# The Fibonacci Quarterly

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# The Fibonacci Quarterly

***Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980) and Br. Alfred Brousseau (1907-1988)***

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES*

# THE FACTORIZATION OF $x^5 \pm p^2x - k$ AND FIBONACCI NUMBERS

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## 1. AIM OF THE PAPER

Here we extend a result established by Rabinowitz [6] by considering the fifth-degree polynomials of the so-called Bring-Jerrard form $q(x,h,k) := x^5 \pm h^2x - k$, where $h$ is either 1 or a prime, and $k$ is an integer. More precisely, the principal aim of the paper is to find necessary and sufficient conditions on $k$ for $q(x,h,k)$ to factor over $\mathbb{Z}$.

Since $q(x,h,k)$ factors trivially as

$$x^5 \pm h^2x - (m^5 \pm h^2m) = (x-m)(x^4 + mx^3 + m^2x^2 + m^3x + m^4 \pm h^2) \tag{1.1}$$

if $k = m^5 \pm h^2m$ $(m \in \mathbb{Z})$, we are concerned with the factorizations of $q(x,h,k)$ that have the form

$$q(x,h,k) = (x^2 + ax + b)(x^3 - ax^2 + cx + d) \quad (a,b,c,d \in \mathbb{Z}). \tag{1.2}$$

The case $h = 1$ has been solved brilliantly by Rabinowitz in [6] (see also [3] and [9]), where he shows that $q(x,1,k)$ has the factorization (1.2) iff $k$ assumes some special values depending on square Fibonacci numbers. In the more general situation ($h$ a prime), certain properties of the Fibonacci (and generalized Fibonacci) numbers play a crucial role as well.

After observing that changing the sign of $k$ implies nothing but the sign change of $a$ and $d$ in (1.2), we can assume that $k \geq 1$ without loss of generality. Consequently, we shall confine ourselves to studying the factorization (1.2) of the polynomials

$$\begin{cases} r(x,p,k) = x^5 - p^2x - k, \\ s(x,p,k) = x^5 + p^2x - k, \end{cases} \quad (k \geq 1, p \text{ a prime}). \tag{1.3}$$

As will be shown in the sequel, it is necessary to distinguish three cases depending on whether the prime $p$ is either 5, or has the form $5j \pm 2$, or the form $5j \pm 1$. Our approach to this problem will follow [3] and Rabinowitz' argumentation but, to render the paper self-contained, the proofs will be given in full detail. For the sake of completeness, the most significant factorizations will be explicitly shown. A brief discussion on the factorization of $r(x,p,k)$ for certain special primes $p$ concludes our study.

It must be noted that some questions remain unsettled that are related to well-known open problems in number theory. Namely, they concern the existence of infinitely many prime Fibonacci numbers, the occurrence of perfect squares in terms of Fibonacci-like sequences, and the solution of a special Pell equation.

A preliminary version of this paper has been presented by the first author at the XIV Österreichischer Mathematikerkongress [4].

## 2. PRELIMINARY RESULTS

Given the factorization (1.2), by equating the coefficients of like powers of $x$ we obtain the system

$$\begin{cases} b+c-a^2=0, \\ a(b-c)-d=0, \\ ad+bc=\pm p^2, \\ bd=-k, \end{cases} \tag{2.1}$$

whence, by using the first two equations to eliminate $a$ and $d$, we obtain the two equations

$$\begin{cases} b^2+bc-c^2=\pm p^2, \\ b^2(b-c)^2(b+c)=k^2. \end{cases} \tag{2.2}$$

Equations (2.2) show that the couple $(b, c)$ must be chosen among the couples that represent $\pm p^2$ by means of the quadratic form $Q(b,c)=b^2+bc-c^2$, subject to the condition that $b+c$ is a perfect square. Hence, finding the solutions of the quadratic equation $Q(b,c)=\pm p^2$ is clearly a necessary step to solve our problem. From Gauss's general theory of the quadratic forms, it is known (e.g., see [5]) that there is a finite number of classes of solutions. Each class consists of an infinitude of solutions which are referred to as *associated solutions*, and is characterized by a single solution called the *fundamental solution*. The classification of the solutions of $Q(b, c)=M$ is given by Dodd in [2]. It depends on the peculiar properties of $\mathbb{Z}(\alpha)$, the ring of integers in the quadratic field $\mathbb{Q}(\alpha)$ which is the extension of the rational field $\mathbb{Q}$ by means of the golden section $\alpha=(1+\sqrt{5})/2$. Recall that $\mathbb{Z}(\alpha)$ is a unique factorization domain.

Every solution $(x_n, y_n)$ of $Q(b, c)=M$, associated to a given fundamental solution $(x_0, y_0)$, is obtained as

$$x_n+\alpha y_n=\alpha^{2n}(x_0+\alpha y_0). \tag{2.3}$$

Equivalently, we can say that both the sequences $\{x_n\}$ and $\{y_n\}$ are generalized Fibonacci sequences obeying the second-order recurrence

$$G_n=G_{n-1}+G_{n-2}, \tag{2.4}$$

with suitable initial conditions $G_0$ and $G_1$. The number of classes of solutions is obtained as a consequence of Theorem 3.12 and Corollary 3.13 of [2] that we quote as a single theorem for ease of reference.

***Theorem 1 (Dodd):*** The quadratic equation $x^2+xy-y^2=M$ is solvable in $\mathbb{Z}$ iff

$$M=\pm 5^t p_1^{2f_1}\cdots p_s^{2f_s} q_1^{g_1}\cdots q_r^{g_r} \quad (t,\, f_i,\, g_i \text{ nonnegative integers}),$$

where $p_i=5j\pm 2$ $(1\le i\le s)$ and $q_i=5j\pm 1$ $(1\le i\le r)$ are primes. The number of fundamental solutions is given by the product $(g_1+1)(g_2+1)\cdots(g_r+1)$.

Consequently, for our special case $M=\pm p^2$ [see (2.2)], we can summarize the above results as follows.

(i) If $p=5$ or $5j\pm 2$, then there is a unique fundamental solution

$$(x_0, y_0)=\begin{cases} (p, 0) & \text{if } M=p^2, \\ (0, p) & \text{if } M=-p^2. \end{cases} \tag{2.5}$$

(ii) If $p = 5j \pm 1$, then there are three fundamental solutions, one of which is given by (2.5). The additional solutions $(x_0^{(1)}, y_0^{(1)})$ and $(x_0^{(2)}, y_0^{(2)})$ can be derived from a solution $(u_0, v_0)$ of the Pell equation $u^2 - 5v^2 = p$. Namely, we have

$$\begin{cases} x_0^{(1)} = u_0^2 - 2u_0v_0 + 5v_0^2, \\ y_0^{(1)} = 4u_0v_0; \end{cases} \quad \begin{cases} x_0^{(2)} = u_0^2 + 2u_0v_0 + 5v_0^2, \\ y_0^{(2)} = -4u_0v_0, \end{cases} \text{if } M = p^2; \tag{2.6}$$

$$\begin{cases} x_0^{(1)} = 4u_0v_0, \\ y_0^{(1)} = u_0^2 + 2u_0v_0 + 5v_0^2, \end{cases} \quad \begin{cases} x_0^{(2)} = -4u_0v_0, \\ y_0^{(2)} = u_0^2 - 2u_0v_0 + 5v_0^2, \end{cases} \text{if } M = -p^2. \tag{2.7}$$

Apparently, there is no direct technique for solving the Pell equation $u^2 - 5v^2 = p$; the best known method (see [5], p. 206) is to check every $u$ lying within the interval $[\sqrt{p+5}, \sqrt{5p}]$.

## 3. THE FACTORIZATION OF $r(x, p, k)$ WHEN $p = 5j \pm 2$

We state the following theorem.

***Theorem 2:*** If $p = 5j \pm 2$, then the polynomial $r(x, p, k)$ given by (1.3) factors as (1.2) iff

$$p = \begin{cases} 2, \text{ and } k = 96 \text{ or } 11424, \\ 3, \text{ and } k = 27 \text{ or } 2808. \end{cases} \tag{3.1}$$

***Proof:*** The system (2.2) becomes

$$\begin{cases} bc + b^2 - c^2 = -p^2, \\ b^2(b-c)^2(b+c) = k^2. \end{cases} \tag{3.2}$$

Since the couple $(0, p)$ is the fundamental solution [see (2.5)] of $Q(b,c) = -p^2$, from (2.3) we know that all the solutions are given by

$$(b, c) = \pm(pF_{2n}, pF_{2n+1}) \quad (n \in \mathbb{Z}), \tag{3.3}$$

where $F_n$ is the $n^{\text{th}}$ Fibonacci number. We recall that $F_{-n} = (-1)^{n+1}F_n$.

From (3.3) and the second equation of (3.2), we see that

$$k^2 = p^4F_{2n}^2F_{2n-1}^2(\pm pF_{2n+2}), \tag{3.4}$$

where the minus sign in the last factor must occur iff $n < -1$. From (3.4), it is plain that, for $k$ to be an integer, $pF_{2n+2}$ must be a perfect square. In turn, this implies that we must have

$$F_{2n+2} = py^2. \tag{3.5}$$

For $p = 2$, Theorem 4 of [1] tells us that the only nonzero solution to (3.5) is $F_6 = 2 \cdot 2^2$ [i.e., $y = 2$ and $n = 2$ in (3.5)]. Consequently, letting $n = p = 2$ in (3.4) yields

$$k = \sqrt{16F_4^2F_3^2(2F_6)} = 96. \tag{3.6}$$

Further, letting $n = -4$ in (3.4) (so that the last factor therein becomes $-2F_{-6}$) yields

$$k = \sqrt{16F_{-8}^2F_{-9}^2(-2F_{-6})} = 11424. \tag{3.7}$$

For $p \geq 3$, Theorem 1 of [8] tells us that the unique solution to (3.5) is $F_4 = 3 \cdot 1^2$ [i.e., $p = 3$, $y = 1$, and $n = 1$ in (3.5)]. Hence, letting $p = 3$ and $n = 1$ in (3.4) yields

$$k = \sqrt{81 F_2^2 F_1^2 (3F_4)} = 27. \tag{3.8}$$

Further, letting $n = -3$ in (3.4) (so that the last factor therein becomes $-3F_{-4}$) yields

$$k = \sqrt{81 F_{-6}^2 F_{-7}^2 (-3F_{-4})} = 2808. \text{ Q.E.D.} \tag{3.9}$$

By using (1.2), (2.1), and (3.3), the factorizations of $r(x, 2, k)$ and $r(x, 3, k)$ for the above values of $k$ are readily obtained. Namely, we get

$$x^5 - 4x - 96 = (x^2 + 4x + 6)(x^3 - 4x^2 + 10x - 16), \tag{3.10}$$

$$x^5 - 4x - 11424 = (x^2 - 4x + 42)(x^3 + 4x^2 - 26x - 272), \tag{3.11}$$

$$x^5 - 9x - 27 = (x^2 + 3x + 3)(x^3 - 3x^2 + 6x - 9), \tag{3.12}$$

$$x^5 - 9x - 2808 = (x^2 - 3x + 24)(x^3 + 3x^2 - 15x - 117). \tag{3.13}$$

## 4. THE FACTORIZATION OF $s(x, p, k)$ WHEN $p = 5j \pm 2$

We state the following theorem.

***Theorem 3:*** If $p = 5j \pm 2$, then the polynomial $s(x, p, k)$ given by (1.3) factors as (1.2) iff

$$p = F_{2n+1} \text{ is a prime Fibonacci number, and } k = \begin{cases} p^3 F_{2n-1} F_{2n-2}, \\ p^3 F_{2n+3} F_{2n+4}. \end{cases} \tag{4.1}$$

***Remark 1:*** For $F_{2n+1}$ to be a prime, $2n+1$ must necessarily be a prime. The question of whether there exist infinitely many prime Fibonacci numbers is still unsolved ([7], p. 226).

***Proof:*** The system (2.2) becomes

$$\begin{cases} bc + b^2 - c^2 = p^2, \\ b^2(b-c)^2(b+c) = k^2. \end{cases} \tag{4.2}$$

Since the couple $(p, 0)$ is the fundamental solution [see (2.5)] of $Q(b, c) = p^2$, from (2.3) we know that all the solutions are given by

$$(b, c) = \pm(pF_{2n-1}, pF_{2n}) \quad (n \in \mathbb{Z}). \tag{4.3}$$

From (4.3) and the second equation of (4.2), we see that

$$k^2 = p^4 F_{2n-1}^2 F_{2n-2}^2 (pF_{2n+1}), \tag{4.4}$$

where one can observe the absence of the minus sign in the last factor which is due to the fact that the odd-subscripted Fibonacci numbers are always positive. From (4.4), it is plain that, for $k$ to be an integer, $pF_{2n+1}$ must be a perfect square. In turn, this implies that we must have

$$F_{2n+1} = py^2. \tag{4.5}$$

Theorem 2 of [8] ensures us that, if $p = 5j \pm 2$, then all the solutions to (4.5) are given (trivially) by

$$F_{2n+1} = p \cdot 1^2. \tag{4.6}$$

From (4.6), expression (4.4) becomes

$$k^2 = p^6 F_{2n-1}^2 F_{2n-2}^2, \tag{4.7}$$

whence one immediately gets the first equality of (4.1). Since $F_{-(2n+1)} = F_{2n+1}$, we can replace $n$ by $-(n+1)$ in (4.4), thus getting [see (4.6)]

$$k^2 = p^4 F_{-2n-3}^2 F_{-2n-4}^2 (pF_{2n+1}) = p^6 F_{2n+3}^2 F_{2n+4}^2, \tag{4.8}$$

whence the second equality of (4.1) is readily obtained. Q.E.D.

In the first and second cases of (4.1), the factorizations (1.2) of $s(x, p, k)$ have the sets of coefficients

$$\begin{cases} a = p = F_{2n+1}, \\ b = pF_{2n-1}, \\ c = pF_{2n}, \\ d = -p^2 F_{2n-2}, \end{cases} \quad \text{and} \quad \begin{cases} a = -p, \\ b = pF_{2n+3}, \\ c = -pF_{2n+2}, \\ d = -p^2 F_{2n+4}, \end{cases} \tag{4.9}$$

respectively. As a numerical example, the factorizations of $s(x, 13, k)$ [$n = 3$ in (4.1)] are shown below. Namely, we have [cf. (4.9)]:

$$x^5 + 169x - 32955 = (x^2 + 13x + 65)(x^3 - 13x^2 + 104x - 507), \tag{4.10}$$

$$x^5 + 169x - 4108390 = (x^2 - 13x + 442)(x^3 + 13x^2 - 273x - 9295). \tag{4.11}$$

***Remark 2:*** The case $p = 5$ is exceptional because 5 occurs in the definition of the quadratic extension ring $\mathbb{Z}(\alpha)$, but according to Theorem 1, it can be treated as the primes of the form $5j \pm 2$. Equation $Q(b, c) = 5^2$ has only one fundamental solution, and, according to the above discussion, we get the only possible factorizations:

$$x^5 + 25x - 250 = (x^2 + 5x + 10)(x^3 - 5x^2 + 15x - 25), \tag{4.12}$$

$$x^5 + 25x - 34125 = (x^2 - 5x + 65)(x^3 + 5x^2 - 40x - 525). \tag{4.13}$$

## 5. THE FACTORIZATION OF $r(x, p, k)$ WHEN $p = 5j \pm 1$

From Theorem 1, we know that, if $p = 5j \pm 1$, then the equation $Q(b, c) = -p^2$ has the three fundamental solutions

$$(b, c) = \begin{cases} \pm(pF_{2n}, pF_{2n+1}), \\ \pm(A_{2n}, A_{2n+1}), \\ \pm(B_{2n}, B_{2n+1}), \end{cases} \quad n \in \mathbb{Z}, \tag{5.1}$$

where the generalized Fibonacci sequences $\{A_n\}$ and $\{B_n\}$ obey the recurrence (2.4) with initial conditions [$A_0 = x_0^{(1)}$; $A_1 = y_0^{(1)}$] and [$B_0 = x_0^{(2)}$; $B_1 = y_0^{(2)}$] that can be obtained from (2.7).

We now state the following theorem.

***Theorem 4:*** If $p = 5j \pm 1$, then the polynomial $r(x, p, k)$ given by (1.3) factors as (1.2) iff $A_{2n}$ and/or $B_{2n}$ are perfect squares for some $n$.

***Proof:*** On the basis of the previously used arguments [see (3.2)-(3.4)], from (5.1) it is clear that we must have

$$k^2 = \begin{cases} p^4F_{2n}^2F_{2n-1}^2(\pm pF_{2n+2}), \\ A_{2n}^2A_{2n-1}^2(\pm A_{2n+2}), \qquad n \in \mathbb{Z}. \\ B_{2n}^2B_{2n-1}^2(\pm B_{2n+2}), \end{cases} \tag{5.2}$$

The equation $\pm F_{2n+2} = py^2$ [cf. (3.5)] has no solutions by virtue of Theorem 1 of [8]. Therefore, the only possibilities for $k$ to be an integer are that $A_{2n+2}$ and/or $B_{2n+2}$ are perfect squares for some $n$. Q.E.D.

As a numerical example, let us find values of $k$ for which $r(x, 11, k)$ factors as (1.2). If $p = 11$, then $(u_0, v_0) = (4, 1)$ is a solution of the Pell equation at point (ii) of Section 2, so that expressions (2.7) give the initial conditions $[A_0 = 16; A_1 = 29]$ and $[B_0 = -16; B_1 = 13]$. From (5.2) and the argument in the proof of Theorem 4 (namely, Theorem 1 of [8]), we have

$$k^2 = \begin{cases} A_{2n}^2A_{2n-1}^2(\pm A_{2n+2}) & \text{(the minus sign when } n \le -3), \\ B_{2n}^2B_{2n-1}^2(\pm B_{2n+2}) & \text{(the minus sign when } n \le 0). \end{cases} \tag{5.3}$$

For $n = -1$, we have that $A_{2n+2} = A_0 = 16$ is a perfect square. Letting $n = -1$ in the first equation of (5.3) yields

$$k = A_{-2}A_{-3}\sqrt{A_0} = 3 \cdot 10 \cdot 4 = 120. \tag{5.4}$$

For the same value of $n$, we see that $B_{2n+2} = B_0 = -16$. Letting $n = -1$ in the second equation of (5.3) and choosing the proper signs yields

$$k = -B_{-2}B_{-3}\sqrt{-B_0} = 45 \cdot 74 \cdot 4 = 13320. \tag{5.5}$$

***Remark 3:*** The occurrence of further even-subscripted terms of $\{A_n\}$ and/or $\{B_n\}$ that are perfect squares would allow us to find further values of $k$ for which $r(x, 11, k)$ factors as (1.2).

The factorizations of $r(x, 11, k)$ for the values of $k$ given by (5.4) and (5.5) are

$$x^5 - 121x - 120 = (x^2 + 4x + 3)(x^3 - 4x^2 + 13x - 40) \tag{5.6}$$

and

$$x^5 - 121x - 13320 = (x^2 - 4x + 45)(x^3 + 4x^2 - 29x - 296), \tag{5.7}$$

respectively.

## 6. THE FACTORIZATION OF $s(x, p, k)$ WHEN $p = 5j \pm 1$

From Theorem 1, we know that, if $p = 5j \pm 1$, then the equation $Q(b, c) = p^2$ has the three fundamental solutions

$$(b, c) = \begin{cases} \pm(pF_{2n-1}, pF_{2n}), \\ \pm(A_{2n}, A_{2n+1}), \qquad n \in \mathbb{Z}, \\ \pm(B_{2n}, B_{2n+1}), \end{cases} \tag{6.1}$$

where the initial conditions for $\{A_n\}$ and $\{B_n\}$ can be obtained from (2.6).

Now, let us state the following theorem.

***Theorem 5:*** If $p = 5j \pm 1$, then the polynomial $s(x, p, k)$ given by (1.3) factors as (1.2) iff either ***(i)*** (4.1) is satisfied (with $p = 5j \pm 1$) or ***(ii)*** $A_{2n+1}$ and/or $B_{2n+1}$ are perfect squares for some $n$.

**N.B.** There is a unique exception to point (i). Namely, $s(x, 3001, k)$ factors as (1.2) for $k = 6858699844168435635$ or $k = 8435643157247893914990$.

After observing that, on the basis of previously used arguments [see (4.2)-(4.4)], we must have

$$k^2 = \begin{cases} p^4F_{2n-1}^2F_{2n-2}^2(pF_{2n+1}), \\ A_{2n}^2\, A_{2n-1}^2(\pm A_{2n+2}), \qquad n \in \mathbb{Z}, \\ B_{2n}^2B_{2n-1}^2(\pm B_{2n+2}), \end{cases} \tag{6.2}$$

it is clear that the proofs of points (i) and (ii) are similar to those of Theorems 3 and 4, respectively. Therefore, we shall confine ourselves to proving the exception to point (i) mentioned in the N.B. above.

As an example of application of the last two equations of (6.2), we invite the reader to prove that $s(x, 19, k)$ factors as (1.2) for $k = 765$ or $26390$ or $37704147$.

**Hint:** After assuming that $(u_0, v_0) = (12, 5)$ is a solution to $u^2 - 5v^2 = 19$, use (2.6) to find $[A_0 = 149, A_1 = 240]$ and $[B_0 = 389, B_1 = -240]$, and observe that $A_{-4} = B_6 = 25$ and $B_{20} = 2809$ are perfect squares.

***Proof of the Exception to Point (i):*** Theorem 2 of [8] tells us that the unique exception to (4.6) occurs when $n = 12$, $p = 3001$, and $y = 5$ in (4.5). If we let these values of $n$ and $p$ in the first equation of (6.2), then we get $k^2 = p^4F_{23}^2F_{22}^2(pF_{25})$ $(p = 3001)$, whence

$$k = 3001^2F_{23}F_{22}\sqrt{3001F_{25}} = 6858699844168435635. \tag{6.3}$$

Further, letting $n = -13$ and $p = 3001$ in the same equation yields $k^2 = p^4F_{-27}^2F_{-28}^2(pF_{-25})$ $(p = 3001)$, whence

$$k = 3001^2F_{27}F_{28}\sqrt{3001F_{25}} = 8435643157247893914990. \text{ Q.E.D.} \tag{6.4}$$

The factorizations of $s(x, 3001, k)$ for the values of $k$ given by (6.3) and (6.4) are:

$$\begin{aligned} &x^5 + 9006001x - 6858699844168435635 \\ &= (x^2 + 15005x + 85999657)(x^3 - 15005x^2 + 139150368x - 797526418555), \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} &x^5 + 9006001x - 8435643157247893914990 \\ &= (x^2 - 15005x + 589450418)(x^3 + 15005x^2 - 364300393x - 14311030919055), \end{aligned} \tag{6.6}$$

respectively.

## 7. CONCLUSIONS

First, we wish to point out that the technique used in Sections 3-6 allows us to obtain the factorization of fifth-degree polynomials that are similar to those considered in this paper. In every case, Fibonacci and Fibonacci-like sequences play a fundamental role, and suggest the existence of an even deeper connection between these sequences and the factorization of fifth-degree polynomials. For example, it is not hard to prove that if

$$k = \begin{cases} n^{5m}F_4F_5, \\ 12n^{5m}F_9F_{10}, \\ 12n^{5m}F_{14}F_{15}, \end{cases} \tag{7.1}$$

then the polynomials $x^5 - n^{4m}x - k$ $(n, m \in \mathbb{N})$ factor as (1.2) (for $n = 1$, cf. (3.8) of [3]). The proof of (7.1) is based on the well-known fact [1] that $F_{2j}$ is a perfect square iff $j = 0, 1$, or 6. Further, the interested reader might enjoy using the above technique for proving that, if

$$k = F_j^3 F_{j\pm2} F_{j\pm3}, \tag{7.2}$$

then the polynomials $x^5 - (-1)^j F_j^2 x - k$ factor as (1.2).

Then, let us conclude our study by considering a special class of primes $p$ such that a couple $(k_1, k_2)$ of values of $k$ for which $r(x, p, k)$ factors as (1.2) can be expressed merely in terms of $p$. Namely, consider the set of all primes $p$ such that $p + 5 = z^4$ is a fourth power. Since $z$ must be an even integer not divisible by 5, it can be readily proven that $p$ has the form $5j + 1$. It is likely that there exist an infinitude of primes belonging to the above defined set. We found 15 of them within the interval $[2, 10^8]$, the smallest (resp. largest) being 11 (resp. 78074891).

***Theorem 6:*** If $p \geq 251$ is a prime such that

$$p + 5 = z^4 \tag{7.3}$$

is a fourth power, and

$$k_{1,2} = 4(p+5)^{1/4}[p^2 + 44p \pm 10(p+5)^{1/2}(p+10) + 220], \tag{7.4}$$

then both $r(x, p, k_1)$ and $r(x, p, k_2)$ factor as (1.2).

***Remark 4:*** For $p = 11$, see (5.4) and (5.5).

***Proof (for $k = k_2$):*** A solution to the Pell equation at point (ii) of Section 2 is clearly $(u, v) = (z^2, 1)$. Hence, from the first system of (2.7), we have

$$\begin{cases} x_0^{(1)} = A_0 = 4z^2 \quad \text{(a perfect square)}, \\ y_0^{(1)} = A_1 = z^4 + 2z^2 + 5, \end{cases} \tag{7.5}$$

and, from (5.2),

$$k^2 = A_{2n}^2 A_{2n-1}^2 A_{2n+2}. \tag{7.6}$$

Letting $n = -1$ in (7.6) yields

$$k^2 = A_{-2}^2 A_{-3}^2 A_0 = A_{-2}^2 A_{-3}^2 4z^2 \quad \text{[from (7.5)]}. \tag{7.7}$$

On calculation, we get

$$\begin{cases} A_{-2} = -z^4 + 6z^2 - 5, \\ A_{-3} = 2z^4 - 8z^2 + 10. \end{cases} \tag{7.8}$$

From (7.7) and (7.8) above, on choosing the signs properly to ensure the positiveness of $k$, one gets

$$k = \begin{cases} 2z(z^4 - 6z^2 + 5)(2z^4 - 8z^2 + 10) & \text{for } z \geq 4, \\ 2z(-z^4 + 6z^2 - 5)(2z^4 - 8z^2 + 10) = 120 & \text{for } z = 2 \text{ (i.e., } p = 11). \end{cases} \tag{7.9}$$

For $z \geq 4$ (i.e., $p \geq 251$), from (7.9) and (7.3), we obtain

$$k = k_2 = 4(p+5)^{1/4}[p^2 + 44p - 10(p+5)^{1/2}(p+10) + 220]$$

as desired. By using the second system of (2.7), the proof for $k = k_1$ can be obtained in a similar way. Q.E.D.

The factorizations (1.2) of $r(x, p, k)$ have the sets of coefficients

$$\begin{cases} a = -2(p+5)^{1/4}, \\ b = p + 6(p+5)^{1/2} + 10, \\ c = -p - 2(p+5)^{1/2} - 10, \\ d = -k/b, \end{cases} \quad \text{and} \quad \begin{cases} a = -2(p+5)^{1/4}, \\ b = -p + 6(p+5)^{1/2} - 10, \\ c = p - 2(p+5)^{1/2} + 10, \\ d = -k/b, \end{cases} \tag{7.10}$$

for $k = k_1$ and $k_2$, respectively. As a numerical example, the factorizations of $r(x, 1291, k_{1,2})$ are shown below. Namely, we have [cf. (7.10)]

$$x^5 - 1291^2x - 52609560 = (x^2 - 12x + 1517)(x^3 + 12x^2 - 1373x - 34680), \tag{7.11}$$

$$x^5 - 1291^2x - 30128280 = (x^2 - 12x - 1085)(x^3 + 12x^2 - 1229x + 27768). \tag{7.12}$$

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# ON IRRATIONAL VALUED SERIES INVOLVING GENERALIZED FIBONACCI NUMBERS

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## 1. INTRODUCTION

There are many rational termed convergent series in analysis that sum to an irrational number. One well-known example can be found via the Taylor expansion of the exponential function, where in particular the base of the natural logarithm is represented as an infinite sum of the reciprocals of $n!$. The irrationality of $e$ can be deduced directly from this series via an argument of Euler's (see [2]). In recent times, a number of authors [3], [4] have noted that other irrational valued series may be constructed by replacing $n!$ in the series for $e$ by the product $v_1v_2\dots v_n$, where $\{v_n\}$ is a strictly monotone increasing sequence of positive integers. However, in such cases, one needed to impose the additional assumption that $n \mid v_1v_2\dots v_n$ for each $n$. In this paper we shall demonstrate that irrational valued series may similarly be constructed from the terms of a generalized Fibonacci sequence, which are generated via the recurrence relation

$$U_n = PU_{n-1} - QU_{n-2},$$

where $P, Q \in \mathbf{Z}$ with $|P| > 1$, $|Q| = 1$, and $U_0 = 0$, $U_1 = 1$. The goal here is to establish the most general result possible by focusing attention on the following factorial-like expression

$$I(n) = U_kU_{k+1}\cdots U_{k+f(n)},$$

where $k \in \mathbf{N}\setminus\{0\}$ and $f : \mathbf{N} \to \mathbf{N}$ is an arbitrary strictly monotone increasing function. Such an expression will naturally reduce to the type of products considered above when $k = 1$ and $f(n) = n-1$. One advantage in dealing with the sequence $\{U_n\}$ is that we no longer need to impose the previous divisibility assumption, as this can be avoided by exploiting a fundamental property of generalized Fibonacci sequences that concerns the occurrence of a given prime factor in the sequence $\{U_n\}$. Unfortunately, the application of this property together with the argument used will require us to restrict the values of the ordered pairs $(P, Q)$ to those prescribed above. To prove the desired result, we will employ here (as in [3]) an argument similar to that used by Euler in establishing the irrationality of $e$. However, before reaching this point, it will be necessary in Section 2 to acquaint ourselves with a few preliminary results, beginning with the aforementioned property of generalized Fibonacci sequences.

## 2. MAIN RESULT

In establishing the irrationality of the series in question, we shall need to invoke within our argument the following technical result: For any given $m \in \mathbf{N}\setminus\{0\}$, there exists a positive integer $N(m) > 0$ such that $m \mid I(n)$ whenever $n \ge N(m)$. This result, which holds irrespectively of the choice of $k$ and $f(n)$, can be deduced directly from a divisibility property of generalized Fibonacci sequences. In order to state this property succinctly, we shall employ a number-theoretic function $\lambda_{rs}(n)$ as introduced in [1], which is defined below.

Let $rs$ and $r+s$ be the roots of any quadratic equation of the form $x^2-ux+v=0$, where $u$ and $v$ are integers. Noting by the Symmetric Function Theorem that $(r-s)^{p-1}$ is an integer for an odd prime $p$, define the symbol $(\frac{r,s}{p})$ by the congruence

$$(r-s)^{p-1} \equiv \left(\frac{r,s}{p}\right) \bmod p,$$

where it is understood that $(\frac{r,s}{p})$ is the residue of least absolute value; whence $(\frac{r,s}{p}) = 0, +1$, or $-1$ according as $(r-s)^{p-1}$ is divisible by $p$, is a quadratic residue of $p$, or is a quadratic non-residue of $p$. In the case $p=2$, the symbol $(\frac{r,s}{2})$ is defined by:

$$\left(\frac{r,s}{2}\right) = \begin{cases} 1 & \text{if } rs \text{ is even,} \\ 0 & \text{if } rs \text{ is odd and } r+s \text{ is even,} \\ -1 & \text{if } rs \text{ and } r+s \text{ are both odd.} \end{cases}$$

Now, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are the different prime factors of $n$, define the functional value of $\lambda_{rs}(n)$ as the least common multiple of the numbers

$$p_i^{\alpha_i - 1}\left[p_i - \left(\frac{r,s}{p_i}\right)\right], \quad i = 1, 2, \ldots, k.$$

The important divisibility property that appeared as Theorem XIII in [1] can now be stated as follows.

***Theorem 2.1:*** Suppose $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to the relatively prime pair $(P, Q)$. If the number $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are the different prime factors of $n$, is relatively prime to $rs = Q$ and if $\lambda = \lambda_{rs}(n)$, then $U_\lambda \equiv 0 \bmod n$.

It is clear from Theorem 2.1, when $\{U_n\}$ is generated with respect to the relatively prime pair $(P, Q)$ with $|Q|=1$, that given any $m \in \mathbf{N}\setminus\{0\}$ the sequence contains an element divisible by $m$. This fact does not allow us automatically to deduce the above technical result, since the product $I(n)$ in certain cases may never contain the term $U_\lambda$ [i.e., when $\lambda_Q(m) < k$]. To deal with situations such as these, observe first from the above definition of $\lambda_{rs}$ that

$$\lambda_Q(n) \geq p_i^{\alpha_i - 1}\left[p_i - \left(\frac{r,s}{p_i}\right)\right] \geq p_i^{\alpha_i-1}[p_i - 1] \geq p_i^{\alpha_i - 1}.$$

Now, if given any positive integer $t$ that contains a prime factor $p_i \geq k$ with $\alpha_i \geq 2$, then $\lambda_Q(tm) \geq p_i^{\alpha_i - 1} \geq p_i \geq k$. So, provided that $(tm, Q) = 1$, the product $I(n)$ will contain, for $n$ large, a term divisible by $tm$, namely $_{\lambda_Q(tm)}$. Thus, we can guarantee in the present case, as $|Q|=1$, that $I(n)$ is divisible by $m$ for suitably chosen $n$. Indeed, if one formally sets

$$t(m) = \min\{t \in \mathbf{N}\setminus\{0\} : \lambda_Q(tm) \geq k\},$$

then it is clear that $m \mid I(n)$ for $n \geq N(m)$, where

$$N(m) = \min\{n \in \mathbf{N}\setminus\{0\} \mid f(n) + k \geq \lambda_Q(t(m)m)\}.$$

Having established the required divisibility property of $I(n)$, we now need only introduce two further preliminary results before reaching the main theorem of this section. Both of these results will be used throughout the main argument of Theorem 2.2.

***Lemma 2.1:*** Suppose $f:\mathbf{N}\to\mathbf{N}$ is a strictly monotone increasing function, then for all $r, m\in\mathbf{N}$, we have $f(m+r)-f(m)\geq r$.

***Proof:*** Due to the strict monotonicity of *f*, it is clear that $f(m+i)-f(m+i-1)\geq 1$ for $i=1,2,\dots,r$. Adding these $r$ inequalities together and noting that the left-hand side is a telescoping sum equal to $f(m+r)-f(r)$, one deduces the desired inequality. □

***Lemma 2.2:*** Suppose $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to the relatively prime pair (*P*, *Q*), where $|P|>1$ and $|Q|=1$. Then the terms $|U_n|$ form a strictly monotone increasing sequence of integers.

***Proof:*** We argue using induction. Clearly $|U_2|-|U_1|=|P|-1>0$. Now suppose the result holds for an integer $n=m>1$, that is, $|U_m|>|U_{m-1}|$. Now, by an application of the reverse triangle inequality, observe that

$$\begin{aligned}|U_{m+1}|-|U_m| &= |PU_m-QU_{m-1}|-|U_m|\\ &\geq |P||U_m|-|U_{m-1}|-|U_m|\\ &=(|P|-1)|U_m|-|U_{m-1}|>0,\end{aligned}$$

noting here that the final inequality follows from the inductive assumption and the fact that $|P|-1\geq 1$. Consequently, the result holds for $n=m+1$. □

***Remark 2.1:*** With the above restrictions placed on the values of the ordered pairs (*P*, *Q*), it is clear from Lemma 2.2 that $I(n)\neq 0$ for $n\geq 1$. Thus, the terms of the series in question are well defined.

***Theorem 2.2:*** Let $\{a_n\}$ be a bounded sequence of integers with the property that $a_n\neq 0$ for infinitely many *n*. Suppose further that $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to the relatively prime pair (*P*, *Q*), where $|P|>1$ and $|Q|=1$. If $I(n)=U_kU_{k+1}\dots U_{k+f(n)}$, where $k\in\mathbf{N}\setminus\{0\}$ and $f:\mathbf{N}\to\mathbf{N}$ is a strictly monotone increasing function, then the resulting series $\sum_{n=1}^{\infty}a_n/I(n)$ converges to an irrational sum.

***Proof:*** We first establish convergence of the series. Observe that by setting $r=n-1$ for $n\in\mathbf{N}\setminus\{0\}$ and $m=1$ in Lemma 2.1, we have $f(n)\geq n-1+f(1)\geq n-1$. If $k\neq 1$, then each of the $f(n)+1$ terms in the product $|I(n)|$ are by Lemma 2.2 greater than or equal to $|U_2|=|P|$. Thus, $|I(n)|\geq |P|^{f(n)+1}\geq |P|^n$, while, if $k=1$ we have $|I(n)|=|U_2||U_3|\cdots|U_{1+f(n)}|\geq |P|^{f(n)}\geq |P|^{n-1}$. In any case, we have $|a_n|/|I(n)|\leq D/|P|^{n-1}$, where $D$ is the upper bound for $\{a_n\}$; consequently, the series is absolutely convergent.

Suppose now, to the contrary, that the sum of the series is a rational number given by $A/B$, where $A,B\in\mathbf{Z}$ with $B\neq 0$. By the above technical result, there exists an $N(|B|)>0$, with $B\mid I(m)$ whenever $m\geq N(|B|)$. Choose $m\geq N(|B|)$ such that $D+1<|U_{k+f(m)+1}|$ and consider the following equality,

$$I(m)\frac{A}{B} - I(m)\sum_{n=1}^{m} \frac{a_n}{I(n)} = \sum_{n=m+1}^{\infty} a_n \frac{I(m)}{I(n)} = C. \tag{1}$$

Since by definition $I(n) | I(m)$ for $n \le m$, it is clear that $C \in \mathbf{Z}$. We now determine upper and lower bounds for $|C|$. First, note that the modulus of $I(m)/I(n)$ for $n \ge m+1$ in the series on the right of (1) is given by

$$\left|\frac{I(m)}{I(m+r)}\right| = (|U_{k+f(m)+1} \cdots U_{k+f(m+r)}|)^{-1},$$

where $r = 1, 2, \ldots$. Now, by Lemma 2.2, each of the $f(m+r) - f(m)$ terms in the denominator of the above expression are in modulus greater than or equal to $|U_{k+f(m)+1}|$, so by Lemma 2.1,

$$|U_{k+f(m)+1} \cdots U_{k+f(m+r)}| \ge |U_{k+f(m)+1}|^{f(m+r)-f(m)} \ge |U_{k+f(m)+1}|^r. \tag{2}$$

Hence, using the triangle inequality and (2), we have

$$\begin{aligned} |C| = \left|\sum_{r=1}^{\infty} a_{m+r} \frac{I(m)}{I(m+r)}\right| &\le \sum_{r=1}^{\infty} D \left|\frac{I(m)}{I(m+r)}\right| \\ &\le \sum_{r=1}^{\infty} \frac{D}{|U_{k+f(m)+1}|^r} = \frac{D}{|U_{k+f(m)+1}| - 1} < 1, \end{aligned}$$

noting here that the last inequality follows from our initial choice of $m$. To obtain a lower bound for $|C|$ set $p = \min\{n \ge m+1 : a_n \ne 0\}$ so $|a_p| \ge$ . Then, by an application of the triangle and reverse triangle inequality, observe that

$$\begin{aligned} |C| = \left|\sum_{r=0}^{\infty} a_{p+r} \frac{I(m)}{I(p+r)}\right| &\ge \left| |a_p| \left|\frac{I(m)}{I(p)}\right| - \left|\sum_{r=1}^{\infty} a_{p+r} \frac{I(m)}{I(p+r)}\right| \right| \\ &\ge \left|\frac{I(m)}{I(p)}\right| - \sum_{r=1}^{\infty} D \left|\frac{I(m)}{I(p+r)}\right| = J. \end{aligned}$$

Clearly, from the definition, $p \ge m+1 > m$, thus, as in (2), we have for each $r \ge 1$,

$$\begin{aligned} \left|\frac{I(p)}{I(p+r)}\right| &= (|U_{k+f(p)+1} \cdots U_{k+f(p+r)}|)^{-1} \\ &\le \frac{1}{|U_{k+f(p)+1}|^r} < \frac{1}{|U_{k+f(m)+1}|^r}. \end{aligned}$$

Consequently,

$$\begin{aligned} |C| \ge J = \left|\frac{I(m)}{I(p)}\right| \left\{1 - \sum_{r=1}^{\infty} D \left|\frac{I(p)}{I(p+r)}\right|\right\} &\ge \left|\frac{I(m)}{I(p)}\right| \left\{1 - \sum_{r=1}^{\infty} \frac{D}{|U_{k+f(m)+1}|^r}\right\} \\ &= \left|\frac{I(m)}{I(p)}\right| \left\{1 - \frac{D}{|U_{k+f(m)+1}| - 1}\right\} > 0, \end{aligned}$$

where again the last inequality follows from the initial choice of $m$. Therefore, we have produced a $C \in \mathbf{Z}$ such that $0 < |C| < 1$, this obvious contradiction implies that the original assumption is false. Hence, the sum of the series in question is irrational. □

***Remark 2.2:*** It was noted in [1] that no simple analog of Theorem 2.1 exists for the sequence of generalized Lucas numbers; thus, the above argument cannot readily be extended to establish a similar result involving these number sequences.

We now consider a simple consequence of Theorem 2.2.

***Corollary 2.1:*** The base of the natural logarithm $e$ is a nonalgebraic number of degree two.

***Proof:*** Suppose that there exist $a, b, c \in \mathbf{Z}$ with $a \neq 0$ such that $ae^2 + be + c = 0$; then

$$ae + ce^{-1} = -b. \tag{3}$$

Now set $(P, Q) = (2, 1)$, $k = 1$, and $f(n) = n$ in Theorem 2.2. If $a_n = a + c(-1)^{n+1}$ we deduce, as $U_n = n$, that

$$\sum_{n=1}^{\infty} \frac{a_n}{I(n)} = a\sum_{n=1}^{\infty} \frac{1}{(n+1)!} + c\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} = a(e-2) + ce^{-1}$$

is an irrational number. Consequently, the number on the left of (3) is also irrational while the number on the right is clearly rational. This obvious contradiction thus establishes the above result. □

In view of Theorem 2.2, one may suspect that a similar result may hold for such factorial-like expressions as $I(n) = U_{f(n)} \cdots U_{f(n)+k}$, where $k \in \mathbf{N}$ and $f: \mathbf{N} \to \mathbf{N} \setminus \{0\}$ is a strictly monotone increasing function. At present, the author has been unable to supply an argument establishing, or a counterexample refuting, this conjecture. However, in the case of $f(n) = 2^n$ and $k = 0$, the author has been able to verify the irrationality of the series sum by direct calculation. To conclude, we now outline the derivation of the sum of these series.

***Proposition 2.1:*** Suppose $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to the relatively prime pair $(P, Q)$, where $P \geq 1$, $Q = -1$; then

$$\sum_{n=1}^{\infty} \frac{1}{U_{2^n}} = \frac{P^2 + 4 - P\sqrt{P^2+4}}{2P}.$$

***Proof:*** Consider the following telescoping sum

$$\sum_{n=1}^{N} \frac{x^{2^n}}{1 - x^{2^{n+1}}} = \sum_{n=1}^{N} \left( \frac{1}{1 - x^{2^n}} - \frac{1}{1 - x^{2^{n+1}}} \right) = \frac{1}{1-x^2} - \frac{1}{1 - x^{2^{N+1}}}.$$

If $|x| > 1$, then the above partial sums tend to a finite limit given by

$$\sum_{n=1}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}} = \frac{1}{1-x^2}.$$

Now $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, where $\alpha$ and $\beta$ are the roots of $x^2 - Px - 1 = 0$. Consequently, as $\alpha\beta = -1$, we have, for $|\beta| > 1$,

$$\frac{1}{\alpha - \beta} \sum_{n=1}^{\infty} \frac{1}{U_{2^n}} = \sum_{n=1}^{\infty} \frac{1}{\alpha^{2^n} - \beta^{2^n}} = \sum_{n=1}^{\infty} \frac{\beta^{2^n}}{1 - \beta^{2^{n+1}}} = \frac{1}{1-\beta^2}.$$

By setting $\alpha = (P-\sqrt{P^2+4})/2$ and $\beta = (P+\sqrt{P^2+4})/2$ in the above (noting that $|\alpha|<1$ and $|\beta|>1$) one obtains the desired sum. Note here that the irrationality of the series sum follows from the presence of the term $\sqrt{P^2+4}$, since $P^2+4$ is never a perfect square for $|P|\geq 1$. □

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# SOME ANALOGS OF THE IDENTITY $F_n^2 + F_{n+1}^2 = F_{2n+1}$

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## 1. INTRODUCTION

The charming identity

$$\sum_{j=0}^{k}(-1)^{\frac{j(j+3)}{2}}\begin{bmatrix}k\\j\end{bmatrix}F_{n+k-j}^{k+1} = F_1 \dots F_k F_{(k+1)(n+\frac{k}{2})} \tag{1.1}$$

is a special case of identity (5) of Torretto and Fuchs [7]. Here $\left[\begin{smallmatrix}k\\j\end{smallmatrix}\right]$ is the Fibonomial coefficient defined for integers $0 \le j \le k$ by

$$\begin{bmatrix}k\\j\end{bmatrix} = \frac{F_k F_{k-1} \dots F_{k-j+1}}{F_1 F_2 \dots F_j}, \quad \begin{bmatrix}k\\0\end{bmatrix} = 1.$$

According to H. W. Gould, generalized binomial coefficients were first suggested by Georges Fontené in 1915, and were rediscovered by Morgan Ward in 1936. These writers simply replaced the natural numbers by an arbitrary sequence $\{A_n\}$ of real or complex numbers. The idea of considering $A_n = F_n$ seems to have originated with Dov Jarden in 1949. For an excellent discussion on these matters, and a comprehensive list of references, see Gould [3].

For $k = 1, 2, 3$, and 4, identity (1.1) becomes, respectively,

$$F_{n+1}^2 + F_n^2 = F_{2n+1}, \tag{1.2}$$

$$F_{n+2}^3 + F_{n+1}^3 - F_n^3 = F_{3n+3}, \tag{1.3}$$

$$F_{n+3}^4 + 2F_{n+2}^4 - 2F_{n+1}^4 - F_n^4 = 2F_{4n+6}, \tag{1.4}$$

$$F_{n+4}^5 + 3F_{n+3}^5 - 6F_{n+2}^5 - 3F_{n+1}^5 + F_n^5 = 6F_{5n+10}. \tag{1.5}$$

To make the right sides of (1.3) and (1.5) more compact, we may replace $n$ by $n-1$ and $n-2$, respectively.

In this paper we present analogs of (1.2)–(1.5) for the so-called Tribonacci and Tetranacci sequences, which we define in Sections 3 and 4. We consider more general third- and fourth-order sequences, and identities associated with them, in Section 5. Our method of discovering these identities is outlined in Section 2, and generalizations and proofs are given in Section 6.

## 2. THE METHOD

To demonstrate our method, we use it to "discover" identities (1.2) and (1.3). To arrive at (1.2), we consider the sequence

$$\{F_n^2 - F_{n+1}F_{n-1}\}_0^\infty = \{-1, 1, -1, 1, -1, \dots\}.$$

This sequence satisfies the recurrence $r_n = -r_{n-1}$, and so we have

$$F_n^2 - F_{n+1}F_{n-1} = -(F_{n-1}^2 - F_nF_{n-2})$$

or

$$F_n^2 + F_{n-1}^2 = F_{n+1}F_{n-1} + F_nF_{n-2}. \tag{2.1}$$

Finally, we observe by trial that the right side of (2.1) is $F_{2n-1}$, and this yields (1.2).

To obtain (1.3), we consider the sequence

$$\{F_n^3 - F_{n+1}F_nF_{n-1}\}_0^\infty = \{0, 1, -1, 2, -3, 5, -8, \ldots\}.$$

This sequence satisfies the recurrence $r_n = -r_{n-1} + r_{n-2}$, so that

$$F_n^3 - F_{n+1}F_nF_{n-1} = -(F_{n-1}^3 - F_nF_{n-1}F_{n-2}) + (F_{n-2}^3 - F_{n-1}F_{n-2}F_{n-3})$$

or

$$F_n^3 + F_{n-1}^3 - F_{n-2}^3 = F_{n+1}F_nF_{n-1} + F_nF_{n-1}F_{n-2} - F_{n-1}F_{n-2}F_{n-3}. \tag{2.2}$$

Again, after making several substitutions, we see that the right side of (2.2) is $F_{3n-3}$, and this yields (1.3).

To obtain (1.4), we could consider the sequence generated by $F_n^4 - F_{n+1}F_n^2F_{n-1}$, or perhaps $F_n^4 - F_{n+3}F_nF_{n-1}F_{n-2}$, or many other such expressions. To decide which product to subtract, we consider two things. First, the product must have "degree" four. Second, the sum of the subscripts of the terms which make up the product must be $4n$. To obtain the analogous identities which involve higher powers, we proceed in a similar manner.

## 3. THE TRIBONACCI SEQUENCE

As a third-order analog of the Fibonacci sequence, Feinberg [2] considered the Tribonacci sequence, defined for all integers by

$$p_n = p_{n-1} + p_{n-2} + p_{n-3}, \quad p_0 = 0, p_1 = 1, p_2 = 1.$$

Proceeding as in Section 2, and with the help of the computer algebra package Mathematica 3.0, we have obtained identities analogous to (1.2)–(1.5) for the Tribonacci sequence. We have found the following:

$$p_{n+3}^2 + p_{n+2}^2 + p_{n+1}^2 - p_n^2 = 2p_{2n} + 3p_{2n+1} + 3p_{2n+2}, \tag{3.1}$$

$$\begin{aligned} &p_{n+7}^3 + 3p_{n+6}^3 + 7p_{n+5}^3 + p_{n+4}^3 - p_{n+3}^3 - 7p_{n+2}^3 - 3p_{n+1}^3 - p_n^3 \\ &= 6758p_{3n} + 10432p_{3n+1} + 12430p_{3n+2}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &p_{n+12}^4 + 4p_{n+11}^4 + 16p_{n+10}^4 - 26p_{n+9}^4 - 5p_{n+8}^4 - 128p_{n+7}^4 \\ &+ 100p_{n+6}^4 + 4p_{n+5}^4 + 43p_{n+4}^4 - 44p_{n+3}^4 + 4p_{n+2}^4 - 2p_{n+1}^4 + p_n^4 \\ &= 27720670104p_{4n} + 42792093864p_{4n+1} + 50986261368p_{4n+2}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &p_{n+18}^5 + 8p_{n+17}^5 + 59p_{n+16}^5 - 126p_{n+15}^5 - 154p_{n+14}^5 - 2758p_{n+13}^5 \\ &+ 2142p_{n+12}^5 + 2394p_{n+11}^5 + 6552p_{n+10}^5 - 7182p_{n+9}^5 - 4284p_{n+8}^5 - 2394p_{n+7}^5 \\ &+ 1386p_{n+6}^5 + 686p_{n+5}^5 + 322p_{n+4}^5 - 98p_{n+3}^5 - 9p_{n+2}^5 - 2p_{n+1}^5 + p_n^5 \\ &= 125288677521300479558 4p_{5n} + 193406754904352278329 6p_{5n+1} \\ &+ 230441805143226167500 8p_{5n+2}. \end{aligned} \tag{3.4}$$

We have found the next identity in this list. The left side has 26 sixth powers, and following the pattern of the previous identities the coefficients are 1, 15, 204, –724, –1946, –58710, 65968, 182480, 921767, –1448495, –2215192, –2814392, 1090180, 2032604, 2528400, –9744, –25313, –238687, –15828, –4372, 9814, 1786, 224, –32, –7, –1. On the right side, the coefficients of $p_{6n}$, $p_{6n+1}$, and $p_{6n+2}$ are, respectively,

$$32119103347966496693731741070891558 40,$$

$$49581906935777165672226969703584998 40,$$

and

$$59076241377269597102082587261723481 60.$$

We have been unable to discern a pattern to the coefficients in the identities above. However, on the basis of our results, we predict that the next identity will involve 34 seventh powers. More generally, we conjecture that for $k \geq 2$ such identities for the Tribonacci sequence involve $\frac{1}{2}(k^2+3k-2)$ $k^{\text{th}}$ powers.

## 4. THE TETRANACCI SEQUENCE

As a fourth-order analog of the Fibonacci sequence, Feinberg [2] also considered the Tetranacci sequence, defined for all integers by

$$q_n = q_{n-1} + q_{n-2} + q_{n-3} + q_{n-4}, \quad q_0 = 0, q_1 = 1, q_2 = 1, q_3 = 2.$$

In the same manner, for the Tetranacci sequence, we have found

$$\begin{aligned} &q_{n+6}^2 + q_{n+5}^2 + 2q_{n+4}^2 + 2q_{n+3}^2 - 2q_{n+2}^2 + q_{n+1}^2 - q_n^2 \\ &\quad = 46q_{2n} + 70q_{2n+1} + 82q_{2n+2} + 88q_{2n+3} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &q_{n+16}^3 + 3q_{n+15}^3 + 8q_{n+14}^3 + 18q_{n+13}^3 - 26q_{n+12}^3 - 35q_{n+11}^3 - q_{n+10}^3 - 56q_{n+9}^3 \\ &\quad + 36q_{n+8}^3 + 88q_{n+7}^3 - 21q_{n+6}^3 + 21q_{n+5}^3 - 16q_{n+4}^3 - 6q_{n+3}^3 + 2q_{n+2}^3 - q_{n+1}^3 + q_n^3 \\ &= 273507715816q_{3n} + 415400801120q_{3n+1} + 489013523880q_{3n+2} + 527203073008q_{3n+3}. \end{aligned} \tag{4.2}$$

The next identity involves 32 fourth powers whose coefficients are 1, 7, 38, 174, –154, –1150, –1368, –7226, –1926, 32582, 22851, 56387, 36788, –34100, –23540, –78932, –56080, 6372, 18724, 50476, 39447, 13621, 2822, –2234, –2290, –910, –280, –10, 34, 14, 5, 1. On the right side, the coefficients of $q_{4n}$, $q_{4n+1}$, $q_{4n+2}$, and $q_{4n+3}$ are, respectively,

$$4029347100326473175037256543628 80,$$

$$6119732339077083643781858779055 36,$$

$$7204203430195641290730114099398 40,$$

and

$$7766816256611693452461325103668 48.$$

We have found that the next identity in this list involves 53 fifth powers. On the basis of our observations, we conjecture that for $k \geq 2$ such identities for the Tetranacci sequence involve $\frac{1}{6}(k^3+6k^2+11k-12)$ $k^{\text{th}}$ powers.

## 5. MORE GENERAL SEQUENCES

Consider now the more general sequence $\{U_n\}$ defined for all integers by

$$U_n = aU_{n-1} + bU_{n-2} + cU_{n-3}, \quad U_0 = 0, U_1 = 1, U_2 = a, \tag{5.1}$$

where $a$, $b$, and $c$ are complex numbers with $c \neq 0$. The sequence $\{U_n\}$ is one of the three *fundamental* sequences (as in Bell [1]) generated by the recurrence in (5.1). We have found that

$$U_{n+3}^2 + bU_{n+2}^2 + acU_{n+1}^2 - c^2U_n^2 = bU_1U_{2n+3} + U_2U_{2n+4}. \tag{5.2}$$

We accomplished this by considering many instances of $(a, b, c)$ and constructing the corresponding identity. This process was tedious, to say the least.

More generally, let $\{R_n\}$ be any sequence generated by the recurrence in (5.1) and with arbitrary initial terms $R_0, R_1, R_2$. Then, in the same manner, we have found that

$$\begin{aligned} &R_{n+3}^2 + bR_{n+2}^2 + acR_{n+1}^2 - c^2R_n^2 \\ &= ((ac-b^2)R_0 - abR_1 + bR_2)R_{2n+2} + (-abR_0 + (b-a^2)R_1 + aR_2)R_{2n+3} + (bR_0 + aR_1)R_{2n+4}. \end{aligned} \tag{5.3}$$

It is interesting to note that the coefficients on the left side of (5.2) match those on the left side of (5.3). Horadam [5] proved the analog of (5.3) for second-order sequences very elegantly with the use of generating functions, but we have been unable to adapt his method to prove (5.3). However, we have discovered another method of proof which we demonstrate in the next section.

As the fourth-order analog of $\{U_n\}$, we define the sequence $\{V_n\}$ by

$$V_n = aV_{n-1} + bV_{n-2} + cV_{n-3} + dV_{n-4}, \quad V_0 = 0, V_1 = 1, V_2 = a, V_3 = a^2 + b. \tag{5.4}$$

We have found that

$$\begin{aligned} &V_{n+6}^2 + bV_{n+5}^2 + (ac+d)V_{n+4}^2 + (a^2d - c^2 + 2bd)V_{n+3}^2 - (d^2 + acd)V_{n+2}^2 + bd^2V_{n+1}^2 - d^3V_n^2 \\ &\quad = (a^2d - c^2 + 2bd)V_1V_{2n+5} + (ac+d)V_2V_{2n+6} + bV_3V_{2n+7} + V_4V_{2n+8}. \end{aligned} \tag{5.5}$$

In (5.5), it is interesting to compare the coefficients of $V_{n+3}^2, V_{n+4}^2, V_{n+5}^2$, and $V_{n+6}^2$ with those of $V_{2n+5}, V_{2n+6}, V_{2n+7}$, and $V_{2n+8}$, respectively. Similar comparisons should be made in (5.2), and also in the known identity

$$u_{n+1}^2 + bu_n^2 = u_{2n+1} = u_1u_{2n+1}. \tag{5.6}$$

Here $\{u_n\}$ is the second-order sequence defined by $u_n = au_{n-1} + bu_{n-2}$, $u_0 = 0, u_1 = 1$.

Our attempt to construct identities similar to those in this section for sequences of order five has proved fruitless. The polynomial coefficients became unwieldy, as can be appreciated when we compare (5.2) with (5.5). The same can be said for higher powers. However, our work with specific examples suggests that identities analogous to those that we have constructed in this paper exist for all sequences, and for all powers. We have looked only at sequences generated by linear recurrence relations with constant coefficients.

We mention that further experimentation with specific examples suggests that, for linear recurrences of order $m$, identities analogous to (1.2) contain $\frac{1}{2}(m^2 - m + 2)$ squares, and identities analogous to (1.3) contain $\frac{1}{6}(m^3 + 3m^2 - 4m + 6)$ cubes.

## 6. GENERALIZATIONS AND PROOFS

At the beginning of Section 2 we started with the identity $F_n^2 - F_{n+1}F_{n-1} = (-1)^n$. Instead, suppose we consider the more general identity

$$F_{n+a}F_{n+b} - F_nF_{n+a+b} = (-1)^n F_aF_b. \tag{6.1}$$

Then, considered as a function of $n$, the sequence $\{F_{n+a}F_{n+b} - F_nF_{n+a+b}\}$ satisfies the recurrence $r_n = -r_{n-1}$. Hence,

$$F_{n+1+a}F_{n+1+b} - F_{n+1}F_{n+1+a+b} = -(F_{n+a}F_{n+b} - F_nF_{n+a+b})$$

or

$$F_{n+1+a}F_{n+1+b} + F_{n+a}F_{n+b} = F_{n+1}F_{n+1+a+b} + F_nF_{n+a+b}. \tag{6.2}$$

With $m$ in place of $n+a$, and $n$ in place of $n+b$, the left side of (6.2) becomes $F_{m+1}F_{n+1} + F_mF_n$. But, by $I_{26}$ in [4], we know that

$$F_{m+1}F_{n+1} + F_mF_n = F_{m+n+1}, \tag{6.3}$$

which generalizes (1.2).

This suggests that to generalize (1.3) we might try

$$F_{k+2}F_{m+2}F_{n+2} + F_{k+1}F_{m+1}F_{n+1} - F_kF_mF_n = F_{k+m+n+3}, \tag{6.4}$$

which is indeed the case. In fact, this mode of generalization extends to (1.1), where the corresponding generalization is a special case of identity (5) of Torretto and Fuchs [7].

Based on numerical evidence, the method of generalization we have just described seems to carry over to all the identities in Sections 3-5. For example, we now prove that

$$p_{m+3}p_{n+3} + p_{m+2}p_{n+2} + p_{m+1}p_{n+1} - p_mp_n = 2p_{m+n} + 3p_{m+n+1} + 3p_{m+n+2}, \tag{6.5}$$

which generalizes (3.1).

***Proof of (6.5):*** Fix $m$. Each of the sequences $\{p_{n+k}\}$, where $k \in \mathbf{Z}$ is fixed, satisfies the recurrence for the Tribonacci numbers. Hence, by linearity, the sequences

$$\{p_{m+3}p_{n+3} + p_{m+2}p_{n+2} + p_{m+1}p_{n+1} - p_mp_n\} \text{ and } \{2p_{m+n} + 3p_{m+n+1} + 3p_{m+n+2}\} \tag{6.6}$$

also satisfy this recurrence. So, to prove that these sequences are identical, it suffices to prove that they have the same initial terms. That is, it suffices to show that

$$\begin{cases} p_{m+3}p_3 + p_{m+2}p_2 + p_{m+1}p_1 - p_mp_0 = 2p_m + 3p_{m+1} + 3p_{m+2}, \\ p_{m+3}p_4 + p_{m+2}p_3 + p_{m+1}p_2 - p_mp_1 = 2p_{m+1} + 3p_{m+2} + 3p_{m+3}, \\ p_{m+3}p_5 + p_{m+2}p_4 + p_{m+1}p_3 - p_mp_2 = 2p_{m+2} + 3p_{m+3} + 3p_{m+4}. \end{cases}$$

We prove only the last of these, since the proofs of the others are similar. Using the recurrence satisfied by the Tribonacci numbers, we see that $p_{m+3} = p_{m+2} + p_{m+1} + p_m$ and $p_{m+4} = 2p_{m+2} + 2p_{m+1} + p_m$. Also, since $p_2 = 1$, $p_3 = 2$, $p_4 = 4$, and $p_5 = 7$, we substitute and observe that both sides reduce to $11p_{m+2} + 9p_{m+1} + 6p_m$. Since $m$ is arbitrary, this proves (6.5) and hence also (3.1). □

This method of proof applies also to identities (4.1), (5.2), (5.3), and (5.5), since they involve squares. As shown above, we proceed by proving the more general identities obtained by introducing the parameter $m$. The proof of the generalized version of (5.3), for example, is not much more complicated than the proof demonstrated above. With $m$ fixed, we need to prove

$$\begin{aligned} R_{m+3}R_{n+3} + bR_{m+2}R_{n+2} + acR_{m+1}R_{n+1} - c^2R_mR_n \\ = AR_{m+n+2} + BR_{m+n+3} + CR_{m+n+4}, \end{aligned} \tag{6.7}$$

where $A$, $B$, and $C$ are as in (5.3). As in the proof of (6.5), our task is to show that (6.7) holds for $n = 0, 1$, and 2. Thus, for $n = 2$, we need to show

$$R_5 R_{m+3} + bR_4 R_{m+2} + acR_3 R_{m+1} - c^2 R_2 R_m = AR_{m+4} + BR_{m+5} + CR_{m+6}. \tag{6.8}$$

Using the recurrence in (5.1), we express $R_3$, $R_4$, and $R_5$ in terms of $R_0$, $R_1$, and $R_2$. Likewise, we express $R_{m+j}$ for $3 \le j \le 6$ in terms of $R_m$, $R_{m+1}$, and $R_{m+2}$. Finally, making these substitutions and using a suitable computer algebra package (in our case Mathematica 3.0), it is straightforward to verify the validity of (6.8). The verifications for $n = 0$ and 1 are treated similarly.

Now to the identities which involve higher powers. We tried to prove (3.2) by first proving

$$\sum_{i=0}^{7} a_i p_{k+i} p_{m+i} p_{n+i} = \sum_{i=0}^{2} b_i p_{k+m+n+i}, \tag{6.9}$$

where the $a_i$ and $b_i$ are given in (3.2). Our attempts failed because of the presence of an extra parameter. However, we found that we could prove the following "intermediate" identity:

$$\sum_{i=0}^{7} a_i p_{m+i} p_{n+i}^2 = \sum_{i=0}^{2} b_i p_{m+2n+i}. \tag{6.10}$$

Our proof, which is similar to the proofs demonstrated previously, requires the following lemma which is contained in [6].

***Lemma:*** Let $\{w_n\}$ be a sequence of complex numbers defined by

$$w_n = \sum_{i=1}^{k} c_i w_{n-i}, \tag{6.11}$$

where $c_1, \dots, c_k$ and $w_0, \dots, w_{k-1}$ are given complex numbers with $c_k \ne 0$. Let $h \ge 1$ be an integer. Then $\{w_n^h\}$ is generated by a linear recurrence of order $\binom{h+k-1}{h}$.

Using the lemma with $h = 2$ and $k = 3$, we see that $\{p_n^2\}$ satisfies a linear recurrence of order 6, and, by solving a system of linear equations, we find that this recurrence is

$$r_n = 2r_{n-1} + 3r_{n-2} + 6r_{n-3} - r_{n-4} - r_{n-6}. \tag{6.12}$$

Furthermore, $\{p_{2n}\}$ satisfies the recurrence

$$r_n = 3r_{n-1} + r_{n-2} + r_{n-3}, \tag{6.13}$$

and, since the auxiliary polynomial of (6.13) divides the auxiliary polynomial of (6.12), the sequence $\{p_{2n}\}$ is also generated by (6.12). To complete the proof, we proceed as before. That is, we fix $m$ and verify the validity of (6.10) for six consecutive values of $n$.

By using this approach, we have also succeeded in proving (3.3), (3.4), and (4.2) by first proving the more general identities obtained by the introduction of the parameter $m$. From the lemma, the number of verifications required to prove each of these identities is 10, 15, and 10, respectively.

While we acknowledge that this method of proof is tedious for identities that involve higher powers, given the nature of these identities, it seems unreasonable to expect anything else.

## ACKNOWLEDGMENT

We are indebted to an anonymous referee whose suggestions led to the generalizations in Section 6, which in turn inspired us to discover our method of proof.

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# AN EXTENSION OF AN OLD PROBLEM OF DIOPHANTUS AND EULER

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Diophantus studied the following problem: Find three (rational) numbers such that the product of any two increased by the sum of those two gives a square. He obtained the solutions $\{4, 9, 28\}$ and $\{\frac{3}{10}, \frac{21}{5}, \frac{7}{10}\}$ (see [3]). Euler treated the same problem with four numbers (see [2]). He found the solution $\{\frac{65}{224}, \frac{9}{224}, \frac{9}{56}, \frac{5}{2}\}$. Indeed, we have

$$\frac{65}{224}\cdot\frac{9}{224}+\frac{65}{224}+\frac{9}{224}=\left(\frac{131}{224}\right)^2, \quad \frac{65}{224}\cdot\frac{9}{56}+\frac{65}{224}+\frac{9}{56}=\left(\frac{79}{112}\right)^2,$$

$$\frac{65}{224}\cdot\frac{5}{2}+\frac{65}{224}+\frac{5}{2}=\left(\frac{15}{8}\right)^2, \quad \frac{9}{224}\cdot\frac{9}{56}+\frac{9}{224}+\frac{9}{56}=\left(\frac{51}{112}\right)^2,$$

$$\frac{9}{224}\cdot\frac{5}{2}+\frac{9}{224}+\frac{5}{2}=\left(\frac{13}{8}\right)^2, \quad \frac{9}{56}\cdot\frac{5}{2}+\frac{9}{56}+\frac{5}{2}=\left(\frac{7}{4}\right)^2.$$

In the present paper we will construct the set of **five** numbers with the above property.

Let $\{x_1, \dots, x_m\}$ be the set of rational numbers such that $x_ix_j + x_i + x_j$ is a perfect square for all $1 \le i < j \le m$. Since

$$x_ix_j + x_i + x_j = (x_i + 1)(x_j + 1) - 1,$$

if we put $x_i + 1 = a_i$, $i = 1, \dots, m$, we obtain the set $\{a_i, \dots, a_m\}$ with the property that the product of its any two distinct elements diminished by 1 is a perfect square. Such a set is called *a (rational) Diophantine m-tuple with the property* $D(-1)$ (see [4], p. 75). If $a_i$'s are positive integers, such a set is also called *a $P_{-1}$-set of size m*. The conjecture is that there does not exist a $P_{-1}$-set of size 4. Let us mention that in [1], [6], and [7] it was proved that some particular $P_{-1}$-sets of size 3 cannot be extended to a $P_{-1}$-set of size 4. In [5], some consequences of the above conjecture were considered.

We will derive a two-parametric formula for Diophantine quintuples and, as a consequence, we will obtain a rational Diophantine quintuple with the property $D(-1)$.

We will consider quintuples of the form $\{A, B, C, D, x^2\}$ with the property $D(\alpha x^2)$, where $A$, $B$, $C$, $D$, $x$, and $\alpha$ are integers. Furthermore, we will use the following simple result known already to Euler: If $BC + n = k^2$, then the set $\{B, C, B + C \pm 2k\}$ has the property $D(n)$.

Therefore, if we assume that

$$BC + \alpha x^2 = k^2, \quad A = B + C - 2k, \quad D = B + C + 2k,$$

then the set $\{A, B, C, D, x^2\}$ has the property $D(\alpha x^2)$ if and only if $AD + \alpha x^2$ is a perfect square. Hence, we reduced the original $\binom{5}{2} = 10$ conditions to only two conditions:

$$(b^2 - \alpha)(c^2 - \alpha) + \alpha x^2 = k^2, \tag{1}$$

$$(a^2 - \alpha)(d^2 - \alpha) + \alpha x^2 = y^2. \tag{2}$$

Our assumptions

$$(b^2-\alpha)+(c^2-\alpha)-2k=a^2-\alpha, \quad (b^2-\alpha)+(c^2-\alpha)+2k=d^2-\alpha$$

imply that $4k=(d+a)(d-a)$. Let $d+a=2p$ and $d-a=2r$. This implies that $k=pr$ and

$$b^2+c^2-\alpha=\frac{1}{2}(a^2+d^2)=p^2+r^2. \tag{3}$$

Let us rewrite condition (2) in the form $(ad-\alpha)^2-\alpha(d-a)^2=y^2-\alpha x^2$. Thus, we may take

$$y=ad-\alpha, \quad x=d-a=2r. \tag{4}$$

Substituting (3) and (4) into (1), we obtain

$$p^2r^2-b^2c^2=4\alpha r^2-\alpha(b^2+c^2-\alpha)=\alpha(3r^2-p^2). \tag{5}$$

At this point we make the further assumption [motivated by (3) and (5)]:

$$b+c=p+r. \tag{6}$$

Now (3) implies

$$pr-bc=\frac{\alpha}{2}, \tag{7}$$

and (5) implies

$$pr+bc=2(3r^2-p^2). \tag{8}$$

Adding (7) and (8) yields

$$\alpha=4p^2+4pr-12r^2. \tag{9}$$

From (6) and (7), we conclude that $b$ and $c$ are the solutions of the quadratic equation

$$z^2-(p+r)z+\left(pr-\frac{\alpha}{2}\right)=0.$$

The discriminant of this equation has to be a perfect square. Thus,

$$(p-r)^2+2\alpha=q^2. \tag{10}$$

Substituting (9) into (10) we have, finally,

$$(3p+r)^2-24r^2=q^2. \tag{11}$$

Hence, we reduce our problem to the solving of (11). However, the general solution of the equation $u^2-24v^2=w^2$ with $(u,v,w)=1$ is given by

$$u=e^2+6f^2, \quad v=ef, \quad w=|e^2-6f^2|$$

or

$$u=2e^2+3f^2, \quad v=ef, \quad w=|2e^2-3f^2|$$

(see [8], p. 225). Thus, we have proved

***Theorem 1:*** If $e\equiv 0 \pmod 3$ or $e\equiv f \pmod 3$, then the set

$$\begin{aligned}&\{\tfrac{1}{3}(e^2+6ef-18f^2)(2f^2+2ef-e^2), \tfrac{1}{3}e^2(e+5f)(3f-e),\\ &f^2(e-2f)(5e+6f), \tfrac{1}{3}(e^2+4ef-6f^2)(6f^2+4ef-e^2), 4e^2f^2\}\end{aligned} \tag{12}$$

has the property $D\left(\frac{16}{9}e^2f^2(e^2-ef-3f^2)(e^2+2ef-12f^2)\right)$, and the set

$$\begin{aligned}&\{\tfrac{1}{3}(9f^2+6ef-2e^2)(2e^2+2ef-f^2), \tfrac{1}{3}e^2(5f-2e)(2e+3f),\\ &f^2(e+f)(5e-3f), \tfrac{1}{3}(3f^2+4ef-2e^2)(2e^2+4ef-3f^2), 4e^2f^2\}\end{aligned} \tag{13}$$

has the property $D\left(\frac{16}{9}e^2f^2(e^2-ef-3f^2)(4e^2+2ef-3f^2)\right)$.

Substituting $e=5$ and $f=2$ in (12), we obtain the following two corollaries.

***Corollary 1:*** The set $\{\frac{13}{40}, \frac{25}{8}, \frac{37}{10}, 10, \frac{533}{40}\}$ is a rational Diophantine quintuple with the property $D(-1)$.

***Corollary 2:*** The five numbers $-\frac{27}{40}, \frac{17}{8}, \frac{27}{10}, 9, \frac{493}{40}$ have the property that the product of any two of them increased by the sum of those two gives a perfect square.

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# FAMILIES OF IDENTITIES INVOLVING SUMS OF POWERS OF THE FIBONACCI AND LUCAS NUMBERS

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## 1. INTRODUCTION

The well-known identity

$$F_{n+1}^2 + F_n^2 = F_{2n+1} \tag{1.1}$$

has as its Lucas counterpart

$$L_{n+1}^2 + L_n^2 = 5F_{2n+1}. \tag{1.2}$$

Indeed, since $L_{n+1} = F_{n+2} + F_n = F_{n+1} + 2F_n$ and $L_n = F_{n+1} + F_{n-1} = 2F_{n+1} - F_n$, (1.2) follows from (1.1).

As analogs of (1.1) and (1.2) we have

$$F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n} \quad \text{(see [7], p. 11),} \tag{1.3}$$

and

$$L_{n+1}^3 + L_n^3 - L_{n-1}^3 = 5L_{3n} \quad \text{(see [4], p. 165).} \tag{1.4}$$

One aim of this paper is to generalize (1.1)-(1.4). These identities belong to a family of similar identities that involve sums of $m^{\text{th}}$ powers ($m \in \mathbf{Z}$, $m \geq 2$) of Fibonacci (Lucas) numbers. As usual, $\mathbf{Z}$ denotes the set of integers. Our second aim is to state a conjecture that proposes a generalization of this family of identities. We state our conjecture in Section 4.

## 2. PRELIMINARY RESULTS

We require some preliminary results. For $m, n \in \mathbf{Z}$,

$$F_{-n} = (-1)^{n+1} F_n, \tag{2.1}$$

$$L_{-n} = (-1)^n L_n, \tag{2.2}$$

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_m F_n, \tag{2.3}$$

$$L_{m+n+1} = F_{m+1}L_{n+1} + F_m L_n, \tag{2.4}$$

$$F_{3(n+2)} = 4F_{3(n+1)} + F_{3n}, \tag{2.5}$$

$$F_{n+4}^3 = 3F_{n+3}^3 + 6F_{n+2}^3 - 3F_{n+1}^3 - F_n^3. \tag{2.6}$$

Identities (2.1)-(2.4) can be found on pages 28 and 59 in Hoggatt [3]. Identity (2.5) is a special case of (2.3) in Shannon and Horadam [5], and (2.6) occurs as (40) in Long [4]. The recurrences in (2.5) and (2.6) are also satisfied by the Lucas numbers.

We also require the following lemmas.

***Lemma 1:*** $3F_{3(m+3)} + 6F_{3(m+2)} - 3F_{3(m+1)} - F_{3m} = F_{3(m+4)}$, $m \in \mathbf{Z}$.

***Proof:*** By (2.5) we have

$$\begin{aligned}
&3F_{3(m+3)}+6F_{3(m+2)}-3F_{3(m+1)}-F_{3m}\\
&=3F_{3(m+3)}+6F_{3(m+2)}-3F_{3(m+1)}-(F_{3(m+2)}-4F_{3(m+1)})\\
&=3F_{3(m+3)}+5F_{3(m+2)}+F_{3(m+1)}\\
&=3F_{3(m+3)}+5F_{3(m+2)}+F_{3(m+3)}-4F_{3(m+2)}\\
&=4F_{3(m+3)}+F_{3(m+2)}=F_{3(m+4)}. \quad \square
\end{aligned}$$

***Lemma 2:*** Let $k, n \in \mathbf{Z}$ with $0 \le n \le 3$. Then

$$F_{3k+1}F^3_{n+k+1}+F_{3k+2}F^3_{n+k}-F^3_{n-2k-1}=F_{3k+1}F_{3k+2}F_{3n}. \tag{2.7}$$

***Proof:*** We give the proof only for $n=3$, since the proofs of the remaining cases are similar. For the case $n=3$, identity (2.1) shows that we need to prove

$$F_{3k+1}F^3_{k+4}+F_{3k+2}F^3_{k+3}+F^3_{2k-2}-34F_{3k+1}F_{3k+2}=0. \tag{2.8}$$

This is easily proved by using a powerful technique developed recently by Dresel [1]. Following Dresel, we see that (2.8) is a homogeneous equation of degree 6 in the variable $k$. Therefore, to prove its validity for all integers $k$, it suffices to verify its validity for seven different values of $k$, say $0 \le k \le 6$. But (2.8) is easily verified for these values, and so it is true for all integers $k$. $\square$

## 3. THE MAIN RESULTS

Our generalizations of (1.1) and (1.2) are contained in the following theorem.

***Theorem 1:*** For $k, n \in \mathbf{Z}$,

$$F^2_{n+k+1}+F^2_{n-k}=F_{2k+1}F_{2n+1} \tag{3.1}$$

and

$$L^2_{n+k+1}+L^2_{n-k}=5F_{2k+1}F_{2n+1}. \tag{3.2}$$

***Proof:*** Using (2.1) and (2.3), we obtain $F_{n+k+1}=F_nF_k+F_{n+1}F_{k+1}$ and $F_{n-k}=F_{n-(k+1)+1}=(-1)^k(F_nF_{k+1}-F_{n+1}F_k)$, so that

$$\begin{aligned}
F^2_{n+k+1}+F^2_{n-k}&=F_k^2F_n^2+F_k^2F^2_{n+1}+F^2_{k+1}F^2_n+F^2_{k+1}F^2_{n+1}\\
&=(F^2_{k+1}+F^2_k)(F^2_{n+1}+F^2_n)=F_{2k+1}F_{2n+1}.
\end{aligned}$$

To prove (3.2), we use (2.4) to show that $L_{n+k+1}=L_nF_k+L_{n+1}F_{k+1}$ and $L_{n-k}=L_{-(k+1)+n+1}=(-1)^k(L_nF_{k+1}-L_{n+1}F_k)$, and proceed similarly. $\square$

When $k=0$, (3.1) and (3.2) reduce to (1.1) and (1.2), respectively. For our next theorem, we use a "traditional" approach to prove the first part and, in contrast, the method of Dresel to prove the second part.

***Theorem 2:*** For $k, n \in \mathbf{Z}$,

$$F_{3k+1}F^3_{n+k+1}+F_{3k+2}F^3_{n+k}-F^3_{n-2k-1}=F_{3k+1}F_{3k+2}F_{3n} \tag{3.3}$$

and

$$F_{3k+1}L^3_{n+k+1}+F_{3k+2}L^3_{n+k}-L^3_{n-2k-1}=5F_{3k+1}F_{3k+2}L_{3n}. \tag{3.4}$$

***Proof:*** We proceed by induction on $n$. Suppose identity (3.3) holds for $n=m$, $m+1$, $m+2$, and $m+3$. Then

$$
\begin{aligned}
-(F_{3k+1}F^3_{m+k+1}+F_{3k+2}F^3_{m+k}-F^3_{m-2k-1}) &= -F_{3k+1}F_{3k+2}F_{3m},\\
-3(F_{3k+1}F^3_{(m+1)+k+1}+F_{3k+2}F^3_{(m+1)+k}-F^3_{(m+1)-2k-1}) &= -3F_{3k+1}F_{3k+2}F_{3(m+1)},\\
6(F_{3k+1}F^3_{(m+2)+k+1}+F_{3k+2}F^3_{(m+2)+k}-F^3_{(m+2)-2k-1}) &= 6F_{3k+1}F_{3k+2}F_{3(m+2)},\\
3(F_{3k+1}F^3_{(m+3)+k+1}+F_{3k+2}F^3_{(m+3)+k}-F^3_{(m+3)-2k-1}) &= 3F_{3k+1}F_{3k+2}F_{3(m+3)}.
\end{aligned}
$$

Adding, and making use of (2.6), we obtain

$$
\begin{aligned}
&F_{3k+1}F^3_{(m+4)+k+1}+F_{3k+2}F^3_{(m+4)+k}-F^3_{(m+4)-2k-1}\\
&=F_{3k+1}F_{3k+2}[3F_{3(m+3)}+6F_{3(m+2)}-3F_{3(m+1)}-F_{3m}]\\
&=F_{3k+1}F_{3k+2}F_{3(m+4)} \quad \text{(by Lemma 1)},
\end{aligned}
$$

and so (3.3) is true for $n=m+4$. But Lemma 2 shows that (3.3) holds for $n=0,1,2$, and 3, and so it holds for $n=4$ and, by induction, for all integers $n\geq 0$.

To establish (3.3) for all integers $n<0$, it suffices to replace $n$ by $-n$, and to prove that the resulting identity holds for all integers $n>0$. That is, it suffices to prove that

$$F^3_{n+2k+1}+(-1)^kF_{3k+2}F^3_{n-k}+(-1)^{k+1}F_{3k+1}F^3_{n-k-1}=F_{3k+1}F_{3k+2}F_{3n} \tag{3.5}$$

holds for all integers $n>0$. After making use of (2.1) to simplify (3.5) for $0\leq n\leq 3$, the equivalent of Lemma 2 is established as before, and the proof proceeds as above.

Following Dresel, we see that (3.4) is a homogeneous equation of degree 3 in the variable $n$. Therefore, to prove its validity for all integers $n$, it suffices to verify its validity for four different values of $n$, say $0\leq n\leq 3$. For $n=3$, (3.4) becomes

$$F_{3k+1}L^3_{k+4}+F_{3k+2}L^3_{k+3}-L^3_{2k-2}=380F_{3k+1}F_{3k+2} \quad \text{[by (2.2)]}. \tag{3.6}$$

But (3.6) is a homogeneous equation of degree 6 in the variable $k$. Therefore, to prove its validity for all integers $k$, it suffices to verify its validity for seven different values of $k$, say $0\leq k\leq 6$. This is easy to verify. We proceed similarly for the other three values of $n$, and this completes the proof of Theorem 2. □

When $k=0$, (3.3) and (3.4) reduce to (1.3) and (1.4), respectively.

## 4. A CONJECTURE FOR HIGHER POWERS

The identity

$$\sum_{j=0}^{m}(-1)^{\frac{j(j+3)}{2}}\begin{bmatrix} m \\ j \end{bmatrix}F^{m+1}_{n+m-j}=F_1\dots F_mF_{(m+1)(n+\frac{m}{2})} \tag{4.1}$$

is a special case of identity (5) of Torretto and Fuchs [6]. Here $\left[{m \atop j}\right]$ is the Fibonomial coefficient defined for integers $m\geq 0$ by

$$\begin{bmatrix} m \\ j \end{bmatrix}=\begin{cases} 0 & j<0 \text{ or } j>m,\\ 1 & j=0, m,\\ \frac{F_mF_{m-1}\dots F_{m-j+1}}{F_1F_2\dots F_j} & 0<j<m. \end{cases}$$

For an excellent discussion on generalized binomial coefficients, and a comprehensive list of references, see Gould [2].

In identity (4.1), $m=1$ yields (1.1) and $m=2$ yields an identity equivalent to (1.3). For $m=3$ and $m=4$, identity (4.1) becomes, respectively,

$$F_{n+3}^4+2F_{n+2}^4-2F_{n+1}^4-F_n^4=2F_{4n+6}, \tag{4.2}$$

and

$$F_{n+4}^5+3F_{n+3}^5-6F_{n+2}^5-3F_{n+1}^5+F_n^5=6F_{5n+10}. \tag{4.3}$$

Our generalizations in Theorems 1 and 2 prompted us to search for similar generalizations of (4.2) and (4.3) and their Lucas counterparts. We accomplished this by introducing the parameter $k$, assuming the existence of an identity of the required shape, and solving systems of simultaneous equations to find the coefficients. Indeed, after employing this constructive approach on several more instances of (4.1), we were led to a conjecture on a generalization of (4.1). First, we need some notation. Write, for example, $(F_5)_{(4)}=F_5F_4F_3F_2$ and $(F_4)_{(6)}=F_4F_3F_2F_1F_{-1}F_{-2}$. In general, we take $(F_n)_{(m)}$ to be the "falling" factorial, which begins at $F_n$ for $n\neq 0$, and is the product of $m$ Fibonacci numbers *excluding* $F_0$. Define $(F_0)_{(0)}=1$ and, for $m\geq 1$, $(F_0)_{(m)}=F_{-1}\dots F_{-m}$. We now state our conjecture in two parts.

***Conjecture:*** Let $k,m,n\in\mathbf{Z}$ with $m\geq 1$. Then:

***(a)***
$$\sum_{j=0}^{m-1}\frac{F_{n+k+m-j}^{m+1}}{(F_{m-1-j})_{(m-1)}F_{(m+1)k+m-j}}+(-1)^{\frac{m(m+3)}{2}}\frac{F_{n-mk}^{m+1}}{\prod_{j=1}^{m}F_{(m+1)k+j}}=F_{(m+1)\left(n+\frac{m}{2}\right)}.$$

***(b)*** To obtain the Lucas counterpart of (a), we first replace each occurrence of $F$ in the numerators on the left by $L$. Then, if $m$ is even, we replace the right side by

$$5^{\frac{m}{2}}L_{(m+1)\left(n+\frac{m}{2}\right)}.$$

If $m$ is odd, we replace the right side by

$$5^{\frac{m+1}{2}}F_{(m+1)\left(n+\frac{m}{2}\right)}$$

When $m=1$ our conjecture yields (3.1) and (3.2), and when $m=2$ it yields (3.3) and (3.4). For $k=0$ we claim that part (a) of our conjecture reduces to (4.1), but this is not obvious. It is useful to consider an example. If we take $m=4$, part (a) becomes

$$\frac{F_{n+k+4}^5}{2F_{5k+4}}+\frac{F_{n+k+3}^5}{F_{5k+3}}-\frac{F_{n+k+2}^5}{F_{5k+2}}-\frac{F_{n+k+1}^5}{2F_{5k+1}}+\frac{F_{n-4k}^5}{F_{5k+1}\dots F_{5k+4}}=F_{5n+10}. \tag{4.4}$$

Now putting $k=0$ we see that (4.4) reduces to (4.3). Indeed, we have performed similar verifications for $1\leq m\leq 9$. With these values of $m$ we have verified that our conjecture is true for a wide selection of the parameters $k$ and $n$.

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❖❖❖

# REPRESENTATION GRIDS FOR CERTAIN MORGAN-VOYCE NUMBERS


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## 1. BACKGROUND

Properties of representation number sequences $\{\mathcal{B}_n\}$, $\{\mathcal{C}_n\}$ associated with the Morgan-Voyce polynomials $B_n(x)$ and the related polynomials $C_n(x)$ were recently investigated in [1]. Hopefully, the notation and references in [1] will be accessible to the reader.

Complementary properties of the number sequences $\{\mathbf{b}_n\}$, $\{\mathbf{c}_n\}$ associated with the Morgan-Voyce polynomials $b_n(x)$ and the related polynomials $c_n(x)$ are now explored.

With $x = 1$ in these just-mentioned polynomials, we define the resulting numbers by

$$b_n = 3b_{n-1} - b_{n-2}, \qquad b_0 = 1,\ b_1 = 1, \tag{1.1}$$

and

$$c_n = 3c_{n-1} - c_{n-2}, \qquad c_0 = -1,\ c_1 = 1. \tag{1.2}$$

Accordingly, these numbers are

$$\begin{array}{rcrrrrrrrrrl} n &=& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots, \\ b_n &=& 1 & 1 & 2 & 5 & 13 & 34 & 89 & 233 & 610 & \ldots, \\ c_n &=& -1 & 1 & 4 & 11 & 29 & 76 & 199 & 521 & 1364 & \ldots. \end{array} \tag{1.3}$$

Consider now the unit coefficient representation sums for $b_n$, $c_n$ analogous to those for $B_n$, $C_n$ [1]. Irrespective of the uniqueness or otherwise of the representations (and of questions of minimality or maximality), we may assert that, for the representation number sequences $\{\mathbf{b}_n\}$, $\{\mathbf{c}_n\}$,

$$\mathbf{b}_n = \sum_{i=1}^{n} b_n = F_{2n} = F_n L_n \tag{1.4}$$

and

$$\mathbf{c}_n = \sum_{i=1}^{n} c_n = L_{2n} - 2 = \begin{cases} L_n^2 & n \text{ odd}, \\ 5F_n^2 & n \text{ even}, \end{cases} \tag{1.5}$$

in terms of the Fibonacci and Lucas numbers $F_n$, $L_n$.

Elements of $\{\mathbf{b}_n\}$, $\{\mathbf{c}_n\}$ are thus

$$\begin{array}{rcrrrrrrrrrl} n &=& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots, \\ \mathbf{b}_n &=& 0 & 1 & 3 & 8 & 21 & 55 & 144 & 377 & 987 & \ldots, \\ \mathbf{c}_n &=& 0 & 1 & 5 & 16 & 45 & 121 & 320 & 841 & 2205 & \ldots. \end{array} \tag{1.6}$$

Why, we may ask, are these numbers worthy of our consideration? Firstly, as mathematical constructs they have an inherent interest to the inquiring mind ("because they are there"!). Secondly, as the theory—necessarily compact—unfolds, they add a little, however modest, to our knowledge of number relationships. Moreover, they complete the theme initiated in [1].

## 2. PROPERTIES OF $\mathbf{b}_n$, $\mathbf{c}_n$

One may readily establish the fundamental infrastructure of these two number systems, details of which are herewith reported (in pairs, for comparison).

***Recurrences:***

$$\mathbf{b}_n = 3\mathbf{b}_{n-1} - \mathbf{b}_{n-2}, \tag{2.1}$$

$$\mathbf{c}_n = 3\mathbf{c}_{n-1} - \mathbf{c}_{n-2} + 2. \tag{2.2}$$

***Binet forms:***

$$\mathbf{b}_n = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \quad (\alpha\beta = 1,\ \alpha\beta = -1), \tag{2.3}$$

$$\mathbf{c}_n = \alpha^{2n} + \beta^{2n} - 2. \tag{2.4}$$

***Generating functions:***

$$\sum_{i=1}^{\infty} \mathbf{b}_i x^{i-1} = [1 - (3x - x^2)]^{-1}, \tag{2.5}$$

$$\sum_{i=1}^{\infty} \mathbf{c}_i x^{i-1} = (1+x)[1 - (4x - 4x^2 + x^3)]^{-1}. \tag{2.6}$$

***Simson formulas:***

$$\mathbf{b}_{n+1}\mathbf{b}_{n-1} - \mathbf{b}_n^2 = -1, \tag{2.7}$$

$$\mathbf{c}_{n+1}\mathbf{c}_{n-1} - \mathbf{c}_n^2 = 1 - 2\mathbf{c}_n. \tag{2.8}$$

***Summations:***

$$\sum_{i=1}^{n} \mathbf{b}_i = F_{2n+1} - 1, \tag{2.9}$$

$$\sum_{i=1}^{n} \mathbf{c}_i = L_{2n+1} - (2n+1), \tag{2.10}$$

$$\sum_{i=1}^{n} \mathbf{b}_{2i} = \frac{1}{5}(L_{2n+1}^2 - 5), \tag{2.11}$$

$$\sum_{i=1}^{n} \mathbf{c}_{2i} = F_{4n+2} - (2n+1), \tag{2.12}$$

$$\sum_{i=1}^{n} \mathbf{b}_{2i-1} = F_{2n}^2, \tag{2.13}$$

$$\sum_{i=1}^{n} \mathbf{c}_{2i-1} = F_{4n} - 2n, \tag{2.14}$$

$$\sum_{i=1}^{n} (-1)^{i+1}\mathbf{b}_i = \frac{1}{5}[1 - (-1)^n L_{2n+1}], \tag{2.15}$$

$$\sum_{i=1}^{n} (-1)^{i+1}\mathbf{c}_i = (-1)^n[1 - F_{2n+1}]. \tag{2.16}$$

***Other simple properties:***

$$\mathbf{b}_{-n} = -\mathbf{b}_n, \tag{2.17}$$

$$\mathbf{c}_{-n} = \mathbf{c}_n, \tag{2.18}$$

$$\mathbf{b}_n - \mathbf{b}_{n-1} = F_{2n-1}, \tag{2.19}$$

$$\mathbf{c}_n - \mathbf{c}_{n-1} = L_{2n-1}, \tag{2.20}$$

$$\mathbf{b}_n + \mathbf{b}_{n+1} = L_{2n-1} \text{ also,} \tag{2.21}$$

$$\mathbf{c}_n + \mathbf{c}_{n+1} = 5L_{2n-1} - 4, \tag{2.22}$$

$$\mathbf{b}_n - \mathbf{b}_{n-2} = L_{2n-2}, \tag{2.23}$$

$$\mathbf{c}_n - \mathbf{c}_{n-2} = 5F_{2n-2}, \tag{2.24}$$

$$\mathbf{b}_n^2 - \mathbf{b}_{n-1}^2 = F_{4n-2} = F_{2n-1}L_{2n-1}. \tag{2.25}$$

## 3. THE REPRESENTATION GRIDS

Next, we introduce the concepts

$$\begin{aligned} \mathbf{b}'_n &= \mathbf{b}_{n+1} + \mathbf{b}_{n-1} \\ &= 3\mathbf{b}_n = 3F_{2n} = \mathcal{B}'_n - \mathcal{B}'_{n-1}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \mathbf{c}'_n &= \mathbf{c}_{n+1} + \mathbf{c}_{n-1} \\ &= 3\mathbf{c}_n + 2 = 3L_{2n} - 4 = \mathcal{B}'_n + \mathcal{B}'_{n+1}, \end{aligned} \tag{3.2}$$

on invoking [1].

Repeating the summation process developed in [1], i.e., $\mathbf{b}''_n = \mathbf{b}'_{n+1} + \mathbf{b}'_{n-1}$, we eventually arrive at the more general notations

$$\mathbf{b}_n^{(m)} = \mathbf{b}_{n+1}^{(m)} + \mathbf{b}_{n-1}^{(m)} \quad (\mathbf{b}_n^{(0)} = \mathbf{b}_n), \tag{3.3}$$

and

$$\mathbf{c}_n^{(m)} = \mathbf{c}_{n+1}^{(m)} + \mathbf{c}_{n-1}^{(m)} \quad (\mathbf{c}_n^{(0)} = \mathbf{c}_n). \tag{3.4}$$

As in [1], these data can be organized in (representation) *grids* for $\mathbf{b}_n^{(m)}$ and $\mathbf{c}_n^{(m)}$, where $m$ denotes columns and $n$ rows.

Various approaches allow us to validate the properties recorded below, some of which are readily obtainable from the patterns in the rectangular grids, which the reader should construct for visual emphasis and clarification of the theory.

***Zero subscripts:***

$$\mathbf{b}_0^{(m)} = 0 = \mathcal{B}_0^{(0)}, \tag{3.5}$$

$$\mathbf{c}_0^{(m)} = 2(3^m - 2^m) = 2\mathcal{B}_0^{(m)} = -2\mathcal{C}_0^{(m)} \quad \text{by [1].} \tag{3.6}$$

***Negative subscripts:***

$$\mathbf{b}_{-n}^{(m)} = -\mathbf{b}_n^{(m)}, \tag{3.7}$$

$$\mathbf{c}_{-n}^{(m)} = \mathbf{c}_n^{(m)}. \tag{3.8}$$

***Recurrences:***

$$\text{(columns)} \quad \begin{cases} \mathbf{b}_n^{(m)} = 3\mathbf{b}_{n-1}^{(m)} - \mathbf{b}_{n-2}^{(m)}, & (3.9) \\ \mathbf{c}_n^{(m)} = 3\mathbf{c}_{n-1}^{(m)} - \mathbf{c}_{n-2}^{(m)} + 2^{m+1}, & (3.10) \end{cases}$$

$$\text{(rows)} \quad \begin{cases} \mathbf{b}_n^{(m)} = 3^m \mathbf{b}_n = 3^m F_{2n} \quad (= 3\mathbf{b}_n^{(m-1)}), & (3.11) \\ \mathbf{c}_n^{(m)} = 3^m \mathbf{c}_n + \mathbf{c}_0^{(m)}. & (3.12) \end{cases}$$

***Binet forms:***

$$\mathbf{b}_n^{(m)} = 3^m(\alpha^{2n} - \beta^{2n}) / (\alpha - \beta), \tag{3.13}$$

$$\mathbf{c}_n^{(m)} = 3^m(\alpha^{2n} + \beta^{2n}) - 2^{m+1}. \tag{3.14}$$

***Generating functions:***

$$\sum_{i=1}^{\infty} \mathbf{b}_i^{(m)} x^{i-1} = 3^m[1 - (3x - x^2)]^{-1}, \tag{3.15}$$

$$\sum_{i=1}^{\infty} \mathbf{c}_i^{(m)} x^{i-1} = 3^m(3 - 2x)[1 - (3x - x^2)]^{-1} - 2^{m+1}. \tag{3.16}$$

***Simson formulas:***

$$\mathbf{b}_{n+1}^{(m)} \mathbf{b}_{n-1}^{(m)} - (\mathbf{b}_n^{(m)})^2 = -3^{2m}, \tag{3.17}$$

$$\mathbf{c}_{n+1}^{(m)} \mathbf{c}_{n-1}^{(m)} - (\mathbf{c}_n^{(m)})^2 = 3^m\{3^m \cdot 5 - 2^{m+1} L_{2n}\}. \tag{3.18}$$

***Summations:***

$$\sum_{i=1}^{n} \mathbf{b}_i^{(m)} = 3^m(F_{2n+1} - 1), \tag{3.19}$$

$$\sum_{i=1}^{n} \mathbf{c}_i^{(m)} = 3^m(L_{2n+1} - 1) - 2^{m+1} n. \tag{3.20}$$

***Other simple properties:***

$$\mathbf{b}_n^{(m)} - \mathbf{b}_{n-1}^{(m)} = 3^m F_{2n-1}, \tag{3.21}$$

$$\mathbf{c}_n^{(m)} - \mathbf{c}_{n-1}^{(m)} = 3^m L_{2n-1}, \tag{3.22}$$

$$\mathbf{b}_n^{(m)} + \mathbf{b}_{n-1}^{(m)} = 3^m L_{2n-1} \quad \text{also}, \tag{3.23}$$

$$\mathbf{c}_n^{(m)} + \mathbf{c}_{n-1}^{(m)} = 3^m \cdot 5F_{2n-1} - 2(3^m + 2^m), \tag{3.24}$$

$$\mathbf{b}_n^{(m)} + \mathbf{c}_n^{(m)} = 2\mathcal{B}_n^{(m)}, \tag{3.25}$$

$$\mathbf{c}_n^{(m)} - \mathbf{b}_n^{(m)} = 2\mathcal{B}_{n-1}^{(m)}. \tag{3.26}$$

## 4. FOREGROUND

### 1. Augmented Sequence

Let us now recall, as in [1], the *augmented sequence* $\{\mathcal{B}_n^*(a, b, k) \equiv \mathcal{B}_n^*\}$ defined by

$$\mathcal{B}_{n+2}^*(a, b, k) = 3\mathcal{B}_{n+1}^*(a, b, k) - \mathcal{B}_n^*(a, b, k) + k. \tag{4.1}$$

Initially, assume

$$\mathcal{B}_1^*(a,b,k)=a,\quad \mathcal{B}_2^*(a,b,k)=b. \tag{4.2}$$

Hence,

$$\mathcal{B}_{n+1}^*(1,3,0)=\mathbf{b}_n, \tag{4.3}$$

and

$$\mathcal{B}_{n+1}^*(1,5,2)=\mathbf{c}_n, \tag{4.4}$$

while

$$\mathcal{B}_{n+1}^*(1,2,0)=b_n, \tag{4.5}$$

and

$$\mathcal{B}_{n+1}^*(1,4,0)=c_n. \tag{4.6}$$

## 2. Brahmagupta Polynomials

Very recently, Suryanarayan showed in [4] and [5] how, by means of the *Brahmagupta matrix*, to generate polynomials $x_n$ and $y_n$ (*Brahmagupta polynomials*) which include *inter alia* Fibonacci, Pell, and Pell-Lucas polynomials, as well as the Morgan-Voyce polynomials $B_n(x)=x_n$ and $b_n(x)=y_n$ described in [1] and [2].

Suppose we express the vital difference equations [4, eqn. (8)], [5, eqn. (9)] in a slightly varied notation as

$$x_{n+1}=Px_n-Qx_{n-1},\quad y_{n+1}=Py_n-Qy_{n-1}. \tag{4.7}$$

Selecting $P=x+2$, $Q=1$, $x_1=2$, $x_2=P$, and $y_1=-1$, $y_2=1$ (so $y_3=x+3$) in (4.7), we readily come to the polynomials $C_n(x)=x_n$ and $c_n(x)=y_n$, which [1], [2] are adjunct to $B_n(x)$ and $b_n(x)$.

## 3. Further Developments

These might profitably include, for instance,

**a)** properties of $b_{-n}, c_{-n}$ $(n>0)$,

**b)** extension of the theory to polynomials $\mathbf{b}_n(x)$, $\mathbf{c}_n(x)$ (and also $\mathcal{B}_n(x)$, $\mathcal{C}_n(x)$ [1]),

**c)** construction of a representation table of sufficient scope to afford numerical enhancement of the patterns contained therein,

**d)** uniqueness or otherwise of the representation, and

**e)** any additional Brahmagupta properties.

## 4. Associated Legendre Polynomials

The author has become aware that the Morgan-Voyce polynomials $b_n(x)$ defined in (1.1) are essentially the *associated Legendre polynomials* $\rho_n(x)$ described by Riordan [3, p. 85]. In fact, $b_{n+1}(x)=\rho_n(x)$, e.g., $b_3(x)=\rho_2(x)=1+3x+x^2$. Properties of $\rho_n(x)$ listed in [3] may then be cast in the $b_n(x)$ notation. Essential links for the equality of $\rho_n(x)$ and $b_{n+1}(x)$ are the closed forms and Chebyshev polynomials results in [3, p. 85] and [2, (2.21) and (4.14)].

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❖❖❖

# ON THE SEQUENCES $T_n = T_{n-1} + T_{n-2} + hn + k$


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## 1. INTRODUCTION

A colleague of ours who needed to evaluate the computational complexity of certain algorithms for optimal traffic routing on multi-service networks asked us for a closed-form expression for the sum of the first $N$ terms of the sequences $\{X_n\}$ and $\{Y_n\}$ obeying the second-order nonhomogeneous recurrence relations

$$X_n = X_{n-1} + X_{n-2} + k \quad (k,\, X_0, \text{ and } X_1 \text{ arbitrary}) \tag{1.1}$$

and

$$Y_n = Y_{n-1} + Y_{n-2} + n \quad [Y_0 = 0;\ Y_1 = 1], \tag{1.2}$$

respectively. His request led us to investigate the main properties of the more general sequences $\{T_n(h, k; a, b)\}$ (or simply $\{T_n\}$ if no misunderstanding can arise) defined as

$$T_n = T_{n-1} + T_{n-2} + hn + k \quad [T_0 = a;\ T_1 = b], \tag{1.3}$$

where $h$, $k$, $a$, and $b$ are arbitrary integers. In doing so, beyond answering the question posed by our colleague, we generalize some results established in [1] and [7]. It is worth pointing out that $T_n$ can be expressed either as the third-order inhomogeneous recurrence relation

$$T_n = 2T_{n-1} - T_{n-3} + h \tag{1.4}$$

with initial conditions

$$T_0,\ T_1, \text{ and } T_2 = T_0 + T_1 + 2h + k, \tag{1.4'}$$

or as the fourth-order homogeneous recurrence relation

$$T_n = 3T_{n-1} - 2T_{n-2} - T_{n-3} + T_{n-4} \tag{1.5}$$

with initial conditions given by (1.4') and the additional condition $T_3 = T_0 + 2T_1 + 5h + 2k$.

As usual, throughout the paper, $F_n$ and $L_n$ will denote the $n^{\text{th}}$ Fibonacci and Lucas number, respectively.

## 2. CLOSED-FORM EXPRESSION FOR $T_n$

The closed-form expression for $T_n$ is, quite obviously, a powerful tool for discovering properties of these numbers. From the definition (1.3), by using standard methods (e.g., see [4]), we found that

$$T_n = AF_{n-1} + BF_n - h(n+3) - k \tag{2.1}$$

where

$$\begin{cases} A = 3h + k + a, \\ B = 4h + k + b. \end{cases} \tag{2.2}$$

The reader can immediately check that (2.1) and (2.2) satisfy both the recurrence and the initial conditions in (1.3). This fact proves the validity of the above expressions.

By using (1.3) or (2.1) and the well-known identity $F_{-n} = (-1)^{n-1}F_n$, the extension of $T_n$ through negative values of the subscript $n$ is readily obtained. Namely, we get

$$T_{-n} = T_n + 2hn - \begin{cases} (5h+k+2b-a)F_n & (n \text{ even}), \\ (3h+k+a)L_n & (n \text{ odd}). \end{cases} \tag{2.3}$$

Observe that the second identity of (2.3) is formally independent of $b$.

***Special cases:***

$T_n(0, 0; 0, 1) = F_n,$ (2.4)

$T_n(0, 0; 2, 1) = L_n,$ (2.5)

$T_n(0, k; a, b) = X_n$ (see (1.1) and [1]), (2.6)

$T_n(1, 0; 0, 1) = Y_n = F_{n+4} - n - 3$ (see (1.2) and Seq. 1053 of [5]), (2.7)

$T_n(h, h; h, h) = h(L_{n+3} - F_{n-2} - n - 4),$ (2.8)

$T_n(0, k; 0, 0) = k(F_{n+1} - 1).$ (2.9)

## 3. SOME SPECIAL PROPERTIES OF THE SEQUENCES $\{T_n\}$

Here we point out three properties of the sequences $\{T_n\}$ that seem especially interesting to us. Their proofs are given in full detail. Let us state the following.

***Proposition 1:*** For an arbitrarily given integer $m$, we have

$$T_n(h, k; a, b) = T_{n-m}(h, k+mh;\ T_m(h, k; a, b),\ T_{m+1}(h, k; a, b)). \tag{3.1}$$

***Proof:*** Use (2.1) and (2.2) to rewrite the right-hand side of (3.1) as

$$\begin{aligned} &[3h+k+mh+AF_{m-1}+BF_m-h(m+3)-k]F_{n-m-1} \\ &\quad +[4h+k+mh+AF_m+BF_{m+1}-h(m+4)-k]F_{n-m}-h(n-m+3)-k-mh \\ &=[AF_{m-1}+BF_m]F_{n-m-1}+[AF_m+BF_{m+1}]F_{n-m}-h(n+3)-k \\ &=A(F_{m-1}F_{n-m-1}+F_mF_{n-m})+B(F_mF_{n-m-1}+F_{m+1}F_{n-m})-h(n+3)-k \\ &=AF_{n-1}+BF_n-h(n+3)-k=T_n(h, k; a, b) \quad [\text{from } \mathrm{I}_{26} \text{ of [3] and (2.1)}]. \ \square \end{aligned}$$

***Proposition 2:*** For given integers $h$, $k$, $a$, $b$, $h_1$, and all $n$, we have

$$T_n(h, k; a, b) = T_n(h_1, k-(n+3)s; a+ns, b+(n-1)s), \tag{3.2}$$

where $s = h_1 - h$.

***Proof:*** From (2.1), it is patent that identity (3.2) can be obtained by solving the system

$$\begin{cases} A = 3h_1 + k_1 + a_1, \\ B = 4h_1 + k_1 + b_1, \\ h(n+3) + k = h_1(n+3) + k_1. \end{cases} \tag{3.3}$$

Put $h_1 = h + s$ in (3.3) and use the third equation to obtain $k_1 = k - (n+3)s$. Then use (2.2), and replace the above expression for $k_1$ in the first two equations of (3.3) to get $a_1 = a + ns$ and $b_1 = b + (n-1)s$. $\square$

Finally, we observe [see (2.7)] that there exist values of the parameters $(h, k; a, b)$ for which $T_n$ has the form

$$T_n = F_m + p(n), \tag{3.4}$$

where $p(n)$ is a first-degree polynomial in $n$. Whereas such a problem is put forward: find conditions on $(h, k; a, b)$ for $T_n$ to have the form (3.4). We give the following proposition.

***Proposition 3:*** For arbitrarily given integers $a$, $b$, and $s$, we have

$$T_n(F_{s-1} + a - b, 3b - L_{s-2} - 4a; a, b) = F_{n+s} - n(F_{s-1} + a - b) - F_s + a. \tag{3.5}$$

***Proof:*** From (2.1), (2.2), and $I_{26}$ of [3], it is evident that (3.4) can be obtained if

$$\begin{cases} A = 3h + k + a = F_s, \\ B = 4h + k + b = F_{s+1}. \end{cases} \tag{3.6}$$

Subtracting the first equation of (3.6) from the second equation, one obtains

$$h = F_{s-1} + a - b, \tag{3.7}$$

and from the first equation,

$$\begin{aligned} k = F_s - 3h - a &= F_s - 3F_{s-1} - 4a + 3b \quad \text{[from (3.7)]} \\ &= 3b - L_{s-2} - 4a. \end{aligned} \tag{3.8}$$

Expressions (3.7) and (3.8) give the left-hand side of (3.5). Its right-hand side can be obtained from (2.1) and (2.2), after some manipulation involving the use of the identities $3F_{s-1} - L_{s-2} = F_s$, $4F_{s-1} - L_{s-2} = F_{s+1}$, and $I_{26}$ of [3]. □

***Examples:*** The right-hand side of (2.7) emerges from the choice $(a, b, s) = (0, 1, 4)$. As a further example, the choice $(a, b, s) = (10, 7, 8)$ yields the numbers $T_n(16, -37; 10, 7) = F_{n+8} - 16n - 11$.

## 4. BASIC IDENTITIES INVOLVING THE NUMBERS $T_n$

Here we give a brief account of the basic identities involving the numbers $T_n$. To save space, the number of detailed proofs will be kept to a minimum (Subsection 4.2).

### 4.1 Results

***Generating function***

By using (2.1), (2.2), and [6, p. 53], we get

$$\sum_{n=0}^{\infty} x^n T_n = \frac{(h + k + a - b)x^3 - (2h + k + 3a - 2b)x^2 + (3a - b)x - a}{x^4 - x^3 - 2x^2 + 3x - 1}. \tag{4.1}$$

Observe that, for $T_n \equiv Y_n$ [see (1.2)], the numerator on the right-hand side of (4.1) collapses to $-x$.

***Simson formula analog***

$$\begin{aligned} \sigma(T_n) &:= T_n^2 - T_{n-1}T_{n+1} \\ &= (-1)^n C + (hn + k)(T_{n-3} + hn + k) - 2h(T_{n-4} + k) - h^2(2n - 3), \end{aligned} \tag{4.2}$$

where

$$C = A^2 + AB - B^2 = 5h(h + k + 2a - b) + k(k + 3a - b) + a^2 + ab - b^2. \tag{4.3}$$

From (4.2), (4.3), and (2.7), we see that

$$\sigma(Y_n) = nF_{n+1} - 2F_n + 1 - (-1)^n. \tag{4.4}$$

***Sums and differences***

$$T_{n+m} + T_{n-m} = \begin{cases} L_m[T_n + h(n+3) + k] - 2[h(n+3)+k] & (m \text{ even}), \\ F_m[T_n + 2T_{n-1} + h(3n+7) + 3k] - 2[h(n+3)+k] & (m \text{ odd}). \end{cases} \tag{4.5}$$

$$T_{n+m} - T_{n-m} = \begin{cases} F_m[T_n + 2T_{n-1} + h(3n+7) + 3k] - 2hm & (m \text{ even}), \\ L_m[T_n + h(n+3) + k] - 2hm & (m \text{ odd}). \end{cases} \tag{4.6}$$

***Duplication formula***

For $n$ even (resp. odd), let $m=n$ in the first (resp. second) identity of (4.5) [resp. (4.6)] to obtain

$$T_{2n} = L_n[T_n + h(n+3) + k] - (-1)^n a - \begin{cases} 2[h(n+3)+k] & (n \text{ even}), \\ 2hn & (n \text{ odd}). \end{cases} \tag{4.7}$$

Observe that (4.7) is formally independent of $b$.

***Finite sums***

$$S_N(T) := \sum_{n=0}^{N} T_n = T_{N+2} - \frac{(N+1)[h(N+4)+2k]}{2} - b. \tag{4.8}$$

From (4.8), (1.1), (1.2), (2.1), and (2.2), we obtain the special identities

$$S_N(T \equiv X) = k(F_{N+3} - N - 2) + X_0 F_{N+1} + X_1(F_{N+2} - 1) \tag{4.9}$$

and

$$S_N(T \equiv Y) = F_{N+6} - (N^2 + 7N + 16)/2, \tag{4.10}$$

which answer the questions that gave rise to our study.

Further, we get the identities:

$$\sum_{n=0}^{N} nT_n = NT_{N+2} - T_{N+3} - \frac{h(N^3 + 3N^2 - 7N - 15)}{3} - \frac{k(N^2 - N - 4)}{2} + a + 2b, \tag{4.11}$$

$$\sum_{n=0}^{N} \binom{N}{n} T_n = T_{2N} - h[2^{N-1}(N+6) - 2N - 3] - k(2^N - 1), \tag{4.12}$$

$$\sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} 2^n T_{2n} = T_{3N} - hN, \tag{4.13}$$

the last of which generalizes (19) of [7].

***Convolution (for $T_n \equiv Y_n$)***

$$\sum_{n=0}^{N} Y_n Y_{N-n} = \frac{NL_{N+8} + F_{N+10}}{5} - L_{N+9} + \frac{N^3 + 18N^2 + 131N}{6} + 65 \tag{4.14}$$

$$= F_{N+8}^{(1)} - 4F_{N+9} + \frac{N^3 + 18N^2 + 131N}{6} + 65, \tag{4.14'}$$

where $F_n^{(1)}$ denotes the $n^{\text{th}}$ term of the Fibonacci first derivative sequence [2].

### 4.2 Proofs

***Proof of (4.2) (a sketch):*** From (2.1) and (2.2), after a good deal of calculation involving the use of some well-known Fibonacci identities ([3], [6]), one gets

$$\begin{aligned}\sigma(T_n) &= (-1)^n C + h^2 + A[(hn+k)F_{n-4} - 2hF_{n-5}] + B[(hn+k)F_{n-3} - 2hF_{n-4}]\\ &= (-1)^n C + h^2 + (hn+k)(AF_{n-4} + BF_{n-3}) - 2h(AF_{n-5} + BF_{n-4})\\ &= (-1)^n C + (hn+k)T_{n-3} - 2hT_{n-4} + h^2 + (hn+k)^2 - 2h^2(n-1) - 2hk,\end{aligned}$$

whence (4.2) is immediately obtained. □

***Proof of (4.6) (for m even):*** By using (2.1) and (2.2), rewrite the left-hand side of (4.6) as

$$\begin{aligned}&A(F_{n+m-1} - F_{n-m-1}) + B(F_{n+m} - F_{n-m}) - 2hm\\ &= AL_{n-1}F_m + BL_nF_m - 2hm \quad (\text{from } \mathrm{I}_{24} \text{ of } [3])\\ &= F_m(AL_{n-1} + BL_n) - 2hm = F_m(AF_{n-2} + BF_{n-1} + AF_n + BF_{n+1}) - 2hm\\ &= F_m\{T_{n-1} + T_{n+1} + 2[h(n+3)+k]\} - 2hm\\ &= F_m[T_n + 2T_{n-1} + h(3n+7) + 3k] - 2hm \quad [\text{from } (1.3)]. \ \square\end{aligned}$$

***Proofs of (4.8), (4.11), and (4.12):*** From (4.8) and the recurrence (1.3), write

$$\begin{aligned}S_N(T) &= \sum_{n=0}^{N} T_{n-1} + \sum_{n=0}^{N} T_{n-2} + \sum_{n=0}^{N}(hn+k)\\ &= \sum_{n=-1}^{N-1} T_n + \sum_{n=-2}^{N-2} T_n + H \quad \left(H := \sum_{n=0}^{N}(hn+k)\right)\\ &= S_N(T) - T_N + T_{-1} + S_N(T) - T_N - T_{N-1} + T_{-1} + T_{-2} + H,\end{aligned}$$

whence

$$\begin{aligned}S_N(T) &= 2T_N + T_{N-1} - 2T_{-1} - T_{-2} - h\\ &= T_N + T_{N+1} - h(N+1) - k - (T_{-1} + T_0 - k) - H \quad [\text{from } (1.3)]\\ &= T_{N+2} - h(N+2) - k - h(N+1) - k - (T_1 - h - 2k) - H\\ &= T_{N+2} - 2h(N+1) - b - H. \qquad (4.15)\end{aligned}$$

Take the meaning of $H$ into account and use (4.15) to obtain (4.8). The identities (4.11) and (4.12) can be proved by means of a similar technique. □

***Proof of (4.13) (Hint):***

***(i)*** Identity (4.13) can be proved by means of the technique used by Zhang [7] after replacing (18) of [7] by the identity $T_n = 2T_{n-1} - T_{n-3} + h$, which can be obtained readily from (1.3).

***(ii)*** Alternatively, use (2.1) to rewrite the left-hand side of (4.13) as

$$(-1)^N\left\{A\sum_{n=0}^{N}\binom{N}{n}(-2)^n F_{2n-1} + B\sum_{n=0}^{N}\binom{N}{n}(-2)^n F_{2n} - \sum_{n=0}^{N}\binom{N}{n}(-2)^n[h(2n+3)+k]\right\},$$

and use the Binet form for Fibonacci numbers along with (3.3) and (3.4) of [2]. □

***Proofs of (4.14) and (4.14'):*** First, use (2.7) and the Binet form for Fibonacci numbers to get

$$
\begin{aligned}
Y_n Y_{N-n} &= (F_{n+4} - n - 3)[F_{N-n+4} + n - (N+3)] \\
&= \frac{L_{N+8} - (-1)^n L_{N-2n}}{5} + nF_{n+4} - (N+3)F_{n+4} - nF_{N-n+4} \\
&\qquad\qquad\qquad - n^2 + nN - 3F_{N-n+4} + 3(N+3).
\end{aligned}
\tag{4.16}
$$

Then, after denoting the left-hand side of (4.14) by $C_N$ and letting $S[x(n)] := \sum_{n=0}^{N} x(n)$ for notational convenience, use (4.16) to write

$$
\begin{aligned}
C_N &= \tfrac{1}{5}S[L_{N+8}] - \tfrac{1}{5}S[(-1)^n L_{N-2n}] + S[nF_{n+4}] - (N+3)S[F_{n+4}] \\
&\quad - S[nF_{N-n+4}] - S[n^2] + NS[n] - 3S[F_{N-n+4}] + 3(N+3)S[1] \\
&:= S_1 - S_2 + S_3 - S_4 - S_5 - S_6 + S_7 - S_8 + S_9.
\end{aligned}
\tag{4.17}
$$

By using the Binet forms for Fibonacci and Lucas numbers, the geometric series formula and some well-known identities ($I_1$ and $I_{40}$ of [3] inclusive), one obtains the partial results,

*(i)* $S_1 = (N+1)L_{N+8}/5$,

*(ii)* $S_2 = 2F_{N+1}/5$,

*(iii)* $S_3 = NF_{N+6} - F_{N+7} + 13$,

*(iv)* $S_4 = (N+3)(F_{N+6} - 5)$,

*(v)* $S_5 = F_{N+7} - 5(N+4) + 7$,

*(vi)* $S_6 = N(N+1)(2N+1)/6$,

*(vii)* $S_7 = N^2(N+1)/2$,

*(viii)* $S_8 = 3(F_{N+6} - 5)$,

*(ix)* $S_9 = 3(N+3)(N+1)$,

among which *(ii)* is quite interesting *per se*. Finally, from (4.17) and *(i)-(ix)*, one finds

$$
C_N = \frac{(N+1)L_{N+8} - 2F_{N+1}}{5} - 6F_{N+6} - 2F_{N+7} + \frac{N^3 + 18N^2 + 131N}{6} + 65,
$$

from which, by applying properties of Fibonacci-Lucas sequences, (4.14) can be obtained immediately. The right-hand side of (4.14') can be found by using (2.5) of [2] to rewrite the first two addends on the right-hand side of (4.14) as

$$
\begin{aligned}
F_{N+8}^{(1)} + (F_{N+8} + F_{N+10} - 8L_{N+8} - 5L_{N+9})/5 &= F_{N+8}^{(1)} - 4(L_{N+10} + L_{N+8})/5 \\
&= F_{N+8}^{(1)} - 4F_{N+9} \quad \text{(from } I_9 \text{ of [3])}. \ \square
\end{aligned}
$$

## 5. FURTHER WORK

From (4.14), one may observe that

$$
Q_n := (nL_{n+8} + F_{n+10})/5 \tag{5.1}
$$

is an integer for all $n$. In fact, it is immediate to check that $Q_n$ obeys the recurrence

$$
Q_n = Q_{n-1} + Q_{n-2} + F_{n+7} \quad [Q_0 = 11;\ Q_1 = 33]. \tag{5.2}
$$

This fact suggests the idea of studying properties of the more general sequences $\{Q_n(k)\}$, defined by

$$
Q_n(k) = Q_{n-1}(k) + Q_{n-2}(k) + F_{n+k} \quad [Q_0(k) = a;\ Q_1(k) = b], \tag{5.3}
$$

the elements of which have the closed-form expression

$$
Q_n(k) = aF_{n-1} + bF_n + (nL_{n+k+1} - L_{k+2}F_n)/5. \tag{5.4}
$$

Much more generally, one might investigate properties of the sequences $\{R_n\}$, defined by

$$R_n = R_{n-1} + R_{n-2} + f_n \quad [R_0 = a;\ R_1 = b], \tag{5.5}$$

where $f_n$ is *any* integer-valued function of $n$. This study will be the aim of a future paper. For the time being, we confine ourselves to showing a compact form for $R_n$. Namely, we get

$$R_n = aF_{n-1} + bF_n + \sum_{r=2}^{n} f_r F_{n-r+1}. \tag{5.6}$$

Observe that, as special cases, $R_n = Q_n(k)$ (resp. $T_n$) for $f_n = F_{n+k}$ (resp. $hn+k$). It can be noted that letting $f_n = hn+k$ in (5.6) yields the expression

$$R_n = T_n = aF_{n-1} + bF_n + h(L_{n+2} - n - 3) + k(F_{n+1} - 1), \tag{5.7}$$

which can be proved easily to be an equivalent form for (2.1). As further special cases, we urge the interested reader to prove that, if $f_n = X^n$, then

$$R_n = (a-1)F_{n-1} + (b-X-1)F_n + \frac{X^{n+2} - XF_{n+2} - F_{n+1}}{X^2 - X - 1}, \tag{5.8}$$

whereas, if $f_n = F_n$ (resp. $L_n$) and $(a, b) = (0, 1)$, then

$$R_n = Q_n(0) = \frac{nL_{n+1} + 2F_n}{5} = \frac{(n+1)L_{n+1} - F_{n+1}}{5} = F_{n+1}^{(1)} \tag{5.9}$$

(see (5.4) and [2]), and

$$R_n = nF_{n+1}, \tag{5.10}$$

respectively.

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# SOME REMARKS ON FIBONACCI MATRICES

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In [1], Dazheng studies Fibonacci matrices, namely matrices $M$ such that every entry of every positive power of $M$ is either 0 or plus or minus a Fibonacci number. He gives 40 such four-by-four matrices. In the following, we give an interpretation of these matrices, from which we give simpler proofs of several of his theorems. We also determine all two-by-two Fibonacci matrices.

Let $\zeta = e^{2\pi i/5}$ be a primitive fifth root of unity. Then $\zeta$ is a root of the irreducible polynomial $X^4+X^3+X^2+X+1$, so the field $\mathbb{Q}(\zeta)$ is a vector space of dimension 4 over $\mathbb{Q}$ with basis $B=\{1,\zeta,\zeta^2,\zeta^3\}$. The ring of algebraic integers in $\mathbb{Q}(\zeta)$ is $\mathbb{Z}[\zeta]$. The units of this ring are of the form $(-\zeta)^m\phi^n$, $0\le m\le 9$, $n\in\mathbb{Z}$, where $\phi=(1+\sqrt{5})/2=-(\zeta^2+\zeta^{-2})$.

If $\alpha\in\mathbb{Q}(\zeta)$, then multiplication by $\alpha$ gives a linear transformation of $\mathbb{Q}(\zeta)$, regarded as a vector space over $\mathbb{Q}$, and hence a matrix $M(\alpha)$ with respect to the basis $B$. For example, let $\alpha=\phi=-(\zeta^2+\zeta^3)$. Then

$$\begin{aligned}\phi\cdot 1 &= -1\cdot\zeta^2-1\cdot\zeta^3,\\ \phi\cdot\zeta &= -\zeta^3-\zeta^4 = 1\cdot 1+1\cdot\zeta+1\cdot\zeta^2,\\ \phi\cdot\zeta^2 &= -\zeta^4-1 = 1\cdot\zeta+1\cdot\zeta^2+1\cdot\zeta^3,\\ \phi\cdot\zeta^3 &= -1\cdot 1-1\cdot\zeta.\end{aligned}$$

Therefore,

$$M(\phi)=\begin{pmatrix}0&1&0&-1\\0&1&1&-1\\-1&1&1&0\\-1&0&1&0\end{pmatrix}.$$

This is the transpose of the matrix $\overline{F}_{10}$ of [1]. Similarly, we have the following matrices:

$$M(\zeta\phi)=\begin{pmatrix}1&0&-1&0\\1&1&-1&-1\\1&1&0&-1\\0&1&0&0\end{pmatrix},\quad M(\zeta^2\phi)=\begin{pmatrix}0&-1&0&0\\1&-1&-1&0\\1&0&-1&-1\\1&0&0&-1\end{pmatrix},\quad M(\zeta^3\phi)=\begin{pmatrix}-1&0&0&1\\-1&-1&0&1\\0&-1&-1&1\\0&0&-1&0\end{pmatrix},$$

$$M(\zeta^4\phi)=\begin{pmatrix}0&0&1&0\\-1&0&1&1\\-1&1&0&0\\-1&0&1&-1\end{pmatrix},\quad M(\phi^{-1})=\begin{pmatrix}-1&1&0&-1\\0&0&1&-1\\-1&1&0&0\\-1&-1&0&-1\end{pmatrix},\quad M(\zeta\phi^{-1})=\begin{pmatrix}1&0&-1&1\\0&1&-1&0\\1&0&0&0\\0&1&-1&1\end{pmatrix},$$

$$M(\zeta^2\phi^{-1})=\begin{pmatrix}0&-1&1&-1\\1&-1&0&0\\0&0&0&-1\\1&-1&1&-1\end{pmatrix},\quad M(\zeta^3\phi^{-1})=\begin{pmatrix}-1&1&-1&1\\-1&0&0&0\\0&0&-1&1\\-1&1&-1&0\end{pmatrix},\quad M(\zeta^4\phi^{-1})=\begin{pmatrix}1&-1&1&0\\0&0&0&1\\0&-1&1&0\\1&-1&0&1\end{pmatrix}.$$

In the notation of [1], these are the transposes of the matrices $\overline{F}_{20}$, $\overline{F}_{14}$, $\overline{F}_3$, $\overline{F}_8$, $\overline{F}_4$, $\overline{F}_{16}$, $\overline{F}_{12}$, $\overline{F}_2$, and $\overline{F}_{15}$, respectively. Letting $\overline{F}_{21-i}=-\overline{F}_i$ gives a set of 20 matrices corresponding to the numbers

$\pm\zeta^m\phi^n$, $0 \le m \le 4$, $n = \pm 1$. Note that any one of these numbers (often called fundamental units), together with $-\zeta$, generates the group of units of $\mathbb{Z}[\zeta]$.

Various properties of the matrices $\overline{F}_i$ follow immediately from the above. The following four propositions can be proved by straightforward calculations, but it is perhaps more interesting to see "conceptual" proofs.

***Proposition 1 (= Proposition 4 of [1]):*** Let $1 \le i \le 20$. There exists $k$ such that $\overline{F}_i^{-1} = \overline{F}_k$.

***Proof:*** Let $\overline{F}_i$ correspond to $\varepsilon = \pm\zeta^m\phi^n$. Let $\overline{F}_k$ correspond to $\varepsilon^{-1} = \pm\zeta^{-m}\phi^{-n}$. Then $\overline{F}_i\overline{F}_k$ corresponds to multiplication by $\varepsilon^{-1}\varepsilon = 1$, so $\overline{F}_i\overline{F}_k = I$. □

***Proposition 2 (= Proposition 5 of [1]):*** Let $1 \le i \le 20$. Then $\det(\overline{F}_i) = 1$.

***Proof:*** The determinant is the norm of the corresponding number (see [3]). It is well known that the norm of a unit (of the ring of algebraic integers) is $\pm 1$. Since the norm of a number from $\mathbb{Q}(\zeta)$ can be expressed as a product of two numbers times the product of their complex conjugates, the norm must be nonnegative. Therefore, the norm of a unit is 1. Since the numbers $\pm\zeta^m\phi^n$ are units, the determinants of the corresponding matrices must be 1. □

***Proposition 3 (= Proposition 6 of [1]):*** Let $1 \le i,\ j \le 20$. Then $\overline{F}_i\overline{F}_j = \overline{F}_j\overline{F}_i$.

***Proof:*** Multiplication in $\mathbb{Q}(\zeta)$ is commutative; therefore, multiplication of the corresponding matrices is commutative. □

Define the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

An easy calculation shows that $A$ is the transpose of $M(-\zeta^4)$. Note that the powers of $-\zeta^4$ give all ten tenth roots of unity in $\mathbb{Q}(\zeta)$.

***Proposition 4 (= Proposition 7 of [1]):*** Let $\mathcal{F}_1 = \{\overline{F}_k \mid k = 1, 3, 7, 8, 10, 11, 13, 14, 18, 20\}$ and let $\mathcal{F}_2 = \{\overline{F}_k \mid k = 2, 4, 5, 6, 9, 12, 15, 16, 17, 19\}$.

***(a)*** Let $i = 1$ or 2. Given $\overline{F}_h, \overline{F}_k \in \mathcal{F}_i$, there exists $\overline{F}_n \in \mathcal{F}_i$ such that $\overline{F}_h\overline{F}_k = \pm\overline{F}_n^2$.

***(b)*** If $\overline{F}_h \in \mathcal{F}_1$ and $\overline{F}_k \in \mathcal{F}_2$, then there exists $n$ such that $\overline{F}_h\overline{F}_k = A^n$.

***(c)*** Let $i = 1$ or 2. If $\overline{F}_h, \overline{F}_k \in \mathcal{F}_i$, then $\overline{F}_h^{10n} = \overline{F}_k^{10n}$ for all $n \in \mathbb{Z}$.

***Proof:*** The matrices in $\mathcal{F}_1$ correspond to numbers of the form $\pm\zeta^m\phi$ and those in $\mathcal{F}_2$ correspond to numbers of the form $\pm\zeta^m\phi^{-1}$. The properties of the matrices now follow from the form of these numbers. □

We now come to the main theorem. It was proved in [1] by fixing indices $1 \le h \le 20$ and $0 \le i \le 9$ and expressing the entries of $\overline{F}_h^{10k+i}$ in terms of Fibonacci numbers of the form $\pm F_{ak+b}$ or 0 for $k = 0, 1, 2, \ldots$. This gives the additional information that, for each index $h$, each Fibonacci number occurs in $\overline{F}_h^n$ for some $n$ (in fact, this property was included in the definition of a Fibonacci matrix in [1]). With a little more care, this can be deduced from the following proof.

***Theorem 1 (= Proposition 1 of [1]):*** Let $1 \le h \le 20$ and let $n$ be a positive integer. Every entry of $\overline{F}_h^n$ is either 0 or $\pm F_m$ for some Fibonacci number $F_m$, where $m = n-1, n,$ or $n+1$.

***Proof:*** Fix $n \ge 1$. For each $a$ mod 5, let

$$g_n(a) = \sum_{\substack{j=0 \\ j \equiv a \pmod 5}}^{n} \binom{n}{j}.$$

***Lemma 1:*** $5g_n(a) = \sum_{i=0}^4 \zeta^{-ai}(1+\zeta^i)^n$.

***Proof:*** The right side is

$$\sum_{j=0}^{n} \binom{n}{j} \sum_{i=0}^{4} \zeta^{i(j-a)}.$$

Since $\sum_{i=0}^4 \zeta^{ib} = 0$ when $b \not\equiv 0 \pmod 5$ and equals 5 when $b \equiv 0 \pmod 5$, the result follows. □

***Lemma 2:*** For any values of $a$ and $b$, the difference $g_n(a) - g_n(b)$ is either 0 or $\pm F_m$ for some Fibonacci number $F_m$, where $m = n-1, n,$ or $n+1$.

***Proof:*** Using the fact that $1+\zeta = -\zeta^{-2}\phi$, $1+\zeta^2 = \zeta\phi^{-1}$, $1+\zeta^3 = \zeta^{-1}\phi^{-1}$, and $1+\zeta^4 = -\zeta^2\phi$, we find that

$$\begin{aligned}5g_n(a) - 5g_n(b) &= \sum_{i=0}^{4} \zeta^{-ai}(1+\zeta^i)^n - \sum_{i=0}^{4} \zeta^{-bi}(1+\zeta^i)^n \\ &= (-\phi)^n(\zeta^{a+2n} + \zeta^{-a-2n} - \zeta^{b+2n} - \zeta^{-b-2n}) \\ &\quad + (\phi^{-1})^n(\zeta^{n-2a} + \zeta^{-n+2a} - \zeta^{n-2b} - \zeta^{-n+2b}).\end{aligned}$$

Since $a + 2n \equiv 2(n-2a) \pmod 5$, we find that we have the following cases:

(1) $\zeta^{a+2n} + \zeta^{-a-2n} = \zeta + \zeta^{-1} = \phi^{-1}$ and $\zeta^{n-2a} + \zeta^{-n+2a} = \zeta^2 + \zeta^3 = -\phi$,

(2) $\zeta^{a+2n} + \zeta^{-a-2n} = \zeta^2 + \zeta^3 = -\phi$ and $\zeta^{n-2a} + \zeta^{-n+2a} = \zeta + \zeta^{-1} = \phi^{-1}$,

(3) $\zeta^{a+2n} + \zeta^{-a-2n} = 2$ and $\zeta^{n-2a} + \zeta^{-n+2a} = 2$.

Similarly, we have three cases for the terms involving $b$.

The coefficient of $(-\phi)^n$ is therefore 0 or one of the following:

(a) $\pm(\phi^{-1} - (-\phi)) = \pm\sqrt{5}$,

(b) $\pm(\phi^{-1} - 2) = \mp\sqrt{5}\phi^{-1}$,

(c) $\pm(-\phi - 2) = \mp\sqrt{5}\phi$.

The corresponding coefficients of $\phi^{-n}$ are 0 and $\mp\sqrt{5}, \mp\sqrt{5}\phi$, and $\mp\sqrt{5}\phi^{-1}$, respectively.

Putting everything together, we find that $5g_n(a) - 5g_n(b)$ is, up to sign, either 0 or one of the following:

$$\sqrt{5}((-\phi)^n - \phi^{-n}) = (-1)^n 5F_n,$$
$$\sqrt{5}((-\phi)^{n-1} - \phi^{-n+1}) = (-1)^{n-1} 5F_{n-1},$$
$$\sqrt{5}((-\phi)^{n+1} - \phi^{-n-1}) = (-1)^{n-1} 5F_{n+1}.$$

This proves the lemma. □

We can now prove Theorem 1. The matrix $\overline{F}_h^n$ corresponds to a number of the form $(\pm\zeta^m\phi^{\pm 1})^n$, which is of the form $\pm\zeta^a(1+\zeta)^n$ or of the form $\pm\zeta^a(1+\zeta^2)^n$. We may ignore the $\pm$.

Consider first $\zeta^a(1+\zeta)^n$. We must multiply this times a power of $\zeta$ and then express the result as a linear combination of elements of the basis $B$. Since the exponent $a$ is already arbitrary, we need only show that when we express a number of the form $^a(1+\zeta)^n$ in terms of $B$ the coefficients are Fibonacci numbers (up to sign) or 0. By the binomial theorem, we have

$$\zeta^a(1+\zeta)^n = \sum_{j=0}^{n}\binom{n}{j}\zeta^{j+a} = \sum_{i=0}^{4} g_n(i-a)\zeta^i = \sum_{i=0}^{3}(g_n(i-a)-g_n(4-a))\zeta^i.$$

Lemma 2 yields the result in this case.

Now consider $\zeta^a(1+\zeta^2)^n$, which equals

$$\sum_{j=0}^{n}\binom{n}{j}\zeta^{2j+a} = \sum_{i=0}^{4} g_n(3i-3a)\zeta^i \quad [\text{since } 2j+a \equiv i \pmod 5 \Rightarrow j \equiv 3i-3a]$$

$$= \sum_{i=0}^{3}(g_n(3i-3a)-g_n(2-3a))\zeta^i.$$

The result again follows from Lemma 2. □

**The Two-by-Two Case**

***Theorem 2:*** Let $M$ be a two-by-two matrix such that each entry of $M^n$ for $n = 1, 2, 3, \dots$ is either 0 or plus or minus a Fibonacci number. Suppose in addition that not all of the entries of $M^n$ are bounded as $n \to \infty$. Then $\pm M$ is a power of one of the following matrices:

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & \pm 1 \\ \mp 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & \pm 1 \\ \mp 1 & 2 \end{pmatrix}.$$

***Remark:*** It is well known, and will follow from the proof of the theorem, that

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$$

and that

$$\begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} (-1)^n F_{n-2} & -F_n \\ F_n & F_{n+2} \end{pmatrix}.$$

From the point of view used above, the first matrix arises from multiplication by $\phi$ with respect to the basis $\{1, \phi\}$ of $\mathbb{Q}(\sqrt{5})$, and the second matrix arises from multiplication by $\phi$ with respect to the basis $\{1, \phi^2\}$.

***Proof:*** We start with the following.

***Lemma 3:*** Suppose $a_n$, $n = 1, 2, \dots$, is a sequence of nonzero integers such that each $a_n$ is plus or minus a Fibonacci number and such that $\lambda = \lim a_{n+1}/a_n$ exists. Then $\lambda$ is of the form $\pm\phi^r$ for some integer $r \geq 0$. If the sequence $a_n$ is unbounded, $r \geq 1$.

***Proof:*** Let $a_n = \pm F_{m_n}$. The limit $\lambda$ cannot be of absolute value less than 1 since the $a_n$ are integers. Clearly, $\lambda = \pm 1$ is equivalent to the boundedness of $a_n$, so henceforth assume the sequence $a_n$ is unbounded. It follows easily that $\lim F_{m_n} = \infty$, hence $\lim m_n = \infty$. Therefore, $\lim F_{m_n} / \phi^{m_n} = 1/\sqrt{5}$, so

$$|\lambda| = \lim \frac{F_{m_{n+1}} \phi^{m_n}}{F_{m_n} \phi^{m_{n+1}}} \phi^{m_{n+1}-m_n} = \lim \phi^{m_{n+1}-m_n}.$$

Since the powers of $\phi$ are discrete in the positive reals, $m_{n+1} - m_n$ must eventually be constant, say $r$. Since $m_n \to \infty$, $r \geq 1$. This proves the lemma. □

Since the elements of the powers of a matrix satisfy a second-order recursion, we need the following result. Recall that we can define Fibonacci numbers for negative indices by $F_{-n} = (-1)^{n+1} F_n$.

***Lemma 4:*** Let $a_1, a_2, \ldots$ be an unbounded sequence of integers satisfying a second-order linear recursion with constant coefficients: $a_{n+2} = ua_{n+1} + va_n$. Suppose each $a_n$ is either 0 or $\pm F_{m_n}$ for some Fibonacci numbers $F_{m_n}$. Then there are integers $r$ and $s$ (possibly negative) and a choice $\delta = \pm 1$ of sign, independent of $n$, such that $a_n = \delta F_{rn+s}$ for all $n$ (we allow Fibonacci numbers with negative indices; see above).

***Remark:*** This result follows, for example, from work of van der Poorten (see the remarks at the end of this article). However, it seems reasonable to give a self-contained proof.

***Proof:*** We have not assumed that the coefficients $u$, $v$ of the recursion are rational numbers, so we first show that this must be the case. The recursion shows that each vector $(a_{n+1}, a_n)$ is a linear combination of $(a_2, a_1)$ and $(a_3, a_2)$. Suppose $\det\left(\begin{smallmatrix} a_2 & a_1 \\ a_3 & a_2 \end{smallmatrix}\right) = 0$. If $a_1 = 0$, then $a_2 = 0$, so $a_n = 0$ for all $n$, contrary to our assumptions. Therefore, assume $a_1 \neq 0$. Then all these vectors are multiples of $(a_2, a_1)$, which implies that

$$a_n = a_1 \left( \frac{a_2}{a_1} \right)^{n-1} \tag{1}$$

for all $n \geq 1$. Therefore, $|a_n| \to \infty$ (otherwise $|a_2 / a_1| \leq 1$ and the sequence is bounded) and $a_{n+1} / a_n = a_2 / a_1$. Since $|a_n|$ is a Fibonacci number [it cannot be 0 by (1)], Lemma 3 implies that $a_2 / a_1 = \pm \phi^r$ for some $r \geq 1$. Since all positive powers of $\phi$ are irrational, this is impossible. This contradiction shows that the determinant is nonzero.

Since

$$\begin{pmatrix} a_2 & a_1 \\ a_3 & a_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a_3 \\ a_4 \end{pmatrix},$$

and the matrix is invertible, the rationality of $a_1, a_2, a_3, a_4$ implies that $u$ and $v$ are rational.

***Remark:*** The recursion $a_{n+2} = \pi a_{n+1} + (4 - 2\pi) a_n$, which is satisfied by the rational numbers $a_n = 2^n$, shows that the rationality of the numbers $a_n$ is not sufficient to guarantee that $u$, $v$ are rational.

Let $\alpha$ and $\beta$ be the two roots of $X^2 - uX - v$. If $\alpha \neq \beta$, then there are constants $A$ and $B$ such that $a_n = A\alpha^n + B\beta^n$. There are several cases to consider, depending on the relative magnitudes of $\alpha$ and $\beta$.

**Case 1.** $|\alpha| > |\beta|$

If $A = 0$, we can replace the pair $(\alpha, \beta)$ by $(\beta, 0)$ in the following argument (and eventually conclude that $A \neq 0$). Therefore, assume $A \neq 0$, so $\lim a_{n+1} / a_n = \alpha$. Since the sequence is unbounded, $|\alpha| > 1$, so $a_n \neq 0$ for all sufficiently large $n$, and each $a_n$ is plus or minus a Fibonacci number. By Lemma 3, $\alpha = \pm\phi^r$ for some $r \geq 1$. Therefore, $\alpha$ is irrational, so the polynomial $X^2 - uX - v$ is irreducible in $\mathbb{Q}[X]$. Since $\beta$ is also a root, it must be the conjugate $\pm(-\phi)^{-r}$ of $\alpha$.

Let $\sigma = \operatorname{sign}(\alpha)$, so $\alpha = \sigma\phi^r$. Let $\delta_n$ be the sign of $a_n$. Note that

$$A = \lim \frac{a_n}{(\sigma\phi^r)^n}.$$

This implies that $\delta_n = \operatorname{sign}(A)\sigma^n$ for $n$ sufficiently large. Also, $a_n = \delta_n F_{m_n}$, so

$$\lim \frac{\delta_n a_n}{\phi^{m_n}} = \frac{1}{\sqrt{5}}.$$

Therefore,

$$A = \lim \frac{\delta_n a_n}{\phi^{m_n}} \delta_n \sigma^{-n} \phi^{m_n - nr} = \frac{\operatorname{sign}(A)}{\sqrt{5}} \lim \phi^{m_n - nr}.$$

Since the powers of $\phi$ are discrete, eventually $m_n - nr$ must stabilize: there exists $s \in \mathbb{Z}$ such that $m_n - nr = s$ for all sufficiently large $n$. This also yields $A = \pm\phi^s / \sqrt{5}$. Since the terms with $\phi^{rn}$ cancel in the equation

$$\begin{aligned} A(\sigma\phi^r)^n + B(\sigma(-\phi)^{-r})^n = a_n = \delta_n F_{m_n} &= \operatorname{sign}(A)\sigma^n F_{rn+s} \\ &= \frac{\operatorname{sign}(A)\sigma^n}{\sqrt{5}} (\phi^{rn+s} - (-\phi^{-1})^{rn+s}), \end{aligned}$$

it follows that $B = -\operatorname{sign}(A)(-\phi)^{-s} / \sqrt{5}$. We have proved that $a_n = \pm(\pm 1)^n F_{rn+s}$. By changing the signs of $r$, $s$ if necessary, we can absorb the $(\pm 1)^n$. This yields the result of the lemma in Case 1.

It remains to show that the other cases do not occur.

**Case 2.** $\alpha = -\beta$

In this case, $u = \alpha + \beta = 0$, so the recursion is $a_{n+2} = va_n$. Since the sequence is assumed to be unbounded, $|v| > 1$ and some $a_{n_0} \neq 0$. Therefore, $a_{n_0+2k+2} / a_{n_0+2k} = v^2 \in \mathbb{Q}$. Since the numbers $a_{n_0+2k} = a_{n_0} v^{2k}$ are nonzero, they are Fibonacci numbers up to sign. Lemma 3 implies that $v^2 = \phi^r$ for some $r \geq 1$. This is impossible.

**Case 3.** $\alpha = \beta$

In this case, $a_n = A\alpha^n + Bn\alpha^n$. Hence, $a_n \neq 0$ for sufficiently large $n$, and $\lim a_{n+1} / a_n = \alpha$. By Lemma 3, $\alpha = \pm\phi^r$ for some $r \geq 1$. Since $\alpha = u / 2 \in \mathbb{Q}$, this is impossible.

**Case 4.** $\overline{\alpha} = \beta$, $\alpha \neq \beta$

Since $a_n = A\alpha^n + B\overline{\alpha}^n \in \mathbb{Q}$ for all $n$, we must have $B = \overline{A}$. Write $A = Re^{i\gamma}$ and $\alpha = \rho e^{i\theta}$. Then

$$a_n = R\rho^n e^{in\theta + i\gamma} + R\rho^n e^{-in\theta - i\gamma} = 2R\rho^n \cos(n\theta + \gamma).$$

Suppose first that $\theta / 2\pi \notin \mathbb{Q}$. By a theorem of Weyl (see [2], Theorem 445), the sequence of fractional parts of $n\theta / 2\pi$ is uniformly distributed in the interval [0, 1]. In particular, there is a sequence of integers $n_i$ such that $n_i\theta / 2\pi + (\theta + 2\gamma)/4\pi - k_i$ is very small for some integers $k_i$, and the limit is 0 as $i \to \infty$. Therefore, $\cos(n_i\theta + \gamma)$ is very close to $\cos(2\pi k_i - \theta/2) = \cos(-\theta/2)$ and $\cos((n_i+1)\theta + \gamma)$ is very close to $\cos(\theta/2) = \cos(-\theta/2)$. Therefore, $\lim a_{n_i+1}/a_{n_i} = \rho$. Lemma 3 shows that $\rho = \phi^r$ for some $r \geq 0$. But $v = \alpha\beta = \rho^2$, so $\phi^{2r} \in \mathbb{Q}$, which implies $r = 0$. Therefore, the sequence $a_n$ is bounded, contrary to assumption.

Now suppose that $\theta/2\pi = w/z \in \mathbb{Q}$, where $w, z \in \mathbb{Z}$. Choose $n_0$ such that $a_{n_0} \neq 0$. Then $a_{n_0+(k+1)z}/a_{n_0+kz} = \rho^z$ for $k = 0, 1, 2, \ldots$. Lemma 3 implies that $\rho^z = \phi^r$ for some $r \geq 0$. Therefore, $\phi^{2r} = \rho^{2z} = v^z \in \mathbb{Q}$, so $r = 0$, which is impossible.

It is easy to see that Cases 1-4 exhaust all possibilities for $\alpha, \beta$. This concludes the proof of Lemma 4. □

***Corollary:*** Suppose $A, B, \alpha, \beta$ are complex numbers such that for each $n \geq 1$ the number $a_n = A\alpha^n + B\beta^n$ is either 0 or plus or minus a Fibonacci number, and such that the sequence $a_n$ is unbounded. Then there are integers $r \geq 1$ and $s$ such that (assume $|\alpha| \geq |\beta|$)

$$\alpha = \pm\phi^r, \qquad \beta = \pm(-\phi)^{-r}$$

and

$$A = \pm\frac{\phi^s}{\sqrt{5}}, \qquad B = \mp\frac{(-\phi)^{-s}}{\sqrt{5}}.$$

***Proof:*** This is a restatement of what was proved above, combined with the fact that the sequence $a_n$ uniquely determines the numbers $A, B, \alpha, \beta$. □

We can now prove Theorem 2. Suppose the matrix $M$ is as in the statement of the theorem, and let $\alpha, \beta$ be the roots of the characteristic polynomial of $M$. The case $\alpha = \beta$ corresponds to Case 3 in the proof of Lemma 4, and the reasoning below shows that it cannot occur, so we assume $\alpha \neq \beta$. Then $M$ is diagonalizable, so there are complex numbers $a$, $b$, $c$, and $d$ with $ad = bc \neq 0$ such that

$$M = \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Therefore,

$$M^n = \frac{1}{ad-bc}\begin{pmatrix} ad\alpha^n - bc\beta^n & -ab(\alpha^n - \beta^n) \\ cd(\alpha^n - \beta^n) & ad\beta^n - bc\alpha^n \end{pmatrix}.$$

We assume $|\alpha| \geq |\beta|$. By the Corollary,

$$\alpha = \pm\phi^r, \quad \beta = \pm(-\phi)^{-r},$$

for some integer $r$. Since not all entries are bounded, $r \geq 1$. If $ad = 0$ then $bc \neq 0$; looking at the first entry in the matrix yields $\beta^n \in \mathbb{Z}$ for all $n$, which is impossible. Similarly, $bc \neq 0$. By the Corollary,

$$\frac{ad}{ad-bc} = \pm\frac{\phi^s}{\sqrt{5}} \quad \text{and} \quad \frac{-bc}{ad-bc} = \mp\frac{(-\phi)^{-s}}{\sqrt{5}}$$

for some integer $s$. Therefore,

$$1 = \frac{ad}{ad-bc} - \frac{bc}{ad-bc} = \pm\frac{\phi^s - (-\phi)^{-s}}{\sqrt{5}} = \pm F_s,$$

so $s = \pm 1, \pm 2$.

Consider the upper right corner of $M^n$. Since $ab \neq 0$, the only possibility allowed by the Corollary is $ab/(ad-bc) = \pm 1/\sqrt{5}$. Similarly, $cd/(ad-bc) = \pm 1/\sqrt{5}$. Therefore, $ab = \pm cd$.

Since the matrix $\left(\begin{smallmatrix} 1/a & 0 \\ 0 & 1/d \end{smallmatrix}\right)$ commutes with $\left(\begin{smallmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{smallmatrix}\right)$, we can replace the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1/a & 0 \\ 0 & 1/d \end{smallmatrix}\right)$ and, therefore, assume $a = d = 1$. This makes the calculations simpler. We now have the following equations (the choices of signs are independent):

$$b = \pm c, \quad \frac{1}{bc} = \frac{ad/(ad-bc)}{bc/(ad-bc)} = (-\phi^2)^s, \quad s = \pm 1, \pm 2.$$

Since $\alpha, \beta \in \mathbb{Q}(\sqrt{5})$, the diagonalizing matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ may be assumed to have entries in $\mathbb{Q}(\sqrt{5})$. Therefore, the case $b = c$, $s = \pm 1$ and the case $b = -c$, $s = \pm 2$ cannot occur. Checking all solutions in the remaining cases and substituting into the formula for $M$ shows that $\pm M$ is the $r^{\text{th}}$ power of one of the matrices in the statement of the theorem. The same calculation yields that each entry of the powers of the matrices in the theorem is plus or minus a Fibonacci number. This completes the proof of Theorem 2. □

In [1], the problem is posed to find all four-by-four Fibonacci matrices. This can be attacked by the above method. One difficulty is proving the analog of Lemma 4 for fourth-order recurrences. A result of van der Poorten ([4], pp. 514-15) says that if an infinite sequence of elements $\{b_0, b_1, \dots\}$ chosen from the members of a nondegenerate (i.e., no ratio of characteristic roots of the recurrence is a root of unity) recurrent sequence $\{a_0, a_1, \dots\}$ again forms a recurrent sequence, then there is an integer $d > 0$, and a set $R$ of integers $r$ with $0 \le r < d$, such that for all $h$ we have $b_h = a_{r_h + hd}$ and $r_h \in R$ is periodic mod $d$. Since the entries in the powers of a matrix form a recurrent sequence, and the Fibonacci numbers form a nondegenerate sequence, this result applies, and we find that the eigenvalues of the matrix must be roots of unity times powers of $\phi$. This reduces the problem to the consideration of several cases for the characteristic roots.

The other difficulty is the calculation involving the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, since it must be replaced by a four-by-four matrix. The calculations are probably possible, but surely would be more difficult.

To conclude, we give a few more four-by-four Fibonacci matrices. They are not as good examples as $\overline{F}_1, \dots, \overline{F}_{20}$ since they all have powers that are reducible. However, they indicate various possibilities that can arise. They were chosen using the fact that their eigenvalues must be roots of unity times powers of $\phi$.

**1.** Let

$$M = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

This is obtained by considering multiplication by $\zeta\phi$ on the basis $(\{1, \phi, \zeta, \zeta\phi\}$. This matrix is almost reducible in the sense that

$$M^5 = \begin{pmatrix} 3 & 5 & 0 & 0 \\ 5 & 8 & 0 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 5 & 8 \end{pmatrix}.$$

This of course can be predicted from the fact that $(\zeta\phi)^5 = \phi^5$.

**2.** The matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ -1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$$

is obtained from multiplication by $\zeta\phi$ on the basis $\{1, \phi^2, \zeta, \zeta\phi^2\}$. The fifth power of this matrix is reducible.

**3.** The matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

is obtained from multiplication by $\zeta_3\phi$ on the basis $\{1, \phi, \zeta_3, \zeta_3\phi\}$ of $\mathbb{Q}(\phi, \zeta_3)$, where $\zeta_3$ is a primitive third root of unity. The third power of this matrix is reducible.

**4.** The matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 2 \\ -1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$$

is obtained from multiplication by $i\phi$ on the basis $\{1, \phi, i, i\phi\}$ of $\mathbb{Q}(\phi, i)$. More generally, any Fibonacci matrix tensored with a permutation matrix, in this case $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, will give a Fibonacci matrix.

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# ON SECOND-ORDER LINEAR RECURRENCE SEQUENCES: WALL AND WYLER REVISITED

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## 1. INTRODUCTION

Sequences of integers satisfying linear recurrence relations have been studied extensively since the time of Lucas [5], notable contributions being made by Carmichael [2], Lehmer [4], Ward [11], and more recently by many others. In this paper we obtain a unified theory of the structure of recurrence sequences by examining the ratios of recurrence sequences that satisfy the same recurrence relation. Results of previous authors usually derived from many complicated identities. Evidently, our method is simple and more conceptual.

The method of using ratios modulo $p$ or over a finite ring has been used in several papers including [1], [3], [6], and [7] among several others. Many known results due to Lucas [5], Lehmer [4], Vinson [9], Wall [10], and Wyler [13] can be derived easily from the well-known method of utilizing ratios. However, our point of view is really different from that of previous authors, so that we obtain our main result (Theorem 3.6(iii)), which improves on a result of Wyler [13], and we also get new information (Corollary 4.3) concerning Wall's question [10].

## 2. PRELIMINARIES AND CONVENTIONAL NOTATIONS

Given $a$ and $b$ in the ring $\mathfrak{R}$ with $b$ a unit, we consider all the second-order linear recurrence sequences $\{u_n\}$ in $\mathfrak{R}$ satisfying $u_n = au_{n-1} + bu_{n-2}$. (However, in this paper we exclude the case $u_n = 0$ for all $n \in \mathbb{Z}$.) We call the sequence $\{u_n\}$ a second-order recurrence sequence with parameters $(a, b)$.

Our idea comes from the following observation: Let $\{u_n\}$ and $\{u'_n\}$ be a pair of sequences in $\mathfrak{R}$ that satisfy the same recurrence relation defined above. Suppose that there exists a unit $c$ in $\mathfrak{R}$ such that $u_t = cu'_{t+s}$ and $u_{t+1} = cu'_{t+s+1}$ for some integers $t$ and $s$. Then, since $b \in \mathfrak{R}^*$, by the recurrence formula, we have that $u_n = cu'_{n+s}$ for all $n \in \mathbb{Z}$. Recall that the two elements $(x_0, x_1)$ and $(y_0, y_1)$ in the projective space $\mathbb{P}^1(\mathfrak{R})$ are the same if $x_0 = cy_0$ and $x_1 = cy_1$ for some $c \in \mathfrak{R}^*$. Hence, if we consider $(u_n, u_{n+1})$ as in the projective space $\mathbb{P}^1(\mathfrak{R})$, then $(u_t, u_{t+1}) = (u'_{t+s}, u'_{t+s+1})$ in $\mathbb{P}^1(\mathfrak{R})$ for some $t$ implies $(u_n, u_{n+1}) = (u'_{n+s}, u'_{n+s+1})$ in $\mathbb{P}^1(\mathfrak{R})$ for all $n$. We have the following definition.

***Definition:*** Let $\{u_n\}$ be a second-order linear recurrence sequence defined over $\mathfrak{R}$. Consider $r_n = (u_n, u_{n+1})$ as an element in the projective space $\mathbb{P}^1(\mathfrak{R})$. We call $r_n$ the $n^{\text{th}}$ ratio of $\{u_n\}$ and we call the sequence $\{r_n\}$ the ratio sequence of $\{u_n\}$.

We say that two sequences $\{u_n\}$ and $\{u'_n\}$ which both satisfy the same recurrence relation are equivalent if there is $c \in \mathfrak{R}^*$ and an integer $s$ such that $u_{n+s} = cu'_s$ for all $n$. Let $\{r_n\}$ and $\{r'_n\}$ be the ratio sequences of $\{u_n\}$ and $\{u'_n\}$, respectively. Then $\{u_n\}$ and $\{r_n\}$ are equivalent if and only if there exist integers $s$ and $t$ such that $r_s = r'_t$ in $\mathbb{P}^1(\mathfrak{R})$.

In particular, suppose that $u_t = u_{t+s}$ and $u_{t+1} = u_{t+s+1}$ for some integers $t$ and $s$. Then we have that $u_n = u_{n+s}$ for all $n$. In this case, we say that $\{u_n\}$ is periodic and the least positive integer $k$ such that $u_0 = u_k$ and $u_1 = u_{k+1}$ is called the period of $\{u_n\}$. When $\{u_n\}$ is periodic, the ratio sequence $\{r_n\}$ of $\{u_n\}$ is also periodic. The least positive integer $z$ such that $r_0 = r_z$ in $\mathbb{P}^1(\mathfrak{R})$ is called the rank of $\{u_n\}$. Suppose that the period of $\{u_n\}$ is $k$ and the rank of $\{u_n\}$ is $z$. It is clear that $z \mid k$ and $r_i \neq r_j$ in $\mathbb{P}^1(\mathfrak{R})$ for all $1 \le i \ne j \le z$.

We remark that, if $\mathfrak{R}$ is a finite ring, then the linear recurrence sequence $\{u_n\}$ is periodic.

## 3. RECURRENCE SEQUENCE MODULO *p*

In this section, we will extend our method to treat general second-order linear recurrence sequences. Results in Lucas [5], Lehmer [4], and Wyler [13] can be derived easily using our method.

Fix $a$ and $b \in \mathbb{Z}$. We consider second-order recurrence sequences of parameters $(a, b)$. Thus, we consider the sequences of integers $\{u_n\}_{n=0}^{\infty}$ defined by $u_n = au_{n-1} + bu_{n-2}$ for all integers $n \ge 2$, where $u_0$ and $u_1$ are given integers. In the case in which $u_0 = 0$ and $u_1 = 1$, the sequence $\{u_n\}_{n=0}^{\infty}$ is called the *generalized Fibonacci sequence* and we denote its terms by $f_0, f_1, \ldots$.

Fix a prime number $p$. We consider the recurrence sequence of parameters $(a, b)$ modulo $p$. Suppose that $p \mid b$. Then it is easy to see that $u_n \equiv a^{n-1}u_1 \pmod p$. Therefore, for the remainder of this section, we always assume that $p \nmid b$ and, hence, $\{u_n\}$ is periodic modulo $p$.

The positive integer $z$ is called the *rank of apparition* of the generalized Fibonacci sequence modulo $p$ if it is the smallest positive integer such that $f_z \equiv 0 \pmod p$. Let $r_i = (f_i, f_{i+1})$ in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$ be the $i^{\text{th}}$ ratio of $\{f_n\}$ modulo $p$. Since $r_0 = (0, 1) = r_z$ in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$ and $z$ is the least positive integer such that $r_z = (0, 1)$ in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$, it is clear that the rank of apparition of the generalized Fibonacci sequence modulo $p$ is equal to the rank of the generalized Fibonacci sequence modulo $p$.

Given a sequence $\{u_n\}$, there exists $r \in \mathbb{Z}$ such that $\{u_n\}$ modulo $p$ is equivalent to the sequence $\{u'_n\}$ modulo $p$ with $u'_0 = 1$ and $u'_1 = r$. Therefore, without loss of generality, we only consider the sequence with $u_0 = 1$ and $u_1 = r$.

***Lemma 3.1*** Let $\{u_n\}$ be the recurrence sequence with parameters $(a, b)$ and $u_0 = 1$, $u_1 = r$. Then the rank of $\{u_n\}$ modulo $p^i$ equals the rank of $\{f_n\}$ modulo $p^i$ if $p \nmid r^2 - ar - b$.

***Proof:*** Suppose that the rank of $\{u_n\}$ modulo $p^i$ is $t$ and the rank of $\{f_n\}$ modulo $p^i$ is $z$. Set $u'_n = bf_{n-1} + rf_n$. We have that $u'_n \equiv au'_{n-1} + bu'_{n-2} \pmod{p^i}$ and $u'_0 \equiv 1$ and $u'_1 \equiv r \pmod{p^i}$. Thus, $u'_n \equiv u_n \pmod{p^i}$ for all $n$. Hence, $u_{z+1} \equiv rf_{z+1} \equiv ru_z \pmod{p^i}$ because $f_z \equiv 0 \pmod{p^i}$ and $bf_{z-1} \equiv f_{z+1} \pmod{p^i}$. This says that $(u_z, u_{z+1}) = (u_0, u_1)$ in $\mathbb{P}^1(\mathbb{Z}/p^i\mathbb{Z})$ and, hence, $t \mid z$. On the other hand, we have that $bf_t + rf_{t+1} \equiv r(bf_{t-1} + rf_t) \pmod{p^i}$, by the assumption that $u_{t+1} \equiv ru_t \pmod{p^i}$. Substituting $f_{t+1} = af_t + bf_{t-1}$, we have that $(r^2 - ar - b)f_t \equiv 0 \pmod{p^i}$. Therefore, $(r^2 - ar - b, p) = 1$ implies that $f_t \equiv 0 \pmod{p^i}$. This says that $z \mid t$. □

***Remark:*** Suppose that $r^2 - ar - b \equiv 0 \pmod p$ and $\{u_n\}$ is the sequence with parameters $(a, b)$ and $u_0 = 1$, $u_1 = r$. Then we can easily obtain $u_n \equiv r^n u_0 \pmod p$. Hence, the rank of $\{u_n\}$ modulo $p$ is 1.

***Proposition 3.2 (Lucas):*** Let $z$ be the rank of the generalized Fibonacci sequence with parameters $(a, b)$ modulo $p$. Let $D = a^2 + 4b$ and denote $(\,/\,)$ to be the Legendre symbol. Then

***(i)*** $z \mid p+1$, if $(D/p) = -1$.

***(ii)*** $z \mid p-1$, if $(D/p) = 1$.

***(iii)*** $z = p$, if $p \mid D$.

***Proof:*** ***(i)*** Suppose that $(D/p) = -1$. Then $x^2 - ax - b \equiv 0 \pmod{p}$ has no solution. Thus, by Lemma 3.1, every recurrence sequence with parameters $(a, b)$ has the same rank modulo $p$. Let $t$ be the number of distinct equivalence classes of recurrence sequence of parameters $(a, b)$ modulo $p$. Let $\{\{u_{i,n}\} \mid 1 \le i \le t\}$ be a representative of these equivalence classes and $\{\{r_{i,n}\} \mid 1 \le i \le t\}$ be their ratio sequences in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$. By definition, we have that $r_{i,s} \ne r_{i,t}$ in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$ for all $1 \le s \ne t \le z$ and, if $i \ne j$, $\{r_{i,n}\}$ and $\{r_{j,n}\}$ are disjoint. Since, for any $r \in \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$, $(u_0, u_1) = r$ gives a sequence $\{u_n\}$, we have that $\{r_{1,1}, \ldots, r_{1,z}\} \cup \cdots \cup \{r_{t,1}, \ldots, r_{t,z}\} = \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$. It follows that $zt = p+1$ because the number of elements in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$ is $p+1$.

***(ii)*** For $(D/p) = 1$, there exist two distinct solutions to $x^2 - ax - b \equiv 0$ (mod $p$). By the Remark following Lemma 3.1, these two solutions give us sequences of rank 1. Consider all the distinct equivalence classes of sequences that have the same rank as the Fibonacci sequence modulo $p$. As in the above argument, their ratio sequences form disjoint subsets of equal numbers of elements of $^1(\mathbb{Z}/p\mathbb{Z})$. Since the number of these ratios is $p+1-2$, our claim follows.

***(iii)*** Since, for $p/D$, there exists exactly one solution to $x^2 - ax - b \equiv 0$ (mod $p$), by the above argument, our claim follows. □

***Remark:*** From the proof of Proposition 3.2, the number of distinct equivalence classes of recurrence sequences with parameters $(a, b)$ that have the same rank $z$ as the generalized Fibonacci sequence modulo $p$ is $(p+1)/z$ (resp. $(p-1)/z, 1$) if $(D/p) = -1$ (resp. $(D/p) = 1, p \mid D$).

***Lemma 3.3:*** Let $\{u_n\}$ and $\{u_n'\}$ be two recurrence sequences with parameters $(a, b)$. Then

$$bu_r u_s' + u_{r+1} u_{s+1}' = bu_{r+1} u_{s-1}' + u_{r+2} u_s'.$$

***Proof:*** By the recurrence formula, we have that

$$bu_{r+1}u_{s-1}' + u_{r+2}u_s' = u_{r+1}(u_{s+1}' - au_s') + (au_{r+1} + bu_r)u_s' = u_{r+1}u_{s+1}' + bu_r u_s'. \ \square$$

Let $r = (a, b)$ and $r' = (a', b')$ be two elements in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$ with $a, a', b$, and $b' \not\equiv 0 \pmod{p}$. Then we define $r \cdot r' = (aa', bb')$ in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$. Let $\{r_n\}$ be the ratio sequence of the generalized Fibonacci sequence modulo $p$ and let $z$ be the rank of the generalized Fibonacci sequence modulo $p$. Since $bf_{z-2} + af_{z-1} = f_z \equiv 0$ (mod $p$) and $f_1 = 1, f_2 = a$, we have that $bf_i f_{z-i-1} + f_{i+1} f_{z-i} \equiv 0$ (mod $p$) by Lemma 3.3 and by induction. This says that $r_i \cdot r_{z-i-1} = (1, -b)$ in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$ for $1 \le i \le z-2$. Because $r_1 \cdot r_2 \cdot \cdots \cdot r_{z-2} = (f_1, f_{z-1})$ in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$, we have the following Lemma.

***Lemma 3.4:*** Let $\{f_n\}$ be the generalized Fibonacci sequence with parameters $(a, b)$ and let $z$ be the rank of $\{f_n\}$ modulo $p$.

***(i)*** If $z$ is even, then $f_{z-1} \equiv (-b)^{(z-2)/2} \pmod{p}$.

***(ii)*** If $z$ is odd, then $f_{z-1} \equiv r(-b)^{(z-3)/2} \pmod{p}$, where $r^2 \equiv -b \pmod{p}$.

We remark that in Lemma 3.4(ii), $r \equiv f_{(z+1)/2} / f_{(z-1)/2} \pmod p$.

Since $f_{z+1} \equiv af_z + bf_{z-1} \equiv bf_{z-1}f_1 \pmod p$ and $f_z \equiv 0 \equiv bf_{z-1}f_0 \pmod p$, it follows that $f_{n+z} \equiv bf_{z-1}f_n \pmod p$ for all $n$ and, hence, $f_{n+\lambda z} \equiv (bf_{z-1})^{\lambda} f_n \pmod p$. Suppose that $\{u_n\}$ is a recurrence sequence with parameters $(a, b)$. Then, since $u_n = bu_0 f_{n-1} + u_1 f_n$, we also have that $u_{n+z} \equiv bf_{z-1}u_n \pmod p$ for all $n$. Furthermore, suppose that $\{u_n\}$ modulo $p$ is not equivalent to $\{f_n\}$ modulo $p$ and suppose that $\{r_n'\}$ is the ratio sequence of $\{u_n\}$ modulo $p$. Since $u_n \not\equiv 0 \pmod p$ (otherwise $\{u_n\}$ is equivalent to $\{f_n\}$), it follows that $r_1' \cdot r_2' \cdot \cdots \cdot r_z' = (u_1, u_{z+1})$ in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$ and, by the above argument, $(u_1, u_{z+1}) = (1, bf_{z-1})$ in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$.

Now we consider the product of all the ratios of nonequivalent sequences modulo $p$ with the exception of $(f_{z-1}, f_z)$ and $(f_z, f_{z+1})$. By Proposition 3.2, we have the following:

(i) If $x^2 - ax - b \equiv 0 \pmod p$ is not solvable, then

$$(1, b^{-1}(bf_{z-1})^{(p+1)/z}) = (1, (p-1)!) = (1, -1) \text{ in } \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}).$$

Hence, $(bf_{z-1})^{(p+1)/z} \equiv -b \pmod p$.

(ii) If $x^2 - ax - b \equiv 0 \pmod p$ is solvable with a double root $\gamma$, then

$$(1, f_{p-1}) \cdot (1, \gamma) = (1, (p-1)!) = (1, -1) \text{ in } \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}).$$

Notice that $\gamma^2 \equiv -b \pmod p$.

(iii) If $x^2 - ax - b \equiv 0 \pmod p$ is solvable with two distinct solutions $\alpha$ and $\beta$, then

$$(1, b^{-1}(bf_{z-1})^{(p-1)/z}) \cdot (1, \alpha) \cdot (1, \beta) = (1, (p-1)!) = (1, -1) \text{ in } \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}).$$

Since $\alpha\beta \equiv -b \pmod p$, it follows that $(bf_{z-1})^{(p-1)/z} \equiv 1 \pmod p$.

Notice that in (ii), since $z = p$ is odd, by Lemma 3.4, $f_{p-1} \equiv (-b)^{(p-3)/2} r \pmod p$, where $r \equiv f_{(z+1)/2} / f_{(z-1)/2} \pmod p$ and $r^2 \equiv -b \pmod p$. We have that $r \equiv -\gamma$ or $r \equiv \gamma \pmod p$. Since $-1 \equiv f_{p-1}\gamma \equiv r^{p-2}\gamma \pmod p$, it follows that $\gamma \equiv -r \pmod p$. Thus, $-f_{(p+1)/2} / f_{(p-1)/2}$ is the double root to $x^2 - ax - b \equiv 0 \pmod p$.

Using a similar argument, by considering (i) and (iii), we can improve the results in Proposition 3.2.

***Proposition 3.5 (Lehmer):*** Let $z$ be the rank of the generalized Fibonacci sequence with parameters $(a, b)$ modulo $p$ and let $D = a^2 + 4b$. Suppose that $p$ is an odd prime such that $p \nmid D$. Then $(-b/p) = 1$ if and only if $z \mid \frac{p-(D/p)}{2}$.

***Proof:*** If $z$ is odd, then, by Lemma 3.4(ii), we have that $(-b/p) = 1$. Since $p - (D/p)$ is even, we have that $2z \mid p - (D/p)$. Suppose that $z$ is even. Then, by (i) and (iii) above, and by Lemma 3.4, we have that

$$(-1)^{\frac{p-(D/p)}{z}} (-b)^{\frac{p-1}{2}} \equiv 1 \pmod p.$$

Our proof is complete because $(-b/p) \equiv (-b)^{(p-1)/2} \pmod p$. □

From Lemma 3.4, we realize that the relation between the period and the rank of $\{f_n\}$ modulo $p$ depends on the order of $-b$ modulo $p$. Denote $\text{ord}_p(d)$ to be the least positive integer $x$ such that $d^x \equiv 1 \pmod p$. We begin with the following easy observation. For $n \in \mathbb{N}$, we have

$$\mathrm{ord}_p(d^n) = \frac{\mathrm{ord}_p(d)}{\gcd(n, \mathrm{ord}_p(d))}.$$

It is also easy to check that

$$\mathrm{ord}_p(-d) = \begin{cases} \mathrm{ord}_p(d), & \text{if } \mathrm{ord}_p(d) \equiv 0 \pmod 4, \\ \frac{1}{2}\mathrm{ord}_p(d), & \text{if } \mathrm{ord}_p(d) \equiv 2 \pmod 4, \\ 2\mathrm{ord}_p(d), & \text{if } \mathrm{ord}_p(d) \equiv 1 \pmod 2. \end{cases}$$

Furthermore, suppose that $x^2 \equiv d \pmod p$ is solvable and suppose that $\lambda$ is one of its solutions. Then

$$\mathrm{ord}_p(\lambda) = \begin{cases} 2\mathrm{ord}_p(d), & \text{if } \mathrm{ord}_p(d) \equiv 0 \pmod 2, \\ 2\mathrm{ord}_p(d) \text{ or } \mathrm{ord}_p(d), & \text{if } \mathrm{ord}_p(d) \equiv 1 \pmod 2. \end{cases}$$

We remark that, if $(d/p) = 1$ and $\mathrm{ord}_p(d)$ is odd, then the order of one of the roots of $x^2 \equiv d \pmod p$ is odd and the order of another root is even.

***Theorem 3.6:*** Let $\{f_n\}$ be the generalized Fibonacci sequence with parameters $(a, b)$ and let $z$ be the rank and $k$ be the period of $\{f_n\}$ modulo $p$, respectively. Let $z = 2^\nu z'$ and $\mathrm{ord}_p(-b) = 2^\mu h$, where $z'$ and $h$ are odd integers.

***(i)*** If $\nu \neq \mu$, then $k = 2\,\mathrm{lcm}[z, \mathrm{ord}_p(-b)]$.

***(ii)*** If $\nu = \mu > 0$, then $k = \mathrm{lcm}[z, \mathrm{ord}_p(-b)]$.

***(iii)*** In the case $\nu = \mu = 0$.

$$k = \begin{cases} 2\,\mathrm{lcm}[z,\ \mathrm{ord}_p(-b)], & \text{if } \mathrm{ord}_p(f_{(z+1)/2} / f_{(z-1)/2}) \text{ is odd}, \\ \mathrm{lcm}[z, \mathrm{ord}_p(-b)], & \text{if } \mathrm{ord}_p(f_{(z+1)/2} / f_{(z-1)/2}) \text{ is even}. \end{cases}$$

***Proof:*** First, we consider the case $\nu > 0$. Since $z$ is even, by Lemma 3.4 and the discussion following Lemma 3.4, we have that $k/z = \mathrm{ord}_p(b(-b)^{(z-2)/2}) = \mathrm{ord}_p(-(-b)^{z/2})$. Suppose $\nu > \mu$. Then $\mathrm{ord}_p((-b)^{z/2}) = h/\gcd(z', h) \equiv 1 \pmod 2$. Hence, $k/z = 2h/\gcd(z', h)$. Therefore, $k = 2\,\mathrm{lcm}[z,\ \mathrm{ord}_p(-b)]$. On the other hand, suppose $\mu = \nu$. Then $\mathrm{ord}_p((-b)^{z/2}) = 2h/\gcd(z', h) \equiv 2 \pmod 4$. Thus, $k/z = h/\gcd(z', h)$, and hence, $k = \mathrm{lcm}[z, \mathrm{ord}_p(-b)]$. Similarly, suppose $\mu > \nu$. Then $\mathrm{ord}_p((-b)^{z/2}) = 2^{\mu-\nu+1}h/\gcd(z', h) \equiv 0 \pmod 4$. Therefore, $k/z = 2^{\mu-\nu+1}h/\gcd(z', h)$, and hence, $k = 2\,\mathrm{lcm}[z, \mathrm{ord}_p(-b)]$. Now we consider the case $\nu = 0$. Since $z$ is odd, we have $k/z = \mathrm{ord}_p(b(-b)^{(z-3)/2}r)$, where $r = f_{(z+1)/2} / f_{(z-1)/2}$ and $r^2 \equiv -b \pmod p$. Hence, $k/z = \mathrm{ord}_p(-r^z)$. Suppose $\mu > \nu$. Then $\mathrm{ord}_p(r) = 2\mathrm{ord}_p(-b)$; hence, $\mathrm{ord}_p(r^z) = 2\mathrm{ord}_p(-b)/\gcd(z, h) \equiv 0 \pmod 4$. Therefore, $k/z = 2\mathrm{ord}_p(-b)/\gcd(z, h)$; hence, $k = 2\,\mathrm{lcm}[z, \mathrm{ord}_p(-b)]$. Finally, suppose $\mu = \nu = 0$. Then either $\mathrm{ord}_p(r) = \mathrm{ord}_p(-b)$ or $\mathrm{ord}_p(r) = 2\mathrm{ord}_p(-b)$. Suppose $\mathrm{ord}_p(r) = \mathrm{ord}_p(-b)$ (that is, $\mathrm{ord}_p(r)$ is odd). Then $\mathrm{ord}_p(r^z) = \mathrm{ord}_p(-b)/\gcd(z, h) \equiv 1 \pmod 2$. Therefore, $k/z = 2\mathrm{ord}_p(-b)/\gcd(z, h)$; hence, $k = 2\,\mathrm{lcm}[z, \mathrm{ord}_p(-b)]$. On the other hand, suppose $\mathrm{ord}_p(r) = 2\mathrm{ord}_p(-b)$ (that is, $\mathrm{ord}_p(r)$ is even). Then $\mathrm{ord}_p(r^z) = 2\mathrm{ord}_p(-b)/\gcd(z, h) \equiv 2 \pmod 4$. Thus, $k/z = \mathrm{ord}_p(-b)/\gcd(z, h)$ and hence, $k = \mathrm{lcm}[z, \mathrm{ord}_p(-b)]$. □

***Remark:*** Cases (i) and (ii) of Theorem 3.6 above are stated as Wyler's main theorem in [13]. However, our approach is different and Wyler does not settle the case in which both $z$ and $\mathrm{ord}_p(-b)$ are odd (that is, case (iii) of our theorem).

## 4. RECURRENCE SEQUENCE MODULO $p^t$

We now treat the case of the generalized Fibonacci sequence modulo $p^t$ for $t \geq 2$.

Let us denote by $z(p^i)$ the rank of $\{f_n\}$ modulo $p^i$. We begin with an easy observation: If $\{u_n\}$ is equivalent to $\{f_n\}$ modulo $p^t$, then the number of possible ratios of $\{u_n\}$ modulo $p^{t+1}$ is $z(p^t)$ or $pz(p^t)$. By Lemma 3.1, the rank of such a sequence modulo $p^{t+1}$ equals $z(p^{t+1})$. Therefore, the rank of $z(p^{t+1})$ divides $pz(p^t)$. Since $z(p^t) \mid z(p^{t+1})$, it follows that either $z(p^{t+1}) = z(p^t)$ or $z(p^{t+1}) = pz(p^t)$.

***Theorem 4.1:*** The rank of apparition of the generalized Fibonacci sequence modulo $p^t$ equals the rank of apparition of the generalized Fibonacci sequence modulo $p^{t+1}$ if and only if there exists a sequence which is equivalent to $\{f_n\}$ modulo $p^t$ but is not equivalent to $\{f_n\}$ modulo $p^{t+1}$.

***Proof:*** $\{u_n\}$ is equivalent to $\{f_n\}$ modulo $p^t$ if and only if $(u_1, u_2) = (f_i, f_{i+1})$ in $\mathbb{P}^1(\mathbb{Z}/p^t\mathbb{Z})$ for some $i$. On the other hand, by the above argument, $z(p^{t+1}) = z(p^t)$ if and only if there exists $r \in \mathbb{Z}$ such that $(1, r) = (f_i, f_{i+1})$ in $\mathbb{P}^1(\mathbb{Z}/p^t\mathbb{Z})$ for some $i$ but $(1, r) \neq (f_j, f_{j+1})$ in $\mathbb{P}^1(\mathbb{Z}/p^{t+1}\mathbb{Z})$ for all $j \in \mathbb{Z}$. Combining these two statements, our proof is complete. $\square$

We remark that $\{u_n\}$ is equivalent to $\{f_n\}$ modulo $p^t$ if and only if $u_i \equiv 0 \pmod{p^t}$ for some $i$.

***Example:*** Consider the Fibonacci sequence

$$\{F_n\}_1^\infty \equiv \{1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, \dots\} \pmod 8$$

and the Lucas sequence

$$\{L_n\}_1^\infty \equiv \{1, 3, 4, 7, 3, 2, 5, 7, 4, 3, 7, 2, 1, 3, \dots\} \pmod 8.$$

The rank of apparition of the Fibonacci sequence modulo 2, 4, and 8 is 3, 6, and 6, respectively. We have that $\{L_n\}$ is equivalent to $\{F_n\}$ modulo 4 because $L_3 \equiv 0 \pmod 4$ but $\{L_n\}$ is not equivalent to $\{F_n\}$ modulo 8 because $L_n \not\equiv 0 \pmod 8$ for all $n$.

For every $t \in \mathbb{N}$, we denote $k(p^t)$ to be the period of $\{f_n\}$ modulo $p^t$. By considering the "Binet form" of $\{f_n\}$, Lehmer [4] proves that, for $p \neq 2$, if $k(p^l) = k(p)$ but $k(p^{l+1}) \neq k(p)$, then $k(p^t) = p^{t-l}k(p)$ for all $t \geq l$. Let $z(p^t)$ denote the rank of apparition of $\{f_n\}$ modulo $p^t$. By a similar method, we can prove that, for $p \neq 2$, if $z(p^l) = z(p)$ but $z(p^{l+1}) \neq z(p)$, then $z(p^t) = p^{t-l}z(p)$ for all $t \geq l$. We note that this result was also proved by Lucas [5] and by Carmichael [2]. We remark that $z(p^{l+1}) \neq z(p^l)$ implies $k(p^{l+1}) \neq k(p^l)$, but the converse is not always true.

***Corollary 4.2:*** Let $\{f_n\}$ be the generalized Fibonacci sequence with parameters $(a, b)$. Let $p$ be an odd prime and, for every $t \in \mathbb{N}$, denote $z(p^t)$ to be the rank of $\{f_n\}$ modulo $p^t$. Suppose that

$z(p^l) \neq z(p^{l+1})$. If $\{u_n\}$ is a recurrence sequence with parameters $(a, b)$ such that $u_i \equiv 0 \pmod{p^l}$ for some $i$, then, for every $t \geq l$, there exists $j_t$ such that $u_{j_t} \equiv 0 \pmod{p^t}$.

***Proof:*** $z(p^l) \neq z(p^{l+1})$ implies $z(p^t) \neq z(p^{t+1})$ for all $t \geq l$. Therefore, according to Theorem 4.1, every sequence that is equivalent to $\{f_n\}$ modulo $p^l$ is also equivalent to $\{f_n\}$ modulo $p^t$ for all $t \geq l$. □

Now we restrict ourselves to considering only the Fibonacci sequence $\{F_n\}$. For every $t \in \mathbb{N}$, we denote $K(p^t)$ to be the period of $\{F_n\}$ modulo $p^t$. In [10], Wall asked whether $K(p) = K(p^2)$ is always impossible; until this day, it remains an open question. According to Williams [12], $K(p) \neq K(p^2)$ for every odd prime $p$ less than $10^9$. Z.-H. Sun and Z.-W. Sun [8] proved that the affirmative answer to Wall's question implies the first case of Fermat's last theorem.

Let $Z(p^t)$ denote the rank of apparition of the Fibonacci sequence modulo $p^t$ for every $t \in \mathbb{N}$. We have that, for $p \neq 2$, $K(p^t) = K(p^{t+1})$ if and only if $Z(p^t) = Z(p^{t+1})$. What makes Theorem 4.1 so interesting is the following Corollary.

***Corollary 4.3:*** Let $p$ be an odd prime and, for every $t \in \mathbb{N}$, denote $K(p^t)$ to be the period of $\{F_n\}$ modulo $p^t$. Suppose that $K(p^l) \neq K(p^{l+1})$. Let $\{u_n\}$ be a sequence satisfying the same recurrence relation as $\{F_n\}$ such that $u_i \equiv 0 \pmod{p^l}$ for some $i$. Then, for every $t \geq l$, there exists $j_t$ such that $u_{j_t} \equiv 0 \pmod{p^t}$.

***Remark:*** In particular, let $p$ be an odd prime such that $K(p) \neq K(p^2)$. Suppose that $\{u_n\}$ is a recurrence sequence with parameters $(1, 1)$ and $u_i \equiv 0 \pmod{p}$ for some $i$. Then Corollary 4.3 implies that, for every positive integer $t$, there exists $j_t$ such that $u_{j_t} \equiv 0 \pmod{p^t}$. This is true for $p < 10^9$ according to Williams [12].

Unlike the Fibonacci case, we have examples for which $k(p) = k(p^2)$ and $z(p) \neq z(p^2)$.

***Example:*** Let $a = 8$ and $b = -7$. Then $\{f_n\}_{n=0}^{\infty} \equiv \{0, 1, 3, 2, 0, 1, \ldots\} \pmod{5}$ and $\{f_n\}_{n=0}^{\infty} \equiv \{0, 1, 8, 7, 0, 1, \ldots\} \pmod{25}$. Consider the sequence $\{u_n\}$ with $u_0 = 5$, and $u_1 = 1$ which satisfies $u_n = 8u_{n-1} - 7u_{n-2}$. We have that $\{u_n\}_{n=0}^{\infty} \equiv \{0, 1, 3, 2, 0, 1, \ldots\} \pmod{5}$ and $\{u_n\}_{n=0}^{\infty} \equiv \{5, 1, 23, 2, 5, 1, \ldots\} \pmod{25}$.

One might ask for what kind of parameters $(a, b)$ the generalized Fibonacci sequence has $k(p) = k(p^2)$, and whether or not $k(p) = k(p^2)$ for infinitely many primes. From our construction above, we understand that this question is related to: "For a given integer $x$, does there exist a prime $p$ such that $\mathrm{ord}_p(x) = \mathrm{ord}_{p^2}(x)$?" and "Are there infinitely many primes $p$ such that, for a given integer $x$, $\mathrm{ord}_p(x) = \mathrm{ord}_{p^2}(x)$?". Of course, we can suppose that $f(x) = x^2 - ax - b$ is irreducible over the integers. Then we have to consider our question over the ring of integers of $\mathbb{Q}[\sqrt{a^2 + 4b}]$. We have not taken up in this paper the question of whether a given recurrence $\{u_n\}$ has zeros modulo a given integer $m$ or not. Ward [11] has shown that $\{u_n\}$ has zeros modulo $p$ for infinitely many primes $p$. For given parameters $(a, b)$, suppose that we know there are only finitely many primes such that $z(p) = z(p^2)$. Then, by Corollary 4.2, it follows that there exist infinitely many primes $p$ such that $\{u_n\}$ has zeros modulo $p^t$ for every $t \in \mathbb{N}$.

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# NETWORK PROPERTIES OF A PAIR OF GENERALIZED POLYNOMIALS


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## 1. INTRODUCTION

Ladder networks have been studied extensively using Fibonacci numbers, Chebyshev polynomials, Morgan-Voyce polynomials, Jacobsthal polynomials, etc. ([10], [11], [2], [14], [9]. [5], [3], and [4]). All these polynomials are, in fact, particular cases of the generalized polynomials defined by

$$U_n(x, y) = xU_{n-1}(x, y) + yU_{n-2}(x, y), \quad (n \geq 2) \tag{1a}$$

with

$$U_0(x, y) = 0, \quad U_1(x, y) = 1, \tag{1b}$$

and

$$V_n(x, y) = xV_{n-1}(x, y) + yV_{n-2}(x, y), \quad (n \geq 2) \tag{2a}$$

with

$$V_0(x, y) = 2, \quad V_1(x, y) = x. \tag{2b}$$

We first show that rational functions derived from the ratios of these polynomials may in fact be synthesized using two-element-kind electrical networks. As particular cases, we will show that the networks realized using Fibonacci and Lucas polynomials, or Pell and Pell-Lucas polynomials are reactance of LC-networks, while those using Jacobsthal polynomials are RC or RL networks. Based on these results, we will establish some elegant relations among the various polynomials, as well as some results regarding the location of the zeros of these polynomials, and also their derivative polynomials. One of the results we need for our development is the following:

$$V_n(x, y) = U_{n+1}(x, y) + yU_{n-1}(x, y) = xU_n(x, y) + 2yU_{n-1}(x, y), \quad (n \geq 1), \tag{3}$$

which can be established easily by induction. We may also show that $U_{2n}(x, y)$ is an odd polynomial in $x$ of degree $(2n-1)$ and a polynomial in $y$ of degree $(n-1)$, while $U_{2n+1}(x, y)$ is an even polynomial in $x$ of degree $2n$ and a polynomial in $y$ of degree $n$. Further, $V_{2n}(x, y)$ is an even polynomial in $x$ of degree $2n$ and a polynomial in $y$ of degree $n$, while $V_{2n+1}(x, y)$ is an odd polynomial in $x$ of degree $(2n+1)$ and a polynomial in $y$ of degree $n$.

## 2. SYNTHESIS WITH $U_n(x, y)$ AND $V_n(x, y)$

Consider the function $\frac{U_{2n+1}(x, y)}{U_{2n}(x, y)}$; we will express this as a continued fraction.

$$\begin{aligned}\frac{U_{2n+1}(x, y)}{U_{2n}(x, y)} &= \frac{xU_{2n}(x, y) + yU_{2n-1}(x, y)}{U_{2n}(x, y)} \\ &= x + \cfrac{1}{\cfrac{U_{2n}(x, y)}{yU_{2n-1}(x, y)}} = x + \cfrac{1}{\cfrac{xU_{2n-1}(x, y) + yU_{2n-2}(x, y)}{yU_{2n-1}(x, y)}}\end{aligned}$$

$$=x+\cfrac{1}{\cfrac{x}{y}+\cfrac{1}{\cfrac{U_{2n-1}(x,y)}{U_{2n-2}(x,y)}}}=\cdots=x+\cfrac{1}{\cfrac{x}{y}+\cfrac{1}{x+\cfrac{1}{\cfrac{x}{y}+\cdots+\cfrac{1}{x+\cfrac{1}{\cfrac{x}{y}}}}}}. \tag{4}$$

If we now consider $\frac{U_{2n+1}(x,y)}{U_{2n}(x,y)}$ as the driving point impedance (DPI) of a one-port network consisting of two kinds of elements, whose impedances are proportional to $x$ and $(y/x)$, then the function $\frac{U_{2n+1}(x,y)}{U_{2n}(x,y)}$ given by (4) may be realized by the network of Fig. 1(a), where there are $n$ elements whose impedances are proportional to $x$, and $n$ other elements whose impedances are proportional to $(y/x)$. It is observed that, if $y$ equals a positive constant, say $\alpha$, and $x=s$ (the complex frequency variable), then the element $x$ corresponds to an inductor of value 1H, while the element $(y/x)$ corresponds to a capacitor of value $(1/\alpha)$F. On the other hand, if $y=s$ and $x$ is a positive constant, say $\beta$, then they correspond, respectively, to a resistor of $\beta$ Ohms and an inductor of value$(1/\beta)$H.

We may similarly express

$$\frac{U_{2n}(x,y)}{U_{2n-1}(x,y)},\quad \frac{V_{2n+1}(x,y)}{V_{2n}(x,y)},\quad \text{and}\quad \frac{V_{2n}(x,y)}{V_{2n-1}(x,y)}$$

by continued fractions, and realize them as the DPIs of the one-ports shown in Figs. 1(b), 1(c), and 1(d), respectively. Now let us synthesize $\frac{V_{2n}(x,y)}{U_{2n}(x,y)}$ as the DPI of a ladder network. We have, from (3),

$$\begin{aligned} Z_{in}&=\frac{V_{2n}(x,y)}{U_{2n}(x,y)}=\frac{xU_{2n}(x,y)+2yU_{2n-1}(x,y)}{U_{2n}(x,y)}\\ &=x+\cfrac{1}{\cfrac{U_{2n}(x,y)}{2yU_{2n-1}(x,y)}}=x+\cfrac{1}{\cfrac{xU_{2n-1}(x,y)+yU_{2n-2}(x,y)}{2yU_{2n-1}(x,y)}}=x+\cfrac{1}{\cfrac{x}{2y}+\cfrac{1}{2\cfrac{U_{2n-1}(x,y)}{U_{2n-2}(x,y)}}}. \end{aligned} \tag{5}$$

It is observed that $2\frac{U_{2n-1}(x,y)}{U_{2n-2}(x,y)}$ is an impedance and may be realized by the network of Fig. 1(a), where all the impedances are now scaled by a factor of 2. Thus, $\frac{V_{2n}(x,y)}{U_{2n}(x,y)}$ may be realized as the DPI of the ladder network shown in Fig. 1(e). Similarly, $\frac{V_{2n+1}(x,y)}{U_{2n+1}(x,y)}$ may be realized as the DPI of the two-element-kind network of Fig. 1(f).

## 3. FIBONACCI, LUCAS, PELL, AND PELL-LUCAS POLYNOMIALS AND LADDER NETWORKS

Let us first consider the case when $x=s$ and $y=\alpha$, a positive constant; that is, we are dealing with $U_n(s,\alpha)$ and $V_n(s,\alpha)$. When $\alpha=1$, they reduce to the Fibonacci and Lucas polynomials $F_n(s)$ and $L_n(s)$, respectively. Hence, we shall call $U_n(s,\alpha)$ and $V_n(s,\alpha)$ modified Fibonacci and

Lucas polynomials, and denote them by $\tilde{F}_n(s)$ and $\tilde{L}_n(s)$, respectively. It is then evident from the results of the previous section that $\tilde{F}_{2n+1}(s)/\tilde{F}_{2n}(s)$ may be realized as the DPI of the reactance network given by Fig. 1(a), where each of the series elements corresponds to an inductor of value 1H and each of the shunt elements corresponds to a capacitor of value $(1/\alpha)$F. Similarly, $\tilde{F}_{2n}(s)/\tilde{F}_{2n-1}(s)$, $\tilde{L}_{2n+1}(s)/\tilde{L}_{2n}(s)$, $\tilde{L}_{2n}(s)/\tilde{L}_{2n-1}(s)$, $\tilde{L}_{2n}(s)/\tilde{F}_{2n}(s)$, and $\tilde{L}_{2n+1}(s)/\tilde{F}_{2n+1}(s)$ may all be realized by low-pass LC-ladder networks corresponding to Figs. 1(b), 1(c), 1(d), 1(e), and 1(f), respectively. Thus, we have the interesting result that $\tilde{F}_{n+1}(s)/\tilde{F}_n(s)$, $\tilde{L}_{n+1}(s)/\tilde{L}_n(s)$, and $\tilde{L}_n(s)/\tilde{F}_n(s)$ are all reactance functions. It is well known that the zeros and poles of a reactance function are simple, purely imaginary, and interlace [1]. Hence, the zeros of the polynomials $\tilde{F}_n(s)$ and $\tilde{L}_n(s)$ lie on the imaginary axis and are simple; further, the zeros of $\tilde{F}_n(s)$ and $\tilde{L}_n(s)$ interlace. Similar statements hold true for the zeros of $\tilde{F}_{n+1}(s)$ and $\tilde{F}_n(s)$, as well as those of $\tilde{L}_{n+1}(s)$ and $\tilde{L}_n(s)$.

Since, for the Pell and Pell-Lucas polynomials, we have

$$P_n(s) = F_n(2s) \tag{6a}$$

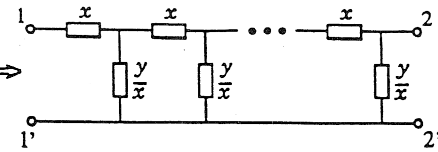
and

$$Q_n(s) = L_n(2s), \tag{6b}$$

it is obvious that $P_{n+1}(s)/P_n(s)$, $Q_{n+1}(s)/Q_n(s)$, and $Q_n(s)/P_n(s)$ are all reactance functions. In fact, using the frequency scaling theorem [1], it is seen that their realizations are the same as those of $F_{n+1}(s)/F_n(s)$, $L_{n+1}(s)/L_n(s)$, and $L_n(s)/F_n(s)$, respectively, except for a scaling of the values of the elements.

We now consider the case when $x=\beta$, a positive constant, and $y=s$; that is, we are dealing with $U_n(\beta,s)$ and $V_n(\beta,s)$. It is observed that when $\beta=1$ they reduce to the Jacobsthal polynomials [7]. Hence, we shall call $U_n(\beta,s)$ and $V_n(\beta,s)$ modified Jacobsthal polynomials and denote them by $\tilde{J}_n(s)$ and $\tilde{j}_n(s)$, respectively. It is then evident from the results of the previous section that $\tilde{J}_{2n+1}(s)/\tilde{J}_{2n}(s)$ may be realized as the DPI of the RL-network given by Fig. 1(a), where each of the series elements is a resistor of value $\beta$ Ohms and each of the shunt elements is an inductor of value $(1/\beta)$H. Similarly, we can realize the functions $\tilde{J}_{2n}(s)/\tilde{J}_{2n-1}(s)$, $\tilde{j}_{2n+1}(s)/\tilde{j}_{2n}(s)$, $\tilde{j}_{2n}(s)/\tilde{j}_{2n-1}(s)$, $\tilde{j}_{2n}(s)/\tilde{J}_{2n}(s)$, and $\tilde{j}_{2n+1}(s)/\tilde{J}_{2n+1}(s)$ as DPIs of the RL-networks corresponding to Figs. 1(b), 1(c), 1(d), 1(e), and 1(f), respectively, where all the series elements are resistors and all the shunt elements are inductors. Thus, we have the result that $\tilde{J}_{n+1}(s)/\tilde{J}_n(s)$, $\tilde{j}_{n+1}(s)/\tilde{j}_n(s)$, and $\tilde{j}_n(s)/\tilde{J}_n(s)$ are all RL-impedance or RC-admittance functions. It is well known that the zeros and poles of an RL-impedance (or an RC-admittance) function lie on the negative real axis, are simple, and interlace; further, the one closest to the origin is a zero of the function [1]. Thus, the zeros of the polynomials $\tilde{J}_n(s)$ and $\tilde{j}_n(s)$ are real and negative; further, the zeros of $\tilde{J}_n(s)$ and $\tilde{j}_n(s)$ interlace, with the zero closest to the origin being that of $\tilde{j}_n(s)$. Similar statements hold true for the zeros of $\tilde{J}_{n+1}(s)$ and $\tilde{J}_n(s)$, as well as those of $\tilde{j}_{n+1}(s)$ and $\tilde{j}_n(s)$. It is also interesting to observe that $\tilde{J}_{n+1}(s)/\tilde{j}_n(s)$ is a ratio of two RC-admittance functions and hence, in general, is not realizable by two-element-kind networks; however, it is a positive real function (PRF), and so is always realizable by an RLC network. In fact, the zeros of $\tilde{J}_{n+1}(s)$ and $\tilde{j}_n(s)$ have a very interesting pairwise alternative relationship on the negative real axis [6].

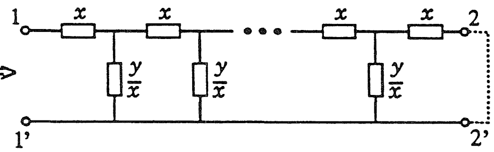
$$Z_{in} = z_{11} = \frac{U_{2n+1}(x,y)}{U_{2n}(x,y)} \Longrightarrow$$




$n$ elements of the type $x$ and $n$ elements of the type $(y/x)$

(a)

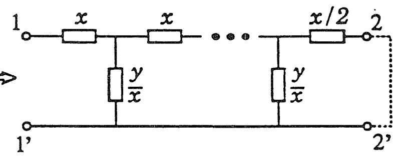
$$Z_{in} = \frac{1}{y_{11}} = \frac{U_{2n}(x,y)}{U_{2n-1}(x,y)} \Longrightarrow$$




$n$ elements of the type $x$ and $(n-1)$ elements of the type $(y/x)$

(b)

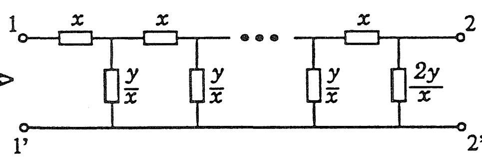
$$Z_{in} = \frac{1}{y_{11}} = \frac{V_{2n+1}(x,y)}{V_{2n}(x,y)} \Longrightarrow$$




$(n+1)$ elements of the type $x$ and $n$ elements of the type $(y/x)$

(c)

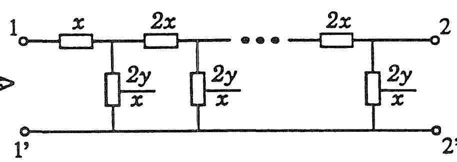
$$Z_{in} = z_{11} = \frac{V_{2n}(x,y)}{V_{2n-1}(x,y)} \Longrightarrow$$




$n$ elements of the type $x$ and $n$ elements of the type $(y/x)$

(d)

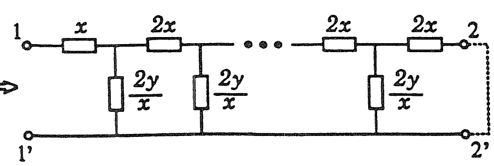
$$Z_{in} = z_{11} = \frac{V_{2n}(x,y)}{U_{2n}(x,y)} \Longrightarrow$$




$n$ elements of the type $x$ and $n$ elements of the type $(y/x)$

(e)

$$Z_{in} = \frac{1}{y_{11}} = \frac{V_{2n+1}(x,y)}{U_{2n+1}(x,y)} \Longrightarrow$$




$(n+1)$ elements of the type $x$ and $n$ elements of the type $(y/x)$

(f)

**FIGURE 1. Various two-element-kind ladder networks.**

## 4. LADDER TWO-PORTS

We will now express the chain parameters (see [2] and [14] for a definition of the chain parameters) of the six ladder two-port networks shown in Figs. 1(a)-1(f) in terms of the polynomials $U_n(x, y)$ and $V_n(x, y)$. First, consider the network of Fig. 1(a). We will now prove by induction that the chain matrix of this $n$-section ladder two-port is given by

$$[a_1]_n = \frac{1}{y^n}\begin{bmatrix} U_{2n+1}(x,y) & yU_{2n}(x,y) \\ U_{2n}(x,y) & yU_{2n-1}(x,y) \end{bmatrix}. \tag{7}$$

It is seen that, for $n=1$, (7) holds since the chain matrix for one section [see Fig. 2(a)] is

$$[a_1]_1 = \begin{bmatrix} 1+(x^2/y) & x \\ (x/y) & 1 \end{bmatrix} = \frac{1}{y}\begin{bmatrix} U_3(x,y) & yU_2(x,y) \\ U_2(x,y) & yU_1(x,y) \end{bmatrix}. \tag{8}$$

The $(n+1)$-section ladder corresponding to Fig. 1(a) is shown in Fig. 2(b). Its chain matrix is

$$[a_1]_{n+1} = \frac{1}{y}\begin{bmatrix} x^2+y & xy \\ x & y \end{bmatrix}[a_1]_n. \tag{9}$$

Hence,

$$\begin{aligned}[a_1]_{n+1} &= \frac{1}{y^{n+1}}\begin{bmatrix} x(xU_{2n+1}+yU_{2n})+yU_{2n+1} & xy(xU_{2n}+yU_{2n-1})+y^2U_{2n} \\ xU_{2n+1}+yU_{2n} & xU_{2n}+yU_{2n-1} \end{bmatrix} \\ &= \frac{1}{y^{n+1}}\begin{bmatrix} xU_{2n+2}+yU_{2n+1} & y(xU_{2n+1}+yU_{2n}) \\ U_{2n+2} & yU_{2n+1} \end{bmatrix} = \frac{1}{y^{n+1}}\begin{bmatrix} U_{2n+3} & yU_{2n+2} \\ U_{2n+2} & yU_{2n+1} \end{bmatrix},\end{aligned}$$

where, for brevity, we have used $U_n$ and $V_n$ for $U_n(x, y)$ and $V_n(x, y)$. Hence, the result is true for $(n+1)$-sections; thus, the result given by (7) is established.

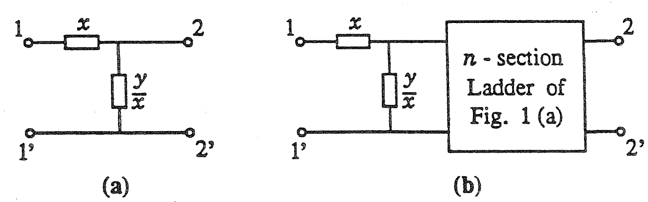



**FIGURE 2. (a) One section of the ladder network of Fig. 1(a).**
**(b) An $(n+1)$-section of the ladder of Fig. 1(a) considered as a cascade of the $L$-section of Fig. 2(a) and the $n$-section ladder of Fig. 1(a).**

We will now obtain the chain matrix for the two-port of Fig. 1(b). This may be considered as a cascade of an $(n-1)$-section ladder of Fig. 1(a) and a single series element shown in Fig. 3. Hence, its chain matrix is given by

$$[a_2]_n = [a_1]_{n-1}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \frac{1}{y^{n-1}}\begin{bmatrix} U_{2n-1}(x,y) & yU_{2n-2}(x,y) \\ U_{2n-2}(x,y) & yU_{2n-3}(x,y) \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

Thus, the chain matrix of the two-port of Fig. 1(b) is given by

$$[a_2]_n = \frac{1}{y^{n-1}} \begin{bmatrix} U_{2n-1}(x,y) & U_{2n}(x,y) \\ U_{2n-2}(x,y) & U_{2n-1}(x,y) \end{bmatrix}. \tag{10}$$

Similarly, we can show that the chain matrix of the two-ports shown in Figs. 1(c) and 1(d) are, respectively, given by

$$[a_3]_n = \frac{1}{y^n} \begin{bmatrix} U_{2n+1}(x,y) & \frac{1}{2}V_{2n+1}(x,y) \\ U_{2n}(x,y) & \frac{1}{2}V_{2n}(x,y) \end{bmatrix} \tag{11}$$

and

$$[a_4]_n = \frac{1}{y^n} \begin{bmatrix} \frac{1}{2}V_{2n}(x,y) & yU_{2n}(x,y) \\ \frac{1}{2}V_{2n-1}(x,y) & yU_{2n-1}(x,y) \end{bmatrix}, \tag{12}$$

where relation (3) has been used.

The network of Fig. 1(e) can be considered as a cascade of an L-section and an $(n-1)$-section ladder of the type shown in Fig. 1(a), except that all the impedances are scaled by a factor of 2, as shown in Fig. 4. Hence, its chain matrix is given by

$$[a_5]_n = \frac{1}{y} \begin{bmatrix} (x^2/2)+y & xy \\ (x/2) & y \end{bmatrix} \frac{1}{y^{n-1}} \begin{bmatrix} U_{2n-1}(x,y) & 2yU_{2n-2}(x,y) \\ \frac{1}{2}U_{2n-2}(x,y) & yU_{2n-3}(x,y) \end{bmatrix}.$$

Thus, the chain matrix of the two-port of Fig. 1(e) may be expressed as

$$[a_5]_n = \frac{1}{y^n} \begin{bmatrix} \frac{1}{2}V_{2n}(x,y) & yV_{2n-1}(x,y) \\ \frac{1}{2}U_{2n}(x,y) & yU_{2n-1}(x,y) \end{bmatrix}. \tag{13}$$

Similarly, we can show that the chain matrix corresponding to the two-port of Fig. 1(f) is

$$[a_6]_n = \frac{1}{y^n} \begin{bmatrix} \frac{1}{2}V_{2n}(x,y) & V_{2n+1}(x,y) \\ \frac{1}{2}U_{2n}(x,y) & U_{2n+1}(x,y) \end{bmatrix}. \tag{14}$$

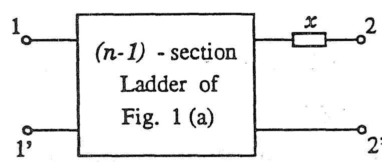



**FIGURE 3. The ladder of Fig. 1(b) considered as a cascade of the *n*-section ladder of Fig. 1(a) and a series element.**

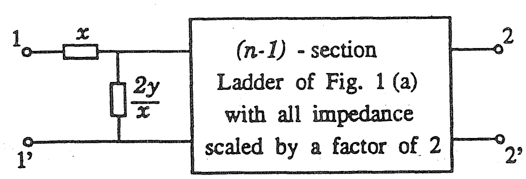



**FIGURE 4. The ladder of Fig. 1(e) considered as a cascade of an *L*-section ladder of Fig. 1(a), which is suitably impedance-scaled.**

As a consequence of the reciprocity property of these ladders, the determinants of the chain matrices given by (7), (10), (11), (12), (13), and (14) are all unity. Hence, we get the following interesting results:

$$U_{n+1}(x,y)U_{n-1}(x,y)-U_n^2(x,y)=(-1)^n y^{n-1}, \tag{15a}$$

$$U_{n+1}(x,y)V_n(x,y)-V_{n+1}(x,y)U_n(x,y)=(-1)^n 2y^n. \tag{15b}$$

As special cases, we also have

$$\widetilde{F}_{n+1}(s)\widetilde{F}_{n-1}(s)-\widetilde{F}_n^2(s)=(-1)^n\alpha^{n-1}, \tag{16a}$$

$$\widetilde{F}_{n+1}(s)\widetilde{L}_n(s)-\widetilde{L}_{n+1}(s)\widetilde{F}_n(s)=(-1)^n 2\alpha^n, \tag{16b}$$

and

$$\widetilde{J}_{n+1}(s)\widetilde{J}_{n-1}(s)-\widetilde{J}_n^2(s)=(-1)^n s^{n-1}, \tag{17a}$$

$$\widetilde{J}_{n+1}(s)\widetilde{j}_n(s)-\widetilde{j}_{n+1}(s)\widetilde{J}_n(s)=(-1)^n 2s^n. \tag{17b}$$

## 5. RELATIONS AMONG THE VARIOUS POLYNOMIALS

We first relate the two-variable polynomials $U_n(x,y)$ and $V_n(x,y)$ to the Morgan-Voyce polynomials $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$ (see [10], [14], [9], [8], [13]). It is known from [14] that the chain matrix of the network of Fig. 1(a) in terms of the Morgan-Voyce polynomials is given by

$$[a_1]_n=\begin{bmatrix} b_n(w) & xB_{n-1}(w) \\ \frac{x}{y}B_{n-1}(w) & b_{n-1}(w)\end{bmatrix}, \tag{18a}$$

where

$$w=x^2/y. \tag{18b}$$

Now, comparing (18a) and (5), we get

$$U_{2n+1}(x,y)=y^n b_n(x^2/y) \tag{19a}$$

and

$$U_{2n}(x,y)=xy^{n-1}B_{n-1}(x^2/y). \tag{19b}$$

Also, from (3), (19b), and (19a), we get

$$V_{2n+1}(x,y)=xy^n\{B_n(x^2/y)+B_{n-1}(x^2/y)\}$$

and

$$V_{2n}(x,y)=y^n\{b_n(x^2/y)+b_{n-1}(x^2/y)\}.$$

Hence,

$$V_{2n+1}(x,y)=xy^n c_n(x^2/y) \tag{19c}$$

and

$$V_{2n}(x,y)=y^n C_n(x^2/y). \tag{19d}$$

Using the above relations, (19a)-(19d), many interesting results for the two-variable polynomials $U_n(x,y)$ and $V_n(x,y)$—including the summation, product, and other formulas—may be derived from the properties of the Morgan-Voyce polynomials. However, we will not pursue it here. Instead, we establish the following relations among the various polynomials.

**Case 1: Modified Fibonacci and Lucas Polynomials**

Let $y = \alpha > 0$. Then, $U_n(x, \alpha) = \widetilde{F}_n(x)$ and $V_n(x, \alpha) = \widetilde{L}_n(x)$. Hence, from (19a)-(19d), we have

$$\widetilde{F}_{2n+1}(x) = \alpha^n b_n(x^2 / \alpha), \quad \widetilde{F}_{2n}(x) = \alpha^{n-1} x B_{n-1}(x^2 / \alpha), \tag{20a}$$

$$\widetilde{L}_{2n+1}(x) = \alpha^n x c_n(x^2 / \alpha), \quad \widetilde{L}_{2n}(x) = \alpha^n C_n(x^2 / \alpha). \tag{20b}$$

Of course, when $\alpha = 1$, the above reduce to the known relations between the Fibonacci, Lucas, and Morgan-Voyce polynomials.

**Case 2: Modified Jacobsthal Polynomials**

Let $\beta > 0$. Then $U_n(\beta, x) = \widetilde{J}_n(x)$ and $V_n(\beta, x) = \widetilde{j}_n(x)$. Hence, from (19a)-(19d), we have

$$\widetilde{J}_{2n+1}(x) = x^n b_n(\beta^2 / x), \quad \widetilde{J}_{2n}(x) = \beta x^{n-1} B_{n-1}(\beta^2 / x), \tag{21a}$$

$$\widetilde{j}_{2n+1}(x) = \beta x^n c_n(\beta^2 / x), \quad \widetilde{j}_{2n} = x^n C_n(\beta^2 / x). \tag{21b}$$

It is clear from (20) and (21) that the modified Fibonacci and Lucas polynomials and, hence, the Fibonacci and Lucas polynomials are directly related to the Jacobsthal polynomials by the simple relations

$$\widetilde{F}_n(x) = x^{n-1} J_n(\alpha / x^2), \quad \widetilde{L}_n(x) = x^n j_n(\alpha / x^2), \tag{22a}$$

and

$$F_n(x) = x^{n-1} J_n(1 / x^2), \quad L_n(x) = x^n j_n(1 / x^2). \tag{22b}$$

The above result could have been obtained from the networks of Figs. 1(e) and 1(f) which, respectively, realize $j_{2n}(s) / J_{2n}(s)$ and $j_{2n+1}(s) / J_{2n+1}(s)$ when $x = 1$ and $y = s$, by first transforming the complex frequency from $s$ to $\alpha / s^2$, and then multiplying all the resulting impedances by $s$.

**Case 3: Modified Chebyshev Polynomials**

We define $\widetilde{G}_n(x)$ and $\widetilde{H}_n(x)$, the modified Chebyshev polynomials of the first and second kind, respectively, by

$$\widetilde{G}_n(x) = U_n(x, -\alpha), \quad \widetilde{H}_n(x) = V_n(x, -\alpha), \tag{23a}$$

where

$$\alpha > 0. \tag{23b}$$

Then, from (19a)-(19d), we have

$$\widetilde{G}_{2n+1}(x) = (-1)^n \alpha^n b_n(-x^2 / \alpha), \quad \widetilde{G}_n(x) = (-1)^{n-1} \alpha^{n-1} x B_{n-1}(x^2 / \alpha) \tag{24a}$$

and

$$\widetilde{H}_{2n+1}(x) = (-1)^n \alpha^n x c_n(-x^2 / \alpha), \quad \widetilde{H}_{2n}(x) = (-1)^n \alpha^n C_n(-x^2 / \alpha). \tag{24b}$$

Now, using (21a) and (21b), we may relate the modified Chebyshev polynomials directly to the Jacobsthal polynomials by

$$\widetilde{G}_n(x) = x^{n-1} J_n(-\alpha / x^2), \quad \widetilde{H}_n(x) = x^n j_n(-\alpha / x^2). \tag{25}$$

Now $\Phi_n(x)$ and $\Theta_n(x)$, the Fermat polynomials of the first and second kinds, respectively, are obtained by letting $\alpha = 2$ in (23). Hence,

$$\Phi_n(x) = x^{n-1}J_n(-2/x^2), \quad \Theta_n(x) = x^n j_n(-2/x^2). \tag{26}$$

Also, the Chebyshev polynomials $S_n(x)$ and $T_n(x)$ are given by

$$S_n(x) = U_n(2x, -1) = (2x)^{n-1}J_n(-1/4x^2) \tag{27a}$$

and

$$T_n(x) = \tfrac{1}{2}V_n(2x, -1) = 2^{n-1}x^n j_n(-1/4x^2). \qquad 27b)$$

**Case 4: Brahmagupta's Polynomials**

Brahmagupta's polynomials $x_n(x, y)$ and $y_n(x, y)$ are defined as follows (see [12]):

$$x_{n+1}(x, y) = 2xx_n(x, y) - \lambda x_{n-1}(x, y), \quad x_0 = 1, \ x_1 = x, \tag{28a}$$

and

$$y_{n+1}(x, y) = 2xy_n(x, y) - \lambda y_{n-1}(x, y), \quad y_0 = 1, \ y_1 = y. \tag{28b}$$

It is known that if $(x_1, y_1)$ is a positive integer set satisfying the relation

$$x_1^2 - ty_1^2 = \lambda \tag{29a}$$

where $t$ is a square-free integer, then the positive integer set $(x_n, y_n)$ is a solution of Brahmagupta-Bhaskara's equation given by [15]:

$$x_n^2 - ty_n^2 = \lambda^n. \tag{29b}$$

The Brahmagupta polynomials are related to $U_n(x, y)$ and $V_n(x, y)$ by

$$x_n(x, y) = \tfrac{1}{2}V_n(2x, -\lambda), \quad y_n(x, y) = yU_n(2x, -\lambda), \tag{30a}$$

and to the Jacobsthal polynomials by

$$x_n(x, y) = 2^{n-1}x^n j_n(-\lambda/4x^2), \quad y_n(x, y) = y(2x)^{n-1}J_n(-\lambda/4x^2). \tag{30b}$$

If $\lambda > 0$, and say $= \alpha$, then

$$x_n(x, y) = \tfrac{1}{2}\widetilde{H}_n(2x), \quad y_n(x, y) = y\widetilde{G}_n(2x). \tag{31}$$

However, if $\lambda < 0$, say $= -\alpha$, $\alpha > 0$, then

$$x_n(x, y) = \tfrac{1}{2}\widetilde{L}_n(2x), \quad y_n(x, y) = y\widetilde{F}_n(2x). \tag{32}$$

Of course, the polynomials $\widetilde{G}_n(x)$, $\widetilde{H}_n(x)$, $\widetilde{F}_n(x)$, and $\widetilde{L}_n(x)$ are related to the Jacobsthal and Morgan-Voyce polynomials, and hence we may relate the Brahmagupta polynomials also to these polynomials. Finally, it is seen that, when $\lambda = 1$,

$$x_n(x, y) = T_n(x), \quad y_n(x, y) = yS_n(x), \tag{33}$$

while, when $\lambda = -1$,

$$x_n(x, y) = \tfrac{1}{2}Q_n(x), \quad y_n(x, y) = yP_n(x). \tag{34}$$

As a consequence of (33) and (34), we can show that

$$Q_{2n}(x) = 2T_n(2x^2 + 1), \quad P_{2n}(x) = 2xS_n(2x^2 + 1). \tag{35}$$

## 6. DERIVATIVE POLYNOMIALS AND THEIR ZEROS

In this section we will show that we can get information about the location of the zeros of the derivative polynomials using the following known results about the nature of the impedance functions of two-element-kind networks.

**Property 1:** If the driving point impedance $Z(s) = N(s) / D(s)$ is a reactance function, then so is $Z_1(s) = N'(s) / D'(s)$, where the prime indicates the derivative with respect to $s$.

**Property 2:** If $Z(s) = N(s) / D(s)$ is an RL-impedance (or an RC-admittance) function, then so is $Z_1(s) = N'(s) / D'(s)$.

Let us first consider the function, $Z(s) = \widetilde{L}_n(s) / \widetilde{F}_n(s)$, a ratio of the modified Fibonacci and Lucas polynomials. We have shown in Section 3 that $Z(s)$ is a reactance function. Hence, from Property 1, the function $Z_1(s) = \widetilde{L}_n'(s) / \widetilde{F}_n'(s)$ is also a reactance function. By successively applying Property 1 $k$ times, we see that the function $Z_k(s) = \widetilde{L}_n^{(k)}(s) / \widetilde{F}_n^{(k)}(s)$, where $(k)$ represents the $k^{\text{th}}$ derivative with respect to $s$, is also a reactance function. Using the property of reactance functions, we see that the zeros of $\widetilde{L}_n^{(k)}(s)$ and $\widetilde{F}_n^{(k)}(s)$ are simple and lie on the imaginary axis, with the two sets of zeros interlacing with each other. Similar statements hold for the zeros of $\widetilde{L}_{n+1}^{(k)}(s)$ and $\widetilde{L}_n^{(k)}(s)$, as well as for those of $\widetilde{F}_{n+1}^{(k)}(s)$ and $\widetilde{F}_n^{(k)}(s)$.

We also proved in Section 3 that the ratios $\widetilde{j}_n(s) / \widetilde{J}_n(s)$, $\widetilde{J}_{n+1}(s) / \widetilde{J}_n(s)$, and $\widetilde{j}_{n+1}(s) / \widetilde{j}_n(s)$ are all RL-impedance functions. Thus, from Property 2, we see that $\widetilde{j}_n^{(k)}(s) / \widetilde{J}_n^{(k)}(s)$, $\widetilde{j}_{n+1}^{(k)}(s) / \widetilde{j}_n^{(k)}(s)$, and $\widetilde{J}_{n+1}^{(k)}(s) / \widetilde{J}_n^{(k)}(s)$ are also RL-impedance functions. Using the property of RL-impedance functions, we see that the zeros of $\widetilde{J}_n^{(k)}(s)$ and $\widetilde{j}_n^{(k)}(s)$ are real and negative. Further, the zeros of $\widetilde{J}_n^{(k)}(s)$ interlace with those of $\widetilde{j}_n^{(k)}(s)$, with the zeros closest to the origin being that of $\widetilde{j}_n^{(k)}(s)$. Similar statements hold true for the zeros of $\widetilde{J}_{n+1}^{(k)}(s)$ and $\widetilde{J}_n^{(k)}(s)$, as well as those of $\widetilde{j}_{n+1}^{(k)}(s)$ and $\widetilde{j}_n^{(k)}(s)$.

Similar results may be established regarding the zeros of the derivatives of the Morgan-Voyce polynomials.

## 7. CONCLUDING REMARKS

It is shown that there exists a close relationship between the network functions of LC, RL, and RC ladder networks and certain generalized polynomials. In view of this, many interesting properties of these polynomials may be derived using the well-known properties of two-element-kind ladder networks, and vice-versa. A few elegant results regarding the location of the zeros of the polynomials such as the Fibonacci, Lucas, Jacobsthal, as well as their derivative polynomials have been derived. Also, the interrelations among these various polynomials and the Morgan-Voyce polynomials have been derived.

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# SIEVE FORMULAS FOR THE GENERALIZED FIBONACCI AND LUCAS NUMBERS

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We present here two sieve-type explicit formulas for $r$-Fibonacci and $r$-Lucas numbers ($r = 2, 3, \dots$) that connect them with families of well-defined combinatorial numbers, and discuss some particular cases.

## 1. DEFINITIONS

We consider the two main families of sequences $\{F_n^{(r)}\}$ and $\{L_n^{(r)}\}$ ($r = 2, 3, \dots$), determined by the simplest general $r^{\text{th}}$-order linear recursion

$$Q_n^{(r)} = \sum_{k=1}^{r} Q_{n-k}^{(r)} \quad (n \geq r) \tag{1}$$

($Q_n^{(r)}$ denotes either $F_n^{(r)}$ or $L_n^{(r)}$) with initial conditions

$$F_0^{(r)} = 0,\ F_1^{(r)} = 1, \dots,\ F_j^{(r)} = 2^{j-2} \quad (2 \leq j \leq r-1); \tag{2}$$

$$L_0^{(r)} = r,\ L_1^{(r)} = 1, \dots,\ L_j^{(r)} = 2^j - 1 \quad (1 \leq j \leq r-1). \tag{3}$$

$F_n^{(r)}$ and $L_n^{(r)}$ are *r-Fibonacci* and *r-Lucas numbers*, respectively (cf. [2], [6], [8], [9]; also [7] with $a_i = 1$ for all $i$)—or the "fundamental" and "primordial" sequences named by Lucas. The sequences $\{F_n^{(r)}\}$ and $\{L_n^{(r)}\}$ differ from the known Tribonacci, Tetranacci, etc., sequences in having a shift $r-2$ places backwards.

The recursion (1) implies a fundamental property—the *subtraction law*

$$Q_n^{(r)} = 2Q_{n-1}^{(r)} - Q_{n-r-1}^{(r)} \quad (n \geq r+2) \tag{4}$$

for sequences of both kinds.

Our aim is to evaluate the differences $2^{n-2} - F_n^{(r)}$ and $2^n - 1 - L_n^{(r)}$ caused by this subtraction. We propose a method of exact calculation of $F_n^{(r)}$ and $L_n^{(r)}$.

As a result, explicit formulas (12) and (18) are obtained, which generalize the known formulas in the particular case $r = 2$ (Section 4).

## 2. THE $r$-FIBONACCI SEQUENCES

The evaluation of $2^{n-2} - F_n^{(r)}$ involves a family of numbers

$$\begin{aligned} d(m,1) &= 1, \\ d(m,n) &= \frac{2m+n-3}{n-1}\binom{m+n-3}{n-2}2^{n-2} \\ &= \frac{2m+n-3}{m-1}\binom{m+n-3}{m-2}2^{n-2} \quad (n \geq 2). \end{aligned} \tag{5}$$

The numbers $d(m,n)$ and $c(m,n)=d(m,n)/2^{n-2}$ for particular values of $m$ and $n$ are well known. For fixed $n$, $c(m,n)$ are the $(n-1)$-dimensional square pyramidal numbers [3] (sequences M3356, M3844, M4135, M4387 in [10]—for $n=2,3,4,5$). There is also

$$c(n-1,n)=(3n-5)C_{n-2}, \tag{6}$$

where $C_n$ is the $n^{\text{th}}$ Catalan number (M2814).

As $c(m,2)=d(m,2)=2m-1$, the array $\{c(m,n)\}$ with $m\geq 2$ may be considered as the "Pascal product" of the sequences $(1,3,5,\ldots,2m-1,\ldots)$ [or $(2,2,\ldots,2,\ldots)$, beginning from $n=1$] and $(1,1,\ldots,1,\ldots)$ with the addition law

$$c(m,n)=c(m,n-1)+c(m-1,n) \tag{7}$$

of the Pascal triangle array $\{\binom{m+n}{n}\}$.

The numbers $c(m,n)$ appear also as coefficients in the Lucas polynomials $L_n(x)$:

$$L_n(x)=\sum_{m=0}^{\lfloor n/2\rfloor+1}c(m+1,n-2m+1)x^{n-2m}. \tag{8}$$

The numbers $d(m,n)$ enter as coefficients in the Chebyshev polynomials $T_n(x)$ [1]:

$$T_n(x)=\sum_{m=1}^{\lfloor n/2\rfloor+1}(-1)^{m-1}d(m,n-2m+3)x^{n-2m+2} \tag{9}$$

(see M2739, M3881, M4405, M4631, M4796, M4907 for $m=2,\ldots,7$).

***Proposition 1:***

$$d(m,n)=2d(m,n-1)+d(m-1,n). \tag{10}$$

***Proof:***

$$\begin{aligned}2d(m,n-1)+d(m-1,n)&=2\frac{2m+n-4}{n-2}\binom{m+n-4}{n-3}2^{n-3}+\frac{2m+n-5}{n-1}\binom{m+n-4}{n-2}2^{n-2}\\&=\left(\frac{2m+n-4}{m-1}+\frac{2m+n-5}{n-1}\right)\binom{m+n-4}{n-2}2^{n-2}\\&=\frac{2m^2+3mn+n^2-9m-6n+9}{(m-1)(n-1)}\cdot\frac{m-1}{m+n-3}\binom{m+n-3}{n-2}2^{n-2}\\&=\frac{2m+n-3}{n-1}\binom{m+n-3}{n-2}2^{n-2}=d(m,n).\quad\square\end{aligned}$$

***Theorem 1:***

$$F_n^{(r)}=\sum_{m=0}^{\lfloor\frac{n-1}{r+1}\rfloor}(-1)^m d(m+1,n-(r+1)m)\quad(n\geq 2). \tag{11}$$

***Proof:*** By induction. We see from (2) that the assertion is true for $n=2,\ldots,r+1$ because $2^{n-2}=d(1,n)$. Also, for $n=r+2$, it follows from (4) that

$$F_{r+2}^{(r)}=2F_{r+1}^{(r)}-F_1^{(r)}=2^r-1=d(1,r+2)-d(2,1).$$

Suppose that (11) holds for all $r$ previous values $n-1, \ldots, n-r-1$. Then, from (4) and (10), we obtain:

$$\begin{aligned}
F_n^{(r)} &= 2F_{n-1}^{(r)} - F_{n-r-1}^{(r)} \\
&= 2\left(2^{n-3} + \sum_{m=1}^{\lfloor \frac{n-1}{r+1} \rfloor} (-1)^m d(m+1, n-(r+1)m-1)\right) \\
&\qquad - \sum_{m=0}^{\lfloor \frac{n-1}{r+1} \rfloor - 1} (-1)^m d(m+1, n-(r+1)(m+1)) \\
&= 2^{n-2} + 2\sum_{m=1}^{\lfloor \frac{n-1}{r+1} \rfloor} (-1)^m d(m+1, n-(r+1)m-1) \\
&\qquad - \sum_{m=1}^{\lfloor \frac{n-1}{r+1} \rfloor} (-1)^{m-1} d(m, n-(r+1)m) \\
&= d(1,n) + \sum_{m=1}^{\lfloor \frac{n-1}{r+1} \rfloor} (-1)^m d(m+1, n-(r+1)m). \quad \square
\end{aligned}$$

From (5) and (11), we obtain the resulting formula.

***Corollary 1:***

$$F_n^{(r)} = 2^{n-2} - \sum_{m=1}^{\lfloor \frac{n-1}{r+1} \rfloor} (-1)^{m-1} \frac{n-m(r-1)-1}{m} \binom{n-mr-2}{m-1} 2^{n-m(r+1)-2} \quad (n \geq 2). \tag{12}$$

***Example:***

$$\begin{aligned}
F_{22}^{(5)} &= d(1,22) - d(2,16) + d(3,10) - d(4,4) \\
&= 2^{20} - 17\binom{15}{0}2^{14} + \frac{13}{2}\binom{10}{1}2^8 - \frac{9}{3}\binom{5}{2}2^2 \\
&= 1048576 - 278528 + 16640 - 120 = 786568.
\end{aligned}$$

## 3. THE $r$-LUCAS SEQUENCES

To evaluate the difference $2^n - 1 - L_n^{(r)}$, we introduce in a similar way the numbers

$$\begin{aligned}
e(r; m, 1) &= r+1, \\
e(r; m, n) &= \frac{(r+1)m+n-1}{m}\binom{m+n-2}{n-1}2^{n-1}.
\end{aligned} \tag{13}$$

The array $\{e(r; m, n)/2^{n-1}\}$ for the given $r$ is a Pascal product of the sequences $(r+1,\ r+1,\ \ldots,\ r+1,\ \ldots)$ and $(1, 1, \ldots, 1, \ldots)$ with the addition law analogous to (7). For the case $r=2$, we find in [10] two sequences from here: M2835 $(m=3)$, M3011 $(m=5)$, explained as coefficients in the expansion of $(1-x-x^2)^{-n}$. The numbers $e(r; m, n)$ show almost no connection with the previous ones; the only common values we can notice are

$$d(2,n) = e(2; 1, n-1) = (n+1)2^{n-2}. \tag{14}$$

However, their addition properties are the same.

***Proposition 2:***

$$e(r;m,n) = 2e(r;m,n-1) + e(r;m-1,n). \tag{15}$$

***Proof:***

$$\begin{aligned}
&2e(r;m,n-1) + e(r;m-1,n)\\
&= 2\frac{(r+1)m+n-2}{m}\binom{m+n-3}{n-2}2^{n-2} + \frac{(r+1)(m-1)+n-1}{m-1}\binom{m+n-3}{n-1}2^{n-1}\\
&= \left(\frac{((r+1)m+n-2)(n-1)}{m(m-1)} + \frac{(r+1)(m-1)+n-1}{m-1}\right)\binom{m+n-3}{n-1}2^{n-1}\\
&= \frac{(r+1)m^2+(r+2)mn+n^2-(2r+3)m-3n+2}{m(m-1)}\cdot\frac{m-1}{m+n-2}\binom{m+n-2}{n-1}2^{n-1}\\
&= \frac{(r+1)m+n-1}{m}\binom{m+n-2}{n-1}2^{n-1} = e(r;m,n). \quad \square
\end{aligned}$$

For the initial value $m = 1$, there obviously is

$$e(r;1,n) = 2e(r;1,n-1) + 2^{n-1}. \tag{16}$$

***Theorem 2:***

$$L_n^{(r)} = 2^n - 1 + \sum_{m=1}^{\lfloor\frac{n}{r+1}\rfloor}(-1)^m e(r;m,n-(r+1)m+1) \quad (n \geq 1). \tag{17}$$

***Proof:*** By induction (as in Theorem 1). The assertion is true for the $r$ initial values (3) (for $n \geq 1$) and $L_n^{(r)} = 2^r - 1$. We also can see that

$$L_{r+1}^{(r)} = 2L_r^{(r)} - L_0^{(r)} = 2^{r+1} - 1 - (r+1) = 2^{r+1} - 1 - e(r;1,1).$$

Performing the induction step $n-1 \to n$, and using (4), (15), and (16), we obtain:

$$\begin{aligned}
L_n^{(r)} &= 2L_{n-1}^{(r)} - L_{n-r-1}^{(r)}\\
&= 2\left(2^{n-1} - 1 + \sum_{m=1}^{\lfloor\frac{n}{r+1}\rfloor}(-1)^m e(r;m,n-(r+1)m)\right)\\
&\qquad - 2^{n-r-1} + 1 - \sum_{m=1}^{\lfloor\frac{n}{r+1}\rfloor-1}(-1)^m e(r;m,n-(r+1)m-r)\\
&= 2^n - 1 + \sum_{m=1}^{\lfloor\frac{n}{r+1}\rfloor}(-1)^m e(r;m,n-(r+1)m)\\
&\qquad - e(r;1,n-r) - \sum_{m=1}^{\lfloor\frac{n}{r+1}\rfloor-1}(-1)^m e(r;m,n-(r+1)(m+1)+1)\\
&= 2^n - 1 + \sum_{m=1}^{\lfloor\frac{n}{r+1}\rfloor}(-1)^m e(r;m,n-(r+1)m) + \sum_{m=2}^{\lfloor\frac{n}{r+1}\rfloor}(-1)^m e(r;m-1,n-(r+1)m+1)\\
&= 2^n - 1 + \sum_{m=1}^{\lfloor\frac{n}{r+1}\rfloor}(-1)^m e(r;m,n-(r+1)m+1). \quad \square
\end{aligned}$$

From (13), (17) follows

***Corollary 2:***

$$L_n^{(r)} = 2^n - 1 - n \sum_{m=1}^{\lfloor \frac{n}{r+1} \rfloor} (-1)^{m-1} \frac{1}{m} \binom{n-mr-1}{m-1} 2^{n-m(r+1)} \quad (n \geq 1). \tag{18}$$

***Example:***

$$\begin{aligned} L_{20}^{(5)} &= 2^{20} - 1 - e(5; 1, 15) + e(5; 2, 9) - e(5; 3, 3) \\ &= 2^{20} - 1 - 20\left(\binom{14}{0} 2^{14} - \frac{1}{2}\binom{9}{1} 2^8 + \frac{1}{3}\binom{4}{2} 2^2\right) \\ &= 1048575 - 20(16384 - 1152 + 8) = 743775. \end{aligned}$$

## 4. FORMULAS IN THE CASE $r = 2$

In the particular case $r = 2$, i.e., for the usual Fibonacci and Lucas numbers $F_n = F_n^{(2)}$, $L_n = L_n^{(2)}$ $(n \geq 3)$, the following formulas are obtained from (5), (12), (13), and (18).

***Corollary 3:***

$$\begin{aligned} F_n &= d(1, n) - d(2, n-3) + d(3, n-6) - \cdots \\ &= 2^{n-2} - \sum_{m=1}^{\lfloor (n-1)/3 \rfloor} (-1)^{m-1} \frac{n-m-1}{m} \binom{n-2m-2}{m-1} 2^{n-3m-2}; \end{aligned} \tag{19}$$

$$\begin{aligned} L_n &= 2^{n-1} - 1 - e(2; 1, n-2) + e(2; 2, n-5) - \cdots \\ &= 2^{n-1} - n \sum_{m=1}^{\lfloor n/3 \rfloor} (-1)^{m-1} \frac{1}{m} \binom{n-2m-1}{m-1} 2^{n-3m}. \end{aligned} \tag{20}$$

Formula (20) in an equivalent form was discovered by Filipponi ([4], formula (2.1)), using a simpler formula of Jaiswal [5], which (with $n$ instead of $n+3$ in the original notation) has the form

$$F_n = 1 + \sum_{m=1}^{\lfloor n/3 \rfloor} (-1)^{m-1} \binom{n-2m-1}{m-1} 2^{n-3m}. \tag{21}$$

This is perhaps the first known example of Fibonacci sieve formulas.

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***Announcement***

# NINTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

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**CALL FOR PAPERS**

Papers on all branches of mathematics and science related to the Fibonacci numbers, number theoretic facts as well as recurrences and their generalizations are welcome. Abstracts, which should be sent in duplicate to F. T. Howard at the address below, are due by June 1, 2000. An abstract should be at most one page in length (preferably half a page) and should contain the author's name and address. New results are especially desirable; however, abstracts on work in progress or results already accepted for publication will be considered. Manuscripts should *not* be submitted. Questions about the conference should be directed to:

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# ON THE DISCOVERY OF THE 38th KNOWN MERSENNE PRIME

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## 1. INTRODUCTION

There is a long, rich history in the search for primes of the form $M_p = 2^p - 1$, named *Mersenne primes*. In the early years the study of Mersenne primes led to several important advances in number theory. In recent decades the computer has become instrumental in the search for large primes leading to several algorithmic advances. This paper provides an overview of the history (see [6]) and techniques used in finding the currently largest known explicit Mersenne prime.

It is helpful to tour briefly some known properties of Mersenne primes. The possible exponents $p$ are restricted by an elementary theorem that says: if $M_p$ is prime, then $p$ is prime. Furthermore, known factors of $M_p$ must of necessity be of the form $q = 2kp+1$; also, when $p > 2$ we must have $q = 1$ or $7 \pmod 8$. These properties can be used effectively to quickly sieve out many composite $M_p$, thus eliminating the need for the comparatively expensive Lucas-Lehmer test, a definitive and rigorous primality test described below. There is also the ancient connection with perfect numbers. A perfect number equals the sum of its positive divisors excluding itself. The first two examples are $6 = 1+2+3$ and $28 = 1+2+4+7+14$. The Euclid-Euler theorem states that an even number $n$ is perfect if and only if it is of the form $n = 2^{p-1}M_p$, where $M_p$ is prime. Therefore, one can say that every new Mersenne prime discovery immediately begets a new perfect number.

Beyond these proven facts there has been a great deal of conjecture and mystery pertaining to the Mersenne primes. Two widely-believed heuristic arguments (see [3] and [8]) say that the probability that $M_p$ be prime is

$$(e^{\gamma} \log ap) / p \log 2), \text{ where } a = 2 \text{ if } p \equiv 3 \pmod 4 \text{ and } a = 6 \text{ if } p \equiv 1 \pmod 4$$

and the number of Mersenne primes less than or equal to $x$ is about

$$(e^{\gamma} / \log 2) * \log\log x,$$

where $\gamma \approx 0.577$ is the Euler constant. It is interesting to compare this heuristic with the occurrence of the currently known Mersenne primes. Chris Caldwell has done just that, providing several graphs examining these conjectures at

http://www.utm.edu/research/primes/notes/faq/NextMersenne.html.

Since 1952, Robinson, Riesel, Hurwitz, Gillies, Tuckerman, and Noll & Nickel all used the most powerful computers of their day to find new Mersenne primes. In 1979, Slowinski, sometimes partnering with Nelson or Gage, began a 17-year reign finding seven ever larger Mersenne primes with Cray supercomputers. Colquitt and Welsh found an overlooked Mersenne prime in 1988. In 1996, Woltman and Kurowski developed a kind of "Internet supercomputer" that has found the last 4 Mersenne primes. Once again, Caldwell's http://www.utm.edu/research/primes/mersenne.shtml

web page provides a superb and more complete history.

## 2. RESULTS

On June 1, 1999, the Mersenne prime $2^{6972593}-1$ was discovered by Nayan Hajratwala on his personal computer using software developed by the author and Scott Kurowski. Hajratwala is one of over 12,000 participants in the Great Internet Mersenne Prime Search (GIMPS). As of this writing, the prime is the largest known explicit prime of any type.

The new prime number is 2098960 decimal digits long. It took 111 days running part-time on a 350 MHz Pentium II computer to prove this number prime. Running non-stop the test would have taken just over three weeks.

The number was subsequently verified as prime by three independent parties, each party employing different hardware *and* software. Gerardo Cisneros used David Slowinski and Paul Gage's program on the 4-CPU CRAY Y-MP at the National Autonomous University of Mexico's General Directorate of Academic Computing Services (DGSCA/UNAM). David Willimore used a program by Ernst Mayer on an Aerial Communications 500 MHz Alpha workstation. Cornelius Caesar used a program by John Sweeney on an IBM RS/6000.

There are now 38 known Mersenne primes. $M_p$ is prime for $p=2$, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593. As we will see later, the new prime number may not be the 38th Mersenne prime by size. In fact, there could be exponents below 2976221 that yield a new Mersenne prime. The current status of the search is frequently updated at

http://www.mersenne.org/status.htm.

## 3. IMPLEMENTATION OF THE LUCAS-LEHMER TEST

One requirement for finding such gargantuan Mersenne primes is a fast and efficient implementation. The search program uses the celebrated Lucas-Lehmer primality test (see [4] and [5]), a convenient variant of which is to define the sequence starting with $x_1 := 4$, and iterate

$$x_{n+1} := x_n^2 - 2 \pmod{M_p}$$

through the index $n = p-1$. $M_p$ is prime if and only if $x_{p-1} = 0 \pmod{M_p}$. It is clear that the main operation in this rigorous test is that of squaring of numbers of size $p$ bits. Schönhage and Strassen [7] showed that a Fast Fourier Transform (FFT) can be used to square a $p$-bit number in $O(p \log p \log\log p)$ bit operations. A very rough description of the procedure is as follows. Because multiplication is essentially what is called "acyclic" convolution, and since acyclic convolution is a cyclic convolution of zero-padded sequences, and since the cyclic case can be handled by Fourier transforms, one may zero-pad a $p$-bit number $x$ to be squared, resulting in a number with about 2p bits, followed by FFT-based convolution to get the desired integer product. In the early 1990s, Richard Crandall [2] observed that, since squaring modulo $M_p$ is a length-$p$ cyclic convolution of the bits of $x$, it is possible to expand $x$ into approximate digits in an irrational base $2^r$, where $r = q/2^k$, and thereby, via cyclic convolution, accomplish two things: eliminate the zero-padding entirely, and allow power-of-two run lengths in the Fourier transforms. In this fashion, the search time for large Mersenne primes was effectively halved. The present author implemented this "irrational base discrete weighted transform" (IBDWT) algorithm in assembly language to take advantage of the pipelining capabilities of the Intel architecture. One extension of the algorithm was to allow convenient, but non-power-of-two run lengths, which further increased search efficiency. The author also used an in-place variant of a David Bailey [1] idea on avoiding power-of-two memory strides that have a devastating impact on memory caches.

The second ingredient for a successful search is a great deal of computing power. Traditionally this involved the use of supercomputers. Today, there is a better alternative—distributed computing. Distributed computing uses the idle cycles on thousands of ordinary computers connected to the Internet. Scott Kurowski, founder of Entropia.com, developed software that makes this easy. An Internet user runs the prime search program which automatically contacts a central server to get a work assignment. The computer works on this assignment off-line at low priority. When the assignment completes the program contacts the server to report its results and get a new work assignment.

The Great Internet Mersenne Prime Search (GIMPS), which was founded in 1996, has a goal of methodically testing all Mersenne numbers up to achievable limits. Slower machines look for small factors of Mersenne numbers, and so perform the kind of sieving mentioned in the Introduction. Faster machines run the Lucas-Lehmer primality tests. Medium speed machines rerun primality tests to make sure the first test ran properly (the last 64 bits of the last Lucas-Lehmer iteration are compared). The distributed nature of the search process means that there are always "gaps" in the testing and double-checking process. There are over 12000 exponents below 6972593 that have not finished testing and over 60000 exponents that have not been double-checked including some below 2976221.

To run a successful distributed project, the central server must do more than hand out assignments and track results. The Entropia.com server must reassign work that is not completed in a timely manner. To help users running on several computers, the server provides reports so the user can keep track of each computer's progress. Finally, many users enjoy seeing where they or their team stand in a "top producers" report. The server also sends out periodic newsletters to interested users. It is important to provide these "extras" to keep an all-volunteer work force enthused and up-to-date on current events.

## 4. THE FUTURE

The Electronic Frontier Foundation, www.eff.org, is offering awards of $100,000 and up for the discovery of primes with 10 million, 100 million, and 1 billion digits. While the 10 million digit award may be within GIMPS reach—it is "only" 125 times more difficult than finding a 2 million digit prime—the larger primes will require significant breakthroughs in primality testing or multiplication algorithms.

There are several factors that affect when the next Mersenne prime will be found. Obviously, the size of the exponent is one. Based on past percentage distances between Mersenne primes, the next exponent could be as high as 28.6 million! The number of computers participating in GIMPS is always growing and the speed of the average computer gets faster every year. While the next Mersenne prime could be found tomorrow, the chances are GIMPS will find the next one sometime during the next 2 years. It will be hard to sustain GIMPS' track record of 4 Mersenne primes found in less than 4 years.

## 5. CONCLUSIONS

It would appear that many supercomputing chores of the future will no doubt be performed by vast networks of computers. Many other fields of computation—ranging from medical data processing to searches for extraterrestrial intelligence—may well benefit from such massive parallelism. It is hoped that this Mersenne discovery serves as a kind of example of what is possible using these techniques.

In many fields, new research will be needed to develop algorithms that can be run on such a massively parallel network using a minimum of bandwidth. Many businesses and universities will eventually harness power of the many small computers they own to do computationally intensive research more economically.

## ACKNOWLEDGMENTS

While there are far too many people to list here, the author wishes to thank the huge number of people that have contributed to the GIMPS project. While most have contributed computer time, many have also contributed by writing software, recruiting others, running mailing lists, helping new users, and hosting web pages.

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AMS Classification Numbers: 11-04, 11A41

# ELEMENTARY PROBLEMS AND SOLUTIONS

## *Edited by*
## Stanley Rabinowitz

*Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to* stanley@tiac.net *on the Internet. All correspondence will be acknowledged.*

*Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.*

### BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

### PROBLEMS PROPOSED IN THIS ISSUE

**B-884** ***Proposed by M. N. Deshpande, Aurangabad, India***

Find an integer $k$ such that the expression

$$F_n^2 F_{n+2}^2 + kF_{n+1}^2 F_{n+2}^2 + F_{n+1}^2 F_{n+3}^2$$

is a constant independent of $n$.

**B-885** ***Proposed by A. J. Stam, Winsum, The Netherlands***

For $n > 0$, evaluate

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{k}{2n-k} \binom{2n-k}{n} F_{k+1}.$$

**B-886** ***Proposed by Peter J. Ferraro, Roselle Park, NJ***

For $n \geq 9$, show that

$$\left\lfloor \sqrt[4]{F_n} \right\rfloor = \left\lfloor \sqrt[4]{F_{n-4}} + \sqrt[4]{F_{n-8}} \right\rfloor.$$

**B-887** ***Proposed by A. J. Stam, Winsum, The Netherlands***

Show that

$$\sum_{k=0}^{n} \binom{y-n-1-k}{n-k} F_{2k+1} = \sum_{k=0}^{n} \binom{y-n-2-k}{n-k} F_{2k+2} = \sum_{j=0}^{n} \binom{y-j}{j}.$$

**B-888** ***Proposed by A. Arya, J. Fellingham, and D. Schroeder, Ohio State University, OH, and J. Glover, Carnegie Mellon University, PA***

For $n \geq 1$, let $A_n = [a_{i,j}]$ denote the symmetric matrix with $a_{i,i} = i+1$ and $a_{i,j} = \min[i, j]$ for all integers $i$ and $j$ with $i \neq j$.

(a) Find the determinant of $A_n$.

(b) Find the inverse of $A_n$.

## SOLUTIONS

### $n$th Derivative

**B-865** ***Proposed by Alexandru Lupas, University Lucian Blaga, Sibiu, Romania (Vol. 36, no. 5, November 1998)***

Let $f(x) = (x^2+4)^{n-1/2}$, where $n$ is a positive integer. Let

$$g(x) = \frac{d^n f(x)}{dx^n}.$$

Express $g(1)$ in terms of Fibonacci and/or Lucas numbers.

***Solution by Richard André-Jeannin, Cosnes et Romain, France***

It is known (Theorem 2 from [1]) that

$$L_n(x) = 2\frac{n!}{(2n)!}\sqrt{x^2+4}\,g(x).$$

From this, we get

$$g(1) = \frac{(2n)!\,L_n}{2\sqrt{5}\,n!}.$$

**Reference**

1. Richard André-Jeannin. "Differential Properties of a General Class of Polynomials." *The Fibonacci Quarterly* **33.5** (1995):453-458.

***Solutions also received by Paul S. Bruckman, H.-J. Seiffert, and the proposer.***

### Divisibility by 25

**B-866** ***Proposed by the editor (Vol. 37, no. 1, February 1999)***

For $n$ an integer, show that $L_{8n+4} + L_{12n+6}$ is always divisible by 25.

***Solution 1 by Pentti Haukkanen, University of Tampere, Tampere, Finland***

It is known [1, (17b)] that

$$L_{n+m} - (-1)^m L_{n-m} = 5F_nF_m.$$

Therefore,

$$L_{12n+6} + L_{8n+4} = 5F_{10n+5}F_{2n+1}.$$

It is well known that $a|b \Rightarrow F_a|F_b$. Therefore, $5|F_{10n+5}$ and, further,

$$25|5F_{10n+5}F_{2n+1} \text{ or } 25|L_{12n+6}+L_{8n+4}.$$

**Reference**

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section.* Chichester: Ellis Horwood Ltd., 1989.

***Solution 2 by Calvin T. Long, Northern Arizona University, Flagstaff, AZ***

More generally, we show that $L_r + L_{r+2s}$ is divisible by 25 if and only if $5|s$ or $5|r+s$. If we then take $r = 8n+4$ and $s = 2n+1$, we have $r+s = 10n+5$ and the above result follows.

It is well known (see, e.g., [1], p. 222) that

$$L_r + L_{r+2s} = \begin{cases} 5F_sF_{r+s} & \text{for } s \text{ odd,} \\ L_sL_{r+s} & \text{for } s \text{ even.} \end{cases}$$

Since $5 \nmid L_n$ for any $n$ and $5|F_n$ if and only if $5|n$, it follows that $25|L_r + L_{r+s}$ if and only if $5|s$ or $5|r+s$.

**Reference**

1. C. Long. "On a Fibonacci Arithmetical Trick." *The Fibonacci Quarterly* **23.3** (1985):221-31.

*Seiffert showed that* $L_{2k} \equiv (-1)^{k-1}(5k^2-2) \pmod{25}$.

***Solutions also received by Richard André-Jeannin, Brian D. Beasley, Paul S. Bruckman, Kathleen E. Lewis, Steve Scarborough, H.-J. Seiffert, Indulis Strazdins, and the proposer.***

## 1999 Belongs

**B-867** ***Proposed by the editor***
***(Vol. 37, no. 1, February 1999)***

Find small positive integers $a$ and $b$ so that 1999 is a member of the sequence $\langle u_n \rangle$, defined by $u_0 = 0, u_1 = 1, u_n = au_{n-1} + bu_{n-2}$ for $n > 1$.

***Solution by Brian D. Beasley, Presbyterian College, Clinton, SC***

Since $u_2 = a$ and $u_3 = a^2 + b$, we may find $a$ and $b$ so that $a^2 + b = 1999$. The solution in positive integers that yields the largest $a$ and hence the smallest $b$ is $(a, b) = (44, 63)$. Such solutions range from $(44, 63)$ and $(43, 150)$ to $(1, 1998)$.

We note that since 1999 is prime and $u_4 = a(a^2 + 2b)$, the only way to achieve $u_4 = 1999$ is to take $(a, b) = (1, 999)$. Also, since $u_5 = a^4 + 3a^2b + b^2$, achieving $u_5 = 1999$ would force $a^4 < 1999$ or $a \in \{1, 2, 3, 4, 5, 6\}$, none of which produces an integer value for $b$.

***Solutions also received by Indulis Strazdins and the proposer.***

## Congruence Mod 25

**B-868** ***Based on a proposal by Richard André-Jeannin, Longwy, France***
***(Vol. 37, no. 1, February 1999)***

Find an integer $a > 1$ such that, for all integers $n$, $F_{an} \equiv aF_n \pmod{25}$.

***Solution 1 by Pentti Haukkanen, University of Tampere, Finland***

Note that $F_{25} = 75025$ is divisible by 25. By the well-known property $c|b \Rightarrow F_c|F_b$, we have $25|(F_{an} - aF_n)$ when $a = 25k$.

***Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC***

We use induction to show that $a=9$ is a solution.

For $n=0$, $F_0=9F_0=0$. For $n=1$, $F_9=34\equiv 9=9F_1 \pmod{25}$. Given $n\ge 2$, we assume $F_{9(n-1)}\equiv 9F_{n-1} \pmod{25}$ and $F_{9(n-2)}\equiv 9F_{n-2} \pmod{25}$. For $n$ an integer, it is straightforward to verify the identity $F_n=76F_{n-9}+F_{n-18}$. Then

$$\begin{aligned} F_{9n} &= 76F_{9(n-1)}+F_{9(n-2)}\equiv 76(9F_{n-1})+9F_{n-2} \pmod{25}\\ &= 675F_{n-1}+9(F_{n-1}+F_{n-2})\equiv 9F_n \pmod{25}. \end{aligned}$$

***Solution 3 (and generalization) by Richard André-Jeannin, Longwy, France***

We shall prove that $F_{an}\equiv aF_n \pmod{25}$ for all integers $n$ if and only if $a\equiv 0 \pmod{25}$ or $a\equiv r \pmod{20}$, where $r\in\{1,5,9\}$.

First, if $a\equiv 0 \pmod{25}$, it is well known that $F_{an}\equiv 0 \pmod{25}$ and the two members are divisible by 25.

Assuming now that $a$ is not divisible by 25, and putting $n=1$ and $n=2$ in the relation $F_{an}\equiv aF_n \pmod{25}$, we get that

$$\begin{cases} F_a\equiv a \pmod{25},\\ F_aL_a=F_{2a}\equiv aF_2=a\equiv F_a \pmod{25}. \end{cases}$$

From the last relation, we get that $F_a(L_a-1)\equiv 0 \pmod{25}$, and thus that $L_a\equiv 1 \pmod 5$ (recall that $F_a$ is divisible by 25 only if $a$ is divisible by 25). It is not hard to prove that the last relation holds if and only if $a\equiv 1 \pmod 4$ or, equivalently, if and only if $a=20k+r$, where $r\in\{1,5,9,13,17\}$.

We now need the following lemma.

***Lemma:*** $F_{20k+r}\equiv 20k+r \pmod{25}$ only if $r\in\{1,5,9\}$.

***Proof:*** The sequence $X_k=F_{20k+r}$ satisfies the recurrence relation

$$X_k=L_{20}X_{k-1}-(-1)^{20}X_{k-2}=15127X_{k-1}-X_{k-2}\equiv 2X_{k-1}-X_{k-2} \pmod{25}.$$

Any sequence of the form $(ck+d)$ is another solution of the recurrence

$$X_k=2X_{k-1}-X_{k-2}.$$

From this, we see that, for every integer $k$, $F_{20k+r}\equiv (F_{20+r}-F_r)k+F_r \pmod{25}$, since the two members satisfy the same recurrence and take the same value for $k=0$ and for $k=1$. Thus, we have to see that $F_{20+r}-F_r\equiv 20 \pmod{25}$ and that $F_r\equiv r \pmod{25}$.

It is readily proven that 4 is the period (mod 25) of the sequence $Z_r=F_{20+r}-F_r$ and that $Z_r\equiv 20 \pmod{25}$ if and only if $r\equiv 1 \pmod 4$ and, particularly, for $r\in\{1,5,9,13,17\}$. On the other hand, we have $F_r\equiv r \pmod{25}$ for $r=1,5,9$ when $F_{13}\equiv 8 \pmod{25}$ and $F_{17}\equiv 22$ (mod 25). This concludes the proof of the lemma.

Now, we have to distinguish two cases. Assuming first that $r=1$ or $r=9$, we see that 20 is the period of the sequence $L_n$ (mod 25) and that $L_{20k+r}\equiv 1 \pmod{25}$ for $r=1$ and $r=9$. Now the sequence $Y_n=F_{(20k+r)n}$ satisfies the recurrence

$$Y_n=L_{20k+r}Y_{n-1}-(-1)^{20k+r}Y_{n-2}\equiv Y_{n-1}+Y_{n-2} \pmod{25},$$

since $r$ is odd. Thus, the two sequences $Y_n$ and $(20k+r)F_n$ satisfy the same recurrence and they take the same value (mod 25) for $n=0$ and for $n=1$ (by the lemma). We deduce from this that $F_{(20k+r)n} \equiv (20k+r)F_n$ (mod 25) for every integer $n$.

Finally, assuming that $r=5$, we see that $L_{20k+5} \equiv 1$ (mod 5) [since $20k+5 \equiv 1$ (mod 4)] and that the sequence $U_n = F_{(20k+5)n}/5$ is a sequence of integers, since 5 divides $20k+5$. Now, the sequence $U_n$ satisfies the recurrence

$$U_n = L_{20k+5}U_{n-1} - (-1)^{20k+5}U_{n-2} \equiv U_{n-1} + U_{n-2} \pmod 5.$$

Thus, the two sequences $U_n$ and $(4k+1)F_n$ satisfy the same recurrence mod (5), and they take the same values (mod 5) for $n=0$ and $n=1$, since, by the lemma, we can write $F_{20k+5}/5 \equiv 4k+1$ (mod 5). We deduce from this that, for every integer $n$, $F_{(20k+5)n}/5 \equiv (4k+1)F_n$ (mod 5) and thus that $F_{(20k+5)n} \equiv (20k+5)F_n$ (mod 25). This concludes the proof.

**Another Generalization of B-868**

Consider the general recurrence $W_n = PW_{n-1} - QW_{n-2}$ with the solutions $U_n$ $(U_0 = 0,\ U_1 = 1)$ and $V_n$ $(V_0 = 2, V_1 = P)$. The sequence of integers

$$X_n = \frac{U_{pn}}{U_p}$$

satisfies the recurrence

$$X_n = V_p X_{n-1} - Q^p X_{n-2}.$$

If $p$ is an odd prime, it is well known that $V_p \equiv P$ (mod $p$) and that $Q^p \equiv Q$ (mod $p$). From this, we see that

$$X_n \equiv PX_{n-1} - QX_{n-2} \pmod p.$$

Reasoning as in the solution of the problem, we get that, for every integer $n$:

$$\frac{U_{pn}}{U_p} \equiv U_n \pmod p \text{ or that } U_{pn} \equiv U_p U_n \pmod{pU_p}.$$

***Solutions also received by H.-J. Seiffert and Indulis Strazdins.***

## A Polynomial for $F$

**B-869** ***Based on a communication by Larry Taylor, Rego Park, NY (Vol. 37, no. 1, February 1999)***

Find a polynomial $f(x)$ such that, for all integers $n$, $2^n F_n \equiv f(n)$ (mod 5).

***Solution by Indulis Strazdins, Riga Technical University, Riga, Latvia***

For $n = 0, 1, 2, 3, 4$ (mod 5), the period of $2^n F_n$ is $(0, 2, 4, 1, 3)$, which coincides with the period of $2n$. Hence, $f(x) = (5m+2)x$ for any integer $m$.

*Seiffert showed that, for n a nonnegative integer,*

$$2^n F_n \equiv \frac{n}{3}(5n^2 - 15n + 16) \pmod{50}.$$

***Solutions also received by Richard André-Jeannin, Brian D. Beasley, Don Redmond, H.-J. Seiffert, and the proposer.***

## Trigonometric Diophantine Equation

**B-870** ***Proposed by Richard André-Jeannin, Longwy, France (Vol. 37, no. 1, February 1999)***

Solve the equation

$$\tan^{-1} y - \tan^{-1} x = \tan^{-1} \frac{1}{x+y}$$

in nonnegative integers $x$ and $y$, expressing your answer in terms of Fibonacci and/or Lucas numbers.

***Solution by the proposer***

Let $\theta = \tan^{-1} y - \tan^{-1} x$. It is clear that $-\pi/2 < \theta < \pi/2$, since $x$ and $y$ are nonnegative. Thus, the original equation is equivalent to

$$\frac{1}{x+y} = \tan\theta = \frac{y-x}{1+xy},$$

which can be written as $y^2 - x^2 = 1 + xy$ or

$$(2y-x)^2 - 5x^2 = 4. \tag{1}$$

It is well known that the nonnegative solutions of the Diophantine equation $Y^2 - 5X^2 = 4$ are given by $X = F_{2n}$ and $Y = L_{2n}$. From this, we see that the solutions of (1) are given by $x = F_{2n}$ and $y = (F_{2n} + L_{2n})/2 = F_{2n+1}$.

***Solutions also received by Charles K. Cook, H.-J. Seiffert, and Indulis Strazdins.***

**Errata:** In the solution to problem B-864 (August 1999), in the line after display (2), insert "and $n = j$" at the end of the line. In the next display after display (2), "since $Q_a = 1$" should read "since $Q_a \equiv 1$".

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# ADVANCED PROBLEMS AND SOLUTIONS

## *Edited by* Raymond E. Whitney

*Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.*

### PROBLEMS PROPOSED IN THIS ISSUE

**H-556** ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada***

Let $f(x)$ and $g(x)$ be continuous and differentiable in the immediate vicinity of $x = a(\neq 0)$ and assume that, for some positive integer *k*,

$$f^{(n)}(a) = g^{(n)}(a) = 0; \quad 0 \le n \le k-1.$$

By definition,

$$f^{(n)}(x) := \frac{d^n}{dx^n} f(x)$$

for any continuous and differentiable function $f(x)$. Further, assume that one of the following conditions holds for $n = k$:

**a.** $f^{(k)}(a) \neq 0,\ g^{(k)}(a) = 0;$

**b.** $f^{(k)}(a) = 0,\ g^{(k)}(a) \neq 0;$

**c.** $f^{(k)}(a) \neq 0,\ g^{(k)}(a) \neq 0;$

Introduce the differential operator $D := x\frac{d}{dx}$ and define, for *m* a nonnegative integer,

$$f_m(x) := D^m f(x), \quad g_m(x) := D^m g(x).$$

Prove that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f_k(a)}{g_k(a)}, \quad a \neq 0.$$

**H-557** ***Proposed by Stanley Rabinowitz, Westford, MA***

Let $\langle w_n \rangle$ be any sequence satisfying the second-order linear recurrence $w_n = Pw_{n-1} - Qw_{n-2}$, and let $\langle v_n \rangle$ denote the specific sequence satisfying the same recurrence but with the initial conditions $v_0 = 2$, $v_1 = P$.

If *k* is an integer larger than 1, and $m = \lfloor k/2 \rfloor$, prove that, for all integers *n*,

$$v_n \sum_{i=0}^{m-1} (-Q^n)^i w_{(k-1-2i)n} = w_{kn} - (-Q^n)^m \times \begin{cases} w_0, & \text{if } k \text{ is even,} \\ w_n, & \text{if } k \text{ is odd.} \end{cases}$$

**Note:** This generalizes problem H-453.

<u>H-558</u> ***Proposed by Paul S. Bruckman, Berkeley, CA***

Prove the following:

$$\pi = \sum_{n=0}^{\infty}(-1)^n\{6\varepsilon_{10n+1} - 6\varepsilon_{10n+3} - 4\varepsilon_{10n+5} - 6\varepsilon_{10n+7} + 6\varepsilon_{10n+9}\}, \tag{$*$}$$

where $\varepsilon_m = \alpha^{-m}/m$.

## SOLUTIONS

### Count on It!

<u>H-540</u> ***Proposed by Paul S. Bruckman, Berkeley, CA***
***(Vol. 36, no. 2, May 1998)***

Consider the sequence $U = \{u(n)\}_{n=1}^{\infty}$, where $u(n) = [n\alpha]$, its characteristic function $\delta_U(n)$, and its counting function $\pi_U(n) \equiv \sum_{k=1}^{n} \delta_U(k)$, representing the number of elements of $U$ that are $\le n$. Prove the following relationships:

**(a)** $\delta_U(n) = u(n+1) - u(n) - 1, \quad n \ge 1;$

**(b)** $\pi_U(F_n) = F_{n-1}, \ n > 1.$

***Solution by H.-J. Seiffert, Berlin, Germany***

Let $v(n) = [n\alpha^2]$, $n \in N$, and $V = \{v(n)\}_{n=1}^{\infty}$. It is known (see [1], p. 472) that $U \cap V = \phi$ and $U \cup V = N$. In [1] it is proved that, for all $n \in N$,

$$u(u(n)+1) - u(u(n)) = 2, \tag{1}$$

$$u(v(n)+1) - u(v(n)) = 1. \tag{2}$$

The equation

$$u(F_n + 1) = F_{n+1} + 1, \ n \in N, \ n > 1, \tag{3}$$

is established on page 311 in [2].

***Proof of (a):*** Let $n \in N$, If $n \in U$, then there exists $k \in N$ such that $n = u(k)$. From (1), we get

$$u(n+1) - u(n) - 1 = u(u(k)+1) - u(u(k)) - 1 = 1 = \delta_U(n).$$

If $n \notin U$, then $n \in V$ and $n = v(k)$, where $k \in N$. Hence, by (2),

$$u(n+1) - u(n) - 1 = u(v(k)+1) - u(v(k)) - 1 = 0 = \delta_U(n).$$

***Proof of (b):*** Summing the equations $\delta_U(k) = u(k+1) - u(k) - 1$ over $k = 1, \ldots, n$ and using $u(1) = 1$ gives

$$\pi_U(n) = u(n+1) - n - 1, \ n \in N. \tag{4}$$

If $n > 1$, then by (3) and (4), $\pi_U(F_n) = u(F_n + 1) - F_n - 1 = F_{n+1} - F_n = F_{n-1}$.

**References:**

1. V. E. Hoggatt, Jr. & A. P. Hillman. "A Property of Wythoff Pairs." *The Fibonacci Quarterly* **16.5** (1978):472.
2. V. E. Hoggatt, Jr., & M. Bicknell-Johnson. "Representations of Integers in Terms of Greatest Integer Functions and the Golden Section Ratio." *The Fibonacci Quarterly* **17.4** (1979):306-18.

***Also solved by the proposer.***

## Just Continue

**H-541** ***Proposed by Stanley Rabinowitz, Westford, MA***
***(Vol. 36, no. 2, May 1998)***

The simple continued fraction expansion for $F_{13}^5 / F_{12}^5$ is

$$11+\cfrac{1}{11+\cfrac{1}{375131+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{9+\cfrac{1}{11}}}}}}}}}}}}}}$$

which can be written more compactly using the notation $[11,11,375131,1,1,1,1,1,1,1,1,1,2,9,11]$. To be even more concise, we can write this as $11^2,375131,1^9,2,9,11]$, where the superscript denotes the number of consecutive occurrences of the associated number in the list.

If $n>0$, prove that the simple continued fraction expansion for $(F_{10n+3}/F_{10n+2})^5$ is $11^{2n},x,$ ${}^{10n-1},2,9,11^{2n-1}]$, where $x$ is an integer, and find $x$.

***Solution by Paul S. Bruckman, Berkeley, CA***

We begin with the well-known isomorphism between simple continued fractions and $2\times 2$ matrices, namely:

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_3 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} p_m & p_{m-1} \\ q_m & q_{m-1} \end{pmatrix},$$

where $p_m/q_m$ is the $m^{\text{th}}$ convergent of the simple continued fraction denoted as $[a_1,a_2,a_3,\ldots,a_m]$. Normally, we will restrict the $a_i$'s to be positive integers. As a particular case, if all the $a_i$'s are equal (say to $a$), this result simplifies to

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}^m=\begin{pmatrix} \Phi_{m+1} & \Phi_m \\ \Phi_m & \Phi_{m-1} \end{pmatrix}, \text{ where } \Phi_m=\Phi_m(a)=\frac{r^m-s^m}{r-s},$$

$$r=r(a)=\tfrac{1}{2}\{a+\theta\},\ s=s(a)=\tfrac{1}{2}\{a-\theta\}, \text{ and } \theta=\theta(a)=(a^2+4)^{\frac{1}{2}}.$$

It is also true that $\Phi_{m+2}=a\Phi_{m+1}+\Phi_m$, $m=0,1,2,\ldots$, with $\Phi_0=0$, $\Phi_1=1$. We note that $\Phi_m(1)=F_m$, and also that $r(11)=\tfrac{1}{2}(11+5\sqrt{5})=\alpha^5$, $s(11)=\tfrac{1}{2}(11-5\sqrt{5})=\beta^5$; therefore,

$$\Phi_m(11)=1/5F_{5m}. \tag{1}$$

We will also utilize the following common identities:

$$5F_uF_v=L_{u+v}-(-1)^uL_{v-u};\ F_uL_v=F_{u+v}-(-1)^uF_{v-u};\ L_uL_v=L_{u+v}+(-1)^uL_{v-u};$$

and

$$25(F_m)^5 = F_{5m} - 5(-1)^m F_{3m} + 10F_m. \tag{2}$$

For brevity, let $\rho_n = (F_{10n+3}/F_{10n+2})^5$, $n = 1, 2, \ldots$. Also, we *assume* that $\rho_n = [11^{2n}, \xi]$, where $\xi = \xi_n$ is not necessarily an integer. This implies that $x = x_n = \lfloor \xi \rfloor$ (here "$\lfloor\ \rfloor$" denotes the "greatest integer" function). Using the formula in (2), we find that

$$\rho_n = (F_{50n+15} + 5F_{30n+9} + 10F_{10n+3}) / (F_{50n+10} - 5F_{30n+6} + 10F_{10n+2}). \tag{3}$$

Also

$$\begin{pmatrix} 11 & 1 \\ 1 & 0 \end{pmatrix}^{2n} = 1/5\begin{pmatrix} F_{10n+5} & F_{10n} \\ F_{10n} & F_{10n-5} \end{pmatrix}, \text{ using (1).}$$

Thus,

$$\begin{pmatrix} 11 & 1 \\ 1 & 0 \end{pmatrix}^{2n}\begin{pmatrix} \xi & 1 \\ 1 & 0 \end{pmatrix} = 1/5\begin{pmatrix} \xi F_{10n+5} + F_{10n} & F_{10n+5} \\ \xi F_{10n} + F_{10n-5} & F_{10n} \end{pmatrix}.$$

Then we require that $\rho_n = (\xi F_{10n+5} + F_{10n}) / (\xi F_{10n} + F_{10n-5})$. Now we substitute the formula in (3), cross-multiply, and simplify, using the multiplication identities previously indicated. After a tedious but straightforward computation, we obtain the following result:

$$\xi_n = 5F_{20n+5} + 6 + (F_{20n+4} + 2) / F_{20n+5}. \tag{4}$$

Note that, if $n > 0$, the fractional part of $\xi_n$ lies in the interval $(0, 1)$, as we would expect. Thus, our earlier assumption is justified, and we conclude that

$$x_n = 5F_{20n+5} + 6. \tag{5}$$

By comparison with the desired expression, it remains to verify that

$$F_{20n+5} / (F_{20n+4} + 2) = [1^{10n-1}, 2, 9, 11^{2n-1}]. \tag{6}$$

In turn, it suffices to show the following:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{10n-1}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 9 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 11 & 1 \\ 1 & 0 \end{pmatrix}^{2n-1} = C\begin{pmatrix} F_{20n+5} & * \\ F_{20n+4}+2 & * \end{pmatrix}, \tag{7}$$

where $C$ is some constant independent of $n$ and the "*" matrix entries are not important to know. Based on our previous results, the left member of (7) is expanded as follows,

$$1/5\begin{pmatrix} F_{10n} & F_{10n-1} \\ F_{10n-1} & F_{10n-2} \end{pmatrix}\begin{pmatrix} 19 & 2 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} F_{10n} & F_{10n-5} \\ F_{10n-5} & F_{10n-10} \end{pmatrix}$$

$$= 1/5\begin{pmatrix} 19F_{10n} + 9F_{10n-1} & F_{10n+2} \\ 19F_{10n-1} + 9F_{10n-2} & F_{10n+1} \end{pmatrix}\begin{pmatrix} F_{10n} & F_{10n-5} \\ F_{10n-5} & F_{10n-10} \end{pmatrix} = 1/25\begin{pmatrix} A_n & * \\ B_n & * \end{pmatrix},$$

where (after simplification):

$$A_n = 19L_{20n} + 9L_{20n-1} + L_{20n-3} = 5F_{20n+5},$$

$$B_n = 19L_{20n-1} + 9L_{20n-2} + L_{20n-4} + 19 - 27 + 18 = 5F_{20n+4} + 10.$$

Thus, (7) is verified (with $C = 1/5$) and the proof is complete. □

**Note:** This result invites generalizations. If $r_n = F_{n+1}/F_n$, we would like to find similar results involving $(r_n)^k$ for various values of $n$ and $k$. The following result (using the proposer's notation) is well known:

$$r_n = [1^n]. \tag{8}$$

The following may also be shown:

$$(r_n)^2 = [2, 1^{n-3}, 3, 1^{n-2}], \quad \text{if } n \geq 3. \tag{9}$$

Also, we may derive the following relations, valid for $n \geq 1$:

$$\begin{aligned} (r_{3n+1})^3 &= [4^n, 1, 1, 1, 4^{n-1}, 2, 2, 1^{3n-2}]; \\ (r_{3n+2})^3 &= [4^n, 10, 4^{n-1}, 2, 2, 1^{3n-1}]; \\ (r_{3n+3})^3 &= [4^n, 3, 2, 3, 4^{n-1}, 2, 2, 1^{3n}]. \end{aligned} \tag{10}$$

These kinds of expressions become more complicated for increasing values of $k$, and apparently require separate treatment for the different values of $n$ (mod $k$). The matrix method indicated above seems to be the most efficient way to handle such problems. □

## Sum Problem!

**H-542** ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 36, no. 4, August 1998)***

Define the sequence $(c_k)_{k\geq 1}$ by

$$c_k = \begin{cases} 1 & \text{if } k \equiv 2 \pmod 5, \\ -1 & \text{if } k \equiv 3 \pmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

Show that, for all positive integers $n$:

$$\frac{1}{n}\sum_{k=1}^{n} k\binom{2n}{n-k}c_k = F_{2n-2}; \tag{1}$$

$$\frac{1}{2n-1}\sum_{k=1}^{2n-1}(-1)^k k\binom{4n-2}{2n-k-1}c_k = 5^{n-1}F_{2n-2}; \tag{2}$$

$$\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k k\binom{4n}{2n-k}c_k = 5^{n-1}L_{2n-1}. \tag{3}$$

***Solution by the proposer***

We consider the Fibonacci polynomials defined by $F_0(x) = 0$, $F_1(x) = 1$, $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, for $n \geq 0$. It is known that

$$\sum_{k=1}^{\infty} F_k(x)z^k = \frac{z}{1-xz-z^2}, \text{ for small } |z|.$$

Replacing $z$ by $iz$, $i = \sqrt{(-1)}$, and taking $x = i\alpha$, resp. $x = i\beta$, gives

$$\sum_{k=1}^{\infty} F_k(i\alpha)(iz)^k = \frac{iz}{1+\alpha z+z^2}, \text{ resp. } \sum_{k=1}^{\infty} F_k(i\beta)(iz)^k = \frac{iz}{1+\beta z+z^2}.$$

Subtracting the first from the second equation and dividing the resulting equation by $i\sqrt{5}$ yields

$$\sum_{k=1}^{\infty}\frac{i^{k-1}}{\sqrt{5}}(F_k(i\beta)-F_k(i\alpha))z^k = \frac{z^2}{1+z+z^2+z^3+z^4}.$$

The sequence $(c_k)_{k\geq 1}$ has the generating function

$$\sum_{k=1}^{\infty} c_k z^k = \sum_{j=0}^{\infty} (z^{5j+2} - z^{5j+3}) = \frac{z^2(1-z)}{1-z^5}, \ |z| < 1.$$

Since $1+z+z^2+z^3+z^4 = (1-z^5)/(1-z)$, comparing coefficients gives

$$c_k = \frac{i^{k-1}}{\sqrt{5}}(F_k(i\beta) - F_k(i\alpha)), \ k \in N. \tag{4}$$

From H-518, we know that, for all complex numbers $x$ and $y$ and all positive integers $n$,

$$\sum_{k=1}^{n} \binom{2n}{n-k} F_k(x) F_k(y) = (x-y)^{n-1} F_n\left(\frac{xy+4}{x-y}\right).$$

Taking $y = 2i$ and using $F_k(2i) = ki^{k-1}$, we find

$$\sum_{k=1}^{n} k\binom{2n}{n-k} i^{k-1} F_k(x) = n(2+ix)^{n-1}. \tag{5}$$

From (4) and (5), we obtain

$$\sum_{k=1}^{n} k\binom{2n}{n-k} c_k = n\frac{(2-\beta)^{n-1} - (2-\alpha)^{n-1}}{\sqrt{5}}.$$

Using $2-\beta = \alpha^2$ and $2-\alpha = \beta^2$ and the Binet form of $F_{2n-2}$ gives the first desired identity (1).

Since $F_k(-x) = (-1)^{k-1} F_k(x)$, we also find

$$\begin{aligned}
&\sum_{k=1}^{n} (-1)^k k\binom{2n}{n-k} c_k \\
&= \sum_{k=1}^{n} k\binom{2n}{n-k} \frac{i^{k-1}}{\sqrt{5}}(F_k(-i\alpha) - F_k(-i\beta)) \\
&= n\frac{(2+\alpha)^{n-1} - (2+\beta)^{n-1}}{\sqrt{5}}.
\end{aligned}$$

Since $2+\alpha = \sqrt{5}\alpha$ and $2+\beta = -\sqrt{5}\beta$, if we replace $n$ by $2n-1$ and $n$ by $2n$, we easily obtain (2) and (3), respectively.

***Also solved by P. Bruckman***

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