The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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VOLUME 38 FEBRUARY 2000 NUMBER 1

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The Fibonacci Quarterly

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SUMS OF CERTAIN PRODUCTS OF FIBONACCI AND LUCAS NUMBERS–PART II

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1. INTRODUCTION

The identities

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1} \tag{1.1}$$

and

$$\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2 = L_n L_{n+1} - L_0 L_1 \tag{1.2}$$

are well known. The right side of (1.2) suggests the notation $[L_j L_{j+1}]_0^n$, which we use throughout this paper in order to conserve space. Each time we use this notation, we take j to be the dummy variable.

In [2], motivated by (1.1) and (1.2), together with

$$\sum_{k=1}^{n} F_k^2 F_{k+1} = \frac{1}{2} F_n F_{n+1} F_{n+2}, \tag{1.3}$$

we obtained several families of similar sums which involve longer products. For example, we obtained

$$\sum_{k=1}^{n} F_k F_{k+1} \dots F_{k+2m}^2 \dots F_{k+4m} = \frac{F_n F_{n+1} \cdots F_{n+4m+1}}{L_{2m+1}}, \tag{1.4}$$

for m a positive integer. By introducing a second parameter, s, we have managed to generalize all of the results in [2], while maintaining their elegance. The object of this paper is to present these generalizations, together with several results involving alternating sums, the like of which were not treated in [2]. In Section 2 we state our results, and in Section 3 we indicate the method of proof. We require the following identities:

$$F_{n+k} + F_{n-k} = F_n L_k, \quad k \text{ even}, \tag{1.5}$$

$$F_{n+k} + F_{n-k} = L_n F_k, \quad k \text{ odd}, \tag{1.6}$$

$$F_{n+k} - F_{n-k} = F_n L_k, \quad k \text{ odd}, \tag{1.7}$$

$$F_{n+k} - F_{n-k} = L_n F_k, \quad k \text{ even}, \tag{1.8}$$

$$L_{n+k} + L_{n-k} = L_n L_k, \quad k \text{ even}, \tag{1.9}$$

$$L_{n+k} + L_{n-k} = 5F_n F_k, \quad k \text{ odd}, \tag{1.10}$$

$$L_{n+k} - L_{n-k} = L_n L_k, \quad k \text{ odd}, \tag{1.11}$$

$$L_{n+k} - L_{n-k} = 5F_n F_k, \quad k \text{ even}, \tag{1.12}$$

$$L_n^2 - L_{2n} = (-1)^n 2 = (-1)^n L_0, \tag{1.13}$$

$$5F_n^2 - L_{2n} = (-1)^{n+1} 2 = (-1)^{n+1} L_0, \tag{1.14}$$

$$5F_{2n}^2 - L_{2n}^2 = -4 = -L_0^2. \tag{1.15}$$

Identities (1.5)-(1.12) occur as (5)-(12) in Bergum and Hoggatt [1], while (1.13)-(1.15) can be proved with the use of the Binet forms. In some of the proofs we need to recall the well-known identity $F_{2n} = F_n L_n$.

2. THE RESULTS

In this section we list our results in eight theorems, in which $s > 0$ and $m \geq 0$ are integers. In some of the theorems the parity of s is important, and the reasons for this become apparent in Section 3. Our numbering of Theorems 1-5 parallels that in [2], so that both sets of results can be easily compared.

Theorem 1:

$$\sum_{k=1}^{n} F_{sk} F_{s(k+1)} \cdots F_{s(k+4m)} L_{s(k+2m)} = \frac{F_{sn} F_{s(n+1)} \cdots F_{s(n+4m+1)}}{F_{s(2m+1)}}, \quad s \text{ even}, \tag{2.1}$$

$$\sum_{k=1}^{n} F_{sk} \cdots F_{s(k+2m)}^2 \cdots F_{s(k+4m)} = \frac{F_{sn} F_{s(n+1)} \cdots F_{s(n+4m+1)}}{L_{s(2m+1)}}, \quad s \text{ odd}. \tag{2.2}$$

Theorem 2:

$$\sum_{k=1}^{n} L_{sk} L_{s(k+1)} \cdots L_{s(k+4m)} F_{s(k+2m)} = \left[\frac{L_{sj} L_{s(j+1)} \cdots L_{s(j+4m+1)}}{5F_{s(2m+1)}} \right]_0^n, \quad s \text{ even}, \tag{2.3}$$

$$\sum_{k=1}^{n} L_{sk} L_{s(k+1)} \cdots L_{s(k+2m)}^2 \cdots L_{s(k+4m)} = \left[\frac{L_{sj} L_{s(j+1)} \cdots L_{s(j+4m+1)}}{L_{s(2m+1)}} \right]_0^n, \quad s \text{ odd}. \tag{2.4}$$

Theorem 3:

$$\sum_{k=1}^{n} F_{sk} F_{s(k+1)} \cdots F_{s(k+4m+2)} L_{s(k+2m+1)} = \frac{F_{sn} F_{s(n+1)} \cdots F_{s(n+4m+3)}}{F_{s(2m+2)}}, \tag{2.5}$$

$$\sum_{k=1}^{n} L_{sk} L_{s(k+1)} \cdots L_{s(k+4m+2)} F_{s(k+2m+1)} = \left[\frac{L_{sj} L_{s(j+1)} \cdots L_{s(j+4m+3)}}{5F_{s(2m+2)}} \right]_0^n. \tag{2.6}$$

Theorem 4:

$$\sum_{k=1}^{n} F_{sk}^2 F_{s(k+1)}^2 \cdots F_{s(k+4m)}^2 F_{s(2k+4m)} = \frac{F_{sn}^2 F_{s(n+1)}^2 \cdots F_{s(n+4m+1)}^2}{F_{s(4m+2)}}, \tag{2.7}$$

$$\sum_{k=1}^{n} L_{sk}^2 L_{s(k+1)}^2 \cdots L_{s(k+4m)}^2 F_{s(2k+4m)} = \left[\frac{L_{sj}^2 L_{s(j+1)}^2 \cdots L_{s(j+4m+1)}^2}{5F_{s(4m+2)}} \right]_0^n. \tag{2.8}$$

Theorem 5:

$$\sum_{k=1}^{n} F_{sk}^2 F_{s(k+1)}^2 \cdots F_{s(k+4m+2)}^2 F_{s(2k+4m+2)} = \frac{F_{sn}^2 F_{s(n+1)}^2 \cdots F_{s(n+4m+3)}^2}{F_{s(4m+4)}}, \tag{2.9}$$

$$\sum_{k=1}^{n} L_{sk}^2 L_{s(k+1)}^2 \cdots L_{s(k+4m+2)}^2 F_{s(2k+4m+2)} = \left[\frac{L_{sj}^2 L_{s(j+1)}^2 \cdots L_{s(j+4m+3)}^2}{5F_{s(4m+4)}}\right]_0^n. \tag{2.10}$$

For $m=0$ we interpret the summands in (2.2) and (2.4) as F_{sk}^2 and L_{sk}^2, respectively. For s odd the corresponding sums are then

$$\sum_{k=1}^{n} F_{sk}^2 = \frac{F_{sn}F_{s(n+1)}}{L_s} \quad \text{and} \quad \sum_{k=1}^{n} L_{sk}^2 = \left[\frac{L_{sj}L_{s(j+1)}}{L_s}\right]_0^n, \tag{2.11}$$

which generalize (1.1) and (1.2), respectively.

Interestingly, for $m=0$, (2.1) and (2.3) provide alternative expressions for the same sum, namely,

$$\sum_{k=1}^{n} F_{2sk} = \frac{F_{sn}F_{s(n+1)}}{F_s} = \left[\frac{L_{sj}L_{s(j+1)}}{5F_s}\right]_0^n, \quad s \text{ even}. \tag{2.12}$$

Theorem 6:

$$\sum_{k=1}^{n} (-1)^k F_{sk}F_{s(k+1)} \cdots F_{s(k+4m)}F_{s(k+2m)} = \frac{(-1)^n F_{sn}F_{s(n+1)} \cdots F_{s(n+4m+1)}}{L_{s(2m+1)}}, \quad s \text{ even}, \tag{2.13}$$

$$\sum_{k=1}^{n} (-1)^k F_{sk}F_{s(k+1)} \cdots F_{s(k+4m)}L_{s(k+2m)} = \frac{(-1)^n F_{sn}F_{s(n+1)} \cdots F_{s(n+4m+1)}}{F_{s(2m+1)}}, \quad s \text{ odd}. \tag{2.14}$$

Theorem 7:

$$\sum_{k=1}^{n} (-1)^k L_{sk}L_{s(k+1)} \cdots L_{s(k+4m)}L_{s(k+2m)} = \left[\frac{(-1)^n L_{sj}L_{s(j+1)} \cdots L_{s(j+4m+1)}}{L_{s(2m+1)}}\right]_0^n, \quad s \text{ even}, \tag{2.15}$$

$$\sum_{k=1}^{n} (-1)^k L_{sk}L_{s(k+1)} \cdots L_{s(k+4m)}F_{s(k+2m)} = \left[\frac{(-1)^n L_{sj}L_{s(j+1)} \cdots L_{s(j+4m+1)}}{5F_{s(2m+1)}}\right]_0^n, \quad s \text{ odd}. \tag{2.16}$$

Theorem 8:

$$\sum_{k=1}^{n} (-1)^k F_{sk}F_{s(k+1)} \cdots F_{s(k+4m+2)}F_{s(k+2m+1)} = \frac{(-1)^n F_{sn}F_{s(n+1)} \cdots F_{s(n+4m+3)}}{L_{s(2m+2)}}, \tag{2.17}$$

$$\sum_{k=1}^{n} (-1)^k L_{sk}L_{s(k+1)} \cdots L_{s(k+4m+2)}L_{s(k+2m+1)} = \left[\frac{(-1)^n L_{sj}L_{s(j+1)} \cdots L_{s(j+4m+3)}}{L_{s(2m+2)}}\right]_0^n. \tag{2.18}$$

Some special cases of these alternating sums are worthy of note. For $m=0$ Theorem 6 yields

$$\sum_{k=1}^{n}(-1)^k F_{sk}^2 = \frac{(-1)^n F_{sn}F_{s(n+1)}}{L_s}, \quad s \text{ even}, \tag{2.19}$$

and

$$\sum_{k=1}^{n}(-1)^k F_{2sk} = \frac{(-1)^n F_{sn}F_{s(n+1)}}{F_s}, \quad s \text{ odd}. \tag{2.20}$$

An alternative formulation for (2.20) is provided by (2.16). For $m = 0$ (2.15) becomes

$$\sum_{k=1}^{n}(-1)^k L_{sk}^2 = \left[\frac{(-1)^n L_{sj}L_{s(j+1)}}{L_s}\right]_0^n, \quad s \text{ even}. \tag{2.21}$$

3. THE METHOD OF PROOF

Each result in Section 2 can be proved with the use of the method in [2]. However, the significance of the parity of s in some of our theorems becomes apparent only when we work through the proofs. For this reason, we illustrate the method of proof once more by proving (2.4).

Proof of (2.4): Let l_n denote the sum on the left side of (2.4) and let

$$r_n = \frac{L_{sn}L_{s(n+1)}\cdots L_{s(n+4m+1)}}{L_{s(2m+1)}}.$$

Then

$$\begin{aligned} r_n - r_{n-1} &= \frac{L_{sn}L_{s(n+1)}\cdots L_{s(n+4m)}}{L_{s(2m+1)}}[L_{s(n+4m+1)} - L_{s(n-1)}] \\ &= \frac{L_{sn}L_{s(n+1)}\cdots L_{s(n+4m)}}{L_{s(2m+1)}}[L_{s(n+2m)+s(2m+1)} - L_{s(n+2m)-s(2m+1)}] \\ &= L_{sn}L_{s(n+1)}\cdots L_{s(n+2m)}^2\cdots L_{s(n+4m)} \quad \text{[by (1.11) since } s(2m+1) \text{ is odd]} \\ &= l_n - l_{n-1}. \end{aligned}$$

Thus $l_n - r_n = c$, where c is a constant.

Now

$$\begin{aligned} c &= l_1 - r_1 \\ &= L_sL_{2s}\cdots L_{s(4m+1)}\left[L_{s(2m+1)} - \frac{L_{s(4m+2)}}{L_{s(2m+1)}}\right] \\ &= L_sL_{2s}\cdots L_{s(4m+1)}\cdot\frac{L_{s(2m+1)}^2 - L_{s(4m+2)}}{L_{s(2m+1)}} \\ &= -\frac{L_0L_sL_{2s}\cdots L_{s(4m+1)}}{L_{s(2m+1)}} \quad \text{[by (1.13)]} \\ &= -r_0, \end{aligned}$$

and this concludes the proof. □

In contrast, when proving (2.3), we are required to factorize $L_{s(n+2m)+s(2m+1)} - L_{s(n+2m)-s(2m+1)}$ for s even, and this requires the use of (1.12).

As in [2], we conclude by mentioning that the results of this paper translate immediately to the sequences defined by

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0, \quad U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2, \quad V_1 = p. \end{cases}$$

We simply replace F_n by U_n, L_n by V_n, and 5 by p^2+4.

ACKNOWLEDGMENT

I would like to express by gratitude to the anonymous referee whose comments have improved the presentation of this paper.

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AMS Classification Numbers: 11B39, 11B37

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GENERALIZATIONS OF MODIFIED MORGAN-VOYCE POLYNOMIALS

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1. INTRODUCTION

In two recent articles [2] and [3], Ferri et al. introduced and studied the properties of two numerical triangles, which they called DFF and DFZ triangles. However, in a subsequent article, André-Jeannin [1] showed that the polynomials generated by the rows of these triangles are indeed the Morgan-Voyce polynomials $B_n(x)$ and $b_n(x)$, whose properties are well known [10] and [11]; in fact, the polynomials $B_n(x)$ and $b_n(x)$ have been used in the study of electrical networks since the 1960s (see, e.g., [8] and [9]). In the same article, André-Jeannin introduced a generalization of the Morgan-Voyce polynomials by defining the sequence of polynomials $\{P_n^{(r)}(x)\}$ by the relation

$$P_n^{(r)}(x) = (x+2)P_{n-1}^{(r)}(x) - P_{n-2}^{(r)}(x), \quad (n \geq 2), \tag{1a}$$

with

$$P_0^{(r)}(x) = 1 \quad \text{and} \quad P_1^{(r)}(x) = x + r + 1. \tag{1b}$$

Subsequently, Horadam [6] defined a closely related sequence of polynomials $\{Q_n^{(r)}(x)\}$ by the relation

$$Q_n^{(r)}(x) = (x+2)Q_{n-1}^{(r)}(x) - Q_{n-2}^{(r)}(x), \quad (n \geq 2), \tag{2a}$$

with

$$Q_0^{(r)}(x) = 2 \quad \text{and} \quad Q_1^{(r)}(x) = x + r + 2, \tag{2b}$$

and studied some of its properties.

The purpose of this article is first to generalize the two sequences of polynomials $\{P_n^{(r)}(x)\}$ and $\{Q_n^{(r)}(x)\}$, and to study some of their properties by first relating them to the parameters of electrical one-ports and then using the properties of such one-ports. Later, following Horadam [7], we will construct and study some of the properties of a composite polynomial which includes the two sets of generalized polynomials introduced in this article.

2. POLYNOMIALS $\tilde{P}_n^{(r)}(x)$ AND $\tilde{Q}_n^{(r)}(x)$

Consider the generalized polynomial $w_n(a, b; x)$ defined by

$$w_n(x) = (x+p)w_{n-1}(x) - w_{n-2}(x), \quad (n \geq 2), \tag{3a}$$

with

$$w_0(x) = a \quad \text{and} \quad w_1(x) = b. \tag{3b}$$

We know that the solution of (3a) and (3b) is given by [5]:

$$w_n(x) = w_1(x)U_n(x) - w_0(x)U_{n-1}(x), \tag{4}$$

where

$$U_n(x) = w_n(0, 1; x). \tag{5}$$

Hence, we may observe that the modified Morgan-Voyce polynomials, $\widetilde{B}_n(x)$, $\widetilde{b}_n(x)$, $\widetilde{C}_n(x)$, and $\widetilde{c}_n(x)$, defined in [12], may be written as

$$\widetilde{B}_n(x) = w_n(1, x+p; x) = U_{n+1}(x), \tag{6a}$$
$$\widetilde{b}_n(x) = w_n(1, x+p-1; x) = U_{n+1}(x) - U_n(x) = \widetilde{B}_n(x) - \widetilde{B}_{n-1}(x), \tag{6b}$$
$$\widetilde{C}_n(x) = w_n(2, x+p; x) = U_{n+1}(x) - U_{n-1}(x) = \widetilde{B}_n(x) - \widetilde{B}_{n-2}(x), \tag{6c}$$
$$\widetilde{c}_n(x) = w_n(1, x+p+1; x) = U_{n+1}(x) + U_n(x) = \widetilde{B}_n(x) + \widetilde{B}_{n-1}(x). \tag{6d}$$

From (6b), (6c), and (6d), we see that

$$\widetilde{C}_n(x) = \widetilde{b}_n(x) + \widetilde{b}_{n-1}(x) = \widetilde{c}_n(x) - \widetilde{c}_{n-1}(x). \tag{7}$$

Let us now define the following two sets of generalized polynomials $\widetilde{P}_n^{(r)}(x)$ and $\widetilde{Q}_n^{(r)}(x)$ as

$$\widetilde{P}_n^{(r)}(x) = w_n(1, x+p+r-1; x) \tag{8a}$$

and

$$\widetilde{Q}_n^{(r)}(x) = w_n(2, x+p+r; x). \tag{8b}$$

Hence, from (4), we have

$$\widetilde{P}_n^{(r)}(x) = U_{n+1}(x) + (r-1)U_n(x) \tag{9a}$$

and

$$\widetilde{Q}_n^{(r)}(x) = U_{n+1}(x) - U_{n-1}(x) + rU_n(x). \tag{9b}$$

Using the relations given in (6a)-(6d), the above may be written as

$$\widetilde{P}_n^{(r)}(x) = \widetilde{b}_n(x) + r\widetilde{B}_{n-1}(x) \tag{10a}$$

and

$$\widetilde{Q}_n^{(r)}(x) = \widetilde{C}_n(x) + r\widetilde{B}_{n-1}(x). \tag{10b}$$

As a consequence of (10a), (10b), and (7), we also have the relation

$$\widetilde{Q}_n^{(r)}(x) = \widetilde{P}_n(x) + \widetilde{b}_{n-1}(x). \tag{10c}$$

It is readily seen that

$$\widetilde{P}_n^{(0)}(x) = \widetilde{b}_n(x), \tag{11a}$$
$$\widetilde{P}_n^{(1)}(x) = \widetilde{B}_n(x), \tag{11b}$$
$$\widetilde{P}_n^{(2)}(x) = \widetilde{c}_n(x), \tag{11c}$$
$$\widetilde{Q}_n^{(0)}(x) = \widetilde{C}_n(x). \tag{11d}$$

It is clear that these results are generalizations of those contained in [1] and [6].

3. $\widetilde{P}_n^{(r)}(x)$, $\widetilde{Q}_n^{(r)}(x)$ AND LADDER ONE-PORTS

In this article we assume that $p \geq 2$ and $r \geq 0$. Consider now the ladder one-port network shown in Figure 1(a), which consists only of resistors and inductors, and thus is an RL-network (see Appendix A), where the series resistors $r_1 = r_2 = r_3 = \cdots = r_n = (p-2)\alpha$ Ohms, the inductors $L_1 = L_2 = L_3 = \cdots = L_n = \alpha$ Henries, and the shunt resistors $R_1 = R_2 = R_3 = \cdots = R_n = \alpha$ Ohms. For such a network, the impedance z_1 of any of the series branches is given by

$$z_1 = (s+p-2)\alpha, \tag{12}$$

where s is the complex frequency variable, while the impedance z_2 of any of the shunt branches is given by

$$z_2 = \alpha. \tag{13}$$

It is known [9] that the driving point impedance (DPI) Z_a of such a network is given by

$$Z_a = z_2 \frac{b_n(w)}{B_{n-1}(w)}, \tag{14}$$

where

$$w = \frac{z_1}{z_2}, \tag{15}$$

and $B_n(w)$ and $b_n(w)$ are the Morgan-Voyce polynomials [8]. Hence,

$$Z_a = \alpha \frac{b_n(s+p-2)}{B_{n-1}(s+p-2)}.$$

However, $b_n(s+p-2) = \widetilde{b}_n(s)$ and $B_n(s+p-2) = \widetilde{B}_n(s)$. Hence, the DPI of the RL-ladder network of Figure 1(a) is given by

$$Z_a = \alpha \frac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)}. \tag{16}$$

Now consider the rational function $\widetilde{P}_n^{(r+k)}(s) / \widetilde{P}_n^{(r)}(s)$, where $k > 0$. Then

$$\frac{\widetilde{P}_n^{(r+k)}(s)}{\widetilde{P}_n^{(r)}(s)} = \frac{\widetilde{b}_n(s) + (r+k)\widetilde{B}_{n-1}(s)}{\widetilde{b}_n(s) + r\widetilde{B}_{n-1}(s)} = 1 + \frac{k\widetilde{B}_{n-1}(s)}{\widetilde{b}_n(s) + r\widetilde{B}_{n-1}(s)} = 1 + \frac{1}{\frac{r}{k} + \frac{1}{k}\frac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)}}. \tag{17}$$

Using (16) and (17), we see that $\widetilde{P}_n^{(r+k)}(s) / \widetilde{P}_n^{(r)}(s)$ may be realized as the driving point admittance (DPA) Y_b of the network shown in Figure 1(b). It is observed that this network also is composed only of resistors and inductors. Thus, $\widetilde{P}_n^{(r+k)}(s) / \widetilde{P}_n^{(r)}(s)$ can be realized as the DPA of an RL-network.

Now consider the rational function $\widetilde{Q}_n^{(r+k)}(s) / \widetilde{Q}_n^{(r)}(s)$, where again $k > 0$. Then

$$\frac{\widetilde{Q}_n^{(r+k)}(s)}{\widetilde{Q}_n^{(r)}(s)} = \frac{\widetilde{C}_n(s) + (r+k)\widetilde{B}_{n-1}(s)}{\widetilde{C}_n(s) + r\widetilde{B}_{n-1}(s)} = 1 + \frac{k\widetilde{B}_{n-1}(s)}{\widetilde{C}_n(s) + r\widetilde{B}_{n-1}(s)} = 1 + \frac{1}{\frac{r}{k} + \frac{1}{k}\frac{\widetilde{C}_n(s)}{\widetilde{B}_{n-1}(s)}}. \tag{18}$$

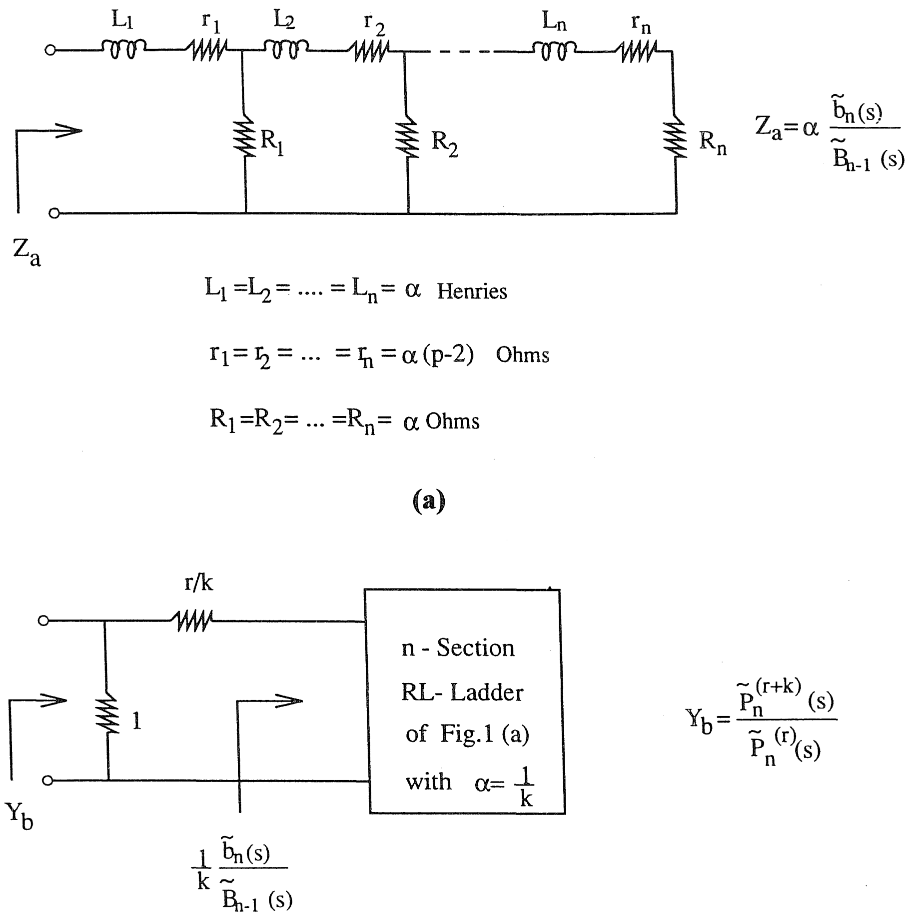
From the results given in [9], it is known that the function

$$\alpha \frac{\widetilde{C}_n(s)}{\widetilde{B}_{n-1}(s)}$$

can be realized as the DPI of the RL-ladder network shown in Figure 2(a). Hence, from (18), we see that $\widetilde{Q}_n^{(r+k)}(s) / \widetilde{Q}_n^{(r)}(s)$ can be realized as the DPA of an RL-network.

Now consider $\widetilde{P}_n^{(r+k)}(s) / \widetilde{Q}_n^{(r)}(s)$, $k \geq 0$. This may be expressed as

$$\frac{\widetilde{P}_n^{(r+k)}(s)}{\widetilde{Q}_n^{(r)}(s)} = \frac{\widetilde{b}_n(s) + (r+k)\widetilde{B}_{n-1}(s)}{\widetilde{C}_n(s) + r\widetilde{B}_{n-1}(s)} = \frac{(r+k) + \frac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)}}{r + \frac{\widetilde{C}_n(s)}{\widetilde{B}_{n-1}(s)}}. \tag{19}$$

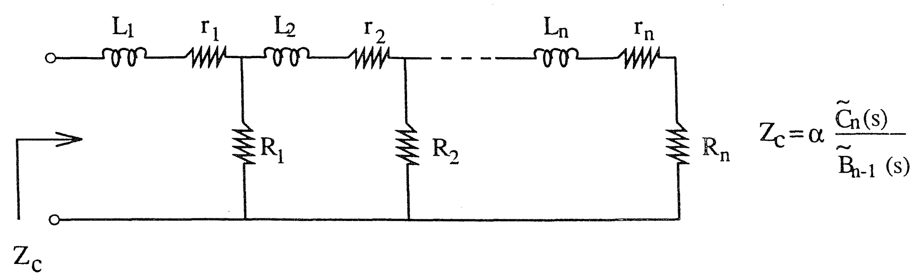


(b)

FIGURE 1

Since both $\tilde{b}_n(s)/\tilde{B}_{n-1}(s)$ and $\tilde{C}_n(s)/\tilde{B}_{n-1}(s)$ are RL-impedance functions, we see from (19) that $\tilde{P}_n^{(r+k)}(s)/\tilde{Q}_n^{(r)}(s)$ is a ratio of two RL-impedance functions. Therefore, in general, it is only a positive real function (see Appendix B) and thus need R, L, and C (capacitors) for its realization [13].

Using the properties of RL-networks (see Appendix A), we may now draw some conclusions regarding the locations of the zeros of $\tilde{P}_n^{(r)}(s)$ and $\tilde{Q}_n^{(r)}(s)$. Since $\tilde{P}_n^{(r+k)}(s)/\tilde{P}_n^{(r)}(s)$ $(k > 0)$ is realizable as the DPA of an RL-network, we see that the zeros of $\tilde{P}_n^{(r)}(s)$ are real, simple, and negative; further, they interlace with those of $\tilde{P}_n^{(r+k)}(s)$, the zero closest to the origin being that of $\tilde{P}_n^{(r)}(s)$. Similar statements hold with regard to the zeros of $\tilde{Q}_n^{(r)}(s)$ and $\tilde{Q}_n^{(r+k)}(s)$ $(k > 0)$, since we have shown that $\tilde{Q}_n^{(r+k)}(s)/\tilde{Q}_n^{(r)}(s)$ is also a DPA of an RL-network. In addition, since $\tilde{P}_n^{(r+k)}(s)/\tilde{Q}_n^{(r)}(s)$ $(k \geq 0)$ is a ratio of two RL-admittance functions, the zeros of $\tilde{P}_n^{(r+k)}(s)$ and $\tilde{Q}_n^{(r)}(s)$ need not interlace; however, their zeros have a very interesting relationship on the negative real axis [4]. In this connection, it may be mentioned that the only known result is the one regarding the zeros of $\widetilde{P}_n^{(0)}(s)$, $\widetilde{P}_n^{(1)}(s)$, $\widetilde{P}_n^{(2)}(s)$, and $\widetilde{Q}_n^{(0)}(s)$, since these are the zeros of $\widetilde{b}_n(s)$, $\widetilde{B}_n(s)$, $\widetilde{c}_n(s)$, and $\widetilde{C}_n(s)$, respectively.

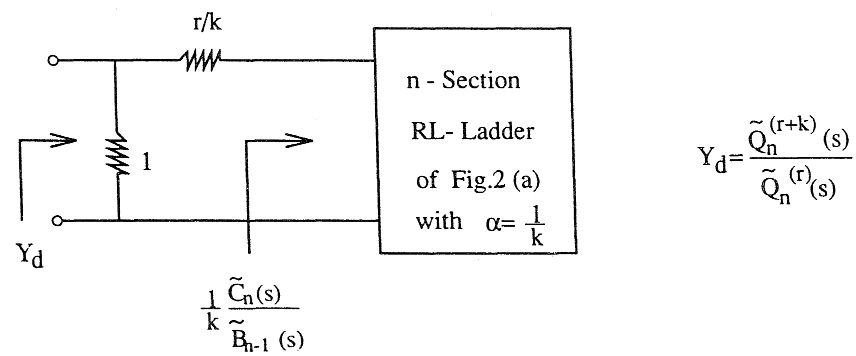


$2L_1 = L_2 = \dots = L_n = 2\alpha$ Henries

$2r_1 = r_2 = \dots = r_n = 2\alpha(p-2)$ Ohms

$R_1 = R_2 = \dots = R_n = 2\alpha$ Ohms

(a)



(b)

FIGURE 2

4. THE COMPOSITE POLYNOMIAL $\widetilde{R}_n^{(r,u)}(x)$

Following Horadam [7], we now define the composite polynomial $\widetilde{R}_n^{(r,u)}(x)$ by the relation

$$\widetilde{R}_n^{(r,u)}(x) = (x+p)\widetilde{R}_{n-1}^{(r,u)}(x) - \widetilde{R}_{n-2}^{(r,u)}(x), \quad (n \ge 2), \tag{20a}$$

with

$$\widetilde{R}_0^{(r,u)}(x) = u \quad \text{and} \quad \widetilde{R}_1^{(r,u)}(x) = x+p+r+u-2, \tag{20b}$$

where r and u are real numbers. It is clear that

$$\begin{aligned} \widetilde{R}_n^{(r,1)}(x) &= \widetilde{P}_n^{(r)}(x), \\ \widetilde{R}_n^{(r,2)}(x) &= \widetilde{Q}_n^{(r)}(x). \end{aligned} \tag{21}$$

Using the results of (3a), (3b), (4), and (5), we see that

$$\begin{aligned}\widetilde{R}_n^{(r,u)}(x) &= (x+p+r+u-2)U_n(x) - uU_{n-1}(x)\\ &= U_{n+1}(x) + (r-1)U_n(x) + (u-1)\{U_n(x) - U_{n-1}(x)\}.\end{aligned}$$

Using (9a) and (6b), the above relation may be rewritten as

$$\widetilde{R}_n^{(r,u)}(x) = \widetilde{P}_n^{(r)}(x) + (u-1)\widetilde{b}_{n-1}(x). \tag{22}$$

Substituting for $\widetilde{b}_{n-1}(x)$ from (10c), equation (22) reduces to

$$\widetilde{R}_n^{(r,u)}(x) = (u-1)\widetilde{Q}_n^{(r)}(x) - (u-2)\widetilde{P}_n^{(r)}(x). \tag{23a}$$

Now using (21), equation (23a) may also be rewritten as

$$\widetilde{R}_n^{(r,u)}(x) = (u-1)\widetilde{R}_n^{(r,2)}(x) - (u-2)\widetilde{R}_n^{(r,1)}(x). \tag{23b}$$

Let us now find the locations of the zeros of $\widetilde{R}_n^{(r,u)}(x)$ for $r \geq 0$ and $u \geq 1$. For this purpose, we first consider the function $\widetilde{R}_n^{(r+k,u)}(s)/\widetilde{R}_n^{(r,u)}(s)$ for $k > 0$. Using (22), we may write

$$\widetilde{R}_n^{(r+k,u)}(s) - \widetilde{R}_n^{(r,u)}(s) = \widetilde{P}_n^{(r+k)}(s) - \widetilde{P}_n^{(r)}(s) = k\widetilde{B}_{n-1}(s), \text{ using (10a)}.$$

Using (22) and (10a), we get

$$\begin{aligned}\frac{\widetilde{R}_n^{(r+k,u)}(s)}{\widetilde{R}_n^{(r,u)}(s)} &= 1 + \frac{k\widetilde{B}_{n-1}(s)}{\widetilde{b}_n(s) + r\widetilde{B}_{n-1}(s) + (u-1)\widetilde{b}_{n-1}(s)}\\ &= 1 + \cfrac{1}{\cfrac{r}{k} + \cfrac{1}{k}\cfrac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)} + \cfrac{u-1}{k}\cfrac{\widetilde{b}_{n-1}(s)}{\widetilde{B}_{n-1}(s)}}.\end{aligned} \tag{24}$$

From the results given in [9], it is known that the function

$$\frac{u-1}{k}\frac{\widetilde{b}_{n-1}(s)}{\widetilde{B}_{n-1}(s)}$$

may be realized as the DPI of the RL-ladder network shown in Figure 3(a), with $\alpha = (u-1)/k$. Further, as already mentioned in Section 3,

$$\frac{1}{k}\frac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)}$$

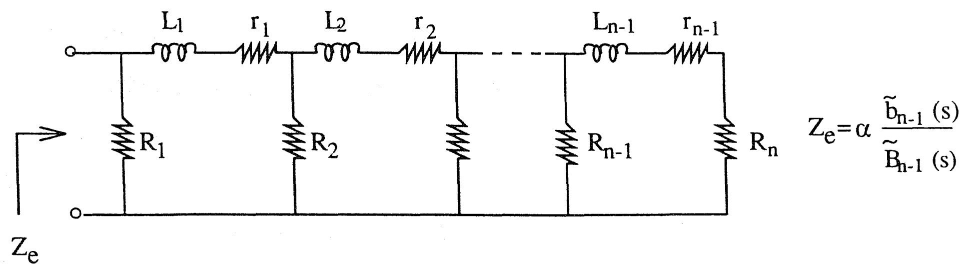
can be realized as the DPI of the RL-ladder network shown in Figure 1(a), with $\alpha = 1/k$. Hence, $\widetilde{R}_n^{(r+k,u)}(s)/\widetilde{R}_n^{(r,u)}(s)$ $(k > 0)$ may be realized as the DPA of the RL-network shown in Figure 3(b).

Again using the properties of RL-networks, we can state that the zeros of $\widetilde{R}_n^{(r,u)}(s)$ are real, simple, and negative; further, the zeros of $\widetilde{R}_n^{(r,u)}(s)$ interlace with those of $\widetilde{R}_n^{(r+k,u)}(s)$, the zero closest to the origin being that of $\widetilde{R}_n^{(r,u)}(s)$.

Now we consider the function $\widetilde{R}_n^{(r+k,u+t)}(s)/\widetilde{R}_n^{(r,u)}(s)$, where $k \geq 0$ and $t > 0$. From (22) and (10a), we have

$$\begin{aligned}\frac{\tilde{R}_n^{(r+k,\,u+t)}(s)}{\tilde{R}_n^{(r,\,u)}(s)} &= \frac{\tilde{P}_n^{(r+k)}(s)+(u+t-1)\tilde{b}_{n-1}(s)}{\tilde{P}_n^{(r)}(s)+(u-1)\tilde{b}_{n-1}(s)} \\ &= \frac{\tilde{b}_n(s)+(r+k)\tilde{B}_{n-1}(s)+(u+t-1)\tilde{b}_{n-1}(s)}{\tilde{b}_n(s)+r\tilde{B}_{n-1}(s)+(u-1)\tilde{b}_{n-1}(s)} \\ &= \frac{(r+k)+\dfrac{\tilde{b}_n(s)}{\tilde{B}_{n-1}(s)}+(u+t-1)\dfrac{\tilde{b}_{n-1}(s)}{\tilde{B}_{n-1}(s)}}{r+\dfrac{\tilde{b}_n(s)}{\tilde{B}_{n-1}(s)}+(u-1)\dfrac{\tilde{b}_{n-1}(s)}{\tilde{B}_{n-1}(s)}}.\end{aligned} \tag{25}$$

Since $\tilde{b}_n(s)/\tilde{B}_{n-1}(s)$ and $\tilde{b}_{n-1}(s)/\tilde{B}_{n-1}(s)$ are both RL-impedance functions, we see from (25) that $\tilde{R}_n^{(r+k,\,u+t)}(s)/\tilde{R}_n^{(r,\,u)}(s)$ $(k \geq 0, t > 0)$ is a ratio of two RL-impedance functions. In view of this, as mentioned earlier in Section 3, the zeros of $\tilde{R}_n^{(r,\,u)}(s)$ and those of $\tilde{R}_n^{(r+k,\,u+t)}(s)$ $(k \geq 0,$ $t > 0)$ need not interlace on the negative real axis [4].

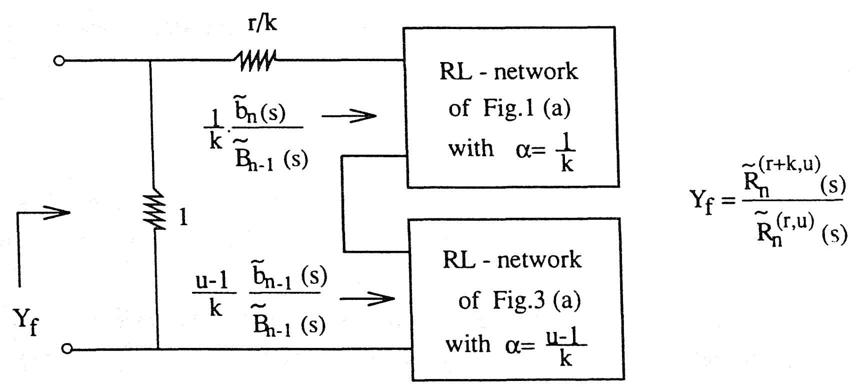


$L_1=L_2=\ldots=L_{n-1}=\alpha$ Henries

$r_1=r_2=\ldots=r_{n-1}=\alpha(p-2)$ Ohms

$R_1=R_2=\ldots=R_n=\alpha$ Ohms

(a)



(b)

FIGURE 3

5. CONCLUDING REMARKS

In this article we have generalized the results of André-Jeannin [1] and Horadam [6] and [7] concerning the sequences $P_n^{(r)}(x)$, $Q_n^{(r)}(x)$, and $R_n^{(r,u)}(x)$. We have also shown that there exist close relationships between these generalized sequences and RL-networks or certain types of RLC-networks. Using these relationships and the properties of such networks, results concerning the locations of the zeros of these generalized sequences have been derived. In view of similar results recently obtained for another pair of polynomials, it is worthwhile exploring such relationships between polynomial sequences and network functions to derive properties of such sequences using the well-known properties of RL, RC, LC, and RLC network functions, and vice-versa.

APPENDIX A

Properties of RL One-Port Networks [14]

A one-port electrical network is a two-terminal network consisting only of two kinds of elements, namely, resistors and inductors.

The driving point impedance $Z(s)$ of such an RL network satisfies the following properties:

(a) All poles and zeros are simple, and are located on the negative real axis of the s-plane.

(b) Poles and zeros interlace.

(c) The lowest critical frequency is a zero which may be located at $s=0$.

(d) The highest critical frequency is a pole which may be at infinity.

(e) $Z(0)<Z(\infty)$.

Also, the driving point admittance of an RL network satisfies the following properties:

(a) All poles and zeros are simple, and are located on the negative real axis of the s-plane.

(b) Poles and zeros interlace.

(c) The lowest critical frequency is a pole which may be located at $s=0$.

(d) The highest critical frequency is a zero which may be at infinity.

(e) $Y(0)>Y(\infty)$.

APPENDIX B

Positive Real Functions [14]

A function $F(s)$, s being a complex variable, is said to be a positive real function if it satisfies the following two conditions:

$$\operatorname{Re}F(s)\geq 0 \text{ for } \operatorname{Re}s\geq 0$$

and

$$F(s) \text{ is real when } s \text{ is real,}$$

where $\operatorname{Re}T$ denotes the real part of T.

A positive real function $F(s)$ can always be realized as the driving point impedance or admittance of a one-port RLC network, that is, a two-terminal network consisting only of resistors, inductors, and capacitors. Conversely, the driving point impedance and admittance functions of an RLC one-port network are always positive real.

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AMS Classification Numbers: 11B39, 33C25

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FIBONACCI FIELDS

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0. INTRODUCTION

In this paper, we consider fields determined by the n^{th} roots of the zeros α and β of the polynomial x^2-x-1; α is the positive zero. The tools for studying these fields will include the Fibonacci and Lucas polynomials. Generalized versions of Fibonacci and Lucas polynomials have been studied in [1], [2], [3], [4], [5], [6], [7], and [12], among others. For the most part, these generalizations consist of considering roots of more general quadratic equations that also satisfy Binet identities. However, it is just the simplest version of these polynomials that we shall need for the results in this paper. (For a far-reaching generalization of all of these generalizations in the context of multiplicative arithmetic functions, see [9].) These polynomials determine many of the properties of the root fields; e.g., they provide the defining polynomials for those fields; they yield a collection of algebraic integers which behave like the Fibonacci numbers and the Lucas numbers in the ring of rational integers; they determine the discriminants of these fields; and, they provide a means of embedding which gives the lattice structure of the fields.

In Part 1, we list properties of these polynomials which we shall need later.

In Part 2, the (odd) m^{th} roots of α and β are discussed; the constant a_m which is, essentially, the sum of two conjugate roots, is introduced. One of two important theorems here is Theorem 2.1, which tells us that the m^{th} Lucas polynomial evaluated at a_m is, up to sign, equal to 1. This will enable us to define a new set of polynomials (by adding a constant to the Lucas polynomial) which, in Part 4, will turn out to be irreducible over the rationals and, hence, will provide us with some useful extension fields (Theorem 4.2). The other important theorem in Part 2 is Theorem 2.2, which tells us that the m^{th} Lucas polynomial evaluated at a_{mn} is a_n. This theorem will lead to an embedding theorem for our fields in Part 4 (Lemma 4.2.2).

In Part 3, we introduce numbers in our extension fields generalizing the Fibonacci numbers, which are algebraic integers in these fields and which turn out to have a peculiar quasi-periodic behavior (Theorem 3.4). (In a sequel to [9], this behavior will be seen to be one typically associated with arithmetic functions.)

In Part 4, the lattice structure of this family of fields is investigated (Lemma 4.2.2, Corollary 4.2.3, Theorem 4.3). Theorem 4.4 tells us that it is the Fibonacci polynomials which provide us with the discriminants of our fields.

The remainder of the paper is occupied with some calculations using a well-known matrix representation of the fields, illustrating computations which produce units and primes in these fields.

The author is indebted to the referee for many helpful suggestions for which he is grateful; especially, he would like to thank the referee for calling to his attention the rich theory of quadratic fields of *Richaud-Degert* type and of R. A. Mollin's book [10]. The fields studied here are extensions of a field of this type.

1. THE POLYNOMIALS $U_n(t)$ AND $V_n(t)$

Here we list some of the well-known properties of the Fibonacci and Lucas polynomials, $U_n(t)$ and $V_n(t)$, that we shall need to use in this paper (see, e.g., [3] and [4]). In [3], [2], and [5], these polynomials were defined explicitly by formulas equivalent to

$$U_m(t) = \sum_{k=0}^{\infty} P_k(m) t^{m-2k-1}, \quad P_k(m) = \binom{m-k-1}{k}, \ k \le \frac{m}{2}, \tag{1.1}$$

$$V_m(t) = \sum_{k=0}^{\infty} \frac{m}{m-2k} P_k(m) t^{m-2k} + \varepsilon_m, \quad \varepsilon_m = \begin{cases} 0, & m \text{ odd}, \\ 2, & m \text{ even}. \end{cases} \tag{1.2}$$

$U_0(t) = 0$, $U_1(t) = 1$, $V_0(t) = 2$, $V_1(t) = t$.

Equivalently, we could have defined $U_n(t)$ and $V_n(t)$ by letting $A(t)$ and $B(t)$ be the roots of the polynomial $p(x) = x^2 - tx - 1$, and setting

$$U_n(t) = \frac{A^n(t) - B^n(t)}{A(t) - B(t)}, \tag{1.3}$$

$$V_n(t) = A^n(t) + B^n(t), \tag{1.4}$$

i.e., the well-known Binet formulas (e.g., see [3] or [6]). From these formulas, it is easy to see that the recursion relation

$$Y_{n+1}(t) = tY_n(t) + Y_{n-1}(t) \tag{1.5}$$

is satisfied by the Fibonacci and Lucas polynomials* [3]. In fact, theses identities provide a painless path for finding most of the identities involving the two sequences of polynomials. Such an identity, which we shall need below, is

$$V_m(V_n(t)) = V_{mn}(t), \quad ([3], 6.2(\text{i})). \tag{1.6}$$

It is, however, equally easy to use the recursion (2.5) to prove that

$$d/dt(V_n(t)) = nU_n(t), \quad ([4], (2.4)), \tag{1.7}$$

which, in turn, gives a short proof using (2.6) of the fact (well known) that U_k divides U_{ks}, with the additional feature of displaying the factors explicitly. To wit:

$$d/dt[V_m(V_n(t))] = mnU_n(t)U_m(V_n(t)) = d/dt[V_{mn}(t)] = mnU_{mn}(t).$$

Thus, the other factor is $U_m(V_n(t))$.

2. THE NUMBERS γ_m, δ_m, a_m

Define γ_m and δ_m up to roots of unity by

$$\gamma_m^m = \alpha, \quad \delta_m^m = \beta.$$

* The first six polynomials in these two sequences are:

$U_0(t)=0$	$U_2(t)=t$	$U_4(t)=t^3+2t$	$V_2(t)=t^2a=2$	$V_0(t)=2$	$V_4(t)=t^4+4t^2+2$
$U_1(t)=1$	$U_3(t)=t^2+1$	$U_5(t)=t^4+3t^2+1$	$V_1(t)=t$	$V_3(t)=t^3+3t$	$V_5(t)=t^5+5t^3+5t$

Since $(\gamma_m\delta_m)^m = \alpha\beta = -1$, we have that $\boxed{\gamma_m\delta_m = \omega_m,}$ where ω_m is a primitive 2^m-th root of unity. When m is odd, then at least one of the γ_m and δ_m is real. Define a_m by $\boxed{\gamma_m + \delta_m = a_m\omega_m^2.}$ Note that $a_1 = 1$. Clearly, $\gamma_1 = \alpha = A(a_1)$ and $\delta_1 = \beta = B(a_1)$. It follows that

$$\gamma_m = \frac{1}{2}(a_m + (a_m^2+4)^{1/2})\omega_m^{(m+1)/2} = A(a_m\omega_m^{(m+1)/2}),$$
$$\delta_m = \frac{1}{2}(a_m - (a_m^2+4)^{1/2})\omega_m^{(m+1)/2} = B(a_m\omega_m^{(m+1)/2})$$

and

$$A(a_m\omega_m^{(m+1)/2}) = \omega_m^{(m+1)/2}A(a_m), \quad B(a_m\omega_m^{(m+1)/2}) = \omega_m^{(m+1)/2}B(a_m).$$

So

$$\boxed{\begin{aligned}\gamma_m &= \omega_m^{(m+1)/2}(a_m),\\ \delta_m &= \omega_m^{(m+1)/2}(a_m).\end{aligned}}$$

Thus,

$$\begin{aligned}A^m(a_m\omega_m^{(m+1)/2}) + B^m(a_m\omega_m^{(m+1)/2}) &= (-1)^{(m+1)/2}(A^m(a_m) + B^m(a_m))\\ &= V_m(a_m) = \gamma_m^m + \delta_m^m = \alpha + \beta = 1,\end{aligned}$$

and so

Theorem 2.1: $(-1)^{(m+1)/2}V_m(a_m) - 1 = 0$, m odd. □

Hence, a_m is a root of the polynomial $D_m(t) = V_m(t) - (-1)^{(m+1)/2}$.

Proposition 2.1.1: $\alpha = \frac{1}{2}(1 + R(a_m))U_m(a_m)$, $\beta = \frac{1}{2}(1 - R(a_m))U_m(a_m)$, $R(t) = (t^2+4)^{1/2}$,

is implied by the next proposition.

Proposition 2.1.2: $A^m(a_m) = \alpha$, $B^m(a_m) = \beta$.

Proposition 2.1.3: $A^m(a_{mn}) = \gamma_n$, $B^m(a_{mn}) = \delta_n$.

Proof: $A^{mn}(a_{mn}) = \alpha_n^n = \gamma_n^n$.

In particular,

Theorem 2.2: $V_m(a_{mn}) = a_n$, up to the roots of unity.

Proof: $A^{mn}(a_{mn}) + B^{mn}(a_{mn}) = V_m(a_{mn}) = \gamma_n + \delta_n = a_n$ (up to roots of unity).

3. GENERALIZED FIBONACCI AND LUCAS NUMBERS

The algebraic numbers $U_k(a_m)$ can be thought of as a generalization of the Fibonacci numbers. However, we need an unambiguous notation for them, so remembering that m is odd in this paper, we pick a fixed real a_m for each natural number m (there is a unique choice), and define

$$\boxed{\Lambda_{m,k} = \Omega_m^k(U_k(a_m)),}$$

where

$$\boxed{\Omega_m = \omega_m^{(m+1)/2}.}$$

Thus,

$$\boxed{\Lambda_{m,k} = \Omega_m^k \frac{A^k(a_m) - B^k(a_m)}{A(a_m) - B(a_m)}}$$

are the generalized Fibonacci numbers (GFN); they are located "between" the number fields $Q(a_m, \omega_m)$ and $Q(\gamma_m, \omega_m)$. However, first observe that $\Lambda_{1,k} = F_k$, i.e., the $\Lambda_{m,k}$ are generalizations of Fibonacci numbers. From (1.5), we see that, for each choice of m, we have a family of GFNs which belong to the field $Q(a_m)$ and which have a functional equation generalizing that in $Q(a_1) = Q$, namely, one which generalizes the usual functional equation for the Fibonacci numbers. Moreover, we have the following interesting quasi-periodic behavior of these numbers, which is manifest only when $m > 1$.*

Theorem 3.4: Let $U_{i,j}(k) = U_{mk+j}(a_m), 0 \le j \le m$, m odd, then

$$\boxed{U_{mj}(k) \equiv F_{k+1}U_j(a_m) + (-1)^j F_k U_{m-j}(a_m) \bmod D_m(t),}$$

F_n, the n^{th} Fibonacci number, and D_m is as defined in Theorem 2.1.

Proof: Assume inductively that the theorem holds for $k < n$ and for $j - 1 \ge 1$. Assume that $U_{m,j}(k-1)$ satisfies the appropriate relation for $j = 0, \ldots, m-1$. We need to compute $U_{m,0}(k)$, but

$$\begin{aligned}
U_{m,0}(k) &= U_{mk}(t) = tU_{mk-1}(t) + U_{mk-2}(t) \\
&= tU_{m,m-1}(k-1) + dU_{m,m-2}(k-1) \\
&= t[F_kU_{m-1} + (-1)^{m-1}F_{k-1}U_1] +]F_kU_{m-2} + (-1)^{m-2}F_{k-1}U_2] \\
&= F_k[tU_{m-1} + U_{m-2}] + F_{k-1}[(-1)^{m-1}tU_1 + (-1)^{m-2}U_2] \\
&= F_kU_m + F_{k-1}[tU_1 - U_2] = F_kU_m,
\end{aligned}$$

since $U_1(t) = 1$, $U_1(t) = t$. But, if the theorem is correct, $U_{m,0}(k) = F_{k+1}U_0 + (-1)^0 F_kU_m = F_kU_m$. Thus, we have shown what is required. Next, we must show that the result holds for a fixed k and $j = 1, 2, \ldots, m-1$. Notice that the theorem is correct for $j = 0, \ldots, m-1$, $k = 0$, and for $j = 0$, $k = 1$. Suppose that it holds for $k < n$ and $j = 0, \ldots, m-1$ and for $k = n$ and $j = 0$. We want to show that it holds for $k = n$, $j = 1, \ldots, m-1$. So consider $U_{mj}(k)$, $k = n$, $1 \le j \le m-1$.

$$\begin{aligned}
U_{mj}(k) &= tU_{m,j-1}(k) + U_{m,j-2}(k) \\
&= t[F_{k+1}U_{j-1} + (-1)^{j-1}F_kU_{m-j+1}] + [F_{k+1}U_{j-2} + (-1)^{j-2}F_kU_{m-j+2}] \\
&= F_{k+1}[tU_{j-1} + U_{j-2}] + (-1)^{j-1}F_k[tU_{m-j+1} - U_{m-j+2}] \\
&= F_{k+1}U_j + (-1)^{j-1}F_k[tU_{m-j+1} - U_{m-j+2}] \\
&= F_{k+1}U_j + (-1)^{j-1}F_k[[tU_{m-j+1} - (tU_{m-j+1} + U_{m-j})] \\
&= F_{k+1}U_j + (-1)^j F_kU_{m-j}. \quad \square
\end{aligned}$$

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* We should point out that this is a special case of a phenomenon which always occurs in the context of a certain class of multiplicative arithmetic functions (see [9]).

The numbers for $m = 3$ are:

$$U_{3,3k} = -F_{k-1}\omega_3(1+a_3^2);\quad U_{3,3k+1} = F_k - F_{k-1}a_3;\quad U_{3,3k+2} = (F_k a_3 + F_{k-1})\omega_3^2.$$

4. THE ALGEBRAIC NUMBER FIELDS $Q(\gamma_m)$, $Q(\delta_m)$, $Q(a_m)$

We assume that m is odd and note that

Proposition 4.1: a_p, γ_p, δ_p, ω_p are units in the ring of integers of $Q(a_p)$.

Proof: $t^{2p} - t^p - 1$ is the minimum polynomial of $Q(\gamma_p)$. Both γ_p and δ_p satisfy this polynomial. Moreover, $a_p = -\omega_p(\gamma_p + \delta_p)$. Note that α and β clearly belong to $Q(\gamma_p)$. □

The most interesting result to come out of the ideas considered in this paper is the way in which the polynomials U_m and V_m provide the structural framework for the algebraic number fields determined by the numbers γ_m, δ_m, a_m. A first example of this fact is contained in the role that the polynomials D_m play. $D_m(t)$ is irreducible over Q for m odd. This can be proved by using earlier propositions and Eisenstein's criterion; however, the following proof is instructive.

Theorem 4.2: $\mathcal{F}_m = Q[t]/\langle D_m(t)\rangle$ is a field for odd m.

Proof: Let p be an odd prime.

Lemma 4.2.1: ***(a)*** $D_p(t)$ is a monic polynomial of degree p with constant term ± 1.

(b) p divides all interior coefficients of $D_p(t)$.

Proof of Lemma: (a) follows from (1.5) by induction and definition. For (b), we need to know that the "interior" coefficients of $D_p(t)$ are given by

$$\boxed{P_k(p+1) + P_{k-1}(p-1) = \binom{p-k-1}{k+1} + \binom{p-k-2}{k}.}$$

But this follows easily from (2.1), (2.2), and (2.5). Then it is straightforward to show that

$$P_k(p+1) + P_{k-1}(p-1) = \frac{(p-k-2)!}{(p-2k-2)!(k+1)!}p.$$

Since p is prime, hence is relatively prime to the denominator, p divides $P_k(p+1) + P_{k-1}(p-1)$. □

Thus, by a standard application of Eisenstein's lemma, $D_p(t)$ is irreducible over Q, so the theorem holds for the case $m = p$, p a prime. Thus, $\mathcal{F}_p$ is a field. We want to show that $\mathcal{F}_{np}$ is a field for any odd prime p and any natural number n. First, we prove a lemma which is of interest in its own right.

Lemma 4.2.2 (The Embedding Lemma): There is a natural embedding of the ring $\mathcal{F}_{p^{n-1}}$ in the ring $\mathcal{F}_{p^n}$.

Proof: It is convenient first to note that the ring $\mathcal{F}_m$ can be represented by elements of the form $\sum_{i=0}^{m-1} m_i a_m^i$, $m_i \in Q$, taken mod $D_m(t)$. Now we consider $(D_{p^{k-1}} \circ V_p)(a_{p^k})$.

$$(D_{p^{k-1}} \circ V_p)(a_{p^k}) = V_{p^{k-1}}(V_p(a_{a^k})) + (-1)^{(p+1)/2} = V_{p^k} a_{p^k} + (-1)^{(p+1)/2} = D_{p^k}(a_{p^k}) = 0.$$

Thus, $V_p(a_{p^k}) = a_{p^{k-1}}$. Now, $V_p(a_{p^k}) \in \mathcal{F}_{p^k}$. Since $a_{p^k} \in \mathcal{F}_{p^k}$, so does a copy of $a_{p^{k-1}}$. Since this element satisfies D_{p^k} and $\mathcal{F}_{p^{k-1}}$ consists of elements of the form $\sum_{i=0}^{p^{k-1}} m_i a_{p^{k-1}}$, so we have an embedding of $\mathcal{F}_{p^{k-1}}$ in $\mathcal{F}_{p^k}$ determined by the polynomials V_k. So assume inductively that $\mathcal{F}_{p^k}$ is a field for $k \le n$, and let I be maximal ideal in the Noetherian ring $\mathcal{F}_{p^k}$. $\mathcal{F}_{p^k}/I$ is a field, one which contains a copy of $\mathcal{F}_{p^{k-1}}$, so the degree of $\mathcal{F}_{p^k}/I$ (over Q) is $\ge p^{k-1}$. Now a_{p^k} is a unit, so $a_{p^k} \notin I$; thus, $a_{p^k} + I \in D_{p^k}/I$ and is not trivial. And so the degree of $D_{p^k}/I > p^{k-1}$, and thus the degree of $D_{p^k}/I = p^k$. Therefore, the minimum polynomial of $\mathcal{F}_{p^k}/I$ is a multiple of D_{p^k}, hence is equal to D_{p^k}, and so $I = O$, and $\mathcal{F}_{p^k}$ is a field. □

Thus, we have proved the theorem for m an odd prime power. This argument applied to $V_m(V_p(a_{mp}), (m, p) = 1$, extends the result to $\mathcal{F}_{mp}$, $(m, p) = 1$. Thus, $\mathcal{F}_n$ is a field for all odd n.

Corollary 4.2.3: If m divides n, m and n both odd, then $\mathcal{F}_m$ is (isomorphic to) a subfield of $\mathcal{F}_n$ under the embedding determined by $(-1)^{(n-1)/2} V_m(V_k(a_{mk})) = a_m$, $n = mk$. □

Since $\gamma_m = \omega_m^{(m+1)/2} A(a_m)$, $\delta_m = \omega_m^{(m+1)/2} B(a_m)$, it follows that $\mathcal{F}_m < Q(a_m, \omega_m) < Q(\gamma_m, \omega_m)$. The last two fields are splitting fields. We thus have the following degree relations.

Theorem 4.3: $[Q(a_m) : Q] = [\mathcal{F}_m : Q] = m$, $[Q(a_m, \omega_m) : \mathcal{F}_m] = \phi(m)$, $[Q(\gamma_m, \omega_m) : Q(a_m, \omega_m)] = 2$, where ϕ is the Euler totient function.

The following theorem is another illustration of how the polynomials U_m and V_m are involved in the structure of the fields $\mathcal{F}_m$.

Theorem 4.4: $\Delta[1, a_m, \ldots, a_m^{m-1}] = (-1)^{m(m-1)/2} m^m N U_m(a_m)$, is the norm of the algebraic number $U_m(a_m)$.

Proof: In any case, since $\frac{d}{dt}(V_m) = \frac{d}{dt}(D_m)$,

$$\Delta[1, a_m, \ldots, a_m^{m-1}] = (-1)^{m(m-1)/2} N\left(\frac{d}{dt}\right)(V_m)(a_m),$$

by (1.7), $d/dtV_m = mU_m$ and $N(mU_m(a_m)) = m^m N(U_m(a_m))$. □

Example: It follows from Theorem 4.4 that, when $m = 3$, $\Delta[1, a_3, a_3^2] = -3^3 \cdot 5$. This can be computed directly by using the representation of $\mathcal{F}_3$ determined by the minimal polynomial. Thus,

$$a_3 = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 0 \end{vmatrix}, \quad \text{and so} \quad 1 + a_3^2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{vmatrix},$$

from which it follows that

$$N\left(\frac{d}{dt}V(t)\bigg|_{t=a_3}\right) = N(3F_3(a_3)) = 3^3 \det(1 + A_3^2) = 3^3 \cdot 5.$$

So $\Delta = -3^3 \cdot 5$ as promised by the theorem. We can write $\Delta[1 \cdot a_m, \ldots, a_m^{m-1}]$ explicitly.

Theorem 4.5: $\Delta[1 \cdot a_m, \ldots, a_m^{m-1}] = (-1)^{m(m-1)/2} m^m \cdot 5^n$, $m = 2n + 1$.

Proof: By Theorem 4.4, we need only compute $N(U_m(a_m)) = 5^n$. To do this, let $\lambda_1, \ldots, \lambda_m$ be the m distinct conjugates of a_m with $a_m = \lambda_1$. Then

$$Nu_m(a_m) = \prod_{k=1}^{m} \frac{A^m(\lambda_k) - B^m(\lambda_k)}{R(\lambda_k)}$$

$$= \prod_{k=1}^{m} \frac{\gamma_{(k)}^m - \delta_{(k)}^m}{R(\lambda_k)} = \prod_{k=1}^{m} \frac{\gamma_{(k)}^m - \delta_{(k)}^m}{\gamma_{(k)} - \delta_{(k)}},$$

where $\gamma_{(k)}$ and $\delta_{(k)}$ are the conjugates of γ_m and δ_m.

$$\prod_{k=1}^{m} \frac{\gamma_{(k)}^m - \delta_{(k)}^m}{\gamma_{(k)} - \delta_{(k)}} = \prod_{k=1}^{m} \frac{(\alpha - \beta)^m}{\gamma_{(k)} - \delta_{(k)}}$$

$$= \frac{(\sqrt{5})^m}{\prod_1^m (\gamma_{(k)} - \delta_{(k)})} = \frac{5^n \sqrt{5}}{\prod_1^m (\gamma_{(k)} - \delta_{(k)})}.$$

Now,

$$\prod_1^m (\gamma_{(k)} - \delta_{(k)}) = \Pi \gamma_{(k)} - \Pi \delta_{(k)} + \sum_{s \geq 1} \gamma_{(k_1)} \cdots \gamma_{(k_s)} \delta_{(k_1)} \cdots \delta_{(k_s)}.$$

Since $\gamma_{(k)}$ satisfies $x^m - \alpha = 0$ and $\delta_{(k)}$ satisfies $x^m - \beta = 0$, $\Pi \gamma_{(k)} = \alpha$ and $\Pi \delta_{(k)} = \beta$, so $\Pi \gamma_{(k)} - \Pi \delta_{(k)} = \alpha - \beta = \sqrt{5}$. The remaining products are symmetric polynomials involving at least two symbols, but not all, so, from the equation satisfied by the γ's and δ's, are 0. □

The significance of the algebraic numbers a_m is now clear. To understand the fields $Q(\alpha_m)$ and $Q(\delta_m)$ and their normal extensions, it is sufficient to understand the fields $\mathcal{F}_m$ (and their normal extensions), for $Q(\gamma_m)$, for example, is an easily understood quadratic extension of $\mathcal{F}_m$. The role that the polynomial sequences U_m and V_m play in determining the structure in these fields is also clear, and surprising. The GFNs are integers in these fields, since a_m and ω_m are. So we are left with the standard questions: the class numbers, the maximal orders, units, primes, etc., of these fields (see, e.g., [11]). It is tempting to believe that, linked as these nonquadratic extensions are to a "base" field which is of the *Richaud-Degert* (R-D) type, some adaptation of the elegant methods used for R-D type fields might be found. Of course, the periodic nature of continued fraction expansions of quadratic irrationalities is an intriguing obstacle in the cases of degree greater than 2.

Some direct computations for small m are possible. We illustrate for $m = 3$. (When $m = 1$, the field is, of course, just $Q(\sqrt{5})$). Therefore, we should start at $m = 3$. (The theory for m even has much in common with the case of m odd, but also some significant differences that occur because the minimal polynomials need not have real roots. Moreover, the sequences $\{U_m\}$ and $\{V_m\}$ are markedly different for m even and for m odd. We postpone this discussion.)

A Computation for $\boldsymbol{m = 3}$**:** Using the faithful representation ρ for a_3 as in the illustration of Theorem 4.4,

$$\rho(a_3) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 0 \end{vmatrix} = M,$$

and letting

$$\rho(k_0+k_1a_3+k_2a_3^2)=k_0I+k_1M+k_2M=\begin{vmatrix} k_0 & k_2 & k_1 \\ k_1 & k_0-3k_2 & k_2-3k_1 \\ k_2 & k_1 & k_0-3k_2 \end{vmatrix}.$$

Then,

$\sum k_ia_3^i \in Z(a_m)$ is an algebraic integer iff $M(k_0, k_1, k_2)$ is an integer matrix;
$\sum k_ia_3^i \in Z(a_m)$ is a unit iff $M(k_0, k_1, k_2) = N(\sum k_ia_3^i) = \pm 1$.
$\sum k_ia_3^i \in Z(a_m)$ is a prime if det $M(k_0, k_1, k_2)$ is a rational prime (e.g., $1-a$ is a prime in $\mathcal{F}_3$).

We know that either a prime ideal in Z is a prime ideal in $\mathcal{F}_3$ or factors into two prime ideals. We can determine this for each prime ideal $\langle p\rangle$ by checking to see if t^3+t+1 is irreducible mod p. For example, 2 is a prime in $\mathcal{F}_3$, while 3 and 5 factor, 7 is prime. Since $\Delta_3(\mathcal{F}_3)=-3^3 5$, 3 and 5 ramify; 3 ramifies totally, $\langle 3\rangle = \langle 1-a\rangle^3$. The ramification index is 3, and the relative degree is 1. For 5, $\langle 5\rangle=\langle 4+a^2\rangle\langle 1+a^2\rangle$ with ramification numbers $e_1=1$ and $e_2=2$ and relative degrees $f_1=1$ and $f_2=1$. Using Minkowski's theorem, we can compute

$$h(\mathcal{F}_3)=\frac{4}{\pi}\frac{3!}{3^3}|\Delta(\mathcal{F}_3)|^{1/2} \le 2,$$

and so the class number of $\mathcal{F}_3$ is 1.

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AMS Classification Numbers: 11A25, 11B39, 11R29, 11R04

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ON DIOPHANTINE APPROXIMATIONS WITH RATIONALS RESTRICTED BY ARITHMETICAL CONDITIONS

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1. INTRODUCTION AND STATEMENT OF RESULTS

One of the most important applications of continued fractions deals with the approximation of real numbers by rationals. The famous approximation theorem of A. Hurwitz [7] states that for every real irrational number ξ there are infinitely many integers u and $v > 0$ such that

$$\left|\xi - \frac{u}{v}\right| \le \frac{1}{\sqrt{5}v^2}.$$

The constant $1/\sqrt{5}$ is well known to be best-possible in general.

S. Hartman [6] was the first to introduce congruence conditions on u and v; the best approximation result of this type up until now is due to S. Uchiyama [12]:

For any irrational number ξ, any $s > 1$, and integers a and b, there are infinitely many integers u and $v \ne 0$ such that

$$\left|\xi - \frac{u}{v}\right| < \frac{s^2}{4v^2} \tag{1.1}$$

and

$$u \equiv a \bmod s, \quad v \equiv b \bmod s, \tag{1.2}$$

provided that a and b are not both divisible by s.

A weaker theorem was proved by J. F. Koksma [9] in 1951. Recently, the author [2] has shown that the constant $1/4$ in (1.1) is best-possible.

But one expects that weaker arithmetical conditions in (1.2) on numerators and denominators will imply smaller constants in (1.1). A result of this kind is proved in [3]:

Let $0 < \varepsilon \le 1$, and let p be a prime with

$$p > \left(\frac{2}{\varepsilon}\right)^2;$$

h denotes any integer that is not divisible by p. Then, for any real irrational number ξ, there are infinitely many integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| \le \frac{(1+\varepsilon)p^{3/2}}{\sqrt{5}v^2} \tag{1.3}$$

and

$$u \equiv hv \not\equiv 0 \bmod p. \tag{1.4}$$

In this paper, we shall improve this result as far as possible, where additionally *coprime* integers u and v are considered.

Theorem 1.1: Let s denote any positive integer having an odd prime divisor p such that $p^\alpha \mid s$ for some positive integer α. Moreover, let h be any integer. Then, for every real irrational number ξ, there are infinitely many integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| \le \frac{s}{\sqrt{5}v^2}$$

and

$$u \equiv hv \mod s, \quad (u, v) \le \frac{s}{p^\alpha}.$$

In general, the constant $1/\sqrt{5}$ is best-possible.

Corollary 1.1: Let $s = p^\alpha$ denote some prime power with an odd prime p. Moreover, let h be any integer. Then, for every real irrational number ξ, there are infinitely many coprime integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| \le \frac{s}{\sqrt{5}v^2} \tag{1.5}$$

and

$$u \equiv hv \mod s. \tag{1.6}$$

By Theorem 3.2 in [1] with $\delta = 1/10$ and $\xi = 12 + \sqrt{145}$, all fractions u/v with odd coprime integers u and $v > 0$ satisfy

$$\left|\xi - \frac{u}{v}\right| > \frac{2}{\sqrt{5}v^2}.$$

Hence, Corollary 1.1 does not hold in the case $s = 2$ and $h = 1$. Also, the bound on the right of (1.5) must be enlarged in the case of moduli s having more than one prime divisor.

Theorem 1.2: Let s be some positive integer having at least two prime divisors. Moreover, h denotes any integer. Then there is a real quadratic irrational number ξ with the following property. For every pair u and v of coprime integers with $|v| > 1$ and $u \equiv hv \mod s$, the inequality

$$\left|\xi - \frac{u}{v}\right| > \frac{s}{2v^2}$$

holds.

It is suggested by the above-mentioned theorems that approximation results with an additional condition like (1.6) depend on arithmetic properties of the modulus s. A general result of this kind is expressed in our final Theorem 1.3. For an integer $s > 1$, the number

$$\delta(s) := \prod_{p \mid s} p$$

is the square-free kernel of s, where p runs through the prime divisors of s. In what follows, p_0 is the smallest prime divisor of s, and

$$S := \min\left\{\frac{s^2}{4}, \frac{s\delta^2(s)}{p_0^2}\right\}.$$

Theorem 1.3: For arbitrary integers $s > 1$ and h and for every real irrational number ξ, there are infinitely many coprime integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| < \frac{S}{v^2}$$

and

$$u \equiv hv \mod s.$$

For further improvements of the bound on the right-hand side of (1.5) in Corollary 1.1 for all numbers ξ from a certain set of measure 1, the author [4] applies the mean value theorem of Gauss-Kusmin-Lévy [10] from the metric theory of continued fractions. This set depends on p. To prove our theorems, we shall need some well-known elementary facts from the theory of continued fractions (see [11] or [5]). By

$$\xi = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}},$$

we denote the continued fraction expansion of a real number ξ.

2. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1: The proof of Theorem 1.1 is based on the following proposition.

Proposition 2.1: Let $p > 2$ be a prime number. Among any six consecutive convergents p_{n+i} / q_{n+i} $(n \geq 0,\ i = 0, 1, 2, 3, 4, 5)$ of a real irrational number η there is at least one fraction, say p_v / q_v, such that

$$\left|\eta - \frac{p_v}{q_v}\right| \leq \frac{1}{\sqrt{5}q_v^2} \tag{2.1}$$

holds and q_v is not divisible by p.

Proof: We denote the set of fractions from $\frac{p_n}{q_n}, \ldots, \frac{p_{n+5}}{q_{n+5}}$ satisfying (2.1) by $\mathcal{A}_n$. From a famous theorem of A. Hurwitz which asserts that at least one of three consecutive convergents satisfies (2.1) (see, e.g., Satz 15, ch. 2 in [11]), we know that $2 \leq |\mathcal{A}_n| \leq 6$. In what follows, we consider several cases according to the distribution of fractions from $\mathcal{A}_n$.

Case 1. There is an integer m such that $\frac{p_m}{q_m}, \frac{p_{m+1}}{q_{m+1}} \in \mathcal{A}_n$.

It is a well-known fact that q_m and q_{m+1} represent coprime integers and, therefore, the prime number p cannot divide both of the numbers q_m and q_{m+1}.

Case 2. There are no consecutive convergents of η in $\mathcal{A}_n$.

Case 2.1. It is $\frac{p_m}{q_m}, \frac{p_{m+2}}{q_{m+2}} \in \mathcal{A}_n$ for some integer m.

Let us assume that p divides both q_m and q_{m+2}. Then the recurrence formula of the q's yields

$$a_{m+2}q_{m+1} = q_{m+2} - q_m \equiv 0 \mod p.$$

From $(q_m, q_{m+1}) = 1$, we know that q_{m+1} is not divisible by p. Therefore, p divides a_{m+2}, and we have $a_{m+2} \geq p > \sqrt{5}$. It follows that

$$\left|\eta - \frac{p_{m+1}}{q_{m+1}}\right| < \frac{1}{a_{m+2}q_{m+1}^2} < \frac{1}{\sqrt{5}q_{m+1}^2},$$

hence $\frac{p_{m+1}}{q_{m+1}} \in \mathcal{A}_n$. But we know that $\frac{p_m}{q_m} \in \mathcal{A}_n$ from the hypothesis of Case 2.1, which is incompatible with the hypothesis of Case 2. We have proved that $p \mid q_m$ and $p \mid q_{m+2}$ cannot hold simultaneously.

Case 2.2. It is $\frac{p_m}{q_m}, \frac{p_{m+3}}{q_{m+3}} \in \mathcal{A}_n$ for some integer m.

As in the preceding case, we assume that p divides both of the denominators q_m and q_{m+3}. We have

$$q_{m+3} = a_{m+3}q_{m+2} + q_{m+1},$$
$$q_{m+2} = a_{m+2}q_{m+1} + q_m,$$

for some positive integers a_{m+2}, a_{m+3} from the continued fraction expansion of η. Putting the second equation into the first one, we obtain the identity

$$q_{m+3} - a_{m+3}q_m = (a_{m+2}a_{m+3} + 1)q_{m+1}.$$

Our assumption on p implies that the integer $(a_{m+2}a_{m+3}+1)q_{m+1}$ is divisible by p. Since q_m and q_{m+1} are coprime, $p \mid q_{m+1}$ is impossible. It follows that p divides $a_{m+2}a_{m+3}+1$ and, consequently, we have $a_{m+2}a_{m+3}+1 \geq p \geq 3$. Hence, it is impossible to have $a_{m+2} = a_{m+3} = 1$. We discuss the remaining cases.

Case 2.2.1. $a_{m+2} \geq 3$ or $a_{m+3} \geq 3$.

From

$$\left|\eta - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2} \quad (n \geq 1),$$

we get

$$\frac{p_m}{q_m}, \frac{p_{m+1}}{q_{m+1}} \in \mathcal{A}_n \qquad (\text{if } a_{m+2} \geq 3),$$

$$\frac{p_{m+2}}{q_{m+2}}, \frac{p_{m+3}}{q_{m+3}} \in \mathcal{A}_n \quad (\text{if } a_{m+3} \geq 3).$$

Again there is a contradiction to the hypothesis of Case 2.

Case 2.2.2. $a_{m+2} = a_{m+3} = 2$.

We have

$$\alpha_{m+2} := [2; 2, a_{m+4}, a_{m+5}, \ldots] > 2 + \frac{1}{2+1} = \frac{7}{3},$$

and finally, it follows that

$$\left|\eta - \frac{p_{m+1}}{q_{m+1}}\right| < \frac{1}{\alpha_{m+2}q_{m+1}^2} < \frac{3}{7q_{m+1}^2} < \frac{1}{\sqrt{5}q_{m+1}^2}.$$

Hence, it is

$$\frac{p_m}{q_m}, \frac{p_{m+1}}{q_{m+1}} \in \mathcal{A}_n,$$

a contradiction.

Case 2.2.3. $a_{m+2} = 2,\ a_{m+3} = 1$.

It is

$$\alpha_{m+2} := [2; 1, a_{m+4}, a_{m+5}, \ldots] > 2 + \frac{1}{1+1} = \frac{5}{2}$$

and

$$\left|\eta - \frac{p_{m+1}}{q_{m+1}}\right| < \frac{2}{5q_{m+1}^2} < \frac{1}{\sqrt{5}q_{m+1}^2}.$$

Again we get

$$\frac{p_m}{q_m}, \frac{p_{m+1}}{q_{m+1}} \in \mathcal{A}_n.$$

Case 2.2.4. $a_{m+2} = 1,\ a_{m+3} = 2$.

First, note that $\alpha_{m+3} := [2; a_{m+4}, a_{m+5}, \ldots] > 2$. We get ,

$$\left|\eta - \frac{p_{m+2}}{q_{m+2}}\right| = \frac{1}{q_{m+2}(\alpha_{m+3}q_{m+2} + q_{m+1})} < \frac{1}{q_{m+2}^2\left(2 + \frac{q_{m+1}}{q_{m+2}}\right)}$$

$$= \frac{1}{q_{m+2}^2\left(2 + \frac{1}{[1; a_{m+1}, \ldots, a_1]}\right)} < \frac{2}{5q_{m+2}^2}$$

by $[1; a_{m+1}, \ldots, a_1] < 2$. The contradiction arises from

$$\frac{p_{m+2}}{q_{m+2}}, \frac{p_{m+3}}{q_{m+3}} \in \mathcal{A}_n.$$

Hence, it is proved that $p \mid q_m$ and $p \mid q_{m+3}$ cannot hold simultaneously. Since for every integer $m \geq 0$ there is at least one fraction among the convergents $\frac{p_{m+1}}{q_{m+1}}$, $\frac{p_{m+2}}{q_{m+2}}$, and $\frac{p_{m+3}}{q_{m+3}}$ satisfying (2.1) by Hurwitz's theorem, we have finished the proof of Proposition 2.1.

By the hypotheses of Theorem 1.1 on ξ, h, and s, we may choose $\eta := (\xi - h)/s$. From Proposition 2.1, we know that there are infinitely many convergents p_m / q_m of η with

$$\left|\frac{\xi - h}{s} - \frac{p_m}{q_m}\right| \leq \frac{1}{\sqrt{5}q_m^2},$$

where p and q_m are coprime integers. Put $u := hq_m + sp_m$ and $v := q_m$. Then, it is $u \equiv hv \bmod s$ and

$$\frac{1}{s}\left|\xi - \frac{u}{v}\right| \leq \frac{1}{\sqrt{5}v^2}.$$

To estimate the greatest common divisor of u and v, we conclude from $(p_m, q_m) = 1$, $p \nmid q_m$, and $p^\alpha \mid s$ that

$$(sp_m, q_m) = (s, q_m) \le \frac{s}{p^\alpha}.$$

By $(u, v) = (hq_m + sp_m, q_m) = (sp_m, q_m)$, the first assertion of Theorem 1.1 follows.

The corresponding assertion of Corollary 1.1 follows immediately. But it remains to show that Theorem 1.1 cannot be improved in general. For this purpose, let $s > 0$ and h be integers. Put $\xi := h + s(1+\sqrt{5})/2$. In what follows, we shall show that for every $\varepsilon > 0$ there are at most finitely many fractions u/v, where $v > 0$,

$$u \equiv hv \mod s \tag{2.2}$$

and

$$\left|\xi - \frac{u}{v}\right| < \frac{(1-\varepsilon)s}{\sqrt{5}v^2}. \tag{2.3}$$

There is nothing to prove in the case in which no fractions u/v satisfy (2.2) and (2.3) simultaneously. Otherwise, we conclude from (2.2) that $u = hv + ws$ holds for a certain integer w. Then we have, by (2.3),

$$\frac{(1-\varepsilon)s}{\sqrt{5}v^2} > s\cdot\left|\frac{1+\sqrt{5}}{2} - \frac{w}{v}\right|,$$

which yields

$$\left|\frac{1+\sqrt{5}}{2} - \frac{w}{v}\right| < \frac{1-\varepsilon}{\sqrt{5}v^2}. \tag{2.4}$$

It is a well-known fact from the theory of continued fractions that there are at most finitely many solutions w/v in (2.4) (see, e.g., Th. 194 in [5]). One knows that every solution of (2.4) satisfies

$$\frac{w}{v} = \frac{F_{n+1}}{F_n} \text{ for some integer } n, \text{ and } v^2 < \frac{1}{5\varepsilon}.$$

Our assertion follows from the inequality $|v\xi - u| < s/\sqrt{5}$, which has at most finitely many solutions for every integer v.

Proof of Theorem 1.2: Let p and q be different primes with $pq \mid s$. Moreover, we define a sequence $(a_n)_{n\ge0}$ of nonnegative integers as follows. Put $a_0 := 0$ and $a_1 := p$. Let a_2 be the unique solution of the congruence

$$a_2 p \equiv -1 \mod q, \tag{2.5}$$

where $1 \le a_2 < q$. Since $(p, q) = 1$, solutions of (2.5) do exist. Finally, put $a_v := p$ for $v = 3, 5, 7, \ldots$ and $a_v := q$ for $v = 4, 6, 8, \ldots$. Then we have $q_0 = 1$, $q_1 = p$, $q_2 = a_2 p + 1 \equiv 0 \mod q$. Applying mathematical induction, we conclude that

$$q_v \equiv \begin{cases} 0 \mod p, & \text{if } v \equiv 1 \mod 2 \\ 0 \mod q, & \text{if } v \equiv 0 \mod 2 \end{cases} \quad (v \ge 1). \tag{2.6}$$

Obviously, $\eta := [a_0; a_1, a_2, \ldots]$ and $\xi := h + s\eta$ represent real quadratic irrational numbers.

Now we assume that integers u and v do exist such that $|v| > 1$, $u \equiv hv \mod s$, and

$$\left|\xi - \frac{u}{v}\right| < \frac{s}{2v^2}.$$

Hence, there is an integer w such that $u = hv + ws$ and

$$\left|\eta - \frac{w}{v}\right| < \frac{1}{2v^2}.$$

It follows from the elementary theory of continued fractions (e.g., see Th. 184 in [5]) that the fraction w/v satisfies

$$\frac{w}{v} = \frac{p_n}{q_n} \tag{2.7}$$

for some convergent p_n/q_n of η. One may exclude the case where $n = 0$, since otherwise it follows from (2.7) and $q_0 = 1$ that $v \mid w$. The integer w was defined by $ws = u - hv$, hence v divides u. This is a contradiction to the hypothesis on u and v, because we have deduced from $n = 0$ that $(u, v) = |v| > 1$. Therefore, we may assume $n > 0$ in (2.7). By (2.6), either p or q divides q_n. Since p_n and q_n are coprime, (2.7) implies that v is divisible by the same primes that also divide q_n. From $pq \mid s$ and $u \equiv hv \bmod s$, it follows that $(u, v) > 1$, a contradiction. It is proved that the integers u and v cannot exist, and the proof of Theorem 1.2 is complete.

3. PROOF OF THEOREM 1.3

Let a and b be integers with $a > 0, b \neq 0$. η denotes any real irrational number. In what follows, we consider two consecutive convergents $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$ of η. For every integer $n \geq 1$ satisfying $aq_n + bq_{n-1} \neq 0$, we define

$$\lambda_n := 1 + \frac{a}{b\alpha_{n+1} - a} - \frac{b}{a\beta_n + b}, \tag{3.1}$$

where $\alpha_{n+1} := [a_{n+1}; a_{n+2}, a_{n+3}, \ldots]$ and $\beta_n := [a_n; a_{n-1}, a_{n-2}, \ldots, a_1]$. From $\alpha_{n+1} \notin \mathbb{Q}$, we have $b\alpha_{n+1} - a \neq 0$; it follows from

$$\beta_n = \frac{q_n}{q_{n-1}} \quad (n \geq 1)$$

and $aq_n + bq_{n-1} \neq 0$ that $a\beta_n + b \neq 0$.

Proposition 3.1: Let $n \geq 1$ and $\gamma := \operatorname{sign}(b\lambda_n)$. Then we have

$$\left|\eta - \frac{ap_n + bp_{n-1}}{aq_n + bq_{n-1}}\right| = \frac{\gamma ab}{\lambda_n (aq_n + bq_{n-1})^2}.$$

This is Proposition 2.1 in [2] apart from different notations concerning α_n, β_n, and η.

At the beginning of the proof of Theorem 1.3, we apply Uchiyama's result mentioned in the Introduction. By (1.1) and (1.2), there are infinitely many integers u_0 and $v_0 \neq 0$ such that

$$\left|\xi - \frac{u_0}{v_0}\right| < \frac{s^2}{4v_0^2} \tag{3.2}$$

and

$$u_0 \equiv h \mod s, \quad v_0 \equiv 1 \mod s. \tag{3.3}$$

Let $d := (u_0, v_0) > 0$. Every common prime divisor p of d and s is a divisor of v_0, too. This is impossible, because $v_0 \equiv 1 \bmod s$. Hence, d and s are coprime, and therefore an integer d_0 exists such that

$$d \cdot d_0 \equiv 1 \bmod s. \tag{3.4}$$

Moreover, there are coprime integers u and $v \neq 0$ satisfying $u_0 = du$ and $v_0 = dv$. Therefore, we have $d_0 u_0 = dd_0 u$ or $u \equiv hd_0 \bmod s$ by (3.3) and (3.4). Similarly, we conclude $v \equiv d_0 \bmod s$. Collecting together, we have proved the existence of infinitely many coprime integers u and $v \neq 0$ with $u \equiv hv \bmod s$ and, by (3.2),

$$\left|\xi - \frac{u}{v}\right| < \frac{s^2}{4v_0^2} \leq \frac{s^2}{4v^2}.$$

If it is $v < 0$, this result is also true for $-u$ and $-v$, and the assertion of the theorem is proved for $S = s^2/4$.

Now let $\eta := \frac{\xi - h}{s} = [a_0; a_1, a_2, \ldots]$, and let $\frac{p_n}{q_n}$ $(n \geq 0)$ denote the convergents of η. In what follows, we assume $n \geq 1$.

Case 1. $(q_{n-1}, s) = 1$.

Put $P_n := p_{n-1}$, $Q_n := q_{n-1}$. Then we have

$$(P_n, Q_n) = 1, \quad (Q_n, s) = 1, \quad \left|\eta - \frac{P_n}{Q_n}\right| < \frac{1}{Q_n^2}. \tag{3.5}$$

Case 2. $(q_{n-1}, s) > 1$ and $\delta(s) \nmid q_{n-1}$.

Let

$$a := \prod_{\substack{p \mid s \\ p \nmid q_{n-1}}} p, \quad P_n := ap_n + p_{n-1}, \quad Q_n := aq_n + q_{n-1}.$$

From the hypothesis of Case 2, we conclude that

$$a > 1. \tag{3.6}$$

By straightforward computations, one gets $q_n P_n - p_n Q_n = (-1)^n$, which implies that

$$(P_n, Q_n) = 1. \tag{3.7}$$

Let p denote any prime divisor of s. If p divides q_{n-1}, we conclude that a is not divisible by p. Moreover, p does not divide q_n because q_n and q_{n-1} are coprime. Finally, we get $p \nmid Q_n$.

Now, let p and q_{n-1} be coprime. Then we have $p \mid a$, and again p does not divide Q_n. Since p is an arbitrary prime divisor of s, we have proved that

$$(Q_n, s) = 1. \tag{3.8}$$

From the hypothesis $(q_{n-1}, s) > 1$, we know that a certain common prime divisor of q_{n-1} and s exists. This and (3.6) imply that

$$1 < a \leq \frac{\delta(s)}{p_0}, \tag{3.9}$$

where p_0 denotes the smallest prime divisor of s. We apply Proposition 3.1 with $b = 1$:

$$\left|\eta-\frac{P_n}{Q_n}\right| = \frac{a}{|\lambda_n|Q_n^2}, \tag{3.10}$$

where

$$\begin{aligned}\frac{1}{|\lambda_n|} &= \left|\frac{a(1+\alpha_{n+1}\beta_n)-a^2\beta_n+\alpha_{n+1}-2a}{a(1+\alpha_{n+1}\beta_n)}\right| \\ &= \left|1-\frac{2+a\beta_n-\frac{\alpha_{n+1}}{a}}{1+\alpha_{n+1}\beta_n}\right| = |1-\rho_n|.\end{aligned} \tag{3.11}$$

We are looking for a suitable upper bound of $|\lambda_n|^{-1}$. For this purpose, we separate the arguments into three cases.

Case 2.1. $2+a\beta_n-\dfrac{\alpha_{n+1}}{a}<0.$

For $n\geq 1$, it is clear that $\alpha_{n+1}>1$ and $\beta_n>1$. It follows from (3.11) that

$$|\lambda_n|^{-1}=1-\rho_n< 1+\frac{\alpha_{n+1}}{a(1+\alpha_{n+1}\beta_n)} < 1+\frac{\alpha_{n+1}}{a(1+\alpha_{n+1})} < 1+\frac{1}{a}<2. \tag{3.12}$$

Case 2.2. $0\leq 2+a\beta_n-\dfrac{\alpha_{n+1}}{a} \leq 1+\alpha_{n+1}\beta_n.$

Then we have $0\leq\rho_n\leq 1$, and consequently

$$|\lambda_n|^{-1}\leq 1. \tag{3.13}$$

Case 2.3. $2+a\beta_n-\dfrac{\alpha_{n+1}}{a}>1+\alpha_{n+1}\beta_n.$

We conclude that

$$|\lambda_n|^{-1}=\rho_n-1<\frac{2+a\beta_n}{1+\alpha_{n+1}\beta_n}-1<\frac{2+a\beta_n}{1+\beta_n}-1<\frac{2}{1+1}+\frac{a\beta_n}{1+\beta_n}-1<a. \tag{3.14}$$

We know that $a\geq 2$, from (3.6). Collecting together from (3.12) through (3.14) we have proved that $|\lambda_n|^{-1}<a$ holds for every integer $n\geq 1$. Hence, (3.10) yields

$$\left|\eta-\frac{P_n}{Q_n}\right| < \frac{a^2}{Q_n^2}. \tag{3.15}$$

Case 3. $\delta(s)\,|\,q_{n-1}.$

Since q_{n-1} and q_n are coprime, it follows from the hypothesis that $(q_n, s)=1$. Put $P_n:=p_n$ and $Q_n:=q_n$. Obviously, the assertions for P_n and Q_n from (3.5) hold.

We collect together the results from (3.5), (3.7), (3.8), and (3.15): For a certain sequence of increasing integers $n\geq 1$, we get a sequence of rationals $(P_\nu/Q_\nu)_{\nu\geq 1}$ with coprime integers P_ν and Q_ν such that $(Q_\nu, s)=1$ $(\nu\geq 1)$,

$$Q_1\leq Q_2\leq Q_3\leq\cdots\leq Q_\nu\to\infty$$

and

$$\left|\frac{\xi-h}{s}-\frac{P_\nu}{Q_\nu}\right|<\frac{a^2}{Q_\nu^2}\quad(\nu\geq 1).$$

Let $u := hQ_v + sP_v,\ v := Q_v$. Then, by the upper bound for a from (3.9), we have

$$\left|\xi - \frac{u}{v}\right| < \frac{s\delta^2(s)}{p_0^2 v^2},$$

where $u \equiv hv \bmod s$. We conclude from $(Q_v, sP_v) = 1$ that u and v are coprime. Since Q_v can be chosen as large as possible, the assertion of Theorem 1.3 is also proved for $S = s \cdot \delta^2(s) \cdot p_0^{-2}$.

4. CONCLUDING REMARK

Using the well-known continued fraction expansion of Euler's number e, the author obtained the following result.

Theorem: For every integer $s \geq 2$ there are infinitely many fractions P/Q with coprime integers $P, Q > 0$ satisfying $P \equiv Q \equiv 1 \bmod s$ and $Q \cdot |Qe - P| = o(1)$ for $Q \to \infty$.

ACKNOWLEDGMENT

I take this opportunity to thank Professor R. C. Vaughan, who suggested to me the basic concept for improving my former results on this topic of diophantine approximation.

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AMS Classification Numbers: 11J04, 11J70

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ON THE DEGREE OF THE CHARACTERISTIC POLYNOMIAL OF POWERS OF SEQUENCES

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In [2], Cooper and Kennedy considered the following question: If $\{U_n\}$ is a sequence satisfying a third-order linear recurrence, what is the degree of the recurrence satisfied by the sequence $\{(U_n)^2\}$? They gave the answer as 6. They then asked if there is a similar result for the sequence $\{(U_n)^3\}$, tossing this question out as a research problem.

In [4], Prodinger answered this latter question in the affirmative, along with the more general question dealing with linear recurrences of any order and arbitrary powers of the original sequence. In the case of the familiar Fibonacci (or Lucas) sequence (where the original sequence satisfies a second-order linear recurrence), Prodinger displayed the recurrences satisfied by $\{(F_n)^k\}$ (or $\{(L_n)^k\}$) for $k = 1, 2, 3, 4, 5, 6$, showing that such recurrences are all linear and of order $(k+1)$. As Cooper and Kennedy had observed in [2], these latter recurrences had been obtained by D. Jarden [3] and are special cases of the following formula:

$$\sum_{j=0}^{k+1}(-1)^{j(j+1)/2}[k+1, j]_F(F_{n-j})^k = 0, \quad k = 1, 2, \dots; n \text{ any integer.} \tag{1}$$

In this formula, the quantities $[k, j]_F$ are the *Fibonomial coefficients* defined by:

$$[k, j]_F \equiv [k!]_F / \{[j!]_F[(k-j)!]_F\},$$

where $0 \le j \le k$, with $[m!]_F \equiv F_1F_2F_3\dots F_m$, $m \ge 1$, and $[0!]_F = 1$. A table of Fibonomial coefficients is provided in Brother Alfred Brousseau's compendium [1]. The formula in (1) is a special case of a more general formula (omitted here) due to Jarden and given in [3], involving certain sequences satisfying a second-order linear recurrence.

It should be added that although Prodinger demonstrated the existence of the order of certain linear recurrences in more general cases than was explored by Cooper and Kennedy, he did not actually derive an exact expression for such order. We rectify this omission in this paper, and extend such result to an even more general situation.

It seems natural to ask whether we can find similar results for the most general type of sequence satisfying a linear recurrence. It will be noted from recurrence theory that any sequence satisfying a linear recurrence possesses a characteristic polynomial of a certain degree with eigenvalues (also known as characteristic roots) of possibly multiple order. In general, such sequence is *nonlinear*. More specifically, we consider a sequence $\{U_n\}$ of the following known form:

$$U_n = \sum_{j=1}^{m}\theta_j(n)(\alpha_j)^n, \tag{2}$$

where the $\theta_j(n)$ are given polynomials in n of degree r_j (with $r_j \ge 0$), and the α_j's are distinct given constants. Such sequences are denoted as *polynomial sequences*. Incidentally, we note that, from the known expression for U_n, we may immediately write the characteristic polynomial $P_1(z)$ of the sequence, namely:

$$P_1(z) = \prod_{j=1}^{m} (z - \alpha_j)^{1+r_j}. \tag{3}$$

Observe that the sequence $\{(U_n)^k\}$ $(k = 1, 2, 3, \dots)$ also possesses a characteristic polynomial, which we denote by $P_k(z)$. We let R_k represent the degree of $P_k(z)$. By definition of the characteristic polynomial, $P_k(z)$ is the *minimum* polynomial such that $P_k(E)(U_n^k) = 0$ (where E is the unit shift operator, i.e., $Ex_n = x_{n+1}$). In other words, R_k is the *order* of the recurrence satisfied by the k^{th} power of the original sequence. Our task is thus to determine k^{th} for $k = 1, 2, 3, \dots$.

Indeed (given (3)), we immediately determine that

$$R_1 = \sum_{j=1}^{m} (1 + r_j). \tag{4}$$

We claim the following main result:

Theorem:

$$R_k = (R_1 - m)\binom{k+m-1}{k-1} + \binom{k+m-1}{k}. \tag{5}$$

In particular, if $r_j = 0$ for $j = 1, 2, \dots, m$, then the characteristic roots are of order one and $R_1 = m$; in this case,

$$R_k = \binom{k+m-1}{k}. \tag{6}$$

This latter result is clearly a corollary of the Theorem. If the original recurrence has characteristic roots of single order, then the characteristic roots of the "power recurrence" are also of single order. For the particular case where $R_1 = m = 2$. we obtain Prodinger's implied result: $R_k = k + 1$.

Proof of (5): We begin by expanding the k^{th} power of the given expression for U_n, using the multinomial theorem:

$$(U_n)^k = \sum_{S(m,k)} \binom{k}{i_1, i_2, \dots, i_m} \{\theta_1(n)(\alpha_1)^n\}^{i_1} \{\theta_2(n)(\alpha_2)^n\}^{i_2} \cdots \{\theta_m(n)(\alpha_m)^n\}^{i_m},$$

where $S(m, k) = \{(i_1, i_2, \dots, i_m) : i_1 + i_2 + \dots + i_m = k,\ 0 \le i_j \le k,\ j = 1, 2, \dots, m\}$, and $\binom{k}{i_1, i_2, \dots, i_m}$ is the multinomial coefficient evaluated as $k!/\{(i_1)!(i_2)!\dots(i_m)!\}$. Note that

$$\text{degree}[\{\theta_1(n)\}^{i_1} \{\theta_2(n)\}^{i_2} \cdots \{\theta_m(n)\}^{i_m}] = \sum_{j=1}^{m} r_j i_j.$$

We see that $P_k(z) = \prod_{S(m,k)} \{z - (\alpha_1)^{i_1} (\alpha_2)^{i_2} \cdots (\alpha_m)^{i_m}\}^{E(i_1, i_2, \dots, i_m)}$, where

$$E(i_1, i_2, \dots, i_m) = 1 + \sum_{j=1}^{m} r_j i_j. \tag{7}$$

Therefore,

$$R_k = \sum_{S(m,k)} E(i_1, i_2, \dots, i_m). \tag{8}$$

It remains to evaluate the last expression. Towards this end, we employ a pair of lemmas. For convenience, we let $U(m, k)$ denote

$$|S(m,k)| = \sum_{S(m,k)} 1,$$

the cardinality of $S(m,k)$, and

$$V(m,k) = \sum_{S(m,k)} i_1.$$

It follows (by symmetry) that

$$V(m,k) = \sum_{S(m,k)} i_j, \ \ j = 1, 2, \dots, m.$$

Therefore, we see from (7) and (8) that $R_k = U(m,k) + V(m,k)\sum_{j=1}^m r_j$, or

$$R_k = U(m,k) + (R_1 - m)V(m,k). \tag{9}$$

Lemma 1:

$$U(m,k) = \binom{k+m-1}{k}. \tag{10}$$

Proof (by induction on m): Let K denote the set of $m \geq 1$ such that (10) is true (k being treated as fixed). Since $S(1,k) = \{k\}$, we see that $U(1,k) = 1 = \binom{k}{k}$; therefore, $1 \in K$. Suppose $1, 2, \dots, m \in K$. Now $S(m+1, k)$ consists of those vectors in ε^{m+1} which have their first component equal to i_1 and the remaining vector (an element of ε^m) equal to a vector in $S(m, k-i_1)$. Since i_1 varies from 0 to k, inclusive, it follows that

$$U(m+1,k) = \sum_{j=0}^{k} U(m,j). \tag{11}$$

By the inductive hypothesis,

$$U(m+1,k) = \sum_{j=0}^{k}\binom{j+m-1}{j} = \sum_{j=m-1}^{k+m-1}\binom{j}{m-1} = \binom{k+m}{m} = \binom{k+m}{k}.$$

We see that this result is the statement of (10) for $(m+1)$. Thus,

$$1, 2, \dots, m \in K \Rightarrow 1, 2, \dots, m, m+1 \in K.$$

Induction completes the proof. □

Lemma 2:

$$V(m,k) = \binom{k+m-1}{k-1}. \tag{12}$$

Proof: Reasoning as in the proof of Lemma 1,

$$V(m,k) = \sum_{j=0}^{k}(k-j)U(m-1,j).$$

Using the result of Lemma 1 and standard combinatorial manipulations,

$$V(m,k) = \sum_{j=0}^{k-1}(k-j)\binom{j+m-2}{j} = k\sum_{j=m-2}^{k+m-3}\binom{j}{m-2} - (m-1)\sum_{j=m-1}^{k+m-3}\binom{j}{m-1}.$$

Then

$$V(m,k) = k\binom{k+m-2}{m-1} - (m-1)\binom{k+m-2}{m} = \binom{k+m-1}{k-1},$$

after simplification. Substituting the results of Lemmas 1 and 2 into (9) yields the Theorem. □

As an illustration of our formula, consider the original sequence to be $\{U_n\} = \{n^2\}$. In this case, $P_1(z) = (z-1)^3$; hence, $m = 1$, $\alpha_1 = 1$, $r_1 = 2$, $R_1 = 3$. In other words, U_n satisfies the third-order linear recurrence: $U_{n+3} - 3U_{n+2} + 3U_{n+1} - U_n = 0$. Then $(U_n)^k = n^{2k}$, for which the characteristic polynomial $P_k(z) = (z-1)^{2k+1}$, and $R_k = 2k+1$. In particular, $R_2 = 5 \neq 6$. Thus, the result of Cooper and Kennedy [2] needs to be modified somewhat. Although it is true that the square of a sequence satisfying a third-order linear recurrence satisfies a linear recurrence of order 6, it may happen that such square sequence in fact satisfies a linear recurrence of order 5; in such case, its characteristic (i.e., *minimal*) polynomial has degree 5, rather than 6. Similar anomalies occur when the original recurrence has characteristic roots or multiplicity greater than one. The main theorem given in this paper treats all such cases with full generality, giving the *minimum* order of the appropriate recurrence. It needs to be emphasized, however, that this order is known only if the characteristic roots of the original sequence and their multiplicities are known in advance (or, equivalently, if the characteristic polynomial is known in advance, along with all of its factors).

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AMS Classification Numbers: 11B39, 11B37, 11B65

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ON t-CORE PARTITIONS

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(*Submitted April 1998-Final Revision July 1998*)

1. INTRODUCTION

Consider the partition of the natural number n given by

$$n = n_1 + n_2 + \cdots + n_s, \tag{I}$$

where $s \geq 1$ and $n_1 \geq n_2 \geq \cdots \geq n_s$. The Young diagram of this partition consists of the nodes (i, j), where $1 \leq i \leq s$, and for each fixed i, $1 \leq j \leq n_i$. The rightmost node in row i, namely (i, n_i), is called a *hand*. The lowest node in a given column is called a *foot*. At least one node, namely (s, n_s) is both a hand and a foot.

A hand (i, n_i) and a foot (k, j) may be connected by what is known as a hook as follows. Let an *arm* consist of the nodes (i, m) such that $j \leq m \leq n_i$; let a *leg* consist of the nodes (h, j) such that $i \leq h \leq k$. The hook is the union of all nodes in the arm and leg. The corresponding hook number (or hook length) is the number of nodes in the hook, namely $n_i - j + k - i + 1$.

Let the integer $t \geq 2$. We say that a partition is *t-core* if none of the hook numbers are divisible by t. Note that t-core partitions arise in the representation theory of the symmetric group (see [5]); such partitions have also been used to provide new proofs of some well-known results of Ramanujan (see [1]). Let $c_t(n)$ denote the number of t-core partitions of n. It is well known that

$$c_2(n) = \begin{cases} 1 & \text{if } n = \frac{1}{2}m(m+1), \\ 0 & \text{otherwise.} \end{cases}$$

If $n = \frac{1}{2}m(m+1)$, then the unique 2-core partition of n is given by

$$n = m + (m-1) + (m-2) + \cdots + 2 + 1. \tag{II}$$

Recently, Granville and Ono [2] have shown that if $t \geq 4$, then $c_t(n) > 0$ for all n.

In this note, we completely characterize 3-core partitions. We show that they are linked to the quadratic form $x^2 + 3y^2$. As a result, we obtain an independent derivation of Granville and Ono's formula for $c_3(n)$ (see [2]). Finally, we derive recurrences that permit the evaluation of $c_4(n)$ and $c_5(n)$. Note that whereas the formula for $c_5(n)$ given by Garvan et al. [1] requires the canonical factorization of $n+1$, our method for computing $c_5(n)$ does not. We also tabulate these three functions, as well as some related functions, in the ranges $1 \leq n \leq 100$ and $1 \leq n \leq 50$.

2. PRELIMINARIES

Let the integer $n \geq 0$, let the integer $t \geq 2$, let p denote an odd prime, and let x be a complex variable with $|x| < 1$.

Definition 1: Let $c_t(n)$ denote the number of t-core partitions of n.

Definition 2: Let $b_t(n)$ denote the number of partitions of n such that no part is divisible by t.

Definition 3: Let $(a/p) = \begin{cases} \text{Legendre symbol} & \text{if } p \nmid a, \\ 0 & \text{if } p \mid a. \end{cases}$

Definition 4: Let $E(n) = \frac{1}{2}n(3n-1)$.

Lemma:

(1) $\sum_{n=0}^{\infty} c_t(n)x^n = \prod_{n=1}^{\infty}(1-x^{tn})^t/(1-x^n)$;

(2) $\sum_{n=0}^{\infty} b_t(n)x^n = \prod_{n=1}^{\infty}(1-x^{tn})/(1-x^n)$;

(3) $b_t(n) = p(n) + \sum_{k\geq 1}(-1)^k(p(n-tE(k)) + p(n-tE(-k)))$;

(4) $\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)x^{\frac{1}{2}k(k+1)}$;

(5) If $n \equiv 4 \pmod 5$, then $p(n) \equiv 0 \pmod 5$.

Remarks: The identities (1) and (2) are well known (see [1], [2], and [7]). Note that (3) follows from (2), (4) is due to Jacobi, and (5) is due to Ramanujan.

Notation: Suppose that a partition of n has r distinct parts and that the summand n_i occurs k_i times, where $1 \leq i \leq r$. Then we occasionally write

$$n = \prod_{i=1}^{r} n_i^{k_i}.$$

Theorem 1: Conjugate partitions have the same hook numbers.

Proof: If $n \geq 1$, consider the map that sends each partition of n to its conjugate. Thus, hands are interchanged with feet, arms with legs, and hooks with hooks having the same hook numbers.

Theorem 2: $c_t(n) = \begin{cases} p(n) & \text{if } n < t, \\ p(t) - t & \text{if } n = t. \end{cases}$

Proof: We define $c_t(0) = p(0) = 1$. If $1 \leq n \leq t-1$, then each hook in a partition of n has length at most $t-1$, so every partition of n is t-core, so $c_t(n) = p(n)$. Now let $n = t$. Each partition $t = (t-j)1^j$, where $0 \leq j \leq t-1$, has a t-hook and thus is not t-core. On the other hand, if the least part in a partition of t is strictly between 1 and t, then each hook number is at most $t-1$, so the partition is t-core. Therefore, $c_t(t) = p(t) - t$.

3. 3-CORE PARTITIONS

By means of Theorems 3 through 8 below, we characterize all 3-core partitions.

Theorem 3: Each of the following partitions is 3-core:

(a) $n = 2m(2m-2)(2m-4)\cdots(4)(2)$;

(b) $n = (2m-1)(2m-3)(2m-5)\ldots(3)(1)$;

(c) $n = m^2(m-1)^2\ldots 2^2 1^2$;

(d) $n = m(m-1)^2(m-2)^2\ldots 2^2 1^2$.

Proof: Since the partitions in (c) and (d) are the conjugates of those in (a) and (b), it suffices, by virtue of Theorem 1, to prove (a) and (b). We first prove (a) by induction on m. The statement is true by Theorem 2 when $m=1$. Let $n'=(2m+2)(2m)(2m-2)\dots(4)(2)$. If we omit the first row or the first two columns in the Young diagram for n', we obtain the Young diagram for n. Therefore, by the induction hypothesis, it suffices to show that all hooks from the new hand, namely $(1, 2m+2)$, to the feet in the last row, namely $(m+1, 1)$ and $(m+1, 2)$, have hook numbers not divisible by 3. These hook numbers are $3m+2$ and $3m+1$, respectively, so we are done.

We sketch the proof of (b), which is similar. Again, the statement is true for $m=1$ by Theorem 2. Let $n''=(2m+1)(2m-1)\dots(3)(1)$. We need only note that the hook from the new hand, namely $(1, 2m+1)$, to the lowest foot, namely $(m+1, 1)$, has hook number $=3m+1$.

Theorem 4: Let $r \geq 1$ and $m \geq 1$. Then each of the following partitions is 3-core:

(a) $n=(m+2r)(m+2r-2)\dots(m+2)m^2(m-1)^2\dots 2^2 1^2$;

(b) $n=(m+2r-1)(m+2r-3)\dots(m+1)m^2(m-1)^2\dots 2^2 1^2$.

Proof: For (a), look at the corresponding Young diagram. By Theorem 3, any hook that occurs entirely in the first r rows or in the last $2m$ rows has length not divisible by 3. Therefore, it suffices to consider hooks from a hand in the first r rows to a foot in the last $2m$ rows. Such a hand has coordinates $(i, m+2r+2-2i)$, where $1 \leq i \leq r$; such a foot has coordinates $(r+2j, m+1-j)$, where $1 \leq j \leq m$. The corresponding hook has length $3(r-i+j)+2$, so we are done.

The proof for (b) is similar. A hand from the first r rows has coordinates $(i, m+2r+1-2i)$, where $1 \leq i \leq r$. Again, a foot from the last $2m$ rows has coordinates $(r+2j, m+1-j)$, where $1 \leq j \leq m$, so the corresponding hook has length $3(r-i+j)+1$.

Theorem 5: Let $n=n_1+n_2+\dots+n_s$ be a 3-core partition of n, where $s \geq 1$. Then the following must hold:

(a) $n_s \leq 2$.

(b) If $n \geq 3$, then $s \geq 2$.

(c) $n_i - n_{i+1} \leq 2$ for all i such that $1 \leq i \leq s-1$.

(d) Each part occurs at most twice.

(e) If $n_{i+1}=n_i$, then either (i) $1 \leq i \leq s-2$ and $n_{i+2}=n_{i+1}-1$ or (ii) $i=s-1$ and $n_{s-1}=1$.

(f) If $n_{i+1}=n_i-1$, then $1 \leq i \leq s-2$ and $n_{i+2}=n_{i+1}$.

Proof: A partition such that any of (a) through (f) fails to hold has a hook of length 3.

Theorem 6: $c_3(n)$ is the number of distinct ways that n can be represented in the form

$$n=r(r+m+k)+m(m+1),$$

where $k=0$ or 1, $r \geq 0, m \geq 0$, and $rm > 0$. For each such representation, the corresponding 3-core partition of n is given by

$$n=(m+2r+k-1)(m+2r+k-3)\dots(m+k+1)m^2(m-1)^2\dots 2^2 1^2.$$

Proof: The conclusion follows from Theorems 4 and 5, and from the hypothesis.

Remark: Note that m is the number of parts that occur twice, while r is the number of parts that occur once.

Theorem 7: n has a self-conjugate 3-core partition iff there exists $r \geq 1$ such that $n = 3(3r \pm 2)$. If such a self-conjugate 3-core partition of r exists, then it is unique.

Proof: If n has a self-conjugate 3-core partition, then the number of parts must equal the largest part. Therefore, by Theorem 6, we must have $r + 2m = m + 2r + k - 1$, with k, m, and r as in the hypothesis of Theorem 6. Thus, $m = r + k - 1$. If $k = 0$, then $n = r(2r-1) + r(r-1) = r(3r-2)$; if $k = 1$, then $n = r(2r+1) + r(r+1) = r(3r+2)$. Conversely, the partitions

$$n = (3r-2)(3r-4)\ldots(r+2)(r(r-1)^2(r-2)^2\ldots 1^2$$

and

$$n = 3r(3r-2)\ldots(r+2)r^2(r-1)^2(r-2)^2\ldots 1^2$$

are 3-core by Theorem 4, and are self-conjugate. Uniqueness follows from the fact that n has at most a single representation, $n = r(3r \pm 2)$.

Corollary 1: $c_3(n) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = r(3r \pm 2), \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$

Proof: This follows from Theorems 1 and 7.

Corollary 2: $c_3(n)$ changes parity infinitely often as n tends to infinity.

Proof: This follows from Corollary 1.

Theorem 8: $c_3(n)$ is the number of solutions of the equation $x^2 + 3y^2 = 12n + 4$ such that $x \geq 1$ and $y \geq [n^{1/2}]$ if $n > 0$.

Proof: By Theorem 6, each 3-core partition of n corresponds to a solution of

$$n = r(r + m + k) + m(m+1),$$

where $k = 0$ or 1, $r \geq 0$, $m \geq 0$, and $rm > 0$. Let $v = m + k$, so $v \geq 0$. Then $n = r(r+v) + v(v \pm 1)$, so that $12n + 4 = (3v \pm 2)^2 + 3(v + 2r)^2$. Let $x = 3v + 2(-1)^k$ and $y = v + 2r$. This yields

$$x^2 + 3y^2 = 12n + 4.$$

If $v = 0$, then $m = k = 0$, so $x = 2$. If $v \geq 1$, then $x \geq 3v - 2 \geq 1$. Thus, in all cases, $x \geq 1$. Now suppose that $y < [n^{1/2}]$. Since $y = v + 2r$ and $r \geq 0$, this implies that $v < [n^{1/2}]$, so $v \leq [n^{1/2}] - 1$. Since $y < [n^{1/2}]$, we must have $x > 3n^{1/2}$, that is, $3v \pm 2 > 3n^{1/2}$, hence $v > n^{1/2} - \frac{2}{3}$. This implies that $n^{1/2} - [n^{1/2}] < -\frac{1}{3}$, an impossibility. Thus, $y \geq [n^{1/2}]$. Conversely, suppose that $x^2 + 3y^2 = 12n + 4$, where $x \geq 1$ and $y \geq [n^{1/2}]$. Since $3 \nmid x$, we may let $x = 3v + 2(-1)^k$, where v is an integer and $k = 0$ or 1. Since $x \equiv y \equiv v \pmod 2$, we may let $y = v + 2r$, where r is an integer.

If $k = 0$, then $v = (x-2)/3$, so $v \geq -\frac{1}{3}$. Since v is an integer, we have $v \geq 0$. If $k = 1$, then $v = (x+2)/3$, so $v \geq 1$. Let $m = v - k$. In either case, we have $m \geq 0$. Since $y \geq [n^{1/2}]$, we have $x^2 \leq 12n - 3[n^{1/2}]^2 + 4$, that is, $x^2 \leq 9n + 3(n - [n^{1/2}]^2) + 4$. But $n - [n^{1/2}]^2 \leq 2[n^{1/2}]$, so we have $x^2 \leq 9n + 6[n^{1/2}] + 4$. Hence $x^2 \leq (3n^{1/2} + 1)^2 + 3$, so that $x \leq ((3n^{1/2}+1)^2 + 3)^{1/2}$, which implies $x \leq 3n^{1/2} + 1$.

If $k=0$, we have $3v+2 \le 3n^{1/2}+1$, hence $v \le n^{1/2}-\frac{1}{3}$. Now $r=\frac{1}{2}(y-v)$, so

$$r \ge \tfrac{1}{2}([n^{1/2}]-n^{1/2}+\tfrac{1}{3}) > \tfrac{1}{2}(-\tfrac{2}{3}) = -\tfrac{1}{3}.$$

If $k=1$, we have $3v-2 \le 3n^{1/2}+1$, hence $v \le n^{1/2}+1$, $v \le [n^{1/2}]+1$. Thus,

$$r \ge \tfrac{1}{2}([n^{1/2}]-[n^{1/2}]-1),$$

that is, $r \ge -\frac{1}{2}$. In either case, since r is an integer, we must have $r \ge 0$.

Finally, if we let $x = 3v \pm 2$ and $y = v+2r$, and substitute into $x^2+3y^2 = 12n+4$, then, after simplifying, we obtain

$$n = v(v \pm 1) + r(r+v).$$

If $k=0$, then $v=m$, so

$$n = m(m+1)+r(r+m).$$

If $k=1$, then $v=m+1$, so

$$n = m(m+1)+r(r+m+1).$$

Thus, we have

$$n = m(m+1)+r(r+m+k).$$

Since $n>0$, we must have $rm>0$.

Lemma 1: Consider the equation

$$x^2+3y^2 = 12n+4. \qquad (*)$$

The number of solutions of $(*)$ such that $|y| \ge [n^{1/2}]$ is $4\sigma(3n+1)$, where $\sigma(n) = \Sigma\{(d/3) : d \mid n\}$. (Here we are following the notation of [2].)

Proof: Let $12n+4 = 2^k m$, where $k \ge 2$ and $2 \nmid m$. According to [4] (p. 308, Ex. 3), if j is the number of solutions of $(*)$, then $j = 6\sigma(3n+1)$. We must show that if j' is the number of solutions of $(*)$ such that $|y| \ge [n^{1/2}]$, then $j' = 4\sigma(3n+1)$.

Suppose that $x=a$, $y=b$ is a solution of $(*)$. Let $\omega = \exp(2\pi i/3)$. Passing to $Q(\omega)$, we have

$$(a+b\sqrt{-3})(a-b\sqrt{-3}) = 12n+4.$$

Let $z_1 = (a+b)+2b\omega = a+b\sqrt{-3}$. Then $N(z_1) = a^2+3b^2 = 12n+4$. However, $Q(\omega)$ has 6 units, namely, $\pm 1, \pm\omega, \pm\omega^2$, so we obtain additional solutions of $(*)$ corresponding to

$$z_2 = \omega z_1,\ z_3 = \omega^2 z_1,\ z_4 = -z_1,\ z_5 = -z_2,\ z_6 = -z_3.$$

Now $z_2 = -2b+(a-b)\omega$ and $z_3 = (b-a)-(a+b)\omega$, so it suffices to show that if $|y| < [n^{1/2}]$, then $|x \pm y| \ge 2[n^{1/2}]$. By hypothesis, we have $|x|^2+3|y|^2 = 12n+4$, so

$$|x|^2 = 12n+4-3|y|^2 > 12n-3[n^{1/2}]^2+4 \ge 9n+4.$$

Thus $|x| > 3n^{1/2}$. Now

$$|x \pm y| \ge |x| - |y| \ge 3n^{1/2} - [n^{1/2}] \ge 2[n^{1/2}],$$

so we are done.

Theorem 9: $c_3(n) = \sigma(3n+1) = \Sigma\{(d/3) : d \mid (3n+1)\}$.

Proof: This follows from Theorem 8 and Lemma 1, omitting solutions of (8) such that $x < 0$ or $y < 0$.

Remark: An alternate proof of Theorem 9, based on the theory of modular forms, was given in [2].

Theorem 10: If there exists $k \geq 1$ such that $3n \equiv 2^{2k-1} - 1 \pmod{2^{2k}}$, then $c_3(n) = 0$.

Proof: By Theorem 8 and [4] (p. 308, Ex. 3), we have $c_3(n) = 0$ if $12n + 4 = 2^{2k+1}m$, where $k \geq 1$ and $2 \nmid m$. That is, $c_3(n) = 0$ if $3n \equiv 2^{2k-1} - 1 \pmod{2^{2k}}$ for some $k \geq 1$.

Corollary 3: $c_3(n) = 0$ if $n \equiv 3 \pmod 4$, $n \equiv 13 \pmod{16}$, $n \equiv 53 \pmod{64}$, etc.

Proof: This follows from Theorem 10.

Theorem 11: $c_3(n)$ is unbounded as n tends to infinity.

Proof: Let $n = (7^{k-1} - 1)/3$. Then $c_3(n) = \sigma(7^{k-1}) = k$. Since k is arbitrary, we are done.

Table 1 below lists $c_3(n)$ for all n such that $1 \leq n \leq 100$.

TABLE 1

n	$c_3(n)$	n	$c_3(n)$	n	$c_3(n)$	n	$c_3(n)$
1	1	26	2	51	0	76	2
2	2	27	0	52	2	77	0
3	0	28	0	53	0	78	0
4	2	29	0	54	0	79	0
5	1	30	4	55	0	80	2
6	2	31	0	56	3	81	2
7	0	32	2	57	2	82	4
8	1	33	1	58	2	83	0
9	2	34	2	59	0	84	0
10	2	35	0	60	2	85	1
11	0	36	2	61	0	86	4
12	2	37	2	62	0	87	0
13	0	38	0	63	0	88	0
14	2	39	0	64	2	89	2
15	0	40	1	65	3	90	2
16	3	41	2	66	2	91	0
17	2	42	2	67	0	92	2
18	0	43	0	68	0	93	0
19	0	44	4	69	2	94	2
20	2	45	0	70	2	95	0
21	1	46	2	71	0	96	1
22	2	47	0	72	4	97	2
23	0	48	0	73	0	98	0
24	2	49	2	74	2	99	0
25	2	50	2	75	0	100	4

4. 4-CORE PARTITIONS

This subject has recently been explored in some detail (see [3] and [8]). The following theorem permits the evaluation of $c_4(n)$.

Theorem 12: $c_4(n)=\sum_{k=0}^{\infty}(1)^k(2k+1)b_4(n-2k(k+1))$.

Proof: Equation (1) implies

$$\begin{aligned}\sum_{n=0}^{\infty}c_4(n)x^n&=\prod_{n=1}^{\infty}(1-x^{4n})^4/(1-x^n)\\&=\prod_{n=1}^{\infty}(1-x^{4n})/(1-x^n)\prod_{n=1}^{\infty}(1-x^{4n})^3\\&=\left(\sum_{n=0}^{\infty}b_4(n)x^n\right)\left(\prod_{n=1}^{\infty}(1-x^{4n})^3\right)\end{aligned}$$

by (2). Let

$$g_4(n)=\begin{cases}(-1)^m(2m+1) & \text{if } n=2m(m+1),\\ 0 & \text{otherwise.}\end{cases}$$

Then (4) implies

$$\begin{aligned}\sum_{n=0}^{\infty}c_4(n)x^n&=\left(\sum_{n=0}^{\infty}b_4(n)x^n\right)\left(\sum_{n=0}^{\infty}g_4(n)x^n\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}b_4(n-k)g_4(k)\right)x^n.\end{aligned}$$

Matching coefficients of like powers of x, we get

$$c_4(n)=\sum_{k=0}^{\infty}b_4(n-k)g_4(k),$$

from which the conclusion follows.

5. 5-CORE PARTITIONS

Garvan, Kim, and Stanton [1] have shown that

$$c_5(n)=\sum_{d\,|\,(n+1)}(d/5)\frac{n+1}{d}.$$

In order to use this formula, one needs to know the divisors (or, equivalently, the canonical factorization) of $n+1$. We now present an alternative method of computing $c_5(n)$ that does not require factorization.

Theorem 13: Let

$$f_5(n)=b_5(n)+\sum_{k\geq 1}(-1)^k(b_5(n-5E(k))+b_5(n-5E(-k))).$$

Then

$$c_5(n)=\sum_{j=0}^{\infty}(-1)^j(2j+1)f_5(n-5j(j+1)/2).$$

Proof: Equation (1) implies

$$\begin{aligned}\sum_{n=0}^{\infty}c_5(n)x^n&=\prod_{n=1}^{\infty}(1-x^{5n})^5/(1-x^n)\\&=\prod_{n=1}^{\infty}(1-x^{5n})^2/(1-x^n)\prod_{n=1}^{\infty}(1-x^{5n})^3.\end{aligned}$$

Now

$$\begin{aligned}\prod_{n=1}^{\infty}(1-x^{5n})^2/(1-x^n)&=\prod_{n=1}^{\infty}(1-x^{5n})/(1-x^n)\prod_{n=1}^{\infty}(1-x^{5n})\\&=\left(\sum_{n=0}^{\infty}b_5(n)x^n\right)\prod_{n=1}^{\infty}(1-x^{5n})\\&=\sum_{n=0}^{\infty}f_5(n)x^n\end{aligned}$$

by (2) and the definition of $f_5(n)$. Also, by (4), we have

$$\prod_{n=1}^{\infty}(1-x^{5n})^3=\sum_{n=0}^{\infty}g_5(n)x^n,$$

where

$$g_5(n)=\begin{cases}(-1)^k(2k+1) & \text{if } n=5k(k+1)/2,\\ 0 & \text{otherwise.}\end{cases}$$

Thus, we have

$$\begin{aligned}\sum_{n=0}^{\infty}c_5(n)x^n&=\left(\sum_{n=0}^{\infty}f_5(n)x^n\right)\left(\sum_{n=0}^{\infty}g_5(n)x^n\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}f_5(n-k)g_5(k)\right).\end{aligned}$$

Matching coefficients of like powers of x, we obtain

$$c_5(n)=\sum_{k=0}^{n}f_5(n-k)g_5(k),$$

from which the conclusion follows.

Table 2 below lists $b_4(n)$, $c_4(n)$, $b_5(n)$, $f_5(n)$, and $c_5(n)$ for each n such that $1\le n\le 50$.

Our final theorem is inspired by examination of Table 2.

Theorem 16: If $n\equiv 4 \pmod 5$, then $b_5(n)\equiv f_5(n)\equiv c_5(n)\equiv 0 \pmod 5$.

Proof: By virtue of Theorem 15 and the definition of $f_5(n)$, it suffices to show that $b_5(n)\equiv 0 \pmod 5$ when $n\equiv 4 \pmod 5$. This follows from (3) and (5).

TABLE 2

n	$b_4(n)$	$c_4(n)$	$b_5(n)$	$f_5(n)$	$c_5(n)$
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	1	5	5	5
5	6	3	6	5	2
6	9	3	10	9	6
7	12	3	13	11	5
8	16	4	19	16	7
9	22	4	25	20	5
10	29	2	34	27	12
11	38	2	44	33	6
12	50	7	60	45	12
13	64	3	76	54	6
14	82	5	100	70	10
15	105	6	127	87	11
16	132	2	164	110	16
17	166	4	205	132	7
18	208	7	262	167	20
19	258	3	325	200	15
20	320	4	409	248	12
21	395	7	505	297	12
22	484	5	628	363	22
23	592	8	769	431	10
24	722	5	950	525	25
25	876	4	1156	621	12
26	1060	4	1414	746	20
27	1280	8	1713	882	18
28	1539	5	2081	1053	30
29	1846	6	2505	1235	10
30	2210	7	3026	1467	32
31	2636	2	3625	1716	21
32	3138	9	4352	2024	24
33	3728	11	5192	2361	16
34	4416	3	6200	2770	30
35	5222	8	7364	3217	21
36	6163	9	8756	3762	36
37	7256	4	10357	4354	20
38	8528	6	12258	5064	24
39	10006	5	14450	5850	25
40	11716	7	17034	6777	42
41	13696	5	20006	7799	12
42	15986	14	23500	9009	42
43	18624	7	27510	10341	36
44	21666	4	32200	11900	35
45	25169	10	37582	13627	22
46	29190	5	43846	15633	46
47	33808	10	51022	17583	22
48	39104	11	59353	20430	43
49	45164	3	68875	23275	25
50	52098	9	79888	26555	32

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AMS Classification Number: 11P81

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EQUATIONS INVOLVING ARITHMETIC FUNCTIONS OF FIBONACCI AND LUCAS NUMBERS

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For any positive integer k, let $\phi(k)$ and $\sigma(k)$ be the number of positive integers less than or equal to k and relatively prime to k and the sum of divisors of k, respectively.

In [6] we have shown that $\phi(F_n) \geq F_{\phi(n)}$ and that $\sigma(F_n) \leq F_{\sigma(n)}$ and we have also determined all the cases in which the above inequalities become equalities. A more general inequality of this type was proved in [7].

In [8] we have determined all the positive solutions of the equation $\phi(x^m - y^m) = x^n + y^n$ and in [9] we have determined all the integer solutions of the equation $\phi(|x^m + y^m|) = |x^n + y^n|$.

In this paper, we present the following theorem.

Theorem:

(1) The only solutions of the equation

$$\phi(|F_n|) = 2^m, \tag{1}$$

are obtained for $n = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 9$.

(2) The only solutions of the equation

$$\phi(|L_n|) = 2^m, \tag{2}$$

are obtained for $n = 0, \pm 1, \pm 2, \pm 3$.

(3) The only solutions of the equation

$$\sigma(|F_n|) = 2^m, \tag{3}$$

are obtained for $n = \pm 1, \pm 2, \pm 4, \pm 8$.

(4) The only solutions of the equation

$$\sigma(|L_n|) = 2^m, \tag{4}$$

are obtained for $n = \pm 1, \pm 2, \pm 4$.

Let $n \geq 3$ be a positive integer. It is well known that the regular polygon with n sides can be constructed with the ruler and the compass if and only if $\phi(n)$ is a power of 2. Hence, the above theorem has the following immediate corollary.

Corollary:

(1) The only regular polygons that can be constructed with the ruler and the compass and whose number of sides is a Fibonacci number are the ones with 3, 5, 8, and 34 sides, respectively.

(2) The only regular polygons that can be constructed with the ruler and the compass and whose number of sides is a Lucas number are the ones with 3 and 4 sides, respectively.

The question of finding all the regular polygons that can be constructed with the ruler and the compass and whose number of sides n has various special forms has been considered by us previously. For example, in [10] we found all such regular polygons whose number of sides n belongs to the Pascal triangle and in [11] we found all such regular polygons whose number of sides n is a difference of two equal powers.

We begin with the following lemmas.

Lemma 1:

(1) $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$.

(2) $2F_{m+n} = F_m L_n + L_n F_m$ and $2L_{m+n} = 5F_m F_n + L_m L_n$.

(3) $F_{2n} = F_n L_n$ and $L_{2n} = L_n^2 + 2(-1)^{n+1}$.

(4) $L_n^2 - 5F_n^2 = 4(-1)^n$.

Proof: See [2]. □

Lemma 2:

(1) Let $p > 5$ be a prime number. If $\left(\frac{5}{p}\right) = 1$, then $p \mid F_{p-1}$. Otherwise, $p \mid F_{p+1}$.

(2) $(F_m, F_n) = F_{(m,n)}$ for all positive integers m and n.

(3) If $m \mid n$ and n/m is odd, then $L_m \mid L_n$.

(4) Let p and n be positive integers such that p is an odd prime. Then $(L_p, F_n) > 2$ if and only if $p \mid n$ and n/p is even.

Proof: (1) follows from Theorem XXII in [1].

(2) follows either from Theorem VI in [1] or from Theorem 2.5 in [3] or from the Main Theorem in [12].

(3) follows either from Theorem VII in [1] or from Theorem 2.7 in [3] or from the Main Theorem in [12].

(4) follows either from Theorem 2.9 in [3] or from the Main Theorem in [12]. □

Lemma 3: Let $k \geq 3$ be an integer.

(1) The period of $(F_n)_{n\geq 0}$ modulo 2^k is $2^{k-1} \cdot 3$.

(2) $F_{2^{k-2}\cdot 3} \equiv 2^k \pmod{2^{k+1}}$. Moreover, if $F_n \equiv 0 \pmod{2^k}$, then $n \equiv 0 \pmod{2^{k-2} \cdot 3}$.

(3) Assume that n is an odd integer such that $F_n \equiv \pm 1 \pmod{2^k}$. Then $F_n \equiv 1 \pmod{2^k}$ and $n \equiv \pm 1 \pmod{2^{k-1} \cdot 3}$.

Proof: (1) follows from Theorem 5 in [13].

(2) The first congruence is Lemma 1 in [4]. The second assertion follows from Lemma 2 in [5].

(3) We first show that $F_n \not\equiv -1 \pmod{2^k}$. Indeed, by (1) above and the Main Theorem in [4], it follows that the congruence $F_n \equiv -1 \pmod{2^k}$ has only one solution $n \pmod{2^{k-1} \cdot 3}$. Since $F_{-2} = -1$, it follows that $n \equiv -2 \pmod{2^{k-1} \cdot 3}$. This contradicts the fact that n is odd.

We now look at the congruence $F_n \equiv 1 \pmod{2^k}$. By (1) above and the Main Theorem in [4], it follows that this congruence has exactly three solutions $n \pmod{2^{k-1} \cdot 3}$. Since $F_{-1} = F_1 = F_2 = 1$, it follows that $n \equiv \pm 1, 2 \pmod{2^{k-1} \cdot 3}$. Since n is odd, it follows that $n \equiv \pm 1 \pmod{2^{k-1} \cdot 3}$. □

Lemma 4: Let $k \geq 3$ be a positive integer. Then

$$L_{2^k} \equiv \begin{cases} 2^{k+1}3 - 1 \pmod{2^{k+4}} & \text{if } k \equiv 1 \pmod 2, \\ 2^{k+1}5 - 1 \pmod{2^{k+4}} & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

Proof: One can check that the asserted congruences hold for $k = 3$ and 4. We proceed by induction on k. Assume that the asserted congruence holds for some $k \geq 3$.

Suppose that k is odd. Then $L_{2^k} = 2^{k+1}3 - 1 + 2^{k+4}l$ for some integer l. Using Lemma 1(3), it follows that

$$\begin{aligned} L_{2^{k+1}} = L_{2^k}^2 - 2 &= ((2^{k+1}3-1)^2 + 2^{k+5}l(2^{k+1}3-1)^2 + 2^{2k+8}l^2) - 2 \\ &\equiv (2^{k+1}3-1)^2 - 2 \pmod{2^{k+5}}. \end{aligned}$$

Hence,

$$L_{2^{k+1}} \equiv 2^{2k+2}9 - 2^{k+2}3 + 1 - 2 \equiv 2^{2k+2}9 + 2^{k+2}(-3) - 1 \pmod{2^{k+5}}.$$

Since $k \geq 3$. it follows that $2k+2 \geq k+5$. Moreover, since $-3 \equiv 5 \pmod{2^3}$, the above congruence becomes

$$L_{2^{k+1}} \equiv 2^{k+2}5 - 1 \pmod{2^{k+5}}.$$

The case k even can be dealt with similarly. □

Proof of the Theorem: In what follows, we will always assume that $n \geq 0$.

(1) We first show that if $\phi(F_n) = 2^m$, then the only prime divisors of n are among the elements of the set $\{2, 3, 5\}$. Indeed, assume that this is not the case. Let $p > 5$ be a prime number dividing n. Since $F_p \mid F_n$, it follows that $\phi(F_p) \mid \phi(F_n) = 2^m$. Hence, $\phi(F_p) = 2^{m_1}$. It follows that

$$F_p = 2^l p_1 \cdot \dots \cdot p_k, \tag{5}$$

where $l > 0$, $k > 0$, and $p_1 < p_2 < \dots < p_k$ are Fermat primes.

Notice that $l = 0$ and $p_1 > 5$. Indeed, since $p > 5$ is a prime, it follows, by Lemma 2(2), that F_p is coprime to F_m for $1 < m \leq 5$. Since $F_3 = 2, F_4 = 3$, and $F_5 = 5$, it follows that $l = 0$ and $p_1 > 5$.

Hence, $p_1 > 5$ for all $i = 1, \dots, k$. Write $p_i = 2^{2^{\alpha_i}} +$ for some $\alpha_i \geq 2$. It follows that

$$p_1 = 4^{2^{\alpha_1 - 1}} + 1 \equiv 2 \pmod 5.$$

Since $\left(\frac{p_1}{5}\right) = \left(\frac{2}{5}\right) = -1$, it follows, by the quadratic reciprocity law, that $\left(\frac{5}{p_1}\right) = -1$. It follows, by Lemma 2(1), that $p_1 \mid F_{p_1+1}$. Hence,

$$p_1 \mid (F_p, F_{p_1+1}) = F_{(p,\, p_1+1)}.$$

The above divisibility relation and the fact that p is prime, forces $p \mid p_1 + 1 = 2(2^{2^{\alpha_1}-1} + 1)$. Hence, $p \mid 2^{2^{\alpha_1}-1} + 1$. Thus,

$$p \leq 2^{2^{\alpha_1}-1} + 1. \tag{6}$$

On the other hand, since

$$F_p = \prod_{i=1}^{k} (2^{2^{\alpha_i}} + 1) \equiv 1 \pmod{2^{2^{\alpha_1}}},$$

it follows, by Lemma 3(3), that $p \equiv \pm 1 \pmod{2^{2^{\alpha_1}-1}3}$. In particular,

$$p \geq 2^{2^{\alpha_1}-1}3-1. \tag{7}$$

From inequalities (6) and (7), it follows that $2^{2^{\alpha_1}-1}3-1 \leq 2^{2^{\alpha_1}-1}+1$ or $2^{2^{\alpha_1}} \leq 2$. This implies that $\alpha_1 = 0$ which contradicts the fact that $\alpha_1 \geq 2$.

Now write $n = 2^a 3^b 5^c$. We show that $a \leq 2$. Indeed, if $a \geq 3$, then $21 = F_8 \mid F_n$, therefore

$$3 \mid 12 = \phi(21) \mid \phi(F_n) = 2^m,$$

which is a contradiction. We show that $b \leq 2$. Indeed, if $b \geq 3$, then $53 \mid F_{27} \mid F_n$, therefore

$$13 \mid 52 = \phi(53) \mid \phi(F_n) = 2^m,$$

which is a contradiction. Finally, we show that $c \leq 1$. Indeed, if $c \geq 2$, then $3001 \mid F_{25} \mid F_n$, therefore

$$3 \mid 3000 = \phi(3001) \mid \phi(F_n) = 2^m,$$

which is again a contradiction. In conclusion, $n \mid 2^2 \cdot 3^2 \cdot 5 = 180$. One may easily check that the only divisors n of 180 for which $\phi(F_n)$ is a power of 2 are indeed the announced ones.

(2) Since $\phi(2) = \phi(1) = 1 = 2^0$ and $\phi(3) = \phi(4) = 2^1$, it follows that $n = 0, 1, 2, 3$ lead to solutions of equation (2). We now show that these are the only ones. One may easily check that $n \neq 4, 5$. Assume that $n \geq 6$. Since $\phi(L_n) = 2^m$, it follows that

$$L_n = 2^l \cdot p_1 \cdot \dots \cdot p_k, \tag{8}$$

where $l \geq 0$ and $p_1 < \dots < p_k$ are Fermat primes. Write $p_i = 2^{2^{\alpha_i}}+1$. Clearly, $p_1 \geq 3$. The sequence $(L_n)_{n \geq 0}$ is periodic modulo 8 with period 12. Moreover, analyzing the terms L_s for $s = 0, 1, \dots, 11$, one notices that $L_s \not\equiv 0 \pmod 8$ for any $s = 0, 1, \dots, 11$. It follows that $l \leq 2$ in equation (8). Since $n \geq 6$, it follows that $L_n \geq 18$. In particular, $p_i \geq 5$ for some $i = 1, \dots, k$. From the equation

$$L_n^2 - 5F_n^2 = (-1)^n \cdot 4, \tag{9}$$

it follows easily that $5 \nmid L_n$. Thus, $p_i > 5$. Hence, $p_i = 2^{2^{\alpha_i}}+1$ for some $\alpha_i \geq 2$. It follows that $p_i \equiv 1 \pmod 4$ and

$$p_i \equiv 4^{2^{\alpha_i - 1}} + 1 \equiv (-1)^{2^{\alpha_i-1}} + 1 \equiv 2 \pmod 5.$$

In particular, $\left(\frac{p_i}{5}\right) = \left(\frac{2}{5}\right) = -1$. Hence, by the quadratic reciprocity law, it follows that $\left(\frac{5}{p_i}\right) = -1$ as well. On the other hand, reducing equation (9) modulo p_i, it follows that

$$5F_n^2 \equiv (-1)^{n-1} \cdot 4 \pmod{p_i}. \tag{10}$$

Since $p_i \equiv 1 \pmod 4$, it follows that $\left(\frac{(-1)^{n-1}}{p_i}\right) = 1$. From congruence (10), it follows that $\left(\frac{5}{p_i}\right) = 1$, which contradicts the fact that $\left(\frac{5}{p_i}\right) = -1$.

(3) Since $\sigma(1) = 1 = 2^0, \sigma(3) = 4 = 2^2$, and $\sigma(21) = 32 = 2^5$, it follows that $n = 1, 2, 4, 8$ are solutions of equation (3). We show that these are the only ones. One can easily check that $n \neq 3, 5, 6, 7$. Assume now that there exists a solution of equation (3) with $n > 8$. Since $\sigma(F_n) = 2^m$, it follows easily that $F_n = q_1 \cdot \dots \cdot q_k$, where $q_1 < \dots < q_k$ are Mersenne primes. Let

$q_i = 2^{p_i} - 1$, where $p_i \geq 2$ is prime. In particular, $q_i \equiv 3 \pmod 4$. Reducing equation (9) modulo q_i, it follows that

$$L_n^2 = (-1)^n \cdot 4 \pmod{q_i}. \tag{11}$$

Since $q_i \equiv 3 \pmod 4$, it follows that $\left(\frac{-1}{q_i}\right) = -1$. From congruence 11, it follows that $2 \mid n$. Let $n = 2n_1$. Since $F_n = F_{2n_1} = F_{n_1}L_{n_1}$ and since F_n is a square free product of Mersenne primes, it follows that F_{n_1} is a square free product of Mersenne primes as well. In particular, $\sigma(F_{n_1}) = 2^{m_1}$. Inductively, it follows easily that n is a power of 2. Let $n = 2^t$, where $t \geq 4$. Then, $n_1 = 2^{t-1}$. Moreover, since $L_{n_1} \mid F_{n_1}L_{n_1} = F_n$, it follows that L_{n_1} is a square free product of Mersenne primes as well. Write

$$L_{n_1} = q_1' \cdot \dots \cdot q_l', \tag{12}$$

where $q_i' < \dots < q_l'$. Let $q_i' = 2^{p_i'} - 1$ for some prime number p_i'. The sequence $(L_n)_{n \geq 0}$ is periodic modulo 3 with period 8. Moreover, analyzing L_s for $s = 0, 1, \dots, 7$, one concludes that $3 \mid L_s$ only for $s = 2, 6$. Hence, $3 \mid L_s$ if and only if $s \equiv 2 \pmod 4$. Since $t \geq 4$, it follows that $8 \mid 2^{t-1} = n_1$. Hence, $3 \nmid L_n$ and $3 \nmid L_{n_1/2}$. In particular, $p_1' > 2$. We conclude that all p_i' are odd and $q_i' = 2^{p_i'} - 1 \equiv 2 - 1 \equiv 1 \pmod 3$. From equation (12), it follows that $L_{n_1} \equiv 1 \pmod 3$. Reducing relation $L_{n_1} = L_{n_1/2}^2 - 2$ modulo 3, it follows that $1 \equiv 1 - 2 \equiv -1 \pmod 3$, which is a contradiction.

(4) We first show that equation (4) has no solutions for which $n > 1$ is odd. Indeed, assume that $\sigma(L_n) = 2^m$ for some odd integer n. Let $p \mid n$ be a prime. By Lemma 2(2), we conclude that $L_p \mid L_n$. Since $\sigma(L_n)$ is a power of 2, it follows that L_n is a square free product of Mersenne primes. Since L_p is a divisor of L_n, it follows that L_p is a square free product of Mersenne primes as well. Write $L_p = q_1 \cdot \dots \cdot q_k$, where $q_1 < \dots < q_k$ are prime numbers such that $q_i = 2^{p_i} - 1$ for some prime $p_i \geq 2$. We show that $p_1 > 2$. Indeed, assume that $p_1 = 2$. In this case, $q_1 = 3$. It follows that $3 \mid L_p$. However, from the proof of (3), we know that $3 \mid L_s$ if and only if $s \equiv 2$ (mod 4). This shows that $p_1 \geq 3$.

Notice that $L_p \equiv \pm 1 \pmod{2^{p_1}}$. It follows that $L_p^2 - 1 \equiv 0 \pmod{2^{p_1+1}}$. Since p is odd, it follows, by Lemma 1(4), that

$$L_p^2 - 5F_p^2 = -4 \tag{13}$$

or $L_p^2 - 1 = 5(F_p^2 - 1)$. It follows that $F_p^2 - 1 \equiv 0 \pmod{2^{p_1+1}}$. Hence, $F_p \equiv \pm 1 \pmod{2^{p_1}}$. From Lemma 3(3), we conclude that $p \equiv \pm 1 \pmod{2^{p_1-1}3}$. In particular,

$$p \geq 2^{p_1-1}3 - 1. \tag{14}$$

On the other hand, reducing equation (13) modulo q_1, we conclude that $5F_p^2 \equiv 4 \pmod{q_1}$, therefore $\left(\frac{5}{q_1}\right) = 1$. By; Lemma 2(1), it follows that $q_1 \mid F_{q_1-1}$. Since $q_1 \mid L_p$ and $F_{2p} = F_pL_p$, it follows that $q_1 \mid F_{2p}$. Hence, $q_1 \mid (F_{2p}, F_{q_1-1}) = F_{(2p, q_1-1)}$. Since $F_2 = 1$, we conclude that $p \mid q_1 = 1 = 2(2^{p_1-1} - 1)$. In particular,

$$p \leq 2^{p_1-1} - 1. \tag{15}$$

From inequalities (14) and (15), it follows that $2^{p_1-1}3 - 1 \leq 2^{p_1-1} - 1$, which is a contradiction.

Assume now that $n > 4$ is even. Write $n = 2^t n_1$, where n_1 is odd. Let

$$L_n = q_1 \cdot \dots \cdot q_k, \tag{16}$$

where $q_1 < \cdots < q_k$ are prime numbers of the Mersenne type. Let $q_i = 2^{p_i} - 1$. Clearly, $q_i \equiv 3$ (mod 4) for all $i = 1, \ldots, k$. Reducing the equation $L_n^2 - 5F_n^2 = 4$ modulo q_i, we obtain that $-5F_n^2 = 4 \pmod{q_i}$. Since $\left(\frac{-1}{q_i}\right) = -1$, it follows that $\left(\frac{5}{q_i}\right) = -1$. From Lemma 2(1), we conclude that $q_i \mid F_{q_i+1} = F_{2^{p_i}}$. We now show that $t \le p_1 - 1$. Indeed, assume that this is not the case. Since $t \ge p_1$, it follows that $2^{p_1} \mid 2^t n_1 = n$. Hence, $q_1 \mid F_{2^{p_1}} \mid F_n$. Since $q_1 \mid L_n$, it follows, by Lemma 1(4), that $q_1 \mid 4$, which is a contradiction. So, $t \le p_1 - 1$. We now show that $n_1 = 1$. Indeed, since $t+1 \le p_1 \le p_i$, $q_i \mid L_n \mid F_{2n}$, and $q_i \mid F_{2^{p_i}}$, it follows, by Lemma 2(2), that $q_i \mid (F_{2n}, F_{2^{p_i}}) = F_{(2n, 2^{p_i})} = F_{2^{t+1}}$. Hence, $q_i \mid F_{2^{t+1}} = F_{2^t} L_{2^t}$. We show that $n_1 = 1$. Indeed, since $t+1 \le p_1 \le p_i$, $q_i \mid L_n \mid F_{2n}$, and $q_i \mid F_{2^{p_i}}$, it follows, by Lemma 2(2), that $q_i \mid (F_{2n}, F_{2^{p_i}}) = F_{(2n, 2^{p_i})} = F_{2^{t+1}}$. Hence, $q_i \mid F_{2^{t+1}} = F_{2^t} L_{2^t}$. We show that $q_i \mid L_{2^t}$. Indeed, for if not, then $q_i \mid F_{2^t}$. Since $2^t \mid n$, it follows that $q_i \mid F_{2^t} \mid F_n$. Since $q_i \mid L_n$, it follows, by Lemma 1(4), that $q_i^2 \mid 4$, which is a contradiction. In conclusion, $q_i \mid L_{2^t}$ for all $i = 1, \ldots, k$. Since q_i are distinct primes, it follows that

$$L_n = q_1 \cdot \cdots \cdot q_k \mid L_{2^t}.$$

In particular, $L_{2^t} \ge L_n = L_{2^t n_1}$. This shows that $n_1 = 1$. Hence, $n = 2^t$.

Since $n > 4$; it follows that $t \ge 3$. It is apparent that $q_1 \ne 3$, since, as previously noted, $3 \mid L_s$ if and only is $s \equiv 2$ (mod 4), whereas $n = 2^t \equiv 0$ (mod 4). Hence, $p_i \ge 3$ for all $i = 1, \ldots, k$. Moreover, since $q_i = 2^{p_i} - 1$ are quadratic nonresidues modulo 5, it follows easily that $p_i \equiv 3$ (mod 4). In particular, if $k \ge 2$, then $p_2 \ge p_1 + 4$.

Now since $t \ge 3$, it follows, by Lemma 4, that

$$L_{2^t} \equiv 2^{t+1} a - 1 \pmod{2^{t+4}}, \tag{17}$$

where $a \in \{3, 5\}$. On the other hand, from formula (16) and the fact that $p_2 \ge p_1 + 4$ whenever $k \ge 2$, it follows that

$$L_{2^t} = \prod_{i=1}^{k} (2^{p_i} - 1) \equiv (-1)^k \cdot (-2^{p_1} + 1) \equiv 2^{p_1} b \pm 1 \pmod{2^{p_1+4}}. \tag{18}$$

where $b \in \{1, 7\}$. One can notice easily that congruences (17) and (18) cannot hold simultaneously for any $t \le p_1 - 1$. This argument takes care of the situation $k \ge 2$. The case $k = 1$ follows from Lemma 3 and the fact that $t \le p_1 - 1$ by noticing that

$$2^{p_1} - 1 = L_{2^t} \equiv 2^{t+1} \cdot 3 - 1 \pmod{2^{t+4}}$$

implies $2^{p_1 - t - 1} \equiv 3 \pmod{2^3}$, which is impossible.

The above arguments show that equation (4) has no even solutions $n > 4$. Hence, the only solutions are the announced ones. □

ACKNOWLEDGMENTS

I would like to thank the editor for his comments on a previous version of this paper and for pointing out various bibliographical references for the preliminary Lemmas 1-3. I would also like to thank an anonymous referee for suggestions which greatly improved the quality of this paper.

Financial support from the Alexander von Humboldt Foundation is gratefully acknowledged.

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AMS Classification Numbers: 11A25, 11B39

A REFINEMENT OF DE BRUYN'S FORMULAS FOR $\sum a^k k^p$

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1. INTRODUCTION

As is known, various methods have been proposed for finding summation formulas for the so-called arithmetic-geometric progression of the form

$$S_{a,p}(n) := \sum_{k=0}^{n} a^k k^p, \tag{1}$$

where a is a real or complex number with $a \neq 0$ and $a \neq 1$, and n and p are nonnegative integers. For some recent papers, see, e.g., de Bruyn [1], Gauthier [4], and Hsu [5]. The object of this note is to show that de Bruyn's formulas expressed in terms of determinants could be given concise explicit forms in terms of Eulerian polynomials. In fact, it is found that the recurrence relations (recursive equations) obtained by de Bruyn for those determinants used in his formulas can be solved by means of Eulerian polynomials.

Let us recall de Bruyn's work briefly. De Bruyn made use of Cramer's rule to develop some explicit formulas for expressing $S_{a,p}(n)$ as $(p+1)\times(p+1)$ determinants. He then gave two formulas for $S_{a,p}(n)$, one in powers of $(n+1)$, the other in powers of n, in which all the coefficients are also expressed as determinants. More precisely, de Bruyn's first formula in powers of $(n+1)$ takes the form

$$S_{a,p}(n) = \frac{a^{n+1}}{a-1}\sum_{r=0}^{p-1}\binom{p}{r} f_r(a)(n+1)^{p-r} + f_p(a)\left(\frac{a^{n+1}-1}{a-1}\right), \tag{2}$$

where $f_p(a) = 1$, and $f_r(a)$ $(r = 1, 2, \ldots, p-1)$ are given by

$$f_r(a) = r!\left(\frac{a}{1-a}\right)^r \det\begin{pmatrix} \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \cdots & \cdots & \frac{1}{1!} & \frac{a-1}{a} \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \cdots & \cdots & \frac{1}{2!} & \frac{1}{1!} \end{pmatrix}, \tag{3}$$

and $f_0(a), f_1(a), f_2(a), \ldots$ satisfy the recurrence relations

$$f_0(a) = 1,\quad a\sum_{j=0}^{r}\binom{r}{j} f_j(a) - f_r(a) = 0,\quad (r = 1, 2, \ldots). \tag{4}$$

De Bruyn observed that if the f_j's are denoted as the Bernoulli numbers B_j, and we put $a = 1$, equation (4) just gives the well-known recurrence formula for the Bernoulli numbers. This led him to call the numbers $f_r(a)$ $(r = 0, 1, 2, \ldots)$ the a-Bernoulli numbers. In the next section, we shall show that $f_r(a)$ are closely related to Eulerian polynomials.

2. SOLUTION OF RECURSIVE EQUATIONS

Evidently the system of equations given by (4) determines $f_r(a)$'s uniquely with $f_0(a) = 1$. Using (4) recursively one may write

$$f_0(a) = \frac{1}{(1-a)^0},\ f_1(a) = \frac{a}{(1-a)^1},\ f_2(a) = \frac{a+a^2}{(1-a)^2},\ f_3(a) = \frac{a+4a^2+a^3}{(1-a)^3},\ \text{etc.}$$

Here it may be verified that the numerators of the $f_r(a)$'s $(r = 0, 1, 2, \dots)$ are precisely the Eulerian polynomials, $A_r(a)$ $(r = 0, 1, 2, \dots)$. In fact, it is known that (cf. Comtet [3], § 6.5)

$$A_0(a) = 1,\ A_1(a) = a,\ A_2(a) = a + a^2,\ A_3(a) = a + 4a^2 + a^3,\ \text{etc.}$$

Thus, one may reasonably conjecture that

$$f_r(a) = \frac{A_r(a)}{(1-a)^r}\ (r = 0, 1, 2, \dots) \tag{5}$$

are the solutions to the recursive equations given by (4). We will prove this below as a lemma.

The historical origin of Eulerian polynomials $A_p(a)$ is the following summation formula for the infinite arithmetic-geometric series

$$\sum_{k=0}^{\infty} a^k k^p = \frac{A_p(a)}{(1-a)^{p+1}},\ |a| < 1,\ a \neq 0, \tag{6}$$

where $A_p(a)$ is a polynomial of degree p in a, $p \geq 0$, and $0^0 := 1$ (see, e.g., Carlitz [2] and Comtet [3; p. 245]). We shall utilize (6) to prove our preceding conjecture given in the following lemma.

Lemma: The functions $f_r(a)$ given by (5) satisfy the recursive equations displayed in (4) for all complex numbers $a \neq 0, 1$.

Proof: Since $A_0(a) = 1 = f_0(a)$, it suffices to consider equation (4) for $r \geq 1$. Clearly these equations may be equivalently replaced by the following:

$$a \sum_{j=0}^{r} \binom{r}{j} \frac{f_j(a)}{1-a} - \frac{f_r(a)}{1-a} = 0\ (r = 1, 2, 3, \dots). \tag{7}$$

Substituting (5) into (7) and using the representation (6) for $A_j(a)/(1-a)^{j+1}$ with $|a| < 1$, it is easily found that the left-hand side (LHS) of (7) becomes

$$\begin{aligned} a\sum_{j=0}^{r}\binom{r}{j}\sum_{k=0}^{\infty} a^k k^j - \sum_{k=0}^{\infty} a^k k^r &= \sum_{k=0}^{\infty} a^{k+1} \sum_{j=0}^{r} \binom{r}{j} k^j - \sum_{k=1}^{\infty} a^k k^r\ (r \geq 1) \\ &= \sum_{k=0}^{\infty} a^{k+1}(k+1)^r - \sum_{k=1}^{\infty} a^k k^r = 0. \end{aligned}$$

This shows that (7) holds for the $f_j(a)$'s given by (5) with $|a| < 1$, $a \neq 0$. Now the LHS of (7) [with $f_j(a)$ given by (5)] is a rational function of a that vanishes for infinitely many values of a; thus, it should vanish identically with the only restrictions $a \neq 0, a \neq 1$. This completes the proof of the Lemma.

3. REFINEMENT OF FORMULA (2)

It is known that the Eulerian polynomial $A_r(a)$ $(r \geq 1)$ may be written in the form (cf. Comtet [3; §6.5])

$$A_r(a) = \sum_{k=1}^{r} A(r, k) a^k, \tag{8}$$

where $A(r, k)$ are called Eulerian numbers given explicitly by

$$A(r, k) = \sum_{j=0}^{k} (-1)^j \binom{r+1}{j} (k-j)^r \quad (1 \leq k \leq r). \tag{9}$$

Using the Lemma, one can express de Bruyn 's formula (2) in a refined form. This is given by the following theorem.

Theorem: For any given integer $p \geq 0$, there holds the summation formula

$$S_{a,\,p}(n) = \frac{1}{a-1}\left[a^{n+1}\sum_{r=0}^{p}\binom{p}{r}\frac{A_r(a)}{(1-a)^r}(n+1)^{p-r} - \frac{A_p(a)}{(1-a)^p}\right], \tag{10}$$

where $A_r(a)$ are given by (8) and (9), $a \neq 0, a \neq 1$.

Remark: De Bruyn's second formula for $S_{a,\,p}(n)$ in powers of n given by

$$S_{a,\,p}(n) = \frac{a^{n+1}}{a-1}n^p + \frac{a^n}{a-1}\sum_{r=1}^{p-1}\binom{p}{r}f_r(a)n^{p-r} + f_p(a)\left(\frac{a^n-1}{a-1}\right), \quad p > 1,$$

can likewise be refined to the form

$$S_{a,\,p}(n) = \frac{a^{n+1}}{a-1}n^p + \frac{1}{a-1}\left[a^n\sum_{r=1}^{p}\binom{p}{r}\frac{A_r(a)}{(1-a)^r}n^{p-r} - \frac{A_p(a)}{(1-a)^p}\right]. \tag{11}$$

This is obtained by means of the Lemma. Surely, both (10) and (11) are useful for practical computations whenever n is much larger than p, say $n >> p^3$. Moreover, it may be worth mentioning that the sum $S_{a,\,p}(n)$ can also be expressed using Stirling numbers of the second kind, and the formula is also available for $n >> p^3$ (cf. [5]).

4. A DIRECT PROOF OF THE THEOREM

Here we shall give a direct computational proof of (10) with the aid of (6). Since (10) is obvious for $p = 0$, it suffices to consider the case $p \geq 1$.

For a given real or complex number a with $a \neq 1$, $a \neq 0$, we shall make use of the simple exponential function ae^θ, θ real or complex. Since $ae^\theta \to a \neq 1$ as $\theta \to 0$, we can find a sufficiently small positive number δ such that $ae^\theta \neq 1$ for $|\theta| < \delta$.

Let us consider the sum

$$S(n, \theta) := \sum_{k=0}^{n} (ae^\theta)^k = \frac{1 - a^{n+1}e^{(n+1)\theta}}{1 - ae^\theta}, \quad (|\theta| < \delta).$$

For given $p \geq 1$, we have the p^{th} derivative with respect to θ:

$$\left(\frac{d^p}{d\theta^p}\right) S(n,\theta) = \sum_{k=1}^{n} a^k k^p e^{k\theta}.$$

Thus, it follows that

$$\begin{aligned} S_{a,p}(n) = \sum_{k=1}^{n} a^k k^p &= \left(\frac{d^p}{d\theta^p}\right)_0 S(n,\theta) \\ &= \left(\frac{d^p}{d\theta^p}\right)_0 \left[(1-a^{n+1}e^{(n+1)\theta})(1-ae^{\theta})^{-1}\right], \end{aligned} \tag{12}$$

where the derivatives are evaluated at $\theta = 0$. Using Leibniz's product formula for differentiation, we easily find that the RHS of (12) equals

$$\sum_{r=0}^{p-1} \binom{p}{r} (-a^{n+1})(n+1)^{p-r} \left(\frac{d^r}{d\theta^r}\right)_0 (1-ae^{\theta})^{-1} + (1-a^{n+1}) \left(\frac{d^p}{d\theta^p}\right)_0 (1-ae^{\theta})^{-1}. \tag{13}$$

It remains to compute

$$\left(\frac{d^r}{d\theta^r}\right)_0 (1-ae^{\theta})^{-1}, \ (0 \le r \le p).$$

This can be done easily by using (6) with $|a| < 1$, $a \ne 0$, as follows:

$$\left(\frac{d^r}{d\theta^r}\right)_0 (1-ae^{\theta})^{-1} = \left(\frac{d^r}{d\theta^r}\right)_0 \left(\sum_{k=0}^{\infty} a^k e^{k\theta}\right) = \sum_{k=0}^{\infty} a^k k^r = \frac{A_r(a)}{(1-a)^{r+1}}. \tag{14}$$

Here it may be noted that the series $\sum_{k=0}^{\infty} a^k e^{k\theta}$ in (14) can be term-wise differentiated any number of times in a neighborhood of $\theta = 0$, say $|\theta| < \delta$, provided that δ is sufficiently small such that $|ae^{\theta}| < \rho = constant < 1$ for $|\theta| < \delta$, which obviously implies the uniform convergence condition for the related series.

Now, recalling (12) and substituting (14) into (13), we obtain

$$S_{a,p}(n) = \frac{1}{a-1}\left[a^{n+1} \sum_{r=0}^{p-1} \binom{p}{r} \frac{A_r(a)}{(1-a)^r} (n+1)^{p-r} + (a^{n+1}-1) \frac{A_p(a)}{(1-a)^p}\right]. \tag{15}$$

This is precisely equivalent to (10).

Finally, note that (15) is an equality between rational functions of a, valid for infinitely many values of a ($|a| < 1$, $a \ne 0$) so that it must be an identity valid for all values of a with the only restrictions $a \ne 1$, $a \ne 0$. This completes the proof of (10).

5. AN EXAMPLE

Consider a pair of trigonometric sums as follows:

$$c(n) = \sum_{k=0}^{n} \alpha^k k^p \cos k\theta, \quad s(n) = \sum_{k=0}^{n} \alpha^k k^p \sin k\theta,$$

where α is a positive real number, $\alpha \ne 1$, p a positive integer, and θ a real number, $0 < \theta < 2\pi$. These sums can be computed precisely using the explicit formulas (10) or (11). Indeed, taking $a = \alpha e^{i\theta}$ ($i^2 = -1$) in (1), we have

$$\sum_{k=0}^{n}(\alpha^k e^{ik\theta})k^p = c(n)+is(n).$$

Denoting the RHS of (10) or of (11) by $\Phi(a, p, n)$, we get

$$c(n) = \operatorname{Re}\Phi(\alpha e^{i\theta}, p, n), \quad s(n) = \operatorname{Im}\Phi(\alpha e^{i\theta}, p, n),$$

where $\operatorname{Re}\Phi$ and $\operatorname{Im}\Phi$ denote the real part and imaginary part of Φ, respectively. Obviously, this follows from the fact that $(\alpha e^{i\theta})^k = \alpha^k \cos k\theta + i\alpha^k \sin k\theta$.

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AMS Classification Numbers: 05A10, 11B37, 11B68, 11B83

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Announcement

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RISING DIAGONAL POLYNOMIALS ASSOCIATED WITH MORGAN-VOYCE POLYNOMIALS

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(*Submitted April 1998-Final Revision January 1999*)

1. INTRODUCTION

Diagonal polynomials have been defined for Chebyshev, Fermat, Fibonacci, Lucas, Jacobsthal and other polynomials, and their properties have been studied (see, e.g., [9]. [5], and [7]). However, these are not applicable to the diagonal polynomials associated with the Morgan-Voyce polynomials (hereafter denoted as MVPs) $B_n(x), b_n(x), c_n(x)$, and $C_n(x)$, defined by:

$$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x) \quad (n \geq 2), \tag{1.1a}$$

with

$$B_0(x) = 1, \quad B_1(x) = x+2; \tag{1.1b}$$

$$b_n(x) = (x+2)b_{n-1}(x) - b_{n-2}(x) \quad (n \geq 2), \tag{1.2a}$$

with

$$b_0(x) = 1, \quad b_1(x) = x+1; \tag{1.2b}$$

$$c_n(x) = (x+2)c_{n-1}(x) - c_{n-2}(x) \quad (n \geq 2), \tag{1.3a}$$

with

$$c_0(x) = 1, \quad c_1(x) = x+3; \tag{1.3b}$$

$$C_n(x) = (x+2)C_{n-1}(x) - C_{n-2}(x) \quad (n \geq 2), \tag{1.4a}$$

with

$$C_0(x) = 2, \quad C_1(x) = x+2. \tag{1.4b}$$

Many interesting results have been proved regarding these MVPs (see [10], [11], [14], [12], [1], [2], [6], and [8]), and some of the important known results are listed in Section 2 for ready reference as well as for establishing the results regarding their associated diagonal polynomials.

2. SOME IMPORTANT PROPERTIES OF THE MORGAN-VOYCE POLYNOMIALS

Interrelations:

$$b_n(x) = B_n(x) - B_{n-1}(x) \quad (n \geq 1), \qquad \text{from [10].} \tag{2.1}$$

$$xB_n(x) = b_{n+1}(x) - b_n(x), \qquad \text{from [10].} \tag{2.2}$$

$$C_n(x) = B_n(x) - B_{n-2}(x) \quad (n \geq 2), \qquad \text{from [14], [13].} \tag{2.3}$$

$$C_n(x) = b_n(x) + b_{n-1}(x) \quad (n \geq 2), \qquad \text{from [14], [13].} \tag{2.4}$$

$$xc_n(x) = b_{n+1}(x) - b_{n-1}(x) \quad (n \geq 1), \qquad \text{from [6].} \tag{2.5}$$

$$C_n(x) = c_n(x) - c_{n-1}(x) \quad (n \geq 1), \qquad \text{from [6], [13].} \tag{2.6}$$

$$xc_n(x) = C_{n+1}(x) - C_n(x), \qquad \text{from (2.4) and (2.5).} \tag{2.7}$$

$$c_n(x) = B_n(x) + B_{n-1}(x) \quad (n \geq 1), \qquad \text{from [13].} \tag{2.8}$$

Closed-Form Expressions:

$$B_n(x) = \sum_{k=0}^{n} \binom{n+k+1}{n-k} x^k, \qquad \text{from [11].} \tag{2.9}$$

$$b_n(x) = \sum_{k=0}^{n} \binom{n+k}{n-k} x^k, \qquad \text{from [11].} \tag{2.10}$$

$$c_n(x) = \sum_{k=0}^{n} \frac{2n+1}{2k+1} \cdot \binom{n+k}{n-k} x^k, \qquad \text{from (2.8) and (2.9).} \tag{2.11}$$

$$C_n(x) = 2 + \sum_{k=1}^{n} \frac{n}{k} \cdot \binom{n+k-1}{n-k} x^k, \qquad \text{from (2.4) and (2.10).} \tag{2.12}$$

It should be noted that (2.12) has been derived earlier (see [2]).

Zeros:

$$B_n(x):\ x_r = -4\sin^2\left\{\frac{r}{n+1} \cdot \frac{\pi}{2}\right\},\ r = 1, 2, \ldots, n, \qquad \text{from [12].} \tag{2.13}$$

$$b_n(x):\ x_r = -4\sin^2\left\{\frac{2r-1}{2n+1} \cdot \frac{\pi}{2}\right\},\ r = 1, 2, \ldots, n, \qquad \text{from [12].} \tag{2.14}$$

$$c_n(x):\ x_r = -4\sin^2\left\{\frac{2r}{2n+1} \cdot \frac{\pi}{2}\right\},\ r = 1, 2, \ldots, n, \qquad \text{from [1].} \tag{2.15}$$

$$C_n(x):\ x_r = -4\sin^2\left\{\frac{2r-1}{2n} \cdot \frac{\pi}{2}\right\},\ r = 1, 2, \ldots, n, \qquad \text{from [14].} \tag{2.16}$$

Generating Functions:

$$B(x, t) = \sum_{0}^{\infty} B_n(x)t^n = [1-(xt+2t-t^2)]^{-1}, \qquad \text{from (1.1a).} \tag{2.17}$$

$$b(x, t) = \sum_{0}^{\infty} b_n(x)t^n = (1-t)B(x, t), \qquad \text{from (2.1) and (2.17).} \tag{2.18}$$

$$c(x, t) = \sum_{0}^{\infty} c_n(x)t^n = (1+t)B(x, t), \qquad \text{from (2.8) and (2.17).} \tag{2.19}$$

$$C(x, t) = \sum_{0}^{\infty} C_n(x)t^n = 1+(1-t^2)B(x, t), \qquad \text{from (2.3) and (2.17).} \tag{2.20}$$

Differential Equations:

$$B_n(x):\ x(x+4)y'' + 3(x+2)y' - n(n+2)y = 0, \qquad \text{from [12].} \tag{2.21}$$

$$b_n(x):\ x(x+4)y'' + 2(x+1)y' - n(n+1)y = 0, \qquad \text{from [12].} \tag{2.22}$$

$$c_n(x):\ x(x+4)y'' + 2(x+3)y' - n(n+1)y = 0, \qquad \text{from [13].} \tag{2.23}$$

$$C_n(x):\ x(x+4)y'' + (x+2)y' - n^2y = 0, \qquad \text{from [3].} \tag{2.24}$$

Orthogonality Property:

$B_n(x)$:	Orthogonal over $(-4, 0)$ with respect to the weight function $\sqrt{-x(x+4)}$,	from [11].	(2.25)
$b_n(x)$:	Orthogonal over $(-4, 0)$ with respect to the weight function $\sqrt{-(x+4)/x}$,	from [11].	(2.26)
$c_n(x)$:	Orthogonal over $(-4, 0)$ with respect to the weight function $\sqrt{-x/(x+4)}$,	from [13].	(2.27)
$C_n(x)$:	Orthogonal over $(-4, 0)$ with respect to the weight function $1/\sqrt{-x(x+4)}$,	from [2].	(2.28)

Simson Formulas:

$$B_{n+1}(x)B_{n-1}(x)-B_n^2(x)=-1, \quad \text{from [11].} \tag{2.29}$$

$$b_{n+1}(x)b_{n-1}(x)-b_n^2(x)=x, \quad \text{from [12].} \tag{2.30}$$

$$c_{n+1}(x)c_{n-1}(x)-c_n^2(x)=-(x+4), \quad \text{from [13].} \tag{2.31}$$

$$C_{n+1}(x)C_{n-1}(x)-C_n^2(x)=x(x+4), \quad \text{from [13].} \tag{2.32}$$

3. RISING DIAGONAL POLYNOMIALS

In order to define the diagonal polynomials associated with the Morgan-Voyce polynomials in a manner similar to the diagonal polynomials defined for Chebyshev, Fermat, Fibonacci, and other polynomials (see [9], [5], [7]), we first need to express the MVPs $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$ in descending powers of x. By letting $i=n-k$ in (2.9), (2.10), (2.11), and (2.12), we get the following expressions for the MVPs:

$$B_n(x)=\sum_{i=0}^{n}\binom{2n+1-i}{i}x^{n-i}; \tag{3.1}$$

$$b_n(x)=\sum_{i=0}^{n}\binom{2n-i}{i}x^{n-i}; \tag{3.2}$$

$$c_n(x)=\sum_{i=0}^{n}\frac{2n+1}{2n+1-2i}\cdot\binom{2n-i}{i}x^{n-i}; \tag{3.3}$$

$$C_n(x)=x^n+\sum_{i=1}^{n-1}\frac{n}{n-i}\cdot\binom{2n-1-i}{i}x^{n-i}+2. \tag{3.4}$$

We now rearrange $C_n(x)$ into a form that will help in formulating a closed-form expression for the corresponding rising diagonal polynomial. It can be shown that

$$\frac{n}{n-i}\cdot\binom{2n-1-i}{i}=\frac{2n}{i}\cdot\binom{2n-1-i}{i-1}.$$

Hence, (3.4) can be rewritten as

$$C_n(x) = x^n + \sum_{i=1}^{n-1} \frac{2n}{i} \cdot \binom{2n-1-i}{i-1} x^{n-i} + 2,$$

or

$$C_n(x) = x^n + \sum_{i=1}^{n} \frac{2n}{i} \cdot \binom{2n-1-i}{i-1} x^{n-i}. \tag{3.5}$$

Let us first consider the rising diagonal polynomial $R_n(x)$ associated with the MVP $B_n(x)$. We see from (3.1) that

$$R_0(x) = 1,\; R_1(x) = x,\; R_2(x) = x^2 + 2,\; R_3(x) = x^3 + 4x, \ldots,$$

$$R_n(x) = x^n + \binom{2n-2}{1} x^{n-2} + \binom{2n-5}{2} x^{n-4} + \binom{2n-8}{3} x^{n-6} + \cdots.$$

The above may be rewritten as

$$R_n(x) = \binom{2n+1}{0} x^n + \binom{2n-2}{1} x^{n-2} + \binom{2n-5}{2} x^{n-4} + \cdots + \binom{2n+1-3\left[\frac{n}{2}\right]}{\left[\frac{n}{2}\right]} x^{n-2\left[\frac{n}{2}\right]}.$$

Hence,

$$R_n(x) = \sum_{i=0}^{[n/2]} \binom{2n+1-3i}{i} x^{n-2i}. \tag{3.6}$$

Similarly, starting with (3.2), (3.3), and (3.5), we may derive the following polynomial expressions for the rising diagonal polynomials $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$ associated, respectively, with the MVPs $b_n(x)$, $c_n(x)$, and $C_n(x)$:

$$r_n(x) = \sum_{i=0}^{[n/2]} \binom{2n-3i}{i} x^{n-2i}; \tag{3.7}$$

$$\rho_n(x) = \sum_{i=0}^{[n/2]} \frac{2n+1-2i}{2n+1-4i} \cdot \binom{2n-3i}{i} x^{n-2i}; \tag{3.8}$$

$$\mathrm{P}_n(x) = x^n + \sum_{i=1}^{[n/2]} \frac{2(n-i)}{i} \cdot \binom{2n-1-3i}{i-1} x^{n-2i}. \tag{3.9}$$

It is readily seen that all the four sets of diagonal polynomials are even for even values of n and odd for odd values of n. Table 1 lists the diagonal polynomials up to $n = 8$.

4. SOME INTERRELATIONS AMONG $R_n(x)$, $r_n(x)$, $\rho_n(x)$ AND $\mathrm{P}_n(x)$

Consider the expression $R_n(x) - R_{n-2}(x)$. Then, from (3.6), we have

$$\begin{aligned}
R_n(x) - R_{n-2}(x) &= \sum_{i=0}^{[n/2]} \binom{2n+1-3i}{i} x^{n-2i} - \sum_{i=0}^{[n/2]-1} \binom{2n-3-3i}{i} x^{n-2-2i} \\
&= x^n + \sum_{i=1}^{[n/2]} \binom{2n+1-3i}{i} x^{n-2i} - \sum_{i=1}^{[n/2]} \binom{2n-3i}{i-1} x^{n-2i} \\
&= x^n + \sum_{i=1}^{[n/2]} \frac{2n-4i+1}{i} \cdot \frac{(2n-3i)\ldots(2n-4i+2)}{(i-1)!} x^{n-2i}
\end{aligned}$$

$$= x^n + \sum_{i=1}^{[n/2]} \binom{2n-3i}{i} x^{n-2i} = \sum_{i=0}^{[n/2]} \binom{2n-3i}{i} x^{n-2i}$$

$$= r_n(x), \text{ using (3.7).}$$

Hence, we have the result that

$$r_n(x) = R_n(x) - R_{n-2}(x) \quad (n \geq 2). \tag{4.1}$$

It is interesting to compare this result with the corresponding one relating the respective MVPs, namely,

$$b_n(x) = B_n(x) - B_{n-1}(x) \quad (n \geq 1).$$

We now prove that

$$xR_n(x) = r_{n+1}(x) - r_{n-1}(x) \quad (n \geq 1), \tag{4.2}$$

a result which corresponds to (2.2) with respect to the original MVPs $B_n(x)$ and $b_n(x)$. First, consider $r_{2n+1}(x) - r_{2n-1}(x)$. Then, from (3.7),

$$\begin{aligned} r_{2n+1}(x) - r_{2n-1}(x) &= \sum_{i=0}^{n} \binom{4n+2-3i}{i} x^{2n+1-2i} - \sum_{i=0}^{n-1} \binom{4n-2-3i}{i} x^{2n-1-2i} \\ &= x^{2n+1} + x\sum_{i=1}^{n} \binom{4n+2-3i}{i} x^{2n-2i} - x\sum_{i=1}^{n} \binom{4n+1-3i}{i-1} x^{2n-2i} \\ &= x^{2n+1} + x\sum_{i=1}^{n} \binom{4n+1-3i}{i} x^{2n-2i} \\ &= x\sum_{i=0}^{n} \binom{4n+1-3i}{i} x^{2n-2i} \\ &= xR_{2n}(x), \text{ using (3.6).} \end{aligned}$$

Similarly, we can show that

$$r_{2n+2}(x) - r_{2n}(x) = xR_{2n+1}(x).$$

Hence, the result (4.2).

Again, from (3.7), we have

$$\begin{aligned} r_{2n+1}(x) + r_{2n-1}(x) &= x^{2n+1} + x\sum_{i=1}^{n} \binom{4n+2-3i}{i} x^{2n-2i} + x\sum_{i=1}^{n} \binom{4n+1-3i}{i-1} x^{2n-2i} \\ &= x^{2n+1} + \sum_{i=1}^{n} \frac{2(2n+1-2i)}{i} \cdot \binom{4n+1-3i}{i-1} x^{2n+1-2i} \\ &= \mathrm{P}_{2n+1}(x), \text{ using (3.9).} \end{aligned} \tag{4.3a}$$

Similarly,

$$r_{2n+2}(x) + r_{2n}(x) = \mathrm{P}_{2n+2}(x). \tag{4.3b}$$

Combining (4.3a) and (4.3b), we get

$$\mathrm{P}_n(x) = r_n(x) + r_{n-2}(x) \quad (n \geq 2), \tag{4.4}$$

a result to be compared with (2.4). Using (4.1), the above relation may be rewritten as

$$\mathrm{P}_n(x) = R_n(x) - R_{n-4}(x) \quad (n \geq 4), \tag{4.5}$$

the corresponding result for the MVPs being (2.3).

Again starting with $R_n(x) + R_{n-2}(x)$ and using (3.6), we can show that

$$\rho_n(x) = R_n(x) + R_{n-2}(x) \quad (n \geq 2), \tag{4.6}$$

which should be compared with relation (2.8) for the corresponding MVPs. Now, using (4.6), we have

$$\rho_n(x) - \rho_{n-2}(x) = R_n(x) - R_{n-4}(x).$$

Hence, from (4.5), we get

$$\mathrm{P}_n(x) = \rho_n(x) - \rho_{n-2}(x) \quad (n \geq 2), \tag{4.7}$$

the corresponding relation for the MVPs being (2.6). Further, using (4.4), we have

$$\begin{aligned}\mathrm{P}_{n+1}(x) - \mathrm{P}_{n-1}(x) &= \{r_{n+1}(x) - r_{n-1}(x)\} + \{r_{n-1}(x) - r_{n-3}(x)\} \\ &= xR_n(x) + xR_{n-2}(x), \text{ using (4.2)}, \\ &= x\rho_n(x), \text{ using (4.6)}.\end{aligned}$$

Hence,

$$x\rho_n(x) = \mathrm{P}_{n+1}(x) - \mathrm{P}_{n-1}(x) \quad (n \geq 1), \tag{4.8}$$

a relation corresponding to (2.7) for the original MVPs.

We may derive a number of such interrelationships among the diagonal polynomials $R_n(x)$, $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$ corresponding to those of the MVPs $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$. We will only list the following:

$$\sum_{i=0}^{n} r_i(x) = R_n(x) + R_{n-1}(x); \tag{4.9}$$

$$x\sum_{i=0}^{n} R_i(x) = r_{n+1}(x) + r_n(x) - 1; \tag{4.10}$$

$$\sum_{i=0}^{n} \mathrm{P}_i(x) = \rho_n(x) + \rho_{n-1}(x) + 1; \tag{4.11}$$

$$x\sum_{i=0}^{n} \rho_i(x) = \mathrm{P}_{n+1}(x) + \mathrm{P}_n(x) - 2. \tag{4.12}$$

5. RECURRENCE RELATIONS AND GENERATING FUNCTIONS

From relation (4.2), we have

$$\begin{aligned}xR_n(x) &= r_{n+1}(x) - r_{n-1}(x) \quad (n \geq 1) \\ &= \{R_{n+1}(x) - R_{n-1}(x)\} - \{R_{n-1}(x) - R_{n-3}(x)\} \quad (n \geq 3), \text{ using (4.1)}.\end{aligned}$$

Hence,

$$R_{n+1}(x) = xR_n(x) + 2R_{n-1}(x) - R_{n-3}(x) \quad (n \geq 3).$$

Therefore, $R_n(x)$ satisfies the recurrence relation

$$R_n(x) = xR_{n-1}(x) + 2R_{n-2}(x) - R_{n-4}(x) \quad (n \geq 4), \tag{5.1a}$$

with

$$R_0(x) = 1,\ R_1(x) = x,\ R_2(x) = x^2 + 2,\ R_3(x) = x^3 + 4x. \tag{5.1b}$$

Similarly, we can deduce that $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$ satisfy the following recurrence relations:

$$r_n(x) = xr_{n-1}(x) + 2r_{n-2}(x) - r_{n-4}(x) \quad (n \geq 4), \tag{5.2a}$$

with

$$r_0(x) = 1,\ r_1(x) = x,\ r_2(x) = x^2 + 1,\ r_3(x) = x^3 + 3x; \tag{5.2b}$$

$$\rho_n(x) = x\rho_{n-1}(x) + 2\rho_{n-2}(x) - \rho_{n-4}(x) \quad (n \geq 4), \tag{5.3a}$$

with

$$\rho_0(x) = 1,\ \rho_1(x) = x,\ \rho_2(x) = x^2 + 3,\ \rho_3(x) = x^3 + 5x; \tag{5.3b}$$

$$\mathrm{P}_n(x) = x\mathrm{P}_{n-1}(x) + 2\mathrm{P}_{n-2}(x) - \mathrm{P}_{n-4}(x) \quad (n \geq 4), \tag{5.4a}$$

with

$$\mathrm{P}_0(x) = 2,\ \mathrm{P}_1(x) = x,\ \mathrm{P}_2(x) = x^2 + 2,\ \mathrm{P}_3(x) = x^3 + 4x. \tag{5.4b}$$

It is interesting to compare the above recurrence relations with those of the corresponding MVPs $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$ given by (1.1), (1.2), (1.3), and (1.4), respectively.

We shall now derive generating functions for these diagonal polynomials using the standard technique. Let $g_n(x)$ represent any one of the diagonal polynomials $R_n(x)$, $r_n(x)$, $\rho_n(x)$, or $\mathrm{P}_n(x)$, and let $G(x, t)$ be the corresponding generating function. Then, from [4], we have

$$\begin{aligned} t^{-4}[G(x,t) - g_0(x) - g_1(x)t - g_2(x)t^2 - g_3(x)t^3] \\ = xt^{-3}[G(x,t) - g_0(x) - g_1(x)t - g_2(x)t^2] \\ + 2t^{-2}[G(x,t) - g_0(x) - g_1(x)t] - G(x,t). \end{aligned}$$

Hence,

$$\begin{aligned} (1 - xt - 2t^2 + t^4)G(x,t) = g_0(x) + \{g_1(x) - xg_0(x)\}t \\ + \{g_2(x) - xg_1(x) - 2g_0(x)\}t^2 + \{g_3(x) - xg_2(x) - 2g_1(x)\}t^4. \end{aligned} \tag{5.5}$$

Therefore, $R(x, t)$, the generating function for the diagonal polynomial $R_n(x)$, is given by

$$\begin{aligned} (1 - xt - 2t^2 + t^4)R(x,t) = 1 + (x - x)t + (x^2 + 2 - x^2 - 2)t^2 \\ + (x^3 + 4x - x^3 - 2x - 2x)t^4 = 1. \end{aligned}$$

Hence,

$$R(x,t) = \sum_0^\infty R_i(x)t^i = [1 - (xt + 2t^2 - t^4)]^{-1}. \tag{5.6}$$

Similarly, by substituting for $g_n(x)$ the diagonal polynomials $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$ in (5.5), we can derive the following generating functions for these polynomials:

$$r(x,t) = \sum_0^\infty r_i(x)t^i = (1 - t^2)R(x,t); \tag{5.7}$$

$$\rho(x,t) = \sum_0^\infty \rho_i(x)t^i = (1 + t^2)R(x,t); \tag{5.8}$$

$$\mathrm{P}(x, t) = \sum_{0}^{\infty} \mathrm{P}_i(x) t^i = 1 + (1 - t^4) R(x, t). \tag{5.9}$$

It is interesting to compare the generating functions (5.6), (5.7), (5.8), and (5.9) of the diagonal polynomials with those of the corresponding MVPs $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$, namely, those given by (2.17), (2.18), (2.19), and (2.20).

Using the generating function (5.6), we will now derive an interesting relation among the derivatives. From (5.6),

$$\frac{\partial R(x, t)}{\partial x} = t R^2(x, t)$$

and

$$\frac{\partial R(x, t)}{\partial t} = (x + 4t - 4t^3) R^2(x, t).$$

Hence,

$$(x + 4t - 4t^3) \frac{\partial R(x, t)}{\partial x} = t \frac{\partial R(x, t)}{\partial t}. \tag{5.10}$$

Thus, from (5.6),

$$x R_n'(x) + 4 R_{n-1}'(x) - 4 R_{n-3}'(x) = n R_n(x). \tag{5.11}$$

However, from (5.1), we have

$$R_{n+1}'(x) = x R_n'(x) + R_n(x) + 2 R_{n-1}'(x) - R_{n-3}'(x). \tag{5.12}$$

Substituting for $x R_n'(x)$ from (5.12) in (5.11) and rearranging the terms, we get

$$(n+1) R_n(x) = \{R_{n+1}'(x) - R_{n-1}'(x)\} + 3\{R_{n-1}'(x) - R_{n-3}'(x)\}.$$

Using (4.1) in the above expression, we have the result

$$(n+1) R_n(x) = r_{n+1}'(x) + 3 r_{n-1}'(x). \tag{5.13}$$

Apart from the above result, it has not been possible to derive any other simple derivative relation for the rising diagonal polynomials.

6. CONCLUDING REMARKS

We have thus defined and obtained polynomial expressions for the four sets of diagonal polynomials associated with the four sets of Morgan-Voyce polynomials $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$. We have also obtained a number of interesting properties of these diagonal polynomials, including the recurrence relations they satisfy. It appears that these diagonal polynomials have a number of other interesting properties.

We would like to mention one such interesting property regarding the location of the zeros of these diagonal polynomials. Using the network properties of two-element-kind electrical networks, it is possible to show that, for $n = 1, 2, \ldots, 8$, the following results hold:

(a) The zeros of $R_n(x)$, $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$ are all simple and lie on the imaginary axis, that is, all the zeros are purely imaginary.

(b) The zeros of $R_{n+1}(x)$ interlace on the imaginary axis with those of $R_n(x)$, $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$. Also, the zeros of $r_{n+1}(x)$ interlace on the imaginary axis with those of $R_n(x)$, $r_n(x)$, and $\mathrm{P}_n(x)$, the zeros of $\rho_{n+1}(x)$ interlace on the imaginary axis with those of $R_n(x)$, $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$, and those of $\mathrm{P}_{n+1}(x)$ interlace on the imaginary axis with those of $R_n(x)$, $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$.

(c) However, the zeros of $r_{n+1}(x)$ and those of $\rho_n(x)$ do not interlace, except for the case of $n=1$.

We conjecture that the above results are true for any value of n.

TABLE 1

Rising Diagonal Polynomials for n = 0, 1, 2, ..., 8

$R_0(x)=1$ $R_1(x)=x$ $R_2(x)=x^2+2$ $R_3(x)=x^3+4x$ $R_4(x)=x^4+6x^2+3$ $R_5(x)=x^5+8x^3+10x$ $R_6(x)=x^6+10x^4+21x^2+4$ $R_7(x)=x^7+12x^5+36x^3+20x$ $R_8(x)=x^8+14x^6+55x^4+56x^2+5$	$r_0(x)=1$ $r_1(x)=x$ $r_2(x)=x^2+1$ $r_3(x)=x^3+3x$ $r_4(x)=x^4+5x^2+1$ $r_5(x)=x^5+7x^3+6x$ $r_6(x)=x^6+9x^4+15x^2+1$ $r_7(x)=x^7+11x^5+28x^3+10x$ $r_8(x)=x^8+13x^6+45x^4+35x^2+1$
$\rho_0(x)=1$ $\rho_1(x)=x$ $\rho_2(x)=x^2+3$ $\rho_3(x)=x^3+5x$ $\rho_4(x)=x^4+7x^2+5$ $\rho_5(x)=x^5+9x^3+14x$ $\rho_6(x)=x^6+11x^4+27x^2+7$ $\rho_7(x)=x^7+13x^5+44x^3+30x$ $\rho_8(x)=x^8+15x^6+65x^4+77x^2+9$	$\mathrm{P}_0(x)=2$ $\mathrm{P}_1(x)=x$ $\mathrm{P}_2(x)=x^2+2$ $\mathrm{P}_3(x)=x^3+4x$ $\mathrm{P}_4(x)=x^4+6x^2+2$ $\mathrm{P}_5(x)=x^5+8x^3+9x$ $\mathrm{P}_6(x)=x^6+10x^4+20x^2+2$ $\mathrm{P}_7(x)=x^7+12x^5+35x^3+16x$ $\mathrm{P}_8(x)=x^8+14x^6+54x^4+50x^2+2$

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AMS Classification Numbers: 11B39, 33C25

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ON PRIMES IN THE FIBONACCI SEQUENCE

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It is well known that primes can occur in the Fibonacci sequence only for prime indices, the only exception being $F_4 = 3$. This follows from the fact that for any two positive integers k and n, F_n divides F_{kn}. I could not locate the earliest reference to that result, but page 111 in [1] contains several proofs of this. Of course, if p is a prime, F_p may very well be composite; the first example of this is $F_{19} = 4181 = 37\cdot 113$. Here is the list of the next few terms F_p that are composite:

$$\begin{aligned}
F_{31} &= 1346269 = 557\cdot 2471,\\
F_{37} &= 24157817 = 73\cdot 149\cdot 2221,\\
F_{41} &= 165580141 = 2789\cdot 59369,\\
F_{53} &= 53316291173 = 953\cdot 55945741.
\end{aligned}$$

In this note we show that in fact F_p is composite for certain primes p. We prove the following result.

Theorem: Let $p > 7$ be a prime satisfying the following two conditions:

I. $p \equiv 2 \pmod 5$ or $p \equiv 4 \pmod 5$;

II. $2p-1$ is also a prime.

Then F_p is composite, in fact, $(2p-1) \mid F_p$.

Proof: We start with the explicit formula for F_p:

$$F_p = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^p - \left(\frac{1-\sqrt{5}}{2}\right)^p\right\}.$$

Multiplying out by $\sqrt{5}$ and squaring, we get

$$5F_p^2 = \frac{1}{2^{2p}}\left\{(1+\sqrt{5})^{2p} + (1-\sqrt{5})^{2p}\right\} + 2$$

or

$$2^{2p-1}\cdot 5F_p^2 = \frac{1}{2}\left\{(1+\sqrt{5})^{2p} + (1-\sqrt{5})^{2p}\right\} + 2^{2p}.$$

When we expand the powers inside the braces, the terms involving $\sqrt{5}$ will cancel out and we get

$$2^{2p-1}\cdot 5F_p^2 = 1 + \binom{2p}{2}5 + \binom{2p}{4}5^2 + \cdots + \binom{2p}{2p-2}5^{p-1} + 5^p + 2^{2p}.$$

Since $2p-1$ is a prime, $2^{2p-1} \equiv 2 \pmod{2p-1}$ and $\binom{2p}{k} \equiv 0 \pmod{2p-1}$ for $k < 2p-1$, so

$$5F_p^2\cdot 2 \equiv 1 + 5^p + 4 \pmod{2p-1}$$

or

$$2F_p^2 \equiv 5^{p-1} + 1 \pmod{2p-1}. \tag{1}$$

Now let $\left(\frac{a}{b}\right)$ denote the Legendre symbol. By Euler's theorem,

$$5^{p-1} \equiv \left(\frac{5}{2p-1}\right) \pmod{2p-1}. \tag{2}$$

Suppose $p \equiv 2 \pmod 5$ so that $p = 5k+2$ for some integer k. Since $2p-1$ is a prime, by Gauss's reciprocity theorem,

$$\left(\frac{5}{2p-1}\right)\left(\frac{2p-1}{5}\right) = (-1)^{\frac{4}{2}\cdot\frac{2p-2}{2}} = 1$$

so that

$$\left(\frac{5}{2p-1}\right) = \left(\frac{2p-1}{5}\right) = \left(\frac{10k+3}{5}\right) = \left(\frac{3}{5}\right) = -1.$$

Hence, by (1) and (2),

$$2F_p^2 \equiv -1+1 = 0 \pmod{2p-1}.$$

This means that $2p-1$ divides F_p, and since $F_p > 2p-1$ for $p > 7$, F_p is composite.

In a similar way, if $p \equiv 4 \pmod 5$, $p = 5k+4$ for some k, and

$$\left(\frac{5}{2p-1}\right) = \left(\frac{2p-1}{5}\right) = \left(\frac{10k+7}{5}\right) = \left(\frac{2}{5}\right) = -1,$$

and again, as before, $2p-1$ divides F_p.

Here is the list of the first 21 prime indices p for which the above Theorem guarantees F_p to be composite: 19, 37, 79, 97, 139,157, 199, 229, 307, 337, 367, 379, 439, 499, 547, 577, 607, 619, 727, 829, 839.

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AMS Classification Numbers: 11A51, 11B39

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ON TOTAL STOPPING TIMES UNDER 3x + 1 ITERATION

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(*Submitted May 1998-Final Revision September 1998*)

1. INTRODUCTION

Let $\mathbf{N}$ denote the nonnegative integers, and let $\mathbf{P}$ denote the positive integers. Define T: $2\mathbf{N}+1 \to 2\mathbf{N}+1$ by $T(x) = \frac{3x+1}{2^j}$, where $2^j \mid 3x+1$ and $2^{j+1} \nmid 3x+1$. The famous $3x+1$ Conjecture asserts that, for any $x \in 2\mathbf{N}+1$, there exists $k \in \mathbf{N}$ satisfying $T^k(x) = 1$. Define the least whole number k for which $T^k(x) = 1$ as the *total stopping time* $\sigma(x)$ of x, and call the sequence of iterates $(x, T(x), T^2(x), \dots)$ the *trajectory* of x. Note that $\sigma(x) = \infty$ if the trajectory of x diverges, and that $\sigma(1) = 0$. Furthermore, if $k \in \mathbf{P}$ is fixed, and x is the smallest positive odd integer satisfying $T^k(x) = 1$, we say that x is *minimal of level k*. In this paper, we employ a specific partition of the positive odd integers to show that if x is minimal of level $k \geq 3$, then $\sigma(x) = \sigma(2x+1)$. In addition, a set of positive integers satisfying $\sigma(x) = \sigma(2x+1)$ is characterized. Using a related partition, we then show that the arithmetic progression (1 *mod* 16) is a "sufficient set," in other words, to prove the $3x+1$ Conjecture, it suffices to prove it for all $x \equiv 1$ *mod* 16. In [4], Korec and Znam proved that the arithmetic progressions (a *mod* p^n), where 2 is a primitive root (*mod* p^2) and $(a, p) = 1$, are sufficient sets; however, this result does not apply when p is a power of 2.

A thorough summary of some known results on the $3x+1$ Conjecture is given in Lagarias [5] and Wirsching [6]. It is important to observe that our formulation of the function $T(x)$ differs from that in [3], in which $T: \mathbf{P} \to \mathbf{P}$ is given by $T(x) = \frac{x}{2}$ if x is even and $T(x) = \frac{3x+1}{2}$ is x is odd. As a consequence, our total stopping times are different. For example, $\sigma(27) = 41$ under our formulation, whereas $\sigma(27) = 70$ in [3].

It is the author's hope that the results of this paper, or perhaps the techniques used in proving the results, will be useful in computing $\pi_a(x)$, which counts the number of positive integers $y \leq x$ such that $T^k(y) = a$ for some nonnegative integer k. The strongest known results along this line are given in Applegate and Lagarias [1].

2. TOTAL STOPPING TIMES OF MINIMAL NUMBERS

We begin by constructing a partition of the positive odd integers. For $a, b \in \mathbf{P}$, denote the arithmetic progression $(\mathrm{am}+\mathrm{b})_{\mathrm{m}=0}^{\infty}$ by $(\mathrm{am}+\mathrm{b})$. Next, define subsets of $2\mathbf{N}+1$ as follows:

$$S_1 = \bigcup_{n \in \mathbf{P}} (2^{2n+1}m + 2^{2n-1} - 1),$$

$$S_2 = \bigcup_{n \in \mathbf{P}} (2^{2n+2}m + 2^{2n+1} + 2^{2n} - 1),$$

$$S_3 = \bigcup_{n \in \mathbf{P}} (2^{2n+1}m + 2^{2n} + 2^{2n-1} - 1),$$

$$S_4 = \bigcup_{n \in \mathbf{P}} (2^{2n+2}m + 2^{2n} - 1).$$

It is easy to verify that $[S_1, S_2, S_3, S_4]$ is a partition of $2\mathbf{N}+1$. We will also need the following two preliminary lemmas, both of which follow directly from the definition of $T(x)$.

Lemma 1: Let $x \in 2\mathbf{N}+1$, and let $k \in \mathbf{N}$ satisfy $k \le \sigma(x)$. Then $\sigma(T^k(x)) = \sigma(x) - k$.

Lemma 2: Let $x \in 2\mathbf{N}+1$ with $x \ne 1$. Then $\sigma(x) = \sigma(4x+1)$.

The following two lemmas give total stopping time properties of certain subsets of the positive integers obtained from our partition. For notational convenience in the upcoming proofs and throughout this paper, we write $2^j \| n$ (2^j *exactly divides* n) if $2^j \mid n$ but $2^{j+1} \nmid n$.

Lemma 3: If $x \in S_1 \cup S_2 - (1)$, then $\sigma(x) = \sigma(2x+1)$.

Proof: First, consider the case in which $x \in S_1$ with $x \ne 1$. By the definition of S_1, x is of the form $2^{2n+1}m + 2^{2n-1} - 1$. Application of the function T yields:

$$T^{2n-1}(x) = \frac{3^{2n-1} \cdot 4m + 3^{2n-1} - 1}{2^j},$$

where $2^j \| 3^{2n-1} \cdot 4m + 3^{2n-1} - 1$. Note that $3^{2n-1} - 1 \equiv 2 \ mod\ 4$, therefore $j = 1$. Furthermore, $T^{2n-1}(2x+1) = 3^{2n-1} \cdot 8m + 3^{2n-1} \cdot 2 - 1$. Thus, $4 \cdot T^{2n-1}(x) + 1 = T^{2n-1}(2x+1)$. Applying Lemma 2, we obtain $\sigma(T^{2n-1}(x)) = \sigma(T^{2n-1}(2x+1))$. Hence, by Lemma 1, it follows that $\sigma(x) = \sigma(2x+1)$.

Next, consider the case $x \in S_2$. By definition of S_2, x is of the form $2^{2n+2}m + 2^{2n+1} + 2^{2n} - 1$. Application of the function T yields:

$$T^{2n}(x) = \frac{3^{2n} \cdot 4m + 3^{2n} \cdot 2 + 3^{2n} - 1}{2^j},$$

where $2^j \| 3^{2n} \cdot 4m + 3^{2n} \cdot 2 + 3^{2n} - 1$. Since $3^{2n} - 1 \equiv 0 \ mod\ 4$ and $3^{2n} \cdot 2 \equiv 2 \ mod\ 4$, we see that $j = 1$. Furthermore, $T^{2n}(2x+1) = 3^{2n} \cdot 8m + 3^{2n} \cdot 4 + 3^{2n} \cdot 2 - 1$. Hence, $4 \cdot T^{2n}(x) + 1 = T^{2n}(2x+1)$. Applying Lemma 2 yields $\sigma(T^{2n}(x)) = \sigma(T^{2n}(2x+1))$, so, using Lemma 1, we conclude that $\sigma(x) = \sigma(2x+1)$. □

Lemma 4: If $x \in S_3 \cup S_4 - (3)$, then there exists $y < x$ satisfying $\sigma(y) = \sigma(x)$.

Proof: First, consider the case in which $x \in S_3$. By definition of S_3, we have $x = 2^{2n+1}m + 2^{2n} + 2^{2n-1} - 1$. If $n = 1$, $x = 8m+5$, so choosing $y = 2m+1$ and applying Lemma 2 gives the result. If $n > 1$, we can choose $y \in 2\mathbf{N}+1$ satisfying $2y+1 = x$. Note that $y \in S_2$, so using a computation similar to that in the proof of Lemma 3, we see that $4 \cdot T^{2n-2}(y) + 1 = T^{2n-2}(x)$. Applying Lemmas 2 and 1, we obtain $\sigma(y) = \sigma(x)$. Now consider the case in which $x \in S_4$ with $x \ne 3$. By definition of S_4, we have $x = 2^{2n+2}m + 2^{2n} - 1$. Again, choose y so that $2y+1 = x$. Clearly, $y \in S_1$, so again by the proof of Lemma 3, it follows that $4 \cdot T^{2n-1}(y) + 1 = T^{2n-1}(x)$. Noting that $y \ne 1$ and applying Lemmas 1 and 2, we obtain $\sigma(y) = \sigma(x)$. □

The following result pertaining to total stopping times of minimal numbers can now be proved.

Theorem 1: If x is minimal of level $k \ge 3$, then $\sigma(x) = \sigma(2x+1)$.

Proof: Let $x \in 2\mathbf{N}+1$ be minimal of level $k \geq 3$. Note that $x \neq 1$ and $x \neq 3$. Using the definition of minimality and Lemma 4, we see that $x \notin S_3 \cup S_4$. Therefore $x \in S_1 \cup S_2$, so Lemma 3 implies that $\sigma(x) = \sigma(2x+1)$. □

Remark: The arguments in Lemmas 3 and 4 actually show that the appropriate trajectories coalesce after a certain number of steps, irrespective of whether or not they converge to 1. This is in part due to the fact that if $f(x) = 4x+1$ and x is odd, then $T(f(x)) = T(x)$. Note also that if $g(x) = 2x+1$, the relation $T(g(x)) = g(T(x))$ holds true for x odd. Furthermore, it can be demonstrated by straightforward computation that if $g_{a,b}(x) = ax+b$ with $a-b=1$ and x is of the form $2^n m + 2^{n-2} - 1$ or $2^n m + 2^{n-1} + 2^{n-2} - 1$ with $n \geq 3$, then $g_{a,b}(T^k(x)) = T^k(g_{a,b}(x))$ for $k \leq n-3$. A study of the interaction of various linear functions $g_{a,b}(x)$ with $T(x)$ under composition deserves further exploration.

3. A SUFFICIENT CONDITION FOR TRUTH OF THE $3x+1$ CONJECTURE

By use of a similar technique, it can now be demonstrated that to prove the $3x+1$ Conjecture, it suffices to prove it for all positive $x \equiv 1 \ mod\ 16$. This improves a result given in Cadogan [2].

Lemma 5: Suppose that for all positive $x \equiv 1 \ mod\ 8$ there exists $k \in \mathbf{N}$ such that $T^k(x) = 1$. Then, for all $x \in 2\mathbf{N}+1$. we can find $k \in \mathbf{N}$ such that $T^k(x) = 1$.

Proof: For $i = 1, 2, 3, 4$, define $T_i = S_i \cap (8m+7)$, where $[S_1, S_2, S_3, S_4]$ is the partition of $2\mathbf{N}+1$ used in Lemmas 3 and 4. We repartition the positive odd integers as follows:

$$2\mathbf{N}+1 = (8m+1) \cup (16m+3) \cup (16m+11) \cup (8m+5) \cup T_1 \cup T_2 \cup T_3 \cup T_4.$$

Now let $x \in 2\mathbf{N}+1$ be given. We can assume that $x \neq 1$ and $x \neq 3$, as the theorem follows trivially for these values of x. We examine the following cases:

Case 1. If $x \in (8m+1)$, by the hypothesis of Lemma 5, there exists $k \in \mathbf{P}$ such that $T^k(x) = 1$.

Case 2. Let $x \in (16m+3)$. Then $x = 2y+1$ for $y \in (8m+1)$. A simple computation shows that $T^2(x) = T^2(y)$. By the hypothesis of Lemma 5, there exists $k \in \mathbf{P}$ such that $T^k(y) = 1$, hence $T^k(x) = 1$.

Case 3. Let $x \in (16m+11)$. Then $T(x) \in (8m+1)$, so the hypothesis of Lemma 5 guarantees that there exists $k \in \mathbf{P}$ satisfying $T^k(T(x)) = 1$. Thus, $T^{k+1}(x) = 1$.

Case 4. Let $x \in T_1 \cup T_2$. If $x \in T_1$, we can write $x = 2^{2n+1}m + 2^{2n-1} - 1$, where $n \geq 2$. Then $T^{2n-2}(x) = 3^{2n-2} \cdot 8m + 3^{2n-2} \cdot 2 - 1$, and since $3^{2n-2} \equiv 1 \ mod\ 8$, we see that $T^{2n-2}(x) \in (8m+1)$. If $x \in T_2$, we can write $x = 2^{2n+2}m + 2^{2n+1} + 2^{2n} - 1$, where $n \geq 2$. Then $T^{2n-1}(x) = 3^{2n-1} \cdot 8m + 3^{2n-1} \cdot 4 + 3^{2n-1} \cdot 2 - 1$, which simplifies to $T^{2n-1}(x) = 3^{2n-1} \cdot 8m + 2(3^{2n} - 1) + 1$, and since $3^{2n} - 1 \equiv 0 \ mod$ 4, we obtain $T^{2n-1}(x) \in (8m+1)$. Invoking our hypothesis yields $T^k(x) = 1$ for some k.

Case 5. Let $x \in T_3 \cup T_4$. If $x \in T_3$, then x is of the form $2^{2n+1}m + 2^{2n} + 2^{2n-1} - 1$, where $n \geq 2$. Choose y satisfying $2y+1 = x$. By a computation similar to that used in the proof of Lemma 4, we see that $4 \cdot T^{2n-2}(y) + 1 = T^{2n-2}(x)$, hence $T^{2n-1}(y) = T^{2n-1}(x)$. If $n = 2$, $y \in (16m+11)$, and if $n > 2$, $y \in T_2$, so by the proofs of Case 3 and Case 4, respectively, there exists k satisfying

$T^k(y)=1$, hence $T^k(x)=1$. If $x \in T_4$, then x is of the form $2^{2n+2}m+2^{2n}-1$, where $n \geq 2$. Let y satisfy $2y+1=x$. Again, $4 \cdot T^{2n-1}(y)+1=T^{2n-1}(x)$, so $T^{2n}(y)=T^{2n}(x)$. But $y \in T_1$, so by Case 4, there exists k satisfying $T^k(y)=1$, hence $T^k(x)=1$.

Case 6. Finally, let $x \in (8m+5)$. Define $f(w)=4w+1$. Choose the smallest positive y satisfying $f^n(y)=x$ for $n \in \mathbf{P}$. Note that $y \notin (8m+5)$, since $f(2m+1)=8m+5$. If $y \neq 1$ and $y \neq 3$, we can invoke the previous cases to obtain k satisfying $T^k(y)=1$. Since $T(f^n(y))=T(y)$, we obtain $T(y)=T(x)$, and therefore $T^k(y)=T^k(x)=1$. If $y=3$, then $T(f^n(y))=T(y)=T(3)=5$, hence $T^2(f^n(y))=1$, so $T^2(x)=1$. If $y=1$, we have $f^n(y)=1+4+\cdots+4^n=(4^{n+1}-1)/3$, hence $T(f^n(y))=1$, so $T(x)=1$. Thus, in all cases, we have displayed $k \in \mathbf{N}$ for which $T^k(x)=1$. □

According to Lemma 5, the arithmetic progression $(8m+1)$ constitutes a sufficient set. The next theorem improves the sufficient set.

Theorem 2: Suppose that for all positive $x \equiv 1 \ mod\ 16$, there exists $k \in \mathbf{N}$ such that $T^k(x)=1$. Then, for all $x \in 2\mathbf{N}+1$, we can find $k \in \mathbf{N}$ such that $T^k(x)=1$.

Proof: Let $x=8m+1$ be given. A straightforward computation yields

$$T^2(64x+49)=\frac{9x+7}{2^j}=\frac{72m+16}{2^j}=\frac{9m+2}{2^{j-3}},$$

where $2^j \,\|\, 9x+7$, and hence $2^{j-3} \,\|\, 9m+2$. Also,

$$T^2(x)=T^2(8m+1)=\frac{9m+2}{2^k},$$

where $2^k \,\|\, 9m+2$. By unique factorization, $k=j-3$, and hence $T^2(x)=T^2(64x+49)$. Since $64x+49$ is in the arithmetic progression $(16m+1)$, we can invoke the hypothesis of Theorem 2; therefore, there exists k satisfying $T^k(T^2(x))=1$. Thus, $T^{k+2}(x)=1$, and since x was chosen arbitrarily from $(8m+1)$, we can apply Lemma 5 to obtain the result. □

Further strengthening of the result given in Theorem 2 certainly seems possible. An interesting question concerns which progressions of the form $(2^n m+1)$ constitute "sufficient sets" whose convergence to 1 guarantees the truth of the $3x+1$ Conjecture. Perhaps it can be proved that convergence of the set of numbers of the form $\{2^n+1 : n=1,2,3,\ldots\}$ is sufficient.

4. OTHER NUMBERS WITH EQUAL TOTAL STOPPING TIMES

We now characterize an additional set of positive odd integers satisfying $\sigma(x)=\sigma(2x+1)$. Let $L_k=\{x \in 2\mathbf{N}+1 \mid \sigma(x)=k\}$, and define $G_x=\{f^n(x) \mid n \in \mathbf{N}\} \cup \{f^n(2x+1) \mid n \in \mathbf{N}\}$, where $f(w)=4w+1$. For convenience, we set $G_{x_{-1}}=\emptyset$. We inductively define the j^{th} *exceptional number of level k* to be the smallest positive integer x_j satisfying $x_j \in L_k - \bigcup_{i=0}^{j} G_{x_{i-1}}$.

Note that for $j=0$, x_j is simply the minimal number of level k. Also observe that Lemma 2 and Theorem 1 tell us that all numbers in G_{x_0} are of level k, hence x_1 is the smallest positive integer of level k not accounted for by G_{x_0}, x_2 is the smallest positive integer of level k not accounted for by $G_{x_0} \cup G_{x_1}$, and so forth. It turns out that the exceptional numbers share the same total stopping time property as the minimal numbers.

Theorem 3: Let x_j denote the j^{th} exceptional number of level k with $k \geq 2$ and $x_j > 3$. Then $\sigma(x_j) = \sigma(2x_j + 1)$.

To prove Theorem 3, we need the following two preliminary lemmas.

Lemma 6: Let x_j denote the j^{th} exceptional number of level k with $k \geq 2$ and $x_j > 3$. Then $x_j \notin (16m+3) \cup (8m+5)$

Proof: Since x_0 is minimal of level k with $k \geq 2$ and $x_j > 3$, we have $x_0 \notin (16m+3) \cup (8m+5)$, hence the Lemma holds for $j = 0$. Let $j \geq 1$. We prove that $x_j \notin (16m+3)$ by contradiction. If $x_j \in (16m+3)$, pick y satisfying $2y+1 = x_j$. Clearly $\sigma(y) = \sigma(x_j)$, hence $y \in L_k$. Since $y < x_j$ and x_j is the smallest number in $L_k - \bigcup_{i=0}^{j} G_{x_{i-1}}$, we see that $y \in G_{x_i}$ for some $i \leq j-1$. Hence $y = f^p(x_i)$ or $y = f^p(2x_i+1)$ for some $p \in \mathrm{N}$. Since $p \geq 1$ yields $y \in (8m+5)$, which is impossible, we have $p = 0$. Hence $y = x_i$ or $y = 2x_i + 1$. But $y = x_i$ yields $2x_i + 1 = x_j$, so $x_j \in G_{x_i}$ with $i \leq j-1$, contradicting the definition of x_j. Hence $y = 2x_i + 1$. But $y \in (8m+1)$ forces x_i to be even, again a contradiction. If $x_j = 8m+5$, then select $y = 2m+1$. Since $\sigma(y) = \sigma(x_j)$ and $y < x_j$, we see that $y \in G_{x_i}$ for some $i \leq j-1$. But $x_j = f(y)$, hence $x_j \in G_{x_i}$, contradicting the definition of x_j. Hence $x_j \notin (8m+5)$ □

Lemma 7: Let S_3 and S_4 be subsets of $2\mathrm{N}+1$ as defined in Section 2. Let x_j be the j^{th} exceptional number of level k with $k \geq 2$ and $x_j > 3$. Then $x_j \notin S_3 \cup S_4$.

Proof: Suppose $x_j \in S_3 \cup S_4$. Then x_j is of the form $2^{2n+1}m + 2^{2n} + 2^{2n-1} - 1$ or $2^{2n+2}m + 2^{2n} - 1$. Furthermore, by Lemma 6, we have $n \geq 2$. Choose y satisfying $2y + 1 = x_j$. As in the proof of Lemma 4, we have $\sigma(y) = \sigma(x_j)$, therefore, by definition of x_j, we must have $y \in G_{x_i}$ for some $i \leq j-1$. Therefore, $y = f^p(x_i)$ or $y = f^p(2x_i+1)$ for some $p \in \mathrm{N}$. If $p \geq 1$, we have $y \in (8m+5)$, hence $x_j \in (16m+11)$, which contradicts the fact that $S_3 \cup S_4$ and $(16m+11)$ are disjoint. Thus $p = 0$, so either $y = x_i$ or $y = 2x_i + 1$. But $y = x_i$ yields $2x_i + 1 = x_j$, hence $x_j \in G_{x_i}$ for $i \leq j-1$, contradicting the definition of x_j. Thus, we have $y = 2x_i + 1$, so $4x_i + 3 = x_j$.

A simple computation shows that x_i must be in $S_3 \cup S_4$. We therefore have proven that $x_j \in S_3 \cup S_4$ implies there exists $x_i \in S_3 \cup S_4$ with $x_i < x_j$. Applying a simple induction and using the definition of S_3 and S_4 yields $x_p \in (8m+5) \cup (16m+3)$ for some p. But this contradicts Lemma 6, hence $x_j \in S_3 \cup S_4$ is impossible. □

Proof of Theorem 3: Consider the partition of $2\mathrm{N}+1$ as defined in the proof of Lemma 5. By Lemmas 6 and 7, we see that $x_j \notin (16m+3) \cup (8m+5) \cup T_3 \cup T_4$. Hence $x_j \in (8m+1) \cup (16m+11) \cup T_1 \cup T_2$. Applying Lemma 3, we obtain $\sigma(x_j) = \sigma(2x_j + 1)$. □

Our final theorem enables us to conclude that there exists an exceptional number x_j of level k for all $k \geq 2$ and for all $j \geq 0$.

Theorem 4: For all $j \geq 0$ and $k \geq 2$, $L_k - \bigcup_{i=0}^{j} G_{x_{i-1}} \neq \emptyset$.

Proof: We proceed by induction on j. Since $L_k \neq \emptyset$ is well known [3], the result holds true for $j = 0$. Now assume $L_k - \bigcup_{i=0}^{j} G_{x_{i-1}} \neq \emptyset$ for all $j < n$. We wish to show that $L_k - \bigcup_{i=0}^{n} G_{x_{i-1}} \neq \emptyset$. For all $j < n$, let x_j be the smallest integer in $L_k - \bigcup_{i=0}^{j} G_{x_{i-1}}$. Note that the sequence $\{x_j\}$ is strictly increasing, and that $x_j \notin G_{x_i}$ for $i \leq j-1$.

Consider the number $w = 64x_{n-1}+49$. We first prove that $w \notin G_{x_i}$ for all $i \le n-1$ by contradiction. If $w \in G_{x_i}$ for some $i \le n-1$, then $w = f^p(x_i)$ or $w = f^p(2x_i+1)$ for some $p \in \mathbf{N}$. Since $w \in (8m+1)$, we must have $p = 0$. Therefore, $w = x_i$ or $w = 2x_i+1$, and since the latter contradicts oddness of x_i, we have $w = x_i$. But this implies that $x_{n-1} < x_i$, contradicting the fact that $\{x_j\}$ is strictly increasing. Hence $w \notin G_{x_i}$ for all $i \le n-1$. Furthermore, as seen in the proof of Theorem 3, we have $\sigma(w) = \sigma(x_{n-1}) = k$, hence w is in $L_k - \bigcup_{i=0}^{n} G_{x_{i-1}}$, so $L_k - \bigcup_{i=0}^{n} G_{x_{i-1}} \neq \emptyset$. □

Remark: An interesting question concerns whether *all* numbers x satisfying $\alpha(x) = \sigma(2x+1)$ can be identified. The general question of finding all numbers x satisfying $\sigma(x) = \sigma(ax+b)$ for arbitrary whole numbers a and b looks difficult. Development of functions such as $f(w) = 64w+49$ which satisfy the condition $\sigma(x) = \sigma(f(x))$ appears to be a promising approach.

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AMS Classification Number: 11B83

COMBINATORIAL SUMS AND SERIES INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

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(Submitted May 1998-Final Revision January 1999)

0. INTRODUCTION

In this note we deal with several combinatorial sums and series involving inverses of binomial coefficients. Some of them have already been considered by other authors (see, e.g., [3], [4]), but it should be noted that our approach is different. It is based on Euler's well-known Beta function defined by

$$B(m,n)=\int_0^1 t^{m-1}(1-t)^{n-1}dt$$

for all positive integers m and n. Since

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\frac{(m-1)!(n-1)!}{(m+n-1)!},$$

we get

$$\binom{n}{k}^{-1}=(n+1)\int_0^1 t^k(1-t)^{n-k}dt \tag{1}$$

for all nonnegative integers n and k with $n\geq k$.

1. SUMS INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

Theorem 1.1 ([4], Theorem 1): If n is a nonnegative integer, then

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}=\frac{n+1}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}.$$

Proof: Let S_n be the sum of inverses of binomial coefficients. From (1) we get

$$\begin{aligned}S_n&=\sum_{k=0}^{n}(n+1)\int_0^1 t^k(1-t)^{n-k}dt\\&=(n+1)\int_0^1\left\{(1-t)^n\sum_{k=0}^{n}\left(\frac{t}{1-t}\right)^k\right\}dt=(n+1)\int_0^1\frac{(1-t)^{n+1}-t^{n+1}}{1-2t}dt.\end{aligned}$$

Making the substitution $1-2t=x$, we obtain

$$\begin{aligned}S_n&=\frac{n+1}{2^{n+2}}\int_{-1}^{1}\frac{(1+x)^{n+1}-(1-x)^{n+1}}{x}dx=\frac{n+1}{2^{n+2}}\left\{\int_{-1}^{1}\frac{(1+x)^{n+1}-1}{x}dx+\int_{-1}^{1}\frac{1-(1-x)^{n+1}}{x}dx\right\}\\&=\frac{n+1}{2^{n+2}}\sum_{k=0}^{n}\left\{\int_{-1}^{1}(1+x)^k dx+\int_{-1}^{1}(1-x)^k dx\right\}=\frac{n+1}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}.\end{aligned}$$

Theorem 1.2: If n is a positive integer, then

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n}{2k}^{-1}=\frac{4n+1}{2n+1}.$$

Proof: Formula (1) yields

$$\begin{aligned}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n}{2k}^{-1}&=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(4n+1)\int_0^1 t^{2k}(1-t)^{4n-2k}dt\\&=(4n+1)\int_0^1\left\{(1-t)^{4n}\sum_{k=0}^{2n}\binom{2n}{k}\left(\frac{-t^2}{(1-t)^2}\right)^k\right\}dt\\&=(4n+1)\int_0^1(1-t)^{4n}\left(1-\frac{t^2}{(1-t)^2}\right)^{2n}dt\\&=(4n+1)\int_0^1(1-2t)^{2n}dt=\frac{4n+1}{2n+1}.\end{aligned}$$

Theorem 1.3 ([5]): If n is a positive integer, then

$$\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}\binom{2n}{k}^{-1}=-\frac{1}{2n-1}.$$

Proof: Let S_n be the sum to evaluate. From (1) we get

$$\begin{aligned}S_n&=\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}(2n+1)\int_0^1 t^k(1-t)^{2n-k}dt\\&=(2n+1)\int_0^1\left\{\sum_{k=0}^{2n}\binom{4n}{2k}(-1)^k t^k(1-t)^{2n-k}\right\}dt\\&=\frac{2n+1}{2}\int_0^1\left\{\left(\sqrt{1-t}+i\sqrt{t}\right)^{4n}+\left(\sqrt{1-t}-i\sqrt{t}\right)^{4n}\right\}dt.\end{aligned}$$

Since

$$\sqrt{1-t}\pm i\sqrt{t}=\cos\left(\arctan\sqrt{\frac{t}{1-t}}\right)\pm i\sin\left(\arctan\sqrt{\frac{t}{1-t}}\right),$$

it follows that

$$S_n=(2n+1)\int_0^1\cos\left(4n\arctan\sqrt{\frac{t}{1-t}}\right)dt.$$

Making the substitution $\arctan\sqrt{\frac{t}{1-t}}=x$, we obtain

$$\begin{aligned}S_n&=(2n+1)\int_0^{\pi/2}\cos(4nx)\sin(2x)dx\\&=\frac{2n+1}{2}\int_0^{\pi/2}\{\sin(4n+2)x-\sin(4n-2)x\}dx=-\frac{1}{2n-1}.\end{aligned}$$

Theorem 1.4 ([2]): If m, n, and p are nonnegative integers with $p\le n$, then

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{p+k}^{-1}=\frac{n+m+1}{n+1}\binom{n}{p}^{-1}.$$

Proof: Formula (1) yields

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{p+k}^{-1} &= \sum_{k=0}^{m}\binom{m}{k}(n+m+1)\int_0^1 t^{p+k}(1-t)^{n+m-p-k}dt\\ &= (n+m+1)\int_0^1\left\{t^p(1-t)^{n+m-p}\sum_{k=0}^{m}\binom{m}{k}\left(\frac{t}{1-t}\right)^k\right\}dt\\ &= (n+m+1)\int_0^1 t^p(1-t)^{n+m-p}\left(1+\frac{t}{1-t}\right)^m dt\\ &= (n+m+1)\int_0^1 t^p(1-t)^{n-p}dt = \frac{n+m+1}{n+1}\binom{n}{p}^{-1}.\end{aligned}$$

Remark: In the special case $p=n$, from the above theorem we get

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{n+k}^{-1} = \frac{n+m+1}{n+1}.$$

Theorem 1.5: If m and n are nonnegative integers, then

$$\sum_{k=0}^{n}(-1)^k\binom{m+n}{m+k}^{-1} = \frac{m+n+1}{m+n+2}\left(\binom{m+n+1}{m}^{-1}+(-1)^n\right).$$

Proof: We have

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{m+n}{m+k}^{-1} &= \sum_{k=0}^{n}(-1)^k(m+n+1)\int_0^1 t^{m+k}(1-t)^{n-k}dt\\ &= (m+n+1)\int_0^1\left\{t^m(1-t)^n\sum_{k=0}^{n}\left(\frac{-t}{1-t}\right)^k\right\}dt\\ &= (m+n+1)\left(\int_0^1 t^m(1-t)^{n+1}dt+(-1)^n\int_0^1 t^{m+n+1}dt\right)\\ &= \frac{m+n+1}{m+n+2}\left(\binom{m+n+1}{m}^{-1}+(-1)^n\right).\end{aligned}$$

Remark: In the special case $m=n$ we get

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}^{-1} = \frac{2n+1}{2n+2}\left(\binom{2n+1}{n}^{-1}+(-1)^n\right),$$

while in the special case $m=0$ we obtain

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^{-1} = \frac{n+1}{n+2}(1+(-1)^n).$$

Consequently (see [3], p. 343),

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^{-1} = \frac{2n+1}{n+1}.$$

Theorem 1.6: If n is a positive integer, then

$$\frac{1}{3n+1}\sum_{k=0}^{n}\binom{3n}{3k}^{-1}-\frac{1}{3n+2}\sum_{k=0}^{n}\binom{3n+1}{3k+1}^{-1}+\frac{1}{3n+3}\sum_{k=0}^{n}\binom{3n+2}{3k+2}^{-1}$$
$$=\frac{1}{3n+3}\sum_{k=0}^{3n+2}\binom{3n+2}{k}^{-1}.$$

Proof: We have

$$\frac{1}{3n+1}\sum_{k=0}^{n}\binom{3n}{3k}^{-1}-\frac{1}{3n+2}\sum_{k=0}^{n}\binom{3n+1}{3k+1}^{-1}+\frac{1}{3n+3}\sum_{k=0}^{n}\binom{3n+2}{3k+2}^{-1}$$
$$=\sum_{k=0}^{n}\int_0^1\{t^{3k}(1-t)^{3n-3k}-t^{3k+1}(1-t)^{3n-3k}+t^{3k+2}(1-t)^{3n-3k}\}\,dt$$
$$=\int_0^1\left\{(1-t)^{3n}(1-t+t^2)\sum_{k=0}^{n}\left(\frac{t^3}{(1-t)^3}\right)^k\right\}dt=\int_0^1\frac{(1-t)^{3n+3}-t^{3n+3}}{1-2t}dt.$$

On the other hand,

$$\frac{1}{3n+3}\sum_{k=0}^{3n+2}\binom{3n+2}{k}^{-1}=\sum_{k=0}^{3n+2}\int_0^1 t^k(1-t)^{3n+2-k}dt$$
$$=\int_0^1\left\{(1-t)^{3n+2}\sum_{k=0}^{3n+2}\left(\frac{t}{1-t}\right)^k\right\}dt=\int_0^1\frac{(1-t)^{3n+3}-t^{3n+3}}{1-2t}\,dt,$$

completing the proof.

2. SERIES INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

Theorem 2.1: If m and n are positive integers with $m>n$, then

$$\sum_{k=0}^{\infty}\binom{mk}{nk}^{-1}=\int_0^1\frac{1+(m-1)t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2}dt.$$

Proof: From (1) we get

$$\sum_{k=0}^{\infty}\binom{mk}{nk}^{-1}=\sum_{k=0}^{\infty}(mk+1)\int_0^1 t^{nk}(1-t)^{(m-n)k}dt$$
$$=m\sum_{k=1}^{\infty}\int_0^1 k(t^n(1-t)^{m-n})^k dt+\sum_{k=0}^{\infty}\int_0^1(t^n(1-t)^{m-n})^k dt.$$

Let $f:[0,1]\to\mathbf{R}$ be the function defined by $f(t)=t^n(1-t)^{m-n}$. It is immediately seen that f attains its maximum at the point $t_0=n/m$. Since $f(t_0)<1$, it follows that

$$\sum_{k=1}^{\infty}k(t^n(1-t)^{m-n})^k=\frac{t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2}$$

and

$$\sum_{k=1}^{\infty}(t^n(1-t)^{m-n})^k = \frac{1}{1-t^n(1-t)^{m-n}}$$

uniformly on [0, 1]. Therefore, we obtain

$$\sum_{k=0}^{\infty}\binom{mk}{nk}^{-1} = m\int_0^1 \frac{t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2}dt + \int_0^1 \frac{dt}{1-t^n(1-t)^{m-n}},$$

completing the proof.

Remark: As special cases of Theorem 2.1 we get

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1} = \frac{4}{3}+\frac{2\pi\sqrt{3}}{27}, \tag{2}$$

$$\sum_{k=0}^{\infty}\binom{4k}{2k}^{-1} = \frac{16}{15}+\frac{\pi\sqrt{3}}{27}-\frac{2\sqrt{5}}{25}\ln\frac{1+\sqrt{5}}{2}. \tag{3}$$

Indeed, according to Theorem 2.1, we have

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1} = \int_0^1 \frac{1+t-t^2}{(1-t+t^2)^2}dt = 2\int_0^1 \frac{dt}{(1-t+t^2)^2} - \int_0^1 \frac{dt}{1-t+t^2} \tag{4}$$

and

$$\sum_{k=0}^{\infty}\binom{4k}{2k}^{-1} = \int_0^1 \frac{1+3t^2(1-t)^2}{(1-t^2(1-t)^2)^2}dt,$$

respectively. Since

$$\frac{1+3x^2}{(1-x^2)^2} = \frac{1+x}{2(1-x)^2}+\frac{1-x}{2(1+x)^2},$$

we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{4k}{2k}^{-1} &= \int_0^1 \frac{1+t-t^2}{2(1-t+t^2)^2}dt + \int_0^1 \frac{1-t+t^2}{2(1+t-t^2)^2}dt \\ &= \int_0^1 \frac{dt}{(1-t+t^2)^2} - \frac{1}{2}\int_0^1 \frac{dt}{1-t+t^2} + \int_0^1 \frac{dt}{(1+t-t^2)^2} - \frac{1}{2}\int_0^1 \frac{dt}{1+t-t^2}.\end{aligned} \tag{5}$$

Taking into account that

$$\int_0^1 \frac{dt}{1-t+t^2} = \frac{2\pi\sqrt{3}}{9} \quad \text{and} \quad \int_0^1 \frac{dt}{1+t-t^2} = \frac{4\sqrt{5}}{5}\ln\frac{1+\sqrt{5}}{2},$$

and that (see, e.g., [1])

$$\int \frac{dt}{(a+bt+ct^2)^2} = \frac{b+2ct}{(4ac-b^2)(a+bt+ct^2)} + \frac{2c}{4ac-b^2}\int \frac{dt}{a+bt+ct^2},$$

from (4) and (5) one can easily obtain (2) and (3).

Theorem 2.2 ([4], Theorem 2): If $n \geq 2$ is an integer, then

$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1} = \frac{n}{n-1}.$$

Proof: For each positive integer p, we have

$$\begin{aligned} s_p &:= \sum_{k=0}^{p} \binom{n+k}{k}^{-1} = \sum_{k=0}^{p} (n+k+1)\int_0^1 t^k (1-t)^n dt \\ &= \int_0^1 \left\{ (n+1)(1-t)^n \sum_{k=0}^{p} t^k + (1-t)^n \sum_{k=0}^{p} kt^k \right\} dt \\ &= (n+1)\int_0^1 (1-t)^n dt - (n+1)\int_0^1 t^{p+1}(1-t)^{n-1} dt + \int_0^1 t(1-t)^{n-2} dt \\ &\qquad - (p+1)\int_0^1 t^{p+1}(1-t)^{n-2} dt + p\int_0^1 t^{p+2}(1-t)^{n-2} dt. \end{aligned}$$

Formula (1) yields

$$s_p = \frac{n}{n-1} - (n-2)!\frac{(np+p+1)(p+1)!}{(p+n+1)!} - (n+1)(n-1)!\frac{(p+1)!}{(p+n+1)!}.$$

Taking into account that $n \geq 2$, we conclude that $s_p \to \frac{n}{n-1}$ when $p \to \infty$.

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AMS Classification Number: 11B65

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NEW ELEMENTARY PROBLEMS' AND SOLUTIONS' EDITORS AND SUBMISSION OF PROBLEMS AND SOLUTIONS

Starting May 1, 2000, all new problem proposals and corresponding solutions must be submitted to the Problems' Editor:

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stanley@tiac.net *on the Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-889 ***Proposed by Mario DeNobili, Vaduz, Lichtenstein***

Find 17 consecutive Fibonacci numbers whose average is a Lucas number.

B-890 ***Proposed by Stanley Rabinowitz, Westford, MA***

If $F_{-a}F_bF_{a-b} + F_{-b}F_cF_{b-c} + F_{-c}F_aF_{c-a} = 0$, show that either $a = b$, $b = c$, or $c = a$.

B-891 ***Proposed by Aloysius Dorp, Brooklyn, NY***

Let $\langle P_n \rangle$ be the Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Find integers a, b, and m such that $L_n \equiv P_{an+b} \pmod{m}$ for all integers n.

B-892 ***Proposed by Stanley Rabinowitz, Westford, MA***

Show that, modulo 47, $F_n^2 - 1$ is a perfect square if n is not divisible by 16.

B-893 ***Proposed by Aloysius Dorp, Brooklyn, NY***

Find integers a, b, c, and d so that

$$F_xF_yF_z + aF_{x+1}F_{y+1}F_{z+1} + bF_{x+2}F_{y+2}F_{z+2} + cF_{x+3}F_{y+3}F_{z+3} + dF_{x+4}F_{y+4}F_{z+4} = 0$$

is true for all x, y, and z.

B-894 ***Proposed by the editor***

Solve for x:

$$F_{110}^x + 442F_{115}^x + 13F_{119}^x = 221F_{114}^x + 255F_{117}^x.$$

SOLUTIONS

Absolute Sum

B-871 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 37, no. 1, February 1999)

Prove that

$$\sum_{k=0}^{2n}\binom{2n}{k}|n-k|^3 = n^2\binom{2n}{n}.$$

Solution by Indulis Strazdins, Riga Technical University, Latvia

The sum is equal to

$$S(n) = 2\sum_{k=0}^{n-1}(n-k)^3\binom{2n}{k} = 2n^3s_0 - 6n^2s_1 + 6ns_2 - 2s_3,$$

where the expressions

$$s_m = \sum_{k=0}^{n-1}k^m\binom{2n}{k} \quad (m = 0, 1, 2, 3)$$

can be derived from the known formulas

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n,$$

$$\sum_{k=0}^{n}k\binom{n}{k} = n\cdot 2^{n-1},$$

$$\sum_{k=0}^{n}k^2\binom{n}{k} = n(n+1)\cdot 2^{n-2},$$

$$\sum_{k=0}^{n}k^3\binom{n}{k} = n^2(n+3)\cdot 2^{n-3}.$$

The results are

$$s_0 = 2^{2n-1} - \frac{1}{2}\binom{2n}{n},$$

$$s_1 = n\left(2^{2n-1} - \binom{2n}{n}\right),$$

$$s_2 = n\left((2n+1)2^{2n-2} - \frac{3}{2}n\binom{2n}{n}\right),$$

$$s_3 = n^2\left((2n+3)2^{2n-2} - \frac{1}{2}(4n+1)\binom{2n}{n}\right).$$

Thus,

$$S(n) = (4n^3 - 12n^3 + 6n^2(2n+1) - 2n^2(2n+3))2^{2n-2} - (n^3 - 6n^3 + 9n^3 - n^2(4n+1))\binom{2n}{n} = n^2\binom{2n}{n}.$$

Bruckman noted that

$$\sum_{k=0}^{2n}\binom{2n}{k}|n-k| = n\binom{2n}{n}$$

and conjectures that

$$\sum_{k=0}^{2n}\binom{2n}{k}|n-k|^{2r-1} = P_r(n)\binom{2n}{n}$$

for some monic polynomial $P_r(n)$ of degree r.

Solutions also received by H.-J. Seiffert and the proposer.

Rational Recurrence

B-872 ***Proposed by Murray S. Klamkin, University of Alberta, Canada (Vol. 37, no. 2, May 1999)***

Let $r_n = F_{n+1}/F_n$ for $n>0$. Find a recurrence for $t_n = r_n^2$.

Solution 1 by Maitland A. Rose, University of South Carolina, Sumter, SC

$$t_n = \frac{F_{n+1}^2}{F_n^2} = \frac{F_n^2+2F_nF_{n-1}+F_{n-1}^2}{F_n^2} = 1+\frac{2F_{n-1}}{F_n}+\frac{F_{n-1}^2}{F_n^2} = 1+\frac{2}{\sqrt{t_{n-1}}}+\frac{1}{t_{n-1}}.$$

Solution 2 by Kathleen E. Lewis, SUNY, Oswego, NY

The identity $F_{n+1}^2 = 2F_n^2+2F_{n-1}^2-F_{n-2}^2$ is straightforward to prove. Dividing by F_n^2 gives

$$t_n = 2+\frac{2}{t_{n-1}}-\frac{1}{t_{n-1}t_{n-2}}.$$

Klamkin, Morrison, and Seiffert all found the corresponding recurrence for an arbitrary second-order linear recurrence $w_{n+2} = Pw_{n+1}-Qw_n$. If $t_n = (w_{n+1}/w_n)^2$, then

$$t_n = (P^2-Q)-\frac{(P^2-Q)Q}{t_{n-1}}+\frac{Q^3}{t_{n-1}t_{n-2}}.$$

Solutions also received by Brian D. Beasley, Paul S. Bruckman, Leonard A. G. Dresel, John F. Morrison, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

A Property of 3

B-873 ***Proposed by Herta Freitag, Roanoke, VA (Vol. 37, no. 2, May 1999)***

Prove that 3 is the only positive integer that is both a prime number and of the form $L_{3n}+(-1)^nL_n$.

Solution by L. A. G. Dresel, Readng, England

Put $T_n = L_{3n}+(-1)^nL_n$. Since the Binet forms for L_{3n} and L_n give the identity $L_{3n} = L_n^3-3(-1)^nL_n$, we have $T_n = L_n(L_n^2-2(-1)^n) = L_nL_{2n}$. Now $L_n = 1$ only if $n=1$, so that $T_1 = 3$. But when $n\neq 1$, T_n is the product of two integers, each greater than 1. Hence, 3 is the only prime of the form T_n.

Solutions also received by Paul S. Bruckman, Kathleen E. Lewis, John F. Morrison, Jaroslav Seibert, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Another Property of 3

B-874 ***Proposed by David M. Bloom, Brooklyn College, NY (Vol. 37, no. 2, May 1999)***

Prove that 3 is the only positive integer that is both a Fibonacci number and a Mersenne number. [A Mersenne number is a number of the form $2^a - 1$.]

Solution by the proposer

If $F_n = 2^a - 1$ with $a \geq 2$, then $F_n + 1 = 2^a$. But the general identity $F_{a+b} + (-1)^b F_{a-b} = F_a L_b$ shows that

$$\begin{array}{lll} n = 4k & \text{implies} & F_n + 1 = F_{2k-1} L_{2k+1}, \\ n = 4k+1 & \text{implies} & F_n + 1 = F_{2k+1} L_{2k}, \\ n = 4k+2 & \text{implies} & F_n + 1 = F_{2k+2} L_{2k}, \\ n = 4k+3 & \text{implies} & F_n + 1 = F_{2k+1} L_{2k+2}. \end{array}$$

Thus, if $F_n + 1 = 2^a$, the L-factor on the right must be a power of 2. But it must also be less than or equal to 4 since no Lucas number is divisible by 8. Thus, in all cases, $L_{2k} \leq 4$ and $k \geq 1$ since $F_n \geq 3$. Hence, $k = 1$ and the result follows.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, and H.-J. Seiffert.

A Third Property of 3

B-875 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France (Vol. 37, no. 2, May 1999)***

Prove that 3 is the only positive integer that is both a triangular number and a Fermat number. [A triangular number is a number of the form $n(n+1)/2$. A Fermat number is a number of the form $2^a + 1$.]

Solution by H.-J. Seiffert, Berlin

Let n be a positive integer and a a nonnegative integer such that $n(n+1)/2 = 2^a + 1$. Multiplying by 2 and then subtracting 2 on both sides yields $(n-1)(n+2) = 2^{a+1}$. Hence, $n \geq 2$, and $n-1$ and $n+2$ both must be powers of 2. Since $n-1$ and $n+2$ are of opposite parity, we then must have $n - 1 = 2^0$ or $n = 2$. This gives $n(n+1)/2 = 3 = 2^1 + 1$.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Jaroslav Seibert, and the proposer.

Trigonometric Sum

B-876 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 37, no. 2, May 1999)***

Evaluate

$$\sum_{k=1}^{n} \sin\left(\frac{\pi F_{k-1}}{F_k F_{k+1}}\right) \sin\left(\frac{\pi F_{k+2}}{F_k F_{k+1}}\right).$$

Solution by Jaroslav Seibert, University of Education, Czech Republic

For all real numbers x and y, we have

$$\sin\frac{x+y}{2}\sin\frac{x-y}{2}=-\frac{1}{2}(\cos x-\cos y).$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{n}\sin\left(\frac{\pi F_{k-1}}{F_kF_{k+1}}\right)\sin\left(\frac{\pi F_{k+2}}{F_kF_{k+1}}\right)&=\sum_{k=1}^{n}\sin\pi\left(\frac{F_{k+1}-F_k}{F_kF_{k+1}}\right)\sin\pi\left(\frac{F_{k+1}+F_k}{F_kF_{k+1}}\right)\\&=-\frac{1}{2}\sum_{k=1}^{n}\left(\cos 2\pi\frac{F_{k+1}}{F_kF_{k+1}}-\cos 2\pi\frac{F_k}{F_kF_{k+1}}\right)\\&=-\frac{1}{2}\sum_{k=1}^{n}\left(\cos 2\pi\frac{1}{F_k}-\cos 2\pi\frac{1}{F_{k+1}}\right)\\&=-\frac{1}{2}\left(\cos 2\pi\frac{1}{F_1}-\cos 2\pi\frac{1}{F_{n+1}}\right)\\&=\frac{1}{2}\left(\cos\frac{2\pi}{F_{n+1}}-1\right)=-\sin^2\frac{\pi}{F_{n+1}}.\end{aligned}$$

Solutions also received by Paul S. Bruckman, Charles K. Cook, Mario DeNobili, Leonard A. G. Dresel, John F. Morrison, Maitland A. Rose, H.-J. Seiffert, and the proposer.

Determining the Determinant

B-877 ***Proposed by Indulis Strazdins, Riga Technical University, Latvia (Vol. 37, no. 2, May 1999)***

Evaluate

$$\begin{vmatrix}F_nF_{n+1} & F_{n+1}F_{n+2} & F_{n+2}F_{n+3} & F_{n+3}F_{n+4}\\ F_{n+4}F_{n+5} & F_{n+5}F_{n+6} & F_{n+6}F_{n+7} & F_{n+7}F_{n+8}\\ F_{n+8}F_{n+9} & F_{n+9}F_{n+10} & F_{n+10}F_{n+11} & F_{n+11}F_{n+12}\\ F_{n+12}F_{n+13} & F_{n+13}F_{n+14} & F_{n+14}F_{n+15} & F_{n+15}F_{n+16}\end{vmatrix}.$$

Solution by the proposer

Let $P_n=F_nF_{n+1}$. It is straightforward to prove the identity

$$P_{n+3}=2P_{n+2}+2P_{n+1}-P_n.$$

Hence, the 4th column is a linear combination of the first three ones, and therefore the determinant is 0.

Most of the solvers pointed out analogous results for larger determinants. If the determinant contains the product of k Fibonacci numbers, $F_nF_{n+1}\cdots F_{n+k-1}$, then the determinant is 0 when the order of the determinant is at least $k+2$.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

Harmonic Inequality

B-878 ***Proposed by L. A. G. Dresel, Reading, England***
(Vol. 37, no. 3, August 1999)

Show that, for positive integers n, the harmonic mean of F_n and L_n can be expressed as the ratio of two Fibonacci numbers, and that it is equal to $L_{n-1}+R_n$, where $|R_n| \le 1$. Find a simple formula for R_n.

Note: If h is the harmonic mean of x and y, then $2/h = 1/x + 1/y$.

Solution by Harris Kwong, SUNY College at Fredonia, NY

The harmonic mean of F_n and L_n is given by

$$\frac{2F_nL_n}{F_n+L_n} = \frac{2F_{2n}}{F_n+F_{n-1}+F_{n+1}} = \frac{F_{2n}}{F_{n+1}} = L_{n-1} + \frac{(-1)^n}{F_{n+1}},$$

in which $F_{2n} = F_{n+1}L_{n-1} + (-1)^n$ follows from Binet's formulas.

Solutions also received by Paul S. Bruckman, Charles K. Cook, Don Redmond, H.-J. Seiffert, James A. Sellers, Indulis Strazdins, and the proposer.

Addenda. We wish to belatedly acknowledge solutions from the following solvers:

Brian Beasley solved B-854, 855, 857, 860, 862, 863, and 864.
L. A. G. Dresel solved B-866, 867, 868, 869, and 870.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-559 ***Proposed by N. Gauthier, Royal Military College of Canada***

Let n and q be nonnegative integers and show that:

a.
$$S_n(q) := \sum_{k=1}^{n} \frac{1}{2\cos(2\pi k/n) + (-1)^{q+1}L_{2q}} = \frac{(-1)^{q+1}nL_{qn}}{5F_{2q}F_{qn}}.$$

b.
$$s_n(q) := \sum_{k=1}^{n} \frac{1}{0.8\sin^2(2\pi k/n) + F_{2q}^2} = \frac{nL_{2qn}}{F_{2q}L_{2q}F_{qn}L_{qn}}, \quad n \text{ odd},$$
$$= \frac{nL_{qn}}{F_{2q}L_{2q}F_{qn}}, \quad n \text{ even}.$$

L_n and F_n are Lucas and Fibonacci numbers.

H-560 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the sequences of Fibonacci and Lucas polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ \text{and}\ F_{n+1}(x) = xF_n(x) + F_{n-1}(x),\ n \in N,$$

and

$$L_0(x) = 2,\ L_1(x) = x,\ \text{and}\ L_{n+1}(x) = xL_n(x) + L_{n-1}(x),\ n \in N,$$

respectively. Show that, for all complex numbers x and all positive integers n,

$$\sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} x^k F_{3k}(x) = F_{2n}(x) + (-x)^n F_n(x)$$

and

$$\sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} x^k L_{3k}(x) = L_{2n}(x) + (-x)^n L_n(x).$$

SOLUTIONS

Continuing...

H-543 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY (Vol. 36, no. 4, August 1998)***

Find all positive nonsquare integers d such that, in the continued fraction expansion

$$\sqrt{d} = \left[n; \overline{a_1, \ldots, a_{r-1}, 2n}\right],$$

we have $a_1 = \cdots = a_{r-1} = 1$. (This includes the case $r = 1$ in which there are no a's.)

Solution by Charles K. Cook, University of South Carolina Sumter, Sumter, SC

For the case $[n; \overline{2n}]$, it is known (see [1], p. 80) that $x = [\overline{2n}]$ satisfies $x^2 = 2nx + 1$. Thus, $x = n + \sqrt{n^2+1}$ and so

$$\sqrt{d} = n + \frac{1}{n + \sqrt{n^2+1}}$$

which simplifies to $d = n^2 + 1$.

Setting y equal to the periodic expansion and recovering a relationship for y using the usual formal manipulations on the continued fraction representation

$$y = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots + \cfrac{1}{2n + \cfrac{1}{y}}}}$$

yields the following equations for y:

$$\begin{array}{ll}
y = \left[0; \overline{1, 2n}\right] & 2ny^2 - 2ny - 1 = 0 \\
y = \left[0; \overline{1, 1, 2n}\right] & (2n+1)y^2 - 4ny - 2 = 0 \\
y = \left[0; \overline{1, 1, 1, 2n}\right] & (4n+1)y^2 - 6n - 3 = 0 \\
y = \left[0; \overline{1, 1, 1, 1, 2n}\right] & (6n+2)y^2 - 10ny - 5 = 0 \\
y = \left[0; \overline{1, 1, 1, 1, 1, 2n}\right] & (10n+3)y^2 - 16ny - 8 = 0
\end{array}$$

and, in general, if F_m is the m^{th} Fibonacci number, then $y = [0; \overline{m\text{-ones}, 2n}]$ and y satisfies $(2nF_m + F_{m-1})y^2 - 2nF_{m+1}y - F_{m+1} = 0$, which can be shown by a routine inductive argument.

Thus,

$$n^2 + \frac{(2n-1)F_m + F_{m+1}}{F_{m+1}}$$

must be integral. So both

$$n^2 + 1 + \frac{(2n-1)F_m}{F_{m+1}} \quad \text{and} \quad \frac{(2n-1)F_m}{F_{m+1}}$$

are integral.

However, $\gcd(F_m, F_{m+1}) = 1$, so $2n \equiv 1 \pmod{F_{m+1}}$. Hence, F_{m+1} must be odd. Therefore, $\gcd(2, F_{m+1}) = 1$, and the linear congruence $2n \equiv 1 \pmod{F_{m+1}}$ always has a solution. Thus, if m is the number of ones in the continued fraction expansion, it follows that

$$d = n^2 + 1 + \frac{(2n-1)F_m}{F_{m+1}}$$

provided F_{m+1} is odd.

A few solutions are shown in the table below.

m	F_m	$n = n(k), k \geq 1$	n values	$d = d(k)$	d values
0	1	k	$1, 2, 3, \ldots$	$k^2 + 1$	$2, 5, 10, \ldots$
1	1	k	$1, 2, 3, \ldots$	$k^2 + 2k$	$3, 8, 15, \ldots$
2	2	None	None	$k^2 + k + \frac{1}{2}$	None
3	3	$3k - 1$	$2, 5, 8, \ldots$	$9k^2 - 2k$	$7, 32, 75, \ldots$
4	5	$5k - 2$	$3, 8, 13, \ldots$	$25k^2 - 14k + 2$	$13, 74, 185, \ldots$
5	8	None	None	$k^2 + (10k + 3) / 8$	None
6	13	$13k - 6$	$7, 20, 33, \ldots$	$169k^2 - 140k + 29$	$58, 425, 1130, \ldots$
7	21	$21k - 10$	$11, 32, 53, \ldots$	$441k^2 - 394k + 88$	$135, 1064, 2875, \ldots$
8	34	None	None	$k^2 + (42k + 13) / 34$	None

Reference

1. C. D. Olds. *Continued Fractions.* Washington, D.C.: The Mathematical Association of America, 1963.

Also solved by P. Bruckman, A. Tuyl, and the proposer.

Primes and FPP's

H-544 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 36, no. 4, August 1998)

Given a prime $p > 5$ such that $Z(p) = p + 1$, suppose that $q = \frac{1}{2}(p^2 - 3)$ and $r = p^2 - p - 1$ are primes with $Z(q) = q + 1$, $Z(r) = \frac{1}{2}(r - 1)$. Prove that $n = pqr$ is a FPP (see previous proposals for definitions of the Z-function and of FPP's).

Solution by the proposer

For all natural m such that $\gcd(m, 10) = 1$, let ε_m denote the Jacobi symbol $(5/m)$, and $m' = m - \varepsilon_m$. If s is any prime $\neq 2, 5$, it is well known that $Z(s) \mid s'$. We then see that $\varepsilon_p = \varepsilon_q = -1$, $\varepsilon_r = \varepsilon_n = \varepsilon_p \varepsilon_q \varepsilon_r = +1$. Thus, $p \equiv \pm 3$, $q \equiv \pm 3$, $r \equiv \pm 1$, $n \equiv \pm 1 \pmod{10}$.

Now, if s is any prime $\neq 2, 5$ and $a(s) = s' / Z(s)$, then $a(s)$ and $\frac{1}{2}(s-1)$ have the same parity (see this journal, Problem H-494, Vol. 33, no. 1, Feb. 1995; solution in Vol. 34, no. 2, Aug. 1996, pp. 190-91). Since $a(p) = a(q) = 1$, $a(r) = 2$, it follows that $p \equiv q \equiv 3$, $r \equiv n \equiv 1 \pmod 4$. Also, $3^2 - 3 - 1 = 5$, which shows that r cannot be prime if $p \equiv 3 \pmod{20}$. Therefore, $p \equiv 7 \pmod{20}$; this in turn implies that $q \equiv 3$, $r \equiv n \equiv 1 \pmod{20}$.

Next, we see that $Z(q) = \frac{1}{2}(p^2 - 1)$, $Z(r) = \frac{1}{2}(p+1)(p-2)$. Then

$$Z(n) = \operatorname{lcm}\{Z(p), Z(q), Z(r)\} = \tfrac{1}{2}(p^2 - 1)(p - 2).$$

In order to show that n is a FPP, it suffices to show that $n - 1 = n' \equiv 0 \pmod{Z(n)}$. Now

$$pq-r=\tfrac{1}{2}(p^3-2p^2-p+2)=\tfrac{1}{2}(p^2-1)(p-2)=Z(n);$$

hence, $pq \equiv r \pmod{Z(n)}$. Then $n \equiv r^2 \pmod{Z(n)}$. Next, $r+1=p(p-1)$, $r-1=(p+1)(p-2)$, whence $r^2-1=p(p^2-1)(p-2)=2pZ(n)\equiv 0 \pmod{Z(n)}$. Thus, $n'=n-1\equiv r^2-1\equiv 0$ (mod $Z(n)$), which shows that n is a FPP. Q.E.D.

Note: The smallest FPP satisfying the above conditions is $7\cdot 23\cdot 41$ $(p=7)$.

Also solved by H.-J. Seiffert.

An Interesting Equation

H-553 ***Proposed by Paul S. Bruckman, Berkeley, CA (Vol. 37, no. 3, August 1999)***

The following Diophantine equation has the trivial solution $(A, B, C, D)=(A, A, A, 0)$:

$$A^3+B^3+C^3-3ABC=D^k, \text{ where } k \text{ is a positive integer.} \tag{1}$$

Find nontrivial solutions of (1), i.e., with all quantities positive integers.

Solution (1) by the proposer

Let

$$\theta=\exp\left(\tfrac{2}{3}i\pi\right), \tag{2}$$

$$K(a,b,c)=a^3+b^3+c^3-3abc. \tag{3}$$

As we may easily verify:

$$K(a,b,c)=s(a,b,c)\cdot s(a,b\theta,c\theta^2)\cdot s(a,b\theta^2,c\theta), \tag{4}$$

where

$$s(a,b,c)=a+b+c. \tag{5}$$

Given U, V, W positive integers, where at least two of them are distinct, let

$$X=(s(U,V,W))^k,\ Y=(s(U,V\theta,W\theta^2))^k,\ Z=(s(U,V\theta^2,W\theta))^k. \tag{6}$$

From (4), it follows that

$$XYZ=(K(U,V,W))^k. \tag{7}$$

Now define the following quantities:

$$A=\tfrac{1}{3}s(X,Y,Z),\ B=\tfrac{1}{3}s(X,Y\theta^2,Z\theta),\ C=\tfrac{1}{3}s(X,Y\theta,Z\theta^2). \tag{8}$$

Again using (4), we see that

$$27ABC=K(X,Y,Z). \tag{9}$$

We now employ the following well-known expression:

$$\tfrac{1}{3}(1+\theta^r+\theta^{2r})=\begin{cases}1 & \text{if } 3\mid r,\\ 0 & \text{if } 3\nmid r.\end{cases} \tag{10}$$

By trinomial expansion of the quantities defined in (8), implementing (6) and (10), we obtain the following expressions:

$$A=F_0(U,V,W),\ B=F_1(U,V,W),\ C=F_2(U,V,W), \tag{11}$$

where

$$F_j(U,V,W) = \sum_{\substack{f+g+h=k \\ g-h\equiv j \pmod 3}} \binom{k}{f,g,h} U^f V^g W^h, \; j=0,1,2, \tag{12}$$

and $\binom{k}{f,g,h}$ is a trinomial coefficient $= \dfrac{k!}{f!g!h!}$.

From (12), it is clear that A, B, and C are positive integers. We may also easily verify the following inverse relations:

$$X = s(A,B,C),\; Y = s(A,B\theta,C\theta^2),\; Z = s(A,B\theta^2,C\theta). \tag{13}$$

Again using (4), this implies

$$XYZ = K(A,B,C). \tag{14}$$

From (7) and (14), it follows that

$$K(A,B,C) = (K(U,V,W))^k. \tag{15}$$

Thus, by reference to (1), we see that we may set

$$D = K(U,V,W). \tag{16}$$

Accordingly, solutions (A,B,C,D) of (1) are given by (11) and (16); alternatively, A, B, and C may be obtained indirectly from (8) and (6).

Note that the restriction that U, V, and W be not all identical ensures that Y and Z are positive, as of course is X. Then, from (7) and (16), it follows that $D>0$, which avoids trivial solutions.

Solution (2) by John Jaroma and Rajib Rahman, Gettysburg College, Gettysburg, PA

After a brief historical background, we will show that, in fact, there are an infinite number of solutions of (1), subject to (2):

$$A^3+B^3+C^3-3ABC = D^k; \tag{1}$$

$$A,B,C,D \in \{1,2,\ldots\} \quad\text{and}\quad k\in\{2,3,\ldots\}. \tag{2}$$

First, in terms of a historical perspective, it appears that Diophantine equations involving cubic terms have generated considerable interest. For example, in 1847, J. J. Sylvester provided sufficient conditions for the insolubility in integers of the equation

$$Ax^3+By^3+Cz^3 = Dxyz. \tag{3}$$

Moreover, Sylvester was able to prove that whenever (3) is insoluble, there must exist an entire family of related equations equally insoluble. His motivation for studying such equations was to break ground in the area of third-degree equations. Ultimately, Sylvester had hoped to open a new field in connection with Fermat's Last Theorem.

Today, cubic equations continue to command a great deal of attention. For instance, although we know that every number (with the possible exception of those in the form $9n\pm4$) can be expressed as the sum of four cubes, it is still not known whether every number can be expressed as the sum of four cubes with two of the cubes equal. Stated algebraically, we would like to know, if given any k, do integral solutions exist for the Diophantine equation

$$A^3+B^3+2C^3 = k. \tag{4}$$

($k=76$ is the first of many values of k for which an integral solution is not known.)

Perhaps an even more difficult problem exists in the question whether numbers not of the form $9n \pm 4$ can be expressed as the sum of three cubes; that is, does the equation

$$A^3 + B^3 + C^3 = k \tag{5}$$

have a solution in integers $\forall k \neq 9n \pm 4$? The first known value of k for which the problem becomes open is $k = 30$. Furthermore, even if we restrict ourselves to the specific case $k = 3$, we do not know whether (1, 1, 1) and (4, 4, − 5) are the only two solutions of (5).

It is likely that Diophantine equations will continue to be an area of research for some time to come, for we know that, given an arbitrary Diophantine equation, there cannot exist an algorithm which in a finite number of steps will decide its solvability. Hilbert's Tenth Problem was demonstrated to be unsolvable by Yuri Matiyasevich in 1970.

Consider the following infinite sets:

(I) $p \in \{1, 2, \dots\},\ k = 3p + 1,\ n_1, n_2 \in \{1, 2, \dots\} : n_1 / n_2 \in \{2, 3, \dots\},$
$D = 1 + n_1^3 + (n_1 / n_2)^3 - 3(n_1 / n_2)n_1,$
$A = D^p,\ B = (n_1 / n_2)A,\ C = n_1 A = (n_2 B).$

(II) $k = 2,\ n \in \{1, 2, \dots\},$
$B = D = 9n^2,\ A = D - n,\ C = D + n.$

Remark: We have ignored the case where $p = 0$, for this would imply that $k = 1$ and it would then be trivial to produce infinitely many solutions of (1).

Proposition: Sets (I) and (II) represent disjoint families of solutions of (1) satisfying (2).

Proof: We first prove that (I) and (II) are disjoint families of solutions of (1). Since elements of (I) and (II) are ordered 4-tuples of the form (A, B, C, D) and $p \in \{1, 2, \dots\}$, it follows immediately that (I) and (II) are disjoint as $3p + 1 \neq 2$.

Now, to show that (I) represents an infinite set of solutions of (1), we let $n = n_2$. Hence, $n_1 = bn$ for some $b \in \{2, 3, \dots\}$ and

$$D = 1 + b^3 + b^3 n^3 - 3b^2 n,\ B = bA,\ C = nbA = nB. \tag{6}$$

Substituting (6) into (1), we get

$$D^{3p} + b^3 D^{3p} + n^3 b^3 D^{3p} - 3nb^2 D^{3p} = D^{3p+1}. \tag{7}$$

Rewriting (7), we obtain

$$D^{3p}(1 + b^3 + n^3 b^3 - 3nb^2) = D^{3p+1}. \tag{8}$$

Thus, (8) is true if and only if $1 + b^3 + b^3 n^3 - 3b^2 n = D$. By (6), the result follows immediately.

Finally to show that (II) is also an infinite family of solutions of (1), we infer from (II) that $B = D = n + A$ and $C = 2n + A$.

Substituting these quantities and the hypothesis that $k = 2$ into (1), we obtain

$$A^3 + (n + A)^3 + (2n + A)^3 - 3A(n + A)(2n + A) = (n + A)^2. \tag{9}$$

Simplifying (9), we obtain $9n^2(n + A) = (n + A)^2$. It now follows that (II) is a set of solutions of (1) if and only if $9n^2 - n - A = 0$. But, by hypothesis, $A = D - n = 9n^2 - n$, and this produces the desired result.

Also solved by B. Beasley, C. Cook, and H.-J. Seiffert.

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Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972. $23.00

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972. $32.00

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