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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

ON FIBONACCI SEQUENCES, GEOMETRY, AND THE m-SQUARE EQUATION

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1. INTRODUCTION

The general solution of Pythagoras' 3-square equation, namely $x^2+y^2=z^2$, for integer triples (x, y, z), has been known since Euclid in the 4th century B.C. In 1988, Georg Schaake and John Turner discovered a new way of finding all solutions, using the rational number tree to help find and classify them [5]. Their methods were heavily dependent on continued fractions, and hence closely related to the arithmetic of Fibonacci sequences.

Recently, when considering properties of certain triangles in $\mathbf{R}^3$ in relation to Fibonacci sequences, Turner discovered solutions for the 4-square equation $x^2+y^2+z^2=w^2$ in integer quadruples (x, y, z, w), with x, y, z, w each being a function of Fibonacci numbers. A simple number tree helped in this discovery too. The idea easily generalized (see Section 3, below), providing a two-variable identity which defined a more general class of integer solutions to the 4-square equation.

The two-variable identity that we found turned out to be a special case of one given by Catalan in 1877; it was discovered independently by Dainelli, and published also in 1877 (see [1], [2], [3]). Modern treatments of the 4-square equation do not refer to these identities, and the general solutions of it do not point to or suggest relationships with Fibonacci numbers (see, e.g., Sierpinski's book [6]).

We believe that our manner of finding such relationships may be new, and that the story of their discovery, from a triangle which we decided to call an *F-triangle*, will be found interesting. Moreover, we show how the vector geometric approach may be exploited, and discover further interesting results about sequences of F-triangles, presenting us with a variety of Fibonacci identities.

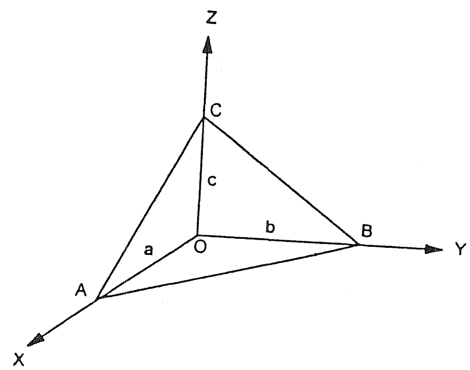
Further study of the 4-square identity led to discovery of infinite classes of integer-solutions, in terms of F_{n-1}, F_n, and F_{n+1}, for both the 3-square equation (Pythagoras') and the 5-square equation. Later we extended this process to provide solution-class formulas for the m-square equation, with $m=6, 7, 8, \ldots$. These formulas for the solutions of the infinite sequence of equations are presented in the final section of the paper.

2. AREAS OF F-TRIANGLES AND 4-SQUARE SOLUTIONS

A triple of positive integers (a, b, c) will be associated with the triangle ABC, where $A=(a, 0, 0)$, $B=(0, b, 0)$, $C=(0, 0, c)$. The diagram below shows the arrangement of axes and triangle. The area of the triangle, denoted by Δ, is easily computed from elementary vector analysis, thus:

$$\Delta = \frac{1}{2}|\underline{AB} \times \underline{AC}| = \frac{1}{2}|(-a, b, 0) \times (-a, 0, c)| = \frac{1}{2}|(bc, ac, ab)|.$$

Thus, $\Delta = \frac{1}{2}\sqrt{S}$, where $S = (ab)^2 + (bc)^2 + (ca)^2$. Any choice of a, b, c for which S is a square provides a solution of the 4-square equation, with $x = ab$, $y = bc$, and $z = ca$.



The case $a = F_{n-1}$, $b = F_n$, $c = F_{n+1}$, where F_n is the n^{th} Fibonacci number, is of special interest. It seems appropriate to call this an *F-Triangle*, where $n = 2, 3, \ldots$. The area of the n^{th} F-triangle is then:

$$\Delta_n = \frac{1}{2}\sqrt{S_n}, \text{ with } S_n = (F_{n-1}F_n)^2 + (F_nF_{n+1})^2 + (F_{n-1}F_{n+1})^2.$$

As the following table indicates, S_n seems to be a perfect square.

n	2	3	4	5
S_n	3^2	7^2	19^2	49^2

The formula $S_n = (F_{2n-1} + F_{n-1}F_n)^2$ (or, equivalently, $S_n = (F_{n+1}^2 - F_nF_{n-1})^2$) was found by inspection. This corresponds to the class of solutions

$$x = F_{n-1}F_n,\ y = F_nF_{n+1},\ z = F_{n-1}F_{n+1},\ w = F_{2n-1} + F_{n-1}F_n$$

of the 4-square equation $x^2 + y^2 + z^2 = w^2$. For $n = 2, 3, 4, 5$, the respective solutions (x, y, z, w) are $(1, 2, 2, 3)$, $(2, 3, 6, 7)$, $(6, 10, 15, 19)$, and $(15, 24, 40, 49)$, all of which happen to be primitive.

The formula given for S_n can be proved to be correct with algebraic manipulation and use of Fibonacci identities. This proof is omitted, however, since in the next section it will be discovered that the success of our approach owes to the fact that $c = a + b$, and not to the particular choice of a and b. This observation will lead to a simple proof of a useful theorem, and to additional classes of solutions of the 4-square equation.

3. MORE GENERAL CLASSES OF SOLUTIONS OF THE 4-SQUARE EQUATION

In Section 2, it was shown that the area Δ of the triangle ABC associated with the positive integer triple (a, b, c) was $\Delta = \frac{1}{2}\sqrt{S}$, where $S = (ab)^2 + (bc)^2 + (ca)^2$. If $c = a + b$, then:

$$\begin{aligned} S &= (ab)^2 + (b(a+b))^2 + ((a+b)a)^2 \\ &= a^4 + b^4 + 3a^2b^2 + 2(a^3b + b^3a)^2 \\ &= (a^2 + b^2 + ab)^2. \end{aligned}$$

This calculation gives us the following theorem.

Theorem: Let a and b be positive integers. Then

$$x = ab,\ \ y = b(a+b),\ \ z = a(a+b),\ \ w = (a^2 + b^2 + ab)$$

is a solution of the 4-square equation $x^2 + y^2 + z^2 = w^2$. □

Any general Fibonacci-type sequence $a, b, a+b, a+2b, 2a+3b, \ldots$ thereby generates an infinite sequence of *general* F-triangles, and a corresponding infinite class of solutions of the 4-square equation.

Applications

(i) The F-triangles of Section 2 give $w = F_{n-1}^2 + F_n^2 + F_{n-1}F_n = F_{2n-1} + F_nF_{n-1}$, proving the result of Section 2.

(ii) The area of the triangle corresponding to the triple $(a, b, a+b)$ is $\Delta = (a^2 + b^2 + ab)/2$. Thus, the area is integral if and only if a and b are both even. Since no two consecutive Fibonacci numbers are even, the Fibonacci F-triangles of Section 2 all have half-integer areas; the first four of these are $3/2$, $7/2$, $19/2$, $49/2$.

(iii) Another example, this time involving Lucas numbers, is the following. Using $(a, b, c) = (L_{n-1}, L_n, L_{n+1})$ gives:

$$(L_{n-1}L_n)^2 + (L_{n-1}L_{n+1})^2 + (L_nL_{n+1})^2 = (2L_{2n} + (-1)^{n-1})^2.$$

Hence, $x = L_{n-1}L_n$, $y = L_{n-1}L_{n+1}$, $z = L_nL_{n+1}$, and $w = 2L_{2n} + (-1)^{n-1}$ is another class of solutions to the 4-square equation.

(iv) Yet another example, this time involving both Fibonacci and Lucas numbers, is the following triangle, and identity. Using $(a, b, c) = (F_{n-1}, F_{n+1}, L_n)$ gives:

$$(F_{n-1}F_{n+1})^2 + (F_{n+1}L_n)^2 + (L_nF_{n-1})^2 = (F_{n-1}^2 + L_nF_{n+1})^2.$$

Hence, $x = F_{n-1}F_{n+1}$, $y = F_{n+1}L_n$, $z = L_nF_{n-1}$, and $w = F_{n-1}^2 + L_nF_{n+1}$ is another class of solutions to the 4-square equation.

4. SOME PROPERTIES OF F-TRIANGLES

There are several interesting geometric properties of the Fibonacci F-triangles defined in Section 2. Similar procedures for the more general F-triangles would lead to other Fibonacci and Lucas identities.

Area Differences: The area of the n^{th} Fibonacci F-triangle is $\Delta_n = (F_{2n-1} + F_{n-1}F_n)/2$.

Then the difference of successive triangle-areas is given by the remarkably simple formula

$$\Delta_{n+1} - \Delta_n = F_nF_{n+1}F_{2n+1}.$$

Area Ratio: The ratio of successive areas is:

$$\frac{\Delta_{n+1}}{\Delta_n} = \frac{F_nF_{n+1} + F_{2n+1}}{F_{n-1}F_n + F_{2n-1}}.$$

This ratio tends to α^2 as n tends to infinity, where α is the golden ratio.

An Application of Hero's Formula: Hero's well-known formula for the area of a triangle of sides u, v, w is $\sqrt{s(s-u)(s-v)(s-w)}$, where $s=(u+v+w)/2$. For the n^{th} Fibonacci triangle, the respective lengths are given (using the Pythagorean theorem and some Fibonacci number identities) by the formulas:

$$u = AB = \sqrt{F_{n-1}^2 + F_n^2} = \sqrt{F_{2n-1}},$$
$$v = BC = \sqrt{F_n^2 + F_{n+1}^2} = \sqrt{F_{2n+1}},$$
$$w = CA = \sqrt{F_{n-1}^2 + F_{n+1}^2}.$$

But we have already derived another formula for Δ_n, so equating the two expressions for this area gives us the rather amazing identity:

$$\frac{1}{2}(F_{n-1}F_n + F_{2n-1}^2) = \sqrt{s\left(s-\sqrt{F_{2n-1}}\right)\left(s-\sqrt{F_{2n+1}}\right)\left(s-\sqrt{F_{n-1}^2+F_{n+1}^2}\right)},$$

where $s=(u+v+w)/2$.

Applying the Law of Cosines: Let θ be the angle between AB and BC. Using the cosine formula $w^2 = v^2 + u^2 - 2vu\cos\theta$, and also direction ratios for AB and BC, we can find expressions for $\cos\theta$ in two ways. Equating these gives the following identity:

$$L_n^2 - L_{2n} = 2(F_{n-1}F_{n+1} - F_n^2).$$

5. CLASSES OF SOLUTIONS FOR THE GENERAL m-SQUARE EQUATIONS

Further study of the 4-square identity found as described in Section 2, together with application of formulas from [5, p. 97], led us to solutions, in terms of Fibonacci numbers, of both the ancient 3-square equation for sides of a right triangle (Pythagoras') and the 5-square equation. We then let m be the number of square terms in an equation, and sought general classes of solutions for $m \in \mathbf{N}$, $m > 2$.

Having obtained solutions for case $m=5$ from those of the case $m=4$, it quickly became obvious how we might extend the process indefinitely; that is, give solution classes for $m = 6$, 7, 8, and so on.

We should point out that the solution to the 3-square equation was obtained from the 4-square one by a backward deductive process, which is not worth spelling out here in detail. The solution's validity can be checked easily using elementary Fibonacci identities. Whereas the solutions for cases $m = 5, 6, 7, \ldots$ follow in sequence, from the 4-square solution, using the method given below under solution (3).*

* If this method were to be applied to the solution of the 3-square equation, another, different infinite list of equation solutions would result for $m = 4, 5, 6, \ldots$. We were concerned only to present one such list, and to concentrate on the $m = 4$ (geometric) case as pivotal.

We end the paper by showing these solution classes (or an algorithm to obtain them sequentially), for the sequence of m-square equations, all expressed in terms of three consecutive Fibonacci numbers.

It must be pointed out that none of these solution classes constitutes a general solution for its equation—each is merely an interesting infinite subclass of the whole set of solutions.

We invite the reader to share the authors' view that there is much poetry in the following list. Might we call the list *An Infinite Ode on Square Equations*, or perhaps, *A Square Dance in Fibonacci Numbers*?

Solutions to m-Square Equations

(1) The equation: $x^2+y^2=z^2,\ m=3$.

Solution:

$$x=F_{n-1}^2+F_n^2+F_{n-1}F_n,$$
$$y=2F_{n-1}F_n^2F_{n+1},$$
$$z=2F_{n-1}F_n^2F_{n+1}+1=y+1.$$

(2) The equation: $x^2+y^2+z^2=w^2,\ m=4$.

Solution:

$$x=F_{n-1}F_n,$$
$$y=F_{n-1}F_{n+1},$$
$$z=F_nF_{n+1},$$
$$w=F_{n-1}^2+F_n^2+F_{n-1}F_n.$$

(3) The equation: $x^2+y^2+z^2+u^2=v^2,\ m=5$.

Solution:

$$x=F_{n-1}F_n,$$
$$y=F_{n-1}F_{n+1},$$
$$z=F_nF_{n+1},$$
$$u=\tfrac{1}{2}[(F_{n-1}^2+F_n^2+F_{n-1}F_n)^2-1],$$
$$v=\tfrac{1}{2}[(F_{n-1}^2+F_n^2+F_{n-1}F_n)^2+1].$$

We can easily extend these solution classes so that, for any given m, we can compute a solution class for the m-square equation in terms of three consecutive Fibonacci numbers. The method is as follows:

Suppose that we have a solution $(x_1, x_2, \ldots, x_{m-2}, x_{m-1})$ for the $(m-1)$-square equation. Then take the function $x_{m-1}=f(F_{n-1}, F_n, F_{n+1})$ from this solution, and form the identity:

$$x_{m-1}^2\equiv\left[\tfrac{1}{2}(x_{m-1}^2+1)\right]^2-\left[\tfrac{1}{2}(x_{m-1}^2-1)\right]^2.$$

Then, if $u=\frac{1}{2}(x_{m-1}^2-1)$ and $v=\frac{1}{2}(x_{m-1}^2+1)$, we can state that $(x_1, x_2, \ldots, x_{m-2}, u, v)$ is an integer solution of the m-square equation, provided that u and v are integers, which they are iff x_{m-1} is odd.

Beginning with the proven case $m=4$, applying this method supplies the solution to case $m=5$ as shown above, since $w=F_{n-1}^2+F_n^2+F_{n-1}F_n$ in (2) is always odd. Then, applying the method again gives a solution for $m=6$; and so on *ad infinitum*.

Note that each of the above solutions is expressed in terms of the three consecutive Fibonacci integers, F_{n-1}, F_n, and F_{n+1}. Of course, we could easily eliminate one of these from the formulas, since $F_{n-1}+F_n=F_{n+1}$; however, that would defeat one of the objectives of the paper—namely, to show how neat identities and solutions arise from consideration of three consecutive numbers taken from Fibonacci sequences.

As a final comment, we note that the method we have just given to extend our classes of solutions to give a general solution for the m-square case uses precisely the same identity which gives that solution to Pythagoras' equation (case $m=3$), which is credited to Pythagoras himself. This same solution was also found by Fibonacci in Proposition 1 of his book *Liber Quadratorum* [Book of Squares] published in 1225 [4]. We are somewhat surprised that Fibonacci did not make our extension to the m-square equation in his book of squares. Perhaps he regarded it as trivially obvious! We wonder whether he would have enjoyed our connection of the solutions with Fibonacci numbers.

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REFERENCES

1. E. Catalan. *Nouv. Correspond. Math.* **3** (1877):29.
2. U. Dainelli. *Giornale di Mat.* **15** (1877):378.
3. L. E. Dickson. *History of the Theory of Numbers.* Vol. 2, p. 266. New York: Chelsea, 1952.
4. Leonardo Pisano. *Liber Quadratorum*, 1225. In English: *The Book of Squares*, trans. L. E. Sigler. New York: Academic Press, 1987.
5. A. G. Schaake & J. C. Turner. *A New Chapter for Pythagorean Triples.* New Zealand: Department of Mathematics, University of Waikato, 1989. 155 pages.
6. W. Sierpinski. *Elementary Theory of Numbers*, MM 42. Warsaw, 1964.

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MAXIMAL SUBSCRIPTS WITHIN GENERALIZED FIBONACCI SEQUENCES

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The generalized Fibonacci sequence $\{H_n\}$, where $H_n = H_{n-1} + H_{n-2} = H_n(a, b)$, $H_1 = a$, $H_2 = b$, a and b integers, has been studied in the classic paper by Horadam [8], and by Hoggatt [7], and Brousseau [1], [2], among others. In this paper we approach the problem of representing a positive integer N as a term in one of these generalized sequences so that, for $N = H_R(A, B)$, the subscript R is as large as possible.

Cohn [5] has solved a similar problem, but part of his theorem statement was omitted. Cohn's problem was: given a large positive integer N, find positive integers A, B such that the sequence $\{w_n\}$ defined by $w_1 = A$, $w_2 = B$, and $w_{n+2} = w_{n+1} + w_n$, $n \geq 1$, contains N, and $A + B$ is minimal.

Cohn's Theorem (Restated): Let $t_n = (-1)^n(NF_{n-1} - tF_n)$, where $t_k = A + B$, $t_{k+1} = B$, $t_{k+2} = A$. Then either $t = [N/\alpha]$ or $t = [N/\alpha] + 1$, where $[x]$ is the greatest integer in x and $\alpha = (1+\sqrt{5})/2$, gives the smallest value for $t_k = A + B$, depending upon n even or n odd.

Our problem has a different approach and allows computation of subscripts. The number $R_{\max}$ of this paper is related to a conjecture made by Hoggatt and proved by Klarner [9] that, for "n sufficiently large," $R(H_n - 1) = R(H_{n+1} - 1)$, where $R(N)$ is the number of representations of N as the sum of distinct Fibonacci numbers; $R_{\max}$ gives the value for n to be "sufficiently large" [3].

1. INTRODUCTION

In order to discuss maximal subscripts, we need a careful analysis of where we want the sequences $\{H_n(A, B)\}$ to begin. The Lucas sequence has $L_0 = 2$, with terms to the right strictly increasing, while $L_{-1} = -1$ is the first negative term in an alternating sequence to the left of L_0. A generalized Fibonacci sequence in which $H_{n+1} = H_n + H_{n-1}$, $H_1 = a \geq 1$, $H_2 = b \geq 1$, has $H_0 = b - a$, where we list terms to the right and the left of H_0 as

$$\ldots, 2b - 3a,\ 2a - b,\ H_0 = b - a,\ a,\ b,\ a + b,\ a + 2b, \ldots.$$

If we want $H_{-1} < 0$ as the first negative term, we need $b > 2a$; then $(2a - b) < 0$ as well as $(b - a) > a$ and $b > a$. Then, $H_1 = a$ is the smallest positive term in the generalized sequence and the terms to the right of H_0 are strictly increasing. The Fibonacci sequence, however, has $F_0 = 0$ with strictly increasing terms to the right of $F_1 = 1$, and the sequences $\{aF_n\}$ are the only sequences $\{H_n\}$ which contain $H_k = 0$. We write $H_{-1} = 0$, $H_0 = a \geq 1$, $H_1 = a$, $H_2 = b = 2a$:

$$\ldots, -3a,\ 2a,\ -a,\ a,\ 0,\ H_0 = a,\ a,\ 2a,\ 3a,\ 5a, \ldots.$$

Notice that the sequence $H_n = aF_{n+1}$ has the same characteristics as the Lucas-like sequence $H_1 = a \geq 1$, $H_2 > 2a$: $H_{-1} = 0$ is the first nonpositive term in an alternating sequence moving left of $H_0 = a$, while terms to the right of H_0 are strictly increasing. $H_1 = H_0 = a$ are the smallest positive terms.

Thus, we define the *standardized generalized Fibonacci (S.G.F.) sequence* $\{H_n(a, B)\}$ by

$$H_{n+1} = H_n + H_{n-1},\ H_1 = A \geq 1,\ H_2 = B \geq 2A.$$

We note that $H_0 = B - A \geq A$, and $H_1 = A$ is the smallest term in the sequence. We will find that H_0 will determine maximal subscripts for the sequence. If $B = 2A$, we will have a Fibonacci-like sequence in which $H_n = AF_{n+1}$. Also, Fibonacci and Lucas numbers are numbered to be consistent with usage in this journal.

We need this careful definition of the beginning terms so that we can identify $H_1 = A$ and $H_2 = B$ given any two adjacent terms somewhere in the sequence. For example, 13 and 17 are adjacent in each of $\{13, 17, 30, 47, \ldots\}$, $\{4, 13, 17, 30, \ldots\}$, and $\{9, 4, 13, 17, 30, \ldots\}$. Note that the S.G.F. sequence will have $A = 4$, $B = 13$. We do not start with $A = 9$, $B = 4$, or with $A = 13$, $B = 17$, since a S.G.F. sequence must have $B \geq 2A$. While $N = H_1$ in an infinite number of such sequences, $N = H_n$, $n \geq 2$, can appear only within a S.G.F. sequence for which $1 \leq H_{n-1} \leq N - 1$. When $N = H_2$, take $1 \leq H_1 \leq [(N-1)/2]$, while $N = H_n$, $n \geq 3$, has $[N/2] + 1 \leq H_{n-1} \leq N - 1$. Thus, the maximal subscript for N can be found by listing possibilities. If $N = 7 = H_n$, examine sequences for which $4 \leq H_{n-1} \leq 6$, giving $1, 3, 4, 7, 11, \ldots$; $2, 5, 7, 12, \ldots$; and $1, 6, 7, 13, \ldots$. The first sequence has $7 = H_4$, and 4 is the maximal subscript for 7. If $N = 6 = H_n$, examine $4 \leq H_{n-1} \leq 5$: $2, 4, 6, 10, \ldots$, and $1, 5, 6, 11, \ldots$. Both sequences have $6 = H_3$, but the first sequence has $B = 2A$ so that $H_3 = 2F_4$; we take the larger subscript, and 4 is the maximal subscript for 6.

Lemma 1.1 gives a second way to compute maximal subscripts.

Lemma 1.1: If $H_n = H_{n-2} + H_{n-1}$, $H_1 = a$, $H_2 = b$, the equation

$$N = H_n(a, b) = aF_{n-2} + bF_{n-1} \tag{1.1}$$

has a solution for any integer N. If (a_0, b_0) is a solution for (1.1), then $a = a_0 - tF_{n-1}, b = b_0 + tF_{n-2}$ is also a solution for (1.1) for any integer t.

Proof: Equation (1.1), which can be proved by mathematical induction, always has solutions [10] for integers a, b as above since $(F_{n-2}, F_{n-1}) = 1$. □

For our purposes in using (1.1), $\{H_n(a, b)\}$ must be a S.G.F. sequence. Note that

$$\{H_n(1, 2)\} = \{F_{n+1}\} \quad \text{and} \quad \{H_n(1, 3)\} = \{L_n\}$$

are S.G.F. sequences since $B \geq 2A$, but while $\{H_n(1, 1\} = \{F_n\}$, this is not a S.G.F. sequence. If $F_{n-1} < N < F_n$, then $(n-2)$ is the largest possible subscript for N in a S.G.F. sequence by examining (1.1). If $N = 31$, since $F_8 < 31 < F_9$, solve $31 = H_7 = AF_5 + BF_6$. We find $31 = H_7(3, 2)$ but $B < 2A$, so we solve $31 = AF_4 + BF_5 = H_6(2, 5)$, where $B > 2A$, obtaining 6 as the maximal subscript for 31. We now have two methods to compute a table of maximal subscripts.

We will say that *a natural number N reaches maximum expansion at R, denoted by* $\rho(N) = R$, if R is the largest subscript possible for N as a member of a S.G.F. sequence or for N as a member of a Fibonacci-like sequence. Let R be the largest subscript such that

$$N = H_R(A, B) = AF_{R-2} + BF_{R-1}$$

for $1 \le A$ and $2A \le B$. Then, if $2A < B$, $\rho(N) = R$; if $2A = B$, $\rho(N) = R+1$. We will find $\rho(F_R) = R = \rho(L_R)$ for $R \ge 3$. For the reader's convenience, we list maximal subscripts $\rho(N)$ in Table 1.

TABLE 1. $N = H_R(A, B)$ with Maximal $R = \rho(N)$

N	R	N	R	N	R	N	R
1	2	26	7	51	6	76	9
2	3	27	5	52	7	77	7
3	4	28	6	53	7	78	7
4	3	29	7	54	6	79	7
5	5	30	5	55	10	80	6
6	4	31	6	56	6	81	8
7	4	32	6	57	6	82	7
8	6	33	6	58	7	83	6
9	4	34	9	59	6	84	8
10	5	35	5	60	8	85	7
11	5	36	6	61	7	86	8
12	4	37	7	62	6	87	7
13	7	38	6	63	8	88	6
14	5	39	7	64	6	89	11
15	5	40	6	65	7	90	7
16	6	41	6	66	7	91	7
17	5	42	8	67	6	92	7
18	6	43	6	68	9	93	7
19	5	44	6	69	7	94	8
20	5	45	7	70	6	95	7
21	8	46	6	71	7	96	6
22	5	47	8	72	6	97	9
23	6	48	6	73	8	98	7
24	6	49	6	74	7	99	8
25	5	50	7	75	6	100	7

2. $\rho(N)$ FOR SOME SPECIAL INTEGERS N

We write $\rho(N)$ for some specialized integers N and consider how many integers N have $R = \rho(N)$ for a given subscript R. If $\rho(N) = R$ in exactly one sequence, N is called a *single*; if in exactly two sequences, N is called a *double*; if in exactly three sequences, N is called a *triple*. The smallest double occurs when $R = 3$, for $N = 4 = H_3(1,3) = 2F_3$, while the smallest triple occurs when $R = 5$, for $N = 35 = 7F_5 = H_5(4,9) = H_5(1,11)$.

Theorem 2.1: For the Fibonacci sequence, $\rho(AF_R) = R$ when $1 \le A < F_{R+1}$, $R \ge 2$. Further, $\rho(AF_R) > R$ when $A > F_{R+1}$.

Proof: $\rho(F_2) = 2$; $\rho(F_3) = 3$. By Lemma 1.1,

$$AF_R = AF_{R-2} + AF_{R-1} = H_R(A + F_{R-1}, A - F_{R-2}),$$

where $A + F_{R-2} > 2(A - F_{R-1})$ when $A < F_{R+1}$ and $\rho(AF_R) \ge R$. Further, $AF_R = 0F_{R-1} + AF_R = H_{R+1}(F_R, A - F_R)$, but a S.G.F. sequence requires that $B > 2A$, and $A - F_{R-1} > 2F_R$ only when $A > F_{R+1}$. Thus, $\rho(AF_R) < R+1$, making $\rho(AF_R) = R$ when $A < F_{R+1}$, and $\rho(AF_R) > R$ when $A > F_{R+1}$. □

Corollary 2.1.1: Let $N = AF_R$, and $\rho(N) = R$, $R \geq 3$. Then N is a single when $A \leq F_{R-1}$; is a double when $F_{R-1} < A \leq 2F_{R-1}$; and is a triple when $2F_{R-1} < A < F_{R+1}$. For each value of R, $\rho(N) = R$ in at most three sequences.

Proof: If $1 \leq A < F_{R+1}$, $\rho(AF_R) = R$ by Theorem 2.1. By Lemma 1.1, any other solutions for $\rho(N) = R$ are found from $N = H_R(A - tF_{R-1}, A + tF_{R-2})$. If $t \leq 0$, $\{H_R\}$ is not a S.G.F. sequence. If $t = 1$, then S.G.F. sequence requirements dictate $A - F_{R-1} \geq 1$, making N a single when $1 \leq A \leq F_{R-1}$, and at least a double if $A > F_{R-1}$. If $t = 2$, we must have $A - 2F_{R-1} \geq 1$, so that N is a double when $F_{R-1} < A \leq 2F_{R-1}$, and a triple when $2F_{R-1} < A < F_{R+1}$. If $t \geq 3$, then $A \geq 1 + tF_{R-1} \geq 1 + F_{R+1}$, and $\rho(N) > R$. □

Corollary 2.1.2: For $R \geq 2$, $\rho(F_R^2) = R$ and $\rho(F_{R+1}F_R) = R+1$; further, $\rho(F_R^2 - 1) = R+1$, R even, and $\rho(F_R^2 + 1) = R+1$, R odd.

Proof: Apply Theorem 2.1 to $F_{R-1}F_{R+1} = F_R^2 + (-1)^R$. □

Corollary 2.1.3: $\rho(F_R L_{R-1}) = R$, $R \geq 2$; $\rho(F_R L_R) = 2R$, $R \geq 1$.

Proof: Since $L_{R-1} = F_R + F_{R-2} < F_{R+1}$, Theorem 2.1 gives $\rho(F_R L_{R-1}) = R$ and $\rho(F_R L_R) = \rho(F_{2R}) = 2R$. □

Corollary 2.1.4: $\rho(L_{n+1}F_n) = n+2$, $n \geq 3$; and $\rho(L_{n+k}F_k) = n+k$, $k \geq 2$, $n \geq 1$.

Proof: Let $N = L_{n+1}F_n = (F_{n-2})F_n + (2F_n)F_{n+1} = H_{n+2}(F_{n-2}, 2F_n)$; as $B > 2A$, $\rho(N) \geq n+2$. Since $N = H_{n+3}(F_{n+1}, F_{n-2})$ has no other positive solutions and $\{H_{n+3}(F_{n+1}, F_{n-2})\}$ is not a S.G.F. sequence, we have $\rho(N) < n+3$, making $\rho(N) < n+2$. Next, let $N = L_{n+k}F_k$. One can derive

$$N = L_{n+k}F_k = (F_k)F_{n+k-2} + (3F_k)F_{n+k-1} = H_{n+k}(F_k, 3F_k),$$

where $B > 2a$ and $\rho(N) \geq n+k$. Also, since $N = H_{n+k+1}(2F_k, F_k)$ has no other positive solutions for A and B, and this solution cannot be used because $A > B$, we have $\rho(N) < n+k+1$; thus, $\rho(L_{n+k}F_k) = n+k$. □

Corollary 2.1.5: $\rho(F_{n+p} + F_{n-p}) = n = \rho(F_{n+p} - F_{n-p})$, $p \geq 2$, $n \geq 2+p$

Proof: Hoggatt (see [7], p. 59] gives

$$F_{n+p} + F_{n-p} = F_n L_p,\ p \text{ even};\quad F_{n+p} + F_{n-p} = L_n F_p,\ p \text{ odd}.$$

If p is even, Corollary 2.1.3 gives $\rho(F_n L_p) = n$; if p is odd, Corollary 2.1.4 gives $\rho(L_n F_p) = n$. Similarly, $F_{n+p} - F_{n-p} = F_n L_p$, p odd, and $F_{n+p} - F_{n-p} = L_n F_p$, p even, yield $\rho(F_{n+p} - F_{n-p}) = n$. □

Corollary 2.1.6: $\rho(F_{2k} - 1) = \rho(F_{2k} + 1) = k+1$, $k \geq 2$.

If k is even, $k \geq 4$, $\rho(F_{2k+1} + 1) = \rho(F_{2k} + 1) = k+1$;
$\rho(F_{2k+1} - 1) = \rho(F_{2k} - 1) + 1 = k+2$.

If k is odd, $k \geq 3$, $\rho(F_{2k+1} - 1) = \rho(F_{2k} - 1) = k+1$;
$\rho(F_{2k+1} + 1) = \rho(F_{2k} + 1) + 1 = k+2$.

Proof: When k is odd, $k \geq 1$:

$$F_{2k}+1 = F_{k+1}L_{k-1}; \qquad F_{2k}-1 = F_{k-1}L_{k+1};$$
$$F_{2k+1}-1 = F_{k+1}L_k. \qquad F_{2k+1}+1 = F_kL_{k+1}.$$

When k is even, $k \geq 2$:

$$F_{2k}+1 = F_{k-1}L_{k+1}; \qquad F_{2k}-1 = F_{k+1}L_{k-1};$$
$$F_{2k+1}-1 = F_kL_{k+1}. \qquad F_{2k+1}+1 = F_{k+1}L_k.$$

Each pair of identities, when summed vertically, gives $F_{2k+2} = F_{k+1}L_{k+1}$, and each can be proved by mathematical induction. Then apply Corollaries 2.1.3 and 2.1.4, which give $\rho(N)$ when $N = F_kL_m$. □

Next, we investigate integers N, where $\rho(N) = R$ and N is not a multiple of F_R. The smallest such double is $N = 83 = H_6(1,16) = H_6(6,13)$.

Theorem 2.2: Let $N = H_R(A,B)$, where $B > 2A$ and $\rho(N) = R$, $R \geq 3$. Then N is a single when $1 \leq A \leq F_{R-1}$ and $B < A + F_R$. N is a double when $F_{R-1} < A \leq F_R - 2$ and $2A < B \leq 2F_R - 3$.

Proof: Select the smallest integer A for which the hypothesis is met. Then $1 \leq A \leq F_{R-1}$. Otherwise, from Lemma 1.1, $N = H_R(A - F_{R-1}, B + F_{R-2})$, contrary to choice of A as smallest. $\{H_R(A + F_{R-1}, B - F_{R-2})\}$ is not a S.G.F. sequence when $B < A + F_R$ because then $A + F_{R-1} > B - F_{R-2}$; thus, the conditions $A \leq F_{R-1}$ and $B < A + F_R$ guarantee a single. When $A \geq 1 + F_{R-1}$ and $B < A + F_R$, $\{H_R(A - F_{R-1}, B + F_{R-2})\}$ is a S.G.F. sequence. Since $2A < B$, rewrite requirements for B as $2A + 1 \leq B \leq F_R + A + 1$, leading to a double when $F_{R-1} + 1 \leq A \leq F_R - 2$ and $2A + 1 \leq B \leq F_R + A + 1 \leq 2F_R - 3$. □

To illustrate Theorem 2.2, consider $H_6(1,16) = H_6(6,13) = 83$. The smallest solution has $A = 1$, where $1 \leq A \leq F_{6-1}$, but $B = 16 > F_6 + A = 9$, allowing a double. Taking $A = 6$, $F_5 < 6 \leq F_6 - 2$ and $B = 13 \leq 2F_6 - 3$.

Corollary 2.2.1: Let $N = H_R(A,B)$, where $B > 2A$. If $1 \leq A \leq F_R - 2$ and $B < A + F_R$, then $\rho(N) = R$, $R \geq 3$.

Proof: By hypothesis, $\rho(N) \geq R$.

$$N = AF_{R-2} + BF_{R-1} = (B-A)F_{R-1} + AF_R = H_{R+1}(B-A, A),$$

but $\{H_{R+1}(B-A, A)\}$ is not a S.G.F. sequence when $B > 2A$, and neither are the other solutions from Lemma 1.1, $N = H_{R+1}(B - A + F_R, A - F_{R-1})$ and $N = H_{R+1}(B - A - F_R, A + F_{R-1})$. Thus, $\rho(N) < R+1$ and $\rho(N) = R$. □

Corollary 2.2.2: $\rho(N) = R$ for $(F_R^2 + F_{R-3})/2$ integers N, $R \geq 3$.

Proof: When $B > 2A$, Corollary 2.2.1 gives $(F_R - 2)$ choices for A. Since $2A + 1 \leq B \leq F_R + A + 1 \leq 2F_R - 3$, when $A = F_R - 2$, there is one choice for B; for $A = F_R - 3$, two choices; ..., for $A = F_R - 1 - k$, k choices. So $\rho(N) = R$ for

$$(F_R - 2)(1 + 2 + 3 + \cdots + (F_R - 2)) = (F_R - 2)(F_R - 1)/2$$

integers N which are not divisible by F_R. If $N = AF_R$, Theorem 2.1 gives $\rho(N) = R$ for $1 \leq A \leq F_{R+1} - 1$, so there are $(F_{R+1} - 1)$ such integers N. Adding and simplifying,

$$(F_R-2)(F_R-1)/2+(F_{R+1}-1)=(F_R^2+F_{R-3})/2$$

as required. □

Theorem 2.3: For the Lucas sequence, $\rho(L_R)=R,\ R\geq 3$.

Proof: $\rho(L_1)=2$ and $\rho(L_2)=4$. For $R\geq 3$, $L_R=H_R(1,3)=1F_{R-2}+3F_{R-1}$, so $\rho(L_R)\geq R$. The only positive solution for $L_R=H_{R+1}(A,B)=AF_{R-1}+BF_R$ is $A=2$ and $B=1$, but this solution cannot be used since $A>B$, so $\rho(L_R)<R+1$, making $\rho(L_R)=R$. Compare with Corollary 2.1.4 for $k=2$. □

Corollary 2.3.1: The smallest integer such that $\rho(N)=R$ is F_R. The smallest integer not divisible by F_R such that $\rho(N)=R$ is L_R.

Theorem 2.4: The largest integer N for which $\rho(N)=R$ is $N=(F_{R+1}-1)F_R$, $R\geq 2$. Also, $N=(F_{R+1}-1)F_R$, $R\geq 5$, is a triple, with the other two occurrences given by

$$N=H_R(F_R-1),2F_R-1)=H_R(F_{R-2}-1,2F_R+F_{R-2}-1).$$

Proof: For $R=2$, $N=(F_3-1)F_2=1$. If $H_R=AF_{R-2}+BF_{R-1}$, where $B\geq 2A$, then $H_R\leq BF_{R-2}+BF_{R-1}=BF_R$, $R\geq 3$. $\rho(BF_R)=R$ when $1\leq B\leq F_{R+1}-1$ by Theorem 2.1. By Corollary 2.1.1, $N=(F_{R+1}-1)F_R$ is a triple that can be calculated using Lemma 1.1. □

Theorem 2.5: If $F_{2k-2}\leq N<F_{2k}$, $k\geq 2$, then $k\leq\rho(N)\leq 2k-1$.

Proof: By Theorem 2.1, the largest possible value for $\rho(N)$ in the interval is $\rho(F_{2k-1})=2k-1$. We show that the smallest value is $\rho(N)=k$ by applying Theorem 2.4. Now, take $N=(F_{k+1}-1)F_k$; then $N<(F_{k+1}+F_{k-1})F_k=L_kF_k=F_{2k}$, while

$$N=(F_k+(F_{k-1}-1))F_k\geq(F_k+F_{k-2})F_{k-1}=L_{k-1}F_{k-1}=F_{2k-2}$$

for $k\geq 4$. Then, by examining $k=2$ and $k=3$, and putting this together,

$$F_{2k-2}\leq N=(F_{k+1}-1)F_k<F_{2k},\ k\geq 2, \tag{2.5.1}$$

where $\rho(N)=k$ and N is the largest integer such that $\rho(N)=k$. Notice that taking $R=k-1$ in Theorem 2.4, $(F_k-1)F_{k-1}<F_{2(k-1)}$ from (2.5.1), so that the largest integer N having $\rho(N)=k-1$ is not in the interval of Theorem 2.5. □

Theorem 2.6: In the interval $F_m<N<F_{m+1}$, $[(m+2)/2]\leq\rho(N)\leq m-1$, $m\geq 4$; and $\rho(F_m+1)\leq[(m+2)/2]+1$, where $[x]$ is the greatest integer in x.

Proof: Since F_m is not in the interval, $\rho(N)\leq m-1$. If m is odd, take $m=2k-1$; if m is even, $m=2k-2$. Either $F_{2k-2}\leq N<F_{2k-1}$ or $F_{2k-1}\leq N<F_{2k}$, so that $\rho(N)\geq k$ from Theorem 2.5. Since either $[(m+2)/2]=(2k-1+2)/2]=k$ or $[(m+2)/2]=(2k-2+2)/2]=k$, $\rho(N)\geq[(m+2)/2]$.

The smallest integer in the interval is F_m+1, and, by Corollary 2.1.6, either $\rho(F_m+1)=[(m+2)/2]$ or $\rho(F_m+1)=[(m+2)/2]+1$. The largest value for $\rho(N)$ is $m-1$, which occurs for $N=2F_{m-1}$ and $N=L_{m-1}$. □

Corollary 2.6.1: For $m \geq 4$,

$$\rho(F_m + F_{m-2}) = \rho(F_{m+1} - F_{m-2}) = m-1;$$
$$\rho(F_m + F_{m-3}) = \rho(F_{m+1} - F_{m-3}) = m-1.$$

Proof: Since $L_{m-1} = F_m + F_{m-2} = F_{m+1} - F_{m-3}$, apply Theorem 2.3 in the first case. Similarly, use Theorem 2.1 with $2F_{m-1} = F_{m+1} - F_{m-2} = F_m + F_{m-3}$. □

3. THE MAXIMUM EXPANSION INDEX OF A S.G.F. SEQUENCE

In this section, we determine when $\rho(H_K) = K$ for the S.G.F. sequence $\{H_n(A, B)\}$. We will call the integer $R_{\max}$ the *maximum expansion index* of the S.G.F. sequence $\{H_n(A, B)\}$ if $\rho(H_K(A, B)) = K$ whenever $K = R_{\max}$. For example, the S.G.F. sequence

$$\{H_n(1, 7)\} = \{1, 7, 8, 15, 23, 38, 61, 99, \ldots\}$$

which has $\rho(H_6) = \rho(38) = 6$ has $R_{\max} = 6$; $\rho(H_K) \neq K$ for $1 \leq K \leq 5$, while $\rho(H_7) = \rho(61) = 7$ as well as $\rho(H_K) = K$ for all $K \geq 6$.

Theorem 3.1: If $F_{R-1} < B - A \leq F_R$ for the S.G.F. sequence $\{H_n(A, B)\}$, then $\rho(H_R(A, B)) = R$. Further, $R = R_{\max}$, and $\rho(H_K(A, B)) = K$ for all $K \geq R$.

Proof: Since $2A \leq B$ in a S.G.F. sequence, $A \leq B - A \leq F_R$ so $1 \leq A \leq F_R$ and $B \leq A + F_R$.. If $B = 2A$, then $N = AF_R$ and $\rho(N) = R$ by Theorem 2.1. Also, $B = A + F_R$ gives a Fibonacci-like case, since $A = F_R - k$, $B = 2F_R - k$ give

$$N = H_R = (F_R - k)F_{r-2} + (2F_R - k)F_{r-1} = (F_{R+1} - k)F_R, \;\; 1 \leq k \leq F_R - 1,$$

where $\rho(N) - R$ by Theorem 2.1, and $k = 0$ gives us $B = 2A$, already discussed.

If $B > 2A$, Corollary 2.2.1 gives $\rho(N) = R$ when $1 \leq A \leq F_R - 2$, $B < A + F_R$, leaving only the cases $A = F_R - 1$ and $A = F_R$. Since cases $B = 2A$ and $B = A + F_R$ were discussed above, we are finished, and $\rho(N) = R$ when $F_{R-1} < B - A \leq F_R$.

Let $K > R$. If

$$N = H_K(A, B) = AF_{K-2} + BF_{K-1} \;\;\text{and}\;\; B \geq 2A,$$

then $\rho(N) \geq K$. Thus,

$$N = H_{K+1}(B - A, A) = (B - A)F_{K-1} + AF_K$$

but this solution cannot be used since $B - A \geq A$ when $B \geq 2A$. Since $F_K > F_R$, $(B - A) - F_K < 0$, and $(B - A) + F_K > A - F_{K-1}$ when $B \geq 2A$, Lemma 1.1 gives no other usable solutions for $N = H_{k+1}$. Thus, $\rho(N) < K + 1$ and $\rho(N) = K$. Putting these cases together, $\rho(H_K(A, B)) = K$ when $K \geq R$, and $R = R_{\max}$. □

Corollary 3.1.1: If $F_{R-1} < 2a \leq F_R$, $a \geq 1$, $R \geq 3$, then $\rho(aL_n) = n$ for $n \geq R$. If $F_{R-1} < A < F_R$, then $\rho(AF_n) = n$ for $n \geq R$, $R \geq 2$.

Proof: $H_n = aL_n$ has $H_0 = 2a$. If $F_{R-1} < B - A = 2a \leq F_R$, apply Theorem 3.1. If $F_{R-1} < 2A - A \leq F_R$, then $\rho(H_{n-1}(A, 2A)) = n - 1$ for $n - 1 \geq R$. Since $B = 2A$ and $AF_n = H_{n-1}(A, 2A)$, $\rho(AF_n) = n$ for $n \geq R$. Compare with Theorem 2.1. □

Theorem 3.2: For $k \ge 2$, $n \ge 2$,

$$\rho(F_{n+2k}+F_n)=\rho(F_{n+2k}-F_n)=n+k;$$
$$\rho(F_{n+2k+1}+F_n)=\rho(F_{n+2k+1}-F_n)+1=n+k+1,\ k \text{ even};$$
$$\rho(F_{n+2k+1}+F_n)=\rho(F_{n+2k+1}-F_n)-1=n+k,\ k \text{ odd}.$$

Proof: $\rho(F_{n+2k}+F_n)=\rho(F_{n+2k}-F_n)=n+k$ by taking $n=n+k$ and $p=k$ in Corollary 2.1.5. Since $N=H_{n+k}=AF_{n+k-2}+BF_{n+k-1}$, where $\rho(H_{n+k})\ge n+k$ if $B\ge 2A$, we derive identities involving F_{n+2k+1} from the identity (see Eq. (8) in [11])

$$F_{m+n}=F_{m-1}F_n+F_mF_{n+1} \tag{2.7.1}$$

to write $N=F_n+F_{n+2k+1}=AF_{n+k-1}+BF_{n+k}$. Take $m=n+k$ and $n=k+1$ for F_{n+2k+1} and $m=n+k$, and $n=(-k)$ for F_n in (2.7.1) to write

$$F_{n+2k+1}=F_{(n+k)+(k+1)}=F_{n+k-1}F_{k+1}+F_{n+k}F_{k+2};$$
$$F_n=F_{(n+k)+(-k)}=F_{n+k-1}F_{-k}+F_{n+k}F_{1-k}=(-1)^{k+1}F_kF_{n+k-1}+(-1)^kF_{k-1}F_{n+k}.$$

Thus,

$$\begin{aligned}N=F_{n+2k+1}+F_n&=(F_{k+1}+(-1)^{k+1}F_k)F_{n+k-1}+(F_{k+2}+(-1)^kF_{k-1})F_{n+k}\\&=H_{n+k+1}(A,B).\end{aligned}$$

If k is even, $A=F_{k+1}-F_k=F_{k-1}$, and $B=F_{k+2}+F_{k-1}=2F_{k+1}$, where $B>2A$. Since $B-A=F_{k+2}$, $R_{\max}=k+2$, where $n+k+1\ge k+2$; $\rho(N)=n+k+1$ by Theorem 3.1. If k is odd, then $A=F_{k+1}+F_k=F_{k+2}$ and $B=F_{k+2}-F_{k-1}=2F_k$ has $A>B$ with no other positive solution, but we find that $N=H_{n+k}(2F_k,4F_k+F_{k-1})$, where $F_{k+1}<B-A\le F_{k+2}$ so that $R_{\max}=k+2$, and again $\rho(N)=n+k, n\ge 2$..

Subtracting the quantities above, $N=F_{n+2k+1}-F_n$ becomes $H_{n+k}(2F_k,4F_k+F_{k-1})$, giving $\rho(N)=n+k$ for k even; for k odd, $N=F_{n+2k+1}-F_n$ becomes $H_{n+k+1}(F_{k-1},2F_{k+1})$, giving $\rho(N)=n+k+1$. □

4. SOLVING $N=H_R(A,B)$ FOR R, A, AND B

Given N, we find A, B, and R so that $N=H_R(A,B)$, where $R=\rho(N)$. Our solution depends upon a greatest integer identity for the S.G.F. sequence $\{H_n(A,B)\}$ which allows us to find H_{n-1} when we are given H_n.

Lemma 4.1: Let $\{H_n(A,B)\}$ be a S.G.F. sequence, where $F_{k-1}<B-A\le F_k$. For $n\le k$, the term preceding $H_n(A,B)$ is $[H_n/\alpha]$ or $[H_n/\alpha]+1$, where $[x]$ is the greatest integer in x, and $\alpha=(1+\sqrt{5})/2$.

Proof: From [4], use Theorem 3.3 when $B>2A$ and Theorem 2.3 when $B=2A$. □

Lemma 4.2: For the S.G.F. sequence $\{H_n(A,B)\}$, if $D=B^2-AB-A^2$:

(i) $F_{n-1}<H_n/\sqrt{D}\le F_{n+1}$;

(ii) $H_n^2-H_nH_{n-1}-H_{n-1}^2=(-1)^nD$;

(iii) $|H_n^2-H_nH_{n-1}-H_{n-1}^2|=K^2$ iff $H_n=KF_{n+1}$, $n\ge 1$.

Proof: (1) Since $B \geq 2A$, $D > 0$; in fact, $D \geq B^2/4$, and $\sqrt{D} \geq B/2$. Thus,

$$H_n = AF_{n-2} + BF_{n-1} \leq (B/2)F_{n-2} + BF_{n-1} = (B/2)F_{n+1}.$$

Dividing by $\sqrt{D}$, $H_n/\sqrt{D} \leq (B/2)F_{n+1}/\sqrt{D} \leq F_{n+1}$, while

$$F_{n+1} \geq H_n/\sqrt{D} = (AF_{n-2} + BF_{n-1})/\sqrt{D} > AF_{n-2}/\sqrt{D} + F_{n-1} > F_{n-1}.$$

For (ii), see [1], [7], and [8]. Lastly, in 1876, Lucas proved that $m^2 - mn - n^2 = \pm 1$ is satisfied by consecutive Fibonacci numbers, and in 1902, Wasteels proved that there are no other solutions (see [6], p. 405). Since $(F_n, F_{n+1}) = 1$, (iii) follows. Note that (iii) is a test for a Fibonacci sequence. □

Lemma 4.3: Let $N = H_n(A, B)$, where n is to be maximized. There are two cases:

(i) $H_{n-1} = [H_n/\alpha]$, $n = n_1$;

(ii) $H_{n-1} = [H_n/\alpha] + 1$, $n = n_2$.

The maximal subscript value for $N = H_R$ occurs for $R = \max(n_1, n_2)$.

Proof: Lemma 4.3 actually is a blueprint for solving for R. By Lemma 4.1, cases (i) and (ii) give the only two possible choices for H_{n-1}. Take case (i). Compute $H_n^2 - H_nH_{n-1} - H_{n-1}^2 = (-1)^n D$ from Lemma 4.2 recalling that $D > 0$. Compute $H_n/\sqrt{D}$ and select n by $F_{n-1} < H_n/\sqrt{D} \leq F_{n+1}$. There are two possibilities for n: if $(-1)^n D > 0$, then n is the even possibility, while n is odd if $(-1)^n D < 0$. Then $n = n_1$ is the solution from case (i). Now take case (ii). Make the same calculations with $H_{n-1} = [H_n/\alpha] + 1$ to find $n = n_2$. Then choose $n = R = \max(n_1, n_2)$. □

Lemma 4.4: If $N = H_n(A, B)$, then

$$A = |H_{n-1}F_{n-1} - NF_{n-2}| \quad \text{and} \quad B = |H_{n-1}F_{n-2} - NF_{n-3}|.$$

Proof: Refer to (1.1) and solve the equations $H_n = AF_{n-2} + BF_{n-1}$ and $H_{n-1} = AF_{n-3} + BF_{n-2}$ simultaneously for A and B. □

Now we can use the four lemmas above to find the S.G.F. sequence $\{H_n(A, B)\}$ with $N = H_R(A, B)$ such that $R = \rho(N)$, given any positive integer N. It is important to note that, if $B = 2A$, $\{H_n\}$ is a Fibonacci-like sequence and the maximal subscript R will increase by 1, since $H_n = AF_{n+1}$. Lemma 4.2 gives a test for a Fibonacci-like sequence, and a shortened solution since, if $|(-1)^n D| = K^2$, then $H_n = KF_{n+1}$.

Example 1: Let $N = 2001 = H_n$. Compute case (i): $[2001/\alpha] = 1236 = H_{n-1}$, and $(-1)^n D = 3069 > 0$, so n_1 is even; next, $F_9 < 2001/\sqrt{3069} \approx 36.1 \leq F_{10}$, so $n_1 = 10$. Compute case (ii) using $[2001/\alpha] + 1 = 1237 = H_{n-1}$ and $(-1)^n D = -1405 < 0$, so n_2 is odd; with $F_9 < 2001/\sqrt{1405} \approx 53.38 \leq F_{10}$, $n_2 = 9$. Take $R = \max(10, 9) = 10 = n$, and use $H_{n-1} = 1236$ from case (i) to compute $a = |1236F_9 - 2001F_8| = 3$, $b = |1236F_8 - 2001F_7| = 57$. Since $b > 2a$, take $N = H_{10}(3, 57)$.

Example 2: Let $N = 357 = H_n$. $[357/\alpha] = 220 = H_{n-1}$ and $(-1)^n D = 509 > 0$, so n_1 is even. Then $F_7 < 357/\sqrt{509} \approx 15.8 \leq F_8$, so $n_1 = 8$. Compute case (ii) for $H_{n-1} = 221$, obtaining $(-1)^n D = -289 < 0$, so n_2 is odd; $F_7 < 357/\sqrt{289} = 21 \leq F_8$, so $n_2 = 7$. We choose $n = n_1 = 8$ and use $H_{n-1} = 220$ to compute $a = |220F_7 - 357F_6| = 4$ and $b = |220F_6 - 357F_5| = 25$. Therefore, $R = 8$,

$A=4$, and $B=25$ yields $N=H_8(4,25)$. Note that $|(-1)^n D|289=17^2$ in case (ii) indicates a Fibonacci-like sequence, $n_2+1=8=R$, giving a double, and $H_{n-1}=221$ for $n_2=7$ yields $a=|221F_6-357F_5|=17=A$ and $b=|221F_5-357F_4|=34=B$, or $N=H_7(17,34)=17F_8$.

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REFERENCES

1. Brother Alfred Brousseau. *Introduction to Fibonacci Discovery*. Santa Clara, CA: The Fibonacci Association, 1965.
2. Brother U. Alfred (Brousseau). "On the Ordering of Fibonacci Sequences." *The Fibonacci Quarterly* **1.4** (1963):43-46.
3. Marjorie Bicknell-Johnson. "A Note on a Representation Conjecture by Hoggatt." In *Appli-cations of Fibonacci Numbers* **7**:39-42. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1998.
4. Marjorie Bicknell-Johnson & David A. Englund. "Greatest Integer Identities for Generalized Fibonacci Sequences $\{H_n\}$, Where $H_n=H_{n-1}+H_{n-2}$." *The Fibonacci Quarterly* **33.1** (1995): 50-58.
5. John H. E. Cohn. "Recurrent Sequences Including *N*." *The Fibonacci Quarterly* **29.1** (1991): 30-36.
6. L. E. Dickson. *History of the Theory of Numbers*. Vol. I. New York: Chelsea, 1971.
7. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1979.
8. A. F. Horadam. "A Generalized Fibonacci Sequence." *Amer. Math. Monthly* **68.5** (1961): 455-59.
9. David A. Klarner. "Representations of *N* as a Sum of Distinct Elements from Special Sequen-ces." *The Fibonacci Quarterly* **4.4** (1966):289-306.
10. William J. LeVeque. *Elementary Theory of Numbers*, p. 32. London: Addison-Wesley, 1962.
11. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Chichester, England: Ellis Horwood, 1989.

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SOME FURTHER PROPERTIES OF ANDRE-JEANNIN AND THEIR COMPANION POLYNOMIALS

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1. INTRODUCTION

In a recent article [4], the author defined two sets of polynomials, $u_n(x)$ and $v_n(x)$, by the relations:

$$u_n(x) = (x+p)u_{n-1}(x) - qu_{n-2}(x), \quad n \ge 2, \tag{1.1a}$$

with

$$u_0(x) = 1, \; u_1(x) = x + p - \sqrt{q}, \tag{1.1b}$$

and

$$v_n(x) = (x+p)v_{n-1}(x) - qv_{n-2}(x), \quad n \ge 2, \tag{1.2a}$$

with

$$v_0(x) = 1, \; v_1(x) = x + p + \sqrt{q}, \tag{1.2b}$$

where $q > 0$, and showed that they are very closely related to two other sets of polynomials $U_n(x)$ and $V_n(x)$ defined by André-Jeannin (see [1] and [2]) by the relations

$$U_n(x) = (x+p)U_{n-1}(x) - qU_{n-1}(x), \quad n \ge 2, \tag{1.3a}$$

with

$$U_0(x) = 0, \; U_1(x) = 1, \tag{1.3b}$$

and

$$V_n(x) = (x+p)V_{n-1}(x) - qV_{n-2}(x), \quad n \ge 2, \tag{1.4a}$$

with

$$V_0(x) = 2, \; V_1(x) = x + p. \tag{1.4b}$$

In the same article, the author derived a few of the properties of the polynomials $u_n(x)$ and $v_n(x)$, as well as some interesting interrelationships. The purpose of this article is to derive further properties of these polynomials and their interrelationships. Since the modified Morgan-Voyce polynomials $\widetilde{B}_n(x)$, $\widetilde{b}_n(x)$, $\widetilde{c}_n(x)$, and $\widetilde{C}_n(x)$ defined in [4] result when $q = 1$, we thus derive a number of interesting properties of these modified Morgan-Voyce polynomials.

Since the polynomials $U_n(x)$ and $V_n(x)$ were defined and a number of their properties were studied for the first time by André-Jeannin, it is appropriate to refer to them as the André-Jeannin polynomials of the first and second kind. The polynomials $u_n(x)$ and $v_n(x)$, which are closely related to the André-Jeannin polynomials, and which exist as real distinct polynomials only when $q > 0$, will be referred to as the companion André-Jeannin polynomials of the first and second kind. We will now list a number of important properties of the polynomials $U_n(x)$, $u_n(x)$, $v_n(x)$, and $V_n(x)$ that are either known or easily derivable from the known properties, since these will be required in establishing the results of the remaining sections.

Simple Interrelations:

$$u_n(x) = U_{n+1}(x) - \sqrt{q}U_n(x), \qquad \text{from [4].} \tag{1.5}$$

$$v_n(x) = U_{n+1}(x) + \sqrt{q}U_n(x), \qquad \text{from [4].} \tag{1.6}$$

$$V_n(x) = U_{n+1}(x) - qU_{n-1}(x), \quad \text{from [2].} \tag{1.7}$$

$$V_n(x) = u_n(x) + \sqrt{q}u_{n-1}(x), \quad \text{from [4].} \tag{1.8}$$

$$V_n(x) = v_n(x) - \sqrt{q}v_{n-1}(x), \quad \text{from [4].} \tag{1.9}$$

$$(x+p-2\sqrt{q})U_n(x) = u_n(x) - \sqrt{q}u_{n-1}(x), \quad \text{by induction.} \tag{1.10}$$

$$(x+p-2\sqrt{q})v_n(x) = u_{n+1}(x) - qu_{n-1}(x), \quad \text{from (1.6) and (1.10).} \tag{1.11}$$

$$(x+p-2\sqrt{q})v_n(x) = V_{n+1}(x) - \sqrt{q}V_n(x), \quad \text{by induction.} \tag{1.12}$$

Simson Formulas:

$$U_{n+1}(x)U_{n-1}(x) - U_n^2(x) = -q^{n-1}, \quad \text{from [1],} \tag{1.13a}$$

$$u_{n+1}(x)u_{n-1}(x) - u_n^2(x) = q^{n-1/2}\Delta_u, \quad \text{from [4],} \tag{1.13b}$$

$$v_{n+1}(x)v_{n-1}(x) - v_n^2(x) = -q^{n-1/2}\Delta_v, \quad \text{from [4],} \tag{1.13c}$$

$$V_{n+1}(x)V_{n-1}(x) - V_n^2(x) = q^{n-1}\Delta_u\Delta_v, \quad \text{from [2],} \tag{1.13d}$$

where

$$\Delta_u = x+p-2\sqrt{q}, \tag{1.14a}$$

$$\Delta_v = x+p+2\sqrt{q}. \tag{1.14b}$$

Binet's Formulas:

$$U_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{from [1],} \tag{1.15a}$$

$$u_n(x) = \frac{\alpha^{n+1/2} + \beta^{n+1/2}}{\alpha^{1/2} + \beta^{1/2}}, \quad \text{from (1.5) and (1.15a),} \tag{1.15b}$$

$$v_n(x) = \frac{\alpha^{n+1/2} - \beta^{n+1/2}}{\alpha^{1/2} - \beta^{1/2}}, \quad \text{from (1.6) and (1.15a),} \tag{1.15c}$$

$$V_n(x) = \alpha^n + \beta^n, \quad \text{from [2],} \tag{1.15d}$$

where

$$\alpha + \beta = x + p, \ \alpha\beta = q, \tag{1.16a}$$

$$\alpha - \beta = \sqrt{\Delta}, \ \Delta = \Delta_u\Delta_v = (x+p)^2 - 4q. \tag{1.16b}$$

2. NEW INTERRELATIONSHIPS

In this section, we will give a number of interesting relations between the André-Jeannin polynomials $U_n(x)$, $V_n(x)$, and their companions $u_n(x)$, $v_n(x)$. In order to present the results in a compact form, we will denote by $A_n(x)$ any one of the polynomials $U_n(x)$, $u_n(x)$, $v_n(x)$, or $V_n(x)$. We will first establish the following Lemma concerning $A_n(x)$ that is extremely useful in establishing certain relations needed to derive the results given in Section 4.

Lemma 1:

$$A_n(x)U_{r-h+2}(x) = A_r(x)U_{n-h+2}(x) - q^{r-h+2}A_{h-2}(x)U_{n-r}(x). \tag{2.1}$$

Proof: We confine ourselves to establishing the result when $A_n(x) \equiv U_n(x)$. Using (1.15a), we have

$$
\begin{aligned}
&U_n(x)U_{r-h+2}(x)-U_r(x)U_{n-h+2}(x)\\
&=\frac{\alpha^n-\beta^n}{\alpha-\beta}\cdot\frac{\alpha^{r-h+2}-\beta^{r-h+2}}{\alpha-\beta}-\frac{\alpha^r-\beta^r}{\alpha-\beta}\cdot\frac{\alpha^{n-h+2}-\beta^{n-h+2}}{\alpha-\beta}\\
&=\frac{-(\alpha\beta)^{r-h+2}}{(\alpha-\beta)^2}[\{\alpha^{n-r+h-2}+\beta^{n-r+h-2}\}-\{\alpha^{n-r}\beta^{h-2}+\beta^{n-r}\alpha^{h-2}\}]\\
&=-(\alpha\beta)^{r-h+2}\cdot\frac{\alpha^{h-2}-\beta^{h-2}}{\alpha-\beta}\cdot\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\\
&=q^{r-h+2}U_{h-2}(x)U_{n-r}(x),\ \text{using (1.16a) and (1.15a).}
\end{aligned}
$$

In a similar manner, Lemma 1 can be established when $A_n(x)\equiv u_n(x)$, $v_n(x)$, and $V_n(x)$ by using (1.15a) along with (1.15b), (1.15c), and (1.15d), respectively.

By letting $r=n-h+1$ and $h=n-r+1$ in (2.1), we get the following result:

$$A_n(x)U_{r-h+2}(x)=A_{n-h+1}(x)U_{r+1}(x)-q^{r-h+2}A_{n-r-1}(x)U_{h-1}(x). \tag{2.2}$$

Now, by equating the right-side expressions in (2.1) and (2.2) and rearranging, we get the determinantal relation

$$\begin{vmatrix} U_{r+1}(x) & U_{n-h+2}(x)\\ A_r(x) & A_{n-h+1}(x)\end{vmatrix}=q^{r-h+2}\begin{vmatrix} U_{h-1}(x) & U_{n-r}(x)\\ A_{h-2}(x) & A_{n-r-1}(x)\end{vmatrix}. \tag{2.3}$$

Now, letting $r=m$, $h=m-r+2$, and $n=m+n+1-r$ in (2.3), we get the interesting relation

$$\begin{vmatrix} U_{m+1}(x) & U_{n+1}(x)\\ A_m(x) & A_n(x)\end{vmatrix}=q^{r}\begin{vmatrix} U_{m+1-r}(x) & U_{n+1-r}(x)\\ A_{m-r}(x) & A_{n-r}(x)\end{vmatrix}. \tag{2.4}$$

Now letting $h=2$, $n=n+1$, and $r=n$ in (2.1), we have

$$U_{n+1}(x)A_n(x)-A_{n+1}(x)U_n(x)=q^nA_0(x). \tag{2.5}$$

Also, letting $m=r=n-1$ in (2.4), we get

$$U_{n+1}(x)A_{n-1}(x)-U_n(x)A_n(x)=q^{n-1}[(x+p)A_0(x)-A_1(x)]. \tag{2.6}$$

It is observed that when $A_n(x)\equiv U_n(x)$, (2.6) reduces to (1.13a). Now, by letting $n=m+n+1$, $h=n+2$, and $r=n+1$ in (2.1), we have the relation

$$A_{m+n+1}(x)=U_{m+1}(x)A_{n+1}(x)-qU_m(x)A_n(x). \tag{2.7}$$

Hence,

$$A_{2n+1}(x)=U_{n+1}(x)A_{n+1}(x)-qU_n(x)A_n(x) \tag{2.8}$$

and

$$A_{2n}(x)=U_n(x)A_{n+1}(x)-qU_{n-1}(x)A_n(x)=A_n(x)U_{n+1}(x)-qA_{n-1}(x)U_n(x). \tag{2.9}$$

We may derive a number of other interesting relations. However, we present only a few of these relations that will be useful in deriving the results of Section 4:

$$u_{n-1}(x)v_n(x)-u_n(x)v_{n-1}(x)=2q^{n-1/2}, \tag{2.10a}$$

$$u_{n-1}(x)V_n(x)-u_n(x)V_{n-1}(x)=-q^{n-1}\Delta_u, \tag{2.10b}$$

$$v_{n-1}(x)V_n(x) - v_n(x)V_{n-1}(x) = -q^{n-1}\Delta_v. \tag{2.10c}$$

$$\sum_{r=0}^{n} q^{n-r} A_{2r+1}(x) = A_{n+1}(x)U_{n+1}(x), \tag{2.11a}$$

$$\sum_{r=0}^{n} q^{n-r} A_{2r}(x) = A_n(x)U_{n+1}(x), \tag{2.11b}$$

$$\sum_{r=1}^{n} q^{n-r} A_{2r}(x) = U_n(x)A_{n+1}(x). \tag{2.11c}$$

In passing, it may be mentioned that, if we let $p=2$, $q=1$, and $x=1$, we have

$$U_n(1) = F_{2n},\ u_n(1) = F_{2n+1},\ v_n(1) = L_{2n+1},\ V_n(1) = L_{2n}. \tag{2.12}$$

Using identities (2.1)-(2.11), we may derive a number of interesting identities for Fibonacci and Lucas numbers. One such identity is the following, which may be obtained by letting $A_n(1) = V_n(1)$ in (2.5):

$$F_{2(n+1)}L_{2n} - F_{2n}L_{2(n+1)} = 2. \tag{2.13}$$

3. DERIVATIVES AND DIFFERENTIAL EQUATIONS

We now derive formulas for the derivatives of $U_n(x)$, $u_n(x)$, $v_n(x)$, and $V_n(x)$ with respect to x.

Theorem 1:

$$U_n'(x) = \sum_{r=1}^{n-1} U_r(x)U_{n-r}(x). \tag{3.1}$$

Proof: We establish the theorem by induction. The result is easily verified to be true for $n=1, 2,$ and 3. Now, assuming the theorem to be true for n and $n+1$, we have

$$\begin{aligned} U_{n+2}'(x) &= (x+p)U_{n+1}'(x) - qU_n'(x) + U_{n+1}(x), \text{ using (1.3a)} \\ &= (x+p)\sum_{r=1}^{n} U_r(x)U_{n+1-r}(x) - q\sum_{r=1}^{n-1} U_r(x)U_{n-r}(x) + U_{n+1}(x) \\ &= \sum_{r=1}^{n-1} U_r(x)[(x+p)U_{n+1-r}(x) - qU_{n-r}(x)] + (x+p)U_n(x)U_1(x) + U_{n+1}(x) \\ &= \sum_{r=1}^{n-1} U_r(x)U_{n+2-r}(x) + U_n(x)U_2(x) + U_{n+1}(x) = \sum_{r=1}^{n+1} U_r(x)U_{n+2-r}(x). \end{aligned}$$

Hence the theorem.

Corollary 1:

$$u_n'(x) = \sum_{r=1}^{n} U_r(x)u_{n-r}(x). \tag{3.2}$$

Proof:

$$\begin{aligned} u_n'(x) &= U_{n+1}'(x) - \sqrt{q}U_n'(x), \text{ using (1.5),} \\ &= \sum_{r=1}^{n} U_r(x)U_{n+1-r}(x) - \sqrt{q}\sum_{r=1}^{n-1} U_r(x)U_{n-r}(x), \text{ from Theorem 1,} \end{aligned}$$

$$= U_n(x)U_1(x) + \sum_{r=1}^{n-1} U_r(x)[U_{n+1-r}(x) - \sqrt{q}U_{n-r}(x)]$$

$$= U_n(x)u_0(x) + \sum_{r=1}^{n-1} U_r(x)u_{n-r}(x), \text{ from (1.5),}$$

$$= \sum_{r=1}^{n} U_r(x)u_{n-r}(x).$$

Corollary 2:

$$v_n'(x) = \sum_{r=1}^{n} U_r(x)v_{n-r}(x). \tag{3.3}$$

This corollary can be proved along the same lines as Corollary 1, using relation (1.6) and Theorem 1.

Corollary 3:

$$V_n'(x) = \sum_{r=1}^{n} U_r(x)V_{n-r}(x) - U_n(x). \tag{3.4}$$

This corollary can be established using (1.9) and Corollary 2.

It is also known that (see [2])

$$V_n'(x) = nU_n(x). \tag{3.5}$$

By induction, we may derive the following similar results for the derivatives of $u_n(x)$ and $v_n(x)$ in terms of $U_n(x)$.

Theorem 2:

$$(x + p + 2\sqrt{q})u_n'(x) = nU_{n+1}(x) + \sqrt{q}(n+1)U_n(x). \tag{3.6}$$

Theorem 3:

$$(x + p - 2\sqrt{q})v_n'(x) = nU_{n+1}(x) - \sqrt{q}(n+1)U_n(x). \tag{3.7}$$

In passing, it may be observed that, from (3.4) and (3.5), we have the following interesting relation:

$$\sum_{r=1}^{n} U_r(x)V_{n-r}(x) = (n+1)U_n(x). \tag{3.8}$$

André-Jeannin [3] has shown that $U_n(x)$ and $U_n(x)$ satisfy, respectively, the differential equations

$$U_n(x)\text{: } \Delta y'' + 3(x+p)y' - (n^2 - 1)y = 0 \tag{3.9}$$

and

$$V_n(x)\text{: } \Delta y'' + (x+p)y' - n^2 y = 0, \tag{3.10}$$

where Δ is given by (1.16b). We now establish similar differential equations satisfied by $u_n(x)$ and $v_n(x)$.

Theorem 4: The polynomial $u_n(x)$ satisfies the differential equation

$$\Delta y'' + 2(x + p - \sqrt{q})y' - n(n+1)y = 0.$$

Proof: Since $U_n(x)$ satisfies the differential equation given by (3.9), we have

$$\Delta U_{n+1}''(x) + 3(x+p)U_{n+1}'(x) - n(n+2)U_{n+1}(x) = 0 \tag{3.11}$$

and

$$\Delta U_n''(x) + 3(x+p)U_n'(x) - (n^2-1)U_n(x) = 0. \tag{3.12}$$

Multiplying (3.12) by $\sqrt{q}$, then subtracting it from (3.11) and making use of relation (1.5) in the resulting equation, we get

$$\Delta u_n''(x) + 3(x+p)u_n'(x) - n(n+1)u_n(x) - [nU_{n+1}(x) + (n+1)\sqrt{q}U_n(x)] = 0. \tag{3.13}$$

Use of Theorem 2 reduces (3.13) to

$$\Delta u_n''(x) + 2(x+p-\sqrt{q})u_n'(x) - n(n+1)u_n(x) = 0. \tag{3.14}$$

Hence the theorem.

Similarly, by using (1.6), (3.9), and Theorem 3, we can prove the following result regarding $v_n(x)$.

Theorem 5: The polynomial $v_n(x)$ satisfies the differential equation

$$\Delta y'' + 2(x+p+\sqrt{q})y' - n(n+1)y = 0. \tag{3.15}$$

André-Jeannin [3] has further shown that $U_n^{(k)}(x)$ and $V_n^{(k)}(x)$, $k = 0, 1, 2, \ldots$, where the superscript (k) stands for the k^{th} derivative with respect to x, satisfies the following differential equations:

$$U_n^{(k)}(x):\ \Delta y'' + (2k+3)(x+p)y' + \{(k+1)^2 - n^2\}y = 0, \tag{3.16a}$$

$$V_n^{(k)}(x):\ \Delta y'' + (2k+1)(x+p)y' + (k^2 - n^2)y = 0. \tag{3.16b}$$

Using a similar procedure, and using Theorems 4 and 5, we may also establish that $u_n^{(k)}(x)$ and $v_n^{(k)}(x)$ satisfy the following differential equations:

$$u_n^{(k)}(x):\ \Delta y'' + 2(k+1)(x+p-\sqrt{q})y' + \{k(k+1) - n(n+1)\}y = 0, \tag{3.16c}$$

$$v_n^{(k)}(x):\ \Delta y'' + 2(k+1)(x+p+\sqrt{q})y' + \{k(k+1) - n(n+1)\}y = 0. \tag{3.16d}$$

It may be pointed out that the above two differential equations are, respectively, the generalizations of the corresponding ones for the modified Morgan-Voyce polynomials $\tilde{b}_n(x)$ and $\tilde{c}_n(x)$ given in [4].

4. INTEGRAL PROPERTIES

From (3.5), we have the result

$$\int U_n(x)\,dx = \frac{V_n(x)}{n} + K. \tag{4.1}$$

Hence,

$$\int u_n(x)\,dx = \frac{V_{n+1}(x)}{n+1} - \sqrt{q}\frac{V_n(x)}{n} + K, \text{ from (1.5) and (4.1)}, \tag{4.2}$$

$$\int v_n(x)\,dx = \frac{V_{n+1}(x)}{n+1} + \sqrt{q}\frac{V_n(x)}{n} + K, \text{ from (1.6) and (4.1)}, \tag{4.3}$$

and

$$\int V_n(x)\,dx = \frac{V_{n+1}(x)}{n+1} - q\frac{V_{n-1}(x)}{n} + K, \text{ from (1.7) and (4.1)}. \tag{4.4}$$

Let us denote

$$a=-p-2\sqrt{q},\quad b=-p+2\sqrt{q}. \tag{4.5}$$

Then we can establish by induction that

$$V_n(a)=(-1)^n 2q^{n/2},\quad V_n(b)=2q^{n/2}. \tag{4.6}$$

Using (4.6) in (4.1), (4.2), (3.3), and (4.4), we have the following results:

$$\int_a^b U_{2n}(x)\,dx=0, \tag{4.7a}$$

$$\int_a^b U_{2n+1}(x)\,dx=\frac{4}{2n+1}q^{n+1/2}, \tag{4.7b}$$

$$\int_a^b u_{2n}(x)\,dx=-\int_a^b u_{2n+1}(x)\,dx=\frac{4}{2n+1}q^{n+1}, \tag{4.8}$$

$$\int_a^b v_{2n}(x)\,dx=\int_a^b v_{2n+1}(x)\,dx=\frac{4}{2n+1}q^{n+1}, \tag{4.9}$$

$$\int_a^b V_{2n}(x)\,dx=-\frac{8}{4n^2-1}q^{n+1/2}, \tag{4.10a}$$

$$\int_a^b V_{2n+1}(x)\,dx=0. \tag{4.10b}$$

Letting $A_n(x)\equiv U_n(x)$ in (2.11b) and using (4.7a), we see that

$$\int_a^b U_{n+1}(x)U_n(x)\,dx=0. \tag{4.11a}$$

Also, by letting $A_n(x)\equiv U_n(x)$ in (2.11a) and using (4.7b), we have

$$\int_a^b U_{n+1}^2(x)\,dx=\sum_{r=0}^{n} q^{n-r}\frac{4q^{r+1/2}}{2r+1}=\sum_{r=0}^{n}\frac{4q^{n+1/2}}{2r+1}.$$

Hence,

$$\int_a^b U_n^2(x)\,dx=4q^{n-1/2}\sum_{r=0}^{n}\frac{1}{2r+1}. \tag{4.11b}$$

Now, integrating (1.13a) and using (4.11b), we have

$$\int_a^b U_{n+1}(x)U_{n-1}(x)\,dx=4q^{n-1/2}\sum_{r=0}^{n-1}\frac{1}{2r+1}. \tag{4.11c}$$

Similarly, by successively letting $A_n(x)\equiv u_n(x)$, $v_n(x)$, and $V_n(x)$ in (2.11c), (2.11a), and (2.11b) and using (4.8), (4.9), (4.10a), and (4.10b), we can derive the following relations:

$$\int_a^b u_{n+1}(x)U_n(x)\,dx=\int_a^b v_{n+1}(x)U_n(x)\,dx=4q^{n+1/2}\sum_{r=1}^{n}\frac{1}{2r+1}, \tag{4.12a}$$

$$\int_a^b V_{n+1}(x)U_n(x)\,dx=-\frac{8n}{2n+1}q^{n+1/2}, \tag{4.12b}$$

$$\int_a^b u_n(x)U_n(x)\,dx=-\int_a^b v_n(x)U_n(x)\,dx=-4q^n\sum_{r=0}^{n-1}\frac{1}{2r+1}, \tag{4.13a}$$

$$\int_a^b V_n(x)U_n(x)\,dx=0, \tag{4.13b}$$

$$\int_a^b u_n(x)U_{n+1}(x)\,dx = \int_a^b v_n(x)U_{n+1}(x)\,dx = 4q^{n+1/2}\sum_{r=0}^{n}\frac{1}{2r+1}, \tag{4.14a}$$

$$\int_a^b V_n(x)U_{n+1}(x)\,dx = \frac{8(n+1)}{2n+1}q^{n+1/2}. \tag{4.14b}$$

Corresponding to the relations (4.11a), (4.11b), and (4.11c) for $U_n(x)$, we may derive relations for the polynomials $u_n(x)$, $v_n(x)$, and $V_n(x)$. Substituting (1.5) and (1.6), respectively, for $u_n(x)$ and $v_n(x)$ in the expressions $u_{n+1}(x)u_n(x)$ and $v_{n+1}(x)v_n(x)$, and utilizing the relations (4.12a) and (4.13a), we have

$$\int_a^b u_{n+1}(x)u_n(x)\,dx = -\int_a^b v_{n+1}(x)v_n(x)\,dx = -4q^{n-1}\left[1+2\sum_{r=1}^{n}\frac{1}{2r+1}\right]. \tag{4.15a}$$

Substituting (1.5) and (1.6) in the expressions $u_n^2(x)$ and $v_n^2(x)$ and using (4.11a) and (4.11b), we get

$$\int_a^b u_n^2(x)\,dx = \int_a^b v_n^2(x)\,dx = 4q^{n+1/2}\left[\frac{1}{2n+1}+2\sum_{r=0}^{n-1}\frac{1}{2r+1}\right]. \tag{4.15b}$$

Now, integrating both sides of (1.13b) and (1.13c) and using (4.15b), we derive

$$\int_a^b u_{n+1}(x)u_{n-1}(x)\,dx = \int_a^b v_{n+1}(x)v_{n-1}(x)\,dx = 4q^{n+1/2}\left[\frac{1}{2n+1}+2\sum_{r=1}^{n-1}\frac{1}{2r+1}\right]. \tag{4.15c}$$

The corresponding expressions involving $V_n(x)$ may be derived using (1.8), (1.9), (1.13d), (4.15a), (4.15b), and (4.15c). These are:

$$\int_a^b V_{n+1}(x)V_n(x)\,dx = 0, \tag{4.16a}$$

$$\int_a^b V_n^2(x)\,dx = \frac{16(2n^2-1)}{(4n^2-1)}q^{n+1/2}, \tag{4.16b}$$

$$\int_a^b V_{n+1}(x)V_{n-1}(x)\,dx = -\frac{16(2n^2+1)}{3(4n^2-1)}q^{n+1/2}. \tag{4.16c}$$

In a similar manner, we can derive relations regarding integrals involving $u_n(x)$ and $v_n(x)$, $u_n(x)$ and $V_n(x)$, and $v_n(x)$ and $V_n(x)$. These correspond to relations (4.12a), (4.12b), and (4.12c) which, respectively, involve $u_n(x)$ and $U_n(x)$, $v_n(x)$ and $U_n(x)$, and $V_n(x)$ and $U_n(x)$. These are:

$$\int_a^b u_n(x)v_{n+1}(x)\,dx = -\int_a^b v_n(x)u_{n+1}(x)\,dx = 4q^{n+1}, \tag{4.17a}$$

$$\int_a^b u_n(x)v_n(x)\,dx = \frac{4}{2n+1}q^{n+1/2}, \tag{4.17b}$$

$$\int_a^b u_n(x)V_{n+1}(x)\,dx = -\int_a^b v_n(x)V_{n+1}(x)\,dx = \frac{8n}{2n+1}q^{n+1}, \tag{4.18a}$$

$$\int_a^b u_n(x)V_n(x)\,dx = \int_a^b v_n(x)V_n(x)\,dx = \frac{8(n+1)}{2n+1}q^{n+1/2}, \tag{4.18b}$$

$$\int_a^b u_{n+1}(x)V_n(x)\,dx = -\int_a^b v_{n+1}(x)V_n(x)\,dx = \frac{8(n+1)}{2n+1}q^{n+1}. \tag{4.18c}$$

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REFERENCES

1. R. André-Jeannin. "A Note on a General Class of Polynomials." *The Fibonacci Quarterly* **32.5** (1994):445-54.
2. R. André-Jeannin. "A Note on a General Class of Polynomials, Part II." *The Fibonacci Quarterly* **33.4** (1995):341-51.
3. R. André-Jeannin. "Differential Properties of a General Class of Polynomials." *The Fibonacci Quarterly* **33.5** (1995):453-57.
4. M. N. S. Swamy. "On a Class of Generalized Polynomials." *The Fibonacci Quarterly* **35.4** (1997):329-34.

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THE EIGENVECTORS OF A CERTAIN MATRIX OF BINOMIAL COEFFICIENTS

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1. INTRODUCTION

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers n by

$$U_n = pU_{n-1} - qU_{n-2}, \quad U_0 = 0,\ U_1 = 1,$$
$$V_n = pV_{n-1} - qV_{n-2}, \quad V_0 = 2,\ V_1 = p,$$

where p and q are real numbers with $q(p^2 - 4q) \neq 0$. These sequences were studied originally by Lucas [4], and have subsequently been the subject of much attention.

The Binet forms of U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}$$

are the roots, assumed distinct, of $x^2 - px + q = 0$. We assume further that α / β is not an n^{th} root of unity for any n.

For n greater than or equal to 1, let $S(n)$ be the $n \times n$ matrix defined by

$$S(n) = \begin{pmatrix} 0 & 0 & 0 & \cdots & (-1)^{n-1}\binom{n-1}{n-1}q^{n-1} \\ & & & \cdots & \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ & & & \cdots & \\ 0 & 0 & \binom{2}{2}q^2 & \cdots & \binom{n-1}{2}p^{n-3}q^2 \\ 0 & -\binom{1}{1}q & -\binom{2}{1}pq & \cdots & -\binom{n-1}{1}p^{n-2}q \\ \binom{0}{0} & \binom{1}{0}p & \binom{2}{0}p^2 & \cdots & \binom{n-1}{0}p^{n-1} \end{pmatrix}.$$

The element in its i^{th} row and j^{th} column is

$$(-q)^{n-i}\binom{j-1}{j+i-n-1}p^{i+j-n-1}.$$

The matrix $S(n)$ is the general term in a sequence of matrices $\{S(n)\}_{n=1}^{\infty}$, where the first few terms are

$$S(1)=(1),\ S(2)=\begin{pmatrix}0 & -q\\ 1 & p\end{pmatrix},\ \text{and}\ S(3)=\begin{pmatrix}0 & 0 & q^2\\ 0 & -q & -2pq\\ 1 & p & p^2\end{pmatrix}.$$

It can be proved by induction that

$$S^n(2)=\begin{pmatrix}-qU_{n-1} & -qU_n\\ U_n & U_{n+1}\end{pmatrix}$$

and

$$S^n(3)=\begin{pmatrix}q^2U_{n-1}^2 & q^2U_{n-1}U_n & q^2U_n^2\\ -2qU_{n-1}U_n & -q(U_n^2+U_{n-1}U_{n+1}) & -2qU_nU_{n+1}\\ U_n^2 & U_nU_{n+1} & U_{n+1}^2\end{pmatrix},$$

with similar results for the higher-order matrices in the sequence $\{S(n)\}_{n=1}^{\infty}$. When $p=-q=1$, $S(2)$ becomes essentially the Q-Matrix for the Fibonacci numbers. For applications of $S(3)$ and $S(4)$ to the derivation of certain infinite series, and for more background information on these matrices, see [6] and [7].

Carlitz [1] considered the matrix $S(n)^T$ for the special case $p=-q=1$. Among other things, he found its eigenvalues and its characteristic polynomial, and stated that its eigenvectors were not evident.

Mahon and Horadam [5] worked with the matrix $S(n)$ for the case $q=-1$ and put forward a conjecture stating its characteristic polynomial. This conjecture was later proved by Duvall and Vaughan [3].

More recently, Cooper and Kennedy [2] considered the matrix $S(n)^T$ and proved a result of Jarden by generalizing the work of Carlitz [1]. If we translate their results to our matrix $S(n)$, they proved, among other things:

(i) The eigenvalues of $S(n)$ are $\alpha^{n-1}, \alpha^{n-2}\beta, \alpha^{n-3}\beta^2, \ldots, \alpha\beta^{n-2}, \beta^{n-1}$.

(ii) The characteristic equation of $S(n)$ is

$$\sum_{k=0}^{n}(-1)^k q^{(k-1)k/2}\{n,k\}\lambda^{n-k}=0,$$

where

$$\{n,k\}=\begin{cases}1, & \text{for } k=0,n,\\ \dfrac{\prod_{i=1}^{n}U_i}{\left(\prod_{i=1}^{k}U_i\right)\left(\prod_{i=1}^{n-k}U_i\right)}, & \text{for } 0<k<n.\end{cases}$$

There remains the question of the eigenvectors of $S(n)$. The purpose of this paper is to answer that question.

2. EIGENVECTORS OF $S(n)$

Let $0\le k\le n-1$ be a fixed integer,

$$f(x)=(x-\alpha)^k(x-\beta)^{n-1-k}=\sum_{r=0}^{n-1}v_rx^r,$$

and

$$\mathbf{v}=(v_0, v_1, \ldots, v_{n-1})^T.$$

Lemma 1: Let $m \geq 0$. Then

$$f^{(m)}(x) = m!\frac{f(x)}{(x-\alpha)^m(x-\beta)^m}\sum_{j=0}^{m}\binom{k}{m-j}\binom{n-1-k}{j}(x-\alpha)^j(x-\beta)^{m-j}.$$

Proof: We will use Leibniz's formula for the m^{th} derivative of a product, i.e.,

$$\frac{d^m}{dx^m}g(x)h(x) = \sum_{j=0}^{m}\binom{m}{j}g^{(m-j)}(x)h^{(j)}(x).$$

Using the notation $x^{\underline{n}}$ to denote the falling factorial, it follows that

$$\begin{aligned} f^m(x) &= \sum_{j=0}^{m}\binom{m}{j}k^{\underline{m-j}}(x-\alpha)^{k-m+j}(n-1-k)^{\underline{j}}(x-\beta)^{n-1-k-j} \\ &= m!\frac{f(x)}{(x-\alpha)^m(x-\beta)^m}\sum_{j=0}^{m}\binom{k}{m-j}\binom{n-1-k}{j}(x-\alpha)^j(x-\beta)^{m-j}. \end{aligned}$$

Lemma 2: Let $0 \leq m \leq n-1$ be a fixed integer. Then

$$v_{n-1-m} = \sum_{j=0}^{m}(-1)^m\binom{k}{m-j}\binom{n-1-k}{j}\alpha^{m-j}\beta^j$$

and

$$(S(n)\mathrm{v})_{n-1-m} = \sum_{r=m}^{n-1}(-q)^m\binom{r}{m}p^{r-m}v_r.$$

Proof: The first result follows by computing the coefficient of x^{n-1-m} in $f(x)$ by multiplying $(x-\alpha)^k$ times $(x-\beta)^{n-1-k}$. The second result follows by computing the product of $S(n)$ and v.

Theorem: $S(n)\mathrm{v} = \alpha^{n-1-k}\beta^k\mathrm{v}$.

Proof:

$$\begin{aligned} (S(n)\mathrm{v})_{n-1-m} &= \sum_{r=m}^{n-1}(-q)^m\binom{r}{m}p^{r-m}v_r \\ &= \frac{(-q)^m}{m!}\sum_{r=m}^{n-1}v_r r^{\underline{m}}p^{r-m} = \frac{(-q)^m}{m!}f^{(m)}(p) \\ &= \frac{(-1)^m(\alpha\cdot\beta)^m}{m!}\,\frac{(p-\alpha)^k(p-\beta)^{n-1-k}}{(p-\alpha)^m(p-\beta)^m} \\ &\quad m!\cdot\sum_{j=0}^{m}\binom{k}{m-j}\binom{n-1-k}{j}(p-\alpha)^j(p-\beta)^{m-j} \\ &= \alpha^{n-1-k}\beta^k(-1)^m\sum_{j=0}^{m}\binom{k}{m-j}\binom{n-1-k}{j}\beta^j\alpha^{m-j} \\ &= \alpha^{n-1-k}\beta^k\sum_{j=0}^{m}(-1)^m\binom{k}{m-j}\binom{n-1-k}{j}\alpha^{m-j}\beta^j \\ &= \alpha^{n-1-k}\beta^k v_{n-1-m}. \end{aligned}$$

REFERENCES

1. L Carlitz. "The Characteristic Polynomial of a Certain Matrix of Binomial Coefficients." *The Fibonacci Quarterly* **3.2** (1965):81-89.
2. C. Cooper & R. E. Kennedy. "Proof of a Result by Jarden by Generalizing a Proof of Carlitz." *The Fibonacci Quarterly* **33.4** (1995):304-10.
3. P. Duvall & T. Vaughan. "Pell Polynomials and a Conjecture of Mahon and Horadam." *The Fibonacci Quarterly* **26.4** (1988):344-53.
4. E. Lucas. "Théorie des fonctions numériques simplement periodiques." *Amer. J. Math.* **1** (1878):184-240, 289-321.
5. Bro. J. M. Mahon & A. F. Horadam. "Matrix and Other Summation Techniques for Pell Polynomials." *The Fibonacci Quarterly* **24.4** (1986):290-309.
6. R. S. Melham. "Lucas Sequences and Functions of a 3-by-3 Matrix." *The Fibonacci Quarterly* **37.2** (1999):111-16.
7. R. S. Melham. "Lucas Sequences and Functions of a 4-by-4 Matrix." *The Fibonacci Quarterly*.**37.3** (1999):269-76.

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NONEXHAUSTIVE GENERALIZED FIBONACCI TREES IN UNEQUAL COSTS CODING PROBLEMS

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1. INTRODUCTION AND BACKGROUND

Fibonacci trees and exhaustive generalized Fibonacci trees have been introduced and studied in connection with a particular unequal costs coding application by Horibe [6], Chang [2], and the author [1]. The k^{th} exhaustive generalized Fibonacci tree, $S(k)$, by definition has $S(k-c(i))$ as its i^{th} leftmost subtree descending from the root, $i=1,2,\ldots,r$, where the $c(i)$ are relatively prime positive integers ordered in monotonically nondecreasing order in i, and the initialization is that $S(k)$, $k=1,2,\ldots,c(r)$, are all single root nodes. The term "exhaustive" indicates that each interior node of the r-ary tree has exactly r descendants, and is referred to as a "full" node. For $r=2$, $c(1)=1$, $c(2)=2$, these trees are Horibe's Fibonacci trees [6]. Throughout this paper, the notation $S(k)$ will refer to the k^{th} exhaustive generalized Fibonacci tree for some fixed set of $c(i)$, $i=1,2,\ldots,r$, and $g(k)$ will refer to the number of leaf nodes in $S(k)$.

The exhaustive generalized Fibonacci trees when interpreted as code trees solve Varn's [10] unequal costs coding problem under the requirement that the code trees be exhaustive [1]. In particular, each of the $c(i)$, now interpreted as the cost of the corresponding code symbol, is associated with one of the r code symbols which label the code tree branches successively in left to right order. The i^{th} leftmost branch is labeled with the i^{th} least costly code symbol. A path from the root to a leaf node describes the sequence of code symbols, or, that is, the codeword, which represents the source symbol associated with that leaf node. The cost of a (leaf or interior) node is the sum of the costs of the branches contained in the path from the root to the node. It is assumed that each source symbol arises with equal probability, and Varn's problem is to find the code tree which minimizes the average codeword cost.

A number of authors in addition to Varn have addressed the nonexhaustive unequal costs coding problem for equiprobable source symbols from the algorithmic point of view for general sets of costs [3], [4], [5], [8], or for specific cost assignments [7] or have obtained bounds on the resulting minimum average cost [9]. The term "nonexhaustive" indicates that each interior node of the tree has at least 2 and at most r descendants. The descendant branches which are present are the leftmost or least costly descendants. An interior node with fewer than r descendants is referred to as a "nonfull" node. A nonexhaustive code tree can have lower average codeword cost than if the exhaustive requirement is imposed, hence the interest in the nonexhaustive case. However, the algorithms to construct optimal nonexhaustive code trees are much more complicated than Varn's simple algorithm for the exhaustive case.

Because recognizing the exhaustive generalized Fibonacci trees as Varn code trees for the exhaustive case reveals an elegant structure underlying the sequence of Varn code trees, it would be of interest to identify a similar recursive tree construction for the nonexhaustive case. It turns out that it is possible to do this not for Varn's original problem, but for a close variant of it. While

Varn looks for optimum codes in the minimum average codeword cost sense, the problem of interest here will be to look for optimum codes in the sense of minimizing the maximum codeword cost, called the minimax cost. It is not hard to see that in the exhaustive case, Varn's algorithm finds optimum code trees in both senses, that is, the minimum average codeword cost tree is also the minimax tree. But this is not the case for nonexhaustive codes. Perl et al. [8] give a simple algorithm for the nonexhaustive minimax problem as a "remark" in their paper otherwise concerned with the minimum average codeword cost case. So, as we'll see, it is the minimax version of Varn's problem which has the Fibonacci-like structure.

In the exhaustive case, Varn's [10] algorithm constructs the minimum average codeword cost tree of N leaf nodes, where $(N-1)/(r-1)$ is an integer, M, the number of interior nodes in the tree including the root. Starting with an r-ary tree from whose root node descend r leaf nodes labeled from left to right by $c(1), c(2), \dots, c(r)$, the costs corresponding to the code symbols, select the lowest cost leaf node, say c, and let descend from it r leaf nodes assigned costs from left to right of $c+c(1), c+c(2), \dots, c+c(r)$. Continue by selecting the lowest cost leaf node from the new tree until there are N leaf nodes. (Ties are broken by first selecting leftmost leaf nodes with respect to their equal cost sibling leaf nodes and otherwise arbitrarily.) Clearly the resulting tree is also a minimax exhaustive tree because splitting any other node besides the least cost node will create a leaf node of greater cost.

The nonexhaustive algorithms for the minimum average cost trees of Varn [10], Perl et al. [8], Cot [4], Golin and Young [5], and Choi and Golin [3] are all rather complicated. Perl, Garey, and Even's [8] algorithm for the minimax tree is simpler than any of the minimum average cost algorithms. A new leaf node can be added to a tree by either "branching" or "adding." In branching, a leaf node of least cost, say c, gets descending from it 2 leaf nodes assigned costs from left to right of $c+c(1)$ and $c+c(2)$. In adding, a nonfull interior node of some cost c which has $2 \le i < r$ leaf nodes descending from it gets an additional descendant leaf node of cost $c+c(i+1)$. The minimax algorithm is to branch or add, creating the least cost next leaf node at each stage.

While Varn's [10] original problem statement and algorithms assumed arbitrary positive costs, the recursive method described here applies to arbitrary positive integer code symbol costs $c(1) \le c(2) \le \cdots \le c(r)$, ordered without loss of generality, whose greatest common divisor is 1. In the binary case, all code trees are exhaustive, and the exhaustive case for all r has been treated previously by the author [1]. The nonexhaustive approach reduces to the exhaustive approach for binary problems. We can permit r to be arbitrarily large and thus include the case of r infinite in the limit although the limiting case will never be achieved. Since common factors shared by all costs do not affect the form of the optimal code tree, the costs considered here are essentially all rational costs or all sets of rational costs with a common irrational multiplier.

2. CONSTRUCTING NONEXHAUSTIVE GENERALIZED FIBONACCI TREES RECURSIVELY

First, let's define nonexhaustive generalized Fibonacci trees through a recursive construction. Then, in Theorem 1, an equivalent construction, based on the method of "types" for constructing exhaustive generalized Fibonacci trees, will be given. As in the exhaustive case, the k^{th} tree in the constructed sequence of nonexhaustive trees, $T(k)$, will have $T(k-c(i))$ as its i^{th} leftmost subtree. However, now the initialization will be $T(1)=T(2)=\cdots=T(c(2))$ each consisting of a single root node. Define the k^{th} nonexhaustive generalized Fibonacci tree, $T(k)$, by this recursive construction. Throughout this paper, the notation $T(k)$ will refer to the k^{th} nonexhaustive generalized Fibonacci tree for some fixed set of $c(i)$, $i = 1, 2, \ldots, r$, and $f(k)$ will refer to the number of leaf nodes in $T(k)$.

An example follows in which the $c(i)$ are interpreted as costs and assigned corresponding to the branches present in the tree. Let $c(1) = c(2) = 2$, $c(3) = 5$. The same example will be used throughout the paper. The trees are described by labeling leaf nodes with their costs, listing them in left to right order with sibling nodes separated by + signs, and using parentheses to indicate depth in the tree from the root. Thus, for example, $((4+4)+(4+4)+5)$ denotes a nonexhaustive 3-ary tree with 5 leaves; in left to right order, with the root at the top, they are at depths 2, 2, 2, 2, 1, respectively, and labeled as 4, 4, 4, 4, 5 from left to right. The root node is full but the two non-root interior nodes do not have a rightmost descendant. The first few trees for the example are given in Table 1. The initializing trees, represented by 0 in this notation, do not appear.

TABLE 1. Recursively generated trees for the example, $c(1) = c(2) = 2$, $c(3) = 5$

k	$T(k)$
3	$(2+2)$
4	$(2+2)$
5	$((4+4)+(4+4))$
6	$((4+4)+(4+4)+5)$
7	$(((6+6)+(6+6))+((6+6)+(6+6))+5)$
...	...

Note that the subsequence of trees is not indexed by the number of leaf nodes which it has and that not every tree is distinct in form. The number of leaf nodes in $T(k)$, $f(k)$ is given by

$$f(k) = \sum_{1 \le i \le r} f(k - c(i)),$$

where

$$f(1) = f(2) = \cdots = f(c(1)) = 1.$$

By the method of generating functions,

$$F(x) = \sum_{1 \le k \le \infty} x^k f(k) = (x + x^2 + \cdots + x^{c(1)}) \Big/ \left(1 - \sum_{1 \le i \le r} x^{c(i)}\right)$$

and the $f(k)$ can be read off as coefficients of x^k. For the example,

$$F(x) = (x + x^2)/(1 - 2x^2 - x^5) = x + x^2 + 2x^3 + 2x^4 + 4x^5 + 5x^6 + 9x^7 + \cdots.$$

The nonexhaustive generalized Fibonacci trees can also be generated by a second method which makes use of the method of "types" for generating the exhaustive generalized Fibonacci trees. The k^{th} exhaustive generalized Fibonacci tree $S(k)$ is also obtained in [1] by using the concept of leaf node "types," essentially a mechanism for keeping track of the relative cost of each leaf node until it is one of the lowest cost leaf nodes and one of the next possible leaf nodes to split in Varn's exhaustive algorithm. (All tied leaf nodes "split" at once in the method of types, but split serially in Varn's algorithm. Thus, the set of values for number of leaf nodes in a tree obtained by Varn's algorithm, $N = M(r-1)+1$, can include values of N not equal to $g(k)$ for any k.) Each exhaustive tree will have $c(r)$ "types" of leaf nodes, denoted by $a(1), a(2), \ldots, a(c(r))$. The tree $S(k+1)$ is obtained from $S(k)$ by replacing leaf nodes in $S(k)$ of type $a(1)$ by r descendant nodes of types $a(c(1)), a(c(2)), \ldots, a(c(r))$ in left to right order and by replacing leaf nodes in $S(k)$ of type $a(j)$ by leaf nodes of type $a(j-1)$, $j = 2, 3, \ldots, c(r)$. The construction starts with $S(1)$ which consists of a single root node of type $a(c(r))$. The equivalence between the recursive construction used to define the exhaustive generalized Fibonacci tree $S(k)$ and the construction by the method of types, as well as the fact that the type associated with a leaf node in $S(k-c(i))$ is unchanged in $S(k)$, was proved by Horibe [6] for $r = 2, c(1) = 1, c(2) = 2$, and the argument goes through to the general case straightforwardly. The exhaustive generalized Fibonacci trees corresponding to the nonexhaustive generalized Fibonacci example of Table 1 are given in Table 2, with leaf nodes labeled by type rather than cost.

TABLE 2. Exhaustive trees for the example, $c(1) = c(2) = 2$, $c(3) = 5$

k	$S(k)$
6	$(a(2)+a(2)+a(5))$
7	$(a(1)+a(1)+a(4))$
8	$((a(2)+a(2)+a(5))+(a(2)+a(2)+a(5))+a(3))$
9	$((a(1)+a(1)+a(4))+(a(1)+a(1)+a(4))+a(2))$
10	$(((a(2)+a(2)+a(5))+(a(2)+a(2)+a(5))+a(3))+((a(2)+a(2)+a(5))+(a(2)+a(2)+a(5))+a(3))+a(1))$
...	...

Theorem 1: The nonexhaustive generalized Fibonacci tree $T(k)$ is given by the corresponding exhaustive generalized Fibonacci tree $S(k+c(r)-c(2))$ from which all leaf nodes except those of types $a(1), a(2), \ldots, a(c(2))$ have been deleted.

Proof of Theorem 1: This correspondence can be proved by induction. The induction initialization is immediate from the initialization of the recursively constructed generalized Fibonacci tree sequences $\{T(k)\}$ and $\{S(k)\}$. Let $S^*(k)$ denote $S(k)$ from which all leaf nodes except those of types $a(1), a(2), \ldots, a(c(2))$ have been deleted. Assuming that $T(k-c(i)) = S^*(k-c(i)+c(r)-c(2))$ for $i = 1, \ldots, r$, we need to show that $T(k) = S^*(k+c(r)-c(2))$. First, note that $T(k)$ has $T(k-c(i))$ as its i^{th} leftmost subtree by definition of $\{T(k)\}$ as a recursively generated sequence of trees so that it has $S^*(k-c(i)+c(r)-c(2))$ as its i^{th} leftmost subtree by the induction hypothesis. Then, note that $S^*(k+c(r)-c(2))$ has $S^*(k-c(i)+c(r)-c(2))$ as its i^{th} leftmost subtree by definition of $\{S(k)\}$ as a recursively generated sequence of trees and because deleting leaf nodes of type $a(j)$ from $S(k+c(r)-c(2))$ corresponds to deleting leaf nodes of type $a(j)$ from $S(k-c(i)+c(r)-c(2))$, $i = 1, \ldots, r$. Therefore, $T(k)$ and $S^*(k+c(r)-c(2))$ both have $S^*(k-c(i)+c(r)-c(2))$ as their i^{th} leftmost subtrees and $T(k) = S^*(k+c(r)-c(2))$. □

The maximum codeword cost in the nonexhaustive generalized Fibonacci tree $T(k)$ will be given by the cost of a leaf of type $a(c(2))$ in the corresponding exhaustive generalized Fibonacci tree $S(k+c(r)-c(2))$ if it appears. In the exhaustive tree $S(k)$, a leaf of type $a(j)$ will cost $k+j-(c(r)+1)$. Thus, the maximum codeword cost for $T(k)$ will be $k-1$ assuming a leaf of type $a(c(2))$ appears in $S(k+c(r)-c(2))$. If no leaf of type $a(c(2))$ appears in $S(k+c(r)-c(2))$, as in the example for $S(7)$, then the maximum codeword cost will be given by the cost of a leaf of type $a(c(2)-1)$ or $a(c(2)-2)$ or ... or $a(2)$ whichever is the leaf of type of highest index less than or equal to $c(2)$ which appears. Therefore, the maximum codeword cost for $T(k)$ is in the range $[k-c(2)+1, k-1]$ for $k > c(2)$. For sufficiently large k, there will always be a leaf of type $a(c(2))$ in $T(k)$ and the maximum codeword cost will be $k-1$.

3. MINIMAX OPTIMALITY

Now, let's show that nonexhaustive generalized Fibonacci trees are optimal nonexhaustive minimax trees of the same number of leaf nodes. We will use the characterization of the nonexhaustive generalized Fibonacci tree $T(k)$ as the exhaustive generalized Fibonacci tree $S(k+c(r)-c(2))$ with leaf nodes of type of index greater than $c(2)$ deleted, given in Theorem 1. The demonstration does not use Perl, Garey, and Even's optimal nonexhaustive minimax algorithm [8]; instead, we'll use a new optimal nonexhaustive minimax algorithm which is in the same spirit as Varn's original nonexhaustive algorithm [10] for optimality in the sense of minimum average cost. The algorithm will follow from Theorem 2, and is contained in the corollary to Theorem 2.

In the following, the notation S_0 will refer to an optimal exhaustive minimax code tree, T_0 an optimal nonexhaustive minimax code tree, S_1 some other exhaustive code tree, and T_1 some other nonexhaustive code tree, all for the same fixed set of costs $c(i), i = 1, 2, \dots, r$.

Theorem 2: An exhaustive tree S_0 with $N_S = M(r-1)+1$ leaf nodes and M interior nodes including the root is an optimal code tree in the sense of minimizing the maximum codeword cost if it is obtained from an optimal minimax nonexhaustive tree T_0 with $N_T < N_S$ leaf nodes by adding $r-i$ descendant leaf nodes to each interior node of T_0 from which i leftmost nodes descend, $2 \le i \le r$.

Lemma (Varn [10]): Let S_1 be an exhaustive tree with N_S leaf nodes and M interior nodes including the root where $N_S = M(r-1)+1$. Denote the costs of the M interior nodes of S_1 by $z_1 \le z_2 \le \cdots \le z_M$. Let S_0 be an optimal exhaustive tree with N_S leaf nodes and M interior nodes including the root. Denote the costs of the M interior nodes of S_0 by $y_1 \le y_2 \le \cdots \le y_M$. Then $y_m \le z_m$ for $m = 1, \dots, M$.

Proof of Theorem 2: The proof is by contradiction. Suppose the exhaustive code tree S_0 with N_S leaf nodes, obtained from the optimal nonexhaustive code tree T_0 by adding descendant leaf nodes to nonfull interior nodes of T_0, is not optimal but there is another exhaustive tree S_1 with N_S leaf nodes which is optimal. Construct a nonexhaustive tree T_1 from S_1 by retaining all of the interior nodes of S_1 and only the two leftmost or least cost descendants of each of the interior nodes. Then the maximum codeword cost of T_1 is $w_M + c(2)$ where w_M is the most costly interior node of S_1. If S_1 is optimal, then $w_M + c(2) \le x_M + c(i)$ for $2 \le i \le r$, from the lemma, where x_M is the most costly interior node of S_0, and because $c(2) \le c(i)$. But the maximum codeword cost of T_0, where S_0 is exhaustive and obtained from T_0 by adding descendant leaf nodes to nonfull interior nodes of T_0, is $x_M + c(i)$ for some i, where $2 \le i \le r$. Therefore, the maximum codeword cost of T_1 is less than or equal to the maximum codeword cost of T_0, contradicting the given optimality of T_0. Therefore, it must be that S_0 is optimal. □

Corollary: A nonexhaustive tree T_0 with N_T leaf nodes is an optimal code tree in the sense of minimizing the maximum codeword cost if it is obtained from an optimal exhaustive tree S_0 with $N_S > N_T$ leaf nodes by deleting $r - i_m$ rightmost descendant leaf nodes from each of the M interior nodes including the root of S_0 for some i_m, $2 \le i_m \le r$, $m = 1, \ldots, M$.

Proof of Corollary: Since by Theorem 2 it is possible to obtain exhaustive S_0 from nonexhaustive T_0 by adding descendant leaf nodes to nonfull interior nodes of T_0, it is also possible to obtain T_0 from S_0 by deleting rightmost or greatest cost descendant leaf nodes from full interior nodes of S_0, repeating until the desired number of leaf nodes N_T is left. Note that no more than $r-2$ leaf nodes from any interior node of S_0 can be deleted because, as in Theorem 2, S_0 has the same set of interior nodes as T_0. □

The algorithm implicit in the corollary for optimal nonexhaustive code trees in the minimax sense is completely analogous to Varn's algorithm for optimal nonexhaustive code trees in the minimum average cost sense. In each case, the algorithm is to examine all candidate optimal exhaustive code trees such that deleting leaf nodes, while maintaining each interior node with at least two descendants, leads to a nonexhaustive tree with N_T leaf nodes, and to select the least costly of these. In each case, the question is to determine the appropriate value for N_S, the number of leaf nodes in the optimal exhaustive code tree from which the optimal nonexhaustive tree is constructed. In fact, a search over many candidate optimal exhaustive trees, S_0, is necessary to identify N_S in the minimum average codeword cost problem in Varn's algorithm for the nonexhaustive case. However, in the minimax codeword cost problem, as we'll see in Theorem 3, such a search is not necessary.

Theorem 3: If $N_T = f(k)$, where $f(k)$ is the number of leaf nodes in a nonexhaustive generalized Fibonacci tree $T(k)$ for some k, then a nonexhaustive code tree T_0 with N_T leaf nodes is an optimal code tree in the sense of minimizing the maximum codeword cost if it is obtained from an exhaustive generalized Fibonacci tree $S(k+c(r)-c(2))$ with $N_S = g(k+c(r)-c(2)) > N_T$ leaf nodes, by deleting all leaf nodes except those of types $a(1), a(2), \ldots, a(c(2))$ from $S(k+c(r)-c(2))$.

Proof of Theorem 3: Construct T_0 by the algorithm of the corollary, that is, consider as candidates all optimal exhaustive trees S_0, such that deleting $N_S - N_T$ rightmost leaf nodes from S_0, where S_0 has N_S leaf nodes, leaves a nonexhaustive tree T with the same set of interior nodes as S_0 and with N_T leaf nodes, and select the least costly of the resulting trees T as T_0. Suppose we start with an optimal exhaustive tree S_0 which is not $S(k+c(r)-c(2))$. The number of leaf nodes in S_0 is N_S and either $N_S > g(k+c(r)-c(2)) > N_T$ or $g(k+c(r)-c(2)) > N_S > N_T$. (If $N_S = g(k')$ for some k', then S_0 is an exhaustive generalized Fibonacci tree, but, for other values of $N_S = M(r-1)+1$, S_0 is obtained by using Varn's algorithm but is not an exhaustive Fibonacci tree. For example, $((4+4+7)+2+5)$ is an optimal exhaustive tree, but not an exhaustive generalized Fibonacci tree, for the example.)

Suppose first that $N_S > g(k+c(r)-c(2))$. We will show that the resulting cost of deleting $N_S - N_T$ most costly leaf nodes from S_0 is greater than the resulting cost of deleting $g(k+c(r)-c(2)) - N_T$ most costly leaf nodes from $S(k+c(r)-c(2))$. For $S(k+c(r)-c(2))$, let n be the number of leaf nodes of types of index $\le c(2)$, and let s be the number of non-root interior nodes. Then $S(k+c(r)-c(2))$ has $g(k+c(r)-c(2)) = r-s+sr$ leaf nodes of which $r-s+sr-n$ are of types of index $> c(2)$. Then S_0 has $N_s = r-s+sr+v(r-1)$ leaf nodes and $s+v$ non-root interior nodes for some v which is ≥ 1 since $N_S > g(k+c(r)-c(2))$. Of these leaf nodes, $r-s+sr+v(r-1)-n$ need to be deleted from S_0 in order to leave n leaf nodes. Compare this with the $r-s-sr-n$ leaf nodes of types of index $> c(2)$ which need to be deleted from $S(k+c(r)-c(2))$ in order to leave n leaf nodes. But $S(k+c(r)-c(2))$ and S_0 have $r-s+sr-v$ leaf nodes in common which have the same costs in both trees, and S_0 has vr expensive leaf nodes which do not appear in $S(k+c(r)-c(2))$, and $S(k+c(r)-c(2))$ has v inexpensive leaf nodes which do not appear in S_0. This is because v leaf nodes in $S(k+c(r)-c(2))$ are instead full interior nodes in S_0 and the two trees are otherwise the same since they are both optimal exhaustive trees generated by splitting least cost leaf nodes as in Varn's algorithm. Deleting $r-s+sr+v(r-1)-n$ leaf nodes from S_0 and $r-s-sr-n$ leaf nodes from $S(k+c(r)-c(2))$, we can only delete at most $v(r-2)$ of the expensive leaf nodes from S_0 while maintaining all the interior nodes of S_0. Since we need to delete $v(r-1)$ more leaf nodes from S_0 than from $S(k+c(r)-c(2))$ and only $v(r-2)$ of them can be expensive, we must delete v of the leaf nodes of common cost (or inexpensive leaf nodes). Thus, the resulting cost obtained from $S(k+c(r)-c(2))$ is less than or equal to the resulting cost obtained from S_0.

Similarly, if $g(k+c(r)-c(2)) > N_S$, the resulting cost of deleting $g(k+c(r)-c(2))-N_T$ most costly leaf nodes from $S(k+c(r)-c(2))$ is less than the resulting cost of deleting $N_S - N_T$ most costly leaf nodes from S_0. For $S(k+c(r)-c(2))$, let n be the number of leaf nodes of types of index $\le c(2)$, and let s be the number of non-root interior nodes. Then $S(k+c(r)-c(2))$ has $g(k+c(r)-c(2)) = r-s+sr$ leaf nodes of which $r-s+sr-n$ are of types of index $> c(2)$. Then S_0 has $N_S = r-s+sr-v(r-1)$ leaf nodes and $s-v$ non-root interior nodes for some v which is ≥ 1 since $g(k+c(r)-c(2)) > N_S$. Of these leaf nodes, $r-s+sr-v(r-1)-n$ need to be deleted from S_0 in order to leave n leaf nodes. Compare this with the $r-s+sr-n$ leaf nodes of types of index $> c(2)$ which need to be deleted from $S(k+c(r)-c(2))$ in order to leave n leaf nodes. Thus, $v(r-1)$ fewer leaf nodes need to be deleted from S_0 than from $S(k+c(r)-c(2))$ in order to leave n leaf nodes in each case. But $S(k+c(r)-c(2))$ and S_0 have $r-s+sr-v(r-1)-v$ leaf nodes in common which have the same costs in both trees, and S_0 has v inexpensive leaf nodes which do not appear in $S(k+c(r)-c(2))$, and $S(k+c(r)-c(2))$ has vr expensive leaf nodes which do not appear in S_0. This is because v leaf nodes in S_0 are instead full interior nodes in $S(k+c(r)-c(2))$ and the two trees are otherwise the same since they are both optimal exhaustive trees generated by splitting least cost leaf nodes as in Varn's algorithm. Deleting $v(r-1)$ more leaf nodes from $S(k+c(r)-c(2))$ than from S_0, we delete fewer than $v(r-2)$ of the vr expensive leaf nodes and more than v of the leaf nodes of common cost (or inexpensive leaf nodes), thus resulting in a tree obtained from $S(k+c(r)-c(2))$ with maximum codeword cost less than or equal to that obtained from S_0, and Theorem 3 follows. □

Note that the corresponding argument breaks down for minimum average cost codes, for which all leaf node costs, not just the maximum leaf node cost, enter into the resulting tree cost calculation, and the argument in the proof of Theorem 3 does not hold. For the example, $S(8) = ((4+4+7)+(4+4+7)+5) = ((a(2)+a(2)+a(5))+(a(2)+a(2)+a(5))+a(3))$. We can select

$n=4$ so that deleting the $r-s+st-n=3$ leaf nodes of type of index >2 leaves $T_0=((4+4)+(4+4))$, an optimal minimax tree of $n=4$ leaf nodes. It has maximum codeword cost 4, the minimax optimal value. In contrast, consider $S_0=((4+4+7)+2+5)$, an optimal exhaustive tree generated by splitting least cost leaf nodes as in Varn's algorithm, but not an exhaustive generalized Fibonacci tree. Theorem 3 says that T_0 cannot be obtained from S_0 by deleting leaf nodes. In particular, the $v=1$ leaf node in S_0 of cost 2 is a full interior node $(4+4+7)$ in $S(8)$ of $vr=3$ leaf nodes. Deleting $v(r-1)=2$ fewer leaf nodes from S_0 than from $S(8)$ generates the tree $((4+4)+2+5)$ which, with a maximum codeword cost of 5, is not minimax optimal. However, $((4+4)+2+5)$ is the optimal tree in the minimum average cost sense [8], with an average cost of $15/4$, and compare this with T_0, with an average cost of 4.

Theorem 4: The nonexhaustive generalized Fibonacci trees are the optimal nonexhaustive code trees for the minimax cost problem for the same number of leaf nodes.

Proof of Theorem 4: The proof is an immediate consequence of Theorems 1 and 3. □

Theorem 4 provides an elegant characterization of optimal nonexhaustive minimax code trees in terms of nonexhaustive generalized Fibonacci trees. Its proof makes use of three intermediate results of independent interest: the equivalence between the recursive construction and the construction by the method of types for nonexhaustive generalized Fibonacci trees in Theorem 1; the new Varn-like algorithm for optimal nonexhaustive minimax code trees in the Corollary to Theorem 2; and the result from Theorem 3 that a search over a set of candidate optimal exhaustive code trees from which to generate the optimal nonexhaustive tree is not necessary in the Varn-like algorithm for the minimax codeword cost problem (although it is necessary in the Varn algorithm for the minimum average codeword cost problem). Instead, from Theorem 3 we know exactly which highest cost leaf nodes, those of type of index $>c(2)$, are to be deleted from exactly which optimal exhaustive minimax tree, $S(k+c(r)-c(2))$, in order to obtain the optimal nonexhaustive minimax tree T_0 (when the desired number of leaf nodes in T_0 is $f(k)$).

Under certain conditions, minimax code trees are also minimum average cost code trees, and in these cases the generalized Fibonacci structure of the minimax code trees applies to the minimum average cost code trees as well. The algorithm of Perl et al. [8] for minimum average cost trees involves two stages, extension and mending, and their algorithm for minimax trees is a variant of the extension stage. Thus, whenever mending is unnecessary and whenever the extension stages give the same tree for both problems, the minimax code tree is also the minimum average cost code tree. A sufficient condition on the code symbol costs for the mending stage to be unnecessary in the minimum average cost problem is that any of the following three (in)equalities holds [8]: (i) $r\leq 3$; (ii) $c(1)+c(2)\leq c(3)$; (iii) $c(3)=c(4)=\cdots=c(r)$. In the extension algorithm for minimum average cost trees, a comparison is made between $c(a)+c(1)+c(2)$ and $c(b)+c(i)$ for some $i>2$, where $c(a)$ is the cost of a least cost leaf node a and $c(b)$ is the cost of some nonfull interior node b, in order to choose the next leaf node to introduce in forming the tree of $N+1$ leaf nodes from the tree of N leaf nodes. If $c(a)+c(1)+c(2)$ is least, we branch a, and otherwise we add to b. In the variant of the extension algorithm for minimax cost trees, the equivalent comparison is made between $c(a)+c(2)$ and $c(b)+c(i)$. Thus, whenever any of these sufficient conditions is satisfied by the costs, and the costs are such that the variant of the extension algorithm for minimax codes and the original extension algorithm for minimum average cost codes both yield the same results from their respective comparison stages, that is, whenever $c(a)+c(1)+c(2)$ and $c(a)+c(2)$ are both either less than or greater than all candidate $c(b)+c(i)$ expressions, minimax and minimum average cost code trees are the same, and the minimum average cost tree sequence shares the nice recursive structure of the minimax tree sequence. This is the case for Patt's [7] costs, $c(i)=i$, $i=1,2,\ldots$, when N is an integer power of 2, and his paper includes the recursive tree sequence structure. The example costs of $c(1)=c(2)=2$, $c(3)=5$ satisfy the sufficient conditions (i) or (ii), however not the comparison conditions even for small N. The tree $((4+4)+2+5)$ is minimum average cost [8] but $((4+4)+(4+4))$ is minimax. Both are obtained by extending $((4+4)+2)$, in the former by noting that $c(a)+c(1)+c(2)>c(b)+c(i)$, and in the latter, $c(a)+c(2)<c(b)+c(i)$, where $c(a)=2$, $c(b)=0$, and $c(i)=c(3)$ in both cases.

REFERENCES

1. J. Abrahams. "Varn Codes and Generalized Fibonacci Trees." *The Fibonacci Quarterly* **33.1** (1995):21-25.
2. D. K. Chang. "On Fibonacci *k*-ary Trees." *The Fibonacci Quarterly* **24.3** (1986):258-62.
3. S.-N. Choi & M. J. Golin. "Lopsided Trees: Analyses, Algorithms, and Applications (Extended Abstract)." Hong Kong University of Science and Technology Computer Science Department Preprint, 1994.
4. N. Cot. "Characterization and Design of Optimal Prefix Codes." Thesis, Stanford Electrical Engineering Department, 1977.
5. M. J. Golin & N. Young. "Prefix Codes: Equiprobable Words, Unequal Letter Costs." *SIAM J. Comput.* **25** (1996):1281-92.
6. Y. Horibe. "Notes on Fibonacci Trees and Their Optimality." *The Fibonacci Quarterly* **21.2** (1983):118-28.
7. Y. N. Patt. "Variable Length Tree Structures Having Minimum Average Search Time." *Comm. ACM* **12** (1969):72-76.
8. Y. Perl, M. R. Garey, & S. Even. "Efficient Generation of Optimal Prefix Code: Equiprobable Words Using Unequal Cost Letters." *JACM* **22** (1975):202-14.
9. S. A. Savari. "Some Notes on Varn Coding." *IEEE Trans. Inform. Th.* **40** (1994):181-86.
10. B. F. Varn. "Optimal Variable Length Codes (Arbitrary Symbol Cost and Equal Code Word Probabilities)." *Inform. Contr.* **19** (1971):289-301.

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ON DIOPHANTINE APPROXIMATION BELOW THE LAGRANGE CONSTANT

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1. INTRODUCTION

For an irrational real number α, the *Lagrange* (often called the *Markoff*) *constant* for α, $\mu(\alpha)$, is defined by

$$\mu(\alpha) = \liminf_{q\to\infty} q\|\alpha q\|,$$

where $\| \ \|$ denotes the distance to the nearest integer function (see [3], although there the Lagrange constant is defined to be $\mu(\alpha)^{-1}$). Thus, for any c, $0 < c < \mu(\alpha)$, it follows that there are only finitely many positive integer solutions q to the inequality

$$q\|\alpha q\| < c. \tag{1.1}$$

We define $\lambda(\alpha)$ by $\lambda(\alpha) = \inf_{q>0} q\|\alpha q\|$.

Given α, two natural and fundamental problems are to compute $\lambda(\alpha)$ and, for a given c, $\lambda(\alpha) < c < \mu(\alpha)$, to explicitly determine the complete set of solutions to (1.1). In a series of three papers ([8], [9], [10]), Winley, Tognetti, and Van Ravenstein address these problems for the case in which α equals a *generalized golden ratio* φ_a, that is,

$$\alpha = \varphi_a = \frac{a+\sqrt{a^2+4}}{2},$$

where a is a positive integer. Not surprisingly, their solution involves generalized Fibonacci numbers. We write $\mathcal{F}_n = \mathcal{F}_n(a)$ for the n^{th} *generalized Fibonacci number*. That is, $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = 1$ and, for $n > 1$, $\mathcal{F}_n = a\mathcal{F}_{n-1} + \mathcal{F}_{n-2}$. Using a well-known connection between $\mu(\alpha)$ and the continued fraction expansion of α (see [3]), one can, for these generalized golden ratios, explicitly compute $\mu(\varphi_a) = 1/\sqrt{a^2+4}$. Given this, we may state the main result of Winley et al. [10] as

Theorem 1: For a positive integer a, $\lambda(\varphi_a) = a/\varphi_a^2$. Moreover, for $\lambda(\varphi_a) < c \le 1/\sqrt{a^2+4}$, an integer $q > 0$ is a solution to

$$q\|\varphi_a q\| < c \tag{1.2}$$

if and only if $q = \mathcal{F}_{2m-1}$, where m is any positive integer satisfying

$$1 - c\sqrt{a^2+4} < \varphi_a^{-4m}. \tag{1.3}$$

The key to the proof of Theorem 1 is the observation that the numerators and denominators of the convergents of φ_a, which are the generalized Fibonacci numbers, enjoy a simple second-order recurrence relation.

In this paper we extend Theorem 1 to arbitrary real quadratic irrationals. Fundamental to our method is an important, but not widely known, result on the arithmetical structure of the sequences of numerators and denominators of the convergents of quadratic irrationals. In particular, each sequence may be partitioned into a finite number of simple second-order linear recurrence sequences all of which satisfy the same recurrence relation. We state this result explicitly as Theorem 3 in Section 2. A pleasant consequence of this result is a very simple method for computing the Lagrange constant for quadratic irrationals. This fundamental result is stated in Section 4 as Lemma 7.

As our generalization of inequality (1.3) for arbitrary quadratic irrationals requires the coefficients occurring in the recurrence relations given in Theorem 3, we postpone stating our main results, Theorems 5 and 6, until Section 3. However, below we illustrate our results with a simple extension of Theorem 1. We recall that the continued fraction expansion for φ_a is $[a, a, a, \dots] = [\overline{a}]$. Thus, it is natural to next examine quadratic irrationals having a purely periodic continued fraction expansion of period length 2. In particular, we consider

$$\alpha(a,b) = \frac{ab+\sqrt{a^2b^2+4ab}}{2b} = [\overline{a,b}].$$

It follows by either a direct calculation or an application of Lemma 7, that

$$\mu(\alpha(a,b)) = \frac{\min\{a,b\}}{b(\alpha(a,b)-\overline{\alpha}(a,b))},$$

where $\overline{\alpha}(a,b)$ denotes the algebraic conjugate of $\alpha(a,b)$. If we let p_n/q_n be the n^{th} convergent of $\alpha(a,b)$, then the following is a special case of Theorem 5.

Theorem 2: For positive integers a and b, let $\alpha = \alpha(a,b)$. Then $\lambda(\alpha) = \min\{b^2\overline{\alpha}+b, \mu(\alpha)\}$. Moreover, for $\lambda(\alpha) < c < \mu(\alpha)$, an integer $q > 0$ is a solution to $q\|\alpha q\| < c$ if and only if $q = q_{2n+1}$, where $n \geq 0$ is any integer satisfying

$$((ab+2)(1+b\overline{\alpha})-1)^n < ((ab+2)(1+b\alpha)-1)(1-c(\alpha-\overline{\alpha})).$$

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2. RECURRENCE SEQUENCES AND QUADRATIC IRRATIONALS

For a real number α, we denote its simple continued fraction expansion by $[a_0, a_1, \dots]$. We define the sequence of *convergents*, p_n/q_n, $n = 1, 2, \dots$, by $p_n/q_n = [a_0, a_1, \dots, a_n]$, where gcd $(p_n, q_n) = 1$. If we declare $p_{-2} = 0$, $p_{-1} = 1$, and $q_{-2} = 1$, $q_{-1} = 0$, then it follows that, for all $n \geq 0$, $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$. By the well-known result of Lagrange, $\alpha \in \mathbb{R}$ is a quadratic irrational if and only if the continued fraction expansion for α is eventually periodic (see [4]). For the remainder of this paper, α is assumed to be a real quadratic irrational and thus we may denote its continued fraction as

$$\alpha = [a_0, a_1, \dots, a_{k-1}, \overline{a_k, a_{k+1}, \dots, a_{k+T-1}}].$$

Hence, for each t, $0 \leq t \leq T-1$, we have

$$p_{Tn+t+k} = a_{t+k}p_{Tn+t+k-1} + p_{Tn+t+k-2} \quad \text{and} \quad q_{Tn+t+k} = a_{t+k}q_{Tn+t+k-1} + q_{Tn+t+k-2}$$

for all $n = 0, 1, 2, \dots$. We will require the following result which shows that the sequences $\{p_n\}$ and $\{q_n\}$ may be partitioned into T simple second-order linear recurrence sequences all satisfying the same recurrence relation. Such a result was also noted by Cusick [2] and again by van der Poorten [7] (see the related work of Kiss [5]), and is of some independent interest (see [1]).

Theorem 3: If

$$\alpha = \left[a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+T-1}}\right] \quad \text{and} \quad \mathbf{P}(\alpha) = \left[\overline{a_k, a_{k+1}, \ldots, a_{k+T-1}}\right],$$

then, for each t, $0 \le t \le T-1$,

$$\begin{aligned} p_{Tn+t+k} &= \omega(\alpha) p_{T(n-1)+t+k} + (-1)^{T+1} p_{T(n-2)+t+k}, \\ q_{Tn+t+k} &= \omega(\alpha) q_{T(n-1)+t+k} + (-1)^{T+1} q_{T(n-2)+t+k}, \end{aligned} \tag{2.1}$$

for all $n = 2, 3, \ldots$, where the constant $\omega(\alpha) = p^*_{T-1} + q^*_{T-2}$ and p^*_n / q^*_n denotes the n^{th} convergent of $\mathbf{P}(\alpha)$. Furthermore, if $\tau_1 = \mathbf{P}(\alpha)$ and $\tau_2 = \overline{\mathbf{P}(\alpha)}$, then $\tau_1 > 1$, $\tau_1\tau_2 = (-1)^T$ and, for each t, $0 \le t \le T-1$, there exist real numbers u_t, v_t, r_t, s_t, with $r_t > 0$, such that $p_{Tn+t+k} = u_t\tau_1^n + v_t\tau_2^n$ and $q_{Tn+t+k} = r_t\tau_1^n + s_t\tau_2^n$ for all $n = 0, 1, 2, \ldots$.

As it is difficult to find a proof of Theorem 3 in the literature, we include one here. We begin with the following elementary but useful lemma from linear algebra. The authors wish to thank the referee for suggesting the following elegant proof of Lemma 4.

Lemma 4: Let A, B, C, D be nonnegative integers satisfying $AD - BC = (-1)^T$ for some fixed integer T. If the sequences of integers $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are defined by

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n,$$

then each of the four sequences, $\{a_n\}, \{b_n\}, \{c_n\}$, and $\{d_n\}$, satisfies the same second-order linear recurrence relation. In particular,

$$\begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = (A+D)\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} + (-1)^{T+1}\begin{pmatrix} a_{n-1} & b_{n-1} \\ c_{n-1} & d_{n-1} \end{pmatrix} \tag{2.2}$$

for $n \ge 2$,

Proof: If we write

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then we note that the characteristic polynomial associated with M is given by

$$\det(M - \mathbf{1}_2 x) = x^2 - (A+D)x + AD - BC,$$

where $\mathbf{1}_2$ denotes the 2×2 identity matrix. By the Cayley-Hamilton Theorem, a matrix is a zero of its associated characteristic polynomial. Specifically, we have

$$M^2 = (A+D)M - (AD-BC)\mathbf{1}_2.$$

Multiplying the previous identity by M^{n-1} yields

$$M^{n+1} = (A+D)M^n - (AD-BC)M^{n-1}.$$

In view of our assumption that $AD - BC = (-1)^T$, the above equality becomes

$$M^{n+1} = (A+D)M^n + (-1)^{T+1}M^{n-1},$$

which completes the proof of the lemma.

Proof of Theorem 3: It is enough to prove that the identities in (2.1) hold, as the subsequent assertions of the theorem follow from (2.1) and the basic theory of linear recurrences. We first prove (2.1) in the case when $t = 0$ and then indicate how to modify the argument for $t \geq 1$.

By a well-known correspondence between partial quotients and convergents, if $p_n / q_n = [a_0, a_1, \ldots, a_n]$, then

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix},$$

(see, for example, [6]). Thus, given that $\mathbf{P}(\alpha) = [\overline{a_k, a_{k+1}, \ldots, a_{k+T-1}}]$, we have

$$\begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{k+1} & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_{k+T-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p^*_{T-1} & p^*_{T-2} \\ q^*_{T-1} & q^*_{T-2} \end{pmatrix}, \tag{2.3}$$

where we recall that p^*_n / q^*_n denotes the n^{th} convergent of $\mathbf{P}(\alpha)$. Hence, as $\mathbf{P}(\alpha)$ has a purely periodic continued fraction expansion of period length T, we see that, for all $n \geq 1$,

$$\begin{pmatrix} p^*_{Tn-1} & p^*_{Tn-2} \\ q^*_{Tn-1} & q^*_{Tn-2} \end{pmatrix} = \begin{pmatrix} p^*_{T-1} & p^*_{T-2} \\ q^*_{T-1} & q^*_{T-2} \end{pmatrix}^n.$$

Taking determinants of both sides of (2.3), we see $p^*_{T-1}q^*_{T-2} - p^*_{T-2}q^*_{T-1} = (-1)^T$. Therefore, we may apply Lemma 4 and deduce that, for all $n \geq 2$,

$$\begin{pmatrix} p^*_{Tn-1} & p^*_{Tn-2} \\ q^*_{Tn-1} & q^*_{Tn-2} \end{pmatrix} = \omega(\alpha)\begin{pmatrix} p^*_{T(n-1)-1} & p^*_{T(n-1)-2} \\ q^*_{T(n-1)-1} & q^*_{T(n-1)-2} \end{pmatrix} + (-1)^{T+1}\begin{pmatrix} p^*_{T(n-2)-1} & p^*_{T(n-2)-2} \\ q^*_{T(n-2)-1} & q^*_{T(n-2)-2} \end{pmatrix},$$

where $\omega(\alpha) = p^*_{T-1} + q^*_{T-2}$.

Finally, turning our attention to the partial quotients of α, we note that, for all $n \geq 1$,

$$\begin{aligned}
\begin{pmatrix} p_{Tn+k} & p_{Tn+k-1} \\ q_{Tn+k} & q_{Tn+k-1} \end{pmatrix} &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix}\left(\begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_{k+T-1} & 1 \\ 1 & 0 \end{pmatrix}\right)^n \\
&= \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix}\begin{pmatrix} p^*_{T-1} & p^*_{T-2} \\ q^*_{T-1} & q^*_{T-2} \end{pmatrix}^n \\
&= \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix}\begin{pmatrix} p^*_{Tn-1} & p^*_{Tn-2} \\ q^*_{Tn-1} & q^*_{Tn-2} \end{pmatrix}.
\end{aligned}$$

Thus, the pair (p_{Tn+k}, q_{Tn+k}) is a nonsingular, linear transformation of (p^*_{Tn-1}, q^*_{Tn-1}). Hence, both p_{Tn+k} and q_{Tn+k} each must satisfy the same second-order linear recurrence enjoyed by p^*_{Tn-1} and q^*_{Tn-1}. Specifically, for all $n \geq 2$,

$$\begin{aligned}
p_{Tn+k} &= \omega(\alpha)p_{T(n-1)+k} + (-1)^{T+1}p_{T(n-2)+k}, \\
q_{Tn+k} &= \omega(\alpha)q_{T(n-1)+k} + (-1)^{T+1}q_{T(n-2)+k},
\end{aligned}$$

which proves the theorem when $t = 0$.

For $t \geq 1$, we note that

$$\begin{pmatrix} p_{Tn+t+k} & p_{Tn+k+t-1} \\ q_{Tn+t+k} & q_{Tn+k+t-1} \end{pmatrix} = \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix} \begin{pmatrix} p^*_{Tn-1} & p^*_{Tn-2} \\ q^*_{Tn-1} & q^*_{Tn-2} \end{pmatrix} \begin{pmatrix} p^*_{t-1} & p^*_{t-2} \\ q^*_{t-1} & q^*_{t-2} \end{pmatrix}.$$

Since both the first and third matrices appearing on the right-hand side of this identity are independent of n, we see that p_{Tn+t+k} and q_{Tn+t+k} are each linear combinations of p^*_{Tn-1}, p^*_{Tn-2}, q^*_{Tn-1}, and q^*_{Tn-2}. Thus, p_{Tn+t+k} and q_{Tn+t+k} must satisfy the same linear recurrence as p^*_{Tn-1}, p^*_{Tn-2}, q^*_{Tn-1}, and q^*_{Tn-2}. This fact establishes (2.1) for any t, $0 \leq t \leq T-1$, and completes the proof.

3. OUR MAIN RESULTS ON DIOPHANTINE APPROXIMATION

Given α as in Theorem 3, it will be convenient to define several new but natural constants that will allow us explicitly to determine $\lambda(\alpha)$. For each t, $0 \leq t \leq T-1$, we let $d_t = r_t v_t - s_t u_t$ and define

$$\lambda_t(\alpha) = \begin{cases} |d_t|\left(1+\frac{s_t}{r_t}\right), & \text{if } s_t < 0; \\ |d_t|, & \text{if } s_t > 0 \text{ and } T \text{ is even}; \\ |d_t|\left(1-\frac{s_t}{r_t}\tau_2^2\right), & \text{if } s_t > 0 \text{ and } T \text{ is odd}. \end{cases}$$

We now state our main result in the case when the continued fraction expansion for α is purely periodic.

Theorem 5: Suppose that $\alpha = [\overline{a_0, a_1, \ldots, a_{T-1}}]$; r_t and s_t are as in Theorem 3, and d_t and $\lambda_t(\alpha)$ are as defined above. Then $\lambda(\alpha) = \min\{\lambda_t(\alpha) : 0 \geq t \leq T-1\}$. Moreover, for $\lambda(\alpha) < c < \mu(\alpha)$, an integer $q > 0$ is a solution to

$$q\|\alpha q\| < c \tag{3.1}$$

if and only if $q = q_{Tn+t}$, where $0 \leq t \leq T-1$, $(-1)^{Tn}s_t \leq 0$, $\lambda_t(\alpha) < c$, and $n \geq 0$ satisfies

$$\frac{r_t}{|s_t|}\left(1-\frac{c}{|d_t|}\right) < \overline{\alpha}^{2n}. \tag{3.2}$$

We remark that upon first inspection it may appear undesirable to have n occur in the bound $(-1)^{Tn}s_t \leq 0$. However, as T and t are known, it is only the *parity* of n that is necessary in computing this inequality. Hence, given c and t, one needs to find all even integers n that satisfy the conditions of the theorem and then all such odd integers. That is, implicit in the inequalities of Theorem 5 are the cases of n even and n odd.

Plainly, if the continued fraction expansion for the quadratic irrational α is not purely periodic, then there is no control on the size of the partial quotients occurring before the period; thus, one must examine each of the associated convergents individually. In particular, if

$$\alpha = [a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+T-1}}],$$

then there may be solutions to (3.1) among the first k convergents. With this unavoidable possibility understood, one has

Theorem 6: Suppose that $\alpha = [a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+T-1}}]$; $\mathbf{P}(\alpha)$, r_t, s_t are all as in Theorem 3, and d_t and $\lambda_t(\alpha)$ are as defined above. Let $\lambda_*(\alpha) = \min\{\lambda_t(\alpha) : 0 \le t \le T-1\}$. Then

$$\lambda(\alpha) = \min\{\lambda_*(\alpha), q_n\|\alpha q_n\| : 0 \le n < k\}.$$

Moreover, for $\lambda_*(\alpha) < c < \mu(\alpha)$, an integer $q \ge q_k$ is a solution to

$$q\|\alpha q\| < c \tag{3.3}$$

if and only if $q = q_{Tn+t+k}$, where $0 \le t \le T-1$, $(-1)^{Tn}s_t \le 0$, $\lambda_t(\alpha) < c$, and $n \ge 0$ satisfies

$$\frac{r_t}{|s_t|}\left(1 - \frac{c}{|d_t|}\right) < \overline{\mathbf{P}(\alpha)}^{2n}.$$

As an illustration, we return to $\alpha = \varphi_a = [\overline{a}]$. In this case, we have $T = 1$ and may verify that

$$\tau_1 = \varphi_a, \quad \tau_2 = \overline{\varphi}_a, \quad r_0 = \frac{\varphi_a}{\sqrt{a^2+4}}, \quad s_0 = \frac{-\overline{\varphi}_a}{\sqrt{a^2+4}},$$

$$u_0 = \frac{a\varphi_a + 1}{\sqrt{a^2+4}}, \quad v_0 = \frac{-a\overline{\varphi}_a - 1}{\sqrt{a^2+4}}, \text{ and thus } d_0 = \frac{-1}{\sqrt{a^2+4}}.$$

As $s_0 > 0$ and T is odd, it follows that

$$\lambda(\varphi_a) = \lambda_0(\varphi_a) = \frac{1}{\sqrt{a^2+4}}\left(1 - \left(\frac{-a+\sqrt{a^2+4}}{a+\sqrt{a^2+4}}\right)\frac{1}{\varphi_a^2}\right) = \frac{\varphi_a^2 - 1}{\varphi_a^3} = \frac{a}{\varphi_a^2},$$

as was stated in Theorem 1. We assume now that $\lambda(\varphi_a) < c < \mu(\varphi_a)$. For q_n to be a solution to (3.1), we must have $n > 0$ odd. If we write $n = 2m-1$, then (3.2) becomes

$$\frac{\varphi_a}{\overline{\varphi}_a}\left(1 - c\sqrt{a^2+4}\right) < \overline{\varphi}_a^{4m-2}.$$

Noting that $\varphi_a\overline{\varphi}_a = -1$ and $\varphi_a/|\overline{\varphi}_a| = \varphi_a^2$, the previous inequality is seen to be equivalent to

$$1 - c\sqrt{a^2+4} < \varphi_a^{-4m}. \tag{3.4}$$

Therefore, all the solutions to (3.1) are given by $q = q_{2m-1}$, where $m > 0$ satisfies (3.4) which, in view of the fact that $q_{2m-1} = \mathcal{F}_{2m-1}$, yields the result of Winley, Tognetti, and Van Ravenstein.

An Illustrative Collection of Examples: We briefly consider various numbers α equivalent to $\frac{1+\sqrt{3}}{2} = [\overline{1, 2}]$. For all such numbers, it follows that $\mu(\alpha) = \frac{1}{2\sqrt{3}}$.

1. Let $\alpha = [\overline{1, 2}]$. It follows that $\lambda_0(\alpha) = \frac{1}{2\sqrt{3}} = \mu(\alpha) \approx .288$ and $\lambda_1(\alpha) = 4 - 2\sqrt{3} \approx .535$, so $\lambda(\alpha) = \mu(\alpha)$. Hence, there are no solutions to (3.1) for any c, $0 < c \le \mu(\alpha)$.

2. $\alpha = [\overline{2, 1}] = 1 + \sqrt{3}$. We find that $\lambda_0(\alpha) = \frac{1}{\sqrt{3}}$ and $\lambda_1(\alpha) = 2 - \sqrt{3} \approx .267$. Thus, there are no solutions to (3.1) for $0 < c \le 2 - \sqrt{3}$; and for $2 - \sqrt{3} < c < \mu(\alpha)$ we have that q_{2n+1} is a solution to (3.1) for all integers $n \ge 0$ satisfying $(7 + 4\sqrt{3})(1 - c2\sqrt{3}) < (7 - 4\sqrt{3})^n$.

3. Let $\alpha = [3, 3, \overline{2, 1}] = \frac{38-\sqrt{3}}{11}$. For $t = 1$, we have $(-1)^{Tn}s_t = s_1 = \frac{30-17\sqrt{3}}{6} > 0$. Thus, for $t = 1$, q_{Tn+t+k} is not a solution to (3.3). For $t = 0$, we note that $\lambda_0(\alpha) = \frac{91-49\sqrt{3}}{11} \approx .557 > \mu(\alpha) > c$. Thus, there are no solutions to (3.3) in this case either. A straightforward calculation shows that

$q_0 = 1$ and $q_1 = 3$ are never solutions to (3.3). Hence, for this α, there are no solutions to (3.3) for any $c < \mu(\alpha)$.

4. Let $\alpha = [3, 3, \overline{1, 2}] = 5 - \sqrt{3}$. As in the previous example, $s_1 = \frac{33-19\sqrt{3}}{6} \approx .015 > 0$ so there are no solutions to (3.3) for $t = 1$. Here, $\lambda_1(\alpha) = \frac{1}{\sqrt{3}} \approx .577$, $\lambda_0(\alpha) = 28 - 16\sqrt{3} \approx .287$; hence, $\lambda_*(\alpha) = 28 - 16\sqrt{3}$, while $\lambda(\alpha) = 1\|\alpha 1\| = 2 - \sqrt{3} \approx .267$. Therefore, after a calculation, we conclude that solutions to (3.3) for $28 - 16\sqrt{3} < c < \mu(\alpha)$ are p_{2n+2} / q_{2n+2} for all integers $n \geq 0$ satisfying $(97 + 56\sqrt{3})(1 - c2\sqrt{3}) < (7 - 4\sqrt{3})^n$. We also note that $q_0 = 1$ is the only solution to (3.1) for $0 < c \leq 28 - 16\sqrt{3}$.

4. THE PROOF OF THEOREMS 5 AND 6

Before proceeding with our proof, we recall some elementary facts from the theory of continued fractions (see [4] or [6] for details). For an irrational real number α, the convergents p_n / q_n satisfy

$$\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(\alpha_{n+1} q_n + q_{n-1})}, \tag{4.1}$$

where $\alpha_n = [a_n, a_{n+1}, a_{n+2}, \dots]$ is the n^{th} *complete quotient.* We recall Hurwitz's celebrated result that $\mu(\alpha) \leq 1/\sqrt{5}$ and Legendre's theorem that any rational solution p/q to

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{2q^2},$$

must be a convergent of α.

We will make use of the following basic lemma which may be of some independent interest.

Lemma 7: Suppose that $\alpha = \left[a_0, a_1, \dots, a_{k-1}, \overline{a_k, a_{k+1}, \dots, a_{k+T-1}}\right]$ and $\tau_1, \tau_2, u_t, v_t, r_t, s_t$ are all as in Theorem 3. For each $0 \leq t \leq T-1$, let $d_t = r_t v_t - s_t u_t$. Then $\alpha = u_t / r_t$ for each t and

$$\mu(\alpha) = \min_{0 \leq t \leq T-1} \{|d_t|\}.$$

Proof: By Theorem 3 we have, for each t,

$$\frac{p_{Tn+t+k}}{q_{Tn+t+k}} = \frac{u_t \tau_1^n + v_t \tau_2^n}{r_t \tau_1^n + s_t \tau_2^n} = \frac{u_t + v_t (\tau_2 / \tau_1)^n}{r_t + s_t (\tau_2 / \tau_1)^n}.$$

In view of the fact that $|\tau_2 / \tau_1| < 1$, as $n \to \infty$ the above identity becomes $\alpha = u_t / r_t$.

Next, we observe that

$$\mu(\alpha) = \liminf_{q \to \infty} q\|\alpha q\| = \lim_{n \to \infty} q_n |\alpha q_n - p_n| = \lim_{n \to \infty} q_{n+k} |\alpha q_{n+k} - p_{n+k}|.$$

Finally, the first part of this lemma, together with a simple calculation, reveals

$$q_{Tn+t+k} |\alpha q_{Tn+t+k} - p_{Tn+t+k}| = \left| \left(\frac{u_t}{r_t}\right)(r_t \tau_1^n + s_t \tau_2^n)^2 - (u_t \tau_1^n + v_t \tau_1^n)(r_t \tau_1^n + s_t \tau_2^n) \right|$$

$$= \left| (-1)^{Tn}(s_t u_t - r_t v_t) + \tau_2^{2n}\left(s_t^2 \left(\frac{u_t}{r_t}\right) - s_t v_t \right) \right| = |d_t| \left| (-1)^{Tn+1} - \left(\frac{s_t}{r_t}\right) \tau_2^{2n} \right|.$$

Since $|\tau_2|< 1$, we have

$$\lim_{n\to\infty} q_{Tn+t+k}|\alpha q_{Tn+t+k} - p_{Tn+t+k}| = |d_t|;$$

hence,

$$\mu(\alpha) = \min_{0\le t\le T-1}\{|d_t|\},$$

as desired.

Proof of Theorems 5 and 6: In view of the results of Hurwitz and Legendre, it is clear that any solution q to (3.3) for $0<c<\mu(\alpha)$ must be a denominator of a convergent. We thus consider an arbitrary convergent p_m/q_m and, for a fixed t, $0\le t\le T-1$, we examine all m such that $m-k\equiv t \bmod T$. That is, we write $m=Tn+t+k$.

Plainly, q_{Tn+t+k} is a solution of (3.3) if and only if

$$0<\left|\alpha-\frac{p_{Tn+t+k}}{q_{Tn+t+k}}\right|<\frac{c}{q^2_{Tn+t+k}}.$$

In view of Theorem 3, identity (4.1), and the fact that $\alpha = u_t/r_t$, the above is equivalent to

$$0<(-1)^{Tn+t+k}d_t\left((-1)^{Tn+1}-\frac{s_t}{r_t}\tau_2^{2n}\right)<c,$$

which may be simplified to

$$0<(-1)^{t+k+1}d_t\left(1+(-1)^{Tn}\frac{s_t}{r_t}\tau_2^{2n}\right)<c. \tag{4.2}$$

We note that the left side of the inequality (4.2) holds for all choices of n. Therefore, if we let n approach infinity, then as $|\tau_2|<1$, we conclude that $(-1)^{t+k+1}d_t>0$. Thus, (4.2) becomes

$$0<|d_t|\left(1+(-1)^{Tn}\frac{s_t}{r_t}\tau_2^{2n}\right)<c.$$

The right side of this inequality holds if and only if

$$(-1)^{Tn}\frac{s_t}{r_t}\tau_2^{2n}<\frac{c}{|d_t|}-1. \tag{4.3}$$

Moreover, as $c<\mu(\alpha)$, Lemma 6 reveals $c<|d_t|$. Thus, the upper bound in (4.3) is negative. Since both τ_2^{2n} and r_t are positive, there are no solutions to (4.3) if $(-1)^{Tn}s_t>0$. Thus, n is a solution to (4.3) if and only if $(-1)^{Tn}s_t\le 0$ and n satisfies (3.2).

Finally, we show that, for each t, $0\le t\le T-1$, there are no solutions $q\ge q_k$ to (3.3) for $c\le\lambda_t(\alpha)$. As we have already seen,

$$q_{Tn+t+k}\|\alpha q_{Tn+t+k}\| = |d_t|\left(1+(-1)^{Tn}\frac{s_t}{r_t}\tau_2^{2n}\right). \tag{4.4}$$

If $s_t<0$, then as $0<\tau_2^2<1$, (4.4) is minimized when $n=0$. If $s_t>0$ and T is even, the infimum of (4.4), $|d_t|$, is approached from above as $n\to\infty$. Finally, if $s_t>0$ and T is odd, it is easy to see that (4.4) is minimized when $n=1$, thus giving a minimum value of

$$|d_t|\left(1-\frac{s_t}{r_t}\tau_2^2\right).$$

These observations yield

$$\lambda_t(\alpha)=\inf_{n\geq 0}\{q_{Tn+t+k}\|\alpha q_{Tn+t+k}\|\}.$$

Hence, it follows that $\lambda(\alpha)=\min\{\lambda_*(\alpha), q_n\|\alpha q_n\|: 0\leq n<k\}$, and there are no solutions to (3.3) for $c\leq\lambda_t(\alpha)$ and $q\geq q_k$. This observation completes the proof.

REFERENCES

1. E. B. Burger. "On Real Quadratic Number Fields and Simultaneous Diophantine Approximation." *Monatshefte für Math.* **128** (1999):201-09.
2. T. W. Cusick. Private notes from 1919, personal communication.
3. T. W. Cusick & M. E. Flahive. *The Markoff and Lagrange Spectra.* Mathematical Surveys and Monographs **30**. Providence, RI: A.M.S., 1989.
4. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers.* 5th ed. Oxford: Oxford University Press, 1993.
5. P. Kiss. "On Second Order Recurrences and Continued Fractions." *Bull. of the Malaysian Math. Soc.* (2) **5** (1982):33-41.
6. A. J. van der Poorten. "An Introduction to Continued Fractions." In *Diophantine Analysis,* pp. 99-138. Ed. J. H. Loxton & A. J. van der Poorten. London Math. Soc. Lecture Note Series **109**. Cambridge: Cambridge University Press, 1986.
7. A. J. van der Poorten. "Recurrence Sequences, Continued Fractions and the HQT." In *Proc. of the 23rd Annual Iranian Math. Conference.* Bakhtaran, 1992. Preprint.
8. T. Van Ravenstein, G. Winley, & K. Tognetti. "A Property of Convergents to the Golden Mean." *The Fibonacci Quarterly* **23.2** (1985):155-57.
9. G. Winley, K. Tognetti, & T. Van Ravenstein. "A Property of Numbers Equivalent to the Golden Mean." *The Fibonacci Quarterly* **25.2** (1987):171-73.
10. G. Winley, K. Tognetti, & T. Van Ravenstein. "Hurwitz's Theorem and the Continued Fraction with Constant Terms." *The Fibonacci Quarterly* **27.5** (1989):420-23.

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RANDOM COMBINATIONS WITH BOUNDED DIFFERENCES AND COSPAN

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1. INTRODUCTION

Let

$$1 \le x_1 < x_2 < \cdots < x_r \le n \tag{1.1}$$

be an r combination from n. Moser and Abramson [17] used the terms *differences* for the quantities $d_j = x_{j+1} - x_j$, $j = 1, 2, \dots, r-1$, *span* for $d = x_r - x_1$, and *cospan* for $n-d$. It is clear that $d_j - 1$ is, for $j = 1, 2, \dots, r-1$, the number of integers, from $\{1, 2, \dots, n\}$, which lie "between" x_j and x_{j+1}. Similarly, $(n-d)-1$ is the number of integers from x_r clockwise to x_1.

To the r-combination (1.1) there corresponds a unique *place-indicator vector* $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ defined by

$$\varepsilon_i = \begin{cases} 1 & \text{if } i = x_1, \dots, x_r, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $\varepsilon_i, i = 1, 2, \dots, n$ are generated by a random process in which the outcome of the t^{th} trial depends on the outcomes of the previous trials in a first-order Markovian fashion. Moreover, let

$$p_j(1) = P(\varepsilon_1 = j), \quad j = 0, 1, \tag{1.2}$$

denote the initial probabilities and

$$p_{ij}(t) = P(\varepsilon_t = j \mid \varepsilon_{t-1} = i), \quad i, j \in \{0, 1\}, \ t = 2, \dots, n, \tag{1.3}$$

denote the first-order transition probabilities of the process. By the term *random combination* we shall refer to the combination associated with $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, i.e., integer j $(1 \le j \le n)$ will be selected if and only if $\varepsilon_j = 1$.

Let $A_n(k, k'; l, l')$ denote the event that the random combination associated with the sequence $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ satisfies the conditions

$$k \le d_j \le k', \quad j = 1, 2, \dots, r-1 \text{ and } l \le n-d \le l',$$

where k, k', l, l' are pre-specified integers $(1 \le k \le k',\ 1 \le l \le l')$ and r is the number of nonzero entities in $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$.

The probability of the event $A_n(k, k'; l, l')$ is, in certain special cases, very closely related to some problems of interest in combinatorial analysis, statistical theory of runs and reliability theory. Thus, for the symmetric i.i.d. case, *viz.*,

$$p_j(1) = p_{ij}(t) = \frac{1}{2}, \quad i, j \in \{0, 1\}, \ t = 2, \dots, n,$$

the quantity

$$2^n P[A_n(k, k'; l, l')]$$

enumerates the (nonrandom) combinations from n whose differences and cospan satisfy the conditions $k \le d_1, d_2, \dots, d_{r-1} \le k'$ and $l \le n-d \le l'$. More specifically, if $C_{n,r}(k, k'; l, l')$ is the number of r-combinations (1.1) with $k \le d_j \le k'$, $j = 1, 2, \dots, r-1$, and $l \le n-d \le l'$ (see [17]), then

$$2^n P[A_n(k, k'; l, l')] = \sum_{r=0}^{n} C_{n,r}(k, k'; l, l').$$

Now let us consider the first-order Markov dependence model and assume that l' is sufficiently large so that $n-d \le l'$ poses no restriction on the cospan. Then, $P[A_n(1, k'; 1, l')]$ turns out to be the probability that the *longest run*[1] of 0's, in n Markov dependent trials $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ does not exceed $k'-1$; for related problems, see [1], [2], [8], [16], and the references therein. Similarly, $P[A_n(1, k'; 1, k')]$ turns out to be the corresponding probability for the *longest circular run*, i.e., when the trials $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are arranged in a circular fashion so that ε_1 becomes adjacent to ε_n (see [13], [21]). In the i.i.d. case,

$$p_0(1) = p_{i0}(t) = q, \ p_1(1) = p_{i1}(t) = p, \ i \in \{0, 1\}, \ t = 2, \dots, n, \tag{1.4}$$

the probabilities mentioned above are closely related to the reliability of a linear/circular consecutive-k'-out-of-n:F reliability system with component failure probabilities q; for a review on this topic, one may refer to Chao et al. [3].

Finally, let us assume that both k' and l' are sufficiently large so that $d_j \le k'$ and $n-d \le l'$ are practically no restrictions. Then, the occurrence of the event $A_n(k, k'; 1, l')$ implies that the length of the *shortest run* of 0's in the sequence $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ is at least $k-1$ or, equivalently, there exists no pair of 1's separated by $k-2$ or less 0's. A related waiting time problem was recently studied by Koutras [10]. In the i.i.d. case, $P[A_n(k, k'; 1, l')]$ coincides with the reliability of a 2-within-consecutive-k-out-of-n:F system with component failure probabilities p (see [3], [19], and [20]); it is also related to sliding window probabilities [18] and scan statistics [5].

The purpose of the present paper is to conduct a detailed study of the probability of the event $A_n(k, k'; k, l')$ when k' and l' are sufficiently large. In this case, we shall use the notation $A_n(k)$ for the event. It is clear that the occurrence of $A_n(k)$ implies that the length of the *shortest circular run* of 0's in the sequence $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ is at least $k-1$.

In Section 2, we introduce the necessary notations and develop formulas for the evaluation of $P[A_n(k)]$ in the general case of Markov dependent trials. In Section 3, we restrict ourselves to a homogeneous Markov-dependence model and derive the generating function of the sequence $\{P[A_n(k)]\}_{n \ge k}$. In addition, a set of recurrence relations is established which offers a computationally efficient scheme for the calculation of $P[A_n(k)]$. In Section 4, we focus our attention on the circular 2-within-consecutive-k-out-of-n:F system. Finally, in Section 5, we express $P[A_n(k)]$ in terms of appropriate generalizations of Lucas polynomials and numbers.

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[1] Here, by the term "run of 0's" we mean a string of consecutive 0's preceded and followed by 1's.

2. GENERAL MODEL

Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be a finite sequence of Markov dependent random variables with initial distribution (1.2) and first-order transition probabilities (1.3). By $R_n(k)$, or simply R_n, if no confusion is likely to arise, we shall denote the probability that the differences and cospan of the associated random r-combinations ($r = 0, 1, \ldots, n$) are at least k, i.e., $d_j \geq k$, $j = 1, 2, \ldots, r-1$, and $n - d \geq k$. By convention, for $r = 0$, we assume that there is only one r-combination satisfying the aforementioned conditions (the one that corresponds to the place indicator $(0, 0, \ldots, 0)$). On the other hand, for $r = 1$, we treat all 1-combinations (i.e., the ones associated with the place indicators $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$) as valid choices for $A_n(k)$. It is worth stressing that this setup is slightly different from the one used by Moser and Abramson [17], who assumed that the 1-combinations are acceptable choices for $n \geq k$ and nonacceptable for $n < k$.

Employing the notation of the last section, we may write

$$R_n = R_n(k) = P[A_n(k)] = P[A_n(k, l; k, l')]$$

with l, l' being sufficiently large.

In order to evaluate R_n, we will employ a Markov chain approach similar to the one used by Koutras [10] for the study of several reliability systems; see also [4] and [12] for additional applications of the same method to success runs enumeration problems.

Observe first that, for $n < k$, we have

$$\begin{aligned}
&R_n = 1 \quad \text{for } n = 0, 1,\\
&R_n = 1 - p_1(1)p_{11}(2) \quad \text{for } n = 2,\\
&R_n = p_0(1)\sum_{t=2}^{n} p_{00}(t) + p_1(1)p_{10}(2)\sum_{t=3}^{n} p_{00}(t) + p_0(1)\sum_{i=2}^{n-1}\left[\prod_{t=2,\, t\neq i,\, i+1}^{n} p_{00}(t)\right]\\
&\qquad \times p_{01}(i)p_{10}(i+1) + p_0(1)\left[\prod_{t=2}^{n-1} p_{00}(t)\right]p_{01}(n) \quad \text{for } 3 \leq n < k.
\end{aligned} \tag{2.1}$$

The first two expressions are obvious. For the third one, it is enough to observe that the occurrence of $A_n(k)$, $3 \leq n < k$, secures that at most one trial among $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ resulted in 1. The required formula is then easily gained by conditioning on the position where 1 should be placed.

Next, assume that $n \geq k$. By introducing the events

$$\begin{aligned}
&B_0 : \varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_k = 0,\\
&B_i : \varepsilon_{k-i+1} = 1 \quad \text{and} \quad \varepsilon_j = 0, \ j \in \{1, 2, \ldots, k\} \setminus \{k-i+1\}
\end{aligned}$$

for $i = 1, 2, \ldots, k$, we may write

$$R_n = \sum_{i=0}^{k} \beta_i \, P[A_n(k) \mid B_i], \tag{2.2}$$

where $\beta_i = P(B_i)$ are given by

$$\beta_i = \begin{cases}
p_0(1)\prod_{t=2}^{k} p_{00}(t) & \text{for } i = 0,\\
p_0(1)\left[\prod_{t=2}^{k-1} p_{00}(t)\right]p_{01}(k) & \text{for } i = 1,\\
p_0(1)\left[\prod_{t=2, t\neq k-i+1,\, t\neq k-i+2}^{k} p_{00}(t)\right] & \\
\qquad \times p_{01}(k-i+1)p_{10}(k-i+2) & \text{for } 2 \leq i \leq k-1,\\
p_1(1)p_{10}(2)\prod_{t=3}^{k} p_{00}(t) & \text{for } i = k.
\end{cases} \tag{2.3}$$

For the evaluation of the conditional probabilities $P[A_n(k) \mid B_i]$, $i = 0, 1, \ldots, k$, we introduce a Markov chain $\{Y_t,\ t = 1, 2, \ldots\}$ defined on the finite state space $\Omega = \{1, 2, \ldots, k+2\}$ as follows:

$Y_t = 1$ if $\varepsilon_i = 0$ for $\max(1, t-k+1) \le i \le t$;

$Y_t = j$ if $\varepsilon_{t-j+2} = 1$ and $\varepsilon_i = 0$ for $i \ne t-j+2$, $\max(1, t-k+1) \le i \le t$ $(2 \le j \le k+1)$;

$Y_t = k+2$ if there exist indices t_1, t_2 with $\max(1, t-k+1) \le t_1 \ne t_2 \le t$ such that $\varepsilon_{t_1} = \varepsilon_{t_2} = 1$.

(Note that states $j, 2 \le j \le k+1$, are only reachable after time $t \ge j-1$.) Let us denote by Λ_t the transition probability of the aforementioned Markov chain, i.e.,

$$\Lambda_t = (p(Y_t = j \mid Y_{t-1} = i))_{(k+2)\times(k+2)}.$$

From the description of the states, we may immediately verify that Λ_t is given by

$$\Lambda_t = \begin{bmatrix} p_{00}(t) & p_{01}(t) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{10}(t) & 0 & \cdots & 0 & 0 & 0 & p_{11}(t) \\ 0 & 0 & 0 & p_{00}(t) & \cdots & 0 & 0 & 0 & p_{01}(t) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & p_{01}(t) \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{00}(t) & 0 & p_{01}(t) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{00}(t) & p_{01}(t) \\ p_{00}(t) & p_{01}(t) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The conditional probabilities $P[A(k) \mid B_i]$, $i = 0, 1, \ldots, k$, can now be expressed by means of higher order transition probability matrices or, equivalently, products of Λ_t's. Thus, denoting by $\mathbf{e}_j$ the j^{th} unit (row) vector of the space $\mathbf{R}^{k+2}$, $\mathbf{u} = (1, 1, \ldots, 1, 0) = \sum_{j=1}^{k+1} \mathbf{e}_j$, and $\mathbf{u}_i = \mathbf{u} - \sum_{j=2}^{i} \mathbf{e}_j$, $i = 2, \ldots, k+1$, we obtain

$$\begin{aligned} P(A_n(k) \mid B_i) &= P(Y_n \ne k+2 \mid Y_k = i+1) = \mathbf{e}_{i+1}\left(\prod_{t=k+1}^{n} \Lambda_t\right)\mathbf{u}' && \text{for } i = 0, 1, \\ P(A_n(k) + B_i) &= P(Y_n \in \{1, i+1, \ldots, k+1\} \mid Y_k = i+1) = \mathbf{e}_{i+1}\left(\prod_{t=k+1}^{n} \Lambda_t\right)\mathbf{u}_i' && \text{for } 2 \le i \le k. \end{aligned} \tag{2.4}$$

A combined use of formulas (2.2), (2.3), and (2.4) offer a compact computational scheme for the evaluation of R_n.

It is worth mentioning that, on introducing the convention $\mathbf{u}_0 = \mathbf{u}_1 = \mathbf{u}$, we may write a unified formula for R_n,

$$R_n = \sum_{i=0}^{k} \beta_i \mathbf{e}_{i+1}\left(\prod_{t=k+1}^{n} \Lambda_t\right)\mathbf{u}_i'. \tag{2.5}$$

Note also that, for $n \le 2k-1$, one does not have to use (2.5) for the evaluation of R_n, since the third part of (2.1) is valid for $k \le n \le 2k-1$ as well.

In closing, we mention that the technique employed here for the study of the event $A_n(k) = A_n(k, k'; k, l')$ can be modified effortlessly to capture the probability of the more general event $A_n(k, k'; l, l')$. The consideration of the special case was for typographical convenience only.

3. HOMOGENEOUS MARKOV-DEPENDENCE MODEL

In this section, we study in some detail the special case in which $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ form a homogeneous first-order Markov sequence, i.e., $p_{ij}(t)$ are independent of t. Using the notation $p_j - p_j(1)$, $p_{ij} = p_{ij}(t)$, $t = 2, 3, \ldots, n$, for $i, j \in \{0, 1\}$, we may write (2.1) as follows:

$$\begin{aligned} R_n &= 1 \text{ if } n = 0, 1; \\ R_n &= 1 - p_1 p_{11} \text{ if } n = 2; \\ R_n &= (p_0 + p_1 p_{10}) p_{00}^{n-2} + (n-2) p_0 p_{01} p_{10} p_{00}^{n-3} \text{ if } 3 \le n < k. \end{aligned} \tag{3.1}$$

Since $\Lambda_t = \Lambda$ for all $t = 2, 3, \ldots, n$, we have $P(A_n(k) \mid B_i) = \mathbf{e}_{i+1} \Lambda^{n-k} \mathbf{u}_i'$, $n \ge k$, and (2.5) leads to the expression

$$R_n = \sum_{i=0}^{k} \beta_i \mathbf{e}_{i+1} \Lambda^{n-k} \mathbf{u}_i', \; n \ge k, \tag{3.2}$$

with the β_i's given by

$$\beta_i = \begin{cases} p_0 p_{00}^{k-1} & \text{if } i = 0, \\ p_0 p_{00}^{k-2} p_{01} & \text{if } i = 1, \\ p_0 p_{00}^{k-3} p_{01} p_{10} & \text{if } 2 \le i \le k-1, \\ p_1 p_{10} p_{00}^{k-2} & \text{if } i = k. \end{cases}$$

From (2), we can easily obtain an explicit expression for the generating function

$$G_k(x) = \sum_{n=k}^{\infty} R_n z^n. \tag{3.3}$$

Now, using (3.2) in (3.3), interchanging the order of summation and then substituting the resulting geometric (matrix) series, we obtain the final expression for the generating function as

$$G_k(z) = z^k \sum_{i=0}^{k} \beta_i \mathbf{e}_{i+1} (I - \Lambda z)^{-1} \mathbf{u}_i'.$$

After somewhat lengthy but straightforward algebraic calculations on the matrix $I - \Lambda z$, we get

$$G_k(z) = \frac{\{(p_0 + p_1 p_{10}) p_{00}^{k-2} + (k-2) p_0 p_{01} p_{10} p_{00}^{k-3}\} z^k + p_0 p_{01} p_{10} \sum_{n=k+1}^{2k-1} p_{00}^{n-3} z^n}{1 - p_{00} z - p_{01} p_{10} p_{00}^{k-2} z^k}. \tag{3.4}$$

From (3.4), we can easily derive a recursive scheme for the evaluation of R_n. Multiplying both sides of (3.4) by the denominator, using (3.3) on the left-hand side, and equating the coefficients of z^n on both sides, we obtain

$$\begin{aligned} R_k &= p_0 p_{00}^{k-1} + p_{00}^{k-3} [p_0 p_{01} p_{00} + (k-2) p_0 p_{01} p_{10} + p_1 p_{10} p_{00}], \\ R_n &= p_{00} R_{n-1} + p_0 p_{01} p_{10} p_{00}^{n-3} \quad \text{for } k+1 \le n \le 2k-1, \\ R_n &= p_{00} R_{n-1} + p_{01} p_{10} p_{00}^{k-2} R_{n-k} \quad \text{for } n \ge 2k. \end{aligned} \tag{3.5}$$

Evaluation of R_n through (3.5) is preferable, instead of through (3.2), due to simplicity (no matrix multiplications are necessary) as well as accuracy (less round-off errors). It may also be noted that, for $2 \le n \le 2k-1$, one can use the exact formula

$$R_n = p_0 p_{00}^{n-1} + p_{00}^{n-3}[p_0 p_{01} p_{00} + (n-2) p_0 p_{01} p_{10} + p_1 p_{10} p_{00}]$$

and employ the recursive scheme in (3.5) only for $n \geq 2k$.

4. CIRCULAR 2-WITHIN-CONSECUTIVE-k-OUT-OF-n:F SYSTEM

An r-within-consecutive-k-out-of-n:F system fails if and only if there exist k consecutive components which include among them at least r failed ones. Applications of such structures have been well documented in the literature, which include applications to telecommunications, design of integrated circuits, quality control, and sliding window detectors (see, e.g., [5], [6], [9], [18], [19], [20], and [24]).

Even for the case of linearly arranged components, the evaluation of system's reliability is a very difficult task, and is mainly performed through approximating formulas. A Markov chain approach for this problem can be found in [10] along with recurrence relations for the special case $r = 2$.

The results of Section 2 can be used for the reliability evaluation of a *circular* 2-within-consecutive-k-out-of-n:F system. Let us assume that the n components of the system work independently and denote by $p = 1 - q$ their failure probabilities (i.i.d. model). It is clear, from the definition of the event $A_n(k)$, that in the special case $p_1 = p_{01} = p_{11} = p$, $p_0 = p_{00} = p_{10} = q$, R_n is exactly the same as the reliability of a circular 2-within-consecutive-k-out-of-n:F system. Recurrence relations in (3.5) reduce in this case to

$$\begin{aligned} R_n &= qR_{n-1} + pq^{n-1} && \text{if } k+1 \leq n < 2k, \\ R_n &= qR_{n-1} + pq^{k-1}R_{n-k} && \text{if } n \geq 2k, \end{aligned} \tag{4.1}$$

with initial conditions $R_0 = R_1 = 1$, $R_n = q^n + npq^{n-1}$, $2 \leq n \leq k$. Note again that instead of the first recurrence relation in (4.1), one could use the exact formula $R_n = q^n + npq^{n-1}$ for all $2 \leq n < 2k$. Another interesting observation to be made here is that, for $n \geq 2k$, the reliability of both linear and circular 2-within-consecutive-k-out-of-n:F systems satisfy exactly the same recurrence relation. This is not surprising since, when the system becomes sufficiently large ($n \geq 2k$), the transition from R_{n-1} to R_n is not affected by the topological arrangement (adjacent or not) of components 1, n.

Recurrence relations in (4.1) can be used in conjunction with the obvious inequality $R_{n-k} \geq R_{n-1}$, $k \geq 1$, in order to establish some simple lower bounds for R_n. Thus, for $n \geq 2k$, we have $R_n \geq (q + pq^{k-1})R_{n-1}$; repeated application of this argument yields $R_n \geq (q + pq^{k-1})^{n-2k+1} R_{2k-1}$, $n \geq 2k$, which, when used with the result that $R_{2k-1} = q^{2k-1} + (2k-1)pq^{2k-2}$, gives a lower bound for R_n as

$$R_n \geq (q + pq^{k-1})^{n-2k+1}\{q^{2k-1} + (2k-1)pq^{2k-2}\}, \quad n \geq 2k. \tag{4.2}$$

This bound is very useful when addressing the problem of specifying the (maximum) size n of the system warrantying a prespecified level a $(0 < a < 1)$ of reliability. In view of (4.2), the condition $R_n \geq a$ will be met if we force the right-hand side of (4.2) to exceed a; upon then solving for n, we obtain

$$n \leq (2k-1) + \left(\log \frac{a}{q^{2k-1} + (2k-1)pq^{2k-2}}\right)\{\log(q + pq^{k-1})\}^{-1}.$$

It needs to be mentioned here that if $R_{2k-1} < a$ then appropriate values of n should be sought with the aid of the exact formula $R_n = q^n + npq^{n-1}$, $2 \le n < 2k$.

Papastavridis and Koutras [19] derived upper and lower bounds for both linear and circular r-within-consecutive-k-out-of-n:F systems. In the special case in which $r = 2$, their lower bound for the circular system becomes

$$R_n \ge (q + pq^{k-1})^{n-1}(q^k + kpq^{k-1}), \quad n \ge 2. \tag{4.3}$$

These authors also established a Weibull limit theorem for system's lifetime under quite general assumptions on components' lifetime distributions. A simple adjustment to their proof yields the following asymptotic result: If p depends on n in such a way that $\lim_{n\to\infty} np^2 = \lambda > 0$, then

$$\lim_{n\to\infty} R_n = e^{-(k-1)\lambda}. \tag{4.4}$$

Simple algebraic calculations on the lower bounds in (4.2) and (4.3) reveal that, under the condition $\lim_{n\to\infty} np^2 = \lambda$, they converge to the limiting value given in (4.4).

In Table 1, a numerical comparison of the lower bounds in (4.2) and (4.3) is performed for selected values of n, k, and p. The exact value of R_n and the limiting value $e^{-(k-1)np^2}$ are also provided for comparison purposes.

5. LUCAS POLYNOMIALS AND NUMBERS

Let $\{L_n^{(k)}(x)\}_{n\ge0}$ be the sequence of polynomials defined recursively as follows:

$$\begin{aligned} L_n^{(k)}(x) &= 1 + nx && \text{for } 0 \le n < 2k, \\ L_n^{(k)}(x) &= L_{n-1}^{(k)}(x) + xL_{n-k}^{(k)}(x) && \text{for } n \ge 2k. \end{aligned} \tag{5.1}$$

It may be seen readily that the degree of $L_n^{(k)}(x)$ for $0 \le n < 2k$ is 1; moreover, if $sk \le n < (s+1)k$, $s = 2, 3, \ldots$, the degree of $L_n^{(k)}(x)$ is s.

Next, let us denote by $L_n^{(k)}$ the integers $L_n^{(k)}(1)$, $n \ge 0$. Then $\{L_n^{(k)}\}_{n\ge0}$ will satisfy the recurrence relation

$$L_n^{(k)} = L_{n-1}^{(k)} + L_{n-k}^{(k)}, \quad n \ge 2k, \tag{5.2}$$

with initial conditions

$$L_n^{(k)} = n + 1, \quad 0 \le n < 2k. \tag{5.3}$$

It is clear that, for $k = 2$, the corresponding numbers $L_n^{(2)}$, $n \ge 2$, coincide with the well-known Lucas numbers L_n. Hence, an appropriate name for the numbers $L_n^{(k)}$ seems to be *k-step Lucas numbers*. Likewise, $L_n^{(k)}(x)$ may aptly be called *k-step Lucas polynomials*.

It is worth noting that the recurrence relation in (5.2), under different initial conditions, gives rise to analogous generalizations of Fibonacci numbers. However, they have been studied in the literature under many different names. For example, Mohanty [15] termed them *generalized Fibonacci numbers* (see also Roselle [23] and Moser and Abramson [17]) and proved the existence of minimal and maximal representations of positive integers as sums of such numbers; Hasunuma and Shibata [7] used the name k^{th} *interspaced Fibonacci numbers*, while Koutras [11] employed the term *k-step Fibonacci numbers*.

Hasunuma and Shibata [7] defined a Lucas number analogue as well, by considering the sequence $L_n^{\langle k\rangle}$ satisfying the recurrence $L_n^{\langle k\rangle} = L_{n-1}^{\langle k\rangle} + L_{n-k}^{\langle k\rangle}$, $n \ge 2$, with initial conditions $L_0^{\langle k\rangle} = k$,

$L_1^{\langle k\rangle}=1$, and $L_n^{\langle k\rangle}=0$ for $n<0$. The numbers $L_n^{\langle k\rangle}$, names as k^{th} *interspaced Lucas numbers* by them, arise in a very natural way in an interesting graph theoretic problem. Specifically, in [7], Hasunuma and Shibata proved that the number of labeled graphs which are k-placeable by a given permutation is a product of interspaced k^{th} Lucas numbers. It can be shown that the k-step Lucas numbers $L_n^{(k)}$ (defined above) and the k^{th} interspaced Lucas numbers $L_n^{\langle k\rangle}$ (just defined) coincide for $n\geq k$, that is, $L_n^{(k)}=L_n^{\langle k\rangle}$ for all $n=k,k+1,\ldots$.

The generating function of the sequence $\{L_n^{(k)}(x)\}_{n\geq k}$ given by

$$G(z;x)=\sum_{n=k}^{\infty}L_n^{(k)}(x)z^n$$

is readily determined from (5.1) to be

$$G(z;x)=\frac{z^k[1+kx+xz(1+z+\cdots+z^{k-2})]}{1-z-xz^k}.$$

Comparing $G(z;x)$ to the generating function of R_n for the i.i.d. case, we obtain the relationship

$$\sum_{n=k}^{\infty}R_nz^n=\frac{(qz)^k[1+k\frac{p}{q}+pz(1+qz+\cdots+(qz)^{k-2})]}{1-qz-pq^{k-1}z^k}=G\left(qz;\frac{p}{q}\right).$$

Hence,

$$\sum_{n=k}^{\infty}R_nz^n=\sum_{n=k}^{\infty}L_n^{(k)}\left(\frac{q}{p}\right)(qz)^n$$

and the reliability R_n of a 2-within-consecutive-k-out-of-n:F system can be expressed in terms of k-step Lucas polynomials as follows:

$$R_n=q^nL_n^{(k)}\left(\frac{p}{q}\right),\quad n\geq k.$$

This formula yields an interesting combinatorial interpretation for the k-step numbers $L_n^{(k)}$. More specifically, considering the symmetric case $p=q=1/2$, we note from the above relation that $L_n^{(k)}=2^nR_n=2^nP[A_n(k)]$, which simply proves that $L_n^{(k)}$ is the total number of "*circular*" combinations whose differences and cospan are at least k. Moser and Abramson [17] arrived at the same conclusion by first computing the number $C_r(k,k';k,l')$ of r-combinations with differences and cospan at least k (k' and l' are assumed sufficiently large so that the differences and cospan are practically unbounded from above) and then noting that

$$L_n^{(k)}=\sum_{r=0}^{n}C_r(k,k';k,l') \tag{5.4}$$

satisfies (5.2) and (5.3). It should be stressed that, due to the different conventions used here for the 1-combinations (c.f. first paragraph of Section 2), our numbers $L_n^{(k)}$ coincide with the ones appearing in [17] only for $n\geq k$. Analogous results can be found in [22] and [25].

In closing, we note that the generating function of the k-step Lucas numbers $L_n^{(k)}$, $n\geq k$, is given by

$$\sum_{n=k}^{\infty}L_n^{(k)}z^n=G(z;1)=\frac{z^k[1+k+z(1+z+\cdots+z^{k-2})]}{1-z-z^k}.$$

Expanding the right-hand side in a power series around 0, we easily get an explicit expression for $L_n^{(k)}$ as

$$L_n^{(k)} = 1 + \sum_{r \geq 1} \frac{n}{r} \binom{n - r(k-1) - 1}{r-1}. \tag{5.5}$$

This formula was also derived by Moser and Abramson [17] using direct combinatorial arguments. In fact, they first proved that

$$C_r(k, k'; k, l') = \frac{n}{r} \binom{n - r(k-1) - 1}{r-1}, \quad r \geq 1,$$

and then employed (5.4) in order to derive the explicit expression in (5.5).

TABLE 1. Comparison of Lower Bounds of Section 4 and Exact and Limiting Values of R_n

n	k	p	*Exact Value* R_n	$e^{-(k-1)np^2}$	*Lower Bound* (2)	*Lower Bound* (3)
4	2	0.01	0.9996	0.9996	0.9996	0.9996
		0.05	0.9905	0.9900	0.9903	0.9900
		0.10	0.9639	0.9608	0.9623	0.9606
		0.20	0.8704	0.8521	0.8602	0.8493
		0.50	0.4375	0.3679	0.3750	0.3164
		0.80	0.0784	0.0773	0.0374	0.0168
		0.90	0.0199	0.0392	0.0053	0.0013
		0.95	0.0050	0.0271	0.0007	0.0001
		0.99	0.0002	0.0198	0.0000	0.0000

n	k	p	*Exact Value* R_n	$e^{-(k-1)np^2}$	*Lower Bound* (2)	*Lower Bound* (3)
6	2	0.01	0.9994	0.9994	0.9994	0.9994
		0.05	0.9858	0.9851	0.9853	0.9851
		0.10	0.9462	0.9418	0.9431	0.9415
		0.20	0.8110	0.7866	0.7927	0.7828
		0.50	0.2813	0.2231	0.2109	0.1780
		0.80	0.0190	0.0215	0.0049	0.0022
		0.90	0.0022	0.0078	0.0002	0.0000
		0.95	0.0003	0.0044	0.0000	0.0000
		0.99	0.0000	0.0028	0.0000	0.0000
	3	0.01	0.9988	0.9988	0.9988	0.9987
		0.05	0.9733	0.9704	0.9726	0.9688
		0.10	0.9054	0.8869	0.9011	0.8831
		0.20	0.7045	0.6188	0.6842	0.6167
		0.50	0.1563	0.0498	0.1172	0.0477
		0.80	0.0047	0.0005	0.0016	0.0001
		0.90	0.0003	0.0001	0.0001	0.0000
		0.95	0.0000	0.0000	0.0000	0.0000

TABLE 1 (continued)

n	k	p	*Exact Value* R_n	$e^{-(k-1)np^2}$	*Lower Bound* (2)	*Lower Bound* (3)
10	2	0.01	0.9990	0.9990	0.9990	0.9990
		0.05	0.9764	0.9753	0.9755	0.9753
		0.10	0.9120	0.9048	0.9060	0.9044
		0.20	0.7053	0.6703	0.6733	0.6648
		0.50	0.1201	0.0821	0.0667	0.0563
		0.80	0.0013	0.0017	0.0001	0.0000
		0.90	0.0000	0.0003	0.0000	0.0000
	3	0.01	0.9981	0.9980	0.9980	0.9979
		0.05	0.9562	0.9512	0.9538	0.9500
		0.10	0.8485	0.8187	0.8345	0.8179
		0.20	0.5604	0.4493	0.5074	0.4573
		0.50	0.0449	0.0067	0.0179	0.0073
		0.80	0.0001	0.0000	0.0000	0.0000
		0.90	0.0000	0.0000	0.0000	0.0000
	5	0.01	0.9962	0.9960	0.9962	0.9955
		0.05	0.9222	0.9048	0.9202	0.8988
		0.10	0.7576	0.6703	0.7482	0.6704
		0.20	0.4094	0.2019	0.3847	0.2380
		0.50	0.0156	0.0000	0.0104	0.0006
		0.80	0.0000	0.0000	0.0000	0.0000

n	k	p	*Exact Value* R_n	$e^{-(k-1)np^2}$	*Lower Bound* (2)	*Lower Bound* (3)
100	2	0.001	0.9999	0.9999	0.9999	0.9999
		0.002	0.9996	0.9996	0.9996	0.9996
		0.005	0.9975	0.9975	0.9975	0.9975
		0.009	0.9920	0.9919	0.9919	0.9919
		0.010	0.9901	0.9900	0.9901	0.9901
		0.050	0.7875	0.7788	0.7787	0.7786
		0.100	0.3981	0.3679	0.3667	0.3660
		0.200	0.0304	0.0183	0.0171	0.0169
		0.500	0.0000	0.0000	0.0000	0.0000
	10	0.001	0.9991	0.9991	0.9991	0.9991
		0.002	0.9965	0.9964	0.9964	0.9963
		0.005	0.9791	0.9778	0.9783	0.9773
		0.009	0.9366	0.9297	0.9328	0.9295
		0.010	0.9232	0.9139	0.9181	0.9140
		0.050	0.2400	0.1054	0.1665	0.1441
		0.100	0.0128	0.0001	0.0025	0.0014
		0.200	0.0000	0.0000	0.0000	0.0000
		0.500	0.0000	0.0000	0.0000	0.0000

TABLE 1 (continued)

n	k	p	*Exact Value* R_n	$e^{-(k-1)np^2}$	*Lower Bound* (2)	*Lower Bound* (3)
100	20	0.001	0.9982	0.9981	0.9981	0.9980
		0.002	0.9928	0.9924	0.9927	0.9919
		0.005	0.9591	0.9536	0.9567	0.9517
		0.009	0.8833	0.8574	0.8728	0.8567
		0.010	0.8607	0.8270	0.8471	0.8276
		0.050	0.1139	0.0087	0.0600	0.0321
		0.100	0.0025	0.0000	0.0004	0.0001
		0.200	0.0000	0.0000	0.0000	0.0000
		0.500	0.0000	0.0000	0.0000	0.0000

REFERENCES

1. S. Aki & K. Hirano. "Discrete Distributions Related to Succession Events in a Two-State Markov Chain." In *Statistical Science and Data Analysis*, pp. 467-74. Ed. Matusita et al. Zeist: VSP International Science Publishers, 1993.
2. S. Aki, H. Kuboki, & K. Hirano. "On Discrete Distributions of Order *k*." *Annals of the Inst. of Statistical Math.* **36** (1984):431-40.
3. M. T. Chao, J. C. Fu, & M. V. Koutras. "A Survey of the Reliability Studies of Consecutive-*k*-out-of-*n*:*F* Systems and Its Related Systems." *IEEE Trans. on Reliability* **44** (1995):120-27.
4. J. C. Fu & M. V. Koutras. "Distribution Theory of Runs: A Markov Chain Approach. *J. Amer. Stat. Assoc.* **89** (1994):1050-58.
5. J. Glaz. "Moving Window Detection for Discrete-Data." *IEEE Trans. on Inform. Theory* **29** (1983):457-62.
6. I. Greenberg. "The First Occurrence of *n* Successes in *N* Trials." *Technometrics* **12** (1970): 627-34.
7. T. Hasunuma & Y. Shibata. "The Number of Labeled Graphs Placeable by a Given Permutation." *J. Graph Theory* **21** (1996):11-19.
8. K. Hirano & S. Aki. "On the Number of Occurrences of Success Runs of Specified Length in a Two-State Markov Chain." *Statistica Sinica* **3** (1993):313-20.
9. S. Kounias & M. Sfakianakis. "The Reliability of a Linear System and Its Connection With the Generalized Birthday Problem." *Statistica Applicata* **3** (1991):531-43.
10. M. V. Koutras. "On a Markov Chain Approach for the Study of Reliability Structures." *J. Appl. Probability* **33** (1996):357-67.
11. M. V. Koutras. "On a Waiting Time Distribution in a Sequence of Bernoulli Trials." *Annals of the Inst. of Statistical Math.* **48** (1996):789-806.
12. M. V. Koutras & V. A. Alexandrou. "Runs, Scans and Urn Model Distributions: A Unified Markov Chain Approach." *Annals of the Inst. of Statistical Math.* **47** (1995):743-66.
13. M. V. Koutras, G. K. Papadopoulos, & S. G. Papastavridis. "Circular Overlapping Success Runs." In *Runs and Patterns in Probability*, pp. 287-305. Ed. A. P. Godbole & S. G. Papastavridis. Dordrecht: Kluwer, 1994.
14. M. V. Koutras, G. K. Papadopoulos, & S. G. Papastavridis. "Runs On a Circle." *J. Applied Probability* **32** (1995):396-404.

15. S. G. Mohanty. "On a Partition of Generalized Fibonacci Numbers." *The Fibonacci Quarterly* **6.1** (1968):22-23.
16. S. G. Mohanty. "Success Runs of Length k in Markov Dependent Trials." *Annals of the Inst. of Statistical Math.* **46** (1994):777-96.
17. W. O. J. Moser & M. Abramson. "Enumeration of Combinations With Restricted Differences and Cospan." *J. Comb. Theory* **7A** (1969):162-70.
18. J. B. Nelson. "Minimal-Order Models for False-Alarm Calculations on Sliding Windows." *IEEE Trans. on Aerospace and Electronic Systems* **14** (1978):351-63.
19. S. G. Papastavridis & M. V. Koutras. "Bounds for Reliability of Consecutive k-within-m-out-of-n:F Systems." *IEEE Trans. on Reliability* **42** (1993):156-60.
20. S. G. Papastavridis & M. V. Koutras. "Consecutive-k-out-of-n Systems." In *New Trends in System Reliability Evaluation*, pp. 228-48. Ed. K. B. Misra. The Netherlands: Elsevier Science Publishers B. V., 1994.
21. A. N. Philippou & F. S. Makri. "Longest Circular Runs With an Application to Reliability, via Fibonacci-Type Polynomials of Order k." In *Applications of Fibonacci Numbers*, pp. 281-86. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1990.
22. J. Riordan. *An Introduction to Combinatorial Analysis*. New York: John Wiley & Sons, 1958.
23. D. P. Roselle. "Enumeration of Certain Triangular Arrays." *The Fibonacci Quarterly* **5.3** (1967):235-46.
24. B. Saperstein. "On the Occurrence of n Successes Within N Bernoulli Trials." *Technometrics* **15** (1973):809-18.
25. K. Yamamoto. "Structure Polynomial of Latin Rectangles and Its Application to a Combinatorial Problem." *Mem. Sci. Kyushu Univ.* **10A** (1956):1-13.

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FIBONACCI AND LUCAS NUMBERS AS CUMULATIVE CONNECTION CONSTANTS[1]

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1. SEQUENCES OF CUMULATIVE CONNECTION CONSTANTS

Let us briefly introduce the notion of *cumulative connection constants*. For more details and related topics, the reader is referred to [2], [4], and [5].

Suppose two sequences $\{r_n\}_{n\ge1}$ and $\{s_n\}_{n\ge1}$ of complex numbers are given. Then one can introduce two associated sequences of polynomials $\{q_n(x)\}_{n\ge0}$ and $\{p_n(x)\}_{n\ge0}$ as follows:

- $q_0(x)=p_0(x)=1$, and
- for any $n\ge1$:
  $q_n(x)=q_{n-1}(x)\cdot(x-r_n)$,
  $p_n(x)=p_{n-1}(x)\cdot(x-s_n)$.

For any $n\ge0$, the *connection constants* (or *generalized Lah numbers*) *relating the* (*root*) *sequence* $\{r_n\}_{n\ge1}$ *to* $\{s_n\}_{n\ge1}$ (or, equivalently, relating $\{q_n(x)\}_{n\ge0}$ to $\{p_n(x)\}_{n\ge0}$) are the complex numbers $L_{n,k}$ uniquely defined via the relationship

$$p_n(x)=\sum_{k=0}^{n}L_{n,k}\cdot q_k(x),$$

where we limit the sum to n since, clearly, $L_{n,k}=0$ for any $k>n$. It is also easy to verify that $L_{n,n}=1$ for any $n\ge0$, our polynomials being monic. Moreover, we stipulate that $L_{n,k}=0$ for negative values of k.

For any $n\ge0$, the n^{th} *cumulative connection constant* (**ccc**, for short) is defined as

$$\mathscr{C}_n=\sum_{k=0}^{n}L_{n,k}.$$

We say that $\{\mathscr{C}_n\}_{n\ge1}$ is the sequence of **ccc**'s *relating* $\{r_n\}_{n\ge1}$ *to* $\{s_n\}_{n\ge1}$. Notice that we stipulate not to start the sequence of **ccc**'s with $\mathscr{C}_0$ which always equals 1, as one may easily see.

The following examples provide very well-known sequences of **ccc**'s. For the sake of completeness, in the tables at the end of the paper we sketch the number sequences involved in these examples.

(i) Let $\{r_n\}_{n\ge1}=0,0,0,\ldots$, and $\{s_n\}_{n\ge1}=-1,-1,-1,\ldots$. Here we have

$$(x+1)^n=\sum_{k=0}^{n}\binom{n}{k}\cdot x^k$$

which clearly yields $\mathscr{C}_n=2^n$ (see Table 1).

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(ii) Let $\{r_n\}_{n\geq 1} = 0, 1, 2, \ldots,$ and $\{s_n\}_{n\geq 1} = 0, 0, 0, \ldots.$ Here we have

$$x^n = \sum_{k=0}^{n} S(n, k)\cdot(x)_k,$$

where $(x)_0 \equiv 1$, $(x)_k = \prod_{i=0}^{k-1}(x-i)$ for $k \geq 1$, are the *falling* (or *lower*) *factorials*, and $S(n, k)$ are the *Stirling numbers of the second kind*. Then $\mathscr{C}_n$ is the n^{th} *Bell number* $\mathscr{B}_n$ (see Table 2).

(iii) Conversely, let $\{r_n\}_{n\geq 1} = 0, 0, 0, \ldots,$ and $\{s_n\}_{n\geq 1} = 0, 1, 2, \ldots.$ Here we have

$$(x)_n = \sum_{k=0}^{n} s(n, k)\cdot x^k,$$

where $s(n, k)$ are the *Stirling numbers of the first kind*. Then $\mathscr{C}_1 = 1$ and $\mathscr{C}_n = 0$ for each $n \geq 2$ (see Table 3).

(iv) Let $\{r_n\}_{n\geq 1} = 0, 0, 0, \ldots,$ and $\{s_n\}_{n\geq 1} = 0, -1, -2, \ldots.$ Here we have

$$\langle x\rangle_n = \sum_{k=0}^{n} c(n, k)\cdot x^k,$$

where $\langle x\rangle_0 \equiv 1$, $\langle x\rangle_n = \prod_{i=0}^{n-1}(x+i)$ for $n \geq 1$, are the *rising* (or *upper*) *factorials*, and $c(n, k) = (-1)^{n-k}\cdot s(n, k)$ are the *signless Stirling numbers of the first kind*. Then $\mathscr{C}_n = n!$ (see Table 4).

(v) Let $\{r_n\}_{n\geq 1} = 1, q, q^2, \ldots,$ and $\{s_n\}_{n\geq 1} = 0, 0, 0, \ldots.$ Here we have

$$x^n = \sum_{k=0}^{n} \binom{n}{k}_q \cdot g_k(x),$$

where $g_0(x) \equiv 1$, $g_k(x) = \prod_{i=0}^{k-1}(x-q^i)$ for $k \geq 1$, are the *Gaussian polynomials*, and $\binom{n}{k}_q$ are the *Gaussian binomial coefficients*. In this case, $\mathscr{C}_n$ is the n^{th} *Galois number relative to q*, namely, $\mathscr{G}_{n,q}$, which is known to count the number of subspaces of an n-dimensional vector space over GF(q) (see, e.g., [1], Ch. II, Sec. 4). This example for $q = 2$ is sketched in Table 5.

These and other relevant examples may be found, e.g., in [1], [3], and [6]. In the sequel, we give instances of the notion of **ccc** that involve Fibonacci, Lucas, and other more general sequences.

2. CCC VERSUS FIBONACCI

We are now going to show that Fibonacci numbers can be seen as the sequence of **ccc**'s relating two specific integer sequences. A generalization of this statement is then provided in Proposition 2.3.

To prove our results, we need the following recurrence on the connection constants $L_{n,k}$ relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$.

Theorem 2.1 [5, Prop. 2]: For any n, k,

$$L_{n,k} = L_{n-1,k-1} + (r_{k+1} - s_n)\cdot L_{n-1,k}. \tag{1}$$

This theorem allows us to obtain a nice recurrence relation for the sequence $\{\mathscr{C}_n\}_{n\geq 1}$ of **ccc**'s relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$.

Proposition 2.1: For any $n \geq 1$,

$$\mathscr{C}_n = (1 - s_n)\cdot\mathscr{C}_{n-1} + \sum_{k=0}^{n-1} L_{n-1,k} \cdot r_{k+1}. \tag{2}$$

Proof: Just put "$\sum_{k=0}^{n}$" on both sides of recurrence (1) in Theorem 2.1. Then the claimed result follows by easy computation. □

We can now state our first result.

Proposition 2.2: Let $\{r_n\}_{n\geq 1}$ be the sequence $0, 0, 1, 0, 1, 0, 1, \ldots$, i.e., $r_1 - 0$ and, for any $k \geq 1$, $r_{2\cdot k} = 0$ and $r_{2\cdot k+1} = 1$. Moreover, let $\{s_n\}_{n\geq 1}$ be the null sequence $0, 0, 0, \ldots$. Then the sequence $\{\mathscr{C}_n\}_{n\geq 1}$ of ccc's relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$ is the *Fibonacci sequence*.

Proof: By applying recurrence (1) to the connection constants relating our two sequences, we easily obtain $L_{1,0} = L_{2,0} = L_{2,1} = 0$. By recalling that $L_{n,n} = 1$ for any $n \geq 0$, and by the definition of ccc, we get

$$\mathscr{C}_1 = L_{1,0} + L_{1,1} = 1,$$
$$\mathscr{C}_2 = L_{2,0} + L_{2,1} + L_{2,2} = 1.$$

Let us compute $\mathscr{C}_n$ for $n \geq 3$. Since $\{s_n\}_{n\geq 1}$ is the null sequence, recurrence (2) becomes

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} L_{n-1,k} \cdot r_{k+1},$$

where we can expand $L_{n-1,k}$ according to recurrence (1), and get

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} (L_{n-2,k-1} + r_{k+1} \cdot L_{n-2,k}) \cdot r_{k+1}. \tag{3}$$

Now, note that our sequence $\{r_n\}_{n\geq 1}$ satisfies $r_n^2 = r_n$ for any $n \geq 1$. We can use this fact in (3) to obtain

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} L_{n-2,k-1} \cdot r_{k+1} + \sum_{k=0}^{n-1} L_{n-2,k} \cdot r_{k+1}. \tag{4}$$

The first sum in (4) gives $L_{n-2,1} + L_{n-2,3} + L_{n-2,5} + \cdots + L_{n-2,n-2} \cdot r_n$, while the second expands to $L_{n-2,2} + L_{n-2,4} + L_{n-2,6} + \cdots + L_{n-2,n-2} \cdot r_{n-1}$. Moreover, $L_{n-2,0} = 0$, as one may easily verify by using recurrence (1). Therefore, (4) becomes

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-2} L_{n-2,k} = \mathscr{C}_{n-1} + \mathscr{C}_{n-2},$$

and our claim follows. □

Indeed, this proposition (as well as the others we shall prove) can also be seen in terms of sequences of polynomials. As stated in Section 1, the two root sequences $\{r_n\}_{n\geq 1}$ and $\{s_n\}_{n\geq 1}$ in Proposition 2.2 originate two sequences of polynomials. The former gives $\{\phi_n(x)\}_{n\geq 0}$ with $\phi_0(x) \equiv 1$, $\phi_1(x) \equiv x$, and $\phi_n(x) = \phi_{n-1}(x) \cdot x^{(n+1) \bmod 2} \cdot (x-1)^{n \bmod 2}$ for $n \geq 2$. The latter yields $\{x^n\}_{n\geq 0}$. Thus, for any $n \geq 1$, we have

$$x^n = \sum_{k=0}^{n} L_{n,k} \cdot \phi_k(x) \quad \text{and} \quad \mathcal{F}_n = \sum_{k=0}^{n} L_{n,k},$$

where $\mathcal{F}_n$ is the n^{th} Fibonacci number.

In Table 6 we summarize the sequences we have just coped with in Proposition 2.2. The result in Proposition 2.2 can be generalized as follows.

Proposition 2.3: For fixed integers $d \geq 1$ and $m \geq 1$, let $\{r_n\}_{n\geq 1}$ be the sequence

$$\underbrace{0, 0, \ldots, 0}_{d-1}, 1, \underbrace{0, 0, \ldots, 0}_{m-1}, 1, \underbrace{0, 0, \ldots, 0}_{m-1}, 1, \ldots,$$

i.e., $r_n = 1$ whenever $n = d + h \cdot m$ for $h \geq 0$, and $r_n = 0$ otherwise. Moreover, let $\{s_n\}_{n\geq 1}$ be the null sequence. Then the sequence $\{\mathcal{C}_n\}_{n\geq 1}$ of **ccc**'s relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$ is

(a) $\mathcal{C}_1 = \mathcal{C}_2 = \cdots = \mathcal{C}_{d-1} = 1$,

(b) $\mathcal{C}_{d+i} = 2 + i$ for $0 \leq i \leq m-1$,

(c) $\mathcal{C}_n = \mathcal{C}_{n-1} + \mathcal{C}_{n-m}$ for $n \geq d+m$.

Proof: For (a), it is enough to verify via recurrence (1) that, for $0 \leq k \leq n \leq d-1$, we have $L_{n,k} = \delta_{n,k}$ (Kronecker's symbol). For (b), again recurrence (1) says that, for $0 \leq i \leq m-1$, we get $L_{d+i,k} = 0$ for $0 \leq k \leq d-2$, while $L_{d+i,k} = 1$ for $d-1 \leq k \leq d+i$.

Let us now turn to (c). For $n \geq d+m$, recurrence (2) is easily seen to be equivalent to

$$\mathcal{C}_n = \mathcal{C}_{n-1} + \sum_{h=0}^{\lfloor \frac{n-d}{m} \rfloor} L_{n-1,\, d+h\cdot m-1}. \tag{5}$$

By considering the structure of $\{r_n\}_{n\geq 1}$, and by repeatedly applying recurrence (1), we can write $L_{n-1,\, d+h\cdot m-1}$ in (5) as

$$L_{n-1,\, d+h\cdot m-1} = \sum_{j=1}^{m} L_{n-m,\, d+h\cdot m-j}.$$

We use this in (5) to obtain

$$\mathcal{C}_n = \mathcal{C}_{n-1} + \sum_{h=0}^{\lfloor \frac{n-d}{m} \rfloor} \sum_{j=1}^{m} L_{n-m,\, d+h\cdot m-j}. \tag{6}$$

It is easy to see that the double sum in (6) actually is $\sum_{k=d-m}^{n-m} L_{n-m,k}$. Furthermore, by recalling that $d \geq 1$, $m \geq 1$, and that we are considering the case $n \geq d+m$, it is also easy to see that $L_{n-m,k} = 0$ for $k \leq d-m-1$. In conclusion, we can write (6) as

$$\mathcal{C}_n = \mathcal{C}_{n-1} + \sum_{k=0}^{n-m} L_{n-m,k} = \mathcal{C}_{n-1} = \mathcal{C}_{n-m},$$

whence the result. □

Needless to remark, Proposition 2.2 is just a special case of Proposition 2.3, up to setting $d = 3$ and $m = 2$. More interesting is the case $d = m = 1$, which yields the constant sequence $\{r_n\}_{n\geq 1} = 1, 1, 1, \ldots$. By Proposition 2.3, the sequence $\{\mathcal{C}_n\}_{n\geq 1}$ of **ccc**'s relating such $\{r_n\}_{n\geq 1}$ to the null sequence is $\mathcal{C}_1 = 2$, $\mathcal{C}_n = 2 \cdot \mathcal{C}_{n-1}$, i.e., $\mathcal{C}_n = 2^n$. This is in perfect accordance with the well-known identity

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (x-1)^k .$$

As a further example, in Table 7 we display the case $d = m = 3$.

3. CCC VERSUS LUCAS

In this section we provide a further interesting generalization of the result in Proposition 2.2. As a simple consequence, we obtain two specific integer sequences whose associated sequence of ccc's is exactly the Lucas sequence.

Proposition 3.1: Given two complex numbers a and b, let $\{r_n\}_{n\geq 1}$ be the sequence defined as $r_1 = a$, $r_2 = b$, and $r_n = 1 - r_{n-1}^2$ for any $n \geq 3$. Moreover, let $\{s_n\}_{n\geq 1}$ be the null sequence. Then the sequence $\{\mathscr{C}_n\}_{n\geq 1}$ of ccc's relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$ is

$$\mathscr{C}_n = \begin{cases} 1+a & \text{if } n=1, \\ 1+a+a^2+b & \text{if } n=2, \\ \mathscr{C}_{n-1}+\mathscr{C}_{n-2} & \text{if } n \geq 3. \end{cases}$$

Proof: The first two values of $\{\mathscr{C}_n\}_{n\geq 1}$ are derived at once by definition of ccc. Let us compute $\mathscr{C}_n$ for $n \geq 3$. Since $\{s_n\}_{n\geq 1}$ is the null sequence, recurrence (2) reads

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} L_{n-1,\,k} \cdot r_{k+1} .$$

Now, we use recurrence (1) to expand $L_{n-1,\,k}$. We obtain

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} (L_{n-2,\,k-1} \cdot r_{k+1} + L_{n-2,\,k} \cdot r_{k+1}^2),$$

which, via the relation $r_n = 1 - r_{n-1}^2$, changes to

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} L_{n-2,\,k-1} \cdot r_{k+1} + \sum_{k=0}^{n-1} L_{n-2,\,k} - \sum_{k=0}^{n-1} L_{n-2,\,k} \cdot r_{k+2} .$$

All terms of the first and the third sum cancel out, except the two terms $L_{n-2,\,-1} \cdot r_1$ and $L_{n-2,\,n-1} \cdot r_{n+1}$ that both equal 0, as noticed in Section 1. Since the second sum coincides with $\mathscr{C}_{n-2}$, our claim follows. □

It is easy to observe that Proposition 3.1 has Proposition 2.2 as a simple consequence, up to setting $a = b = 0$. Furthermore, it enables us to immediately get our claim on Lucas numbers as a sequence of ccc's.

Corollary 3.1: Let $\{r_n\}_{n\geq 1}$ be the sequence defined as in Proposition 3.1, up to setting $a = 0$ and $b = 2$. Moreover, let $\{s_n\}_{n\geq 1}$ be the null sequence. Then the sequence $\{\mathscr{C}_n\}_{n\geq 1}$ of ccc's relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$ is the Lucas sequence.

In Table 8 we outline the sequences singled out in this corollary.

4. A FINAL REMARK

For the sake of precision, it is worth noticing that all sequences of ccc's are meant to be determined *up to translation* of the related root sequences. More precisely: if $\{\mathscr{C}_n\}_{n\geq 1}$ relates $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$, then $\{\mathscr{C}_n\}_{n\geq 1}$ also relates the translated sequences $\{r_n+\xi\}_{n\geq 1}$ and $\{s_n+\xi\}_{n\geq 1}$ for any complex number ξ. In fact, from Theorem 2.1, it is easy to verify that the connection constants relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$ are the same that relate $\{r_n+\xi\}_{n\geq 1}$ and $\{s_n+\xi\}_{n\geq 1}$.

TABLE 1. Number of Subsets as Sequences of ccc's Arising from Binomial Coefficients $\binom{n}{k}$ (Ex. (i))

n	r_n	s_n	$\binom{n}{k}$								$\mathscr{C}_n=2^n$
			$k=0$	1	2	3	4	5	6	7	
1	0	−1	1	1							2
2	0	−1	1	2	1						4
3	0	−1	1	3	3	1					8
4	0	−1	1	4	6	4	1				16
5	0	−1	1	5	10	10	5	1			32
6	0	−1	1	6	15	20	15	6	1		64
7	0	−1	1	7	21	35	35	21	7	1	128

TABLE 2. Bell Numbers $\mathscr{B}_n$ as Sequences of ccc's Arising from Stirling Numbers of the Second Kind $S(n,k)$ (Ex. (ii))

n	r_n	s_n	$S(n,k)$								$\mathscr{C}_n=\mathscr{B}_n$
			$k=0$	1	2	3	4	5	6	7	
1	0	0	0	1							1
2	1	0	0	1	3						2
3	2	0	0	1	3	1					5
4	3	0	0	1	7	6	1				15
5	4	0	0	1	15	25	10	1			52
6	5	0	0	1	31	90	65	15	1		203
7	6	0	0	1	63	301	350	140	21	1	877

TABLE 3. The Sequence of ccc's Arising from Stirling Numbers of the First Kind $s(n,k)$ (Ex. (iii))

n	r_n	s_n	$s(n,k)$								$\mathscr{C}_n$
			$k=0$	1	2	3	4	5	6	7	
1	0	0	0	1							1
2	0	1	0	−1	1						0
3	0	2	0	2	−3	1					0
4	0	3	0	−6	11	−6	1				0
5	0	4	0	24	−50	35	−10	1			0
6	0	5	0	−120	274	−225	85	−15	1		0
7	0	6	0	720	−1764	1624	−735	175	−21	1	0

TABLE 4. Factorial Numbers as Sequences of ccc's Arising from Signless Stirling Numbers of the First Kind $c(n,k)=(-1)^{n-k}\cdot s(n,k)$ (Ex. (iv))

n	r_n	s_n	$c(n,k)$								$\mathscr{C}_n = n!$
			$k=0$	1	2	3	4	5	6	7	
1	0	0	0	1							1
2	0	−1	0	1	1						2
3	0	−2	0	2	3	1					6
4	0	−3	0	6	11	6	1				24
5	0	−4	0	24	50	35	10	1			120
6	0	−5	0	120	274	225	85	15	1		720
7	0	−6	0	720	1764	1624	735	175	21	1	5040

TABLE 5. Galois Numbers $\mathscr{G}_{n,2}$ as Sequences of ccc's Arising from Gaussian Binomial Coefficients $\binom{n}{k}_2$ (Ex. (v))

n	r_n	s_n	$\binom{n}{k}_2$								$\mathscr{C}_n = \mathscr{G}_{n,2}$
			$k=0$	1	2	3	4	5	6	7	
1	1	0	1	1							2
2	2	0	1	3	1						5
3	4	0	1	7	7	1					16
4	8	0	1	15	35	15	1				67
5	16	0	1	31	155	155	31	1			374
6	32	0	1	63	651	1395	651	63	1		2825
7	64	0	1	127	2667	11811	11811	2667	127	1	29212

TABLE 6. Fibonacci Numbers $\mathscr{F}_n$ as Sequences of ccc's (Prop. 2.2)

n	r_n	s_n	$L_{n,k}$								$\mathscr{C}_n = \mathscr{F}_n$
			$k=0$	1	2	3	4	5	6	7	
1	0	0	0	1							1
2	0	0	0	0	1						1
3	1	0	0	0	1	1					2
4	0	0	0	0	1	1	1				3
5	1	0	0	0	1	1	2	1			5
6	0	0	0	0	1	1	3	2	1		8
7	1	0	0	0	1	1	4	3	3	1	13

TABLE 7. The Sequence of ccc's Arising from Prop. 2.3, for $d=m=3$

n	r_n	s_n	$L_{n,k}$								$\mathscr{C}_n$
			$k=0$	1	2	3	4	5	6	7	
1	0	0	0	1							1
2	0	0	0	0	1						1
3	1	0	0	0	1	1					2
4	0	0	0	0	1	1	1				3
5	0	0	0	0	1	1	1	1			4
6	1	0	0	0	1	1	1	2	1		6
7	0	0	0	0	1	1	1	3	2	1	9

TABLE 8. Lucas Numbers $\mathscr{L}_n$ as Sequences of ccc's (Cor. 3.1)

n	r_n	s_n	$L_{n,k}$								$\mathscr{C}_n = \mathscr{L}_n$
			$k=0$	1	2	3	4	5	6	7	
1	0	0	0	1							1
2	2	0	0	2	1						3
3	−3	0	0	4	−1	1					4
4	−8	0	0	8	7	−9	1				7
5	−63	0	0	16	−13	79	−72	1			11
6	−3968	0	0	32	55	−645	4615	−4040	1		18
7	−15745023	0	0	64	−133	5215	−291390	16035335	−15749063	1	29

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REFERENCES

1. M. Aigner. *Combinatorial Theory*. New York: Springer-Verlag, 1979.
2. L. Colucci. "Sequenze equicostanti di polinomi." Thesis, Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano, Italy, 1995 (in Italian).
3. L. Comtet. *Advanced Combinatorics*. Dordrecht: Reidel, 1974.
4. E. Damiani, O. D'Antona, & G. Naldi. "On the Connection Constants." *Studies in Applied Math.* **85** (1991):289-302.
5. E. Damiani, O. D'Antona, G. Naldi, & L. Pavarino. "Tiling Bricks with Bricks." *Studies in Applied Math.* **83** (1990):91-110.
6. R. L. Graham, D. E. Knuth, & O. Patashnik. *Concrete Mathematics*. New York: Addison Wesley, 1994.

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TWO PROOFS OF FILIPPONI'S FORMULA FOR ODD-SUBSCRIPTED LUCAS NUMBERS

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In a recent paper [1], Filipponi presented, without proof, a formula for odd-subscripted Lucas numbers which can be equivalently rewritten as

$$L_{2n+1} = \sum_{0\le j\le \frac{n}{2}} (-1)^j \binom{n-j}{j} 3^{n-1-2j} \frac{4n-5j}{n-j}. \tag{1}$$

This form is better suited for our treatment, and we can observe that, for even n, the summand for $j=\frac{n}{2}$ is $(-1)^{n/2}$, explaining the extra term in [1]. The aim of this note is to give two distinct proofs of (1).

First proof of (1). We use the standard forms

$$G_n(w) = \sum_{0\le k\le n} \binom{n-k}{k} w^k. \tag{2}$$

They are studied very well in [2], and we have, for $n\ge 0$,

$$\sum_{0\le j\le \frac{n}{2}} (-1)^j \binom{n-j}{j} 3^{n-1-2j} \frac{4n-5j}{n-j} = 3^n G_n\left(-\tfrac{1}{9}\right) + 3^{n-1} G_{n-1}\left(-\tfrac{1}{9}\right).$$

In Exercise 7.34 of [2] we find also the generating function

$$G(z,w) = \sum_{n\ge 0} G_n(w) z^n = \frac{1}{1-z-wz^2}.$$

Hence,

$$\begin{aligned}
&\sum_{n\ge 0}\left(3^n G_n\left(-\tfrac{1}{9}\right) + 3^{n-1} G_{n-1}\left(-\tfrac{1}{9}\right)\right) z^n \\
&= \sum_{n\ge 0} G_n\left(-\tfrac{1}{9}\right)(3z)^n + z\sum_{n\ge 0} G_n\left(-\tfrac{1}{9}\right)(3z)^n \\
&= \frac{1+z}{1-3z+z^2}.
\end{aligned}$$

Since a trivial computation gives

$$\sum_{n\ge 0} L_{2n+1} z^n = \frac{1+z}{1-3z+z^2},$$

the proof is finished. □

We can even get a general formula for L_{sn+t} with nonnegative integers $0\le t<s$; for that, we set up generating functions:

$$\begin{aligned}
F_{s,t}(z) &= \sum_{n\geq 0} L_{sn+t}z^n = \alpha^t\sum_{n\geq 0}(\alpha^s z)^n + \beta^t\sum_{n\geq 0}(\beta^s z)^n = \alpha^t\frac{1}{1-\alpha^s z}+\beta^t\frac{1}{1-\beta^s z}\\
&= \frac{\alpha^t(1-\beta^s z)+\beta^t(1-\alpha^s z)}{(1-\alpha^s z)(1-\beta^s z)} = \frac{\alpha^t+\beta^t-((-1)^t\alpha^{s-t}+(-1)^t\beta^{s-t})z}{1-(\alpha^s+\beta^s)z+(-1)^s z^2}\\
&= \frac{L_t-(-1)^t L_{s-t}z}{1-L_s z+(-1)^s z^2}.
\end{aligned} \tag{3}$$

By using (2), (3) can be written as

$$F_{s,t}(z) = (L_t-(-1)^t L_{s-t}z)\,G\left(L_s z, \frac{(-1)^{s-1}}{L_s^2}\right)$$

and, therefore, we get the formula:

$$\begin{aligned}
L_{sn+t} &= L_t\sum_{0\leq k\leq n}\binom{n-k}{k}(-1)^{(s-1)k}L_s^{n-2k}\\
&\quad -(-1)^t L_{s-t}\sum_{0\leq k\leq n-1}\binom{n-1-k}{k}(-1)^{(s-1)k}L_s^{n-1-2k}.
\end{aligned} \tag{4}$$

We do not know whether this formula is new, but it is easy to prove and generates infinitely many "Filipponi formulas."

Second proof of (1). For the second (mechanical) proof ("Zeilberger's algorithm"), we note the following (see [3] and [2] for the underlying theory): Set

$$f(n,k) := (-1)^k\binom{n-k}{k}3^{n-1-2k}\frac{4n-5k}{n-k} \quad \text{and} \quad F(n) := \sum_k f(n,k).$$

Furthermore, set

$$g(n,k) := -\frac{9k(n-k)(4n-5k+5)}{(n-2k+2)(n-2k+1)(4n-5k)}f(n,k),$$

then

$$f(n+2,k)-3f(n+1,k)+f(n,k) = g(n,k+1)-g(n,k)$$

(check!!), thus we get, on summing over k, $F(n+2)-3F(n+1)+F(n)=0$. Since the odd-subscripted Lucas numbers also satisfy the recursion $L_{2n+5}-3L_{2n+3}+L_{2n+1}=0$ and two initial values match as well, the proof is finished. □

REFERENCES

1. P. Filipponi. "Combinatorial Expressions for Lucas Numbers." *The Fibonacci Quarterly* **36.1** (1998):63-65.
2. R. L. Graham, D. E. Knuth, & O. Patashnik. *Concrete Mathematics*. 2nd ed. New York: Addison Wesley, 1994.
3. M. Petkovšek, H. Wilf, & D. Zeilberger. $A=B$. London: A. K. Peters, Ltd., 1996.

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GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS AND MULTIPLICATIVE ARITHMETIC FUNCTIONS

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INTRODUCTION

In [7], two families of polynomials $\{F_{k,n}\}$ and $\{G_{k,n}\}$ in k indeterminates were defined:

$$\begin{aligned} &F_{k,0}(\mathbf{t}) = 0, && G_{k,0} = k, \\ &F_{k,1}(\mathbf{t}) = 1, && G_{k,1} = t_1, \\ &F_{k,n}(\mathbf{t}) = F_{k-1,n}(\mathbf{t}), 1 \le n \le k, && G_{k,n} = G_{k-1,n}, 1 \le n < k, \\ &F_{k,n}(\mathbf{t}) = \sum_{j=1}^{k} t_j F_{k,n-j}(t), n > k, && G_{k,n} = \sum_{j=1}^{k} t_j G_{k,n-j}, n \ge k, \end{aligned}$$

where $\mathbf{t} = (t_1, \ldots, t_k)$.

There it was pointed out that these two families generalize the Fibonacci and Lucas polynomials (see [3], e.g.). In the course of [7], they arose in a natural way in the context of a subgroup of the group of multiplicative arithmetic functions (see [8], e.g.); the group operation is the convolution product. The subgroup in question is sometimes called the *rational subgroup* of the group of *multiplicative functions* (RMF) (e.g., see [1]). It is the subgroup generated, under convolution by the *completely multiplicative functions* (CMF), those multiplicative functions γ which satisfy the identity $\gamma(m)\gamma(n) = \gamma(mn)$ $\forall m, n \in \mathbb{N}$, where the product, this time, is the pointwise product. These CMF can also be described as those multiplicative functions which are completely determined by their values at primes. The RMF can be described as those multiplicative functions which are completely determined by their values on a finite number of prime powers for each prime p.

In [7], Corollary 1.3.2, it is shown that the rational group RMF is a(n uncountably generated) free abelian group.[1] The group minus the identity thus splits into two disjoint subsets, the free semigroup generated by the CMF's—call these the *positive* functions, and the set of their inverses—call these the *negative* functions. It is a consequence of the fact that elements are determined locally by their values on finitely many prime powers for each prime p, that there is associated with each pair consisting of a positive function χ and its inverse χ^{-1} a unique monic polynomial of degree k, $P_{\gamma,p}(\mathbf{t})$, $\mathbf{t} = (t_1, \ldots, t_k)$, with complex coefficients, and that k can be chosen to be the same for all primes p [7]; that is, the set of k's is bounded. Moreover, every such polynomial determines such a pair of rational MF's. An RMF determined in this way will be said to be of degree k. It is then clear that the positive functions form a graded semigroup.

[1]A consequence of this result and a result of Carroll and Gioia [2] is that the rational group is embedded in a torsion-free divisible group in the group of multiplicative arithmetic functions.

The role of the (recursive) family of polynomials $\{F_{k,n}\}$ is that, when evaluated at the coefficients $(1, a_1, \dots, a_k)$ of $P_{\gamma,p}(\mathbf{t})$, they give the values of γ at the n^{th} powers of the prime p. Thus, the set $\{F_{k,n}(\mathbf{t})\}$ yields every possible positive RMF of degree k under the evaluation map on k-tuples of complex numbers. A negative RMF (i.e., an inverse of a positive RMF) has a value 0 for all powers of p greater than k, and for powers of p less than or equal to k, the values are just the coefficients of $P_{\gamma,p}(\mathbf{t})$.

The polynomials $\{G_{k,n}(\mathbf{t})\}$ are somewhat more elusive, but are closely related to the $F_{k,n}$. When $k = 2$ and $P_{\gamma,p}(\mathbf{t}) = x^2 - tx - 1$, then $F_{k,n}$ and $G_{k,n}$ are just the Fibonacci and Lucas polynomials, respectively. In general, $\partial G_{k,n} / \partial t_1 = nF_{k,n}$. From [7], we have the following generalization of the Binet formulas which, moreover, gives relations among the roots of $P_{\gamma,p}(x; t_1, \dots, t_k)$, the values in the sequence $\gamma(p^n)$ and the polynomials $F_{k,n}(\mathbf{t})$. Thus, letting

$$\Delta_k = \Delta(\lambda_1, \dots, \lambda_k) = \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_k \\ \cdots & & \\ \lambda_1^{k-1} & \cdots & \lambda_k^{k-1} \end{vmatrix}, \quad \Delta_{k,n} = \Delta_{k,n}(\lambda_1, \dots, \lambda_k) = \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_k \\ \cdots & & \\ \lambda_1^{k-2} & \cdots & \lambda_k^{k-2} \\ \lambda_1^{n+k-2} & \cdots & \lambda_k^{n+k-2} \end{vmatrix},$$

we have that

$$\gamma(p^n) = F_{k,n+1}(t) = \frac{\Delta_{k,n}}{\Delta_k},$$

where the $\lambda_j(\mathbf{t}) = \lambda_j$ are the k roots of the polynomial $P_{\gamma,p}(x; t_1, \dots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k$. This is clearly a pleasant generalization of the Binet equations for $k = 2$; but more is true.

$$\frac{\Delta_{k,n}}{\Delta_k} = \mathrm{CSP}(k, n) = \Sigma \lambda_1^{i_j} \cdots \lambda_k^{i_k}$$

where $\Sigma i_j = n$. These are just the *complete symmetric polynomials of degree n in the* λ_i ([6], pp. 21 ff.). The $G_{k,n}$ now become transparent: $G_{k,n}(\mathbf{t}(\lambda)) = \lambda_1^n + \cdots + \lambda_k^n = \mathrm{PSP}(k, n)$. These are just the *power symmetric polynomials of degree n in the* λ_j (see [6]).

In Section 1, Theorem 2, it is shown that each $F_{k,n+1}$ can be rewritten as a sum of products of the $G_{k,j}$ with rational coefficients; this rewriting process has an inverse which rewrites each of the $G_{k,n}$ as sums of products of the $F_{k,j}$ with integer coefficients (Theorem 3). There is also a map which sends $F_{k,n+1}$ and $G_{k,n}$ to symmetric polynomials in the roots of $P_\gamma(x; \mathbf{t})$, in the first case, a $\mathrm{CSP}(k, n)$, in the second, a $\mathrm{PSP}(k, n)$.

$$\begin{array}{ccc} F_{k,n+1} & \rightarrow & \mathrm{CSP}(k,n) \\ \downarrow & & \downarrow \\ G_{k,n} & \rightarrow & \mathrm{PSP}(k,n) \end{array}$$

All of these maps are invertible. This gives an effective process for rewriting the elements of the ring Λ^n of symmetric polynomials of degree N regarded as the Z-algebra generated by CPS's as elements in Λ^n regarded as the Q-algebra generated by the PSP's, and vice versa. In this way, the $F_{k,n}$ and the $G_{k,n}$ are identified with Schur polynomials in Λ^n.

There is also a number of relations among the two sets of polynomials $F_{k,n}$ and the $G_{k,n}$ which generalize the well-known relations among the Fibonacci and Lucas polynomials (e.g., see [4], or more conveniently [10], pp. 44-46). Those which have appeared in [7] will be listed for reference merely as Result without proof.

In 1995, Glasson [5] showed that the Fibonacci and Lucas polynomials satisfy second-order partial differential equations. We generalize this result in Theorem 4, Section 2. Here we show that the partial differential equation $D_{11} - \sum t_j D_{j2} = mD_2$, where $j = 1, \ldots, k$, is satisfied by $F_{k,n}$ if $m = 2$, and by $G_{k,n}$ if $m = 1$.

In Section 3, Theorem 5, we show that the RMF's are just those arithmetic functions which are locally recursive of finite degree.

1. IDENTITIES

Result 1 ([7], Theorem 3.4): $\dfrac{\partial G_{k,n}}{\partial t_1} = nF_{k,n},\ n \geq 0.$ □

Result 2 ([7], Corollary 3.4.1): $\displaystyle\sum_{j=1}^{k} \frac{\partial G_{k,n}}{\partial t_j} = n\sum_{j=0}^{k-1} F_{k,n-j}.$ □

Result 3 ([7], Corollary 3.4.2): $\displaystyle\sum_{j=1}^{k} \frac{\partial G_{k,n}}{\partial t_j} t_j = n\sum_{j=0}^{k-1} t_j F_{k,n-j} = nF_{k,n+1}.$ □

Result 4 ([7], Corollary 2.1.3 and Theorem 3.2): If the F-polynomials and G-polynomials are regarded as functions of the zeros of the defining monic polynomial $P_\gamma(x)$ of γ, then

(a) $F_{k,n}(\mathbf{t}(\boldsymbol{\lambda})) = \sum \lambda_1^{i_1} \cdots \lambda_k^{i_k}$, where $\sum i_j = n$.

(b) $G_{k,n}(\mathbf{t}(\boldsymbol{\lambda})) = \lambda_1^n + \cdots + \lambda_k^n$,

where $\mathbf{t} = (t_1, \ldots, t_k)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$. □

Theorem 1: $\displaystyle nF_{k,n+1} = \sum_{r=1}^{n} G_{k,r} F_{k,n-r+1}.$

First we prove

Lemma 1.1: $\displaystyle G_{k,0} = k,\ G_{k,n} = F_{k,n+1} + \sum_{j=1}^{k-1} jt_{j+1}F_{k,n-j},\ n \geq 1.$

Proof of Lemma 1.1: By definition, we can write

$$\begin{aligned} G_{k,n} = \sum_{j=1}^{k} t_j G_{k,n-j} &= \sum_{j=1}^{k} t_j \left[\left(\sum_{i=1}^{k-1} it_{i+1}F_{k,n-j-i}\right) + F_{k,n-j+1}\right] \\ &= \sum_{j=1}^{k} t_j F_{k,n+1-j} + \sum_{i=1}^{k-1}\sum_{j=1}^{k} it_j t_{i+1} F_{k,n-j-i}, \end{aligned}$$

and again by definition,

$$\begin{aligned} &= F_{k,n+1} + \sum_{i=1}^{k-1} it_{i+1}\left(\sum_{j=1}^{k} t_j F_{k,(n-i)-j}\right) \\ &= F_{k,n+1} + \sum_{i=1}^{k-1} it_{i+1}F_{k,n-i}. \ \square \end{aligned}$$

Proof of Theorem 1: Again, we proceed by induction noting that the result holds when $n=1$. We need to show that $(n+1)F_{k,n+2}=\sum_{r=1}^{n+1}G_{k,r}F_{k,n-r+2}$. With the understanding that $F_{k,m}=0$ when $m\le 0$, we have

$$\begin{aligned}
\sum_{r=1}^{n+1}G_{k,r}F_{k,n-r+2} &= \sum_{r=1}^{n}G_{k,r}\sum_{j=1}^{k}t_jF_{k,n-r-j+2}+G_{k,n+1}\\
&= \sum_{j=1}^{k}t_j\sum_{r=1}^{n}G_{k,r}F_{k,n-r-j+2}+G_{k,n+1}\\
&= \sum_{j=1}^{k}t_jF_{k,n+1-j}+\sum_{i=1}^{k-1}\sum_{j=1}^{k}it_jt_{i+1}F_{k,n-j-i}\\
&= F_{k,n+1}+\sum_{i=1}^{k-1}it_{i+1}\left(\sum_{j=1}^{k}t_jF_{k,(n-i)-j}\right)\\
&= \sum_{j=1}^{k}t_j(n-j+1)F_{k,n-j+2}+G_{k,n+1}\\
\text{so by Lemma 1.1}\quad &= \sum_{j=1}^{k}t_j(n-j+1)F_{k,n-j+2}+\sum_{j=1}^{k-1}jt_{j+1}F_{k,n-j+1}+F_{k,n+2}\\
&= F_{k,n+2}+t_k(n-k+1)F_{k,n-k+2}+\sum_{j=1}^{k-1}(t_j(n-j+1)F_{k,n-j+2}+jt_{j+1}F_{k,n-j+1})\\
&= F_{k,n+2}+n\sum_{j=1}^{k}t_jF_{k,n-j+2}=F_{k,n+2}+nF_{k,n+2}=(n+1)F_{k,n+2}.\ \square
\end{aligned}$$

The following two theorems give an effective rewriting process for writing products of PSP's in terms of CSP's and vice versa, that is, they will do so once it is explained how to write the PSP's and the CSP's in terms of the F- and G-polynomials. We shall state the theorems first.

Theorem 2:

$$F_{k,n+1}=\sum_{d_i\in d}\frac{1}{z_{d_i}}G_{k,i_1}\dots G_{k,i_{s_i}},\ z_{d_i}=\prod i_j^{\nu(i_j)}\nu(i_j)!$$

where $d=\{d_1,\dots,d_s\}$ is the set of partitions of n, $\nu(i_j)=$ number of times i_j occurs in $d_i=(i_1,\dots,i_{s(d)})$, $d_i\in d, \sum_{i=1}^{s}(1/z_{d_i})=1$.

Proof: Noting that the F-polynomials, when regarded as functions of the roots of the defining polynomial $P_\gamma(x)$ are just the complete symmetric polynomials; the G-polynomials are the power symmetric polynomials (Result 4), each of which is a basis for the space of symmetric polynomials. In particular, each polynomial $F_{k,r}$ can be written uniquely as a polynomial in the G-polynomials. So if the theorem is correct, it is just a statement of this representation. Now, $F_{k,n+1}$ regarded as a polynomial in the roots $\lambda_1,\dots,\lambda_k$, is complete symmetric of degree n; hence, each monomial summand is obtained as a partition of n; so in the language of Pólya's Counting Theorem [9], we let the figure inventory consist of $\lambda_1+\cdots+\lambda_k$ and then the cycle index is given by $(1/z_{d_i})G_{k,i_1}\dots G_{k,i_{s_i}}$, where $G_{k,r}=(\lambda_1^r+\cdots+\lambda_k^r)$. Since $i_1+\cdots+i_{s_i}=n$, $(1/z_{d_i})G_{k,i_1}\dots G_{k,i_{s_i}}$ is just a monomial of total degree n and so the sum is, indeed, $F_{k,n+1}$. Here, of course, $z_{d_i}=\#$ conjugates of the element in S_n whose cycle structure is given by d_i. $\square$

Theorem 3:

$$G_{k,n} = n\sum_{d_i(n)} \frac{(-1)(l(d_i)-1)!}{\prod v_j(d_i)!} F_{k,i_2}, \ldots, F_{k,i_{s+1}},$$

using the notation of Theorem 2, and where $l(d_i)$ = length of d_i. □

The expressions in Theorems 2 and 3 are inverses of one another, which can be shown by direct computation, providing a proof of Theorem 3. Now, to get back and forth between the two sequences of polynomials, we identify the symbols t_j which appear in $G_{k,n}$ and $F_{k,n+1}$ with the elementary symmetric polynomials in the $\lambda_1, \ldots, \lambda_k$ as follows: $t_j = (-1)^{j+1}\sigma_{k,j}$, where $\sigma_{k,j} = \sigma_{k,j}(\lambda_1, \ldots, \lambda_k) =$ the j^{th} symmetric polynomial in the roots of the polynomial $P_{\gamma,p}(x; \mathbf{t})$. This identification is the basis of the proof of Lemma 4 in [7]. The substitution of the σ's for the t's yields the horizontal maps in the diagram in the introduction. The left-hand vertical arrows are just the maps implied by Theorems 1 and 2, Theorem 1 going downhill, Theorem 2 going uphill. For example, $\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2$ "is"

$$F_{2,3} = \tfrac{1}{2}G_{2,2} + \tfrac{1}{2}G_{2,1}^2 = \tfrac{1}{2}(\lambda_1^2 + \lambda_2^2) + \tfrac{1}{2}(\lambda_1 + \lambda_2)^2 = \lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2;$$

$d_1 = [2]$, $d_2 = [1, 1]$.

2. PARTIAL DIFFERENTIAL EQUATIONS

Define a differential operator by $L_{2,m} = D_{11} - t_1D_{12} - t_2D_{22} - mD_2$, $m = 1, 2$, and, more generally, $L_{k,m} = D_{11} - \Sigma_{j=1}^{k} t_jD_{j2} - mD_2$. The following theorem states that the polynomials $G_{n,k}$ and $F_{n,k}$ are solutions of second-order partial differential equations, with the exceptions of the cases for $k = 1$.

Theorem 4: ***(a)*** $L_{k,1}G_{k,n} = 0$, $k > 1$,

(b) $L_{k,2}F_{k,n} = 0$, $k > 1$.

Proof: We proceed by induction.

Lemma: ***(a)*** $L_{2,1}G_{2,n} = 0$,

(b) $L_{2,2}F_{2,n} = 0$.

(These identities were proved in [5]; however, we shall give a proof here that is self-contained using the methods of this paper.) Assuming the result for $1 < r < n+1$, we can write $G_{2,n+1} = t_1G_{2,n} + t_2G_{2,n-1}$, and thus

$$\begin{aligned} L_{2,1}(G_{2,n+1}) &= L_{2,1}(t_1G_{2,n} + t_2G_{2,n-1}) \\ &= t_1L_{2,1}(G_{2,n}) + t_2L_{2,1}(G_{2,n-1}) + 2D_1G_{2,n} - t_1D_2G_{2,n} - t_1D_1G_{2,n-1} - 2t_2D_2G_{2,n-1} - G_{2,n-1} \\ &= 2nF_{2,n} - (2n-1)t_1F_{2,n-1} - 2(n-1)t_2F_{n-2} - F_{2,n} - t_2F_{2,n-2} \\ &= (2n-1)F_{2,n} - (2n-1)F_{2,n} = 0, \end{aligned}$$

equalities which follow from the induction hypothesis, definition of the F- and G- polynomials, and Result 1 and Lemma 1.1. □

The proof of part (b) follows from (a) and Result 1 as follows.

$$\begin{aligned} L_{2,2}(F_{2,n}) &= (D_{11} - t_1D_{12} - t_2D_{22} - 2D_2)F_{2,n} \\ &= (1/n)(D_{11} - t_1D_{12} - t_2D_{22} - 2D_2)D_1G_{2,n} \\ &= 1/n\, D_1L_{2,1}G_{2,n} + (D_{21} - D_{12})G_{2,n} = 0, \end{aligned}$$

using part (a). □

To complete the proof of the theorem, we assume that the result of the theorem holds for all $G_{s,n}$ for which $1 \le s \le k-1$, and note that $G_{k,n} = G_{k-1,n}$ for $1 \le n \le k$. Assume the result for $G_{k,s}$ for $1 \le s \le n$, and consider $L_{k,1}G_{k,n+1} = LG_{k,n+1}$, $n \ge k$. A straightforward, but rather tedious, computation, as in the proof of the lemma, using the inductive definition of $G_{k,n+1}$, which takes hold for this range of n's, and again using Result 1 and Lemma 1.1, and Theorems 2 and 3, we complete the induction. Part (b) now follows by a similar argument. □

3. CONCLUDING REMARKS

Theorem 5: Given the recursion $u_{j+1} = a_1u_j + \cdots + a_ku_{j-k}$ with $u_0 = 0, u_1 = 1$, then

$$u_{j+1} = F_{k,j+1}(\mathbf{a}).$$

Proof: The theorem follows by induction and the definition. □

Notice that this result can be applied to any linear recursion formula, for if the coefficient of u_{j+1} is any nonzero (complex) number, we can divide through by it and apply the theorem.

We define a sequence to be *locally linearly recursive* of degree k if at each prime p the prime powers of the sequence are given by a linear recursive relation involving k independent unknowns, the same k for each prime p.

Corollary 5.1: A sequence is locally linearly recursive of degree k if and only if when regarded as an arithmetic function, it belongs to the positive semigroup of the group of rational multiplicative functions. □

We define a positive rational multiplicative sequence to be *uniform* if at each prime it is determined by the same polynomial, $P_{\gamma,p}(x) = P_{\gamma,p'}(x)$ for all primes p and p'. It is clear that the uniform sequences form a sub-semigroup of the semigroup of positive rational functions. It is also clear from the above corollary that

Corollary 5.2: A sequence is linearly recursive of degree k if and only if it is, as an arithmetic function, uniform. □

Here, *linear recursive of degree k* has the obvious meaning; the same relation holds for all primes.

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REFERENCES

1. T. B. Carroll and A. A. Gioia. "On a Subgroup of the Group of Multiplicative Arithmetic Functions." *J. Austr. Math. Soc.* **20** (Series A) (1974):348-52.
2. T. B. Carroll and A. A. Gioia. "Roots of Multiplicative Functions." *Compositio Mathematica* **65** (1988):349-58.
3. B. G. S. Doman & J. K. Williams. "Fibonacci and Lucas Polynomials." *Mathematical Proceedings of the Cambridge Philosophical Society* **90**, Part 3 (1981):385-87.
4. Günther Frei. "Binary Lucas and Fibonacci Polynomials, I." *Mathematisches Nachrichten* **96** (1980):83-112.
5. Alan R. Glasson. "Remainder Formulas Involving Generalized Fibonacci and Lucas Polynomials." *The Fibonacci Quarterly* **33.3** (1995):268-72.
6. I. G. MacDonald. *Symmetric Functions and Hall Polynomials.* Oxford Science Publications. Oxford: Clarendon Press, 1995.
7. T. MacHenry. "A Subgroup of the Group of Units in the Ring of Arithmetic Functions." *Rocky Mountain Journal of Mathematics* **29** (1999):1055-65.
8. P. J. McCarthy. *Introduction to Arithmetical Functions.* New York: Springer-Verlag, 1985.
9. G. Pólya. "Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen." *Acta Math.* **68** (1937):145-254.
10. P. Ribenboim. *The Book of Prime Number Records.* New York: Springer-Verlag, 1989.

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A GEOMETRIC CONNECTION BETWEEN GENERALIZED FIBONACCI SEQUENCES AND NEARLY GOLDEN SECTIONS

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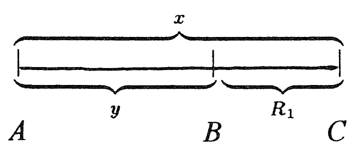
1. INTRODUCTION

We introduce a way to construct generalized golden sections, and demonstrate a geometric connection between these sections and generalized Fibonacci sequences of the form $u_{n+1} = k \cdot u_n + u_{n-1}$, where $u_0 = 0$, $u_1 = 1$. for $k \geq 1$. We let $\phi = (1+\sqrt{5})/2$, the golden ratio, and $F_n^{(k)}$ represent the n^{th} term of the k^{th} generalized Fibonacci sequence, defined above. Our method will employ a geometric version of the Euclidean Algorithm.

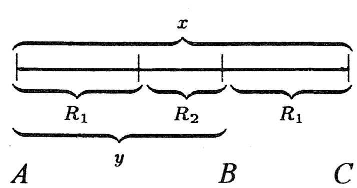
For $k = 1$, the key fact is that if two line segments with lengths x and y satisfy $x/y = \phi$, then $x = y + R_1$, where $R_1 < y$ and y/R_1 is itself equal to ϕ. This follows from the definition of the golden section. See Figure 1 and the mathematical argument given in [3, pp. 9-10].



$$\frac{y}{R_1} = \frac{AB}{BC} = \frac{AC}{AB} = \frac{x}{y} = \phi$$

FIGURE 1

Since $x = y + R_1$, and $R_1 < y$, we can approximate x (badly) by ignoring the remainder R_1, and estimate $x/y = (y+R_1)/y \approx 1$. To refine this estimate, we should use a smaller unit with which to measure. Hence, we now choose R_1. This is shown in Figure 2.



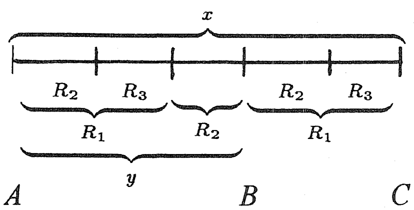
$$\frac{R_1}{R_2} = \frac{y}{R_1} = \frac{x}{y} = \phi$$

FIGURE 2

From Figure 2, a new estimate of x/y, ignoring the remainder R_2, is

$$\frac{x}{y} = \frac{2R_1 + R_2}{R_1 + R_2} \approx 2.$$

If we now lay off R_2 against each R_1, we have the construct in Figure 3.



$$\frac{R_2}{R_3} = \frac{R_1}{R_2} = \frac{y}{R_1} = \frac{x}{y} = \phi$$

FIGURE 3

From Figure 3, a new estimate, this time ignoring the remainder R_3, is

$$\frac{x}{y} = \frac{3R_2 + 2R_3}{2R_2 + R_3} \approx 3/2.$$

If we continue this process, it is easy to see and to prove by induction that

$$\phi = \frac{x}{y} = \frac{F_{n+2} \cdot R_n + F_{n+1} \cdot R_{n+1}}{F_{n+1} \cdot R_n + F_n \cdot R_{n+1}} \approx \frac{F_{n+2}}{F_{n+1}}.$$

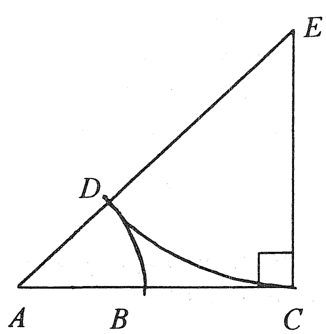
This gives a geometric flavor to the well-known identity

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi.$$

2. NEARLY GOLDEN SECTIONS

We generalize the golden section in a manner not entirely unlike Philip Engstrom's generalization in [2]. We do so by giving a ruler and compass method of locating point B, between A and C, as in Figure 4 below.

Let k be a fixed positive integer. Given a line segment $\overline{AC}$, first bisect the segment. Construct a perpendicular $\overline{EC}$ at point C of length $\frac{k}{2} \cdot AC$. By striking arcs, locate points D and B, as shown in Figure 4, so that $DE = CE$ and $AB = AD$.



$$\overline{EC} \perp \overline{AC}$$
$$EC = \tfrac{k}{2} \cdot AC$$
$$DE = CE,\ AB = AD$$

FIGURE 4

By the Pythagorean Theorem,

$$AE = \sqrt{(AC)^2 + (EC)^2} = \sqrt{(AC)^2 + \frac{k^2}{4}(AC)^2} = \frac{1}{2}\sqrt{k^2+4} \cdot AC.$$

So we have

$$\begin{aligned} AB = AD &= AE - DE = AE - CE \\ &= \frac{1}{2}\sqrt{k^2+4} \cdot AC - \frac{k}{2} \cdot AC = \frac{-k+\sqrt{k^2+4}}{2} \cdot AC. \end{aligned}$$

It follows that

$$\frac{AC}{AB} = \frac{2}{-k+\sqrt{k^2+4}} = \frac{k+\sqrt{k^2+4}}{2}.$$

We shall call this ratio ϕ_k, the k^{th} *generalized golden ratio*. That is,

$$\phi_k = \frac{k+\sqrt{k^2+4}}{2}$$

Letting $t_k = -1/\phi_k$, the other root of the equation $t^2 - kt - 1 = 0$, it is now a simple exercise to follow the reasoning in Hoggatt's book [3, pp. 10-11], to establish the Binet form

$$F_n^{(k)} = \frac{\phi_k^n - t_k^n}{\phi_k - t_k}.$$

Using the notation of Horadam [4, p. 161], $F_n^{(k)} = w(0, 1; k, -1)$, a generalized Fibonacci sequence of the form mentioned in the introduction. In [4] and a large number of other articles appearing in this journal, one can find many formulas for the sequences $F_n^{(k)}$ and the related generalized Lucas sequences given by $L_n^{(k)} = \phi_k^n + t_k^n$. However, one formula we did not find is $(\phi_{2m+1} - m)^2 = (m^2+1) + \phi_{2m+1}$. This formula is easy to prove by using the formula for the value of ϕ_k given above. This identity implies that, for odd k, the decimal part of ϕ_k is the decimal part of a number which differs from its square by a positive integer. The table below gives some examples to illustrate this.

TABLE 1. A Squaring Property

m	$\phi_{2m+1} - m$	$(\phi_{2m+1} - m)^2$
0	1.6180339887...	2.6180339887...
1	2.3027756377...	5.3027756377...
2	3.1925824036...	10.1925824036...
3	4.1400549446...	17.1400549446...

3. THE GEOMETRIC CONNECTION FOR GENERALIZED FIBONACCI SEQUENCES

We now use the construction of Section 2 to emulate the geometric process of Section 1 for approximating ϕ_k for $k \geq 2$. The goal is to demonstrate a geometric connection, similar to the one shown in Section 1, between ratios of generalized Fibonacci numbers and generalized golden ratios.

Definition: To form the k^{th} *nearly golden section*, cut a line segment into $k+1$ pieces such that

1. k of the pieces have equal length,
2. the remaining piece is shorter than the first k pieces, and
3. the ratio of the length of a single larger piece to the smaller piece is equal to the length of the whole segment to that of the larger piece.

The construction of Section 2 tells us how to cut a line segment in this way. A few comments are in order.

With lengths as described in Figure 4, we have

$$\frac{AC-k\cdot AB}{AB}=\frac{AC}{AB}-k=\frac{-k+\sqrt{k^2+4}}{2},$$

and so,

$$\frac{AB}{AC-k\cdot AB}=\frac{2}{-k+\sqrt{k^2+4}}=\frac{k+\sqrt{k^2+4}}{2}=\frac{AC}{AB}=\phi_k.$$

From this calculation, we deduce first that when $AC/AB=\phi_k$, as in the construction, then $k\cdot AB<AC$. So, by duplicating the length AB an additional $k-1$ times on the segment $\overline{AC}$, beginning at point B, we can cut the line segment $\overline{AC}$ in the manner illustrated for $k=2$ and $k=3$ below. (These are generalizations of the cut made in Figure 1.)

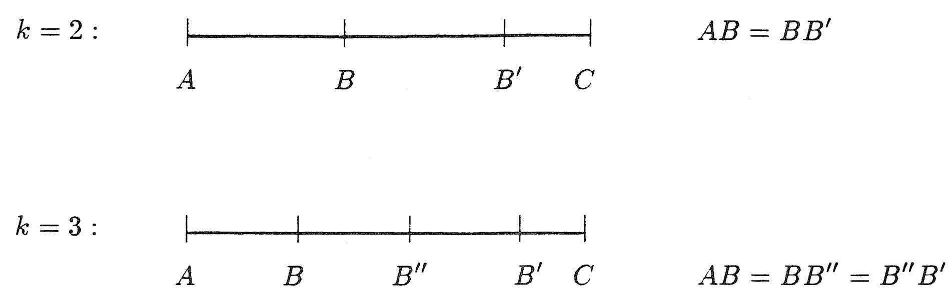


FIGURE 5

Said another way, if A, B, and C are as in Figure 4, with $AC/AB=\phi_k$, then $AC=k\cdot AB+B'C$ (as in Figure 5). Moreover,

$$\frac{AB}{B'C}=\frac{AC}{AB}=\phi_k.$$

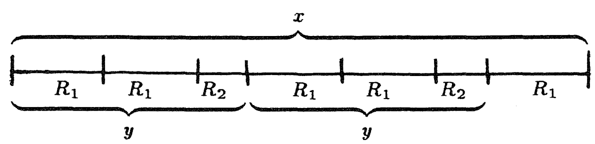
These facts allow us to emulate the geometric process we described in the introduction.

The key fact now, obtained from the preceding discussion, is that if two line segments with lengths x and y satisfy $x/y=\phi_k$ then $x=k\cdot y+R_1$, where $R_1<y$ and $y/R_1=\phi_k$. (In the definition, x is the length of the original segment, y that of one of the larger pieces, and R_1 that of the shorter piece. In Figure 5, $x=AC$, $y=AB$, and $R_1=B'C=AC-k\cdot AB$.) Thus, geometrically, y can be laid off k times against x, with a remainder of length R_1, and the ratio y/R_1 is the same as the original ratio x/y. This means that now R_1 can be laid off k times against *each* y, with remainder $R_2=y-k\cdot R_1$, and $R_1/R_2=y/R_1=x/y=\phi_k$. This process can be repeated indefinitely.

We now estimate ϕ_k. Our first estimate (ignoring the remainder R_1) is

$$\phi_k = \frac{x}{y} = \frac{k \cdot y + R_1}{y} = \frac{F_2^{(k)} \cdot y + F_1^{(k)} \cdot R_1}{F_1^{(k)} \cdot y + F_0^{(k)}} \approx \frac{F_2^{(k)}}{F_1^{(k)}}.$$

Now, as we said above, y / R_1 is also equal to ϕ_k. So $y = k \cdot R_1 + R_2$, where $R_2 < R_1$ and $R_1 / R_2 = y / R_1 = \phi_k$. We can lay off R_1 k times against *each* y. We illustrate for $k = 2$ in Figure 6 below.



$$\frac{y}{R_1} = \frac{x}{y} = \phi_2$$

FIGURE 6

By substitution, we have

$$x = k \cdot y + R_1 = k(k \cdot R_1 + R_2) + R_1 = (k^2 + 1)R_1 + k \cdot R_2,$$
$$y = k \cdot R_1 + R_2.$$

Since $F_1^{(k)} = 1$, $F_2^{(k)} = k$, and $F_3^{(k)} = k^2 + 1$, we may write

$$x = F_3^{(k)} \cdot R_1 + F_2^{(k)} \cdot R_2,$$
$$y = F_2^{(k)} \cdot R_1 + F_1^{(k)} \cdot R_2.$$

So our second estimate, this time ignoring the remainder R_2, is

$$\phi_k = \frac{x}{y} = \frac{F_3^{(k)} \cdot R_1 + F_2^{(k)} \cdot R_2}{F_2^{(k)} \cdot R_1 + F_1^{(k)} \cdot R_2} \approx \frac{F_3^{(k)}}{F_2^{(k)}}.$$

These are the first steps of an iterative process in which, at each step, R_n is laid off k times against *each* R_{n-1} (since $R_{n-1} = k \cdot R_n + R_{n+1}$), and

$$\frac{x}{y} = \frac{y}{R_1} = \frac{R_1}{R_2} = \cdots = \frac{R_n}{R_{n+1}}.$$

At the n^{th} step we have

$$\begin{aligned} x &= a_{n+1} \cdot R_{n-1} + a_n \cdot R_n, \\ y &= a_n \cdot R_{n-1} + a_{n-1} \cdot R_n. \end{aligned} \tag{1}$$

By substitution into (1), since $R_{n-1} = k \cdot R_n + R_{n+1}$, we have

$$\begin{aligned} x &= a_{n+1}(k \cdot R_n + R_{n+1}) + a_n \cdot R_n \\ &= \underbrace{(k \cdot a_{n+1} + a_n)}_{a_{n+2}} R_n + a_{n+1} \cdot R_{n+1}, \end{aligned}$$

$$\begin{aligned} y &= a_n(k \cdot R_n + R_{n+1}) + a_{n-1} \cdot R_n \\ &= \underbrace{(k \cdot a_n + a_{n-1})}_{a_{n+1}} R_n + a_n \cdot R_{n+1}. \end{aligned}$$

We see that the sequence a_n is defined by the rule $a_{n+2} = k \cdot a_{n+1} + a_n$ for all $n \geq 1$. That is, $a_n = F_n^{(k)}$, and

$$\phi_k = \frac{x}{y} = \frac{F_{n+2}^{(k)} \cdot R_n + F_{n+1}^{(k)} \cdot R_{n+1}}{F_{n+1}^{(k)} \cdot R_n + F_n^{(k)} \cdot R_{n+1}} \approx \frac{F_{n+2}^{(k)}}{F_{n+1}^{(k)}}.$$

This is the desired generalization of the geometric approximation in the introduction.

ACKNOWLEDGMENT

The author appreciates the patience and advice of the anonymous referee whose comments and suggestions contributed largely to improving the form and presentation of this article.

REFERENCES

1. Carl B. Boyer. *A History of Mathematics*. Princeton: Princeton University Press, 1985.
2. Philip G. Engstrom. "Sections, Golden and Not So Golden." *The Fibonacci Quarterly* **26.4** (1988):118-27.
3. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1979.
4. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.3** (1965):161-76.
5. N. N. Vorobyov. *The Fibonacci Numbers*. (English translation.) Boston: Heath, 1951.

AMS Classification Number: 11B39

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A MESSAGE OF GRATITUDE TO DR. STANLEY RABINOWITZ

The Editor, Editorial Board, and Board of Directors of The Fibonacci Association wish to express their deep gratitude to Dr. Stanley Rabinowitz for his excellent work as Editor of the Elementary Problems and Solutions section of *The Fibonacci Quarterly*. Our best wishes go with him as he retires from this position after nine years to devote full time to his publishing enterprise, MathPro Press.

Announcement

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CALL FOR PAPERS

Papers on all branches of mathematics and science related to the Fibonacci numbers, number theoretic facts as well as recurrences and their generalizations are welcome. Abstracts, which should be sent in duplicate to F. T. Howard at the address below, are due by June 1, 2000. An abstract should be at most one page in length (preferably half a page) and should contain the author's name and address. New results are especially desirable; however, abstracts on work in progress or results already accepted for publication will be considered. Manuscripts should *not* be submitted. Questions about the conference should be directed to:

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Missouri State University, 800 University Drive, Maryville, MO 64468.

Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column. Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-895 ***Proposed by Indulis Strazdins, Riga Tech. Univ., Latvia***

Find a recurrence for F_{n^2}.

B-896 ***Proposed by Andrew Cusumano, Great Neck, NY***

Find an integer k such that the expression $F_n^4 + 2F_n^3F_{n+1} + kF_n^2F_{n+1}^2 - 2F_nF_{n+1}^3 + F_{n+1}^4$ is a constant independent of n.

B-897 ***Proposed by Brian D. Beasley, Presbyterian College, Clinton, SC***

Define $\langle a_n \rangle$ by $a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n$ for $n \geq 0$ with initial conditions $a_0 = 4$, $a_1 = 2$, and $a_2 = 10$. Express a_n in terms of Fibonacci and/or Lucas numbers.

B-898 ***Proposed by Alexandru Lupaş, Sibiu, Romania***

Evaluate

$$\sum_{k=0}^{s} (-1)^{(n-1)(s-k)} \binom{2s+1}{s-k} F_{n(2k+1)}.$$

B-899 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY***

In a sequence of coin tosses, a *single* is a term (H or T) that is not the same as any adjacent term. For example, in the sequence HHTHHHTH, the singles are the terms in positions 3, 7, and 8. Let $S(n, r)$ be the number of sequences of n coin tosses that contain exactly r singles. If $n \geq 0$ and p is a prime, find the value modulo p of $\frac{1}{2}S(n+p-1, p-1)$.

SOLUTIONS

A Recurrence for nF_n

B-879 ***Proposed by Mario DeNobili, Vaduz, Lichtenstein (Vol. 37, no. 3, August 1999)***

Let $\langle c_n \rangle$ be defined by the recurrence $c_{n+4} = 2c_{n+3} + c_{n+2} - 2c_{n+1} - c_n$ with initial conditions $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, and $c_3 = 6$. Express c_n in terms of Fibonacci and/or Lucas numbers.

Solution by H.-J. Seiffert, Berlin, Germany

Let $\langle c_n \rangle$ satisfy the given recurrence, but with initial conditions $c_0 = 0$, $c_1 = a$, $c_2 = 2b$, and $c_3 = 3(a+b)$, where a and b are any fixed numbers. We shall prove that $c_n = nH_n$, where the sequence $\langle H_n \rangle$ satisfies the recurrence $H_{n+2} = H_{n+1} + H_n$ with initial values $H_1 = a$ and $H_2 = b$. Direct computation shows that the equation $c_n = nH_n$ holds for $n = 0, 1, 2$, and 3. Suppose that it is true for all $k \in \{0, 1, \ldots, n+3\}$, where $n \geq 0$. We then have

$$\begin{aligned} c_{n+4} &= 2c_{n+3} + c_{n+2} - 2c_{n+1} - c_n \\ &= 2(n+3)H_{n+3} + (n+2)H_{n+2} - 2(n+1)H_{n+1} - nH_n \\ &= (n+4)(H_{n+3} + H_{n+2}) + n(H_{n+3} - 2H_{n+1} - H_n) + 2(H_{n+3} - H_{n+2} - H_{n+1}) \\ &= (n+4)H_{n+4}; \end{aligned}$$

note that $H_{n+3} = H_{n+2} + H_{n+1} = 2H_{n+1} + H_n$. This completes the induction proof. It can be shown that this equation holds for negative n as well.

To solve the present proposal, take $a = b = 1$. Then $c_n = nF_n$. With $a = 1$ and $b = 3$, we have $\langle H_n \rangle = \langle L_n \rangle$, and therefore, $c_n = nL_n$.

Solutions also received by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, N. Gauthier, Joe Howard, Harris Kwong, James A. Sellers, Indulis Strazdins, and the proposer.

A Sum for F_{2m+2}

B-880 ***Proposed by A. J. Stam, Winsum, The Netherlands (Vol. 37, no. 3, August 1999)***

Express

$$\sum_{2i \leq m} \binom{m-i}{i} (-1)^i 3^{m-2i}$$

in terms of Fibonacci and/or Lucas numbers.

Solution by Harris Kwong, SUNY College at Fredonia, NY

Denote the given sum by s_m. The generating function of the sequence $\langle s_m \rangle$ is

$$\begin{aligned} \sum_{m=0}^{\infty} s_m z^m &= \sum_{m=0}^{\infty} \sum_{2i \leq m} \binom{m-i}{i} (-1)^i 3^{m-2i} z^m = \sum_{k=0}^{\infty} \sum_{2i \leq k+i} \binom{k}{i} (-1)^i 3^{k-i} z^{k+i} \\ &= \sum_{k=0}^{\infty} \sum_{i \leq k} \binom{k}{i} (-1)^i (3z)^{k-i} z^{2i} = \sum_{k=0}^{\infty} (3z - z^2)^k = \frac{1}{1 - 3z + z^2}. \end{aligned}$$

Since $1 - 3x^2 + x^4 = (1 - x^2)^2 - x^2$, we find

$$\sum_{m=0}^{\infty} s_m x^{2m+2} = \frac{x^2}{(1-x^2)^2 - x^2} = \frac{1}{2}\left\{\frac{x}{1-x-x^2} - \frac{x}{1+x-x^2}\right\}$$
$$= \frac{1}{2}\left\{\sum_{n=0}^{\infty} F_n x^n + \sum_{n=0}^{\infty} F_n(-x)^n\right\} = \sum_{n=0}^{\infty} F_{2n} x^{2n}.$$

Therefore, $s_m = F_{2m+2}$.

Redmond showed that

$$\sum_{2i \le m} \binom{m-i}{i} (-1)^i (2x)^{m-2i} = U_m(x),$$

where $U_m(x)$ is the m^{th} *Chebyshev polynomial of the second kind.*

Cook noted that the problem is a slight variation of a problem posed by Mrs. William Squire in [1], solved by M. N. S. Swamy in [2], and further explored by H. W. Gould in [3].

References

1. Mrs. W. Squire. "Problem H-83." *The Fibonacci Quarterly* **4.1** (1966):57.
2. M. N. S. Swamy. "Solution of H-83." *The Fibonacci Quarterly* **6.1** (1968):54-55.
3. H. W. Gould. "A Fibonacci Formula of Lucas and Its Subsequent Manifestations and Rediscoveries." *The Fibonacci Quarterly* **15.1** (1977):25-29.

Seiffert reported that the identities

$$\sum_{2i \le m} \binom{m-i}{i} (-xy)^i (x+y)^{m-2i} = \frac{x^{m+1} - y^{m+1}}{x-y}, \; m \ge 0,$$

and

$$\sum_{2i \le m} \frac{m}{m-i} \binom{m-i}{i} (-xy)^i (x+y)^{m-2i} = x^m + y^m, \; m \ge 1,$$

are due to E. Lucas (Théorie des Nombres [Paris: Blanchard, 1961], Ch. 18).

Solutions also received by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Don Redmond, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Diophantine Pair

B-881 ***Proposed by Mohammad K. Azarian, University of Evansville, IN (Vol. 37, no. 3, August 1999)***

Consider the two equations

$$\sum_{i=1}^{n} L_i x_i = F_{n+3} \quad \text{and} \quad \sum_{i=1}^{n} L_i y_i = L_3 - F_{n+1}.$$

Show that the number of positive integer solutions of the first equation is equal to the number of nonnegative integer solutions of the second equation.

Solution by L. A. G. Dresel, Reading, England

We have $(L_1 + L_2 + \cdots + L_n) = \Sigma(L_{j+2} - L_{j+1}) = L_{n+2} - L_2$. Subtracting this from the first of the given equations and using the identity $L_{n+2} = F_{n+3} + F_{n+1}$, we obtain $\Sigma L_i(x_i - 1) = L_2 - F_{n+1}$. Putting $y_i = x_i - 1$, it follows that y_i is a nonnegative integer whenever x_i is a positive integer. Similarly, starting with the second of the given equations, we can obtain the first given equation.

Thus, the two given equations are equivalent and have the same number of solutions of the specified kinds.

Generalization by Harris Kwong, SUNY College at Fredonia, NY

We give a generalization. Let $\langle u_n \rangle$ be a sequence that satisfies the recurrence $u_n = u_{n-1} + u_{n-2}$. It can be proved, for instance, by induction, that $\sum_{i=1}^{n} u_i = u_{n+2} - u_2$. Let $\langle v_n \rangle$ be another sequence. Consider the equations

$$\sum_{i=1}^{n} u_i x_i = v_{n+3} \quad \text{and} \quad \sum_{i=1}^{n} u_i y_i = v_{n+3} - u_{n+2} + u_2 .$$

Every positive integer solution $(a_1, a_2, \ldots, a_n)$ of the first equation yields a nonnegative solution $(a_1 - 1, a_2 - 1, \ldots, a_n - 1)$ of the second equation. Conversely, any nonnegative solution $(b_1, b_2, \ldots, b_n)$ of the second equation leads to a positive solution $(b_1 + 1, b_2 + 1, \ldots, b_n + 1)$ of the first equation. Therefore, the positive solutions of the first equation and the nonnegative solutions of the second equation are in one-to-one correspondence. In particular, when $u_n = L_n$ and $v_n = F_n$, the two equations reduce to the ones in the problem statement, because $L_{n+2} = F_{n+3} + F_{n+1}$.

Solutions also received by Paul S. Bruckman, H.-J. Seiffert, Indulis Strazdins, and the proposer.

A Multiple of F_{n+1}

B-882 *Proposed by A. J. Stam, Winsum, The Netherlands (Vol. 37, no. 3, August 1999)*

Suppose the sequence $\langle A_n \rangle$ satisfies the recurrence $A_n = A_{n-1} + A_{n-2}$. Let

$$B_n = \sum_{k=0}^{n} (-1)^k A_{n-2k} .$$

Prove that $B_n = A_0 F_{n+1}$ for all nonnegative integers n.

Solution by H.-J. Seiffert, Berlin, Germany

Let x be any complex number and suppose that the sequence $\langle A_n(x) \rangle$ satisfies the recurrence $A_n(x) = xA_{n-1}(x) + A_{n-2}(x)$. Define

$$B_n(x) = \sum_{k=0}^{n} (-1)^k A_{n-2k}(x).$$

We shall prove that $B_n(x) = A_0(x) F_{n+1}(x)$ for all nonnegative integers n, where $\langle F_n(x) \rangle$ denotes the sequence of Fibonacci polynomials which satisfies the same recurrence as $\langle A_n(x) \rangle$, but with given initial conditions $F_0(x) = 0$ and $F_1(x) = 1$.

If $n \geq 2$, then

$$\begin{aligned} B_n(x) &= x \sum_{k=0}^{n} (-1)^k A_{n-1-2k}(x) + \sum_{k=0}^{n} (-1)^k A_{n-2-2k}(x) \\ &= x \sum_{k=0}^{n-1} (-1)^k A_{n-1-2k}(x) + \sum_{k=0}^{n-2} (-1)^k A_{n-2-2k}(x) \\ &\quad + (-1)^n (xA_{-n-1}(x) + A_{-n-2}(x) - A_{-n}(x)) \end{aligned}$$

or $B_n(x) = xB_{n-1}(x) + B_{n-2}(x)$. Since $B_0(x) = A_0(x) = A_0(x)F_1(x)$ and $B_1(x) = A_1(x) - A_{-1}(x) = xA_0(x) = A_0(x)F_2(x)$, the desired equation now follows by a simple induction argument.

The proposal's result is obtained when taking $x = 1$.

Solutions also received by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, N. Gauthier, Pentti Haukkanen, Joe Howard, Harris Kwong, Don Redmond, James A. Sellers, Indulis Strazdins, and the proposer.

Property of a Periodic Sequence

B-883 *Proposed by L. A. G. Dresel, Reading, England (Vol. 37, no. 3, August 1999)*

Let $\langle f_n \rangle$ be the Fibonacci sequence F_n modulo p, where p is a prime, so that we have $f_n \equiv F_n \pmod{p}$ and $0 \le f_n < p$ for all $n \ge 0$. The sequence $\langle f_n \rangle$ is known to be periodic. Prove that, for a given integer c in the range $0 \le c < p$, there can be at most four values of n for which $f_n = c$ within any one cycle of this period.

Solution by the proposer

From the identities $F_{n+1} + F_{n-1} = L_n$ and $F_{n+1} - F_{n-1} = F_n$, we obtain $2F_{n+1} = L_n + F_n$, and we also have $L_n^2 = 5(F_n)^2 + 4(-1)^n$. We shall assume first that $p \neq 2$, and that $F_n \equiv c \pmod{p}$ for some even value of n. Then it follows that $L_n \equiv \pm\sqrt{(5c^2+4)}$ and $2F_{n+1} \equiv c \pm \sqrt{(5c^2+4)} \pmod{p}$; this gives two possible values for f_{n+1}, say s_1 and s_2. It is possible that we also have $f_n \equiv c$ occurring for some odd value of n, so that we have $f_{n+1} \equiv c \pm \sqrt{(5c^2-4)} \pmod{p}$, giving two further possible values s_3 and s_4, say. These values may not all be distinct, but clearly there are at most four different values of s which can follow c in the sequence $\langle f \rangle$. But if a given consecutive pair of values c, s were to occur a second time, the sequence $\langle f \rangle$ would repeat itself because of the recurrence relation. Hence, the value c can occur at most four times in $\langle f \rangle$ within one cycle of the period, namely, at most twice for an even value of n and at most twice for an odd value of n.

For the special case $p = 2$, we see that a complete cycle is $\langle f \rangle \equiv 0, 1, 1 \pmod{2}$.

Corollary: Since there are only p values of c in the range $0 \le c \le p-1$, it follows that the period $K(p)$ of the sequence F_n modulo p satisfies $K(p) \le 4p$. In the special case $p = 5$, we do in fact obtain $K(5) = 20$.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-561 ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada***

Let n be an integer and set

$$s_{n+1} = \alpha^n + \alpha^{n-1}\beta + \cdots + \alpha\beta^{n-1} + \beta^n,$$

where $\alpha + \beta = a$, $\alpha\beta = b$, with $a \neq 0$, $b \neq 0$ two arbitrary parameters. Then prove that:

a) $$s_p^r s_{qr+n} = \sum_{\ell=0}^{r} \binom{r}{\ell} b^{q(r-\ell)} s_q^\ell s_{p-q}^{r-\ell} s_{p\ell+n};$$

b) $$b^{pr} s_q^r s_n = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} s_p^\ell s_{q+p}^{r-\ell} s_{q\ell+pr+n};$$

c) $$s_{2p+q}^r s_{qr+n} = \sum_{\ell=0}^{r} \binom{r}{\ell} b^{(p+q)(r-\ell)} s_p^{r-\ell} s_{p+q}^\ell s_{(2p+q)\ell-pr+n};$$

where $r \geq 0$, n, $p(\neq 0)$, and $q(\neq 0, \pm p)$ are arbitrary integers.

H-562 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Show that, for all nonnegative integers n,

$$L_{2n+1} = 4^n - 5\sum_{k=0}^{[\frac{n-2}{5}]} \binom{2n+1}{n-5k-2},$$

where [ ] denotes the greatest integer function.

H-563 ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada***

Let $m > 0$, $n \geq 0$, $p \neq 0$, $q \neq -p, 0$, and s be integers and, for $1 \leq k \leq n$, let $(n)_k := n(n-1) \ldots (n-k+1)$ and $S_m^{(k)}$ be a Stirling number of the second kind.

Prove the following identity for Fibonacci numbers:

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} r^m [F_p / F_{p+q}]^r F_{qr+s}$$

$$= (-1)^{np} [F_q / F_{p+q}]^n \sum_{k=1}^{m} (-1)^{(p+1)k} (n)_k S_m^{(k)} [F_p / F_q]^k F_{(p+q)k-np+s}.$$

SOLUTIONS

An Odd Problem

H-545 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 36, no. 5, November 1998)

Prove that, for all odd primes p,

(a) $\sum_{k=1}^{p-1} L_k \cdot k^{-1} \equiv \frac{-2}{p}(L_p - 1) \pmod p$;

(b) $\sum_{k=1}^{p-1} F_k \cdot k^{-1} \equiv 0 \pmod p$.

Solution by the proposer

We first observe that $L_p \equiv 1 \pmod p$ for all primes p; thus, all the expressions indicated in (a) and (b) are well-defined integers (mod p). Now

$$\alpha^p = (1-\beta)^p = \sum_{k=0}^{p}\binom{p}{k}(-\beta)^k = 1 - \beta^p + \sum_{k=1}^{p-1}\frac{p}{k}\binom{p-1}{k-1}(-\beta)^k.$$

Hence,

$$\frac{1}{p}(L_p - 1) = \sum_{k=1}^{p-1}\frac{1}{k}\binom{p-1}{k-1}(-\beta)^k.$$

Now

$$\binom{p-1}{k-1} \equiv \binom{-1}{k-1} = (-1)^{k-1} \pmod p.$$

Thus,

$$\frac{1}{p}(L_p - 1) \equiv -\sum_{k=1}^{p-1} k^{-1} \cdot \beta^k \pmod p.$$

Similarly, it is also true that

$$\frac{1}{p}(L_p - 1) \equiv -\sum_{k=1}^{p-1} k^{-1} \cdot \alpha^k \pmod p.$$

Adding and subtracting the last two congruences yields (a) and (b), respectively.

Note: From (a) and (b), it follows that a necessary and sufficient condition for $p^2 \mid (L_p - 1)$ is that

$$\sum_{k=1}^{p-1} F_{k+n} \cdot k^{-1} \equiv 0 \pmod p, \text{ for all integers } n.$$

Equivalently,

$$\sum_{k=1}^{p-1} L_{k+n} \cdot k^{-1} \equiv 0 \pmod p, \text{ for all } n.$$

Other equivalent forms of such conditions are:

$$\sum_{k=1}^{p-1} F_k \cdot k^{-1} \equiv \sum_{k=1}^{p-1} F_{k+1} \cdot k^{-1} \equiv 0 \pmod p$$

or

$$\sum_{k=1}^{p-1} L_k \cdot k^{-1} \equiv \sum_{k=1}^{p-1} L_{k+1} \cdot k^{-1} \equiv 0 \pmod{p}.$$

In turn, these conditions are equivalent to the condition that $Z(p^2) = Z(p)$, where $Z(m)$ is the "Fibonacci entry-point" of m (i.e., the smallest positive integer n such that $m \mid F_n$).

Also solved by H.-J. Seiffert.

A Strange Triangle

H-546 ***Proposed by André-Jeannin, Longwy, France (Vol. 36, no. 5, November, 1998)***

Find the triangular Mersenne numbers. (The sequence of Mersenne numbers is defined by $M_n = 2^n - 1$.)

Solution by the proposer

We shall prove that the only Mersenne triangular numbers are M_0, M_1, M_2, M_4, and M_{12}. In fact, the equation

$$M_n = 2^n - 1 = \frac{k(k+1)}{2}$$

is clearly equivalent to the equation

$$x^2 = 2^{n+3} - 7, \tag{1}$$

where $x = 2k+1$.

It is known [1] that (1) admits the only positive solutions $(n = 0, x = 1)$, $(n = 1, x = 3)$, $(n = 2, x = 5)$, $(n = 4, x = 11)$, and $(n = 12, x = 181)$. The result follows.

Reference

1. Th. Skolem, P. Chowla, & D. J. Lewis. "The Diophantine Equation $2^{n+2} - 7 = x^2$ and Related Problems." *Proc. Amer. Math. Soc.* **10** (1959):663-69.

Also solved by P. Bruckman and H.-J. Seiffert.

A Prime Problem

H-547 ***Proposed by T. V. Padmakumar, Thycaud, India (Vol. 37, no. 1, February 1999)***

If p is a prime number, then

$$\left[\sum_{n=1}^{P} \frac{1}{(2n-1)}\right]^2 - \left[\sum_{n=1}^{P} \frac{1}{(2n-1)^2}\right] \equiv 0 \pmod{p}.$$

Solution by L. A. G. Dresel, Reading, England

Note: The result is clearly true for $p = 2$. However, when p is an odd prime, each summation contains the undefined term $p^{-1} \pmod{p}$. Therefore, we shall assume that these terms are to be omitted (or, possibly, consider then as *formally* canceling each other). The result is then true for $p \geq 5$ but false for $p = 3$.

Proof for $p \geq 5$: For $1 \leq n \leq p$, the sequence of odd numbers $2n-1 \pmod p$ reproduces the residues 0, 1, 2, ..., $p-1$ in a different order. Omitting the residue 0, as explained above, consider the summations over $1 \leq s \leq p-1$, $A \equiv \sum s^{-1}$, and $B \equiv \sum s^{-2} \pmod p$.

Now consider the reciprocals modulo p of two residues s and t. Then it is easily shown that $s^{-1} \equiv t^{-1} \pmod p$ if and only if $s \equiv t \pmod p$. Hence, all the terms in the summation A are distinct (mod p), so that we have $A \equiv \sum s$ and, similarly, we obtain $B \equiv \sum s^2 \pmod p$.

Finally consider the equation $x^{p-1} - 1 \equiv 0 \pmod p$. By Fermat's theorem, this is satisfied for $x = 1, 2, \ldots, p-1$, so we can write $x^{p-1} - 1 \equiv (x-1)(x-2)\ldots(x-p+1) \pmod p$. If $p > 2$, it follows that the sum of the roots is zero, and if $p > 3$, we also have the sum of the products of the roots taken two at a time is zero (mod p). Hence, we have $A \equiv 0$, and also $A^2 - B \equiv 0 \pmod p$ for $p \geq 5$.

Also solved by P. Bruckman, H. Kwong, and the proposer.

Pell-Mell

H-548 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 37, no. 1, February 1999)

Define the sequence of Pell numbers by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Show that, if q is a prime such that $q \equiv 1 \pmod 8$, then

$$q \mid P_{(q-1)/4} \text{ if and only if } 2^{(q-1)/4} \equiv (-1)^{(q-1)/8} \pmod q.$$

Solution by the proposer

Consider the Lucas polynomials defined by $L_0(x) = 2$, $L_1(x) = x$, and $L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$ for $n \geq 0$. It is well known that

$$L_n(x) = \left(\frac{x + \sqrt{x^2+4}}{2}\right)^n + \left(\frac{x - \sqrt{x^2+4}}{2}\right)^n, \quad n \geq 0. \tag{1}$$

Let $Q_n = L_n(2)$, $n \geq 0$, denote the n^{th} Pell-Lucas number.

Proposition: For all $n \geq 0$, it holds that

$$Q_n = 2^{-[(n-1)/2]} \sum_{\substack{k=0 \\ 4 \nmid n+2k+2}}^{n} (-1)^{[(n-2k+1)/4]} \binom{2n}{2k},$$

where [ ] denotes the greatest integer function.

Proof: If $t \neq 1$ is any complex number, then by (1),

$$L_n\left(2i\frac{1+t}{1-t}\right) = \frac{i^n}{(1-t)^n}\left((1+\sqrt{t})^{2n} + (1-\sqrt{t})^{2n}\right),$$

where $i = \sqrt{(-1)}$. Applying the binomial theorem gives

$$L_n\left(2i\frac{1+t}{1-t}\right) = \frac{2i^n}{(1-t)^n} \sum_{k=0}^{n} \binom{2n}{2k} t^k.$$

Now we take $t=-i$. Since $(1-i)/(1+i)=-i$, $1/(i+1)=(1-i)/2$, $-i=1/i$, and $L_n(2)=Q_n$, we find

$$Q_n = 2^{1-n}\sum_{k=0}^{n}\binom{2n}{2k} i^{n-k}(1-i)^n.$$

Using $i=e^{i\pi/2}$ and $1-i=\sqrt{2}\,e^{-i\pi/4}$ yields

$$Q_n = 2^{1-n/2}\sum_{k=0}^{n}\binom{2n}{2k}\exp\left(i(n-2k)\frac{\pi}{4}\right).$$

Equating the real parts gives

$$Q_n = 2^{1-n/2}\sum_{k=0}^{n}\binom{2n}{2k}A_{n-2k},$$

where $A_j := \cos(j\pi/4)$, $j\in Z$. An elementary calculation shows that

$$A_j = \begin{cases}(-1)^{[(j+1)/4]}2^{[j/2]-j/2} & \text{if } j\not\equiv 2 \pmod 4,\\ 0 & \text{if } j\equiv 2 \pmod 4.\end{cases}$$

The stated identity easily follows. Q.E.D.

The next result is known.

Lemma: If q is a prime, then

$$\binom{q-1}{k}\equiv(-1)^k \pmod q \text{ for } k=1,\dots,q-1.$$

Proof: Since q is a prime, q divides $\binom{q}{k}$ for $k=1,\dots,q-1$. Hence, the equation

$$\binom{q}{k}=\binom{q-1}{k}+\binom{q-1}{k-1}$$

implies that

$$\binom{q-1}{k}\equiv-\binom{q-1}{k-1} \pmod q \text{ for } k=1,\dots,q-1,$$

so that the desired congruence can be proved by a simple induction argument. Q.E.D.

If q is a prime such that $q\equiv 1 \pmod 8$, then $q=8j+1$ for some positive integer j. Using the identity of the proposition with $n=(q-1)/2$ and applying the Lemma, modulo q we find that

$$2^{(q-5)/4}Q_{(q-1)/2}\equiv\sum_{\substack{k=0\\k\text{ even}}}^{4j}(-1)^{[j-(2k-1)/4]}=\sum_{r=0}^{2j}(-1)^{j-r}=(-1)^j \pmod q$$

or

$$2^{(q-5)/4}Q_{(q-1)/2}\equiv(-1)^{(q-1)/8} \pmod q. \tag{2}$$

The well-known identity $8P_n^2=Q_{2n}-2(-1)^n$ with $n=(q-1)/4$ and (2) imply that

$$2^{(q+7)/4}P_{(q-1)/4}^2\equiv(-1)^{(q-1)/8}-2^{(q-1)/4} \pmod q.$$

This proves the desired criterion.

Remark: The two smallest such primes are $q=41$ and $q=113$. In fact, we have $P_{10}=2378=41\cdot 58$ and $P_{28}=18457556052=113\cdot 163341204$.

Also solved by P. Bruckman

Resurrection

H-549 ***Proposed by Paul S. Bruckman, Berkeley, CA (Vol. 37, no. 1, February 1999)***

Evaluate the expression: $\sum_{n\ge 1}(-1)^{n-1}\tan^{-1}(1/F_{2n})$. (1)

Note: A number of readers have pointed out that this problem appeared in the *Quarterly* (Vol. 1, no. 4, 1963) on page 71 as Theorem 5.

Solution by Charles K. Cook, University of South Carolina Sumter, Sumter, SC

Note first that this problem was presented as a theorem by Hoggatt and Ruggles in [2].

Lemma 1: $\tan^{-1}\left(\frac{F_n}{F_{n+1}}\right)-\tan^{-1}\left(\frac{F_{n+1}}{F_{n+2}}\right)=\tan^{-1}\left(\frac{(-1)^{n-1}}{F_{n+2}}\right)$.

Proof: Using (I_{10}), $F_{2n}=F_{n+1}^2-F_{n-1}^2$, and (I_{13}), $F_{n-1}F_{n+1}-F_n^2=(-1)^n$, see Hoggatt [1],

$$\frac{(-1)^{n-1}}{F_{2n+2}}=\frac{(-1)^{n-1}}{F_{(n+1)+1}^2-F_{(n+1)-1}^2}=\frac{(-1)^{n-1}}{F_{n+2}^2-F_n^2}=\frac{(-1)^{n-1}}{(F_{n+2}-F_n)(F_{n+2}+F_n)}$$

$$=\frac{F_nF_{n+2}-F_{n+1}^2}{F_{n+1}(F_{n+2}+F_n)}=\frac{\frac{F_n}{F_{n+1}}-\frac{F_{n+1}}{F_{n+2}}}{1+\frac{F_n}{F_{n+2}}}=\frac{\frac{F_n}{F_{n+1}}-\frac{F_{n+1}}{F_{n+2}}}{1+\left(\frac{F_n}{F_{n+1}}\right)\left(\frac{F_{n+1}}{F_{n+2}}\right)}$$

$$=\tan\left(\tan^{-1}\left(\frac{F_n}{F_{n+1}}\right)-\tan^{-1}\left(\frac{F_{n+1}}{F_{n+2}}\right)\right).$$

The lemma follows by taking inverse tangents.

Lemma 2: $\sum_{m=1}^{n}(-1)^{m-1}\tan^{-1}\left(\frac{1}{F_{2m}}\right)=\tan^{-1}\left(\frac{F_n}{F_{n+1}}\right)$.

Proof: Using Lemma 1, it is seen that the series telescopes:

$$\sum_{m=1}^{n}(-1)^{m-1}\tan^{-1}\left(\frac{1}{F_{2m}}\right)=\tan^{-1}\frac{1}{F_2}-\tan^{-1}\frac{1}{F_4}+\tan^{-1}\frac{1}{F_6}-\tan^{-1}\frac{1}{F_8}+\cdots+(-1)^{m-1}\tan^{-1}\frac{1}{F_{2m}}$$

$$=\tan^{-1}\frac{F_1}{F_2}-\tan^{-1}\frac{F_0}{F_1}+\tan^{-1}\frac{F_2}{F_3}-\tan^{-1}\frac{F_1}{F_2}+\tan^{-1}\frac{F_3}{F_4}-\tan^{-1}\frac{F_2}{F_3}$$

$$+\tan^{-1}\frac{F_4}{F_5}-\tan^{-1}\frac{F_3}{F_4}+\cdots+\tan^{-1}\frac{F_n}{F_{n+1}}-\tan^{-1}\frac{F_{n-1}}{F_n}$$

$$=\tan^{-1}\frac{F_n}{F_{n+1}}-\tan^{-1}\frac{F_0}{F_1}=\tan^{-1}\frac{F_n}{F_{n+1}}.$$

This completes the proof of Lemma 2.

Note that the arctangent function is continuous and increasing on the interval (0, 1), so

$$\tan^{-1}\left(\frac{1}{F_{2n+2}}\right) \le \tan^{-1}\left(\frac{1}{F_{2n}}\right)$$

and that the series

$$\sum_{n=1}^{\infty}(-1)^{m-1}\tan^{-1}\left(\frac{1}{F_{2n}}\right)$$

is alternating with $\lim_{n\to\infty}(1/F_{2n}) = 0$, and thus converges to some value A, say. Therefore,

$$\sum_{m=1}^{n}(-1)^{m-1}\tan^{-1}\left(\frac{1}{F_{2m}}\right) \le A \le \sum_{m=1}^{n+1}(-1)^{m-1}\tan^{-1}\left(\frac{1}{F_{2m+2}}\right)$$

for n an odd integer. So, by Lemma 2,

$$\tan^{-1}\left(\frac{F_{n+1}}{F_{n+2}}\right) \le A \le \tan^{-1}\left(\frac{F_n}{F_{n+1}}\right).$$

Taking limits and using the well-known result that $\lim_{n\to\infty}(F_n / F_{n+1}) = \frac{1}{\alpha} = (\sqrt{5}-1)/2$, the golden number (see Hoggatt, [3]), it follows that

$$\lim_{n\to\infty}\tan^{-1}\left(\frac{F_{n+1}}{F_{n+2}}\right) \le A \le \lim_{n\to\infty}\tan^{-1}\left(\frac{F_n}{F_{n+1}}\right) \Rightarrow \tan^{-1}\frac{1}{\alpha} \le A \le \tan^{-1}\frac{1}{\alpha}.$$

Thus,

$$A = \tan^{-1}\left(\frac{2}{\sqrt{5}+1}\right) = \tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right).$$

A similar argument works for the case in which n is an even integer. In either case, the value of the given expression is

$$\sum_{n\ge 1}(-1)^{n-1}\tan^{-1}\left(\frac{1}{F_{2n}}\right) = \tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right).$$

References

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton-Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1979.
2. V. E. Hoggatt, Jr., & I. D. Ruggles. "A Primer for the Fibonacci Numbers–Part IV." *The Fibonacci Quarterly* **1.4** (1963):65-71.
3. V. E. Hoggatt, Jr., & I. D. Ruggles. "A Primer for the Fibonacci Numbers–Part V." *The Fibonacci Quarterly* **2.1** (1964):61.

Also solved by P. Bruckman, L. A. G. Dresel, H. Kwong, H.-J. Seiffert, and I. Strazdins.

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