The Fibonacci Quarterly

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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

THE NUMBER OF SOLUTIONS TO $ax + by = n$

Amitabha Tripathi
Dept. of Mathematics, Indian Institute of Technology, New Delhi 110016, India
atripath@maths.iitd.ernet.in
(Submitted August 1998-Final Revision May 1999)

In this note we determine an exact formula for the number of solutions, $N(a,b;n)$ in nonnegative integer pairs (x,y) of the equation $ax+by=n$ if $\gcd(a,b)=1$. There is no loss of generality in this since $ax+by=n$ is solvable if and only if $d \doteq \gcd(a,b) \mid n$, so that the number of solutions in general would be given by $N(\frac{a}{d}, \frac{b}{d}; \frac{n}{d})$. It is well known that $N(a,b;n)$ is always one of the two consecutive integers $\lfloor \frac{n}{ab} \rfloor$ or $\lfloor \frac{n}{ab} \rfloor + 1$; see, for instance [3, page 214] or [4, page 90]. A history of this and related problems may be found in [2, pages 64-71]. In this note we shall henceforth assume that a, b are positive, relatively prime integers, that n is a nonnegative integer, and prove the following

Theorem:

$$N(a,b;n) = \frac{n + aa'(n) + bb'(n)}{ab} - 1,$$

where $a'(n) \equiv -na^{-1} \pmod{b}$, $1 \le a'(n) \le b$, $b'(n) \equiv -nb^{-1} \pmod{a}$, $1 \le b'(n) \le a$.

We observe that $n + bb'(n)$ is a multiple of a and that $n + aa'(n)$ is a multiple of b. Therefore, $n + aa'(n) + bb'(n)$ is a multiple of ab, and is at least $n + a + b$. It follows that the expression that represents $N(a,b;n)$ in the theorem is indeed a nonnegative integer.

We prove our result in two ways. Our first method uses generating functions to determine the function $N(a,b;n)$, while the second method verifies the formula just obtained by showing that this function meets the characterizing properties that such a function should satisfy.

We begin our first proof by observing that $N(a,b;n)$ equals the coefficient of x^n in the expansion of $(1-x^a)^{-1}(1-x^b)^{-1}$. Also, since $x^m - 1 = \prod_{k=1}^{m}(x - \zeta_m^k)$, we have

$$1 - x^m = \prod_{k=1}^{m}(1 - \zeta_m^{-k}x),$$

where $\zeta_m \doteq e^{2\pi i/m}$.

We write

$$\begin{aligned} \mathcal{N}(x) \doteq \sum_{n\ge 0} N(a,b;n)x^n &= \frac{1}{(1-x^a)(1-x^b)} \\ &= \frac{c_1}{1-x} + \frac{c_2}{(1-x)^2} + \sum_{k=1}^{a-1} \frac{A_k}{1-\zeta_a^{-k}x} + \sum_{k=1}^{b-1} \frac{B_k}{1-\zeta_b^{-k}x}, \end{aligned} \tag{1}$$

where $\zeta_a \doteq e^{2\pi i/a}$ and $\zeta_b \doteq e^{2\pi i/b}$.

In (1) and elsewhere, we adopt the usual convention of assigning the value 0 to any empty sum and the value 1 to any empty product. Comparing coefficients of x^n, we have

$$N(a,b;n) = c_1 + c_2(n+1) + \sum_{k=1}^{a-1} A_k \zeta_a^{-nk} + \sum_{k=1}^{b-1} B_k \zeta_b^{-nk}. \tag{2}$$

A simple calculation shows that $c_1=(a+b-2)/2ab$ and $c_2=1/ab$. Evaluation of the A_k's and the B_k's is done by multiplying both sides of (1) by the corresponding $1-\zeta^{-k}x$ and taking limits as $x\to\zeta^k$. This yields $A_k=1/a(1-\zeta_a^{bk})$, with a similar expression for the B_k's.

From (2),

$$N(a,b;n)=\frac{n}{ab}+\frac{a+b}{2ab}+\frac{1}{a}\sum_{k=1}^{a-1}\frac{\zeta_a^{-nk}}{1-\zeta_a^{bk}}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{\zeta_b^{-nk}}{1-\zeta_b^{ak}}. \tag{3}$$

We observe that each sum on the right is periodic in n, the first with period a and the second with period b. Since a and b are coprime, the two sums together has a period ab, and the expression for $N(a,b;n)$ is essentially determined modulo ab. The form that the function $N(a,b;n)$ takes is well known; see [4, page 90] and [1, pages 113-14].

Notation: We set $a'(n)\equiv -na^{-1}\pmod{b}$, $b'(n)\equiv -nb^{-1}\pmod{a}$, with $1\le a'(n)\le b$, $1\le b'(n)\le a$.

We note that $\sum_{k=1}^{a-1}\zeta_a^{mk}=\sum_{k=0}^{a-1}\zeta_a^{mk}-1=-1$ for any integer m that is not a multiple of a; it equals $a-1$ otherwise.

From $\sum_{k=0}^{a-1}x^k=\prod_{k=1}^{a-1}(x-\zeta_a^k)$, logarithmic differentiation at $x=1$ gives $\sum_{k=1}^{a-1}(1-\zeta_a^k)^{-1}=(a-1)/2$ if $a\ge 2$. The equation is also trivially valid for $a=1$.

Since $(1-\zeta_a^{bk})(1+\zeta_a^{bk}+\zeta_a^{2bk}+\cdots+\zeta_a^{(b'-1)bk})=1-\zeta_a^{b'bk}=1-\zeta_a^{-nk}$, we have

$$\begin{aligned}\sum_{k=1}^{a-1}\frac{\zeta_a^{-nk}}{1-\zeta_a^{bk}}&=-\sum_{k=1}^{a-1}(1+\zeta_a^{bk}+\zeta_a^{2bk}+\cdots+\zeta_a^{(b'-1)bk})+\sum_{k=1}^{a-1}\frac{1}{1-\zeta_a^{bk}}\\&=-[(a-1)-(b'-1)]+\sum_{k=1}^{a-1}\frac{1}{1-\zeta_a^{k}}\\&=b'-a+\frac{a-1}{2}=b'-\frac{a+1}{2},\quad\text{where } b'=b'(n).\end{aligned}$$

Putting all this into (3), we have

$$\begin{aligned}abN(a,b;n)&=\frac{a+b}{2}+n+b\left(b'(n)-\frac{a+1}{2}\right)+a\left(a'(n)-\frac{b+1}{2}\right)\\&=aa'(n)+bb'(n)-ab+n,\end{aligned} \tag{4}$$

which completes the proof of our result.

We now prove that the following four properties, stated in the Proposition below and which *uniquely* characterize the function $N(a,b;n)$, are satisfied by the expression given in the Theorem, thus providing a second proof of the Theorem. It is well known that the function which counts the number of nonnegative integer solutions of $ax+by=n$ must satisfy these properties; see, for instance, [4, pages 87-91].

Proposition: The function $N(a,b;n)$ is the unique function satisfying the four conditions:

$$\begin{array}{llll} N(a,b;n+k\cdot ab) & = N(a,b;n)+k & \text{if } k\ge 0; \\ N(a,b;n) & = \quad 1 & \text{if } ab-a-b<n<ab; \\ N(a,b;p)+N(a,b;q) & = \quad 1 & \text{if } p+q=ab-a-b,\ p,q\ge 0; \\ N(a,b;n) & = \quad 1 & \text{iff } n=ax_0+by_0<ab-a-b,\ x_0,y_0\ge 0. \end{array}$$

For convenience, we now use the notation

$$N'(a,b;n)=\frac{n+aa'(n)+bb'(n)}{ab}-1.$$

Lemma 1: $N'(a,b;n+k\cdot ab)=N'(a,b;n)+k$ for all integers $k\geq 0$.

Proof: Although this is an immediate consequence of (2), we also give a proof that involves the expression for $N(a,b;n)$ given by the Theorem.

$$\begin{aligned}ab\cdot N'(a,b;n+k\cdot ab)&=(n+k\cdot ab)+aa'(n+k\cdot ab)+bb'(n+k\cdot ab)-ab\\&=(n+aa'(n)+bb'(n)-ab)+k\cdot ab\\&=ab\cdot N'(a,b;n)+k\cdot ab.\ \square\end{aligned}$$

Lemma 2: $N'(a,b;n)=1$ if $ab-a-b<n<ab$.

Proof: If $ab-a-b<n<ab$, then $ab<n+a+b\leq n+aa'(n)+bb'(n)\leq n+ab+ab<3ab$, so that $n+aa'(n)+bb'(n)=2ab$ and $N'(a,b;n)=1$. $\square$

Lemma 3: If p and q are nonnegative integers such that $p+q=ab-a-b$, then $N'(a,b;p)+N'(a,b;q)=1$.

Proof: We note that $a'(p)+a'(q)\equiv 1 \pmod{b}$, so that $a'(p)+a'(q)=b+1$ since each is at least 1; similarly, $b'(p)+b'(q)=a+1$. Therefore,

$$\begin{aligned}ab\cdot N'(a,b;p)+ab\cdot N'(a,b;q)&=(aa'(p)+aa'(q))+(bb'(p)+bb'(q))-2ab+(p+q)\\&=a(b+1)+b(a+1)-(ab+a+b)\\&=ab.\ \square\end{aligned}$$

We observe that Lemma 3 asserts that exactly one of n and $ab-a-b-n$ is of the form ax_0+by_0 with $x_0, y_0\geq 0$, if $0\leq n\leq ab-a-b$. Therefore, any n which is not representable by a and b is of the form $ab-a-b-(ax_1+by_1)$, with $0\leq x_1\leq b-1$, $0\leq y_1\leq a-1$.

Lemma 4: For n such that $0\leq n\leq ab-a-b-1$,

$$N'(a,b;n)=\begin{cases}1 & \text{if } n=ax_0+by_0 \text{ for some } x_0, y_0\geq 0;\\ 0 & \text{otherwise.}\end{cases}$$

Proof: If there exist nonnegative integers x_0, y_0 such that $ax_0+by_0=n$, then $x_0\leq b-1$ and $y_0\leq a-1$, and we have

$$\begin{aligned}ab\cdot N'(a,b;n)&=(ax_0+by_0)+aa'(ax_0+by_0)+bb'(ax_0+by_0)-ab\\&=(ax_0+by_0)+a(b-x_0)+b(a-y_0)-ab\\&=ab.\end{aligned}$$

Otherwise, $n=ab-a-b-(ax_1+by_1)$ with $0\leq x_1\leq b-1$, $0\leq y_1\leq a-1$, and we have

$$\begin{aligned}ab\cdot N'(a,b;n)&=aa'(ab-a-b-ax_1-by_1)+bb'(ab-a-b-ax_1-by_1)\\&\quad-ab+(ab-a-b-ax_1-by_1)\\&=aa'(-a-ax_1)+bb'(-b-by_1)-(a+b+ax_1+by_1)\\&=a(1+x_1)+b(1+y_1)-(a+b+ax_1+by_1)=0.\ \square\end{aligned}$$

Lemmas 1-4 together show that the formula given by our Theorem meets the conditions that $N(a,b;n)$ satisfies, thereby completing our second (and less direct) proof.

An interesting consequence of our result is a solution of a special case of the *Coin Exchange Problem*. If we restrict x, y to be nonnegative, it is well known that the equation $ax+by=n$ always has a solution for all sufficiently large n. This means that the set

$$\mathcal{S}(a,b) \doteq \mathbb{N}\setminus\{ax+by : x, y \geq 0\}$$

is *finite*. The two functions

$$g(a,b) \doteq \max_{n\in\mathcal{S}} n \quad \text{and} \quad n(a,b) \doteq |\mathcal{S}|$$

can be evaluated readily from the function $N(a,b;n)$, as we now show in the following

Corollary:

(*a*) $g(a,b)=ab-a-b$;

(*b*) $n(a,b)=(a-1)(b-1)/2$.

Proof: By Lemma 4, or directly, $N(a,b;0)=1$, so that $N(a,b;ab-a-b)=0$ by Lemma 3, while $N(a,b;n)\geq 1$ if $n>ab-a-b$ by Lemmas 1 and 2. This establishes (a).

Lemma 3 implies that there is a one-to-one correspondence between representable and non-representable integers between 0 and $ab-a-b$, and (b) follows from (a). □

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SUMMATION OF RECIPROCALS WHICH INVOLVE PRODUCTS OF TERMS FROM GENERALIZED FIBONACCI SEQUENCES

R. S. Melham
School of Mathematical Sciences, University of Technology, Sydney
PO Box 123, Broadway, NSW 2007 Australia
(*Submitted August 1998-Final Revision November 1998*)

1. INTRODUCTION

We consider the sequence $\{W_n\}$ defined, for all integers n, by

$$W_n = pW_{n-1} + W_{n-2}, \quad W_0 = a, \quad W_1 = b. \tag{1.1}$$

Here a, b, and p are real numbers with p strictly positive. Write $\Delta = p^2 + 4$. Then it is well known [5] that

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.2}$$

where $\alpha = (p + \sqrt{\Delta})/2$, $\beta = (p - \sqrt{\Delta})/2$, $A = b - a\beta$, and $B = b - a\alpha$. As in [5], we put $e_W = AB = b^2 - pab - a^2$.

We define a companion sequence $\{\overline{W}_n\}$ of $\{W_n\}$ by

$$\overline{W}_n = A\alpha^n + B\beta^n. \tag{1.3}$$

Aspects of this sequence have been treated, for example, in [6] and [7]. In the first of these references $\{W_n\}$ and $\{\overline{W}_n\}$ are denoted by $\{H_n\}$ and $\{K_n\}$, respectively.

For $(W_0, W_1) = (0, 1)$ we write $\{W_n\} = \{U_n\}$, and for $(W_0, W_1) = (2, p)$ we write $\{W_n\} = \{V_n\}$. The sequences $\{U_n\}$ and $\{V_n\}$ are generalizations of the Fibonacci and Lucas sequences, respectively. From (1.2) and (1.3), we see that $\overline{U}_n = V_n$ and $\overline{V}_n = \Delta U_n$. It is clear that $e_U = 1$ and $e_V = -\Delta = -(\alpha - \beta)^2$.

The purpose of this paper is to investigate certain infinite sums. In Section 3 we investigate the sum

$$S_{k,m} = \sum_{n=1}^{\infty} \frac{\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}}, \tag{1.4}$$

and in Section 4 we investigate the sum

$$T_{k,m} = \sum_{n=1}^{\infty} \frac{(-1)^n}{W_{kn}W_{k(n+m)}W_{k(n+2m)}W_{k(n+3m)}}, \tag{1.5}$$

where k and m are taken to be odd positive integers.

Now since $p > 0$, then $\alpha > 1$ and $\alpha > |\beta|$, so that

$$W_n \cong \frac{A}{\alpha - \beta}\alpha^n \quad \text{and} \quad \overline{W}_n \cong A\alpha^n. \tag{1.6}$$

Hence, assuming that a and b are chosen so that no denominator vanishes, we see from the ratio test that $S_{k,m}$ and $T_{k,m}$ are absolutely convergent.

2. PRELIMINARY RESULTS

We require the following, in which k and m are taken to be odd integers.

$$W_{n+k}+W_{n-k}=\overline{W}_n U_k, \tag{2.1}$$

$$W_{n+k}-W_{n-k}=W_n V_k, \tag{2.2}$$

$$\beta W_n + W_{n-1} = B\beta^n, \tag{2.3}$$

$$\alpha^m W_{n+m}+W_n = A\alpha^{n+m}U_m, \tag{2.4}$$

$$W_{k(n+m)}W_{k(n+2m)}-W_{kn}W_{k(n+3m)}=e_W(-1)^n U_{km}U_{2km}, \tag{2.5}$$

$$\sum_{n=n_1}^{n_2}\frac{1}{\alpha^{kn}W_{kn}}=\frac{1}{B}\sum_{n=n_1}^{n_2}(-1)^n\frac{W_{kn-1}}{W_{kn}},\quad n_2-n_1 \text{ odd}, \tag{2.6}$$

$$\frac{1}{\alpha^{kn}W_{kn}}+\frac{1}{\alpha^{k(n+m)}W_{k(n+m)}}=A\frac{U_{km}}{W_{kn}W_{k(n+m)}}. \tag{2.7}$$

Identities (2.1)-(2.5) are readily proved with the use of (1.2) and (1.3). Now, since k is odd, then $\alpha^{-kn}=(-1)^{kn}\beta^{kn}=(-1)^n\beta^{kn}$. Hence,

$$\begin{aligned}\sum_{n=n_1}^{n_2}\frac{1}{\alpha^{kn}W_{kn}} &= \sum_{n=n_1}^{n_2}\frac{(-1)^n\beta^{kn}}{W_{kn}}\\ &=\frac{1}{B}\sum_{n=n_1}^{n_2}\frac{(-1)^n(\beta W_{kn}+W_{kn-1})}{W_{kn}},\quad \text{by (2.3)},\\ &=\frac{1}{B}\sum_{n=n_1}^{n_2}\left((-1)^n\beta+(-1)^n\frac{W_{kn-1}}{W_{kn}}\right),\end{aligned}$$

and since n_2-n_1+1 is even, this yields (2.6). Identity (2.7) is readily established with the use of (2.4).

We also require the following theorem, which follows immediately from (2.7).

Theorem 1: If k and m are odd positive integers, then

$$AU_{km}\sum_{n=1}^{\infty}\frac{1}{W_{kn}W_{k(n+m)}}=2\sum_{n=1}^{\infty}\frac{1}{\alpha^{kn}W_{kn}}-\sum_{n=1}^{m}\frac{1}{\alpha^{kn}W_{kn}}. \tag{2.8}$$

Since $\alpha>1$ and $\alpha>|\beta|$, it follows from the ratio test that the infinite sums in (2.8) are absolutely convergent. For similar infinite sums in which the denominator consists of products of two terms from the sequence $\{W_n\}$, see [2].

3. THE SUM $S_{k,m}$

The first of two theorems in this section is

Theorem 2: If k and m are odd positive integers, then

$$AU_{km}^2 S_{k,m}=4\sum_{n=1}^{\infty}\frac{1}{\alpha^{kn}W_{kn}}-\sum_{n=1}^{2m}\frac{1}{\alpha^{kn}W_{kn}}-2\sum_{n=1}^{m}\frac{1}{\alpha^{kn}W_{kn}}. \tag{3.1}$$

Proof: Consider the expression

$$\frac{1}{\alpha^{kn}W_{kn}}+\frac{1}{\alpha^{k(n+m)}W_{k(n+m)}}+\frac{1}{\alpha^{k(n+2m)}W_{k(n+2m)}}. \tag{3.2}$$

Using (2.7), we can write this as

$$\frac{AU_{km}}{W_{kn}W_{k(n+m)}}+\frac{1}{\alpha^{k(n+2m)}W_{k(n+2m)}}, \tag{3.3}$$

or as

$$\frac{1}{\alpha^{kn}W_{kn}}+\frac{AU_{km}}{W_{k(n+m)}W_{k(n+2m)}}. \tag{3.4}$$

Now

$$\begin{aligned}\frac{AU_{km}}{W_{kn}W_{k(n+m)}}+\frac{AU_{km}}{W_{k(n+m)}W_{k(n+2m)}} &= \frac{AU_{km}}{W_{k(n+m)}}\left[\frac{1}{W_{kn}}+\frac{1}{W_{k(n+2m)}}\right]\\ &= \frac{AU_{km}}{W_{k(n+m)}}\cdot\frac{W_{k(n+2m)}+W_{kn}}{W_{kn}W_{k(n+2m)}}\\ &= AU_{km}^2\cdot\frac{\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}}, \quad \text{by (2.1).}\end{aligned} \tag{3.5}$$

But, from (3.2)-(3.4), we then have

$$\begin{aligned}&2\left[\frac{1}{\alpha^{kn}W_{kn}}+\frac{1}{\alpha^{k(n+m)}W_{k(n+m)}}+\frac{1}{\alpha^{k(n+2m)}W_{k(n+2m)}}\right]\\ &=\frac{1}{\alpha^{kn}W_{kn}}+\frac{1}{\alpha^{k(n+2m)}W_{k(n+2m)}}+AU_{km}^2\cdot\frac{\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}},\end{aligned}$$

so that

$$AU_{km}^2\cdot\frac{\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}}=\frac{1}{\alpha^{kn}W_{kn}}+\frac{1}{\alpha^{k(n+2m)}W_{k(n+2m)}}+\frac{2}{\alpha^{k(n+m)}W_{k(n+m)}}.$$

Now, summing both sides, we obtain (3.1). □

Our next theorem expresses $S_{k,m}$ in terms of $S_{k,1}$.

Theorem 3: Let k and m be odd positive integers with $m>1$. Then

$$AU_{km}^2S_{k,m}=AU_k^2S_{k,1}-\frac{1}{B}\left[\sum_{n=3}^{2m}(-1)^n\frac{W_{kn-1}}{W_{kn}}+2\sum_{n=2}^{m}(-1)^n\frac{W_{kn-1}}{W_{kn}}\right]. \tag{3.6}$$

Proof: From (3.1), we have

$$AU_k^2S_{k,1}=4\sum_{n=1}^{\infty}\frac{1}{\alpha^{kn}W_{kn}}-\sum_{n=1}^{2}\frac{1}{\alpha^{kn}W_{kn}}-\frac{2}{\alpha^kW_k}. \tag{3.7}$$

In (3.7), we solve for

$$4\sum_{n=1}^{\infty}\frac{1}{\alpha^{kn}W_{kn}}$$

and substitute in (3.1) to obtain

$$AU_{km}^2 S_{k,m} = AU_k^2 S_{k,1} - \sum_{n=3}^{2m} \frac{1}{\alpha^{kn} W_{kn}} - 2\sum_{n=2}^{m} \frac{1}{\alpha^{kn} W_{kn}}.$$

From this, we arrive at (3.6) by using (2.6). □

For an application of Theorem 3, take $k=1$ and $m=3$. Then, with $W_n = F_n$ and $W_n = L_n$, (3.6) becomes, respectively,

$$\sum_{n=1}^{\infty} \frac{L_{n+3}}{F_n F_{n+3} F_{n+6}} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{L_{n+1}}{F_n F_{n+1} F_{n+2}} - \frac{143}{480}, \tag{3.8}$$

and

$$\sum_{n=1}^{\infty} \frac{F_{n+3}}{L_n L_{n+3} L_{n+6}} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{F_{n+1}}{L_n L_{n+1} L_{n+2}} - \frac{115}{11088}. \tag{3.9}$$

4. THE SUM $T_{k,m}$

We denote the infinite sum on the left side of (2.8) by

$$t_{k,m} = \sum_{n=1}^{\infty} \frac{1}{W_{kn} W_{k(n+m)}}.$$

Then, from (2.8), we see that

$$\begin{cases} AU_{3km} t_{k,3m} = 2\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} W_{kn}} - \sum_{n=1}^{3m} \frac{1}{\alpha^{kn} W_{kn}}, \\ AU_{km} t_{k,m} = 2\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} W_{kn}} - \sum_{n=1}^{m} \frac{1}{\alpha^{kn} W_{kn}}. \end{cases}$$

Next, we solve for $t_{k,3m}$ and $t_{k,m}$ and consider their difference. Then, making use of (2.2) to factor $U_{3km} - U_{km}$, and noting that $U_{2n} = U_n V_n$, we obtain

$$A(t_{k,3m} - t_{k,m}) = \frac{-2V_{km}^2}{U_{3km}} \sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} W_{kn}} + \frac{1}{U_{km}} \sum_{n=1}^{m} \frac{1}{\alpha^{kn} W_{kn}} - \frac{1}{U_{3km}} \sum_{n=1}^{3m} \frac{1}{\alpha^{kn} W_{kn}}. \tag{4.1}$$

Our main result concerning $T_{k,m}$ can now be given in the following theorem.

Theorem 4: Let k and m be odd positive integers. Then

$$\begin{aligned} e_W A U_{km} U_{2km} T_{k,m} = &\frac{-2V_{km}^2}{U_{3km}} \sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} W_{kn}} + \frac{1}{U_{km}} \sum_{n=1}^{m} \frac{1}{\alpha^{kn} W_{kn}} \\ &- \frac{1}{U_{3km}} \sum_{n=1}^{3m} \frac{1}{\alpha^{kn} W_{kn}} + A \sum_{n=1}^{m} \frac{1}{W_{kn} W_{k(n+m)}}. \end{aligned} \tag{4.2}$$

Proof: Using (2.5), we see that

$$\frac{e_W (-1)^n U_{km} U_{2km}}{W_{kn} W_{k(n+m)} W_{k(n+2m)} W_{k(n+3m)}} = \frac{1}{W_{kn} W_{k(n+3m)}} - \frac{1}{W_{k(n+m)} W_{k(n+2m)}}.$$

If we now sum both sides, we obtain

$$e_W U_{km} U_{2km} T_{k,m} = t_{k,3m} - \left[t_{k,m} - \sum_{n=1}^{m} \frac{1}{W_{kn} W_{k(n+m)}} \right],$$

and (4.2) follows from (4.1). □

We mention that $T_{k,m}$ can be expressed in terms of $T_{k,1}$. We simply write down (4.2) for the case $m = 1$, solve for $\sum_{n=1}^{\infty}(1/\alpha^{kn}W_{kn})$, and then substitute in (4.2). Since the result is rather lengthy, we do not give it here.

As can be seen from Theorems 2 and 4, $S_{k,m}$ and $T_{k,m}$ can be expressed in terms of the infinite sum $\sum_{n=1}^{\infty}(1/\alpha^{kn}W_{kn})$ together with certain finite sums. If we consider specializations $W_n = U_n$ or $W_n = V_n$, this infinite sum can be expressed in terms of the Lambert series, which is defined as

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}, \quad |x| < 1. \text{ In this regard, see [1].}$$

Remark: For the sake of definiteness, we have assumed throughout this paper that $p > 0$, so that $\sum_{n=1}^{\infty}(1/\alpha^{kn}W_{kn})$ is absolutely convergent. However, we can immediately write down parallel results for $p < 0$. For then we see that $\beta < -1$ and $|\beta| > |\alpha|$, so that $W_n \cong (-B/(\alpha-\beta))\beta^n$ and $\overline{W}_n \cong B\beta^n$. It follows from the ratio test that $\sum_{n=1}^{\infty}(1/\beta^{kn}W_{kn})$ is absolutely convergent. We then obtain counterparts of Theorems 1 through 4 if in each theorem we replace $\alpha(\beta)$ by $\beta(\alpha)$ and $A(B)$ by $B(A)$. Indeed, these substitutions are valid in (2.3), (2.4), (2.6), and (2.7), regardless of the sign of p.

Finally, two early references that touch on a wide variety of infinite sums in which the denominators of the summands contain products of Fibonacci and Lucas numbers are [3] and [4].

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FIBONACCI NUMBERS, GENERATING SETS AND HEXAGONAL PROPERTIES

James S. Tanton
Dept. of Mathematics, Merrimack College, 315 Turnpike St., North Andover, MA 01845
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1. INTRODUCTION

It is well known that the entries $p(n, t) = \binom{n}{t}$, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, $0 \le t \le n$, of Pascal's triangle satisfy *equal product* and *equal gcd* (*greatest common divisor*) *hexagonal properties*: the two alternate triads arising from the six binomial coefficients surrounding any given entry in Pascal's array have equal product and equal gcd [7, 8, 10, 14, 15] (see Fig. 1). Pascal's array fails to satisfy an *equal lcm* (*least common multiple*) *hexagonal property*.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & \mathbf{4} & & \mathbf{6} & & 4 & & 1 & & \\
 & 1 & & \mathbf{5} & & 10 & & \mathbf{10} & & 5 & & 1 & \\
1 & & 6 & & \mathbf{15} & & \mathbf{20} & & 15 & & 6 & & 1
\end{array}
$$

FIGURE 1. A Typical Hexagon in Pascal's Array
Notice: $5 \cdot 6 \cdot 20 = 4 \cdot 10 \cdot 15$ and $\gcd(5, 6, 20) = 1 = \gcd(4, 10, 15)$.

These observations of the early 1970s initiated the search for and discovery of many beautiful configurations within Pascal's array satisfying equal product, equal gcd and even equal lcm properties [2, 4, 5, 12]. Similar properties have been discovered for other arrays, such as the Binomial triangle, and for higher-dimensional "pyramids" of multinomials, multi-Fibonomials, and the like [1, 3, 9, 11, 13].

Recently, R. P. Grimaldi [6] discovered hexagonal properties occurring within the array with entries $g(n, t) = F_t F_{n+1-t}$ $(n \in \mathbb{N}, 1 \le t \le n)$. This array arose from a study of *generating sets*, that is, subsets S of $[n] = \{1, 2, \ldots, n\}$ satisfying $S \cup (S+1) = [n+1]$, where $S+1 = \{s+1 : s \in S\} \subseteq [n+1]$. Counting the number of such sets that contain the particular element t produces the quantity $g(n, t)$. It is a surprise to learn that this array satisfies both the equal product and equal gcd hexagonal properties.

In this paper we study higher-order generating sets, that is, subsets S of $[n]$ satisfying $S \cup (S+k) = [n+k]$ for some $k \in \mathbb{N}$, $1 \le k \le n$ (where $S = k = \{s+k : s \in S\}$). We call such a set a k^{th}*-order generating set* and say that S generates $[n+k]$.

Setting $g^k(n, t)$ to be the number of such sets that contain the particular element $t \in \mathbb{N}$, we are thus given, for each $k \in \mathbb{N}$, new arrays with potential hexagonal properties. We will show that the entries $g^k(n, t)$ are products of $k+1$ Fibonacci numbers and explore the extent to which equal product and equal gcd hexagonal properties hold.

In this paper it is convenient to set $F_n = 0$ for n a nonpositive integer. For $x \in \mathbb{R}$, we use $\lceil x \rceil$ to denote the least integer greater than or equal to x and $\lfloor x \rfloor$ the greatest integer less than or equal to x.

We begin by recalling some standard properties of the Fibonacci sequence that will be used throughout this work.

2. THE FIBONACCI NUMBERS

An easy inductive argument establishes:

A. $\gcd(F_t, F_{t+1}) = 1$ for all $t > 0$.

Since $\gcd(F_t, F_{t+2}) = \gcd(F_t, F_{t+1} + F_t) = \gcd(F_t, F_{t+1})$, we have:

B. $\gcd(F_t, F_{t+2}) = 1$ for all $t > 0$.

We also have the key relation:

C. $F_{t+r} = F_r F_{t+1} + F_{r-1} F_t$ for all $t \geq 0, r \geq 1$.

This is easily established by an induction argument on r. With **A** and **B** it has the following consequences:

D. Let $r, t \geq 0, d \in \mathbb{N}$. If $d | F_t$ and $d | F_{t+r}$, then $d | F_r$. Consequently $\gcd(F_t, F_{t+r}) | F_r$.

E. Let $r, t \geq 0, d \in \mathbb{N}$. Suppose $d | F_t$ and $d | F_{t+r}$. If, for $k \in \mathbb{N}$, $d | F_k$, then $d | F_{k \pm r}$.

[By **D**, $d | F_r$. **A** and **C** now show $d | F_{k+r}$. If $r \leq k$, then; $F_k = F_r F_{k+1-r} + F_{r-1} F_{k-r}$, from which it follows that $d | F_{k-r}$. The result is trivial for $r > k$.]

F. Let $r, k \geq 0, d \in \mathbb{N}$. If $d | F_r$ and $d | F_k$, then $d | F_{k \pm mr}$ for any $m \in \mathbb{N}$.

[This follows from repeated application of **E** with $t = 0$.]

G. $F_k | F_{mk}$ for $m, k \in \mathbb{N}$.

[Set $r = k$ in **F**.]

H. Let $d \in \mathbb{N}$ and let F_a be the first Fibonacci number ($a \in \mathbb{N}$) so that $d | F_a$. Let $k \geq 0$. Then $d | F_k \Leftrightarrow a | k$.

[($\Leftarrow$) follows from **G**. For ($\Rightarrow$), write $k = ma + b$ with $0 \leq b < a$, $m \in \mathbb{N}$. **F** shows $d | F_b$, a contradiction unless $b = 0$.]

I. For $r, k \in \mathbb{N}$, $\gcd(F_r, F_k) = F_{\gcd(r,k)}$.

[That $F_{\gcd(r,k)}$ is a common divisor of F_r and F_k follows from **G**, **F**, and the Euclidean algorithm show that $\gcd(F_r, F_k) = F_{\gcd(r,k)}$.]

3. A CONSTRUCTIVE MODEL

We have the following familiar model for constructing Fibonacci numbers: For $n, k, t \in \mathbb{N}$, let

S_n = the set of all n bit sequences beginning and ending with 1 and containing no two consecutive 0's.

S_n^k = the subset of those sequences that contain precisely k 1's.

$S_n(t)$ = the subset of all those sequences containing a 1 in the t^{th} place.

We have:

J. $|S_n| = F_n$ for all $n \in \mathbb{N}$.

Proof: Clearly $|S_1| = |S_2| = 1$ and, by considering the choice of the penultimate term in an n bit sequence, we see that $|S_n| = |S_{n-1}| + |S_{n-2}|$ for $n \geq 3$. □

K. $|S_n^k| = \binom{k-1}{n-k}$ provided $n-k \leq k-1$. (It is zero otherwise.)

Proof: There are $n-k$ zeros "to be placed" in the $k-1$ spaces between the ones. □

This yields:

L. $F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k}$.

Proof: As S_n is the disjoint union of the subsets S_n^k, $0 \leq k \leq n$, $|S_n| = \Sigma_{k=0}^{n} |S_n^k|$. □

M. $|S_n(t)| = F_t F_{n+1-t}$.

Proof: Clearly $S_n(t)$ is isomorphic as a set to $S_t \times S_{n+1-t}$. □

Later, we will denote $|S_n(t)|$ by $g^1(n,t)$. Observe that, by reversing the strings, we have the symmetry, $g^1(n,t) = g^1(n, n+1-t)$. This symmetry appears in many of the tables presented later in this paper.

4. GENERATING SETS

An n bit sequence determines a subset $S \subseteq [n]$ and vice versa (declare $t \in S$ if and only if the t^{th} place of that sequence is a one). Subsets arising from binary sequences as described in the previous section are the generating sets (of order 1). By **J** there are F_n generating subsets of $[n]$ of order 1.

More generally, let $h_k(n)$ be the number of subsets of $[n]$ of order k that generate $[n+k]$. A table for $h_k(n)$ appears in Figure 2.

$h_k(n)$	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	2	1	1							
4	3	1	1	1						
5	5	2	1	1	1					
6	8	4	1	1	1	1				
7	13	6	2	1	1	1	1			
8	21	9	4	1	1	1	1	1		
9	34	15	8	2	1	1	1	1	1	
10	55	25	12	4	1	1	1	1	1	1

FIGURE 2

Theorem 4.1: For $n, k \in \mathbb{N}$, $h_k(n)$ is a product of k Fibonacci numbers. Precisely

$$h_k(n) = \left(F_{\lceil \frac{n}{k} \rceil}\right)^r \left(F_{\lfloor \frac{n}{k} \rfloor}\right)^{k-r},$$

where $n \equiv r \pmod{k}$ with $1 \le r \le k$.

Proof: Clearly $h_k(n) = 0$ if $n < k$ and the theorem is true. Assume then that $n \ge k$. Write $n = mk + r$ with $1 \le r \le k$. (Here, $m = \lceil \frac{n}{k} \rceil - 1$.)

Let $H_k(n)$ be the set of all n bit sequences that correspond to a generating set of order k. Any such sequence must contain a 1 in the first and last k places. It also has the property that if the t^{th} place fails to be a 1 then the $t-k^{\text{th}}$ place must be. Thus, we can partition the sequence into k intertwined subsequences corresponding to those place numbers congruent mod k. Each such subsequence corresponds to a first-order generating set. Counting the lengths of these subsequences, we see that we have a set isomorphism

$$H_k(n) = \underbrace{S_{m+1} \times \cdots \times S_{m+1}}_{r} \times \underbrace{S_m \times \cdots \times S_m}_{k-r}.$$

The theorem follows from **J**. □

5. ARRAYS

For $n, t, k \in \mathbb{N}$, let $g^k(n, t)$ be the number of generating subsets of $[n]$ of order k that contain t. Figures 3, 4, and 5 give tables for $g^k(-,-)$ for $k = 1, 2$, and 3, respectively.

$g^1(n,t)$	t = 1	2	3	4	5	6	7	8	9	10
n = 1	1									
2	1	1								
3	2	1	2							
4	3	2	2	3						
5	5	3	4	3	5					
6	8	5	6	6	5	8				
7	13	8	10	9	10	8	13			
8	21	13	16	15	15	16	13	21		
9	34	21	26	24	25	24	26	21	34	
10	55	34	42	39	40	40	39	42	34	55

FIGURE 3

$g^2(n,t)$	1	2	3	4	5	6	7	8	9	10
1	0									
2	1	1								
3	1	1	1							
4	1	1	1	1						
5	2	2	1	2	2					
6	4	4	2	2	4	4				
7	6	6	4	2	4	6	6			
8	9	9	6	6	6	6	9	9		
9	15	15	9	10	12	10	9	15	15	
10	25	25	15	15	20	20	15	15	25	25

FIGURE 4

$g^3(n,t)$	1	2	3	4	5	6	7	8	9	10
1	0									
2	0	0								
3	1	1	1							
4	1	1	1	1						
5	1	1	1	1	1					
6	1	1	1	1	1	1				
7	2	2	2	1	2	2	2			
8	4	4	4	2	2	4	4	4		
9	8	8	8	4	4	4	8	8	8	
10	11	11	11	7	5	5	7	11	11	11

FIGURE 5

Observation **M** establishes $g^1(n,t)=F_tF_{n+1-t}$ (see also [6]). We now determine the general formula for $g^k(n,t)$.

Theorem 5.1: For $n,k,t\in\mathbb{N}$, if $n\equiv r \pmod k$ and $t\equiv i \pmod k$ with $1\le r.\ i\le k$, then

$$g^k(n,t)=\begin{cases}\left(F_{\lceil\frac{n}{k}\rceil}\right)^{r-1}\left(F_{\lfloor\frac{n}{k}\rfloor}\right)^{k-r}F_{\lceil\frac{t}{k}\rceil}F_{\lceil\frac{n}{k}\rceil-\lceil\frac{t}{k}\rceil+1} & \text{if } i\le r,\\ \left(F_{\lceil\frac{n}{k}\rceil}\right)^{r}\left(F_{\lfloor\frac{n}{k}\rfloor}\right)^{k-r-1}F_{\lceil\frac{t}{k}\rceil}F_{\lfloor\frac{n}{k}\rfloor-\lceil\frac{t}{k}\rceil+1} & \text{if } i>r.\end{cases}$$

Proof: Let $H_k(n,t)$ be the set of n bit sequences arising from k^{th}-order generating sets $S\subseteq[n]$ containing the element t. Thus, we are declaring that a 1 must always appear in the t^{th} place. This 1 occurs in the $\lceil\frac{t}{k}\rceil^{\text{th}}$ place of the i^{th} subsequence corresponding to the i^{th} congruent class of place numbers mod k. Writing $n=mk+r$, we have a set isomophism

$$H_k(n,t)\cong S_{m+1}\times\cdots\times \overset{i^{\text{th}}\text{ place}}{S_{m+1}\left(\lceil\tfrac{t}{k}\rceil\right)}\times\cdots\times S_m$$

if $i>r$. The result follows. □

6. EQUAL PRODUCT HEXAGONAL PROPERTY

In [6], R. P. Grimaldi observed and proved that the $g^1(-,-)$ array (Fig. 3) satisfies the equal product hexagonal property. For example, the two alternate triads in the six entries 6, 4, 3, 5, 10, 9 surrounding $g^1(6,4)=6$ in a (skewed) hexagon have equal products: $6\cdot3\cdot10=4\cdot5\cdot9$.

We call such a hexagon a *hexagon of radius 1*. It is *centered* about $g^1(6,4)$. We observe here that Figure 3 satisfies the equal product property for all hexagons of greater radii. For example, taking two steps outward from the same center $g^1(6,4)$ in the direction of the vertices of the original (skewed) hexagon yields six entries 5, 2, 3, 8, 16, 15 whose alternate triads again satisfy $5\cdot3\cdot16=2\cdot8\cdot15$. We call such a configuration of six entries a *hexagon of radius 2*. The notion of a hexagon of general radius *r* is defined similarly.

Although all hexagons of arbitrary radius in Figure 3 satisfy the equal product property, the same is not true for the arrays in Figures 4 and 5, or in a general array $g^k(-,-)$, $k\ge2$. Only those hexagons with radius divisible by *k* can be guaranteed to satisfy the equal product property.

Theorem 6.1 (Equal Product Hexagonal Property): For $n,k,t,a\in\mathbb{N}$,

$$g^k(n-ak,t-ak)\cdot g^k(n,t+ak)\cdot g^k(n+ak,t)=g^k(n-ak,t)\cdot g^k(n,t-ak)\cdot g^k(n+ak,t+ak).$$

Proof: Suppose $n\equiv r \pmod k$, $t\equiv i \pmod k$ with $1\le i,\ r\le k$. Then

$$n-ak,n,n+ak\equiv r \pmod k,$$
$$t-ak,a,t+ak\equiv i \pmod k.$$

Assume $i\le r$. Then, by Theorem 5.1,

$$g^k(n\pm ak,t)=\left(F_{\lceil\frac{n}{k}\rceil\pm a}\right)^{r-1}\left(F_{\lfloor\frac{n}{k}\rfloor\pm a}\right)^{k-r}F_{\lceil\frac{t}{k}\rceil}F_{\lceil\frac{n}{k}\rceil-\lceil\frac{t}{k}\rceil+1\pm a},$$

$$g^k(n,t\pm ak)=\left(F_{\lceil\frac{n}{k}\rceil}\right)^{r-1}\left(F_{\lfloor\frac{n}{k}\rfloor}\right)^{k-r}F_{\lceil\frac{t}{k}\rceil\pm a}F_{\lceil\frac{n}{k}\rceil-\lceil\frac{t}{k}\rceil+1\mp a},$$

$$g^k(n \pm ak, t \pm ak) = \left(F_{\lceil \frac{n}{k} \rceil \pm a}\right)^{r-1} \left(F_{\lfloor \frac{n}{k} \rfloor \pm a}\right)^{k-r} F_{\lceil \frac{t}{k} \rceil \pm a} F_{\lceil \frac{n}{k} \rceil - \lceil \frac{t}{k} \rceil + 1},$$

and the result is easy to establish. The case $i > r$ is proved similarly. □

7. EQUAL gcd HEXAGONAL PROPERTY

7.1 The $g^1(-,-)$ Array

As established by Grimaldi [6], $g^1(-,-)$ also satisfies the equal gcd hexagonal property for radius 1 hexagons. Inspection of Figure 3 might encourage one to suspect that the equal gcd property also holds for higher sized hexagons but this turns out to be false. Consider the hexagon of radius 2 centered in row $n = 23$ about $g^1(23, 10)$. Here the greatest common divisors of the alternate triads are:

$$\gcd(g^1(21,10), g^1(23,8), g^1(25,12)) = \gcd(15 \cdot 144, 21 \cdot 987, 144 \cdot 377) = 9,$$
$$\gcd(g^1(21,8), g^1(25,10), g^1(23,12)) = \gcd(21 \cdot 377, 55 \cdot 987, 144 \cdot 144) = 3.$$

(Incidentally, this is the first instance of failed equal gcd for this array.) Note, however, that both gcd's are composed of powers of the same prime. We will say an array satisfies a *weak gcd hexagonal property* for hexagons of radius r if the greatest common divisors of the alternate triads in any hexagon of that radius are composed of positive powers of the same primes.

Lemma 7.1: The array $g^1(-,-)$ satisfies the weak gcd hexagonal property for all hexagons of arbitrary size. That is, for $n, t, r \in \mathbb{N}$,

$$\gcd(g^1(n-r, t-r), g^1(n, t+r), g^1(n+r, t))$$

and

$$\gcd(g^1(n-r, t), g^1(n, t-r), g^1(n+r, t+r))$$

are composed of positive powers of the same primes.

Proof: By observation **M**,

$$g^1(n-r, t-r) = F_{t-r} F_{n+1-t},$$
$$g^1(n, t+r) = F_{t+r} F_{n+1-t-r},$$
$$g^1(n+r, t) = F_t F_{n+1-t+r},$$

and

$$g^1(n-r, t) = F_t F_{n+1-t-r},$$
$$g^1(n, t-r) = F_{t-r} F_{n+1-t-r},$$
$$g^1(n+r, t+r) = F_{t+r} F_{n+1-t}.$$

Suppose p, a prime, is a common divisor of the first triad. (The case where p is a common divisor of the second triad is proved similarly.) We have four possibilities:

i) p divides each of F_{t-r}, F_{t+r}, and F_t.
ii) p divides each of $F_{n+1-t-r}$, $F_{n+1-t+r}$, and F_{n+1-t}.
iii) p divides two of F_{t-r}, F_{t+r}, and F_t but not the third.
iv) p divides two of $F_{n+1-t-r}$, $F_{n+1-t+r}$, and F_{n+1-t} but not the third.

It is clear that cases i) and ii) imply that p is a common divisor of the second triad.

Consider case iii). By observation, **E**, it must be the case that $p|F_{t-r}$ and $p|F_{t+r}$ but $p \nmid F_t$. By **D**, $p|F_{2r}$. Since $p|g^1(n+r,t)$, we have $p|F_{n+1-t+r}$. Consequently, by **E**, $p|F_{n+1-t-r}$ and p is a common divisor of the second triad.

Case iv) is established similarly. □

We can now quickly establish Grimaldi's result.

Corollary 7.2: All alternate triads for radius 1 hexagons in the $g^1(-,-)$ array have greatest common divisor equal to 1. Consequently, the equal gcd hexagonal property holds for such hexagons.

Proof: We see from the proof of Lemma 7.1 that any common prime divisor p of a triad satisfies $p|F_{2r}$ (in some instances, we even have $p|F_r$). When $r=1$, $F_{2r}=1$. □

7.2 The $g^2(-,-)$ Array

Consider the $g^2(-,-)$ array derived from generating sets of order 2 (Fig. 4). Hexagons of arbitrary size generally fail to satisfy the weak gcd property. Section 6 suggests we focus on those hexagons whose radii are divisible by $k=2$. We have the following result.

Lemma 7.3: Consider hexagons of radius $r=2a$, $a\in\mathbb{N}$, in the $g^2(-,-)$ array. If $a=1$, then the equal gcd property always holds (and in fact all gcd's of alternate triads equal 1). If $a\geq 2$, the weak gcd property always holds.

Proof: Consider a hexagon of radius $r=2$ centered about $g^2(n,t)$. We will show that each alternate triad has gcd equal to 1.

Consider first the case where both n and t are odd. Set $u=\lceil \frac{n}{k}\rceil$ and $v=\lceil \frac{t}{2}\rceil$. By Theorem 5.1, our alternate triads are:

$$g^2(n-2,t-2)=F_{u-2}F_{v-1}F_{u-v+1},$$
$$g^2(n,t+2)=F_{u+1}F_{v+1}F_{u-v},$$
$$g^2(n+2,t)=F_uF_vF_{u-v+2},$$

and

$$g^2(n-2,t)=F_{u-2}F_vF_{u-v},$$
$$g^2(n,t+2)=F_{u-1}F_{v-1}F_{u-v+2},$$
$$g^2(n+2,t+2)=F_uF_{v+1}F_{u-v+1}.$$

Let p be a common prime divisor for the first triad. By observations **A** and **B**, it is impossible for p to be a common divisor of any two of F_{u-2}, F_{u-1}, or F_u. It must be the case that p divides at least two of $F_{v-1}F_{u-v+1}$, $F_{v+1}F_{u-v}$, and F_vF_{u-v+2}. Again, noting **A** and **B**, this allows six possibilities:

i) $p|F_{v-1}$ and $p|F_{u-v}$ (and consequently $p|F_u$).

ii) $p|F_{u-v+1}$ and $p|F_{v+1}$ (and consequently $p|F_u$).

iii) $p|F_{v-1}$ and $p|F_{u-v+2}$ (and consequently $p|F_{u-1}$).

iv) $p|F_{u-v+1}$ and $p|F_v$ (and consequently $p|F_{u-1}$).

v) $p|F_{v+1}$ and $p|F_{u-v+2}$ (and consequently $p|F_{u-2}$).
vi) $p|F_{u-v}$ and $p|F_v$ (and consequently $p|F_{u-2}$).

Let F_m be the first Fibonacci number such that $p|F_m$, and consider case i). By **H** we have

$$\begin{aligned} v-1 &\equiv 0 \pmod{m}, \\ u-v &\equiv 0 \pmod{m}, \\ u &\equiv 0 \pmod{m}. \end{aligned}$$

Consequently $m=1$ and $p|F_m=1$.

Similarly, the remaining cases yield contradictions. Thus, the greatest common divisor of the first triad must be 1. Similarly for the second triad.

The same argument applies to the cases n even, and n odd, t even.

We will now establish the weak gcd property for hexagons of radius $r=2a$, $a\in\mathbb{N}$. Again, set $u=\lceil\frac{n}{k}\rceil$ and $v=\lceil\frac{t}{2}\rceil$ and consider the case n odd, t odd. The alternate triads are:

$$\begin{aligned} g^2(n-2a, t-2a) &= F_{u-1-a}F_{v-a}F_{u-v+1}, \\ g^2(n, t+2a) &= F_{u-1}F_{v+a}F_{u-v+1-a}, \\ g^2(n+2a, t) &= F_{u-1+a}F_vF_{u-v+1+a}, \end{aligned}$$

and

$$\begin{aligned} g^2(n-2a, t) &= F_{u-1-a}F_vF_{u-v+1-a}, \\ g^2(n, t-2a) &= F_{u-1}F_{v-a}F_{u-v+1+a}, \\ g^2(n+2a, t+2a) &= F_{u-1+a}F_{v+a}F_{u-v+1}. \end{aligned}$$

Let p be a common prime divisor of the first triad. There are 27 possibilities as to which Fibonacci factors it must divide. We must show that each scenario forces p to be a common divisor of the second triad. We will illustrate the four typical arguments used to demonstrate this. We leave the details of applying these arguments to the remaining 23 cases to the diligent reader.

Suppose $p|F_{u-1-a}$, $p|F_{v+a}$, and $p|F_{u-v+1+a}$. Then p is trivially a common divisor of the second triad.

Suppose $p|F_{u-1-a}$, $p|F_{u-1}$. Then, by **E**, $p|F_{u-1+a}$ and so p is a common divisor of the second triad.

Suppose $p|F_{u-1-a}$, $p|F_{v+a}$. Then, by **D**, $p|F_{2a}$ and, by **E**, $p|F_{v+a-2a}=F_{v-a}$ and so p is a common divisor of the second triad.

Suppose $p|F_{u-1+a}$, $p|F_{v+a}$, and $p|F_{u-v+1}$. Let F_m be the first Fibonacci number such that $p|F_m$. Then, by **H**,

$$\begin{aligned} u-a+1 &\equiv 0 \pmod{m}, \\ v+a &\equiv 0 \pmod{m}, \\ u-v+1 &\equiv 0 \pmod{m}. \end{aligned}$$

This is possible only if $m=1$ or $m=2$. But $p|F_m$ yields a contradiction. Therefore, this scenario cannot occur.

The remaining cases n even, and n odd, t even, are proven similarly. □

The following example shows that the equal gcd hexagonal property fails even for the case $a = 2$. Consider the hexagon of radius 4 centered about $g^2(44, 19)$. Then the alternate triads are:

$$g^2(40, 15) = F_{20}F_8F_{13} = 6765 \cdot 21 \cdot 233,$$
$$g^2(44, 23) = F_{22}F_{12}F_{11} = 17711 \cdot 144 \cdot 89,$$
$$g^2(48, 19) = F_{24}F_{10}F_{15} = 46368 \cdot 55 \cdot 610,$$

with gcd $= 9$, and

$$g^2(40, 19) = F_{20}F_{10}F_{11} = 6765 \cdot 55 \cdot 89,$$
$$g^2(44, 15) = F_{22}F_8F_{15} = 17711 \cdot 21 \cdot 610,$$
$$g^2(48, 23) = F_{24}F_{12}F_{13} = 46368 \cdot 144 \cdot 233,$$

with gcd $= 3$.

(Challenge for the reader: Prove that any common prime divisor p of an alternate triad from a hexagon of radius 4 in the $g^2(-,-)$ array must be a divisor of $F_8 = 21$. Consequently $p = 3$ or 7.)

7.3 The $g^k(-,-)$ Array, $k \geq 3$

In general, not even the weak gcd hexagonal property holds for $g^k(-,-)$, $k \geq 3$, arrays, even if the hexagon is of radius divisible by k. One can easily find examples to illustrate this. A simple one is the hexagon of radius 3 in the $g^3(-,-)$ array centered about $n = 14$, $t = 5$. Here the alternate triads are:

$$g^3(11, 2) = F_4F_3F_1F_4 = 18,$$
$$g^3(14, 8) = F_5F_4F_3F_3 = 60,$$
$$g^3(17, 5) = F_6F_5F_2F_5 = 200,$$

with gcd $= 2$, and

$$g^3(13, 5) = F_4F_3F_2F_3 = 12,$$
$$g^3(14, 2) = F_5F_4F_1F_5 = 75,$$
$$g^3(17, 8) = F_6F_5F_3F_4 = 240,$$

with gcd $= 3$.

This completes our analysis of the $g^k(-,-)$ arrays. We summarize our results in the following theorem.

Theorem 7.4: Concerning gcd hexagonal properties for hexagons of radius $r = ka$ in the array $g^k(-,-)$ (with $a, k \in \mathbb{N}$) we have the following:

1) For $k = 1$ and $k = 2$: The equal gcd property folds for $a = 1$. The weak gcd property holds for $a \geq 2$.

2) For $k \geq 3$: The weak gcd property fails.

As a final comment, we note that the equal lcm hexagonal property does not hold for the arrays $g^k(-,-)$.

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EVALUATION OF CERTAIN INFINITE SERIES INVOLVING TERMS OF GENERALIZED SEQUENCES

Piero Filipponi

Fondazione Ugo Bordoni, Via B. Castiglione 59, I-00142, Rome, Italy

e-mail: filippo@fub.it

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1. INTRODUCTION AND PRELIMINARIES

This article deals with the *generalized Fibonacci numbers* $U_k(m)$ and the *generalized Lucas numbers* $V_k(m)$ which have already been considered in [9], [11], and elsewhere. These numbers satisfy the second-order recurrence relation

$$W_k(m) = mW_{k-1}(m) + W_{k-2}(m) \quad (k \geq 2) \tag{1.1}$$

where W stands for either U or V, and m is an arbitrary integer. The initial conditions in (1.1) are $W_0(m) = 0$, $W_1(m) = 1$, or $W_0(m) = 2$, $W_1(m) = m$ depending on whether W is U or V. Whenever no misunderstanding can arise, $U_k(m)$ and $V_k(m)$ will be denoted simply by U_k and V_k, respectively.

Closed-form expressions (Binet forms) for these numbers are

$$\begin{cases} U_k(m) = (\alpha_m^k - \beta_m^k) / \Delta_m, \\ V_k(m) = \alpha_m^k + \beta_m^k, \end{cases} \tag{1.2}$$

where

$$\begin{cases} \Delta_m = (m^2+4)^{1/2}, \\ \alpha_m = (m+\Delta_m)/2, \\ \beta_m = (m-\Delta_m)/2, \end{cases} \tag{1.3}$$

so that

$$\alpha_m\beta_m = -1, \;\; \alpha_m + \beta_m = m, \;\text{ and }\; \alpha_m - \beta_m = \Delta_m. \tag{1.4}$$

It can be proved that the extension through negative values of the subscript leads to

$$\begin{cases} U_{-k} = (-1)^{k-1}U_k, \\ V_{-k} = (-1)^k V_k. \end{cases} \tag{1.5}$$

Observe that $U_k(1) = F_k$ and $V_k(1) = L_k$ (the k^{th} Fibonacci and Lucas number, respectively), while $U_k(2) = P_k$ and $V_k(2) = Q_k$ (the k^{th} Pell and Pell-Lucas number, respectively).

The principal aim of this paper is to generalize [14], and some results established in [1], [4], and [12] (see also [5]), by finding an explicit form for the infinite series

$$S_h(r, s, m) \overset{\text{def}}{=} \sum_{k=1}^{\infty} k^h \frac{r^k}{s^k} V_k, \tag{1.6}$$

where V_k obviously stands for $V_k(m)$ and h, r, and s are positive integers, the last two of which are subject to the restriction,

$$r/s < 1/\alpha_m. \tag{1.7}$$

By using the Binet form (1.2) for V_k, it can be proved readily (e.g., see [6], pp. 266-67]) that the inequality (1.7) is a necessary and sufficient condition for the sum (1.6) to converge.

The paper is set out as follows. An explicit form for the sum (1.6) (the main result) is established in Section 2, while special cases of it are considered in Section 3. A glimpse of some possible extensions is caught in Sections 4 and 5.

2. THE MAIN RESULT

The main result established in this paper reads as follows.

Theorem 1:

$$S_h(r,s,m)=\frac{\sum_{i=0}^{h+1}\sum_{j=1}^{h}\binom{h+1}{i}A_{h,j}s^{i+j}r^{2h-i-j+2}V_{i-j}}{(s^2-srm-r^2)^{h+1}},\tag{2.1}$$

where

$$A_{h,j}=\sum_{n=0}^{j}(-1)^n\binom{h+1}{n}(j-n)^h\tag{2.2}$$

are the *Eulerian numbers* (e.g., see [3]), and the inequality (1.7) is assume to be satisfied. We recall that the numbers $A_{h,j}$ may be expressed equivalently [2] as

$$A_{h,j}=(-1)^j\sum_{n=1}^{j}(-1)^n\binom{h+1}{j-n}n^h.\tag{2.2'}$$

Observe that (2.1) involves the use of (1.5). For illustration only, we show the first few numbers $A_{h,j}$ ($A_{h,j}\neq 0$ for $1\leq j\leq h$; $A_{h,j}=A_{h,h-j+1}$):

$$\begin{aligned}
&A_{1,1}=1,\\
&A_{2,1}=A_{2,2}=1,\\
&A_{3,1}=A_{3,3}=1;\ A_{3,2}=4,\\
&A_{4,1}=A_{4,4}=1;\ A_{4,2}=A_{4,3}=11,\\
&A_{5,1}=A_{5,5}=1;\ A_{5,2}=A_{5,4}=26;\ A_{5,3}=66,\\
&A_{6,1}=A_{6,6}=1;\ A_{6,2}=A_{6,5}=57;\ A_{6,3}=A_{6,4}=302.
\end{aligned}$$

Proof of Theorem 1: First, recall (e.g., see [10]) that

$$\sum_{k=1}^{\infty}k^hy^k=\frac{1}{(1-y)^{h+1}}\sum_{j=1}^{h}A_{h,j}y^{h-j+1}\quad(|y|<1).\tag{2.3}$$

Then use (1.2), (2.3), and (1.4) along with (1.6) to write

$$\begin{aligned}
S_h(r,s,m)&=\sum_{k=1}^{\infty}k^h(r\alpha_m/s)^k+\sum_{k=1}^{\infty}k^h(r\beta_m/s)^k\\
&=\frac{\sum_{j=1}^{h}A_{h,j}(r\alpha_m/s)^{h-j+1}}{(1-r\alpha_m/s)^{h+1}}+\frac{\sum_{j=1}^{h}A_{h,j}(r\beta_m/s)^{h-j+1}}{(1-r\beta_m/s)^{h+1}}
\end{aligned}$$

$$= \frac{(r\alpha_m)^{h+1}\sum_{j=1}^{h} A_{h,j}(r\alpha_m/s)^{-j}}{(s-r\alpha_m)^{h+1}} + \frac{(r\beta_m)^{h+1}\sum_{j=1}^{h} A_{h,j}(r\beta_m/s)^{-j}}{(s-r\beta_m)^{h+1}}$$

$$= \frac{(sr\alpha_m+r^2)^{h+1}\sum_{j=1}^{h} A_{h,j}(r\alpha_m/s)^{-j} + (sr\beta_m+r^2)^{h+1}\sum_{j=1}^{h} A_{h,j}(r\beta_m/s)^{-j}}{(s^2-srm-r^2)^{h+1}}. \tag{2.4}$$

By using the binomial expansions of $(sr\alpha_m+r^2)^{h+1}$ and $(sr\beta_m+r^2)^{h+1}$, the equality (2.4) becomes

$$S_h(r,s,m) = \left[\sum_{i=0}^{h+1}\sum_{j=1}^{h}\binom{h+1}{i}A_{h,j}s^{i+j}r^{2h+2-i-j}\alpha_m^{i-j} + \sum_{i=0}^{h+1}\sum_{j=1}^{h}\binom{h+1}{i}A_{h,j}s^{i+j}r^{2h+2-i-j}\beta_m^{i-j}\right] / (s^2-srm-r^2)^{h+1},$$

whence, by using the Binet form (1.4), one gets the right-hand side of (2.1). Q.E.D.

3. SPECIAL CASES

In this section we consider ordered pairs (r, s) subject to (1.7) for which $S_h(r,s,m)$, beyond being a positive *integer*, has a form that is much more compact than (2.1). First, we need the following two propositions.

Proposition 1:
$$U_{2n+1}^2 - mU_{2n+1}U_{2n} - U_{2n}^2 = 1. \tag{3.1}$$

Proposition 2: If we let δ stand for either α or β, then
$$\delta_m U_{2n+1} + U_{2n} = \delta_m^{2n+1}. \tag{3.2}$$

Proof of Proposition 1: Rewrite the left-hand side of (3.1) as

$$\begin{aligned} &U_{2n+1}^2 - U_{2n}(mU_{2n+1}+U_{2n}) \\ &= U_{2n+1}^2 - U_{2n}U_{2n+2} \quad \text{[by (1.1)]} \\ &= -(-1)^{2n+1} = 1 \qquad \text{\{by the \textit{Simson formula} (e.g., see (2.18) of [9])\}}. \text{ Q.E.D.} \end{aligned}$$

Proof of Proposition 2: For the sake of brevity, we shall prove only the case $\delta = \alpha$. Using (1.2), rewrite the left-hand side of (3.2) as

$$\begin{aligned} &\left[(\alpha_m^{2n+1}-\beta_m^{2n+1})\alpha_m + \alpha_m^{2n} - \beta_m^{2n}\right]/\Delta_m \\ &= (\alpha_m^{2n+2}+\alpha_m^{2n})/\Delta_m \quad \text{[by (1.4)]} \\ &= \alpha^{2n+1}(\alpha_m + \alpha_m^{-1})/\Delta_m \\ &= \alpha^{2n+1}(\alpha_m - \beta_m)/\Delta_m \quad \text{[by (1.4)]} \\ &= \alpha^{2n+1}U_1(m) = \alpha^{2n+1}\cdot 1 = \alpha^{2n+1}. \text{ Q.E.D.} \end{aligned}$$

Now, after observing that $U_{2n}/U_{2n+1} < 1/\alpha_m$ for all m, we state the following theorem.

Theorem 2:

$$S_h(U_{2n}, U_{2n+1}, m) = \sum_{j=1}^{h} A_{h,j} U_{2n+1}^j U_{2n}^{h+1-j} V_{(2n+1)(h+1)-j}\,. \tag{3.3}$$

Proof: Replace r by U_{2n} and s by U_{2n+1} in (2.4), and use Propositions 1 and 2 to write

$$\begin{aligned} S_h(U_{2n}, U_{2n+1}, m) &= (U_{2n}\alpha_m^{2n+1})^{h+1} \sum_{j=1}^{h} A_{h,j}\left(\frac{U_{2n}}{U_{2n+1}}\alpha_m\right)^{-j} \\ &\quad + (U_{2n}\beta_m^{2n+1})^{h+1} \sum_{j=1}^{h} A_{h,j}\left(\frac{U_{2n}}{U_{2n+1}}\beta_m\right)^{-j} \\ &= \sum_{j=1}^{h} A_{h,j} U_{2n+1}^j U_{2n}^{h+1-j}\left[\alpha_m^{(2n+1)(h+1)-j} + \beta_m^{(2n+1)(h+1)-j}\right]. \end{aligned}$$

By using the Binet form (1.2), the right-hand side of (3.3) is immediately obtained. Observe that, by solving in integers the Pell equation (1) on page 100 in [13], it can be proved that the pairs (U_{2n}, U_{2n+1}) are the *only* pairs (r, s) for which the denominator of (2.1) equals 1. Q.E.D.

A very special case $(n = m = 1)$ of (3.3) is

$$S_h(1, 2, 1) = \sum_{k=1}^{\infty} k^h L_k / 2^k = \sum_{j=1}^{h} A_{h,j} 2^j L_{3(h+1)-j}\,. \tag{3.4}$$

The proof of the identity

$$S_h(V_{2n-1}, V_{2n}, m) = \left[\sum_{j=1}^{h} A_{h,j} V_{2n}^j V_{2n-1}^{h+1-j} V_{2n(h+1)-j}\right] / \Delta_m^2, \tag{3.5}$$

which is the Lucas analog of (3.3), is left as an exercise for the interested reader.

4. EXTENSIONS

It is obvious that the result (2.1) allows us to evaluate the more general series

$$\sum_{k=1}^{\infty} p(k)\frac{r^k}{s^k} V_k\,, \tag{4.1}$$

where $p(k)$ is a polynomial in k. As a minor example, we offer the following identity:

$$S(n, m) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} (k^2 + k)\frac{U_{2n}^k}{U_{2n+1}^k} V_k = 2U_{2n} U_{2n+1}^2 V_{6n+1}\,. \tag{4.2}$$

Proof: By (3.3), write

$$\begin{aligned} S(n, m) &= S_1(U_{2n}, U_{2n+1}, m) + S_2(U_{2n}, U_{2n+1}, m) \\ &= A_{1,1} U_{2n+1} U_{2n} V_{2(2n+1)-1} + A_{2,1} U_{2n+1} U_{2n}^2 V_{3(2n+1)-1} + A_{2,2} U_{2n+1}^2 U_{2n} V_{3(2n+1)-2}\,. \end{aligned}$$

Recalling that $A_{1,1} = A_{2,1} = A_{2,2} = 1$,

$$S(n, m) = U_{2n} U_{2n+1}(V_{4n+1} + U_{2n} V_{6n+2} + U_{2n+1} V_{6n+1}). \tag{4.3}$$

After some tedious manipulations involving the use of (1.2), it can be proved that

$$V_{4n+1}+U_{2n}V_{6n+2}=U_{2n+1}V_{6n+1}. \tag{4.4}$$

The identity (4.2) readily follows from (4.3) and (4.4). Q.E.D.

We also tried to extend (1.6) to negative values of h. For $h=-1$, we get

$$\sum_{k=1}^{\infty}\frac{1}{k}\frac{r^kV_k}{s^k}=-\ln\frac{s^2-rsm-r^2}{s^2} \quad (r/s<1/\alpha_m). \tag{4.5}$$

Proof: By using (1.2), (1.4), and the identity 1.513.4 on page 44 in [7],

$$\sum_{k=1}^{\infty}k^{-1}y^k=-\ln(1-y) \quad (|y|<1), \tag{4.6}$$

the left-hand side of (4.5) can be rewritten as

$$-\ln(1-r\alpha_m/s)-\ln(1-r\beta_m/s)=-\ln[(1-r\alpha_m/s)(1-r\beta_m/s)]$$
$$=-\ln\frac{s^2-rsm-r^2}{s^2}. \quad \text{Q.E.D.}$$

Special cases of (4.5) are

$$\sum_{k=1}^{\infty}\frac{1}{k}\frac{U_{2n}^kV_k}{U_{2n+1}^k}=-\ln\frac{1}{U_{2n+1}^2}=2\ln U_{2n+1}. \tag{4.7}$$

For $h=-2$, we should have at our disposal a closed-form expression for $\sum_{k=1}^{\infty}k^{-2}y^k$. By (4.6), it can readily be seen that

$$\sum_{k=1}^{\infty}k^{-2}y^k=-\int\frac{\ln(1-y)}{y}dy \quad (|y|<1). \tag{4.8}$$

Unfortunately, the right-hand side of (4.8) cannot be expressed in terms of elementary transcendental functions.

We conclude this paper by establishing the following identity:

$$\sum_{k=1}^{\infty}\frac{1}{k^2+k}\frac{U_{2n}^kV_k}{U_{2n+1}^k}=2+\frac{U_{2n+1}}{U_{2n}}(m\ln U_{2n+1}-2n\Delta_m\ln\alpha_m)+2\ln U_{2n+1}. \tag{4.9}$$

Proof: By using (1.2) and the identity 1.513.5 on page 45 of [7],

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}y^k=1-\frac{1-y}{y}\ln\frac{1}{1-y} \quad (|y|<1), \tag{4.10}$$

first write

$$X(r,s)\overset{\text{def}}{=}\sum_{k=1}^{\infty}\frac{1}{k^2+k}\frac{r^kV_k}{s^k}=1-\frac{1-r\alpha_m/s}{r\alpha_m/s}\ln\frac{1}{1-r\alpha_m/s}+1-\frac{1-r\beta_m/s}{r\beta_m/s}\ln\frac{1}{1-r\beta_m/s}$$
$$=2-\left[\frac{s-r\alpha_m}{r\alpha_m}\ln\frac{s}{s-r\alpha_m}+\frac{s-r\beta_m}{r\beta_m}\ln\frac{s}{s-r\beta_m}\right]$$
$$=2-\left[\frac{s}{r\alpha_m}\ln\frac{s}{s-r\alpha_m}+\frac{s}{r\beta_m}\ln\frac{s}{s-r\beta_m}\right]+\ln\frac{s^2}{(s-r\alpha_m)(s-r\beta_m)}.$$

After some manipulations involving the use of (1.4), one obtains

$$X(r,s)=2+\frac{sm}{r}\ln s+\ln\frac{s^2}{s^2-rsm-r^2}+\frac{s}{r\alpha_m}\ln(s-r\alpha_m)+\frac{s}{r\beta_m}\ln(s-r\beta_m). \tag{4.11}$$

Then, replace r by U_{2n} and s by U_{2n+1} in (4.11), thus getting

$$\begin{aligned}X(U_{2n},U_{2n+1})&=2+\frac{mU_{2n+1}}{U_{2n}}+(2\ln U_{2n+1}-\ln 1)\\&+\left[\frac{U_{2n+1}}{\alpha_m U_{2n}}\ln(U_{2n+1}-\alpha_m U_{2n})+\frac{U_{2n+1}}{\beta_m U_{2n}}\ln(U_{2n+1}-\beta_m U_{2n})\right].\end{aligned} \tag{4.12}$$

Using (1.2) and (1.4), the expression within square brackets becomes

$$\begin{aligned}\left[\frac{U_{2n+1}}{\alpha_m U_{2n}}\ln\beta_m^{2n}+\frac{U_{2n+1}}{\beta_m U_{2n}}\ln\alpha_m^{2n}\right]&=\frac{U_{2n+1}}{U_{2n}}\left[\frac{1}{\alpha_m}\ln\frac{1}{\alpha_m^{2n}}+\frac{1}{\beta_m}\ln\alpha_m^{2n}\right]\\&=\frac{U_{2n+1}}{U_{2n}}(\beta_m^{-1}-\alpha_m^{-1})\ln\alpha_m^{2n}=-\frac{U_{2n+1}}{U_{2n}}2n\Delta_m\ln\alpha_m.\end{aligned} \tag{4.13}$$

The right-hand side of (4.9) readily follows from (4.12) and (4.13). Q.E.D.

5. CONCLUDING COMMENTS

For the sake of brevity, we confined ourselves to considering series involving only the numbers $V_k(m)$. On the other hand, analogous results for $U_k(m)$ can readily be obtained by paralleling the arguments in Sections 2, 3, and 4. This can be done as an exercise by the interested reader.

The investigation of infinite series involving terms of the more general sequences $\{W_k(a,b;m,q)\}_{k=0}^{\infty}$ (see [8]) seems to be a substantial extension of our study, and will be the object of a future work. For $a=0$ and $b=-q=1$, this investigation might lead to an interesting generalization of Theorem 10 of [11].

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IN MEMORIAM—LEONARD CARLITZ

Leonard Carlitz, a long-time friend and supporter of The Fibonacci Association, passed away on September 17, 1999. For many years Carlitz was on the editorial board of *The Fibonacci Quarterly*, and between 1963 and 1984 he published 72 articles in the *Quarterly* (including 19 joint papers and 7 short notes).

Carlitz was born in 1907 in Philadelphia, and he grew up in that city. He won a scholarship to the University of Pennsylvania where he completed his AB degree in 1927, his MA degree in 1928, and his Ph.D. in 1930—all in mathematics. His Ph.D. thesis advisor was H. H. Mitchell, who had been a student of Oswald Veblen who, in turn, had studied under E. H. Moore. Inspired by earlier research of Emil Artin, Carlitz wrote his dissertation of "Galois Fields of Certain Types." This work appeared under the same title in the 1930 *Transactions of the AMS* (Vol. 32, pp. 451-472).

Carlitz spent the 1930-1931 academic year as a National Research Council Fellow studying with E. T. Bell at the California Institute of Technology, and he spent the 1931-1932 academic year with G. H. Hardy in Cambridge, England, as an International Research Fellow. He taught at Duke University, where he was James B. Duke Professor of Mathematics, from 1932 until his retirement in 1977. At Duke he was research advisor to 44 Ph.D. students and 51 MS students. He was also involved in the early planning for the *Duke Mathematical Journal* (established 1935), and he served for many years as the managing editor. He spent the year 1935-1936 at the Institute for Advanced Study.

In the summer of 1931, between Caltech and Cambridge, Carlitz met and married Clara Skaler. They had two children: Michael (born 1939) and Robert (born 1945). Mrs. Carlitz died in 1990.

Carlitz was a prolific and insightful researcher, with 771 publications in many different areas of mathematics. He will be remembered as a first-class mathematician, an inspiring teacher, and a kind, generous man. More information about him, including some personal anecdotes, can be found in the excellent tribute by Joel Brawley: "Dedicated to Leonard Carlitz: The Man and His Work" [*Finite Fields and Their Applications* **1** (1995):135-151].

F. T. Howard

CERTIFICATES OF INTEGRALITY FOR LINEAR BINOMIALS

David Callan

Dept. of Statistics, Univ. of Wisconsin-Madison, 1210 W. Dayton St., Madison, WI 53706-1693

callan@stat.wisc.edu

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1. INTRODUCTION AND STATEMENT OF MAIN THEOREM

Everyone knows that the familiar binomial coefficients are integers. But it is not so obvious that quotients of binomial coefficients whose parameters are linear in n by factors linear in n also sometimes yield sequences of integers. For example,

$$\left\{\frac{1}{n+1}\binom{2n}{n}\right\}_{n\geq 0} = \{1, 1, 2, 5, 14, 42, 132, \ldots\}$$

is the well-known sequence of Catalan numbers. In the same vein,

$$\left\{\frac{3}{n}\binom{2n}{n-3}\right\}_{n\geq 3} = \{1, 6, 27, 110, 429, \ldots\}$$

is sequence M4177 in Sloane and Plouffe's *Encyclopedia of Integer Sequences* [3],

$$\left\{\frac{3}{n+2}\binom{2n}{n-1}\right\} \text{ is sequence M2809,}$$

$$\left\{\frac{4}{(3n+2)(3n+1)}\binom{3n+2}{n}\right\} \text{ is sequence M1660,}$$

$$\left\{\frac{5}{n+3}\binom{2n}{n-1}\right\} \text{ is sequence M3904.}$$

There are at least another dozen such sequences listed in the *Encyclopedia*, including M1782, M2243, M2926, M2946, M2997, M3483, M3542, M3587, M4198, M4214, M4529, M4721. Incidentally, the smallest-parameter such sequence of integers *not* listed seems to be

$$\left\{\frac{1}{n}\binom{3n}{n+1}\right\} = \left\{2\binom{3n-1}{n} - \binom{3n-1}{n+1}\right\} = \{3, 10, 42, 198, 1001, \ldots\}.$$

Why are these sequences integral while similar sequences such as

$$\frac{k}{n}\binom{2n}{n} \quad \text{and} \quad \frac{k}{2n+1}\binom{2n}{n}$$

are not, no matter what the integer k is? Here we attempt to shed some light on this question. Each of the above sequences is an integer multiple of a sequence of the form

$$\mathbf{w} = \frac{1}{P(n)}\binom{an+b}{cn+d},$$

where $P(n)$ is a product of one or more factors linear in n with integral coefficients and a, b, c, d are integers with $a > c > 0$. Let us call such a sequence $\mathbf{w}$ *linear binomial*. In this paper, we establish a simple and intuitively appealing criterion for a linear binomial sequence **w** to have bounded denominators, equivalently, for the existence of an integer k such that $k\mathbf{w}$ is a sequence of integers. Furthermore, when the criterion is met, the proof consists of verification of an algorithm that produces not only a suitable multiplier k, but also a "Certificate of Integrality" for $k\mathbf{w}$ in the form of an identity expressing it as an integral linear combination of binomial coefficients. For example, the algorithm yields that the Catalan number $\frac{1}{n+1}\binom{2n}{n}$ is equal to

$$\binom{2n}{n}-\binom{2n}{n-1}.$$

For $\frac{1}{n}\binom{2n}{n-3}$, the algorithm returns the identity

$$\frac{3}{n}\binom{2n}{n-3}=\binom{2n-1}{n-3}-\binom{2n-1}{n-4}.$$

A small Mathematica package, DecomposeBinomial, implementing this algorithm, is available from the author's home page at **http://www.stat.wisc.edu/~callan/.**

The criterion for bounded denominators revolves around cancellation of the factors in $P(n)$ with factors in what might be called the symbolic numerator of $\binom{an+b}{cn+d}$. Here cancellation refers to proportional polynomials or, equivalently, division in the polynomial ring $\mathbb{Q}[n]$. Set $e=a-c$ and $f=b-d$. Thus, for any particular n,

$$\binom{an+b}{cn+d}=\frac{(an+b)(an+b-1)\dots(en+f+1)}{(cn+d)!}. \tag{1}$$

Now define the *numerator set* N of this binomial coefficient (considered symbolically) to be $N=U\cup V$, where $U=\{an+b-i\}_{i\geq 0}$ and $V=\{en+f+j\}_{j\geq 1}$. Thus, N contains both "ends" of the range of factors in the numerator in (1) but not the "middle." For example, for $\binom{6n}{2n}$, the numerator set consists of $\{6n,6n-1,6n-2,\dots\}\cup\{4n+1,4n+2,\dots\}$ (but not any term of the form $5n\pm i$). Similarly, define the *denominator set* $D=\{cn+d-i\}_{i\geq 0}$. The desired criterion can now be expressed as follows: *Each linear factor in $P(n)$ must divide a factor in N and if a factor in D is proportional to one in $P(n)$, it too must divide a factor in N (always taking multiplicity into account).*

For example, $\frac{1}{2n+1}\binom{2n}{n}$ fails to meet this criterion because $2n+1$ does not divide any term in $N=\{2n,2n-1,\dots\}\cup\{n+1,n+2,\dots\}$. And $\frac{1}{n}\binom{2n}{n}$ also fails to meet the criterion because $D=\{n,n-1,\dots\}$ includes n, giving two n's that need to divide factors in $N=\{2n,2n-1,\dots\}\cup\{n+1,n+2,\dots\}$ but only one term in N is divisible by n. On the other hand, $\frac{1}{n}\binom{5n}{2n+1}$ does meet the criterion because, although here again D includes a factor proportional to n, namely $2n$, the numerator set $N=\{5n,5n-1,\dots\}\cup\{3n,3n+1,\dots\}$ contains *two* terms proportional to n, and so both offending factors can be canceled. Clearly, no two factors in U (resp. V, resp. D) can be proportional. It follows that the criterion cannot be met if $P(n)$ has two proportional (or repeated) factors. This is because the only way N can contain two proportional factors is if one of them is in U (say in the i^{th} position) and the other in V (say in the j^{th} position). But then a simple calculation shows that the $(i+j)^{\text{th}}$ term in D would also be proportional to both, and "three into two won't go."

To state the criterion (and our main result) succinctly, we make two definitions. Say a linear factor *appears* in a set if it is proportional to a term in the set. Thus, $2n+1$ appears in the numerator set of $\binom{4n+3}{n}$. Also, say a linear binomial sequence $\frac{1}{P(n)}\binom{an+b}{cn+d}$ is *normalized* if each linear factor $gn+h$ in $P(n)$ has relatively prime coefficients g, h.

Using this terminology, our main result can be formulated as follows.

Theorem 1: Suppose $\mathbf{w} = \frac{1}{P(n)}\binom{an+b}{cn+d}$ is a normalized linear binomial sequence. Then $\mathbf{w}$ has bounded denominators if and only if $P(n)$'s linear factors are distinct and each such factor appears in the numerator set N of the binomial coefficient (as defined above), and appears there twice if it also appears in the denominator set D.

Furthermore, if a linear binomial sequence $\mathbf{w}$ has bounded denominators, then there is a positive integer k such that $k\mathbf{w}$ is an integral linear combination of a fixed number (independent of n) of binomial coefficients with parameters linear in n.

Remark: Bearing in mind that a factor can appear at most twice in N, an equivalent but more pithy formulation of the criterion for bounded denominators is: if and only if $P(n)$'s linear factors are distinct, and each appears more often in N than in D.

The "only if" part is proved in §2. It relies on Dirichlet's classic theorem on primes in arithmetic progressions [1, Chap. 7], and Kummer's pretty rule for finding the exact power of a prime p that divides a binomial coefficient; the number of carries when its parameters are subtracted in base p. See [2, Ex. 5.36, p. 245] for a proof of Kummer's rule (in an equivalent formulation in terms of addition in base p). The "if" part is proved in §4. It relies on a neat determinant expansion, of interest in its own right, that is presented in §3. Finally, §5 contains a mild extension of the main theorem, some further remarks, and a conjecture.

2. MAIN THEOREM: PROOF OF "ONLY IF"

We will show that infinitely many primes occur among the denominators in $\frac{1}{P(n)}\binom{an+b}{cn+d}$ when the criterion of Theorem 1 is not met. Let $gn+h$ be a factor in $P(n)$. Suppose $p = gn+h$ is prime (as it will be for infinitely many n by Dirichlet's theorem, since g and h are relatively prime). Write $a = q_1 g + r_1$ with $0 \le r_1 \le g$ and $c = q_2 g + r_2$ with $0 \le r_2 \le g$ (division algorithm). Expressed in base p, the two parameters of the binomial coefficient are then (for sufficiently large n)

$$an+b = \begin{array}{|c|c|} \hline p & 1 \\ \hline q_1 & r_1 n + b - q_1 h \\ q_1 & b - q_1 h \\ q_1 - 1 & p - (q_1 h - b) \\ \hline \end{array} \quad \begin{array}{l} \\ \text{if } r_1 \ne 0, \\ \text{if } r_1 = 0 \text{ and } b \ge q_1 h, \\ \text{if } r_1 = 0 \text{ and } b < q_1 h, \end{array}$$

and similarly,

$$cn+d = \begin{array}{|c|c|} \hline p & 1 \\ \hline q_2 & r_2 n + d - q_2 h \\ q_2 & d - q_2 h \\ q_2 - 1 & p - (q_2 h - d) \\ \hline \end{array} \quad \begin{array}{l} \\ \text{if } r_2 \ne 0, \\ \text{if } r_2 = 0 \text{ and } d \ge q_2 h, \\ \text{if } r_2 = 0 \text{ and } d < q_2 h. \end{array}$$

In particular, since $an+b$ has only two digits in base p, at most one carry can occur in subtracting $cn+d$ from $an+b$ in base p. Thus, $p^2 \nmid \binom{an+b}{cn+d}$ and, if $gn+h$ is a repeated factor in $P(n)$, then p will occur among the denominators in $\mathbf{w}$ (for infinitely many primes p) and $\mathbf{w}$ will have unbounded denominators. Also, no carries occur in subtraction, equivalently $p \nmid \binom{an+b}{cn+d}$ if and only if $(an+b) \bmod p \ge (cn+d) \bmod p$. It is straightforward to verify that $gn+h$ appears (i) in U iff $r_1 = 0$ and $b \ge q_1 h$, (ii) in V iff $r_1 = r_2$ and $(q_1 - q_2)h > b - d$, (iii) in D iff $r_2 = 0$ and $d \ge q_2 h$. Except for one wrinkle, it is now simply a matter of checking cases to verify $p \nmid \binom{an+b}{cn+d}$ unless $gn+h = p$ appears in the numerator set N at least once, and twice if it appears in the denominator set D. This will show that infinitely many primes occur among the denominators in $\mathbf{w}$, as desired. The one wrinkle is that when $0 < r_1 < r_2$ (a subcase where $gn+h$ does not appear in N at all), p *does* divide $\binom{an+b}{cn+d}$ and we proceed as follows. Set $n = (g-1)m - h$ with m variable; thus,

$$\frac{1}{gn+h}\binom{an+b}{cn+d} = \frac{1}{g-1}\,\frac{1}{gm-h}\binom{a(g-1)m-ah+b}{c(g-1)m-ch+d}.$$

Here $r_1' := (a(g-1)) \bmod g = g - r_1$ and $r_2' := (c(g-1)) \bmod g = g - r_2$. Since $r_1' > r_2'$, the case $r_1 > r_2$ applies with m in place of n, $a(g-1)$ in place of a, and the role of p played by $gm-h$. This completes the proof of the "only if" part.

3. A DETERMINANT EXPANSION

The following result is crucial for the "if" part of the main theorem in the next section. Let *coeff* denote the function that produces the row vector of coefficients of a polynomial or the matrix of coefficients of a list of polynomials. Thus,

$$\operatorname{coeff}\left(\sum_{i=0}^{m} c_i x^i\right) = (c_i)_{i=0}^{m}.$$

Let $*$ denote convolution of sequences; thus,

$$\operatorname{coeff}(p(x)q(x)) = \operatorname{coeff}(p(x)) * \operatorname{coeff}(q(x)).$$

Also, for a matrix N, let N° denote the column vector obtained by taking the Hadamard (entry-wise) product of the columns in N. For example, for $N = \left(\begin{smallmatrix}1 & 2\\ 3 & 4\end{smallmatrix}\right)$, $N^\circ = \left(\begin{smallmatrix}2\\ 12\end{smallmatrix}\right)$.

Theorem 2: Let m be a positive integer and let a_j $(1 \le j \le m)$, b_{ji} $(1 \le i \le j \le m)$, c, e, x be indeterminates. Let N be the $m+1$ by m matrix with rows indexed $[0, m]$ and columns indexed $[1, m]$, and (i, j) entry

$$\begin{cases} cx + a_j & \text{if } 0 \le i < j \le m, \\ ex + b_{ij} & \text{if } 1 \le j \le i \le m. \end{cases}$$

Let M be the $m+1$ by $m+1$ matrix $\operatorname{coeff}(N^\circ)$. For example, when $m = 2$,

$$N = \begin{pmatrix} cx+a_1 & cx+a_2 \\ ex+b_{11} & cx+a_2 \\ ex+b_{21} & ex+b_{22} \end{pmatrix} \quad\text{and}\quad M = \begin{pmatrix} a_1a_2 & (a_1+a_2)c & c^2 \\ b_{11}a_2 & b_{11}c + a_2 e & ce \\ b_{21}b_{22} & (b_{21}+b_{22})e & e^2 \end{pmatrix}.$$

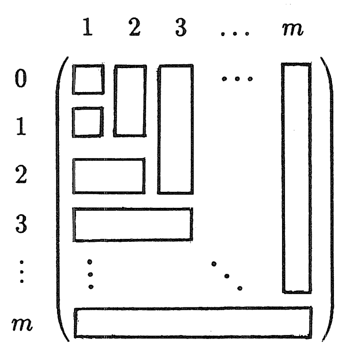
Then $\det M = \prod_{1 \le i \le j \le m}(ea_j - cb_{ji})$.

Proof: We first show, for $1 \le i \le j \le m$, that $ea_j - cb_{ji}$ divides det M in the polynomial ring $\mathbb{Q}(e, c)[a'\text{s}, b'\text{s}]$. To do so, suppose $ea_j = cb_{ji}$ for some i, j. Let N_j denote the submatrix of n consisting of rows 0 through j. Then $p_j := \prod_{j \le i \le m}(cx + a_i)$ is a factor in each entry of N_j; we may write $N_j^\circ = (r_i)_{0 \le i \le j} p_j$ with $\deg r_i = j - 1$ $(0 \le i \le j)$. Now rows 0 through j of M constitute the submatrix $M_j = \text{coeff}\,(N_j^\circ) = (\text{coeff}(r_i))_{0 \le i \le j} * \text{coeff}(p_j)$ [convolution of each coeff(r_i) with coeff(p_j)]. Since $R_j := (\text{coeff}(r_i))_{0 \le i \le j}$ is a $j+1$ by j matrix, its rows are linearly dependent [over $\mathbb{Q}(e, c, a'\text{s}, b'\text{s})$] and there exists a nonzero vector $\mathbf{u} = (u_i)_{0 \le i \le j}$ such that $\mathbf{u}R_j = \mathbf{0}$. Thus,

$$\mathbf{u}M_j = \mathbf{u}(R_j * \text{coeff}(p_j)) = (\mathbf{u}R_j) * \text{coeff}(p_j) = \mathbf{0} * \text{coeff}(p_j) = \mathbf{0}$$

and M is singular. Hence, $ea_j - cb_{ji}$ is a factor of det M. Since each $ea_j - cb_{ji}$ is obviously prime in $\mathbb{Q}(e, c)[a'\text{s}, b'\text{s}]$, their product also divides det M and Theorem 2 follows by confirming the degrees agree and the coefficients of any one term agree.

Corollary 3: Let N be an $m+1$ by m matrix with linear polynomials in one indeterminate as entries. Partition N into offset row and column segments as indicated. (Each vertical column segment sits atop the last position in the corresponding row segment.)



Suppose, for $1 \le j \le m$, that all entries in column segment j are equal and this common entry does not divide any of the entries in row segment j.

Then the $m+1$ by $m+1$ matrix $M = \text{coeff}(N^\circ)$ is invertible.

Proof: The matrix N is of the form in Theorem 2. Clearly, a factor $ea_j - cb_{ji}$ $(1 \le i \le j \le m)$ in det M is 0 if and only if $cx + a_j$ is proportional to $ex + b_{ji}$, that is, divides $ex + b_{ji}$. But these polynomials lie in corresponding row and column segments and thus the hypothesis ensures that one does not divide the other. Hence det $M \ne 0$ and M is invertible.

4. MAIN THEOREM: PROOF OF "IF"

We seek an expression for $\frac{1}{P(n)}\binom{an+b}{cn+d}$ as a rational-coefficient linear combination of binomial coefficients. Due to the basic identity $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$, we can always reduce an upper parameter at the expense of increasing the number of terms in the linear combination. Thus, we look for a combination in which all the upper parameters are the same. It will turn out that a suitable upper parameter is determined by the factors in $P(n)$ that appear in U (the upper range in the numerator set). Specifically, it is $an+b-u$, where u is the location of the last term in U that appears in $P(n)$ (and $u=0$ if there is no such term).

By hypothesis, each (linear) factor of $P(n)$ appears in U or V or possibly both. Let

$$(an+b+1-i)_{i\in I} \cup (en+f+j)_{j\in J}$$

be a complete listing of these appearances, where I and J are finite subsets (one of them may be empty) of the positive integers. Set $u=\max I$ and $v=\max J$ (with $\max\emptyset := 0$). Let $r_1 = an+b+1-i$, $s_1 = en+f+v+1-i$, and $t_i = cn+d-u-v+i$, so that

$$\binom{an+b}{cn+d} = \frac{\overbrace{r_1r_2\ldots r_u}\cdots\overbrace{s_1s_2\ldots s_v}}{\underbrace{t_{u+v}t_{u+v-1}\ldots t_1}\cdots} = \frac{\prod_{i=1}^{u} r_i \prod_{j=1}^{v} s_j}{\prod_{i=1}^{u+v} t_i}\binom{an+b-u}{cn+d-(u+v)}.$$

We claim that all appearances of $P(n)$'s factors in $N\cup D$ occur within the three groupings displayed in the middle expression. This is true for the numerator N by definition of u and v. And if a $P(n)$ factor $gn+h$ appears in D, then by hypothesis it appears in both U and V, say in the i^{th} position in U and the j^{th} position in V. As noted earlier, a simple calculation then shows that the position in D at which $gn+h$ appears is $i+j$. Since $i\le u$ and $j\le v$, it follows that $i+j\le u+v$ and so the $(i+j)$ term in D is one of the displayed t's. Hence, the claim.

Next, we have to determine appropriate lower parameters for the binomial coefficients in the desired linear combination. This turns out to be a little tricky; rather than being consecutive as one might expect, they turn out to form an interval with a hole in it. To this end, define $L=\{i\in[1,u+v]: t_i\,|\,s_j$ in the ring $\mathbb{Q}[n]$ for some j with $1\le j\le i\}$. Since the j here is necessarily unique, we get a map $\phi: L\to[1,u+v]$ satisfying $t_i\,|\,s_{\phi(i)}$ and $\phi(i)\le i$, $i\in L$. Also, it is easy to check that L is either empty or an interval of integers. (The reader might like to look ahead to the illustrative example at the end of this section.) Suitable lower parameters are determined by removing L from the set $[1,u+v]$ and adjoining 0. Thus, we set $K:=[1,u+v]\setminus L$ and the rest of the proof is devoted to showing that there exist (unique) rational numbers $(c_i)_{i\in K\cup\{0\}}$ such that

$$\sum_{i\in K\cup\{0\}} c_i\binom{an+b-u}{cn+d-(u+v)+i} = \frac{1}{P(n)}\binom{an+b}{cn+d}. \tag{2}$$

Factoring out $\binom{an+b-u}{cn+d-(u+v)}/\prod_{j\in K} t_j$ from each side, (2) is equivalent to

$$c_0\prod_{j\in K} t_j + \sum_{i\in K} c_i \frac{s_1\ldots s_i t_{i+1}\ldots t_{u+v}}{\prod_{j\in L} t_j} = \frac{\prod_{i=1}^{u} r_i\prod_{j=1}^{v} s_j}{P(n)\prod_{j\in L} t_j}. \tag{3}$$

We will show that (i) both sides of (3) are polynomials in n, and (ii) equating coefficients of like powers of n in these polynomials yields a system of linear equations for the c_i's with a coefficient matrix to which the Corollary to Theorem 2 applies (and which is therefore invertible).

Consider the right side of (3). All the factors in $P(n)$ appear in its numerator by definition of u and v. For $j\in L$, we have $t_j\,|\,s_{\phi(j)}$. If $\phi(j)\le v$, then $s_{\phi(j)}$ is present in the numerator. If, on the other hand, $\phi(j)>v$, we claim: t_j also divides some r_i with $1\le i\le u$. In fact, $i=u+\phi(j)-j$ works. First, $i\ge 1$ since $i>u+v-j\ge 0$ and $i\le u$ since $\phi(j)\le j$. Second, $t_j\,|\,s_{\phi(j)}$ implies

$$\begin{aligned} t_j\,|\,t_j+s_{\phi(j)} &= (cn+d-(u+v)+j)+(en+f+v+1-\phi(j)) \\ &= an+b+1-u+j-\phi(j) = an+b+1-i = r_i. \end{aligned}$$

Hence, the claim. Thus, every factor in the denominator divides a factor in the numerator. And if a factor in $P(n)$ also appears among $\{t_j\}_{j\in L}$, then by hypothesis it appears twice in N and hence appears twice in the numerator. So the right side of (3) is indeed a polynomial $P_{\mathrm{rhs}}(n)$ and its degree is $u+v-\deg P-|L|=|K|-\deg P$.

As for the left side of (3), it is clearly a polynomial if $L=\emptyset$. Else, since $K=[1,u+v]\setminus L$ and L consists of consecutive integers in $[1,u+v]$, K may be written as a disjoint union of intervals $K_s\cup K_b$ (K_s for the smaller numbers, here one of K_s, K_b may be empty). For $i\in K_s$, summand i is $c_i(\prod_{j=1}^{i}s_j)(\prod_{k\in K,k>i}t_k)$. Now suppose $i\in K_b$. As $t_j\,|\,s_{\phi(j)}$ for $j\in L$ and $\phi(j)\le j\le \max L<i$, each t in the denominator of summand i divides some s in the numerator, leaving a quotient $q:=e/c$ (e and c being the coefficients of n in the s's and t's, respectively). Hence, the left side of (3) is the polynomial

$$P_{\mathrm{lhs}}(n)=c_0\prod_{j\in K}t_j+\sum_{i\in K_s}c_i\left(\prod_{j=1}^{i}s_j\prod_{k\in K,k>i}t_k\right)+\sum_{i\in K_b}c_iq^{|L|}\left(\prod_{j\in[1,i],\,j\notin\mathrm{rng}\,\phi}s_j\times t_{i+1}\cdots t_{u+v}\right)\tag{4}$$

and its degree is $|K|$.

Equating coefficients of powers of n in these polynomials gives a linear system of equations for the linear combination coefficients c_i. To apply Corollary 3 to the coefficient matrix of this system, arrange the factors in the products occurring in $P_{\mathrm{lhs}}(n)$ into a (block) matrix

$$N=\begin{array}{c c}
 & \begin{array}{cc} K_s & K_b \end{array}\\
\begin{array}{c} K_s\cup\{0\}\\ K_b\end{array} & \begin{pmatrix} N_1 & N_2\\ N_3 & N_4\end{pmatrix}
\end{array}$$

with rows and columns indexed as indicated. For blocks N_1 and N_4, the ij entry is t_j if $i<j$ and s_j if $i\ge j$. For N_2, the ij entry is t_j for all i. For N_3, each row is $(s_j)_{j\in K_s\cup L,\,j\notin\phi(L)}$ (order immaterial). Thus, in matrix terms, $P_{\mathrm{lhs}}(n)=\mathbf{c}N^\circ$, where $\mathbf{c}=(c_0,(c_i)_{i\in K_s},(q^{|L|}c_i)_{i\in K_b})$ incorporates the $q^{|L|}$ factors.

Now equate coefficients of powers of n in $P_{\mathrm{lhs}}(n)=P_{\mathrm{rhs}}(n)$, that is, in $\mathbf{c}N^\circ=P_{\mathrm{rhs}}(n)$, by applying the coeff operator of §3, to obtain

$$\mathbf{c}\,\mathrm{coeff}(N^\circ)=\mathrm{coeff}(P_{\mathrm{rhs}}(n)).$$

This is a linear system of $|K|+1$ equations in the $|K|+1$ unknowns $\mathbf{c}$. The coefficient matrix $M=\mathrm{coeff}(N^\circ)$ is invertible because Corollary 3 applies to N. The hypothesis of the Corollary is met because, for all $j\in K=K_s\cup K_b$, all entries of N directly above position (j,j) are equal to t_j, and all entries at or to its left are of the form s_i with $i\le j$. And t_j does not divide any such s_i or else j would lie in L whereas, by the definition of K, j does not lie in L.

To illustrate, for $\frac{1}{(6n+14)(4n+13)}\binom{6n+15}{2n+8}$, we have $u=2$, $v=6$, $r_i=6n+16-i$, $s_i=4n+14-i$, $t_i=2n+i$. Since $t_3\,|\,s_8$, $t_4\,|\,s_6$, $t_5\,|\,s_4$, and $t_6\,|\,s_2$, we have $L=\{5,6\}$ with $\phi(5)=4$, $\phi(6)=2$. This makes $K_s=[1,4]$ and $K_b=[7,8]$. The common factor in (2) is

$$\binom{6n+13}{2n}\Big/((2n+1)(2n+2)(2n+3)(2n+4)(2n+7)(2n+8)).$$

After dividing this out, the polynomial remaining on the right side is

$$2^2(6n+15)(4n+8)(4n+9)(4n+11)$$

while that on the left side is

$$(c_0, c_1, c_2, c_3, c_4, 4c_7, 4c_8)N^\circ,$$

where $N =$

$$\begin{array}{c|c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & 8 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 7 \\ 8 \end{array} & \begin{pmatrix} 2n+1 & 2n+2 & 2n+3 & 2n+4 & 2n+7 & 2n+8 \\ 4n+13 & 2n+2 & 2n+3 & 2n+4 & 2n+7 & 2n+8 \\ 4n+13 & 4n+12 & 2n+3 & 2n+4 & 2n+7 & 2n+8 \\ 4n+13 & 4n+12 & 4n+11 & 2n+4 & 2n+7 & 2n+8 \\ 4n+13 & 4n+12 & 4n+11 & 4n+10 & 2n+7 & 2n+8 \\ 4n+13 & 4n+8 & 4n+11 & 4n+9 & 4n+7 & 2n+8 \\ 4n+13 & 4n+8 & 4n+11 & 4n+9 & 4n+7 & 4n+6 \end{pmatrix}. \end{array}$$

5. CONCLUDING REMARKS

Theorem 1 enables one to tell by inspection if a linear binomial sequence $\frac{1}{P(n)}\binom{an+b}{cn+d}$ has bounded denominators. The theorem readily extends to sequences of the form $\frac{Q(n)}{P(n)}\binom{an+b}{cn+d}$, where both p and Q have linear factors. Indeed, if $gn+h$ is a factor in $P(n)$ with g and h relatively prime, and if $g'n+h'$ is a factor in $Q(n)$, then the prime values of $gn+h$ can divide $g'n+h'$ for only finitely many values of n unless $gn+h$ divides $g'n+h'$ (as polynomials in n over $\mathbb{Q}$), in which case they can be canceled. Thus, the criterion of Theorem 1 also applies to $\frac{Q(n)}{P(n)}\binom{an+b}{cn+d}$.

The algorithm of Theorem 1 often yields the "smallest" sequence of integers among all multiples of the original sequence that are integral. But it does not always do so. It does not necessarily even yield the smallest sequence expressible as an integral linear combination of binomials. For example, $\binom{5n}{2n}$ will be returned unchanged, whereas

$$\frac{1}{5}\binom{5n}{2n} = \binom{5n-1}{2n} - \binom{5n-1}{2n-1}$$

Here is another phenomenon: $\binom{4n}{2n-1}$ is also returned unchanged, while

$$\frac{1}{8}\binom{4n}{2n-1} = n^3\binom{4n-1}{2n-1} - n^3\binom{4n-1}{2n-2} - (4n-1)(4n-3)\binom{4n-5}{2n-3}$$

is clearly a sequence of integers. We conjecture that every such rational multiple of a linear binomial that yields a sequence of integers is similarly expressible as a linear combination of binomial coefficients with polynomial coefficients in $\mathbb{Z}[n]$. It would be interesting to characterize those cases where the coefficients can be taken to be constants, to extend the algorithm of Theorem 1 to sums

$$\sum_i \frac{P_i(n)}{Q_i(n)}\binom{a_in+b_i}{c_in+d_i},$$

and to sharpen it to yield "smallest" sequences.

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Mr. Mark Bowron
Director of Operations, MathPro Press
P.O. Box 713
Westford, MA 01886 USA
bowron@my-deja.com

CONVERGENT ∞-GENERALIZED FIBONACCI SEQUENCES

Walter Motta
Departamento de Matemática, CETEC-UFU,
Campus Santa Mônica, 38400-902 Uberlândia, MG, Brazil
wmotta@ufu.br

Mustapha Rachidi
Département de Mathématiques, Faculté des Sciences,
Université Mohammed V, B. P. 1014, Rabat, Morocco
rachidi@fsr.ac.ma

Osamu Saeki
Department of Mathematics, Faculty of Science,
Hiroshima University, Higashi-Hiroshima 739-8526, Japan
saeki@math.sci.hiroshima-u.ac.jp

1. INTRODUCTION

In [4], the authors have defined ∞-generalized Fibonacci sequences, which are defined by recurrence formulas involving infinitely many terms and which are generalizations of weighted r-generalized Fibonacci sequences with r finite as defined in [1]. In this paper we study the convergence property of such sequences and their associated series.

Let us first recall the definition of ∞-generalized Fibonacci sequences. Take an infinite sequence $\{a_i\}_{i=0}^{\infty}$ of complex numbers. We set $h(z)=\sum_{i=0}^{\infty}a_i z^i$ for $z\in\mathbf{C}$ and $u(x)=\sum_{i=1}^{\infty}|a_i|x^i$ for $x\in\mathbf{R}$. Let R denote the radius of convergence of the power series h, which coincides with that of u. We assume the following:

$$0<R\leq\infty. \tag{1.1}$$

Let X be the set of the sequences $\{x_i\}_{i=0}^{\infty}$ of complex numbers such that there exist $C>0$ and T with $0<T<R$ satisfying $|x_i|\leq CT^i$ for all i. Note that X is an infinite dimensional vector space over $\mathbf{C}$, which will be the set of initial sequences for ∞-generalized Fibonacci sequences associated with the weight sequence $\{a_i\}_{i=0}^{\infty}$. Define $f:X\to\mathbf{C}$ by $f(x_0,x_1,\ldots)=\sum_{i=0}^{\infty}a_i x_i$. Since the series $\sum_{i=0}^{\infty}a_i CT^i$ converges absolutely, the series defining f also converges absolutely. Then, for a sequence $\{y_0,y_{-1},y_{-2},\ldots\}\in X$, we define the sequence $\{y_1,y_2,y_3,\ldots\}$ by

$$y_n=f(y_{n-1},y_{n-2},y_{n-3},\ldots)=\sum_{i=1}^{\infty}a_{i-1}y_{n-i}\quad(n=1,2,3,\ldots),$$

which is well defined as is shown in [4]. The sequence $\{y_i\}_{i\in\mathbf{Z}}$ is called an *∞-generalized Fibonacci sequence* associated with the weight sequence $\{a_i\}_{i=0}^{\infty}$. Note that if there exists an integer $r\geq 1$ such that $a_i=0$ for all $i\geq r$, then the sequence $\{a_i\}_{i=0}^{\infty}$ satisfies condition (1.1) and the above definition coincides with that of weighted r-generalized Fibonacci sequences with r finite (see [1]).

Note that, as far as the authors know, this is a new generalization of the usual Fibonacci sequences and almost nothing has been known about such sequences until now, except those results obtained in [4]. For example, the following questions naturally arise.

(Q1) Are they combinations of geometric progressions, as in the finite case?
(Q2) Are they asymptotically geometric?
(Q3) Do they converge to limits?
(Q4) Do their sums converge to limits?
(Q5) Is it possible to express the n^{th} term of such a sequence as a function of n in some nice way?

We briefly recall the results obtained in [4], which are fundamental for the present paper and which give an answer to (Q2) above.

Lemma 1.2 ([4], Lemma 2.3): ***(1)*** Suppose that each a_i is a nonnegative real number and that there exists an S with $0 < S < R$ satisfying

$$a_0 > S^{-1} - u(S) \text{ (or, equivalently, } Sh(S) > 1). \tag{1.2.1}$$

Then there exists a unique $q \in \mathbf{R}$ such that $q > S^{-1}$, $\{q^{-(i+1)}\}_{i=0}^{\infty} \in X$, and $f(q^{-1}, q^{-2}, q^{-3}, \ldots) = 1$.

(2) Suppose there exists an S with $0 < S < R$ satisfying

$$|a_0| > S^{-1} + u(S). \tag{1.2.2}$$

Then there exists a unique $q \in \mathbf{C}$ such that $|q| > S^{-1}$, $\{q^{-(i+1)}\}_{i=0}^{\infty} \in X$, and $f(q^{-1}, q^{-2}, q^{-3}, \ldots) = 1$.

Note that, in the finite case [1], the above q corresponds to the root of the characteristic polynomial of maximal modulus. As has been seen in [4], the existence of such a q plays an important role in studying the asymptotic behavior of ∞-generalized Fibonacci sequences [see (Q2) above] and hence in exploring questions (Q3) and (Q4) above (see §5). More precisely, the following has been proved in [4].

Theorem 1.3 ([4], Theorem 3.10): Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of complex numbers that satisfies (1.1) and admits an S with $0 < S < R$ satisfying (1.2.1) or (1.2.2), and

$$S^2 u'(S) < 1. \tag{1.3.1}$$

Then $\lim_{n\to\infty} y_n / q^n$ exists and is equal to

$$\frac{\sum_{m=0}^{\infty} b_m q^m y_{-m}}{\sum_{m=0}^{\infty} b_m} \quad \text{with} \quad b_m = \sum_{i=m}^{\infty} \frac{a_i}{q^{i+1}}.$$

In the following, *we always assume that the conditions of Theorem 1.3 are satisfied.* These conditions demand that the modulus of the leading weight coefficient a_0 should be sufficiently large (see also §5). There are many sequences that satisfy these conditions. For example, take an arbitrary holomorphic function $h_1(z)$ defined in a neighborhood of zero. Then the sequence appearing as the coefficients of the power series expansion of the holomorphic function $h(z) = h_1(z) + a$ at $z = 0$ satisfies the above conditions for all $a \in \mathbf{C}$ with sufficiently large modulus $|a|$.

In this paper we consider questions (Q3) and (Q4) mentioned above and prove the following results, which give answers to the questions in certain situations.

Theorem 1.4: Suppose that each a_i is a nonnegative real number and that $\sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$.

1. The following three are equivalent.
   (a) The sequence $\{y_n\}_{n=1}^{\infty}$ does not converge.
   (b) $\sum_{i=0}^{\infty} a_i > 1$.
   (c) $q > 1$.
2. The following three are equivalent.
   (a) The sequence $\{y_n\}_{n=1}^{\infty}$ converges to a nonzero real number.
   (b) $\sum_{i=0}^{\infty} a_i = 1$.
   (c) $q = 1$.

Furthermore, in case 2(a), we have

$$\lim_{n\to\infty} y_n = \frac{\sum_{m=0}^{\infty} b_m y_{-m}}{\sum_{m=0}^{\infty} b_m} \quad \text{with} \quad b_m = \sum_{i=m}^{\infty} a_i .$$

3. The following three are equivalent.
   (a) The sequence $\{y_n\}_{n=1}^{\infty}$ converges to zero.
   (b) $\sum_{i=0}^{\infty} a_i < 1$.
   (c) $q < 1$.

Theorem 1.5: Suppose that each a_i is a nonnegative real number and that $\sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$.

1. The following three are equivalent.
   (a) The series $\sum_{n=1}^{\infty} y_n$ does not converge.
   (b) $\sum_{i=0}^{\infty} a_i \geq 1$.
   (c) $q \geq 1$.
2. The following three are equivalent.
   (a) The series $\sum_{n=1}^{\infty} y_n$ converges.
   (b) $\sum_{i=0}^{\infty} a_i < 1$.
   (c) $q < 1$.

Furthermore, in case 2(a), we have

$$\sum_{n=1}^{\infty} y_n = \frac{\sum_{j=0}^{\infty}\sum_{i=j}^{\infty} a_i y_{j-i}}{1-\sum_{i=0}^{\infty} a_i} = \frac{\sum_{m=0}^{\infty}\left(\sum_{i=m}^{\infty} a_i\right) y_{-m}}{1-\sum_{i=0}^{\infty} a_i}.$$

When a_i are general complex numbers, we have the following theorem.

Theorem 1.6: Suppose that $\sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$.

1. We have the implications ***(b)*** $\Rightarrow$ ***(c)*** $\Leftrightarrow$ ***(a)*** among the following:
   (a) The sequence $\{|y_n|\}_{n=1}^{\infty}$ converges to ∞.
   (b) $|a_0| - \sum_{i=1}^{\infty} |a_i| > 1$.
   (c) $|q| > 1$.

2. The following two are equivalent.
   (a) The sequence $\{|y_n|\}_{n=1}^{\infty}$ converges to a nonzero real number.
   (b) $|q|=1$.

Furthermore, in case 2(a), we have

$$\lim_{n\to\infty}|y_n| = \left|\frac{\sum_{m=0}^{\infty} b_m q^m y_{-m}}{\sum_{m=0}^{\infty} b_m}\right|.$$

3. We have the implications $(b) \Rightarrow (c) \Leftrightarrow (a)$ among the following:
   (a) The sequence $\{y_n\}_{n=1}^{\infty}$ converges to zero.
   (b) $\sum_{i=0}^{\infty}|a_i|<1$.
   (c) $|q|<1$.

Theorem 1.7: Suppose that $\sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$.

1. The following two are equivalent.
   (a) The series $\sum_{n=1}^{\infty}|y_n|$ does not converge.
   (b) $|q|\geq 1$.

2. The following two are equivalent.
   (a) The series $\sum_{n=1}^{\infty}|y_n|$ converges.
   (b) $|q|<1$.

Furthermore, in case 2(a), we have

$$\sum_{n=1}^{\infty} y_n = \frac{\sum_{j=0}^{\infty}\sum_{i=j}^{\infty} a_i y_{j-i}}{1-\sum_{i=0}^{\infty} a_i} = \frac{\sum_{m=0}^{\infty}\left(\sum_{i=m}^{\infty} a_i\right) y_{-m}}{1-\sum_{i=0}^{\infty} a_i}.$$

Note that the above results generalize some of the results of Gerdes [2], [3], concerning weighted r-generalized Fibonacci sequences with $r=2$ and 3.

The paper is organized as follows: In §2 we prove the convergence result, Theorem 1.4, for the nonnegative real case. In §3 we prove the convergence result, Theorem 1.6, for the general case. In §4 we give an explicit formula for the generating functions of ∞-generalized Fibonacci sequences that generalize a result of Raphael [5], and prove the convergence results for the series, i.e., Theorems 1.5 and 1.7. Finally, in §5 we give some remarks concerning questions (Q1)-(Q5) mentioned above.

2. CONVERGENCE OF SEQUENCE—NONNEGATIVE REAL CASE

In this section, we prove Theorem 1.4.

Lemma 2.1: If $K=\sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$ and $|q|>1$, then $\lim_{n\to\infty}|y_n|=\infty$.

Proof: By Theorem 1.3, there exists an integer N such that, for all $n \geq N$, we have

$$|(y_n / q^n) - K| < |K|/2.$$

In particular, we have

$$|y_n| \geq |K||q^n| - |K - (y_n / q^n)||q^n| > |K||q|^n/2$$

for all $n \geq N$. Then the result is obvious. □

Lemma 2.2: If $|q| < 1$, then $\lim_{n\to\infty}|y_n| = 0$.

Proof: By Theorem 1.3, there exists an integer N such that, for all $n > N$, we have

$$|(y_n / q^n) - K| < 1.$$

Then we have

$$|y_n / q^n| \leq |(y_n / q^n) - K| + |K| < |K| + 1;$$

hence, $|y_n| < (|K| + 1)|q|^n$ for all $n \geq N$. Then the result is obvious. □

Lemma 2.3: Suppose that each a_i is a nonnegative real number.

(1) If $\sum_{i=0}^{\infty} a_i > 1$, then $q > 1$.

(2) If $\sum_{i=0}^{\infty} a_i = 1$, then $q = 1$.

(3) If $\sum_{i=0}^{\infty} a_i < 1$, then $q < 1$.

Proof: Let $\varphi : [0, R) \to \mathbf{R}$ be the function defined by $\varphi(x) = xh(x)$, which is strictly increasing. Note that $\varphi(x) = 1$ if and only if $f(x, x^2, x^3, \ldots) = 1$.

(1) When $S \leq 1$, we have nothing to prove, since $q > S^{-1}$. When $S > 1$, we have $1 < S < R$ by our assumption and

$$\varphi(1) = h(1) = \sum_{i=0}^{\infty} a_i > 1 = \varphi(q^{-1})$$

by Lemma 1.2. Thus, we have $1 > q^{-1}$, which implies that $q > 1$.

(2) Since $\sum_{i=0}^{\infty} a_i$ converges, we have $R \geq 1$. If $R > 1$, then $x = 1$ is the unique solution of the equation $\varphi(x) = 1$ on the interval $[0, R)$. Thus, $q^{-1} = 1$ by Lemma 1.2. If $R = 1$, then φ can be extended to a strictly increasing function on $[0, R]$ with $\varphi(1) = 1$. This contradicts the assumption that $\varphi(S) > 1$ for some $S \in [0, R)$.

(3) Since $\sum_{i=0}^{\infty} a_i$ converges, we have $R \geq 1$. By the same argument as in (2), we have $R > 1$. Then we have $\varphi(1) < 1 = \varphi(q^{-1})$ by Lemma 1.2. Thus, we have $1 < q^{-1}$, which implies that $q < 1$. □

Theorem 1.4 follows from the above lemmas together with Theorem 1.3.

The condition that $\sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$ is satisfied, for example, for $y_i = g_i$, where $\{g_i\}_{i \in \mathbf{Z}}$ is the sequence as defined in [4, §3]. Thus, we have the following corollary.

Corollary 2.4: Let $\{g_i\}_{i=1}^{\infty}$ be the ∞-generalized Fibonacci sequence associated with the initial sequence $\{g_{-i}\}_{i=0}^{\infty}$, where $g_0 = 1$ and $g_{-i} = 0$ for $i \geq 1$. If all a_i are nonnegative real numbers and if $\sum_{i=0}^{\infty} a_i = 1$, then the sequence $\{g_i\}_{i=1}^{\infty}$ converges to

$$\left(\sum_{m=0}^{\infty} b_m\right)^{-1} = \left(\sum_{i=0}^{\infty} (i+1)a_i\right)^{-1} = \frac{1}{h(1) + h'(1)}.$$

3. CONVERGENCE OF SEQUENCE—GENERAL CASE

In this section we prove Theorem 1.6.

Lemma 3.1: If $|a_0| - \sum_{i=1}^{\infty}|a_i| > 1$, then $|q| > 1$.

Proof: When $S \le 1$, we have $|q| > 1$ by Lemma 1.2. When $S > 1$, we have $1 < S < R$. Consider the function $t: [0, R) \to \mathbf{R}$ defined by $t(x) = x^2 u'(x)$, which is strictly increasing. Then we have $t(1) < t(S) < 1$ by condition (1.3.1). Furthermore, by our assumption, we have $|a_0| > 1 + u(1)$. Hence, in (1.2.2) and (1.3.1), we may assume that $S = 1$. Thus, we have $|q| > 1$ by Lemma 1.2. This completes the proof. □

Lemma 3.2: If $\sum_{i=0}^{\infty}|a_i| < 1$, then $|q| < 1$.

Proof: We have

$$\sum_{i=0}^{\infty}|a_i| < 1 = f(q^{-1}, q^{-2}, q^{-3}, \ldots) = \left|\sum_{i=0}^{\infty} a_i q^{-(i+1)}\right| \le \sum_{i=0}^{\infty}|a_i||q^{-1}|^{i+1}.$$

Thus, we have $|q^{-1}| > 1$. This completes the proof. □

Theorem 1.6 follows from the above lemmas together with the lemmas in §2 and Theorem 1.3.

Corollary 3.3: Let $\{g_i\}_{i=1}^{\infty}$ be the ∞-generalized Fibonacci sequence associated with the initial sequence $\{g_{-i}\}_{i=0}^{\infty}$, where $g_0 = 1$ and $g_{-i} = 0$ for $i \ge 1$. Then the sequence $\{g_i\}_{i=1}^{\infty}$ converges to a nonzero complex number if and only if $|q| = 1$.

4. GENERATING FUNCTION AND CONVERGENT SERIES

First, we prove the following formula for the generating function of ∞-generalized Fibonacci sequences, which generalizes a result of Raphael [5].

Theorem 4.1: Suppose that the sequence $\{a_i\}_{i=0}^{\infty}$ satisfies the condition in Lemma 1.2. Then the generating function of the sequence $\{y_i\}_{i=1}^{\infty}$ is equal to

$$\sum_{i=0}^{\infty} y_{i+1} z^i = \frac{k(z)}{1 - zh(z)},$$

where $h(z) = \sum_{i=0}^{\infty} a_i z^i$ and

$$k(z) = \sum_{j=0}^{\infty}\left(\sum_{i=j}^{\infty} a_i y_{j-i}\right) z^j.$$

More precisely, the above equality holds for all $z \in \mathbf{C}$ with $|z| < |q|^{-1}$.

Proof: First, consider the power series $k(z)$. Let the radius of convergence of k be denoted by R'. Then we have

$$R' = \left(\limsup_{j\to\infty} \sqrt[j]{\left|\sum_{i=j}^{\infty} a_i y_{j-i}\right|}\right)^{-1}.$$

Since the sequence $\{y_0, y_{-1}, y_{-2}, \ldots\}$ is an element of X, there exist $C > 0$ and T with $0 < T < R$ such that $y_{-i} \le CT^i$ for all $i \ge 0$. Thus, we have

$$\limsup_{j\to\infty} \sqrt[j]{\left|\sum_{i=j}^{\infty} a_i y_{j-i}\right|} \le \limsup_{j\to\infty} \sqrt[j]{\sum_{i=j}^{\infty} |a_i| C T^{i-j}} = \limsup_{j\to\infty} \sqrt[j]{\frac{C}{T^j}\sum_{i=j}^{\infty} |a_i| T^i}$$
$$\le \limsup_{j\to\infty} \frac{\sqrt[j]{C}}{T}\sqrt[j]{u(T)} = \frac{1}{T}.$$

Thus, we have $R' \ge T$. Since we can choose T as close to R as we want, we have $R' \ge R$. Thus, in particular, for $z \in \mathbf{C}$ with $|z| < R$, the series $k(z)$ converges absolutely.

Therefore, for z with $|z| < R$, we have

$$\begin{aligned}(1 - zh(z))\left(\sum_{i=0}^{\infty} y_{i+1} z^i\right) &= \left(1 - \sum_{i=0}^{\infty} a_i z^{i+1}\right)\left(\sum_{i=0}^{\infty} y_{i+1} z^i\right) \\ &= y_1 + (y_2 - a_0 y_1) z + (y_3 - a_0 y_2 - a_1 y_1) z^2 \\ &\quad + (y_4 - a_0 y_3 - a_1 y_2 - a_2 y_1) z^3 + \cdots \\ &= \sum_{j=0}^{\infty}\left(\sum_{i=j}^{\infty} a_i y_{j-i}\right) z^j = k(z),\end{aligned}$$

where we have changed the order of addition appropriately, which is allowed since all the series above converge absolutely. Thus, as long as $1 - zh(z) \neq 0$, we have

$$\sum_{i=0}^{\infty} y_{i+1} z^i = \frac{k(z)}{1 - zh(z)}. \tag{4.1.1}$$

On the other hand, we have $q^{-1} h(q^{-1}) = 1$ and that q^{-1} is the solution for $zh(z) = 1$ which has the smallest modulus by Lemma 1.2. Hence, for $|z| < |q|^{-1}$, we have (4.1.1). This completes the proof of Theorem 4.1. □

Now Theorem 1.5 follows from Theorems 1.4 and 4.1.

Proof of Theorem 1.7: By Theorem 1.3 and Lemma 2.1, if $|q| \ge 1$, then the series $\sum_{n=1}^{\infty} |y_n|$ does not converge. Suppose that $|q| < 1$. The radius of convergence of the power series

$$c(z) = \sum_{i=0}^{\infty} |y_{i+1}| z^i$$

is equal to the radius of convergence R'' of the power series

$$\sum_{i=0}^{\infty} y_{i+1} z^i.$$

By Theorem 4.1 together with our assumption, we have $R'' \ge |q|^{-1} > 1$. Thus, the series $c(z)$ for $z = 1$ converges. Then the rest of Theorem 1.7 follows from Theorem 4.1. This completes the proof. □

Corollary 4.2: Let $\{g_i\}_{i=1}^{\infty}$ be the ∞-generalized Fibonacci sequence associated with the initial sequence $\{g_{-i}\}_{i=0}^{\infty}$, where $g_0 = 1$ and $g_{-i} = 0$ for $i \ge 1$. If $|q| < 1$, then the series $\sum_{i=1}^{\infty} g_i$ converges to

$$\sum_{i=0}^{\infty} a_i \Big/ \left(1 - \sum_{i=0}^{\infty} a_i\right).$$

5. CONCLUDING REMARKS

In this section we give some remarks about questions (Q1)-(Q5) raised in §1.

About (Q1), in the finite case, the answer to this question is given by a Binet-type formula (e.g., see [1]). The question in the infinite case is also posed in [4, Problem 4.5]. In a forthcoming paper we will consider approximation of ∞-generalized Fibonacci sequences by finitely generalized ones and will give an asymptotic Binet formula which will give an answer to the question in a certain sense. This study is also closely related to question (Q2).

About (Q2), in the finite case, it has been shown that if the characteristic polynomial has a simple root of maximal modulus then the sequence is asymptotically geometric (see [1]). This condition is satisfied as long as the leading weight coefficient a_0 has sufficiently large modulus (see Theorem 15 and Remark 16 of [1]). In [4], the authors have shown that a statement similar to this also holds in the infinite case as well, which is nothing but Theorem 1.3 of the present paper.

About (Q3) and (Q4), Theorems 1.4 and 1.5, respectively, give satisfactory answers in the nonnegative real coefficient case under our assumption. In the general case, Theorems 1.6 and 1.7, respectively, give partial answers to the questions.

About (Q5), in a forthcoming paper, combinatorial expressions for the general terms of an ∞-generalized Fibonacci sequence will be studied.

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DERIVATIVE SEQUENCES OF GENERALIZED JACOBSTHAL AND JACOBSTHAL-LUCAS POLYNOMIALS

Gospava B. Djordjevic
University of Niš, Faculty of Technology, 16000 Leskovac, Yugoslavia
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1. INTRODUCTION

In this note we define two sequences $\{J_{n,m}(x)\}$—the generalized Jacobsthal polynomials, and $\{j_{n,m}(x)\}$—the generalized Jacobsthal-Lucas polynomials, by the following recurrence relations:

$$J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x), \quad n \ge m, \tag{1.1}$$

with starting polynomials $J_{0,m}(x) = 0$, $J_{n,m}(x) = 1$, $n = 1, 2, \ldots, m-1$, and

$$j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x), \quad n \ge m, \tag{1.2}$$

with starting polynomials $j_{0,m}(x) = 2$, $j_{n,m}(x) = 1$, $n = 1, 2, \ldots, m-1$.

For $m = 2$, these polynomials are studied in [1], [2], and [3].

From (1.1) and (1.2), using the standard method, we find that the polynomials $\{J_{n,m}(x)\}$ and $\{j_{n,m}(x)\}$ have, respectively, the following generating functions:

$$F(x,t) = (1-t-2xt^m)^{-1} = \sum_{n=1}^{\infty} J_{n,m}(x)t^{n-1} \tag{1.3}$$

and

$$G(x,t) = \frac{1+4xt^{m-1}}{1-t-2xt^m} = \sum_{n=1}^{\infty} j_{n,m}(x)t^{n-1}. \tag{1.4}$$

From (1.3) and (1.4), we find the following explicit representations for the polynomials $\{J_{n,m}(x)\}$ and $\{j_{n,m}(x)\}$:

$$J_{n,m}(x) = \sum_{k=0}^{[(n-1)/m]} \binom{n-1-(m-1)k}{k}(2x)^k \tag{1.5}$$

and

$$j_{n,m}(x) = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k}\binom{n-(m-1)k}{k}(2x)^k. \tag{1.6}$$

Differentiating (1.5) and (1.6) with respect to x, we get

$$J^{(1)}_{n,m}(x) = \sum_{k=1}^{[(n-1)/m]} 2k\binom{n-1-(m-1)k}{k}(2x)^{k-1} \tag{1.7}$$

and

$$j^{(1)}_{n,m}(x) = \sum_{k=1}^{[n/m]} 2k\frac{n-(m-2)k}{n-(m-1)k}\binom{n-(m-1)k}{k}(2x)^{k-1}, \tag{1.8}$$

with

$$J^{(1)}_{n,m}(x) = j^{(1)}_{n,m}(x) = 0, \quad n = 0, 1, \ldots, m-1. \tag{1.9}$$

For $x = 1$ in (1.5), (1.6), (1.7), and (1.8), we have, respectively: $\{J_{n,m}(1)\}$—the generalized Jacobsthal numbers, $\{j_{n,m}(1)\}$—the generalized Jacobsthal-Lucas numbers, $\{J^{(1)}_{n,m}(1)\}$—the generalized Jacobsthal derivative sequence, and $\{j^{(1)}_{n,m}(1)\}$—the generalized Jacobsthal-Lucas derivative sequence.

The aim of this note is to study some characteristic properties of the sequences of numbers $\{J^{(1)}_{n,m}(1)\}$ and $\{j^{(1)}_{n,m}(1)\}$. We shall use the notations $H^1_{n,m}$ instead of $J^{(1)}_{n,m}(1)$ and $K^1_{n,m}$ instead of $j^{(1)}_{n,m}(1)$.

The first few members of the sequences $\{J_{n,m}(x)\}$, $\{J^{(1)}_{n,m}(x)\}$, and $\{H^1_{n,m}\}$ are presented in Table 1, and the first few members of the sequences $\{j_{n,m}(x)\}$, $\{j^{(1)}_{n,m}(x)\}$, and $\{K^1_{n,m}\}$ are given in Table 2.

TABLE 1

n	$J_{n,m}(x)$	$J^{(1)}_{n,m}(x)$	$H^1_{n,m}$
0	0	0	0
1	1	0	0
2	1	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	1	0	0
m	1	0	0
$m+1$	$1+2x$	2	2
$m+2$	$1+4x$	4	4
$m+3$	$1+6x$	6	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2m-1$	$1+2(m-1)x$	$2(m-1)$	$2(m-1)$
$2m$	$1+2mx$	$2m$	$2m$
$2m+1$	$1+2(m+1)x+4x^2$	$2(m+1)+8x$	$2m+10$
$2m+2$	$1+2(m+2)x+12x^2$	$2(m+2)+24x$	$2m+28$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 2

n	$j_{n,m}(x)$	$j^{(1)}_{n,m}(x)$	$K^1_{n,m}$
0	2	0	0
1	1	0	0
2	1	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	1	0	0
m	$1+4x$	4	0
$m+1$	$1+6x$	6	0
$m+2$	$1+8x$	8	0
$m+3$	$1+10x$	10	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2m-1$	$1+2(m+1)x$	$2(m+1)$	$2(m+1)$
$2m$	$1+2(m+2)x+8x^2$	$2(m+2)+16x$	$2m+20$
$2m+1$	$1+2(m+3)x+20x^2$	$2(m+3)+40x$	$2m+46$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

From Table 1 and Table 2, we can prove by induction and (1.1) the following relation:

$$\begin{aligned} j_{n,m}(x) &= J_{n,m}(x) + 4xJ_{n+1-m,m}(x) \\ &= J_{n+1,m}(x) + 2xJ_{n+1-m,m}(x), \quad \text{[by (1.1)]}. \end{aligned} \tag{1.10}$$

Observe that the first equation in (1.10) is a direct consequence of (1.3) and (1.4).

2. SOME PROPERTIES OF $H^1_{n,m}$ AND $K^1_{n,m}$

Differentiating (1.3) and (1.4) with respect to x, we get the following generating functions, respectively:

$$\sum_{n=0}^{\infty} J^{(1)}_{n,m}(x)t^n = \frac{2t^{m+1}}{(1-t-2xt^m)^2} \tag{2.1}$$

and

$$\sum_{n=0}^{\infty} j^{(1)}_{n,m}(x)t^n = \frac{2t^m(2-t)}{(1-t-2xt^m)^2}. \tag{2.2}$$

Hence, for $x=1$ in (2.1) and (2.2), we get the generating functions for $H^1_{n,m}$ and $K^1_{n,m}$, respectively:

$$\sum_{n=0}^{\infty} H^1_{n,m}t^n = \frac{2t^{m+1}}{(1-t-2t^m)^2} \tag{2.3}$$

and

$$\sum_{n=0}^{\infty} K^1_{n,m}t^n = \frac{2t^m(2-t)}{(1-t-2t^m)^2}. \tag{2.4}$$

If we substitute $x=1$ in (1.1) and (1.2), we get the sequences of numbers $\{J_{n,m}\}$ and $\{j_{n,m}\}$, which satisfy the following relations:

$$j_{n,m} = J_{n,m} + 4J_{n+1-m,m} = J_{n+1,m} + 2J_{n+1-m,m} \quad \text{[by (1.10)]}, \tag{2.5}$$

$$j_{n+1,m} + j_{n,m} = 3J_{n+1,m} + 4J_{n+2-m,m} - J_{n,m} \quad \text{[by (2.5), (1.1)]}, \tag{2.6}$$

$$j_{n+1,m} - j_{n,m} = 4J_{n+2-m,m} + J_{n,m} - J_{n+1,m} \quad \text{[by (2.5), (1.1)]}, \tag{2.7}$$

$$j_{n+1,m} - 2j_{n,m} = 4J_{n+2-m,m} + 2J_{n,m} - 3J_{n+1,m} \quad \text{[by (2.5), (1.1)]}, \tag{2.8}$$

$$J_{n,m} + j_{n,m} = 2J_{n+1,m}. \tag{2.9}$$

For $m=2$, relations (2.5)-(2.9) yield the following relations:

$$\begin{aligned} j_n &= J_{n+1} + 2J_{n-1} && \text{((2.10) in [2])}, \\ j_{n+1} + j_n &= 3(J_{n+1} + J_n) && \text{((2.12) in [2])}, \\ j_{n+1} - j_n &= 5J_n - J_{n+1}, && \\ j_{n+1} - 2j_n &= 3(2J_n - J_{n+1}) && \text{((2.14) in [2])}, \\ J_n + j_n &= 2J_{n+1} && \text{((2.20) in [2])}, \end{aligned}$$

where $J_{n,2} = J_n$ and $j_{n,2} = j_n$.

Differentiating (1.1) and (1.2) with respect to x, and substituting $x=1$. we get the following recurrence relations:

$$H^1_{n,m} = H^1_{n-1,m} + 2H^1_{n-m,m} + 2J_{n-m,m}, \quad n \geq m, \tag{2.10}$$

with $H^1_{n,m} = 0$, $n = 0, 1, \ldots, m-1$ and

$$K^1_{n,m} = K^1_{n-1,m} + 2K^1_{n-m,m} + 2j_{n-m,m}, \quad n \geq m, \tag{2.11}$$

with $K^1_{n,m} = 0$, $n = 0, 1, \ldots, m-1$.

In a similar way, from (1.10), we get

$$K^1_{n,m} = H^1_{n,m} + 4H^1_{n+1-m,m} + 4J_{n+1-m,m}, \quad n \geq m-1. \tag{2.12}$$

For $m=2$, relations (2.10)-(2.12) become

$$H^1_{n+2} = H^1_{n+1} + 2H^1_n + 2J_n \quad ((3.3) \text{ in } [1]),$$

$$K^1_{n+2} = K^1_{n+1} + 2K^1_n + 2j_n \quad ((3.4) \text{ in } [1]),$$

$$K^1_{n+1} = H^1_{n+1} + 4H^1_n + 4J_n.$$

From (2.10) and (2.12), we get $K^1_{n,m} + H^1_{n,m} = 2H^1_{n+1,m}$.

For $m=2$, the last equality yields the known relation (3.8) in [1].

Again, from (2.10) and (2.12), we find

$$K^1_{n,m} - H^1_{n,m} = 2H^1_{n+1,m} - 2H^1_{n,m}. \tag{2.13}$$

For $m=2$, (2.13) becomes (3.9) in [1].

Theorem 2.1: The polynomials $\{J_{n,m}(x)\}$ and $\{j_{n,m}(x)\}$ satisfy the following relations, respectively,

$$\sum_{i=0}^{n} J_{i,m}(x) = \frac{J_{n+m,m}(x)-1}{2x} \tag{2.14}$$

and

$$\sum_{i=0}^{n} j_{i,m}(x) = \frac{j_{n+m,m}(x)-1}{2x}. \tag{2.15}$$

Proof: From (1.1) and (1.2), by induction on n, we can prove (2.14) and (2.15).

Corollary 2.1: For $m=2$ in (2.14) and (2.15), we get the known relations (2.7) and (2.8) in [2].

Theorem 2.2: The numbers $H^1_{i,m}$ and $K^1_{n,m}$ satisfy the following relations, respectively,

$$\sum_{i=0}^{n} H^1_{i,m} = 1/2(H^1_{n+m,m} - J_{n+m,m} + 1) \tag{2.16}$$

and

$$\sum_{i=0}^{n} K^1_{i,m} = 1/2(K^1_{n+m,m} - j_{n+m,m} + 1). \tag{2.17}$$

Proof: Differentiating (2.14) and (2.15), respectively, with respect to x, and substituting $x=1$, we get (2.16) and (2.17).

Corollary 2.2: For $m = 2$, from (2.16) and (2.17), we have

$$\sum_{i=0}^{n} H_i^1 = 1/2(H_{n+2}^1 - J_{n+2} + 1) \quad \text{and} \quad \sum_{i=0}^{n} K_i^1 = 1/2(K_{n+2}^1 - j_{n+2} + 1).$$

Furthermore, from (1.7), we get

$$H_{n+m,\,m}^1 + 2(m-1)H_{n,\,m}^1 - 2nJ_{n,\,m}. \tag{2.18}$$

For $m = 2$ in (2.18), we have ((3.6′) in [1]), $H_{n+2}^1 + 2H_n^1 = 2nJ_n$.
In a similar way, from (1.8), we get

$$K_{n,\,m}^1 = 2(n+2-m)J_{n+1-m,\,m} - 2(m-2)H_{n+1-m,\,m}^1. \tag{2.19}$$

For $m = 2$ in (2.19), we obtain ((2.4) in [1]), $K_n^1 = 2nJ_{n-1}$.

GENERALIZATION

Differentiating (1.1), (1.2), and (1.10) k times with respect to x, we get

$$J_{n,\,m}^{(k)}(x) = J_{n-1,\,m}^{(k)}(x) + 2kJ_{n-m,\,m}^{(k-1)}(x) + 2xJ_{n-m,\,m}^{(k)}(x), \quad k \ge 1,\ n \ge m,$$

$$j_{n,\,m}^{(k)}(x) = j_{n-1,\,m}^{(k)}(x) + 2kj_{n-m,\,m}^{(k-1)}(x) + 2xj_{n-m,\,m}^{(k)}(x), \quad k \ge 1,\ n \ge m,$$

$$j_{n,\,m}^{(k)}(x) = J_{n-1,\,m}^{(k)}(x) + 4kJ_{n+1-m,\,m}^{(k-1)}(x) + 4xJ_{n+1-m,\,m}^{(k)}(x), \quad k \ge 1,\ n \ge m,$$

respectively.

From the last equalities, using the notations $J_{n,\,m}^{(k)}(1) \equiv H_{n,\,m}^k$ and $j_{n,\,m}^{(k)}(1) \equiv K_{n,\,m}^k$, we can prove the following relations:

$$H_{n,\,m}^k = H_{n-1,\,m}^k + 2kH_{n-m,\,m}^{k-1} + 2H_{n-m,\,m}^k, \quad k \ge 1, n \ge m,$$

$$K_{n,\,m}^k = K_{n-1,\,m}^k + 2kK_{n-m,\,m}^{k-1} + 2K_{n-m,\,m}^k, \quad k \ge 1, n \ge m-1,$$

$$K_{n,\,m}^k = H_{n-1,\,m}^k + 4kH_{n+1-m,\,m}^{k-1} + 4H_{n+1-m,\,m}^k, \quad k \ge 1, n \ge m-1.$$

The sequences $\{H_{n,\,m}^k\}$ and $\{K_{n,\,m}^k\}$ have the following generating functions:

$$\sum_{n=0}^{\infty} H_{n,\,m}^k t^n = \frac{2^k k!\, r^{mk+1}}{(1-t-2t^m)^{k+1}} \quad \text{and} \quad \sum_{n=0}^{\infty} K_{n,\,m}^k t^n = \frac{2^k k!(2-t)t^{mk}}{(1-t-2t^m)^{k+1}}.$$

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A REMARK ON THE PAPER OF A. SIMALARIDES: "CONGRUENCES MOD p^n FOR THE BERNOULLI NUMBERS"

I. Slavutskii
4 Hamarva Str., P.O.B. 23393, Akko, Israel
(Submitted October 11998-Final Revision June 1999)

In the paper under discussion, the author presented interesting p^n-divisibility criteria for Bernoulli numbers (B.n.) of the form $B_{(2k-1)p^n+1}$, with an odd prime p, $k = 1, 2, \ldots, (p-3)/2$, and $n \in \mathbf{N}$. However, the central part of the work (Theorem 2) can be proved directly in a short and elementary way by relying on the classical methods of G. F. Voronoi. In [2] the author first proves a p-adic analog of Voronoi's congruence (Theorem 1) using Fourier analysis, then derives Theorem 2 from this proof as a corollary by reducing mod p^n the Teichmüller character involved in Theorem 1.

Theorem ([2]): Let p be a prime > 3. If a is an integer with $(a, p) = 1$, then

$$\{a - a^{p^{n-1}(p-2k)}\}B_{(2k-1)p^n+1} \equiv \sum_{i=1}^{p-1} i^{p^{n-1}(2k-1)}[ai/p] \pmod{p^n}$$

for every $k \geq 1$ such that $p-1$ does not divide $2k$. Here $[x]$ is the greatest integer $\leq x$.

Remark: By von Staudt-Clausen's theorem and Kummer's congruence for B.n., we will rewrite the above congruence in the equivalent form

$$\{a - a^{p^{n-1}(p-2k)}\}B_z/z \equiv \sum_{i=1}^{p-1} i^{z-1}[ai/p] \pmod{p^n} \tag{1}$$

with $z = (2k-1)p^{n-1}+1$, $p > 3$.

Indeed, $(2k-1)p^n + 1 = (2k-1)p^{n-1}(p-1) + z$, and $p-1$ does not divide $(2k-1)p^m + 1 = 2kp^m - (p^m - 1)$ for an integer $m \geq 0$. Hence, $B_{(2k-1)p^n+1} \equiv ((2k-1)p^n + 1)B_z/z \equiv B_z/z \pmod{p^n}$. Thus, we can give the proof of the theorem in the form (1).

Proof: Let $S := \sum_{i=1}^{p-1} i^z$ with $z = (2k-1)p^{n-1}+1$, $n \in \mathbf{N}$. Then, by Voronoi's idea (see, e.g., [8] or [3]), we have

$$\begin{aligned} S &= \sum_{i=1}^{p-1}(ai - [ai/p]p)^z \\ &= a^z\sum_{i=1}^{p-1} i^z - pz\sum_{i=1}^{p-1}(ai)^{z-1}[ai/p] + \sum_{j=2}^{z}(-1)^j\binom{z}{j}p^j\sum_{i=1}^{p-1}(ai)^{z-j}([ai/p])^j \end{aligned}$$

or

$$S(a^z - 1)/z = p\sum_{i=1}^{p-1}(ai)^{z-1}[ai/p] + \sum_{j=2}^{z}(-1)^{j-1}\binom{z-1}{j-1}(p^j/j)\sum_{i=1}^{p-1}(ai)^{z-j}([ai/p])^j.$$

Consequently,

$$S(a^z - 1)/z \equiv p\sum_{i=1}^{p-1}(ai)^{z-1}[ai/p] \pmod{p^{n+1}}, \tag{2}$$

because

$$\operatorname{ord}_p\left\{\binom{z-1}{j-1}p^j/j\right\} = \operatorname{ord}_p\left\{\binom{z-2}{j-2}(z-1)p^j/(j(j-1))\right\}$$
$$\geq \operatorname{ord}_p\{p^{n+1}p^{j-2}/(j(j-1))\} \geq n+1 \text{ for } j \geq 2 \text{ and } p \geq 3.$$

On the other hand, $S = (B_{z+1}(p) - B_{z+1})/(z+1)$ or

$$S(a^z-1)/z = (a^z-1)B_z p/z + pB_{z-1}(a^z-1)/2$$
$$+\sum_{j=3}^{z+1}(a^z-1)(z-1)\binom{z-2}{j-3}p^j B_{z+1-j}/(j(j-1)(j-2)),$$

if we assume that $\binom{0}{0} = 1$ and that an empty sum is equal to zero.

Further, since by the Staudt-Clausen theorem, pB_{z+1-j} is p-integral, we obtain

$$\operatorname{ord}_p\{(z-1)p^j B_{z+1-j}/(j(j-1)(j-2))\} \geq \operatorname{ord}_p\{p^{j-3}/(j(j-1)(j-2))\} + n+1 \geq n+1$$

for $j \geq 3$ and $p > 3$. Hence, it follows that

$$S(a^z-1)/z \equiv (a^z-1)B_z p/z \pmod{p^{n+1}}. \tag{3}$$

With the help of $a^{p^{n-1}(p-1)} \equiv 1 \pmod{p^n}$, $(a, p) = 1$, we conclude that

$$(a^z-1)/a^{z-1} \equiv a - a^{p^{n-1}(p-1)-(2k-1)p^{n-1}} \equiv a - a^{p^{n-1}(p-2k)} \pmod{p^n}. \tag{4}$$

Note that the above transformation is useful for applications considered by the author (in the case $1 \leq k \leq (p-3)/2$, $p > 3$).

Congruences (2), (3), and (4) yield the interesting form (1) of Voronoi's congruence (with a short interval of summation in the right-hand side part).

Remark 1: It should be noted that Voronoi has proved his famous congruence (a) for an arbitrary modulus > 1 (not only prime power!) and (b) without the restriction that $p-1$ does not divide $2k$ (see [8] and [3]).

Remark 2: There is an interesting equivalent variant of Voronoi's congruence due to Vandiver (see [7] and [5]).

Remark 3: It is clear from what has been said here that a congruence similar to (1) can be obtained for generalized Bernoulli numbers $B_{n,\chi}$ belonging to a Dirichlet character (with the corresponding conductor). For relevant facts, see [4], and [9, chs. 4 and 5].

Remark 4: Finally, for more information on the history of the Voronoi congruence, see [6] or [1].

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HYPERGEOMETRIC FUNCTIONS AND FIBONACCI NUMBERS

Karl Dilcher
Department of Mathematics and Statistics, Dalhousie University
Halifax, Nova Scotia, B3H 3J5, Canada
dilcher@mscs.dal.ca
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1. INTRODUCTION

Hypergeometric functions are an important tool in many branches of pure and applied mathematics, and they encompass most special functions, including the Chebyshev polynomials. There are also well-known connections between Chebyshev polynomials and sequences of numbers and polynomials related to Fibonacci numbers. However, to my knowledge and with one small exception, direct connections between Fibonacci numbers and hypergeometric functions have not been established or exploited before.

It is the purpose of this paper to give a brief exposition of hypergeometric functions, as far as is relevant to the Fibonacci and allied sequences. A variety of representations in terms of finite sums and infinite series involving binomial coefficients are obtained. While many of them are well known, some identities appear to be new.

The method of hypergeometric functions works just as well for other sequences, especially the Lucas, Pell, and associated Pell numbers and polynomials, and also for more general second-order linear recursion sequences. However, apart from the final section, we will restrict our attention to Fibonacci numbers as the most prominent example of a second-order recurrence.

The idea and "philosophy" behind this paper is similar to that of R. Roy in [42] concerning binomial identities, though somewhat more limited in scope. It can be seen as an attempt to bring some partial order into the confusing abundance of formulas satisfied by Fibonacci numbers. For reasons of brevity and clarity, no attempt has been made to be complete, or to classify the many identities in the literature that are similar to, but still different from, those obtained in this paper. After each hypergeometric transformation, only the most immediate Fibonacci formula is given.

Statements that a certain identity is apparently new should be taken with the necessary caution. Only *The Fibonacci Quarterly* has been checked to any degree of completeness, and even there it may be possible for some identities to have been overlooked. The author apologizes in advance for any missed or incomplete references.

In spite of the relative absence of hypergeometric series from the pages of *The Fibonacci Quarterly* or related papers published elsewhere, it should be mentioned that they were occasionally used in somewhat different connections. The four papers that make most extensive use of hypergeometric functions are, to the best of my knowledge, by P. S. Bruckman [8], L. Carlitz [12], [13], and H. W. Gould [25]. To this we should add the article-length solution [44] by P. S. Bruckman to a problem in *The Fibonacci Quarterly*. The one direct connection to Fibonacci numbers that I could find is in the solution (by the proposer) of a problem by H.-J. Seiffert [43].

2. HYPERGEOMETRIC FUNCTIONS

Almost all of the most common special functions in mathematics and mathematical physics are particular cases of the *Gauss hypergeometric series* defined by

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{2.1} $$

where the *rising factorial* $(a)_k$ is defined by $(a)_0 = 1$ and

$$ (a)_k = a(a+1)\cdots(a+k-1) \quad (k \geq 1), \tag{2.2} $$

for arbitrary $a \in \mathbf{C}$. The series (2.1) is not defined when $c = -m$, with $m = 0, 1, 2, \ldots$, unless a or b are equal to $-n, n = 0, 1, 2, \ldots$, and $n < m$. It is also easy to see that the series (2.1) reduces to a polynomial of degree n in z when a or b is equal to $-n, n = 0, 1, 2, \ldots$. In all other cases, the series has radius of convergence 1; this follows from the ratio test and (2.2). The function defined by the series (2.1) is called the Gauss hypergeometric function. When there is no danger of confusion with other types of hypergeometric series, (2.1) is commonly denoted simply by $F(a, b; c; z)$ and called the hypergeometric series, resp. function.

Most properties of the hypergeometric series can be found in the well-known reference works [1], [37], and [19] (in increasing order of completeness). Proofs of many of the more important properties can be found, e.g., in [40]; see also the important works [5] and [47].

At this point we mention only the special case

$$ F(a,b;b;z) = (1-z)^{-a}, \tag{2.3} $$

the binomial formula. The case $a = 1$ yields the geometric series; this gave rise to the term *hypergeometric.*

More properties will be introduced in later sections, as the need arises.

3. FIBONACCI NUMBERS

We will use two different (but related) connections between Fibonacci numbers and hypergeometric functions. The first one is Binet's formula

$$ F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right], \tag{3.1} $$

which allows us to use the identity

$$ F\left(a, \frac{1}{2}+a; \frac{3}{2}; z^2\right) = \frac{1}{2z(1-2a)} [(1+z)^{1-2a} - (1-z)^{1-2a}] \tag{3.2} $$

(see, e.g., [1], (15.1.10)). If we take $a = (1-n)/2$, $z = \sqrt{5}$, and compare (3.2) with (3.1), we obtain

$$ F_n = \frac{n}{2^{n-1}} F\left(\frac{1-n}{2}, \frac{2-n}{2}; \frac{3}{2}; 5 \right). $$

Note that one of the numbers $(1-n)/2$, $(2-n)/2$ is always a negative integer (or zero) for $n \geq 1$, so (3.3) is in fact a finite sum and we need not worry about convergence (see, however, the remark following (4.28)).

Our second approach will be via the well-known connection between Fibonacci numbers and the Chebyshev polynomials of the second kind, namely,

$$F_n = (-i)^{n-1} U_{n-1}\left(\frac{i}{2}\right). \tag{3.4}$$

This follows directly from the recurrence relation for the polynomials $U_n(x)$ (see, e.g., [1], [19], or [37]). But also

$$U_n(x) = (n+1)F\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right) \tag{3.5}$$

(see, e.g., [1], (22.5.48), or any of the other books mentioned above; but note that identity (25) in [19], p. 186 is incorrect). Comparing (3.4) and (3.5), we get

$$F_n = (-i)^{n-1} nF\left(1-n, 1+n; \frac{3}{2}; \frac{2-i}{4}\right); \tag{3.6}$$

again, this hypergeometric series is a finite sum.

It is worth mentioning that the Chebyshev polynomials $U_n(x)$ are special cases of the ultra-spherical (or Gegenbauer) polynomials $C_n^{\lambda}(x)$ and the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, namely,

$$U_n(x) = C_n^1(x) = \frac{4^n}{\binom{2n+1}{n}} P_n^{(1/2,1/2)}(x) \tag{3.7}$$

(see, e.g., [1], Ch. 22). Now, there is a variety of known representations by hypergeometric series for the Gegenbauer and Jacobi polynomials; see, e.g., [37], pp. 212, 220. These, in combination with (3.7) and (3.4), can be used to obtain more representations for the Fibonacci numbers by hypergeometric series. However, all of these can be obtained from (3.3) and (3.6) by way of linear and quadratic transformations, as in the following section.

Before continuing, we rewrite the representations (3.3) and (3.6) as combinatorial sums. The rising factorials involved are easily seen to be

$$\left(\frac{3}{2}\right)_k = \frac{(2k+1)!}{4^k k!}, \tag{3.8}$$

$$\left(\frac{1-n}{2}\right)_k \left(\frac{2-n}{2}\right)_k = \frac{(n-1)!}{4^k(n-1-2k)!}, \tag{3.9}$$

$$(1-n)_k = (-1)^k \frac{(n-1)!}{(n-1-k)!}, \tag{3.10}$$

$$(1+n)_k = \frac{(n+k)!}{n!}, \tag{3.11}$$

and with (2.1), the representation (3.3) becomes

$$F_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} 5^k, \tag{3.12}$$

which is a well-known formula due to Catalan (see, e.g., [29], p. 150). Formula (3.6) can be rewritten as

$$F_n = (-i)^{n-1} \sum_{k=0}^{n-1} \binom{n+k}{2k+1} (i-2)^k. \tag{3.13}$$

4. LINEAR AND QUADRATIC TRANSFORMATIONS

In this section we will use the well-known linear and quadratic transformations for the hypergeometric functions to derive a large number of representations from (3.3) and (3.6). In each case we will obtain, as immediate consequences, combinatorial sums (or series) of the form (3.12) or (3.13).

We begin with the pair of linear transformation formulas,

$$F(a,b;c;z)=(1-z)^{-a}F\left(a,c-b;c;\frac{z}{z-1}\right) \tag{4.1}$$

and

$$F(a,b;c;z)=(1-z)^{-b}F\left(b,c-a;c;\frac{z}{z-1}\right), \tag{4.2}$$

that are linked together by the relation

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z), \tag{4.3}$$

which is due to Euler (see, e.g., [1], p. 559). We also have the obvious relationship $F(a,b;c;z)=F(b,a;c;z)$ which will be invoked without special mention; it follows from the definition (2.1).

Some case must be taken on the question of convergence and the range of validity of the transformation formulas used, especially since the argument of the hypergeometric function in (3.3) is larger than 1. If we apply (4.1) to (3.3), then the right-hand side is a finite sum only when n is odd. In this case, we get

$$F_{2n+1}=(-1)^n(2n+1)F\left(-n,n+1;\frac{3}{2};\frac{5}{4}\right). \tag{4.4}$$

In general, (4.1) is valid only when both $|z|<1$ and $|z/(z-1)|<1$ (see, e.g., [40], p. 59]), but when both sides are finite sums, then by analytic continuation, (4.1) is valid on all of $\mathbb{C}$, with the possible exception of $z=1$. (In this case, there is a removable singularity at $z=1$). The situation is, of course, similar for all other transformation formulas.

We get a companion relationship to (4.4) by applying (4.2) to (3.3). In this case, n has to be even:

$$F_{2n}=(-1)^n nF\left(1-n,1+n;\frac{3}{2};\frac{5}{4}\right). \tag{4.5}$$

The next linear transformation formula in the list in [1], p. 559, is

$$\begin{aligned} F(a,b;c;z)=&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z)\\ &+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;1-z).\end{aligned} \tag{4.6}$$

However, since $a+b-c=-n$ in (3.3), one of the gamma terms in the numerator is not defined. Instead, we have to use formula (15.3.11) in [1], p. 559, which in the special case where a or b is a negative integer and m is a nonnegative integer becomes

$$F(a,b;a+b+m;z)=\frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)}F(a,b;1-m;1-z). \tag{4.7}$$

(For the general case, see [1], (15.3.11), p. 559.) This, applied to (3.3), gives

$$F_n = F\left(\frac{1-n}{2}, \frac{2-n}{2}; 1-n; -4\right). \tag{4.8}$$

Here we have evaluated the gamma terms in (4.7) as follows, using the duplication formula for $\Gamma(z)$ (see, e.g., [1], p. 256):

$$\begin{aligned}\frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)} &= \frac{\Gamma(n)\Gamma(\frac{3}{2})}{\Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(\frac{n}{2}+1)} \\ &= \frac{(2\pi)^{-1/2}2^{n-1/2}\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+\frac{1}{2})\frac{1}{2}\sqrt{\pi}}{\Gamma(\frac{n}{2}+\frac{1}{2})\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{2^{n-1}}{n}.\end{aligned}$$

Another transformation formula similar to (4.6) is

$$\begin{aligned}F(a,b;c;z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right).\end{aligned} \tag{4.9}$$

We apply this to (3.3) and note that b is a negative integer or 0 when $n \geq 2$ is even, and a is a negative integer or 0 when n is odd, while $c = 3/2$ and $b-a = 1/2$ are not integers. Using the fact that $\Gamma(z)$ has poles at the nonpositive integers, we see that one of the two terms in (4.9) always disappears. The gamma terms in the remaining expression can be evaluated as above, and we obtain

$$F_{2n+1} = \left(\frac{5}{4}\right)^n F\left(-n, -\frac{1}{2}-n; \frac{1}{2}; \frac{1}{5}\right), \tag{4.10}$$

$$F_{2n} = n\left(\frac{5}{4}\right)^{n-1} F\left(1-n, \frac{1}{2}-n; \frac{3}{2}; \frac{1}{5}\right). \tag{4.11}$$

Euler's formula (4.3) can be applied to both, and we get

$$F_{2n+1} = \left(\frac{4}{5}\right)^{n+1} F\left(\frac{1}{2}+n, 1+n; \frac{1}{2}; \frac{1}{5}\right), \tag{4.12}$$

$$F_{2n} = n\left(\frac{4}{5}\right)^{n+1} F\left(\frac{1}{2}+n, 1+n; \frac{3}{2}; \frac{1}{5}\right). \tag{4.13}$$

These two formulas are interesting because they give us the first infinite series representations for the Fibonacci numbers; see the following section.

Next we note that (4.10) and (4.11) satisfy the hypotheses of the transformation formula (4.7), which gives

$$F_{2n+1} = 5^n F\left(-n, -\frac{1}{2}-n; -2n; \frac{4}{5}\right), \tag{4.14}$$

$$F_{2n} = 5^{n-1} F\left(1-n, \frac{1}{2}-n; 1-2n; \frac{4}{5}\right). \tag{4.15}$$

The next transformation formula,

$$\begin{aligned}F(a,b;c;z) &= (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}F\left(a,c-b;a-b+1;\frac{1}{1-z}\right)\\ &+(1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}F\left(b,c-a;b-a+1;\frac{1}{1-z}\right),\end{aligned} \tag{4.16}$$

applied to (3.3), gives the representations

$$F_{2n+1} = F\left(-n,1+n;\frac{1}{2};\frac{-1}{4}\right), \tag{4.17}$$

$$F_{2n} = nF\left(1-n,1+n;\frac{3}{2};-\frac{1}{4}\right), \tag{4.18}$$

and (4.3) applied to these,

$$F_{2n+1} = \frac{2}{\sqrt{5}}F\left(\frac{1}{2}+n,-\frac{1}{2}-n;\frac{1}{2};\frac{-1}{4}\right), \tag{4.19}$$

$$F_{2n} = \frac{2n}{\sqrt{5}}F\left(\frac{1}{2}+n,\frac{1}{2}-n;\frac{3}{2};\frac{-1}{4}\right). \tag{4.20}$$

Identity (4.17) was explicitly stated in the solution of [43].

To obtain further representations with rational arguments of the hypergeometric function, we have to use quadratic transformations. The first such formula we need is

$$F(a,b;a-b+1;z) = (1+z)^{-a}F\left(\frac{a}{2},\frac{a}{2}+\frac{1}{2};a-b+1;\frac{4z}{(1+z)^2}\right) \tag{4.21}$$

(see, e.g., [1], (15.3.26), p. 561). This, applied to (4.10) and (4.11), gives, respectively

$$F_{2n+1} = \left(\frac{3}{2}\right)^n\sqrt{\frac{6}{5}}F\left(\frac{-1-2n}{4},\frac{1-2n}{4};\frac{1}{2};\frac{5}{9}\right), \tag{4.22}$$

$$F_{2n} = n\left(\frac{3}{2}\right)^{n-1}F\left(\frac{1-n}{2},\frac{2-n}{2};\frac{3}{2};\frac{5}{9}\right). \tag{4.23}$$

Again we can apply (4.3) and obtain

$$F_{2n+1} = \frac{2}{\sqrt{5}}\left(\frac{2}{3}\right)^{n+\frac{1}{2}}F\left(\frac{3+2n}{4},\frac{1+2n}{4};\frac{1}{2};\frac{5}{9}\right), \tag{4.24}$$

$$F_{2n} = n\left(\frac{2}{3}\right)^{n+1}F\left(\frac{2+n}{2},\frac{1+n}{2};\frac{3}{2};\frac{5}{9}\right). \tag{4.25}$$

We can now apply linear transformation formulas again to obtain further representations. It is easy to see that (4.6) applies to (4.24) and (4.7) to (4.25). The gamma function terms can be evaluated as before, and we obtain

$$F_{2n+1} = \frac{3^{n+\frac{1}{2}}}{\sqrt{5}}F\left(\frac{-1-2n}{4},\frac{1-2n}{4};\frac{1}{2}-n;\frac{4}{9}\right)+\frac{1}{3^{n+\frac{1}{2}}\sqrt{5}}F\left(\frac{3+2n}{4},\frac{1+2n}{4};\frac{3}{2}+n;\frac{4}{9}\right), \tag{4.26}$$

$$F_{2n} = 3^{n-1} F\left(\frac{1-n}{2}, \frac{2-n}{2}; 1-n; \frac{4}{9}\right). \tag{4.27}$$

Euler's formula (4.3) can be applied to (4.26) to give

$$F_{2n+1} = 3^{n-\frac{1}{2}} F\left(\frac{3-2n}{4}, \frac{1-2n}{4}; \frac{1}{2}-n; \frac{4}{9}\right) + \frac{1}{3^{n+\frac{3}{2}}} F\left(\frac{3+2n}{4}, \frac{5+2n}{4}; \frac{3}{2}+n; \frac{4}{9}\right). \tag{4.28}$$

Remark: A word of caution is in order at this point. As was the case with several other identities before, (4.27) cannot be transformed by (4.3), even though $|z| < 1$ and both sides of (4.3) would be finite sums (either the first or the second parameter is a negative integer in this and the other cases). The reason for this lies in the fact that the proof of (4.1) (see, e.g., [40], pp. 58ff.) breaks down when a and c and negative integers with $c < a$, while b is not a negative integer. This can be remedied by simply interchanging the order of the two parameters a and b, which means that one of the identities (4.1), (4.2) is true, while the other is not. Since (4.3) and (4.1) imply (4.2) (and similarly, (4.2) and (4.1) imply (4.3)), the identity (4.3) cannot be used under the circumstances in question.

That (4.3) is actually false in this case can be seen as follows. If a and c are nonpositive integers, $c < a$, then the hypergeometric series on both sides of (4.3) are actually polynomials in z. However, $(1-z)^{c-a-b}$ is an infinite series since $c-a-b$ cannot be a positive integer or zero. This is a contradiction.

To obtain further hypergeometric series representations, we apply (4.1) to (4.26):

$$F_{2n+1} = 5^{\frac{n}{2}-\frac{1}{4}} F\left(\frac{-1-2n}{4}, \frac{1-2n}{4}; \frac{1}{2}-n; \frac{-4}{5}\right) + \frac{3}{5^{\frac{n}{2}+\frac{5}{4}}} F\left(\frac{3+2n}{4}, \frac{5+2n}{4}; \frac{3}{2}+n; \frac{-4}{5}\right). \tag{4.29}$$

To transform (4.27), we have to distinguish between even and odd n. For n odd, we apply (4.1), and (4.2) when n is even, to obtain

$$F_{4n+2} = 5^n F\left(-n, -\frac{1}{2}-n; -2n; -\frac{4}{5}\right), \tag{4.30}$$

$$F_{4n} = 3 \cdot 5^{n-1} F\left(1-n, \frac{1}{2}-n; 1-2n; -\frac{4}{5}\right). \tag{4.31}$$

In accordance with the remark following (4.28), identity (4.29) can be further transformed by formula (4.3), while this is not possible for (4.30) and (4.31). We get

$$F_{2n+1} = 3 \cdot 5^{\frac{n}{2}-\frac{3}{4}} F\left(\frac{3-2n}{4}, \frac{1-2n}{4}; \frac{1}{2}-n; \frac{-4}{5}\right) + 5^{-\frac{n}{2}+\frac{3}{4}} F\left(\frac{3+2n}{4}, \frac{1+2n}{4}; \frac{3}{2}+n; \frac{-4}{5}\right). \tag{4.32}$$

Next we apply (4.16) to identity (4.23). We have to distinguish between the cases n even and n odd. If we determine the gamma function terms as before, we obtain

$$F_{4n+2} = (-1)^n F\left(-n, 1+n; \frac{1}{2}; \frac{9}{4}\right), \tag{4.33}$$

$$F_{4n} = (-1)^{n+1} 3n F\left(1-n, 1+n; \frac{3}{2}; \frac{9}{4}\right). \tag{4.34}$$

Transformed with formula (4.1), these two identities lead to

$$F_{4n+2}=\left(\frac{5}{4}\right)^n F\left(-n,-\frac{1}{2}-n;\frac{1}{2};\frac{9}{5}\right), \tag{4.35}$$

$$F_{4n}=3n\left(\frac{5}{4}\right)^{n-1} F\left(1-n,\frac{1}{2}-n;\frac{3}{2};\frac{9}{5}\right). \tag{4.36}$$

We note that the last four formulas are all for even-index Fibonacci numbers. The transformation formula (4.16), and other appropriate transformations, will only give rise to divergent series. It appears doubtful that there exist simple expressions for odd-index Fibonacci numbers in terms of hypergeometric series (necessarily finite sums) with arguments $9/4$, $9/5$, or $-5/4$ (below).

Finally in this section, we use the following two related quadratic transformation formulas:

$$F(a,b;a-b+1;z)=(1-z)^{-a}F\left(\frac{a}{2},\frac{a-2b+1}{2};a-b+1;\frac{-4z}{(1-z)^2}\right), \tag{4.37}$$

$$F(a,b;a-b+1;z)=\frac{1+z}{(1-z)^{a+1}}F\left(\frac{1+a}{2},\frac{a}{2}-b+1;a-b+1;\frac{-4z}{(1-z)^2}\right). \tag{4.38}$$

The first one of these is formula (15.3.28) in [1], p. 561, and both can be found in [19], p. 113. We apply them to (3.3) with the first two parameters interchanged, i.e., with $a=(2-n)/2$ and $b=(1-n)/2$. For the right-hand sides of (4.37) and (4.38) to be convergent, the series have to be terminating, and this occurs when $n\equiv 2 \pmod 4$, resp. $n\equiv 0 \pmod 4$. Thus, we get

$$F_{4n+2}=(2n+1)F\left(-n,n+1;\frac{3}{2};-\frac{5}{4}\right), \tag{4.39}$$

$$F_{4n}=3nF\left(1-n,1+n;\frac{3}{2};-\frac{5}{4}\right). \tag{4.40}$$

We have thus obtained numerous representations of Fibonacci numbers in terms of hypergeometric functions with rational arguments. In fact, twelve different rational arguments occurred, and in Section 9 below we will discuss the question of whether these are all.

5. EXPLICIT FORMULAS

In this section we will simply rewrite the formulas obtained above in terms of combinatorial sums, using (2.1) and (2.2). The easy identities (3.8)-(3.11) will help with this task; other similar such identities which may be required below will not be stated explicitly.

It should be noted that the same formula may come in different guises. First, there is the obvious relationship $\binom{n}{k}=\binom{n}{n-k}$ between binomial coefficients. Then, reversing the order of summation (in finite sums) leads to a new sum that is a bit different in appearance, but in most cases is easily seen to be equivalent. As a rule, we will state below only one of these obviously equivalent forms. For a general discussion of this lack of uniqueness in combinatorial sums, see the introductions of [42] and [28].

We begin with finite sums, ordered according to powers that may occur. Many of these formulas are well known; in these cases only one or two easily accessible references will be given.

Identities (4.8), (4.17), and (4.18), respectively, lead to the sums

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k}, \tag{5.1}$$

$$F_{2n+1} = \sum_{k=0}^{n} \binom{n+k}{2k}, \tag{5.2}$$

$$F_{2n} = \sum_{k=0}^{n-1} \binom{n+k}{2k+1}. \tag{5.3}$$

Formula (5.1) is probably the best known of all. It is the "rising diagonal sum" property that links the Fibonacci sequence closely to the Pascal triangle; it can be found in most references on Fibonacci numbers, e.g., [31], p. 50. Formula (5.2) is listed in [24] as identity (1.76), and both (5.2) and (5.3) can be found in [17]. For generalizations of (5.2) and (5.3), see [18].

Identities (3.3), (4.40), and (4.5) give rise to

$$F_n = 2^{1-n} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} 5^k, \tag{5.4}$$

$$F_{4n} = 3\sum_{k=0}^{n-1} \binom{n+k}{2k+1} 5^k, \tag{5.5}$$

$$F_{2n} = (-1)^{n-1} \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} 5^k. \tag{5.6}$$

Catalan's well-known identity (5.4), repeated here for completeness, was already mentioned in (3.12). Identities (4.10) and (4.11) give only special cases (even, resp. odd n) of (5.4). While (5.6) appears in [11], the author was unable to find (5.5) in the literature. Identities (4.31) and (4.15) also lead to (5.5) and (5.6), respectively.

Both (4.4) and (4.14) lead to the second, and (4.30) and (4.39) to the first of the following identities:

$$F_{4n+2} = (2n+1)\sum_{k=0}^{n} \binom{n+k}{2k} \frac{5^k}{2k+1}, \tag{5.7}$$

$$F_{2n+1} = (-1)^n (2n+1) \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k} \frac{5^k}{2k+1}. \tag{5.8}$$

Identity (5.8) can be found in [11], while (5.7) appears to be new.

Next, we obtain from (4.23) and (4.27), respectively,

$$F_{2n} = \left(\frac{3}{2}\right)^{n-1} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \left(\frac{5}{9}\right)^k, \tag{5.9}$$

$$F_{2n} = 3^{n-1} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-1-k}{k} \left(\frac{1}{9}\right)^k. \tag{5.10}$$

Identity (5.9) follows from formula (1.95) in [24]; for references on (5.10), see [26]. Special cases of (5.9) for n even, resp. odd, follow from (4.35) and (4.36). If we distinguish between the cases n even and odd also in (5.10) and reverse the orders of summation, we obtain

$$F_{4n} = 3(-1)^{n-1}\sum_{k=0}^{n-1}(-1)^k\binom{n+k}{2k+1}9^k, \tag{5.11}$$

$$F_{4n+2} = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n+k}{2k}9^k. \tag{5.12}$$

These last two identities also follow from (4.34) and (4.33), respectively.

Numerous other identities of types (5.1)-(5.12) can be found in the literature, especially in articles and problems in *The Fibonacci Quarterly*. Among the multitude of different methods used to obtain and prove these results, only a few general methods seem to have emerged. One of them can be found in [34]; see also the discussion at the end of that article concerning discovering as opposed to proving identities.

In the second half of this section we list infinite series representations for Fibonacci numbers as direct consequences of the remaining hypergeometric identities in Section 4. We will make use of the generalized binomial coefficient which, for arbitrary real or complex a and positive integer k, is defined by

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!} = \frac{(a-k+1)_k}{k!} = \frac{\Gamma(a+1)}{\Gamma(a-k+1)\Gamma(k+1)}. \tag{5.13}$$

The restriction on k could actually be relaxed, but in what follows, k will always be a positive integer.

Using, as before, (3.8)-(3.11) and other similar relationships, we obtain from (4.19), (4.20), (4.12), and (4.13), respectively,

$$F_{2n+1} = \frac{2}{\sqrt{5}}\left(1+\left(n+\frac{1}{2}\right)\sum_{k=1}^{\infty}\binom{n+k-\frac{1}{2}}{2k-1}\frac{1}{2k}\right), \tag{5.14}$$

$$F_{2n} = \frac{2n}{\sqrt{5}}\sum_{k=0}^{\infty}\binom{n+k-\frac{1}{2}}{2k}\frac{1}{2k+1}, \tag{5.15}$$

$$F_{2n+1} = \left(\frac{4}{5}\right)^{n+1}\sum_{k=0}^{\infty}\binom{2n+2k}{2k}\left(\frac{1}{5}\right)^k, \tag{5.16}$$

$$F_{2n} = \frac{1}{2}\left(\frac{4}{5}\right)^{n+1}\sum_{k=0}^{\infty}\binom{2n+2k}{2k+1}\left(\frac{1}{5}\right)^k. \tag{5.17}$$

While (5.14) and (5.15) appear to be new, (5.16) and (5.17) follow immediately from the identity

$$F_{n+1} = \sum_{2k\geq n}\binom{2k}{n}2^{n+1}5^{-k-1}, \tag{5.18}$$

which is an exercise in [41], p. 240.

The next three identities follow from (4.22), (4.24), and (4.25), respectively:

$$F_{2n+1}=\left(\frac{3}{2}\right)^n\sqrt{\frac{6}{5}}\sum_{k=0}^{\infty}\binom{n+\frac{1}{2}}{2k}\left(\frac{5}{9}\right)^k, \tag{5.19}$$

$$F_{2n+1}=\frac{2}{\sqrt{5}}\left(\frac{2}{3}\right)^{n+\frac{1}{2}}\sum_{k=0}^{\infty}\binom{n+2k-\frac{1}{2}}{2k}\left(\frac{5}{9}\right)^k, \tag{5.20}$$

$$F_{2n}=\left(\frac{2}{3}\right)^{n+1}\sum_{k=0}^{\infty}\binom{n+2k}{2k+1}\left(\frac{5}{9}\right)^k. \tag{5.21}$$

Identity (5.19) could be considered the odd-index analog of (5.9). None of (5.19)-(5.21) seem to have occurred before in the literature.

The last four identities in this section involve two infinite series each; they follow from (4.26), (4.28), (4.29), and (4.32), respectively.

$$\begin{aligned}F_{2n+1}&=\frac{3^{n+\frac{1}{2}}}{\sqrt{5}}\left[1+\left(n+\frac{1}{2}\right)\sum_{k=1}^{\infty}\binom{n-k-\frac{1}{2}}{k-1}\frac{1}{k}\left(\frac{-1}{9}\right)^k\right]\\&+\frac{3^{-n-\frac{1}{2}}}{\sqrt{5}}\left[1+\left(n+\frac{1}{2}\right)\sum_{k=1}^{\infty}\binom{n+2k-\frac{1}{2}}{k-1}\frac{1}{k}\left(\frac{1}{9}\right)^k\right],\end{aligned} \tag{5.22}$$

$$F_{2n+1}=3^{n-\frac{1}{2}}\sum_{k=0}^{\infty}\binom{n-k-\frac{1}{2}}{k}\left(\frac{-1}{9}\right)^k+3^{-n-\frac{3}{2}}\sum_{k=0}^{\infty}\binom{n+2k+\frac{1}{2}}{k}\left(\frac{1}{9}\right)^k, \tag{5.23}$$

$$\begin{aligned}F_{2n+1}&=5^{\frac{n}{2}-\frac{1}{4}}\left[1+\left(n+\frac{1}{2}\right)\sum_{k=1}^{\infty}\binom{n-k-\frac{1}{2}}{k-1}\frac{1}{k}\left(\frac{1}{5}\right)^k\right]\\&+3\cdot5^{-\frac{n}{2}-\frac{5}{4}}\sum_{k=1}^{\infty}\binom{n+2k+\frac{1}{2}}{k}\left(\frac{-1}{5}\right)^k,\end{aligned} \tag{5.24}$$

$$\begin{aligned}F_{2n+1}&=3\cdot5^{\frac{n}{2}-\frac{3}{4}}\sum_{k=0}^{\infty}\binom{n-k-\frac{1}{2}}{k}\left(\frac{1}{5}\right)^k\\&+5^{-\frac{n}{2}-\frac{3}{4}}\left[1+\left(n+\frac{1}{2}\right)\sum_{k=0}^{\infty}\binom{n+2k-\frac{1}{2}}{k-1}\frac{1}{k}\left(\frac{-1}{5}\right)^k\right].\end{aligned} \tag{5.25}$$

Again, these four identities appear to be new.

6. MORE TRANSFORMATIONS: IRRATIONAL ARGUMENTS

In this section we will use the linear and quadratic transformations of Section 4, and a few new ones, to derive more representations of Fibonacci numbers in terms of hypergeometric functions. Here the arguments will all be irrational.

We will need the additional quadratic transformation formulas:

$$F\left(a, b; a+b-\frac{1}{2}; z\right) = (1-z)^{-\frac{1}{2}} F\left(2a-1, 2b-1; a+b-\frac{1}{2}; \frac{1-\sqrt{1-z}}{2}\right), \tag{6.1}$$

$$F\left(a, a+\frac{1}{2}; c; z\right) = (1\pm\sqrt{z})^{-2a} F\left(2a, c-\frac{1}{2}; 2c-1; \pm\frac{2\sqrt{z}}{1\pm\sqrt{z}}\right); \tag{6.2}$$

they are listed as formulas (15.3.24), resp. (15.3.20) in [1], p. 560f. Formula (6.1) applied to (4.17) and (4.18) immediately gives

$$F_{2n+1} = \frac{\pm 2}{\sqrt{5}} F\left(-1-2n, 1+2n; \frac{1}{2}; \frac{2\mp\sqrt{5}}{4}\right), \tag{6.3}$$

$$F_{2n} = \frac{\pm 2n}{\sqrt{5}} F\left(1-2n, 1+2n; \frac{3}{2}; \frac{2\mp\sqrt{5}}{4}\right). \tag{6.4}$$

Originally, we obtain the "upper signs" in the $\pm$ or $\mp$ pairs. However, since the Fibonacci numbers are integers, the hypergeometric functions are rational multiples of $\sqrt{5}$. Therefore, changing the sign of $\sqrt{5}$ in the argument will also change the sign of the function value.

Now, applying the linear transformation (4.1) to (6.3) and (6.4), we get, respectively,

$$F_{2n+1} = \frac{\pm 2}{\sqrt{5}} \left(\frac{2\pm\sqrt{5}}{4}\right)^{2n+1} F\left(-1-2n, -\frac{1}{2}-2n; \frac{1}{2}; 9\mp 4\sqrt{5}\right), \tag{6.5}$$

$$F_{2n} = \frac{\pm 2}{\sqrt{5}} \left(\frac{2\pm\sqrt{5}}{4}\right)^{2n+1} F\left(-1-2n, \frac{1}{2}-2n; \frac{3}{2}; 9\mp 4\sqrt{5}\right). \tag{6.6}$$

Euler's transformation (4.3) can be applied to (6.3)-(6.5), and we obtain, respectively,

$$F_{2n+1} = \frac{\pm 1}{\sqrt{5}} (2\pm\sqrt{5})^{\frac{1}{2}} F\left(\frac{3}{2}+2n, -\frac{1}{2}-2n; \frac{1}{2}; \frac{2\mp\sqrt{5}}{4}\right), \tag{6.7}$$

$$F_{2n} = \frac{\pm 4n}{\sqrt{5}} (-2\pm\sqrt{5})^{\frac{1}{2}} F\left(\frac{1}{2}+2n, \frac{1}{2}-2n; \frac{3}{2}; \frac{2\mp\sqrt{5}}{4}\right), \tag{6.8}$$

$$F_{2n+1} = \pm\frac{2^{4n+3}}{\sqrt{5}} (-2\pm\sqrt{5})^{2n+1} F\left(\frac{3}{2}+2n, 1+2n; \frac{1}{2}; 9\mp 4\sqrt{5}\right), \tag{6.9}$$

$$F_{2n} = \pm\frac{n2^{4n+3}}{\sqrt{5}} (-2\pm\sqrt{5})^{2n+1} F\left(\frac{1}{2}+2n, 1+2n; \frac{3}{2}; 9\mp 4\sqrt{5}\right). \tag{6.10}$$

Next, we note that (6.5) and (6.6) satisfy the hypothesis of the linear transformation (4.7). We easily obtain

$$F_{2n+1} = \frac{\mp 1}{\sqrt{5}} (2\mp\sqrt{5})^{2n+1} F\left(-\frac{1}{2}-2n, -1-2n; -1-4n; -8\mp 4\sqrt{5}\right), \tag{6.11}$$

$$F_{2n} = \frac{\mp 1}{\sqrt{5}} (2\mp\sqrt{5})^{2n-1} F\left(1-2n, \frac{1}{2}-2n; 1-4n; -8\mp 4\sqrt{5}\right). \tag{6.12}$$

It is interesting to note that the six different arguments above relate to each other as the six rational arguments in Section 4 (up to (4.20)) relate to each other, and so do the six further arguments in the second half of Section 4. More on this in Section 9 below.

To obtain another set of hypergeometric representations, we apply (6.2) to (3.3),

$$F_n = n\left(\frac{1\pm\sqrt{5}}{2}\right)^{n-1} F\left(1-n, 1; 2; \frac{5\mp\sqrt{5}}{2}\right); \tag{6.13}$$

then we apply (4.7) to this, and get

$$F_n = \left(\frac{1\pm\sqrt{5}}{2}\right)^{n-1} F\left(1-n, 1; 1-n; \frac{-3\pm\sqrt{5}}{2}\right). \tag{6.14}$$

Finally, we apply (4.1) to (6.14) to obtain

$$F_n = (\pm\sqrt{5})^{n-1} F\left(1-n, n; 1-n; \frac{5\mp\sqrt{5}}{2}\right). \tag{6.15}$$

Neither one of these three identities allows the use of Euler's transformation (4.3): the identity (6.13) would lead to a divergent series, and (6.14), (6.15) have nonpositive integers as first and third parameters (for $n \geq 1$).

Finally in this section, we use the following quadratic transformation:

$$\begin{aligned} F\left(a, b; \frac{3}{2}; z\right) &= \frac{\Gamma(a-\frac{1}{2})\Gamma(b-\frac{1}{2})}{2\Gamma(-\frac{1}{2})\Gamma(a+b-\frac{1}{2})} z^{-\frac{1}{2}} \\ &\times \left\{F\left(2a-1, 2b-1; a+b-\frac{1}{2}; \frac{1-\sqrt{z}}{2}\right) - F\left(2a-1, 2b-1; a+b-\frac{1}{2}; \frac{1+\sqrt{z}}{2}\right)\right\}. \end{aligned} \tag{6.16}$$

(This is formula (9) in [19], p. 111.) We apply this to (3.3); the gamma functions term is easily evaluated to be $2^{n-1}/n$, and we get

$$F_n = \frac{1}{\sqrt{5}}\left\{F\left(-n, 1-n; 1-n; \frac{1-\sqrt{5}}{2}\right) - F\left(-n, 1-n; 1-n; \frac{1+\sqrt{5}}{2}\right)\right\}. \tag{6.17}$$

7. RELATIONSHIPS AMONG FIBONACCI NUMBERS

Just as we did in Section 5, we can rewrite the various hypergeometric representations from the previous section in terms of finite combinatorial sums and infinite series. For example, (6.3) leads to

$$F_{2n+1} = \pm\frac{2(2n+1)}{\sqrt{5}} \sum_{k=0}^{2n+1} \binom{2n+k+1}{2k} \frac{(\pm\sqrt{5}-2)^k}{2n+k+1}. \tag{7.1}$$

We will not explicitly write down the remaining such series (with the exception of four infinite series), but instead use Binet's formulas (3.1) and

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n, \tag{7.2}$$

and the obvious relations

$$2 \pm \sqrt{5} = \left(\frac{1 \pm \sqrt{5}}{2}\right)^3, \quad 9 \pm 4\sqrt{5} = \left(\frac{1 \pm \sqrt{5}}{2}\right)^6, \tag{7.3}$$

to express Fibonacci numbers as sums of other Fibonacci or Lucas numbers. (Recall that the Lucas numbers L_n, which could be defined by (7.2), satisfy the recursion $L_0 = 2$, $L_1 = 1$, and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$.)

We simply add the two versions of (7.1) and use (7.3) and (3.1) to obtain

$$F_{2n+1} = (2n+1) \sum_{k=0}^{2n+1} (-1)^{k+1} \binom{2n+k+1}{2k} \frac{F_{3k}}{2n+k+1}. \tag{7.4}$$

In a similar fashion, the companion relation (6.4) gives

$$F_{2n} = \frac{1}{2} \sum_{k=0}^{2n-1} (-1)^{k+1} \binom{2n+k}{2k+1} F_{3k}. \tag{7.5}$$

We note that (6.11) and (6.12) also lead to (7.4) and (7.5), respectively. Similarly, the pair of relations (6.5) and (6.6) is easily transformed into

$$F_{2n+1} = 2^{-4n-1} \sum_{k=0}^{n} \binom{4n+2}{2k} F_{6n+6k+3}, \tag{7.6}$$

$$F_{2n} = 2^{1-4n} \sum_{k=0}^{n-1} \binom{4n}{2k+1} F_{6n-6k-3}. \tag{7.7}$$

What is probably the simplest such formula follows directly from (6.17):

$$F_n = \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} F_k. \tag{7.8}$$

(Here, the term F_n also occurs on the right-hand side, with coefficient $(-1)^{n+1}$).

Next, we obtain a few formulas that involve both Fibonacci and Lucas numbers. Once (6.14) has been rewritten as a finite sum, we have to distinguish between the cases n odd and n even. Using (7.3) and (7.2), we easily obtain

$$F_{2n+1} = (-1)^n + \sum_{k=1}^{n} (-1)^{n-k} L_{2k}, \tag{7.9}$$

$$F_{2n} = \sum_{k=1}^{n} (-1)^{n-k} L_{2k-1}. \tag{7.10}$$

To obtain the last formula of this kind, we write (6.15) as a sum, reverse the order of summation, and separate even and odd indices of summation; (3.1), (7.2), and (7.3) then give

$$F_n = \frac{1}{2} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} 5^k L_{n-2k-1} - \frac{1}{2} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 5^k F_{n-2k}. \tag{7.11}$$

The same formula is also obtained from (6.13).

As is to be expected, not all of the formulas (7.4)-(7.11) are new. In fact, (7.4) is, up to an easy transformation, identical with a formula in [46], p. 190. Identity (7.8) was a problem in *The Fibonacci Quarterly* [7]; it also follows from an earlier, more general result in [23], p. 13. Identities (7.9) and (7.10) follow from formula (97) in [48], p. 183; there, use is made of negative-index

Lucas numbers which allow for the two cases to be written as one. Identities (7.6) and (7.7) follow easily from one of the three identities in [45]; they are also very similar to some general formulas in [32], but do not appear to be special cases of these formulas. Many more formulas of this type appear in [50], [6], [21], [36], [15], [30], [14], and throughout the problem sections of *The Fibonacci Quarterly*.

Finally in this section, we rewrite the representations (6.7)-(6.10) as infinite series involving binomial coefficients:

$$F_{2n+1}=\left(\frac{2+\sqrt{5}}{5}\right)^{1/2}\sum_{k=0}^{\infty}\binom{2n+\frac{1}{2}+k}{2k}(-2+\sqrt{5})^k, \tag{7.12}$$

$$F_{2n}=4n\left(\frac{-2+\sqrt{5}}{5}\right)^{1/2}\sum_{k=0}^{\infty}\binom{2n-\frac{1}{2}+k}{2k}\frac{(-2+\sqrt{5})^k}{2k+1}, \tag{7.13}$$

$$F_{2n+1}=-\frac{2^{4n+3}}{\sqrt{5}}\sum_{k=0}^{\infty}\binom{4n+2k+1}{2k}\left(\frac{1-\sqrt{5}}{2}\right)^{6n+6k+3}, \tag{7.14}$$

$$F_{2n}=-\frac{2^{4n+1}}{\sqrt{5}}\sum_{k=0}^{\infty}\binom{4n+2k}{2k+1}\left(\frac{1-\sqrt{5}}{2}\right)^{6n+6k+3}. \tag{7.15}$$

The conjugates of these expressions would be divergent, so (7.12)-(7.15) will not lead to any obvious formulas involving Fibonacci or Lucas numbers on the right-hand side, as in the finite cases. However, a formula of that type occurs in [20] as identity (4.19).

A different type of identity can be derived as follows. Using formula (7.2), we see that for odd integers n we have

$$\frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n=\frac{1}{5F_n+\sqrt{5}\left(\frac{1-\sqrt{5}}{2}\right)^n}. \tag{7.16}$$

This leads in a natural way to a continued fraction and, from (7.14), for example, we obtain the curious expansion

$$F_{2n+1}=4^{4n+3}\sum_{k=0}^{\infty}\cfrac{\binom{4n+2k+1}{2k}}{5F_{6n+6k+3}-\cfrac{1}{F_{6n+6k+3}-\cfrac{1}{5F_{6n+6k+3}-\cdots}}}. \tag{7.17}$$

By truncating the continued fraction, or the infinite series (or both), one obtains approximate expressions with easily quantifiable error terms.

8. COMPLEX ARGUMENTS

It will now be clear that the transformations in Sections 4 and 6 lead to a variety of hypergeometric representations with complex arguments. One such formula, namely (3.6), has already been encountered, and it was rewritten as a combinatorial sum in (3.13). While it will be left to the reader to derive other formulas, we examine (3.13) a little further.

First, we use the fact that $(2+i)/\sqrt{5}=\exp(\tan^{-1}(1/2))$. Adding (3.13) to its complex conjugate, we obtain

$$F_{2n+1}=(-1)^n\sum_{k=0}^{2n}\binom{2n+1+k}{2k+1}(-\sqrt{5})^k\cos\left(k\tan^{-1}\left(\frac{1}{2}\right)\right), \tag{8.1}$$

$$F_{2n}=(-1)^n\sum_{k=0}^{2n-1}\binom{2n+k}{2k+1}(-\sqrt{5})^k\sin\left(k\tan^{-1}\left(\frac{1}{2}\right)\right). \tag{8.2}$$

The cosine and sine terms in these expressions show that there is a connection with Chebyshev polynomials; this will not be taken further at this point.

Second, we define the sequences $u_k=(i-2)^k+(-i-2)^k$ and $v_k=i[(i-2)^k-(-i-2)^k]$ for integers $k\ge 0$. Using standard methods for dealing with second-order recurrences, we find that the characteristic polynomial for both sequences is $(x-(i-2))(x-(-i-2))=x^2+4x+5$, so that we have

$$u_{k+1}=-4u_k-5u_{k-1},\ u_0=2, u_1=-4; \tag{8.3}$$

$$v_{k+1}=-4v_k-5v_{k-1},\ v_0=0, v_1=-2. \tag{8.4}$$

The first few terms of these sequences are $2,-4,6,-4,-14$ and $0,-2,8,-22,48$, respectively. The sign behaviors, by the way, are explained by the cosine and sine terms in (8.1) and (8.2). Again adding (3.13) to its complex conjugate, we finally obtain

$$F_{2n+1}=\frac{(-1)^n}{2}\sum_{k=0}^{2n}\binom{2n+k+1}{2k+1}u_k, \tag{8.5}$$

$$F_{2n}=\frac{(-1)^n}{2}\sum_{k=0}^{2n-1}\binom{2n+k}{2k+1}v_k. \tag{8.6}$$

Numerous other related formulas can be obtained in this way.

9. THE SET OF POSSIBLE ARGUMENTS

TABLE 1. Possible Arguments

z	$\frac{z}{z-1}$	$1-z$	$1-\frac{1}{z}$	$\frac{1}{z}$	$\frac{1}{1-z}$
5	$\frac{5}{4}$	-4	$\frac{4}{5}$	$\frac{1}{5}$	$-\frac{1}{4}$
$\frac{5}{9}$	$-\frac{5}{4}$	$\frac{4}{9}$	$-\frac{4}{5}$	$\frac{9}{5}$	$\frac{9}{4}$
$\frac{1+\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{3-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
$\frac{2-\sqrt{5}}{4}$	$9-4\sqrt{5}$	$\frac{2+\sqrt{5}}{4}$	$9+4\sqrt{5}$	$-8-4\sqrt{5}$	$-8+4\sqrt{5}$
$\frac{-3+\sqrt{5}}{2}$	$\frac{5-\sqrt{5}}{10}$	$\frac{5-\sqrt{5}}{2}$	$\frac{5+\sqrt{5}}{2}$	$\frac{-3-\sqrt{5}}{2}$	$\frac{5+\sqrt{5}}{10}$
$\frac{-5+3\sqrt{5}}{2}$	$\frac{-5-3\sqrt{5}}{2}$	$\frac{7-3\sqrt{5}}{2}$	$\frac{5-3\sqrt{5}}{10}$	$\frac{5+3\sqrt{5}}{10}$	$\frac{7+3\sqrt{5}}{2}$
$\frac{2-i}{4}$	$\frac{-3+4i}{5}$	$\frac{2+i}{4}$	$\frac{-3-4i}{5}$	$\frac{8+4i}{5}$	$\frac{8-4i}{5}$
$\frac{2+i\sqrt{5}}{4}$	$\frac{1-4i\sqrt{5}}{9}$	$\frac{2-i\sqrt{5}}{4}$	$\frac{1+4i\sqrt{5}}{9}$	$\frac{8-4i\sqrt{5}}{9}$	$\frac{8+4i\sqrt{5}}{9}$

It will be interesting to know whether the set of twelve rational arguments considered in Section 4 is exhaustive, and what would be the complete set of real irrational and of complex arguments. First we note that, starting with the argument z, all linear transformations lead to the set of arguments $\{z, z/(z-1), 1-z, 1-1/z, 1/z, 1/(1-z)\}$; see e.g., [19], pp. 105ff. This means that, given one argument, linear transformations will lead to at most five more arguments.

Things are somewhat more complicated in the case of quadratic transformations. However, since not all parameter triples (a, b, c) are permissible (see, e.g., [19], pp. 110ff.), the number of possible arguments remains quite limited. It is possible to find them all by inspection; they are listed in Table 1 above.

The arguments listed in the rows of Table 1 can be obtained from each other by linear transformations. To go to different rows, appropriate quadratic transformations have to be used. The entries in the "z" column are arbitrary; only the entry "5", as the "original" argument of (3.3), has been placed in the upper left corner.

10. FURTHER APPLICATIONS

Hypergeometric functions have long been part of a well-developed theory, and they have been generalized in several important directions. Therefore, it is not surprising that further properties of Fibonacci numbers and related numbers and functions can be obtained rather easily by applying classical results on hypergeometric functions. However, in the confines of this article, it is not possible (or even desirable) to give a full account. Instead, I will conclude this paper by making brief remarks on a number of topics not yet considered.

1. ***Integral representations:*** While it does not appear possible to apply Euler's integral or other related integrals (see, e.g., [19], pp. 114ff.) directly to the representation in Sections 4 and 6, the transformed integral representation

$$F\left(a, a-b+\frac{1}{2}; b+\frac{1}{2}; z^2\right) = \frac{\Gamma(b+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(b)} \int_0^\pi \frac{(\sin\phi)^{2b-1}}{(1+2z\cos\phi+z^2)^a} d\phi, \tag{10.1}$$

valid when Re $b > 0$ and $|z| < 1$ (see [19], p. 81) can be applied to (4.9). We get immediately

$$F_{2n} = \frac{n}{2}\left(\frac{3}{2}\right)^{n-1} \int_0^\pi \left(1+\frac{\sqrt{5}}{3}\cos\phi\right)^{n-1} \sin\phi\, d\phi. \tag{10.2}$$

This integral, of course, can be verified easily by a simple substitution and reduction to the combinatorial identity (5.9).

2. ***Double sums:*** Whenever the argument of the hypergeometric representation is of the form $a+b\sqrt{5}$ or $a+bi$, with rational a, b (see Section 9), we can use a binomial expansion of $(a+b\sqrt{5})^k$, resp. $(a+bi)^k$ to obtain a double sum for F_n. For example, (3.13) easily leads to

$$F_{2n+1} = \sum_{k=0}^{2n} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{2n+1+k}{2k+1}\binom{k}{2j}(-1)^{n+k+j}2^{k-2j}, \tag{10.3}$$

with an analogous identity for F_{2n}. Such formulas have occurred in the problem sections of *The Fibonacci Quarterly*; see, e.g., [9]. Mostly they involve the product of two binomial coefficients, as in (10.3). However, a double sum expansion for F_n^2 with a single binomial coefficient can be found in [35].

*3. **Contiguous hypergeometric functions:*** An important set of relations between hypergeometric functions, totally neglected so far in this article, are the eighteen possible relations between contiguous functions; see, e.g., [1], p. 558, or [19], p 103. As an illustration of their use, we take the following two:

$$(c-a-1)F(a,b;c;z)+aF(a+1,b;c;z)-(c-1)F(a,b;c-1;z)=0, \tag{10.4}$$

$$(c-a-b)F(a,b;c;z)-(c-a)F(a-1,b;c;z)+b(1-z)F(a,b+1;c;z)=0. \tag{10.5}$$

With $a=-n,\ b=-1/2-n,\ c=1/2$, and $z=1/5$, we obtain, from (10.4),

$$F\left(-n,-\frac{1}{2}-n;\frac{1}{2};\frac{1}{5}\right)=(2n+1)F\left(-n,-\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right)-2nF\left(1-n,-\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right), \tag{10.6}$$

and (10.5) with $a=1-n, c=3/2$, and b, z as before gives

$$F\left(1-n,-\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right)=\frac{1}{2}F\left(-n,-\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right)+\frac{2}{5}F\left(1-n,\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right). \tag{10.7}$$

Combining (10.6) and (10.7), we obtain

$$F\left(-n,-\frac{1}{2}-n;\frac{1}{2};\frac{1}{5}\right)=(n-1)F\left(-n,-\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right)-\frac{4}{5}nF\left(1-n,\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right), \tag{10.8}$$

and this, by way of (4.10) and (4.11), is just the recurrence $F_{2n+1}=F_{2n+2}-F_n$. In general, relations such as (10.4) and (10.5) are often useful in obtaining recurrence relations.

*4. **Generalized hypergeometric functions:*** In direct analogy to (2.1), the generalized hypergeometric functions are defined by

$${}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p; \\ \beta_1,\dots,\beta_q;\end{matrix}\,z\right]=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{z^k}{k!}, \tag{10.9}$$

with the rising factorials as defined in (2.2). For convergence and other properties, see, e.g., [19], pp. 182ff., [40], pp. 73ff., or [5]. Many relations and transformations are known; most of them are analogous to those satisfied by the Gaussian hypergeometric functions. Among those relations that connect ${}_2F_1$ functions with generalized hypergeometric functions, we quote only the following identity due to T. Clausen:

$$\left[F\left(a,b;a+b+\frac{1}{2};z\right)\right]^2={}_3F_2\left[\begin{matrix}2a,a+b,2b; \\ a+b+\frac{1}{2},2a+2b;\end{matrix}\,z\right]. \tag{10.10}$$

Note that this is incorrectly stated in [19], p. 185, and in [37], p. 63; it is correct in [27], p. 253. While this identity in itself is interesting and very important (for instance, it helped provide the final step in the proof of the famous Bieberbach conjecture, see, e.g., [4]), we give only a brief application to Fibonacci numbers.

First, if we take $a=-n,\ b=n+1$, and $z=5/4$, then (4.4) and (10.10) give

$$F_{2n+1}^2=(2n+1)^2\,{}_3F_2\left[\begin{matrix}-2n,1,2n+2; \\ \frac{3}{2},2;\end{matrix}\,\frac{5}{4}\right]. \tag{10.11}$$

Similarly, with $a = 1/2+n$, $b = 1/2-n$, and $z = -1/4$, we get with (4.20),

$$F_{2n}^2 = \frac{4}{5}n^2 \, {}_3F_2\left[\begin{matrix}1+2n, 1, 1-2n; \\ \frac{3}{2}, 2;\end{matrix} \; -\tfrac{1}{4}\right]. \tag{10.12}$$

Both are easily rewritten, by way of (10.9), as

$$F_{2n+1}^2 = (2n+1)\sum_{k=0}^{2n}\binom{2n+k+1}{2k+1}\frac{(-5)^k}{k+1}, \tag{10.13}$$

$$F_{2n}^2 = \frac{2n}{5}\sum_{k=0}^{2n-1}\binom{2n+k}{2k+1}\frac{1}{k+1}. \tag{10.14}$$

Formula (10.13) is a special case of an identity in [46], p. 190. It is not so surprising that these formulas resemble those in Section 5; the identities $5F_{2n+1}^2 = L_{4n+2}+2$ and $5F_{2n}^2 = L_{4n}-2$ (see, e.g., [31], p. 59), show a close connection to Lucas numbers which can be treated very much like the Fibonacci numbers in Sections 4 and 5.

5. ***Other generalizations:*** These include a double hypergeometric function, used in [2] in connection with Lucas numbers and a p-adic version which can be found in [49]. Although there exist several definitions of q-analogs of Fibonacci and Lucas numbers, one of the most important extensions of hypergeometric functions, namely, *basic hypergeometric functions* (see, e.g., [22] or [47]), have not been encountered in connection with Fibonacci and Lucas numbers.

6. ***Other second-order recurrences:*** The most important one of these is the Lucas sequence, already used in Section 7. A companion identity to (3.2) is

$$F\left(a, \frac{1}{2}+a; \frac{1}{2}; z^2\right) = \frac{1}{2}[(1+z)^{-2a}+(1-z)^{-2a}]; \tag{10.15}$$

see, e.g., [1], (15.1.9). If we set $a = -n/2$, $z = \sqrt{5}$, and compare (10.15) with (7.2), we obtain

$$L_n = 2^{1-n}F\left(-\frac{n}{2}, \frac{1}{2}-\frac{n}{2}; \frac{1}{2}; 5\right), \tag{10.16}$$

a direct analog to (3.3). Using the transformations and other hypergeometric formulas mentioned in this paper, a large number of identities for the L_n, as well as identities connecting Lucas and Fibonacci numbers, can be obtained. The only identity of the type (10.16) which the author was able to find in the literature (see [16], p. 427) is

$$L_{2n} = 2F\left(-2n, 2n; \frac{1}{2}; \frac{2+\sqrt{5}}{4}\right). \tag{10.17}$$

More generally, a second-order linear recurrence with constant coefficients has a Binet-type representation and can thus be rewritten in terms of hypergeometric functions, via (3.2) or (10.15) or a combination of both. Finally, the same is true for many polynomial sequences, such as the Fibonacci and Lucas polynomials which are, in any case, closely related to the Chebyshev polynomials of both kinds.

7. ***Fibonacci function:*** Many authors have extended the Fibonacci sequence to arbitrary real or complex subscripts or, in other words, defined a Fibonacci function. A discussion of earlier results can be found in [10]; later papers include [3] and [33]. The most natural way to define such a function is by

$$F(\alpha)=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{\alpha}-\left(\frac{1-\sqrt{5}}{2}\right)^{\alpha}\right], \tag{10.18}$$

and similarly for a Lucas function and various generalizations. Some care has to be taken in the use of (3.2) because, in general, all the hypergeometric series will now be infinite and convergence is more of an issue than before. Numerous identities and series representations for the $F(\alpha)$ and related functions can be obtained.

8. ***Computer algebra:*** Most modern computer algebra systems are capable of manipulating and evaluating hypergeometric functions, sometimes in closed form. The author used "Maple" to check the hypergeometric identities in this paper for misprints (and did find and correct a few). Also, numerical experimentation is easy; new (and not so new) identities are easily discovered and can then be proved by standard methods. For example, the identity

$$(-1)^n\frac{2\sqrt{5}}{3^{\frac{n}{2}+1}}F\left(\frac{2+n}{4},\frac{4+n}{4};\frac{2+n}{2};\frac{4}{9}\right)=L_n-F_n\sqrt{5} \tag{10.19}$$

was discovered as a result of a misprint. It can be proved, for example, by using the fact that the left-hand side of (10.19) has to satisfy the same recurrence as the Fibonacci and Lucas numbers. Using properties of the binomial coefficients will probably be easier than the use of contiguous relations such as (10.4) and (10.5).

In this connection, it should be mentioned that S. Rabinowitz has developed algorithms for manipulating Fibonacci identities as well as identities for other, more general sequences. These algorithms have been implemented and are available as "Mathematica" programs (see [39]).

Finally, the powerful algorithms of Gosper, of Wilf and Zeilberger, and other related ones must be mentioned here. For general as well as more detailed discussions, see [27] and especially [38].

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∞-GENERALIZED FIBONACCI SEQUENCES AND MARKOV CHAINS

Mehdi Mouline and Mustapha Rachidi
Département de Mathématiques et Informatique, Faculté des Sciences,
Université Mohammed V, B.P. 1014, Rabat, Morocco
(Submitted October 1998-Final Revision August 1999)

1. INTRODUCTION

Let $\{a_j\}_{j=0}^{r-1}$ $(r \geq 2, a_{r-1} \neq 0)$ be a sequence of real numbers. An r-generalized Fibonacci sequence $\{V_n\}_{n=0}^{+\infty}$ is defined by the following linear recurrence relation of order r:

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_{r-1} V_{n-r+1} \quad \text{for } n \geq r-1,$$

where $V_0, \ldots, V_{r-1}$ are specified by the initial conditions. Such sequences are largely studied in the literature (see, e.g., [2], [3], [6], and [7]). Let $\{a_j\}_{j \geq 0}$ be a sequence of real numbers and consider the sequence $\{V_n\}_{n \in \mathbb{Z}}$ defined by the following linear recurrence relation of order ∞:

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_m V_{n-m} + \cdots, \text{ for } n \geq 0, \tag{1}$$

where $\{V_{-j}\}_{j \geq 0}$ are specified by the initial conditions. Such sequences, called *∞-generalized Fibonacci sequences*, were introduced and studied in [8]. We shall refer to them in the sequel as *sequences (1)*.

The aim of this paper, motivated by [8] and [10], is to study the connection between sequences (1) and Markov chains when the coefficients $\{a_j\}_{j \geq 0}$ are nonnegative. Such a connection is a generalization of those considered in [9] for r-generalized Fibonacci sequences. As in [8], we consider some hypotheses on $\{a_j\}_{j \geq 0}$ and $\{V_{-j}\}_{j \geq 0}$ in order to ensure the existence of the general term V_n for any $n \geq 1$, and then we extend results of [3] and [9] to the case of sequences (1). More precisely, using some Markov chain properties (see [1], [4], and [5]), we give a necessary and sufficient condition on the convergence of the ratio $\frac{V_n}{q^n}$, where $q > 0$ is a specified real number. This result extends the sufficient conditions of [8], under the hypotheses considered on the two sequences $\{a_j\}_{j \geq 0}$ and $\{V_{-j}\}_{j \geq 0}$. We also give the expression $\lim_{n \to +\infty} \frac{V_n}{q^n}$.

This paper is organized as follows. In Section 2 we study the case of sequences (1) in connection with Markov chains, when the coefficients $\{a_j\}_{j \geq 0}$ are nonnegative with $\sum_{j \geq 0} a_j = 1$. We also give a necessary and sufficient condition for the convergence of V_n and the expression of $\lim_{n \to +\infty} V_n$. In Section 3 we extend the results of Section 2 to the case of arbitrary nonnegative coefficients.

2. SEQUENCES (1) AND MARKOV CHAINS

2.1 Fundamental Hypotheses and Existence of the General Term

Let $\{V_n\}_{n \in \mathbb{Z}}$ be a sequence (1). Its general term V_n does not exist in general for any $n \geq 1$. For example, suppose that $\{a_j\}_{j \geq 0}$ and $\{V_{-j}\}_{j \geq 0}$ are defined by

$$a_0 = 1,\ a_j = j^j \text{ for } j \geq 1 \text{ and } V_0 = 1,\ V_{-j} = j^{-(j+2)} \text{ for } j \geq 1.$$

Then, by a direct computation, we obtain

$$V_1 = 1 + \sum_{j \geq 1} \frac{1}{j^2} \quad \text{and} \quad V_2 = V_1 + V_0 + \sum_{m \geq 2} \frac{m^m}{(m-1)^{m+1}} = +\infty.$$

Thus, to ensure the existence of V_n for any $n \geq 0$, we need some hypotheses on the two sequences $\{a_j\}_{j \geq 0}$ and $\{V_{-j}\}_{j \geq 0}$. More precisely, suppose that the following hypotheses are satisfied:

(H.1) For any $m \geq 0$, there exists $k \geq m$ such that $a_k > 0$.

(H.2) There exists $C > 0$ such that $a_k \leq C$.

(H.3) The series $\sum_{m \geq 0} |V_{-m}|$ is convergent.

The two hypotheses (H.2) and (H.3) are trivially satisfied in the case of r-generalized Fibonacci sequences. These three hypotheses are more convenient with a Markov chain formulation of sequences (1). They are not necessary for the existence of the general term V_n. Other conditions are considered in [8].

2.2 Sequences (1) Whose Coefficients Are Nonnegative with Sum 1

Suppose that the coefficients $\{a_j\}_{j \geq 0}$ of the sequence (1) satisfy hypothesis (H.1) and the following condition:

$$\sum_{j \geq 0} a_j = 1. \tag{2}$$

It is obvious that identity (2) implies (H.2) is trivially satisfied. Consider the following matrix:

$$P = \begin{pmatrix} a_0 & a_1 & \cdots & a_n & \cdots & & \\ 1 & 0 & 0 & \cdots & 0 & \cdots & \\ 0 & 1 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & & & & \\ 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots \\ \vdots & & & & & & \end{pmatrix}. \tag{3}$$

Condition (2) shows that the matrix P defined by (3) is a stochastic matrix. Then P is a transition matrix of a Markov chain $(\mathcal{T})$, whose state space is $\mathbf{N} = \{0, 1, \ldots\}$. Set $P = (P(n, m))_{n, m \in \mathbf{N}}$, then $P(0, m) = a_m$ and $P(n, m) = \delta_{n-1, m}$ for $n \geq 1$, where $\delta_{k,s}$ is the Kronecker symbol. Set $P^k = P \cdots P$ (k times), then $P^k = (P^{(k)}(n, m))_{n, m \in \mathbf{N}}$ for any $k \geq 1$, where $P^{(k)}(n, m)$ is the probability to go from the state n to the state m after k transitions. Since $P(n, n-1) = 1$, we derive

$$P^{(n-m)}(n, m) = 1 \text{ for any } m < n. \tag{4}$$

Then we have the following proposition.

Proposition 2.1: Let $\{a_j\}_{j \geq 0}$ be a sequence of nonnegative real numbers such that hypothesis (H.1) and condition (2) are satisfied. Let $(\mathcal{T})$ be the Markov chain associated to the matrix P defined by (3). Then:

(i) The chain $(\mathcal{T})$ is irreducible.

(ii) The chain $(\mathcal{T})$ is recurrent positive if $\sum_{m \geq 0}(m+1)a_m < +\infty$ and it is recurrent null if $\sum_{m \geq 0}(m+1)a_m = +\infty$.

Proof: (i) Let n and m be two states of $(\mathcal{T})$. Suppose, for example, that $m < n$. Hypothesis (H.1) and relation (4) imply that there exists $n_0 > n$ such that $a_{n_0} > 0$ and thus

$$P^{(m+n_0+1-n)}(n,m) \geq P^{(m)}(m,0)P(0,n_0)P^{(n_0-n)}(n_0,n)$$

which implies that $P^{(m+n_0+1-n)}(n,m) \geq a_{n_0} > 0$. Hence, the Markov chain $(\mathcal{T})$ is irreducible.

(ii) To study the nature of $(\mathcal{T})$, it is sufficient to study the nature of the state 0. Starting from 0, the Markov process associated to $(\mathcal{T})$ will go at the first transition to a state m with probability a_m. And it will be back to 0 with probability 1 after m transitions. Therefore, a_m is the probability of going from 0 and coming back to this state after $m+1$ transitions. The probability of coming back to 0 is $\sum_{m=0}^{+\infty} a_m = 1$. Therefore, $(\mathcal{T})$ is recurrent. Let T_0 be the real random variable which defines the first instant of return of the process to 0. We have established that $a_m = \Pr\{T_0 = m+1\}$; thus, the mean value of T_0 is $E(T_0) = \sum_{m=0}^{+\infty}(m+1)a_m$. Then $(\mathcal{T})$ is recurrent positive if $\sum_{m\geq 0}(m+1)a_m < +\infty$ and it is recurrent null if $\sum_{m\geq 0}(m+1)a_m = +\infty$. □

Remark 2.1: Let R be the radius of convergence of the series $\sum_{m\geq 0} a_m X^m$. Hypothesis (H.2) implies that $R \geq 1$. R is also the radius of convergence of the series $\sum_{m\geq 0} ma_m X^m$. Hence, if $R > 1$, we have $\sum_{m\geq 0} ma_m < +\infty$. Then $(\mathcal{T})$ is recurrent positive.

Recall that the period $d(m)$ of a given state m of $(\mathcal{T})$ is defined by

$$d(m) = \text{CGD}\{k \in \mathbf{N};\ P^{(k)}(n,m) > 0\}.$$

It is well known that, for an irreducible Markov chain $(\mathcal{T})$, we have $d(m) = d(0) = d$ for any m in $(\mathcal{T})$ (see, e.g., [4]). We recall here a very well-known theorem on the asymptotic behavior of a Markov chain.

Theorem 2.2: (See, e.g., [4].) Let $P = (P(n,m))_{n,m\in\mathbf{N}}$ be the transition matrix of an irreducible Markov chain $(\mathcal{T})$. Then:

(i) The sequence of matrices $\{P^k\}_{k\geq 0}$ converges if and only if the Markov chain $(\mathcal{T})$ is aperiodic or identically $d = 1$.

(ii) If $(\mathcal{T})$ is recurrent null, then $\lim_{k\to+\infty} P^{(k)}(n,m) = 0$ for any states n and m in $(\mathcal{T})$.

(iii) If $(\mathcal{T})$ is recurrent positive, then $\lim_{k\to+\infty} P^{(k)}(n,m)$ does not depend on n and we have $\lim_{k\to+\infty} P^{(k)}(n,m) = \Pi(m)$, where $\Pi(m) > 0$ for any m. And the stationary distribution vector $\Pi = (\Pi(0), \Pi(1), \ldots, \Pi(m), \ldots)$ is the solution of the following matrix equation

$$\Pi = \Pi \cdot P, \tag{5}$$

where $\sum_{m=0}^{+\infty} \Pi(m) = 1$.

Let $\{V_n\}_{n\in\mathbf{Z}}$ be a sequence (1) and consider the infinite column vector $X_n = (V_n, V_{n-1}, \ldots, V_{n-k}, \ldots)^t$, where R^t means the transpose of R. We can show easily that expression (1) may be written as follows:

$$X_{n+1} = PX_n \quad \text{or} \quad X_{n+1} = P^{n+1}X_0 \tag{6}$$

for any $n \geq 0$, where $X_0 = (V_0, V_{-1}, \ldots, V_{-k}, \ldots)^t$ is the infinite vector of the initial conditions. With the use of (6), Proposition 2.1, and Theorem 2.2, we can extend the necessary and sufficient condition of convergence established in [3] and [9] for r-generalized Fibonacci sequences to the case of sequences (1) as follows.

Theorem 2.3: Let $\{a_j\}_{j\geq 0}$ and $\{V_{-j}\}_{j\geq 0}$ be two sequences of real numbers such that hypotheses (H.1) and (H.3) and condition (2) are satisfied. Then the associated sequence (1) converges if and only if the following condition $(\mathscr{C})$: $\text{CGD}\{j+1; a_j > 0\} = 1$ is satisfied, where CGD means the common great divisor.

Proof: From (6), we derive that the sequence (1) converges for any choice of the initial conditions $\{V_{-j}\}_{j\geq 0}$ if and only if the sequence of matrices $\{P^k\}_{k\geq 0}$ converges. Then, let us study the aperiodicity of the Markov chain $(\mathscr{T})$ associated with the matrix P. We search the period of the state 0. Starting from state 0, the process will go at the first transition to a state m with probability $a_m > 0$. And it comes back to 0 after m transitions with probability 1. Hence, the process returns to 0 after $m+1$ transitions. Starting from the state 0 after crossing k_1 times the state m_1, k_2 times the state $m_2, \ldots$. Then the process has made $k_1(m_1+1)+k_2(m_2+1)+\cdots$ transitions. Thus, $P^{(n)}(0,0) > 0$ if and only if n is of the following form: $n = k_1(m_1+1)+k_2(m_2+1)+\cdots$, where k_1, $k_2, \ldots$ are in N. Hence, we have $\{n; P^{(n)}(0,0)>0\} = \{n; n = k_1(m_1+1)+k_2(m_2+1)+\cdots;\ k_1,\ k_2, \ldots \in \mathbb{N}\}$, which implies that $\text{CGD}\{n;\ P^{(n)}(0,0)>0\} = \text{CGD}\{m+1;\ a_m > 0\}$. Then $(\mathscr{T})$ is aperiodic if and only if condition $(\mathscr{C})$ is satisfied. Thus, the sequence (1) converges if and only if condition $(\mathscr{C})$ is satisfied. □

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.4: Under the hypotheses of Theorem 2.3 and if $a_0 > 0$, then the sequence (1) converges.

Now we shall find the expression of the limit of the sequence (1) when condition $(\mathscr{C})$ is satisfied.

Lemma 2.5: Let $\{a_j\}_{j\geq 0}$ be a sequence of real numbers such that hypothesis (H.1) and the two conditions (2) and $(\mathscr{C})$ are verified. Let $P = (P(n,m))_{n,m\geq 0}$ be the stochastic matrix (3). Then we have:

(i) $\lim_{n\to+\infty} P^{(k)}(n,m) = 0$ if $\sum_{j=0}^{+\infty}(j+1)a_j = +\infty$.

(ii) $\lim_{n\to+\infty} P^{(k)}(n,m) = \Pi(m)$ if $\sum_{j=0}^{+\infty}(j+1)a_j < +\infty$, where

$$\Pi(m) = \frac{\sum_{l=m}^{+\infty} a_l}{\sum_{l=0}^{+\infty}(l+1)a_l}. \tag{7}$$

Proof: Proposition 2.1 shows that $(\mathscr{T})$ is irreducible. And condition $(\mathscr{C})$ implies that $(\mathscr{T})$ is aperiodic and $\lim_{n\to+\infty} P^{(k)}(n,m)$ exists. Proposition 2.1, Theorem 2.2, and condition $(\mathscr{C})$ allow us to see that (i) $(\mathscr{T})$ is recurrent null with $\lim_{n\to+\infty} P^{(k)}(n,m) = 0$ if $\sum_{m=0}^{+\infty}(m+1)a_m = +\infty$, and (ii) $(\mathscr{T})$ is recurrent positive with $\lim_{n\to+\infty} P^{(k)}(n,m) = \Pi(m)$ if $\sum_{m=0}^{+\infty}(j+1)a_j < +\infty$, where $\Pi(m)$ is the $(m+1)^{\text{th}}$ component of the stationary distribution vector $\Pi = (\Pi(0), \Pi(1), \ldots, \Pi(k), \ldots)$ which satisfies

$$\Pi = \Pi \cdot P \quad \text{and} \quad \sum_{m=0}^{+\infty} \Pi(m) = 1. \tag{8}$$

The first equation of (8) is equivalent to an infinite system of equations whose unknown variables are $\Pi(m)$. By taking into consideration the second equation of (8), we derive

$$\Pi(m) = \frac{\sum_{l=m}^{+\infty} a_l}{\sum_{l=0}^{+\infty}(l+1)a_l}. \quad \square$$

Theorem 2.6: Under the hypotheses of Theorem 2.3 and if condition ($\mathscr{C}$) is satisfied, we have:

(i) $\lim_{n\to+\infty} V_n = 0$ if $\sum_{j=0}^{+\infty}(j+1)a_j = +\infty$.

(ii) $\lim_{n\to+\infty} V_n = \sum_{j=0}^{+\infty} \Pi(m)V_{-m}$ if $\sum_{j=0}^{+\infty}(j+1)a_j < +\infty$, where the $\Pi(m)$ are given by (7).

Proof: Expression (6) shows that $V_k = \sum_{m=0}^{+\infty} P^{(k)}(0,m)V_{-m}$. The inequality $|P^{(k)}(0,m)V_{-m}| \le |V_{-m}|$ and hypothesis (H.3) imply that

$$\lim_{k\to+\infty} V_k = \sum_{m=0}^{+\infty}(\lim_{k\to+\infty} P^{(k)}(0,m))V_{-m}.$$

Hence, using Lemma 2.5, we derive the result. $\square$

Theorem 2.6 is a generalization of Theorem 2.4 of [9] to the case of sequences (1) under hypotheses (H.1), (H.2), and (H.3).

2.3 The Case of CGD$\{m+1; a_m > 0\} \ge 2$

Let $\{a_j\}_{j\ge 0}$ be a sequence of nonnegative real numbers which satisfies hypothesis (H.1) and condition (2). Suppose that $\text{CGD}\{m+1; a_m > 0\} \ge 2$. Let P be the stochastic matrix (3), and consider the Markov chain ($\mathscr{T}$) associated with P. Then we have the following proposition.

Proposition 2.7: (See, e.g., [5].) Let ($\mathscr{T}$) be an irreducible recurrent positive Markov chain. Let d be the period of the states of ($\mathscr{T}$). Suppose that $d \ge 2$. Then the state space E of ($\mathscr{T}$) may be written as follows: $E = D_1 \cup D_2 \cup \cdots \cup D_d$, where $D_i \cap D_j = \emptyset$ for $i \ne j$, such that if the process is in the class D_i at the instant n, then it can go to the class D_{i+1} after one transition, with probability 1 (for $i = d$, it goes from D_d to D_1). Each class D_i ($1 \le i \le d$) is called a cyclic class. For any k, l with $k \le l \le d$ and i, j in E, the following limit, $\lim_{n\to+\infty} P^{(nd+l)}(i,j)$, exists and for any $i \in D_k$ we have

$$\lim_{n\to+\infty} P^{(nd+l)}(i,j) = \begin{cases} d\Pi(j), & \text{if } j \in D_{k+l} \pmod{d}, \\ 0, & \text{if not,} \end{cases}$$

where $\Pi(j)$ is the $(j+1)^{\text{th}}$ component of the stationary distribution vector of P.

In our case, we have $P(i+1,i) = 1$; hence, the cyclic classes are given by $D_j = \{nd+j; n \ge 0\}$, $j = 0, 1, \ldots, d$. We derive the following result from Proposition 2.7.

Theorem 2.8: Under the hypotheses of Theorem 2.3, suppose that the Markov chain ($\mathscr{T}$) associated with P is irreducible recurrent positive. Suppose that $\text{CGD}\{m+1; a_m > 0\} \ge 2$. Then $\{V_n\}_{n\in\mathbf{Z}}$, the sequence (1) has d subsequences defined by $\{V_{nd+l}\}_{n\in\mathbf{Z}}$, where $l = 0, 1, \ldots, d-1$, and each of these subsequences, which is also a sequence (1), is convergent. More precisely, for any arbitrary initial conditions and a fixed l ($0 \le l \le d-1$), we have

$$\lim_{n\to+\infty} V_{nd+l} = d\sum_{k=0}^{+\infty} \Pi(kd+l)V_{-(kd+l)},$$

where the $\Pi(m)$ are given by expression (7).

Proof: We have $V_{nd+l} = \sum_{m=0}^{+\infty} P^{(nd+l)}(0,m)V_{-m}$. Hypothesis (H.3) implies that

$$\lim_{n\to+\infty} V_{nd+l} = \sum_{m=0}^{+\infty}(\lim_{n\to+\infty} P^{(nd+l)}(0,m))V_{-m},$$

and the result is derived from Proposition 2.7. □

Theorem 2.8 is an extension of Theorem 4.2 of [9] to sequences (1), whose nonnegative coefficients satisfy condition (2), under hypotheses (H.1) and (H.3).

3. SEQUENCES (1) WHOSE COEFFICIENTS ARE NONNEGATIVE

In this section we consider that the nonnegative coefficients $\{a_j\}_{j\geq 0}$ are of arbitrary finite sum.

Let $\{V_n\}_{n\in\mathbf{Z}}$ be a sequence (1) such that hypotheses (H.1), (H.2), and (H.3) are satisfied. Let R be the radius of convergence of the power series

$$f(x) = \sum_{m=0}^{+\infty} a_m x^{m+1}. \tag{9}$$

Hypothesis (H.2) implies that $R \geq 1$.

Consider the following limit $L = \lim_{x\to R^-} f(x)$. The study of sequences (1) depends on the following three cases: $L<1$, $L=1$, and $L>1$.

Study of the Case $L < 1$. In this case, we have $\sum_{m=0}^{+\infty} a_m < 1$ because $R\geq 1$ and the function f is not decreasing. Then we have the following result.

Proposition 3.1: Let $\{V_n\}_{n\in\mathbf{Z}}$ be a sequence (1) such that hypotheses (H.1), (H.2), and (H.3) are satisfied. Then, if $\sum_{m=0}^{+\infty} a_m < 1$, we have $\lim_{n\to+\infty} V_n = 0$ for any choice of the initial conditions.

Proof: Let $S_N = \sum_{n=1}^{N}|V_n|$. We have $S_N = \sum_{n=1}^{N}|\sum_{m=0}^{+\infty} a_m V_{n-m-1}| \leq \sum_{n=1}^{N}\sum_{m=0}^{+\infty} a_m |V_{n-m-1}|$, which implies that $S_N \leq \sum_{m=0}^{+\infty} a_m(\sum_{k=0}^{+\infty}|V_{-k}| + S_N)$. Thus, $S_N \leq (\sum_{m=0}^{+\infty} a_m)(\sum_{k=0}^{+\infty}|V_{-k}|)(1-\sum_{m=0}^{+\infty} a_m)^{-1}$. And from hypothesis (H.3), we derive $\lim_{n\to+\infty} V_n = 0$. □

Study of the Case $L = 1$. In this case, we have the following two subcases: $\sum_{m=0}^{+\infty} a_m = 1$ if $R=1$ and $\sum_{m=0}^{+\infty} a_m < 1$ if $R>1$. The first one is studied in Section 2 and the second one is nothing but the preceding case.

Study of the Case $L > 1$. In this case, the analytic power series of f defined by (9) is a continuous and not decreasing function on $]0,R[$ that satisfies $f(0)=0$. Then there exists $x_0 \in\,]0,R[$ such that $f(x_0)=1$. Set $q = 1/x_0$ and $b_m = a_m/q^{m+1}$ for any $m\in\mathbf{N}$. Then we have

$$b_m \geq 0,\ \sum_{m=0}^{+\infty} b_m = 1,\ \text{and}\ \mathrm{CGD}\{m+1;\, a_m>0\} = \mathrm{CGD}\{m+1;\, b_m>0\}. \tag{10}$$

Now consider the sequence $\{W_n\}_{n\in\mathbf{Z}}$ defined by $W_n = \frac{V_n}{q^n}$. From relation (1), we derive

$$W_{n+1} = \sum_{m=0}^{+\infty} b_m V_{n-m} \tag{11}$$

for any $n \geq 0$. Thus, $\{W_n\}_{n\in Z}$ is also a sequence (1) that satisfies the two hypotheses (H.1) and (H.2) and condition (2). Hypothesis (H.3) is not satisfied in general by the initial conditions $\{W_{-m}\}_{m\geq 0}$. From Theorems 2.3 and 2.6 and expressions (10) and (11), we can formulate the extension of Theorems 5 and 9 of [3] and Theorems 3.1 and 3.3 of [9] as follows.

Theorem 3.2: Let $\{V_n\}_{n\in Z}$ be a sequence (1) such that hypotheses (H.1), (H.2), and (H.3) are satisfied. Let $\{W_n\}_{n\in Z}$ be the sequence defined by (11) and suppose that the initial conditions $\{W_{-m}\}_{m\geq 0}$ satisfy hypothesis (H.3). Then:

(a) $\lim_{n\to+\infty} W_n = \lim_{n\to+\infty} \frac{V_n}{q^n}$ exists if and only if condition $(\mathscr{C})$ is satisfied.

(b) If condition $(\mathscr{C})$ is satisfied, we have:

(i) $\lim_{n\to+\infty} \frac{V_n}{q^n} = 0$ if $\sum_{m=0}^{+\infty}(m+1)b_m = +\infty$;

(ii) $\lim_{n\to+\infty} \frac{V_n}{q^n} = \sum_{m=0}^{+\infty} \Pi(m)V_{-m}q^m$ if $\sum_{m=0}^{+\infty}(m+1)b_m < +\infty$, where

$$\Pi(m) = \frac{\sum_{s=m}^{+\infty} b_s}{\sum_{s=0}^{+\infty}(s+1)b_s}. \tag{12}$$

The second expression of $\lim_{n\to+\infty} \frac{V_n}{q^n}$ given in Theorem 3.2 is identical to the expression of Theorem 3.10 in [8].

For $q \leq 1$ or $\sum_{s=0}^{+\infty} \leq 1$, we have $|W_m| = |V_m q^m| \leq |V_m|$. Thus, hypothesis (H.3) is satisfied by $\{W_{-m}\}_{m\geq 0}$. But for $q > 1$ or $\sum_{s=0}^{+\infty} a_s > 1$, such hypothesis is not satisfied in general by $\{W_{-m}\}_{m\geq 0}$.

Case $d = \mathrm{CGD}\{m+1;\ a_m > 0\} \geq 2$. In this case, we derive from expression (10) that $\mathrm{CGD}\{m+1;\ b_m > 0\} \geq 2$. Thus, we can extend Theorem 4.2 of [9] as follows.

Theorem 3.3: Under the hypotheses of Theorem 3.2, suppose that $d = \mathrm{CGD}\{m+1;\ a_m > 0\} \geq 2$. Then $\{W_n\}_{n\in Z}$ has d subsequences defined by $\{W_{nd+l}\}_{n\in Z}$, $l = 0, 1, \ldots, d-1$ that are also sequences (1). And each subsequence $\{W_{nd+l}\}_{n\in Z}$ converges for any choice of those initial conditions with

$$\lim_{n\to+\infty} W_{nd+l} = \lim_{n\to+\infty} \frac{V_{nd+l}}{q^{nd+l}} = d\sum_{k=0}^{+\infty} \Pi(kd+l)W_{-(kd+l)},$$

where the $\Pi(kd+l)$ are given by expression (12).

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2001. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

Dedication: The problems in this issue are dedicated to Dr. Stanley Rabinowitz in recognition of his nine years of devoted service as Editor of the Elementary Problems Section.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

<u>**B-900**</u> ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***

Show that $\tan(2n\arctan(\alpha))$ is a rational number for every $n \geq 0$.

<u>**B-901**</u> ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***

Let A_n be the sequence defined by $A_0 = 1$, $A_1 = 0$, $A_n = (n-1)(A_{n-1} + A_{n-2})$ for $n \geq 2$. Find

$$\lim_{n\to+\infty} \frac{A_n}{n!}.$$

<u>**B-902**</u> ***Proposed by H.-J. Seiffert, Berlin, Germany***

The Pell polynomials are defined by $P_0(x) = 0$, $P_1(x) = 1$, and $P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$ for $n \geq 2$. Show that, for all nonzero real numbers x and all positive integers n,

$$\sum_{k=1}^{n} \binom{n}{k} (1-x)^{n-k} P_k(x) = x^{n-1} P_n(1/x).$$

<u>**B-903**</u> ***Proposed by the editor***

Find a closed form for $\sum_{n=0}^{\infty} F_n^2 x^n$.

B-904 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***

Find the positive integers n and m such that $F_n = L_m$.

B-905 ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain***

Let n be a positive integer greater than or equal to 2. Determine

$$\frac{(F_n^2+1)F_{n+1}F_{n+2}}{(F_{n+1}-F_n)(F_{n+2}-F_n)}+\frac{F_n(F_{n+1}^2+1)F_{n+2}}{(F_n-F_{n+1})(F_{n+2}-F_{n+1})}+\frac{F_nF_{n+1}(F_{n+2}^2+1)}{(F_n-F_{n+2})(F_{n+1}-F_{n+2})}.$$

SOLUTIONS

A Constant Summation

B-884 ***Proposed by M. N. Deshpande, Aurangabad, India (Vol. 37, no. 4, November 1999)***

Find an integer k such that the expression $F_n^2F_{n+2}^2+kF_{n+1}^2F_{n+2}^2+F_{n+1}^2F_{n+3}^2$ is a constant independent of n.

Composite solution by L. A. G. Dresel, Reading, England, and Maitland A. Rose, University of South Carolina (independently)

Denoting the given expression by Q_n, we have

$$Q_{n+1}-Q_n = k(F_{n+2})^2\{(F_{n+3})^2-(F_{n+1})^2\}+(F_{n+2})^2\{(F_{n+4})^2-(F_n)^2\}.$$

Now

$$(F_{n+3})^2-(F_{n+1})^2=(F_{n+3}-F_{n+1})(F_{n+3}+F_{n+1})=F_{n+2}L_{n+2}$$

and

$$(F_{n+4})^2-(F_n)^2=(F_{n+4}-F_n)(F_{n+4}+F_n)=(F_{n+3}+F_{n+1})(3F_{n+2})=3F_{n+2}L_{n+2},$$

so that $Q_{n+1}-Q_n=(k+3)(F_{n+2})^3L_{n+2}$. Since we require $Q_{n+1}-Q_n=0$ for all n, we must have $k=-3$, giving the identity

$$(F_nF_{n+2})^2-3(F_{n+1}F_{n+2})^2+(F_{n+1}F_{n+3})^2=1 \text{ for all } n.$$

The proposer noted that a similar result holds for the Lucas numbers. The constant k would still be -3 but the value of the analogous expression would be 25.

Also solved by Gerald Heuer, H.-J. Seiffert, James A. Sellers, Indulis Strazdins, and the proposer.

A Unit Summation

B-885 ***Proposed by A. J. Stam, Winsum, The Netherlands (Vol. 37, no. 4, November 1999)***

For $n>0$, evaluate

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{k}{2n-k}\binom{2n-k}{n}F_{k+1}.$$

Solution 1 by Kuo-Jye Chen, National Changhua University of Education, Taiwan

We rewrite the sum as

$$\sum_{k=0}^{n}(-1)^k\frac{n-k}{n+k}\binom{n+k}{k}F_{n-k+1}:=A_n,$$

and claim that $A_n = 1$ for $n > 0$.

It is readily seen that

$$A_1 = 1. \tag{1}$$

Next, we show that

$$A_n - A_{n-1} = 0 \text{ for } n > 1. \tag{2}$$

Write

$$A_n - A_{n-1} = d_0 + d_1 + d_2 + \cdots + d_{n-1}, \tag{3}$$

where

$$d_k := (-1)^k \left\{ \frac{n-k}{n+k} \binom{n+k}{k} F_{n-k+1} - \frac{n-k-1}{n+k-1} \binom{n+k-1}{k} F_{n-k} \right\}.$$

We compute the following partial sums of (3):

$$d_0 = F_{n-1},$$
$$d_0 + d_1 = -\frac{n-1}{n+1}\binom{n+1}{1} F_{n-2},$$
$$d_0 + d_1 + d_2 = \frac{n-2}{n+2}\binom{n+2}{2} F_{n-3},$$
$$d_0 + d_1 + d_2 + d_3 = -\frac{n-3}{n+3}\binom{n+3}{3} F_{n-4},$$

and, in general,

$$d_0 + d_1 + d_2 + \cdots + d_k = (-1)^k \frac{n-k}{n+k}\binom{n+k}{k} F_{n-k-1}, \text{ for } 0 \le k \le n-1. \tag{4}$$

In particular, when $k = n-1$, formula (4) reduces to $d_0 + d_1 + d_2 + \cdots + d_{n-1} = 0$, which completes the proof of (2).

Combining (1) and (2), we obtain, for $n > 0$,

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{k}{2n-k}\binom{2n-k}{n} F_{k+1} = \sum_{k=0}^{n} (-1)^k \frac{n-k}{n+k}\binom{n+k}{k} F_{n-k+1} = 1.$$

Solution 2 by H.-J. Seiffert, Berlin, Germany

Define the Fibonacci polynomials and the Lucas polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_{n+1}(x) = xF_n(x) + F_{n-1}(x),\ n \ge 1,$$

and

$$L_0(x) = 2,\ L_1(x) = x,\ L_{n+1}(x) = xL_n(x) + L_{n-1}(x),\ n \ge 1,$$

respectively. Differentiating the known identity (see [1])

$$\sum_{k=1}^{n} (-1)^{n-k}\binom{2n-1-k}{n-1} x^k L_k(x) = x^{2n}$$

with respect to x, using the fact that $L_k'(x) = kF_k(x)$ and $L_k(x) + xF_k(x) = 2F_{k+1}(x)$, and multiplying by $x/2$ gives

$$\sum_{k=1}^{n} (-1)^{n-k}\binom{2n-1-k}{n-1} kx^k F_{k+1}(x) = nx^{2n}.$$

Hence, by $\binom{2n-1-k}{n-1} = \frac{n}{2n-k}\binom{2n-k}{n}$,

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{k}{2n-k}\binom{2n-k}{n}x^k F_{k+1}(x) = x^{2n}.$$

Now, take $x=1$ to see that the value of the sum in question is 1.

Reference

1. R. André-Jeannin & Paul S. Bruckman. "Problem H-479." *The Fibonacci Quarterly* **32.5** (1994):477-78.

Also solved by Indulis Strazdins and the proposer. One incomplete solution was received.

Some Sum

B-887 ***Proposed by A. J. Stam, Winsum, The Netherlands (Vol. 37, no. 4, November 1999)***

Show that

$$\sum_{k=0}^{n}\binom{y-n-1-k}{n-k}F_{2k+1} = \sum_{k=0}^{n}\binom{y-n-2-k}{n-k}F_{2k+2} = \sum_{j=0}^{n}\binom{y-j}{j}.$$

Solution by H.-J. Seiffert, Berlin, Germany

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, and $F_{n+1}(x)=xF_n(x)+F_{n-1}(x)$ for $n\in\mathbb{Z}$. It is known (see [1], identity (60)) that

$$F_{n+1}(x) = \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-j}{j}x^{n-2j},\ n\in\mathbb{N}_0, \tag{1}$$

and (see [1], identity (39))

$$x^m F_r(x) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}F_{m+r-2j}(x),\ m\in\mathbb{N}_0,\ r\in\mathbb{Z}. \tag{2}$$

For the nonnegative integer n, consider the polynomials

$$P(z) = \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{z}{j}F_{n-2j+1}(x) \quad\text{and}\quad Q(z) = \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-z-j}{j}x^{n-2j}.$$

If $z=m$ is an integer such that $0\le m\le\lfloor n/2\rfloor$, then by (1) and (2), $P(m)=x^mF_{n-m+1}(x)=Q(m)$. Since $P(z)$ and $Q(z)$ are both polynomials in z of degree not greater than $\lfloor n/2\rfloor$, we then must have $P(z)=Q(z)$ for all complex numbers z. This proves the identity

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{z}{j}F_{n-2j+1}(x) = \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-z-j}{j}x^{n-2j}, \tag{3}$$

valid for all $n\in\mathbb{N}_0$ and all complex numbers x and z.

To deduce the desired identities, we will use the well-known relation

$$(-1)^j\binom{-a-1}{j} = \binom{a+j}{j},\ j\in\mathbb{N}_0,\ a\in\mathbb{C}.$$

Now, using (3) with n replaced by $2n$ and $z=2n-y$ and then reindexing on the left-hand side $j=n-k$ gives

$$\sum_{k=0}^{n}\binom{y-n-1-k}{n-k}F_{2k+1}(x)=\sum_{j=0}^{n}\binom{y-j}{j}x^{2n-2j}.$$

Similarly, using (3) with n replaced by $2n+1$ and then taking $z=2n+1-y$ yields

$$\sum_{k=0}^{n}\binom{y-n-2-k}{n-k}F_{2k+2}(x)=\sum_{j=0}^{n}\binom{y-j}{j}x^{2n-2j+1}.$$

Finally, take $x=1$.

Reference

1. S. Rabinowitz. "Algorithmic Manipulation of Second-Order Linear Recurrences." *The Fibonacci Quarterly* **37.2** (1999):162-77.

Also solved by the proposer.

Determine the Determinant

B-888 ***Proposed by A. Arya, J. Fellingham, and D. Schroeder, Ohio State University, OH, and J. Glover, Carnegie Mellon University, PA***
(Vol. 37, no. 4, November 1999)

For $n\geq 1$, let $A_n=[a_{i,j}]$ denote the symmetric matrix with $a_{i,i}=i+1$ and $a_{i,j}=\min[i,j]$ for all integers i and j with $i\neq j$.

(a) Find the determinant of A_n.

(b) Find the inverse of A_n.

Composite solution to part (a) by L. A. G. Dresel, C. Libis, I. Strazdins, and the proposers.

Let D_n denote the determinant of A_n. Then we have $D_1=2$ and $D_2=5$, and for $n>2$,

$$D_n=\begin{vmatrix}2&\cdot&\cdot&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&n&n-1\\ 1&\cdot&\cdot&n-1&n+1\end{vmatrix}=\begin{vmatrix}2&\cdot&\cdot&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&n&n-1\\ 0&0&0&-1&2\end{vmatrix}=\begin{vmatrix}2&\cdot&\cdot&1&0\\ \cdot&\cdot&\cdot&\cdot&0\\ \cdot&\cdot&\cdot&\cdot&0\\ 1&\cdot&\cdot&n&-1\\ 0&0&0&-1&3\end{vmatrix},$$

since the determinant is unchanged if we first subtract the penultimate row from the last row, and then subtract the penultimate column from the last column. Expanding the resulting determinant by its last row, we obtain $D_n=3D_{n-1}-D_{n-2}$. But we have $D_1=2=F_3$ and $D_2=5=F_5$ so that, if we assume that $D_n=F_{2n+1}$ for $n\leq N$, we obtain $D_{N+1}=3D_N-D_{N-1}=3F_{2N+1}-F_{2N-1}=F_{2N+3}$. Hence, by induction, we have $D_n=F_{2n+1}$ for all $n\geq 1$.

No detailed solution to part (b) was received. The proposers stated that the inverse of A_n is $[b_{ij}]$, where

$$b_{ii}=\left(\frac{1}{F_{2n+1}}\right)\left(F_{2(n-i)+1}F_{2i-1}+F_{2(n-i)}F_{2i}\right)\ \textit{and}\ b_{ij}=-\left(\frac{1}{F_{2n+1}}\right)\left(F_{2(n-\max\{i,j\})+1}\right)\left(F_{2\min\{i,j\}}\right),\ i\neq j.$$

However, showing that $[a_{ij}][b_{ij}]=I_n$ involves tedious algebra.

Addenda: We wish to belatedly acknowledge solutions from the following solvers:

Charlie Cook—Problems B-873, B-875, and B-877; Maitland A. Rose—Problem B-878.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-564 ***Proposed by Stanley Rabinowitz, Westford, MA***

Let k be a positive integer and let $a_0 = 1$. Find integers $a_1, a_2, \dots, a_k$ and $b_0, b_1, b_2, \dots, b_k$ such that

$$\sum_{i=0}^{k} a_i L_{n+i}^{2k} = \sum_{i=0}^{k} b_i F_{n+i}^{2k}$$

is true for all integers n. Prove that your answer is unique.

For example, when $k = 4$, we have the identity

$$L_n^8 + 21L_{n+1}^8 + 56L_{n+2}^8 + 21L_{n+3}^8 + L_{n+4}^8 = 625(F_n^8 + 21F_{n+1}^8 + 56F_{n+2}^8 + 21F_{n+3}^8 + F_{n+4}^8).$$

H-565 ***Proposed by Paul S. Bruckman, Berkeley, CA***

Let p be a prime with $p \equiv -1 \pmod{2m}$, where $m \ge 3$ is an odd integer. Prove that all residues are m^{th} powers (mod p).

H-566 ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada***

Let $\phi_n := \pi / 2n$, where n is a positive integer, and set $L_n = a^n + b^n$, $F_n = (a^n - b^n)/(a-b)$, where $a = \frac{1}{2}(u + \sqrt{u^2 - 4})$, $b = \frac{1}{2}(u - \sqrt{u^2 - 4})$, $u \ne \pm 2$, and show that, for $n \ge 2$,

$$\begin{aligned} S_n(u) &:= \sum_{k=1}^{n-1} \frac{1}{1 + \left(\frac{u+2}{u-2}\right) tg^2(k\phi_n)} \\ &= -\frac{1}{2} + \frac{n}{2(u+2)^2 F_n}[L_{n+1} + 3L_n + 3L_{n-1} + L_{n-2}]. \end{aligned}$$

SOLUTIONS

A Very Odd Problem

H-550 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 37, no. 2, May 1999)

Suppose n is an odd integer, p an odd prime $\ne 5$. Prove that $L_n \equiv 1 \pmod{p}$ if and only if (i) $\alpha^n \equiv \alpha$, $\beta^n \equiv \beta \pmod{p}$, or (ii) $\alpha^n \equiv \beta$, $\beta^n \equiv \alpha \pmod{p}$.

Solution by H.-J. Seiffert, Berlin, Germany

Recall that the integers of $Q(\sqrt{5})$ have the form $(a+\sqrt{5}b)/2$, where $a, b \in \mathbf{Z}$ such that $a \equiv b \pmod 2$.

Since n is odd, we have $L_n^2 - 5F_n^2 = -4$. Clearly, $L_n \equiv F_n \pmod 2$.

Suppose that $L_n \equiv 1 \pmod p$. Then $F_n^2 \equiv 1 \pmod p$, because p is a prime $\neq 5$. Hence, since p is an odd prime, either $F_n \equiv 1 \pmod p$ or $F_n \equiv -1 \pmod p$. If $F_n \equiv 1 \pmod p$, then

$$\alpha^n = \frac{L_n + \sqrt{5}F_n}{2} \equiv \frac{1+\sqrt{5}}{2} = \alpha \pmod p \quad \text{and} \quad \beta^n = \frac{L_n - \sqrt{5}F_n}{2} \equiv \frac{1-\sqrt{5}}{2} = \beta \pmod p.$$

Similarly, if $F_n \equiv -1 \pmod p$, then

$$\alpha^n = \frac{L_n + \sqrt{5}F_n}{2} \equiv \frac{1-\sqrt{5}}{2} = \beta \pmod p \quad \text{and} \quad \beta^n = \frac{L_n - \sqrt{5}F_n}{2} \equiv \frac{1+\sqrt{5}}{2} = \alpha \pmod p.$$

Conversely, suppose that either (i) or (ii) holds, then in each case $L_n = \alpha^n + \beta^n \equiv \alpha + \beta = 1 \pmod p$.

Also solved by L. A. G. Dresel and the proposer.

Some Restriction!

H-551 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 37, no. 2, May 1999)***

Let k be a nonnegative integer and define the following restricted double-sum,

$$S_k := \sum_{\substack{r=0 \\ br+as<ab}}^{a-1} \sum_{s=0}^{b-1} (br+as)^k,$$

where a and b are relatively prime positive integers.

a. Show that $S_{k-1} = \dfrac{1}{kb}\left[\sum_{r=0}^{b-1}((ab+r)^k - a^k r^k) - \sum_{m=2}^{k}\binom{k}{m} b^m S_{k-m}\right]$ for $k \geq 1$.

The convention that $\binom{k}{m} = o$ if $m > k$ is adopted.

b. Show that $S_2 = \dfrac{ab}{12}[3a^2b^2 + 2a^2b + 2ab^2 - a^2 - b^2 - 9ab + a + b + 2]$.

Solution by Paul S. Bruckman, Berkeley, CA

In the rs-plane, let L denote the line segment $br + as = ab$ (i.e., $r/a + s/b = 1$), with $0 \leq r \leq a$, $0 \leq s \leq b$. Also, let T denote the triangular first-quadrant region bounded by the axes and L, including points on the axes that are not on L, and excluding points on L.

Note that S_k considers only the lattice points $\{r, s\}$ that are elements of T. The only lattice points that lie on L are the points $\{a, 0\}$ and $\{0, b\}$, a consequence of the fact that $\gcd(a, b) = 1$. For brevity, we write

$$S_k = \sum_T \sum (br+as)^k,$$

where $\{r, s\}$ are lattice points of T.

We will first derive the following identity:

$$\sum_{\substack{r=0 \\ ab\le br+as<ab+b}}^{a-1}\sum_{s=0}^{b-1}(br+as)^k = \sum_{m=0}^{b-1}(ab+m)^k - \sum_{m=0}^{[(b-1)/a]}(ab+am)^k. \tag{1}$$

Proof of (1): Concentrating on the left member of (1), we see that the value of the variable $br+as$, subject to the indicated restriction, is $ab+m$, where m is an element of the set $\{0, 1, 2, \ldots, b-1\}$. Since a and b are coprime, there exist positive integers u and v such that $au-bv=1$. Assuming that $m>0$ is given and not a multiple of a, the equation $br+as=ab+m$ has solutions $\{r, s\}$ given by: $r=a-mv+at$, $s=mu-bt$, where t is an arbitrary integer. However, since we require $0\le r\le a-1$, $0\le s\le b-1$, this forces t to have a unique value, and the solutions $\{r, s\}$ are therefore unique.

On the other hand, if $m=am'$ is given, the equation $br+as=ab+am'$ implies $br\equiv 0 \pmod{a}$, hence $a|r$. This, in turn, can only occur if $r=0$, in which case $br+as<ab$, falling outside of the range of restricted values allowed. Thus, m assumes each value in $\{0, 1, 2, \ldots, b-1\}$ exactly once, with the exception of the multiples of a, which do not occur at all (it is seen at once that m, likewise, cannot be a multiple of b). These latter values of m to be excluded therefore comprise the set $\{0, a, 2a, \ldots, [(b-1)/a]\}$. Putting these facts together establishes the identity in (1). □

Next, consider the sum $\sum_{m=0}^{k} {}_kC_m b^m S_{k-m}$, which may also be written symbolically as $(b+S)^k$, it being understood that, in such a binomial expansion, "exponents" of S are translated to subscripts. We see that, provided $k\ge 2$, such sum equals

$$(b+S)^k = S_k + kbS_{k-1} + \sum_{m=2}^{k} {}_kC_m b^m S_{k-m}. \tag{2}$$

Note that this is also true for $k=1$ if we define the sum in (2) to vanish for $k=1$. Also, however,

$$\begin{aligned}(b+S)^k &= \sum_T\sum(b+br+as)^k \\ &= \sum_{\substack{r=1 \\ br+as<ab+b}}^{a}\sum_{s=0}^{b-1}(br+as)^k = \sum_{\substack{r=0 \\ br+as<ab+b}}^{a-1}\sum_{s=0}^{b-1}(br+as)^k + \sum_{\substack{s=0 \\ as<b}}^{b-1}(ab+as)^k - \sum_{\substack{s=0 \\ as<ab+b}}^{b-1}(as)^k \\ &= S_k + U_k + \sum_{s=0}^{[(b-1)/a]}(ab+as)^k - \sum_{s=0}^{b-1}(as)^k,\end{aligned}$$

where U_k is the sum indicated in the left member of (1). Then, using the result of (1), we obtain the following:

$$(b+S)^k = S_k + \sum_{m=0}^{b-1}(ab+m)^k - \sum_{m=0}^{b-1}(am)^k.$$

Now, substituting the result in (2) and simplifying yields the result of Part a. □

We simply substitute $k=1, 2,$ or 3 in the recurrence formula just derived in order to compute S_{k-1}. Incidentally, it is to be noted that although this recurrence is not symmetric in a and b, it should be evident that the expression for S_k must be symmetric in a and b. Thus, a comparable recurrence, with a and b switched, is also true. For $k=1$, we obtain

$$bS_0 = \sum_{m=0}^{b-1} \{ab - (a-1)m\} = ab^2 - (a-1)b(b-1)/2;$$

hence, $S_0 = ab - (a-1)(b-1)/2$ or

$$S_0 = (ab + a + b - 1)/2. \tag{3}$$

Next, setting $k = 2$, we have

$$\begin{aligned} 2bS_1 &= \sum_{m=0}^{b-1} \{a^2b^2 + 2abm - (a^2-1)m^2\} - b^2S_0 \\ &= a^2b^3 + ab^2(b-1) - (a^2-1)b(b-1)(2b-1)/6 - b^2(ab+a+b-1)/2; \end{aligned}$$

hence,

$$\begin{aligned} 12S_1 &= 6a^2b^2 + 6ab^2 - 6ab - 2a^2b^2 + 3a^2b - a^2 + 2b^2 - 3b + 1 - 3ab^2 - 3ab - 3b^2 + 3b \\ &= 4a^2b^2 + 3ab^2 + 3a^2b - a^2 - 9ab - b^2 + 1 \end{aligned}$$

or

$$S_1 = (4a^2b^2 + 3ab^2 + 3a^2b - a^2 - 9ab - b^2 + 1)/12. \tag{4}$$

Finally, setting $k = 3$, we obtain

$$\begin{aligned} 3bS_2 &= \sum_{m=0}^{b-1} \{a^3b^3 + 3a^2b^2m + 3abm^2 - (a^3-1)m^3\} - 3b^2S_1 - b^3S_0 \\ &= a^3b^4 + 3a^2b^3(b-1)/2 + 3ab^2(b-1)(2b-1)/6 - (a^3-1)b^2(b-1)^2/4 \\ &\quad - b^2(4a^2b^2 + 3ab^2 + 3a^2b - a^2 - 9ab - b^2 + 1)/4 - b^3(ab+a+b-1)/2, \end{aligned}$$

which, after simplification, reduces to the expression in Part b. □

Also solved by H.-J. Seiffert and the proposer.

Be Determinant

H-552 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 37, no. 2, May 1999)

Given $m \geq 2$, let $\{U_n\}_{n=0}^{\infty}$ denote a sequence of the following form:

$$U_n = \sum_{i=1}^{m} a_i(\theta_k)^n,$$

where the a_i's and θ_i's are constants, with the θ_i's distinct, and the U_n's satisfy the initial conditions $U_n = 0$, $n = 0, 1, \ldots, m-2$; $U_{m-1} = 1$.

Part A. Prove the following formula for the U_n's:

$$U_n = \sum_{S(n-m+1,\, m)} (\theta_1)^{i_1}(\theta_2)^{i_2} \cdots (\theta_m)^{i_m}, \tag{a}$$

where

$$S(N, m) = \{(i_1, i_2, \ldots, i_m) : i_1 + i_2 + \cdots + i_m = N,\ 0 \leq i_j < N,\ j = 1, 2, \ldots, m\}. \tag{b}$$

Part B. Prove the following determinant formula for the U_n's:

$$U_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_m \\ (\theta_1)^2 & (\theta_2)^2 & (\theta_3)^3 & \cdots & (\theta_m)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\theta_1)^{m-2} & (\theta_2)^{m-2} & (\theta_3)^{m-2} & \cdots & (\theta_m)^{m-2} \\ (\theta_1)^n & (\theta_2)^n & (\theta_3)^n & \cdots & (\theta_m)^n \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_m \\ (\theta_1)^2 & (\theta_2)^2 & (\theta_3)^3 & \cdots & (\theta_m)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\theta_1)^{m-2} & (\theta_2)^{m-2} & (\theta_3)^{m-2} & \cdots & (\theta_m)^{m-2} \\ (\theta_1)^{m-1} & (\theta_2)^{m-1} & (\theta_3)^{m-1} & \cdots & (\theta_m)^{m-1} \end{vmatrix}.$$

Solution by H.-J. Seiffert, Berlin, Germany

If $V(x_1, \ldots, x_m) = \det((x_k)^{j-1})_{j,k=1,\ldots,m}$ denotes the Vandermonde determinant of m distinct numbers $x_1, \ldots, x_m$, then, as is well known,

$$V(x_1, \ldots, x_m) = \prod_{i \le j < k \le m} (x_k - x_j). \tag{1}$$

From $U_n = 0$, $n = 0, \ldots, m-2$, and $U_{m-1} = 1$, we have the following system of linear equations:

$$\begin{array}{llll} a_1 & +a_2 & +\cdots+a_m & = 0, \\ a_1(\theta_1) & +a_2(\theta_2) & +\cdots+a_m(\theta_m) & = 0, \\ \vdots & \vdots & \vdots & \vdots \\ a_1(\theta_1)^{m-2} & +a_2(\theta_2)^{m-2} & +\cdots+a_m(\theta_m)^{m-2} & = 0, \\ a_1(\theta_1)^{m-1} & +a_2(\theta_2)^{m-1} & +\cdots+a_m(\theta_m)^{m-1} & = 1. \end{array}$$

Solving by Cramer's rule and expanding the nominator determinant occurring there by the i^{th} column gives

$$a_i = (-1)^{m-i} \frac{V(\theta_1, \ldots, \hat{\theta}_i, \ldots, \theta_m)}{V(\theta_1, \ldots, \theta_m)}, \ i = 1, \ldots, m, \tag{2}$$

where $\hat{\theta}_i$ indicates that θ_i is released. Hence, by (1),

$$a_i = \prod_{\substack{j=1 \\ j \ne i}}^{m} \frac{1}{\theta_i - \theta_j}, \ i = 1, \ldots, m. \tag{3}$$

For sufficiently small $|x|$, we consider the generating functions

$$U(x) = \sum_{n=0}^{\infty} U_n x^n \quad \text{and} \quad W(x) = \sum_{n=0}^{\infty} \sum_{S(n-m+1,\, m)} (\theta_1)^{i_1} (\theta_2)^{i_2} \cdots (\theta_m)^{i_m} x^n.$$

Then,

$$U(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{m} a_i (\theta_i)^n x^n = \sum_{i=1}^{m} a_i \sum_{n=0}^{\infty} (\theta_i x)^n,$$

or

$$U(x) = \sum_{i=1}^{m} \frac{a_i}{1 - \theta_i x}, \tag{4}$$

and

$$W(x) = \sum_{n=0}^{\infty} \sum_{S(n-m+1,\, m)} (\theta_1 x)^{i_1} (\theta_2 x)^{i_2} \cdots (\theta_m x)^{i_m} x^{m-1} = x^{m-1} \prod_{j=1}^{m} \left(\sum_{i=0}^{\infty} (\theta_j x)^i \right),$$

or

$$W(x)=x^{m-1}\prod_{j=1}^{m}\frac{1}{1-\theta_j x}. \tag{5}$$

From Lagrange's interpolation formula, we have

$$\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{x-\theta_j}{\theta_i-\theta_j}=1 \text{ for all } x.$$

Replacing x by $1/x$ and then multiplying by $x^{m-1}\prod_{j=1}^{m}(1-\theta_j x)^{-1}$ yields

$$\sum_{i=1}^{m}\frac{1}{1-\theta_i x}\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{1}{\theta_i-\theta_j}=x^{m-1}\prod_{j=1}^{m}\frac{1}{1-\theta_j x},$$

valid for all sufficiently small $|x|$. Now, from (3), (4), and (5), we see that $U(x)=W(x)$ for all sufficiently small $|x|$. Comparing coefficients of these generating functions gives the desired equation (a) of Part A.

Expanding the nominator determinant of the requested equation of Part B by the last row, we see that we must show:

$$U_n=\sum_{i=1}^{m}(-1)^{m-i}\frac{V(\theta_1,\ldots,\hat{\theta}_i,\ldots,\theta_m)}{V(\theta_1,\ldots,\theta_m)}(\theta_i)^n.$$

However, this holds by (2) and the definition of U_n. This completes the solution.

Also solved by the proposer.

Lotsa Terms

H-554 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 37, no. 3, August 1999)***

Let k, a, and b be positive integers with a and b relatively prime to each other, and define

$$N_k:=(1+(-1)^k-L_k)^{-1}=\begin{cases}(2-L_k)^{-1} & \text{if } k \text{ is even,}\\ -L_k^{-1} & \text{if } k \text{ is odd.}\end{cases}$$

a. Show that

$$\sum_{\substack{r=0\\ br+as<ab}}^{a-1}\sum_{s=0}^{b-1}L_q(br+as)=N_{qa}N_{qb}[2+L_{q(a+b)}-L_{qa}-L_{qb}-L_{qab}+(-1)^{qa}L_{qa(b-1)}$$
$$+(-1)^{qb}L_{qb(a-1)}+(-1)^{q(a+b)+1}L_{q(ab-a-b)}]+N_q[(-1)^qL_{q(ab-1)}-L_{qab}],$$

where q is a positive integer.

b. Show that

$$\sum_{\substack{r=0\\ br+as<ab}}^{a-1}\sum_{s=0}^{b-1}F_q(br+as)=N_{qa}N_{qb}[(-1)^{q(a+b)+1}F_{q(ab-a-b)}+F_{qa}+F_{qb}-F_{qab}+(-1)^{qa}F_{qa(b-1)}$$
$$+(-1)^{qb}F_{qb(a-1)}-F_{q(a+b)}]+N_q[(-1)^qF_{q(ab-1)}-F_{qab}],$$

where q is a positive integer.

Solution by Paul S. Bruckman, Berkeley, CA

In the rs-plane, let L denote the line segment $br+as=ab$ (i.e., $r/a+s/b=1$), with $0\le r\le a$, $0\le s\le b$. Let S denote the set of lattice points included in the triangular first-quadrant region bounded by the axes and L, including points on the axes, but excluding points on L. Also, let T denote the triangular "mirror-image" of S about L, and let $R=S\cup T\cup(0,b)\cup(a,0)$. Thus, R is the set of all lattice points included in the rectangular region with $0\le r\le a$, $0\le s\le b$. Since $\gcd(a,b)=1$, we see that the only lattice points of $R\cap L$ are at the end-points of L, namely, at $(0,b)$ and $(a,0)$; moreover, these end-points are neither in S nor in T.

For brevity, write $N=br+as$, and $N\in S$ to mean $(r,s)\in S$, with similar notation for T and for R. Make the following definitions, valid for arbitrary $x\ne 1$:

$$S(x)=\sum_{N\in S}x^N,\quad T(x)=\sum_{N\in T}x^N,\quad R(x)=\sum_{N\in R}x^N. \tag{1}$$

We see that

$$\begin{aligned}R(x)&=(1+x^a+x^{2a}+\cdots+x^{ba})(1+x^b+x^{2b}+\cdots+x^{ab})\\&=(1-x^{(b+1)a})(1-x^{(a+1)b})/\{(1+x^a)(1-x^b)\}.\end{aligned}$$

We also see that $R(x)=S(x)+T(x)+2x^{ab}$ and, moreover, that $T(x)=\sum_{N\in S}x^{2ab-N}=x^{2ab}S(x^{-1})$. This yields the following symmetrical relation (upon division by x^{ab}):

$$x^{-ab}S(x)+x^{ab}S(x^{-1})+2=x^{-ab}R(x)=U(x)\quad\text{(say)}. \tag{2}$$

As we may verify, $U(x)=x^{-ab}(1-x^{(b+1)a})(1-x^{(a+1)b})/\{(1-x^a)(1-x^b)\}=U(x^{-1})$.

We also observe, due to the fact that $\gcd(a,b)=1$, that each value of N occurring as an exponent in the sum $S(x)$ occurs but once.

We may evaluate the sum $S(x)$ by means of certain manipulations. Thus,

$$\begin{aligned}S(x)&=\sum_{N\in S}x^N=\sum_{\substack{r=0\\br+as<ab}}^{a-1}\sum_{s=0}^{b-1}x^{br+as}=\sum_{\substack{r=1\\b(r-1)+as<ab}}^{a}\sum_{s=0}^{b-1}x^{b(r-1)+as}\\&=x^{-b}\sum_{\substack{r=0\\br+as<b(a+1)}}^{a-1}\sum_{s=0}^{b-1}x^{br+as}-x^{-b}\sum_{s=0}^{b-1}x^{as}+x^{-b}\sum_{s=0}^{[(b-1)/a]}x^{ba+as};\end{aligned}$$

assuming that $a\ge b$, then

$$\begin{aligned}x^bS(x)&=\sum_{\substack{r=0\\br+as<ab}}^{a-1}\sum_{s=0}^{b-1}x^{br+as}+\sum_{\substack{r=0\\ab<br+as<ab+b}}^{a-1}\sum_{s=0}^{b-1}x^{br+as}-(x^{ab}-1)/(x^a-1)+x^{ab}\\&=S(x)+\sum_{k=ab+1}^{ab+b-1}x^k-(x^{ab}-1)/(x^a-1)+x^{ab}\\&=S(x)+(x^{ab+b}-x^{ab+1})/(x-1)-(x^{ab}-1)/(x^a-1)+x^{ab}.\end{aligned}$$

After simplification, we obtain the symmetric expression (valid also if $a\le b$):

$$S(x)=(1-x^{ab})/\{(1-x^a)(1-x^b)\}-x^{ab}/(1-x). \tag{3}$$

Now the sums given in the problem are seen to equal the following expressions:

$$\sum_{N \in S} L_{qN} = S(\alpha^q) + S(\beta^q); \quad \sum_{N \in S} F_{qN} = 5^{-1/2}\{S(\alpha^q) - S(\beta^q)\}. \tag{4}$$

Therefore, it remains to show that the expressions in (4) may be simplified to the expressions given in the statement of the problem.

Note that $(\alpha^{qk} - 1)(\beta^{qk} - 1) = (-1)^{qk} + 1 - L_{qk} = 1/N_{qk}$. Then

$$\begin{aligned} S(\alpha^q) &= (1-\alpha^{qab})/\{(1-\alpha^{qa})(1-\alpha^{qb})\} - \alpha^{qab}/(1-\alpha^q) \\ &= N_{qa}N_{qb}(1-\alpha^{qab})(1-\beta^{qa})(1-\beta^{qb}) - N_q\alpha^{qab}(1-\beta^q) \\ &= N_{qa}N_{qb}(1-\alpha^{qab})(1-\beta^{qa}-\beta^{qb}+\beta^{q(a+b)}) - N_q\alpha^{qab}(1-\beta^q) \end{aligned}$$

or

$$\begin{aligned} S(\alpha^q) = N_{qa}N_{qb}\{&1-\beta^{qa}-\beta^{qb}+\beta^{q(a+b)}-\alpha^{qab}+(-1)^{qa}\alpha^{qab-qa} \\ &+(-1)^{qb}\alpha^{qab-qb}-(-1)^{qa+qb}\alpha^{qab-qa-qb}\} - N_q(\alpha^{qab}-(-1)^q\alpha^{qab-q}). \end{aligned} \tag{5}$$

Likewise,

$$\begin{aligned} S(\beta^q) = N_{qa}N_{qb}\{&1-\alpha^{qa}-\alpha^{qb}+\alpha^{q(a+b)}-\beta^{qab}+(-1)^{qa}\beta^{qab-qa} \\ &+(-1)^{qb}\beta^{qab-qb}-(-1)^{qa+qb}\beta^{qab-qa-qb}\} - N_q(\beta^{qab}-(-1)^q\beta^{qab-q}). \end{aligned} \tag{6}$$

Now, adding the expressions in (5) and (6) (and using (4)), we obtain

$$\begin{aligned} \sum_{N \in S} L_{qN} = N_{qa}N_{qb}\{&2-L_{qa}-L_{qb}+L_{q(a+b)}-L_{qab}+(-1)^{qa}L_{qa(b-1)} \\ &+(-1)^{qb}L_{qb(a-1)}-(-1)^{q(a+b)}L_{q(ab-a-b)}\} - N_q\{L_{qab}-(-1)^qL_{q(ab-1)}\}, \end{aligned}$$

which is seen to be equivalent to the result of Part a.

Subtracting the two expressions in (5) and (6) and dividing by $5^{1/2}$ yields

$$\begin{aligned} \sum_{N \in S} F_{qN} = N_{qa}N_{qb}\{&F_{qa}+F_{qb}-F_{q(a+b)}-F_{qab}+(-1)^{qa}F_{qa(b-1)} \\ &+(-1)^{qb}F_{qb(a-1)}-(-1)^{q(a+b)}F_{q(ab-a-b)}\} - N_q\{F_{qab}-(-1)^qF_{q(ab-1)}\}, \end{aligned}$$

which is seen to be equivalent to the result of Part b, except for a small error.

We note that the statement of the problem in the August 1999 issue of this journal contains a typographical error: the term in Part b that reads "$+F_{qb(a-1)}$" should read "$+(-1)^{qb}F_{qb(a-1)}$". The problem is stated correctly above. □

Also solved by H.-J. Seiffert and the proposer.

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