The Fibonacci Quarterly

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The Fibonacci Quarterly

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*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

r-GENERALIZED FIBONACCI SEQUENCES AND THE LINEAR MOMENT PROBLEM

Bouazza El Wahbi
Département de Mathématiques, Faculté des Sciences
Université Abdelmalek Essadi, B.P. 2121, Tétouan-Morocco

Mustapha Rachidi
Département de Mathématiques et Informatique, Faculté des Sciences
Université Mohammed V, B.P. 1014, Rabat-Morocco

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1. INTRODUCTION

Let $a_0, a_1, \ldots, a_{r-1}$ $(r \geq 2)$ be real numbers with $a_{r-1} \neq 0$. An r-generalized Fibonacci sequence $\{V_n\}_{n\geq 0}$ is defined by the initial conditions $(V_0, V_1, \ldots, V_{r-1})$ and the following linear recurrence relation of order r,

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_{r-1} V_{n-r+1} \text{ for } n \geq r-1. \tag{1}$$

In the sequel we shall refer to such sequences as *sequences* (1). When $a_j = 1$ for all j $(0 \leq j \leq r-1)$ and $V_0 = \cdots = V_{r-2} = 0$, $V_{r-1} = 1$, sequence (1) defines the well-known r-generalized Fibonacci numbers introduced by Miles in [9], which have been studied extensively in the literature. Let $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ be the characteristic polynomial of sequence (1) and set $\sigma_P = \{\lambda \in \mathbf{C};\ P(\lambda) = 0\}$.

Let $(E, (.,.))$ be a unitary real vector space of finite dimension m, and consider $\Lambda(E)$, the space of linear self-adjoint operators on E. An operator $S \in \Lambda(E)$ is called *simple* if its spectrum $\sigma(S) = \{\mu_1, \mu_2, \ldots, \mu_m\}$ is such that $\mu_i \neq \mu_j$ for $i \neq j$. Set $\Lambda_s(E) = \{S \in \Lambda(E);\ S \text{ is simple}\}$ and $\Lambda_s^P(E) = \{S \in \Lambda_s(E);\ \sigma(S) \cap \sigma_P \neq \emptyset\}$. For any $S \in \Lambda(E)$, the sequence $\{V_n\}_{0\leq n\leq p}$ defined by $V_n = (S^n x, x)$ for $n = 0, 1, \ldots, p \leq \infty$, where $x \neq 0$ is a vector of E, is called a *sequence of moments* of S on the vector x. The *linear moment problem* associated to a sequence $\{V_n\}_{0\leq n\leq p}$ consists of finding $S \in \Lambda_s(E)$ such that

$$V_n = (S^n x, x) \text{ for } n = 0, 1, \ldots, p \leq \infty, \tag{2}$$

where $x \neq 0$ is a vector of E (see [5] and [6]). In expression (2), the vector x is considered as fixed because it is not unique.

The aim of this paper is to study the linear moment problem for a given sequence (1). More precisely, we give a necessary and sufficient condition for the sequence (1) to be a sequence of moments of $S \in \Lambda_s(E)$. Applications and examples are given. In particular, we can characterize sequences (1) which are linear combinations of geometric sequences. We also consider an application to the study of a linear system of Vandermonde type.

2. LINEAR MOMENT PROBLEM FOR SEQUENCES (1)

2.1 Moments of an Operator and Sequences (1)

Let $(E, (.,.))$ be a unitary real vector space of dimension m and $S \in \Lambda_s(E)$ such that $\sigma(S) = \{\lambda_1, \dots, \lambda_m\}$. We have $E = \oplus_{j=1}^m E_j$, where E_j is the eigenspace $E_j = \{x \in E;\ Sx = \lambda_j x\}$. Let $\{e_1, e_2, \dots, e_m\}$ be an orthogonal basis of E, where $e_j \in E_j$. Set $\{V_n\}_{n\geq 0}$, the sequence of moments of S on a fixed (nonvanishing) vector $x = \sum_{j=1}^m \mu_j e_j$ of E. Let $P_c(X) = \prod_{j=1}^m (X - \lambda_j)$ be the characteristic polynomial of S, and consider a polynomial $Q(X) = X^r - a_0 X^{r-1} - \dots - a_{r-1}$ such that $P_c(X)$ is a divisor of $Q(X)$; we derive from the Cayley-Hamilton theorem that $Q(S) = 0$, and then $S^{n+1} = a_0 S^n + \dots + a_{r-1} S^{n-r+1}$ for any $n \geq r-1$. Thus, the sequence of moments $V_n = (S^n x, x)$ $(n \geq 0)$ of S on x is a sequence (1). If $r \leq m-1$, we suppose that $\sigma(S) \cap \sigma_Q = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and set $S_1 = S_{|L}$, where $L = \oplus_{j=1}^k E_j$. It is clear that $S_1 \in \Lambda_s(L)$ and $Q(S_1) = 0$. Then, for any $x \in L$ (with $x \neq 0$), the sequence of moments $V_n = (S_1^n x, x)$ $(n \geq 0)$ is again a sequence (1).

In the sequel, we study the converse question. More precisely, we study the linear moment problem (2) for a given sequence (1).

2.2 Reduction of the Linear Moment Problem for Sequences (1)

Let $P(X) = X^r - a_0 X^{r-1} - \dots - a_{r-1}$ be the characteristic polynomial of sequence (1) and let $\sigma_P = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ be the set of characteristic roots of the sequence (1).

Lemma 2.1: Let $\{V_n\}_{n\geq 0}$ be a sequence (1). Suppose that there exists $S \in \Lambda_s(E)$ such that $V_n = (S^n x, x)$ for all $n \geq 0$, where $x \neq 0$ is a (fixed) vector of E. Then $\sigma(S) \cap \sigma_P = \{\lambda_1, \dots, \lambda_l\} \neq \emptyset$ and $x = \sum_{j=1}^l x_j$, where $x_j \in E_j$.

Proof: Let $S \in \Lambda_s(E)$ and, for any $n \geq r-1$, set $R_n = S^{n+1} - a_0 S^n - \dots - a_{r-1} S^{n-r+1}$. Then we have $R_n x = \sum_{j=1}^m \mu_j \lambda_j^{n-r+1} P(\lambda_j) e_j$ for any $x = \sum_{j=1}^m \mu_j e_j \in E$. Using equation (2), we obtain the following system of m linear equations in the unknown variables $\mu_1^2, \mu_2^2, \dots, \mu_m^2$,

$$\sum_{j=1}^m \lambda_j^k P(\lambda_j) \mu_j^2 = 0;\ k = 0, 1, \dots, m-1,$$

by taking $n = r-1, r, \dots, r+m-2$. The determinant of this system of Vandermonde type is $\Delta = P(\lambda_1) \dots P(\lambda_m) \prod_{1\leq i<j\leq m} (\lambda_j - \lambda_i)$. The operator S is simple, so $\lambda_j \neq \lambda_i$ for $i \neq j$, and because $x \neq 0$ we must have $\Delta = 0$, which implies that $\sigma(S) \cap \sigma_P = \{\lambda_1, \dots, \lambda_l\} \neq \emptyset$ and $x = \sum_{j=1}^l \mu_j e_j$. □

If $P(X)$ does not have a real root, Lemma 2.1 shows that the sequence (1) is not a sequence of moments of an operator $S \in \Lambda_s^P(E)$. Let $S \in \Lambda_s(E)$; if $\sigma(S) \cap \sigma_P = \emptyset$, then the sequence (1) cannot be a sequence of moments of the operator S. A partial converse of Lemma 2.1 is given by Lemma 2.2.

Lemma 2.2: Let $\{V_n\}_{n\geq 0}$ be a sequence (1). Suppose that there exists $S \in \Lambda_s^P(E)$ with $\sigma(S) \cap \sigma_P = \{\lambda_1, \dots, \lambda_l\}$. Then there exists a vector $x \neq 0$ in E such that $(S^{n+1}x, x) = \sum_{j=0}^{r-1} a_j (S^{n-j} x, x)$ for all $n \geq r-1$. More precisely, we have $x = \sum_{j=1}^l x_j$, where $x_j \in E_j$.

Proof: Let $S \in \Lambda_s(E)$ and set $\sigma(S) = \{\lambda_1, \ldots, \lambda_m\}$. Then we have the orthogonal decomposition $E = \oplus_{j=1}^m E_j$, where $E_j = \{x \in E;\ Sx = \lambda_j x\}$. Suppose that $\sigma(S) \cap \sigma_P = \{\lambda_1, \ldots, \lambda_s\}$. For any λ_j $(0 \le j \le s)$ and $x_j \in E_j$, we have $S^k x_j = \lambda_j^k x_j$ $(k \ge 0)$ and $\lambda_j^{k+1} = a_0\lambda_j^k + \cdots + a_{r-2}\lambda_j^{k-r+2} + a_{r-1}\lambda_j^{k-r+1}$ $(k \ge r-1)$. Thus, we have $(S^{n+1}x_j, x_j) = \sum_{j=0}^{r-1} a_j(S^{n-j}x_j, x_j)$ for all $n \ge r-1$. Because the decomposition $E = \oplus_{j=1}^m E_j$ is orthogonal, we derive $(S^{n+1}x, x) = \sum_{j=0}^{r-1} a_j(S^{n-j}x, x)$ for any $n \ge r-1$, where $x = \sum_{j=1}^{l} x_j$ $(x_j \in E_j)$. □

Example 2.1—Characterization of geometric r-generalized Fibonacci sequences which are sequences of moments: If $E = \mathbf{R}$, a simple self-adjoint operator S on E is defined by $S(x) = \lambda x$, where $\lambda = S(1)$ and $\sigma(S) = \{\lambda\}$. Let $\{V_n\}_{n\ge 0}$ be a sequence (1). If $V_n = (S^n x, x)$ for all $n \ge 0$, we derive that $x^2 = V_0$ and $V_n = (V_1/V_0)^n V_0$, $n = 1, 2, \ldots, r-1$. For $n \ge r$, expression (1) allows us to have $V_1^r = \sum_{j=0}^{r-1} a_j V_0^{j+1} V_1^{r-j-1}$. Then $\{V_n\}_{n\ge 0}$ is a sequence of moments of $S \in \Lambda_s(\mathbf{R})$ on $x \ne 0$ if and only if $x^2 = V_0$, $V_n = (V_1/V_0)^n V_0$, $n = 1, 2, \ldots, r-1$, and $Q(V_0, V_1)=0$, where $Q(X, Y) = Y^r - \sum_{j=0}^{r-1} a_j X^{j+1} Y^{r-j-1}$. Geometrically, sequence (1) is a sequence of moments of $S \in \Lambda_s(\mathbf{R})$ on $x \ne 0$ if and only if $x^2 = V_0$, $V_n = (V_1/V_0)^n V_0$, $n = 1, 2, \ldots, r-1$, and (V_0, V_1) is a point of the algebraic curve of equation $Q(X, Y) = 0$. □

A subspace L of E is called *invariant* under $S \in \Lambda(E)$ (or *S-invariant*) if $Sx \in L$ for all $x \in L$, and we denote by $S_{|L}$ the restriction of S to L.

Lemma 2.3: Let $S \in \Lambda_s(E)$ and let L be a nontrivial S-invariant subspace of E. Then $P(S_{|L}) = 0$ if and only if $L \subseteq \oplus_{\lambda \in \sigma(S) \cap \sigma_P} E_\lambda$.

The proof of this lemma my be deduced using the fact that any operator $S \in \Lambda_s(E)$ defines a basis of eigenvectors of E and its restriction to any nontrivial S-invariant subspace L is an operator of $\Lambda_s(L)$.

From Lemmas 2.1, 2.2, and 2.3, we can derive the following proposition.

Proposition 2.1: Let $\{V_n\}_{n\ge 0}$ be a sequence (1). Suppose that there exists $S \in \Lambda_s(E)$ such that $V_n = (S^n x, x)$ for all $n \ge 0$, where $x \ne 0$ is a vector of an S-invariant subspace L of E. Then we have $V_n = (S_{|L}^n x, x)$ for any $n \ge 0$.

With the aid of Lemma 2.2 and Proposition 2.1, the linear moment problem for a sequence (1) may be reduced as follows: Find $S \in \Lambda_s(E)$ such that $\sigma(S) \cap \sigma_P \ne \emptyset$ and $V_n = (S^n x, x)$ for $n = 0, 1, \ldots, r-1$, with $x \ne 0$ in $L = \oplus_{\lambda \in \sigma(S) \cap \sigma_P} E_\lambda$, where $E_\lambda = \{x \in E;\ Sx = \lambda x\}$. Thus, from Lemmas 2.1-2.2 and Proposition 2.1, we derive the following result.

Theorem 2.1: Let $\{V_n\}_{n\ge 0}$ be a sequence (1). Then $\{V_n\}_{n\ge 0}$ is a sequence of moments of $S \in \Lambda_s(E)$ on a vector $x \ne 0$ of E if and only if $S \in \Lambda_s^P(E)$ and S is a solution of the reduced moment problem

$$V_n = (S^n x, x) \text{ for } n = 0, 1, \ldots, r-1, \tag{3}$$

where $x \in L = \oplus_{\lambda \in \sigma(S) \cap \sigma_P} E_\lambda$.

Suppose the reduced linear moment problem (3) has a solution $S \in \Lambda_s(E)$ with $x \ne 0$ in L, an S-invariant subspace of E. If $P(S_{|L}) = 0$, then S is a solution of the linear moment problem (2).

Proposition 2.2: Let $\{V_n\}_{n\geq 0}$ be a sequence (1). Suppose that $S \in \Lambda_s(E)$ is a solution of the linear moment problem (3) with $x \neq 0$ in L, an S-invariant subspace of E, such that $P(S_{|L}) = 0$. Then any extension $S_1 \in \Lambda_s(E)$ of $S_{|L}$ to E satisfies $V_n = (S_1^n x, x)$ for all $n \geq 0$.

Example 2.2: Let $\{V_n\}_{n\geq 0}$ be a sequence (1) with $V_0 > 0$ and $S \in \Lambda_s(E)$. Suppose $\sigma(S) \cap \sigma_P = \{\lambda\}$; it is obvious that $\lambda \neq 0$ because $a_{r-1} \neq 0$. Let e_1 be a basis of E_λ with $(e_1, e_1) = 1$. Then $\{V_n\}_{n\geq 0}$ is a sequence of moments of S on $x = \sqrt{V_0}$ (or $-\sqrt{V_0}$) if and only if $V_k = (S^k x, x) = \lambda^k V_0$ for any $k = 0, 1, \ldots, r-1$. This example is an extension of Example 2.1. Thus, we have the same geometrical interpretation. □

Example 2.3: Let $(E, (.,.))$ be a unitary real vector space of dimension m and let $\{V_n\}_{n\geq 0}$ be a sequence (1) with $r \geq 2$. Suppose that the reduced linear moment problem, $V_k = (S^k x, x)$ for $k = 0, 1, \ldots, r-1$, where $x \neq 0$, has a solution $S \in \Lambda_s(E)$. Let $\sigma(S) \cap \sigma_P = \{\lambda_1, \lambda_2\}$, with $\lambda_1 < \lambda_2$. Let e_j be a basis of E_j with $(e_j, e_j) = 1$ $(j = 1, 2)$. It is obvious that $(e_1, e_2) = 0$. Then we have $V_k = (S^k x, x) = \lambda_1^k a^2 + \lambda_2^k b^2$ for any $k \geq 0$, where $x = ae_1 + be_2$. If $r = 2$, we have

$$a^2 = \frac{\lambda_2 V_0 - V_1}{\lambda_2 - \lambda_1} > 0 \quad \text{and} \quad b^2 = \frac{V_1 - \lambda_1 V_0}{\lambda_2 - \lambda_1} > 0.$$

If $r \geq 3$, we have

$$a^2 = \frac{1}{\lambda_2^{k-1}} \frac{\lambda_2 V_{k-1} - V_k}{\lambda_2 - \lambda_1} > 0 \quad \text{and} \quad b^2 = \frac{1}{\lambda_2^{k-1}} \frac{V_k - \lambda_1 V_{k-1}}{\lambda_2 - \lambda_1} > 0$$

for any $k = 1, \ldots, r-1$. These expressions imply that

$$V_k = \frac{1}{\lambda_2 - \lambda_1}[(\lambda_2 V_0 - V_1)\lambda_1^k + (V_1 - \lambda_1 V_0)\lambda_2^k]$$

for all $k \geq 0$. □

2.3 Sequences (1) and Associated Matrix S

For the construction of $S \in \Lambda_s(E)$ associated to a given sequence (1), it is more convenient to consider a unitary real vector space $(E, (.,.))$ of dimension $m = card\{\lambda_j \in \mathbf{R} \cap \sigma_P\} \leq r$. In this case, we set $\sigma(S) \cap \sigma_P = \{\lambda_1, \ldots, \lambda_l\} \subset \mathbf{R}$ and consider an orthogonal basis $\{e_1, \ldots, e_m\}$ of E, where $Se_j = \lambda_j e_j$ for $j = 1, \ldots, l$. Then S may be identified with the diagonal matrix $D = diag(\lambda_1, \ldots, \lambda_l)$. If $m \geq r+1$, Theorem 2.1 and Proposition 2.2 allow us to see that we can consider a self-adjoint extension S_1 of S and $x \neq 0$, $x \in L = \oplus_{\lambda \in \sigma(S_1) \cap \sigma_P} E_\lambda$.

3. REDUCED LINEAR MOMENT PROBLEM OF SEQUENCES (1) AND HANKEL FORMS

3.1 Hankel Matrices and Hankel Forms

A real (or complex) matrix $M = (a_{jk})_{0 \leq j, k \leq p}$, where $0 \leq p \leq +\infty$, is called *positive semi-definite* (resp. *positive definite*) if $\sum_{0 \leq j, k \leq m} a_{jk} \eta_j \overline{\eta}_k \geq 0$ (resp. > 0) for any finite sequence $\eta = \{\eta_j\}_{0 \leq j \leq m}$, where $\overline{z}$ denotes the complex conjugate of z. Let $\gamma = \{\gamma_j\}_{j \geq 0}$ be a sequence of real or complex numbers. The family of matrices defined by $H(m) = (\gamma_{j+k})_{0 \leq j, k \leq m-1}$, where $m = 1, 2, \ldots,$

are called *Hankel matrices* associated with $\gamma = \{\gamma_j\}_{j\geq 0}$, and the family of quadratic forms defined by $\mathcal{H}_m(\eta, \eta) = \sum_{0\leq j,k\leq m-1} \gamma_{j+k}\eta_j\overline{\eta}_k$, where $\eta = \{\eta_j\}_{0\leq j\leq m-1}$, are called *Hankel forms*. An infinite Hankel matrix, $H = (\gamma_{j+k})_{j,k\geq 0}$, is called *positive semidefinite* (resp. *positive definite*) if, for any m, the Hankel form $\mathcal{H}_m$ is positive semidefinite (resp. positive definite) or, equivalently, the Hankel matrices $H(m) = (\gamma_{j+k})_{0\leq j,k\leq m-1}$ $(m = 1, 2, \ldots)$ are positive semidefinite (resp. positive definite). Hankel matrices and forms play an important role in the theory of moment problems (see, e.g., [1]-[6]).

3.2 Linear Moment Problem of Sequences (1) and Hankel Forms

Let $(E, (.,.))$ be a unitary real vector space of finite dimension m and fix an orthogonal basis $\{e_1, e_2, \ldots, e_m\}$ of E. Let $A = (V_0, \ldots, V_{r-1})$ $(r \geq 2)$ be a sequence of real numbers, and consider the real quadratic Hankel forms on E defined by $\mathcal{H}_p^A(x,x) = \sum_{0\leq j,k\leq p-1} V_{j+k}\xi_j\xi_k$ $(p \geq 1)$ for $x = \sum_{j=1}^m \xi_j e_j$. Suppose $r = 2m-1$ and that the Hankel form $\mathcal{H}_m^A$ is positive definite, and consider the scalar product on the $\mathbf{K}$-vector space $\mathbf{K}_{m-1}[X]$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$) of polynomials of degree $\leq m-1$, defined by $(P,Q) = \sum_{0\leq j,k\leq m-1} \gamma_{j+k}\zeta_j\overline{\eta}_k$, where $P = \sum_{0\leq j\leq m-1}\zeta_j X^j$ and $Q = \sum_{0\leq j\leq m-1}\eta_j X^j$. Let $S: \mathbf{K}_{m-1}[X] \to \mathbf{K}_{m-1}[X]$ be the linear operator defined by $S(X^j) = X^{j+1}$. Then S is a simple hermitian operator of defect 1 which satisfies $V_k = (S^k x, x)$ for $k = 0, 1, \ldots, r-1$, where $x = P(X) = 1$ (see [5], pp. 348-51; [6], pp. 443-48). More generally, it was shown in [5] and [6] that the linear moment problem $V_k = (S^k x, x)$ for $k = 0, 1, \ldots, r-1$ has a solution $S \in \Lambda_s(E)$ on $x \neq 0$ if and only if the Hankel form $\mathcal{H}_{[\frac{r+1}{2}]}^A$ is positive semidefinite and $rk\mathcal{H}_p^A = \min(p, m)$ for $p = 1, 2, \ldots, [\frac{r+1}{2}]$, where $rk\mathcal{H}_p^A$ is the rank of $\mathcal{H}_p^A$ and $[z]$ is the integer defined by $[z] \leq z < [z]+1$ for $z \in \mathbf{R}$. Let Ω_r be the set of $A = (V_0, \ldots, V_{r-1}) \in \mathbf{R}^r$ such that $\mathcal{H}_{[\frac{r+1}{2}]}^A$ is positive semidefinite and $rk\mathcal{H}_p^A = \min(p, m)$ for $p = 1, \ldots, [\frac{r+1}{2}]$. Then, for a sequence (1), we derive the following result from Theorem 2.1.

Theorem 3.1: Let $\{V_n\}_{n\geq 0}$ be a sequence (1) and set $A = (V_0, \ldots, V_{r-1})$. Then the following statements are equivalent:

(i) $A = (V_0, \ldots, V_{r-1}) \in \Omega_r$.

(ii) The reduced linear moment problem (3), $V_n = (S^n x, x)$ for $0 \leq n \leq r-1$, has a solution $S \in \Lambda_s^P(E)$ on a nonvanishing vector $x \in E$.

(iii) The linear moment problem (2), $V_n = (S^n x, x)$ for $n \geq 0$, has a solution $S \in \Lambda_s^P(E)$ on a nonvanishing vector $x \in E$.

In (ii) and (iii), we have $x \neq 0$ and $x \in L$, an S-invariant subspace of E.

3.3 The Case of Fibonacci and Lucas Numbers

Let $(E, (.,.))$ be a unitary real vector space of dimension 2. Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 2$. Then the associated Hankel matrix

$$H = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

is positive semidefinite and has rank 2. Thus, the reduced linear moment problem,

$$L_0 = 2 = (x, x),\ L_1 = 1 = (Sx, x),\ L_2 = 3 = (S^2x, x),$$

is solvable. Then Theorem 2.1 implies that the linear moment problem is solvable for all $L_n = (S^n x, x)$, and because

$$\sigma_P = \left\{\phi_+ = \frac{1+\sqrt{5}}{2},\ \phi_- = \frac{1-\sqrt{5}}{2}\right\},$$

we derive from the method of construction of S (see Subsection 2.3) that we can choose

$$S = \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix} \quad \text{and} \quad x = (1, 1).$$

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Then the associated Hankel matrix

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is not positive semidefinite. Therefore, even the reduced linear moment problem $F_0 = 0 = (x, x)$, $F_1 = 1 = (Sx, x)$, and $F_2 = 1 = (S^2x, x)$ is not solvable.

Because of Theorem 2.1 and Proposition 2.2, we can consider $(E, (.,.))$ as a unitary real vector space of dimension $m \geq 2$.

4. DEFINITE AND INDEFINITE LINEAR MOMENT PROBLEM FOR SEQUENCES (1)

Let $(E, (.,.))$ be a unitary real vector space of finite dimension m, and consider a sequence of real numbers $(V_n)_{0\leq n\leq p}$, where $p \leq \infty$. The linear moment problem (2) is called *definite* if it has a unique (up to conjugation by a unitary operator) solution S and *indefinite* if not. It was shown in [5] and [6] that the linear moment problem (2) is definite if and only if $p \geq 2m-1$. Let $\{V_n\}_{n\geq 0}$ be a sequence (1). Theorem 2.1 shows that if the linear moment problem (2) is solvable, it is reduced to the linear moment problem (3), $V_n = (S^n x, x)$ for $n = 0, 1, \ldots, r-1$.

Suppose that the reduced linear moment problem (3) has a solution $S \in \Lambda_s(E)$. Then the Hankel form $\mathcal{H}^A_{[\frac{r+1}{2}]}$ is positive semidefinite, and $rk\mathcal{H}^A_p = \min(p, m)$ for $p = 1, 2, \ldots, [\frac{r+1}{2}]$ (see [5], [6]), and from Theorem 2.1, we also have $S \in \Lambda^P_s(E)$. Therefore, in this case, the definite or indefinite reduced linear moment problem (3) depends on the cardinality l of the set $\sigma(S) \cap \sigma_P = \{\lambda_1, \ldots, \lambda_l\}$. Even more precisely, let $S \in \Lambda^P_s(E)$ and $\{e_1, e_2, \ldots, e_l\}$ be an orthogonal basis of $L = \oplus^l_{j=1} E_j$, where $e_j \in E_j$. Then the scalars α_j of the vector $x = \sum^l_{j=1} \alpha_j e_j$ in the reduced linear moment problem (3) satisfy the following linear system of r equations,

$$\lambda_1^j y_1 + \cdots + \lambda_l^j y_l = V_j,\ j = 0, \ldots, r-1,$$

with $y_j = \alpha_j^2$. Using Propositions 2.1 and 2.2 and Theorem 3.1, we derive the following result.

Theorem 4.1: Let $S \in \Lambda_s(E)$ and $(V_n)_{n\geq 0}$ be a sequence (1) with $r \geq 3$. Suppose $\sigma(S) \cap \sigma_P = \{\lambda_1, \ldots, \lambda_t\}$, where $l \geq 2$. Then, if the linear moment problem (3) has a solution, it is definite if $l = m \leq [\frac{r+1}{2}]$ and indefinite if $l \leq [\frac{r+1}{2}] < m$.

Example 4.1: Let $(V_n)_{n\geq 0}$ be a sequence (1) with $r \geq 3$. Suppose that the Hankel form $\mathcal{H}^A_{[\frac{r+1}{2}]}$ is positive semidefinite and $rk\mathcal{H}^A_p = \min(p, m)$ for $p = 1, 2, \ldots, [\frac{r+1}{2}]$. Then, by Theorem 3.1, the reduced moment problem (3) has a solution $S \in \Lambda_s(E)$. If $\sigma(S) \cap \sigma_P = \{\lambda_1, \lambda_2\}$ with $\lambda_1 < \lambda_2$, we set $L = E_1 \oplus E_2$. Then $P(S_{|L}) = 0$, and we have $V_n = (S^n x, x) = \lambda_1^n a^2 + \lambda_2^n b^2$ for $n = 0, 1$, where $x = ae_1 + be_2$. For $m = 2$, Theorem 4.1 shows that the operator S is unique and that the linear moment problem (2) for the sequence (1) is definite. If $3 \leq m \leq [\frac{r+1}{2}]$, Proposition 2.1 implies that, for any self-adjoint extension S_1 of $S_{|L}$ such that $\sigma(S_1) \cap \sigma_P = \{\lambda_1, \lambda_2\}$, we also have that $V_n = (S_1^n x, x)$ for $n \geq 0$. Thus, the operator S is not a unique solution of the reduced linear moment problem (2). Hence, the linear moment problem (2) for the sequence (1) is indefinite. □

5. APPLICATIONS AND CONCLUDING REMARKS

5.1 Application 1: Sequences (1) Which Are Linear Combinations of Geometric Sequences

Let $\{V_n\}_{n\geq 0}$ be a sequence (1). It is well known that $V_n = \sum_{l=1}^{k} \sum_{j=0}^{m_l-1} \beta_{l,j} \lambda_l^n$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are roots of the characteristic polynomial of the sequence (1), with multiplicities $m_1, m_2, \ldots, m_k$, respectively $(m_1 + m_2 + \cdots + m_k = r)$, and $\beta_{l,j}$ are obtained from the initial conditions $(V_0, V_1, \ldots, V_{r-1})$ (see [7] and [8]). Then $\{V_n\}_{n\geq 0}$ is a linear combination of real geometric sequences if and only if

$$\beta_{l,j} = 0;\; j = 1, \ldots, m_l - 1,\; l = 1, \ldots, k, \tag{4}$$

and $\beta_{l,0} = 0$ if λ_l is a complex root. The choice of the initial conditions $(V_0, V_1, \ldots, V_{r-1})$ such that the $\beta_{l,j}$ satisfies the system of equations (4) implies that $V_n = \sum_{l=1}^{k} \beta_{l,0} \lambda_l^n$, with $\beta_{l,0} = 0$ if λ_l is a complex root. It seems difficult to find such $(V_0, V_1, \ldots, V_{r-1})$ by a direct computation from the system of equations (4). Meanwhile, Theorems 2.1 and 3.1 allow us to answer this question, as was shown in Examples 2.1-2.3.

5.2 Application 2: Sequences (1) and Linear Systems of Vandermonde Type

Consider the linear system of r equations and m unknowns $y_1, \ldots, y_m$ of Vandermonde type

$$\lambda_1^j y_1 + \cdots + \lambda_m^j y_m = b_j,\; j = 0, 1, \ldots, r-1, \tag{5}$$

where $r \geq m$ and $\lambda_j \in \mathbf{R}$ with $\lambda_i \neq \lambda_j$ if $i \neq j$. The preceding results may be used to study this system. More precisely, we can associate to this linear system of equations a sequence (1) such that $(V_0, \ldots, V_{r-1}) = (b_0, \ldots, b_{r-1})$ and whose coefficients $a_0, \ldots, a_{r-1}$ are given by the characteristic polynomial $P(X) = (X - \lambda_1) \cdots (X - \lambda_m) Q(X)$, where $Q(X)$ is a polynomial of degree $r - m$. We now consider the linear moment problem (2) for $\{V_n\}_{n\geq 0}$ with $S \in \Lambda_s(E)$, where $(E, (.,.))$ is a unitary real vector space of dimension m such that $\sigma(S) = \{\lambda_1, \ldots, \lambda_m\}$. Hence, if $\{V_n\}_{n\geq 0}$ is a sequence of moments of an operator $S \in \Lambda_s(E)$, the linear system (5) has a solution $(y_1, \ldots, y_m)$ with $y_j \geq 0$. Conversely, suppose that the system (5) has a solution $(y_1, \ldots, y_m)$ with $y_j \geq 0$. Let $(E, (.,.))$ be a unitary real vector space of dimension m and set $S \in \Lambda_s(E)$ such that $\sigma(S) = \{\lambda_1, \ldots, \lambda_m\}$. Let $\{e_1, e_2, \ldots, e_m\}$ be an orthogonal basis of E, where $Se_j = \lambda_j e_j$. Then we can verify that $V_n = (S^n x, x)$ for all $n \geq 0$, where $x = \sum_{j=1}^{m} \mu_j e_j$ with $\mu_j^2 = y_j$.

5.3 Relation with Scalar Spectral Measures

Let $(E,(.,.))$ be a unitary real vector space of dimension m and set $S \in \Lambda_s(E)$ such that $\sigma(S)=\{\lambda_1, \ldots, \lambda_m\}$. Consider the spectral decomposition $E=\oplus_{j=1}^{m} E_j$ of E. For all $x=\sum_{j=1}^{m} x_j$ and $y=\sum_{j=1}^{m} y_j$ in E, where x_j and y_j are in E_j $(1 \le j \le m)$, the scalar spectral measure $\nu_{x,y}$ is defined by

$$\int_{\sigma(S)} f(t)\,d\nu_{x,y}(t)=(f(S)x, y), \tag{6}$$

where f is a continuous function on $\sigma(S)$, which may be identified with a finite sequence $(\alpha_1, \ldots, \alpha_m)$. From expression (6), we derive $\nu_{x,y}=\sum_{j=1}^{m} \nu_{x_i, y_j}$, and it is easy to see that

$$\int_{\sigma(S)} f(t)\,d\nu_{x_i, y_i}(t)=(f(S)x_i, y_i)=f(\lambda_i)(x_i, y_i).$$

Thus, $\nu_{x_i, y_i}=(x_i, y_i)\delta_{\lambda_i}$, where δ_λ is the Dirac measure. In particular, for $f(z)=z^n$, we have $(f(S)x_i, y_i)=(x_i, y_i)\lambda_i^n$. Let $\{V_n\}_{n\ge 0}$ be a sequence (1) and suppose that it is a sequence of moments of the operator S on a vector $x=\sum_{j=1}^{l} \mu_j e_j$. Then we have $(x_j, y_j)=\mu_j^2$, μ_j^2 satisfies the linear system of equations of Vandermonde type (5), and $\{V_n\}_{n\ge 0}$ is a sequence of moments of the positive measure $\nu_{x,x}=\sum_{j=1}^{m} \mu_j^2 \delta_{\lambda_i}$ on $\sigma(S)$. This measure is unique if the moment problem (2) (or (3)) is definite.

In general, we can consider the measure moment problem for sequences (1) on the interval [0, 1]; it can be formulated as follows: Characterize sequences (1) that are sequences of moments ${}_n=\int_0^1 t^n d\nu(t)$ of a (unique) positive Borel measure ν on [0, 1] (see, e.g., [1]-[4]). We have found some results on this question using techniques presented in [1]-[4].

5.4 Complex Case

Suppose that $(E,(.,.))$ is a unitary complex vector space of finite dimension m. Then all results still hold.

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IN MEMORIAM

Herta Taussig Freitag

December 1908-January 2000

Herta Taussig Freitag, long-time teacher, mathematician, and Fibonacci enthusiast, died January 25 at the age of 91. Her radiant smile and articulate speech reflecting her native Austria were unforgettable to hundreds of colleagues and friends. She remained an active participant in The Fibonacci Association until shortly before her death, assisting in the presentation of four papers at its 8th International Conference in 1998.

Herta's life story is one of triumph over adversity. Born in Vienna, she pursued her education there with a major interest in mathematics. When Hitler took over Austria in 1938, an event for which she had vivid memories, she began a six-year struggle to emigrate to the United States. It became clear that the only way out of Nazi Austria without a financial guarantor was to obtain employment in England as a domestic servant, an experience her brother describes as "Dickensian." A more complete account of her journey to freedom is included in the book *One-Way Ticket* by former student Mary Ann Johnson.

Upon arrival in the United States in 1944, Herta first taught at Greer School in New York State where she met her husband, Arthur H. Freitag. She began her long career at Hollins College in 1948 and completed her Ph.D. at Columbia in 1953. Among her numerous teaching awards were the Hollins Medal and the Virginia College Mathematics Teacher of the Year Award.

In her lifetime, Herta experienced prejudice in several forms but was never embittered by it. When she received the Humanitarian Award from the National Conference of Christians and Jews in 1997, the nomination read, in part: "What would have been a life-shattering experience for many set her on a course of personal professional achievement directed toward helping everyone, regardless of race, sex, color, ethnic background, religious persuasion or social class reach their maximum potential. And she does it in such a way as to make one feel that she is traveling with you, rather than leading the way."

Herta is survived by a brother, Walter Taussig, an associate conductor with the Metropolitan Opera, and a niece, Lynn Taussig. She will also be greatly missed by her Fibonacci "family" and a host of friends.

Margie Ribble

DUAL FORM OF COMBINATORIAL PROBLEMS AND LAPLACE TECHNIQUES

Lorenz Halbeisen
Department of Mathematics, Evans Hall 938
University of California at Berkeley, Berkeley, CA 94720
E-mail: halbeis@math.berkeley.edu

Norbert Hungerbühler
Department of Mathematics, University of Alabama at Birmingham
452 Campbell Hall, 1300 University Boulevard, Birmingham, AL 35294-1170
E-mail: buhler@uab.edu

1. INTRODUCTION

One of the central tools in enumerative combinatorics is that of generating functions. Generating functions can, e.g., be used to find the asymptotic behavior of the enumerating sequence (e.g., the Hardy-Ramanujan estimate for the partition function $P(n)$, see [3]) or even may yield an explicit formula for the solution (e.g., Rademacher's famous explicit formula for $P(n)$, see [6]).

Given a combinatorial problem, there are numerous ways to find the corresponding generating function. One possibility is to start with a recurrence relation, as, e.g., the recurrence for the Fibonacci numbers $(a_n)_{n\in\mathbb{N}_0} = (0, 1, 1, 2, 3, 5, 8, \ldots)$, which we write in the following form:

$$\begin{array}{ll} a_n = a_{n-2} + a_{n-1} + \delta_{1,n} & \forall n \in \mathbb{Z}, \\ a_n = 0 & \forall n < 0 \end{array} \tag{1}$$

($\delta_{k,n}$ denotes the Kronecker symbol). The z-transformation method requires multiplying (1) by z^n and summing over n. This yields an algebraic equation for the generating function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, namely,

$$f(z) = z^2 f(z) + zf(z) + z,$$

which is easily solved, giving $f(z) = \frac{z}{1-z-z^2}$. The Taylor expansion of this function yields

$$f(z) = \frac{z}{1-z-z^2} = \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5}},$$

i.e., we obtain the explicit Euler-Binet* formula for the Fibonacci numbers

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

A second way to find a generating function is to use Polya's index theorem. For example, let M be the set of all syntactic bracket figures with index n equal to the number of bracket pairs. For $n = 3$, we have the set M_3 of three bracket pairs:

$$M_3 = \{[][][], [[[]]], [[][]], [[]][], [][[]]\}.$$

* This formula was derived by Jacques P. M. Binet in 1843, although the result was known to Euler and to Daniel Bernoulli more than a century earlier.

By

$$\begin{aligned} M &\to M_1 \times M \times M \cup M_0 \\ [a]b &\mapsto ([], a, b) \\ \varnothing &\mapsto \varnothing \end{aligned}$$

we have a bijection between the sets M and $M_1 \times M \times M \cup M_0$ which is additive, that is, $\mathrm{ind}([a]b) = 1 + \mathrm{ind}(a) + \mathrm{ind}(b)$. Then, by Polya's theorem, the relation between the sets translates directly into a relation for the generating function for the numbers $c_n = \mathrm{card}(M_n)$, namely,

$$f(z) = zf^2(z) + 1.$$

Taylor expansion of the solution $f(z) = \frac{1}{2z}(1 - \sqrt{1-4z}) = \sum_{n=0}^{\infty} c_n z^n$ yields the Catalan numbers

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

A third way is to use methods from the theory of difference equations, which reach from continued fractions to Laplace transformation. As an example, we mention a recent theorem of Oberschelp (see [5]) that allows us to transform a difference equation into a differential equation for the exponential generating function by a formal procedure. For example, the Sloane-Plouffe sequence M1497 in [7], f_n, which counts the number of ways to build a sequence without repetition with n variables satisfies the recurrence $f_{n+1} = (n+1)f_n + 1$. Oberschelp's theorem requires the exchange

$$\binom{n}{k} f_{n+s-k} \leftrightarrow \frac{z^k}{k!} f^{(s)},$$

i.e., to replace f_{n+1} by f', nf_n by zf', f_n by f, and 1 by e^z. This procedure yields the ordinary differential equation $(1-z)f' - f = e^z$ with the solution $f(z) = \frac{e^z}{1-z}$ determined by $f(0) = 1$. Since $f(z)$ is the exponential generating function, we get in fact $f_n = n!(1 + \frac{1}{1!} + \cdots + \frac{1}{n!})$.

Experience shows that the situation becomes considerably more delicate as soon as the problem requires solving partial difference equations. In this article we want to describe methods which allow us to calculate the generating function from a recurrence relation. The idea is to link the Laplace transform directly to generating functions by interpreting the Fourier formula for the inverse Laplace transform as a residual integral. The reader who is not familiar with the Laplace or Fourier transformation might consult [1] or [8]. The idea is certainly not new; however, we would like to show that it applies also to more complicated (e.g., non-local) partial difference equations.

2. AUXILIARY RESULTS

2.1 Laplace Transformation

Let $(a_n)_{n \in Z}$, $a_n = 0$ for $n < 0$, be a sequence of real numbers with generating function $f(z) = \sum_{n \in Z} a_n z^n$. We call

$$A(z) := \sum_{n \in Z} a_n \chi_{[n, n+1[}(z)$$

the associated step-function. Here, χ_I denotes the characteristic function of the set I. Then the following theorem holds.

Theorem 1: If the Laplace transform $\mathscr{L}[A]$ of the associated step-function A exists; it is related to the generating function f by

$$\mathscr{L}[A](s) = \frac{1}{s}(1-e^{-s})f(e^{-s}).$$

Proof: Since we assume A to have at most exponential growth, we may transform term by term and obtain

$$\mathscr{L}[A](s) = \sum_{n=0}^{\infty} a_n \, \mathscr{L}[\chi_{[n,\,n+1[}].$$

Writing $\chi_{[n,\,n+1[} = H(\cdot - n) - H(\cdot - (n+1))$, where $H = \chi_{[0,\,\infty[}$ denotes the Heaviside function, and using the fact that $\mathscr{L}[H](s) = \frac{1}{s}$, we obtain, by applying the basic rules for the Laplace transformation,

$$\mathscr{L}[A](s) = \sum_{n=0}^{\infty} a_n \frac{1}{s} e^{-ns}(1-e^{-s}),$$

which is what we claimed. □

The following calculation provides a useful variant of Theorem 1: If $\frac{1}{z}g(e^{-z})$ is the Laplace transform of a piecewise smooth function G, we have by Fourier's formula for the inverse Laplace transformation that, for every point $x \in \mathbb{R}_+$ where G is continuous,

$$G(x) = \frac{1}{2\pi i}\,\mathrm{pv}\int_\Gamma \frac{1}{z} g(e^{-z}) e^{xz}\, dz.$$

Here, Γ is the curve $\Gamma : \mathbb{R} \to \mathbb{C}$, $t \mapsto s + it$, with $s \in \mathbb{R}$ large enough, and "pv" denotes the principal value. If we denote $\Gamma_n : [0, 2\pi[\to \mathbb{C}$, $t \mapsto z := s + i(t + 2n\pi)$, we have

$$G(x) = \frac{1}{2\pi i}\,\mathrm{pv}\sum_{n \in \mathbb{Z}} \int_{\Gamma_n} \frac{1}{z} g(e^{-z}) e^{xz}\, dz. \tag{2}$$

Observe that, by the Fourier-series expansion, we have, for $x \notin \mathbb{Z}$,

$$\sum_{n \in \mathbb{Z}} \frac{1}{s + i(t + 2n\pi)} e^{x(s+i(t+2n\pi))} = \frac{e^{\lceil x \rceil (s+it)}}{e^{s+it} - 1},$$

where $\lceil \cdot \rceil$ denotes the ceiling function, i.e., $\lceil x \rceil$ is the smallest integer larger than or equal to x. Hence, by substituting $u = e^{-z}$, we obtain from (2) with $n = \lfloor x \rfloor$,

$$G(x) = \frac{1}{2\pi i} \int_\gamma \frac{g(u)}{1-u} \frac{du}{u^{n+1}}, \tag{3}$$

where $\gamma : [0, 2\pi[\to \mathbb{C}$, $t \mapsto e^{-s}e^{it}$, and where $\lfloor \cdot \rfloor$ denotes the floor function, i.e., $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. Thus, if g is analytic in a neighborhood of 0, we may interpret the integral in (3) as the Cauchy residue integral for the n^{th} Taylor coefficient of the function $\frac{g(u)}{1-u}$. Thus, we have the following corollary.

Corollary 1: Assume f and g_n are analytic functions in a neighborhood of 0 and a_n is given by

$$a_n = \frac{1}{2\pi i}\,\mathrm{pv}\int_\Gamma \frac{1}{z} g_n(e^{-z}) e^{xz}\, dz \tag{4}$$

for some (and hence any) $x \in]n, n+1[$ and Γ as above. If $\lim_{z\to 0} \frac{f(z)-g_n(z)}{z^n} = 0$ for all $n \in \mathbb{N}_0$, then $\frac{f(z)}{1-z}$ is the generating function of the sequence a_n.

Let us briefly mention some advantages that the use of the Laplace transformation provides: Suppose we are given a generating function $f(u)$. Only in simple cases is it possible to use direct Taylor expansion to obtain a formula for the coefficient a_n of u^n. Also, the Cauchy residue $a_n = \operatorname{Res}_{u=0} \frac{f(u)}{u^{n+1}}$ or (in case of a meromorphic function f) $a_n = -\sum \operatorname{Res}_{u\neq 0} \frac{f(u)}{u^{n+1}}$ is often difficult to calculate. In such a situation, it may be helpful to split the residues via the Laplace transformation (as in the calculation preceding Corollary 1) in order to obtain an expansion (or at least an asymptotic formula) for the a_n. To illustrate this, let us consider the example of the generating function of the Bernoulli numbers

$$f(u) = u \cot u = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} u^n.$$

According to Theorem 1, the Laplace transform of the associated step-function G is

$$g(s) = \frac{1-e^{-s}}{s} f(e^{-s}),$$

and we may use the Fourier formula to invert g: $\mathscr{L}^{-1}[g](t) = \sum \operatorname{Res} g(s)e^{ts}$. The singularities of $g(s)e^{ts}$ are located at $s_{k,m} = m\pi i - \log(k\pi)$, $k \in \mathbb{N}$, $m \in \mathbb{Z}$. For $t \in \mathbb{Z}$ we have

$$\operatorname{Res}_{s_{k,m}} g(s)e^{ts} = \begin{cases} -\frac{1-k\pi}{s_{k,m}(k\pi)^t} & \text{if } m \text{ is even,} \\ -\frac{1+k\pi}{s_{k,m}(-k\pi)^t} & \text{if } m \text{ is odd.} \end{cases}$$

Combining residues for m and $-m$, we can easily sum the residues for fixed k over all m and obtain

$$\mathscr{L}^{-1}[g](t) = -\sum_{k=1}^{\infty} \frac{1}{(k\pi)^{2\lfloor t/2 \rfloor}}.$$

(Notice that one obtains a formula for $\sum_{m=1}^{\infty} \frac{1}{a^2+m^2}$ by expanding e^{ax} on $]-\pi, \pi[$ in a Fourier series.) Since $t \in \mathbb{Z}$ (G jumps in $\mathbb{Z}$), we finally get the zeta-function formula for the Bernoulli numbers:

$$B_{2n} = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}}.$$

A second benefit of the Laplace transformation are the various rules. For example, by the rule $\mathscr{L}[f'](s) = s\mathscr{L}[f](s) - f(0)$, we have, for $f_z(t) := t^z$, that

$$\mathscr{L}[f_z'](s) = s\mathscr{L}[f_z](s) = z\mathscr{L}[f_{z-1}](s).$$

Hence, for fixed s, the analytic function

$$h_s(z) := \mathscr{L}[f_z](s) = \int_0^{\infty} t^z e^{-st} dt$$

solves the difference equation $sh_s(z) = zh_s(z-1)$. In particular, for $s = 1$, we obtain Euler's integral representation of the Gamma-function. It is a particular feature of the Laplace-transformation method that it can be used to determine the analytic continuation of a discrete function. The

Laplace transformation also yields a functional connection between the exponential generating function $e(x)$ and the ordinary generating function $f(x)$ of a sequence a_n. In fact, we have

$$\mathscr{L}[e](s) = \mathscr{L}\left[\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right](s) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \underbrace{\mathscr{L}[x^n](s)}_{\frac{n!}{s^{n+1}}} = \frac{1}{s} f\left(\frac{1}{s}\right).$$

The translation-rule $\mathscr{L}[f(t-c)(s) = e^{-sc}\mathscr{L}[f(t)](s)$ for $c \geq 0$ allows us to transform a (linear) difference equation into an algebraic equation for the transformed function (this feature is similar to the z-transformation). In particular, it is possible to reduce a linear partial difference equation with n variables to an equation with $n-1$ variables. For an example, see Section 3.4 or 3.5.

Another virtue of the Laplace transformation appears when one looks for an asymptotic expansion of a sequence or (which is a similar thing) when one treats difference equations which show oscillation and damping effects. If one is only interested in the stationary state, one can already, at the level of the transformed function, identify terms which lead to exponentially decaying terms in the solution and drop them for the rest of the calculation.

2.2 The Dual of a Linear Difference Equation

Many combinatorial problems lead to partial difference equations. As a prototype example, we investigate the two-dimensional case.

Let $X \subset \mathbb{Z}^2$. For a map $p: X \to \mathbb{R}$, we consider the linear equation

$$p(z) = \sum_{\{\zeta \in X : \zeta \in \operatorname{spt} a_z\}} a_z(\zeta) p(\zeta), \qquad (*)$$

where we assume that the cardinality of the support of a_z (spt $a_z \subset X$) is finite for all $z \in X$, i.e., that the sum in $(*)$ is always finite. A set $A \subset X$ is called *stable* if for all maps $f: A \to \mathbb{R}$ there exists a unique solution p of $(*)$ such that $p|_A = f$. A triple $(X, A, *)$ is called *triangular* if X can be written as $X = (x_i)_{i \in \mathbb{N}}$ in such a way that, for all $i \in \mathbb{N}$, there holds spt $a_{x_i} \subset A \cup \{x_1, \ldots, x_{i-1}\}$, and for all $z \in A$: spt $a_z = \{z\}$ and $a_z(z) = 1$. In particular we have that, for a triangular triple $(X, A, *)$, the set A is stable.

Now, let $(X, A, *)$ be triangular and $f: A \to \mathbb{R}$ be given. Then, for any fixed $x = x_i \in X$, the solution p of $(*)$ in x is a finite linear combination of the values of f on A, i.e.,

$$p(x) = \sum_{\zeta \in A} \alpha_x(\zeta) f(\zeta).$$

In order to determine the weights $\alpha_x(\zeta)$, we proceed as follows:

(i) Put a red mark on x.

(ii) Replace each red mark on $y \in X \setminus A$ by a blue one on y and by $a_y(\zeta)$ many red marks on ζ for all $\zeta \in \operatorname{spt} a_y$.

(iii) Iterate (ii) until no more red marks on $X \setminus A$ exist.

If n denotes the maximum of the set $\{i$: there is a red mark on $x_i\}$, then, in each iteration step, n decreases at least by one due to the triangular structure. Hence, the iteration process terminates. If we denote by $\tilde{q}(\zeta)$ the number of red marks on ζ, the quantity

$$\sum_{\zeta \in X} \tilde{q}(\zeta) p(\zeta)$$

is invariant during the iteration. Hence, we obtain the result that after the iteration is completed the number of (red) marks on $\zeta \in A$, i.e., $\tilde{q}(\zeta)$, equals the weight $\alpha_x(\zeta)$.

If we denote by $q(\zeta)$ the final number of marks (blue or red) on ζ (i.e., after termination of the iteration), the iteration process described above translates into a partial difference equation for the function q:

$$q(z) = \sum_{\{\zeta \in A_x : z \in \operatorname{spt} a_\zeta\}} a_\zeta(z) q(\zeta) \tag{$**$}$$

with $q(x) = 1$ and with $A_x := \operatorname{tr} x \setminus A$, where $\operatorname{tr} x$ is the equivalence class of x with respect to the transitive hull of the relation $u \sim v :\Leftrightarrow u \in \operatorname{spt} a_v, v \notin A$. Notice that $(A_x, \{x\}, **)$ is triangular and finite. Let us summarize this result in a theorem.

Theorem 2: If $(X, A, *)$ is triangular with prescribed values f on A, then the weights α_x in the solution formula $p(x) = \sum_{\zeta \in A} \alpha_x(\zeta) f(\zeta)$ can be determined by the iteration scheme (i)-(iii) or, equivalently, by solving the dual linear recursion $(**)$ with initial value $q(x) = 1$.

Many transformation problems (for example, the Boustrophedon transformation in [4]) can be described as follows: Let $(X, A, *)$ be triangular; then we fix sets $A' = \{a_1, a_2, \dots\} \subset A$ and $X' = \{b_1, b_2, \dots\} \subset X$ and prescribe $f(a_i) = \phi_i$ and $f = 0$ on $A \setminus A'$. If we denote the solu-tion $\psi_i = p(b_i)$, the mapping $\Psi_{X,X',A,A',*} : (\phi_i) \mapsto (\psi_i)$ is a linear transformation of sequences, the associated linear mapping (ALM). The problem of finding its matrix (or the matrix of the inverse transformation) can often be solved by using the Laplace transformation technique for the partial difference equation for the *weights* $(**)$ even in cases where it is not possible to use directly the Laplace transformation in the *original* partial difference equation $(*)$. We will see some examples in the following section.

Before we discuss the examples, we will close this section by stating a simple path-counting lemma.

Lemma 1: Suppose the coefficient functions a in $(*)$ satisfy the following invariance property for all $z = (n, k)$ and $z' = (n, k')$ in $X = \mathbb{Z}^2$:

$$a_z(n+i, k+j) = a_{z'}(n+i, k'+j), \quad \forall i, j \in \mathbb{Z}. \tag{5}$$

Suppose, furthermore, that the column $\{(0, k) : k \in \mathbb{Z}\}$ is stable and that p denotes the solution of $(*)$ with prescribed values α_k on $(0, k)$. Then the column $\{(N, k) : k \in \mathbb{Z}\}$ is stable for

$$\tilde{p}(z) = \sum_{\{\zeta \in X\}} \bar{a}_z(\zeta) \tilde{p}(\zeta) \tag{$\dagger$}$$

where $\bar{a}_{u+v}(u) := a_u(u + \bar{v})$ and $\overline{(i, j)} := (i, -j)$. Finally, if we prescribe the values α_k on (N, k) for the equation $(\dagger)$, then $\tilde{p}(0, k) = p(N, k)$.

Proof of Lemma 1: We may interpret $(*)$ as a directed graph G with $a_z(\zeta)$ many edges from ζ to z. If we set $\alpha_k := \delta_{k,k_0}$, then $p(N, k)$ is the number of paths in G from $(0, k_0)$ to (N, k). If we flip the graph horizontally by $z \mapsto \bar{z}$ and invert the orientation of the edges, we obtain a graph G'. Now, $(\dagger)$ describes G' and $\tilde{p}(0, k)$ is the number of paths in G' from (N, k_0) to $(0, k)$ which equals, by construction, the number of paths in G from $(0, k_0)$ to (N, k).

For general (α_k), the claim follows by linearity. □

3. EXAMPLES AND APPLICATIONS

3.1 The Fibonacci Numbers and a Variant of Faulhaber's Formula

Let $X = \{(k,n): n \geq k \geq 0\}$ and $A = \{(k,n) \in X : n \in \{k, k+1\}\}$. Further, let

$$a_{(k,n)}(i,j) = \begin{cases} \delta_{k,i}\delta_{n-1,j} + \delta_{k+1,i}\delta_{n-1,j} & \text{for } (k,n) \notin A, \\ \delta_{k,i}\delta_{n,j} & \text{otherwise,} \end{cases}$$

in the equation $(*)$. This is easily seen to be triangular. For the sets $A' = \{(k, k+1) \in A\}$ and $X' = \{(0,n) \in X : n \geq 0\}$, we have that the ALM $\Psi_{X,X',A,A',*}$ applied to the sequence $(1,1,\ldots)$ yields the Fibonacci sequence $(f(n))_n$. Let us calculate the weights via $(**)$:

$$q(k,n) = q(k,n+1) + q(k-1,n+1)$$

with $q(0,l) = 1$. This is (up to renumbering) just the recursion for the binomial numbers, i.e., we get the "shallow diagonal" sum formula connecting Pascal's triangle to the Fibonacci numbers:

$$f(n+1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{n-2k}.$$

The binomial weights always occur for this type of equation: For another example, let $p(k,n) := \sum_{i=1}^{n} i^k$. Obviously, for fixed k, p is a polynomial in n of degree $k+1$. Faulhaber's famous formula expresses this polynomial in the basis $\{1, n, n^2, n^3, \ldots\}$, and the coefficients in this basis involve the Bernoulli numbers. Here, we want to express the polynomial in the basis

$$\left\{ \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots \right\}.$$

Consider again the "binomial" difference equation $f(k,n) = f(k,n-1) + f(k+1,n-1)$, this time on $X = \mathbb{N}_0^2$, with initial data $f(0,n) = p(k,n-1)$ for fixed k. The weights for the dual equation clearly are, as above, the binomial coefficients; hence, $p(k,n-1) = \sum_{i=1}^{n} \binom{n}{i} f(i,0)$, and it remains to find $f(i,0)$. Since $f(1,n) = n^k$, we use

$$\sum_{i=0}^{n} \binom{n}{i} i!\, S_2(k,i) = n^k, \tag{6}$$

where S_2 denotes the Stirling number of the second kind (see next section). Indeed, each term in the sum may be interpreted as the number of sequences in $\{1,\ldots,n\}^k$ with exactly i different numbers. Thus, $f(i+1,0) = i!\, S_2(k,i)$ and we recover the well-known formula

$$p(k,n) = \sum_{i=1}^{n} \binom{n+1}{i+1} i!\, S_2(k,i),$$

which one also gets by summing (6).

3.2 The Stirling Numbers

The Stirling numbers of the first kind $S_1(n,k)$ count the permutations of n distinct objects that can be written with exactly k disjoint cycles (cf. [2]). They can be computed recursively as follows: $S_1(n+1,k) := n \cdot S_1(n,k) + S_1(n,k-1)$, where $S_1(1,k) := \delta_{1,k}$.

Let $\widetilde{S}_n(k) := S_1(n,k)$; then $\widetilde{S}_n(k)$ satisfies the recurrence $\widetilde{S}_{n+1}(k) = n\widetilde{S}_n(k) + \widetilde{S}_n(k-1)$. Let $L_n(s)$ denote the Laplace transform of the associated step-function of $\widetilde{S}_n(k)$. Then we get $L_{n+1}(s) = nL_n(s) + e^{-s}L_n(s) = L_n(s)(n+e^{-s})$, with $L_1(s) = \frac{1}{s}(1-e^{-s})$. Hence,

$$L_n(s) = \frac{1}{s}(1-e^{-s})\prod_{j=1}^{n-1}(j+e^{-s}).$$

Thus, by Theorem 1, we find that

$$f_n(u) = \prod_{j=1}^{n-1}(j+u)$$

is the generating function for $(S_1(n,k))_k$.

The Stirling numbers of the second kind $S_2(n,k)$ count the number of groupings of n distinct objects into k disjoint (nonempty) groups. They can be computed recursively as follows: $S_2(n+1,k) := k \cdot S_2(n,k) + S_2(n,k-1)$, where $S_2(1,k) := \delta_{1,k}$.

Let $\widetilde{S}_k(n) := S_2(n,k)$; then $\widetilde{S}_k(n)$ satisfies the recurrence $\widetilde{S}_k(n) = k\widetilde{S}_k(n-1) + \widetilde{S}_{k-1}(n-1)$. Let $L_k(s)$ denote the Laplace transform of the associated step-function of $\widetilde{S}_k(n)$. Then we obtain $L_k(s) = ke^{-s}L_k(s) + e^{-s}L_{k-1}(s)$. Therefore,

$$L_k(s) = L_{k-1}(s)\frac{e^{-s}}{1-ke^{-s}} = L_1(s)\prod_{j=2}^{k}\frac{e^{-s}}{1-je^{-s}}$$

with $L_1(s) = \frac{1}{s}$. Thus, by Theorem 1, we get that

$$f_k(u) = \prod_{j=1}^{k}\frac{u}{1-ju}$$

is the generating function for $(S_2(n,k))_n$.

It is well known that the matrix of the Stirling numbers of the first and second kind are inverse in the sense that

$$f(n) = \sum_{i=1}^{n} S_1(n,i)e(i)$$

if and only if

$$e(n) = \sum_{i=1}^{n}(-1)^{n-i}S_2(n,i)f(i).$$

Instead of proving this rather special formula, we now investigate more general conditions which still imply an inversion formula of the above type.

3.3 An Inversion Formula

We consider the following situation: Given a linear equation $(*)$ with $X = \mathbb{N}_0 \times \mathbb{Z}$, which satisfies the invariance property (5), we suppose that with $A := \{(0,k) : k \in \mathbb{Z}\}$ the triple $(X, A, *)$ is triangular. We set $A' := \{(0,k) : k \in \mathbb{N}_0\}$ and $X' := \{(n,0) : n \in \mathbb{N}_0\}$ and consider the mapping $\Psi_{X,X',A,A',*} : (\phi_i) \mapsto (\psi_i)$. Notice that the equation $(**)$ for the weights inherits the invariance property (5), and hence we can apply Lemma 1 to $(**)$ and obtain

$$\widetilde{p}(z) = \sum_{\{\zeta \in X\}} \overline{a}_z(\zeta)\widetilde{p}(\zeta), \tag{††}$$

with $\tilde{p}(n, 0) = \delta_{n,0}$, where $\bar{a}_{u+v}(u) := a_{u+\bar{v}}(u)$. Then we have

$$\psi_n = \sum_{i=0}^{\infty} \tilde{p}(n, i)\phi_i. \tag{7}$$

Now we invert the previous equation: Let $Y := \mathbb{N}_0 \times \mathbb{N}_0$ and $Y' := \{(0, k) : k \in \mathbb{N}_0\}$. For any fixed $z \in X$, we can replace $(*)$ equivalently by the equation

$$p(\zeta_0) = \frac{1}{a_z(\zeta_0)} p(z) - \sum_{\{\zeta \in \operatorname{spt} a_z \setminus \{\zeta_0\}\}} \frac{a_z(\zeta)}{a_z(\zeta_0)} p(\zeta) =: \sum_{\{\zeta \in \operatorname{spt} a'_{\zeta_0}\}} a'_{\zeta_0}(\zeta) p(\zeta) \tag{$*'$}$$

for arbitrary $\zeta_0 \in \operatorname{spt} a_z$. Assume that for any $z \in X$ we can—by choosing a suitable ζ_0—replace $(*)$ by $(*')$ in such a way that

- the coefficients a'_z respect the invariance relation (5),
- the triple $(Y, Y', *')$ is triangular.

The equation for the weights for $(*')$ is

$$q(z) = \sum_{\{\zeta \in A_{(0,0)} : z \in \operatorname{spt} a'_\zeta\}} a'_\zeta(z) q(\zeta), \tag{$**'$}$$

with initial condition $q(0, 0) = 1$ (because $(**')$ satisfies (5)). Then we have

$$\phi_n = \sum_{i=0}^{\infty} q(i, -n)\psi_i. \tag{8}$$

Hence, in view of (8) and (7), q and $\tilde{p}$ are inverse matrices, where q and $\tilde{p}$ satisfy certain difference equations which are related in the described manner. Notice also that, by choosing ζ_0 (see above), there is a certain freedom in the coefficients a' which can be useful sometimes.

As an example of the previous result, we investigate a generalization of the Stirling numbers.

Let us define $a_{(n,k)}(i, j) := c(i)\delta_{i,n-1}\delta_{j,k} + d(i)\delta_{i,n-1}\delta_{j,k+1}$, where c and d are nonvanishing functions. Then the procedure described above yields the following proposition.

Proposition 1: The numbers $s_1(n, k)$ and $s_2(n, k)$ for $(n, k) \in \mathbb{Z} \times \mathbb{Z}$, defined by

$$s_1(n, k) = c(n-1)s_1(n-1, k) + d(n-1)s_1(n-1, k-1)$$

and

$$s_2(n, k) = -\frac{c(n)}{d(n)} s_2(n, k-1) + \frac{1}{d(n-1)} s_2(n-1, k-1)$$

with $s_1(0, m) = s_2(m, 0) = \delta_{m,0}$, are inverse in the sense that

$$\psi_n = \sum_{i=0}^{\infty} s_1(n, i)\phi_i \Leftrightarrow \phi_n = \sum_{i=0}^{\infty} s_2(i, n)\psi_i.$$

For special choices of the functions c and d, one easily gets, e.g., the inversion formulas for the Stirling numbers $(c(n) = n, d(n) = 1)$, the binomial numbers $(c(n) = 1, d(n) = 1)$, or the numbers $Q_l(n) := \binom{n}{l} l!$ counting the number of ways to build sequences of length l with n objects without repetitions $(c(l) = -\frac{1}{l}, d(l) = \frac{1}{l})$—guess what the inverse numbers are!

3.4 The Partition Numbers

As a further example, we consider the number $p(n, k)$ of partitions of an integer n into parts larger than or equal to k. This leads to the (non-local) partial difference equation

$$p(n, k) = p(n-k, k) + p(n, k+1), \tag{9}$$

with $p(n, k) = 0$ for $k > n > 0$ and $p(n, n) = 1$. In the above setting, the problem reads as follows: $X = \mathbb{N}^2$, $A = \{(n, k) : k \geq n\}$, $A' = \{(n, n) : n \in \mathbb{N}\}$, $X' = \{(n, 1) : n \in \mathbb{N}\}$. Also, for $(n, k) \in X \setminus A$, we have

$$p(n, k) = \sum_{i, j \in \mathbb{N}} (\delta_{i,n-k}\delta_{j,k} + \delta_{i,n}\delta_{j,k+1})p(i, j). \tag{10}$$

The ALM $\Psi_{X,X',A,A',(10)}$ maps the sequence $(1, 1, \ldots)$ into the sequence $p(n, 1) = P(n)$ of the partition numbers. The equation for the weights is given by $q(n, k) = q(n, k-1) + q(n+k, k)$ with initial conditions $q(n, 1) = 1$ for $n \leq N$ and $q(n, k) = 0$ for $n > N$. Then we have $P(N) = \sum_{i=1}^{N} q(i, i)$. By renumbering, this is equivalent to saying

$$\tilde{q}(n, k) = \tilde{q}(n, k-1) + \tilde{q}(n+k, k) \tag{11}$$

with $\tilde{q}(n, 1) = 1$ for all n, $\tilde{q}(n, k) = 0$ for $n \leq 0$, and $P(N) = \sum_{i=1}^{N} \tilde{q}(i, N-i+1)$. Note that $\tilde{q}(n, k)$ no longer depends on N. Laplace transformation of (11) with respect to the first variable with k fixed yields

$$r_k(s) = \frac{1}{1-e^{-sk}} r_{k-1}(s)$$

with initial value $r_1(s) = \frac{1}{s}$ (since $\tilde{q}(1, k) = 1$ for $k \in \mathbb{N}$). Thus, we have

$$r_k(s) = \frac{1}{s} \prod_{j=2}^{k} \frac{1}{1-e^{-js}}$$

and, by Theorem 1, the generating function $g_k(u)$ of $(\tilde{r}_k(n))_n$ is given by

$$g_k(u) = \prod_{j=1}^{k} \frac{1}{1-u^j}.$$

From this, it is easy to derive Euler's classical generating function $E(u)$ of the partition numbers $P(N)$. But, by interpreting $\tilde{q}(n, k)$ as the number of partitions of $n-1$ into k or less parts (and hence $P(n-1) = \tilde{q}(n, n-1) = \tilde{q}(n, n)$), we immediately get from the above calculation together with Corollary 1 that

$$E(u) = \prod_{j=1}^{\infty} \frac{1}{1-u^j}. \tag{12}$$

Also, if $f(s)$ denotes the Laplace transform of E, it follows from (12) that

$$\frac{1}{s}(1-e^{-s}) \prod_{j=1}^{\infty} (1-e^{-js}) = f(s) \sum_{j=1}^{\infty} (-1)^{\lfloor \frac{j}{2} \rfloor} e^{-st_j},$$

where $t_j = 0, 1, 2, 5, 7, \ldots$ are the pentagonal numbers. Laplace inversion of the last equation yields Euler's formula $\sum_{j=1}^{\infty} (-1)^{\lfloor \frac{j}{2} \rfloor} P(n-t_j) = \delta_{n,0}$.

What about counting weighted partitions? Let $f:\mathbb{N}\to\mathbb{R}$ be a weight function with the meaning that we count partitions into i parts $f(i)$ many times, or—what is the same thing by considering Ferrers diagram—count partitions which largest part of size i, $f(i)$ many times. Then the calculation above gives the generating function for this problem:

$$\sum_{i=1}^{\infty}\frac{f(i)u^i}{\prod_{j=1}^{i}1-u^j}.$$

So, choosing, e.g., f as the characteristic function of the even numbers, we compute $(e(n))_n = (0, 1, 1, 3, 3, 6, 7, 12, 14, \ldots)$.

To conclude this section, let us compute the inverse of the ALM $\Psi_{X,X',A,A',(10)}$. Let us put a red mark on (L, L). In view of (10), we can replace a red mark on (n, k) (for $n \geq k > 1$) by a red mark on $(n, k-1)$, a negative red mark on $(n-k+1, k-1)$, and a blue mark on (n, k). This game terminates when all red marks are in $A\setminus A'$ (these marks are multiplied by 0) or in X' (where a mark on $(i, 1)$ is multiplied by ψ_i). Hence, $\phi_L = \sum_{n=1}^{L}\psi_n\omega(L, n)$, where $\omega(L, n)$ denotes the number of red marks on $(n, 1)$.

To compute $\omega(L, n)$, we consider the directed, finite graph G_L with vertices $\{(n, k): L \geq n \geq k \geq 1\}$ and an edge from (n, k) to (n', k') if $k' = k-1$ and $n' = n$ (these edges are called v-edges) or if $k' = k$ and $n' = n-k$ (these edges are called h-edges of length k). Now let $W_L(n)$ be the number of paths through the graph G_L from the vertex (L, L) to $(n, 1)$, such that all h-edges have different length and each path is weighted by +1 if the number of h-edges contained in the path is even, otherwise it is weighted by -1. It is easy to see that $W_L(n) = \omega(L, n)$. To compute $W_L(n)$, let us first define the function $w(m, l, s)$, which is the number of weighted paths from (m, m) to $(m-l, 1)$, such that the maximum of the lengths of h-edges contained in the path equals s (where $s = 0$ means that the path contains no h-edge). For the function $w(m, l, s)$, we have

$$w(m,l,s)=\begin{cases}1 & \text{if } l=s=0,\\ 0 & \text{if } s>l \text{ or } s>\lfloor\frac{m}{2}\rfloor,\\ -\sum_{j=1}^{s}w(m-s,l-s,s-j) & \text{otherwise.}\end{cases}$$

Now, by construction, we obtain

$$W_L(n)=\sum_{s=0}^{\lfloor\frac{L}{2}\rfloor}w(L,L-n,s).$$

For example, for $L = 12$, we get $(W_{12}(n))_n = (1, -1, -2, 0, 2, 0, 1, 0, 0, -1, -1, 1)$ and, in fact,

$$\begin{aligned}&P(12)-P(11)-P(10)+P(7)+2P(5)-2P(3)-P(2)+P(1)\\&=77-56-42+15+2\cdot 7-2\cdot 3-2+1=1.\end{aligned}$$

3.5 A Path Counting Problem

We consider paths in a three-dimensional lattice: The starting point of the paths is a point $(x, 0, 0)$, $x \in \mathbb{N}_0$, on the x-axis. If (x, y, z) is a point on the path, then a unit step in the positive y or z direction is allowed or a step of length $y+z+1$ in the negative x direction. We want to count the number $H_M(x)$ of allowed paths starting in $(x, 0, 0)$ which end in a given set $M \subset \mathbb{Z}^3$.

The dual of this problem is given by the non-local linear difference equation

$$q_{z,y}(x) = q_{z-1,y}(x) + q_{z,y-1}(x) + q_{z,y}(x-y-z-1) \tag{13}$$

with $q_{z,y}(x) := 0$ if one of the numbers x, y, or z is negative and $q_{0,0}(0) := 1$. We already used an index notation because we want to Laplace-transform equation (13) with respect to the variable x. First, we have $Q_{0,0}(s) = \frac{1}{s}$, since $q_{0,0}(x) = 1$ for $x \geq 0$. Laplace transformation of (13) yields

$$Q_{z,y}(s) = Q_{z-1,y}(s) + Q_{z,y-1}(s) + e^{-s(y+z+1)}Q_{z,y}(s).$$

Considering s as a parameter, the solution of this difference equation in y and z is given by

$$Q_{z,y}(s) = \frac{1}{s}\binom{z+y}{z}\frac{1}{\prod_{j=2}^{z+y+1}(1-e^{-js})}.$$

Thus, the generating function of $q_{z,y}(x)$ is

$$f_{z,y}(u) = \binom{z+y}{z}\prod_{j=1}^{z+y+1}\frac{1}{1-u^j}.$$

Hence, using the notation of Section 3.4,

$$q_{z,y}(x) = \tilde{r}_{z+y+1}(x)\binom{z+y}{z}.$$

Finally, the solution to our path counting problem is given by the formula

$$H_M(\xi) = \sum_{(\xi-x,\, y,\, z)\in M} \tilde{r}_{z+y+1}(x)\binom{z+y}{z}.$$

For example, let us count the paths starting in $(\xi, 0, 0)$ with at most h unit steps in the z direction and such that the total number of unit steps in the negative x and in the positive y direction equals ξ. This corresponds to the set $M = \{(x, y, z) \in \mathbb{Z}^3 : x = y, z \leq h\}$, and the solution formula yields

$$H_M(\xi) = \sum_{z\leq h,\, x\leq \xi} \tilde{r}_{z+\xi-x+1}(x)\binom{z+\xi-x}{z}.$$

3.6 Local Linear Difference Equations

For $X = \{(k, l) : 0 \leq k \leq l\}$ and $A = \{(k, l) : l \in \{k, k+1, k+2\}\}$, we consider the model equation

$$z(k, l) = a_1 z(k, l-1) + a_2 z(k+1, l-1) + a_3 z(k+2, l-1). \tag{14}$$

$(X, A, (14))$ is triangular and, for $X' = \{(0, l) : l \geq 3\}$, the equation for the weights is

$$q(k, l) = a_1 q(k, l+1) + a_2 q(k-2, l+1) + a_3 q(k-1, l+1) \tag{15}$$

with initial condition $q(k, L) = \delta_{k,0}$ for a fixed $L \geq 0$. Laplace transformation of (15) with respect to the variable k with l fixed gives $Q_l(s) = Q_{l+1}(s)(a_1 + a_2 e^{-s} + a_3 e^{-2s})$ with initial condition $Q_L(s) = \frac{1}{s}(1-e^{-s})$. The solution is

$$Q_l(s) = \frac{1}{s}(1-e^{-s})(a_1 + a_2 e^{-s} + a_3 e^{-2s})^{L-l},$$

and Theorem 1 gives, for the generating function of the sequence $(q(k, l))_k$, the function $(a_1 + a_2 u + a_3 u^2)^{L-l}$. Multinomial expansion yields

$$q(k,l)=\sum_{k_2+2k_3=k}\binom{L-l}{L-l-k_2-k_3,\,k_2,\,k_3}a_1^{L-l-k_2-k_3}a_2^{k_2}a_3^{k_3}.$$

Since (15) does not stop the iteration when a mark lies on A, we have to compensate by setting

$$\begin{aligned}&\tilde{q}(k,k+2)=q(k,k+2),\\ &\tilde{q}(k,k+1)=q(k,k+1)-a_1q(k,k+2),\ \text{and}\\ &\tilde{q}(k,k)=q(k,k)-a_1q(k,k+1)-a_2q(k-1,k+1).\end{aligned}$$

Then, if α_z is given on $z\in A$ as initial data for (14), we get the solution

$$z(0,l)=\sum_{i=2}^{l}\sum_{j=0}^{2}\alpha_{(i-j,i)}\tilde{q}(i-j,i). \tag{16}$$

In particular, if $\alpha_{(k+j,k)}=x_j$ (for $j=0,1,2$), $z(0,l)$ is the solution of $x_n=a_1x_{n-1}+a_2x_{n-2}+a_3x_{n-k}$ with initial values x_0,x_1,x_2 and (16) is a root-free representation of the solution.

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ON POLYNOMIALS RELATED TO POWERS OF THE GENERATING FUNCTION OF CATALAN'S NUMBERS

Wolfdieter Lang

Institut für Theoretische Physik, Universität Karlsruhe
Kaiserstrasse 12, D-76128 Karlsruhe, Germany
E-mail: wolfdieter.lang@physik.uni-karlsruhe.de

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1. INTRODUCTION AND SUMMARY

Catalan's sequence of numbers $\{C_n\}_0^\infty = \{1, 1, 2, 5, 14, 42, \ldots\}$ (nr.1459 and $A000108$ of [14]) emerges in the solution of many combinatorial problems (see [2], [4], [5], and [16] for further references). The moments μ_{2k} of the normalized weight function of Chebyshev's polynomials of the second kind are given by $C_k/2^k$ (see, e.g., [3], Lemma 4.3, p. 160 for $l=0$, and [17], p. II-3). This sequence also shows up in the asymptotic moments of zeros of scaled Laguerre and Hermite polynomials (see [9], eqs. (3.34) and (3.35)). The generating function $c(x) = \sum_{n=0}^\infty C_n x^n$ is the solution of the quadratic equation $xc^2(x) - c(x) + 1 = 0$ with $c(0) = 1$. Therefore, every positive integer power of $c(x)$ can be written as

$$c^n(x) = p_{n-1}(x)1 + q_{n-1}(x)c(x), \tag{1}$$

with certain polynomials p_{n-1} and q_{n-1}, both of degree $(n-1)$, in $1/x$. In Section 2, they are shown to be related to Chebyshev polynomials of the second kind:

$$p_{n-1}(x) = -\left(\frac{1}{\sqrt{x}}\right)^n S_{n-2}\left(\frac{1}{\sqrt{x}}\right),\ q_{n-1}(x) = \left(\frac{1}{\sqrt{x}}\right)^{n-1} S_{n-1}\left(\frac{1}{\sqrt{x}}\right) = -xp_n(x), \tag{2}$$

with $S_n(y) = U_n(y/2)$. Therefore, it is possible to extend the range of the power n to negative integers (or to real or complex numbers). Tables for the $U_n(x)$ polynomials can be found, e.g., in [1]. Because powers of a generating function correspond to convolutions of the generated number sequence, the given decomposition of $c^n(x)$ will determine convolutions of the Catalan sequence. In passing, an explicit expression for general convolutions in the form of nested sums will also be given. Contact with the works of [6], [12], [18], and [5] will be made.

Together with the known (e.g., [4], [11]) result (valid for real n),

$$c^n(x) = \sum_{k=0}^\infty C_k(n)x^k,\ \text{with}\ C_k(n) = \frac{n}{n+2k}\binom{n+2k}{k} = \frac{n}{k+n}\binom{n-1+2k}{k}, \tag{3}$$

one finds, from the alternative expression (1) for positive n, two sets of identities:

$$(P1) \qquad \sum_{l=0}^{p}(-1)^l\binom{n+1-p+l}{p-l}C_l = \binom{n-p}{p} \tag{4}$$

for $n \in \mathbf{N}_0$, $p \in \{0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, and

$$(P2) \qquad \sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^l\binom{n-1-l}{l}C_{k+n-1-l} = C_k(n) \tag{5}$$

for $n \in \mathbf{N}$, $k \in \mathbf{N}_0$.

For negative powers in (1), two other sets of identities result:

$$(P3) \qquad \sum_{l=0}^{\min\left(\lfloor\frac{n-1}{2}\rfloor, k-1\right)} (-1)^l \binom{n-1-l}{l} C_{k-1-l} = (-1)^{k+1}\binom{n-k-1}{k-1} \tag{6}$$

for $n \in \mathbf{N}$, $k \in \{0, 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor\}$ (for $k = 0$, both sides are by definition zero), and

$$(P4) \qquad \sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor} (-1)^l \binom{n-1-l}{l} C_{k-1-l} = -C_k(-n) = \frac{n}{k}\binom{2k-n-1}{k-1} \tag{7}$$

for $n \in \mathbf{N}$, $k \in \mathbf{N}$ with $k \geq \lfloor\frac{n}{2}\rfloor + 1$. These identities can be continued for appropriate values of real n.

Another expression for the coefficients of negative powers of $c(x)$ is

$$C_k(-n) = \sum_{l=1}^{\min(n,k)} (-1)^l \binom{n}{l} C_{k-l}(n) \tag{8}$$

for $n, k \in \mathbf{N}$, and $C_0(-n) = 1$, $C_n(0) = \delta_{n,0}$. Also, from (3), $C_k(-n) = -C_{k-n}(n)$ for $n, k \in \mathbf{N}$ with $k \geq n$.

The remainder of this paper provides proofs for the above given statements. Section 2 deals with integer (and real) powers of the generating function $c(x)$. Convolutions of general sequences are expressed there in terms of nested sums. In Section 3 some families of integer sequences related to the polynomials $q_n(x)$ (2) evaluated for $x = 1/m$ for $m = 4, 5, \ldots$ and $(-1)^n q_n(x)$ evaluated at $x = -1/m$, $m \in \mathbf{N}$, are considered.

2. POWERS

The equation $xc^2(x) - c(x) + 1 = 0$ whose solution defines the generating function of Catalan's numbers if $c(0) = 1$ can be considered as a characteristic equation for the recursion relation

$$xr_{n+1} - r_n + r_{n-1} = 0, \quad n = 0, 1, \ldots, \tag{9}$$

with arbitrary inputs $r_{-1}(x)$ and $r_0(x)$. A basis of two linearly independent solutions is given by the Lucas-type polynomials $\{u_n\}$ and $\{v_n\}$ with standard inputs $u_{-1} = 0$, $u_0 = 1$, $(u_{-2} = -x)$, and $v_{-1} = 1$, $v_0 = 2$, $(v_1 = 1/x)$, in the Binet form

$$u_{n-1}(x) = \frac{c_+^n(x) - c_-^n(x)}{c_+(x) - c_-(x)}, \tag{10}$$

$$v_n(x) = c_+^n(x) + c_-^n(x) = \frac{1}{x}(u_{n-1}(x) - 2u_{n-2}(x)), \tag{11}$$

with the two solutions of the characteristic equation, *viz* $c_\pm(x) := (1 \pm \sqrt{1-4x})/(2x)$. $c(x) := c_-(x)$ satisfies $c(0) = 1$, and $c_+(x) = 1/(xc(x))$, as well as $c_+(x) + c(x) = 1/x$. From the recurrence (9), it is clear that, for positive $n \neq 0$, u_n is a polynomial in $1/x$ of degree $n-1$. If $c_+(x) - c_-(x) = 0$, i.e., $x = 1/4$, equation (10) is replaced by $u_n(1/4) = 2^n(n+1)$. The second equation in (11) holds because both sides of the equation satisfy recurrence (9) and the inputs for v_0 and v_1 match. One may associate with the recurrence relation (9) a transfer matrix

$$\mathbf{T}(x)=\begin{pmatrix}1/x & -1/x\\ 1 & 0\end{pmatrix},\quad \det\mathbf{T}(x)=1/x. \tag{12}$$

With this matrix, one can rewrite (9) as

$$\begin{pmatrix}r_n\\ r_{n-1}\end{pmatrix}=\mathbf{T}(x)\begin{pmatrix}r_{n-1}\\ r_{n-2}\end{pmatrix}=\mathbf{T}^n(x)\begin{pmatrix}r_0(x)\\ r_{-1}(x)\end{pmatrix}. \tag{13}$$

Because $\mathbf{T}^n=\mathbf{T}\,\mathbf{T}^{n-1}$ with input $\mathbf{T}^1=\mathbf{T}(x)$ given by (12), one finds from the recurrence relation (9) with $r_n=\mathcal{U}_n$ that

$$\mathbf{T}^n(x)=\begin{pmatrix}\mathcal{U}_n(x) & -\frac{1}{x}\mathcal{U}_{n-1}(x)\\ \mathcal{U}_{n-1}(x) & -\frac{1}{x}\mathcal{U}_{n-2}(x)\end{pmatrix}. \tag{14}$$

Note that, for $x=1$, one has $c_{\pm}(1)=(1\pm i\sqrt{3})/2$, which are 6th roots of unity, and the related period 6 sequences are $\{\mathcal{U}_n(1)\}_{-1}^{\infty}=\{\overline{0,1,1,0,-1,-1}\}$, as well as $\{\mathcal{V}_n(1)\}_0^{\infty}=\{\overline{2,1,-1,-2,-1,1}\}$. This follows from equations (10) and (11). It is convenient to map the recursion relation (9) to the familiar one for Chebyshev's $S_n(x)=U_n(x/2)$ polynomials of the second kind, *viz*

$$S_n(x)=xS_{n-1}(x)-S_{n-2}(x),\ S_{-1}=0,\ S_0=1, \tag{15}$$

with characteristic equation $\lambda^2-x\lambda+1=0$ and solutions $\lambda_{\pm}(x)=\frac{x}{2}(1\pm\sqrt{1-(2/x)^2})$, satisfying $\lambda_+(x)\lambda_-(x)=1$ and $\lambda_+(x)+\lambda_-(x)=x$. The relation to $c_{\pm}(x)$ is

$$\sqrt{x}\,c_{\pm}(x)=\lambda_{\pm}(1/\sqrt{x}). \tag{16}$$

The Binet form of the corresponding two independent polynomial systems is

$$S_{n-1}(x)=\frac{\lambda_+^n(x)-\lambda_-^n(x)}{\lambda_+(x)-\lambda_-(x)}, \tag{17}$$

$$2T_n(x/2)=\lambda_+^n(x)+\lambda_-^n(x), \tag{18}$$

and $T_n(x/2)=(S_n(x)-S_{n-2}(x))/2$ are Chebyshev polynomials of the first kind. Tables of Chebyshev polynomials can be found in [1]. The coefficient triangles of the $S_n(x)$, $U_n(x)$, and $T_n(x)$ polynomials can also be viewed under the numbers A049310, A053117, and A053120, respectively, in the on-line database [14].

The extension to negative indices runs as follows:

$$\mathcal{U}_{-n}(x)=-x^{n-1}\mathcal{U}_{n-2}(x), \tag{19}$$

$$S_{-(n+2)}(x)=-S_n(x). \tag{20}$$

This follows from (10) and (17). Note that from (9), $\mathcal{U}_n$ is for positive n a monic polynomial in $1/x$ of degree n, and for negative n in general, a nonmonic polynomial in x of degree $\lfloor -\frac{n}{2}\rfloor$. It is possible to extend the range of n to complex numbers using the Binet forms.

A connection between both systems of polynomials is made, using (10), (16), and (17), by

$$\mathcal{U}_n(x)=\left(\frac{1}{\sqrt{x}}\right)^n S_n(1/\sqrt{x}). \tag{21}$$

This holds for $n\in\mathbf{Z}$ in accordance with (19) and (20).

After these preliminaries, we are ready to state the following proposition.

Proposition 1: The n^{th} power of $c(x)$, the generating function of Catalan numbers can, for $n \in \mathbf{Z}$, be written as

$$c^n(x) = -\frac{1}{x}\mathcal{U}_{n-2}(x) + \mathcal{U}_{n-1}(x)c(x), \tag{22}$$

$$= -\left(\frac{1}{\sqrt{x}}\right)^n S_{n-2}(1/\sqrt{x}) + \left(\frac{1}{\sqrt{x}}\right)^{n-1} S_{n-1}(1/\sqrt{x})c(x). \tag{23}$$

Proof: Due to $c^2(x) = (c(x)-1)/x$ and $c^{-1}(x) = 1 - xc(x)$, one can write

$$c^n(x) = p_{n-1}(x) + q_{n-1}(x)c(x)$$

for $n \in \mathbf{Z}$. From $c^n(x) = c(x)c^{n-1}(x)$, one is led to $q_{n-1} = p_{n-2} + \frac{1}{x}q_{n-2}$ and $p_{n-1} = -\frac{1}{x}q_{n-2}$, or $q_{n-1} = (q_{n-2} - q_{n-3})/x$ with input $q_{-1} = 0$, $q_0 = 1$. So $q_{n-1}(x) = \mathcal{U}_{n-1}(x)$ and $p_{n-1}(x) = -\mathcal{U}_{n-2}(x)/x$. Equation (23) then follows from (21). □

Note 1: Because

$$S_n(y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} y^{n-2j},$$

the explicit form of these polynomials (2) is

$$p_{n-1}(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} (-1)^{j+1} \binom{n-2-j}{j} x^{-(n-1-j)}, \quad p_{-1} = 1, p_0 = 0,$$

and

$$q_{n-1}(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \binom{n-1-j}{j} x^{-(n-1-j)}, \quad q_{-1} = 0.$$

For negative index one has, due to (20),

$$p_{-(n+1)}(x) = (\sqrt{x})^n S_n(1/\sqrt{x}) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} x^j$$

and

$$q_{-(n+1)}(x) = -(\sqrt{x})^{n+1} S_{n-1}(1/\sqrt{x}) = -x \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \binom{n-1-j}{j} x^j.$$

In the Table, one can find the coefficient triangle for the polynomials $\{p_n(x)\}_{-1}^{12}$ with column m corresponding to $(\frac{1}{x})^m$, $m \geq 0$.

Note 2: An alternative proof of Proposition 1 can be given starting with (17) and (18) which show, together with $\lambda_+(x) - \lambda_-(x) = \sqrt{x^2-4}$, that

$$\lambda_\pm^n(x) = T_n(x/2) \pm \sqrt{(x/2)^2 - 1}\, S_{n-1}(x), \tag{24}$$

or, from $\pm\sqrt{(x/2)^2-1} = \lambda_\pm(x) - x/2$ and the S_n recurrence relation (15),

$$\lambda_\pm^n(x) = T_n(x/2) - \frac{1}{2}(S_n(x) + S_{n-2}(x)) + S_{n-1}(x)\lambda_\pm(x) \tag{25}$$

$$= -S_{n-2}(x) + S_{n-1}(x)\lambda_{\pm}(x). \tag{26}$$

Now (23) follows from (16). This also proves that, in Proposition 1, one may replace $c(x)$ by $c_+(x) = 1/(xc(x))$, from which one recovers the c^{-n} formula for $n \in \mathbf{N}$ in accordance with (19) and (20).

TABLE. $p(n, m) = [1/x^m]p\_\{n\}(x)$ Coefficient Matrix
$n = -1, \ldots, 12,\ m = 0, \ldots, 12$

$n\backslash m$	0	1	2	3	4	5	6	7	8	9	10	11	12
−1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	−1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	−1	0	0	0	0	0	0	0	0	0	0
3	0	0	1	−1	0	0	0	0	0	0	0	0	0
4	0	0	0	2	−1	0	0	0	0	0	0	0	0
5	0	0	0	−1	3	−1	0	0	0	0	0	0	0
6	0	0	0	0	−3	4	−1	0	0	0	0	0	0
7	0	0	0	0	1	−6	5	−1	0	0	0	0	0
8	0	0	0	0	0	4	−10	6	−1	0	0	0	0
9	0	0	0	0	0	−1	10	−15	7	−1	0	0	0
10	0	0	0	0	0	0	−5	20	−21	8	−1	0	0
11	0	0	0	0	0	0	1	−15	35	−28	9	−1	0
12	0	0	0	0	0	0	0	6	−35	56	−36	10	−1

Note 3: For the transfer matrix $\mathbf{T}(x)$, defined in (12), one can prove for $n \in \mathbf{N}$, in an analogous manner, that

$$\mathbf{T}^n = -\left(\frac{1}{\sqrt{x}}\right)^n S_{n-2}(1/\sqrt{x})\mathbf{1} + \left(\frac{1}{\sqrt{x}}\right)^{n-1} S_{n-1}(1/\sqrt{x})\mathbf{T}(x), \tag{27}$$

by employing the Cayley-Hamilton theorem for the 2×2 matrix $\mathbf{T}$ with $\operatorname{tr}\mathbf{T} = \frac{1}{x} = \det\mathbf{T}$, which states that $\mathbf{T}$ satisfies the characteristic equation $\mathbf{T}^2 - \frac{1}{x}\mathbf{T} + \frac{1}{x}\mathbf{1} = 0$.

Powers of a function which generates a sequence generate convolutions of this sequence. Therefore, Proposition 1 implies that convolutions of the Catalan sequence can be expressed in terms of Catalan numbers and binomial coefficients. Before giving this result, we shall present an explicit formula for the n^{th} convolution of a general sequence $\{C_l\}$ generated by $c(x) = \sum_{l=0}^{\infty} C_l x^l$. Usually, the convolution coefficients $C_l(n)$, defined by $c^n(x) = \sum_{l=0}^{\infty} C_l(n)x^l$, are written as

$$C_l(n) = \sum_{\sum_{j=1}^{n} i_j = l} C_{i_1}C_{i_2}\cdots C_{i_n}, \text{ with } i_j \in \mathbf{N}_0. \tag{28}$$

An explicit formula with $(l-1)$ nested sums is the content of the next lemma.

Lemma 1–General convolutions: For $l = 2, 3, \ldots,$

$$C_l(n) = C_0^{n-l}C_1^l\left(\prod_{k=2}^{l}\sum_{i_k=a_k}^{\lfloor b_k \rfloor}\right)\langle n, l, \{i_j\}_2^l\rangle \prod_{j=2}^{l}\left(\left(\frac{C_j C_0^{j-1}}{C_1^j}\right)^{i_j}\frac{1}{i_j!}\right), \tag{29}$$

with

$$b_2 = l/2, \ b_k = \left(l - \sum_{j=2}^{k-1} j i_j \right) / k, \tag{30}$$

$$a_k = 0 \ \text{ for } k = 2, 3, \ldots, l-1; \ a_l = \max\left(0, \left\lceil \frac{l - n - \sum_{j=2}^{l-1} (j-1) i_j}{l-1} \right\rceil \right), \tag{31}$$

$$\langle n, l, \{i_j\}_2^l \rangle = \frac{n!}{\left(n - l + \sum_{j=2}^{l} (j-1) i_j\right)! \left(l - \sum_{j=2}^{l} j i_j\right)!}. \tag{32}$$

The first product in (29) is understood to be ordered such that the sums have indices $i_2, i_3, \ldots, i_l$ when written from the left to the right. In addition: $C_0(n) = C_0^n$ and $C_1(n) = nC_0^{n-1}C_1$.

Proof: $C_l(n)$ of (28) is rewritten first as

$$C_l(n) = \Sigma(n, l, \{i_j\}_0^l) C_0^{i_0} C_1^{i_1} \cdots C_l^{i_l}, \ i_j \in \mathbf{N}_0, \tag{33}$$

where the sum is restricted by

$$(i)\text{: } \sum_{j=0}^{l} j i_j = l \ \text{ and } \ (ii)\text{: } \sum_{j=0}^{l} i_j = n. \tag{34}$$

$(n, l, \{i_j\}_2^l)$ is a combinatorial factor to be determined later on. (E.g., for $n = 3, l = 5$, one has five terms in the sum: $i_5 = 1, i_0 = 2$; $i_4 = 1, i_1 = 1, i_0 = 1$; $i_3 = 1, i_2 = 1, i_0 = 1$; $i_3 = 1, i_1 = 2$; $i_2 = 2, i_1 = 1$, with other indices vanishing, and the combinatorial factors are 3, 6, 6, 3, 3, respectively.) (ii) restricts the sum to terms with n factors, and (i) produces the correct weight l. These restrictions are solved by

$$(i')\text{: } i_1 = l - \sum_{j=2}^{l} j i_j \ \text{ and } \ (ii')\text{: } i_0 = n - i_1 - \sum_{j=2}^{l} i_j = n - l + \sum_{j=2}^{l} (j-1) i_j.$$

From $i_1 \geq 0$, i.e., $l - \sum_{j=2}^{l} j i_j \geq 0$, one infers $i_2 \leq \lfloor \frac{l}{2} \rfloor$; thus, $i_2 \in [0, \lfloor \frac{l}{2} \rfloor]$. For given i_2 in this range, $i_3 \leq \lfloor \frac{l - 2i_2}{2} \rfloor$, etc. In general,

$$0 \leq i_k \leq \left\lfloor \left(l - \sum_{j=2}^{k-1} j i_j \right) k \right\rfloor \ \text{ for } k = 2, 3, \ldots, l$$

with the sum replaced by zero for $k = 2$. This accounts for the upper boundaries $\lfloor b_k \rfloor$ in (30). Now, because $i_0 \geq 0$, (ii') implies a lower bound for i_l, the index of the last sum, *viz*

$$i_l \geq \left\lceil \left(l - n - \sum_{j=2}^{l-1} (j-1) i_j \right) / (l-1) \right\rceil$$

with the ceiling function $\lceil \cdot \rceil$. In any case $i_l \geq 0$; therefore, the lower boundary for the i_l-sum is a_l as given in (31). All restrictions have then been solved and the lower boundaries of the other sums are given by $a_k = 0$ for $k = 2, \ldots, l-1$. As to the combinatorial factor, it now depends only on $n, l, \{i_j\}_2^l$ and is written as $\langle n, l, \{i_j\}_2^l \rangle$. It counts the number of possibilities for the occurrence of the considered term of the sum which is given by

$$\binom{n}{i_0}\binom{n - i_0}{i_1} \cdots \binom{n - \sum_{j=0}^{l-1} i_j}{i_l} = n! / \left(\prod_{j=0}^{l} i_j! \right) \left(n - \sum_{j=0}^{l} i_j \right)!.$$

Inserting i_0 and i_1 from (ii') and (i'), respectively, remembering (ii), produces $\langle n, l, \{i_j\}_2^l\rangle$ as given in (32). Finally, $\sum\langle n, l, \{i_j\}_2^l\rangle C_0^{i_0}C_1^{i_1}\cdots C_l^{i_l}$ is transformed into $(l-1)$ nested sums with boundaries a_k and $\lfloor b_k\rfloor$ after replacement of i_1 and i_0. This completes the proof of (29) for the nontrivial $l\geq 2$ cases. □

Corollary 1–Catalan convolutions: For Catalan's sequence $\{C_n\}_0^\infty$, the n^{th} convolution sequence for $n\in\mathbf{N}$ is given by $C_0(n)=1$, $C_1(n)=n$ and, for $l=2,3,\ldots$, by

$$C_l(n)=\left(\prod_{k=2}^{l}\sum_{i_k=a_k}^{\lfloor b_k\rfloor}\right)\langle n, l, \{i_j\}_2^l\rangle\prod_{j=2}^{l}\left(\frac{C_j^{i_j}}{i_j!}\right), \tag{35}$$

with (30), (31), and (32).

Proof: This is Lemma 1 with $C_0=1=C_1$. □

Example 1: $C_4(3)=3C_4+5C_3+3C_2^2+3C_2=90.$

Corollary 2: With the Catalan generating function $c(x)$ and the definition, one has, for $n\in\mathbf{N}$, $c^{-n}(x)=:\sum_{l=0}^\infty C_l(-n)x^l$, for $l=2,3,\ldots$,

$$C_l(-n)=(-1)^l\left(\prod_{k=2}^{l}\sum_{i_k=a_k}^{\lfloor b_k\rfloor}\frac{(-1)^{(k-1)i_k}}{i_k!}\right)\langle n, l, \{i_j\}_2^l\rangle\prod_{j=2}^{l-1}C_j^{i_{j+1}}, \tag{36}$$

with (30), (31), (32), and Catalan numbers C_k. In addition, $C_0(-n)=1$, $C_1(-n)=-n$.

Proof: Lemma 1 is used for powers of $c(x)$ replaced by those of $c^{-1}(x)=1-xc(x)$, with the Catalan generating function $c(x)$. Hence, $c^{-1}(x)=\sum_{k=0}^\infty C_k(-1)x^k$ with

$$C_k(-1)=\begin{cases}1 & \text{for } k=0,\\ -C_{k-1} & \text{for } k=1,2,\ldots.\end{cases}$$ Then, in Lemma 1, C_k is replaced by $C_k(-1)$. □

Example 2: $C_4(-3)=-3C_3+6C_2-3+3=-3.$

Convolutions of Catalan's sequence have been encountered in various contexts, for example, in the enumeration of nonintersecting path pairs on a square lattice (see [12], [18], [5]), and in the problem of inverting triangular matrices with Pascal triangle entries (see [6] and earlier works cited there; they also appear in [15], p. 148).

Note 4: Shapiro's Catalan triangle has entries

$$B_{n,k}=\frac{k}{n}\binom{2n}{n-k}\ \text{for } n\geq k\geq 1,\ \text{and}\ B_{n,k}=[x^n](x^k\hat{c}^k(x)),$$

with $[x^n]f(x)$ denoting the coefficient of x^n in the expansion of $f(x)$ around $x=0$. In this case, $\hat{c}(x)=(c(x)-1)/x=c^2(x)$. (See [12], Propositions (2.1) and (3.3), with $i_j\in\mathbf{N}$, *not* $\mathbf{N}_0$.) In [18] this triangle of numbers from [12] reappears as $b(n,k)$, and it is shown there that $B_{n,k}\equiv b(n,k)=[x^n](xc^2(x))^k$, in accordance with the identity $\hat{c}(x)=c^2(x)$. Therefore, only even powers of $c(x)$ appear in Shapiro's Catalan triangle. In [5], $C_l(n)$ appears as special case ${}_2d_{2-n,l+1}$. In [6], all powers of $c(x)$ show up as convolutions for the special case of the S_1 sequence there. The entries of the S_1-array ([6], p. 397) are $[x^n]c^{k+1}(x)$ for $n,k\in\mathbf{N}_0$.

The anonymous referee of this paper noticed that the inverse of the lower triangular matrix $S_{n,k} = [x^k]S_n(x)$, for $n, k \in \mathbf{N}_0$, with Chebyshev's $S_n(x) = U_n(x/2)$ polynomials is the lower triangular convolution matrix obtained from its first ($k = 0$) column sequence generated by $c(x^2)$ (Catalan numbers alternating with zeros). This follows from the fact that the **S**-matrix is also a lower triangular convolution matrix with generating function $1/(1+x^2)$ of its first column. See [13] for such type of matrices **M** and the relation between the generating functions of the first columns of **M** and $\mathbf{M}^{-1}$. The head of this Catalan triangle can be viewed under number A053121 in the on-line database [14]. See also [6] for inverses of Pascal-type arrays.

Lemma 2–Explicit form of Catalan convolutions [12], [18], [6], [4], [11], and [5]:

For $n \in \mathbf{R}$, $l \in \mathbf{N}_0$:

$$C_l(n) = \frac{n}{l}\binom{2l+n-1}{l-1} = \frac{n}{n+2l}\binom{n+2l}{l} = \frac{n}{l+n}\binom{2l+n-1}{l}. \tag{37}$$

Proof: Three equivalent expressions have been given for convenience. See [4], page 201, equation (5.60), with $\mathcal{B}_2(z) = c(z)$, $t \to 2, k \to l, r \to n$. The proof of (5.60) appears as (7.69) on page 349 of [4], with $m = 2$, $n = l \in \mathbf{R}$.

The same formula occurs as Exercise 213 in Vol. 1 of [11] for $\beta = 2$ as a special instance of Exercises 211 and 212. Put $\alpha = n$ and $n = l$ in the solution of Exercise 213 on page 301.

In order to prove this lemma from [12] or [18], one can use

$$C_l(n) = \sum_{j=0}^{\min(l,\,n)} \binom{n}{j} \hat{C}_l(j)$$

obtained from $c(x) =: 1 + \hat{c}(x)$ with

$$\hat{c}^n(x) =: \sum_{k=n}^{\infty} \hat{C}_k(n) x^{k-n}.$$

The result in [12] and [18] is, with this notation,

$$\hat{C}_l(j) = B_{l,j} = b(l, j) = \frac{j}{l}\binom{2l}{l-j}.$$

Inserting this in the given sum, making use of the identity $j\binom{n}{j} = n\binom{n-1}{j-1}$ and the Vandermonde convolution identity, leads to Lemma 2 at least for positive integer n, but one can continue this formula to real (or complex) n.

In [6], one finds this result as equation (3.1), page 402, for $i = 1$: $s_1(l, n) = C_l(n)$.

In [5], ${}_2d_{2-n,\,l+1} = C_l(n)$, with the result given in Theorem 2.3, equation (2.6), page 71. □

Note 5: As a side remark we mention that, from (37), $E_l(x) := l!\,C_l(x)$ (with real $n = x$) is a polynomial of degree l, *viz* $\prod_{j=0}^{l-1}(x + l + 1 + j)$. These polynomials, which are not the subject of this work, are known (see [8] and references given there) as exponential convolution polynomials satisfying $E_l\,(x+y) = \sum_{k=0}^{l}\binom{l}{k}E_k(x)E_{l-k}(y)$.

We now compute the coefficients $C_l(n) = [x^l]c^n(x)$ (see Note 4 for this notation) from our formula given in Proposition 1. This can be done for $n \in \mathbf{Z}$.

First, consider $n \in \mathbf{N}_0$. For $n = 0$ and $n = 1$, there is nothing new due to the inputs $S_{-2} = -1$, $S_{-1} = 0$, and $S_0 = 1$. $C_l(n) = 0$ for negative integer l. Therefore, terms proportional to $1/x^l$ with

$l \in \mathbf{N}$ have to cancel in (23), or in (1). For $n = 2, 3, \ldots$, terms of the type $1/x^{n-j}$ occur for $j \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The coefficient of $1/x^{n-j}$ in $p_{n-1}(x)$ is $(-1)^j \binom{n-1-j}{j-1}$ (see Note 1 for the explicit form of p_{n-1}). For the $1/x^{n-j}$ coefficient in $q_{n-1}(x)c(x)$, one finds the convolution

$$\sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{n-(j-l)}{j-l-1} C_l.$$

Compensation of both coefficients leads to identity $(P1)$ given in (4) after $(j-1)$ has been traded for p. Thus, after a shift $n \to n+2$,

Proposition 2–Identity (P1): For $n \in \mathbf{N}_0$ and $p = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, identity $(P1)$, given in equation (4), holds.

Example 3: $n = 2(k-1)$, $p = k-1$, and $n = 2k-1$, $p = k-1$ for $k \in \mathbf{N}$;

$$\sum_{l=0}^{k-1} (-1)^l \binom{k+l}{2l+1} C_l = 1, \quad \sum_{l=0}^{k-1} (-1)^l \binom{k+l+1}{2(l+1)} C_l = k;$$

e.g., $k = 3$: $3C_0 - 4C_1 + 1C_2 = 1$, $6C_0 - 5C_1 + 1C_2 = 3$.

For $n = 2, 3, \ldots$, terms in (1), or in (23), proportional to x^k with $k \in \mathbf{N}_0$ arise only from $q_{n-1}(x)c(x)$, and they are given by the convolution (cf. Note 1),

$$\sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^l \binom{n-1-l}{l} C_{k+n-1-l}.$$

For $n = 1$, this is C_k. The left-hand side of (1) contributes $C_k(n)$, and $C_k(1) = C_k$. Therefore,

Proposition 3–Identity (P3): For $n \in \mathbf{N}$, $k \in \mathbf{N}_0$, identity $(P2)$, given in equation (5) with (3) holds.

Example 4: $k = 0$, $(n-1) \to n$:

$$\sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{l+1} \binom{n-l}{l} C_{n-l} = C_n - 1;$$

e.g., $n = 3$: $2C_2 = C_3 - 1$, $n = 4$: $3C_3 - 1C_2 = C_4 - 1$.

Now consider negative powers in (1), i.e., $c^{-n}(x)$, $n \in \mathbf{N}$. No negative powers of x appear (cf. Note 1 for the explicit form of $p_{-(n+1)}(x)$ and $q_{-(n+1)}(x)$). The coefficient of x^k, $k \in \mathbf{N}_0$, of the right-hand side of (1) is

$$(-1)^k \binom{n-k}{k} - \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^l \binom{n-1-l}{l} C_{k-1-l},$$

where the first term, arising from $p_{-(n+1)}(x)$, contributes only for $k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. In the summation, one also needs $l \le k-1$ because no Catalan numbers with negative index occur in (1). The left-hand side of (1) has $[x^k]c^{-n}(x) = C_k(-n)$. From the last equation in (37), one finds

$$C_k(-n) = \frac{n}{n-k} \binom{2k-n-1}{k} = (-1)^k \frac{n}{n-k} \binom{n-k}{k}.$$

In the last equation, the upper index in the binomial has been negated (cf. [4], (5.14)). Two sets of identities follow, depending on the range of k.

Proposition 4–Identity (P3): For $n \in \mathbf{N}$, $k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, identity $(P3)$, given in equation (6), holds.

Example 5: $k = 3, n \geq 6$: $C_2 - (n-2)C_1 + \binom{n-3}{2}C_0 = \binom{n-4}{2}$.

Proposition 5–Identity (P4): For $n \in \mathbf{N}$, $k \in \mathbf{N}$, with $k \geq \lfloor \frac{n}{2} \rfloor + 1$, identity $(P4)$, given in equation (7), holds.

In $(P4)$, only the $q_{-(n+1)}(x)c(x)$ part of (1) contributed, and we used the first expression for $C_k(-n)$ in (37). In $(P3)$, where $p_{-(n+1)}(x)$ also contributed, we used the negated binomial coefficient for $C_l(-n)$ and absorption in the resulting one.

Note that (37) implies $C_k(-n) = -C_{k-n}(n)$ for $k, n \in \mathbf{N}$, $k \geq n$, and $C_k(0) = \delta_{k,0}$.

Example 6: $n = 5, k \geq 3$:

$$C_{k-1} - 3C_{k-2} + C_{k-3} = \frac{5}{k}\binom{2k-6}{k-1};$$

e.g., $k = 7$: $C_6 - 3C_5 + C_4 = 20$.

If one uses the binomial formula for $c^{-n}(x) = (1 - xc(x))^n$ and $c^n(x) = \sum_{k=0}^{\infty} C_k(n)x^k$, one arrives at equation (8).

3. SOME FAMILIES OF INTEGER SEQUENCES

In this section we present some sequences of positive integers which are defined with the help of the $\mathcal{U}_n$ polynomials (10).

$$u_n(m) := \mathcal{U}_n(1/m) = (\sqrt{m})^n S_n(\sqrt{m}). \tag{38}$$

The last equation is due to (21). It will be shown that $u_n(m)$ is a nonnegative integer for each $m = 4, 5, \ldots$ and $n = -1, 0, \ldots$. Also negative integers $-m$, $m \in \mathbf{N}$ are of interest. In this case, we add a sign factor:

$$v_n(m) := (-1)^n \mathcal{U}_n(-1/m) = (-i\sqrt{m})^n S_n(i\sqrt{m}). \tag{39}$$

From the S_n recursion relation (15), one infers those for the $u_n(m)$ and $v_n(m)$ sequences:

$$u_n(m) = m(u_{n-1}(m) - u_{n-2}(m)), \quad u_{-1}(m) \equiv 0, \ u_0(m) \equiv 1; \tag{40}$$

$$v_n(m) = m(v_{n-1}(m) + v_{n-2}(m)), \quad v_{-1}(m) \equiv 0, \ v_0(m) \equiv 1. \tag{41}$$

This shows that $v_n(m)$ constitutes a nonnegative integer sequence for positive integer m. It describes certain generalized Fibonacci sequences (see, e.g., [7] with $v_n(m) = W_{n+1}(0, 1; m, m)$). For example, $v_n(m)$ counts the length of the binary word $W(m; n)$ obtained at step n from the substitution rule $1 \to 1^m 0$, $0 \to 1^m$, starting at step $n = 0$ with 0. The number of 1's, resp. 0's, in $W(m; n)$ is $mv_{n-1}(m)$, resp. $mv_{n-2}(m)$. E.g., $W(2; 3) = (110)^2 1^2 (110)^2 1^2$ and $v_3(2) = 16$, $2v_2(2) = 12$, and $2v_1(2) = 4$. For $m = 1$, this substitution rule produces the well-known Fibonacci-tree. Of course, one can define in a similar manner generalized Lucas sequences using the polynomials $\{\mathcal{V}_n\}$ given in (11). Each $u_n(m)$ sequence (which is identified with $W_{n+1}(0, 1; m, -m)$ of [7]) turns out to be composed of two simpler sequences, *viz* $u_{2k}(m) =: m^k \alpha_k(m)$ and $u_{2k-1}(m) =: m^k \beta_k(m)$, $k \in \mathbf{N}_0$. These new sequences, which are due to (38), given by $\alpha_k = S_{2k}(\sqrt{m})$ and $\beta_k(m) = S_{2k-1}(\sqrt{m}) / \sqrt{m}$, satisfy therefore the following relations:

$$\beta_{k+1}(m) = (m-2)\beta_k(m) - \beta_{k-1}(m), \; \beta_0(m) \equiv 0, \beta_1(m) \equiv 1, \tag{42}$$

and

$$\alpha_{k-1}(m) = \beta_k(m) + \beta_{k-1}(m). \tag{43}$$

From (42) it is now clear that $\beta_n(m)$ is a nonnegative integer sequence for $m = 4, 5, \ldots$. (In [7], $\beta_n(m) = W_n(0, 1; m-2, -1)$.) This property is then inherited by the $\alpha_n(m)$ sequences due to (43), and then by the composed sequence $u_n(m)$.

The ordinary generating functions are:

$$g_\beta(m; x) := \sum_{n=0}^{\infty} \beta_n(m) x^n = \frac{x}{x^2 - (m-2)x + 1}, \; g_\alpha(m; x) := \sum_{n=0}^{\infty} \alpha_n(m) x^n = \frac{1+x}{x^2 - (m-2)x + 1}; \tag{44}$$

$$g_u(m; x) := \sum_{n=0}^{\infty} u_n(m) x^n = \frac{1}{1 - mx + mx^2}, \qquad g_v(m; x) := \sum_{n=0}^{\infty} v_n(m) x^n = \frac{1}{1 - mx - mx^2}. \tag{45}$$

Note 6: The $\{\beta_n(m)\}$ sequences for $m = 4, 5, 6, 7, 8, 10$ appear in the book [14]. The case $m = 4$ produces the sequence of nonnegative integers; $m = 5$ are the even-indexed Fibonacci numbers. The $m = 9$ sequence appears in Sloane's "On-Line-Encyclopedia" [14] as A004187. The $\{\alpha_n(m)\}$ sequences for $m = 4, 5, 6, 8$ also appear in the book [14]. The case $m = 4$ yields the positive odd integer sequence; $m = 5$ is the odd-indexed Lucas number sequence. The $m = 7$ sequence appears in the database [14] as A030221. The composed sequences $\{u_n(m)\}$ do not appear in the book [14], but some of them are found in the database [14]. $m = 4$ is the sequence $(n+1)2^n$, A001787, and $m = 5, 6, 7$ appear as A030191, A030192, and A030140, respectively. As mentioned above, $\{v_{n-1}(1)\}$ is the Fibonacci sequence. The instances $m = 2$ and 3 appear as A002605 and A030195, respectively, in the database [14].

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THE INTEGRITY OF SOME INFINITE SERIES

Feng-Zhen Zhao
Department of Applied Mathematics, Dalian University of Technology
116024 Dalian, China

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1. INTRODUCTION

Consider a sequence $\{W_n\}$ defined by the recurrence relation

$$W_n = pW_{n-1} - qW_{n-2},\ n \geq 2,\ W_0 = a,\ W_1 = b,$$

where a, b, p, and q are integers with $p > 0$, $q \neq 0$, and $\Delta = p^2 - 4q > 0$. We are interested in the following two special cases of $\{W_n\}$: $\{U_n\}$ is defined by $U_0 = 0$, $U_1 = 1$, and $\{V_n\}$ is defined by $V_0 = 2$, $V_1 = p$. It is well known that $\{U_n\}$ and $\{V_n\}$ can be expressed in the form

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ V_n = \alpha^n + \beta^n, \tag{1.1}$$

where $\alpha = (p + \sqrt{\Delta})/2$, $\beta = (p - \sqrt{\Delta})/2$. Especially, if $p = -q = 1$, $\{U_n\}$ and $\{V_n\}$ are the usual Fibonacci and Lucas sequences.

Recently, André-Jeannin studied the infinite sum (see [1])

$$S_V(x) = S_V(x; p, q) = \sum_{n=0}^{\infty} \frac{V_n}{x^n}.$$

At the end of his paper, he presented a problem to study the integrity of an infinite sum

$$T_k(x) = \sum_{n=0}^{\infty} \frac{V_{kn}}{x^n},\ k > 0.$$

In this paper, we solve this problem completely for the case in which $k \geq 2$ and $q = \pm 1$.

By using the Binet forms (1.1) and the geometric series formula, we have

$$T_k(x) = \frac{x(2x - V_k)}{x^2 - V_k x + q^k},\ |x| > \alpha^k.$$

In what follows, we shall make use of the identities:

$$U_{n+2k} - q^k U_n = U_k V_{n+k}, \tag{1.2}$$

$$U_{n+2k} + q^k U_n = V_k U_{n+k}, \tag{1.3}$$

$$V_{2n} + 2q^n = V_n^2, \tag{1.4}$$

$$V_{2n} - 2q^n = \Delta U_n^2, \tag{1.5}$$

$$U_{2n} = U_n V_n, \tag{1.6}$$

$$V_k V_{2k(n+1)} + \Delta U_k U_{2k(n+1)} = 2V_{k(2n+3)}. \tag{1.7}$$

All the identities can be obtained by the Binet form (1.1).

2. MAIN RESULTS

Let

$$V_n' = V_{kn} = \alpha^{kn} + \beta^{kn},\ U_n' = \frac{\alpha^{kn} - \beta^{kn}}{\alpha^k - \beta^k} = \frac{U_{kn}}{U_k}. \tag{2.1}$$

In fact, the sequences $\{U_n'\}$ and $\{V_n'\}$ satisfy the recurrence relation $W_n = V_k W_{n-1} - q^k W_{n-2}$. From (2.1) and applying Theorems 2-4 in [1] to the sequence $\{V_n'\}$, we can obtain the main results of this paper.

Theorem 1: If $q = \pm 1$ and $k \geq 2$, there do not exist negative rational values such that $T_k(x)$ is an integer.

Theorem 1 is a direct consequence of Theorem 2 in [1], and its proof is omitted.

Theorem 2: If $q = -1$ and $r \geq 0$, the positive rational values of x for which $T_{2r+1}(x)$ is integral are given by

$$x = \frac{U_{(2r+1)(2m+1)}}{U_{2m(2r+1)}} \quad (m = 1, 2, \ldots)$$

and

$$x = \frac{V_{(2r+1)(2m+2)}}{V_{(2r+1)(2m+1)}} \quad (m = 0, 1, 2, \ldots).$$

The corresponding values of $T_{2r+1}(x)$ are given by

$$T_{2r+1}\left(\frac{U_{(2r+1)(2m+1)}}{U_{2m(2r+1)}}\right) = \frac{U_{(2r+1)(2m+1)}V_{2m(2r+1)}}{U_{2r+1}}$$

and

$$T_{2r+1}\left(\frac{V_{(2r+1)(2m+2)}}{V_{(2r+1)(2m+1)}}\right) = \frac{U_{(2r+1)(2m+1)}V_{(2r+1)(2m+2)}}{U_{2r+1}}.$$

Proof: Since $q = -1$, we can apply Theorem 3 in [1] to the sequence $\{V_n' = V_{(2r+1)n}\}$. Therefore, the positive rational values of x for which $T_{2r+1}(x)$ is integral are given by

$$x = \frac{U_{2m+1}'}{U_{2m}'} \quad (m = 1, 2, \ldots)$$

and

$$x = \frac{V_{2m+2}'}{V_{2m+1}'} \quad (m = 0, 1, 2, \ldots).$$

The corresponding values of $T_{2r+1}(x)$ are given by

$$T_{2r+1}\left(\frac{U_{2m+1}'}{U_{2m}'}\right) = U_{2m+1}'V_{2m}'$$

and

$$T_{2r+1}\left(\frac{V_{2m+2}'}{V_{2m+1}'}\right) = U_{2m+1}'V_{2m+2}'.$$

From (2.1), we can obtain the results. □

Theorem 3: If $q=1$, $p\geq 3$, and $r\geq 0$, the positive rational values of x for which $T_{2r+1}(x)$ is integral are given by

$$x=\frac{U_{(2r+1)(m+1)}}{U_{m(2r+1)}}\quad (m=1,2,\dots)$$

and

$$x=\frac{X_{(2r+1)(m+1)}}{X_{m(2r+1)}}\quad (m=0,1,2,\dots),$$

where $X_{m(2r+1)}=U_{(2r+1)(m+1)}+U_{m(2r+1)}$. The corresponding values of $T_{2r+1}(x)$ are given by

$$T_{2r+1}\left(\frac{U_{(2r+1)(m+1)}}{U_{m(2r+1)}}\right)=\frac{U_{(2r+1)(m+1)}V_{m(2r+1)}}{U_{2r+1}}$$

and

$$T_{2r+1}\left(\frac{X_{(2r+1)(m+1)}}{X_{m(2r+1)}}\right)=\frac{(U_{(2r+1)(m+1)}-U_{m(2r+1)})X_{(2r+1)(m+1)}}{U_{2r+1}^2}.$$

Proof: By $q=1$ and $p\geq 3$, we have that

$$V_{2r+1}=\alpha^{2r+1}+\alpha^{-(2r+1)}\geq 3.$$

Similarly to the proof of the last theorem, we can prove the conclusion from Theorem 4 in [1]. □

Theorem 4: If $q=-1$ and $r\geq 1$, the positive rational values of x for which $T_{2r}(x)$ is integral are given by

$$x=\frac{U_{r(m+2)}}{U_{rm}}\quad (r \text{ or } m \text{ even and } m\geq 1)$$

and

$$x=\frac{V_{r(2m+3)}}{V_{r(2m+1)}}\quad (r \text{ odd and } m\geq 0).$$

The corresponding values of $T_{2r}(x)$ are given by

$$T_{2r}\left(\frac{U_{r(m+2)}}{U_{rm}}\right)=\frac{U_{r(m+2)}V_{rm}}{U_{2r}}$$

and

$$T_{2r}\left(\frac{V_{r(2m+3)}}{V_{r(2m+1)}}\right)=\frac{U_{r(2m+1)}V_{r(2m+3)}}{U_{2r}}.$$

Proof: Apply Theorem 4 in [1] to the sequence $\{V'_n=V_{2rn}\}$. Therefore, the positive rational values of x for which $T_{2r}(x)$ is integral are given by

$$x=\frac{U'_{m+1}}{U'_m}\quad (m=1,2,\dots)$$

and

$$x=\frac{X'_{m+1}}{X'_m}\quad (m=0,1,\dots),$$

where $X'_m=U'_{m+1}+U'_m$. The corresponding values of $T_{2r}(x)$ are given by

$$T_{2r}\left(\frac{U'_{m+1}}{U'_m}\right) = U'_{m+1}V'_m$$

and

$$T_{2r}\left(\frac{X'_{m+1}}{X'_m}\right) = X'_{m+1}(U'_{m+1} - U'_m).$$

From (2.1), we have

$$\frac{U'_{m+1}}{U'_m} = \frac{U_{2r(m+1)}}{U_{2rm}} \quad \text{and} \quad X'_{m+1} = \frac{U_{2r(m+2)} + U_{2r(m+1)}}{U_{2r}}.$$

It follows from (1.2) and (1.3) that

$$X'_{m+1} = \frac{U_{2r}V_{2r(m+1)} + V_{2r}U_{2r(m+1)} + 2U_{2r(m+1)}}{2U_{2r}}.$$

From (1.2)-(1.7), we have

$$X'_{m+1} = \begin{cases} \frac{U_{r(2m+3)}}{U_r} & r \text{ is even,} \\ \frac{V_{r(2m+3)}}{V_r} & r \text{ is odd.} \end{cases}$$

Using a similar method, we have

$$X'_m = \begin{cases} \frac{U_{r(2m+1)}}{U_r} & r \text{ is even,} \\ \frac{V_{r(2m+3)}}{V_r} & r \text{ is odd.} \end{cases}$$

It follows from (1.2) and (1.6) that

$$U'_{m+1} - U'_m = \frac{U_{2r(m+1)} - U_{2rm}}{U_{2r}} = \frac{U_rV_{r(2m+1)} + ((-1)^r - 1)U_{2rm}}{U_{2r}}$$

$$= \begin{cases} \frac{V_{r(2m+1)}}{V_r} & r \text{ is even,} \\ \frac{U_rV_{r(2m+1)} - 2U_{2rm}}{U_{2r}} & r \text{ is odd.} \end{cases}$$

When $q = -1$ and r is odd, from (1.2) and (1.3) we have

$$U'_{m+1} - U'_m = \begin{cases} \frac{V_{r(2m+1)}}{V_r} & r \text{ is even,} \\ \frac{U_{r(2m+1)}}{U_r} & r \text{ is odd.} \end{cases}$$

Therefore, the positive rational values of x for which $T_{2r}(x)$ is integral are given by

$$x = \frac{U_{2r(m+1)}}{U_{2rm}} \ (m \geq 1), \tag{2.2}$$

$$x = \frac{U_{r(2m+3)}}{U_{r(2m+1)}} \ (r \text{ even and } m \geq 0), \tag{2.3}$$

and

$$x = \frac{V_{r(2m+3)}}{V_{r(2m+1)}} \ (r \text{ odd and } m \geq 0).$$

The corresponding values of $T_{2r}(x)$ are given by

$$T_{2r}\left(\frac{U_{2r(m+1)}}{U_{2rm}}\right)=\frac{U_{2r(m+1)}V_{2rm}}{U_{2r}}, \tag{2.4}$$

$$T_{2r}\left(\frac{U_{r(2m+3)}}{U_{r(2m+1)}}\right)=\frac{U_{r(2m+3)}V_{r(2m+1)}}{U_{2r}} \quad (r \text{ even}), \tag{2.5}$$

and

$$T_{2r}\left(\frac{V_{r(2m+3)}}{V_{r(2m+1)}}\right)=\frac{U_{r(2m+1)}V_{r(2m+3)}}{U_{2r}} \quad (r \text{ odd}).$$

It is clear that (2.2) and (2.3) can be rewritten as

$$x=\frac{U_{r(m+2)}}{U_{rm}}.$$

Similarly, (2.4) and (2.5) can be rewritten as

$$T_{2r}\left(\frac{U_{r(m+2)}}{U_{rm}}\right)=\frac{U_{r(m+2)}V_{rm}}{U_{2r}}.$$

On the other hand, since $q=-1$, $\frac{U_{r(m+2)}}{U_{rm}}>\alpha^{2r}$ holds when m or r is even. Hence, the conclusions are valid. □

Theorem 5: If $q=1$, $p\geq 3$, and $r\geq 1$, the positive rational values of x for which $T_{2r}(x)$ is integral are given by

$$x=\frac{U_{r(m+2)}}{U_{rm}} \quad (m=1,2,\dots),$$

and the corresponding values of $T_{2r}(x)$ are given by

$$T_{2r}\left(\frac{U_{r(m+2)}}{U_{rm}}\right)=\frac{U_{r(m+2)}V_{rm}}{U_{2r}}.$$

Proof: Since $q=1$ and $V_{2r}\geq 3$, we can apply Theorem 4 in [1] to the sequence $\{V_n'=V_{2rn}\}$. The proof is similar to the one of the last theorem. □

Clearly, André-Jeannin's results are special cases of Theorems 2 and 3.

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EXTRACTION PROBLEM OF THE PELL SEQUENCE

Wai-Fong Chuan

Dept. of Math., Chung-yuan Christian University, Chung Li, Taiwan 32023, R.O.C.

Fei Yu

Yuanpei Technical College, 306 Yuanpei St., Hsinchu City, Taiwan, R.O.C.

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1. INTRODUCTION

Let A be an alphabet and let A^* be the free monoid over A. Let $A^+ = A^* \setminus \{\varepsilon\}$, where ε denotes the empty word. For $w \in A^*$, let $|w|$ denote the length of w. Let $|\varepsilon| = 0$. A word x is said to be a *prefix* of a finite or infinite word w over A if $x \in A^+$ and there is a word y such that $w = xy$. The finite or infinite word y is called a *suffix* of w. Let R be the *reversion operator* on A^+ defined by $R(c_1c_2\dots c_n) = c_n\dots c_2c_1$, where $c_i \in A$, $1 \le i \le n$, $n \ge 1$.

Let α be an irrational number between 0 and 1. The *characteristic sequence* (or word) of α is an infinite binary sequence f whose n^{th} term is $[(n+1)\alpha] - [n\alpha]$, $n \ge 1$. It will be regarded as an infinite word over the alphabet $\{0, 1\}$. Let s_m denote the prefix of f of length m and let f_m denote the suffix of f with $f = s_m f_m$, $m > 0$. Let $f_0 = f$. The characteristic sequence of $(\sqrt{5}-1)/2$ (resp., $\sqrt{2}-1$) is called the *golden sequence* (resp., *Pell sequence*).

Hofstadter [9] introduced the concept of aligning two words u and v over A (see also [3], [8]). The idea is to try to match each term (letter) in v with a term in u. After a term in v has been matched with a term in u, one looks for the earliest match to the next term in v. Those terms in u that are skipped over form the extracted word $\langle u, v\rangle$. The following example illustrates this concept.

$$\begin{array}{rccccccccccc} u: & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ v: & & 1 & 1 & & 0 & & 0 & & 1 & & 0 \\ \langle u, v\rangle: & 0 & & & 1 & & 1 & & 0 & & 1 & \end{array}$$

Originally, Hofstadter considered the problem of aligning f_m with f, where f is the characteristic sequence of an irrational number α. He conjectured that $\langle f_m, f\rangle = f_{m-2}$, $m \ge 2$. For $\alpha = (\sqrt{5}-1)/2$, Hendel and Monteferrante [8] solved this problem completely. They determined the set M of all integers $m \ge 2$ for which $\langle f_m, f\rangle = f_{m-2}$ and they proved that, if $m \ge 2$ and $m \notin M$, then $\langle f_m, f\rangle = 0f_{m-1}$. For example, $\langle f_5, f\rangle = f_3$ and $\langle f_4, f\rangle = 0f_3 \ne f_2$. The extraction problems $\langle f, f_m\rangle$ and $\langle f_m, f_n\rangle$ were first considered by Chuan [3] who proved that $\langle f, f_m\rangle = R(s_m)$, $m \ge 1$, and that $\langle f_m, f_n\rangle$ differs either from $\langle f_{m-n}, f\rangle$ (if $m > n$) or from $\langle f, f_{n-m}\rangle$ (if $m < n$) by at most the first letter. Using a concatenation lemma (Lemma 3 of [8]) and some representation theorems (Section IV of [7]), Hendel [7] also formulated and proved an extraction conjecture for $\langle f_m, f\rangle$ and $\langle f, f_m\rangle$ when $\alpha = \sqrt{2}-1$, for an infinite set of m.

In this paper, we shall use a special case of a powerful representation theorem that Chuan discovered recently [5] to prove that the following conjecture is true when $\alpha = \sqrt{2}-1$.

Conjecture: Let α be an irrational number between 0 and 1 and let f be its characteristic sequence. Then $\langle f, f_m\rangle = R(s_m)$, $m \ge 1$.

It follows from the representation lemmas in Section 2 that this conjecture has an equivalent formulation described below. Let $[0, a_1+1, a_2, \ldots]$ be the continued fraction expansion of α. Define the sequence $\{u_n\}$ of words over the alphabet $\{0, 1\}$ by

$$u_0 = 0,\ u_1 = 10^{a_1},\ u_n = u_{n-2}u_{n-1}^{a_n} \quad (n \geq 2).$$

Equivalent Formulation (Subtraction Rule of Exponents): If $n \geq 1$, $r_1, r_2, \ldots$ is an infinite sequence of integers with $0 \leq r_i \leq a_i$ $(i \geq 1)$ and $r_i = 0$ $(i > n)$, then

$$\left\langle u_0u_1^2u_2^2 \ldots,\ u_0^{1-r_1}u_1^{2-r_2}u_2^{2-r_3} \ldots \right\rangle = u_0^{r_1}u_1^{r_2} \ldots u_{n-1}^{r_n}.$$

2. PRELIMINARIES

Let $u = a_1a_2 \ldots a_n$, $v = b_1b_2 \ldots b_m$, and $e = c_1c_2 \ldots c_p$, where a_i, b_j, $c_k \in A$, n, $m > 0$, $p \geq 0$, and $n = m + p$. As in [8], we say that *u aligns with v with extraction e* if there are integers $j_1, j_2, \ldots, j_p$ such that

$$u = (b_1 \ldots b_{j_1})c_1(b_{j_1+1} \ldots b_{j_2})c_2 \ldots c_p(b_{j_p+1} \ldots b_m),$$

where $0 \leq j_1 \leq j_2 \leq \cdots \leq j_p < m$ and $c_i \neq b_{j_i+1}$ for $1 \leq i \leq p$. Here $b_1 \ldots b_k = \varepsilon$ if $k < i$. This relationship is called an *alignment* and is denoted by $\langle u, v\rangle = e$. Clearly, we have $\langle u, u\rangle = \varepsilon$.

Let u, v, and e be (possibly infinite) words over A. If $\{u_n\}$, $\{v_n\}$, and $\{e_n\}$ are sequences of finite words such that $\langle u_n, v_n\rangle = e_n$, $\lim_{n\to\infty} u_n = u$, $\lim_{n\to\infty} v_n = v$, and $\lim_{n\to\infty} e_n = e$, we say that *u aligns with v with extraction e*. This alignment is also denoted by $\langle u, v\rangle = e$.

The goal of this paper is to prove the following theorem.

Theorem 2.1: ***(a)*** Let $\alpha = \sqrt{2} - 1$ and let f be the characteristic sequence of α. Then $\langle f, f_m\rangle = R(s_m)$ for all $m \geq 1$.

(b) (Subtraction rule of exponents) If $n \geq 1$, $r_1, r_2, \ldots$ is an infinite sequence of integers with $0 \leq r_1 \leq 1$, $0 \leq r_i \leq 2$ $(2 \leq i \leq n)$, and $r_i = 0$ $(i > n)$, then

$$\left\langle u_0u_1^2u_2^2 \ldots, u_0^{1-r_1}u_1^{2-r_2}u_2^{2-r_3} \cdots \right\rangle = u_0^{r_1}u_1^{r_2} \ldots u_{n-1}^{r_n}.$$

To prove this theorem, we need the following concatenation lemma and three basic representation lemmas (Lemmas 2.3-2.5).

Lemma 2.2 (see [8]): If $p > 1$, u_n, v_n, $e_n \in A^+$ and $\langle u_n, v_n\rangle = e_n$, $1 \leq n \leq p$, then

$$\left\langle \prod_{n=1}^{p} u_n, \prod_{n=1}^{p} v_n \right\rangle = \prod_{n=1}^{p} e_n.$$

Here $\prod_{n=1}^{p} x_n$ denotes $x_1x_2 \ldots x_p$, where $x_1, x_2, \ldots, x_p \in A^+$. The result also holds if u_p and v_p are infinite words.

Throughout the rest of this section, let α be an irrational number between 0 and 1 with continued fraction $\alpha = [0, a_1+1, a_2, \ldots]$ and let f be its characteristic sequence. Let

$$\begin{aligned} &q_0 = 1, && q_1 = a_1 + 1, && q_n = a_nq_{n-1} + q_{n-2}, \\ &x_0 = 0, && x_1 = 0^{a_1}1, && x_n = x_{n-1}^{a_n}x_{n-2}, \\ &u_0 = 0, && u_1 = 10^{a_1}, && u_n = u_{n-2}u_{n-1}^{a_n},\ n \geq 2. \end{aligned}$$

Note that $\{q_n\}$ is a sequence of positive integers and $\{x_n\}$ and $\{u_n\}$ are sequences of α-words over the alphabet $\{0, 1\}$ (see [4] for a definition of α-word) and $u_n = R(x_n)$, $n \geq 0$.

Lemma 2.3 (see [6]): Every positive integer m can be expressed uniquely as $m = \sum_{i=1}^{n} r_i q_{i-1}$, where $0 \leq r_i \leq a_i$ $(1 \leq i \leq n)$, $r_n \neq 0$, and $r_{i-1} = 0$ whenever $r_i = a_i$ $(2 \leq i \leq n)$.

The expression of m in Lemma 2.3 is called the *generalized Zeckendorf representation* of m in the q_i's. When $\alpha = (\sqrt{5}-1)/2 = [0, 1, 1, \ldots]$, it is the *Zeckendorf representation* and $q_i = F_{i+1}$. When $\alpha = \sqrt{2}-1 = [0, 2, 2, \ldots]$, it is also called the *Pellian representation* of m in the Pell numbers [2, 10, 11]. If $m = \sum_{i=1}^{n} r_i q_{i-1}$, where $0 \leq r_i \leq a_i$ $(1 \leq i \leq n)$, the sequence $r_1 r_2 \ldots r_n$ is called a *code* of m with respect to α (or the q_i's).

A representation of prefixes s_m of f in terms of the x_i's is given in the following lemma.

Lemma 2.4 (see [5]): Let $m = \sum_{i=1}^{n} r_i q_{i-1}$, where $0 \leq r_i \leq a_i$ $(1 \leq i \leq n)$. Then

$$s_m = x_{n-1}^{r_n} \ldots x_1^{r_2} x_0^{r_1} = R(u_0^{r_1} u_1^{r_2} \ldots u_{n-1}^{r_n}).$$

We remark that a special case of Lemma 2.4 in which the representation of m is the generalized Zeckendorf representation has been obtained by Brown [1].

In the following lemma, f and its suffixes f_m are expressed in terms of the u_n's.

Lemma 2.5 (see [5]): Let $m = \sum_{i=1}^{\infty} r_i q_{i-1}$, where $0 \leq r_i \leq a_i$ $(i \geq 1)$. Then

$$f = u_0^{a_1} u_1^{a_2} u_2^{a_3} \ldots,$$
$$f_m = u_0^{a_1 - r_1} u_1^{a_2 - r_2} u_2^{a_3 - r_3} \ldots.$$

Note that when $\alpha = (\sqrt{5}-1)/2$, the representations of f and f_m given here reduce to the ones used in [3] and [8].

3. PROOF OF THEOREM 2.1

In this section we restrict our attention to the irrational number $\alpha = \sqrt{2}-1 = [0, 2, 2, \ldots]$. The sequences $\{q_n\}$, $\{x_n\}$, and $\{u_n\}$ defined in Section 2 now become

$$\begin{aligned} &q_0 = 1, \quad q_1 = 2, \quad q_n = 2q_{n-1} + q_{n-2}, \\ &x_0 = 0, \quad x_1 = 01, \quad x_n = x_{n-1}^2 x_{n-2}, \\ &u_0 = 0, \quad u_1 = 10, \quad u_n = u_{n-2} u_{n-1}^2, \; n \geq 2. \end{aligned} \tag{1}$$

We first prove some alignments that involve the u_n's.

Lemma 3.1:

(a) $\langle u, u \rangle = \varepsilon$ for all finite or infinite word u over $\{0, 1\}$.
(b) $\langle u_{n-1} u_n, u_n \rangle = u_{n-1}$ $(n \geq 1)$.
(c) $\langle u_n^2, u_{n-1} u_n \rangle = u_{n-2} u_{n-1}$ $(n \geq 2)$.
(d) $\langle u_{n-1}^2 u_n^2, u_n^2 \rangle = u_{n-1}^2$ $(n \geq 2)$.
(e) $\langle u_0 u_1^2 \ldots u_n^2, u_1 \ldots u_{n-1} u_n^2 \rangle = u_0 u_1 \ldots u_{n-1}$ $(n \geq 1)$.
(f) $\langle u_n^2 u_{n+1}^2 \ldots u_{n+p}^2, u_n \ldots u_{n+p-1} u_{n+p}^2 \rangle = u_n u_{n+1} \ldots u_{n+p-1}$ $(n \geq 1, p \geq 1)$.
(g) $\langle u_n^2 u_{n+1}^2 \ldots u_{n+p}^2, u_{n+1} \ldots u_{n+p-1} u_{n+p}^2 \rangle = u_n^2 u_{n+1} \ldots u_{n+p-1}$ $(n \geq 1, p \geq 2)$.

Proof:

(a) By definition.

(b)-(d) Clearly, the results hold for $n \le 3$. Let $k \ge 3$. Suppose that (b)-(d) hold for all $n \le k$. Then:

(i) $\langle u_k u_{k+1}, u_{k+1} \rangle$
$= \langle u_{k-2}u_{k-1}, u_{k-1} \rangle \langle u_{k-1}u_{k-1}u_k u_k, u_k u_k \rangle$ [by (1) and Lemma 2.2]
$= u_{k-2}u_{k-1}u_{k-1}$ [by the inductive hypothesis of (b) and (d)]
$= u_k$.

(ii) $\langle u_{k+1}u_{k+1}, u_k u_{k+1} \rangle$
$= \langle u_{k-1}u_k, u_k \rangle \langle u_k u_{k+1}, u_{k+1} \rangle$ [by (1) and Lemma 2.2]
$= u_{k-1}u_k$ [by (i) and the inductive hypothesis of (b)].

(iii) $\langle u_k^2 u_{k+1}^2, u_{k+1}^2 \rangle$
$= \langle u_{k-2}u_{k-1}, u_{k-1} \rangle \langle u_{k-1}u_k, u_k \rangle \langle u_{k+1}^2, u_k u_{k+1} \rangle$ [by (1) and Lemma 2.2]
$= u_{k-2}u_{k-1}u_{k-1}u_k$ [by the inductive hypothesis of (b) and (ii)]
$= u_k^2$.

Therefore, (b)-(d) hold.

(e) $\langle u_0 u_1^2 \ldots u_n^2, u_1 u_2 \ldots u_{n-1} u_n^2 \rangle$
$= \langle u_0 u_1, u_1 \rangle \langle u_1 u_2, u_2 \rangle \cdots \langle u_{n-1}u_n, u_n \rangle \langle u_n, u_n \rangle$ [by (1) and Lemma 2.2]
$= u_0 u_1 \ldots u_{n-1}$ [by (b) and (a)].

(f) $\langle u_n^2 u_{n+1}^2 \ldots u_{n+p}^2, u_n u_{n+1} \ldots u_{n+p-1} u_{n+p}^2 \rangle$
$= \langle u_n, u_n \rangle \langle u_n u_{n+1}, u_{n+1} \rangle \langle u_{n+1}u_{n+2}, u_{n+2} \rangle$
$\cdots \langle u_{n+p-1}u_{n+p}, u_{n+p} \rangle \langle u_{n+p}, u_{n+p} \rangle$ [by (1) and Lemma 2.2]
$= u_n u_{n+1} \ldots u_{n+p-1}$ [by (a) and (b)].

(g) $\langle u_n^2 u_{n+1}^2 \ldots u_{n+p}^2, u_{n+1} \ldots u_{n+p-1} u_{n+p}^2 \rangle$

$$= \langle u_n^2, u_{n-1}u_n \rangle \left(\prod_{i=n}^{n+p-2} (\langle u_{i-1}u_i, u_i \rangle \langle u_i u_{i+1}, u_i u_{i+1} \rangle) \right) \langle u_{n+p}^2, u_{n+p-1}u_{n+p} \rangle \text{ [by (1) and Lemma 2.2]}$$

$$= x \left(\prod_{i=n}^{n+p-2} u_{i-1} \right) u_{n+p-2}u_{n+p-1} \text{ [by (a), (b), and (c)]}$$

$= u_n^2 u_{n+1} \ldots u_{n+p-1}$.

Here

$$x = \begin{cases} 1 & \text{if } n = 1, \\ u_{n-2}u_{n-1} & \text{if } n > 1. \end{cases}$$

Lemma 3.2: Let $n \ge 1$. Let $0 \le r_1 \le 1$, $0 \le r_i \le 2$ $(2 \le i \le n)$, $r_n \ne 0$, and $r_{i-1} = 0$ whenever $r_i = 2$ $(2 \le i \le n)$. Then

$$\langle u_0 u_1^2 \ldots u_n^2, u_0^{1-r_1} u_1^{2-r_2} \cdots u_{n-1}^{2-r_n} u_n^2 \rangle = u_0^{r_1} u_1^{r_2} \ldots u_{n-1}^{r_n}.$$

Proof: Write $r_1r_2\dots r_n = 0^{s_1}C_10^{s_2}C_2\dots 0^{s_m}C_m$, where $s_1 \ge 0$, $s_j \ge 1$ $(2 \le j \le m)$, each C_j is of the form 1^{t_j}, 2, or 21^{t_j} $(t_j \ge 1)$ and $C_1 = 1^{t_1}$ if $s_1 = 0$. We proceed by induction on m.

Let $m = 1$. For simplicity, write s for s_1 and t for t_1. There are four cases according to the values of s and t.

(i) $r_1r_2\dots r_{s+t} = 0^s1^t$ $(s > 0, t > 0)$:

$$\begin{aligned}&\langle u_0u_1^2\dots u_{s+t}^2,\, u_0u_1^2\dots u_{s-1}^2u_s\dots u_{s+t-1}u_{s+t}^2\rangle\\ &= \langle u_0u_1^2\dots u_{s-1}^2,\, u_0u_1^2\dots u_{s-1}^2\rangle\langle u_s^2\dots u_{s+t}^2,\, u_s\dots u_{s+t-1}u_{s+t}^2\rangle \quad \text{[by Lemma 2.2]}\\ &= u_su_{s+1}\dots u_{s+t-1} \quad \text{[by (a) and (f) of Lemma 3.1].}\end{aligned}$$

(ii) $r_1r_2\dots r_{s+1} = 0^s2$ $(s > 0)$:

$$\begin{aligned}&\langle u_0u_1^2\dots u_{s+1}^2,\, u_0u_1^2\dots u_{s-1}^2u_{s+1}^2\rangle\\ &= \langle u_0u_1^2\dots u_{s-1}^2,\, u_0u_1^2\dots u_{s-1}^2\rangle\langle u_s^2u_{s+1}^2,\, u_{s+1}^2\rangle \quad \text{[by Lemma 2.2]}\\ &= u_s^2 \quad \text{[by (a) and (d) of Lemma 3.1].}\end{aligned}$$

(iii) $r_1r_2\dots r_{s+t+1} = 0^s21^t$ $(s > 0, t > 0)$:

$$\begin{aligned}&\langle u_0u_1^2\dots u_{s+t+1}^2,\, u_0u_1^2\dots u_{s-1}^2u_{s+1}\dots u_{s+t}u_{s+t+1}^2\rangle\\ &= \langle u_0u_1^2\dots u_{s-1}^2,\, u_0u_1^2\dots u_{s-1}^2\rangle\langle u_s^2\dots u_{s+t+1}^2,\, u_{s+1}\dots u_{s+t}u_{s+t+1}^2\rangle \quad \text{[by Lemma 2.2]}\\ &= u_s^2u_{s+1}\dots u_{s+t} \quad \text{[by (a) and (g) of Lemma 3.1].}\end{aligned}$$

(iv) $r_1r_2\dots r_t = 1^t$ $(t > 0)$:

$$\begin{aligned}&\langle u_0u_1^2\dots u_t^2,\, u_1\dots u_{t-1}u_t^2\rangle\\ &= u_0u_1\dots u_{t-1} \quad \text{[by (e) of Lemma 3.1].}\end{aligned}$$

Thus, the result holds for $m = 1$. Now, suppose that the result holds for $m = k$, that is,

$$r_1r_2\dots r_n = 0^{s_1}C_10^{s_2}C_2\dots 0^{s_k}C_k, \text{ and}$$

$$\langle u_0u_1^2\dots u_n^2,\, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\rangle = u_0^{r_1}u_1^{r_2}\dots u_{n-1}^{r_n},$$

where $C_1, \dots, C_k$ satisfy the above-mentioned conditions. Let $s_{k+1} \ge 1$ and let $C_{k+1} = 1^t$, 2, or 21^t for some $t \ge 1$. There are three cases to consider:

(i) $C_{k+1} = 1^t$: Let $r_{n+1}r_{n+2}\dots r_p = 0^{s_{k+1}}1^t$, where $p = n + s_{k+1} + t$. Then

$$\begin{aligned}&\langle u_0u_1^2\dots u_p^2,\, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\dots u_{p-t-1}^2u_{p-t}\dots u_{p-1}u_p^2\rangle\\ &= \langle u_0u_1^2\dots u_n^2,\, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\rangle\langle u_{n+1}^2\dots u_{p-t-1}^2,\, u_{n+1}^2\dots u_{p-t-1}^2\rangle\\ &\qquad \langle u_{p-t}^2\dots u_p^2,\, u_{p-t}\dots u_{p-1}u_p^2\rangle \quad \text{[by Lemma 2.2]}\\ &= u_0^{r_1}u_1^{r_2}\dots u_{n-1}^{r_n}\varepsilon u_{p-t}\dots u_{p-1} \quad \text{[by (a), (f) of Lemma 3.1 and the inductive hypothesis]}\\ &= u_0^{r_1}u_1^{r_2}\dots u_{p-1}^{r_p}.\end{aligned}$$

(ii) $C_{k+1} = 2$: Let $r_{n+1}r_{n+2}\dots r_p = 0^{s_{k+1}}2$, where $p = n + s_{k+1} + 1$. Then

$$\begin{aligned}&\langle u_0u_1^2\dots u_p^2,\, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\dots u_{p-2}^2u_p^2\rangle\\ &= \langle u_0u_1^2\dots u_n^2,\, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\rangle\langle u_{n+1}^2\dots u_{p-2}^2,\, u_{n+1}^2\dots u_{p-2}^2\rangle\langle u_{p-1}^2u_p^2,\, u_p^2\rangle\end{aligned}$$

$$= u_0^{r_1} u_1^{r_2} \dots u_{n-1}^{r_n} \varepsilon u_{p-1}^2 \text{ [by (a), (d) of Lemma 3.1 and the inductive hypothesis]}$$
$$= u_0^{r_1} u_1^{r_2} \dots u_{p-1}^{r_p}.$$

(iii) $C_{k+1} = 21^t$: Let $r_{n+1} r_{n+2} \dots r_p = 0^{s_{k+1}} 21^t$, where $p = n + s_{k+1} + t + 1$. Then

$$\langle u_0 u_1^2 \dots u_p^2, u_0^{1-r_1} u_1^{2-r_2} \dots u_{n-1}^{2-r_n} u_n^2 \dots u_{p-t-2}^2 u_{p-t} \dots u_{p-1} u_p^2 \rangle$$
$$= \langle u_0 u_1^2 \dots u_n^2, u_0^{1-r_1} u_1^{2-r_2} \dots u_{n-1}^{2-r_n} u_n^2 \rangle \langle u_{n+1}^2 \dots u_{p-t-2}^2, u_{n+1}^2 \dots u_{p-t-2}^2 \rangle$$
$$\langle u_{p-t-1}^2 \dots u_p^2, u_{p-t} \dots u_{p-1} u_p^2 \rangle \text{ [by Lemma 2.2]}$$
$$= u_0^{r_1} u_1^{r_2} \dots u_{n-1}^{r_n} \varepsilon u_{p-t-1}^2 u_{p-t} \dots u_{p-1} \text{ [by (a), (g) of Lemma 3.1 and the inductive hypothesis]}$$
$$= u_0^{r_1} u_1^{r_2} \dots u_{p-1}^{r_p}.$$

This completes the proof.

Proof of Theorem 2.1: (a) Let $m = \sum_{i=1}^n r_i q_{i-1}$ be the generalized Zeckendorf representation of m in the q_i's. Define $r_k = 0$ $(k > n)$. Then

$$\langle f, f_m \rangle$$
$$= \langle u_0 u_1^2 u_2^2 \dots, u_0^{1-r_1} u_1^{2-r_2} u_2^{2-r_3} \dots \rangle \text{ [by Lemma 2.5]}$$
$$= \langle u_0 u_1^2 \dots u_n^2, u_0^{1-r_1} u_1^{2-r_2} \dots u_{n-1}^{2-r_n} u_n^2 \rangle \left\langle \prod_{k=n+1}^{\infty} u_k^2, \prod_{k=n+1}^{\infty} u_k^2 \right\rangle \text{ [by Lemma 2.2]}$$
$$= u_0^{r_1} u_1^{r_2} \dots u_{n-1}^{r_n} \varepsilon \text{ [by Lemma 3.2 and (a) of Lemma 3.1]}$$
$$= R(x_{n-1}^{r_n} \dots x_1^{r_2} x_0^{r_1}) \text{ } [u_i = R(x_i),\ i \ge 0]$$
$$= R(s_m) \text{ [by Lemma 2.4].}$$

(b) Let $m = \sum_{i=1}^n r_i q_{i-1}$. Then, by Lemmas 2.4-2.5 and the fact that $u_i = R(x_i)$ for all i, we have that

$$\langle u_0 u_1^2 u_2^2 \dots, u_0^{1-r_1} u_1^{2-r_2} u_2^{2-r_3} \dots \rangle = u_0^{r_1} u_1^{r_2} \dots u_{n-1}^{r_n}$$

is another way of writing $\langle f, f_m \rangle = R(s_m)$.

Example: If m is a positive integer having a code 0211020111 with respect to $\sqrt{2} - 1$, then $\langle f, f_m \rangle = u_1^2 u_2 u_3 u_5^2 u_7 u_8 u_9$, in view of part (b) of Theorem 2.1. Thus, the extracted word $\langle f, f_m \rangle$ can be found by computing $u_1, u_2, \dots, u_9$. There is no need to compute $m, f,$ and f_m.

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SUFFIXES OF FIBONACCI WORD PATTERNS

Wai-Fong Chuan, Chih-Hao Chang, and Yen-Liang Chang
Department of Mathematics, Chung-yuan Christian University, Chung Li, Taiwan 32023, R.O.C.
(Submitted November 1998-Final Revision June 1999)

1. INTRODUCTION

Let $\mathcal{A}$ be an alphabet. Let $\mathcal{A}^*$ be the monoid of all words over $\mathcal{A}$. Let ε denote the empty word, and let $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$. If $w = a_1a_2 \ldots a_n$, where $a_i \in \mathcal{A}$, the positive integer n is called the *length* of w, denoted by $|w|$. Let $|\varepsilon| = 0$. A word x is said to be a *prefix* (resp., *suffix*) of w, denoted by $x <_p w$ (resp., $x <_s w$), if there is a word $y \in \mathcal{A}^+$ such that $w = xy$ (resp., $w = yx$). We write $x \leq_p w$ (resp., $x \leq_s w$) if $x <_p w$ (resp., $x <_s w$) or $x = w$. Prefixes and suffixes of an infinite word are defined similarly.

Let f be an infinite word over $\mathcal{A}$. For $j \geq 0$, let $S^j f$ denote the suffix of f obtained from f by deleting the first j letters of f. For simplicity we write Sf for $S^1 f$. This defines an operator S acting on infinite words over $\mathcal{A}$. The *cyclic shift operator* T on $\mathcal{A}^+$ is given by $T(a_1a_2 \ldots a_n) = a_2 \ldots a_n a_1$, where $a_i \in \mathcal{A}$. For $j \geq 1$, let $T^j = T(T^{j-1})$, where T^0 denotes the identity operator on $\mathcal{A}^+$. Clearly, each operator T^j has an inverse T^{-j}.

Let $u, v \in \mathcal{A}^+$, $x_1 = u$, $x_2 = v$, and $x_n = x_{n-2}x_{n-1}$ $(n \geq 3)$. The infinite word $x_1x_2x_3 \ldots$ is called a *Fibonacci word pattern* generated by u and v and is denoted by $F(u, v)$. The words u and v are called the *seed words* of $F(u, v)$. Let $\mathcal{F}^{m,n}$ denote the set of all Fibonacci word patterns $F(u, v)$ with $|u| = m$ and $|v| = n$. Let $\mathcal{F}$ denote the set of all Fibonacci word patterns.

Given $u, v \in \mathcal{A}^+$, $|u| = m$, $|v| = n$, Turner [17] proved that $F(u, v) \in \mathcal{F}^{r,s}$, where $(r, s) = (F_{2i-1}m + F_{2i}n,\ F_{2i}m + F_{2i+1}n)$ for all $i \geq 1$. In Section 2 of this paper we find necessary and sufficient conditions for $F(u, v)$ to be a member of $\mathcal{F}^{n,m+n}$ (resp., $\mathcal{F}^{n-m,m}$, $\mathcal{F}^{2m-n,n-m}$) (Theorems 2.2-2.4). We also find necessary and sufficient conditions for $SF(u, v)$ to be a member of $\mathcal{F}^{m,n}$ (resp., $\mathcal{F}^{n,m+n}$) (Theorems 2.5-2.6). The fact that $\mathcal{F}$ is invariant under S is a consequence of Theorem 2.7, which asserts that $SF(u, v)$ always belongs to $\mathcal{F}^{m+n,m+2n}$. The Fibonacci word patterns over $\{0, 1\}$ are called *Fibonacci binary patterns* (see [5], [17]). The most famous Fibonacci binary pattern is the *golden sequence* $F(1, 01)$, which is identical to the binary word $c_1c_2 \ldots$, where $c_n = [(n+1)\alpha] - [n\alpha]$, $n \geq 1$, and $\alpha = (\sqrt{5} - 1)/2$. See, for example, [2], [3], and [5]-[18]. In Section 3 we use the above results and Lemma 3.1 to compute the possible lengths of the seed words of the suffixes $S^j F(1, 01)$, $j \geq 0$ (Theorem 3.2 and Table 1). It turns out that all these possible pairs of seed words of $S^j F(1, 01)$ have Fibonacci lengths and are pairs of Fibonacci words, the notion of which was introduced by Chuan [4] (see Definition in Section 4). They can be determined by different representations of j in Fibonacci numbers (Theorems 4.5 and 4.6). This gives another proof of Corollary 3.3 of [9] for the case $\alpha = (\sqrt{5} - 1)/2$.

2. FIBONACCI WORD PATTERNS AND THEIR SUFFIXES

Throughout this section, let $u, v \in \mathcal{A}^+$, $|u| = m$, $|v| = n$.

Theorem 2.1 (see [17]): $F(u, v) = F(uv, uvv) \in \mathcal{F}^{m+n,m+2n}$.

Theorem 2.2:

(a) Let $m \le n$. Then $F(u, v) \in \mathcal{F}^{n, m+n}$ if and only if $u \le_s v$. Moreover, $F(u, xu) = F(ux, uux)$ for all $x \in \mathcal{A}^*$.

(b) Let $m > n$, $u = xy$, where $x, y \in \mathcal{A}^+$, $|x| = n$. Then $F(u, v) \in \mathcal{F}^{n, m+n}$ if and only if $xy = yv$. In this case, $F(u, v) = F(x, xyx)$.

Proof: ***(a)*** $(m \le n)$ Suppose that $F(u, v) \in \mathcal{F}^{n, m+n}$. Let $v = xy$, where $x, y \in \mathcal{A}^*$, $|y| = m$. Then

$$F(u, v) = F(u, xy) = (u)(xy)(uxy)(xyuxy)\cdots$$
$$= (ux)(yux)(yxyux)\cdots.$$

Since $F(u, v) \in \mathcal{F}^{n, m+n}$, it follows that

$$F(u, v) = F(ux, yux) = (ux)(yux)(uxyux)\cdots.$$

By comparing the two expressions of $F(u, v)$ and using the assumption that $|y| = |u| = m$, we have $u = y$. This proves that $u \le_s v$, $v = xu$, and $F(u, xu) = F(ux, uux)$.

Conversely, let $v = xu$, where $x \in \mathcal{A}^*$. We claim that $F(u, xu) = F(ux, uux)$. Let

$$x_1 = u, \quad x_2 = v = xu, \quad x_n = x_{n-2}x_{n-1},$$
$$y_1 = ux, \quad y_2 = uux, \quad y_n = y_{n-2}y_{n-1}, \ n \ge 3.$$

Clearly, $u \le_s x_n$, $n \ge 1$. Write $x_n = z_n u$, where $z_n \in \mathcal{A}^*$. Since $x_n = x_{n-2}x_{n-1}$, we have $z_n = z_{n-2}uz_{n-1}$, $n \ge 3$. Now it is easy to see that $y_{n-1} = uz_n$, $n \ge 2$. Therefore,

$$F(u, v) = F(u, xu) = x_1x_2x_3\cdots = u(z_2u)(z_3u)\cdots$$
$$= (uz_2)(uz_3)(uz_4)\cdots = y_1y_2y_3\cdots = F(ux, uux).$$

(b) $(m > n)$ The proof is similar to part (a). □

We note that the condition $xy = yv$ holds if and only if there are words $z_1, z_2 \in \mathcal{A}^*$ and an integer $r \ge 0$ such that $x = z_1z_2$, $y = (z_1z_2)^r z_1$, and $v = z_2z_1$ (see [15]).

Corollary: Let $u \le_s v$ and let $u_k, v_k \in \mathcal{A}^+$ be such that $|u_k| = F_{k-1}m + F_k n$, $|v_k| = F_k m + F_{k+1}n$, and $u_k v_k <_p F(u, v)$, $k \ge 0$. Then $F(u, v) = F(u_k, v_k) \in \mathcal{F}^{|u_k|, |v_k|}$ and $u_k \le_s v_k$. Here $F_{-1} = 1$, $F_0 = 0$.

Theorem 2.3: Let $m < n \le 2m$. Then $F(u, v) \in \mathcal{F}^{n-m, m}$ if and only if u and v have a common prefix of length $n-m$ and $u <_s v$.

Proof: Suppose that $F(u, v) = F(x, z)$, where $|x| = n-m$ and $|z| = m$. It follows from part (a) of Theorem 2.2 that $x \le_s z$, i.e., $z = yx$ for some $y \in \mathcal{A}^*$. Also, $u = xy$ and $v = xxy$. Hence, x is a common prefix of u and v of length $n-m$ and $u <_s v$.

Conversely, suppose that u and v have a common prefix x of length $n-m$ and $u <_s v$. Then $u = xy$, $v = xxy$, where $y \in \mathcal{A}^*$. Then, according to part (a) of Theorem 2.2, we have $F(x, yx) = F(xy, xxy)$. Hence, $F(u, v) \in \mathcal{F}^{n-m, m}$. □

Theorem 2.4 follows from Theorem 2.1.

Theorem 2.4: Let $m < n < 2m$. Then $F(u, v) \in \mathcal{F}^{2m-n, n-m}$ if and only if u and v have a common suffix of length $n-m$ and $u <_p v$.

Theorem 2.5: Let $1 \le k \le \min(m, n)$. Then $S^j F(u, v) \in \mathcal{F}^{m,n}$ for all j, $0 \le j \le k$, if and only if u and v have a common prefix of length k. In this case, $S^j F(u, v) = F(T^j(u), T^j(v))$. If, in addition, $u \le_s v$, then $T^j(u) \le_s T^j(v)$.

Proof: Suppose that $S^k F(u, v) \in \mathcal{F}^{m,n}$. Let $u = wx$, $v = w_1 y$, where w, w_1, x, and y are words and $|w| = |w_1| = k$. Then it is clear that $S^k F(u, v) = F(xw_1, yw)$ and $w = w_1$. Thus, w is a common prefix of both u and v.

Conversely, suppose that u and v have a common prefix az, where $a \in \mathcal{A}$, $z \in \mathcal{A}^*$. Write $u = azx$, $v = azy$, where $x, y \in \mathcal{A}^*$. Then $SF(u, v) = F(zxa, zya) \in \mathcal{F}^{m,n}$. Moreover, z is a common prefix of the seed words zxa, zya of $SF(u, v)$, $|z| = k-1$, $zxa = T(u)$, and $zya = T(v)$. If $u \le_s v$, then clearly $zxa \le_s zya$. Now the result follows by inductive argument. □

The following theorem can be proved in a similar way.

Theorem 2.6:

(a) Let $m \le n$. Then $SF(u, v) \in \mathcal{F}^{n, m+n}$ if and only if u and v have a common suffix of length $m-1$. Moreover, $F(ax, zx) = aF(xz, xaxz)$ for all $a \in \mathcal{A}$, x, $z \in \mathcal{A}^+$.

(b) Let $m > n$, $u = axy$, where $a \in \mathcal{A}$, x, $y \in \mathcal{A}^*$, $|x| = n$. Then $SF(u, v) \in \mathcal{F}^{n, m+n}$ if and only if $xy = yv$. In this case, $F(axy, v) = aF(x, yvax)$.

Corollary: Let $j \ge 0$, u_j, $v_j \in \mathcal{A}^+$, $u_j v_j <_p S^j F(u, v)$, $|u_j| = F_{j-1}m + F_j n$, $|v_j| = F_j m + F_{j+1} n$. If $u \le_s v$, then $S^j F(u, v) = F(u_j, v_j) \in \mathcal{F}^{|u_j|, |v_j|}$ and $u_j \le_s v_j$.

Theorem 2.7: $SF(u, v) \in \mathcal{F}^{m+n, m+2n}$.

Proof: According to Theorem 2.1, $F(u, v) = F(uv, uvv) \in \mathcal{F}^{m+n, m+2n}$. Since uv and uvv have the same first letter, it follows from Theorem 2.5 that $SF(u, v) = SF(uv, uvv) \in \mathcal{F}^{m+n, m+2n}$. □

Corollary: All suffixes of $F(u, v)$ belong to $\mathcal{F}$. More precisely, for $j \ge 0$, $S^j F(u, v) \in \mathcal{F}^{r,s}$, where $(r, s) = (F_{2j-1}m + F_{2j}n, F_{2j}m + F_{2j+1}n)$.

3. THE GOLDEN SEQUENCE $F(1, 01)$

Let $\mathcal{A} = \{0, 1\}$. Consider the golden sequence $f = F(1, 01)$. For each $j \ge 0$, we shall show how to compute pairs of positive integers (r, s) for which $S^j f \in \mathcal{F}^{r,s}$. A key observation is the following lemma.

Lemma 3.1: Let $n \ge 2$ and $F_n - 1 \le j \le F_{n+1} - 2$. Then $S^j f = F(u_j, v_j)$, where $u_j, v_j \in \{0, 1\}^+$, $|u_j| = F_n$, $|v_j| = F_{n+1}$, $u_j <_s v_j$, and u_j, v_j have a common prefix of largest length $F_{n+1} - 2 - j$. (When $n = 2$ and $j = 0$, u_0, v_0 have different first letters.)

Proof: The result clearly holds when $n = 2, 3$. Suppose that it holds for $n = k$. Let $i = F_{k+1} - 2$. It follows from Theorems 2.5 and 2.6 that $S^{i+1} f \in \mathcal{F}^{F_{k+1}, F_{k+2}} \backslash \mathcal{F}^{F_k, F_{k+1}}$. Moreover, $S^{i+1} f = F(u_{i+1}, v_{i+1})$, where $|u_{i+1}| = F_{k+1}$, $|v_{i+1}| = F_{k+2}$, $u_{i+1} <_s v_{i+1}$, and u_{i+1}, v_{i+1} have a common prefix of largest length $F_k - 1$. According to Theorem 2.5, if $1 \le m \le F_k$ and $j = i + m$, then

$S^j f = F(u_j, v_j)$, where $|u_j| = F_{k+1}$, $|v_j| = F_{k+2}$, $u_j <_s v_j$, and u_j, v_j have a common prefix of largest length $F_k - m = F_{k+2} - 2 - j$. Thus, the result holds for all $n \geq 2$. □

Theorem 3.2: Let $n \geq 2$ and $F_n - 1 \leq j \leq F_{n+1} - 2$. Then $S^j f \in \mathscr{F}^{F_k, F_{k+1}}$ if $k \geq n$, and $S^j f \notin \mathscr{F}^{F_k, F_{k+1}}$ if $1 \leq k \leq n-1$.

Proof: The first part is a consequence of Lemma 3.1, Theorem 2.5, and the Corollary to Theorem 2.2. The second part follows from Lemma 3.1 and Theorems 2.1, 2.3, and 2.4. □

For example, when $n = 6$ and $7 \leq j \leq 11$, Theorem 3.2 implies that $S^j f \in \mathscr{F}^{r,s}$, where $(r, s) = (8,13), (13,21), (21,34), \ldots$ and $S^j f \notin \mathscr{F}^{r,s}$, where $(r, s) = (1,2), (2,3), (3,5), (5,8)$. This completes the part of Table 1 corresponding to $7 \leq j \leq 11$.

TABLE 1

j	(r, s) for which $S^j f \in \mathscr{F}^{r,s}$
0	(1,2), (2,3), (3,5), (5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
1	(2,3), (3,5), (5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
2	(3,5), (5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
3	(3,5), (5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
4	(5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
5	(5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
6	(5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
7	(8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
8	(8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
9	(8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
10	(8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
11	(8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
12	(13,21), (21,34), (34,55), (55,89), (89,144)
13	(13,21), (21,34), (34,55), (55,89), (89,144)

4. SEED WORDS OF $S^j F(1, 01)$ ARE FIBONACCI WORDS

Again we let $f = F(1, 01)$. We have seen in Theorem 3.2 that, if $n \geq 2$ and $F_n - 1 \leq j \leq F_{n+1} - 2$, then $S^j f \in \mathscr{F}^{F_k, F_{k+1}}$ for all $k \geq n$. Now let (u_{jk}, v_{jk}) denote the pair of seed words of $S^j f$ such that $|u_{jk}| = F_k$ and $|v_{jk}| = F_{k+1}$. We shall show in Theorem 4.5 that u_{jk} and v_{jk} are Fibonacci words, as defined below, whose labels can be determined. Special cases can be found in [5].

Fibonacci words over the alphabet {0, 1} are defined as follows: Let

$$w(0) = 10, \; w(1) = 01,$$
$$w(00) = 101, \; w(01) = 110, \; w(10) = 011, \; w(11) = 101.$$

For any binary sequence $r_1, r_2, \ldots, r_n$, $n \geq 3$, the word $w(r_1 r_2 \ldots r_n)$ is defined recursively by

$$w(r_1 r_2 \ldots r_k) = \begin{cases} w(r_1 r_2 \ldots r_{k-1}) w(r_1 r_2 \ldots r_{k-2}) & \text{if } r_k = 0, \\ w(r_1 r_2 \ldots r_{k-2}) w(r_1 r_2 \ldots r_{k-1}) & \text{if } r_k = 1, \end{cases}$$

$3 \le k \le n$. The word $w(r_1r_2\dots r_n)$ is called a *Fibonacci word* generated by the pair of words $(0, 1)$. The sequence $r_1, r_2, \dots, r_n$ is called a *label* of $w(r_1r_2\dots r_n)$. It describes how the Fibonacci word $w(r_1r_2\dots r_n)$ is generated. A Fibonacci word may have several different labels. For example, $10101101 = w(0010) = w(1100) = w(1111)$. The words 0 and 1 are also Fibonacci words. For convenience, we write $1 = w(\lambda)$, where λ denotes the empty label. The above notion of Fibonacci word was introduced by Chuan [4] and was later generalized to the notion of α-word by her [8]. :Many known results in the literature involve Fibonacci words (see, e.g., [4]-[12], [16]-[18]).

We need the following properties of Fibonacci words, the proofs of which can be found in [4]. Let $y_1 = 0$, $y_2 = 1$, $y_n = y_{n-2}y_{n-1}$ (i.e., $y_n = w(11\dots1)$), $n \ge 3$.

Lemma 4.1: Let $n \ge 1$, $r_1, r_2, \dots, r_n$, $s_1, s_2, \dots, s_n \in \{0, 1\}$. Then:

(a) $|w(r_1r_2\dots r_n)| = F_{n+2}$.

(b) If $j = \sum_{i=1}^{n} r_iF_{i+1}$, then $w(r_1r_2\dots r_n) = T^{-k}(y_{n+2})$, where $k = F_{n+3} - 2 - j$.

(c) If $\sum_{i=1}^{n} s_iF_{i+1} \equiv \sum_{i=1}^{n} r_iF_{i+1} \pmod{F_{n+2}}$, then $w(r_1r_2\dots r_n) = w(s_1s_2\dots s_n)$.

Let $u, x \in \mathcal{A}^+$. Then

$$F(u, xu) = F(ux, uux) = uF(xu, uxu) = uxF(uux, uxuux).$$

The first equality follows from part (a) of Theorem 2.2; the second one is trivial; the third one can be proved in a similar way as Theorem 2.2(a). It follows that, if $|u| = m$ and $|x| = t$, then

$$S^mF(u, xu) = F(xu, uxu),$$
$$S^{m+t}F(u, xu) = F(uux, uxuux).$$

In particular, we have the following lemma. Part (d) follows from Theorem 2.1.

Lemma 4.2: Let $n \ge 1$, $r_1, r_2, \dots, r_n$, $r_{n+1} \in \{0, 1\}$. Let $u = w(r_1r_2\dots r_n)$, $v = w(r_1r_2\dots r_n1)$. Then:

(a) $F(u, v) = F(w(r_1r_2\dots r_n0), w(r_1r_2\dots r_n01))$.

(b) $S^{F_{n+2}}F(u, v) = F(w(r_1r_2\dots r_n1), w(r_1r_2\dots r_n11))$.

(c) $S^{F_{n+3}}F(u, v) = F(w(r_1r_2\dots r_n01), w(r_1r_2\dots r_n011))$.

(d) $F(w(r_1r_2\dots r_n), w(r_1r_2\dots r_{n+1})) = F(w(r_1\dots r_{n+1}1), w(r_1\dots r_{n+1}10))$.

Lemma 4.3 (see [1]): Each positive integer *j* is uniquely expressed as $j = \sum_{i=1}^{n} r_iF_{i+1}$, where $r_n = 1$, $r_i \in \{0, 1\}$, and $\max(r_i, r_{i+1}) = 1$ $(1 \le i \le n-1)$.

The representation $j = \sum_{i=1}^{n} r_iF_{i+1}$ in Lemma 4.3 is called the *maximal representation* of *j*. The code $\langle r_1r_2\dots r_n\rangle$ is called the *maximal code* of *j*. The number *n* is called the *length* of the maximal code of *j*. For convenience, the maximal code of the integer 0 is defined to be the empty code λ. It has length 0. We note that $F_{n+2} - 1 \le j \le F_{n+3} - 2$ if and only if the length of the maximal code of *j* is *n*.

Lemma 4.4: For each $j \ge 0$, let $\langle r_1r_2\dots r_n\rangle$ be the maximal code of *j*. Then $S^jf = F(w(r_1r_2\dots r_n), w(r_1r_2\dots r_n1))$.

Proof: The result clearly holds for $0 \le j \le 3$. Now suppose that $k > 3$ and that the result is true for all j, $0 \le j < k$. We show that it is also true for $j = k$. Let $n \ge 3$ be such that $F_{n+2} - 1 \le k \le F_{n+3} - 2$.

(a) If $F_{n+2} - 1 \le k \le 2F_{n+1} - 2$, then $F_n - 1 \le k - F_{n+1} \le F_{n+1} - 2$. By the inductive hypothesis,

$$S^{k-F_{n+1}} f = F(w(r_1 r_2 \dots r_{n-2}), w(r_1 r_2 \dots r_{n-2} 1)),$$

where $\langle r_1 r_2 \dots r_{n-2} \rangle$ is the maximal code of $k - F_{n+1}$. Clearly, $\langle r_1 r_2 \dots r_{n-2} 01 \rangle$ is the maximal code of k. Also,

$$\begin{aligned} S^k f &= S^{F_{n+1}} S^{k-F_{n+1}} f = S^{F_{n+1}} F(w(r_1 r_2 \dots r_{n-2}), w(r_1 r_2 \dots r_{n-2} 1)) \\ &= F(w(r_1 r_2 \dots r_{n-2} 01), w(r_1 r_2 \dots r_{n-2} 011)), \end{aligned}$$

according to part (c) of Lemma 4.2.

(b) If $2F_{n+1} - 1 \le k \le F_{n+3} - 2$ and if $\langle r_1 r_2 \dots r_{n-1} \rangle$ is the maximal code of $k - F_{n+1}$, then the inductive hypothesis implies that

$$S^{k-F_{n+1}} f = F(w(r_1 r_2 \dots r_{n-1}), w(r_1 r_2 \dots r_{n-1} 1)).$$

Therefore, $\langle r_1 r_2 \dots r_{n-1} 1 \rangle$ is the maximal code of k and

$$S^k f = F(w(r_1 r_2 \dots r_{n-1} 1), w(r_1 r_2 \dots r_{n-1} 11)),$$

according to part (b) of Lemma 4.2. □

Using Lemma 4.4 and part (a) of Lemma 4.2, the seed words of $S^j f$ can now be determined.

Theorem 4.5: Let $j \ge 0$ and let $\langle r_1 r_2 \dots r_n \rangle$ be the maximal code of j. Let $k \ge n+2$. Then $u_{jk} = w(r_1 r_2 \dots r_n 0 \dots 0)$ and $v_{jk} = w(r_1 r_2 \dots r_n 0 \dots 01)$ (there are $k - n - 2$ zeros right after r_n).

For example, since $3 = F_2 + F_3$ is the maximal representation of 3, we have $u_{36} = w(1100)$, $v_{36} = w(11001)$. As observed before, the labels for u_{jk} and v_{jk} may not be unique.

Corollary: Let $j \ge 0$ and let n be the smallest integer ≥ 2 such that $j \le F_{n+1} - 2$. If $k \ge n$, then $S^j f = F(T^{-i_k}(y_k), T^{-i_k}(y_{k+1}))$, where $i_k = F_{k+1} - 2 - j$.

Proof: The result follows from Theorem 4.5 and part (b) of Lemma 4.1. □

Note that this corollary contains part (b) of Theorem 8 of [5].

Theorem 4.6: Let $j = \sum_{i=1}^{k-2} s_i F_{i+1}$, where $s_i \in \{0, 1\}$ $(1 \le i \le k-2)$ and $k \ge 3$, then

$$S^j f = F(w(s_1 s_2 \dots s_{k-2}), w(s_1 s_2 \dots s_{k-2} 1)).$$

Proof: If $j = 0$, then the result is contained in Theorem 4.5. Now let $j \ge 1$ and let $\langle r_1 r_2 \dots r_n \rangle$ be the maximal code of j. Clearly, $n \le k-2$. Define $r_i = 0$ if $n < i \le k-2$. Then

$$j = \sum_{i=1}^{k-2} r_i F_{i+1} = \sum_{i=1}^{k-2} s_i F_{i+1},$$

$$j + F_k = \sum_{i=1}^{k-2} r_i F_{i+1} + F_k = \sum_{i=1}^{k-2} s_i F_{i+1} + F_k.$$

Hence,

$$\begin{aligned}(u_{jk}, v_{jk}) &= (w(r_1r_2\ldots r_{k-2}), w(r_1r_2\ldots r_{k-2}1)) \text{ [by Theorem 4.5]}\\ &= (w(s_1s_2\ldots s_{k-2}), w(s_1s_2\ldots s_{k-2}1)) \text{ [by part (c) of Lemma 4.1].}\end{aligned}$$

This completes the proof. □

For example, since $3 = F_2 + F_3 = F_4$, we have $u_{36} = w(1100) = w(0010)$ and $v_{36} = w(11001) = w(00101)$. It also follows from Theorem 4.6 that the Fibonacci word pattern generated by a pair of Fibonacci words of the form $w(r_1r_2\ldots r_n)$, $w(r_1r_2\ldots r_n1)$ is a suffix of f.

Corollary: For every binary sequence $r_1, r_2, \ldots, r_n$, the Fibonacci word pattern $F(w(r_1r_2\ldots r_n), w(r_1r_2,\ldots r_n1))$ is a suffix of f. More precisely,

$$F(w(r_1r_2\ldots r_n), w(r_1r_2,\ldots r_n1)) = S^j f,$$

where $j = \sum_{i=1}^n r_iF_{i+1}$.

We remark that Theorem 4.6 is a special case of Corollary 3.3 of [9], which was proved by a general representation theorem. In our proof given here, only elementary properties of Fibonacci word patterns and Fibonacci words are used.

Seed words of the Fibonacci word pattern $F(0,1)$ can also be obtained easily. Let $w_1 = 0$, $w_2 = 1$, and for $n \geq 3$, let $w_n = w_{n-2}w_{n-1}$ if n is odd and $w_n = w_{n-1}w_{n-2}$ if n is even [that is, $w_n = w(r_1r_2\ldots r_{n-2})$, where r_i equals 1 if n is odd and equals 0 if n is even $(n \geq 3)$]. It follows immediately from part (d) of Lemma 4.2 that $F(0,1) = F(w_{2n-1}, w_{2n}) \in \mathcal{F}^{F_{2n-1}, f_{2n}}$ $(n \geq 1)$. Since w_{2n-1} and the suffix of w_{2n} having length $|w_{2n-1}|$ $(= F_{2n-1})$ have different first letters (see [6]), it follows that $F(0,1) \notin \mathcal{F}^{F_{2n}, F_{2n+1}}$ $(n \geq 1)$, according to part (c) of Theorem 2.2.

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ON THE k-ARY CONVOLUTION OF ARITHMETICAL FUNCTIONS

Pentti Haukkanen

Department of Mathematical Sciences, University of Tampere
PO Box 607, FIN-33101 Tampere, Finland
mapehau@uta.fi

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1. INTRODUCTION

A divisor d of n is said to be a unitary divisor of n if the greatest common divisor of d and n/d is 1 (see [4], [9]), and a divisor d of n is said to be a biunitary divisor of n if the greatest common unitary divisor of d and n/d is 1 (see [11], [12]). It is easy to see that the unitary divisors of a prime power p^a $(a \geq 1)$ are 1 and p^a, and the biunitary divisors of p^a $(a \geq 1)$ are $1, p, p^2, \ldots, p^a$, except for $p^{a/2}$ when a is even. Cohen [5] extends the above notions inductively.

Definition 1.1: If $d \mid n$, then d is a 0-ary divisor of n. For $k \geq 1$, a divisor d of n is a k-ary divisor of n if the greatest common $(k-1)$-ary divisor of d and n/d is 1.

Remark: Different extensions of the concept of a unitary divisor have been developed by Suryanarayana [10] (who also used the term k-ary divisor) and Alladi [1]. We do not consider these extensions here.

We write $d \mid_k n$ to mean that d is a k-ary divisor of n, and $(m, n)_k$ to stand for the greatest common k-ary divisor of m and n. Thus, for $k \geq 1$, $d \mid_k n$ if and only if $d \mid n$ and $(d, n/d)_{k-1} = 1$ with the convention that $(d, n/d)_0 = (d, n/d)$. In particular, $d \mid_1 n$ (resp. $d \mid_2 n$) means that d is a unitary (resp. biunitary) divisor of n.

Definition 1.2: We say that p^b is an infinitary divisor of p^a $(a \geq 1)$ (written as $p^b \mid_\infty p^a$) if $p^b \mid_{a-1} p^a$. In addition, 1 is the only infinitary divisor of 1. Further, $d \mid_\infty n$ if $p^{d(p)} \mid_\infty p^{n(p)}$ for all primes p, where $d = \prod_p p^{d(p)}$ and $n = \prod_p p^{n(p)}$ are the canonical forms of d and n.

The justification for Definition 1.2 is that, for $k \geq a-1 \geq 0$, $p^b \mid_k p^a \Leftrightarrow p^b \mid_{a-1} p^a$ (see [5]). Thus, for $k \geq a-1 \geq 0$,

$$p^b \mid_k p^a \Leftrightarrow p^b \mid_\infty p^a. \tag{1.1}$$

This means that, for $a = 0, 1, 2, \ldots, k+1$, the k-ary divisors of p^a are the same as the infinitary divisors of p^a. For example, for $a = 0, 1, 2, \ldots, 101$, the 100-ary divisors of p^a are the infinitary divisors of p^a.

Cohen and Hagis ([5], [6], [7]) give an elegant method for determining infinitary divisors. Let $I = \{p^{2^\alpha} \mid p \text{ is a prime, } \alpha \text{ is a nonnegative integer}\}$. It follows from the fundamental theorem of arithmetic and the binary representation that every n (>1) can be written in exactly one way (except for the order of factors) as the product of distinct elements of I. Each element of I in this product is called an I-component of n. Cohen and Hagis ([5], [6], [7]) also note that $d \mid_\infty n$ if and only if every I-component of d is also an I-component of n with the convention that $1 \mid_\infty n$ for all n. For example, if $n = 2^3 3^5 = 2 \cdot 2^2 \cdot 3 \cdot 3^4$, then the I-components of n are $2, 2^2, 3, 3^4$. Note that this method makes it possible to compute the k-ary divisors of the prime powers $1, p, p^2, \ldots, p^{k+1}$. A general formula for the k-ary divisors of p^a for $a \geq k+2$ is not known.

The concept of divisor is related to the Dirichlet convolution of arithmetical functions. The concepts of unitary and biunitary divisor lead to the unitary and biunitary convolution. This suggests we define the k-ary convolution of arithmetical functions f and g as

$$(f *_k g)(n) = \sum_{d \mid_k n} f(d)g(n/d)$$

for $k \geq 0$. In particular, the 0-ary, 1-ary, and 2-ary convolution is the Dirichlet, unitary, and biunitary convolution, respectively.

The purpose of this paper is to represent the basic algebraic properties of the k-ary convolution and to study the Möbius function under the k-ary convolution.

2. BASIC PROPERTIES OF THE k-ARY CONVOLUTION

In this section we represent the basic algebraic properties of the k-ary convolution. Particular attention is paid to multiplicative functions. An arithmetical function f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$, and an arithmetical function f is said to be completely multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all m and n. Cohen and Hagis [6] say that an arithmetical function f is I-multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m, n)_\infty = 1$, where $(m, n)_\infty$ is the greatest common infinitary divisor of m and n. It is easy to see that

$$\begin{aligned} f \text{ is completely multiplicative } &\Rightarrow f \text{ is } I\text{-multiplicative} \\ &\Rightarrow f \text{ is multiplicative.} \end{aligned} \tag{2.1}$$

Theorem 2.1: Let $k \geq 0$.

1) The k-ary convolution is commutative.

2) The function δ serves as the identity under the k-ary convolution, where $\delta(1) = 1$ and $\delta(n) = 0$ for $n \geq 2$.

3) An arithmetical function f possesses an inverse under the k-ary convolution if and only if $f(1) \neq 0$. The inverse $(f^{-1})_k$ is given recursively as $(f^{-1})_k(1) = 1/f(1)$ and, for $n \geq 2$,

$$(f^{-1})_k(n) = \frac{-1}{f(1)} \sum_{\substack{d \mid_k n \\ d>1}} f(d)(f^{-1})_k(n/d). \tag{2.2}$$

4) The k-ary convolution preserves multiplicativity, that is, if f and g are multiplicative, so is their k-ary convolution.

5) If f is multiplicative, so is $(f^{-1})_k$.

Proof: Theorem 2.1 can be proved by adopting the standard argument (see, e.g., [2], [9]). As part 5 is needed later, we present the details of the proof of part 5. Assume that $(m, n) = 1$. If $mn = 1$, then $(f^{-1})_k(mn) = 1 = (f^{-1})_k(m)(f^{-1})_k(n)$. Assume that $mn \neq 1$ and that $(f^{-1})_k(m'n') = (f^{-1})_k(m')(f^{-1})_k(n')$ whenever $(m', n') = 1$ and $m'n' < mn$. If $m = 1$ or $n = 1$, then $(f^{-1})_k(mn) = (f^{-1})_k(m)(f^{-1})_k(n)$. Assume that $m, n \neq 1$. With the aid of (2.2), we obtain

$$
\begin{aligned}
(f^{-1})_k(mn) &= -\sum_{\substack{d\,|_k\, mn\\ d>1}} f(d)(f^{-1})_k(mn/d) = -\sum_{\substack{d_1\,|_k\, m\\ d_2\,|_k\, n\\ d_1d_2>1}} f(d_1d_2)(f^{-1})_k((m/d_1)(n/d_2))\\
&= -\sum_{\substack{d_1\,|_k\, m\\ d_2\,|_k\, n\\ d_1d_2>1}} f(d_1)f(d_2)(f^{-1})_k(m/d_1)(f^{-1})_k(n/d_2)\\
&= -(f^{-1})_k(m)\sum_{\substack{d_2\,|_k\, n\\ d_2>1}} f(d_2)(f^{-1})_k(n/d_2) - (f^{-1})_k(n)\sum_{\substack{d_1\,|_k\, m\\ d_1>1}} f(d_1)(f^{-1})_k(m/d_1)\\
&\qquad -\sum_{\substack{d_1\,|_k\, m\\ d_1>1}} f(d_1)(f^{-1})_k(m/d_1)\sum_{\substack{d_2\,|_k\, n\\ d_2>1}} f(d_2)(f^{-1})_k(n/d_2)\\
&= (f^{-1})_k(m)(f^{-1})_k(n) + (f^{-1})_k(m)(f^{-1})_k(n) - (f^{-1})_k(m)(f^{-1})_k(n)\\
&= (f^{-1})_k(m)(f^{-1})_k(n).
\end{aligned}
$$

This completes the proof. □

Remark: The k-ary convolution is not associative in general. For example, the biunitary convolution is not associative (see [8]).

The infinitary convolution [6] of arithmetical functions f and g is defined as

$$(f *_\infty g)(n) = \sum_{d\,|_\infty\, n} f(d)g(n/d).$$

The infinitary convolution possesses the properties given in Theorem 2.1. In addition, it is associative and possesses basic properties with respect to I-multiplicative functions. We present these results in the following theorem.

Theorem 2.2:

1) The infinitary convolution is associative.

2) The infinitary convolution is commutative.

3) The function δ serves as the identity under the infinitary convolution, where $\delta(1) = 1$ and $\delta(n) = 0$ for $n \ge 2$.

4) An arithmetical function f possesses an inverse under the infinitary convolution if and only if $f(1) \ne 0$. The inverse $(f^{-1})_\infty$ is given recursively as $(f^{-1})_\infty(1) = 1/f(1)$ and, for $n \ge 2$,

$$(f^{-1})_\infty(n) = \frac{-1}{f(1)}\sum_{\substack{d\,|_\infty\, n\\ d>1}} f(d)(f^{-1})_\infty(n/d). \tag{2.3}$$

5) The infinitary convolution preserves multiplicativity.

6) If f is multiplicative, so is $(f^{-1})_\infty$.

7) The infinitary convolution preserves I-multiplicativity.

8) If f is I-multiplicative, so is $(f^{-1})_\infty$.

Theorem 2.2 is given in Cohen and Hagis [6] except for equation (2.3) and parts 5 and 6. Cohen and Hagis [6] do not prove their results. We do not prove these results either, since the standard argument (see, e.g., [2], [9]) can be applied.

Remark: It is easy to see that the k-ary convolution for all k and the infinitary convolution do not preserve complete multiplicativity.

Remark: Theorem 2.2 shows that I-multiplicative functions possess two basic properties under the infinitary convolution. This leads us to propose the following unsolved research problem. Define k-ary multiplicative functions so that they possess basic properties under the k-ary convolution.

3. THE k-ARY MÖBIUS FUNCTION

We define the k-ary Möbius function μ_k as the inverse of the constant function 1, denoted by ζ, under the k-ary convolution. In particular, μ_0 is the classical number-theoretic Möbius function and μ_1 is the unitary Möbius function (see [4], [9]). Since ζ is a multiplicative function, so is μ_k. Therefore, μ_k is completely determined by its values at prime powers. The values of μ_k at prime powers are obtained recursively as $\mu_k(1)=1$ and, for $a \geq 1$,

$$\mu_k(p^a) = -\sum_{\substack{p^b \mid_k p^a \\ 0 \leq b < a}} \mu_k(p^b). \tag{3.1}$$

A general explicit formula for μ_k is not known.

We define the infinitary Möbius function μ_∞ as the inverse of the function ζ under the infinitary convolution. An explicit formula for μ_∞ is known. Let $s_2(a)$ denote the number of nonzero terms in the binary representation of a with the convention that $s_2(0)=0$, and let $J(n)$ denote the arithmetical function defined as $J(1)=0$ and, for $n \geq 2$, $J(n) = \sum_p s_2(n(p))$, where $n = \prod_p p^{n(p)}$ is the canonical form of n. Note that $J(n)$ is the number of I-components of n. Cohen and Hagis [6] show that

$$\mu_\infty(n) = (-1)^{J(n)}. \tag{3.2}$$

It follows from (1.1) that

$$\mu_k(p^a) = \mu_\infty(p^a) \quad \text{for } a = 0, 1, 2, \ldots, k+1. \tag{3.3}$$

Therefore, in a sense, μ_k comes closer to μ_∞ as k increases.

It is interesting that

$$\mu_2 = \mu_\infty. \tag{3.4}$$

This is a consequence of Theorem 3.1 given below and equation (3.2).

Theorem 3.1: If f is completely multiplicative, then

$$(f^{-1})_2(n) = (-1)^{J(n)} f(n). \tag{3.5}$$

Proof: Since both sides of (3.5) are multiplicative functions in n, we may confine ourselves to prime powers p^a. By (2.2), and knowing the biunitary divisors of p^a, we have $(f^{-1})_2(1)=1$ and, for $a \geq 1$,

$$\begin{cases} \sum_{i=0}^{a} (f^{-1})_2(p^i) f(p^{a-i}) = 0 & \text{if } a \text{ is odd,} \\ \sum_{i=0}^{a} (f^{-1})_2(p^i) f(p^{a-i}) - (f^{-1})_2(p^{a/2}) f(p^{a/2}) = 0 & \text{if } a \text{ is even.} \end{cases}$$

Therefore, for $a \geq 0$,

$$\begin{cases} \sum_{i=0}^{2a+1} (f^{-1})_2(p^i) f(p^{2a+1-i}) = 0, \\ \sum_{i=0}^{2a} (f^{-1})_2(p^i) f(p^{2a-i}) - (f^{-1})_2(p^a) f(p^a) = 0. \end{cases}$$

This shows that the function $(f^{-1})_2$ at prime powers is completely determined by the recurrence relation

$$\begin{cases} (f^{-1})_2(p^{2a+1}) + f(p^{a+1})(f^{-1})_2(p^a) = 0, \\ (f^{-1})_2(p^{2a+2}) - f(p^{a+1})(f^{-1})_2(p^{a+1}) = 0, \end{cases}$$

for $a \geq 0$, with the initial condition $(f^{-1})_2(1) = 1$.

We show that the function $g(n) = (-1)^{J(n)} f(n)$ satisfies the same recurrence relation at prime powers. In fact,

$$\begin{aligned} g(p^{2a+1}) + f(p^{a+1}) g(p^a) &= (-1)^{s_2(2a+1)} f(p^{2a+1}) + f(p^{a+1})(-1)^{s_2(a)} f(p^a) \\ &= (-1)^{s_2(a)+1} f(p)^{2a+1} + f(p)^{a+1}(-1)^{s_2(a)} f(p)^a = 0 \end{aligned}$$

and

$$\begin{aligned} g(p^{2a+2}) - f(p^{a+1}) g(p^{a+1}) &= (-1)^{s_2(2a+2)} f(p^{2a+2}) - f(p^{a+1})(-1)^{s_2(a+1)} f(p^{a+1}) \\ &= (-1)^{s_2(a+1)} f(p)^{2a+2} - f(p)^{a+1}(-1)^{s_2(a+1)} f(p)^{a+1} = 0, \end{aligned}$$

for $a \geq 0$, with initial condition $g(1) = 1$. This completes the proof. □

Remark: The idea for the recurrence relation in the proof of Theorem 3.1 is developed from [3].

Cohen and Hagis [6] show that, if f is I-multiplicative, then

$$(f^{-1})_\infty(n) = (-1)^{J(n)} f(n).$$

On the basis of equations (2.1) and (3.5), we see that, if f is completely multiplicative, then

$$(f^{-1})_2 = (f^{-1})_\infty. \tag{3.6}$$

Since the function ζ is completely multiplicative, we obtain equation (3.4).

Remark: It is an open question whether (3.6) holds for all I-multiplicative functions f.

It is known [5] that the 3-ary divisors of p^a are 1 and p^a, except for the cases $a = 3$ and $a = 6$. The 3-ary divisors of p^3 are $1, p, p^2, p^3$, and the 3-ary divisors of p^6 are $1, p^2, p^4, p^6$. Using this result and (3.1), we conclude that

$$\mu_3(p^a) = \begin{cases} 1 & \text{if } a = 0, 3, 6, \\ -1 & \text{otherwise.} \end{cases}$$

Thus, in the case $k=3$, we have $\mu_k(p^a)=\mu_\infty(p^a)$ for $a=0,1,2,\ldots,k+1$ (cf. (3.3)), but $\mu_k(p^{k+2})=-\mu_\infty(p^{k+2})$ or $\mu_3(p^5)=-1=-\mu_\infty(p^5)$. Further evaluations of μ_k for small values of k could be derived using the results on k-ary divisors given in [5].

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ON AN OBSERVATION OF D'OCAGNE CONCERNING THE FUNDAMENTAL SEQUENCE

R. S. Melham
School of Mathematical Sciences, University of Technology, Sydney
PO Box 123, Broadway, NSW 2007 Australia
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1. INTRODUCTION

Following the notation in [3], we consider the sequence $\{W_n\} = \{W_n(a, b; p, q)\}$ defined, for all integers n, by

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = a,\ W_1 = b. \tag{1.1}$$

Throughout this paper we take a, b, p, and q to be arbitrary integers with $q \neq 0$.

Distinguished among all the sequences generated by the recurrence in (1.1) is the pair $U_n = W_n(0, 1; p, q)$ and $V_n = W_n(2, p; p, q)$, whose importance was first recognized by Lucas [4]. The sequences $\{U_n\}$ and $\{V_n\}$ are often referred to as the *fundamental* and *primordial* sequences, respectively [13]. Because of their special properties, $\{U_n\}$ and $\{V_n\}$ continue to be the focus of much attention [2], [5], [9], [12]. Our interest in this paper is in a property of $\{U_n\}$ which, according to Dickson ([1], p. 409), was first observed by D'Ocagne. D'Ocagne observed that there exist **integers** c_0 and c_1, independent of n, such that

$$W_n = c_0U_n + c_1U_{n+1}, \quad n \in \mathbf{Z}. \tag{1.2}$$

Indeed, it can be proved by induction that

$$W_n = (W_1 - pW_0)U_n + W_0U_{n+1}, \quad n \in \mathbf{Z}. \tag{1.3}$$

In this sense $\{U_n\}$ can be regarded as a "basis" for the sequences generated by the recurrence in (1.1). In fact, as stated in the reference of Dickson mentioned above, D'Ocagne observed this property for the higher-order analogs of $\{U_n\}$.

It is natural to ask if there are other sequences generated by the recurrence in (1.1) which also possess this property of $\{U_n\}$. To be more precise, we make the following definition.

Property of D'Ocagne: An integer sequence $\{S_n\} = \{W_n(S_0, S_1; p, q)\}$ is said to have the property of D'Ocagne if there exist integers c_0 and c_1, independent of n, such that $W_n = c_0S_n + c_1S_{n+1}$, $n \in \mathbf{Z}$.

For $q = \pm 1$ we have characterized all sequences which have the property of D'Ocagne. The object of this paper is to present our results.

2. PRELIMINARY RESULTS

For the remainder of the paper we take $\{S_n\} = \{W_n(S_0, S_1; p, q)\}$ to be an integer sequence. In order to make the paper self-contained, we now list several known results which will be required in the sequel.

Lemma 1:
$$D_n = \begin{vmatrix} W_n & S_n & S_{n+1} \\ W_1 & S_1 & S_2 \\ W_0 & S_0 & S_1 \end{vmatrix} = 0, \quad n \in \mathbf{Z}.$$

Lemma 2: The points with integer coordinates on the conics $y^2 - 3xy + x^2 = \pm 1$ are precisely the pairs $(x, y) = \pm(F_n, F_{n+2})$.

Lemma 3: In (1.1) suppose $p \neq 0$ and $q = -1$. Then the points with integer coordinates on the conics $y^2 - pxy - x^2 = \pm 1$ are precisely the pairs $(x, y) = \pm(U_n, U_{n+1})$.

Lemma 4: In (1.1) suppose $|p| > 2$ and $q = 1$. Then the points with integer coordinates on the conic $y^2 - pxy + x^2 = 1$ are precisely the pairs $(x, y) = \pm(U_n, U_{n+1})$.

Lemma 1 is a special case of Theorem 1 in [7]. Lemmas 2, 3, and 4 are special cases of Theorems 1, 2, and 5, respectively, in [6].

We also require several well-known theorems concerning the integer solutions of the Pell equation

$$x^2 - dy^2 = 1, \tag{2.1}$$

and its generalization

$$x^2 - dy^2 = N. \tag{2.2}$$

Here we assume that d is a positive integer that is not a perfect square and N is an integer.

Theorem 1 (see Theorem 11.5 in [11]): Let h_m / k_m denote the m^{th} convergent of the simple continued fraction of $\sqrt{d}$, $m = 0, 1, 2, \ldots$, and let l be the period length of this continued fraction. If l is even, then $(x, y) = (h_{l-1}, k_{l-1})$ is a solution of (2.1).

Theorem 2 (see Theorem 11.3 in [11]): Suppose $|N| < \sqrt{d}$. If (x, y), with x and y positive, is a solution of (2.2), then x / y is a convergent of the simple continued fraction of $\sqrt{d}$.

Theorem 3 (see Theorem 3.3, p. 128, in [10]): If (2.2) has a solution, then it has infinitely many solutions. At least one of these solutions satisfies

$$0 < x < \sqrt{((x_0 + 1)|N|/2)},$$

where (x_0, y_0) is the fundamental solution of (2.1).

Finally, we require the following lemma. For part (a), see page 389 of [11]. Indeed, both parts can be established with the use of the standard method for developing a surd as a continued fraction. See, for example, page 176 of [8].

Lemma 5: Let d be a positive integer.

(a) If $d > 3$ is odd, the simple continued fraction of $\sqrt{d^2 - 4}$ is

$$\left[d - 1; \overline{1, (d-3)/2, 2, (d-3)/2, 1, 2d - 2}\right].$$

(b) If $d > 4$ is even, the simple continued fraction of $\sqrt{d^2 - 4}$ is

$$\left[d - 1; \overline{1, (d-4)/2, 1, 2d - 2}\right].$$

3. THE MAIN RESULTS

Our first theorem gives necessary and sufficient conditions for the sequence $\{S_n\}$ to have the property of D'Ocagne.

Theorem 4: Suppose $S_1^2 - S_0S_2 \neq 0$. Then $\{S_n\}$ has the property of D'Ocagne if and only if $S_1^2 - S_0S_2 = \pm 1$.

Proof: From Lemma 1 we have

$$(S_1^2 - S_0S_2)W_n = (S_1W_1 - S_2W_0)S_n + (S_1W_0 - S_0W_1)S_{n+1}, \quad n \in \mathbf{Z}. \tag{3.1}$$

Hence, if $S_1^2 - S_0S_2 = \pm 1$, then $\{S_n\}$ has the property of D'Ocagne.

Conversely, suppose $\{S_n\}$ has the property of D'Ocagne. Then there exist integers c_0 and c_1 such that

$$U_n = c_0S_n + c_1S_{n+1}, \quad n \in \mathbf{Z}. \tag{3.2}$$

Putting $n = 0$ and $n = 1$, we see from Cramer's rule that c_0 and c_1 are unique. Now, by (3.1), we have

$$U_n = \frac{S_1}{S_1^2 - S_0S_2}S_n - \frac{S_0}{S_1^2 - S_0S_2}S_{n+1}, \quad n \in \mathbf{Z}. \tag{3.3}$$

But, by the uniqueness of c_0 and c_1 we have

$$c_0 = \frac{S_1}{S_1^2 - S_0S_2} \quad \text{and} \quad c_1 = -\frac{S_0}{S_1^2 - S_0S_2},$$

which means that $S_1^2 - S_0S_2$ divides S_n, $n \geq 0$. Consequently, putting $n = 1$ in (3.2), we see that $S_1^2 - S_0S_2$ divides 1, and this completes the proof. □

Our next theorem characterizes those sequences $\{S_n\} = \{W_n(S_0, S_1; p, -1)\}$ that have the property of D'Ocagne.

Theorem 5: If $p \neq 0$, then $\{S_n\} = \{W_n(S_0, S_1; p, -1)\}$ has the property of D'Ocagne if and only if $(S_0, S_1) = \pm(U_m, U_{m+1})$ for some integer m.

Proof: We first prove that $S_1^2 - S_0S_2 \neq 0$. On the contrary, suppose $S_1^2 - S_0S_2 = 0$. If one of S_0, S_1, or S_2 is zero, one of the others must be zero, which means that $\{S_n\}$ is the zero sequence. So we can assume that $S_0S_1S_2 \neq 0$. Now

$$\frac{S_1}{S_0} = \frac{S_2}{S_1} = \frac{pS_1 + S_0}{S_1} = p + \frac{S_0}{S_1},$$

and this implies that

$$\frac{S_1}{S_0} = \frac{p \pm \sqrt{p^2 + 4}}{2}.$$

But since $p^2 + 4$ is not a perfect square, S_1 / S_0 is irrational, which is a contradiction. Hence, $S_1^2 - S_0S_2 \neq 0$. Then, by Theorem 4, $\{S_n\}$ has the property of D'Ocagne if and only if $S_1^2 - S_0S_2 = S_1^2 - pS_0S_1 - S_0^2 = \pm 1$. Theorem 5 now follows from Lemma 3. □

Our final theorem characterizes those sequences $\{S_n\} = \{W_n(S_0, S_1; p, 1)\}$ that have the property of D'Ocagne.

Theorem 6: Let $|p| > 2$ and let $\{S_n\} = \{W_n(S_0, S_1; p, 1)\}$.

(a) If $p = 3$, then $\{S_n\}$ has the property of D'Ocagne if and only if $(S_0, S_1) = \pm(F_m, F_{m+2})$ for some integer m.

(b) If $p=-3$, then $\{S_n\}$ has the property of D'Ocagne if and only if $(S_0, S_1)=\pm(F_m, -F_{m+2})$ for some integer m.

(c) If $|p|>3$, then $\{S_n\}$ has the property of D'Ocagne if and only if $(S_0, S_1)=\pm(U_m, U_{m+1})$ for some integer m.

Proof: As in the proof of Theorem 5, it is straightforward to prove that $S_1^2 - S_0S_2 \neq 0$. Since $S_1^2 - S_0S_2 - S_1^2 - pS_0S_1 + S_0^2$, we see from Theorem 4 that $\{S_n\}$ has the property of D'Ocagne if and only if

$$S_1^2 - pS_0S_1 + S_0^2 = \pm 1. \tag{3.4}$$

Now part (a) follows immediately from Lemma 2. Writing $S_1^2 + 3S_0S_1 + S_0^2$ as $(-S_1)^2 - 3S_0(-S_1) + S_0^2$, we see that part (b) also follows from Lemma 2.

To prove part (c), we consider first the equation

$$S_1^2 - pS_0S_1 + S_0^2 = 1, \quad |p|>3. \tag{3.5}$$

By Lemma 4, the solutions of (3.5) are precisely the pairs $(S_0, S_1)=\pm(U_m, U_{m+1})$. Next we consider the equation

$$S_1^2 - pS_0S_1 + S_0^2 = -1, \quad |p|>3, \tag{3.6}$$

and solve for S_1 to obtain

$$S_1 = \frac{pS_0 \pm \sqrt{(p^2-4)S_0^2 - 4}}{2}, \quad |p|>3. \tag{3.7}$$

To complete the proof of (c), it is enough to prove that (3.7) yields no integer pairs (S_0, S_1). We accomplish this by proving that the generalized Pell equation

$$x^2 - (p^2-4)y^2 = -4, \quad |p|>3, \tag{3.8}$$

has no solutions. It suffices to consider only $p>3$.

To begin we assume that p is odd. Using Lemma 5, part (a), we find the convergents h_m/k_m, $0 \le m \le 5$, of the continued fraction expansion of $\sqrt{p^2-4}$ from the following table. In the table, the a_m are the partial quotients.

TABLE 1

m	a_m	h_m	k_m
0	$p-1$	$p-1$	1
1	1	p	1
2	$(p-3)/2$	$(p^2-p-2)/2$	$(p-1)/2$
3	2	p^2-2	p
4	$(p-3)/2$	$(p^3-2p^2-3p+4)/2$	$(p^2-2p-1)/2$
5	1	$(p^3-3p)/2$	$(p^2-1)/2$

Now by Theorem 1 and Lemma 5, part (a), and as is easily verified by substitution, (h_5, k_5) is a solution of $x^2-(p^2-4)y^2=1$. For integers $x_0 \ge 3$, $(x_0-1)^2>3$. This implies $x_0^2 > 2(x_0+1)$ which in turn implies $x_0 > \sqrt{2(x_0+1)}$. Consequently, taking $x_0 = (p^3-3p)/2 > 3$, we can replace the inequality in Theorem 3 by the more generous inequality $0<x<x_0$. But by trial we find that none of the pairs (h_m, k_m), $0 \le m \le 4$, is a solution of (3.8). Hence, by Theorems 2 and 3, (3.8) has no solutions.

To complete the proof, we consider (3.8) for $p \geq 4$, p even. For $p = 4$, equation (3.8) has no solutions since it has no solutions modulo 3. For $p > 4, p$ even, we use Lemma 5, part (b), to construct the following table for the continued fraction expansion of $\sqrt{p^2 - 4}$.

TABLE 2

m	a_m	h_m	k_m
0	$p-1$	$p-1$	1
1	1	p	1
2	$(p-4)/2$	$(p^2-2p-2)/2$	$(p-2)/2$
3	1	$(p^2-2)/2$	$p/2$

Now (h_3, k_3) is a solution of $x^2 - (p^2 - 4)y^2 = 1$, but, as is easily verified, none of the pairs (h_m, k_m), $0 \leq m \leq 2$, is a solution of (3.8). Hence, by the same reasoning as before, (3.8) has no solutions for $p > 4, p$ even. This completes the proof of Theorem 6. □

Our attempts to obtain analogs of Theorems 5 and 6 for $q \neq \pm 1$ have, to this point, been unsuccessful. This will continue to be the subject of our endeavors.

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CONVOLUTION SUMMATIONS FOR PELL AND PELL-LUCAS NUMBERS

A. F. Horadam
The University of New England, Armidale, Australia 2351
(*Submitted December 1998-Final Revision April 1999*)

1. RATIONALE

Pell and Pell-Lucas Convolution Numbers

Pell and Pell-Lucas polynomials $P_n(x)$ and $Q_n(x)$, respectively, were investigated in some detail in [3], which was followed up with a study of the properties [4] of the m^{th} convolution polynomials $P_n^{(m)}(x)$ and $Q_n^{(m)}(x)$.

These convolution polynomials may be defined [4] by generating functions, thus:

$$\sum_{n=0}^{\infty} P_{n+1}^{(m)}(x)y^n = (1-2xy-y^2)^{-(m+1)} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} Q_{n+1}^{(m)}(x)y^n = \left(\frac{2x+2y}{1-2xy-y^2}\right)^{m+1} \tag{1.2}$$

Putting $x=1$ yields the m^{th} convolution Pell and Pell-Lucas numbers $P_n^{(m)}(1)$ and $Q_n^{(m)}(1)$, respectively. Furthermore, if also $m=0$, then we have the Pell numbers $P_n^{(0)}(1)=P_n$ and the Pell-Lucas numbers $Q_n^{(0)}(1)=Q_n$.

Recurrence relations are given in (2.1) and (2.2) for $P_n^{(m)}$, and in (3.1) with (3.2) for $Q_n^{(m)}$ ($m \geq 1$ in both cases). Further specific work on P_n and Q_n was related to Morgan-Voyce numbers in [2].

Morgan-Voyce and Quasi Morgan-Voyce Polynomials

Morgan-Voyce polynomials $X_n(x)=B_n(x)$, $b_n(x)$, $C_n(x)$, and $c_n(x)$, and the four associated quasi Morgan-Voyce polynomials $Y_n(x)=\mathcal{B}_n(x)$, $\mathbf{b}_n(x)$, $\mathcal{C}_n(x)$, and $\mathbf{c}_n(x)$ are defined [1], [2] recursively by

$$X_{n+2}(x)=X_{n+1}(x)-3X_n(x),\ X_0(x)=a,\ X_1(x)=b, \tag{1.3}$$

and

$$Y_{n+2}(x)=Y_{n+1}(x)+3Y_n(x),\ Y_0(x)=a,\ Y_1(x)=b, \tag{1.4}$$

(a, b integers), in accordance with the following tabulation:

$X_n(x)$	a	b	$Y_n(x)$
$B_n(x)$	0	1	$\mathcal{B}_{n+1}(x)$
$b_n(x)$	1	1	$\mathbf{b}_{n+1}(x)$
$C_n(x)$	2	$2+x$	$\mathcal{C}_n(x)$
$c_n(x)$	-1	1	$\mathbf{c}_{n+1}(x)$

(1.5)

Only $\mathcal{B}_n(x)$ is required in this paper.

Our Challenge

Yet remaining for attention are some additional data to be obtained for $P_n^{(m)}(x)$ in Section 2, to be complemented by a corresponding, and slightly more thorough, analysis of properties of $Q_n^{(m)}(x)$ in Section 3.

In particular, our study of the row sums and column sums of $P_n^{(m)}$ and $Q_n^{(m)}$, as well as the rising diagonal sums $\sum_{m=1}^{n} P_m^{(n-m)}$ and $\sum_{m=1}^{n} Q_m^{(n-m)}$ will reveal some pleasing features.

For ease of reference and calculation, the short table of Pell number convolutions $P_n^{(m)}(1)$ which appeared in [4] will necessarily have to be repeated here as Table 1. Furthermore, a new table for Pell-Lucas number convolutions $Q_n^{(m)}(1)$, not previously recorded, will have to be incorporated as Table 2. Extensions of Tables 1 and 2 may be effected by employing the recurrence relations (2.1) and (3.1).

2. NEW PROPERTIES OF PELL CONVOLUTIONS

Prompted by an observation made by a colleague at the Rochester, New York State, meeting of the Fibonacci Association (July 1998)—an observation actually covered in [2]—we begin an investigation of certain summation properties of the Pell convolutions (Table 1).

Crucial to our presentation is the *recurrence relation* [4] for Pell convolutions,

$$P_n^{(m)} = 2P_{n-1}^{(m)} + P_{n-2}^{(m)} + P_n^{(m-1)} \quad (m \geq 1), \tag{2.1}$$

with

$$P_0^{(m)} = 0. \tag{2.2}$$

An abbreviated table for these convolutions, given in [2] and [4], is repeated here for the reader's convenience.

TABLE 1. Pell Convolution Numbers $P_n^{(m)}$

$n \backslash m$	0	1	2	3	4
1	1	1	1	1	1
2	2	4	6	8	10
3	5	14	27	44	65
4	12	44	104	200	340
5	29	131	366	810	1555

When required for formal algebraic purposes, values of $P_n^{(m)}$ could be extended for negative n in (2.1).

Basically, our concern is with **three** summation formulas, namely, those for rows, columns, and rising diagonals in Table 1.

Row Sums

Theorem 1: $\displaystyle\sum_{k=0}^{m} P_n^{(k)} = \frac{1}{2}\left\{P_{n+1}^{(m)} - \sum_{k=0}^{m} P_{n-1}^{(k)}\right\}$ (**n fixed**).

Proof: Write out (2.1) for successive values of m $(=0,1,\ldots,k)$ with n **fixed**. Add (the columns) to obtain

$$\sum_{k=0}^{m} P_n^{(k)} = 2\sum_{k=0}^{m} P_{n-1}^{(k)} + \sum_{k=0}^{m} P_{n-2}^{(k)} + \sum_{k=0}^{m-1} P_n^{(k)},$$

$$P_n^{(m)} + \sum_{k=0}^{m-1} P_n^{(k)} = 2\sum_{k=0}^{m} P_{n-1}^{(k)} + \sum_{k=0}^{m} P_{n-2}^{(k)} + \sum_{k=0}^{m-1} P_n^{(k)},$$

whence the result enunciated for k follows on replacing n by $n+1$.

***Example* ($n=3, m=4$):** Theorem $1 \to 2\times 155 = 340-30(=310)$.

Column Sums

Theorem 2: $\sum_{i=1}^{n} P_i^{(m)} = \frac{1}{2}\left\{P_{n+1}^{(m)} + P_n^{(m)} - \sum_{i=1}^{n+1} P_i^{(m-1)}\right\}$ (**m fixed**).

Proof: Proceed as in Theorem 1 (**m fixed**). Quickly it follows that

$$\begin{aligned} 2\sum_{i=1}^{n} P_i^{(m)} &= P_{n+2}^{(m)} - P_{n+1}^{(m)} - \sum_{i=1}^{n+2} P_i^{(m-1)} \\ &= P_{n+1}^{(m)} + P_n^{(m)} + P_{n+2}^{(m-1)} - \sum_{i=1}^{n+2} P_i^{(m-1)} \quad \text{by (2.1)} \\ &= P_{n+1}^{(m)} + P_n^{(m)} - \sum_{i=1}^{n+1} P_i^{(m-1)}. \end{aligned}$$

Hence, the theorem is demonstrated.

***Example* ($m=3, n=4$):** Theorem $2 \to 253 = \frac{1}{2}\{810+200-504\}$.

Note: For $m=0$ (excluded from Theorem 2), we have [3, (2.11)] where $x=1$,

$$\sum_{i=0}^{n} P_i = \frac{1}{2}\{P_{n+1} + P_n - 1\}. \tag{2.3}$$

Rising Diagonal Sums

Upward slanting (i.e., rising) diagonals are to be imagined in the mind's eye in Table 1. Accordingly, we seek $\sum_{m=1}^{n} P_m^{(n-m)}$. Specifically, these convolution number sums $\sum_{m=1}^{n} P_m^{(n-m)}$ turn out empirically to be the sequence

$$(0), 1, 3, 10, 33, 109, 360, \ldots = F_n(3), \tag{2.4}$$

where $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ $(F_0(x)=0, F_1(x)=1)$ are the Fibonacci polynomials.

Why is this so?

Theorem 3: $\sum_{m=1}^{n} P_m^{(n-m)} = F_n(3)$.

Proof (by induction): For small values $n = 1, 2, 3, 4$ (say), the validity of the theorem is clearly verifiable. Suppose it is true for $n = N$ (fixed). That is, assume

$$P_1^{(N-1)} + P_2^{(N-2)} + P_3^{(N-3)} + \cdots + P_{N-2}^{(2)} + P_{N-1}^{(1)} + P_N^{(0)} = F_n(3). \tag{A}$$

Apply the recurrence relation (2.1) repeatedly for $m = 1, 2, \ldots, N+1$. Arrange the summations in three columns, in accordance with (2.1). Then

$$\begin{aligned}\sum_{m=1}^{N+1} P_m^{(N+1-m)} &= P_1^{(N)} + P_2^{(N-1)} + P_3^{(N-2)} + \cdots + P_{N-1}^{(2)} + P_N^{(1)} + P_{N+1}^{(0)} \\ &= 2F_N(3) + F_{N-1}(3) + F_N(3) \quad \text{by (2.1) and (A)} \\ &= 3F_N(3) + F_{N-1}(3) \\ &= F_{N+1}(3) \text{ by the definition of } F_n(x) \text{ above.}\end{aligned}$$

Hence, the theorem is valid for $n = N+1$.

Consequently, Theorem 3 has been demonstrated for all n.

Indeed [2]

$$F_n(3) = \mathcal{B}_n(1) \equiv \mathcal{B}_n, \tag{2.5}$$

where $\mathcal{B}_n$ are quasi Morgan-Voyce numbers (of one kind) formed from the quasi Morgan-Voyce polynomials $\mathcal{B}_n(x)$ when $x = 1$.

Now the *Binet form* for these quasi Morgan-Voyce numbers is [2]

$$\mathcal{B}_n = (\alpha^n - \beta^n) / \Delta, \tag{2.6}$$

where α, β are the roots of the characteristic quasi Morgan-Voyce equation

$$\lambda^2 - 3\lambda - 1 = 0, \tag{2.7}$$

whence

$$\alpha = \frac{3+\sqrt{13}}{2},\ \beta = \frac{3-\sqrt{13}}{2},\ \alpha\beta = -1,\ \alpha+\beta = 3,\ \alpha-\beta = \Delta = \sqrt{13}. \tag{2.8}$$

Combining these ideas, we deduce that

Theorem 3a: $\displaystyle\sum_{m=1}^{n} P_m^{(n-m)} = \mathcal{B}_n = \frac{\alpha^n - \beta^n}{\Delta}$, where α, β, Δ are defined in (2.8).

Example $(\boldsymbol{n = 5})$***:*** $\displaystyle\sum_{m=1}^{5} P_m^{(5-m)} \equiv \frac{\alpha^5 - \beta^5}{\alpha - \beta} = 109 = \mathcal{B}_5$.

As an extension, the sum of the $\mathcal{B}_n$ (i.e., the sum of the sums of the rising diagonal convolutions) reduces, after algebraic maneuvering, to

Theorem 4: $\displaystyle\sum_{n=1}^{k} \mathcal{B}_n = \frac{1}{3}(\mathcal{B}_{k+1} + \mathcal{B}_k - 1)$.

Example $(\boldsymbol{k = 5})$***:*** Theorem $4 \to 156 = \frac{1}{3}(360 + 109 - 1)$.

Properties of the quasi Morgan-Voyce numbers $\mathcal{B}_n$ which are well documented in [2] may, because of Theorem 3a, be conceived in terms of sums of rising diagonal Pell convolutions. Recall that $\mathcal{B}_n = \mathcal{B}_n(x)$ when $x = 1$.

One might compare the forms on the right-hand side in Theorem 4 and equation (2.3).

3. NEW PROPERTIES OF PELL-LUCAS CONVOLUTIONS

Recurrence Relation

Coming now to the Pell-Lucas convolution polynomials $Q_n^{(m)}$, we must first discover their recurrence relation, a fundamental requirement which was not incorporated into [4].

Ordinarily, one might reasonably anticipate that the form of this recurrence relation would closely resemble that in (2.1). However, there is an unexpected scorpion-like twist to the tail of this formula.

Empirical evidence enables us to spot the following recurrence relation, cf. (2.1),

$$Q_n^{(m)} = 2Q_{n-1}^{(m)} + Q_{n-2}^{(m)} + 2(Q_n^{(m-1)} + Q_{n-1}^{(m-1)}) \quad (m \geq 1) \tag{3.1}$$

with

$$Q_0^{(m)} = 2. \tag{3.2}$$

Substituting $m = 1$ in (3.1) reduces the bracketed "tail" to $4P_n$.

On the basis of (3.1) and (3.2), we can construct a shortened convolution array for $Q_n^{(m)}$ (Table 2). Recall that a few simple values ($m = 1, 2$; $n = 1, 2, 3, 4, 5$) could readily have been calculated from the data in the table on page 68 in [4].

TABLE 2. Pell-Lucas Convolution Numbers $Q_n^{(m)}$

$n \backslash m$	0	1	2	3	4
1	2	4	8	16	32
2	6	24	72	192	480
3	14	92	384	1312	4004
4	34	304	1632	6848	24810
5	82	932	6120	30512	128344

Extension Example: $Q_6^{(1)} = 2Q_5^{(1)} + Q_4^{(1)} + 2(Q_6 + Q_5) = 1864 + 304 + 2(198 + 82) = 2728.$

Paralleling the triad of Theorems 1-3 in Section 2, we now explore the new territory for $Q_n^{(m)}$. Not unexpectedly, the forms of the corresponding enunciations are not quite so pleasing to the eye, because of (3.1).

Row Sums

Theorem 5: $\sum_{k=0}^{m} Q_n^{(k)} = Q_{n-1}^{(m+1)} - 2Q_{n-1}^{(m+1)} - 4\sum_{k=0}^{m} Q_{n-1}^{(k)} - 2(2^{m+1} - 1)$ (n **fixed**).

Proof: Proceed as for Theorem 1.

Example **($m = 3$, $n = 3$)*:*** $\sum_{k=0}^{3} Q_3^{(k)} = 4004 - 964 - 1176 - 62 (= 1802).$

Column Sums

Aesthetically, we are blessed with no more joy here than we were in Theorem 5.

Theorem 6: $\sum_{k=2}^{n-2} Q_k^{(m)} = \frac{1}{2}\{Q_n^{(m)} - Q_{n-1}^{(m)}\} - 2\sum_{k=2}^{n-1} Q_k^{(m-1)} - Q_n^{(m-1)} - 2^{m+2} \Big\}$ m **fixed**, $n \geq 2$.

Proof: As for Theorem 2.

Example **($m = 2$, $n = 5$)*:*** $456 = \frac{1}{2}\{6120 - 1632\} - 840 - 932 - 16.$

The requirements of realism necessitate the lower summation bound to be at $k = 2$. This is because $k = 0$ and $k = 1$, from (3.1), will yield terms $Q_0^{(m)}$ and $Q_{-1}^{(m)}$ which do not exist in Table 2.

Rising Diagonal Sums

Upward slanting (rising) diagonal sums are of the form $\sum_{m=1}^{n} Q_m^{(n-m)}$. Denote this by $\mathcal{Q}_n$ so that $\mathcal{Q}_1 = 2$. Then Table 2 reveals that

$$\{\mathcal{Q}_n\} = 2, 10, 46, 214, 994, 4618, \ldots, \tag{3.3}$$

whence one can spot the *recurrence relation*

$$\mathcal{Q}_{n+2} = 4\mathcal{Q}_{n+1} + 3\mathcal{Q}_n. \tag{3.4}$$

What can we know about this new sequence? Elementary procedures enable us to establish the relation

$$\mathcal{Q}_n = Z_n + Z_{n-1} \tag{3.5}$$

where the *Binet form* for Z_n is

$$Z_n = \frac{2}{\Delta_1}(\gamma^n - \delta^n), \tag{3.6}$$

in which γ, δ are the roots of the characteristic equation for (3.4), namely,

$$t^2 - 4t - 3 = 0; \tag{3.7}$$

so that

$$\gamma + \delta = 4, \ \gamma\delta = -3, \ \gamma - \delta = 2\sqrt{7} = \Delta_1. \tag{3.8}$$

Consequently, we have $(Z_0 = 0)$

$$\{Z_n\} = 2, 8, 38, 176, 818, \ldots, \tag{3.9}$$

with the same form of the recurrence relation for Z_n as that for $\mathcal{Q}_n$, i.e.,

$$Z_{n+2} = 4Z_{n+1} + 3Z_n. \tag{3.10}$$

Since $\mathcal{Q}_n$ is a composite of two Z-numbers, it is simpler to concentrate our energies on Z_n.

Generating Functions

One may readily obtain the generating function for the Z-numbers, to wit,

$$\sum_{k=1}^{\infty} Z_k x^k = 2(1-4x-3x^2)^{-1}, \tag{3.11}$$

thence (3.5) engenders

$$\sum_{k=1}^{\infty} \mathcal{Q}_n x^k = (2+2x)(1-4x-3x^2)^{-1}. \tag{3.12}$$

Summations

The Binet form (3.6) leads to

$$\sum_{k=1}^{n} Z_k = \frac{1}{6}\{Z_{n+1} + 3Z_n - 2\} \tag{3.13}$$

which, by (3.5) with (3.8), produces

$$\sum_{k=1}^{n} \mathcal{Q}_k = \frac{1}{3}(Z_{n+1} - 2). \tag{3.14}$$

Example: $\sum_{k=1}^{5} \mathcal{Q}_k = \frac{1}{3}(3800-2) = 1266.$

Simson Formulas

Invoking the application of (3.6) with (3.8), we derive the Simson formula

$$Z_{n+1}Z_{n-1} - Z_n^2 = -4(-3)^{n-1} \tag{3.15}$$

while employing (3.5) with (3.8) yields the Simson formula

$$\mathcal{Q}_{n+1}\mathcal{Q}_{n-1} - \mathcal{Q}_n^2 = -8(-3)^{n-2}. \tag{3.16}$$

Example $(\boldsymbol{n} = \mathbf{4})$***:*** Both sides of (3.14) have the value -72.

Observe, in passing, that

$$\mathcal{Q}_{n+1} - \mathcal{Q}_n = Z_{n+1} - Z_{n-1}. \tag{3.17}$$

Limits

From (3.6) and (3.5),

$$\lim_{n\to\infty} \frac{Z_{n+1}}{Z_n} = \lim_{n\to\infty} \frac{\mathcal{Q}_{n+1}}{\mathcal{Q}_n} = \gamma = 2+\sqrt{7}\,(\approx 4.646), \tag{3.18}$$

whereas by (2.6) and (2.8),

$$\lim_{n\to\infty} \frac{\mathcal{B}_{n+1}}{\mathcal{B}_n} = \alpha = \frac{3+\sqrt{13}}{2}(\approx 3.303). \tag{3.19}$$

Merely for curiosity we record that

$$\frac{\gamma}{\alpha} \approx 1.4 \quad \text{(one decimal place)}. \tag{3.20}$$

4. END-PIECE

Though the properties of the $Q_n^{(m)}$ will, by their very nature, be necessarily more complicated than those for $P_n^{(m)}$, it is nevertheless pleasing to unearth the rather unexpected conjunction of the Z's in (3.5). While other facets of the convolution numbers $P_n^{(m)}$ and $Q_n^{(m)}$ might be pursued, it seems reasonable to halt at this stage.

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A COMPOSITE OF GENERALIZED MORGAN-VOYCE POLYNOMIALS

Gospava B. Djordjević
University of Niš, Faculty of Technology, 16000 Leskovac, Yugoslavia
(Submitted December 1998-Final Revision January 2000)

1. INTRODUCTION

In this note we shall study a class of polynomials $\{R_{n,m}^{(r,u)}(x)\}$, where r and u are integers, n and m are nonnegative integers. The polynomials $\{R_{n,1}^{(r,u)}(x)\}$ and $\{R_{n,2}^{(r,u)}(x)\}$ were studied in [2]. Furthermore, the class of polynomials $\{R_{n,m}^{(r,u)}(x)\}$ involve a great number of known polynomials. Some of these polynomials are (see [2]):

$$\begin{aligned}
&R_{n,1}^{(0,1)}(x) = b_{n+1}(x),\\
&R_{n,1}^{(1,1)}(x) = B_{n+1}(x),\\
&R_{n,1}^{(2,1)}(x) = c_{n+1}(x),\\
&R_{n,1}^{(0,2)}(x) = C_n(x),\\
&R_{n,1}^{(0,0)}(x) = xB_n(x),\\
&R_{n,2}^{(1,0)}(x) = \phi_n(2,-1;x) \ \text{(see [3])},\\
&R_{n,m}^{(r,1)}(x) = P_{n,m}^{(r)}(x) \ \text{(see [4])},
\end{aligned}$$

where $B_n(x)$ and $b_n(x)$ are Morgan-Voyce polynomials (see [1]). In this paper we also consider the sequence of numbers $\{R_{n,3}^{(r,u)}(2)\}$.

2. POLYNOMIALS $\{R_{n,m}^{(r,u)}(x)\}$

First, we define the polynomials $\{R_{n,m}^{(r,u)}(x)\}$ by the following recurrence relation:

$$R_{n,m}^{(r,u)}(x) = 2R_{n-1,m}^{(r,u)}(x) - R_{n-2,m}^{(r,u)}(x) + xR_{n-m,m}^{(r,u)}(x), \ n \geq m, \tag{2.1}$$

with

$$R_{n,m}^{(r,u)}(x) = (n+1)r + u, \ n = 0, 1, \ldots, m-2, \ R_{m-1,m}^{(r,u)}(x) = mr + u + x. \tag{2.2}$$

From (2.1) and (2.2), we get

$$\begin{aligned}
R_{m,m}^{(r,u)}(x) &= u + (m+1)r + (2+u+r)x,\\
R_{m+1,m}^{(r,u)}(x) &= u + (m+2)r + (3+3u+4r)x,\\
R_{m+2,m}^{(r,u)}(x) &= u + (m+3)r + (4+6u+10r)x, \ldots.
\end{aligned} \tag{2.3}$$

Hence, we see that there is a sequence of numbers $\{c_{n,k}^{(r,u)}\}$ such that

$$R_{n,m}^{(r,u)}(x) = \sum_{k=0}^{[(n+1)/m]} c_{n+1,k}^{(r,u)} x^k, \tag{2.4}$$

where $c_{n,k}^{(r,u)} = 0$ for $k < 0$ or $k > [(n+1)/m]$.

If we take $x=0$ in (2.4), then we have

$$R_{n,m}^{(r,u)}(0)=c_{n+1,0}^{(r,u)}. \tag{2.5}$$

Furthermore, from (2.1), (2.2), and (2.4), we get

$$c_{n+1,k}^{(r,u)}=2c_{n,0}^{(r,u)}-c_{n-1,0}^{(r,u)},\ n\geq 1, m\geq 2, \tag{2.6}$$

with

$$c_{0,0}^{(r,u)}=u+r,\ c_{1,0}^{(r,u)}=u+2r. \tag{2.7}$$

The solution of the difference equation (2.6), using (2.7), is

$$c_{n,0}^{(r,u)}=u+(n+1)r,\ n\geq 0. \tag{2.8}$$

Again from (2.1) and (2.4), we have

$$c_{n,k}^{(r,u)}=2c_{n-1,k}^{(r,u)}-c_{n-2,k}^{(r,u)}+c_{n-m,k-1}^{(r,u)},\ k\geq 1, n\geq m. \tag{2.9}$$

3. COEFFICIENTS $c_{n,k}^{(r,u)}$

The main purpose in this section is to determinate the coefficients $\{c_{n,k}^{(r,u)}\}$ for $k\geq 1$. First of all, we shall write the coefficients $\{c_{n,k}^{(r,u)}\}$ in the following form:

TABLE 1

$n\setminus k$	0	1	2	$\dots$
0	$r+u$	$\dots$	$\dots$	$\dots$
1	$2r+u$	$\dots$	$\dots$	$\dots$
2	$3r+u$	$\dots$	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	$mr+u$	1	$\dots$	$\dots$
m	$(m+1)r+u$	$2+u+r$	$\dots$	$\dots$
$m+1$	$(m+2)r+u$	$3+3u+4r$	$\dots$	$\dots$
$m+2$	$(m+3)r+u$	$4+6u+10r$	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Now we shall prove the following theorem, using induction.

Theorem 3.1: The coefficients $c_{n,k}^{(r,u)}$ are given by

$$c_{n+1,k}^{(r,u)}=u\binom{n-(m-2)k}{2k}+r\binom{n+1-(m-2)k}{2k+1}+\binom{n-(m-2)k}{2k-1}, \tag{3.1}$$

where $n\geq 0$ and $0\leq k\leq [(n+1)/m]$.

Proof: For $n=0,1,\dots,m-2$, we obtain $k=0$. Then, from (2.8), we see that (3.1) is true. We shall assume that (3.1) is true for $n\ (n\geq 1)$. Then, by (2.9) and (3.1), we get

$$\begin{aligned}c_{n,k}^{(r,u)}&=2c_{n-1,k}^{(r,u)}-c_{n-2,k}^{(r,u)}+c_{n-m,k-1}^{(r,u)}\\&=\alpha_{n,k}+r\beta_{n,k}+u\gamma_{n,k},\end{aligned} \tag{3.2}$$

where

$$\alpha_{n,k} = 2\binom{n-1-(m-2)k}{2k-1} - \binom{n-2-(m-2)k}{2k-1} + \binom{n-m-(m-2)(k-1)}{2k-3},$$
$$\beta_{n,k} = 2\binom{n-(m-2)k}{2k+1} - \binom{n-1-(m-2)k}{2k+1} + \binom{n+1-m-(m-2)(k-1)}{2k-1}, \tag{3.3}$$
$$\gamma_{n,k} = 2\binom{n-1-(m-2)k}{2k} - \binom{n-2-(m-2)k}{2k} + \binom{n-m-(m-2)(k-1)}{2k-2}.$$

Using the known equality $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$, by equalities (3.3) we have

$$\alpha_{n,k} = \binom{n-(m-2)k}{2k-1},\ \beta_{n,k} = \binom{n+1-(m-2)k}{2k+1},\ \gamma_{n,k} = \binom{n-(m-2)k}{2k}. \tag{3.4}$$

Hence, (3.1) is true for all $n \geq 0$.

Corollary 3.1: If $m=1$, then (3.1) becomes (see [2], eq. (2.11)):

$$c_{n+1,k}^{(r,u)} = \binom{n+k}{2k-1} + r\binom{n+k+1}{2k+1} + u\binom{n+k}{2k}.$$

Corollary 3.2: If $m=2$ and n is even, then by (3.1) we have (see [2], eq. (6.3)):

$$c_{n/2+1,\,n/2}^{(r,u)} = u+r+n. \tag{3.5}$$

Using the standard methods, from (2.1) and (2.2), we can prove the following theorem.

Theorem 3.2: The polynomials $\{R_{n,m}^{(r,u)}(x)\}$, for $m \geq 2$, possess the following generating function:

$$\sum_{n=0}^{\infty} R_{n,m}^{(r,u)}(x)t^n = \frac{u+r-ut+xt^{m-1}}{1-2t+t^2-xt^m}. \tag{3.6}$$

Corollary 3.3: For $m=2$ in (3.6), we get the generating function of the polynomials $\{R_n^{(r,u)}(x)\}$,

$$\sum_{n=0}^{\infty} R_n^{(r,u)}(x)t^n = \frac{u+r+(x-u)t}{1-2t+t^2(1-x)}, \tag{3.7}$$

(see [2]).

The Binet Form for $R_{n,3}^{(r,u)}(x)$

For $m=3$ in (2.1) and (2.2), we get the polynomials $\{R_{n,3}^{(r,u)}(x)\}$ such that

$$R_{n,3}^{(r,u)}(x) = 2R_{n-1,3}^{(r,u)}(x) - R_{n-2,3}^{(r,u)}(x) + xR_{n-3,3}^{(r,u)}(x),\ n \geq 3, \tag{3.8}$$

with

$$R_{0,3}^{(r,u)}(x) = r+u,\ R_{1,3}^{(r,u)}(x) = 2r+u,\ R_{2,3}^{(r,u)}(x) = 3r+u+x. \tag{3.9}$$

Using the known methods, by (3.8) and (3.9), we find the Binet form for $\{R_{n,3}^{(r,u)}(x)\}$. That is, we can prove the following theorem.

Theorem 3.3: The Binet form for $\{R_{n,3}^{(r,u)}(x)\}$ is given by

$$R_{n,3}^{(r,u)}(x) = C_1\lambda_1^n + C_2\lambda_2^n + C_3\lambda_3^n, \tag{3.10}$$

where

$$\begin{aligned}\lambda_1 &= \frac{2}{3}+\frac{1}{3}(\alpha(x)+\beta(x)),\\ \lambda_2 &= \frac{2}{3}+\frac{\varepsilon_1}{3}(\alpha(x)+\varepsilon_1\beta(x)),\\ \lambda_3 &= \frac{2}{3}+\frac{\varepsilon_1}{3}(\varepsilon_1\alpha(x)+\beta(x)),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\alpha(x) &= \left(\frac{\Delta(x)+\sqrt{\Delta(x)^2-4}}{2}\right)^{1/3},\\ \beta(x) &= \left(\frac{\Delta(x)-\sqrt{\Delta(x)^2-4}}{2}\right)^{1/3},\\ \Delta(x) &= 27x-2,\\ \varepsilon_1 &= \frac{-1+i\sqrt{3}}{2},\end{aligned} \tag{3.12}$$

for $(\Delta(x))^2-4\geq 0$.

The coefficients C_1, C_2, and C_3 are the solutions of the equation

$$\lambda^3-2\lambda^2+\lambda-x=0, \tag{3.13}$$

with starting values (3.9). Namely, we get

$$C_i=\frac{\lambda_i x+(r+u)(\lambda_i^2+x-\lambda_i)+r\lambda_i^2}{2x+\lambda_i^3-\lambda_i},\ i=1,2,3. \tag{3.14}$$

4. SEQUENCE OF NUMBERS

The sequence of numbers $\{R_{n,3}^{(r,u)}(1)\}$ was studied in [2]. In this section we shall consider the sequence of numbers $\{R_{n,3}^{(r,u)}(2)\}$. Namely, for $m=3$ and $x=2$, from (2.1) and (2.2), we get the following difference equation,

$$a_{n+3}=2a_{n+2}-a_{n+1}+2a_n,\ n\geq 0, \tag{4.1}$$

with

$$a_0=r+u,\ a_1=2r+u,\ a_2=3r+u+2, \tag{4.2}$$

where $R_{n,3}^{(r,u)}(2)\equiv a_n$.

The characteristic equation for (4.1) is

$$\lambda^3-2\lambda^2+\lambda-2=0, \tag{4.3}$$

whose roots are

$$\lambda_1=2,\ \lambda_2=i,\ \lambda_3=-i, \tag{4.4}$$

with

$$\lambda_1+\lambda_2+\lambda_3=2,\ \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3=1,\ \lambda_1\lambda_2\lambda_3=2. \tag{4.5}$$

Hence, we have the following representation:

$$a_n=C_1\cdot 2^n+C_2\cdot i^n+C_3\cdot(-i)^n. \tag{4.6}$$

From (4.6) and (4.2), we get the following system of linear equations:

$$\begin{aligned} C_1 + C_2 + C_3 &= r + u, \\ 2C_1 + i\cdot C_2 - i\cdot C_3 &= 2r + u, \\ 4C_1 - C_2 - C_3 &= 3r + u + 2, \end{aligned}$$

whose solutions are

$$\begin{aligned} C_1 &= \frac{2}{5}(2r + u + 1), \\ C_2 &= \frac{1}{10}(r + 3u - 2 - i(2r + u - 4)), \\ C_3 &= \frac{1}{10}(r + 3u - 2 + i(2r + u - 4)). \end{aligned} \tag{4.7}$$

Hence, from (4.6), it follows that

$$a_n = \frac{2^{n+1}}{5}(2r + u + 1) + \frac{i^n}{10}(r + 3u - 2 - i(2r + u - 4)) + \frac{(-i)^n}{10}(r + 3u - 2 + i(2r + u - 4)). \tag{4.8}$$

Using (4.8), we find that

$$\begin{aligned} a_{4n} &= \frac{1}{5}(2^{4n+1}(2r + u + 1) + r + 3u - 2), \\ a_{4n+1} &= \frac{1}{5}(2^{4n+2}(2r + u + 1) + 2r + u - 4), \\ a_{4n+2} &= \frac{1}{5}(2^{4n+3}(2r + u + 1) - r - 3u + 2), \\ a_{4n+3} &= \frac{1}{5}(2^{4(n+1)}(2r + u + 1) - 2r - u + 4). \end{aligned} \tag{4.9}$$

Remark: It is interesting to consider a generalized numerical sequence $\{R_{n,m}(2^{m-2})\}$, $m \geq 2$, For example, if $m = 4$, we have

$$R_{n,4}(4) = C_1(-1)^n + C_2 2^n + C_3\left(\frac{1+i\sqrt{7}}{2}\right)^n + C_4\left(\frac{1-i\sqrt{7}}{2}\right)^n,$$

where the coefficients C_i, $i = 1, 2, 3, 4$, are the solutions of the following system:

$$\begin{aligned} &C_1 + C_2 + C_3 + C_4 = r + u, \\ &-C_1 + 2C_2 + \frac{1+i\sqrt{7}}{2}C_3 + \frac{1-i\sqrt{7}}{2}C_4 = 2r + u, \\ &C_1 + 4C_2 + C_3\left(\frac{1+i\sqrt{7}}{2}\right)^2 + C_4\left(\frac{1-i\sqrt{7}}{2}\right)^2 = 3r + u, \\ &-C_1 + 8C_2 + C_3\left(\frac{1+i\sqrt{7}}{2}\right)^3 + C_4\left(\frac{1-i\sqrt{7}}{2}\right)^3 = 4(r + 1) + u. \end{aligned}$$

ACKNOWLEDGMENT

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AMS Classification Numbers: 11B39, 26A24, 11B83

LETTER TO THE EDITOR

February 14, 2000

Professor Cooper,

I would like to bring your attention to an error in Paul S. Bruckman's article entitled "On the Degree of the Characteristic Polynomial of Powers of Sequences," *The Fibonacci Quarterly* **38.1** (2000):35-38. In particular, the following counterexample illustrates the error. In the notation of the article, let $U_n = 1+2^n+4^n$ with the characteristic polynomial $P_1(z) = (z-1)(z-2)(z-4)$ of degree $R_1 = 3$ with $m = 3$ roots. The main theorem then predicts

$$R_3 = \binom{5}{3} = 10.$$

However, on expanding $(U_n)^3$ we get $(U_n)^3 = 1+3*2^n+6*4^n+7*8^n+6*16^n+3*32^n+64^n$ with a characteristic polynomial of degree only 7, namely, $P_3(z) = (z-1)(z-2)(z-4)(z-8)(z-16)(z-32)(z-64)$.

The particular reasoning error in Bruckman's article revolves around the assumption that the products of the powers of the original eigenvalues are all distinct, indicated implicitly in the equation for $P_k(z)$ right before equation (7) on page 36 of the article. I brought attention to this issue in my recent article in your quarterly, noting that *if each root has a unique prime divisor that distinguishes it from the other roots*, then the final order of the power $(U_n)^k$ can be determined easily.

Regrettably, the general result stated in Bruckman's paper is erroneous.

For your careful consideration.

Adam Stinchcombe
Adirondack Community College
640 Bay Road
Queensbury, NY 12804

ON THE EXTENDIBILITY OF THE SET {1, 2, 5}

Omar Kihel
Département de mathématiques de statistiques, Pav. Vachon, Université Laval, G1K7P4, Québec, Canada
(*Submitted January 1999-Final Revision July 1999*)

Let t be a nonzero integer and S a set of positive integers. We say that S is a P_t-set if, for any two distinct elements x and y of S, the integer $xy+t$ is a perfect square. A P_t-set is extendible if there exists a positive integer $a \notin S$ such that $S \cup \{a\}$ is still a P_t-set.

The problem of extending P_t-sets is very old and dates back to the time of Diophantus (see Dickson [5], p. 513). The most spectacular result in this area is due to Baker and Davenport [3] who showed that the P_1-set $\{1, 3, 8, 120\}$ is nonextendible. Since then, several authors have made efforts to give a characterization of the P_t-sets (see references).

The P_{-1}-set $\{1, 2, 5\}$ was studied by Brown [4] who proved that this set is nonextendible. His method is based on deep results of Baker [3] and techniques of Grinstead [10]. In this paper we give another proof of the nonextendibility of the P_{-1}-set $\{1, 2, 5\}$ using only elementary number theory.

Suppose that there exists an integer a such that $\{1, 2, 5, a\}$ is a P_{-1}-set. Then the following system of equations

$$\begin{cases} a-1=Y^2, \\ 2a-1=Z^2, \\ 5a-1=X^2, \end{cases} \tag{1}$$

has integral solutions X, Y, Z, in $\mathbf{Z}$. Without loss of generality, we can suppose X, Y, Z are in $\mathbb{N}^*$. Elimination of a in system (1) yields

$$\begin{cases} Z^2-2Y^2=1, \\ 2X^2-5Z^2=3. \end{cases} \tag{2}$$

Lemma 1: If system (1) admits a solution a, then there exists an integer k such that $a = 12k+1$.

Proof: From system (1), it is clear that $a \equiv 1 \pmod 4$. The first equation in system (1) implies that $a \equiv \pm 1 \pmod 3$. If $a \equiv -1 \pmod 3$, then the second and third equations in system (1) imply that X and Z are both divisible by 3, which is impossible from the second equation in system (2). This gives $a \equiv 1 \pmod 3$. Then there exists an integer k such that $a = 12k+1$. □

After replacing a by $12k+1$ in system (1), we obtain

$$\begin{cases} 12k=Y^2, \\ 24k+1=Z^2, \\ 60k+4=X^2. \end{cases} \tag{3}$$

System (3) yields

$$\begin{cases} 3k=y^2, \\ 24k+1=z^2, \\ 15k+1=x^2, \end{cases} \tag{4}$$

where $X = 2x$, $Y = 2y$, and $Z = z$. Therefore,

$$x^2 + 3y^2 = z^2, \text{ where } (x, y, z) = 1. \tag{5}$$

It is well known that the solutions of equation (5) are $x = \pm(n^2 - 3m^2)$, $y = 2nm$, $z = n^2 + 3m^2$, with n and m two relatively prime integers.

The equation $y^2 = 3k$ implies $4n^2m^2 = 3k$ and $n^2 = \frac{3k}{4m^2}$. Therefore,

$$24k + 1 = z^2 = (n^2 + 3m^2)^2 = \left(\frac{3k}{4m^2} + 3m^2\right)^2$$

and

$$(24k+1)16m^4 = 9k^2 + 144m^8 + 72m^4k.$$

Hence,

$$9k^2 - 312m^4k - 16m^4(1 - 9m^4) = 0. \tag{6}$$

Equation (6) is of the second degree in k with integer coefficients. Since k is an integer, the discriminant $12^2 13^2 m^8 + 144m^4(1 - 9m^4) = 144m^4(160m^4 + 1)$ of the left side in (6) should be the square of an integer. That is, $160m^4 + 1 = t^2$ for some $t \in \mathbb{N}$.

Lemma 2: The only solution of $160m^4 + 1 = t^2$ is $(m, t) = (0, \pm 1)$.

Proof: Clearly $m = 0$, $t = \pm 1$ is a solution for the equation $160m^4 + 1 = t^2$. Without loss of generality, we can suppose $m > 0$ and $t > 0$ [of course, if (m, t) is a solution, $(\pm m, \pm t)$ is also a solution for our equation]. Put $M = 2m$, then we obtain the equation

$$10M^4 + 1 = t^2, \ M > 0, \ t > 0. \tag{7}$$

From $(t-1)(t+1) = 10M^4$, we have either

$$\begin{cases} t - 1 = 2a^4, \ t + 1 = 80b^4, \ M = 2ab \\ \text{or} \\ t - 1 = 80b^4, \ t + 1 = 2a^4, \ M = 2ab \end{cases} \tag{8}$$

or

$$\begin{cases} t - 1 = 10a^4, \ t + 1 = 16b^4, \ M = 2ab \\ \text{or} \\ t - 1 = 16b^4, \ t + 1 = 10a^4, \ M = 2ab, \end{cases} \tag{9}$$

where a and b are two positive integers.

System (8) gives

$$a^4 - 40b^4 = \pm 1. \tag{10}$$

A congruence mod 4 shows that the minus sign on the left side of equation (10) can be rejected, and from $(a^2 - 1)(a^2 + 1) = 40b^4$, since $a^2 + 1$ and $a^2 - 1$ are not squares in $\mathbb{N}$ and $a^2 + 1$ is not divisible by 4, we have $a^2 + 1 = 2c^4$, $a^2 - 1 = 20d^4$, and $b = cd$, which gives

$$10d^4 + 1 = C^2, \text{ where } C = c^2. \tag{11}$$

Equation (11) is of the same type as equation (7), and since $d < a < M$, one can apply the method of descent.

System (9) gives

$$5a^4 - 8b^4 = \pm 1. \tag{12}$$

A congruence mod 8 shows that this is impossible. □

Theorem 1: The P_{-1}-set $\{1, 2, 5\}$ is nonextendible.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2001. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-906 *Proposed by N. Gauthier, Royal Military College of Canada*

Consider the following $n \times n$ determinants,

$$\Delta_1(n) := \begin{vmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix};$$

$$\Delta_2(n) := \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix};$$

n is taken to be a positive integer and $\Delta_1(0)=1$, $\Delta_2(0)=0$, by definition. Prove the following:

a. $\Delta_1(n)=F_{2n+1}$;

b. $\Delta_2(n)=F_{2n}$.

B-907 ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia***

Prove that

$$F_1^{F_1}\cdot F_2^{F_2}\cdot F_3^{F_3}\cdot\ldots\cdot F_n^{F_n}\le e^{(F_n-1)(F_{n+1}-1)}.$$

B-908 ***Proposed by Indulis Strazdins, Riga Tech. University, Latvia***

The Fibonacci polynomials, $F_n(x)$, are defined by

$$F_0(x)=0,\ F_1(x)=1,\text{ and } F_{n+2}(x)=xF_{n+1}(x)+F_n(x)\quad\text{for } n\ge 0.$$

Prove the identity

$$F_{n+1}^2(x)-4xF_n(x)F_{n-1}(x)=x^2F_{n-2}^2(x)+(x^2-1)F_{n-1}(x)(xF_n(x)-F_{n-3}(x)).$$

B-909 ***Proposed by J. Cigler, Universität Wien, Austria***

Consider an arbitrary sequence of polynomials $p_k(x)$ of the form $p_k(x)=x^{a_k}(x-1)^{b_k}$, where a_k and b_k are integers satisfying $a_k+b_k=k$ and $a_k\ge b_k+1\ge 0$. Let $L_{n,k}$ be the uniquely determined numbers such that $x^n=\sum L_{n,k}p_k(x)$. Show that

$$F_n=\sum L_{n,k}F_{a_k-b_k},$$

where F_n are the Fibonacci numbers.

If all $a_k-b_k\in\{1,2\}$, then we have $F_n=\sum L_{n,k}$. This generalizes Proposition 2.2 of the paper "Fibonacci and Lucas Numbers as Cumulative Connection Constants" in *The Fibonacci Quarterly* **38.2** (2000):157-64.

B-910 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***

Solve the equation $p^n+1=\frac{k(k+1)}{2}$, where p is a prime number and k is a positive integer.

Remark: The case $p=2$ is Problem B-875 (*The Fibonacci Quarterly*, May 1999; see February 2000 for the solution).

SOLUTIONS

A Fibonacci Average Which Is a Lucas Number

B-889 ***Proposed by Mario DeNobili, Vaduz, Lichtenstein***
(Vol. 38, no. 1, February 2000)

Find 17 consecutive Fibonacci numbers whose average is a Lucas number.

Solution by Richard André-Jeannin, Cosnes et Romain, France

The identity $S_n=F_n+\cdots+F_{n+16}=F_{n+18}-F_{n+1}$ follows easily from Binet's formulas. Therefore, we have to find integers n and m such that $F_{n+18}-F_{n+1}=17L_m$. This relation implies that $F_{n+18}-F_{n+1}\equiv 0 \pmod{17}$. By induction, one can verify that $F_{n+18}\equiv -F_n \pmod{17}$; thus, we have $F_{n+18}-F_{n+1}\equiv -F_n-F_{n+1}=-F_{n+2}\equiv 0 \pmod{17}$, which implies that $n+2\equiv 0 \pmod 9$, since 9 is the rank of apparition of 17.

Let us define the sequence T_k by

$$T_k = \frac{S_{9K-2}}{17} = \frac{F_{9k+16} - F_{9k-1}}{17}.$$

Then we have

$$T_{-1} = \frac{S_{-11}}{17} = \frac{F_7 - F_{-10}}{17} = \frac{F_7 + F_{10}}{17} = 4 = L_3.$$

We shall prove that this is the only solution. It is straightforward to see that sequences such that F_{9k+r}, L_{9k+r}, or T_k satisfy the recurrence

$$X_k = 76X_{k-1} + X_{k-2}. \tag{1}$$

Assuming first that $k \geq 0$, we see that $L_8 < T_0 < L_9$ *and that* $L_{17} < T_1 < L_{18}$. By this and (1), it is clear that $L_{9k+8} < T_k < L_{9k+9}$ for every $k \geq 0$. Thus, T_k is not a Lucas number for $k \geq 0$.

On the other hand, we have

$$T_{-k} = (-1)^{k+1}\left(\frac{F_{9k-16} + F_{9k+1}}{17}\right).$$

by the formula $F_{-k} = (-1)^{k+1}F_k$. From this, we have

$$|T_{-k}| = \frac{F_{9k-16} + F_{9k+1}}{17}$$

for $k \geq 1$, and one can verify that $L_{11} < |T_{-2}| < L_{12}$ *and that* $L_{20} < |T_{-3}| < L_{21}$. Using this and (1), it is clear that $L_{9k-7} < |T_{-k}| < L_{9k-6}$ for $k \geq 2$. This concludes the proof.

Also solved by Brian D. Beasley, David M. Bloom, Paul S. Bruckman, L. A. G. Dresel, H.-J. Seiffert, Indulis Strazdins, and the proposer.

A Sum of Products Equals Zero

B-890 ***Proposed by Stanley Rabinowitz, Westford, MA***
(Vol. 38, no. 1, February 2000)

If $F_{-a}F_bF_{a-b} + F_{-b}F_cF_{b-c} + F_{-c}F_aF_{c-a} = 0$, show that either $a = b$, $b = c$, or $c = a$.

Solution by Paul S. Bruckman, Berkeley, CA

Let $U(a, b, c)$ denote the expression given in the statement of the problem. We prove the following identity:

$$U(a, b, c) = F_{a-b}F_{b-c}F_{c-a}. \tag{1}$$

Note that $F_n = 0$ iff $n = 0$; hence (assuming (1) is true), $U(a, b, c) = 0$ iff $a = b$, $b = c$, or $c = a$. Therefore, it suffices to prove (1).

The following known identities are employed:

$$5F_mF_n = L_{m+n} - (-1)^n L_{m-n}; \tag{2}$$

$$F_mL_n = F_{m+n} + (-1)^n F_{m-n}. \tag{3}$$

These lead to the following (symmetric) identity:

$$5F_xF_yF_z = F_{x+y+z} - (-1)^x F_{-x+y+z} - (-1)^y F_{x-y+z} - (-1)^z F_{x+y-z}. \tag{4}$$

In particular, $5F_{-a}F_bF_{a-b} = -(-1)^a F_{2a} + (-1)^b F_{2b} - (-1)^{a+b} F_{2b-2a}$.

Combining similar terms and simplifying, we obtain several cancellations, along with the following result:

$$5U(a,b,c) = -(-1)^{a+b}F_{2b-2a} - (-1)^{b+c}F_{2c-2b} - (-1)^{a+c}F_{2a-2c}. \tag{5}$$

On the other hand, $5F_{a-b}F_{b-c} = L_{a-c} - (-1)^{b+c}L_{a-2b+c}$; hence,

$$\begin{aligned} 5F_{a-b}F_{b-c}F_{c-a} &= F_{c-a}(L_{a-c} - (-1)^{b+c}L_{a-2b+c}) \\ &= F_0 + (-1)^{a+c}F_{2c-2a} - (-1)^{b+c}\{F_{2c-2b} + (-1)^{a+c}F_{2b-2z}\} \\ &= -(-1)^{a+b}F_{2b-2a} - (-1)^{b+c}F_{2c-2b} - (-1)^{a+c}F_{2a-2c}, \end{aligned}$$

which is seen to be the same expression as in (5). This establishes (1), hence the desired result.

*L. A. G. Dresel gave a solution similar to the featured one. However, he had a different proof for identity (1) based on his paper "Transformations of Fibonacci-Lucas Identities" in Applications of Fibonacci Numbers **5**:169-84.*

Also solved by Brian D. Beasley, L. A. G. Dresel, Hradec Králové, and the proposer.

A Lucas-Pell Congruence

B-891 ***Proposed by Aloysius Dorp, Brooklyn, NY (Vol. 38, no. 1, February 2000)***

Let $\langle P_n \rangle$ be the Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Find integers a, b, and m such that $L_n \equiv P_{an+b} \pmod{m}$ for all integers n.

Solution by H.-J. Seiffert, Berlin, Germany

Extend the recursion of the Pell numbers to $n \in \mathbf{Z}$, and define the Pell-Lucas numbers by $Q_0 = 2$, $Q_1 = 2$, and $Q_{n+2} = 2Q_{n+1} + Q_n$ for $n \in \mathbf{Z}$. For the integers a and b, where a is odd, let

$$m = \gcd(Q_a - 1, P_b - 2, P_{a+b} - 1).$$

We claim that

$$L_n \equiv P_{an+b} \pmod{m} \text{ for all } n \in \mathbf{Z}.$$

Since a is odd, we have [see A. F. Horadam & Bro. J. M. Mahon, "Pell and Pell-Lucas Polynomials," *The Fibonacci Quarterly* **23.1** (1985):7-20, equation (3.29)]

$$P_{a(n+2)+b} = Q_a P_{a(n+1)+b} + P_{an+b}, \quad n \in \mathbf{Z},$$

which by $Q_a \equiv 1 \pmod{m}$ implies that

$$P_{a(n+2)+b} \equiv P_{a(n+1)+b} + P_{an+b} \pmod{m}, \quad n \in \mathbf{Z}.$$

Hence, if $A_n = L_n - P_{an+b}$, $n \in \mathbf{Z}$, then $A_{n+2} \equiv A_{n+1} + A_n \pmod{m}$, $n \in \mathbf{Z}$. Since $A_0 = 2 - P_b \equiv 0 \pmod{m}$ and $A_1 = 1 - P_{a+b} \equiv 0 \pmod{m}$, it now easily follows that $A_n \equiv 0 \pmod{m}$ for all $n \in \mathbf{Z}$. This proves the above stated congruence.

Examples:

(a) With $a = -3$ and $b = 2$, we have $m = \gcd(-15, 0, 0) = 15$. The above result gives $L_n \equiv P_{-3n+2}$ for all $n \in \mathbf{Z}$.

(b) Taking $a = 5$ and $b = 2$, we have $m = \gcd(81, 0, 168) = 3$. Hence, $L_n \equiv P_{5n+2} \pmod{3}$ for all $n \in \mathbf{Z}$.

(c) With $a=5$ and $b=-6$, we have $m=\gcd(81,-72,0)=9$, so that $L_n \equiv P_{5n-6} \pmod 9$ for all $n \in \mathbf{Z}$.
(d) Take $a=7$ and $b=-6$. Then, $m=\gcd(477,-72,0)=9$. Hence, $L_n \equiv P_{7n-6} \pmod 9$ for all $n \in \mathbf{Z}$.
(e) With $a=9$ and $b=-8$, we have $m=\gcd(2785,-410,0)=5$, so that $L_n \equiv P_{9n-8} \pmod 5$ for all $n \in \mathbf{Z}$.

The featured solution contains the solutions given by the other solvers.

Also solved by Richard André-Jeannin, Brian D. Beasley, Paul S. Bruckman, L. A. G. Dresel, and the proposer.

A Perfect Square Only When Modulo 47

B-892 ***Proposed by Stanley Rabinowitz, Westford, MA***
(Vol. 38, no. 1, February 2000)

Show that, modulo 47, F_n^2-1 is a perfect square if n is not divisible by 16.

Solution by L. A. G. Dresel, Reading, England

We note that for $n=1$ to 8, $(F_n)^2-1$ is given successively by 0, 0, 3, 8, 24, 63, 168, 440, and that modulo 47 these numbers are congruent to the squares of 0, 0, 12, 14, 20, 4, 11, and 8, respectively. The identity (15b) of [1] gives $F_{8+m}-(-1)^m F_{8-m}=F_m L_8$, and since $L_8=47$ this gives $(F_{8+m})^2 \equiv (F_{8-m})^2 \pmod{47}$. Hence, modulo 47, we have $(F_n)^2-$ as a perfect square for $n=1$ to 15. Finally, the identity (15a) of [1] gives $F_{16+m}+(-1)^m F_{16-m}=L_m F_{16} \equiv 0 \pmod{47}$, since $F_{16}=F_8 L_8$. This completes the proof that $(F_n)^2-$ is a perfect square modulo 47 if n is not divisible by 16. When n is divisible by 16, $F_n \equiv 0 \pmod{47}$, and -1 is not a perfect square modulo 47.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section.* Chichester: Ellis Horwood Ltd., 1989.

Also solved by Richard André-Jeannin, Brian D. Beasley, Paul S. Bruckman, H.-J. Seiffert, and the proposer.

A Sum of Product of Fibonacci Numbers That Is Identically Zero

B-893 ***Proposed by Aloysius Dorp, Brooklyn, NY***
(Vol. 38, no. 1, February 2000)

Find integers a, b, c, and d so that

$$F_xF_yF_z+aF_{x+1}F_{y+1}F_{z+1}+bF_{x+2}F_{y+2}F_{z+2}+cF_{x+3}F_{y+3}F_{z+3}+dF_{x+4}F_{y+4}F_{z+4}=0$$

is true for all x, y, and z.

Solution by L. A. G. Dresel, Reading, England

Let

$$T(x,y,z)=F_xF_yF_z+aF_{x+1}F_{y+1}F_{z+1}+bF_{x+2}F_{y+2}F_{z+2}+cF_{x+3}F_{y+3}F_{z+3}+dF_{x+4}F_{y+4}F_{z+4},$$

giving

$$T(x,y,-4)=-3F_xF_y+2aF_{x+1}F_{y+1}-bF_{x+2}F_{y+2}+cF_{x+3}F_{y+3}$$

and

$$T(x, y, -3) = 2F_xF_y - aF_{x+1}F_{y+1} + bF_{x+2}F_{y+2} + dF_{x+4}F_{y+4}.$$

But, from the recurrences for F_x and F_y, we obtain

$$F_{x+3}F_{y+3} = (F_{x+2} + F_{x+1})(F_{y+2} + F_{y+1})$$

and

$$F_xF_y = (F_{x+2} - F_{x+1})(F_{y+2} - F_{y+1}).$$

Adding these together gives the identity

$$F_{x+3}F_{y+3} - 2F_{x+2}F_{y+2} - 2F_{x+1}F_{y+1} + F_xF_y = 0.$$

Denoting the left side of this by $D(x, y)$, we have $D(x, y) = 0$ for all x and y. We can now choose values for a, b, and c to make $T(x, y, -4) = -3D(x, y)$ identically, namely $a = 3$, $b = -6$, $c = -3$. If, in addition, we choose $d = 1$, we find that $T(x, y, -3) = 2D(x, y) + D(x+1, y+1)$. It follows that with these values of a, b, c, and d we have $T(x, y, -4) = 0$ and $T(x, y, -3) = 0$. Furthermore, since the recurrence for F_z gives

$$T(x, y, z+2) = T(x, y, z+1) + T(x, y, z),$$

we can prove by induction on z that $T(x, y, z) = 0$ for all x, y, and z. Hence, we have the identity

$$F_xF_yF_z + 3F_{x+1}F_{y+1}F_{z+1} - 6F_{x+2}F_{y+2}F_{z+2} - 3F_{x+3}F_{y+3}F_{z+3} + F_{x+4}F_{y+4}F_{z+4} = 0.$$

Paul Bruckman noted that the coefficients 3, -6, -3, and 1 correspond to the coefficients appearing in the recurrence relation satisfied by the cubes of the Fibonacci numbers. This is also seen by setting $x = y = z$ in the equation.

Also solved by Brian D. Beasley, Paul S. Bruckman, Hradec Králové, H.-J. Seiffert, and the proposer.

Addendum: We wish to belatedly acknowledge solutions from Paul S. Bruckman to Problems B-884, B-885, B-887, and B-888, and from H.-J. Seiffert to Problems B-878, B-879, B-880, B-881, and B-882.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-567 ***Proposed by Ernst Herrmann, Siegburg, Germany***

Let F_n denote the n^{th} Fibonacci number. For any natural number $n \geq 3$, the four inequalities

$$\frac{1}{F_n}+\frac{1}{F_{n+a_1}}<\frac{1}{F_{n-1}} \leq \frac{1}{F_n}+\frac{1}{F_{n+a_1-1}}, \tag{1}$$

$$\frac{1}{F_n}+\frac{1}{F_{n+a_1}}+\frac{1}{F_{n+a_1+a_2}}<\frac{1}{F_{n-1}} \leq \frac{1}{F_n}+\frac{1}{F_{n+a_1}}+\frac{1}{F_{n+a_1+a_2-1}}, \tag{2}$$

determine uniquely two natural numbers a_1 and a_2.

Find the numbers a_1 and a_2 dependent on n.

H-568 ***Proposed by N. Gauthier, Royal Military College of Canada, Kingston, Ontario***

The following was inspired by Paul S. Bruckman's Problem B-871 in *The Fibonacci Quarterly* (proposed in Vol. 37, no. 1, February 1999; solved in Vol. 38, no. 1, February 2000).

"For integers $n, m \geq 1$, prove or disprove that

$$f_m(n) \equiv \frac{1}{\binom{2n}{n}^2} \sum_{k=0}^{2n} \binom{2n}{n}^2 |n-k|^{2m-1}$$

is the ratio of two polynomials with integer coefficients

$$f_m(n) = P_m(n) / Q_m(n),$$

where $P_m(n)$ is of degree $\lfloor \frac{3m}{2} \rfloor$ in n and $Q_m(n)$ is of degree $\lfloor \frac{m}{2} \rfloor$; determine $P_m(n)$ and $Q_m(n)$ for $1 \leq m \leq 5$."

H-569 ***Proposed by Paul S. Bruckman, Berkeley, CA***

Let $\tau(n)$ and $\sigma(n)$ denote, respectively, the number of divisors of the positive integer n and the sum of such divisors. Let $e_2(n)$ denote the highest exponent of 2 dividing n. Let p be any odd prime, and suppose $e_2(p+1)=h$. Prove the following for all odd positive integers a:

$$e_2(\sigma(p^a)) = e_2(\tau(p^a)) + h - 1. \tag{*}$$

SOLUTIONS

Bi-Nomial

H-555 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 37, no. 3, August 1999)

Prove the following identity:

$$\begin{aligned}&(x^n+y^n)(x+y)^n\\&= -(-xy)^n + \sum_{k=0}^{[n/3]}(-1)^k C_{n,k}[xy(x+y)]^{2k}(x^2+xy+y^2)^{n-3k},\\&n=1,2,\dots,\end{aligned} \tag{1}$$

where

$$C_{n,k} = \binom{n-2k}{k}\cdot n/(n-2k).$$

Using (1), prove the following:

(a) $5^{n/2}L_n = -1 + \sum_{k=0}^{[n/3]}(-1)^k C_{n,k}5^k 4^{n-3k}, \quad n=2,4,6,\dots;$

(b) $5^{(n+1)/2}F_n = 1 + \sum_{k=0}^{[n/3]}(-1)^k C_{n,k}5^k 4^{n-3k}, \quad n=1,3,5,\dots;$

(c) $L_n = -1 + \sum_{k=0}^{[n/3]}(-1)^k C_{n,k}2^{n-3k}, \quad n=1,2,3,\dots.$

Solution by Reiner Martin, New York, NY

Let us write

$$P_n(x,y) = (x^n+y^n)(x+y)^n + (-xy)^n.$$

We have

$$P_{n+3}(x,y) = P_{n+2}(x,y)(x^2+xy+y^2) - P_n(x,y)[xy(x+y)]^2.$$

Our goal is to show that the corresponding recursion holds for the sum in (1).

Next, note that

$$C_{n+3,k} = C_{n+2,k} + C_{n,k-1}.$$

Using this identity, we get

$$\sum_{k=0}^{[(n+3)/3]}(-1)^k C_{n+3,k}[xy(x+y)]^{2k}(x^2+xy+y^2)^{n+3-3k}$$
$$=(x^2+xy+y^2)\sum_{k=0}^{[(n+2)/3]}(-1)^k C_{n+2,k}[xy(x+y)]^{2k}(x^2+xy+y^2)^{n+2-3k}$$
$$+[xy(x+y)]^{2k}\sum_{k=1}^{[n/3]+1}(-1)^k C_{n,k-1}[xy(x+y)]^{2(k-1)}(x^2+xy+y^2)^{n-3(k-1)}.$$

So, the sum in (1) satisfies the same recursion as $P_n(x, y)$. Since the cases $n = 1, 2, 3$ are trivial, identity (1) follows for all $n \geq 1$.

Finally, (a) and (b) follow from (1) by specializing to $x = \alpha$ and $y = -\beta$, while (c) follows by using $x = \alpha$ and $y = \beta$.

Also solved by H.-J. Seiffert and the proposer.

Some Operator

H-556 ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada (Vol. 37, No. 4, November 1999)***

Let $f(x)$ and $g(x)$ be continuous and differentiable in the immediate vicinity of $x = a (\neq 0)$ and assume that, for some positive integer k,

$$f^{(n)}(a) = g^{(n)}(a) = 0; \quad 0 \leq n \leq k-1.$$

By definition,

$$f^{(n)}(x) := \frac{d^n}{dx^n} f(x)$$

for any continuous and differentiable function $f(x)$. Further, assume that one of the following conditions holds for $n = k$:

a. $f^{(k)}(a) \neq 0,\ g^{(k)}(a) = 0$;

b. $f^{(k)}(a) = 0,\ g^{(k)}(a) \neq 0$;

c. $f^{(k)}(a) \neq 0,\ g^{(k)}(a) \neq 0$;

Introduce the differential operator $D := x\frac{d}{dx}$ and define, for m a nonnegative integer,

$$f_m(x) := D^m f(x), \quad g_m(x) := D^m g(x).$$

Prove that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f_k(a)}{g_k(a)}, \quad a \neq 0.$$

Solution by the proposer

Note first that

$$D^2 = x\frac{d}{dx} + x^2\frac{d^2}{dx^2}; \quad D^3 = x\frac{d}{dx} + 3x^2\frac{d^2}{dx^2} + x^3\frac{d^3}{dx^3}; \text{ etc.,}$$

the general term being, for $m \geq 1$,

$$D^m = \sum_{s=1}^{m} a_s(m) x^s \frac{d^s}{dx^s}, \tag{1}$$

as can readily be shown by induction on m. The set of coefficients $\{a_s(m): 1 \le s \le m; 1 \le m\}$ can be determined recursively, as follows. Consider

$$D^{m+1} = \sum_{s=1}^{m+1} a_s(m+1) x^s \frac{d^s}{dx^s}, \tag{2}$$

which follows from (1) with m replaced by $m+1$. But one also has that $D^{m+1} = D(D^m)$, where D^m is given by (1), so

$$\begin{aligned} D^{m+1} &= D \sum_{s=1}^{m} a_s(m) x^s \frac{d^s}{dx^s} \\ &= \sum_{s=1}^{m} a_s(m) \left[s x^s \frac{d^s}{dx^s} + x^{s+1} \frac{d^{s+1}}{dx^{s+1}} \right] \\ &= \sum_{s=1}^{m+1} [s a_s(m) + a_{s-1}(m)] x^s \frac{d^s}{dx^s}. \end{aligned} \tag{3}$$

The third line follows by introducing the definitions

$$a_s(m) = 0 : s > m; \quad a_0(m) = 0 : 1 \le m.$$

Equating the last line of (3) to (2) then gives the desired recurrence:

$$a_s(m+1) = s a_s(m) + a_{s-1}(m) : 1 \le s \le m+1, 1 \le m; \tag{4}$$

$a_0(m) = a_{m+1}(m) = 0$, $a_1(1) = 1$; $a_s(m)$ is thus a Stirling number of the second kind. Putting $s = m+1$ gives

$$a_{m+1}(m+1) = a_{m+1}(m) + a_m(m) = a_m(m), \tag{5}$$

so that $a_m(m) = 1$ by induction on m, since $a_1(1) = 1$. Now consider, for $k \ge 1$,

$$f_k(x) := D^k f(x) = \sum_{s=1}^{k} a_s(k) x^s \frac{d^s}{dx^s} f(x) = \sum_{s=1}^{k} a_s(k) x^s f^{(s)}(x)$$

and, similarly,

$$g_k(x) = \sum_{s=1}^{k} a_s(k) x^s g^{(s)}(x).$$

Evaluating at $x = a (\ne 0)$ and using $f^{(n)}(a) = g^{(n)}(a) = 0 : n = 0, 1, \ldots, k-1$ for some k, with either one of conditions (a), (b), or (c) in the statement of the problem assumed to hold, then gives

$$f_k(a) = \sum_{s=1}^{k} a_s(k) a^s f^{(s)}(a) = a_k(k) a^k f^{(k)}(a) = a^k f^{(k)}(a); \ a \ne 0,$$

with an equivalent result for $g_k(a)$:

$$g_k(a) = a^k g^{(k)}(a); \ a \ne 0.$$

Finally, invoke l'Hôpital's rule to find the limit of f/g as $x \to a$ to get

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(k)}(a)}{g^{(k)}(a)} = \frac{f_k(a)}{g_k(a)}; \ a \ne 0.$$

This completes the required proof. Note, in passing, that this new formulation of l'Hôpital's rule makes it much easier to resolve indeterminate forms when $f(x)$ and $g(x)$ are polynomials. This is due to the fact that $D^k x^v = v^k x^v$ for v arbitrary and k a nonnegative integer.

Also solved by P. Bruckman and H.-J. Seiffert

Generalize

H-557 ***Proposed by Stanley Rabinowitz, Westford, MA***
(Vol. 37, no. 4, November 1999)

Let $\langle w_n \rangle$ be any sequence satisfying the second-order linear recurrence $w_n = Pw_{n-1} - Qw_{n-2}$, and let $\langle v_n \rangle$ denote the specific sequence satisfying the same recurrence but with the initial conditions $v_0 = 2$, $v_1 = P$.

If k is an integer larger than 1, and $m = \lfloor k/2 \rfloor$, prove that, for all integers n,

$$v_n \sum_{i=0}^{m-1} (-Q^n)^i w_{(k-1-2i)n} = w_{kn} - (-Q^n)^m \times \begin{cases} w_0, & \text{if } k \text{ is even,} \\ w_n, & \text{if } k \text{ is odd.} \end{cases}$$

Note: This generalizes problem H-453.

Solution by Paul S. Bruckman, Berkeley, CA

Let

$$F(x; k, n) = \sum_{i=0}^{m-1} (-Q^n)^i x^{(k-1-2i)m}, \text{ where } m = [k/2]. \tag{1}$$

Then, after simplification,

$$F(x; k, n) = x^{(k-m)n}\{x^{mn} - (-1)^m Q^{mn} x^{-mn}\} / \{x^n + Q^n x^{-n}\}. \tag{2}$$

Let $\langle u_n \rangle$ denote the *fundamental sequence* associated with the given recurrence, that is, the sequence satisfying this same recurrence but with initial conditions $u_0 = 0$, $u_1 = 1$. Then $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, $v_n = \alpha^n + \beta^n$, where $\alpha = (P+\theta)/2$, $\beta = (P-\theta)/2$, and $\theta = (P^2 - 4Q)^{1/2}$. Note that $\alpha + \beta = P$ and $\alpha\beta = Q$.

We readily determine the following results:

$$F(\alpha; k, n) = \alpha^{(k-m)n}(\alpha^{mn} - (-1)^m \beta^{mn}) / v_n; \tag{3}$$

$$F(\beta; k, n) = (-1)^{m+1} \beta^{(k-m)n}(\alpha^{mn} - (-1)^m \beta^{mn}) / v_n. \tag{4}$$

Next, we define the following sums:

$$G_w(k, n) = v_n \sum_{i=0}^{m-1} (-Q^n)^i w_{(k-1-2i)n}, \tag{5}$$

$$G_v(k, n) = v_n \sum_{i=0}^{m-1} (-Q^n)^i v(k-1-2i)n. \tag{6}$$

Note that $G_v(k, n) = v_n\{F(\alpha; k, n) + F(\beta; k, n)\} = \{\alpha^{(k-m)n} - (-1)^m \beta^{(k-m)n}\}\{\alpha^{mn} - (-1)^m \beta^{mn}\}$ or, after simplification,

$$G_v(k, n) = v_{kn} - (-1)^m Q^{mn} v_{(k-2m)n}. \tag{7}$$

Note that $k = 2m$ if k is even, while $k = 2m+1$ if k is odd. Thus, we see that (7) is a special case of the statement of the problem, with $\langle w_n \rangle = \langle v_n \rangle$.

We now use the following relation between the general sequence $\langle w_n \rangle$ and the particular sequence $\langle v_n \rangle$:

$$w_N = \{w_n u_N - Q^n w_0 u_{N-n}\} / u_n. \tag{8}$$

This may be verified by noting that $u_{-N} = -Q^{-N} u_N$. In particular, we obtain

$$w_{kn} = \{w_n u_{kn} - Q^n w_0 u_{(k-1)n}\} / u_n. \tag{9}$$

Substituting the expression from (8) into the sum in (5), we obtain

$$u_n G_w(k,n) = v_n \sum_{i=0}^{m-1} (-Q^n)^i \{w_n u_{(k-1-2i)n} - Q^n w_0 u_{n(k-2-2i)n} \quad \text{or}$$

$$u_n G_w(k,n) = w_n G_u(k,n) - Q^n w_0 G_u(k-1,n), \tag{10}$$

where

$$G_u(k,n) = v_n \sum_{i=0}^{m-1} (-Q^n)^i u_{(k-1-2i)n}. \tag{11}$$

Note that

$$\begin{aligned} G_u(k,n) &= v_n\{F(\alpha;k,n) - F(\beta;k,n)\} / (\alpha - \beta) \\ &= \{\alpha^{(k-m)n} + (-1)^m \beta^{(k-m)n}\}\{\alpha^{mn} - (-1)^m \beta^{mn}\} / (\alpha - \beta) \\ &= \{\alpha^{kn} - \beta^{kn} - (-1)^m Q^{mn}(\alpha^{(k-2m)n} - \beta^{(k-2m)n})\} / (\alpha - \beta) \quad \text{or} \end{aligned}$$

$$G_u(k,n) = u_{kn} - (-1)^m Q^{mn} u_{(k-2m)n}. \tag{12}$$

We observe that (12) is another special case of the statement of the problem, with $\langle w_n \rangle = \langle u_n \rangle$.

Now, substituting the result of (12) into the expression in (10), we obtain the following:

$$\begin{aligned} u_n G_w(k,n) = {} & w_n u_{kn} - (-1)^m Q^{mn} w_n u_{(k-2m)n} \\ & - Q^n w_0 \{u_{(k-1)n} - (-1)^{m'} Q^{m'n} u_{(k-1-2m')n}\}, \end{aligned}$$

where $m' = [(k-1)/2]$.

(a) If $k = 2m$, then $m' = m-1$ and $k = 2m'+2$. Then

$$\begin{aligned} u_n G_w(k,n) &= w_n u_{kn} - Q^n w_0 u_{(k-1)n} - (-1)^m Q^{mn} w_0 u_n \\ &= u_n w_{kn} - (-1)^m Q^{mn} w_0 u_n, \end{aligned}$$

using the result in (9). Hence, $G_w(k,n) = w_{kn} - (-1)^m Q^{mn} w_0$ if k is even.

(b) If $k = 2m+1$, then $m' = m$ and $k = 2m'+1$. Then

$$\begin{aligned} u_n G_w(k,n) &= w_n u_{kn} - Q^n w_0 u_{(k-1)n} - (-1)^m Q^{mn} w_n u_n \\ &= u_n w_{kn} - (-1)^m Q^{mn} w_n u_n, \end{aligned}$$

using the result in (9). Hence, $G_w(k,n) = w_{kn} - (-1)^m Q^{mn} w_n$ if k is odd.

We may combine both formulas into one, as follows:

$$G_w(k,n) = w_{kn} - (-1)^m Q^{mn} w_{(k-2m)n}. \tag{13}$$

Also solved by H.-J. Seiffert and the proposer.

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