ denote the derivatives with respect to x and y, respectively.

$$D_y^{(r)}U_{n+1}(x,y)=D_x^{(r)}U_{n-r+1}(x,y), \tag{3.17}$$

$$(n-r+1)D_y^{(r)}V_{n+1}(x,y)=(n+1)D_x^{(r)}V_{n-r+1}(x,y). \tag{3.18}$$

We will now derive the linear differential equations satisfied by $U_n(x,y)$ and $V_n(x,y)$. From (2.12), we have $\Delta\dot{U}_{n-1}+4U_{n-1}=2\dot{V}_n-x\dot{V}_{n-1}$. Hence,

$$\begin{aligned}\frac{1}{n}\Delta\ddot{V}_n+4U_{n-1} &= 2nU_{n-1}-x(n-1)U_{n-2},\ \text{using (3.11),}\\ &= 2nU_{n-1}-(n-1)[2U_{n-1}-V_{n-2}],\ \text{using (2.9)}\\ &= 2U_{n-1}+(n-1)(\Delta U_{n-1}-V_n)/y,\ \text{using (2.11).}\end{aligned}$$

Therefore,

$$y\Delta\ddot{V}_n+\{2y-(n-1)\Delta\}nU_{n-1}+n(n-1)V_n=0. \tag{3.19}$$

Substituting (3.11) in (3.19), we see that $V_n(x,y)=z$ satisfies the differential equation given by

$$y(x^2+4y)\ddot{z}+\{2y-(n-1)(x^2+4y)\}\dot{z}+n(n-1)z=0. \tag{3.20}$$

Differentiating (3.19) again with respect to y and again making use of the result (3.11), we get

$$y(x^2+4y)n\ddot{U}_{n-1}+\{6y-(n-2)(x^2+4y)\}n\dot{U}_{n-1}+(n-2)(n-3)nU_{n-1}=0.$$

Now, replacing $(n-1)$ by n, we see that $U_n(x,y)=z$ satisfies the differential equation

$$y(x^2+4y)\ddot{z}+\{6y-(n-1)(x^2+4y)\}\dot{z}+(n-1)(n-2)z=0. \tag{3.21}$$

Since the Jacobsthal polynomials [7] $J_n(x)$ and $j_n(x)$ are given by

$$J_n(x) = U_n(1, 2x) \quad \text{and} \quad j_n(x) = V_n(1, 2x), \tag{3.22}$$

we see, from (3.21), that $J_n(x)$ satisfies the differential equation

$$x(8x+1)z'' - \{4(2n-5)x + (n-1)\}z' + 2(n-1)(n-2)z = 0, \tag{3.23}$$

while, from (3.20), we see that $j_n(x)$ satisfies the equation

$$x(8x+1)z'' - \{4(2n-3)x + (n-1)\}z' + 2n(n-1)z = 0. \tag{3.24}$$

Recall that $y = 2x$ implies that $\dot{z} = \frac{1}{2}z'$ and $\ddot{z} = \frac{1}{4}z''$.

In a similar way, differentiating both sides of (2.11) with respect to x, and utilizing (3.10), (2.8), and (1.1a), we can show that $V_n(x, y)$ satisfies the equation

$$(x^2 + 4y)z'' + xz' - n^2 z = 0. \tag{3.25}$$

Differentiating (3.25) with respect to x once and making use of (3.10), we see that $U_n(x, y)$ satisfies the equation

$$(x^2 + 4y)z'' + 3xz' + (1 - n^2)z = 0. \tag{3.26}$$

It should be noted that equations (3.25) and (3.26) appear in [1] as equations (1.11) and (2.6), respectively, in a slightly varied notation. It should also be noted that, after the submission of this article, an article by Horadam [9] appeared generalizing the results given in (3.20), (3.21), (3.25), and (3.26).

Also, from (3.11) and (1.2), we can show that

$$\dot{V}_n(x, y) = \frac{n}{n-1}x\dot{V}_{n-1}(x, y) + \frac{n}{n-2}y\dot{V}_{n-2}(x, y) \quad (n \geq 3), \tag{3.27a}$$

while, from (3.10) and (1.2), we can prove that

$$V_n'(x, y) = \frac{n}{n-1}xV_{n-1}'(x, y) + \frac{n}{n-2}yV_{n-2}'(x, y) \quad (n \geq 3). \tag{3.28a}$$

Thus, we see that both $\dot{V}_n(x, y)$ and $V_n'(x, y)$ satisfy the same recurrence relation, but with different initial conditions as given by

$$\dot{V}_1(x, y) = 0, \quad \dot{V}_2(x, y) = 2, \tag{3.27b}$$

$$V_i'(x, y) = 1, \quad V_2'(x, y) = 2x. \tag{3.28b}$$

4. RISING DIAGONAL POLYNOMIALS

Let us first consider the rising diagonal polynomials $R_n(x, y)$ associated with $U_n(x, y)$; these polynomials are formed the same way as the rising diagonal polynomials associated with Fermat, Chebyshev, Jacobsthal, and other similar polynomials (see [2], [4], [5], [6], and [7]). Thus, from (2.21), we see that $R_0(x, y) = 0$, $R_1(x, y) = 1$, $R_2(x, y) = x, \ldots,$

$$R_n(x, y) = x^{n-1} + \binom{n-3}{1}x^{n-4}y + \binom{n-5}{2}x^{n-7}y^2 + \binom{n-7}{3}x^{n-10}y^3 + \cdots.$$

The above may be rewritten as

$$R_n(x, y) = x^{n-1} + \binom{n-1-2\cdot 1}{1}x^{n-1-3\cdot 1}y + \binom{n-1-2\cdot 2}{2}x^{n-1-3\cdot 2}y^2 +$$

$$+\binom{n-1-2\cdot 3}{3}x^{n-1-3\cdot 3}y^3+\cdots+\binom{n-1-2\cdot\left[\frac{n-1}{3}\right]}{\left[\frac{n-1}{3}\right]}y^{\left[\frac{n-1}{3}\right]}.$$

Hence,

$$R_n(x,y)=\sum_{i=0}^{[(n-1)/3]}\binom{n-1-2i}{i}x^{n-1-3i}y^i \quad (n\ge 1),\quad R_0(x,y)=0. \tag{4.1}$$

Similarly, starting with (2.22), we may show that the rising polynomials $r_n(x,y)$ associated with $V_n(x,y)$ are given by

$$r_n(x,y)=\sum_{i=0}^{[n/3]}\frac{n-i}{n-2i}\binom{n-2i}{i}x^{n-3i}y^i \quad (n\ge 1),\quad r_0(x,y)=2. \tag{4.2}$$

We now derive some interesting relationships for these rising polynomials including the recurrence relations. From (4.1) and (4.2), we have

$$\begin{aligned} r_n(x,y)+xR_n(x,y) &= \sum_{i=0}^{[n/3]}\frac{n-i}{n-2i}\binom{n-2i}{i}x^{n-3i}y^i+\sum_{i=0}^{[(n-1)/3]}\binom{n-1-2i}{i}x^{n-3i}y^i \\ &= 2\sum_{i=0}^{[n/3]}\binom{n-2i}{i}x^{n-3i}y^i=2R_{n+1}(x,y). \end{aligned}$$

Hence,

$$r_n(x,y)+xR_n(x,y)=2R_{n+1}(x,y). \tag{4.3}$$

Similarly, we can show that

$$r_n(x,y)-xR_n(x,y)=2yR_{n-2}(x,y)\quad (n\ge 2). \tag{4.4}$$

Hence, from (4.3) and (4.4),

$$\begin{aligned} r_n(x,y)&=R_{n+1}(x,y)+yR_{n-2}(x,y)\quad (n\ge 2),\\ R_{n+1}(x,y)&=xR_n(x,y)+yR_{n-2}(x,y)\quad (n\ge 2). \end{aligned} \tag{4.5}$$

Thus, we see that $R_n(x,y)$ satisfies the recurrence relation

$$R_n(x,y)=xR_{n-1}(x,y)+yR_{n-3}(x,y)\quad (n\ge 3), \tag{4.6a}$$

with

$$R_0(x,y)=0,\quad R_1(x,y)=1,\quad R_2(x,y)=x. \tag{4.6b}$$

Similarly, using (4.3), (4.4), and (4.5), we can deduce that $r_n(x,y)$ satisfies the recurrence relation

$$r_n(x,y)=xr_{n-1}(x,y)+yr_{n-3}(x,y)\quad (n\ge 3), \tag{4.7a}$$

with

$$r_0(x,y)=2,\quad r_1(x,y)=x,\quad r_2(x,y)=x^2. \tag{4.7b}$$

It is interesting to compare the relations (4.6), (4.7), (4.5), (4.3), and (4.4) with their counterparts for $U_n(x,y)$ and $V_n(x,y)$ given, respectively, by (1.1), (1.2), (2.8), (2.9), and (2.10).

The generating functions for $R_n(x,y)$ and $r_n(x,y)$ may be found by following the usual technique. They are given by

$$R(x,y,t)=\sum_{i=1}^{\infty}R_i(x,y)t^{i-1}=\{1-t(x+yt^2)\}^{-1}, \tag{4.8}$$

$$r(x,y,t)=\sum_{i=0}^{\infty}r_i(x,y)t^i=(2-xt)\{1-t(x+yt^2)\}^{-1} \tag{4.9}$$

$$=1+(1+yt^3)\{1-t(x+yt^2)\}^{-1}. \tag{4.10}$$

Using these generating functions, we may now derive a number of results concerning the derivatives of $R(x,y,t)$ and $r(x,y,t)$ where, for the sake of convenience, R and r are used for $R(x,y,t)$ and $r(x,y,t)$. A few of these results are:

$$\frac{\partial R}{\partial y}=t^2\frac{\partial R}{\partial x}, \tag{4.11}$$

$$\frac{\partial r}{\partial y}=t^2\frac{\partial r}{\partial x}+t^3R, \tag{4.12}$$

$$x\frac{\partial R}{\partial x}+3y\frac{\partial R}{\partial y}=t\frac{\partial R}{\partial t}, \tag{4.13}$$

$$x\frac{\partial r}{\partial x}+3y\frac{\partial r}{\partial y}=t\frac{\partial r}{\partial t}. \tag{4.14}$$

The above results may be established in a way similar to those given in (3.1) to (3.6). From the above results, we may derive the following relationships for the derivatives of $R_n(x,y)$ and $r_n(x,y)$, where again the primes indicate partial derivatives with respect to x and dots those with respect to y:

$$\dot{R}_{n+2}(x,y)=R_n'(x,y), \tag{4.15}$$

$$\dot{r}_{n+2}(x,y)=r_n'(x,y)+R_n(x,y), \tag{4.16}$$

$$xR_n'(x,y)+3y\dot{R}_n(x,y)=(n-1)R_n(x,y) \tag{4.17}$$

$$xr_n'(x,y)+3y\dot{r}_n(x,y)=nr_n(x,y). \tag{4.18}$$

Again, it is interesting to compare the relationships (4.11), (4.12), (4.13), (4.14), (4.15), (4.16), (4.17), and (4.18) with their counterparts for $U_n(x,y)$ and $V_n(x,y)$, namely, the relations (3.3), (3.4), (3.5), (3.6), (3.12), (3.14), (3.15), and (3.16), respectively.

5. DESCENDING DIAGONAL POLYNOMIALS

Let us now consider the descending diagonal polynomials $D_n(x,y)$ and $d_n(x,y)$ associated with the polynomials $U_n(x,y)$ and $V_n(x,y)$, respectively; these are formed the same way as the descending diagonal functions associated with Chebyshev, Fermat, Jacobsthal, and other similar polynomials (see [2], [4], [5], [6], and [7]). Thus, from (2.21), we see that the descending polynomial $D_n(x,y)$ associated with $U_n(x,y)$ is given by

$$D_0(x,y)=0,\quad D_1(x,y)=1,\quad D_2(x,y)=x+y,\ldots,$$

$$D_n(x,y)=\binom{n-1}{0}x^{n-1}+\binom{n-1}{1}x^{n-2}y+\cdots+\binom{n-1}{n-1}y^{n-1}=(x+y)^{n-1}.$$

Hence,

$$D_n(x,y)=\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^i=(x+y)^{n-1}\quad(n\ge 1),\quad D_0(x,y)=0. \tag{5.1}$$

Similarly, starting with relation (2.22), we can obtain the descending polynomial $d_n(x, y)$ associated with the polynomial $V_n(x, y)$ to be

$$d_n(x,y)=\sum_{i=0}^{n}\frac{n+i}{n}\binom{n}{i}x^{n-i}y^i \quad (n\geq 1),\quad d_0(x,y)=2. \tag{5.2}$$

Now consider

$$D_{n+1}(x,y)+yD_n(x,y)=\sum_{i=0}^{n}\left\{\binom{n}{i}+\binom{n-1}{i-1}\right\}x^{n-i}y^i=\sum_{i=0}^{n}\frac{n+i}{n}\binom{n}{i}x^{n-i}y^i \quad (n\geq 1).$$

Hence

$$d_n(x,y)=D_{n+1}(x,y)+yD_n(x,y) \quad (n\geq 1). \tag{5.3}$$

Thus,

$$d_n(x,y)=(x+2y)(x+y)^{n-1} \quad (n\geq 1). \tag{5.4}$$

We also have, from (5.1) and (5.4),

$$\frac{D_{n+1}(x,y)}{D_n(x,y)}=\frac{d_{n+1}(x,y)}{d_n(x,y)}=x+y \quad (n\geq 1), \tag{5.5}$$

$$d_{n+1}(x,y)+yd_n(x,y)=(x+2y)^2D_n(x,y) \quad (n\geq 1). \tag{5.6}$$

We may also formulate the following generating functions for the descending polynomials $D_n(x, y)$ and $d_n(x, y)$ by following the usual procedure:

$$D(x,y,t)=\sum_{i=1}^{\infty}D_i(x,y)t^{i-1}=\{1-(x+y)t\}^{-1}, \tag{5.7}$$

$$d(x,y,t)=\sum_{i=1}^{\infty}d_i(x,y)t^{i-1}=(x+2y)\{1-(x+y)t\}^{-1}. \tag{5.8}$$

From the above generating functions, we may deduce the following relations for the derivatives of $D(x, y, t)$ and $d(x, y, t)$ where, for the sake of convenience, D and d are used for $D(x, y, t)$ and $d(x, y, t)$:

$$\frac{\partial D}{\partial y}=\frac{\partial D}{\partial x}, \tag{5.9}$$

$$\frac{\partial d}{\partial y}=\frac{\partial d}{\partial x}+D, \tag{5.10}$$

$$x\frac{\partial D}{\partial x}+y\frac{\partial D}{\partial y}=t\frac{\partial D}{\partial t}, \tag{5.11}$$

$$x\frac{\partial d}{\partial x}+y\frac{\partial d}{\partial y}=t\frac{\partial d}{\partial t}+d, \tag{5.12}$$

$$(x+y)\frac{\partial D}{\partial y}=(x+y)\frac{\partial D}{\partial x}=t\frac{\partial D}{\partial t}, \tag{5.13}$$

$$(x+y)\frac{\partial d}{\partial y}x=t\frac{\partial d}{\partial t}+(x+y)D, \tag{5.14}$$

$$(x+y)\frac{\partial d}{\partial y}=t\frac{\partial d}{\partial t}+2(x+y)D. \tag{5.15}$$

Using the above relations, we may write the corresponding interrelations for the derivatives of the polynomials $D_n(x, y)$ and $d_n(x, y)$ with respect to x and y as has been done for $R_n(x, y)$ and $r_n(x, y)$.

6. CONCLUDING REMARKS

We have generalized all the known results concerning the diagonal functions associated with Fibonacci, Lucas, Chebyshev, Fermat, Pell, and Jacobsthal polynomials to the case of diagonal functions associated with the generalized polynomials given by (1.1) and (1.2). We have also derived a number of new interesting results concerning the derivatives of $U_n(x, y)$ and $V_n(x, y)$ with respect to y, the differential equations satisfied by these polynomials, as well as the interrelations between their derivatives with respect to x and y. Similar results have also been derived for both the rising and the descending diagonal polynomials associated with $U_n(x, y)$ and $V_n(x, y)$; however, we have not been able to find the differential equations satisfied by $R_n(x, y)$, $r_n(x, y)$, and $d_n(x, y)$ with respect to either x or y. It may also be observed that the descending (rising) polynomials associated with the rising (descending) polynomials of $U_n(x, y)$ and $V_n(x, y)$ are, respectively, $U_n(x, y)$ and $V_n(x, y)$. This answers one of the questions raised by Horadam [7] regarding the rising polynomials of the descending polynomials of $J_n(x)$ and $j_n(x)$ as well as the descending polynomials of the rising polynomials of $J_n(x)$ and $j_n(x)$.

ACKNOWLEDGMENT

The author would like to thank the anonymous referee for many valuable comments and suggestions that have enhanced the quality and presentation of this article.

REFERENCES

1. R. André-Jeannin. "Differential Properties of a General Class of Polynomials." *The Fibonacci Quarterly* **33.5** (1995):453-58.
2. D. V. Jaiswal. "On Polynomials Related to Tchebichef Polynomials of the Second Kind." *The Fibonacci Quarterly* **12.3** (1974):263-65.
3. V. E. Hoggatt, Jr., & C. T. Long. "Divisibility Properties of Generalized Fibonacci Polynomials." *The Fibonacci Quarterly* **12.2** (1974):113-20.
4. A. F. Horadam. "Diagonal Functions." *The Fibonacci Quarterly* **16.1** (1978):33-36.
5. A. F. Horadam. "Chebyshev and Fermat Polynomials for Diagonal Functions." *The Fibonacci Quarterly* **17.4** (1979):328-33.
6. A. F. Horadam. "Extensions of a Paper on Diagonal Functions." *The Fibonacci Quarterly* **18.1** (1980):3-8.
7. A. F. Horadam. "Jacobsthal Representation Polynomials." *The Fibonacci Quarterly* **35.2** (1997):137-48.
8. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.3** (1965):161-76.
9. A. F. Horadam. "Rodrigues' Formula for Jacobsthal-Type Polynomials." *The Fibonacci Quarterly* **35.4** (1997):361-70.
10. E. Lucas. *Théorie des Nombres*. Paris: Blanchard, 1961.
11. M. N. S. Swamy. "Further Properties of Morgan-Voyce Polynomials." *The Fibonacci Quarterly* **6.2** (1968):167-75.

AMS Classification Numbers: 11B39, 33C25

❖❖❖

ON ∞-GENERALIZED FIBONACCI SEQUENCES

Walter Motta

Departamento de Matemática, CETEC-UFU,
Campus Santa Mônica, 38400-902 Uberlândia, MG, Brazil
e-mail: wmotta@ufu.br

Mustapha Rachidi

Départment de Mathématiques, Faculté des Sciences,
Université Mohammed V, B.P. 1014, Rabat, Morocco
e-mail: rachidi@fsr.ac.ma

Osamu Saeki

Department of Mathematics, Faculty of Science,
Hiroshima University, Higsashi-Hiroshima 739-8526, Japan
e-mail: saeki@math.sci.hiroshima-u.ac.jp

1. INTRODUCTION

Let $a_0, a_1, \dots, a_{r-1}$ be arbitrary complex numbers with $a_{r-1} \neq 0$ $(1 \le r < \infty)$. For a given sequence of complex numbers $A = (\alpha_{-r+1}, \alpha_{-r+2}, \dots, \alpha_0)$, we define the *weighted r-generalized Fibonacci sequence* $\{y_A(n)\}_{n=-r+1}^{\infty}$ by using a recurrence formula involving $r+1$ terms* as follows:

$$y_A(n) = \alpha_n \quad (n = -r+1, -r+2, \dots, 0);$$

$$y_A(n) = \sum_{i=1}^{r} a_{i-1} y_A(n-i) \quad (n = 1, 2, 3, \dots).$$

When $a_i = 1$ for all i and $A = (0, 0, \dots, 0, 1)$, we get the r-generalized Fibonacci numbers (see [4]). A Binet-type formula and a combinatorial expression of weighted r-generalized Fibonacci sequences are given in [3]. Furthermore, in [2], the convergence of the sequence $\{y_A(n) / n^{\nu-1} q^n\}$ has been studied, where q is a root of the characteristic polynomial $P(x) = x^r - a_0 x^{r-1} - \cdots - a_{r-2} x - a_{r-1}$ of multiplicity ν.

The purpose of this paper is to generalize the weighted r-generalized Fibonacci sequences with $1 \le r < \infty$ to a class of sequences which are defined by recurrence formulas involving *infinitely many* terms, and to analyze their asymptotic behavior. We call such sequences *∞-generalized Fibonacci sequences*. This is a new generalization of the usual Fibonacci sequences and almost nothing has been known about such sequences until now. For example, there has been no theory of difference equations for such sequences.

More precisely, an ∞-generalized Fibonacci sequence is defined as follows. We suppose that two infinite sequences of complex numbers are given, one for the initial sequence and the other for the weight sequence. Then a member of the ∞-generalized Fibonacci sequence is determined by the weighted series of its preceding members (for a precise definition, see §2). Since the recurrence formula always involves infinitely many terms, we always have to worry about the convergence of the series corresponding to the recurrence formula and hence we need auxiliary conditions on the initial sequence and the weight sequence.

---

* This is called an *r-th order linear recurrence* in [3].

One of the striking results of this paper is that, under certain conditions, an ∞-generalized Fibonacci sequence behaves very much like a weighted r-generalized Fibonacci sequence with r finite, as far as its asymptotic behavior is concerned.

The paper is organized as follows. In §2 we give a precise definition of the ∞-generalized Fibonacci sequences. In §3 we analyze their asymptotic behavior under certain conditions. In §4 we give some explicit examples in order to illustrate our results.

2. ∞-GENERALIZED FIBONACCI SEQUENCES

Take an infinite sequence $\{a_i\}_{i=0}^{\infty}$ of complex numbers, which will later be the weight sequence of ∞-generalized Fibonacci sequences. We set $h(z)=\sum_{i=0}^{\infty}a_i z^i$ for $z\in\mathbf{C}$ and $u(x)=\sum_{i=1}^{\infty}|a_i|x^i$ for $x\in\mathbf{R}$. Let R denote the radius of convergence of the power series h, which coincides with the radius of convergence of u. We assume the following condition:

$$0<R\le\infty. \tag{2.1}$$

Let X be the set of sequences $\{x_i\}_{i=0}^{\infty}$ of complex numbers such that there exist $C>0$ and T with $0<T<R$ satisfying $|x_i|\le CT^i$ for all i. Note that X is an infinite dimensional vector space over $\mathbf{C}$; it will be the set of initial sequences for ∞-generalized Fibonacci sequences associated with the weight sequence $\{a_i\}_{i=0}^{\infty}$. Define $f:X\to\mathbf{C}$ by $f(x_0,x_1,\dots)=\sum_{i=0}^{\infty}a_ix_i$. Since the series $\sum_{i=0}^{\infty}a_iCT^i$ converges absolutely, the series defining f also converges absolutely.

Lemma 2.2: If $\{y_0,y_{-1},y_{-2},\dots\}\in X$, then the sequence $\{y_m,y_{m-1},y_{m-2},\dots,y_1,y_0,y_{-1},y_{-2},\dots\}$ is an element of X for every finite sequence of complex numbers $y_m,y_{m-1},\dots,y_1$ $(m\ge1)$.

Proof: By our assumption, there exist $C>0$ and T with $0<T<R$ such that $|y_{-i}|\le CT^i$ for all $i\ge0$. Then we have $|y_{-i}|\le(CT^{-m})T^{i+m}$ for all $i\ge0$. On the other hand, there exists $C'>0$ such that $|y_{m-j}|\le C'T^j$ for $j=0,1,\dots,m-1$. Putting $C''=\max\{C',CT^{-m}\}$, we have $|y_{m-j}|\le C''T^j$ for all $j\ge0$. This completes the proof. □

Now we define an ∞-generalized Fibonacci sequence as follows. For a sequence $\{y_0,y_{-1},y_{-2},\dots\}\in X$, we define the sequence $\{y_1,y_2,y_3,\dots\}$ by

$$y_n=f(y_{n-1},y_{n-2},y_{n-3},\dots)=\sum_{i=1}^{\infty}a_{i-1}y_{n-i}\quad(n=1,2,3,\dots).$$

This is well defined by Lemma 2.2. The sequence $\{y_i\}_{i\in\mathbf{Z}}$ is called an *∞-generalized Fibonacci sequence* associated with the weight sequence $\{a_i\}_{i=0}^{\infty}$. Note that if there exists an integer $r\ge1$ such that $a_i=0$ for all $i\ge r$, then the sequence $\{a_i\}_{i=0}^{\infty}$ satisfies the condition (2.1) and the above definition coincides with that of weighted r-generalized Fibonacci sequences. Thus ∞-generalized sequences generalize weighted r-generalized Fibonacci sequences with r finite.

Lemma 2.3:

(1) Suppose that each a_i is a nonnegative real number and that there exists an S with $0<S<R$ satisfying

$$a_0>S^{-1}-u(S)\ \text{ (or, equivalently, } Sh(S)>1). \tag{2.3.1}$$

Then there exists a unique $q\in\mathbf{R}$ such that $q>S^{-1}$, $\{q^{-(i+1)}\}_{i=0}^{\infty}\in X$, and $f(q^{-1},q^{-2},q^{-3},\dots)=1$.

(2) Suppose that there exists an S with $0 < S < R$ satisfying

$$|a_0| > S^{-1} + u(S). \tag{2.3.2}$$

Then there exists a unique $q \in \mathbf{C}$ such that $|q| > S^{-1}$, $\{q^{-(i+1)}\}_{i=0}^{\infty} \in X$, and $f(q^{-1}, q^{-2}, q^{-3}, \dots) = 1$.

Proof:

(1) For $x > R^{-1}$, set $\varphi(x) = f(x^{-1}, x^{-2}, x^{-3}, \dots) = x^{-1}h(x^{-1})$. Note that φ is a differentiable function. Then we have $\lim_{x\to\infty}\varphi(x) = 0$ and $\varphi'(x) = f(-x^{-2}, -2x^{-3}, \dots) < 0$ for all $x > R^{-1}$. Furthermore, we have $\varphi(S^{-1}) > 1$ by (2.3.1). Then the intermediate value theorem implies that there exists a unique $q > S^{-1}$ such that $\varphi(q) = 1$.

(2) Define the holomorphic function $v(z)$ by $v(z) = 1 - \sum_{i=1}^{\infty} a_i z^{i+1}$ for z with $|z| < R$. Then, for z with $|z| = S$, we have

$$|v(z)| \le 1 + \sum_{i=1}^{\infty} |a_i|\,|z|^{i+1} = 1 + Su(S) \; < \; |a_0|S = |a_0 z|$$

by (2.3.2). Hence, by Rouché's theorem, $a_0 z - v(z)$ and $a_0 z$ have the same number of zeros in the region $|z| < S$. Note that $a_0 z - v(z) = 0$ if and only if $zh(z) = 1$. Since $a_0 z$ has a unique zero in the region, we have the conclusion. □

Remark 2.4: For a weighted r-generalized Fibonacci sequence of nonnegative real numbers with r finite, condition (2.3.1) is always satisfied, and the real number q as in Lemma 2.3(1) is the unique positive real root of the characteristic polynomial (not necessarily asymptotically simple in the terminology of [2]).

Remark 2.5: In the situation of the above lemma, if $\{y_0, y_{-1}, y_{-2}, \dots\} = \{1, q^{-1}, q^{-2}, q^{-3}, \dots\}$, then we can check easily that $y_n = q^n$ for all $n \in \mathbf{Z}$.

Note that if condition (2.3.1) or (2.3.2) is not satisfied, then, in general, there exists no $q \neq 0$ such that $\{q^{-(i+1)}\}_{i=0}^{\infty} \in X$ and $f(q^{-1}, q^{-2}, q^{-3}, \dots) = 1$. For example, consider $a_i = -1/(i+1)!$. The sequence $\{a_i\}_{i=0}^{\infty}$ satisfies condition (2.1) with $R = \infty$. However, $1 - zh(z) = e^z$ and there exists no $q \neq 0$ with $q^{-1}h(q^{-1}) = 1$.

3. CONVERGENCE RESULT FOR $\lim_{n\to\infty} y_n / q^n$

Our aim in this section is to prove a convergence theorem for the sequence $\{y_n / q^n\}$ (Theorem 3.10), where $\{y_n\}$ is an ∞-generalized Fibonacci sequence as defined in §2 and q is as in Lemma 2.3.

We first define the auxiliary sequence $\{g_n\}$ as follows. We set $g_0 = 1$, $g_n = 0$ for $n \le -1$, and define $\{g_n\}_{n=1}^{\infty}$ as the ∞-generalized Fibonacci sequence associated with the weight sequence $\{a_i\}_{i=0}^{\infty}$ and the initial sequence $\{g_n\}_{n=0}^{-\infty}$; i.e., $g_1 = f(g_0, g_{-1}, g_{-2}, \dots)$, $g_2 = f(g_1, g_0, g_{-1}, \dots)$, etc.

Lemma 3.1: For all $n \ge 1$, we have

$$y_n = g_n y_0 + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n} g_{n-j} a_{i+j-1}\right) y_{-i}.$$

Furthermore, the series on the right-hand side converges absolutely; i.e., the following series converges:

$$|g_n y_0| + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n}|g_{n-j}a_{i+j-1}|\right)|y_{-i}|.$$

Proof: Note that $g_1 = a_0$. Then the equality for $n=1$ together with the absolute convergence is easily checked. Now assume that, for n, $n-1, n-2, \ldots, 1$, the right-hand side of the equality converges absolutely and that the equality is valid. Then we have

$$\begin{aligned}
y_{n+1} &= \sum_{i=0}^{\infty} a_i y_{n-i} \\
&= \sum_{i=0}^{n-1} a_i\left(g_{n-i}y_0 + \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n-i} g_{n-i-j}a_{k+j-1}\right)y_{-k}\right) + \sum_{i=n}^{\infty} a_i y_{n-i} \\
&= \left(\sum_{i=0}^{n} a_i g_{n-i}\right)y_0 + \sum_{i=0}^{n-1} a_i \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n-i} g_{n-i-j}a_{k+j-1}\right)y_{-k} + \sum_{k=1}^{\infty} a_{n+k}y_{-k} \\
&= g_{n+1}y_0 + \sum_{k=1}^{\infty}\left(\sum_{i=0}^{n-1} a_i \sum_{j=1}^{n-i} g_{n-i-j}a_{k+j-1} + a_{n+k}\right)y_{-k} \\
&= g_{n+1}y_0 + \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n}\left(\sum_{i=0}^{n-j} a_i g_{n-j-i}\right)a_{k+j-1} + g_0 a_{n+k}\right)y_{-k} \\
&= g_{n+1}y_0 + \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n+1} g_{n+1-j}a_{k+j-1}\right)y_{-k}.
\end{aligned}$$

Note that we can change the order of addition, since each of the series appearing in the second line converges absolutely. Thus, the equality is valid also for $n+1$ and the right-hand side converges absolutely. □

Set

$$b_m = \sum_{i=m}^{\infty}\frac{a_i}{q^{i+1}} \quad (m \geq 0),$$

where q is as in Lemma 2.3. Note that $b_0 = f(q^{-1}, q^{-2}, q^{-3}, \ldots) = 1$. By the previous lemma combined with Remark 2.5, we have, for $n \geq 1$,

$$q^n = g_n + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n} g_{n-j}a_{i+j-1}\right)q^{-i}.$$

Hence, we have

$$1 = \frac{g_n}{q^n} + \sum_{i=1}^{\infty}\sum_{j=1}^{n}\frac{g_{n-j}}{q^{n-j}}\cdot\frac{a_{i+j-1}}{q^{i+j}} = \frac{g_n}{q^n} + \sum_{j=1}^{n}\left(\sum_{i=1}^{\infty}\frac{a_{i+j-1}}{q^{i+j}}\right)\frac{g_{n-j}}{q^{n-j}} = \frac{g_n}{q^n} + \sum_{j=1}^{n} b_j\frac{g_{n-j}}{q^{n-j}}.$$

In other words, we have $1 = b_0c_n + b_1c_{n-1} + b_2c_{n-2} + \cdots + b_nc_0$ for all $n \geq 0$, where $c_n = g_n / q^n$. We will show that $\lim_{n\to\infty} c_n$ exists. Set $k_n = c_n - c_{n-1}$.

Lemma 3.2: For all $n \geq 1$, we have

$$k_n = \sum_{(i_1, \ldots, i_s) \in \Theta_n} (-1)^s b_{i_1} \cdots b_{i_s},$$

where Θ_n is the finite set defined by

$$\Theta_n = \left\{ (i_1, \ldots, i_s) : i_j \in \mathbf{Z},\ i_j \geq 1,\ s \geq 1, \sum_{j=1}^{s} i_j = n \right\}.$$

Proof: First, note that $k_0 b_0 = 1$ and that $k_0 b_n + k_1 b_{n-1} + \cdots + k_n b_0 = 0$ $(n \geq 1)$. The equality is easily checked for $n = 1$. Suppose that the equality is valid for $n, n-1, \ldots, 1$. We put $\Theta_0 = \{\emptyset\}$ and adopt the convention that the sum over Θ_0 is equal to 1. Then we have

$$k_{n+1} = -b_1 k_n - b_2 k_{n-1} - \cdots - b_{n+1} k_0 = -\sum_{i=1}^{n+1} b_i \sum_{(i_1, \ldots, i_r) \in \Theta_{n+1-i}} (-1)^r b_{i_1} \cdots b_{i_r}.$$

On the other hand, we have

$$\Theta_{n+1} = \bigcup_{i=1}^{n+1} \{ (i, i_1, \ldots, i_r) : (i_1, \ldots, i_r) \in \Theta_{n+1-i} \}.$$

Then it follows that

$$k_{n+1} = \sum_{(i_1, \ldots, i_r) \in \Theta_{n+1}} (-1)^r b_{i_1} \cdots b_{i_r}.$$

This completes the proof. □

Lemma 3.3: If $\sum_{m=1}^{\infty} |b_m| < 1$, then the series $\sum_{n=0}^{\infty} k_n$ converges absolutely and is equal to $(\sum_{m=0}^{\infty} b_m)^{-1}$.

Proof: First, note that the series $\sum_{i=0}^{\infty} (-1)^i z^i$ converges absolutely for $|z| < 1$ and is equal to $(1+z)^{-1}$. Since $\sum_{m=1}^{\infty} |b_m| < 1$ by our assumption, we see that the series $\sum_{i=0}^{\infty} (-1)^i (\sum_{m=1}^{\infty} b_m)^i$ converges absolutely and is equal to $(1 + \sum_{m=1}^{\infty} b_m)^{-1} = (\sum_{m=0}^{\infty} b_m)^i$. Hence, we can change the order of addition in the series $\sum_{i=0}^{\infty} (-1)^i (\sum_{m=1}^{\infty} b_m)^i$. Then, using Lemma 3.2, it is not hard to verify that, changing the order of addition appropriately, this series coincides with the series $\sum_{n=0}^{\infty} k_n$. This completes the proof. □

Note that Lemma 3.3 is an analog of Lemma 13 and Theorem 14 of [2]. However, the method in [2] cannot be applied directly to our case.

Proposition 3.4: Suppose that there exists an S with $0 < S < R$ satisfying (2.3.1) or (2.3.2), and

$$S^2 u'(S) < 1. \tag{3.4.1}$$

Then $\lim_{n \to \infty} c_n = \lim_{n \to \infty} g_n / q^n$ exists and is equal to $(1 + 1^{-2} h'(q^{-1}))^{-1} = (\sum_{m=0}^{\infty} b_m)^{-1}$.

Proof: Since $k_0 b_0 = 1$ and $k_0 b_n + k_1 b_{n-1} + \cdots + k_n b_0 = 0$ for all $n \geq 1$, we see that

$$\sum_{j=0}^{\infty} \sum_{i=0}^{j} k_i b_{j-i} = 1.$$

On the other hand, we have

$$\sum_{m=1}^{\infty}|b_m| = \sum_{m=1}^{\infty}\left|\sum_{i=m}^{\infty}\frac{a_i}{q^{i+1}}\right| \le \sum_{m=1}^{\infty}\sum_{i=m}^{\infty}|a_i|S^{i+1} = \sum_{i=1}^{\infty} i|a_i|S^{i+1} = S^2u'(S) < 1$$

by (3.4.1). Thus, $\lim_{n\to\infty} c_n = \sum_{n=0}^{\infty} k_n$ converges absolutely by Lemma 3.3. Therefore, we have $(\sum_{n=0}^{\infty} k_n)(\sum_{m=0}^{\infty} b_m) = 1$, since $\sum_{m=0}^{\infty} b_m$ converges absolutely. On the other hand, we have

$$q^{-2}h'(q^{-1}) = q^{-2}\left(\sum_{i=0}^{\infty} ia_i q^{-(i-1)}\right) = \sum_{i=1}^{\infty} ia_i q^{-(i+1)},$$

and $\sum_{m=0}^{\infty} b_m = 1 + \sum_{i=1}^{\infty} ia_i q^{-(i+1)}$. This completes the proof. □

Note that the limit as in Proposition 3.4 does not always exist in general as is seen in [2] if we drop the condition (3.4.1). When there exists an r with $a_i = 0$ $(i \ge r)$, the above lemma shows that the sequence is asymptotically simple with dominant root q and dominant multiplicity 1 in the terminology of [2].

Remark 3.5: Note that it is easy to construct sequences which satisfy condition (2.1) and which admit a real number S with $0 < S < R$ satisfying (2.3.1) or (2.3.2), and (3.4.1). For example, take an arbitrary holomorphic function $h_1(z)$ defined in a neighborhood of zero. Then the sequence appearing as the coefficients of the power series expansion of the holomorphic function $h(z) = h_1(z) + a$ at $z = 0$ satisfies the above conditions for all $a \in \mathbf{C}$ with sufficiently large modulus $|a|$.

Remark 3.6: Suppose that each a_i is a nonnegative real number and that there exists an S with $0 < S < R$ satisfying (2.3.1). Then the condition in Lemma 3.3 is equivalent to each of the following:

(1) $$\sum_{i=1}^{\infty}\frac{ia_i}{q^{i+1}} < 1;$$

(2) $$\sum_{i=1}^{\infty}(i-1)\frac{a_i}{q^i} < a_0;$$

(3) $$q^{-2}u'(q^{-1}) = q^{-2}h'(q^{-1}) < 1;$$

(4) $$e'(q^{-1}) < 2q,$$

where $e(x) = xh(x)$ for $x > R^{-1}$. Note that $h(q^{-1}) = q$ and $e(q^{-1}) = 1$. In particular, each of the above conditions is equivalent to (3.4.1).

Problem 3.7: Suppose that the sequence $\{a_i\}_{i=0}^{\infty}$ admits an S with $0 < S < R$ satisfying (2.3.1) or (2.3.2). If $\sum_{m=1}^{\infty}|b_m| \ge 1$, what happens? Does it happen that $\lim_{n\to\infty} g_n / q^n$ exists and is not equal to the value as in Proposition 3.4?

Remark 3.8: Suppose that each a_i is a nonnegative real number and that there exists an S with $0 < S < R$ satisfying (2.3.1). If $a = \lim_{n\to\infty} g_n / q^n$ exists, then we have $1 \le a\sum_{m=0}^{\infty} b_m \le 2$. This is seen as follows. First, we see easily that

$$\sum_{j=0}^{n}\sum_{m=0}^{j} b_m c_{j-m} \le \left(\sum_{m=0}^{n} b_m\right)\left(\sum_{l=0}^{n} c_l\right) \le \sum_{j=0}^{2n}\sum_{m=0}^{j} b_m c_{j-m}.$$

This implies that

$$n+1 \le \left(\sum_{m=0}^{n} b_m\right)\left(\sum_{l=0}^{n} c_l\right) \le 2n+1,$$

since $\sum_{m=0}^{j} b_m c_{j-m} = 1$ $(j \ge 0)$, as we have seen in the paragraph just before Lemma 3.2. Hence, we have

$$1 \le \left(\sum_{m=0}^{n} b_m\right)\cdot\frac{1}{n+1}\left(\sum_{l=0}^{n} c_l\right) \le \frac{2n+1}{n+1}.$$

Thus, if $a = \lim_{n\to\infty} c_n = \lim_{n\to\infty} g_n / q^n$ exists, then we have $1 \le a\sum_{m=0}^{\infty} b_m \le 2$.

Now we proceed to the study of the asymptotic behavior of the sequence $\{y_n\}_{n=1}^{\infty}$. By Lemma 3.1, for all $n \ge 1$, we have

$$y_n = g_n y_0 + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n} g_{n-j} a_{i+j-1}\right) y_{-i}.$$

Thus, we have $d_n = c_n y_0 + \sum_{i=1}^{\infty} e_i^{(n)} y_{-i}$ $(n \ge 1)$, where $d_n = y_n / q^n$ and $e_i^{(n)} = \sum_{j=1}^{n}(c_{n-j} a_{i+j-1} / q^j)$. Since the above series converges absolutely by Lemma 3.1, we have

$$d_n = c_n y_0 + \sum_{j=1}^{n} \frac{c_{n-j}}{q^j}\left(\sum_{i=1}^{\infty} a_{i+j-1} y_{-i}\right) = c_n y_0 + \sum_{j=1}^{n} c_{n-j} p_j,$$

where $p_j = q^{-j}\sum_{i=1}^{\infty} a_{i+j-1} y_{-i}$ $(j \ge 1)$. Putting $p_0 = y_0$, we have $d_n = \sum_{j=0}^{n} c_{n-j} p_j$ $(n \ge 0)$. Set $t_n = d_n - d_{n-1}$ $(n \ge 1)$ and $t_0 = y_0 = d_0$. Then we have $t_n = p_0 k_n + p_1 k_{n-1} + \cdots + p_{n-1} k_1 + p_n k_0$ for all $n \ge 0$. Thus, if the series $\sum_{i=0}^{\infty} p_i$ converges absolutely, then the series $\sum_{n=0}^{\infty} t_n$ converges absolutely and is equal to the product $(\sum_{i=0}^{\infty} p_i)(\sum_{i=0}^{\infty} k_i)$, since the series $\sum_{i=0}^{\infty} k_i$ converges absolutely under the condition of Lemma 3.3. Note that $\sum_{i=0}^{n} t_i = d_n$ and that $\lim_{n\to\infty} d_n = \sum_{i=0}^{\infty} t_i$.

Lemma 3.9: If there exists an S with $0 < S < R$ satisfying (2.3.1) or (2.3.2), and (3.4.1), then the series $\sum_{i=0}^{\infty} p_i$ converges absolutely and is equal to $\sum_{i=0}^{\infty} q^i b_i y_{-i}$.

Proof: First, consider the series $\sum_{i=0}^{\infty} q^i b_i y_{-i}$. Since the sequence $\{y_{-i}\}_{i=0}^{\infty}$ is an element of X, there exist $C > 0$ and T with $0 < T < R$ satisfying $|y_{-i}| \le CT^i$ for all i. If $T|q| \le 1$, then we have

$$|q^i b_i y_{-i}| \le C(T|q|)^i |b_i| \le C|b_i|$$

and, hence, the series $\sum_{i=0}^{\infty} q^i b_i y_{-i}$ converges absolutely by the proof of Proposition 3.4. When $T|q| > 1$, we have

$$|q^i b_i y_{-i}| = |q|^i \left|\sum_{j=i}^{\infty} \frac{a_j}{q^{j+1}}\right| |y_{-i}| \le |q|^i \left(\sum_{j=i}^{\infty} \frac{|a_j|}{|q|^{j+1}}\right) |y_{-i}| \le C|q|^{-1}(T|q|)^i \sum_{j=i}^{\infty} \frac{|a_j|}{|q|^j}. \tag{3.9.1}$$

Now consider the series

$$\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}(T|q|)^i\right)\frac{|a_j|}{|q|^j} = \sum_{j=0}^{\infty}\left(\frac{(T|q|)^{j+1}-1}{T|q|-1}\right)|a_j|(|q|^{-1})^j. \tag{3.9.2}$$

The radius of convergence of the power series

$$w(z)=\sum_{j=0}^{\infty}\left(\frac{(T|q|)^{j+1}-1}{T|q|-1}\right)|a_j|z^j$$

is equal to

$$\left(\limsup_{j\to\infty}\sqrt[j]{\left|\frac{(T|q|)^{j+1}-1}{T|q|-1}\right|}\sqrt[j]{|a_j|}\right)^{-1}=\frac{R}{T|q|}.$$

Since we always have $|q|^{-1}<R/(T|q|)$, we see that the series (3.9.2) converges (absolutely). Changing the order of the addition, we see that the series

$$\sum_{i=0}^{\infty}C|q|^{-1}(T|q|)^i\sum_{j=i}^{\infty}\frac{|a_j|}{|q|^j}$$

converges (absolutely). Thus, by (3.9.1), we see that the series $\sum_{i=0}^{\infty}q^ib_iy_{-i}$ converges absolutely.

Then we have

$$\sum_{i=1}^{\infty}q^ib_iy_{-i}=\sum_{i=1}^{\infty}q^i\left(\sum_{j=i}^{\infty}\frac{a_j}{q^{j+1}}\right)y_{-i}=\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{a_{i+j-1}}{q^j}\right)y_{-i}$$
$$=\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\frac{a_{i+j-1}}{q^j}y_{-i}\right)=\sum_{j=1}^{\infty}p_j.$$

Here we have changed the order of addition, which is allowed since the series in the first line converges absolutely as we have seen above. This completes the proof. □

Thus, we have proved the following.

Theorem 3.10: Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of complex numbers which satisfies (2.1) and which admits an S with $0<S<R$ satisfying (2.3.1) or (2.3.2), and (3.4.1). Then $\lim_{n\to\infty}y_n/q^n$ exists and is equal to

$$\left(\sum_{m=0}^{\infty}b_m\right)^{-1}\left(\sum_{i=0}^{\infty}p_i\right)=\frac{\sum_{m=0}^{\infty}b_mq^my_{-m}}{\sum_{m=0}^{\infty}b_m},$$

where q is as in Lemma 2.3.

Note that the above limiting value can be calculated by using only a_i $(i\geq 1)$, y_{-j} $(j\geq 0)$, and q. We also note that the above result coincides with the results in [2] concerning the case where there exists an integer r such that $a_i=0$ for all $i\geq r$. Furthermore, we note that, using our results of this section, we can obtain convergence results for the ratio of two ∞-generalized Fibonacci sequences and for the ratio of successive terms of an ∞-generalized Fibonacci sequence. For details, see [2, §3]. As to the ratio of two successive terms Dence [1] has obtained a similar result for weighted r-generalized Fibonacci sequences with r finite; however, Dence uses all the roots of the characteristic polynomial, while we obtain a formula in terms of only one root q.

Problem 3.11: For a given sequence $\{a_i\}_{i=0}^{\infty}$ as above, characterize those sequences $\{y_n\}_{n\in\mathbf{Z}}$ such that $y_n=f(y_{n-1},y_{n-2},y_{n-3},\ldots)$ for all $n\in\mathbf{Z}$ (not just for $n\geq 1$). Note that $y_n=q^n$ is such an example. When $a_0=a_1=1$ and $a_i=0$ for all $i\geq 2$, then the sequence $\{y_n\}_{n\in\mathbf{Z}}$ defined by

$$y_n = \begin{cases} F_n & n \geq 1, \\ 0 & n = 0, \\ (-1)^{n+1} F_{-n} & n \leq -1, \end{cases}$$

is also such an example, where $\{F_n\}_{n=1}^{\infty}$ is the usual Fibonacci sequence.

4. EXAMPLES

In this section we give some examples that will help us to understand general phenomena.

Example 4.1: Let b and α be positive real numbers and consider the sequence $\{a_i\}_{i=0}^{\infty}$ defined by $a_i = b\alpha^i$. Then it is not difficult to see that $q = b + \alpha$, $\sum_{m=1}^{\infty} b_m = \alpha / b$, $g_0 = 1$, $g_1 = b$, and that $g_{n+1} = qg_n$ for all $n \geq 1$. Thus, we see that $\lim_{n\to\infty} g_n / q^n$ exists and is equal to $b/q = b/(b+\alpha) = (1 + \sum_{m=1}^{\infty} b_m)^{-1}$. This shows that, even if condition (3.4.1) is not satisfied, the conclusion of Proposition 3.4 holds in this case. In fact, condition (3.4.1) is equivalent to $\alpha / b < 1$ in this example. (When $\alpha / b < 1$, choose $r > 1$ with $r - 1 < b/\alpha < (r-1)^2$ and set $R = \alpha^{-1}$ and $S = (r\alpha)^{-1}$. Then condition (3.4.1) is satisfied.)

Example 4.2: We consider the sequence $\{a_i\}_{i=0}^{\infty}$ defined by $a_0 = 0$ and $a_i = b\alpha^i$ for $i \geq 1$ for some positive real numbers b and α, which is a slight modification of Example 4.1. It is easy to see that $q = (\alpha + \sqrt{\alpha^2 + 4b\alpha})/2$, $\sum_{m=1}^{\infty} b_m = b\alpha / (q-\alpha)^2 = b(b - (q-\alpha))^{-1} > 1$, $g_0 = 1$, $g_1 = 0$, and $g_{n+1} = \alpha g_n + b\alpha g_{n-1}$ for $n \geq 1$. Set $\xi_n = g_{n+1}$. Then we see that $\xi_{-1} = 1$, $\xi_0 = 0$, and $\xi_{n+1} = a_0' \xi_n + a_1' \xi_{n-1}$ $(n \geq 0)$, where $a_0' = \alpha$ and $a_1' = b\alpha$. Note that the number associated with the finite sequence $\{a_0', a_1'\}$ as in Lemma 2.3 coincides with the number q associated with $\{a_i\}_{i=0}^{\infty}$. Since conditions (2.1), (2.3.1), and (3.4.1) are satisfied for the sequence $\{a_0', a_1'\}$, we see that $\lim_{n\to\infty} \xi_n / q^n$ exists and is equal to $qb\alpha / (q^2 + b\alpha)$ by Theorem 3.10. Thus, we see that $\lim_{n\to\infty} g_n / q^n$ exists and is equal to $b\alpha / (q^2 + b\alpha)$. (This can also be obtained by a direct computation as in the previous example.) Note that we always have $\sum_{m=1}^{\infty} b_m = b(b-(q-\alpha))^{-1} > 1$ and that $(1 + \sum_{m=1}^{\infty} b_m)^{-1} = b\alpha / (q^2 + b\alpha)$. In other words, although condition (3.4.1) is not satisfied, the conclusion of Proposition 3.4 holds in this case.

Example 4.3: We consider the sequence $\{a_i\}_{i=0}^{\infty}$ defined by $a_i = a\alpha^i + b\beta^i$ for some positive real numbers a, b, α, and β. Then we see that $q > \alpha, \beta$ and that $q^2 - (\alpha + \beta + a + b)q + (b\alpha + a\beta + \alpha\beta) = 0$. Furthermore, we see that $g_0 = 1$, $g_1 = a + b$, $g_2 = (a+b)^2 + (a\alpha + g\beta)$, and $g_{n+1} = (\alpha + \beta + a + b)g_n - (b\alpha + a\beta + \alpha\beta)g_{n-1}$ for $n \geq 2$. Therefore, we have $g_n = Aq^n + Br^n$ $(n \geq 1)$ for some real numbers A and B, where r is the solution of the equation $r^2 - (\alpha + \beta + a + b)r + (b\alpha + a\beta + \alpha\beta) = 0$ with $r \neq q$. Since $|r| < q$, we see that $\lim_{n\to\infty} g_n / q^n$ exists and is equal to A. The value of A can be calculated by using g_1 and g_2. After tedious but elementary computations, we see that $A = (1 + a\alpha / (q-\alpha)^2 + b\beta / (1-\beta)^2)^{-1} = (1 + \sum_{m=1}^{\infty} b_m)^{-1}$. Note that the value $\sum_{m=1}^{\infty} b_m$ can be greater than 1. For example, for $(\alpha, \beta, a, b) = (1, 1/2, 1, 1)$, the sum is smaller than 1 while, for $(\alpha, \beta, a, b) = (3, 1, 1, 1)$, it is greater than 1.

Example 4.4: Consider the sequence $\{a_i\}_{i=0}^{\infty}$ with $a_i = 1/(i+1)!$. Note that, for this sequence, we have $h(x) = (e^x - 1)/x$ and $e(x) = e^x - 1$. Hence, the radius of convergence R is equal to ∞. In this case, we can easily check that $q = (\log 2)^{-1}$. Hence, we have $e'(q^{-1}) = 2 < 2(\log 2)^{-1} = 2q$, which implies that the condition in Lemma 3.3 is satisfied by Remark 3.6. Thus, by an easy calculation, we see that the sequence $\{g_n\}_{n=1}^{\infty}$ behaves like $(\log 2)^{-(n+1)}/2$ when n goes to ∞. More generally, the sequence $\{y_n\}_{n=1}^{\infty}$ behaves like $b(\log 2)^{-n}$, where

$$b = \frac{1}{2}(\log 2)^{-1}\left(y_0 + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(\log 2)^j}{(i+j)!}y_{-i}\right).$$

Problem 4.5: For an ∞-generalized Fibonacci sequence, the function $H(z) = z^{-1}h(z^{-1}) - 1$ seems to be the analog to the characteristic polynomial in the finite case. This raises the question as to a possible analog to Binet-type formulas for the finite case (see [3] and [2, Th. 1], for example). If $H(z)$ has finitely many zeros, Examples 4.1 through 4.3 seem to suggest that Binet-type formulas hold as in the finite case. If $H(z)$ has infinitely many zeros, as in Example 4.4, then will there be a Binet-type formula that is an infinite series involving powers of the zeros?

ACKNOWLEDGMENT

The authors would like to thank François Dubeau for his helpful comments and suggestions. They also would like to express their thanks to the referee for invaluable comments and suggestions. In particular, Problem 4.5 is due to his/her observation. W. Motta and O. Saeki have been partially supported by CNPq, Brazil. The work of M. Rachidi has been done in part while he was a visiting professor at UFMS, Brazil. O. Saeki has also been partially supported by the Grant-in-Aid for Encouragement of Young Scientists (No. 08740057), Ministry of Education, Science and Culture, Japan, and by the Anglo-Japanese Scientific Exchange Programme, run by the Japan Society for the Promotion of Science and the Royal Society.

REFERENCES

1. T. P. Dence. "Ratios of Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **25.2** (1987):137-43.
2. F. Dubeau, W. Motta, M. Rachidi, and O. Saeki. "On Weighted r-Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **35.2** (1997):102-10.
3. C. Levesque. "On m-th Order Linear Recurrences." *The Fibonacci Quarterly* **23.4** (1985): 290-93.
4. E. P. Miles. "Generalized Fibonacci Numbers and Associated Matrices." *Amer. Math. Monthly* **67** (1960):745-52.

AMS Classification Numbers: 40A05, 40A25

❖❖❖

GENERALIZED BRACKET FUNCTION INVERSE PAIRS

Temba Shonhiwa
Department of Mathematics, The University of Zimbabwe
PO Box MP 167, Mt. Pleasant, Harare, Zimbabwe
e-mail: temba@maths.uz.ac.zw
(*Submitted September 1997-Final Revision December 1997*)

The aim of this paper is to prove the existence of inverse pairs for a certain class of number-theoretic functions. An application of the result is also illustrated. The motivation comes from the study of functions such as

$$C_k(n) = \sum_{\substack{a_1+\cdots+a_k=n \\ a_i\geq 1}} 1 \quad \text{and} \quad R_k(n) = \sum_{\substack{a_1+\cdots+a_k=n \\ (a_1,\ldots,a_k)=1}} 1 .$$

Gould [1] showed that $C_k(n) = \sum_{d|n} R_k(d)$ and that $R_k(n)$ has an inverse. In [5] a pair of functions similarly related is also studied and similar results obtained.

We start our investigation by giving the following theorem due to Gould [2].

Theorem 1 (The Bracket Function Transform): Define

$$S(n) = \sum_{k=1}^{n} \left[\frac{n}{k}\right] A_k = \sum_{j=1}^{n} \sum_{d|j} A_d, \tag{1}$$

$$A(x) = \sum_{n=1}^{\infty} x^n A_n, \tag{2}$$

and

$$S(x) = \sum_{n=1}^{\infty} x^n S_n. \tag{3}$$

Then

$$S(x) = \frac{1}{1-x} \sum_{n=1}^{\infty} A_n \frac{x^n}{1-x^n}. \tag{4}$$

From this it follows that

$$(1-x)\,S(x) = \sum_{n=1}^{\infty} x^n S_n - \sum_{n=1}^{\infty} x^{n+1} S_n = \sum_{n=1}^{\infty} (S_n - S_{n-1})\, x^n, \text{ where } S_0 = 0. \tag{5}$$

That is

$$\sum_{n=1}^{\infty} (S_n - S_{n-1})\, x^n = \sum_{n=1}^{\infty} A_n \frac{x^n}{(1-x^n)},$$

a result equivalent to

$$S_n - S_{n-1} = \sum_{d|n} A_d \quad \text{(see Hardy \& Wright [4], p. 257).} \tag{6}$$

But relation (6), in turn, implies that

$$A(n) = \sum_{d|n} (S(d) - S(d-1))\mu\left(\frac{n}{d}\right). \tag{7}$$

A result also obtained by Gould [2], albeit through an entirely different argument. For completeness, we also include here Gould's [2] elegant formulation of the above result.

Theorem 2:

$$a(n,k)=\sum_{j=1}^{n}\left[\frac{n}{j}\right]b(j,k)=\sum_{j=1}^{n}\sum_{d\mid j}b(d,k)$$

if and only if

$$b(n,k)=\sum_{d\mid n}(a(d,k)-a(d-1,k))\mu\left(\frac{n}{d}\right).$$

We now prove our next result.

Lemma 3: Define

$$H(x)=\sum_{n=1}^{\infty}x^nH_n,\quad\text{where } H_n=\sum_{d\mid n}A_d.$$

Then

$$S(x)=\frac{H(x)}{1-x}. \tag{8}$$

Proof:

$$S(x)=\sum_{n=1}^{\infty}S_nx^n=\sum_{n=1}^{\infty}x^n\sum_{j=1}^{n}\sum_{d\mid j}A_d=\sum_{n=1}^{\infty}x^n\sum_{j=1}^{n}H_j$$
$$=\sum_{j=1}^{\infty}H_j\sum_{n=j}^{\infty}x^n=\sum_{j=1}^{\infty}H_jx^j\sum_{n=j}^{\infty}x^{n-j}=\frac{H(x)}{1-x},\ |x|<1.$$

Next, we prove our main result.

Theorem 4: Define

$$H(n,k)=\sum_{d\mid n}A(d,k),$$
$$S(n,k)=\sum_{j=k}^{n}\left[\frac{n}{j}\right]A(j,k)=\sum_{j=k}^{n}\sum_{d\mid j}A(d,k),\ \text{and}$$
$$B(n,k)=\left[\frac{n}{k}\right]-\sum_{j=k}^{n-1}S(n,j)B(j,k),$$

where $A(n,k)=B(n,k)=0$ if $n<k$ and $A(k,k)=B(k,k)=1$.

Then the functions $A(n,k)$ and $B(n,k)$ satisfy the orthogonality relations

$$\sum_{j=k}^{n}A(j,k)B(n,j)=\delta_k^n\ \text{and}\ \sum_{j=k}^{n}B(j,k)A(n,j)=\delta_k^n,\ \text{where } \delta_k^n=\begin{cases}1, & \text{if } n=k,\\ 0, & \text{otherwise.}\end{cases}$$

Proof: Consider

$$\sum_{n=1}^{\infty}B(n,k)x^n=\sum_{n=1}^{\infty}\left[\frac{n}{k}\right]x^n-\sum_{n=1}^{\infty}x^n\sum_{j=k}^{n-1}S(n,j)B(j,k)$$

$$= \frac{x^k}{(1-x)(1-x^k)} + \sum_{j=k}^{\infty} B(j,k)x^j - \sum_{n=k}^{\infty} x^n \sum_{j=k}^{n} S(n,j)B(j,k),$$

since $S(k,k)=1$ by hypothesis.

That is,

$$\sum_{n=k}^{\infty} x^n \sum_{j=k}^{n} S(n,j)B(j,k) = \frac{x^k}{(1-x)(1-x^k)} \tag{9}$$

or

$$\sum_{j=k}^{\infty} B(j,k) \sum_{n=j}^{\infty} x^n S(n,j) = \sum_{j=k}^{\infty} B(j,k)S(x) = \frac{x^k}{(1-x)(1-x^k)}. \tag{10}$$

From the last equality in (10) and Theorem 1, we have

$$\sum_{j=k}^{\infty} B(j,k) \sum_{n=j}^{\infty} A(n,j)\frac{x^n}{1-x^n} = \sum_{n=k}^{\infty}\sum_{j=k}^{n} B(j,k)A(n,j)\frac{x^n}{1-x^n} = \frac{x^k}{1-x^k},$$

from which it follows that

$$\sum_{j=k}^{n} B(j,k)A(n,j) = \delta_k^n.$$

Also, from $H(n,k) = \sum_{d|n} A(d,k)$, Theorem 1, and Lemma 3, we have that

$$\sum_{n=1}^{\infty} A(n,k)\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} H(n,k)x^n,$$

and, hence, that

$$\sum_{n=1}^{\infty} A(n,k)\frac{x^n}{(1-x)(1-x^n)} = \frac{H(x)}{1-x}.$$

We may now use relation (9) and rewrite this last equation in the form

$$\sum_{n=1}^{\infty} A(n,k) \sum_{i=n}^{\infty} x^i \sum_{j=n}^{i} S(i,j)B(j,n) = \sum_{n=1}^{\infty} S(n,k)x^n,$$

that is,

$$\sum_{j=1}^{\infty}\sum_{n=1}^{j} B(j,n)A(n,k)\sum_{i=j}^{\infty} x^i S(i,j) = \sum_{j=1}^{\infty}\sum_{i=j}^{\infty} x^i S(i,j)\sum_{n=1}^{j} B(j,n)A(n,k)$$

$$= \sum_{n=1}^{\infty} S(n,k)x^n,$$

which implies that $\sum_{n=1}^{j} B(j,n)A(n,k) = \delta_k^j$, and, hence, the result $\sum_{j=1}^{n} A(j,k)B(n,j) = \delta_k^n$..

Theorem 4, in turn, implies the following result.

Theorem 5: For any ordered pair of functions $\langle f(n,k), g(n,k)\rangle$, the following holds:

$$f(n,k) = \sum_{j=k}^{n} g(n,j)A(j,k) \ \text{ if and only if } \ g(n,k) = \sum_{j=k}^{n} f(n,j)B(j,k),$$

where $A(n, k)$ and $B(n, k)$ are as defined in Theorem 4.

Of interest are the function pairs $\langle f(n, k), g(n, k)\rangle$ satisfying Theorem 5. One such class may be obtained from the following result.

Theorem 6: Let

$$g(n, k) = \begin{cases} h(n, k), & \text{if } k / n, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f(n, k) = \sum_{d|n} h(n, d) A(d, k).$$

Then $\langle f(n, k), g(n, k)\rangle$ satisfies Theorem 5, where $h(n, k)$ is any number-theoretic function.

Proof: If $f(n, k) = \Sigma_{d|n} h(n, d) A(d, k)$, then

$$\begin{aligned} g(n, k) &= \sum_{j=k}^{n} \sum_{d|n} h(n, d) A(d, j) B(j, k) = \sum_{j=k}^{n} \sum_{\substack{d=j \\ d|n}}^{n} h(n, d) A(d, j) B(j, k) \\ &= \sum_{\substack{d=k \\ d|n}}^{n} \sum_{j=k}^{d} h(n, d) A(d, j;) B(j, k) = \sum_{\substack{d=k \\ d|n}}^{n} h(n, d) \delta_k^d \\ &= \begin{cases} h(n, k), & \text{if } k / n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The converse is trivial.

Similarly, it may be shown that the functions

$$f(n, k) = \begin{cases} h(n, k), & \text{if } k / n, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(n, k) = \sum_{d|n} h(n, d) B(d, k)$$

satisfy Theorem 5.

As an application, we shall consider some of the results in [5]. There it was established that

$$\binom{n}{k} = \sum_{d|n} T_k^d(d), \tag{11}$$

where

$$T_k^n(n) = \sum_{\substack{1 \le a_1 < a_2 < \cdots < a_k \le n \\ (a_1, a_2, \ldots, a_k, n) = 1}} 1 \quad , \quad n \ge k. \tag{12}$$

It follows from equation (11) that

$$\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1} = \sum_{j=k}^{n} \sum_{d|j} T_k^d(d) = \sum_{j=k}^{n} \left[\frac{n}{j}\right] T_k^j(j). \tag{13}$$

Therefore, by Theorem 2,

$$T_k^n(n) = \sum_{d|n} \left\{ \binom{d+1}{k+1} - \binom{d}{k+1} \right\} \mu\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \binom{n}{k}. \tag{14}$$

Further, if $A(n,k)=T_k^n(n)$ and $H(n,k)=\sum_{d|n}T_k^d(d)=\binom{n}{k}$, we may also apply Theorem 1 and Lemma 3 to find the equivalent $S(x)$ and $H(x)$.

And, by Theorem 4, the function $T_k^n(n)$ has an inverse given by

$$K_k(n)=\left[\frac{n}{k}\right]-\sum_{j=k}^{n-1}\binom{n+1}{j+1}K_k(j). \tag{15}$$

Clearly,

$$K_k(k)=\left[\frac{k}{k}\right],\quad K_k(k+1)=\binom{k+2}{k+2}\left[\frac{k+1}{k}\right]-\binom{k+2}{k+1}\left[\frac{k}{k}\right],$$

and

$$\begin{aligned}K_k(k+2)&=\binom{k+3}{k+3}\left[\frac{k+2}{k}\right]-\binom{k+3}{k+2}\left[\frac{k+1}{k}\right]+\left[\frac{k}{k}\right]\left\{\binom{k+3}{k+2}\binom{k+2}{k+1}-\binom{k+3}{k+1}\right\}\\&=\binom{k+3}{k+3}\left[\frac{k+2}{k}\right]-\binom{k+3}{k+2}\left[\frac{k+1}{k}\right]+\binom{k+3}{k+1}\left[\frac{k}{k}\right]\\&=\sum_{j=k}^{k+2}\binom{(k+2)+1}{j+1}\left[\frac{j}{k}\right](-1)^{(k+2)-j}.\end{aligned}$$

We may, therefore, generalize and obtain the following explicit form for $K_k(n)$.

Theorem 7:

$$K_k(n)=\sum_{j=k}^{n}(-1)^{n-j}\binom{n+1}{j+1}\left[\frac{j}{k}\right]\quad\text{where }K_k(n)=0\text{ if }n<k.$$

Proof: We prove the result by induction on n. We shall assume the result holds for k, $k+1$, ..., n and consider

$$\begin{aligned}K_k(n+1)&=\left[\frac{n+1}{k}\right]-\sum_{j=k}^{n}\binom{n+2}{j+1}K_k(j)\\&=\left[\frac{n+1}{k}\right]-\sum_{j=k}^{n}\binom{n+2}{j+1}\sum_{i=k}^{j}(-1)^{j-i}\binom{j+1}{i+1}\left[\frac{i}{k}\right]\quad\text{by the inductive hypothesis}\\&=\left[\frac{n+1}{k}\right]-\sum_{i=k}^{n}\sum_{j=i}^{n}(-1)^{j-i}\binom{n+2}{j+1}\binom{j+1}{i+1}\left[\frac{i}{k}\right]\\&=\left[\frac{n+1}{k}\right]-\sum_{i=k}^{n}\left[\frac{i}{k}\right]\sum_{j=i+1}^{n+2}\binom{n+2}{j}\binom{j}{i+1}(-1)^{j-(i+1)}-\sum_{i=k}^{n}\left[\frac{i}{k}\right]\binom{n+2}{i+1}(-1)^{n-i}\\&=\sum_{j=k}^{n+1}(-1)^{n+1-j}\binom{n+2}{j+1}\left[\frac{j}{k}\right],\end{aligned}$$

where we have used the identity

$$\sum_{j=i}^{n}(-1)^{j-i}\binom{n}{j}\binom{j}{i}=\delta_i^n\quad\text{(see Gould [3, (3.119)], p. 36).}$$

And, from Theorem 5, it follows that

$$f(n,k)=\sum_{j=k}^{n} g(n,j)T_k^j(j) \text{ if and only if } g(n,k)=\sum_{j=k}^{n} f(n,j)K_k(j). \tag{16}$$

The functions

$$f(n,k)=\binom{n+1}{k+1} \text{ and } g(n,k)=\left[\frac{n}{k}\right]$$

are particular cases of this result.

Also, from Theorem 6, we may obtain other such function pairs for given $h(n,k)$; in particular, with $h(n,k)=1$, we obtain the functions

$$g(n,k)=\begin{cases}1, & \text{if } k/n,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$f(n,k)=\sum_{d|n} T_k^d(d)=\binom{n}{k}.$$

Further, using the techniques in [6], we may prove the following result.

Theorem 8: Let

$$f(n,k)=\frac{(-1)^k}{k+1}$$

and

$$g(n,k)=\sum_{i=k}^{n}\frac{(-1)^i}{i+1}\left[\frac{i}{k}\right]\binom{n+1}{i+1}.$$

Then $\langle f(n,k), g(n,k)\rangle$ satisfies Theorem 5, where $A(n,k)=T_k^n(n)$ and $B(n,k)=K_k(n)$.

Proof: Suppose $f(n,k)=\frac{(-1)^k}{k+1}$, then

$$\begin{aligned} g(n,k) &= \sum_{j=k}^{n} f(n,j)\sum_{i=k}^{j}(-1)^{j-i}\binom{j+1}{i+1}\left[\frac{i}{k}\right]=\sum_{i=k}^{n}(-1)^i\left[\frac{i}{k}\right]\sum_{j=i}^{m}\frac{(-1)^{2j}}{j+1}\binom{j+1}{i+1}\\ &= \sum_{i=k}^{n}\frac{(-1)^i}{i+1}\left[\frac{i}{k}\right]\sum_{j=i}^{n}\binom{j}{i}=\sum_{i+1}^{n}\frac{(-1)^i}{i+1}\left[\frac{i}{k}\right]\binom{n+1}{i+1}. \end{aligned}$$

Conversely, assuming this form for $g(n,k)$, we obtain that

$$\begin{aligned} f(n,k) &= \sum_{j=k}^{n}\sum_{i=j}^{n}\frac{(-1)^i}{i+1}\left[\frac{i}{j}\right]\binom{n+1}{i+1}T_k^j(j)=\sum_{i=k}^{n}\frac{(-1)^i}{i+1}\binom{n+1}{i+1}\sum_{j=k}^{i}\left[\frac{i}{j}\right]T_k^j(j)\\ &= \sum_{i=k}^{n}\frac{(-1)^i}{i+1}\binom{n+1}{i+1}\binom{i+1}{k+1}=\frac{1}{k+1}\sum_{i=k}^{n}(-1)^i\binom{n+1}{i+1}\binom{i}{k} \end{aligned}$$

from equation (13). We now use the relation

$$\sum_{j=i}^{n}\binom{j}{i}=\binom{n+1}{i+1} \quad \text{(see Gould [3, (1.52)])}$$

to obtain that

$$\sum_{i=k}^{n}(-1)^i\binom{n+1}{i+1}\binom{i}{k}=\sum_{i=k}^{n}(-1)^i\binom{i}{k}\sum_{j=i}^{n}\binom{j}{i}$$
$$=\sum_{j=k}^{n}(-1)^k\sum_{i=k}^{j}(-1)^{k-i}\binom{j}{i}\binom{i}{k}=(-1)^k$$

and, hence, the result.

Clearly, many more such function pairs can be found by use of the right Binomial Identities. And, as in [6], generalizations of such functions are also possible.

ACKNOWLEDGMENT

The author is most grateful for the anonymous referee's insightful comments which improved the presentation of this paper.

REFERENCES

1. H. W. Gould. "Binomial Coefficients, the Bracket Function, and Compositions with Relatively Prime Summands." *The Fibonacci Quarterly* **2.4** (1964):241-60.
2. H. W. Gould. "A Bracket Function Transform and Its Inverse." *The Fibonacci Quarterly* **32.2** (1994):176-79.
3. H. W. Gould. *Combinatorial Identities*. Published by the author. Morgantown, West Virginia, 1972.
4. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 5th ed. London: Clarendon Press, 1979.
5. T. Shonhiwa. "A Generalization of the Euler and Jordan Totient Functions." To appear in *The Fibonacci Quarterly*.
6. T. Shonhiwa. "Investigations in Number Theoretic Functions." Ph.D. Dissertation, West Virginia University, Morgantown, West Virginia, 1996.

AMS Classification Numbers: 11A25, 11B65

PARTIAL FIBONACCI AND LUCAS NUMBERS

Indulis Strazdins
Riga Technical University, Riga LV-1658, Latvia
(Submitted September 1997)

0. INTRODUCTION

The well-known Lucas formula

$$F_{n+1}=\sum_{r=0}^{\lfloor n/2\rfloor}\binom{n-r}{r} \tag{1}$$

connects the Fibonacci numbers with binomial coefficients. Our interest is to find out what kind of numbers are obtained by taking every number r in (1) from a fixed residue class modulo m $(m=2,3,\dots)$. As a result, a new family of sequences is introduced: the partial, or $1/m$-Fibonacci numbers. We give here a primary description of these numbers and their generating functions. By a similar construction, partial Lucas, Pell, and other specialized Fibonacci-type sequences can be obtained. Properties of these number systems will be explained in many respects.

1. THE BASIC RECURSION

Given a modulo m $(m=1,2,3,\dots)$, we define the (m,k)*-Fibonacci numbers* as follows:

$$F_{n+1}^{(m,k)}=\sum_{r=0}^{\ell}\binom{n-mr-k}{mr+k}\quad(k=0,1,\dots,m-1), \tag{2}$$

where $\ell=\lfloor(n-2k)/2m\rfloor$; $n=2k,2k+1,\dots$. For $n=1,\dots,2k$, $F_n^{(m,k)}=0$ $(k>0)$. Irrespective of the value of k or even of m, these numbers may be called *1/m-Fibonacci numbers* or *partial Fibonacci numbers*. For every natural n, according to (1),

$$\sum_{k=0}^{m-1}F_n^{(m,k)}=F_n=F_n^{(1,0)}. \tag{3}$$

For $n\le 2m$, there is $F_n^{(m,k)}=\binom{n-k-1}{k}$ for all k. We usually disregard (except in §4) the all-zero case $n=0$.

Theorem 1: For every m, the sequence $\{F_n^{(m,k)}\}$ is the difference sequence of $\{F_n^{(m,k+1)}\}$ over k in cyclic order, i.e.,

$$\begin{aligned}F_{n+1}^{(m,k)}&=F_{n+3}^{(m,k+1)}-F_{n+2}^{(m,k+1)}\quad&(k<m-1),\\ F_{n+1}^{(m,m-1)}&=F_{n+3}^{(m,0)}-F_{n+2}^{(m,0)}\quad&(k=m-1).\end{aligned} \tag{4}$$

Proof: As $\binom{n-1}{k-1}=\binom{n}{k}-\binom{n-1}{k}$, for the r^{th} summand in (2) there obviously is

$$\binom{n-mr-k}{mr+k}=\binom{n+2-mr-k-1}{mr+k+1}-\binom{n+1-mr-k-1}{mr+k+1}\quad(k<m-1),$$

$$\binom{n-mr-m+1}{mr+m-1}=\binom{n+2-m(r+1)}{m(r+1)}-\binom{n+1-m(r+1)}{m(r+1)}\quad(k=m-1).$$

In the last case, for $r = 0$ the right side is

$$\binom{n+2}{0} - \binom{n+1}{0} = 0. \quad \square$$

Thus, all m sequences $\{F_n^{(m,k)}\}$ form a cyclic set with respect to the difference operator Δ_2 (see [3]).

Theorem 2: For every m and k, the recurrence

$$F_n^{(m,k)} = \sum_{s=0}^{m} (-1)^s \binom{m}{s} F_{n+2m-s}^{(m,k)} \tag{5}$$

of order $2m$ holds.

Proof: From (4), with n instead of $n+1$, by consecutive forward substitutions

$$F_n^{(m,k+1)} \to F_n^{(m,k)} \ (k < m-1), \quad F_n^{(m,0)} \to F_n^{(m,m-1)},$$

and with $k = 0$ instead of $k = m$ for the transition step (addition modulo m), we have

$$\begin{aligned} F_n^{(m,k)} &= F_{n+4}^{(m,k+2)} - 2F_{n+3}^{(m,k+2)} + F_{n+2}^{(m,k+2)} \\ &= F_{n+6}^{(m,k+3)} - 3F_{n+5}^{(m,k+3)} + 3F_{n+4}^{(m,k+3)} - F_{n+3}^{(m,k+3)} = \cdots, \end{aligned}$$

so that (5) follows after $m-1$ steps. This can be proved easily by induction. $\square$

2. FIBONACCI CYCLOTOMIC POLYNOMIALS

From the recurrence (5), we obtain the characteristic polynomial

$$\sum_{s=0}^{m} (-1)^s \binom{m}{s} x^{2m-s} - 1 = (x^2 - x)^m - 1 = p_m(x) \tag{6}$$

of degree $2m$. The polynomials (6) can be called *Fibonacci cyclotomic polynomials*, as the substitution $u = x(x-1)$ turns them into the classical cyclotomic polynomials (see [4]). Hence, they admit the following factorization over $\mathbb{C}$:

$$p_m(x) = \prod_{j=0}^{m-1} (x^2 - x - \varepsilon^j), \tag{7}$$

where $\varepsilon^j = \cos\frac{2\pi j}{m} + i \sin\frac{2\pi j}{m}$ are the values of $\sqrt[m]{1}$. The factor $x^2 - x - 1$ (for $j = 0$) whose zeros are $\alpha = \frac{1}{2}(1+\sqrt{5})$, $\beta = 1 - \alpha$, is present in all $p_m(x)$. The quotient polynomial

$$q_m(x) = \frac{p_m(x)}{x^2 - x - 1} = \sum_{j=0}^{m-1} (x^2 - x)^j \tag{8}$$

has the first m lower terms $(-1)^h F_{h+1} x^h$ $(h = 0, 1, \ldots, m-1)$ and its (pairwise conjugate) zeros are

$$\begin{aligned} &\zeta_j,\ \bar{\zeta}_j = \tfrac{1}{2}\left(1 \pm \sqrt{1 + 4\varepsilon_j}\right); \\ &\left|\zeta_j\right| = \sqrt{17 + 8\cos\frac{2\pi j}{m}} \quad (j = 1, 2, \ldots, m-1). \end{aligned} \tag{9}$$

Examples:

$$q_1(x)=1;\quad q_2(x)=x^2-x+1;\quad q_3(x)=x^4-2x^3+2x^2-x+1;$$
$$q_4(x)=x^6-3x^5+4x^4-3x^3+2x^2-x+1=q_2(x)(x^4-2x^3+x^2+1);$$
$$q_5(x)=x^8-4x^7+7x^6-7x^5+5x^4-3x^3+2x^2-x+1;$$
$$q_6(x)=x^{10}-5x^9+11x^8-14x^7+12x^6-8x^5+5x^4-3x^3+2x^2-x+1$$
$$=q_2(x)q_3(x)(x^4-2x^3+x+1).$$

The final factorization to quadratic trinomials over $\mathbb{R}$ is more difficult:

$$q_3(x)=(x^2-(1+A)x+M)(x^2-(1-A)x+1/M),$$
$$\frac{q_4(x)}{q_2(x)}=(x^2-(1+B)x+N)(x^2-(1-B)x+1/N),$$

where

$$A=\sqrt{\tfrac{1}{2}(\sqrt{13}-1)},\qquad M=\tfrac{1}{4}\left(\sqrt{13}+1+\sqrt{2(\sqrt{13}-1)}\right);$$
$$B=\sqrt{\tfrac{1}{2}(\sqrt{17}+1)},\qquad N=\tfrac{1}{4}\left(\sqrt{17}+1+\sqrt{2(\sqrt{17}+1)}\right).$$

Solutions of the equation $q_m(x)=0$ for $m\le 6$ involve radicals $\sqrt{3}, \sqrt{5}, \sqrt{13}, \sqrt{17}$, and $\sqrt{21}$.

3. GENERATING FUNCTIONS

Theorem 3: The generating function of the sequence $\{F_n^{(m,k)}\}$,

$$f^{(m,k)}(x)=\sum_{n=2k}^{\infty}F_{n+1}^{(m,k)}x^n=\frac{x^{2k}(1-x)^{m-k-1}}{r_m(x)},\tag{10}$$

where

$$r_m(x)=x^{2m}p_m(1/x)=(1-x)^m-x^{2m}.$$

Proof: In the case $k=m-1$,

$$f^{(m,m-1)}(x)=\frac{x^{2m-2}}{r_m(x)},\tag{11}$$

i.e., the series $\sum_{n=0}^{\infty}F_{2m+n+1}^{(m,m-1)}x^n$ with shifted coefficient sequence (with $F_{2m+1}^{(m,m-1)}=1$ being the first one) is the inverse for $r_m(x)$:

$$\frac{1}{x^{2m-2}}f^{(m,m-1)}(x)r_m(x)=1,$$

as can be seen from the convolution formulas (see [2], [3])

$$\sum_{j=0}^{\ell}(-1)^j\binom{m}{j}\binom{m+\ell-j-1}{m-1}=\begin{cases}1 & (\ell=0),\\ 0 & (\ell=1,\dots,m).\end{cases}$$

Further, it follows from (4) that

$$f^{(m,k)}(x) = \frac{1-x}{x^2} f^{(m,k+1)}(x) \quad (k = 0, 1, \ldots, m-2). \tag{12}$$

From this, we obtain (10). In particular,

$$f^{(m,0)}(x) = \frac{(1-x)^{m-1}}{r_m(x)}. \quad \square \tag{13}$$

Now we can verify the identity (3) in terms of generating functions. Indeed,

$$r_m(x) = (1-x-x^2)s_m(x),$$

where

$$s_m(x) = x^{2m} q_m(1/x) = \sum_{k=0}^{m-1} x^{2k}(1-x)^{m-k-1}$$

is exactly the sum of numerators in (10) over all k. Hence,

$$\sum_{k=0}^{m-1} f^{(m,k)}(x) = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1}x^n = f(x).$$

4. EXPLICIT EXPRESSIONS: $m = 2$

In some simplest cases, it is possible to express the numbers $F_n^{(m,k)}$ directly as functions of n, thus giving generalizations of the Binet formula

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n). \tag{14}$$

For $m = 2$, denote

$$F_n^{(2,0)} = \sum_{r=0}^{\lfloor (n-1)/4 \rfloor} \binom{n-1-2r}{2r} = E_n$$

and

$$F_n^{(2,1)} = \sum_{r=0}^{\lfloor (n-3)/4 \rfloor} \binom{n-2-2r}{2r+1} = D_n$$

(the *even* and *odd semi-Fibonacci numbers*). Then, from (6) and (7), the characteristic equation

$$p_2(x) \equiv (x^2 - x - 1)(x^2 - x + 1) = 0$$

is obtained, whose roots are $\alpha, \beta = 1 - \alpha$, and $\varepsilon, \bar{\varepsilon} = \frac{1}{2}(1 \pm i\sqrt{3})$. As $\varepsilon^6 = 1$, there is

$$\varepsilon^2 = \varepsilon - 1, \quad \varepsilon^3 = -1, \quad \varepsilon^4 = -\varepsilon, \quad \varepsilon^5 = 1 - \varepsilon = \bar{\varepsilon}.$$

Using the (extended) initial conditions $E_0 = D_0 = D_1 = D_2 = 0$ and $E_1 = E_2 = E_3 = D_3 = 1$ in the general solution

$$E_n, D_n = A\alpha^n + B(1-\alpha)^n + C\varepsilon^n + D(1-\varepsilon)^n,$$

we obtain for both E_n and D_n,

$$A = -B = \frac{2\alpha - 1}{10} = \frac{1}{2(2\alpha - 1)} = \frac{1}{2\sqrt{5}},$$

and for E_n and D_n, respectively (instead of C and D),

$$C' = -D' = \frac{1}{2(2\varepsilon-1)} \quad\text{and}\quad C'' = -D'' = -\frac{1}{2(2\varepsilon-1)}.$$

Hence,

$$E_n, D_n = \frac{1}{2(2\alpha-1)}(\alpha^n - (1-\alpha)^n) \pm \frac{1}{2(2\varepsilon-1)}(\varepsilon^n - (1-\varepsilon)^n), \tag{15}$$

and, in accordance to (3), $E_n + D_n = F_n$. The first summand in (15) is exactly $F_n/2$, whereas the differences

$$\delta_n = E_n - D_n = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r \binom{n-r-1}{r} = \frac{1}{2\varepsilon-1}(\varepsilon^n - (1-\varepsilon)^n)$$

form a periodic sequence $(0, 1, 1, 0, -1, -1)$ modulo 6. (See also [1].)

The generating functions (11) and (13) are

$$f^{(2,0)}(x) = \sum_{n=0}^{\infty} E_{n+1}x^n = (1-x)/r_2(x) = e(x)$$

and

$$f^{(2,1)}(x) = \sum_{n=0}^{\infty} D_{n+1}x^n = x^2/r_2(x) = d(x),$$

where $r_2(x) = (1-x-x^2)(1-x+x^2)$. Then

$$e(x) + d(x) = \frac{1}{1-x-x^2} = f(x),$$

$$e(x) - d(x) = \frac{1}{1-x+x^2} = \sum_{n=0}^{\infty}(x-x^2)^n = 1 + x - x^3 - x^4 + x^6 + x^7 - \cdots.$$

5. PARTIAL LUCAS NUMBERS

Next we apply our approach to the Lucas numbers

$$L_n = F_{n-1} + F_{n+1} = 1 + \sum_{r=1}^{\lfloor (n-1)/2 \rfloor}\left(\binom{n-r-1}{r-1} + \binom{n-r}{r}\right). \tag{16}$$

Then a definition of the (m, k)-*Lucas numbers*, parallel to (2), is

$$L_{n+1}^{(m,k)} = 1 + \sum_{r=0}^{\ell}\left(\binom{n-mr-k}{mr+k} + \binom{n-mr-k+1}{mr+k+1}\right) \quad (k = 0, 1, \ldots, m-1), \tag{17}$$

where $\ell = \lfloor (n-2k)/2m \rfloor$; $n = 2k, 2k+1, \ldots$. For $n = 0, 1, \ldots, 2k$, $L_n^{(m,k)} = 0$ $(k > 0)$, and $L_0^{(m,0)} = 2$, $L_0^{(m,k)} = 0$ $(k > 0)$. The formula

$$\sum_{k=0}^{m-1} L_n^{(m,k)} = L_n = L_n^{(1,0)} \tag{18}$$

corresponds to (3).

The numbers $L_n^{(m,k)}$ satisfy conditions analogous to (4) and, consequently, also the basic recursion (5). The particular solutions differ from the previous Fibonacci case only because of another initial conditions. Thus, for $m = 2$ (the *semi-Lucas numbers*), we obtain, instead of (15),

$$L_n^{(2,0)},\ L_n^{(2,1)} = \tfrac{1}{2}L_n \pm \tfrac{1}{2}(\varepsilon^n + (1-\varepsilon)^n). \tag{19}$$

The differences $\delta'_n = L_n^{(2,0)} - L_n^{(2,1)}$ form a periodic sequence $(2, 1, -1, -2, -1, 1)$ modulo 6. The generating functions are

$$\ell^{(2,0)}(x) = \sum_{n=0}^{\infty} L_{n+1}^{(2,0)} x^n = \frac{2-3x+x^2}{r_2(x)}$$

and

$$\ell^{(2,1)}(x) = \sum_{n=0}^{\infty} L_{n+1}^{(2,1)} x^n = \frac{2x^2-x^3}{r_2(x)},$$

and their sum (18) is

$$\frac{2-x}{1-x-x^2} = \sum_{n=0}^{\infty} L_{n+1}x^n = \ell(x).$$

The general formula that corresponds to (10) here is

$$\ell^{(m,k)}(x) = \sum_{n=2k}^{\infty} L_{n+1}^{(m,k)} x^n = \frac{x^{2k}(1-x)^{m-k-1}(2-x)}{r_m(x)}. \tag{20}$$

6. NUMERICAL RESULTS

We give the values of $F_n^{(m,k)}$ and $L_n^{(m,k)}$ for $m \le 4$ in Tables 1 and 2 below. For the negative subscripts (in Table 1), formulas (4) were used.

7. SOME PROPERTIES

We mention here without proof the following appealing properties of $F_n^{(m,k)}$ and $L_n^{(m,k)}$, discovered after short observations:

1) $F_{-n}^{(m,k)} = (-1)^{n+1} F_n^{(m,k\ominus r)}$; (21)

2) $L_{-n}^{(m,k)} = (-1)^n L_n^{(m,k\ominus r)}$ $(n = mq + r > 0,\ r = 0, 1, \ldots, m-1)$, (22)
   where $\ominus$ is subtraction modulo m;

3) $L_n^{(m,k)} = F_{n-1}^{(m,k\ominus 1)} + F_{n+1}^{(m,k)}$; (23)

4) $L_n^{(m,k)} = F_{n+2}^{(m,k)} - F_{n-2}^{(m,k\oplus(m-2))}$, (24)
   where $\oplus$ is addition modulo m;

5) $$\sum_{j=1}^{n} F_j^{(m,k)} = \begin{cases} F_{n+2}^{(m,k+1)} & (k = 0, 1, \ldots, m-2), \\ F_{n+2}^{(m,0)} - 1 & (k = m-1); \end{cases} \tag{25}$$

6) $$\sum_{j=1}^{n} L_j^{(m,k)} = \begin{cases} L_{n+2}^{(m,1)} - 2 & (k = 0); \\ L_{n+2}^{(m,k+1)} & (k = 1, \ldots, m-2), \\ L_{n+2}^{(m,0)} - 1 & (k = m-1). \end{cases} \tag{26}$$

These examples reveal a remarkable variety of repetition patterns, including the "rotation" (twisting) phenomenon. The usual Fibonacci-type formulas are obtained by summation over all k.

TABLE 1. Numbers $F_n^{(m,k)}$

n	F_n	$m=2$		δ_n	$m=3$			$m=4$			
		$k=0$	1		$k=0$	1	2	$k=0$	1	2	3
−10	−55	−27	−28	1	−13	−21	−21	−21	−20	−6	−8
−9	34	17	17	0	11	8	15	7	15	10	2
−8	−21	−11	−10	−1	−10	−5	−6	−1	−6	−10	−4
−7	13	6	7	−1	5	6	2	1	1	5	6
−6	−8	−4	−4	0	−1	−4	−3	−3	0	−1	−4
−5	5	3	2	1	1	1	3	3	1	0	1
−4	−3	−1	−2	1	−2	0	−1	−1	−2	0	0
−3	2	1	1	0	1	1	0	0	1	1	0
−2	−1	−1	0	−1	0	−1	0	0	0	−1	0
−1	1	0	1	−1	0	0	1	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	1	1	0	0	1	0	0	0
2	1	1	0	1	1	0	0	1	0	0	0
3	2	1	1	0	1	1	0	1	1	0	0
4	3	1	2	−1	1	2	0	1	2	0	0
5	5	2	3	−1	1	3	1	1	3	1	0
6	8	4	4	0	1	4	3	1	4	3	0
7	13	7	6	1	2	5	6	1	5	6	1
8	21	11	10	1	5	6	10	1	6	10	4
9	34	17	17	0	11	8	15	2	7	15	10
10	55	27	28	−1	21	13	21	6	8	21	20
11	89	44	45	−1	36	24	29	16	10	28	35
12	144	72	72	0	57	45	42	36	16	36	56
13	233	117	116	1	86	81	66	71	32	46	84
14	377	189	188	1	128	138	111	127	68	62	120
15	610	305	305	0	194	224	192	211	139	94	166
16	987	493	494	−1	305	352	330	331	266	162	228
17	1597	798	799	−1	497	546	554	497	477	301	322

TABLE 2. Numbers $L_n^{(m,k)}$

n	F_n	$m=2$		δ'_n	$m=3$			$m=4$			
		$k=0$	1		$k=0$	1	2	$k=0$	1	2	3
0	2	2	0	2	2	0	0	2	0	0	0
1	1	1	0	1	1	0	0	1	0	0	0
2	3	1	2	−1	1	2	0	1	2	0	0
3	4	1	3	−2	1	3	0	1	3	0	0
4	7	3	4	−1	1	4	2	1	4	2	0
5	11	6	5	1	1	5	5	1	5	5	0
6	18	10	8	2	3	6	9	1	6	9	2
7	29	15	14	1	8	7	14	1	7	14	7
8	47	23	24	−1	17	10	20	3	8	20	16
9	76	37	39	−2	31	18	27	10	9	27	30
10	123	61	62	−1	51	35	37	26	12	35	50
11	199	100	99	1	78	66	55	56	22	44	77
12	322	162	160	2	115	117	90	106	48	56	112
13	521	261	260	1	170	195	156	183	104	78	156
14	843	421	422	−1	260	310	273	295	210	126	212
15	1364	681	683	−2	416	480	468	451	393	230	290

ACKNOWLEDGMENTS

The author was supported in part by the Science Council of Latvia, Grant No. 96.0196. He also wishes to thank Dr. Valdemars Linis for valuable discussions.

REFERENCES

1. P. Filipponi. Problem B-828, *The Fibonacci Quarterly* **35.2** (1997):181; Solution, *The Fibonacci Quarterly* **36.1** (1998):87-88.
2. H. W. Gould. *Combinatorial Identities*. Morgantown: West Virginia University, 1972.
3. J. Riordan. *Combinatorial Identities*. New York: John Wiley, 1968.
4. B. L. van der Waerden. *Algebra I*. Berlin: Springer-Verlag, 1971.

AMS Classification Numbers: 11B39, 05A15, 12D05

SUMS OF CERTAIN PRODUCTS OF FIBONACCI AND LUCAS NUMBERS

R. S. Melham
School of Mathematical Sciences, University of Technology, Sydney
PO box 123, Broadway, NSW 2007, Australia
(Submitted September 1997-Final Revision February 1998)

1. INTRODUCTION

Inspired by the charming result

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}, \tag{1.1}$$

Clary and Hemenway [3] discovered factored closed-form expressions for all sums of the form $\sum_{k=1}^{n} F_{rk}^3$, where r is an integer. One of their main aims was to find sums that could be expressed neatly as products of Fibonacci and Lucas numbers. At the end of their paper they mentioned the result

$$\sum_{k=1}^{n} F_k^2 F_{k+1} = \frac{1}{2} F_n F_{n+1} F_{n+2}, \tag{1.2}$$

published by Block [2] in 1953.

Motivated by (1.1) and (1.2), we have discovered an infinity of similar identities which we believe are new. For example, we have found

$$\sum_{k=1}^{n} F_k F_{k+1} F_{k+2}^2 F_{k+3} F_{k+4} = \frac{1}{4} F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4} F_{n+5}, \tag{1.3}$$

and

$$\sum_{k=1}^{n} F_k F_{k+1} F_{k+2} F_{k+3} F_{k+4}^2 F_{k+5} F_{k+6} F_{k+7} F_{k+8} = \frac{1}{11} F_n F_{n+1} \dots F_{n+9}. \tag{1.4}$$

In Section 2 we prove a theorem involving a sum of products of Fibonacci numbers, and in Section 3 we prove the corresponding theorem for the Lucas numbers. In Section 4 we present three additional theorems, two of which involve sums of products of squares of Fibonacci and Lucas numbers.

We require the following identities:

$$F_{n+k} + F_{n-k} = F_n L_k, \ k \text{ even}, \tag{1.5}$$
$$F_{n+k} + F_{n-k} = L_n F_k, \ k \text{ odd}, \tag{1.6}$$
$$F_{n+k} - F_{n-k} = F_n L_k, \ k \text{ odd}, \tag{1.7}$$
$$F_{n+k} - F_{n-k} = L_n F_k, \ k \text{ even}, \tag{1.8}$$
$$L_{n+k} + L_{n-k} = L_n L_k, \ k \text{ even}, \tag{1.9}$$
$$L_{n+k} + L_{n-k} = 5F_n F_k, \ k \text{ odd}, \tag{1.10}$$
$$L_{n+k} - L_{n-k} = L_n L_k, \ k \text{ odd}, \tag{1.11}$$
$$L_{n+k} - L_{n-k} = 5F_n F_k, \ k \text{ even}, \tag{1.12}$$
$$L_n^2 - L_{2n} = -2 = -L_0, \ n \text{ odd}, \tag{1.13}$$

$$5F_{2n}^2 - L_{2n}^2 = -4 = -L_0^2, \tag{1.14}$$
$$5F_{2n}^2 - L_{4n} = -2 = -L_0. \tag{1.15}$$

Identities (1.5)-(1.8) occur on page 59 of Hoggatt [4], while (1.9)-(1.12) occur as (9)-(12), respectively, in Bergum and Hoggatt [1]. Identities (1.13)-(1.15) can be proved with the use of the Binet forms.

2. A FAMILY OF SUMS FOR THE FIBONACCI NUMBERS

Theorem 1: Let m be a positive integer. Then

$$\sum_{k=1}^{n} F_k F_{k+1} \dots F_{k+2m}^2 \dots F_{k+4m} = \frac{F_n F_{n+1} \dots F_{n+4m+1}}{L_{2m+1}}. \tag{2.1}$$

Proof: We use the elegant method described on page 135 in [3] to prove (1.2). Let l_n and r_n denote the left and right sides, respectively, of (2.1). Then $l_n - l_{n-1} = F_n F_{n+1} \dots F_{n+2m}^2 \dots F_{n+4m}$. Also,

$$\begin{aligned} r_n - r_{n-1} &= \frac{F_n F_{n+1} \dots F_{n+4m}}{L_{2m+1}} [F_{n+4m+1} - F_{n-1}] \\ &= \frac{F_n F_{n+1} \dots F_{n+4m}}{L_{2m+1}} [F_{(n+2m)+(2m+1)} - F_{(n+2m)-(2m+1)}] \\ &= l_n - l_{n-1} \quad \text{using (1.7).} \end{aligned}$$

Hence, to prove that $l_n = r_n$ it suffices to show that $l_1 = r_1$. But

$$\begin{aligned} r_1 &= \frac{F_1 F_2 \dots F_{4m+1} F_{2m+1} L_{2m+1}}{L_{2m+1}} \quad (\text{since } F_{2n} = F_n L_n) \\ &= l_1, \quad \text{and this completes the proof. } \square \end{aligned}$$

When $m = 1$ and 2, identity (2.1) reduces to (1.3) and (1.4), respectively. However, while (1.1) and (1.2) can be proved in a similar way, they are not special cases of (2.1).

3. CORRESPONDING RESULTS FOR THE LUCAS NUMBERS

Corresponding to (1.1) we have

$$\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2, \tag{3.1}$$

which occurs as I_4 in Hoggatt [4]. The Lucas counterpart to (1.2) is

$$\sum_{k=1}^{n} L_k^2 L_{k+1} = \frac{1}{2} L_n L_{n+1} L_{n+2} - 3. \tag{3.2}$$

The constants on the right sides of (3.1) and (3.2) can be obtained by trial, and also in the same manner as in our next theorem, demonstrating a certain unity.

Theorem 2: Let m be a positive integer. Then

$$\sum_{k=1}^{n} L_k L_{k+1} \dots L_{k+2m}^2 \dots L_{k+4m} = \frac{L_n L_{n+1} \dots L_{n+4m+1}}{L_{2m+1}} - R_0, \tag{3.3}$$

where

$$R_n = \frac{L_n L_{n+1} \dots L_{n+4m+1}}{L_{2m+1}}, \quad n = 0, 1, 2, \dots.$$

Proof: Again, let l_n denote the left side of (3.3). Then

$$\begin{aligned} R_n - R_{n-1} &= \frac{L_n L_{n+1} \dots L_{n+4m}}{L_{2m+1}} [L_{n+4m+1} - L_{n-1}] \\ &= \frac{L_n L_{n+1} \dots L_{n+4m}}{L_{2m+1}} [L_{(n+2m)+(2m+1)} - L_{(n+2m)-(2m+1)}] \\ &= L_n L_{n+1} \dots L_{n+2m}^2 \dots L_{n+4m} \quad \text{[by (1.11)]} \\ &= l_n - l_{n-1}. \end{aligned}$$

From this we see that $l_n - R_n = c$, where c is a constant. Now,

$$\begin{aligned} c &= l_1 - R_1 \\ &= L_1 L_2 \dots L_{4m+1} \left[L_{2m+1} - \frac{L_{4m+2}}{L_{2m+1}} \right] \\ &= L_1 L_2 \dots L_{4m+1} \cdot \frac{L_{2m+1}^2 - L_{4m+2}}{L_{2m+1}} \\ &= -\frac{L_0 L_1 L_2 \dots L_{4m+1}}{L_{2m+1}} \quad \text{[by (1.13)]} \\ &= -R_0. \end{aligned}$$

This concludes the proof. □

Since this method of proof applies to (3.1) and (3.2), we see that the appropriate constants on the right sides are $-2 = -L_0 L_1$ and $-3 = -\frac{1}{2} L_0 L_1 L_2$, respectively. Accordingly, we write (3.1), for example, as

$$\sum_{k=0}^{n} L_k^2 = [L_k L_{k+1}]_0^n.$$

We use this notation throughout the remainder of the paper.

Remark: If for $m = 0$ we interpret the summands in (2.1) and (3.3) to be F_k^2 and L_k^2, respectively, then we can realize (1.1) and (3.1) within the framework of our two theorems. However, the same cannot be said for (1.2) and (3.2).

4. MORE SUMS OF PRODUCTS

In this section we state three additional theorems, two of which involve sums of products of squares. Using (1.5)-(1.15), they can be proved in the same manner as Theorems 1 and 2, and so we leave this task to the reader. In each theorem, m is assumed to be a nonnegative integer.

Theorem 3:

$$\sum_{k=1}^{n} F_k F_{k+1} \dots F_{k+4m+2} L_{k+2m+1} = \frac{F_n F_{n+1} \dots F_{n+4m+3}}{F_{2m+2}}, \tag{4.1}$$

$$\sum_{k=1}^{n} L_k L_{k+1} \dots L_{k+4m+2} F_{k+2m+1} = \left[\frac{L_k L_{k+1} \dots L_{k+4m+3}}{5F_{2m+2}}\right]_0^n. \tag{4.2}$$

Theorem 4:

$$\sum_{k=1}^{n} F_k^2 F_{k+1}^2 \dots F_{k+4m}^2 F_{2k+4m} = \frac{F_n^2 F_{n+1}^2 \dots F_{n+4m+1}^2}{F_{4m+2}}, \tag{4.3}$$

$$\sum_{k=1}^{n} L_k^2 L_{k+1}^2 \dots L_{k+4m}^2 F_{2k+4m} = \left[\frac{L_k^2 L_{k+1}^2 \dots L_{k+4m+1}^2}{5F_{4m+2}}\right]_0^n. \tag{4.4}$$

In the proof of (4.3), when finding $r_n - r_{n-1}$, we obtain the expression $F_{n+4m+1}^2 - F_{n-1}^2$, which by (1.6) and (1.7) can be written as

$$[F_{(n+2m)+(2m+1)} - F_{(n+2m)-(2m+1)}][F_{(n+2m)+(2m+1)} + F_{(n+2m)-(2m+1)}]$$
$$= F_{n+2m} L_{2m+1} \cdot L_{n+2m} F_{2m+1} = F_{2n+4m} F_{4m+2}.$$

Similar expressions that arise in the proof of (4.4), and in the proof of the next theorem, can be treated in the same manner.

A simple special case of (4.3), which occurs for $m = 0$, is $\sum_{k=1}^{n} F_k^2 F_{2k} = F_n^2 F_{n+1}^2$.

Theorem 5:

$$\sum_{k=1}^{n} F_k^2 F_{k+1}^2 \dots F_{k+4m+2}^2 F_{2k+4m+2} = \frac{F_n^2 F_{n+1}^2 \dots F_{n+4m+3}^2}{F_{4m+4}}, \tag{4.5}$$

$$\sum_{k=1}^{n} L_k^2 L_{k+1}^2 \dots L_{k+4m+2}^2 F_{2k+4m+2} = \left[\frac{L_k^2 L_{k+1}^2 \dots L_{k+4m+3}^2}{5F_{4m+4}}\right]_0^n. \tag{4.6}$$

To conclude we mention that, for p real, the sequences $\{U_n\}$ and $\{V_n\}$, defined for all integers n by

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0,\ U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2,\ V_1 = p, \end{cases}$$

generalize the Fibonacci and Lucas numbers, respectively. The results contained in Theorems 1-5 translate immediately to U_n and V_n. The reason is that if we replace F_n by U_n, L_n by V_n, and 5 by $p^2 + 4$, then U_n and V_n satisfy (1.5)-(1.15).

REFERENCES

1. G. E. Bergum & V. E. Hoggatt, Jr. "Sums and Products for Recurring Sequences." *The Fibonacci Quarterly* **13.2** (1975):115-20.
2. D. Block. "Curiosum #330: Fibonacci Summations." *Scripta Mathematica* **19.2-3** (1953): 191.
3. S. Clary & P. D. Hemenway. "On Sums of Cubes of Fibonacci Numbers." In *Applications of Fibonacci Numbers* **5:**123-36. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1993.
4. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton-Mifflin, 1969.

AMS Classification Numbers: 11B39, 11B37

❖❖❖

FIBONACCI NUMBERS AND HARMONIC QUADRUPLES

Georg Johann Rieger
Institut für Mathematik, Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
(Submitted September 1997-Final Revision May 1998)

Here, we combine number theory (Fibonacci numbers) and projective geometry (harmonic fourth).

Let the real numbers

$$A<B<C<D \tag{1}$$

form a harmonic quadruple (see [1], pp. 159-60), i.e.,

$$\frac{B-C}{B-A}:\frac{D-C}{D-A} \text{ (cross ratio) } =-1$$

or

$$D(2B-A-C)=BC-2CA+AB. \tag{2}$$

The number D is also called a harmonic fourth. The affine map $x \mapsto \alpha x+\beta$ with real numbers $\alpha>0$ and β does not change equations (1) and (2). Especially, with $\alpha=2/(C-A)$ and $\beta=-(C+A)/(C-A)$, we get $A_1=-1$, $C_1=1$ and, therefore, $B_1D_1=1$, $0<B_1<1<D_1$. Then, $B_1=(2B-A-C)/(C-A)>0$ implies, from (1), that

$$2B>A+C. \tag{3}$$

It is easy to find harmonic quadruples of squares and also of primes like

$$1^2<3^2<4^2<11^2, \quad 1^2<11^2<15^2<41^2,$$
$$3^2<11^2<13^2<17^2, \quad 4^2<9^2<11^2<17^2,$$

and

$$3<11<17<59, \quad 3<23<41<383,$$
$$5<13<19<61, \quad 7<19<29<139;$$

also, the number 0 together with any three consecutive terms $(n+2)^{-1}$, $(n+1)^{-1}$, n^{-1} of the harmonic series form a harmonic quadruple.

Theorem: There are no harmonic quadruples of Fibonacci numbers.

Proof (by contradiction): For integers $2\le a<b<c<d$, we replace (1) by

$$F_a<F_b<F_c<F_d \tag{4}$$

and (2) by

$$F_d(2F_b-F_a-F_c)=F_bF_c-2F_cF_a+F_aF_b. \tag{5}$$

By (3), we must have $2F_b>F_a+F_c\ge 1+F_c$ and, hence, $c=b+1$; however, $2F_b\ge 2+F_{b+1}$ or $F_{b-2}\ge 2$ holds exactly for $b\ge 5$. Inequality (3) now says $F_{b-2}\ge 1+F_a$. By $b\ge 5$, this is satisfied exactly for $2\le a\le b-3$. Consequently, instead of (5), we have to look at

$$F_d(F_{b-2}-F_a)=F_bF_{b+1}-2F_aF_{b+1}+F_aF_b \tag{6}$$

or

$$F_d F_{b-2} - F_b F_{b+1} = F_a(F_d - 2F_{b+1} + F_b) \tag{7}$$

for $b \geq 5,\ 2 \leq a \leq b-3,\ d \geq b+2$. We observe that

$$F_d - 2F_{b+1} + F_b \geq F_{b+2} - 2F_{b+1} + F_b = F_{b-2} > 0.$$

For $a = 2$ and $a = b-3$, we obtain "$\geq$" and "$\leq$", respectively, in (7) and thus in (6). This means

$$F_d(F_{b-2} - 1) \geq F_b F_{b+1} - 2F_{b+1} + F_b, \tag{8}$$

and

$$F_d F_{b-4} \leq F_b F_{b+1} - 2F_{b-3}F_{b+1} + F_{b-3}F_b. \tag{9}$$

But

$$F_b F_{b+1} - 2F_{b+1} + F_b + F_{b+3}(1 - F_{b-2}) = 2(F_b + (-1)^b) > 0 \quad (b \geq 3)$$

and (8) imply $d \geq b+4$. Furthermore,

$$F_{b+5}F_{b-4} - F_b F_{b+1} + 2F_{b-3}F_{b+1} - F_{b-3}F_b = (18F_b - 11F_{b+1})F_{b+2} > 0 \quad (b \geq 5)$$

and (9) imply $d \leq b+4$. This leaves $d = b+4$.

We found that $b \geq 5$, and $2 \leq a \leq b-3$, and that (4) and (7) can be replaced, respectively, by $F_a < F_b < F_{b+1} < F_{b+4}$ and

$$F_{b+4}F_{b-2} - F_b F_{b+1} = F_a(F_{b+4} - 2F_{b+1} + F_b)$$

or, equivalently,

$$(F_a - 12F_b + 3F_{b+1})(3F_b + F_{b+1}) = -32F_b^2.$$

This implies $(3F_b + F_{b+1}) \mid 32F_b^2$. Since $1 = (F_{b+1}, F_b) = (3F_b + F_{b+1}, F_b)$, we obtain

$$(3F_b + F_{b+1}) \mid 32. \tag{10}$$

But, $3F_5 + F_6 = 23$ and $3F_b + F_{b+1} \geq 37$ $(b \geq 6)$. Hence (10) is impossible. □

REFERENCE

1. Oswald Veblen & John Wesley Young. *Projective Geometry*. New York-Toronto-London: Blaisdell Publishing Company, 1910, 1938.

AMS Classification Numbers: 11B39, 51M04

❖❖❖

NOTES ON RECIPROCAL SERIES RELATED TO FIBONACCI AND LUCAS NUMBERS

Feng-Zhen Zhao
Dalian University of Technology, 116024 Dalian, China
(Submitted September 1997)

1. INTRODUCTION

As usual, the Fibonacci and Lucas numbers are defined by

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}}, \quad L_n = \alpha^n + (-1)^n \alpha^{-n},$$

where $\alpha = (\sqrt{5}+1)/2$. Sums of the form $\Sigma 1/(F_{an+b}+c)$ or $\Sigma 1/(L_{an+b}+c)$ have been computed in many publications for certain values of a, b, and c (see, for instance, [2]-[5]). For example: Backstrom [3] obtained

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}+3} = \frac{2\sqrt{5}+1}{10}, \quad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1}+1} = \frac{\sqrt{5}}{2};$$

André-Jeannin [2] proved that

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}+\sqrt{5}} = \frac{1}{\alpha}, \quad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1}+3/\sqrt{5}} = 1,$$

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}+2/\sqrt{5}} \approx \frac{\sqrt{5}}{4\log\alpha} - \frac{\sqrt{5}\pi^2}{(\log\alpha)^2 (e^{\pi^2 (\log\alpha)^{-1}}+2)};$$

and Almkvist [1] also gave an estimate of another series, i.e.,

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}+2} \approx \frac{1}{8} + \frac{1}{4\log\alpha} + \frac{\pi^2}{(\log\alpha)^2 (e^{\pi^2 (\log\alpha)^{-1}}-2)}.$$

In this paper, we continue this work and obtain some new results of similar kinds.

In Section 2, some identities related to Fibonacci and Lucas numbers, which may be compared with the ones of [2] and [3], are established. In Section 3, following Almkvist's method, we express the series $\Sigma_{n=0}^{\infty} 1/(F_{2n+1} - 2/\sqrt{5})$ and $\Sigma_{n=1}^{\infty} 1/(L_{2n}-2)$ in terms of the theta functions and give their estimates.

2. MAIN RESULTS

The following lemma will be used later.

Lemma: Let q be a real number with $|q| > 1$, s and a be positive integers, and b be a nonnegative integer. Then one has that

$$\sum_{n=0}^{\infty} \frac{1}{q^{2an+b} + q^{-2an-b} - (q^{as}+q^{-as})} = \frac{1}{q^{-as}-q^{as}} \sum_{n=0}^{s-1} \frac{1}{1-q^{2an+b-as}} \tag{1}$$

$(b \neq as,\ 2a \nmid (as-b))$.

Proof: One can readily verify that

$$\frac{1}{q^{2n+b}+q^{-2an-b}-(q^{as}+q^{-as})}=\frac{1}{q^{as}-q^{-as}}\left(\frac{1}{1-q^{2an+b+as}}-\frac{1}{1-q^{2an+b-as}}\right)$$

holds for $n>s$. Hence, by the telescoping effect, one has that

$$\sum_{n=0}^{N}\frac{1}{q^{2an+b}+q^{-2an-b}-(q^{as}+q^{-as})}=\frac{1}{q^{as}-q^{-as}}\left(\sum_{n=N-s+1}^{N}\frac{1}{1-q^{2an+b+as}}-\sum_{n=0}^{s-1}\frac{1}{1-q^{2an+b-as}}\right)$$

for all $N>s$. Letting $N\to+\infty$, we obtain the equality (1). □

From the Lemma, some reciprocal series related to Fibonacci and Lucas numbers can be computed.

Theorem 1: Assume that a and b are integers with $a\geq 1$ and $b\geq 0$. Then

$$\sum_{n=0}^{\infty}\frac{1}{F_{2an+b}-L_a/\sqrt{5}}=\frac{1}{F_a(\alpha^{b-a}-1)}\quad (a\text{ even, }b\text{ odd}),\tag{2}$$

$$\sum_{n=0}^{\infty}\frac{1}{L_{2an+b}-\sqrt{5}F_a}=\frac{1}{L_a(\alpha^{b-a}-1)}\quad (a\text{ odd, }b\text{ even}),\tag{3}$$

$$\sum_{n=0}^{\infty}\frac{1}{L_{2an+b}-L_a}=\frac{1}{\sqrt{5}F_a(\alpha^{b-a}-1)}\quad (a,\ b\text{ even, }a\neq b\text{, and }2a\nmid(a-b)),\tag{4}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{F_{2an+b}-F_a}=\frac{\sqrt{5}}{L_a(\alpha^{b-a}-1)}\quad (a,\ b\text{ odd, }a\neq b\text{, and }2a\nmid(a-b)).\tag{5}$$

Proof: Set $q=\alpha$. It follows from (1) that

$$\sum_{n=0}^{\infty}\frac{1}{\alpha^{2an+b}+\alpha^{-2an-b}-(\alpha^{as}+\alpha^{-as})}=\frac{1}{\alpha^{-as}-\alpha^{as}}\sum_{n=0}^{s-1}\frac{1}{1-\alpha^{2an-as+b}}.\tag{6}$$

Examine different cases according to the values of a, b, and s. If a is even, b is odd, and $s=1$, then we have (2) from (6). If a is odd, b is even, and $s=1$ in (6), then (3) holds. On the other hand, assume that $s=1$, $2a\nmid(a-b)$, and $a\neq b$ in (6), then we have the equalities (4) and (5) if both a and b are even or odd, respectively. □

Theorem 2: Suppose that s is a positive integer. Then

$$\sum_{n=0}^{\infty}\frac{1}{L_{2n}-\sqrt{5}F_s}=\frac{1}{L_s}\left(\frac{1-s}{2}+\frac{1}{\alpha^{-s}-1}\right)\quad (s\text{ odd})\tag{7}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}-L_s/\sqrt{5}}=\frac{s}{2F_s}\quad (s\text{ even}).\tag{8}$$

Proof: Letting $a=1$ and $b=0$ in (6), we have

$$\sum_{n=0}^{\infty}\frac{1}{L_{2n}-\sqrt{5}F_s}=-\frac{1}{L_s}\sum_{n=0}^{s-1}\frac{1}{1-\alpha^{2n-s}}\quad (s\text{ odd}).$$

Due to

$$\sum_{n=0}^{2m}\frac{1}{1-\alpha^{2n-2m-1}}=\frac{1}{1-\alpha^{-2m-1}}+\sum_{n=1}^{m}\left(\frac{1}{1-\alpha^{-2n+1}}+\frac{1}{1-\alpha^{-2n-1}}\right)=\frac{1}{1-\alpha^{-2m-1}}+m,$$

we obtain the equality (7). On the other hand, if $a=b=1$ in (6), then

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}-L_s/\sqrt{5}}=-\frac{1}{F_s}\sum_{n=0}^{s-1}\frac{1}{1-\alpha^{2n+1-s}}\qquad (s\text{ even}).$$

Noticing that

$$\sum_{n=0}^{2m-1}\frac{1}{1-\alpha^{2n+1-2m}}=\sum_{n=1}^{m}\left(\frac{1}{1-\alpha^{-2n+1}}+\frac{1}{1-\alpha^{2n-1}}\right)=m,$$

we have the equality (8). □

Remark: Consider the recurrence relation $W_n=pW_{n-1}+W_{n-2}$, $n\ge 2$, $p>0$, and the solutions

$$U_n=\frac{\alpha^n-(-1)^n\alpha^{-n}}{\sqrt{\Delta}},\quad V_n=\alpha^n+(-1)^n\alpha^{-n},$$

where $\Delta=p^2+4$, $\alpha=(p+\sqrt{\Delta})/2>1$. $\{U_n\}$ and $\{V_n\}$ are the generalizations of $\{F_n\}$ and $\{L_n\}$. Clearly, the conclusions of Theorems 1 and 2 can be generalized to the case in which F_n, L_n, and $\sqrt{5}$ are replaced by U_n, V_n, and $\sqrt{\Delta}$, respectively.

The identities given in the above theorems may be compared with the ones in [2]. In addition, we can also obtain some interesting equalities. For example, letting $a=2$, $b=1$ in (2) and $s=1$ in (7), respectively, we have

$$\sum_{n=0}^{\infty}\frac{1}{F_{4n+1}-3/\sqrt{5}}=\sum_{n=0}^{\infty}\frac{1}{L_{2n}-\sqrt{5}}=-1-\alpha,$$

and letting $s=2$ in (8), we have

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}-3/\sqrt{5}}=-1.$$

3. ESTIMATES OF TWO SERIES

In this section, the summation $\sum_n$ is over all integers n.

Putting $a=b=1$ and $s=0$ in the left-hand side of (6), we have

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}-2/\sqrt{5}}=\sqrt{5}\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(q^{2n+1}-1)^2},$$

where $q=\alpha^{-1}$. From a classical result (see, e.g., [1] or [6]), we know that

$$\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(q^{2n+1}-1)^2}=\frac{\vartheta_4''}{8\pi^2\vartheta_4},$$

where

$$\vartheta_4=\sqrt{-\frac{\pi}{\log q}}\sum_n e^{\pi^2(n-0.5)^2(\log q)^{-1}}.$$

Through simple computation, we obtain

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}-2/\sqrt{5}}=\frac{\pi^2\sqrt{5}}{8(\log\alpha)^2}-\frac{\sqrt{5}}{4\log\alpha}-\frac{\pi^2\sum_n(n^2-n)e^{-\pi^2(n-0.5)^2(\log\alpha)^{-1}}}{2(\log\alpha)^2\sum_n e^{-\pi^2(n-0.5)^2(\log\alpha)^{-1}}}.$$

Thus, a good estimate of the series $\sum_{n=0}^{\infty}1/(F_{2n+1}-2/\sqrt{5})$ is given by

$$\frac{\pi^2\sqrt{5}}{8(\log\alpha)^2}-\frac{\sqrt{5}}{4\log\alpha}.$$

Using a similar method, we obtain an estimate of another series. From the following results,

$$\sum_{n=1}^{\infty}\frac{1}{L_{2n}-2}=\sum_{n=1}^{\infty}\frac{q^{2n}}{(q^{2n}-1)^2},$$

where $q=\alpha^{-1}$, and

$$\sum_{n=1}^{\infty}\frac{q^{2n}}{(q^{2n}-1)^2}=\frac{1}{24\pi^2}\left(\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_4''}{\vartheta_4}\right)+\frac{1}{24},$$

where (see [1] or [6])

$$\vartheta_2=\sqrt{-\frac{\pi}{\log q}}\sum_n(-1)^n e^{\pi^2n^2(\log q)^{-1}},\quad \vartheta_3=\sqrt{-\frac{\pi}{\log q}}\sum_n e^{\pi^2n^2(\log q)^{-1}},$$

we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{L_{2n}-2}&\approx\frac{1}{24}-\frac{1}{4\log\alpha}+\frac{\pi^2}{3(\log\alpha)^2}\left(\frac{1}{e^{\pi^2(\log\alpha)^{-1}}+2}-\frac{1}{e^{\pi^2(\log\alpha)^{-1}}-2}+\frac{1}{8}\right)\\&\approx\frac{1}{24}-\frac{1}{4\log\alpha}+\frac{\pi^2}{24(\log\alpha)^2}.\end{aligned}$$

ACKNOWLEDGMENT

The author wishes to thank the anonymous referees for their helpful comments. The last section is heavily due to their suggestions.

REFERENCES

1. G. Almkvist. "A Solution to a Tantalizing Problem." *The Fibonacci Quarterly* **24.4** (1986): 316-22.
2. R. André-Jeannin. "Summation of Certain Reciprocal Series Related to Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **29.3** (1991):200-04.
3. R. Backstrom. "On Reciprocal Series Related to Fibonacci Numbers with Subscripts in Arithmetic Progression." *The Fibonacci Quarterly* **19.1** (1981):14-21.
4. B. Popov. "On Certain Series of Reciprocals of Fibonacci Numbers." *The Fibonacci Quarterly* **22.3** (1982):261-65.
5. B. Popov. "Summation of Reciprocal Series of Numerical Functions of Second Order." *The Fibonacci Quarterly* **24.1** (1986)17-21.
6. E. T. Whittaker & G. N. Watson. *A Course of Modern Analysis*. Cambridge: Cambridge University Press, 1984.

AMS Classification Number: 11B39

ON THE FORM OF SOLUTIONS OF MARTIN DAVIS' DIOPHANTINE EQUATION

Anatoly S. Izotov
Mining Institute, Novosibirsk, Russia
(Submitted September 1997-Final Revision February 1998)

1. INTRODUCTION

M. Davis proved in [1] that, if the Diophantine equation

$$9(u^2+7v^2)^2-7(r^2+7s^2)^2=2 \tag{1}$$

had no nontrivial solutions other than $u=v=1$, $r=s=0$, in nonnegative integers, then Hilbert's Tenth Problem would be unsolvable. J. Robinson proved that Hilbert's Tenth Problem would be unsolvable if (1) had only finitely many solutions.

In [3], O. Herrmann proved the existence of nontrivial solutions of (1) and D. Shanks [5] solved (1) explicitly.

D. Shanks and S. S. Wagstaff [6] found 48 more solutions of (1). They also conjectured that this equation has infinitely many solutions and gave an elaborate argument in this direction.

In this note, it is proved that solutions of (1) are members of a certain Lucas sequence and its form is described.

2. REPRESENTATION OF A_nB_n AS A MEMBER OF A RECURRENCE OF ORDER TWO

Herrmann [2] considered the Pell-like equation

$$9A_n^2-7B_n^2=2. \tag{2}$$

He proved that, if $A_0=1$, $B_0=1$, then

$$A_{n+1}=8A_n+7B_n \quad\text{and}\quad B_{n+1}=9A_n+8B_n \tag{3}$$

give all positive solutions of (2).

If A_n has the form u^2+7v^2 and B_n has the form r^2+7s^2, then every solution of (2) is a solution of (1).

By the first equation of (3), we have

$$7B_n=A_{n+1}-8A_n \quad\text{and}\quad 7B_{n+1}=A_{n+2}-8A_{n+1}, \tag{4}$$

while, by the second equation of (3), we see that $7B_{n+1}=63A_n+8\cdot 7B_n$ and, by (4), that

$$A_{n+2}-8A_{n+1}=63A_n+8(A_{n+1}-8A_n)$$

or

$$A_{n+2}=16A_{n+1}-A_n,\ A_0-1,\ A_1=15. \tag{5}$$

Analogously,

$$B_{n+2}=16B_{n+1}-B_n,\ B_0=1,\ B_1=17. \tag{6}$$

Now, consider the recurrence

$$U_{n+2}=16U_{n+1}-U_n,\ U_0=0,\ U_1=1. \tag{7}$$

By the theory of integer linear recurrences of order two, $A_n = U_{n+1} - U_n$, $B_n = U_{n+1} + U_n$, and

$$A_n B_n = U_{n+1}^2 - U_n^2 = U_{2n+1}. \tag{8}$$

Let S be the set of all odd positive numbers that have the form $x^2 + 7y^2$ for any integers x, y. The criterion for an odd $z \in S$ is:

$$\begin{aligned} &z \in S \text{ if and only if, for some prime } p,\ p^k \| z, \text{ then } p^k \in S; \\ &p^k \in S \text{ if and only if } p^k \equiv 0, 1, 2, \text{ or } 4 \pmod 7. \end{aligned} \tag{9}$$

By the criterion for integer $z \in S$, we have

$$\begin{aligned} &\text{if } z_1 \in S,\ z_2 \in S, \text{ then } z_1 z_2 \in S, \\ &\text{if } z = z_1 z_2,\ (z_1, z_2) = 1, \text{ and } z \in S, \text{ then } z_1 \in S,\ z_2 \in S. \end{aligned} \tag{10}$$

It is clear that a solution A_n, B_n of (2) is a solution of (1) if and only if $A_n \in S$, $B_n \in S$. Since $(A_n, B_n) = 1$, we see by (8) that A_n, B_n is a solution of (2) if and only if $U_{2n+1} \in S$.

Later on, we shall say that $U_{2n+1} = A_n B_n$ is a solution of (2), and accordingly of (1), if A_n, B_n is a solution of (2). The notations U_n and $U(n)$ are considered to be equivalent.

It is known that $\{U_i\}$ is periodic modulo 7 and its period is $\{1, 2, 3, 4, 5, 6, 0\}$. By (9) and (10), if $U_m \in S$ and m is odd, then

$$m \equiv 1, 7, 9, 11 \pmod{14}. \tag{11}$$

3. SOME PROPERTIES OF $U(m)$

In what follows, we shall need some properties of recurrence (7), which we give here without proofs, since they can be found in [2] and [4].

Let m_1 and m_2 be positive integers. Then

$$U(m_i) | U(m_1 m_2),\ i = 1, 2, \tag{A}$$

$$(U(m_1 m_2) / U(m_2), U(m_2)) = (m_1, U(m_2)). \tag{B}$$

If p_1 and p_2 are primes not equal to 3 or 7 and $p_1 \le p_2$, then, for $k > 0$,

$$(p_1, U(p_2^k)) = 1. \tag{C}$$

If the prime q has the form $q = 2 \cdot N + (7/q)$, where $(7/q)$ is the Legendre symbol, then

$$q | U(N). \tag{D}$$

4. "PRIME" AND "COMPOSITE" SOLUTIONS

Let U_m be a solution of (1). We say that m is a "prime" solution if m is prime and a "composite" solution if m is a composite number. By the properties of integer linear recurring sequences of order two, if m is a "composite" solution, then there exists a "prime" solution.

Theorem 1: Let m be an odd composite number, $m = p_1^{k_1} \cdots p_d^{k_d}$, $2 < p_1 < p_2 < \cdots < p_d$, $k_i > 0$. If $d = 1$, then $k_1 > 1$. Let $U(m) \in S$. Then, for all $i = 1, 2, \ldots, d$, $p_i \in S$, and for all k, $1 \le k \le k_i$, $U(p_i^k) \in S$.

Proof: We shall prove the theorem by induction on d.

(i) Let $d=1$ and $m=p_1^{k_1}$, $k_1>1$. For $0<k<k_1$, by (A) we have

$$U(p_1^{k_1}) = U(p_1^{k_1})/U(p_1^k)\cdot U(p_1^k).$$

By (B),

$$(U(p_1^{k_1})/U(p_1^k),\ U(p_1^k)) = (p_1^{k_1-k}, U(p_1^k)) = 1 \quad \text{for } p_1 \neq 3 \text{ or } 7.$$

Since $U(m)\in S$, we have $U(p_1^k)\in S$ for $k=1,2,\ldots,k_1-1$. So, for $k=1$, $U(p_1)\in S$ and, by (11), $p_1\in S$.

If $p_1=3$ or 7 and $U(p_1^{k_1})\in S$, then

$$U(p_1^{k_1}) = p_1 U(p_1^{k_1})/U(p_1)\cdot U(p_1)/p_1 \quad \text{and} \quad (p_1U(p_1^{k_1})/U(p_1),\ U(p_1)/p_1) = 1.$$

Therefore, $U(p_1)/p_1\in S$, which is impossible since

$$U(3)/3 = 5\cdot 17 \notin S \quad \text{and} \quad U(7)/7 = 13\cdot 293\cdot 617 \notin S.$$

(ii) Let $d=2$, $m=p_1^{k_1}p_2^{k_2}$, $p_1<p_2$, and $U(m)\in S$. Then, $U(m)=U(m)/U(p_2^{k_2})\cdot U(p_2^{k_2})$. Since $(U(m)/U(p_2^{k_2})\cdot U(p_2^{k_2})) = (p_1^{k_1}, U(p_2^{k_2})) = 1$ by (C), we have $U(p_2^{k_2})\in S$ and, by (i), $p_2\in S$, $U(p_2)\in S$.

Furthermore, $U(m)=U(m)/U(p_1^{k_1})\cdot U(p_1^{k_1})$. Let

$$(U(m)/U(p_1^{k_1}), U(p_1^{k_1})) = (p_2^{k_2}\cdot U(p_1^{k_1})) = p_2^c, \text{ for } 0<c\le k_2.$$

Then, $U(m)=p_2^c\cdot U(m)/U(p_1^{k_1})\cdot U(p_1^{k_1})/p_2^c$ and $M=U(p_1^{k_1})/p_2^c\in S$. Since $p_2\in S$, $p_2^c\in S$, and $Mp_2^c=U(p_1^{k_1})\in S$. By (i), we have $U(p_1)\in S$, $p_1\in S$.

(iii) Assume that the statements of Theorem 1 are true for $1<t<d$. Then, for $t=d$, let $m=p_1^{k_1}\cdot p_2^{k_2}\cdot\cdots\cdot p_d^{k_d}$, $p_1<p_2<\cdots<p_d$, and $U(m)\in S$. Also, let $m=m_1p_d^{k_d}$. By (ii), $p_d>7$. Furthermore, $U(m)=U(m)/U(p_d^{k_d})\cdot U(p_d^{k_d})$.

Since $(U(m)/U(p_d^{k_d}), U(p_d^{k_d})) = (m_1, U(p_d^{k_d})) = 1$ by (C), we have $U(p_d^{k_d})\in S$ and, by (i), $U(p_d)\in S$, $p_d\in S$.

Consider $U(m)=U(m)/U(m_1)\cdot U(m_1)$. Let $D=(U(m)/U(m_1), U(m_1)) = (p_d^{k_d}, U(m_1)) = p_d^c$, where $0\le c\le k_d$. Then, $U(m)=D\cdot U(m)/U(m_1)\cdot U(m_1)/D$ and $M=U(m_1)/D\in S$. Since $D\in S$, we have $U(m_1)=M\cdot D\in S$ and, by the induction statements, for all i, $0<i<d$, $p_i\in S$, $U(p_i^k)\in S$, $0<k<k_i$, and the theorem is proved.

In [6], Daniel Shanks and Samuel S. Wagstaff conjectured that equation (1) has infinitely many solutions. Theorem 2 gives some information on the form of these solutions.

Theorem 2: If there are infinitely many solutions of (1), but only finitely many "prime" solutions, then there is at least one prime $q\in S$ such that $U(q^k)\in S$ for all $k>0$.

Inversely, if there are infinitely many solutions of (1) and, for each prime $p\in S$, there exists $k=k(p)$ such that $U(p^k)\notin S$, then there are infinitely many "prime" solutions.

Proof: Let $\{m_i\}$, $i=1,2,\ldots$ be the set of solutions of (1). If $p_1,p_2,\ldots,p_d$ is the finite set of "prime" solutions of (1), then, by Theorem 1, $m_i=p_1^{k_{1i}}p_2^{k_{2i}}\cdots p_d^{k_{di}}$. Since $m_{i_{i\to\infty}}\to\infty$, there exists at least one $p_j, 0<j\le d$ such that $k_{ji}\to\infty$ as $i\to\infty$. By Theorem 1, $U(p_j^k)\in S$ for all $k>1$.

Inversely, if for each prime $p \in S$ there exists $k = k(q)$ and $U(p^k) \notin S$, then, by Theorem 1, $U(p^d) \notin S$ for $d \geq k(p)$. If there are finitely many "prime" solutions, then, by Theorem 1, there exist finitely many "composite" solutions only.

It is more probable that there exist infinitely many "prime" solutions. Indeed, if for each prime $p \in S$ there exists at least one prime q of the form $q = 2p^n + (7/q)$, where $(7/q)$ is the Legendre symbol, then $(q/7) = -1$ and so $q \notin S$. By (D), $q|U(p^n)$ and $(p^n) \notin S$.

5. ON "COMPOSITE" SOLUTIONS

In [6], the solution $p_1 = 53$ was given and two new solutions of equation (1) were found: $p_2 = 67$ and $p_3 = 71$. By Theorem 1, the corresponding "composite" solutions have the form $m = p_1^a p_2^b p_3^c$, $a, b, c \geq 0$. To test whether there are "composite" solutions, it is sufficient to consider $m_1 = p_1^2, m_2 = p_1 p_2, m_3 = p_1 p_3, m_4 = p_2^2, m_5 = p_2 p_3$, and $m_6 = p_3^2$.

A computer examination produced the following:

For $m_1 = 53^2 = 2809$, $U(m_1)$ has no prime divisors up to $1 \cdot 10^9$.

For $m_2 = 53 \cdot 67 = 3551$, $7103 \| U(m_2)$, and $7103 \notin S$, so $U(m_2) \notin S$.

For $m_3 = 53 \cdot 71 = 3763$, $1979339 \| U(m_3)$, and $1979339 \notin S$, so $U(m_3) \notin S$.

For $m_4 = 67^2 = 4489$, $673349 \| U(m_4)$, and $673349 \notin S$, so $U(m_4) \notin S$.

For $m_5 = 67 \cdot 71 = 4757$, $332989 \| U(m_5)$, and $332989 \notin S$, so $U(m_5) \notin S$.

For $m_6 = 71^2 = 5041$, $46427611 \| U(m_6)$, and $46427611 \notin S$, so $U(m_6) \notin S$.

Note that:

For $m_7 = 53^3 = 148877$, $893261 \| U(m_7)$, and $893261 \notin S$, so $U(m_7) \notin S$.

Perhaps the only "composite" solution of (1) of the form $m = p_1^a p_2^b p_3^c$ is $m_1 = 53^2 = 2809$, and it is the least "composite" solution.

REFERENCES

1. Martin Davis. "One Equation To Rule Them All." *Trans. New York Acad. Sci.* (II) **30** (1968):766-73.
2. H. J. A. Duparc. "Periodicity Properties of Recurring Sequences, II." *Proc. Koninkl. Nederl. Acad. Wetensch*, A57, **4** (1954):473-85.
3. Oskar Herrmann. "A Non-Trivial Solution of the Diophantine Equation $9(x^2+y^2)^2 - 7(u^2+v^2)^2 = 2$." In *Computers in Number Theory*, pp. 207-12. London: Academic Press, 1971.
4. Dov Jarden. *Recurring Sequences*. Jerusalem: Riveon Lematematika, 1966.
5. Daniel Shanks. "Five Number-Theoretic Algorithms." In *Proc. of the Second Manitoba Conference on Numerical Mathematics*, pp. 51-70. Winnipeg: Univ. of Manitoba, 1972.
6. Daniel Shanks & Samuel S. Wagstaff. "48 More Solutions of Martin Davis's Quaternary Quartic Equation." *Math. Comp.* **64** (1995):1717-31.

AMS Classification Numbers: 11B25, 11B37

ON THE INTEGERS OF THE FORM $n(n-1)-1$

Piero Filipponi and Odoardo Brugia
Fondazione Ugo Bordoni, Via B. Castiglione 59, I-00142 Rome, Italy
e-mail: filippo@fub.it
(*Submitted October 1997-Final Revision May 1998*)

1. AIM OF THE NOTE

The principal aim of this short note is to put into evidence a quite interesting property of the integers M_n given by the left-hand side of the Fibonacci characteristic equation

$$x^2-x-1=0 \tag{1.1}$$

taken at integers. More precisely, let us define the odd numbers M_n as

$$M_n := n(n-1)-1 = n^2-n-1 \quad (n \geq 2 \text{ an integer}). \tag{1.2}$$

After establishing two marginal properties of the numbers M_n, we prove their main property: namely, for $n \geq 3$, their canonical decomposition does not contain primes of the form $10h \pm 3$. A brief discussion on which numbers M_n are also Fibonacci or Lucas numbers concludes our note.

2. MARGINAL PROPERTIES OF THE NUMBERS M_n

Proposition 1:

$$M_n \equiv \begin{cases} 1 \pmod{10} \\ 5 \pmod{10} \\ 9 \pmod{10} \end{cases} \text{if} \quad n \equiv \begin{cases} 2, 4, 7, \text{ or } 9 & \pmod{10} \\ 3 \text{ or } 8 & \pmod{10} \\ 0, 1, 5, \text{ or } 6 & \pmod{10}. \end{cases} \tag{2.1}$$

Proposition 1 can be proved by simply computing (1.2) modulo 10.

Proposition 2: For $n \geq 2$, M_n is not divisible by 25.

Proof: From (2.1), we see that, for M_n to be divisible by 5, one must have $n = 5h+3$ ($h = 0$, 1, 2, ...). Consequently, from (1.2), we have $M_{5h+3} = 25h^2+25h+5 \equiv 5 \pmod{25}$.

3. MAIN RESULT

Proposition 3: For $n \geq 3$, the canonical decomposition of M_n has the form

$$M_n = 5^t \prod_{k=1}^{\infty} p_k^{s_k}, \tag{3.1}$$

where t is either 0 or 1 and p_k is a prime of the form $10h \pm 1$ with s_k a nonnegative integer. In particular, the canonical decomposition of M_n does not contain primes of the form $10h \pm 3$.

Remark: If M_n is a prime, then the statement of Proposition 3 and that of Proposition 1 coincide.

Proof of Proposition 3: From (1.2) and Proposition 2, it is sufficient to prove that the incongruence

$$n^2-n-1 \not\equiv 0 \pmod{10h \pm 3} \quad (10h \pm 3 \text{ a prime}) \tag{3.2}$$

holds true for all n. Let D $(=5)$ be the discriminant of the equation $x^2-x-1=0$. In [3, p. 223] it is shown how the solution of the congruence $x^2-x-1\equiv 0 \pmod q$ (q a prime) is given by the solution of the congruence $z^2\equiv D \pmod q$. It follows that a sufficient condition for the incongruence (3.2) to be satisfied is that the congruence $z^2\equiv 5 \pmod{10h\pm 3}$ has no solutions. In other words, denoting by (m/p) (p an odd prime, m an integer not divisible by p) the Legendre symbol, to prove (3.2) we have to prove that

$$(5/10h\pm 3)=-1. \tag{3.3}$$

To obtain (3.3), first use the reciprocity law for (m/p) (e.g., see [3, p. 322]), thus getting

$$\begin{cases}(5/10h+3)(10h+3/5)=(-1)^{(5-1)/2\cdot(10h+2)/2}=(-1)^{10h+2}\\(5/10h-3)(10h-3/5)=(-1)^{(5-1)/2\cdot(10h-4)/2}=(-1)^{10h-4}\end{cases}$$

whence

$$(5/10h\pm 3)(10h\pm 3/5)=1. \tag{3.4}$$

Then, on using the property $(m/p)\equiv m^{(p-1)/2} \pmod p$ (see [3, p. 315]), write

$$\begin{aligned}(10h\pm 3/5)&\equiv(10h\pm 3)^{(5-1)/2} \pmod 5\\&\equiv(\pm 3)^2\equiv 9\equiv -1 \pmod 5\end{aligned}$$

whence

$$(10h\pm 3/5)=-1. \tag{3.5}$$

The validity of (3.3) follows necessarily from (3.5) and (3.4). □

An Observation: At first sight, we were amazed at the relatively large number of prime M_n (cf. Sequences 179 and 1558 of [4]): we found 48 of them for $3\le n\le 100$ and 311 of them for $3\le n\le 1000$, whereas it can be seen readily [2] that the expected number of primes in a set of 1000 odd numbers randomly chosen in $[3, 10^6]$ is 157. Actually, the fact that there are so many prime M_n is not surprising, for we know, from Proposition 3, that M_n is not divisible by 3 (or by 7), and that most of the composite numbers are.

4. A QUESTION ABOUT THE NUMBERS M_n

We observed that

$$\begin{aligned}&M_2=F_1=F_2=L_1, && M_6=L_7,\\&M_3=F_5, && M_8=F_{10},\\&M_4=L_5, && M_{10}=F_{11}.\end{aligned} \tag{4.1}$$

A computer experiment allows us to ascertain that, for $11\le n\le 10^{10}$, no numbers M_n are Fibonacci or Lucas numbers. This experiment was carried out by seeking values of k for which the discriminant $4F_k+5$ (resp. $4L_k+5$) of the equation $n^2-n-1=F_k$ (resp. $=L_k$) is a perfect square.

Question: Do there exist numbers M_n that are Fibonacci or Lucas numbers besides those given in (4.1)?

Remark: By virtue of the identity $4L_{2k}+(-1)^k 8=(2L_k)^2$ (see identities I_{15} and I_{18} of [1]), it is not hard to prove that M_n cannot equal an even-subscripted Lucas number.

REFERENCES

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton Mifflin, 1969.
2. H. Riesel. *Prime Numbers and Computer Methods for Factorization.* Stuttgart: Birkhäuser, 1985.
3. W. Sierpinski. *Elementary Theory of Numbers.* Panstwowe Wydawnictwo Naukowe, 1964.
4. N. J. A. Sloane. *A Handbook of Integer Sequences.* San Diego, CA: Academic Press, 1973.

AMS Classification Numbers: 11A07, 11A51, 11B39

❖❖❖

Announcement

NINTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

July 17-July 22, 2000

Institut Supérieur de Technologie
Grand Duché de Luxembourg

LOCAL COMMITTEE

J. Lahr, Chairman
R. Andre-Jeannin
M. Malvetti
C. Molitor-Braun
M. Oberweis
P. Schroeder

INTERNATIONAL COMMITTEE

A.F. Horadam (Australia), Co-chair
A. N. Philippou (Cyprus), Co-chair
C. Cooper (U.S.A.)
P. Filipponi (Italy)
H. Harborth (Germany)
Y. Horibe (Japan)
M. Johnson (U.S.A.)
P. Kiss (Hungary)
G.M. Phillips (Scotland)
J. Turner (New Zealand)
M. E. Waddill (U.S.A.)

LOCAL INFORMATION

For information on local housing, food, tours, etc., please contact:

Professor Joseph Lahr
Institut Supérieur de Technologie
6, rue R. Coudenhove-Kalergi
L-1359 Luxembourg
joseph.lahr@ist.lu
FAX-(00352) 432124 PHONE-(00352) 420101-1

CALL FOR PAPERS

Papers on all branches of mathematics and science related to the Fibonacci numbers and generalized Fibonacci numbers, as well as papers related to recurrences and their generalizations are welcome. Abstracts, which should be sent in duplicate to F. T. Howard at the address below, are due by June 1, 2000. An abstract should be at most one page in length (preferably half a page) and should contain the author's name and address. New results are especially desirable, but abstracts on work in progress or results already accepted for publication will be considered. Manuscripts should *not* be submitted. Questions about the conference may be directed to:

Professor F.T. Howard
Wake Forest University
Box 7388 Reynolda Station
Winston-Salem, NC 27109 (USA)
howard@mthcsc.wfu.edu

ARITHMETIC FUNCTIONS OF FIBONACCI NUMBERS

Florian Luca
Syracuse University, Syracuse, NY 13244-1150
(Submitted November 1997-Final Revision February 1998)

For any integers $n \geq 1$ and $k \geq 0$, let $\phi(n)$ and $\sigma_k(n)$ be the Euler totient function of n and the sum of the k^{th} powers of the divisors of n, respectively. In this note, we present the following inequalities.

Theorem:

(1) $\phi(F_n) \geq F_{\phi(n)}$ for all $n \geq 1$. Equality is obtained only if $n = 1, 2, 3$.

(2) $\sigma_k(F_n) \leq F_{\sigma_k(n)}$ for all $n \geq 1$ and $k \geq 1$. Equality is obtained only if $n = 1$ or $(k, n) = (1, 3)$.

(3) $\sigma_0(F_n) \geq F_{\sigma_0(n)}$ for all $n \geq 1$. Equality is obtained only if $n = 1, 2, 4$.

Proof:

(1) See [2] for a more general result. □

(2) Let $k \geq 1$. Notice that $\sigma_k(F_1) = F_{\sigma_k(1)} = 1$ for all $k \geq 1$. Moreover, as $\sigma_k(2) = 1 + 2^k \geq 3$ for $k \geq 1$, it follows that $F_{\sigma_k(2)} = F_{1+2^k} \geq F_3 = 2 > 1 = \sigma_k(1) = \sigma_k(F_2)$. Now let $n = 3$. Notice that $F_{\sigma_1(3)} = F_4 = 3 = \sigma_1(2) = \sigma_1(F_3)$. However, if $k \geq 2$, then $\sigma_k(3) = 1 + 3^k \geq 10$. Since $F_n > n$ for $n \geq 6$, it follows that $F_{\sigma_k(3)} = F_{1+3^k} > 1 + 3^k > 1 + 2^k = \sigma_k(2) = \sigma_k(F_3)$ for $k \geq 2$. From this point on, we assume that $n \geq 4$.

Moreover, assume that

$$\sigma_k(F_n) \geq F_{\sigma_k(n)} \tag{1}$$

for some $n \geq 4$ and some $k \geq 1$. First, we show that if inequality (1) holds, then n is prime. Indeed, assume that n is not prime.

Since $n^k \geq nk$ for all $n \geq 4$ and $k \geq 1$, and since $F_{u+v} \geq F_u \cdot F_v$ for all integers u and v, it follows that

$$F_{n^k} \geq F_{nk} \geq F_n^k \quad \text{for } n \geq 4 \text{ and } k \geq 1. \tag{2}$$

Clearly

$$\frac{m}{\phi(m)} > \frac{\sigma_k(m)}{m^k} \quad \text{for } m \geq 2 \text{ and } k \geq 1. \tag{3}$$

If $n \leq 41$, then $F_n \leq F_{41} < 2 \cdot 10^9$. By Lemma 4.2 in [3], it follows that

$$6 > \frac{F_n}{\phi(F_n)}, \tag{4}$$

and by inequalities (1)-(4), it follows that

$$F_6 = 8 > 6 > \frac{F_n}{\phi(F_n)} > \frac{\sigma_k(F_n)}{F_n^k} \geq \frac{F_{\sigma_k(n)}}{F_{n^k}} \geq F_{\sigma_k(n) - n^k}. \tag{5}$$

Hence, $6 > \sigma_k(n) - n^k$. Since n is not prime, it follows that

$$\sigma_k(n) - n^k \geq \sqrt{n}^k. \tag{6}$$

Therefore, $6 > \sqrt{n}^k$. Since $n \geq 4$, it follows that $6 > \sqrt{4}^k = 2^k$ or $k < 3$. The only pairs (k, n) satisfying the inequality $6 > \sqrt{n}^k$ for which $4 \leq n \leq 40$ is not prime are $(k, n) = (2, 4)$ and $(1, n)$, where $4 \leq n \leq 35$ is not prime. One can check using Mathematika, for example, that $F_{\sigma_k(n)} > \sigma_k(F_n)$ for all the above pairs (k, n).

Suppose now that inequality (1) holds for some $k \geq 1$ and some $n \geq 42$ that is not a prime. Since $F_n \geq F_{42} > 2 \cdot 10^9$, it follows by Lemma 4.1 in [3] that

$$\log(F_n) > \frac{F_n}{\phi(F_n)}. \tag{7}$$

By inequalities (1), (2), (3), and (7), it follows that

$$\log(F_n) > \frac{F_n}{\phi(F_n)} > \frac{\sigma_k(F_n)}{F_n^k} \geq \frac{F_{\sigma_k(n)}}{F_{n^k}} \geq F_{\sigma_k(n)-n^k}. \tag{8}$$

Since

$$\left(\frac{1+\sqrt{5}}{2}\right)^n > F_n > \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - 1\right) \quad \text{for all } n \geq 1, \tag{9}$$

it follows from inequalities (6) and (9) that

$$n \log\left(\frac{1+\sqrt{5}}{2}\right) > \log F_n > F_{\sigma_k(n)-n^k} > \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^{\sqrt{n}^k} - 1\right). \tag{10}$$

If $k \geq 2$, then $\sqrt{n}^k \geq n$, and inequality (10) implies that

$$n \log\left(\frac{1+\sqrt{5}}{2}\right) > \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - 1\right). \tag{11}$$

Inequality (11) implies that $n < 3$, which contradicts the fact that $n \geq 42$. Hence $k = 1$. Inequality (10) becomes

$$n \log\left(\frac{1+\sqrt{5}}{2}\right) > \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^{\sqrt{n}} - 1\right),$$

which implies that $n < 92$. One can check using Mathematika, for example, that $F_{\sigma_1(n)} > \sigma_1(F_n)$ for all $42 \leq n \leq 91$.

From the above arguments, it follows that if inequality (1) holds for some $n \geq 4$ and some $k \geq 1$, then n is prime. In particular, $n \geq 5$,

Write $F_n = q_1^{\gamma_1} \cdot \dots \cdot q_t^{\gamma_t}$, where $q_1 < \dots < q_t$ are prime numbers, and $\gamma_i \geq 1$ for $i = 1, \dots, t$. We show that q_1, q_2, and t satisfy the following conditions:

(a) $q_1 \geq n$;

(b) If $t > 1$, then $q_2 \geq 2n - 1$;

(c) $t - 1 > 2(n-1)\log\left(\frac{3}{2} \cdot e^{-1/(n-1)}\right)$.

Indeed, let q be one of the primes dividing F_n. From Lemma II and Theorem XII in [1], it follows that $q | F_{q^2} \cdot F_{q^2-1}$.

Suppose first that $q | F_{q^2}$. We conclude that $q | (F_n, F_{q^2}) = F_{(n, q^2)}$. Since $F_1 = 1$, we conclude that $(n, q^2) \neq 1$. Since both q and n are prime, it follows that $q = n$.

Suppose now that $q | F_{q^2-1}$. We conclude that $q | (F_n, F_{q^2-1}) = F_{(n, q^2-1)} \cdot q \equiv \pm 1 \pmod{n}$. Then, clearly, $q \neq n \pm 1$ because q and n are both prime and $n \geq 5$. Hence, $q \geq 2n-1$ in this case.

Now (a) and (b) follow immediately from the above arguments.

For (c), notice that by inequalities (1), (2), and (3),

$$\prod_{i=1}^{t}\left(1+\frac{1}{q_i-1}\right) = \frac{F_n}{\phi(F_n)} > \frac{\sigma_k(F_n)}{F_n^k} \geq \frac{F_{\sigma_k(n)}}{F_n^k} = \frac{F_{1+n^k}}{F_n^k} \geq \frac{F_{1+n^k}}{F_{n^k}} \geq \frac{3}{2}, \tag{12}$$

because $F_{m+1} / F_m \geq 3/2$ for all $m \geq 3$. Taking logarithms in inequality (12), and using the fact that $\log(1+x) < x$ for all $x > 0$, we conclude that

$$\sum_{i=1}^{t} \frac{1}{q_i - 1} > \log\left(\frac{3}{2}\right).$$

From (a) and (b), it follows that

$$\frac{1}{n-1} + \frac{t-1}{2(n-1)} > \log\left(\frac{3}{2}\right). \tag{13}$$

Inequality (13) is obviously equivalent to the inequality asserted at (c) above.

From inequality (10) and inequalities (a)-(c) above, it follows that

$$\begin{aligned} n\log\left(\frac{1+\sqrt{5}}{2}\right) &> \log F_n \geq \sum_{i=1}^{t} \log q_i \geq \log n + (t-1)\log(2n-1) \\ &> (t-1)\log(2n-1) > 2(n-1)\log(2n-1)\log\left(\frac{3}{2}\cdot e^{-1/(n-1)}\right). \end{aligned}$$

Hence,

$$\frac{n}{2(n-1)\log(2n-1)} \cdot \log\left(\frac{1+\sqrt{5}}{2}\right) - \log\left(\frac{3}{2}\right) + \frac{1}{n-1} > 0. \tag{14}$$

Inequality (14) implies that $n < 5$, which contradicts the fact that $n \geq 5$. □

(3) Let $k = 0$. For any positive integer m, let $\tau(m)$ and $\nu(m)$ be the number of divisors of m and the number of prime divisors of m, respectively. Notice that $\tau(m) = \sigma_0(m)$. Therefore, the inequality asserted at (3) is equivalent to $\tau(F_n) \geq F_{\tau(n)}$ for $n \geq 1$.

Let n be a positive integer. Recall that a *primitive divisor* of F_n is a prime number q, such that $q | F_n$, but $q \nmid F_m$ for any $1 \leq m < n$. From Theorem XXIII in [1], we know that F_n has a primitive divisor for all $n \geq 1$ except $n = 1, 2, 6, 12$. We distinguish the following cases.

Case 1. $6 \nmid n$. Since $F_d | F_n$ for all $d | n$, and F_d has a primitive divisor for all d except $d = 1$, 2, it follows that $\nu(F_n) \geq \tau(n) - 2$. Hence,

$$\tau(F_n) \geq 2^{\nu(F_n)} \geq 2^{\tau(n)-2}. \tag{15}$$

Since $2^{k-2} > F_k$ for all $k \geq 4$, it follows that the inequality asserted by (3) holds for all n such that $\tau(n) \geq 4$.

If $\tau(n) = 1$, then $n = 1$ and $\tau(F_1) = F_{\tau(1)} = 1$.

If $\tau(n) = 2$, then $n = p$ is a prime and $\tau(F_p) \geq 1 = F_2 = F_{\tau(p)}$. Obviously, equality holds only if $p = 2$.

If $\tau(n) = 3$, then $n = p^2$, where p is a prime. Moreover, $\tau(F_{p^2}) \geq 2 = F_3 = F_{\tau(p^2)}$, and equality certainly holds for $p = 2$. If $p > 2$, then both F_p and F_{p^2} have a primitive divisor; therefore,

$$\tau(F_{p^2}) \geq 4 > 2 = F_3 = F_{\tau(p^2)}.$$

Case 2. $6 \mid n$, but $12 \nmid n$. In this case, $\nu(F_n) \geq \tau(n) - 3$. Moreover, since $F_6 = 8 \mid F_n$, it follows that the exponent at which 2 appears in the prime factor decomposition of F_n is at least 3. Hence,

$$\tau(F_n) \geq 2^{\nu(n)-1} \cdot (3+1) \geq 2^{\tau(n)-4} \cdot 4 = 2^{\tau(n)-2} > F_{\tau(n)},$$

because $\tau(n) \geq 4 = \tau(6)$.

Case 3. $12 \mid n$. In this case, $\nu(F_n) \geq \tau(n) - 4$. Moreover, since $2^4 \cdot 3^2 = F_{12} \mid F_n$, it follows that the exponents at which 2 and 3 appear in the prime factor decomposition of F_n are at least 4 and 2, respectively. Thus,

$$\tau(F_n) \geq 2^{\nu(n)-2} \cdot (4+1) \cdot (2+1) \geq 2^{\tau(n)-6} \cdot 15. \tag{16}$$

Moreover, since $12 \mid n$, it follows that $\tau(n) \geq 6 = \tau(12)$. By inequality (15), it follows that it suffices to show that

$$15 \cdot 2^{k-6} > F_k \quad \text{for } k \geq 6. \tag{17}$$

This can be proved easily by induction. □

This completes the proof of the Theorem. □

REFERENCES

1. R. D. Carmichael. "On the Numerical Factors of Arithmetic Forms." *Ann. of Math.* **15** (1913-1914):30-70.
2. F. Luca. "Euler Indicators of Lucas Sequences." To appear in *Bull. Math. Soc. Sci. Math. Roumanie.*
3. M. Ward. "The Intrinsic Divisors of Lehmer Numbers." *Ann. of Math.* **62** (1955):230-236.

AMS Classification Numbers: 11A25, 11B39

LUCAS SEQUENCES AND FUNCTIONS OF A 4-BY-4 MATRIX

R. S. Melham
School of Mathematical Sciences, University of Technology, Sydney,
PO Box 123, Broadway, NSW 2007 Australia
(Submitted November 1997-Final Revision April 1998)

1. INTRODUCTION

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers n by

$$\begin{cases} U_n = pU_{n-1} - qU_{n-2}, & U_0 = 0,\ U_1 = 1, \\ V_n = pV_{n-1} - qV_{n-2}, & V_0 = 2,\ V_1 = p, \end{cases} \tag{1.1}$$

where p and q are real numbers with $q(p^2-4q) \neq 0$. These sequences were studied originally by Lucas [6], and have subsequently been the subject of much attention.

The Binet forms for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \tag{1.2}$$

where

$$\alpha = \frac{p + \sqrt{p^2-4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2-4q}}{2} \tag{1.3}$$

are the roots, assumed distinct, of $x^2 - px + q = 0$. We assume further that α / β is not an n^{th} root of unity for any n.

A well-known relationship between U_n and V_n is

$$V_n = U_{n+1} - qU_{n-1}, \tag{1.4}$$

which we use subsequently.

Recently, Melham [7] considered functions of a 3-by-3 matrix and obtained infinite sums involving squares of terms from the sequences (1.1). Here, using a similarly defined 4-by-4 matrix, we obtain new infinite sums involving cubes, and other terms of degree three, from the sequences (1.1). For example, closed expressions for

$$\sum_{n=0}^{\infty} \frac{U_n^3}{n!} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{U_n^2 U_{n+1}}{n!}$$

arise as special cases of results in Section 3 [see (3.4) and (3.5)]. Since the above mentioned paper of Melham contains a comprehensive list of references, we have chosen not to repeat them here.

Unfortunately, one of the matrices which we need to record does not fit comfortably on a standard page. We overcome this difficulty by simply listing elements in a table. Following convention, the (i, j) element is the element in the i^{th} row and j^{th} column.

2. THE MATRIX $A_{k,x}$

By lengthy but straightforward induction on n, it can be shown that the 4-by-4 matrix

$$A=\begin{pmatrix} 0 & 0 & 0 & -q^3 \\ 0 & 0 & q^2 & 3pq^2 \\ 0 & -q & -2pq & -3p^2q \\ 1 & p & p^2 & p^3 \end{pmatrix} \tag{2.1}$$

is such that, for nonnegative integers n, A^n is as follows:

$$\begin{pmatrix} -q^3U_{n-1}^3 & -q^3U_{n-1}^2U_n & -q^3U_{n-1}U_n^2 & -q^3U_n^3 \\ 3q^2U_{n-1}^2U_n & q^2(2U_n^2U_{n-1}+U_{n+1}U_{n-1}^2) & q^2(U_n^3+2U_{n-1}U_nU_{n+1}) & 3q^2U_n^2U_{n+1} \\ -3qU_{n-1}U_n^2 & -q(U_n^3+2U_{n-1}U_nU_{n+1}) & -q(2U_n^2U_{n+1}+U_{n-1}U_{n+1}^2) & -3qU_nU_{n+1}^2 \\ U_n^3 & U_n^2U_{n+1} & U_nU_{n+1}^2 & U_{n+1}^3 \end{pmatrix}.$$

To complete the proof by induction, we make repeated use of the recurrence for $\{U_n\}$. For example, performing the inductive step for the (2, 2) position, we have

$$\begin{aligned} &-q^3(U_n^3+2U_{n-1}U_nU_{n+1})+3pq^2U_n^2U_{n+1} \\ &= q^2U_n[U_n(-qU_n)+2U_{n+1}(-qU_{n-1})+3pU_nU_{n+1}] \\ &= q^2U_n[U_n(U_{n+2}-pU_{n+1})+2U_{n+1}(U_{n+1}-pU_n)+3pU_nU_{n+1}] \\ &= q^2U_n[2U_{n+1}^2+U_nU_{n+2}] \\ &= q^2[2U_{n+1}^2U_n+U_{n+2}U_n^2], \text{ which is the required expression.} \end{aligned}$$

When $p=1$ and $q=-1$, the matrix A becomes

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

which is a 4-by-4 *Fibonacci matrix*. Other 4-by-4 Fibonacci matrices have been studied, for example, in [3] and [4].

The characteristic equation of A is

$$\lambda^4-p(p^2-2q)\lambda^3+q(p^2-2q)(p^2-q)\lambda^2-pq^3(p^2-2q)\lambda+q^6=0.$$

Since $p=\alpha+\beta$ and $q=\alpha\beta$, it is readily verified that α^3, $\alpha^2\beta$, $\alpha\beta^2$, and β^3 are the eigenvalues λ_j $(j=1,2,3,4)$ of A. These eigenvalues are nonzero and distinct because of our assumptions in Section 1.

Associated with A, we define the matrix $A_{k,x}$ by

$$A_{k,x}=xA^k, \tag{2.2}$$

where x is an arbitrary real number and k is a nonnegative integer. From the definition of an eigenvalue, it follows immediately that $x\alpha^{3k}$, $x\alpha^{2k}\beta^k$, $x\alpha^k\beta^{2k}$, and $x\beta^{3k}$ are the eigenvalues of $A_{k,x}$. Again, they are nonzero and distinct.

3. THE MAIN RESULTS

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series whose domain of convergence includes the eigenvalues of $A_{k,x}$. Then we have, from (2.2),

$$f(A_{k,x})=\sum_{n=0}^{\infty}a_nA_{k,x}^n=\sum_{n=0}^{\infty}a_nx^nA^{kn}. \tag{3.1}$$

The final sum in (3.1) can be expressed as a 4-by-4 matrix whose entries we record in the following table.

(i,j)	(i,j) element of $f(A_{k,x})$
(1, 1)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn-1}^3$
(1, 2)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn-1}^2U_{kn}$
(1, 3)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn-1}U_{kn}^2$
(1, 4)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn}^3$
(2, 1)	$3q^2\sum_{n=0}^{\infty}a_nx^nU_{kn-1}^2U_{kn}$
(2, 2)	$q^2\sum_{n=0}^{\infty}a_nx^n(2U_{kn}^2U_{kn-1}+U_{kn+1}U_{kn-1}^2)$
(2, 3)	$q^2\sum_{n=0}^{\infty}a_nx^n(U_{kn}^3+2U_{kn-1}U_{kn}U_{kn+1})$
(2, 4)	$3q^2\sum_{n=0}^{\infty}a_nx^nU_{kn}^2U_{kn+1}$
(3, 1)	$-3q\sum_{n=0}^{\infty}a_nx^nU_{kn-1}U_{kn}^2$
(3, 2)	$-q\sum_{n=0}^{\infty}a_nx^n(U_{kn}^3+2U_{kn-1}U_{kn}U_{kn+1})$
(3, 3)	$-q\sum_{n=0}^{\infty}a_nx^n(2U_{kn}^2U_{kn+1}+U_{kn-1}U_{kn+1}^2)$
(3, 4)	$-3q\sum_{n=0}^{\infty}a_nx^nU_{kn}U_{kn+1}^2$
(4, 1)	$\sum_{n=0}^{\infty}a_nx^nU_{kn}^3$
(4, 2)	$\sum_{n=0}^{\infty}a_nx^nU_{kn}^2U_{kn+1}$
(4, 3)	$\sum_{n=0}^{\infty}a_nx^nU_{kn}U_{kn+1}^2$
(4, 4)	$\sum_{n=0}^{\infty}a_nx^nU_{kn+1}^3$

On the other hand, from the theory of functions of matrices ([2] and [5]), it is known that

$$f(A_{k,x}) = c_0 I + c_1 A_{k,x} + c_2 A_{k,x}^2 + c_3 A_{k,x}^3, \tag{3.2}$$

where I is the 4-by-4 identity matrix, and where c_0, c_1, c_2, and c_3 can be obtained by solving the system

$$\begin{cases} c_0 + c_1 x\alpha^{3k} + c_2 x^2 \alpha^{6k} + c_3 x^3 \alpha^{9k} = f(x\lambda_1^k) = f(x\alpha^{3k}), \\ c_0 + c_1 x\alpha^{2k}\beta^k + c_2 x^2 \alpha^{4k}\beta^{2k} + c_3 x^3 \alpha^{6k}\beta^{3k} = f(x\lambda_2^k) = f(x\alpha^{2k}\beta^k), \\ c_0 + c_1 x\alpha^{k}\beta^{2k} + c_2 x^2 \alpha^{2k}\beta^{4k} + c_3 x^3 \alpha^{3k}\beta^{6k} = f(x\lambda_3^k) = f(x\alpha^{k}\beta^{2k}), \\ c_0 + c_1 x\beta^{3k} + c_2 x^2 \beta^{6k} + c_3 x^3 \beta^{9k} = f(x\lambda_4^k) = f(x\beta^{3k}). \end{cases}$$

With the use of Cramer's rule, and making use of the Binet form for U_n, we obtain, after much tedious algebra,

$$\begin{aligned} c_0 = {} & \frac{-f(x\alpha^{3k})\beta^{6k}}{U_k U_{2k} U_{3k}(\alpha-\beta)^3} + \frac{f(x\alpha^{2k}\beta^k)\alpha^k\beta^{3k}}{U_k^2 U_{2k}(\alpha-\beta)^3} \\ & - \frac{f(x\alpha^k\beta^{2k})\alpha^{3k}\beta^k}{U_k^2 U_{2k}(\alpha-\beta)^3} + \frac{f(x\beta^{3k})\alpha^{6k}}{U_k U_{2k} U_{3k}(\alpha-\beta)^3}, \end{aligned}$$

$$\begin{aligned} c_1 = {} & \frac{f(x\alpha^{3k})\beta^{3k}(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x\alpha^{2k}U_k U_{2k} U_{3k}(\alpha-\beta)^3} - \frac{f(x\alpha^{2k}\beta^k)(\alpha^{3k}+\beta^{3k}+\alpha^{2k}\beta^k)}{x\alpha^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} \\ & + \frac{f(x\alpha^k\beta^{2k})(\alpha^{3k}+\beta^{3k}+\alpha^k\beta^{2k})}{x\beta^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} - \frac{f(x\beta^{3k})\alpha^{3k}(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x\beta^{2k}U_k U_{2k} U_{3k}(\alpha-\beta)^3}, \end{aligned}$$

$$\begin{aligned} c_2 = {} & \frac{-f(x\alpha^{3k})\beta^{k}(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x^2\alpha^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3} + \frac{f(x\alpha^{2k}\beta^k)(\alpha^{3k}+\beta^{3k}+\alpha^{k}\beta^{2k})}{x^2\alpha^{3k}\beta^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} \\ & - \frac{f(x\alpha^k\beta^{2k})(\alpha^{3k}+\beta^{3k}+\alpha^{2k}\beta^{k})}{x^2\alpha^{2k}\beta^{3k}U_k^2 U_{2k}(\alpha-\beta)^3} + \frac{f(x\beta^{3k})\alpha^{k}(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x^2\beta^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3}, \end{aligned}$$

$$\begin{aligned} c_3 = {} & \frac{f(x\alpha^{3k})}{x^3\alpha^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3} - \frac{f(x\alpha^{2k}\beta^k)}{x^3\alpha^{3k}\beta^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} \\ & + \frac{f(x\alpha^k\beta^{2k})}{x^3\alpha^{2k}\beta^{3k}U_k^2 U_{2k}(\alpha-\beta)^3} - \frac{f(x\beta^{3k})}{x^3\beta^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3}. \end{aligned}$$

The symmetry in these expressions emerges if we compare the coefficients of $f(x\alpha^{3k})$ and $f(x\beta^{3k})$ and the coefficients of $f(x\alpha^{2k}\beta^k)$ and $f(x\alpha^k\beta^{2k})$.

Now, if we consider (3.1) and (3.2) and the expressions for the entries of A^n, and equate entries in the (4, 1) position, we obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^3 = c_1 x U_k^3 + c_2 x^2 U_{2k}^3 + c_3 x^3 U_{3k}^3. \tag{3.3}$$

Finally, with the values of c_1, c_2, and c_3 obtained above, we obtain, with much needed help from the software package "Mathematica":

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^3 = \frac{f(x\alpha^{3k}) - 3f(x\alpha^{2k}\beta^k) + 3f(x\alpha^k\beta^{2k}) - f(x\beta^{3k})}{(\alpha-\beta)^3}. \tag{3.4}$$

In precisely the same manner, we equate appropriate entries in (3.1) and (3.2) to obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^2 U_{kn+1} = \frac{\alpha f(x\alpha^{3k}) - (2\alpha+\beta) f(x\alpha^{2k}\beta^k) + (\alpha+2\beta) f(x\alpha^k\beta^{2k}) - \beta f(x\beta^{3k})}{(\alpha-\beta)^3}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn} U_{kn+1}^2 = \frac{\alpha^2 f(x\alpha^{3k}) - (\alpha^2+2\alpha\beta) f(x\alpha^{2k}\beta^k) + (\beta^2+2\alpha\beta) f(x\alpha^k\beta^{2k}) - \beta^2 f(x\beta^{3k})}{(\alpha-\beta)^3}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn+1}^3 = \frac{\alpha^3 f(x\alpha^{3k}) - 3\alpha^2\beta f(x\alpha^{2k}\beta^k) + 3\alpha\beta^2 f(x\alpha^k\beta^{2k}) - \beta^3 f(x\beta^{3k})}{(\alpha-\beta)^3}, \tag{3.7}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn}^2 = \frac{\beta f(x\alpha^{3k}) - (\alpha+2\beta) f(x\alpha^{2k}\beta^k) + (2\alpha+\beta) f(x\alpha^k\beta^{2k}) - \alpha f(x\beta^{3k})}{\alpha\beta(\alpha-\beta)^3}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} a_n x^n (U_{kn}^3 + 2U_{kn-1}U_{kn}U_{kn+1}) = \frac{3\alpha\beta(f(x\alpha^{3k}) - f(x\beta^{3k})) - (\alpha+2\beta)(2\alpha+\beta)(f(x\alpha^{2k}\beta^k) - f(x\alpha^k\beta^{2k}))}{\alpha\beta(\alpha-\beta)^3}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} a_n x^n (2U_{kn}^2 U_{kn+1} + U_{kn-1}U_{kn+1}^2) = \frac{3\alpha^2\beta f(x\alpha^{3k}) - \alpha(\alpha+2\beta)^2 f(x\alpha^{2k}\beta^k) + \beta(2\alpha+\beta)^2 f(x\alpha^k\beta^{2k}) - 3\alpha\beta^2 f(x\beta^{3k})}{\alpha\beta(\alpha-\beta)^3}, \tag{3.10}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1}^2 U_{kn} = \frac{\beta^2 f(x\alpha^{3k}) - \beta(2\alpha+\beta) f(x\alpha^{2k}\beta^k) + \alpha(\alpha+2\beta) f(x\alpha^k\beta^{2k}) - \alpha^2 f(x\beta^{3k})}{\alpha^2\beta^2(\alpha-\beta)^3}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} a_n x^n (2U_{kn}^2 U_{kn-1} + U_{kn+1}U_{kn-1}^2) = \frac{3\alpha\beta^2 f(x\alpha^{3k}) - \beta(2\alpha+\beta)^2 f(x\alpha^{2k}\beta^k) + \alpha(\alpha+2\beta)^2 f(x\alpha^k\beta^{2k}) - 3\alpha^2\beta f(x\beta^{3k})}{\alpha^2\beta^2(\alpha-\beta)^3}, \tag{3.12}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1}^3 = \frac{\beta^3 f(x\alpha^{3k}) - 3\alpha\beta^2 f(x\alpha^{2k}\beta^k) + 3\alpha^2\beta f(x\alpha^k\beta^{2k}) - \alpha^3 f(x\beta^{3k})}{\alpha^3\beta^3(\alpha-\beta)^3}. \tag{3.13}$$

From (3.4) and (3.9), we obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn} U_{kn+1} = \frac{\alpha\beta(f(x\alpha^{3k}) - f(x\beta^{3k})) - (\alpha^2+\alpha\beta+\beta^2)(f(x\alpha^{2k}\beta^k) - f(x\alpha^k\beta^{2k}))}{\alpha\beta(\alpha-\beta)^3}. \tag{3.14}$$

Similarly, (3.5) and (3.10) and then (3.8) and (3.12) yield, respectively,

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn+1}^2 = \frac{\alpha^2\beta f(x\alpha^{3k}) - \alpha(\alpha^2+2\beta^2) f(x\alpha^{2k}\beta^k) + \beta(2\alpha^2+\beta^2) f(x\alpha^k\beta^{2k}) - \alpha\beta^2 f(x\beta^{3k})}{\alpha\beta(\alpha-\beta)^3}, \tag{3.15}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn+1} U_{kn-1}^2 = \frac{\alpha\beta^2 f(x\alpha^{3k}) - \beta(2\alpha^2+\beta^2) f(x\alpha^{2k}\beta^k) + \alpha(\alpha^2+2\beta^2) f(x\alpha^k\beta^{2k}) - \alpha^2\beta f(x\beta^{3k})}{\alpha^2\beta^2(\alpha-\beta)^3}. \tag{3.16}$$

Finally, from (1.2), we have $V_{kn}^3 = U_{kn+1}^3 - 3qU_{kn+1}^2 U_{kn-1} + 3q^2 U_{kn+1} U_{kn-1}^2 - q^3 U_{kn-1}^3$. This, together with (3.7), (3.13), (3.15), and (3.16), yields

$$\sum_{n=0}^{\infty} a_n x^n V_{kn}^3 = f(x\alpha^{3k}) + 3f(x\alpha^{2k}\beta^k) + 3f(x\alpha^k\beta^{2k}) + f(x\beta^{3k}) \tag{3.17}$$

after some tedious manipulation involving the use of the equality $\alpha\beta = q$.

4. APPLICATIONS

We now specialize (3.4) and (3.17) to the Chebyshev polynomials to obtain some attractive sums involving third powers of the sine and cosine functions.

Let $\{T_n(t)\}_{n=0}^{\infty}$ and $\{S_n(t)\}_{n=0}^{\infty}$ denote the Chebyshev polynomials of the first and second kinds, respectively. Then

$$\left.\begin{aligned} S_n(t) &= \frac{\sin n\theta}{\sin\theta} \\ T_n(t) &= \cos n\theta \end{aligned}\right\}, \quad t = \cos\theta,\ n \geq 0.$$

Indeed, $\{S_n(t)\}_{n=0}^{\infty}$ and $\{2T_n(t)\}_{n=0}^{\infty}$ are the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$, respectively, generated by (1.1), where $p = 2\cos\theta$ and $q = 1$. Thus,

$$\alpha = \cos\theta + i\sin\theta = e^{i\theta} \quad \text{and} \quad \beta = \cos\theta - i\sin\theta = e^{-i\theta},$$

which are obtained from (1.3). Further information about Chebyshev polynomials can be found, for example, in [1].

We use the following well-known power series, each of which has the complex plane as its domain of convergence:

$$\sin z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!},\tag{4.1}$$

$$\cos z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!},\tag{4.2}$$

$$\sinh z=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!},\tag{4.3}$$

$$\cosh z=\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}.\tag{4.4}$$

Now, in (3.4), taking $U_n=\sin n\theta/\sin\theta$ and replacing f by the functions in (4.1)-(4.4), we obtain , after replacing all occurrences of $k\theta$ by ϕ,

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\sin^3(2n+1)\phi}{(2n+1)!}=\frac{3\cos(x\cos\phi)\sinh(x\sin\phi)-\cos(x\cos 3\phi)\sinh(x\sin 3\phi)}{4},\tag{4.5}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}\sin^3 2n\phi}{(2n)!}=\frac{-3\sin(x\cos\phi)\sinh(x\sin\phi)+\sin(x\cos 3\phi)\sinh(x\sin 3\phi)}{4},\tag{4.6}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}\sin^3(2n+1)\phi}{(2n+1)!}=\frac{3\cosh(x\cos\phi)\sin(x\sin\phi)-\cosh(x\cos 3\phi)\sin(x\sin 3\phi)}{4},\tag{4.7}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n}\sin^3 2n\phi}{(2n)!}=\frac{3\sinh(x\cos\phi)\sin(x\sin\phi)-\sinh(x\cos 3\phi)\sin(x\sin 3\phi)}{4}.\tag{4.8}$$

Similarly, in (3.17), taking $V_n=2\cos n\theta$ and replacing f by the functions in (4.1)-(4.4), we obtain, respectively,

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\cos^3(2n+1)\phi}{(2n+1)!}=\frac{3\sin(x\cos\phi)\cosh(x\sin\phi)+\sin(x\cos 3\phi)\cosh(x\sin 3\phi)}{4},\tag{4.9}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}\cos^3 2n\phi}{(2n)!}=\frac{3\cos(x\cos\phi)\cosh(x\sin\phi)+\cos(x\cos 3\phi)\cosh(x\sin 3\phi)}{4},\tag{4.10}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}\cos^3(2n+1)\phi}{(2n+1)!}=\frac{3\sinh(x\cos\phi)\cos(x\sin\phi)+\sinh(x\cos 3\phi)\cos(x\sin 3\phi)}{4},\tag{4.11}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n}\cos^3 2n\phi}{(2n)!}=\frac{3\cosh(x\cos\phi)\cos(x\sin\phi)+\cosh(x\cos 3\phi)\cos(x\sin 3\phi)}{4}.\tag{4.12}$$

Finally, we mention that much of the tedious algebra in this paper was accomplished with the help of "Mathematica".

ACKNOWLEDGMENT

The author gratefully acknowledges the input of an anonymous referee, whose suggestions have improved the presentation of this paper.

REFERENCES

1. M. Abramowitz & I. A. Stegun. *Handbook of Mathematical Functions*. New York: Dover, 1972.
2. R. Bellman. *Introduction to Matrix Analysis*. New York: McGraw-Hill, 1970.
3. O. Brugia & P. Filipponi. "Functions of the Kronecker Square of the Matrix Q." In *Applications of Fibonacci Numbers* **2**:69-76. Ed. A. N. Philippou et al. Dordrecht: Kluwer, 1988.
4. P. Filipponi. "A Family of 4-by-4 Fibonacci Matrices." *The Fibonacci Quarterly* **35.4** (1997):300-08.
5. F. R. Gantmacher. *The Theory of Matrices*. New York: Chelsea, 1960.
6. E. Lucas. "Théorie des Fonctions Numériques Simplement Periodiques." *Amer. J. Math.* **1** (1878):184-240, 289-321.
7. R. S. Melham. "Lucas Sequences and Functions of a 3-by-3 Matrix." *The Fibonacci Quarterly* **37.2** (1999):111-16.

AMS Classification Numbers: 11B39, 15A36, 30B10

❖❖❖

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stanley@tiac.net *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-878 ***Proposed by L. A. G. Dresel, Reading, England***

Show that, for positive integers n, the harmonic mean of F_n and L_n can be expressed as the ratio of two Fibonacci numbers, and that it is equal to $L_{n-1} + R_n$ where $|R_n| \le 1$. Find a simple formula for R_n.

Note: If h is the harmonic mean of x and y, then $2/h = 1/x + 1/y$.

B-879 ***Proposed by Mario DeNobili, Vaduz, Lichtenstein***

Let $\langle c_n \rangle$ be defined by the recurrence $c_{n+4} = 2c_{n+3} + c_{n+2} - 2c_{n+1} - c_n$ with initial conditions $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, and $c_3 = 6$. Express c_n in terms of Fibonacci and/or Lucas numbers.

B-880 ***Proposed by A. J. Stam, Winsum, The Netherlands***

Express

$$\sum_{2i \le m} \binom{m-i}{i} (-1)^i 3^{m-2i}$$

in terms of Fibonacci and/or Lucas numbers.

B-881 ***Proposed by Mohammad K. Azarian, University of Evansville, IN***

Consider the two equations

$$\sum_{i=1}^{n} L_i x_i = F_{n+3} \quad \text{and} \quad \sum_{i=1}^{n} L_i y_i = L_2 - F_{n+1}.$$

Show that the number of positive integer solutions of the first equation is equal to the number of nonnegative integer solutions of the second equation.

B-882 ***Proposed by A. J. Stam, Winsum, The Netherlands***

Suppose the sequence $\langle A_n \rangle$ satisfies the recurrence $A_n = A_{n-1} + A_{n-2}$. Let

$$B_n = \sum_{k=0}^{n} (-1)^k A_{n-2k}.$$

Prove that $B_n = A_0 F_{n+1}$ for all nonnegative integers n.

B-883 ***Proposed by L. A. G. Dresel, Reading, England***

Let $\langle f_n \rangle$ be the Fibonacci sequence F_n modulo p, where p is a prime, so that we have $f_n \equiv F_n$ (mod p) and $0 \le f_n < p$ for all $n \ge 0$. The sequence $\langle f_n \rangle$ is known to be periodic. Prove that, for a given integer c in the range $0 \le c < p$, there can be at most four values of n for which $f_n = c$ within any one cycle of this period.

SOLUTIONS

A Perfect Square

B-860 ***Proposed by Herta T. Freitag, Roanoke, VA***
(Vol. 36, no. 5, November 1998)

Let k be a positive integer. The sequence $\langle A_n \rangle$ is defined by the recurrence $A_{n+2} = 2kA_{n+1} - A_n$ for $n \ge 0$ with initial conditions $A_0 = 0$ and $A_1 = 1$. Prove that $(k^2 - 1)A_n^2 + 1$ is a perfect square for all $n \ge 0$.

Solution by Don Redmond, Southern Illinois University, Carbondale, IL

We give a generalization. Let p and q be integers and let $A_0 = 0$ and $A_1 = 1$. Define, for $n \ge 0$, the sequence $\langle A_n \rangle$ by $A_{n+2} = 2pA_{n+1} - qA_n$. Then, for $n \ge 0$,

$$q^n + (p^2 - q)\, A_n^2$$

is a perfect square. If we let $q = 1$ and $p = k$, we obtain the desired result.

Let s and t be the roots of the polynomial $x^2 - 2px + q = 0$. Then we know that we can write, for $n \ge 0$,

$$A_n = \frac{s^n - t^n}{s - t}.$$

Now $s + t = 2p$, $st = q$, and $s - t = 2\sqrt{p^2 - q}$. Thus,

$$4q^n + 4(p^2 - q)\, A_n^2 = 4(st)^n + (s-t)^2 \left(\frac{s^n - t^n}{s - t} \right)^2 = 4(st)^n + (s^n - t^n)^2 = (s^n + t^n)^2.$$

Hence

$$q^n + (p^2 - q)\, A_n^2 = \left(\frac{s^n + t^n}{2} \right)^2.$$

Finally, $(s^n + t^n)/2$ is indeed an integer because it satisfies the same recurrence as A_n but with initial values 1 and p.

Seiffert also found this generalization. Lord showed that, for the original sequence,

$$(k^2-1)A_n^2+1=(kA_n-A_{n-1})^2.$$

Redmond noted that another generalization can be found on page 501 of [1]: If a_0 *is any integer,* $a_1=ka_0+p$ *and, for* $n\ge 0$, $a_{n+2}=2ka_{n+1}-a_n$, *then* $(k^2-1)(a_n^2-a_0^2)+p^2$ *is a perfect square. The problem at hand is the case* $a_0=0$ *and* $p=1$.

Reference:

1. Don Redmond. *Number Theory: An Introduction*. New York: Marcel Dekker, 1996.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Steve Edwards, N. Gauthier, Joe Howard, Hans Kappus, Harris Kwong, Graham Lord, Maitland A. Rose, H.-J. Seiffert, Indulis Strazdins, Andràs Szilàrd, and the proposer.

Integer Coefficients?

B-861 ***Proposed by the editor***
(Vol. 36, no. 5, November 1998)

The sequence $w_0, w_1, w_2, w_3, w_4, \dots$ satisfies the recurrence $w_n=Pw_{n-1}-Qw_{n-2}$ for $n>1$. If every term of the sequence is an integer, must P and Q both be integers?

Counterexample by Steve Edwards, Southern Polytechnic State University, Marietta, GA

The sequence $w_n=k$, where k is an integer, is a counterexample when p is not an integer and $P-Q=1$.

Solution by L. A. G. Dresel, Reading, England

We shall prove that P and Q must both be integers provided that $w_1^2-w_0w_2\ne 0$.

Let $D_n=w_{n+1}^2-w_nw_{n+2}$. Eliminating P from the equations $w_{n+2}=Pw_{n+1}-Qw_n$ and $w_{n+3}=Pw_{n+2}-Qw_{n+1}$, we have $D_{n+1}=QD_n$ for $n\ge 0$. Therefore, $D_1=QD_0$, and by induction $D_n=Q^nD_0$ for $n\ge 0$.

If $D_0\ne 0$, then $Q=D_1/D_0$ is the ratio of two integers, and then $D_n=Q^nD_0$ for all $n\ge 1$ implies that Q must be an integer.

It remains to prove that P must also be an integer in this case. Suppose, on the contrary, that P is a rational fraction, $P=p/d$, where $\gcd(p,d)=1$. Consider the recurrence in the form $Pw_n=w_{n+1}+Qw_{n-1}$ for $n\ge 1$. It follows that d divides w_n, so that d divides $w_1, w_2, w_3, \dots$. Therefore, for $n\ge 2$, the right side of the recurrence is divisible by d, and we have d^2 divides w_2, w_3, $w_4, \dots$. Continuing in this way (let us call it the escalator principle), we find that for each n, d^n divides w_n. Hence d^{2n+2} divides $D_n=Q^nD_0$ for all n, and it follows that Q is divisible by d^2.

Returning again to the recurrence $Pw_n=w_{n+1}+Qw_{n-1}$ for $n\ge 1$, we see that the right side is divisible by d^2, and therefore d^3 divides $w_1, w_2, w_3, \dots$. Then, for $n\ge 2$, the right side of the recurrence is divisible by d^5, so that d^6 divides $w_2, w_3, w_4, \dots$. Continuing with this escalator principle, we find that, for each n, d^{3n} divides w_n. Hence, d^{6n+6} divides $D_n=Q^nD_0$ for all n, and it follows that Q is divisible by d^6. Returning again (and again) to the recurrence formula and applying the escalator principle, we require even higher powers of d dividing both Q and D_0, so that we cannot construct the sequence of integers unless $d=1$.

This implies that when $D_0\ne 0$, both P and Q must be integers.

Counterexamples also received by Richard André-Jeannin, Paul S. Bruckman, H.-J. Seiffert, and Andràs Szilàrd.

Large LCM

B-862 ***Proposed by Charles K. Cook, University of South Carolina, Sumter, SC (Vol. 36, no. 5, November 1998)***

Find a Fibonacci number and a Lucas number whose sum is 114,628 and whose least common multiple is 567,451,586.

Solution by Scott H. Brown, Auburn University at Montgomery, Montgomery, AL

Since the sum of the two numbers ends in 8, we test the combinations in the one's digits: $0+8$, $1+7$, $2+6$, $3+5$, and $4+4$. Testing these combinations and observing that they must add up to 114,628, many of these combinations are eliminated with the exception of the following:

(a) $F_{21} = 10{,}946 \quad L_{24} = 103{,}682$;

(b) $F_{25} = 75{,}025 \quad L_{22} = 39{,}603$.

These values were found on pages 83 and 84 in [1].

Factoring the integers in question, we find $10946 = 2\cdot 13\cdot 421$, $103682 = 2\cdot 47\cdot 1103$, $75025 = 5^2\cdot 3001$, and $39603 = 3\cdot 43\cdot 307$.

Checking the LCM we find, in case (a), $\operatorname{lcm}(F_{21}, L_{24}) = 2\cdot 13\cdot 421\cdot 47\cdot 1103 = 567451586$ and, in case (b), $\operatorname{lcm}(F_{25}, L_{22}) = 2971215075$. Case (b) does not give the desired LCM.

Hence, the answer is F_{21} and L_{24}.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Leonard A. G. Dresel, Steve Edwards, Daina Krigens, Carl Libis, H.-J. Seiffert, Indulis Strazdins, Andràs Szilàrd. and the proposer.

Matrix Lucas Sequence

B-863 ***Proposed by Stanley Rabinowitz, Westford, MA (Vol. 36, no. 5, November 1998)***

Let

$$A = \begin{pmatrix} -9 & 1 \\ -89 & 10 \end{pmatrix},\quad B = \begin{pmatrix} -10 & 1 \\ -109 & 11 \end{pmatrix},\quad C = \begin{pmatrix} -7 & 5 \\ -11 & 8 \end{pmatrix},\quad \text{and}\quad D = \begin{pmatrix} -4 & 19 \\ -1 & 5 \end{pmatrix},$$

and let n be a positive integer. Simplify $30A^n - 24B^n - 5C^n + D^n$.

Solution by Hans Kappus, Rodersdorf, Switzerland

It is easily checked that the matrix equation $X^2 = X + I$, where I is the identity matrix, is true for $X = A, B, C$, and D. Hence, the matrices $M_n = 30A^n - 24B^n - 5C^n + D^n$, $n = 0, 1, 2, \ldots$, satisfy the recurrence $M_{n+2} = M_{n+1} + M_n$. Furthermore, $M_0 = 2I$ and $M_1 = I$. Therefore,

$$M_n = L_n I = \begin{pmatrix} L_n & 0 \\ 0 & L_n \end{pmatrix},\quad n = 0, 1, 2, \ldots.$$

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Carl Libis, Maitland A. Rose, H.-J. Seiffert, Andràs Szilàrd, and the proposer.

Confound Those Congruences

B-864 ***Proposed by Stanley Rabinowitz, Westford, MA (Vol. 36, no. 5, November 1998)***

The sequence $\langle Q_n \rangle$ is defined by $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n > 1$ with initial conditions $Q_0 = 2$ and $Q_1 = 2$.

(a) Show that $Q_{7n} \equiv L_n \pmod{159}$ for all n.
(b) Find an integer $m > 1$ such that $Q_{11n} \equiv L_n \pmod{m}$ for all n.
(c) Find an integer a such that $Q_{an} \equiv L_n \pmod{31}$ for all n.
(d) Show that there is no integer a such that $Q_{an} \equiv L_n \pmod{7}$ for all n.
(e) Extra credit: Find an integer $m > 1$ such that $Q_{19n} \equiv L_n \pmod{m}$ for all n.

Solution by David M. Bloom, Brooklyn College of CUNY, Brooklyn , NY

For $k \geq 0$, we have

$$Q_{n+k} + (-1)^k Q_{n-k} = Q_n Q_k \tag{1}$$

by induction on k. (The cases $k = 0$ and $k = 1$ are easy.) Next, we show the following:

$$\text{If } a \text{ is odd and } m \mid (Q_a - 1), \text{ then } Q_{an} \equiv L_n \pmod{m} \text{ for all } n. \tag{2}$$

Indeed, (2) holds for $n = 0$ trivially and for $n = 1$ by hypothesis on m, and, if true for $n = j - 1$, then

$$\begin{aligned} Q_{a(j+1)} &= Q_{a(j-1)} + Q_{aj}Q_a && \text{[by (1)]} \\ &\equiv Q_{a(j-1)} + Q_{aj} \pmod{m} && \text{(since } Q_a = 1\text{)} \\ &\equiv L_{j-1} + L_j \pmod{m} && \text{(by the induction hypothesis)} \\ &\equiv L_{j+1} \pmod{m}, \end{aligned}$$

so that (2) holds for $n = j + 1$. Hence, (2) holds for all n by induction.

Part (a) of the problem now follows from (2) with $a = 7$ since $Q_7 - 1 = 477$ is divisible by 159.

Part (b) holds with $m = 13$ since 13 divides $Q_{11} - 1$.

Part (c) holds with $a = 17$ since 31 divides $Q_{17} - 1$.

Part (e) holds with $m = Q_{19} - 1 = 18738637$.

Finally, if $Q_{an} \equiv L_n \pmod{7}$ for all n, then, in particular, $Q_a \equiv L_1 = 1 \pmod{7}$. However, this is impossible since we have (mod 7)

$$(Q_0, Q_1, Q_2, \ldots) \equiv (2, 2, 6, 0, 6, 5, 2, 2, \ldots),$$

which clearly repeats with period 6 and never assumes the value 1. Thus, part (d) is proved.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Leonard A. G. Dresel, H.-J. Seiffert, Andràs Szilàrd, and the proposer.

Belated Acknowledgment: Brian Beasley was inadvertently omitted as a solver of Problems B-854, B-855, and B-857.

❖❖❖

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-553 ***Proposed by Paul Bruckman, Berkeley, CA***

The following Diophantine equation has the trivial solution $(A, B, C, D) = (A, A, A, 0)$:

$$A^3 + B^3 + C^3 - 3ABC = D^k, \text{ where } k \text{ is a positive integer.} \tag{1}$$

Find nontrivial solutions of (1), i.e., with all quantities positive integers.

H-554 ***Proposed by N. Gauthier, Royal Military College of Canada***

Let k, a, and b be positive integers, with a and b relatively prime to each other, and define

$$\begin{aligned} N_k &:= (1+(-1)^k - L_k)^{-1} \\ &= (2-L_k)^{-1}, \ k \text{ even}; \\ &= -L_k^{-1}, \ k \text{ odd}. \end{aligned}$$

a. Show that

$$\begin{aligned} \sum_{\substack{r=0 \\ br+as<ab}}^{a-1} \sum_{s=0}^{b-1} L_{q(br+as)} = {} & N_{qa}N_{qb}[2 + L_{q(a+b)} - L_{qa} - L_{qb} - L_{qab} + (-1)^{qa} L_{qa(b-1)} \\ & + (-1)^{qb} L_{qb(a-1)} + (-1)^{q(a+b)+1} L_{q(ab-a-b)}] \\ & + N_q[(-1)^q L_{q(ab-1)} - L_{qab}], \end{aligned}$$

where q is a positive integer.

b. Show that

$$\begin{aligned} \sum_{\substack{r=0 \\ br+as<ab}}^{a-1} \sum_{s=0}^{b-1} F_{q(br+as)} = {} & N_{qa}N_{qb}[(-1)^{q(a+b)+1} F_{q(ab-a-b)} + F_{qa} + F_{qb} \\ & -F_{qab} + (-1)^{qa} F_{qa(b-1)} + F_{qb(a-1)} - F_{q(a+b)}] \\ & + N_q[(-1)^q F_{q(ab-1)} - F_{qab}], \end{aligned}$$

where q is a positive integer.

H-555 ***Proposed by Paul S. Bruckman, Berkeley, CA***

Prove the following identity:

$$(x^n+y^n)(x+y)^n=-(-xy)^n+\sum_{k=0}^{[n/3]}(-1)^k C_{n,k}[xy(x+y)]^{2k}(x^2+xy+y^2)^{n-3k},\ n=1,2,\ldots, \tag{1}$$

where

$$C_{n,k}=\binom{n-2k}{k}\cdot n/(n-2k).$$

Using (1), prove the following:

(a) $$5^{n/2}L_n=-1+\sum_{k=0}^{[n/3]}(-1)^k C_{n,k}5^k 4^{n-3k},\ n=2,4,6,\ldots; \tag{2}$$

(b) $$5^{(n+1)/2}F_n=1+\sum_{k=0}^{[n/3]}(-1)^k C_{n,k}5^k 4^{n-3k},\ n=1,3,5,\ldots; \tag{3}$$

(c) $$L_n=-1+\sum_{k=0}^{[n/3]}(-1)^k C_{n,k}2^{n-3k},\ n=1,2,3,\ldots. \tag{4}$$

SOLUTIONS

An Odd Problem

H-536 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 36, no. 1, February 1998)

Given an odd prime p, integers n and r with $n\ge 1$, let $m=2\left[\frac{1}{2}n\right]-1$,

$$S_{n,r,p}=\sum_{k=1}^{p-1}F_m^k\cdot\frac{F_{nk+r}}{k},\quad T_{n,r,p}=\sum_{k=1}^{p-1}F_m^k\cdot\frac{L_{nk+r}}{k}.$$

Prove the following congruences:

(a) $$S_{n,r,p}\equiv\frac{F_n^pF_{mp+r}-F_m^pF_{np+r}+F_r}{p}\pmod p;$$

(b) $$T_{n,r,p}\equiv\frac{F_n^pL_{mp+r}-F_m^pL_{np+r}+L_r}{p}\pmod p.$$

Solution by the proposer

Proof: We begin with the following identity:

$$F_n\alpha^m=F_m\alpha^n-1. \tag{$*$}$$

We may verify $(*)$ by dealing with the cases n even or n odd separately, then expanding the Binet formulas. A similar identity holds with the α's replaced by β's.

Raising each side of $(*)$ to the power p, we obtain:

$$F_n^p\alpha^{mp}=F_m^p\alpha^{np}-1+\sum_{k=1}^{p-1}\binom{p}{k}(-1)^{p-k}F_m^k\alpha^{nk}=F_m^p\alpha^{np}-1+\sum_{k=1}^{p-1}\binom{p-1}{k-1}\frac{p}{k}(-1)^{k-1}F_m^k\alpha^{nk}.$$

For $1 \le k \le p-1$,

$$\binom{p-1}{k-1} \equiv \binom{-1}{k-1} \pmod p$$
$$= (-1)^{k-1}.$$

Then, multiplying throughout by α^r, we obtain:

$$F_n^p \alpha^{mp+r} - F_m^p \alpha^{np+r} + \alpha^r \equiv p \sum_{k=1}^{p-1} F_m^k \frac{\alpha^{nk+r}}{k} \pmod{p^2}.$$

Note that the quantities "$1/k$" are the uniquely determined inverses k^{-1} (mod p^2); upon division throughout by p, these become the uniquely determined inverses k^{-1} (mod p). A similar congruence holds with the α's replaced by β's. Subtracting these two congruences and dividing throughout by $p\sqrt{5}$ yields the result in (a). Adding these two congruences and dividing throughout by p yields (b).

Note: Using these results, it may be shown that a necessary and sufficient condition for $Z(p^2) = Z(p)$ is that

$$S_{1,1,p} = \sum_{k=1}^{p-1} \frac{F_{k+1}}{k} \equiv 0 \pmod p.$$

Also solved by H.-J. Seiffert

A Recurrent Theme

H-537 ***Proposed by Stanley Rabinowitz, Westford, MA (Vol. 36, no. 1, February 1998)***

Let $\langle w_n \rangle$ be any sequence satisfying the recurrence

$$w_{n+2} = Pw_{n+1} - Qw_n.$$

Let $e = w_0 w_2 - w_1^2$ and assume $e \ne 0$ and $Q \ne 0$.

Computer experiments suggest the following formula, where k is an integer larger than 1:

$$w_{kn} = \frac{1}{e^{k-1}} \sum_{i=0}^{k} c_{k-i} \binom{k}{i} (-1)^i w_n^i w_{n+1}^{k-i},$$

where

$$c_i = \sum_{j=0}^{k-2} \binom{k-2}{j} (-Qw_0)^j w_1^{k-2-j} w_{i-j}.$$

Prove or disprove this conjecture.

Solution by Paul S. Bruckman, Berkeley, CA

We may express the w_n's in terms of the "fundamental" sequence $\langle \phi_n \rangle$, defined as follows:

$$\phi_n = (u^n - v^n)/(u-v), \tag{1}$$

where

$$u = \tfrac{1}{2}(P+\theta), \quad v = \tfrac{1}{2}(P-\theta), \quad \theta = (P^2 - 4Q)^{\frac{1}{2}}. \tag{2}$$

Note that $u+v=P$, $u-v=\theta$, and $uv=Q$. Also note that the ϕ_n's satisfy the same recurrence relation as the w_n,s, but have the initial values:

$$\phi_0=0, \quad \phi_1=1. \tag{3}$$

Also, $\phi_{-1}=-1/Q$, $\phi_2=P$. The formula for w_n is then as follows:

$$w_n=w_1\phi_n-Qw_0\phi_{n-1}. \tag{4}$$

We proceed to obtain closed form expressions for the indicated sums. First, we obtain a closed formula for the c_i's, substituting the expressions in (4):

$$\begin{aligned} c_i &= \theta^{-1}(uw_1-Qw_0)u^{i-1}\sum_{j=0}^{k-2} {}_{k-2}Cj(w_1)^{k-2-j}(-Qw_0/u)^j \\ &\quad - \theta^{-1}(vw_1-Qw_0)v^{i-1}\sum_{j=0}^{k-2} {}_{k-2}Cj(w_1)^{k-2-j}(-Qw_0/v)^j \\ &= \theta^{-1}(uw_1-Qw_0)u^{i-1}(w_1-vw_0)^{k-2}-\theta^{-1}(vw_1-Qw_0)v^{i-1}(w_1-uw_0)^{k-2} \end{aligned}$$

or

$$ci=\theta^{-1}u^i(w_1-vw_0)^{k-1}-\theta^{-1}v^i(w_1-uw_0)^{k-1}. \tag{5}$$

Next, let

$$S_{n,k}=\sum_{i=0}^{k} {}_kC_i(w_n)^{k-i}(-w_{n+1})^i c_{k-i}.$$

Note that this last expression differs from the sum given in the statement of the problem (with the roles of w_n and w_{n+1} interchanged). Substituting the expression in (5) yields:

$$S_{n,k}=\theta^{-1}\sum_{i=0}^{k} {}_kC_i(w_n)^{k-i}(-w_{n+1})^i\{u^{k-i}(w_1-vw_0)^{k-1}-v^{k-i}(w_1-uw_0)^{k-1}\}$$

or

$$S_{n,k}=\theta^{-1}(w_1-vw_0)^{k-1}(uw_n-w_{n+1})^k-\theta^{-1}(w_1-uw_0)^{k-1}(vw_n-w_{n+1})^k. \tag{6}$$

The problem (as corrected) asks us to verify or refute the relation

$$S_{n,k}=e^{k-1}w_{kn}. \tag{7}$$

Next, we employ the following relations [easily verified from the preceding relations, including (4)]:

$$uw_n-w_{n+1}=(uw_0-w_1)v^n, \tag{8}$$

$$vw_n-w_{n+1}=(vw_0-w_1)u^n. \tag{9}$$

It is also easily verified that

$$(uw_0-w_1)(vw_0-w_1)=-e. \tag{10}$$

Putting these facts together, we obtain (after simplification):

$$\begin{aligned} S_{n,k} &= \theta^{-1}(w_1-vw_0)^{k-1}(uw_n-w_{n+1})^k-\theta^{-1}(w_1-uw_0)^{k-1}(vw_n-w_{n+1})^k \\ &= e^{k-1}(w_1\phi_{kn}-Qw_0\phi_{kn-1})=e^{k-1}w_{kn}. \quad \text{Q.E.D.} \end{aligned}$$

Thus, there is a typographical error in the statement of the problem; the result is true only if the quantities w_n and w_{n+1} occurring in the first sum given are interchanged.

Also solved by H.-J. Seiffert

An Elementary Result

H-538 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 36, no. 1, February 1998)

Define the sequence of integers $(B_k)_{k\geq 0}$ by the generating function:

$$(1-x)^{-1}(1+x)^{-\frac{1}{2}} = \sum_{k\geq 0} B_k \frac{\left(\frac{1}{2}x\right)^k}{k!}, \quad |x|<1 \quad (\text{see [1]}).$$

Show that

$$\sum_{k\geq 0} B_k^2 \cdot \frac{1}{(2k+2)!} = \frac{\pi^2}{8} - \frac{1}{4}\log^2 u, \quad \text{where } u = 1+\sqrt{2}.$$

Reference

1. P. S. Bruckman. "An Interesting Sequence of Numbers Derived from Various Generating Functions." *The Fibonacci Quarterly* **10.2** (1972):169-81.

Solution by the proposer

In [1], it is shown that

$$\tan^{-1} x \cdot (1-x^2)^{-1/2} = \sum_{k\geq 0} B_k^2 \frac{x^{2k+1}}{(2k+1)!}.$$

The following result is Elementary Problem E3140, Part (b)(ii), proposed by Khristo Boyadzhiev in *The American Math Monthly* **93.3** (1986):216:

$$\int_0^1 \tan^{-1} x \cdot (1-x^2)^{-1/2} dx = \pi^2/8 - \tfrac{1}{4}\log^2 u.$$

(The notation is modified to conform to our own.) The result follows immediately, by integrating the series given in [1] term by term and evaluating it at the integral's limits.

Beta Version

H-539 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 36, no. 2, May 1998)

Let

$$H_m(p) = \sum_{j=1}^{m} B\left(\frac{j}{2}, p\right), m \in N, \ p>0,$$

where

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

denotes the Beta function. Show that for all positive reals p and all positive integers n,

$$\sum_{k=1}^{n} (-1)^{k-1}\binom{n}{k} H_{2k}(p) = 4^{n+p-1}B(n+p, n+p-1) + \frac{1}{n+p-1}. \tag{1}$$

From (1), deduce the identities

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{4^k}\binom{n}{k}\binom{2k}{k}=\frac{2}{4^n}\binom{2n-2}{n-1} \tag{2}$$

and

$$\sum_{k=1}^{n}(-1)^{k-1}4^k\binom{n}{k}\Big/\binom{2k}{k}=\frac{2n}{2n-1}. \tag{3}$$

Solution by the proposer

Since

$$B\left(\frac{j}{2},p\right)=\int_0^1 t^{(j-2)/2}(1-t)^{p-1}dt=2\int_0^1 u^{j-1}(1-u^2)^{p-1}du,$$

it easily follows that

$$H_m(p)=2\int_0^1\frac{1-u^m}{1-u}(1-u^2)^{p-1}du,\ m\in N.$$

If $S_n(p)$ denotes the left side of the stated identity (1), then, by the Binomial theorem,

$$S_n(p)=2\int_0^1\left(\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(1-u^{2k})\right)\frac{(1-u^2)^{p-1}}{1-u}du$$

$$=2\int_0^1\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}u^{2k}\right)\frac{(1-u^2)^{p-1}}{1-u}du=2\int_0^1(1-u^2)^n\frac{(1-u^2)^{p-1}}{1-u}du$$

or

$$\frac{1}{2}S_n(p)=\int_0^1(1-u)^{n+p-2}(1+u)^{n+p-1}du.$$

Substituting $u=1-2v$ yields

$$\frac{1}{2}S_n(p)=4^{n+p-1}\int_0^{1/2}v^{n+p-2}(1-v)^{n+p-1}dv. \tag{4}$$

Integrating (4) by parts, we find

$$\frac{1}{2}S_n(p)=\frac{1}{n+p-1}+4^{n+p-1}\int_0^{1/2}v^{n+p-1}(1-v)^{n+p-2}dv.$$

Replacing v by $1-v$ in the latter integral, we get

$$\frac{1}{2}S_n(p)=\frac{1}{n+p-1}+4^{n+p-1}\int_{1/2}^1 v^{n+p-2}(1-v)^{n+p-1}dv. \tag{5}$$

Now, the desired identity (1) follows by adding (4) and (5).

Interestingly, (2) and (3) will follow from (1), simultaneously, when taking $p=1/2$. Since, as is well known,

$$B\left(r+\frac{1}{2},\frac{1}{2}\right)=\frac{\pi}{4^r}\binom{2r}{r},\ r\in N_0,\ \text{ and }\ B\left(r,\frac{1}{2}\right)=\frac{4^r}{r}\Big/\binom{2r}{r},\ r\in N,$$

we have

$$H_{2k}\left(\frac{1}{2}\right)=\sum_{r=0}^{k-1}B\left(r+\frac{1}{2},\frac{1}{2}\right)+\sum_{r=1}^{k}B\left(r,\frac{1}{2}\right)=\pi\sum_{r=0}^{k-1}\frac{1}{4^r}\binom{2r}{r}+\sum_{r=1}^{k}\frac{4^r}{r}\Big/\binom{2r}{r}.$$

Each of the equations

$$\sum_{r=0}^{k-1}\frac{1}{4^r}\binom{2r}{r}=\frac{2k}{4^k}\binom{2k}{k},\ k\in N,\quad\text{and}\quad \sum_{r=1}^{k}\frac{4^r}{r}\Big/\binom{2r}{r}=2\left(4^k\Big/\binom{2k}{k}-1\right),\ k\in N,$$

can be proved by a simple induction argument. Hence,

$$H_{2k}\left(\frac{1}{2}\right)=\frac{2k}{4^k}\binom{2k}{k}\pi+2\left(4^k\Big/\binom{2k}{k}-1\right),\ k\in N.$$

Using

$$B\left(n+\frac{1}{2},n-\frac{1}{2}\right)=\frac{2\pi}{4^{2n-1}}\binom{2n-2}{n-1}$$

and observing that π is an irrational number, from (1) with $p=1/2$, we find the two equations

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{2k}{4^k}\binom{n}{k}\binom{2k}{k}=\frac{1}{4^{n-1}}\binom{2n-2}{n-1}\tag{6}$$

and

$$2\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\left(4^k\Big/\binom{2k}{k}-1\right)=\frac{2}{2n-1}.\tag{7}$$

Obviously, (2) is equivalent to (6). Dividing (7) by 2 and adding

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}=1$$

to both sides of the resulting equation gives (3).

With $p=1$, identity (1), after dividing by 2, gives

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{2k}=2^{2n-1}\frac{n!(n-1)!}{(2n)!}+\frac{1}{2n},$$

where $H_m=H_m(1)/2=\sum_{j=1}^{m}1/j$ is the m^{th} harmonic number. This equation (including a generalization in another direction) was obtained in [1].

Reference

1. L. C. Hsu & H. Kappus. Problem B-818. *The Fibonacci Quarterly* **35.3** (1997):280-81.

Also solved by P. Bruckman and partially by A. Stam.

❖❖❖

SUSTAINING MEMBERS

*H.L. Alder
G.L. Alexanderson
P. G. Anderson
S. Ando
R. Andre-Jeannin
*J. Arkin
D.C. Arney
C. Ashbacher
J.G. Bergart
G. Bergum
*M. Bicknell-Johnson
M.W. Bowron
P.S. Bruckman
M.F. Bryn
G.D. Chakerian
C. Chouteau
C.K. Cook
M.J. DeBruin
M.J. DeLeon
J. De Kerf
E. Deutsch
L.A.G. Dresel
U. Dudley
L.G. Ericksen, Jr.
D.R. Farmer
D.C. Fielder
P. Filipponi
C.T. Flynn
E. Frost
Fondazione Ugo Bordoni
*H.W. Gould
P. Hagis, Jr.
H. Harborth
J. Herrera
*A.P. Hillman
*A.F. Horadam
Y. Horibe
F.T. Howard
R.J. Howell
S. Kasparian
R.E. Kennedy
C.H. Kimberling
Y.H.H. Kwong
J.C. Lagarias
J. Lahr
B. Landman
*C.T. Long
G. Lord
W.L. McDaniel
F.U. Mendizabal
J.L. Miller
M.G. Monzingo
J.F. Morrison
H. Niederhausen
S.A. Obaid
J. Pla
A. Prince
B.M. Romanic
S. Sato
J.A. Schumaker
H.J Seiffert
A.G. Shannon
L.W. Shapiro
L. Somer
P. Spears
W.R. Spickerman
P.K. Stockmeyer
J. Suck
M.N.S. Swamy
*D. Thoro
J.C. Turner
C. Vanden Eynden
T.P. Vaughan
J.N. Vitale
M.J. Wallace
J.E. Walton
W.A. Webb
V. Weber
R.E. Whitney
B.E. Williams

*Charter Members

INSTITUTIONAL MEMBERS

BIBLIOTECA DEL SEMINARIO MATHEMATICO
Padova, Italy

CALIFORNIA STATE UNIVERSITY SACRAMENTO
Sacramento, California

CHALMERS UNIVERSITY OF TECHNOLOGY AND UNIVERSITY OF GOTEBORG
Goteborg, Sweeden

ETH-BIBLIOTHEK
Zurich, Switzerland

GONZAGA UNIVERSITY
Spokane, Washington

HOWELL ENGINEERING COMPANY
Bryn Mawr, Califoria

KLEPCO, INC.
Sparks, Nevada

KOBENHAVNS UNIVERSITY
Matematisk Institut
Copenhagen, Denmark

MISSOURI SOUTHERN STATE COLLEGE
Joplin, Missouri

SAN JOSE STATE UNIVERSITY
San Jose, California

SANTA CLARA UNIVERSITY
Santa Clara, California

UNIVERSITY OF NEW ENGLAND
Armidale, N.S.W. Australia

WASHINGTON STATE UNIVERSITY
Pullman, Washington

YESHIVA UNIVERSITY
New York, New York

JOVE STATISTICAL TYPING SERVICE
2088 Orestes Way
Campbell, California 95008

BOOKS AVAILABLE
THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau, Fibonacci Association (FA), 1965. $18.00

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972. $23.00

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972. $32.00

Fibonacci's Problem Book, Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974. $19.00

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969. $6.00

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971. $6.00

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972. $30.00

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973. $39.00

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965. $14.00

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965. $14.00

A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980. $38.00

Applications of Fibonacci Numbers, Volumes 1-7. Edited by G.E. Bergum, A.F. Horadam and A.N. Philippou. Contact Kluewer Academic Publisher for price.

Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger. FA, 1993. $37.00

Fibonacci Entry Points and Periods for Primes 100,003 through 415,993 by Daniel C. Fielder and Paul S. Bruckman. $20.00

Handling charges will be $4.00 for the first book and $1.00 for each additional book in the United States and Canada. For foreign orders, the handling charge will be $8.00 for the first book and $3.00 for each additional book.

Please write to the Fibonacci Association, P.O. Box 320, Aurora, S.D. 57002-0320, U.S.A., for more information.