The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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REFEREES

In addition to the members of the Board of Directors and the members of the Editorial Board, the following mathematicians, engineers, and physicists have assisted *The Fibonacci Quarterly* by refereeing papers during the past year. Their special efforts are sincerely appreciated, and we apologize in advance for any names that inadvertently may have been overlooked or misspelled.

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REPORT ON THE NINTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

George M. Phillips

The Ninth International Conference on Fibonacci Numbers and Their Applications was held in Luxembourg 17-22 July 2000. It was the first conference without our dear Herta Freitag, who died on 25 January 2000. Herta graced every one of our previous eight conferences with her warm and friendly presence, and enlivened our sessions with her own contributions, all presented in her unique masterly fashion. In those eight conferences she was author or coauthor of sixteen papers, at least one at every conference, finishing on a high note at her last conference in Rochester with three papers. This previous and very special lady, whose humility was so natural that we took it for granted, loved mathematics and her fellow human beings, and brought out the best in us all. She was just as fascinated with those she met for the first time as those she had known longer, and her magical influence will remain with all who met her until the end of their days. Herta was already 75 years of age when she came to our first conference in Patras, Greece, accompanied by Debbie Harrell. We will always remember with gratitude that either Debbie or Adair Robertson, and usually both of them, came with Herta to every conference. No Ruth was more faithful to her Naomi than Debbie and Adair were to Herta, and these three friends shared many happy times together.

It has often been said that these conferences keep getting better and better, and this monotonic property continued to hold for the Ninth Conference. The overall quality of the papers has surely improved, and there were several really outstanding papers at this conference. As at previous conferences, the greatest representation was from the USA, which contributed 18 to number of those attending. There were four from Japan, three each from Germany and Hungary, two each from Australia, England, and Luxembourg, and one each from Austria, Brazil, Brunei, Canada, Cyprus, Finland, France, Italy, Latvia, New Zealand, Poland, Romania, Russia, Scotland, and Ukraine.

At our welcoming reception, we sampled some fine wines of Luxembourg, graciously provided by our main host, Professor Joseph Lahr. On the Tuesday evening we enjoyed a reception at the beautiful town hall of Luxembourg as guests of the Mayor, Mr. Paul Helminger. This was followed by a most interesting guided walking tour of the city. Our Wednesday afternoon and evening excursion took us on a journey through the Luxembourg countryside to the Castle of Vianden, followed by a visit to the nearby hydro-electric power station and a splendid meal at the Hotel Victor Hugo, generously provided by the "Société Electrique de l'Our" and its Director, Mr. Hubert Weis. The magnificent conference banquet, which was entirely sponsored by Madame Erna Hennicot-Schoebges, Minister for Culture, Higher Education and Research, was held in the stylish and elegant Castle of Bourglinster. After dinner, we had the especial pleasure of hearing an outstanding pianist, Matylda, daughter of our much respected colleague Andrzej Rotkiewicz, play pieces by Chopin and Magin. Following the conference, there was a find excursion to the German towns of Trier and Bernkastel along the river Moselle, including a visit to the library of the mathematician and theologian Nikolaus Cusanus.

We are most deeply indebted to Dr. Prosper Schroeder and members of the Institut Superieur de Technologie of the Grand Duchy of Luxembourg and, in particular, to Professors Joseph Lahr and Massimo Malvetti, and to Mrs. Josiane Meissner. This was indeed a smooth-running conference, thanks to the organizers, the local support and, as at several previous conferences, the cheerful and efficient work of "our own" Shirley Bergum and Patricia Solsaa.

Since our last conference, Cal Long has demitted office as President of the Fibonacci Association, to be succeeded by Fred Howard. Our Association is indeed very fortunate: one could not reasonably have expected a Washington to be followed by a Jefferson. God willing, we look forward to accompanying Fred to Flagstaff, Arizona, for our Tenth Conference in 2002, and to being welcomed there by Cal.

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HARMONIC SEEDS: ERRATA

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A natural number n is harmonic if its positive divisors have integral harmonic mean $H(n)$. In our paper [1], we gave an algorithm for determining all harmonic squarefree multiples of a given harmonic number, based on our concept of a harmonic seed, but our program to implement the algorithm was faulty.

Table 1 in [1] listed all harmonic seeds less than 10^{12}. The most prolific of these in producing harmonic squarefree multiples is 513480135168, as stated in [1], but there are **227** such multiples, not 216. The largest is N_1, as given in [1].

We wrote also of the harmonic squarefree multiples of the largest known 4-perfect number,

$$N_2 = 2^{37}3^{10}7 \cdot 11 \cdot 23 \cdot 83 \cdot 107 \cdot 331 \cdot 3851 \cdot 43691 \cdot 174763 \cdot 524287.$$

This has **320** harmonic squarefree multiples (not 169 as given in [1]), the largest of which (replacing the corresponding statement in [1]) is

$$\begin{aligned} N_3 &= N_2 \cdot 31 \cdot 37 \cdot 43 \cdot 61 \cdot 487 \cdot 3181 \cdot 25447 \cdot 50893 \cdot 49569781 \cdot 99139561 \\ &\approx 1.93 \cdot 10^{82}, \end{aligned}$$

with $H(N_3) = 99139561$.

Reference

1. G. L. Cohen & R. M. Sorli. "Harmonic Seeds." *The Fibonacci Quarterly* **36.5** (1998):386-90.

ON r-GENERALIZED FIBONACCI SEQUENCES AND HAUSDORFF MOMENT PROBLEMS

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1. INTRODUCTION

Let $a_0, a_1, \ldots, a_{r-1}$ with $a_{r-1} \neq 0$ $(r \geq 2)$ be fixed complex numbers. For any sequence of complex numbers $A = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})$, we define the r-generalized Fibonacci sequence $\{Y_A(n)\}_{n\geq 0}$ as follows: $Y_A(n) = \alpha_n$ for $n = 0, 1, \ldots, r-1$ and

$$Y_A(n+1) = a_0 Y_A(n) + a_1 Y_A(n-1) + \cdots + a_{r-1} Y_A(n-r+1) \tag{1}$$

for all $n \geq r-1$. Such sequences have been studied in the literature (see, e.g., [5], [6], and [8]-[12]).

Let $\gamma = \{\gamma_n\}_{0\leq n\leq p}$, where $p \leq +\infty$, be a sequence of real numbers. The Hausdorff moment problem associated with γ consists of finding a positive Borel measure μ such that

$$\gamma_n = \int_a^b t^n d\mu(t) \text{ for all } n\ (0 \leq n \leq p) \text{ and } Supp(\mu) \subset [a, b], \tag{2}$$

where $Supp(\mu)$ is the support of μ. If this problem has a solution μ, we say that μ is the *representing measure* of $\gamma = \{\gamma_n\}_{0\leq n\leq p}$. For $p = +\infty$, problem (2) is called the *full Hausdorff moment problem* (see, e.g., [1] and [2]). When $p < +\infty$, problem (2) is called the *truncated Hausdorff moment problem*, and it has been studied by Curto-Fialkow in [3], [4], and [7].

The aim of this paper is to study the Hausdorff moment problem on $[a, b]$ associated with an r-generalized Fibonacci sequence $\gamma = \{Y_A(n)\}_{n\geq 0}$. Some necessary and sufficient conditions for the existence of a positive Borel measure μ satisfying (2) are derived from those established for the full or truncated Hausdorff moment problem (see [1]-[4] and [7]).

This paper is organized as follows: In Section 2 we study the connection between the discrete positive measure and sequences (1). We also give two fundamental lemmas on representing measures of sequences (1). Section 3 deals with the full Hausdorff moment problem for sequences (1) using Cassier's method (see [2]). In Section 4 the Hausdorff moment problem for sequences (1) is studied using Curto-Fialkow's method (see [3]). Section 5 concerns the extension property of the truncated Hausdorff moment sequence to sequences (1).

2. SEQUENCES (1) AND REPRESENTING MEASURES

2.1. Discrete Positive Measure and Sequences (1)

Let $[a, b]$ be an interval of **R** and consider the following discrete positive measure

$$\mu = \sum_{j=0}^{r-1} \rho_j \delta_{x_j},$$

where $\rho_j \in \mathbf{R}$ and $Supp(\mu) \subset [a, b]$. Let $\{\alpha_n\}_{n\geq 0}$ be the sequence of moments of μ. Hence,

$$\alpha_n = \int_a^b t^n d\mu(t) = \sum_{j=0}^{r-1} \rho_j x_j^n \quad \text{for all } n \geq 0.$$

Consider the polynomial $P_\mu(X) = \prod_{j=0}^{r-1}(X - x_j) = X^r - a_0 X^{r-1} - a_1 X^{r-2} - \cdots - a_{r-1}$. It is clear that $x_0, x_1, \ldots, x_{r-1}$ are simple roots of $P_\mu(X)$. Thus, we have $x_j^{n+1} = a_0 x_j^n + a_1 x_j^{n-1} + \cdots + a_{r-1} x_j^{n-r+1}$ $(0 \leq j \leq r-1)$ for any $n \geq r-1$. This implies that

$$\alpha_{n+1} = a_0 \alpha_n + a_1 \alpha_{n-1} + \cdots + a_{r-1} \alpha_{n-r+1} \quad \text{for all } n \geq r-1.$$

Then the moment sequence $\{\alpha_n\}_{n\geq 0}$ of $\mu = \sum_{j=0}^{r-1} \rho_j \delta_{x_j}$ is sequence (1) with coefficients $a_0, \ldots, a_{r-1}$ and initial conditions $A = (\alpha_0, \ldots, \alpha_{r-1})$.

We can see then that problem (2) for sequences (1) is nothing more than the converse of the preceding assertions.

2.2. Two Fundamental Lemmas on Representing Measures of Sequences (1)

Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1) and suppose that μ is a representing measure of $\{Y_A(n)\}_{0\leq n\leq 2r}$. Then, for any n $(0 \leq n \leq r)$, we have

$$Y_A(n+r) = \int_a^b t^{n+r} d\mu(t) = \int_a^b t^n [a_0 t^{r-1} + a_1 t^{r-2} + \cdots + a_{r-1}] d\mu(t).$$

Thus, we have $\int_a^b t^n P(t) d\mu(t) = 0$ for all n $(0 \leq n \leq r)$, where $P(X)$ is the characteristic polynomial of sequence (1). The preceding relation implies that $\int_a^b P(t)^2 d\mu(t) = 0$. Since μ is a positive Borel measure, it follows that $Supp(\mu) \subset Z(P) = \{x \in [a, b]; P(x) = 0\}$. Hence, we have the following lemma.

Lemma 2.1: Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1). Suppose that μ is a representing measure of $\{Y_A(n)\}_{0\leq n\leq 2r}$. Then $Supp(\mu) \subset Z(P) = \{x \in [a, b]; P(x) = 0\}$, where P is the characteristic polynomial of $\{Y_A(n)\}_{n\geq 0}$.

We note that the proof of Lemma 2.1 is identical to the proof of Lemma 3.6 of [3], but in our case $P(X)$ is the characteristic polynomial of sequence (1). It follows from Lemma 2.1 that, if sequence (1) is a moment sequence of a positive Borel measure μ on $[a, b]$, then μ is a discrete measure with $Supp(\mu) \subset Z(P)$.

Using Lemma 2.1, we can prove the following property.

Lemma 2.2 (Lemma of Reduction): Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1) and let $P(X)$ be its characteristic polynomial. Let μ be a Borel measure on $[a, b]$. Then the following statements are equivalent.

(i) μ is a representing measure of $\{Y_A(n)\}_{n\geq 0}$ on $[a, b]$.
(ii) μ is a representing measure of $\{Y_A(n)\}_{0\leq n\leq 2r}$ on $[a, b]$.
(iii) μ is a representing measure of $A = (\alpha_0, \ldots, \alpha_{r-1})$ with $Supp(\mu) \subset Z(P) = \{x \in [a, b]; P(x) = 0\}$.

Proof: It is easy to see that $(i) \Rightarrow (ii)$. From Lemma 2.1, we derive that $(ii) \Rightarrow (iii)$. If $\alpha_j = \int_a^b t^j d\mu(t)$ for $0 \le j \le r-1$ and $Supp(\mu) \subset Z(P)$, then

$$Y_A(r) = \int_a^b [a_0 t^{r-1} + a_1 t^{r-2} + \cdots + a_{r-1}] d\mu(t) = \int_a^b t^r d\mu(t).$$

By induction we have $Y_A(n) = \int_a^b t^n d\mu(t)$ for any $n \ge r$. Consequently, μ is a representing measure of $\{Y_A(n)\}_{n\ge0}$ on $[a, b]$. □

Lemma 2.2 has two important consequences. First, we can use it to see that the full Hausdorff moment problem for sequences (1) may be reduced to the truncated Hausdorff moment problem studied in [3]. Second, we shall also see that the truncated Hausdorff moment problem for a sequence $\gamma = \{\gamma_j\}_{0\le j\le n}$ can be extended to sequence (1).

3. SEQUENCES (1) AND FULL HAUSDORFF MOMENT PROBLEM

Let $\{Y_A(n)\}_{n\ge0}$ be sequence (1) in [0, 1] and $\mathbf{R}[X]$ the $\mathbf{R}$-vector space of polynomials. Consider the linear functional $L: \mathbf{R}[X] \to \mathbf{R}$ defined by $L(X^n) = Y_A(n)$ for $n \ge 0$. From relation (1), we derive that $L(X^k P(X)) = 0$ for all $k \ge 0$, which implies that $L(QP) = 0$ for any Q in $\mathbf{R}[X]$. Hence, $I = (P)$ is an ideal of $\mathbf{R}[X]$ with $(P) \subset \ker L$. Conversely, let $\{V_n\}_{n\ge0}$ be a sequence of real numbers and $L: \mathbf{R}[X] \to \mathbf{R}$ a linear functional defined by $L(X^n) = V_n$. If there exists $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ such that $L(X^k P(X)) = 0$ for all $k \ge 0$, then $\{V_n\}_{n\ge0}$ is given by sequence (1) with coefficients $a_0, \ldots, a_{r-1}$ and initial conditions $A = (V_0, \ldots, V_{r-1})$.

Proposition 3.1: Let $\{V_n\}_{n\ge0}$ be a sequence of real numbers and $L: \mathbf{R}[X] \to \mathbf{R}$ a linear functional defined by $L(X^n) = V_n$ for $n \ge 0$. Then:

(i) If $\{V_n\}_{n\ge0}$ is given by sequence (1) with characteristic polynomial P, we have $I = (P) \subset \ker L$.

(ii) If there exists a polynomial $P = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ $(r \ge 2)$ such that $I = (P) \subset \ker L$, then $\{V_n\}_{n\ge0}$ is given by sequence (1) with coefficients $a_0, \ldots, a_{r-1}$ and initial conditions $A = (V_0, \ldots, V_{r-1})$.

Let $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ $(a_{r-1} \ne 0)$ and let $R(X)$ be in $\mathbf{R}[X]$ such that $R(X) \ge 0$ for all x in [0, 1]. It is well known that there exists A and B in $\mathbf{R}[X]$ such that $R(X) = A(X)^2 + X(1-X)B(X)^2$ (see, e.g., [2]). Since $A = Q_1 P + A_1$ and $B = Q_2 P + B_1$, where Q_1, Q_2, A_1, and B_1 are in $\mathbf{R}[X]$, with $\deg A_1 \le r-1$ and $\deg B_1 \le r-1$, we derive the following lemma.

Lemma 3.2: Let $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ $(a_{r-1} \ne 0)$ and $R(X) \in \mathbf{R}[X]$ such that $R(x) \ge 0$ for all x in [0, 1]. Then there exist Q, A_1, A_2 in $\mathbf{R}[X]$ such that $R(X) = Q(X)P(X) + A_1^2 + X(1-X)B_1^2$, where $\deg A_1 \le r-1$ and $\deg B_1 \le r-1$.

We recall that a real matrix $M = [m_{ij}]_{0\le i,\, j\le k}$ $(k \le +\infty)$ is *positive* if, for all (finite) real sequences $\{\xi_j\}_{0\le j\le p}$, we have

$$\sum_{i,\, j=0}^{k} m_{ij} \xi_i \xi_j \ge 0.$$

Note that $M \ge 0$. It was proved in [2] (see Theorem 1.2.3) that a real sequence $\{Y_n\}_{n\ge0}$ is a moment sequence of Borel positive measure μ on [0, 1] if and only if the two matrices

$$M = [Y_{i+j}]_{i,\, j\geq 0} \quad \text{and} \quad N = [Y_{i+j+1} - Y_{i+j+2}]_{i,\, j\geq 0}$$

are positive. From Proposition 3.1 and Lemma 3.2, we derive the following result.

Theorem 3.3: Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1). Then $\{Y_A(n)\}_{n\geq 0}$ is a moment sequence of a unique positive Borel measure μ on [0, 1] if and only if the following two matrices,

$$H(r) = [Y_A(i+j)]_{0\leq i,\, j\leq r-1} \quad \text{and} \quad K(r) = [Y_A(i+j+1) - Y_A(i+j+2)]_{0\leq i,\, j\leq r-1}, \tag{3}$$

are positive.

Proof: Suppose that the two matrices $H(r)$ and $K(r)$ as defined in (3) are positive. Let $L:\mathbf{R}[X] \to \mathbf{R}$ be a linear functional defined by $L(X^n) = V_n$ for $n \geq 0$. For any $R \in \mathbf{R}[X]$ such that $R(x) \geq 0$ for any $x \in [0, 1]$, Lemma 3.2 implies that $R = QP + A_1^2 + X(1-X)B_1^2$, where Q, A_1, $B_1 \in R[X]$ with $\deg A_1 \leq r-1$ and $\deg B_1 \leq r-1$. If $A_1(X) = \sum_{j=0}^{r-1} \lambda_j X^j$ and $B_1(X) = \sum_{j=0}^{r-1} \beta_j X^j$, then

$$L(R) = \sum_{0\leq i,\, j\leq r-1} Y_A(i+j)\lambda_i\lambda_j + \sum_{0\leq i,\, j\leq r-1} [Y_A(i+j+1) - Y_A(i+j+2)]\beta_i\beta_j.$$

Since $H(r)$ and $K(r)$ are positive, we obtain $L(R) \geq 0$. Consider the Banach space

$$(C([0, 1], \mathbb{R}),\ \|\cdot\|_{[0,1]})$$

of continuous functions on [0, 1], where $\|f\|_{[0,1]} = \sup_{x\in[0,1]} |f(x)|$. Then $|L(R)| \leq \|R\|_{[0,1]} L(1)$. This allows us to extend the linear functional L to a positive measure μ on [0, 1], where $L(f) = \int_0^1 f(t)d\mu(t)$ for any $f \in C([0, 1], \mathbb{R})$. Thus, we have $Y_A(n) = \int_0^1 t^n d\mu(t)$ for any $n \geq 0$. Conversely, if $A(X) = \sum_{j=0}^{r-1} \lambda_j X^j$, then $A(x)^2 \geq 0$ and $x(1-x)A(x)^2 \geq 0$ for any $x \in [0, 1]$. Thus,

$$\int_0^1 A(t)^2 d\mu(t) = \sum_{0\leq i,\, j\leq r-1} Y_A(i+j)\lambda_i\lambda_j \geq 0$$

and

$$\int_0^1 t(1-t)A(t)^2 d\mu(t) = \sum_{0\leq i,\, j\leq r_1} [Y_A(i+j+1) - Y_A(i+j+2)]\lambda_i\lambda_j \geq 0.$$

Therefore, the two matrices $H(r)$ and $K(r)$ are positive. □

Using an affine transformation, it was established in [2] (see Corollary 1.2.4) that a real sequence $\{Y_n\}_{n\geq 0}$ is a moment sequence of a positive Borel measure on $[a, b]$ if and only if the two matrices

$$M = [Y_{i+j}]_{i,\, j\geq 0} \quad \text{and} \quad N = [(a+b)Y_{i+j+1} - Y_{i+j+2} - abY_{i+j}]_{i,\, j\geq 0}$$

are positive. Thus, for sequences (1), we derive the following corollary from Theorem 3.3.

Corollary 3.4: Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1). Then $\{Y_A(n)\}_{n\geq 0}$ is a moment sequence of a unique positive Borel measure μ on $[a, b]$ if and only if the two matrices

$$H(r) = [Y_A(i+j)]_{0\geq i,\, j\leq r-1} \quad \text{and} \quad K(r) = [(a+b)Y_A(i+j+1) - Y_A(i+j+2) - abY_A(i+j)]_{0\leq i,\, j\leq r-1}$$

are positive.

Theorem 3.3 and Corollary 3.4 allow us to see that the full Hausdorff moment problem for sequence (1) can be reduced to the truncated Hausdorff moment problem, which is conformable with the result of Lemma 2.2.

4. SEQUENCES (1) AND TRUNCATED HAUSDORFF MOMENT PROBLEM

The Hankel matrices associated with a given real sequence $\gamma = \{\gamma_j\}_{j\geq 0}$ are defined by $H(n) = [\gamma_{i+j}]_{0\leq i,\, j\leq n}$, where $n \geq 0$. The (*Hankel*) *rank* of the Hankel matrix $A = [\gamma_{i+j}]_{0\leq i,\, j\leq k}$, where $(\gamma_0, \dots, \gamma_{2k})$ in $\mathbf{R}^{2k+1}$, denoted by $rank(\gamma)$ is defined as follows: If A is nonsingular, $rank(\gamma) = k+1$, and if A is singular, $rank(\gamma)$ is the smallest integer i $(1 \leq i \leq k)$ such that $V_i \in span\{V_1, \dots, V_{i-1}\}$, where $V_j = (\gamma_{j+l})_{l=0}^{k}$ is the j^{th} column of A. Thus, if A is singular, there exists a unique $(\phi_0, \dots, \phi_{i-1})$ in $\mathbf{R}^i$ such that $V_i = \phi_{i-1}V_0 + \cdots + \phi_0 V_{i-1}$. The polynomial $g_\gamma(X) = X^i - \phi_0 X^{i-1} + \cdots + \phi_{i-1}$ is called the *generating function* of $\gamma = (\gamma_0, \dots, \gamma_{2k})$ (see [3]).

Let $Y_A = \{Y_A(n)\}_{n\geq 0}$ be given by the sequence (1) and consider the full Hausdorff moment problem (2) for Y_A on $[a, b]$. From Lemma 2.2, this problem may be reduced to the following truncated Hausdorff moment problem: Find necessary and sufficient conditions for the existence of a positive Borel measure μ such that

$$Y_A(n) = \int_a^b t^n d\mu(t), \ (0 \leq n \leq 2r) \quad \text{and} \quad Supp(\mu) \subset [a, b].$$

The general case for the truncated Hausdorff moment problem has been studied in [3]. Consider the two Hankel matrices

$$A(r) = [Y_A(i+j)]_{0\leq i,\, j\leq r} \quad \text{and} \quad B(r) = [Y_A(i+j+1)]_{0\leq i,\, j\leq r}.$$

Since $Y_A(n+1) = \Sigma_{j=0}^{r-1} a_j Y_A(n-j)$ for $n \geq r-1$, the column vector $V(r+1, r) = (Y_A(r+1+j))_{j=0}^{r}$ is an element of the range of $A(r)$ and the (Hankel) $rank(Y_A)$ is equal to $rank(Y_A^{(r)})$, where $Y_A^{(r)} = (Y_A(0), \dots, Y_A(2r))$. Thus, we have $s := rank(Y_A) \leq r$. Hence, for sequence (1), the preceding Lemma 2.2 and Theorem 4.3 of [3] imply that the following are equivalent.

(i) There exists a Borel positive measure μ such that $Supp(\mu) \subset [a, b]$ and $Y_A(n) = \int_a^b t^n d\mu(t)$, $0 \leq n \leq 2r$.

(ii) There exists an r-atomic representing measure μ for Y_A such that $Supp(\mu) \subset [a, b]$.

(iii) $A(r) \geq 0$ and $bA(r) \geq B(r) \geq aA(r)$.

Consequently, we have the following result.

Theorem 4.1: Let $Y_A = \{Y_A(n)\}_{n\geq 0}$ be given by sequence (1), where $A = (\alpha_0, \dots, \alpha_{r-1})$ with $\alpha_0 > 0$ and let $s := rank(Y_A) = rank(Y_A^{(r)})$. The following statements are equivalent.

(i) There exists a Borel positive measure μ with $Supp(\mu) \subset [a, b]$ such that $Y_A(n) = \int_a^b t^n d\mu(t)$ for all $n \geq 0$.

(ii) There exists an s-atomic representing measure μ for Y_A such that $Supp(\mu) \subset [a, b]$.

(iii) $A(r) \geq 0$ and $bA(r) \geq B(r) \geq aA(r)$.

Suppose that $\{Y_A(n)\}_{n\geq 0}$ is a moment sequence of a positive Borel measure μ on $[a,b]$. Then, from Theorem 4.1, we derive that $\mu=\sum_{j=1}^{s}\rho_j\delta_{x_j}$, where $p_j\geq 0$ and $\{x_1,\dots,x_s\}\subset[a,b]\cap Z(P)$. The real numbers ρ_j are given by the following linear system of r equations

$$x_1^j\rho_1+x_2^j\rho_2+\cdots+x_s^j\rho_s=\alpha_j,\ 0\leq j\leq r-1.$$

5. FIBONACCI EXTENSION OF γ

Let $\gamma=(Y_0,\dots,Y_m)\in\mathbf{R}^{m+1}$ with $(Y_0>0)$. In [3], Curto-Fialkow give necessary and sufficient conditions for the existence of a positive Borel measure μ such that

$$Y_j=\int_a^b t^j d\mu(t)\ \text{ for } j=0,1,\dots,m \ \text{ and } \ Supp(\mu)\subset[a,b]. \tag{4}$$

Let $V_i=(Y_{i+j})_{0\leq j\leq k}$ $(i=0,\dots,k+1)$ be the i^{th} column vector of $A(k)$ and $r=rank(\gamma)$. Thus, $\{V_1,\dots,V_{r-1}\}$ are linearly independent, and there exists $(b_0,\dots,b_{r-1})\in\mathbf{R}^r$ such that $V_r=b_0V_{r-1}+\cdots+b_{r-1}V_0$. If $V(r,r-1)=(Y_{r+j})_{j=0}^{r-1}$, then we have $(b_0,\dots,b_{r-1})=A(r-1)^{-1}V(r,r-1)$. For $m=2k$ or $2k+1$, Curto-Fialkow proved in [3] that there exists a positive Borel measure μ satisfying (4) and $Supp(\mu)\subset[a,b]\cap Z(P_\gamma)$, where $r=rank(\gamma)$ and P_γ is the generating function of γ (see Theorem 4.1 and 4.3 of [3]). Since $Supp(\mu)\subset Z(P_\gamma)$, we derive that

$$Y_{j+1}=b_0Y_j+\cdots+b_{r-1}Y_{j-r+1}\ \text{ for } r-1\leq j\leq 2k.$$

Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1) defined by $A=(Y_0,\dots,Y_{r-1})$ and $Y_A(n+1)=b_0Y_A(n)+\cdots+b_{r-1}Y_A(n-r+1)$ for $n\geq 0$. This sequence, called the *Fibonacci extension of the truncated Hausdorff moment problem of* γ, satisfies

$$Y_A(n)=\int_a^b t^n d\mu(t)\ \text{ for all } n\geq 0.$$

Proposition 5.1: Let $\gamma=(Y_0,\dots,Y_m)$ with $Y_0>0$. Suppose that there exists a positive Borel measure μ which is a representing measure of γ. Then γ owns an extension $\{Y_A(n)\}_{n\geq 0}$ which is a sequence (1), where $r=rank(\gamma)$, $A=(Y_0,\dots,Y_{r-1})$ and the coefficients $b_0,\dots,b_{r-1}$ are given by the characteristic polynomial P_μ of μ.

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REFERENCES

1. N. I. Akhiezer. *The Classical Moment Problem*. New York: Hafner Publ. Co., 1965.
2. G. Cassier. "Problème des moments n-dimensionnel, mesures quasi-spectrales et semi-groupes." Thèse de 3ième Cycle, Université C. Bernard-Lyon **1** (1983) [or Publ. Dép. Math. (NS)-Lyon, 1983].
3. R. Curto & L. Fialkow. "Recursiveness, Positivity, and Truncated Moment Problems." *Houston J. Math.* **17.4** (1991):603-35.

4. R. Curto & L. Fialkow. "Solution of the Truncated Complex Moment Problem for Flat Data." *Mem. Amer. Math. Soc.* **119.568** (1996).
5. T. P. Dence. "On r-Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **27.2** (1987):137-43.
6. F. Dubeau, W. Motta, M. Rachici, & O. Saeki. "On Weighted r-Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **35.2** (1997):102-10.
7. L. Fialkow. "Positivity, Extensions and the Truncated Complex Moment Problem." In *Multivariable Operator Theory, Contemporary Mathematics* **185:**133-50. Providence, RI: American Math. Soc., 1995.
8. J. A. Jesk. "Linear Recurrence Relations, Part I." *The Fibonacci Quarterly* **1.1** (1963):69-74.
9. W. G. Kelly & A. C. Peterson. *Difference Equations: An Introduction with Application.* San Diego, CA: Academic Press, 1991.
10. C. Levesque. "On m^{th}-Order Linear Recurrences." *The Fibonacci Quarterly* **23.4** (1985): 290-95.
11. E. P. Miles. "Generalized Fibonacci Numbers and Associated Matrices." *Amer. Math. Monthly* **67** (1960):745-52.
12. M. Mouline & M. Rachidi. "Suites de Fibonacci Généralisées et Chaines de Markov." *Revista de la Real Academia de Ciencias Exatas, Fisicas y Naturales de Madrid* **89.1-2** (1995):61-77.

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ON THE LUCAS CUBES

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1. INTRODUCTION

A Lucas cube $\mathcal{L}_n$ can be defined as the graph whose vertices are the binary strings of length n without either two consecutive 1's or a 1 in the first and in the last position, and in which the vertices are adjacent when their Hamming distance is exactly 1. A Lucas cube $\mathcal{L}_n$ is very similar to the Fibonacci cube Γ_n which is the graph defined as $\mathcal{L}_n$ except for the fact that the vertices are binary strings of length n without two consecutive ones. The Fibonacci cube has been introduced as a new topology for the interconnection of parallel multicomputers alternative to the classical one given by the Boolean cube [4]. An attractive property of the Lucas cube of order n is the decomposition, which can be carried out recursively into two disjoint subgraphs isomorphic to Fibonacci cubes of order $n-1$ and $n-3$; on the other hand, the Lucas cube of order n can be embedded in the Boolean cube of order n. This implies that certain topologies commonly used, as the linear array, particular types of meshes and trees and the Boolean cubes, directly embedded in the Fibonacci cube, can also be embedded in the Lucas cube. Thus, the Lucas cube can also be used as a topology for multiprocessor systems.

Among many different interpretations, F_{n+2} can be regarded as the cardinality of the set formed by the subsets of $\{1, \ldots, n\}$ which do not contain a pair of consecutive integers; i.e., the set of the binary strings of length n without two consecutive ones, the *Fibonacci strings*.

If C_n is the set of the Fibonacci strings of order n, then $C_{n+2} = 0C_{n+1} + 10C_n$ and $|C_n| = F_{n+2}$.

A Lucas string is a Fibonacci string with the further condition that there is no 1 in the first and in the last position simultaneously. If $\hat{C}_n$ is the set of Lucas strings of order n, then $|\hat{C}_n| = L_n$, where L_n are the Lucas numbers for every $n > 0$. For $n \geq 1$, L_n can be regarded as the cardinality of the family of the subsets of $\{1, \ldots, n\}$ without two consecutive integers and without the couple $1, n$. We have

$$L_n = \sum_{k \geq 0} \binom{n-k}{k} \cdot \frac{n}{n-k}. \tag{1}$$

The Fibonacci cube Γ_n of order n is the bipartite graph whose vertices are the Fibonacci strings and two strings are adjacent when their Hamming distance is 1. Based on the decomposition of C_n, a Fibonacci cube of order n can be decomposed into a subgraph Γ_{n-1}, a subgraph Γ_{n-2} and F_{n-2} edges between the two subgraphs; this decomposition is represented by the relation $\Gamma_n = \Gamma_{n-1} \hat{+} \Gamma_{n-2}$. In a similar way, it is easy to decompose the set $\hat{C}_{n+3}$ into the sum $0C_{n+2} + 10C_n0$ and, therefore, to write $\mathcal{L}_n = \Gamma_{n-1} \hat{+} \Gamma_{n-3}$.

In Figure 1, we draw $\mathcal{L}_n$ for the first values of n; the circled vertices denote the vertices in Γ_n that are not in $\mathcal{L}_n$.

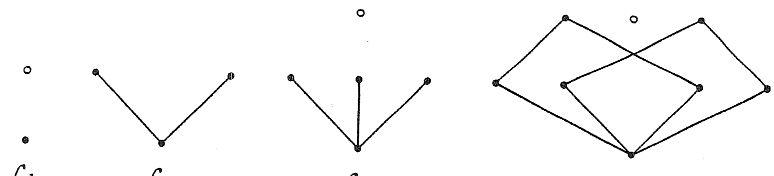

FIGURE 1

In this paper we determine structural and enumerative properties of the Lucas cubes such as the independence numbers of edges and vertices, the radius, the center, the generating function of a sequence of numbers connected to the partite sets, the asymptotic behavior of the ratio of the numbers of edges and vertices. A consequence of the properties on the independence numbers is that $\mathscr{L}_n$ is not Hamiltonian. Moreover, we obtain some identities involving Fibonacci and Lucas numbers which seem to be new. Finally, we introduce the Lucas semilattice and found its characteristic polynomial.

2. GENERAL PROPERTIES

The following identities hold: $L_n = F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2}$. For each of them there exists an immediate combinatorial interpretation in terms of Lucas cubes. The first says that the Lucas strings of length n beginning with 0 consist of the element 0 followed by any Fibonacci string of length $n-1$, while the Lucas strings beginning with 1 must start with the couple 10 and end with 0, and have any Fibonacci string of length $(n-3)$ between 10 and 0.

The second equality says that the Lucas n-strings are merely the Fibonacci n-strings not beginning and ending with the couple 10 and 01 simultaneously, and consisting of any Fibonacci $(n-4)$-string between these two extremal couples.

Using the first construction, we notice that the edges of $\mathscr{L}_n$ connecting pairs of vertices of Γ_{n-1} (resp. Γ_{n-3}) are just the edges of Γ_{n-1} (resp. Γ_{n-3}); moreover, for any vertex v of Γ_{n-3} there is exactly one edge connecting it to a vertex of Γ_{n-1}, i.e., the edge connecting $10v0$ to $00v0$. Let f_n and l_n denote the cardinalities of the edge sets of Γ_n and $\mathscr{L}_n$, respectively. Thus,

$$l_n = f_{n-1} + f_{n-3} + F_{n-1}$$

for $n \geq 3$; moreover, by direct computation we have $l_1 = 0$, $l_2 = 2$.

Since $f_n = f_{n-1} + f_{n-2} + F_n$, where $n \geq 2$, $f_0 = 0$, $f_1 = 1$, we have immediately $f_{n-1} < l_n < f_n$.

We will prove the following properties, analogous to the ones proved in [6] for the Fibonacci cubes.

The *eccentricity* of a vertex v in a connected graph G is the maximum distance between v and the other vertices, i.e., the number

$$e(v) := \max_{v \in V(G)} d(u, v);$$

the *diameter* of G is the maximal eccentricity when v runs in G, i.e.,

$$\operatorname{diam}(G) := \max_{u \in V(G)} e(v) = \max_{u,v \in V(G)} d(u, v);$$

the *radius* of G is the minimum eccentricity of the vertices of G, i.e.,

$$\operatorname{rad}(G) := \min_{u \in V(G)} e(v).$$

A vertex v is *central* if $e(v) = \operatorname{rad}(G)$; the *center* $Z(G)$ of G is the set of all central vertices; a string $\alpha = [a_1, \ldots, a_n]$ is said to be *symmetric* if $a_i = a_{n-i}$ for $i = 1, \ldots, n$.

For every n, we have that the diameter of $\mathcal{F}_n$ if equal to n; it is easy to prove that

$$\operatorname{diam}(\mathcal{L}_n) = \begin{cases} n & \text{for } n \text{ even,} \\ n-1 & \text{for } n \text{ odd.} \end{cases}$$

Moreover, we have the following proposition.

Proposition 1: The number of pairs of vertices at distance equal to the diameter is 1 for n even, $n-1$ for n odd.

Proof: Let n be even. The strings having 1 in all the odd or even positions are clearly at distance n and they are the only possible strings at distance n.

Let n be odd. We partition the strings having $\frac{n-1}{2}$ ones into two sets A and B, depending on whether the first element is 1 or 0. Assume that a string starts with 1; then it is possible to decompose it into $\frac{n-1}{2}$ subsequences 10 and one 0. This element 0 can be put after a subsequence 10 into $\frac{n-1}{2}$ ways. Clearly, similar considerations hold for the strings starting with 0. The difference now is that there are $\frac{n-1}{2}$ subsequences 01 and one 0 and the 0 can be put after the subsequences 01 in $\frac{n-1}{2}$ positions and also before the first 01, i.e., into $\frac{n+1}{2}$ positions. In any case, every string contains only one substring 00. A string of the first set has two strings of the second set at distance $n-1$, according to the position of 1 in the subsequence corresponding to 00. Thus, we obtain $2 \cdot \frac{n-1}{2}$ pairs of vertices at distance $n-1$.

Theorem 1: For $n \geq 1$, any Lucas cube $\mathcal{L}_n$ satisfies the following properties:

(i) $\operatorname{rad}(\mathcal{L}_n) = \lfloor \frac{n}{2} \rfloor$.

(ii) $Z(\mathcal{L}_n) = \{\hat{0}\}$.

Proof: ***(i)*** The distance $d(v, \hat{0})$ is the number of the elements 1 in the string v; hence, $e(\hat{0}) = \frac{n}{2}$ if n is even and $e(\hat{0}) = \frac{n-1}{2}$ if n is odd.

If $v \neq \hat{0}$, let k denote the number of the elements 1 in the string v. The set of the 0's (with the order induced by v) can be regarded as a Lucas string of length $n-k$ and precisely as the $\hat{0} \in \mathcal{L}_{n-k}$. In order to prove that $e(v) \geq \lfloor \frac{n}{2} \rfloor$, we consider the string v^* obtained by replacing the k elements 1 with 0 and the set of the 0's with a Lucas string of length $n-k$ at maximal distance from $\hat{0}$. Then $v^* \in \mathcal{L}_n$ and we have

$$d(v, v^*) = k + \left\lfloor \frac{n-k}{2} \right\rfloor = \begin{cases} \frac{n+k}{2} > \frac{n}{2} \geq \lfloor \frac{n}{2} \rfloor & n-k \text{ even,} \\ \frac{n+k-1}{2} \geq \frac{n}{2} \geq \lfloor \frac{n}{2} \rfloor & n-k \text{ odd.} \end{cases}$$

(ii) The previous construction of v^* shows that $e(v) > e(\hat{0})$ for $k > 1$ or for n odd. If $k = 1$ and n is even, we replace v^* with the string v^{**} defined in the following way: let h be the number of 0's on the left of the element 1 and l the number of 0's on the right. Without loss of generality, we can assume that h is even and l is odd. Let us replace the l-sequence of 0's, regarded as $\hat{0} \in \Gamma_l$, by a Fibonacci string β with maximal distance from $\hat{0}$ and replace the h-sequence of 0's, regarded as $\hat{0} \in \mathcal{L}_h$, by the Lucas string α obtained by concatenating $\frac{h}{2}$ couples 01.

The sequence $v^{**} = (\alpha 0 \beta)$ is again a Lucas string whose distance from v is greater than $\lfloor \frac{n}{2} \rfloor$. Indeed,

$$d(v, v^{**}) = \frac{h}{2} + 1 + \left\lceil \frac{l}{2} \right\rceil = \frac{h}{2} + 1 + \frac{l+1}{2} = \frac{h+l+1}{2} + 1 = \frac{n}{2} + 1. \quad \square$$

We have already noticed that the distance $d(v, \hat{0})$ is the number of the 1's in the string v. Thus, the summands in equality (1) can be regarded as the cardinalities of the sets of the n-strings at distance k from $\hat{0}$. Now, if n is odd, in $\mathcal{L}_n$ there are $\frac{n-1}{2}$ strings starting with 1 and $\frac{n+1}{2}$ strings starting with 0 at maximum distance $\frac{n-1}{2}$ from $\hat{0}$. Hence, the number N of Lucas strings of order n odd having maximal eccentricity is

$$N = \frac{n-1}{2} + \frac{n+1}{2} = n.$$

Then, in equality (1), the summand for $k = \frac{n-1}{2}$ becomes

$$\binom{\frac{n+1}{2}}{\frac{n-1}{2}} \cdot \frac{n}{\frac{n+1}{2}} = n$$

and we obtain a new combinatorial interpretation of the well-known identity

$$\binom{\frac{n+1}{2}}{\frac{n-1}{2}} = \frac{n+1}{2}.$$

Theorem 2: The number of symmetric Lucas strings of $\mathcal{L}_n$ is sim $\mathcal{L}_n = F_{\lfloor \frac{n}{2} \rfloor + 1 - (-1)^n}$.

Proof: Let n be even. In this case, we will write $n = 2m+2$, $m > 0$. Any symmetric string must begin and end with 0 and have in its center a couple 00; hence, sim $\mathcal{L}_{2m+2} = F_{m+1}$. Now let n be odd, $n = 2m+3$, $m > 0$. The symmetric strings having at the center 1 must have as center the triple 010 and two other 0's as extremal. The symmetric strings having at the center 0 satisfy the only condition of having two 0's as extremals; hence, sim $\mathcal{L}_{2m+3} = F_{m+1} + F_{m+2} = F_{m+3}$. In both cases, the statement holds. $\square$

3. ENUMERATIVE PROPERTIES

In [6] we denoted by E_n and O_n the sets of Fibonacci strings having an even or odd number of 1's, the partite sets of Γ_n, and by e_n, o_n their cardinalities. Now we use analogous notations. Thus, we denote by $\hat{E}_n$ and $\hat{O}_n$ the sets of vertices of $\mathcal{L}_n$ having an even or odd number of ones. Their cardinalities $\hat{e}_n$ and $\hat{o}_n$ are

$$\hat{e}_n := |\hat{E}_n| = \sum_{k \geq 0} \binom{n-2k}{2k} \frac{n}{n-2k}$$

and (2)

$$\hat{o}_n := |\hat{O}_n| = \sum_{k \geq 0} \binom{n-2k-1}{2k+1} \frac{n}{n-2k-1},$$

where $n \geq 2$ and obviously $\hat{e}_n + \hat{o}_n = L_n$.

Remark: The Lucas cubes $\mathcal{L}_n$ are defined properly only for $n \geq 1$; however, we shall define also $\hat{C}_0$ as the set formed by the string of length 0, i.e., the empty set. Since in an empty string there are no 1's, we set $\hat{e}_0 = 1$ and $\hat{o}_0 = 0$.

Using the construction related to the equality $L_n = F_{n+1} + F_{n-1}$, we see that the even (odd) vertices of Γ_{n-1} remain even (resp. odd) also in $\mathcal{L}_n$. In fact, by adjoining 0 before the strings of Γ_{n-1}, the number of 1's is not changed. On the contrary, the vertices of Γ_{n-3} becoming vertices of $\mathcal{L}_n$ change parity, because one element 1 is adjoined to their strings.

Furthermore, we have immediately the following relations:

$$\hat{e}_n = e_{n-1} + o_{n-3} \quad \text{and} \quad \hat{o}_n = o_{n-1} + e_{n-3}. \tag{3}$$

In [6] it was proved that

$$h_{n+2} = h_{n+1} - h_n, \; h_{n+3} = -h_n, \text{ and } h_{n+6} = h_n. \tag{4}$$

Consider $\hat{h}_n := \hat{e}_n - \hat{o}_n$. From (3) and (4), it follows immediately that $\hat{h}_{n+3} = h_{n+2} - h_n = -\hat{h}_n$ and $\hat{h}_n = h_{n-1} - h_{n-3} = \hat{h}_{n-1} - \hat{h}_{n-2}$. Moreover, we have the following theorem.

Theorem 3: The sequence $\{\hat{h}_n\}$ satisfies the properties:

(i) $\hat{h}_{n+6} = \hat{h}_n$, $n \geq 1$, and the repeated values are $1, -1, -2, -1, 1, 2$.

(ii) The generating function of $\hat{h}_n$ is $\hat{H}(x) = \frac{1}{1-x+x^2}$.

Proof: ***(i)*** $\hat{h}_{n+6} = -\hat{h}_{n+3} = \hat{h}_n$. By direct computation, we have: $\hat{e}_1 = 1$, $\hat{o}_1 = 0$; thus, $\hat{h}_1 = 1$. $\hat{e}_2 = 1$, $\hat{o}_2 = 2$; thus, $\hat{h}_2 = -1$. $\hat{e}_3 = 1$, $\hat{o}_3 = 3$; thus, $\hat{h}_3 = -2$. Also, $\hat{h}_4 = -\hat{h}_1 = -1$, $\hat{h}_5 = -\hat{h}_2 = 1$, and $\hat{h}_6 = -\hat{h}_3 = 2$. From the settings in the Remark, we have $\hat{h}_0 = 1$.

(ii) Let $\hat{H}(x) := \sum_{n=0}^{\infty} \hat{h}_n x^n$. We have

$$x\hat{H}(x) = \sum_{n=0}^{\infty} \hat{h}_n x^{n+1} \quad \text{and} \quad x^2\hat{H}(x) = \sum_{n=0}^{\infty} \hat{h}_n x^{n+2}.$$

Then it follows that

$$(1 - x + x^2)\hat{H}(x) = \hat{h}_0 + (\hat{h}_1 - \hat{h}_0)x + \sum_{n=2}^{\infty} (\hat{h}_n - \hat{h}_{n-1} + \hat{h}_{n-2})x^n = \hat{h}_0 + (\hat{h}_1 - \hat{h}_0)x = 1. \quad \square$$

The first values of these sequences are

n	0	1	2	3	4	5	6	7	8	9	10
F_n		1	1	2	3	5	8	13	21	34	55
L_n		1	3	4	7	11	18	29	47	76	123
$\hat{e}_n$	(1)	1	1	1	3	6	10	15	23	37	61
$\hat{o}_n$	(0)	0	2	3	4	5	8	14	24	39	62
$\hat{h}_n$	(1)	1	−1	−2	−1	1	2	1	−1	−2	−1

Remark: A standard argument enables us to obtain identities concerning positive integers starting from generating functions. In fact, we have, identically,

$$\frac{1}{1-x+x^2}=\frac{A}{1-\alpha x}+\frac{B}{1-\beta x},$$

where

$$\alpha=\frac{1+\sqrt{-3}}{2},\ \beta=\frac{1-\sqrt{-3}}{2},\ A=\frac{3-\sqrt{-3}}{6},\ B=\frac{3+\sqrt{-3}}{6},$$

which implies

$$\begin{aligned}\frac{1}{1-x+x^2}&=A(1+\alpha x+\alpha^2x^2+\cdots)+B(1+\beta x+\beta^2x^2+\cdots)\\&=1+(A\alpha+B\beta)x+(A\alpha^2+B\beta^2)x^2+\cdots;\end{aligned}$$

hence, $\hat{h}_n=A\alpha^n+B\beta^n$ for all n. Thus, for any $n\in\mathbf{N}$, we have

$$\begin{aligned}&A\alpha^{6m+1}+B\beta^{6m+1}=1,\ A\alpha^{6m+2}+B\beta^{6m+2}=-1,\ A\alpha^{6m+3}+B\beta^{6m+3}=-2,\\&A\alpha^{6m+4}+B\beta^{6m+4}=-1,\ A\alpha^{6m+5}+B\beta^{6m+5}=1,\ A\alpha^{6m}+B\beta^{6m}=2\end{aligned}$$

(in accord with the fact that $\alpha^3=\beta^3=-1$). Combining the equalities $\hat{h}_n=\hat{e}_n-\hat{o}_n$ and $\hat{e}_n+\hat{o}_n=L_n$, we obtain

$$\begin{cases}\hat{e}_n=\frac{L_n+\hat{h}_n}{2},\\ \hat{o}_n=\frac{L_n-\hat{h}_n}{2}.\end{cases}\tag{5}$$

From (2) and (5), we have immediately the following identities concerning the Lucas numbers.

Proposition 2:

$$L_n=2\sum_{k\ge0}\binom{n-2k}{2k}\frac{n}{n-2k}-\hat{h}_n;\ \ L_n=2\sum_{k\ge0}\binom{n-2k-1}{2k+1}\frac{n}{n-2k-1}+\hat{h}_n;$$

$$L_n=2\sum_{k\ge0}\binom{n-2k}{2k}\frac{n}{n-2k}-(A\alpha^n+B\beta^n);\ L_n=2\sum_{k\ge0}\binom{n-2k-1}{2k+1}\frac{n}{n-2k-1}+A\alpha^n+B\beta^n.$$

In [6] it was proved that

$$h_n=2e_n-F_{n+2}.^*\tag{6}$$

Furthermore, from (4) and (6) we can obtain the following proposition.

Proposition 3:

(i) $F_{n+2}=\sum_{k\ge0}\binom{n+3-2k}{2k}\frac{n+3}{n+3-2k}-\sum_{k\ge0}\binom{n+3-2k}{2k}+\sum_{k\ge0}\binom{n+1-2k}{2k}.$

(ii) $F_{n+2}=\sum_{k\ge0}\binom{n-2k}{2k+1}\frac{n+1}{n-2k}+\sum_{k\ge0}\binom{n+1-2k}{2k}-\sum_{k\ge0}\binom{n-1-2k}{2k}.$

Proof: Let

$$\Sigma=\sum_{k\ge0}\binom{n-2k}{2k}\frac{n}{n-2k},\ \ \Sigma'=\sum_{k\ge0}\binom{n-2k-1}{2k+1}\frac{n}{n-2k-1}.$$

---

* Indeed in [6] this equality is written $h_n=2e_n-F_n$ because in [6] the Fibonacci numbers are defined by the recurrence $F_0=1,\ F_1=2,\ F_{n+2}=F_{n+1}+F_n$.

We have $L_n = 2\Sigma - h_{n-1} + h_{n-3} = 2\Sigma - 2e_{n-1} + F_{n+1} + 2e_{n-3} - F_{n-1}$,

$$F_{n-1} = \Sigma - e_{n-1} + e_{n-3} = \sum_{k\geq 0}\binom{n-2k}{2k}\frac{n}{n-2k} - \sum_{k\geq 0}\binom{n-2k}{2k} + \sum_{k\geq 0}\binom{n-2-2k}{2k}$$

and so the first statement is proved; moreover, we have $L_n = 2\Sigma' + h_{n-1} - h_{n-3} = 2\Sigma' + 2e_{n-1} - F_{n+1} - 2e_{n-3} + F_{n-1}$, thus we have

$$F_{n+1} = \Sigma' + e_{n-1} - e_{n-3} = \sum_{k\geq 0}\binom{n-2k-1}{2k+1}\frac{n}{n-2k-1} + \sum_{k\geq 0}\binom{n-2k}{2k} - \sum_{k\geq 0}\binom{n-2-2k}{2k},$$

hence the second statement is proved. □

4. INDEPENDENCE NUMBERS

Recall that in a connected graph the *vertex independence number* $\beta_0(G)$ is the maximum among all cardinalities of independent sets of vertices of G, the *edge independence number* $\beta_1(G)$ is the maximum among all cardinalities of independent sets of edges of G. We have the following theorem.

Theorem 4: Let $\beta_1(\mathscr{L}_n)$ be the edge independence number of $\mathscr{L}_n$. Then

$$\beta_1(\mathscr{L}_n) = \left\lfloor \frac{L_n - 1}{2} \right\rfloor.$$

Proof: Let L_n be odd. Since $L_n = F_{n+1} + F_{n-1}$, then F_{n+1} and F_{n-1} have different parities. In [5] it was proved that the Fibonacci cubes have a Hamiltonian cycle in the case of an even number of vertices and a cycle containing all the vertices but one in the odd case [5]. Thus, it is possible to determine $\frac{L_n-1}{2}$ independent edges; since this is the maximum, the result holds.

When L_n is even, it follows from the sequences of Fibonacci and Lucas numbers that F_{n+1} and F_{n-1} are both odd. In this case, the Fibonacci cubes Γ_{n-1} and Γ_{n-3} have cycles of length $F_{n+1} - 1$ and $F_{n-1} - 1$, respectively, and we can find $\frac{L_n-2}{2}$ independent edges. By Theorem 3, we have $|\hat{e}_n - \hat{o}_n| = 2$ when L_n is even. Then the order of one of the partite sets is $\frac{L_n-2}{2}$, which coincides with the maximal number of independent edges. Thus, the maximum number of independent edges is exactly $\frac{L_n-2}{2}$. □

We immediately have the following.

Corollary 1: $\mathscr{L}_n$ is not Hamiltonian.

Proof: It is obvious in the case of L_n odd. In the even case, it follows from Theorem 4 that the maximum number of independent edges is $\frac{L_n-2}{2}$. This excludes that $\mathscr{L}_n$ is Hamiltonian. □

Corollary 2: $\beta_1(\mathscr{L}_n) = \min(\hat{e}_n, \hat{o}_n)$.

Proof: From Theorem 3, it follows that $|\hat{e}_n - \hat{o}_n|$ is equal to 1 or 2, depending on whether L_n is odd or even. Since $L_n = \hat{e}_n + \hat{o}_n$, $\lfloor \frac{L_n-1}{2} \rfloor$ coincides with $\min(\hat{e}_n, \hat{o}_n)$. The result follows from Theorem 4. □

We are now able to prove the following theorem, analogous to the one in [6].

Theorem 5: Let $\beta_0(\mathcal{L}_n)$ be the vertex independence number of $\mathcal{L}_n$. Then $\beta_0(\mathcal{L}_n) = \max(\hat{e}_n, \hat{o}_n)$.

Proof: By Theorem 3, $\hat{e}_n$ and $\hat{o}_n$ are always distinct. Without loss of generality, we can assume $\hat{e}_n < \hat{o}_n$. Thus, by Theorem 4 and Corollary 2, $\mathcal{L}_n$ contains $\hat{e}_n$ independent edges and every vertex $v \in \hat{E}_n$ can be paired with a vertex $v' \in \hat{O}_n$. This implies that a set A of independent vertices cannot have cardinality greater than $\hat{o}_n$, because both v and v' cannot belong to A. □

5. ASYMPTOTIC BEHAVIOR

For the applications, it seems to be useful to consider the indices

$$i(\Gamma_n) := \frac{f_n}{F_{n+2}}, \quad i(\mathcal{L}_n) := \frac{l_n}{L_n}$$

and their asymptotic behavior. In order to prove that $\lim_{n\to\infty} i(\mathcal{L}_n) = +\infty$, it is convenient to express f_n and l_n in a direct way instead of by recurrence, for instance, by writing

Proposition 4: The following equalities hold:

(i) $f_n = \dfrac{nF_{n+1} + 2(n+1)F_n}{5}$ for $n \geq 2$;

(ii) $l_n = nF_{n-1}$ for $n \geq 3$.

Proof: ***(i)*** Indeed,

$$\frac{2F_3 + 2(2+1)F_2}{5} = 2 = f_2 \quad \text{and} \quad \frac{3F_4 + 2(3+1)F_3}{5} = 5 = f_3.$$

Now assume by induction that

$$f_{n+1} = \frac{(n-1)F_n + 2nF_{n-1}}{5} \quad \text{and} \quad f_{n-2} = \frac{(n-2)F_{n-1} + 2(n-1)F_{n-2}}{5}.$$

Then

$$f_n = f_{n-1} + f_{n-2} + F_n = \frac{(n+4)F_n + nF_{n-1} + (2n-2)(F_{n-1} + F_{n-2})}{5} = \frac{(2n+2)F_n + n(F_n + F_{n-1})}{5}.$$

(ii)
$$\begin{aligned} l_n &= f_{n-1} + f_{n-3} + F_{n-1} \\ &= \frac{(n-1)F_n + 2nF_{n-1} + (n-3)F_{n-2} + 2(n-2)F_{n-3} + 5F_{n-1}}{5} \\ &= \frac{(3n+4)F_{n-1} + (2n-4)F_{n-2} + (2n-4)F_{n-3}}{5} = nF_{n-1}. \quad \square \end{aligned}$$

Furthermore, we recall that

$$F_n = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}} \quad \text{and} \quad L_n = \phi^n + \hat{\phi}^n, \tag{7}$$

(where $\phi = (1+\sqrt{5})/2$ and $\hat{\phi} = (1-\sqrt{5})/2$). Then we have

Theorem 6:

(i) $i(\Gamma_{n-1}) < i(\mathcal{L}_n) < i(\Gamma_n)$.

(ii) $\lim_{n\to\infty} i(\mathcal{L}_n) = +\infty$.

Proof: ***(i)*** We have to prove that $f_{n-1}L_n < l_nF_{n+1}$ and $l_nF_{n+2} < f_nL_n$ and that these inequalities are part of an increasing sequence of positive integers:

$$\ldots f_{n-1}L_n < l_nF_{n+1} < l_nF_{n+2} < f_nL_n < f_nL_{n+1} < l_{n+1}F_{n+2} < l_{n+1}F_{n+3} < f_{n+1}L_{n+1}\cdots.$$

Now let $a_n := f_nF_{n+1} - f_{n-1}F_{n+2}$. We begin by showing that $a_n > 0$. Indeed, by direct computation we have $a_1 = f_1F_2 - f_0F_3 = 1$, $a_2 = f_2F_3 - f_1F_4 = 1$, and for $n \geq 3$,

$$a_n = f_{n-2}F_{n+1} + F_{n+1}F_n - f_{n-1}F_n = f_{n-2}F_{n-1} - f_{n-3}F_n + F_n^2 = a_{n-2} + F_n^2 > a_{n-2}.$$

In order to prove the first inequality, we have

$$\begin{aligned} l_nF_{n+1} - f_{n-1}L_n &= (f_{n-1} + f_{n-3} + F_{n-1})F_{n+1} - f_{n-1}(F_{n+1} + F_{n-1}) \\ &= (f_{n-3} + F_{n-1})F_{n+1} - f_{n-1}F_{n-1} \\ &= (f_{n-3} + F_{n-1})(F_n + F_{n-1}) - (f_{n-2} + f_{n-3} + F_{n-1})F_{n-1} \\ &= (f_{n-3} + F_{n-1})F_n - f_{n-2}F_{n-1} = a_{n-1} > 0. \end{aligned}$$

The second inequality is immediate for $n < 4$; for $n > 4$ we have

$$\begin{aligned} f_nL_n - l_nF_{n+2} &= -f_{n-1}F_{n-2} + f_{n-2}(F_{n+1} + F_{n-1}) - f_{n-3}F_{n+2} + F_{n+1}F_{n-2} \\ &= -(f_{n-2} + f_{n-3} + F_{n-1})F_{n-2} + f_{n-2}(3F_{n-1} + F_{n-2}) - f_{n-3}(3F_{n-1} + 2F_{n-2}) + F_{n+1}F_{n-2} \\ &= (-3f_{n-3}F_{n-2} - 3f_{n-3}F_{n-1}) + 3f_{n-2}F_{n-1} + F_{n+1}F_{n-2} - F_{n-1}F_{n-2} \\ &= 3(f_{n-2}F_{n-1} - f_{n-3}F_n) + F_nF_{n-2} = 3a_{n-2} + F_nF_{n-2} > 0. \end{aligned}$$

(ii) From (7) it follows that

$$i(\mathscr{L}_n) = \frac{l_n}{L_n} = \frac{nF_{n-1}}{L_n} \sim \frac{n}{\sqrt{5}\phi}. \quad \square$$

6. LUCAS SEMILATTICES

In [3] we studied a poset connected to Γ_n. In a similar way, the set of Lucas strings can be partially ordered with respect to the relation $\leq$ defined by $[a_1, \ldots, a_n] \leq [b_1, \ldots, b_n]$ if and only if $a_i \leq b_i$ for $i = 1, \ldots, n$ for all Lucas strings $[a_1, \ldots, a_n]$, $[b_1, \ldots, b_n]$. Moreover,

$$[a_1, \ldots, a_n] \vee [b_1, \ldots, b_n] = [c_1, \ldots, c_n],$$

where $c_1 = \max(a_i, b_i)$ for $i = 1, \ldots, n$ if $[c_1, \ldots, c_n]$ exists. The minimal element is $\hat{0} = [0, \ldots, 0]$. The poset $(\hat{C}_n, \leq)$ is closed under inf, where $[a_1, \ldots, a_n] \wedge [b_1, \ldots, b_n] = [\min(a_1, b_1), \ldots, \min(a_n, b_n)]$ and $\hat{0} = [0, \ldots, 0]$. Thus, $(\hat{C}_n, \leq)$ is a meet-semilattice L_n.

By Theorem 1, the height of L_n, i.e., the maximum number of 1's in a Lucas string of length n is $\lfloor \frac{n}{2} \rfloor$.

Recall that in a semilattice S an *atom* is an element covering $\hat{0}$; the set of atoms is denoted by $\mathrm{Atom}(S)$. A semilattice is *atomic* if for each $x \in S$ there exists a subset $A \subseteq \mathrm{Atom}(S)$ such that $x = \vee A$; it is *strictly atomic* when for each element $x \in S$ there exists a unique $A \subseteq \mathrm{Atom}(S)$ such that $x = \vee A$.

A semilattice is *simplicial* where every interval is isomorphic to a Boolean lattice. In [3] we proved that a finite semilattice S with $\hat{0}$ is strictly atomic if and only if it is simplicial. Moreover, every finite strictly atomic semilattice S is ranked, where the *rank* is the function $r: S \to \mathbf{N}$

defined by $r(x) = |A|$ if and only if $x = \vee A$. Finally, we proved that the characteristic polynomial of a finite strictly atomic semilattice S is $\chi(S, x) = \Sigma(-1)^k W_k(S) \cdot x^{h(S)-k}$, where W_k is a Whitney number of the second kind (i.e., the number of elements of S of rank k) and $h(S)$ is the height of S. All the properties of the Fibonacci semilattices also hold in this case. The difference concerns W_k and the height. Now it is

$$W_k(L_n) = \binom{n-k}{k} \cdot \frac{n}{n-k} \quad \text{and} \quad h(L_n) = \left\lfloor \frac{n}{2} \right\rfloor;$$

then we have

$$\chi(L_n, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \cdot \frac{n}{n-k} \cdot (-1)^k \cdot x^{\lfloor \frac{n}{2} \rfloor - k}.$$

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REFERENCES

1. M. Aigner. *Combinatorial Theory*. New York: Springer-Verlag, 1979.
2. G. Chartrand & L. Lesniak. *Graphs and Digraphs*. Belmont, CA: Wadsworth and Brooks, 1986.
3. S. Fontanesi, E. Munarini, & N. Zagaglia Salvi. "On the Fibonacci Semilattices." *Algebras and Combinatorics* (1997):237-46.
4. W. J. Hsu. "Fibonacci Cubes–A New Interconnection Topology." *IEEE Trans. on Parallel and Distributed Systems* **4** (1993).
5. J. Liu, W. J. Hsu, & M. J. Chung. "Generalized Fibonacci Cubes Are Mostly Hamiltonian." *J. Graph Theory* **18** (1994):817-29.
6. E. Munarini & N. Zagaglia Salvi. "Structural and Enumerative Properties of the Fibonacci Cubes." *Intellectual Systems* **2.1-4** (1997):265-74 (in Russian).

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APPLICATION OF THE ε-ALGORITHM TO THE RATIOS OF r-GENERALIZED FIBONACCI SEQUENCES

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1. INTRODUCTION

Let $a_0, a_1, \ldots, a_{r-1}$ $(r \geq 2)$ be some real or complex numbers with $a_{r-1} \neq 0$. An r-generalized Fibonacci sequence $\{V_n\}_{n\geq 0}$ is defined by the linear recurrence relation of order r,

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_{r-1} V_{n-r+1} \quad \text{for } n \geq r-1, \tag{1}$$

where $V_0, V_1, \ldots, V_{r-1}$ are specified by the initial conditions. Such sequences are widely studied in the literature (see, e.g., [5], [6], [9], [10], [11], and [13]). We shall refer to them in the sequel as *sequences (1)*. It is well known that, if the limit $q = \lim_{n\to+\infty} \frac{V_{n+1}}{V_n}$ exists, then q is a root of the characteristic equation $x^r = a_0 x^{r-1} + \cdots + a_{r-2}x + a_{r-1}$. Hence, sequences (1) may also be used as a tool in the approximation of roots of algebraic equations (see [12]), like Newton's method or the secant method as it was considered in [7].

The Aitken acceleration $\{x_n^*\}_{n\geq 0}$ associated with a convergent sequence $\{x_n\}_{n\geq 0}$ is defined by

$$x_n^* = \frac{x_{n+1}x_n - x_n^2}{x_{n+1} - 2x_n + x_{n-1}}. \tag{2}$$

For numerical analysis, this process is of practical interest in those cases in which $\{x_n^*\}_{n\geq 0}$ converges faster than $\{x_n\}_{n\geq 0}$ to the same limit (see, e.g., [1], [2], [3], [4], and [8]). In the case of sequences (1) with $r = 2$, McCabe and Philips had considered a theoretical application of Aitken acceleration for the accelerability of convergence of $\{x_n\}_{n\geq 0}$, where $x_n = \frac{V_{n+1}}{V_n}$ (see [12]). This is nothing more than the application of Aitken acceleration to the solution of the quadratic equation $x^2 - a_0 x - a_1 = 0$ by an iterative method (see [12]).

The main purpose of this paper is to apply the method of the ε-algorithm (see [3], [4]), which generalizes the Aitken acceleration, to accelerate the convergence of $\{x_n\}_{n\geq 0}$, where $x_n = \frac{V_{n+1}}{V_n}$, for any sequence (1). Hence, we extend the idea of McCabe and Philips [12] to the general case of sequences (1). Thus, we get the acceleration of the solution of algebraic equations.

This paper is organized as follows. In Section 2 we give a preliminary connection between sequences (1) and the ε-algorithm. In Section 3 we apply the ε-algorithm to the sequence of the ratios $x_n = \frac{V_{n+1}}{V_n}$. Some concluding remarks are given in Section 4.

2. SEQUENCES (1) AND THE ε-ALGORITHM

Let $\{x_n\}_{n\geq 0}$ be a convergent sequence of real numbers with $x = \lim_{n\to+\infty} x_n$. The ε-algorithm is a particular case of the extrapolation method (see [2], [3], [4]). The main idea is to consider a sequence transformation T of $\{x_n\}_{n\geq 0}$ into a sequence $\{T_n\}_{n\geq 0}$, which converges very quickly to the same limit x, this means that $\lim_{n\to+\infty}\frac{T_n-x}{x_n-x}=0$ (see [3] and [4] for more details). The kernel of the transformation T, defined by

$$\mathcal{K}_T=\{\{x_n\}_{n\geq 0};\ \exists N>0,\ T_n=x,\ \forall n>N\},$$

is of great interest for an extrapolation method like Richardson or ε-algorithm (see [3], [4]). In summary, the ε-algorithm associated with the convergent sequence $\{x_n\}_{n\geq 0}$ consists in considering the following sequence $\{\varepsilon_k^{(n)}\}_{k\geq -1,\, n\geq 0}$, where

$$\varepsilon_{-1}^{(n)}=0,\ \varepsilon_1^{(n)}=x_n;\ n\geq 0, \tag{3}$$

$$\varepsilon_{k+1}^{(n)}=\varepsilon_{k-1}^{(n)}+\frac{1}{\varepsilon_k^{(n+1)}-\varepsilon_k^{(n)}},\ n,k\geq 0. \tag{4}$$

This algorithm can be applied when $\varepsilon_k^{(n)}\neq\varepsilon_k^{(n+1)}$ for any n, k. The ε-algorithm theory also shows that the only interesting quantities are $\varepsilon_{2k}^{(n)}$, the quantities $\varepsilon_{2k+1}^{(n)}$ are used only for intermediate computations (see [2], [3], [4]). For $k=2$, we can derive from expressions (3) and (4) that $\varepsilon_2^{(n)}$ is nothing but the Aitken acceleration associated with $\{x_n\}_{n\geq 0}$ as defined by (2) (see [3], [4]).

For any convergent sequence $\{x_n\}_{n\geq 0}$ with $x=\lim_{n\to+\infty}x_n$, Theorem 35 of [3] and Theorem 2.18 of [4] show that there exists $N>0$ such that $\varepsilon_{2k}^{(n)}=x$ for any $n\geq N$ if and only if there exists $a_0,\dots,a_k$ with $\sum_{j=0}^k a_j\neq 0$ such that $\sum_{j=0}^k a_j(x_{n+j}-x)=0$ for any $n\geq N$. It is easy to see that we can suppose in the last preceding sum that $a_0\neq 0$ and $a_k\neq 0$. Hence, we derive the following property.

Proposition 2.1: Let $\{x_n\}_{n\geq 0}$ be a convergent sequence such that $x=\lim_{n\to+\infty}x_n$. Then the following are equivalent:

(a) There exists $N>0$ such that $\varepsilon_{2k}^{(n)}=x$ for any $n\geq N$.

(b) The sequence $\{V_n\}_{n\geq 0}$ defined by $V_n=x_{n+N}-x$ is a sequence (1) corresponding to $r=k$, whose coefficients and initial conditions are, respectively,

$$b_0=-\frac{a_{k-1}}{a_k},\dots,b_{k-1}=-\frac{a_0}{a_k}\quad\text{and}\quad V_0=x_N-x,\dots,V_{k-1}=x_{N+k-1}-x.$$

(c) The sequence $\{x_n\}_{n\geq N}$ is a sequence (1) corresponding to $r=k+1$ such that $\lambda=1$ is a simple characteristic root, $V_0=x_N,\dots,V_k=x_{N+k}$ are its conditions, and its coefficients $a_0,\dots,a_k$ are the coefficients of the characteristic polynomial $P(X)=(X-1)Q(X)$, where $Q(X)$ is the characteristic polynomial of $\{V_n\}_{n\geq 0}$ defined in (b).

Proposition 2.1 shows that, in the case of the ε-algorithm, the kernel $\mathcal{K}_T$ may be expressed using sequences (1).

3. APPLICATION OF THE ε-ALGORITHM TO $\lim_{n\to+\infty}\frac{V_{n+1}}{V_n}$

Let $\{V_n\}_{n\geq 0}$ be a sequence (1) and $\lambda_0,\dots,\lambda_l$ be the roots of the characteristic polynomial $P(X)=X^r-a_0X^{r-1}-\cdots-a_{r-1}$. Suppose that λ_0 is a simple root and

$$0<|\lambda_l|\leq|\lambda_{l-1}|\leq\cdots\leq|\lambda_1|<|\lambda_0|.$$

Thus, the Binet formula of the sequence (1) is

$$V_n = \beta_{00}\lambda_0^n + \sum_{s=1}^{l}\left(\sum_{j=0}^{s_j-1}\beta_{js}n^n\right)\lambda_s^n,$$

where the β_{js} are given by the initial conditions and s_j is the multiplicity of λ_j $(0 \le j \le l)$ (see, e.g., [9] and [10]). Suppose that $V_0, \ldots, V_{r-1}$ are such that $\beta_{00} \neq 0$. Then we can derive that $\lim_{n\to+\infty}\frac{V_{n+1}}{V_n} = \lambda_0$.

It is known that if we applied the Aitken acceleration process to a convergent sequence $\{x_n\}_{n\ge0}$ with $x = \lim_{n\to+\infty} x_n$ and if $\lim_{n\to+\infty}\frac{x_{n+1}-x}{x_n-x} = \rho \neq 1$, then the sequence $\{\varepsilon_2^{(n)}\}_{n\ge0}$ converges more quickly than $\{x_n\}_{n\ge0}$ to x (see [3], Theorem 32, p. 37). In the case of $x_n = \frac{V_{n+1}}{V_n}$, a direct computation using the Binet formula results in

$$\lim_{n\to+\infty}\frac{x_{n+1}-\lambda_0}{x_n-\lambda_0} = \frac{\lambda_1}{\lambda_0} \neq 1$$

because $|\lambda_1| < |\lambda_0|$. Hence, we have derived the following property.

Proposition 3.1: Let $\{V_n\}_{n\ge0}$ be a sequence (1). Suppose that the characteristic roots $\{\lambda_j\}_{j=0}^{l}$ are such that $0 < |\lambda_l| \le |\lambda_{l-1}| \le \cdots \le |\lambda_1| < |\lambda_0|$ with λ_0 simple. Apply the Aitken acceleration to

$$\left\{x_n = \frac{V_{n+1}}{V_n}\right\}_{n\ge0}.$$

Then, the sequence $\{\varepsilon_2^{(n)}\}_{n\ge0}$ converges faster than $\{x_n\}_{n\ge0}$ to λ_0.

Let $\{x_n\}_{n\ge0}$ be a convergent sequence with $x = \lim_{n\to+\infty} x_n$. If $x_n = f(x_{n-1}, \ldots, x_{n-k})$, where $x_0, \ldots, x_{k-1}$ are given and

$$\sum_{i=0}^{r-1}\frac{\partial f}{\partial y_i}(x, \ldots, x) \neq 1,$$

then $\lim_{n\to+\infty}\varepsilon_{2k}^{(n)} = x$ (see [3], Theorem 52, p. 70). Let f be the function $f: \mathbf{D} \subset \mathbf{R}^{r-1} \to \mathbf{R}$, where $\mathbf{D} = \{(y_1, \ldots, y_{r-1}) \in \mathbf{R}^{r-1};\ y_j \neq 0, \forall j\ (1 \le j \le r-1)\}$, defined by

$$f(y_1, \ldots, y_{r-1}) = a_0 + \frac{a_1}{y_1} + \frac{a_2}{y_1y_2} + \cdots + \frac{a_{r-1}}{y_1\cdots y_{r-1}}.$$

Consider the ratio $x_n = \frac{V_{n+1}}{V_n}$. Then, from expression (1), we derive that $x_n = f(x_{n-1}, \ldots, x_{n-r+1})$. It is clear that f is a class C^1 on $\mathbf{D}$. By direct computation we obtain

$$\sum_{i=1}^{r-1}\frac{\partial f}{\partial y_i}(\lambda_0, \ldots, \lambda_0) = 1 - \frac{1}{\lambda_0^{r-1}}\frac{dP}{dx}(\lambda_0).$$

Then we have derived the following result.

Proposition 3.2: Let $\{V_n\}_{n\ge0}$ be a sequence (1). Suppose that the characteristic roots $\{\lambda_j\}_{j=0}^{l}$ are such that λ_0 is simple and $0 < |\lambda_l| \le |\lambda_{l-1}| \le \cdots \le |\lambda_1| < |\lambda_0|$. Apply the ε-algorithm to the sequence $\left\{x_n = \frac{V_{n+1}}{V_n}\right\}_{n\ge0}$. Then we have $\lim_{n\to+\infty}\varepsilon_{2(r-1)}^{(n)} = \lambda_0$.

More precisely, we have the following result.

Proposition 3.3: Let $\{V_n\}_{n\ge 0}$ be a sequence (1). Suppose that the characteristic roots $\{\lambda_j\}_{j=0}^{l}$ are such that λ_0 is simple and $0<|\lambda_l|\le|\lambda_{l-1}|\le\cdots\le|\lambda_1|<|\lambda_0|$. Apply the ε-algorithm to the sequence $\left\{x_n=\frac{V_{n+1}}{V_n}\right\}_{n\ge 0}$. Then the sequence $\{\varepsilon_{2(r-1)}^{(n)}\}_{n\ge 0}$ converges faster than $\{x_{n+r-1}\}_{n\ge 0}$ to λ_0.

Proof: Let $b_j=\frac{\partial f}{\partial y_j}(\lambda_0,\dots,\lambda_0)$. Then there exists $b_j^{(n)}$ $(1\le j\le r-1)$ such that

$$(\varepsilon_{2(r-1)}^{(n)}-\lambda_0)\left(-1+\sum_{j=1}^{r-1}b_j^{(n)}\right)=\sum_{j=1}^{r-1}(b_j^{(n)}-b_j)(x_{n-j}-\lambda_0)-R_n, \tag{$*$}$$

where

$$R_n=(x_n-\lambda_0)-b_1(x_{n-1}-\lambda_0)-\cdots-b_{r-1}(x_{n-r+1}-\lambda_0). \tag{$**$}$$

The application $(x_{n-r+1},\dots,x_{n+r-1})\to(b_1^{(n)},\dots,b_{r-1}^{(n)})$ is continuous (see [3] and [4]). Hence, for any $\varepsilon>0$, there exists $N>0$ such that $|b_j^{(n)}-b_j|<\varepsilon$ for any $n>N$ with $j=1,\dots,r-1$. Then, from $(*)$, we derive that

$$\lim_{n\to+\infty}\frac{\varepsilon_{2(r-1)}^{(n)}-\lambda_0}{x_{n+r-1}-\lambda_0}=\frac{1}{-1+\sum_{j=1}^{r-1}b_j}\lim_{n\to+\infty}\frac{R_n}{x_{n+r-1}-\lambda_0}.$$

From expression $(**)$ of R_n, we obtain that

$$\lim_{n\to+\infty}\frac{R_n}{x_{n+r-1}-\lambda_0}=\left(\frac{\lambda_0}{\lambda_1}\right)^r-b_1\left(\frac{\lambda_0}{\lambda_1}\right)^{r+1}-\cdots-b_{r-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{2r-2}.$$

A direct computation using the expression

$$b_j=-\frac{a_j}{\lambda_0^{j+1}}-\frac{a_{j+1}}{\lambda_0^{j+2}}-\cdots-\frac{a_{r-1}}{\lambda_0^{r}}\quad(1\le j\le r-1),$$

results in $\lim_{n\to+\infty}\frac{R_n}{x_{n+r-1}-\lambda_0}=0$. Thus, we have

$$\lim_{n\to+\infty}\frac{\varepsilon_{2(r-1)}^{(n)}-\lambda_0}{x_{n+r-1}-\lambda_0}=0.\ \square$$

The proof of Proposition 3.3 is nothing more than an adaptation of the proof of Theorem 52 of [3] to the case in which

$$f(y_1,\dots,y_r)=a_0+\frac{a_1}{y_1}+\frac{a_2}{y_1y_2}+\cdots+\frac{a_{r-1}}{y_1\cdots y_{r-1}}.$$

4. CONCLUDING REMARKS

Note that the ε-algorithm may also be used to accelerate the convergence of sequences (1). More precisely, for a convergent sequence (1), the Binet formula results in $|\lambda_j|\le 1$ for any characteristic root λ_j $(0\le j\le l)$. Suppose that $0<|\lambda_1|<\cdots<|\lambda_1|<|\lambda_0|\le 1$. Then the Binet formula and expression (1) imply that $\lim_{n\to+\infty}V_n=0$ for $|\lambda_j|<1$ for any j or $\lim_{n\to+\infty}V_n=\beta_{00}$ if $|\lambda_j|<1$ for any $j\ne 0$, and $\lambda_0=1$ is a simple characteristic root.

For $\lim_{n\to+\infty} V_n = 0$, we show by direct computation that $\lim_{n\to+\infty} \frac{V_{n+1}}{V_n} = \lambda_j$, depending on the choice of the initial conditions $\{V_k\}_{k=0}^{r-1}$. Then, by applying the ε-algorithm, we can derive that $\{\varepsilon_{2p}^{(n)}\}_{n\geq 0}$ converges to 0 faster than $\{V_n\}_{n\geq 0}$, for any $p = 1, \ldots, r-j$.

For $\lim_{n\to+\infty} V_n = \beta_{00} = S \neq 0$, we can derive by direct computation that $\lim_{n\to+\infty} \frac{V_{n+1}-S}{V_n-S} = \lambda_j$, depending on the choice of the initial conditions $\{V_k\}_{k=0}^{r-1}$. Then, by applying the ε-algorithm, we also derive that $\{\varepsilon_{2p}^{(n)}\}_{n\geq 0}$ converges to S faster than $\{V_n\}_{n\geq 0}$ for any $p = 1, \ldots, r-j$. In particular, this case may be used to accelerate the convergence of the ratios $\frac{V_n}{q^n}$ when the a_j are nonnegative and CGD$\{j+1; a_j > 0\} = 1$ (see [6] and [14]).

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REFERENCES

1. A. C. Aitken. "Studies in Practical Mathematics II." *Proc. Roy. Soc. Edin.* **57** (1937).
2. C. Brezinski. "A General Extrapolation Algorithm." *Numerical Math.* **35** (1980).
3. C. Brezinski. *Accélération de convergence en Analyse Numérique*. Publication du Laboratoire de Calcul, Université des Sciences et Techniques de Lille, 1973.
4. C. Brezinski & M. Redivo Zaglia. "Extrapolation Methods: Theory and Practice." *Studies in Computational Mathematics* **2**. Ed. C. Brezinski & L. Wuytack. Amsterdam, London, New York, Tokyo: North-Holland, 1991.
5. T. P. Dence. "On r-Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **27.2** (1987):137-43.
6. F. Dubeau, W. Motta, M. Rachidi, & O. Saeki. "On Weighted r-Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **35.2** (1997):102-10.
7. J. Gill & G. Miller. "Newton's Method and Ratios of Fibonacci Numbers." *The Fibonacci Quarterly* **19.1** (1981):1.
8. P. Henrici. *Elements of Numerical Analysis*. New York: Wiley & Son, 1964.
9. J. A. Jesk. "Linear Recurrence Relations, Part I." *The Fibonacci Quarterly* **1.1** (1963):69-74.
10. W. G. Kelly & A. C. Peterson. *Difference Equations: An Introduction with Application*. San Diego: Academic Press, 1991.
11. C. Levesque. "On m^{th}-Order Linear Recurrences." *The Fibonacci Quarterly* **23.4** (1985): 290-95.
12. J. H. McCabe & G. M. Philips. "Fibonacci and Lucas Numbers and Aitken Acceleration." *Fibonacci Numbers and Their Applications* **1**:181-84. Ed. A. N. Philippou et al. Dordrecht: Reidel, 1986.
13. E. P. Miles. "Generalized Fibonacci Numbers and Associated Matrices." *American. Math. Monthly* **67** (1960):745-52.
14. M. Mouline & M. Rachidi. "Suites de Fibonacci généralisées et chaines de Markov." *Real Academia de Ciencias Exatas, Fisicas y Naturales de Madrid* **89.1-2** (1995):61-77.

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THE ZECKENDORF NUMBERS AND THE INVERSES OF SOME BAND MATRICES

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1. INTRODUCTION

For the tridiagonal matrix

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}_{(n-1)\times(n-1)},$$

it is known (see [1]) that $A^{-1} = [a_{ij}]$ has elements

$$a_{ij} = \begin{cases} \frac{i(n-j)}{n}, & i \le j, \\ \frac{j(n-i)}{n}, & i > j. \end{cases}$$

We will find the inverse of the matrix A_p, $p \in \mathbb{N}$, where

$$A_p = \begin{bmatrix} 2 & -1 & & & & & & & & & & \\ 0 & 2 & -1 & & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & & & & & & & & \\ \cdot & & \cdot & \cdot & \cdot & & & & & & & \\ 0 & \cdot & \cdot & 0 & 2 & -1 & & & & & & \\ -1 & 0 & \cdot & \cdot & 0 & 2 & -1 & & & & & \\ & -1 & 0 & \cdot & \cdot & 0 & 2 & -1 & & & & \\ & & \cdot & \cdot & & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & & \cdot & \cdot & \cdot & & \\ & & & & -1 & 0 & \cdot & \cdot & 0 & 2 & -1 \\ & & & & & -1 & \underbrace{0 \ \cdot \ \cdot \ 0}_{p-1} & & & & 2 \end{bmatrix}_{(n-1)\times(n-1)}.$$

It is evident that, for $p = 1$, $A_p = A$.

Let $F_j^{(p)}$, $p \in \mathbb{N}$, $j \in \mathbb{N} \cup \{0\}$, be the Zeckendorf numbers given in [2] with

$$F_0^{(p)} = 0, \ p \in \mathbb{N},$$
$$F_1^{(p)} = 1, \ p \in \mathbb{N},$$

and

$$F_j^{(p)} = \begin{cases} 1, & p = 1, j \in \mathbb{N}, \\ 2^{j-2}, & p \geq 2, j \in \{2, 3, \ldots, p\}, \\ F_{j-1}^{(p)} + F_{j-2}^{(p)} + \cdots + F_{j-p}^{(p)}, & p \geq 2, j > p, \end{cases}$$

and let us define

$$\begin{aligned} u_i^{(0)} &= \sum_{j=1}^{i} F_j^{(p)}, \quad i \in \{1, 2, \ldots, n\}, \\ u_i^{(p)} &= 0, \quad i \leq 0. \end{aligned} \tag{1}$$

Theorem (Main Result): If the numbers $u_i^{(p)}$ are as in (1), define the matrix $A_p' = [a_{ij}]$, $p \geq 2$, by

$$a_{ij} = \frac{1}{u_n^{(p)}} \left(u_i^{(p)} u_{n-j}^{(p)} - u_n^{(p)} u_{i-j}^{(p)} \right), \quad i, j \in \{1, 2, \ldots, n-1\}. \tag{2}$$

Then $A_p' = A_p^{-1}$.

It is important to mention that, since $p \geq 2$, n must not be less than 4.

2. PROOF OF THE MAIN RESULT

First, we will establish some properties of the numbers $u_i^{(p)}$.

Lemma 1: For $i, p \in \mathbb{N}$, $p \geq 2$, and $i \leq p+1$,

$$u_i^{(p)} = 2^{i-1}.$$

Proof: If $i \leq p$, then

$$u_i^{(p)} = \sum_{j=1}^{i} F_j^{(p)} = 1 + (1 + 2 + \cdots + 2^{i-2}) = 1 + \frac{2^{i-1} - 1}{2 - 1} = 2^{i-1},$$

and if $i = p + 1$, then

$$u_{p+1}^{(p)} = u_p^{(p)} + F_{p+1}^{(p)} = 2 \sum_{j=1}^{p} F_j^{(p)} = 2^p. \quad \square$$

Lemma 2: If δ_{kl} is the Kronecker delta symbol, then

$$u_{k-p-l}^{(p)} - 2u_{k-l}^{(p)} + u_{k+1-l}^{(p)} = \delta_{kl}, \tag{3}$$

for $p, k \in \mathbb{N} \setminus \{1\}$ and $l \in \mathbb{N} \cup \{0\}$.

Proof: Let us consider two different cases: ***(a)*** $l \geq k$; ***(b)*** $l < k$.

(a) For $l \geq k$m we have $k - p - l < 0$, $k - l \leq 0$, and $k + 1 - l \leq 1$. Hence, by (1),

$$u_{k-p-l}^{(p)} = u_{k-l}^{(p)} = 0,$$

$$u_{k+1-l}^{(p)} = \begin{cases} 0, & k < l, \\ F_1^{(p)} = 1, & k = l, \end{cases}$$

and (3) is valid.

(b) For $l < k$, first let $2 \le k \le p$. Then $k-p-l \le 0$ and $k-l < k+1-l \le p+1$. Hence, by (1) and Lemma 1,

$$u_{k-p-l}^{(p)} = 0, \; u_{k-l}^{(p)} = 2^{k-l-1}, \quad \text{and} \quad u_{k+1-l}^{(p)} = 2^{k-l}.$$

It follows that (3) is true.

If $k \ge p+1$, then for: ***(i)*** $0 < k-l \le p$,

$$u_{k-p-l}^{(p)} - 2u_{k-l}^{(p)} + u_{k+1-l}^{(p)} = 0 - 2 \cdot 2^{k-i-1} + 2^{k-l} = 0;$$

(ii) $k-l > p$, let $k-l = p+t$, $t \ge 1$, then

$$\begin{aligned} u_{k-p-l}^{(p)} - 2u_{k-l}^{(p)} + u_{k-l+1}^{(p)} &= \sum_{j=1}^{t} F_j^{(p)} - 2\sum_{j=1}^{p+t} F_j^{(p)} + \sum_{j=1}^{p+t+1} F_j^{(p)} \\ &= -\sum_{j=t+1}^{p+t} F_j^{(p)} + F_{p+t+1}^{(p)} = 0. \; \square \end{aligned}$$

Proof of the Main Result: Since A_p is a square matrix, it is sufficient to prove that A_p' is a right sided inverse. Using Lemma 2, we will prove statement (2). If we set $A_p = [c_{ij}]$, then

$$c_{ij} = \begin{cases} 2, & j = i, \\ -1, & j = i+1 \text{ or } j = i-p, \\ 0, & \text{otherwise.} \end{cases}$$

For $k = 1$ and $j \in \{1, 2, \ldots, n-1\}$, we have

$$\begin{aligned} (A_p A_p')_{1j} &= \sum_{l=1}^{n-1} c_{1l} a_{lj} = 2a_{1j} - a_{2j} \\ &= \frac{u_{n-j}^{(p)}}{u_n^{(p)}} \left(2u_1^{(p)} - u_2^{(p)}\right) + \left(-2u_{1-j}^{(p)} + u_{2-j}^{(p)}\right). \end{aligned}$$

Using (1) and the definition of $F_j^{(p)}$,

$$2u_1^{(p)} - u_2^{(p)} = 2F_1^{(p)} - (F_1^{(p)} + F_2^{(p)}) = 0$$

and

$$-2u_{1-j}^{(p)} + u_{2-j}^{(p)} = u_{2-j}^{(p)} = \begin{cases} u_1^{(p)} = 1, & j = 1, \\ 0, & j \in \{2, \ldots, n-1\}. \end{cases}$$

Therefore, $(A_p A_p')_{1j} = \delta_{1j}$ for $j \in \{1, 2, \ldots, n-1\}$.

For $k \in \{2, \ldots, p\}$ and $j \in \{1, 2, \ldots, n-1\}$, we have

$$\begin{aligned} (A_p A_p')_{kj} &= \sum_{l=1}^{n-1} c_{kl} a_{lj} = 2a_{kj} - a_{k+1,j} \\ &= \frac{u_{n-j}^{(p)}}{u_n^{(p)}} \left(2u_k^{(p)} - u_{k+1}^{(p)}\right) + \left(-2u_{k-j}^{(p)} + u_{k+1-j}^{(p)}\right), \end{aligned}$$

and from (1) and (3) it follows that

$$2u_k^{(p)} - u_{k+1}^{(p)} = -\left(u_{k-p}^{(p)} - 2u_k^{(p)} + u_{k+1}^{(p)}\right) = -\delta_{k0} = 0$$

and

$$-2u_{k-j}^{(p)} + u_{k+1-j}^{(p)} = u_{k-p-j}^{(p)} - 2u_{k-j}^{(p)} + u_{k+1-j}^{(p)} = \delta_{kj}.$$

For $k \in \{p+1, \ldots, n-2\}$ and $j \in \{1, 2, \ldots, n-1\}$, let $k = p+t$, $t \geq 1$. Then

$$\begin{aligned}
(A_p A_p')_{kj} &= \sum_{l=1}^{n-1} c_{kl} a_{lj} = c_{p+t,t} a_{tj} + c_{kk} a_{kj} + c_{k,k+1} a_{k+1,j} \\
&= -a_{tj} + 2a_{kj} - a_{k+1,j} = -a_{k-p,j} + 2a_{kj} - a_{k+1,j} \\
&= \frac{u_{n-j}^{(p)}}{u_n^{(p)}}\left[-u_{k-p}^{(p)} + 2u_k^{(p)} - u_{k+1}^{(p)}\right] + \left(u_{k-p-j}^{(p)} - 2u_{k-j}^{(p)} + u_{k+1-j}^{(p)}\right) \\
&= \frac{u_{n-j}^{(p)}}{u_n^{(p)}}\delta_{k0} + \delta_{kj} = \delta_{kj}.
\end{aligned}$$

For $k = n-1$ and $j \in \{1, 2, \ldots, n-1\}$, we have

$$\begin{aligned}
(A_p A_p')_{n-1,j} &= -a_{n-p-1,j} + 2a_{n-1,j} \\
&= -\frac{1}{u_n^{(p)}}\left(u_{n-p-1}^{(p)} u_{n-j}^{(p)} - u_n^{(p)} u_{n-p-1-j}^{(p)}\right) + \frac{2}{u_n^{(p)}}\left(u_{n-1}^{(p)} u_{n-j}^{(p)} - u_n^{(p)} u_{n-1-j}^{(p)}\right) \\
&= -\frac{u_{n-j}^{(p)}}{u_n^{(p)}}\left(u_{n-p-1}^{(p)} - 2u_{n-1}^{(p)}\right) + u_{n-1-p-j}^{(p)} - 2u_{n-1-j}^{(p)} \\
&= -\frac{u_{n-j}^{(p)}}{u_n^{(p)}}\left(-u_n^{(p)} + \delta_{n1}\right) + \delta_{n-1,j} - u_{n-j}^{(p)} = \delta_{n-1,j}. \quad \square
\end{aligned}$$

Using the previous theorem, we can now easily find inverses for the following band matrices A, with $A^{-1} = [a_{ij}]$:

- For the matrix

$$\begin{array}{c}
\qquad\qquad \overbrace{\hphantom{0 \quad \cdot \quad \cdot \quad 0}}^{p-1} \\
A = \begin{bmatrix}
2 & 0 & \cdot & \cdot & 0 & -1 & & & & & \\
-1 & 2 & 0 & \cdot & \cdot & 0 & -1 & & & & \\
 & \cdot & \cdot & \cdot & & & \cdot & \cdot & & & \\
 & & \cdot & \cdot & \cdot & & & \cdot & \cdot & & \\
 & & & -1 & 2 & 0 & \cdot & \cdot & 0 & -1 & \\
 & & & & -1 & 2 & 0 & \cdot & \cdot & 0 & -1 \\
 & & & & & -1 & 2 & 0 & \cdot & \cdot & 0 \\
 & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & -1 & 2 & 0 \\
 & & & & & & & & & -1 & 2
\end{bmatrix}_{(n-1)\times(n-1)},
\end{array}$$

$$a_{ij} = \frac{1}{u_n^{(p)}}\left(u_j^{(p)} u_{n-i}^{(p)} - u_n^{(p)} u_{j-i}^{(p)}\right), \quad i, j \in \{1, 2, \ldots, n-1\}.$$

- For the matrix

$$A=\begin{bmatrix}
 & & & & & -1 & \multicolumn{4}{c}{\overbrace{0\quad\cdot\quad\cdot\quad 0}^{p-1}} & 2 \\
 & & & & -1 & 0 & \cdot & \cdot & 0 & 2 & -1 \\
 & & & \cdot & \cdot & & & & \cdot & \cdot & \cdot \\
 & & \cdot & \cdot & & & & \cdot & \cdot & \cdot & \\
 & -1 & 0 & \cdot & \cdot & 0 & 2 & -1 & & & \\
 -1 & 0 & \cdot & \cdot & 0 & 2 & -1 & & & & \\
 0 & \cdot & \cdot & 0 & 2 & -1 & & & & & \\
 \cdot & & \cdot & \cdot & \cdot & & & & & & \\
 \cdot & \cdot & \cdot & \cdot & & & & & & & \\
 0 & 2 & -1 & & & & & & & & \\
 2 & -1 & & & & & & & & &
\end{bmatrix}_{(n-1)\times(n-1)},$$

$$a_{ij}=\frac{1}{u_n^{(p)}}\left(u_i^{(p)}u_j^{(p)}-u_n^{(p)}u_{i+j-n}^{(p)}\right),\quad i,j\in\{1,2,\ldots,n-1\}.$$

- For the matrix

$$A=\begin{bmatrix}
 & & & & & & & & & -1 & 2 \\
 & & & & & & & & -1 & 2 & 0 \\
 & & & & & & & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & -1 & 2 & 0 & \cdot & \cdot & 0 \\
 & & & & -1 & 2 & 0 & \cdot & \cdot & 0 & -1 \\
 & & & -1 & 2 & 0 & \cdot & \cdot & 0 & -1 & \\
 & & \cdot & \cdot & \cdot & & & \cdot & \cdot & & \\
 & \cdot & \cdot & \cdot & & & \cdot & \cdot & & & \\
 -1 & 2 & 0 & & & 0 & -1 & & & & \\
 2 & \multicolumn{4}{c}{\underbrace{0\quad\cdot\quad\cdot\quad 0}_{p-1}} & -1 & & & & &
\end{bmatrix}_{(n-1)\times(n-1)},$$

$$a_{ij}=\frac{1}{u_n^{(p)}}\left(u_{n-i}^{(p)}u_{n-j}^{(p)}-u_n^{(p)}u_{n-i-j}^{(p)}\right),\quad i,j\in\{1,2,\ldots,n-1\}.$$

REFERENCES

1. R. S. Varga. *Matrix Iterative Analysis.* Englewood Cliffs, NJ: Prentice-Hall, Inc., 1962.
2. E. Zeckendorf. "Representations des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas." *Bulletin de la Societe Royale des Sciences de Liege* **41** (1972):179-82.

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FUNCTION DIGRAPHS OF QUADRATIC MAPS MODULO p

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1. INTRODUCTION

In this paper we will consider geometric representations of the iteration of quadratic polynomials modulo p. This is a discrete analog of the classical quadratic Julia sets which have been the subject of much study (e.g., [3], [4]). In particular let $\mathrm{fd}_m(\mathrm{u}(x))$ denote the function digraph which has $\mathbf{Z}_m$ as vertices and edges of the form $(x, \mathrm{u}(x))$, where x is an element of $\mathbf{Z}_m$. This digraph geometrically represents the function $\mathrm{u}(x)$ and paths correspond to iteration of $\mathrm{u}(x)$. The function digraphs resulting from squaring mod m, $\mathrm{fd}_m(x^2)$, have been studied when m is prime or has a primitive root (see [1], [2], [5], [10]). In particular, the cycle and tree structures have been classified. In [8], these results were generalized from $\mathrm{fd}_p(x^2)$ to $\mathrm{fd}_p(x^k)$ and a correspondence between geometric subsets of the function digraph and subgroups of the group of units was established. Subsequently, most of the results were generalized to general moduli in [12].

The aim of our paper is to explore these same ideas for the iteration of general quadratic functions instead of powers. In other words, we will consider $\mathrm{fd}_p(a_0 + a_1x + a_2x^2)$, where a_0, $a_1 \in \mathbf{Z}_p$ and $a_2 \in \mathbf{Z}_p^*$. It is easy to enumerate the four function digraphs for $p = 2$, so we will study the case when p is an odd prime. Although these digraphs do not contain nearly as much symmetry as the previously studied cases, it is possible to observe some aspects of their structure. Consider Figure 1 which shows the digraphs resulting from the iteration of x^2 and $x^2 + 1$ mod 13.

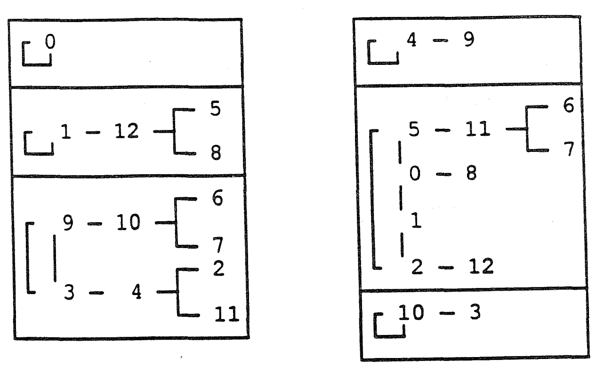

FIGURE 1. The Function Digraphs for x^2 and $x^2 + 1$ Modulo 13

Each of those digraphs breaks into three components. In reading the digraphs, note that the cycle contained in each component appears at the left and the cycles progress clockwise; that is, $\mathrm{u}(x)$ appears below x, except for the lowest cycle element where $\mathrm{u}(x)$ appears at the top. For noncycle elements, $\mathrm{u}(x)$ appears to the left of x in accordance with the indicated tree structure. Notice that the trees associated with each cycle element are uniform for x^2 but not for x^2+1. While it seems very difficult to completely determine the tree and cycle structure without enumerating the entire digraph, we can determine various things about the structure.

In particular, basic results for general function digraphs are given in Section 2. There it is established that, from the $p^2(p-1)$ function digraphs there are at most p digraphs distinct up to isomorphism. In Section 3 we investigate what appear to be tight bounds on number of cycles of a given length. The occurrence of cycles containing exactly one and two elements is completely classified. In Section 4 we empirically compare these quadratic digraphs to "random" digraphs and this motivates our conjecture that there are exactly p distinct quadratic digraphs mod p except, remarkably, for $p=17$. The quadratic function x^2-2 plays a special role in real dynamics [4] and in the theory of Mersenne primes [7], [9]. In Section 5 we investigate the corresponding family of function digraphs $\mathrm{fd}_p(x^2-2)$. The geometric form of these digraphs is very structured. We will see there are remarkable identities involving geometric position, addition, and multiplication for these digraphs that lead to that rich structure.

2. BASIC RESULTS

We begin by discussing properties that are common to function digraphs on $\mathbf{Z}_m$, and then turn to our quadratic function digraphs.

Proposition 1: Let $\mathrm{u}:\mathbf{Z}_m \to \mathbf{Z}_m$ be a function.

(a) The out-degree of any vertex $\mathrm{fd}_m(\mathrm{u}(x))$ is exactly one.

(b) The path in $\mathrm{fd}_m(\mathrm{u}(x))$ resulting from repeated iteration of any given element will eventually lead to a cycle.

(c) Every component of $\mathrm{fd}_m(\mathrm{u}(x))$ contains exactly one cycle.

Proof:

(a) This follows from the fact that $\mathrm{u}(x)$ is a function.

(b) Since the function maps points in a finite set, any path must eventually return to a previously visited vertex.

(c) If a component has more than one cycle, then somewhere on the undirected path connecting two cycles there would need to be a vertex with out-degree 2, contradicting (a). □

Theorem 2: The function digraphs $\mathrm{fd}_m(\mathrm{u}(x))$ and $\mathrm{fd}_m(\mathrm{v}(x))$ are isomorphic if and only if there exists a permutation r such that $\mathrm{r}^{-1}\circ\mathrm{u}\circ\mathrm{r}\equiv\mathrm{v} \bmod m$.

Proof: ($\Rightarrow$) Let r denote an isomorphism between $\mathrm{fd}_m(\mathrm{v}(x))$ and $\mathrm{fd}_m(\mathrm{u}(x))$; r gives a bijection between the vertices. The isomorphism of edges implies that, for all $x\in\mathbf{Z}_m$, the edge $(x,\mathrm{v}(x))$ in $\mathrm{fd}_m(\mathrm{v}(x))$ is mapped by r to the edge $(\mathrm{r}(x),\mathrm{u}(\mathrm{r}(x)))$ in $\mathrm{fd}_m(\mathrm{u}(x))$; hence, $\mathrm{u}(\mathrm{r}(x))\equiv$ $\mathrm{r}(\mathrm{v}(x)) \bmod m$, which gives $\mathrm{r}^{-1}\circ\mathrm{u}\circ\mathrm{r}\equiv\mathrm{v}$.

($\Leftarrow$) Let r denote a permutation such that $\mathrm{r}^{-1} \circ \mathrm{u} \circ \mathrm{r} \equiv \mathrm{v}$. Now r gives a bijection between the vertices; hence, we need to check this bijection respects the edges. Since $\mathrm{r}^{-1} \circ \mathrm{u} \circ \mathrm{r}(x) \equiv \mathrm{v}(x)$ for all $x \in \mathbf{Z}_m$, we have $\mathrm{u}(\mathrm{r}(x)) \equiv \mathrm{r}(\mathrm{v}(x))$, which implies that the edge $(x, \mathrm{v}(x))$ in $\mathrm{fd}_m(\mathrm{v}(x))$ is mapped to the edge $(\mathrm{r}(x), \mathrm{u}(\mathrm{r}(x)))$ in $\mathrm{fd}_m(\mathrm{u}(x))$ as required. □

Theorem 3: Let $m \geq 3$ be odd, and $\gcd(a_2, m) = 1$. The quadratic function digraph $\mathrm{fd}_m(a_0 + a_1x + a_2x^2)$ is isomorphic to the function digraph of the canonical form quadratic $\mathrm{fd}_m(x^2 + \gamma)$, where $\gamma = a_0a_2 + 2^{-1}a_1 - 2^{-2}a_1^2$.

Proof: First note that since m is odd, 2^{-1} exists and hence γ is well defined. Let $\mathrm{u}(x) = a_0 + a_1x + a_2x^2$, $\mathrm{v}(x) = x^2 + \gamma$, and $\mathrm{r}(x) = a_2^{-1}x - 2^{-1}a_1a_2^{-1}$. Note that a_2^{-1} is well defined since $\gcd(a_2, m) = 1$. By direct computation, we can check $\mathrm{r}^{-1} \circ \mathrm{u} \circ \mathrm{r}(x) \equiv \mathrm{v}(x) \bmod m$ as required. □

Corollary 4: Let $m \geq 3$ be odd. There are, up to isomorphism, at most m quadratic function digraphs mod m with leading coefficient relatively prime to m.

Proof: By Theorem 3, every quadratic function digraph with $\gcd(a_2, m) = 1$ in $\mathbf{Z}_m$ is isomorphic to that of a quadratic in the canonical form $x^2 + \gamma$. Since there are m distinct quadratics in the canonical form, up to isomorphism, there are no more than m quadratic function digraphs mod m with leading coefficient relatively prime to m. □

The proviso on leading coefficients is necessary. For example, when $m = 4$, the eight polynomials x^2, $x^2 + 1$, $2x^2$, $2x^2 + x$, $2x^2 + 2x$, $2x^2 + 3x$, $2x^2 + x + 1$, $2x^2 + 3x + 1$ all have non-isomorphic function digraphs. Our interest lies primarily with odd prime moduli. Of course, Theorem 3 and Corollary 4 hold for odd prime moduli, p. Hereafter in the paper, we will let p denote an odd prime.

There is a situation when, up to isomorphism, there are fewer than p quadratic function digraphs mod p. Figure 2 shows two canonical form function digraphs that are isomorphic mod 17. However, we conjecture that this is the only example where there are fewer than p quadratic function digraphs; this will be discussed further in Section 4 after we have established certain facts about arbitrary digraphs that satisfy the conclusion of the next theorem. This theorem is the first that requires the modulus to be an odd prime.

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┌ 3 - 14
└┘
---------------------------
           ┌ 4
┌ 9 - 10 ──┤
│          └ 13 ──┬ 6
│                 └ 11 - 0
└ 7 - 8
---------------------------
           ┌ 5
┌ 15 - 2 ──┤
└┘         └ 12 ──┬ 1
                  └ 16
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┌ 13 - 4
└┘
---------------------------
                 ┌ 3
             6 ──┤
┌ 15 - 16 ──┤    └ 14 - 0
│ │         └ 11
└ 1 - 2
---------------------------
             ┌ 7
┌ 5 - 12 ───┤
└┘           └ 10 ──┬ 8
                    └ 9
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FIGURE 2. The Function Digraph $\mathbf{fd_{17}(x^2 + 11)}$ Is Isomorphic to $\mathbf{fd_{17}(x^2 + 14)}$

Theorem 5: Let $\mathrm{u}(x)$ be a quadratic function modulo p. In the function digraph $\mathrm{fd}_p(\mathrm{u}(x))$, there are exactly $(p-1)/2$ vertices of in-degree 0, one vertex of in-degree 1, and $(p-1)/2$ vertices of in-degree 2.

Proof: There are $(p-1)/2$ quadratic residues and nonresidues mod p. Note that we need only consider the digraphs of quadratics in canonical form. In order to determine the in-degree of a vertex y we need to know the number of solutions to $y \equiv x^2 + \gamma$. Note that a vertex has in-degree 2 if and only if $y-\gamma$ is a quadratic residue, it has in-degree 0 if and only if $y-\gamma$ is a quadratic nonresidue, and it has in-degree 1 if and only if $y-\gamma \equiv 0$. Thus, there are exactly $(p-1)/2$ vertices of in-degree 0, 1 vertex of in-degree 1, and $(p-1)/2$ vertices of in-degree 2. $\square$

3. CYCLES

Notice that each element in an n-cycle of $\mathrm{fd}_p(\mathrm{u}(x))$ must be a solution to the congruence $\mathrm{u}^n(x) \equiv x \bmod p$, where $\mathrm{u}^n(x)$ denotes the composition of the function $\mathrm{u}(x)$ with itself n times. In contrast, we will use $\mathrm{u}(x)^n$ to denote the n^{th} power of the function $\mathrm{u}(x)$. Since the congruence $\mathrm{u}^n(x) \equiv x$ has degree 2^n when $\mathrm{u}(x)$ is quadratic, it is a standard result that there can be at most 2^n solutions since the modulus is prime [11]. Thus, there are at most $2^n/n$ cycles of length n. It turns out that we can establish a better bound on the number of cycles of length n when $n = p^e$ as we will see in Corollary 8. A heuristic argument suggests this bound works for general n, and empirical evidence indicates the bound is tight. In order to establish the bound, we use the following lemma and theorem. We will then consider our heuristic and empirical evidence and finish this section by classifying the number of one and two cycles that appear.

Lemma 6: Let $\mathrm{v}(x) = x^2 + \gamma$, $\gamma \in \mathbf{R}$, and $\mathrm{P}_n(x) = \mathrm{v}^n(x) - x$. Then $\mathrm{P}_n(x)$ divides $\mathrm{P}_{kn}(x)$ as a polynomial in $\mathbf{R}[x]$. Moreover, the quotient is in $\mathbf{Z}[x]$ if $\gamma \in \mathbf{Z}$.

Proof: Since both $\mathrm{P}_n(x)$ and $\mathrm{P}_{kn}(x)$ are monic, it suffices to show that every complex root of $\mathrm{P}_n(x)$ is also a root of $\mathrm{P}_{kn}(x)$ with at least as high a multiplicity. Note that if x_0 is a root of $\mathrm{P}_n(x)$ then $\mathrm{v}^n(x_0) = x_0$, and hence $\mathrm{v}^{kn}(x_0) = x_0$, which implies that x_0 is a root of $\mathrm{P}_{kn}(x)$; this takes care of the single roots. Note that x_0 is a root of $\mathrm{P}_n(x)$ of multiplicity m if and only if it is a root of $P_n(x)$ and a root of the derivative of $\mathrm{P}_n(x)$ with mul-tiplicity $m-1$. Using the chain rule repeatedly and the fact that $\mathrm{v}'(x) = 2x$, we see that

$$\mathrm{P}_n'(x) = 2^n \mathrm{v}^{n-1}(x)\mathrm{v}^{n-2}(x) \dots \mathrm{v}(x)x - 1,$$

and hence,

$$\mathrm{P}_{kn}'(x) = 2^{kn} \mathrm{v}^{kn-1}(x)\mathrm{v}^{kn-2}(x) \dots \mathrm{v}(x)x - 1.$$

Now we want to consider the derivative $\mathrm{P}_{kn}'(x)$ modulo $\mathrm{P}_n(x)$. Note that by definition $\mathrm{v}^n(x) \equiv x \bmod \mathrm{P}_n(x)$, and hence $\mathrm{v}^{jn+i}(x) \equiv \mathrm{v}^i(x) \bmod \mathrm{P}_n(x)$, so that

$$\mathrm{P}_{kn}'(x) \equiv (2^n \mathrm{v}^{n-1}(x)\mathrm{v}^{n-2}(x) \dots \mathrm{v}(x)x)^k - 1 \mod \mathrm{P}_n(x).$$

If we let $\mathrm{w}(x) = 2^n \mathrm{v}^{n-1}(x)\mathrm{v}^{n-2}(x) \dots \mathrm{v}(x)x$, then $\mathrm{P}_n'(x) = \mathrm{w}(x) - 1$ and

$$\begin{aligned} \mathrm{P}_{kn}'(x) \equiv \mathrm{w}(x)^k - 1 &= (w(x)-1)(w(x)^{k-1} + w(x)^{k-2} + \dots + w(x) + 1) \\ &= \mathrm{P}_n'(x)(w(x)^{k-1} + \dots + 1) \mod \mathrm{P}_n(x). \end{aligned}$$

Now suppose x_0 is a root of $\mathrm{P}_n(x)$ of multiplicity m, and hence is also a root of $\mathrm{P}_n'(x)$ of multiplicity $m-1$. By the above, it is also a root of $\mathrm{P}_{kn}'(x)$ of multiplicity $m-1$ and hence is a root of $P_{kn}(x)$ of multiplicity m. Thus, $\mathrm{P}_n(x)$ divides $\mathrm{P}_{kn}(x)$.

To see that the quotient is in $\mathbf{Z}[x]$ if $\gamma \in \mathbf{Z}$, consider the following. Since $\mathrm{P}_n(x) \in \mathbf{Z}[x]$ is monic of degree 2^n, we can write $\mathrm{P}_n(x) = b_0 + b_1 x + \cdots + b_{2^n-1} x^{2^n-1} + x^{2^n}$, where $b_i \in \mathbf{Z}$. Since $\mathrm{P}_{kn}(x) \in \mathbf{Z}[x]$ is also monic, we can also write the quotient in the form

$$\mathrm{f}(x) = a_0 + a_1 x + \cdots + a_{K-1} x^{K-1} + x^K,$$

where $K = 2^{kn} - 2^n$ is the degree of the quotient. Suppose $\mathrm{f}(x) \notin \mathbf{Z}[x]$. Let m be the largest integer such that $a_m \notin \mathbf{Z}$. Now the coefficient of x^{2^n+m} in the product $\mathrm{P}_{kn}(x) = \mathrm{P}_n(x)\mathrm{f}(x)$ is a finite sum of the form $a_m + a_{m+1} b_{2^n-1} + a_{m+2} b_{2^n-2} + \cdots$. This coefficient is an integer because $\mathrm{P}_{kn}(x) \in \mathbf{Z}[x]$ and each factor in the second and higher terms of the finite sum are integers; thus, a_m is also an integer that contradicts $a_m \notin \mathbf{Z}$ and proves the claim. □

The elements $x_0 \in \mathbf{R}$ such that $\mathrm{v}^n(x_0) = x_0$ are said to be cyclic of period n. Any root of $\mathrm{P}_n(x)$ which is of period n and not of any shorter period is said to be of *prime period n*. Any complex root with nonprime period n will be a root of some $\mathrm{P}_d(x)$, where d divides n though it is possible that $\mathrm{P}_n(x)$ does not have roots of prime period n. For example, when $\gamma = -3/4$, then $\mathrm{P}_1(x) = (x+1/2)(x-3/2)$ and $\mathrm{P}_2(x) = (x+1/2)^3(x-3/2)$ which has no new roots; hence, there are no points of prime period 2 for this γ.

The following theorem and conjecture involve a factorization similar to the classical factorization of $x^n - 1$ in terms of cyclotomic polynomials [10], yet it is quite different in that $\mathrm{P}_n(x) = \mathrm{v}^n(x) - x$ involves function iteration, not ordinary powers.

Theorem 7: Let $\mathrm{P}_n(x) = \mathrm{v}^n(x) - x$ be as above and let $n = q^k$ be a power of a prime. Also let

$$\mathrm{Q}_1(x) = \mathrm{P}_1(x) \quad \text{and} \quad \mathrm{Q}_n(x) = \frac{\mathrm{P}_n(x)}{\prod_{d|n,\, d<n} \mathrm{Q}_d(x)},$$

then $\mathrm{Q}_n(x)$ is a polynomial in $\mathbf{R}[x]$ for $\gamma \in \mathbf{R}$ and it is in $\mathbf{Z}[x]$ for $\gamma \in \mathbf{Z}$.

Proof: Since $n = q^k$ is a power of a prime, it is easy to check that

$$\mathrm{Q}_n(x) = \mathrm{Q}_{q^k}(x) = \frac{\mathrm{P}_{q^k}(x)}{\mathrm{Q}_{q^{k-1}}(x)\mathrm{Q}_{q^{k-2}}(x)\ldots\mathrm{Q}_q(x)\mathrm{Q}_1(x)} = \frac{\mathrm{P}_{q^k}(x)}{\mathrm{P}_{q^{k-1}}(x)},$$

which is a polynomial by Lemma 6. The remark about the quotient being in $\mathbf{Z}[x]$ for $\gamma \in \mathbf{Z}$ follows as in the previous Lemma. □

We conjecture that this property holds for general n.

Conjecture A: Let $\mathrm{P}_n(x) = \mathrm{v}^n(x) - x$ be as above and let

$$\mathrm{Q}_1(x) = \mathrm{P}_1(x) \quad \text{and} \quad \mathrm{Q}_n(x) = \frac{\mathrm{P}_n(x)}{\prod_{d|n,\, d<n} \mathrm{Q}_d(x)},$$

then $\mathrm{Q}_n(x)$ is a polynomial in $\mathbf{R}[x]$ for $\gamma \in \mathbf{R}$ and it is in $\mathbf{Z}[x]$ for $\gamma \in \mathbf{Z}$.

Consider the following heuristic argument in favor of the conjecture. Solving for $P_n(x)$, we see that $P_n(x) = \prod_{d|n} Q_d(x)$. We can obtain a sum over the divisors of n by taking logarithms and then we can apply the Möbius inversion formula. On rewriting the result as a product, we see that $Q_n(x) = \prod_{d|n} P_d(x)^{\mu(n/d)}$, where $\mu(n)$ is the Möbius function. In the case when $n = q_1^{k_1} q_2^{k_2}$ is the product of powers of two primes, this amounts to

$$Q_n(x) = Q_{q_1^{k_1} q_2^{k_2}}(x) = \frac{P_n(x) P_{\frac{n}{q_1 q_2}}(x)}{P_{\frac{n}{q_1}}(x) P_{\frac{n}{q_2}}(x)}.$$

Now, if $P_{n/q_1}(x)$ and $P_{n/q_2}(x)$ have no roots in common, then all their roots with all their multiplicity are also roots of $P_n(x)$, and hence $Q_n(x)$ is a polynomial. If they have a common root x_0 and it is a single root of at least one factor of the denominator, then the factor with the higher multiplicity divides $P_n(x)$ by Lemma 6. Since x_0 is a root of $P_{n/q_1}(x)$ and $P_{n/q_2}(x)$, it has period n/q_1 and also has period n/q_2; hence, it has period

$$\gcd\left(\frac{n}{q_1}, \frac{n}{q_2}\right) = \frac{n}{q_1 q_2}.$$

That is, it is a root of $P_{n/q_1 q_2}(x)$. Thus, the factors arising from the root x_0 will cancel except possibly some factors in the numerator. As long as common roots of factors appearing in the denominator do not have common multiplicity over 1, this argument would generalize to any number of prime factors. We expect that, for a generic choice of γ, the roots of $P_n(x)$ would all be single roots. Thus, common multiplicity would be one, so the above argument would work. However, once the result is true for some generic γ, it should be true for the formal parameter γ as well.

We can formally compute $Q_n(x)$ for small n. Notice that these are polynomials in x and γ.

$$\begin{aligned}
&Q_1(x) = x^2 - x + \gamma, \\
&Q_2(x) = x^2 + x + \gamma + 1, \\
&Q_3(x) = x^6 + x^5 + (1+3\gamma)x^4 + (1+2\gamma)x^3 + (1+3\gamma+3\gamma^2)x^2 + (1+\gamma)^2 x + \gamma^3 + 2\gamma^2 + \gamma + 1, \\
&Q_4(x) = x^{12} + 6\gamma x^{10} + x^9 + (15\gamma^2 + 3\gamma)x^8 \\
&\qquad\qquad + \cdots + (2\gamma + \gamma^2 + 2\gamma^3 + \gamma^4)x + (1 + 2\gamma^2 + 3\gamma^3 + 3\gamma^4 + 3\gamma^5 + \gamma^6).
\end{aligned}$$

Using symbolic manipulation software, we have verified that $Q_6(x)$ is a formal polynomial in x and γ.

Corollary 8: Let $u(x)$ be a quadratic function and $n = q^k$ be a power of a prime. In $\mathrm{fd}_p(u(x))$, the number of cycles of length n is less than or equal to

$$\frac{1}{n}\deg(Q_n(x)) = \frac{1}{n}\left(2^n - \sum_{d|n,\, d<n} \deg(Q_d(x))\right).$$

Proof: The number of elements with prime period n is less than or equal to the degree of $Q_n(x)$, which can be computed recursively from its definition given in Theorem 7. □

Of course, if Conjecture A is true, we would have also established Corollary 8 for general n. Hence, we have the following conjecture.

Conjecture B: Let $\mathrm{u}(x)$ be a quadratic function and n be a positive integer. In $\mathrm{fd}_p(\mathrm{u}(x))$, the number of cycles of length n is less than or equal to

$$\frac{1}{n}\deg(\mathrm{Q}_n(x)) = \frac{1}{n}\left(2^n - \sum_{d|n,\, d<n} \deg(\mathrm{Q}_d(x))\right).$$

Note that the bounds given in the corollary and conjecture may be ugly in the sense that they are recursively defined, but they are easy to compute. Table 1 gives some examples illustrating primes where these bounds are achieved. Notice the bounds seem to be tight even though they get large. It seems remarkable that the theoretic bound on 11-cycles is 186 occurrences and this happens for a relatively small prime. The fact that these bounds are indeed the maximal number of occurrences we found for some additional cases where n is not a prime power provides additional evidence for the correctness of Conjectures A and B. It is also interesting to compare these bounds which are computed algebraically here with the number of orbits of prime period arising from the genealogy of periodic points in classical real dynamics [3]. The next theorems allow us to determine when there are 1-cycles and 2-cycles.

TABLE 1. Minimal Odd Prime p Such that the Function Digraph $\mathrm{fd}_p(x^2)$ Achieves the Maximal Repetition of Cycle Lengths

Cycle Length	Bound on Repetitions	Odd Prime p
1	2	3
2	1	7
3	2	29
4	3	31
5	6	311
6	9	127
7	18	509
8	30	1,021
9	56	3,067
10	99	4,093
11	186	36,847
12	335	8,191

Theorem 9: The number of 1-cycles in $\mathrm{fd}_p(x^2+\gamma)$ is $1+\left(\frac{2^{-2}-\gamma}{p}\right)$.

Proof: Recall that since p is an odd prime, 2^{-1} exists modulo p. By completing the square of $\mathrm{Q}_1(x) = x^2 - x + \gamma \equiv 0$, we have $(x-2^{-1})^2 \equiv 2^{-2} - \gamma$. Thus, $\mathrm{fd}_p(x^2+\gamma)$ has two, one, or zero 1-cycles if and only if $2^{-2}-\gamma$ is a quadratic residue, 0, or a nonresidue, respectively. □

Theorem 10: There is exactly one 2-cycle in $\mathrm{fd}_p(x^2+\gamma)$ if and only if $\left(\frac{2^{-2}-\gamma-1}{p}\right) = 1$.

Proof: Notice that if $\mathrm{Q}_2(x)$ has a repeated root mod p, the root is a 1-cycle; moreover, if $\mathrm{Q}_1(x)$ and $\mathrm{Q}_2(x)$ have a shared root, then $\mathrm{Q}_2(x) - \mathrm{Q}_1(x) = 2x+1 \equiv 0$, from which we see that $x \equiv -2^{-1}$ is the only possible shared root. In such a case, the other root of $\mathrm{Q}_2(x)$ must be a 1-cycle, hence both roots must be -2^{-1}. Thus, the function digraph $\mathrm{fd}_p(x^2+\gamma)$ has exactly one

2-cycle if and only if $Q_2(x) = x^2 + x + \gamma + 1 \equiv 0$ has two distinct solutions in $\mathbf{Z}_p$. Completing the square in that congruence yields $(x+2^{-1})^2 \equiv 2^{-2} - \gamma -$ which has two distinct solutions in $\mathbf{Z}_p$ if and only if $2^{-2} - \gamma - 1$ is a quadratic residue mod p. □

4. RANDOM QUASIQUADRATIC DIGRAPHS

We have seen that it is difficult to predict the structure of $\mathrm{fd}_p(\mathrm{u}(x))$ for quadratic functions $u(x)$, yet we have been able to give some restrictions on the behavior of these function digraphs. In this section we will compare the structure of the quadratic function digraphs $\mathrm{fd}_p(\mathrm{u}(x))$ with those of "random" functions whose function digraphs have the same number of vertices with in-degree 0, 1, and 2 as have the quadratic function digraphs. In particular, we will call a function $\mathrm{q}:\mathbf{Z}_p \to \mathbf{Z}_p$ *quasi-quadratic* if it is 2-to-1 for all of its domain except that it is 1-to-1 for one element in its domain; e.g., Figure 3 shows a randomly chosen quasiquadatic function digraph on $\mathbf{Z}_{17}$. Notice it has the same random appearance of the quadratic function digraphs modulo 17 but it has two 2-cycles, which is impossible for a quadratic function digraph by Corollary 8.

0	->	7
1	->	1
2	->	6
3	->	15
4	->	6
5	->	13
6	->	4
7	->	12
8	->	11
9	->	15
10	->	1
11	->	9
12	->	7
13	->	9
14	->	11
15	->	4
16	->	13

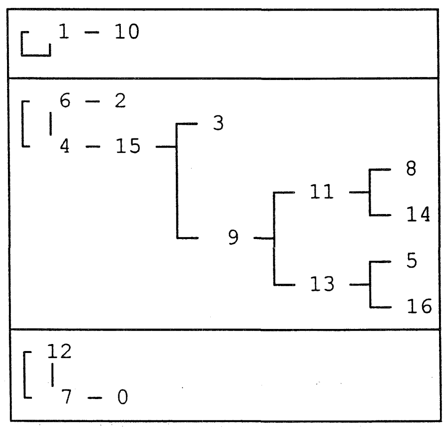

FIGURE 3. A Random Quasiquadratic Function and Its Digraph Which Contains Two 2-Cycles

We begin our investigation by counting the number of quasiquadratic functions.

Theorem 11: Given a prime modulus $p \geq 3$,

(a) the number of quasiquadratic functions is $\frac{p+1}{2}\binom{p}{(p+1)/2}\binom{p}{2\,2\ldots2\,2\,1}$ and

(b) the number of quasiquadratic digraphs that are nonisomorphic is $\binom{p}{(p+1)/2}$.

Proof: ***(a)*** There are $\binom{p}{(p+1)/2}$ ways to choose the $\frac{p+1}{2}$ range elements of the quasiquadratic functions and there are $\frac{p+1}{2}\binom{p}{2\,2\ldots2\,2\,1}$ permutations what would result in distinct rearrangements since the multinomial $\binom{p}{2\,2\ldots2\,2\,1}$ gives the number of ways to partition p elements into classes of size 2, 2, ..., 2, 1 and there are $\frac{p+1}{2}$ ways to position the 1.

(b) An isomorphism between quasiquadratic digraphs must map each pair of the range of the first digraph to a pair in the range of the second digraph; the isomorphism must also map the singleton of the range of the first digraph to the singleton in the range of the second digraph.

Since there are $\binom{p}{2\,2\ldots 2\,2\,1}$ ways to pick the pairs and singleton and $\frac{p+1}{2}$ ways to place the singleton, there are $\frac{p+1}{2}\binom{p}{2\,2\ldots 2\,2\,1}$ such permutations. Dividing into this total number of quasiquadratic digraphs, we see the number of quasiquadratic digraphs that are nonisomorphic is $\left(\binom{p}{(p+1)/2}\right)$.

Notice that we really only used the fact that the modulus is odd, not that it is prime. □

We can easily generate random quasiquadratic digraphs and compare their structure with the structure of quadratic digraphs. Figure 4 shows the frequency that cycles of specified length appear in 10,000 random choices of quasiquadratic digraphs modulo 1009. These quasiquadratic frequencies are shown with the connected lines. The isolated points show the same information for the 1009 quadratic function digraphs. Likewise, Figure 5 shows the average frequency that specified numbers of components occur for quasiquadratic and quadratic function digraphs modulo 1009. While the fits are not perfect, they are remarkably good and this provides empirical support for the heuristic view that the quadratic function digraphs are nearly "random."

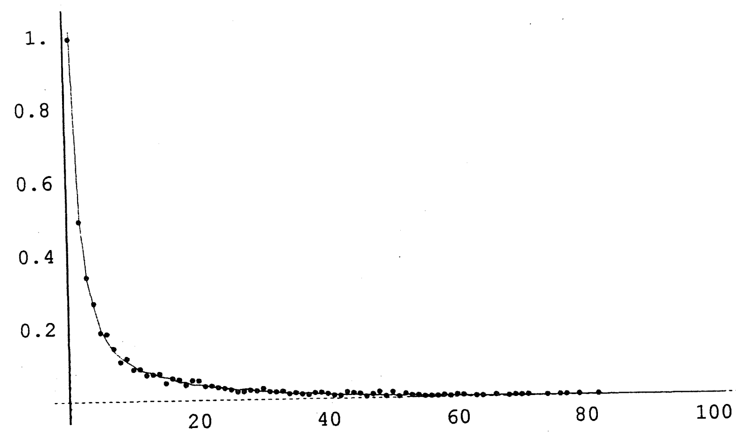

FIGURE 4. The Average Frequency of Cycle Lengths for Quadratic and Quasiquadratic Digraphs Modulo 1009

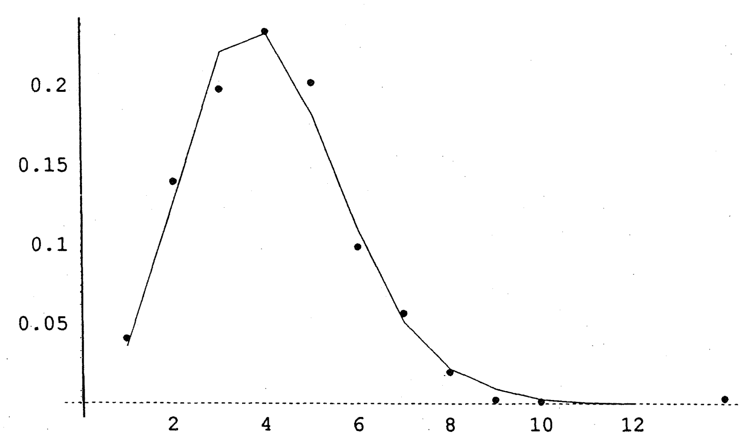

FIGURE 5. The Average Number of Components for Quadratic and Quasiquadratic Digraphs Modulo 1009

In Section 2 we noted an example of quadratic function digraphs in canonical form that are isomorphic: $\mathrm{fd}_{17}(x^2+11) \cong \mathrm{fd}_{17}(x^2+14)$. If we assume that the p quadratic function digraphs are randomly distributed over the quasiquadratic function digraphs, then we can estimate the expected number of pairs of quadratic function digraphs that will be isomorphic by multiplying the number of pairs $\binom{p}{2}$ by the reciprocal of the number of distinct quasiquadratic function digraphs. Table 2 shows the expected number of isomorphic pairs implied by that estimate. One might choose to use $\binom{p-2}{2}$ instead of $\binom{p}{2}$ since $\mathrm{fd}_p(x^2)$ and $\mathrm{fd}_p(x^2-2)$ are special; see [8] and Section 5, respectively, for how those digraphs are special. Using $\binom{p-2}{2}$ would reduce the expected numbers, especially for small p. However, the main point is that these expected numbers approach 0 very quickly since the number of pairs is quadratic but the number of quasiquadratic functions is exponential in p. Hence, we make the following conjecture.

TABLE 2. The Expected Number of Isomorphic Quadratic Function Digraphs for Small Odd Primes

p	Expected Isomorphisms
3	1.0
5	1.0
7	0.6
11	0.119
13	0.0455
17	0.00559
19	0.00185
23	0.000187
29	0.00000523
31	0.00000154

Conjecture C–Quadratic Digraph Isomorphism Conjecture: The only occurrence of isomorphic quadratic function digraphs in canonical form is $\mathrm{fd}_{17}(x^2+11) \cong \mathrm{fd}_{17}(x^2+14)$.

In addition to the heuristic argument in favor of this conjecture given above, we have computationally verified the conjecture for all primes up to 1009.

5. FUNCTION DIGRAPHS $\mathrm{fd}_p(x^2-2)$

In classical dynamics, the dynamics of the function x^2-2 are special (see [4]) because the Julia set is unusually simple. A similar statement can be made in the theory of numbers where iteration of this function plays a role in whether Mersenne numbers $p=2^{q-1}-1$ are prime (see [7] and [9]). We investigate the family of function digraphs $\mathrm{fd}_p(x^2-2)$ for general odd prime modulus which has far more structure than typical quadratic function digraphs. This structure seems to be as deep as, but more complicated than, the structure of $\mathrm{fd}_p(x^2)$. Indeed, we will see that the identities we use involve both multiplication and addition. Figure 6 shows $\mathrm{fd}_{239}(x^2-2)$. This example is rather large but serves to illustrate all the properties that we want to observe without requiring several examples. We see that all the cycle elements have one leaf or a binary tree attached. The nonleaf trees all have the same depth and are isomorphic except for one vertex of in-degree one. Our goal in this section is to show that those claims are true in general. Also notice that the cycle lengths seem to have some coherence. Readers who would like to see examples of the remarkable arithmetic/structure identities before considering the general theory may preview the examples that follow Theorem 19.

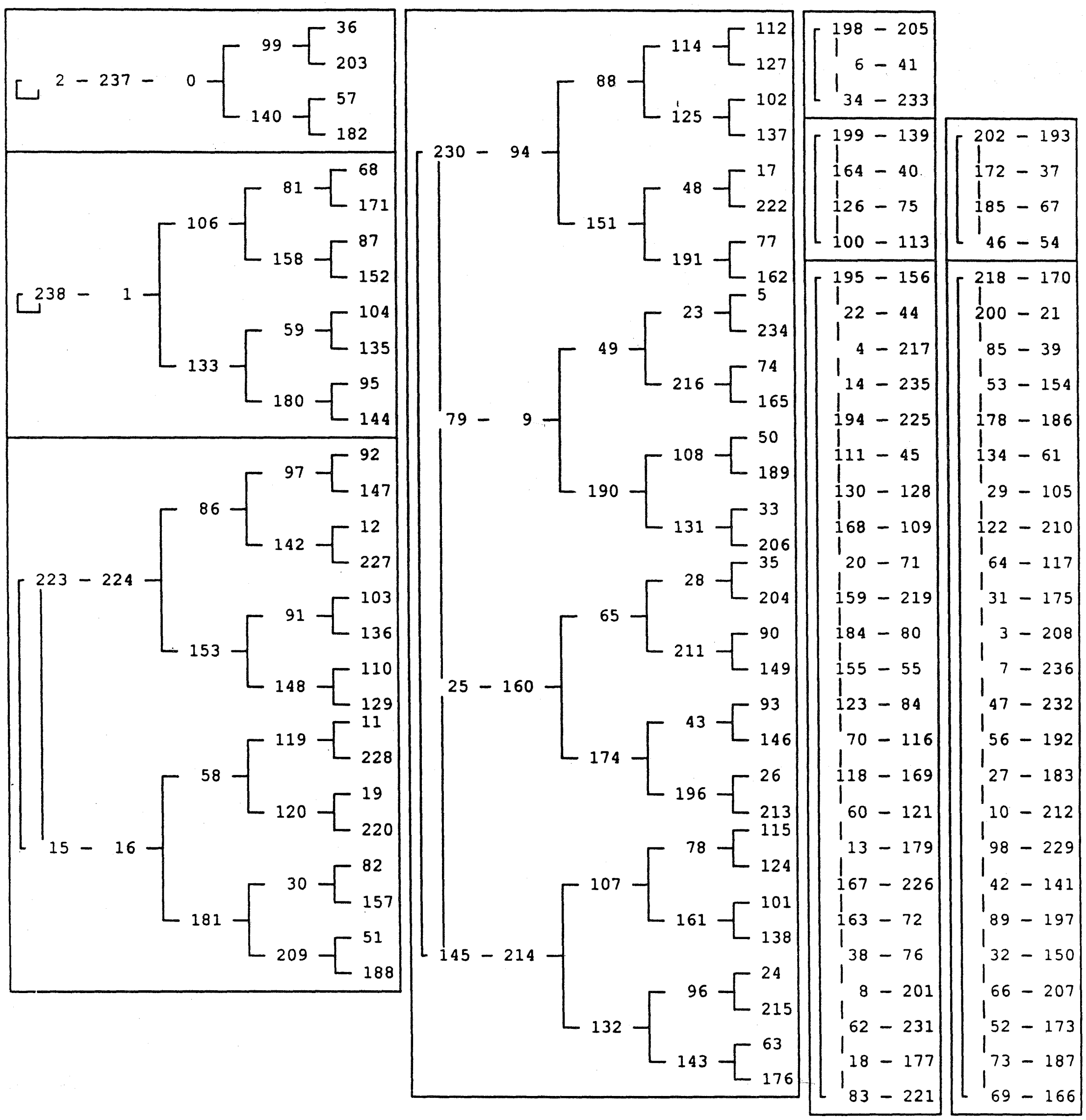

FIGURE 6. The Function Digraph $\mathrm{fd}_{239}(x^2-2)$

In this section we will let $s(x)=x^2-2$. The *level* of a vertex x measured from its cycle is given by the smallest k such that $s^k(x)$ is a cycle element. Thus, cycle elements are at level 0. Components with at least one vertex at level 2 are called *branched components*. Other components are called *stumpy components*. We say that two distinct vertices M and N are *k-ancestors* if

k is the smallest positive integer such that $s^k(M) = s^k(N)$. For example, M and N are 1-ancestors if and only if $M = -N$ and they are 2-ancestors if and only if $s(M) = -s(N)$; namely, $M^2 - 2 = 2 - N^2$.

Our first lemma in this section shows that multiplying two 2-ancestors gives a nearby vertex. We think of this theorem as giving enough structure to the digraphs so that we can establish a base case for our eventual induction. It also establishes enough structure so that in the subsequent lemma we can discuss leaves and cycles and distinguish two fundamentally different types of digraph components: those that reach level two and those that do not.

Lemma 12: If M and N are 2-ancestors in $\mathrm{fd}_p(s(x))$, then MN and $s(M)$ are 2-ancestors and MN and $s(N)$ are 2-ancestors as well.

Proof: Since M and N are 2-ancestors, $s(M) = -s(N)$ so $M^2 - 2 = 2 - N^2$ and, therefore, $N^2 = 4 - M^2$. Now

$$s^2(M) = M^4 - 4M^2 + 2 = 2 - M^2(4 - M^2) = 2 - (MN)^2 = -s(MN).$$

Thus, MN and $s(M)$ are 2-ancestors and by symmetry so are MN and $s(N)$. □

As an aside, we notice that if we try to generalize this to $v(x) = x^2 + \gamma$, we see that M and N are 2-ancestors means $M^2 + \gamma = -\gamma - N^2$, and hence $-2\gamma = N^2 + M^2$. Thus,

$$v^2(M) = M^4 + 2\gamma M^2 + \gamma^2 + \gamma = M^4 + (-N^2 - M^2)M^2 + \gamma^2 + \gamma = -v(MN) + \gamma^2 + 2\gamma,$$

and hence, $v^2(M) = -v(MN)$ if and only if $\gamma^2 + 2\gamma = 0$. This gives the special cases $\gamma = 0$ and $\gamma = -2$ mentioned in Section 4.

We will refer to Figure 6 to provide an illustration of Lemma 12 in $\mathrm{fd}_{239}(s(x))$. Notice that $M = 230$ and $N = 65$ are 2-ancestors appearing in the component of $\mathrm{fd}_{239}(s(x))$ that has a 4-cycle. We see that $s(M) = s(230) = 79$ while $MN = 230 * 65 \equiv 132 \bmod 239$. We can observe that 79 and 132 are also 2-ancestors.

If x is a noncycle element, we define the *tree leading to* x to be the union of all paths leading to x. More precisely, the tree leading to x is $\{y \in \mathbf{Z}_p \mid s^k(y) = x \text{ for some } k \geq 0\}$. Notice that, for each p, the function digraph $\mathrm{fd}_p(s(x))$ contains a component where 0 maps to -2 which maps to 2 and where 2 maps back onto itself. The vertex $x = -2$ is the single vertex of in-degree 1 and it is at level one. Therefore, all cycle elements have in-degree 2. Thus, each cycle element has a unique noncycle parent. If c is a cycle element, we define the *tree associated with* c to be the tree leading to the noncycle parent of c. In particular, x is an element of the tree leading to x but c is not an element of the tree associated with c.

A vertex $x \neq -2$ has parents if and only if there are two solutions y to $y^2 - 2 = x$, and this occurs exactly when the Legendre symbol $\left(\frac{2+x}{p}\right) = 1$. In particular, the tree leading to 0 contains more than the vertex 0 if and only if $p = 1, 7 \bmod 8$, since those are the cases when 2 is a quadratic residue. We call this component the 0-component. Eventually, we will see that the existence and depth of the three leading to 0 influences the structure of the branched components.

We say a tree is a *complete binary tree up to level* k if the tree has a root and each vertex at level less than k from the root has exactly two parents.

The next lemma describes the structure of the components up to level 2 which gives a starting point for our structure theorem.

Lemma 13: In $\mathrm{fd}_p(\mathrm{s}(x))$:

(a) the tree associated with a cycle element in a stumpy component consists of one leaf at level one;

(b) the tree associated with a cycle element in a branched component, except the vertex 2, is a tree structure that is a complete binary tree up to level 2.

Proof: Recall that -2 is the only vertex of in-degree one and it is not a cycle element. Thus, every cycle element has in-degree 2.

(a) By definition, stumpy components cannot have any elements at level 2 or higher and we have noted every cycle element in a stumpy component will have in-degree 2; this gives the result.

(b) We know that each cycle element has a single noncycle parent. By definition, in every branched component there is some vertex M at level 2. Let N be the cycle element such that M and N are 2-ancestors. Lemma 12 implies that MN is at level 2 leading to the cycle element after $\mathrm{s}^2(M)$. Repeating the process on MN and proceeding around the entire cycle implies there is a vertex at level 2 in the tree associated with every cycle element. Since the in-degree of all the level 1 vertices must be 2, except for at -2 in the 0-component, we see the trees associated with such a cycle element from a branched component must be a complete binary tree up to level 2. □

We now show that, in the branched components, any vertex has parents if and only if its additive inverse has parents. We already noted that ± 2 both have parents.

Lemma 14: If x is a vertex other than ± 2 in a branched component, then x has parents if and only if $-x$ has parents. That is, if $\left(\frac{2+x}{p}\right)=\left(\frac{2-x}{p}\right)$.

Proof: We have seen that this is true for levels 0 and 1 since all such vertices have parents [Lemma 13(b)], and it is trivially true for $x=0$. Suppose that this lemma is not true in general. Assume x is a vertex at the lowest level such that $\left(\frac{2+x}{p}\right)\neq\left(\frac{2-x}{p}\right)$. Now

$$\left(\frac{2+x}{p}\right)\left(\frac{2-x}{p}\right)=\left(\frac{4-x^2}{p}\right)=\left(\frac{2-\mathrm{s}(x)}{p}\right).$$

Since $\mathrm{s}(x)$ is at a lower level, the result it true for $\mathrm{s}(x)$; hence,

$$\left(\frac{2-\mathrm{s}(x)}{p}\right)=\left(\frac{2+\mathrm{s}(x)}{p}\right)=\left(\frac{x^2}{p}\right)=1.$$

Therefore, $\left(\frac{2+x}{p}\right)=\left(\frac{2-x}{p}\right)$, which contradicts the supposition and completes the proof. □

Note that, if any vertex x appears at level 3 or higher, then $s(x)$ and $-s(x)$ will both have parents; hence, we get at least four vertices at the level of x.

We already know that 2 is a 1-cycle element and -2 is a nonleaf leading to that cycle. Thus, ± 2 are nonleaves in a branched component. The next theorem shows that we can use the two Legendre symbols from Lemma 14 to classify all the other vertices into four geometric classes.

Theorem 15: Suppose x is a vertex other than ± 2 in $\mathrm{fd}_p(\mathrm{s}(x))$.

(a) If $\left(\frac{2+x}{p}\right)=1$ and $\left(\frac{2-x}{p}\right)=1$, then x is a nonleaf in a branched component.

(b) If $\left(\frac{2+x}{p}\right)=1$ and $\left(\frac{2-x}{p}\right)=-1$, then x is a cycle element in a stumpy component.

(c) If $\left(\frac{2+x}{p}\right) = -1$ and $\left(\frac{2-x}{p}\right) = 1$, then x is a leaf in a stumpy component.

(d) If $\left(\frac{2+x}{p}\right) = -1$ and $\left(\frac{2-x}{p}\right) = -1$, then x is a leaf in a branched component.

Proof: Lemma 14 showed that the vertices in the branched components have equal Legendre symbols. We saw that, for $x \neq -2$, $\left(\frac{2+x}{p}\right) =$ if and only if x has parents; hence, $\left(\frac{2+x}{p}\right) = -1$ if and only if x is a leaf. Checking the Legendre symbol values for the leaf and nonleaf positions gives the results. □

To count the actual number of vertices in each of the geometric classes, we use the following results on sums of the Jacobi symbol on quadratic forms.

Lemma 16:

(a) Let $p > 2$ and $a^2 - 4b \neq 0 \bmod p$, then $\sum_{x=1}^{p}\left(\frac{x^2+ax+b}{p}\right) = -1$.

(b) Let $p > 2$, then $\sum_{x=1}^{p}\left(\frac{4-x^2}{p}\right) = -1\left(\frac{-1}{p}\right)$.

Proof: Part (a) is Theorem 8.2 in [6] and (b) is $\left(\frac{-1}{p}\right)$ times a special case. □

Theorem 17: In the function digraph $\mathrm{fd}_p(\mathrm{s}(x))$:

(a) the total number of nonleaf vertices in the branched components is $1 + \frac{1}{4}\left(p - \left(\frac{-1}{p}\right)\right)$;

(b) the total number of cycle vertices in the stumpy components is $\frac{1}{4}\left(p - 2 + \left(\frac{-1}{p}\right)\right)$;

(c) the total number of leaf vertices in the stumpy components is $\frac{1}{4}\left(p - 2 + \left(\frac{-1}{p}\right)\right)$;

(d) the total number of leaf vertices in the branched components is $\frac{1}{4}\left(p - \left(\frac{-1}{p}\right)\right)$.

Proof: First consider (d). The total number of leaf vertices in the branched components is

$$\frac{1}{4}\sum_{x=1}^{p}\left(1 - \left(\frac{2+x}{p}\right)\right)\left(1 - \left(\frac{2-x}{p}\right)\right),$$

where we take care to notice that the terms are zero for $x = \pm 2$ and those vertices are not leaves. Expanding, we see

$$\frac{1}{4}\sum_{x=1}^{p}\left(1 - \left(\frac{2+x}{p}\right) - \left(\frac{2-x}{p}\right) + \left(\frac{4-x^2}{p}\right)\right) = \frac{1}{4}\left(p - \left(\frac{-1}{p}\right)\right),$$

where we use Theorem 16(b) and the fact that $\sum_{x=1}^{p}\left(\frac{x}{p}\right) = 0$. Next consider (a). The total number of nonleaf vertices in the branched components is

$$1 + \frac{1}{4}\sum_{x=1}^{p}\left(1 + \left(\frac{2+x}{p}\right)\right)\left(1 + \left(\frac{2-x}{p}\right)\right),$$

where we take care to notice that the terms of the sum are 2 for $x = \pm 2$ and those vertices are not leaves, hence we need to add 1 to get the correct count. Expanding as above gives the desired result. We can handle (b) and (c) in a similar way or note that we already know from Lemma 13(a) that these numbers must be equal and thus are half of the vertices not accounted for in (a) and (d). □

For example, consider $p = 239$. Since $\left(\frac{-1}{239}\right) = -1$, we see that by Theorem 17(d) the number of leaves in the branched components is $\frac{1}{4}[239 - (-1)] = 60$

The next lemma gives a technical identity that provides the key inductive step in the theorem that follows the lemma. Informally speaking, it shows that we can follow a chain of sums of paths multiplied by inverses of elements in the tree leading to 0 to get a path in the "next" tree of the appropriate size.

Lemma 18: Suppose r, $\mathrm{s}(r)$, and $\mathrm{s}^2(r)$ are nonzero elements in $\mathrm{fd}_p(\mathrm{s}(x))$. If $\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N))$ is a parent of $\frac{1}{\mathrm{s}^2(r)}(\mathrm{s}^2(M)+\mathrm{s}^2(N))$, then either $\frac{1}{r}(M+N)$ or $\frac{1}{r}(M-N)$ is a parent of $\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N))$.

Proof: Notice that a vertex x is a parent of y if and only if $\mathrm{s}(x)-y$ is zero. Direct computation verifies that

$$r^4\left(\mathrm{s}\left(\frac{1}{r}(M+N)\right)-\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N))\right)\left(\mathrm{s}\left(\frac{1}{r}(M-N)\right)-\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N))\right)$$

is

$$\frac{4(M^2+2MN+N^2-4r^2-MNr^2+r^4)(M^2-2MN+N^2-4r^2+MNr^2+r^4)}{\mathrm{s}(r)^2},$$

which is identical to

$$-2\mathrm{s}^2(r)\left(\mathrm{s}\left(\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N))\right)-\frac{1}{\mathrm{s}^2(r)}(\mathrm{s}^2(M)+\mathrm{s}^2(N))\right).$$

Now the last expression is zero by the hypothesis; hence, one of the factors of the first expression is zero. This gives the claim. □

Lemma 12 gave a multiplicative relationship between vertices that were 2-ancestors. The following result involves both addition of k-ancestors and multiplication by inverses of tree elements in the 0-component. This connects the existence of tree elements in the 0-component to the existence of vertices with higher ancestry.

Theorem 19: If M and N are k-ancestors in $\mathrm{fd}_p(\mathrm{s}(x))$ for some $k\geq 2$ and if r is a predecessor of 0 such that $\mathrm{s}^{k-1}(r)=0$, then M and something of the form $\frac{1}{r}(M+N')$ are $k+1$-ancestors, where N' is a vertex such that N' and M are k-ancestors. Moreover, if M is at level $k+2$ or higher, then there are 2^k vertices that are $k+1$-ancestors with M.

Proof: Proceed by induction on k. When $k=2$, we claim that $\frac{1}{r}(M+N)$ is a 3-ancestor of M. We are assuming $\mathrm{s}(r)=0$ and hence $r^2=2$; from Lemma 12, we saw that MN is a 2-ancestor of $\mathrm{s}(M)$ and hence we need only show $\mathrm{s}(\frac{1}{r}(M+N))=MN$. Notice that

$$\mathrm{s}\left(\frac{1}{r}(M+N)\right)=\frac{1}{r^2}(M^2+2MN+N^2)-2=MN+\frac{1}{2}(M^2+N^2-4)=MN,$$

where the last equality holds since M and N are 2-ancestors implies that $M^2+N^2=4$. Also, if M is at level 4 or higher, we know that $\frac{1}{r}(M+N)$ is also at level 4 or higher since $\mathrm{s}^3(M)$ is not a cycle element. Therefore, $\frac{1}{r}(M+N)$ is its own "0-ancestor," it has one 1-ancestor (its additive inverse) and two 2-ancestors from the reasoning in the remark after Lemma 14. All of those are 3-ancestors of M; hence, we have four 3-ancestors of M.

When $k=3$, we know that M and N are 3-ancestors means $\mathrm{s}(M)$ and $\mathrm{s}(N)$ are 2-ancestors. By the $k=2$ induction step, we can assume that $\mathrm{s}(M)$ has a 3-ancestor of the form $\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N))$. Now we need to show that

$$\mathrm{s}\left(\frac{1}{r}(M+N)\right)=\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N)) \quad \text{or} \quad \mathrm{s}\left(\frac{1}{r}(M-N)\right)=\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N)).$$

First, note that M and N are 3-ancestors if and only if $\mathrm{s}^2(M)=-\mathrm{s}^2(N)$, which is true if and only if $M^4-4M^2+4-4N^2+N^4=0$. Now direct computation using the fact that $r^4=4r^2-2$ gives

$$\left(\mathrm{s}\left(\frac{1}{r}(M+N)\right)-\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N))\right)\left(\mathrm{s}\left(\frac{1}{r}(M-N)\right)-\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N))\right)$$
$$=\frac{4}{r^4\mathrm{s}(r)^2}(M^4-4M^2+4-4N^2+N^4)$$

which is zero since M and N are 3-ancestors. Hence, one of the two conditions required must hold. Thus, we have found a 4-ancestor $A=\frac{1}{r}(M\pm N)$ of M and we can rename N if desired to avoid the minus sign. Now suppose M is at level at least $k+2$. We see that A must be at the same level since $\mathrm{s}^{k+1}(M)$ is not a cycle element. We know that A is its own "0-ancestor," it has one 1-ancestor, two 2-ancestors, and four 3-ancestors by induction. All of those are 4-ancestors of M, and hence M has eight 4-ancestors.

Now suppose we have shown the theorem up to $k-1$ and want to show it for k. By renaming N if need be (to avoid minus signs), we can assume that $\mathrm{s}(M)$ and $\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N))$ are k-ancestors and $\mathrm{s}^2(M)$ and $\frac{1}{\mathrm{s}^2(r)}(\mathrm{s}^2(M)+\mathrm{s}^2(N))$ are $k-1$-ancestors with

$$\mathrm{s}\left(\frac{1}{\mathrm{s}(r)}(\mathrm{s}(M)+\mathrm{s}(N))\right)=\frac{1}{\mathrm{s}^2(r)}(\mathrm{s}^2(M)+\mathrm{s}^2(N)).$$

We can now apply Lemma 18 to get a $k+1$-ancestor of M of the desired form. When M is at level at least $k+2$, using the induction steps to complete the tree surrounding this new vertex gives the desired 2^k vertices which are $k+1$-ancestors with M. □

We will refer to Figure 6 to provide some illustrations of this theorem in $\mathrm{fd}_{239}(\mathrm{s}(x))$. Notice that $\mathrm{s}(99)=0$ and the multiplicative inverse of $r=99$ is 169. Now $M=112$ and $N=102$ are 2-ancestors appearing at level 4 in the branched component containing a 4-cycle. Note that $\frac{1}{r}(M+N)=169(112+102)=77$, which is a 3-ancestor with M. Also, 65, which appears at level 2 and 230, which is a cycle element, are 2-ancestors. Note that $\frac{1}{r}(M+N)=169(230+65)=143$ is at level 3 in the next tree; hence, 65 and 143 are 3-ancestors. Also, $\mathrm{s}^2(36)=0$ and the multiplicative inverse of $r=36$ is 166. Therefore, we are now able to lift to level 4 via $\frac{1}{r}(M+N)=166(230+143)=17$; hence, 17 and 143 are 4-ancestors.

Observe that we are able to use Theorem 19 to find elements at the same level in a tree associated with a cycle element and we are also able to use the theorem to find elements at a higher level in the next tree associated with a cycle element having more distant ancestry. Thus, it can be used both to complete trees and to lift levels. We put these ideas together in our main theorem about the tree structure in $\mathrm{fd}_p(\mathrm{s}(x))$.

Theorem 20: The tree leading to any vertex at level 2 in $\mathrm{fd}_p(\mathrm{s}(x))$ is a complete binary tree and is isomorphic to the tree leading to the vertex 0.

Proof: Suppose the leaves in the 0-component reach level d. Then, if r is such a leaf in that component, $\mathrm{s}^{d-2}(r)=0$. Now, if $d\geq 3$, we can use Theorem 19 on vertices at height 2 with a cycle element that gives a 2-ancestor to produce a 3-ancestor at level 3. If $d\geq 4$, we can use this vertex and a cycle vertex to get a vertex at level 4. We can repeat this $d-2$ times resulting in a vertex at height d. We can then use Theorem 19 and the vertices at height d to see that all the trees in the branched components are complete binary trees from level 2 up to height d.

Lastly, we need to show that, if any component has reached level $d+1$, then we can reverse the identity used to raise to level $d+1$ (in Lemma 18) to solve for an r that leads to 0 in one more step, contradicting our choice of d. In particular, we can assume that r is at level d in the tree leading to 0 and that there is a vertex R at level $d+1$ in some other branched component. By the induction to level d, we know the trees to level d are complete; in fact, trees rooted to depth d or less from any vertex are complete. Now $\mathrm{s}(R)$ is at level d and the trees are complete to that level. Thus, $\mathrm{s}(R)$ must be obtainable from the process of lifting described in Theorem 19. In particular, we can find a cycle vertex M and a vertex N at level d so that $\mathrm{s}(M)$ and $\mathrm{s}(N)$ are d-ancestors lifting to $\mathrm{s}(R)$. That is,

$$\frac{1}{r}(\mathrm{s}(M)+\mathrm{s}(N))=\mathrm{s}(R) \tag{$*$}$$

and

$$\frac{1}{\mathrm{s}(r)}(\mathrm{s}^2(M)+\mathrm{s}^2(N))=\mathrm{s}^2(R),$$

from which it follows that

$$\mathrm{s}\left(\frac{1}{r}(\mathrm{s}(M)+\mathrm{s}(N))\right)=\frac{1}{\mathrm{s}(r)}(\mathrm{s}^2(M)+\mathrm{s}^2(N)).$$

We need only show that $\mathrm{s}(\frac{1}{R}(M+N))=r$ or $\mathrm{s}(\frac{1}{R}(M-N))=r$ to show that the tree leading to 0 rises to level $d+1$. Now an identity similar to that appearing in Lemma 18 is

$$\begin{aligned}&\mathrm{s}\left(\frac{1}{r}(\mathrm{s}(M)+\mathrm{s}(N))\right)-\frac{1}{\mathrm{s}(r)}(\mathrm{s}^2(M)+\mathrm{s}^2(N))\\&\qquad=\frac{2}{r^2\mathrm{s}(r)}(4-M^2-N^2-rMN-r^2)(-4+M^2+N^2-rMN+r^2).\end{aligned}$$

We noted above that the left-hand side must be zero. If we assume the first factor of the right-hand side is zero and simplify using $(*)$, we get $\mathrm{s}(\frac{1}{R}(M+N))=r$ and the other possibility arises from the other factor. In this way, we see that all the trees in branched components have the same height; this completes the proof. □

Notice that knowing the trees are uniform complete binary trees, along with the knowledge of the number of leaves in the branched components, now allows us to compute the number of branched cycle elements, c, and the depth, d, of the trees. For example, when $p=239$, we checked that there are 60 leaves in the branched components. Since there are $2c-1$ trees associated with level 2 vertices each of which will have 2^{d-2} leaves, we see that

$$(2c-1)2^{d-2}=60=15(2^2);$$

by equating the odd factors and powers of two, we see that $2c-1=15$ and $2^{d-2}=2^2$, so $c=8$ and $d=4$, which is correct.

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REFERENCES

1. E. L. Blanton, Jr., S. P. Hurd, & J. S. McCranie. "On a Digraph Defined by Squaring Modulo *n*." *The Fibonacci Quarterly* **30.4** (1992):322-34.
2. E. L. Blanton, Jr., S. P. Hurd, & J. S. McCranie. "On the Digraph Defined by Squaring Mod *m*, When *m* Has Primitive Roots." *Congressus Numerantium* **82** (1991):167-77.
3. R. L. Devaney. *An Introduction to Chaotic Dynamical Systems*. Reading, Mass.: Addison-Wesley, 1987.
4. R. L. Devaney. *A First Course in Chaotic Dynamical Systems*. Reading, Mass.: Addison-Wesley, 1992.
5. A. Flores. "Geometry of Numeric Iterations." *PRIMUS* **4.1** (1994):29-38.
6. L. K. Hua. *Introduction to Number Theory*. Berlin: Springer-Verlag, 1982.
7. S. Kravitz. "The Lucas-Lehmer Test for Mersenne Numbers." *The Fibonacci Quarterly* **8.1** (1970):1-3.
8. C. Lucheta, E. Miller, & C. Reiter. "Digraphs from Powers Modulo *p*." *The Fibonacci Quarterly* **34.3** (1996):226-39.
9. H. Riesel. *Prime Numbers and Computer Methods for Factorization*. Boston: Birkhäuser, 1985.
10. T. D. Rogers. "The Graph of the Square Mapping on the Prime Fields." *Discrete Math.* **148** (1996):317-24.
11. J. Strayer. *Elementary Number Theory*. Boston: PWS Publishing Co., 1994.
12. B. Wilson. "Power Digraphs Modulo *p*." *The Fibonacci Quarterly* **36.3** (1998):229-39.

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CONGRUENCES RELATING RATIONAL VALUES OF BERNOULLI AND EULER POLYNOMIALS

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1. INTRODUCTION

For $n \in \mathbf{N}$, where $\mathbf{N} = \{0, 1, 2, \ldots\}$, the Bernoulli polynomials, $B_n(t)$, are defined by means of the generating function

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}, \quad |x| < 2\pi. \tag{1}$$

Some of the more important properties of these polynomials include

$$B_n(t+1) - B_n(t) = nt^{n-1}, \tag{2}$$

$$B_n(1-t) = (-1)^n B_n(t), \tag{3}$$

each of which follows from (1). From (2) we can derive

$$B_n(m) - B_n(0) = n\sum_{j=0}^{m-1} j^{n-1},$$

holding for all positive integers m. We define the Bernoulli numbers, B_n, by $B_n = B_n(0)$, from which (1) allows us to write

$$B_n(t) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m} t^m.$$

Note that we obtain the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, ..., and $B_n = 0$ for odd $n \geq 3$. For even $n \geq 2$, we have

$$B_n = -\frac{1}{n+1} \sum_{m=0}^{n-1} \binom{n+1}{m} B_m.$$

Perhaps the most fundamental property of Bernoulli numbers is the von Staudt-Clausen theorem which states that, for even positive n, the quantity

$$B_n + \sum_{\substack{p \text{ prime} \\ (p-1)|n}} \frac{1}{p}$$

is an integer. This implies that, for such n, the denominator of B_n is square-free.

The Euler polynomials, $E_n(t)$, $n \in \mathbf{N}$, are defined by means of the generating function

$$\frac{2e^{tx}}{e^x + 1} = \sum_{n=0}^{\infty} E_n(t) \frac{x^n}{n!}, \quad |x| < \pi. \tag{4}$$

Each can be expanded in terms of Bernoulli numbers according to

$$E_n(t) = \sum_{m=0}^{n} \binom{n}{m} 2(1-2^{m+1}) \frac{B_{m+1}}{m+1} t^{n-m}.$$

Euler polynomials satisfy the identities

$$E_n(t+1) + E_n(t) = 2t^n, \tag{5}$$

$$E_n(1-t) = (-1)^n E_n(t), \tag{6}$$

each following from (4). From (5) we see that, for positive integers m,

$$E_n(m) - (-1)^m E_n(0) = 2\sum_{j=0}^{m-1} (-1)^{m-1-j} j^n.$$

The Euler numbers, E_n, are defined by $E_n = 2^n E_n(1/2)$. Each $E_n \in \mathbb{Z}$ (see [10], p. 53), and as a result of (6) we must have $E_n = 0$ whenever n is odd.

There are three particular identities, known as multiplication identities, associated with the Bernoulli and Euler polynomials. They enable one to rewrite a particular value of one of these polynomials in terms of a sum of a variety of values of either the same or another such polynomial. We present them as follows with the assumption that, for each, q is a positive integer. For the Bernoulli polynomials, we have Raabe's identity [12],

$$B_n(qt) = q^{n-1} \sum_{j=0}^{q-1} B_n\left(t + \frac{j}{q}\right), \tag{7}$$

which follows from (1). For q odd, the Euler polynomials satisfy

$$E_n(qt) = q^n \sum_{j=0}^{q-1} (-1)^j E_n\left(t + \frac{j}{q}\right), \tag{8}$$

which follows from (4). Finally, for q even,

$$E_{n-1}(qt) = -\frac{2q^{n-1}}{n} \sum_{j=0}^{q-1} (-1)^j B_n\left(t + \frac{j}{q}\right), \tag{9}$$

which follows from (1) and (4).

The problem of studying Bernoulli and Euler polynomials at values in $\mathbb{R}$ is tantamount to that of considering the polynomials in certain intervals of $\mathbb{R}$. From (2) and (5) we see that we can reduce this problem to that of considering the polynomials in $[0, 1)$. Utilizing (3) and (6) allows us further to reduce this to the interval $[0, 1/2]$. Because of this it becomes a point of interest to consider the polynomials at various "special" values of t in $[0, 1/2]$, especially at rational t. Applications of (7), (8), and (9) enable us to find relations between values of these polynomials at several rationals within this interval.

Let us now consider the known values of these polynomials. As we have seen, $B_n(0) = B_n$ and $E_n(1/2) = 2^{-n} E_n$ for each $n \geq 0$. The following can be derived from (7)-(9) for all $n \geq 0$:

$$E_n(0) = -\frac{1}{n+1}(2^{n+2} - 2)B_{n+1},$$

$$B_n\left(\frac{1}{2}\right) = -(1 - 2^{1-n})B_n.$$

In addition, for even $n \geq 0$, the following can also be derived from (7)-(9):

$$B_n\left(\frac{1}{3}\right) = -\frac{1}{2}(1-3^{1-n})B_n,$$

$$B_n\left(\frac{1}{4}\right) = -2^{-n}(1-2^{1-n})B_n,$$

$$B_n\left(\frac{1}{6}\right) = \frac{1}{2}(1-2^{1-n})(1-3^{1-n})B_n,$$

$$E_n\left(\frac{1}{6}\right) = 2^{-n-1}(1+3^{-n})E_n.$$

Also, for odd $n \geq 1$, we have

$$E_n\left(\frac{1}{3}\right) = -\frac{1}{n+1}(2^{n+1}-1)(1-3^{-n})B_{n+1},$$

$$B_n\left(\frac{1}{4}\right) = -n4^{-n}E_{n-1}.$$

Each of these can be found in [11]. Similar expressions have been found for each of $B_n(1/3)$ and $B_n(1/6)$ when n is odd, but these are in terms of a sequence of rational values I_n, whose denominators consist of certain powers of 3 (see [5], [6]).

Bernoulli and Euler numbers and polynomials have numerous applications in mathematics. Because of this, they have been studied quite extensively. Besides the study of these polynomials at specific rational points, efforts have also been made to find congruence relations that describe specific Bernoulli polynomials at arbitrary rational points. A. Granville and Z.-W. Sun [7] have shown that if an integer $q \geq 3$ is odd and $1 \leq a \leq q$, with $(a, q) = 1$, then for p prime,

$$B_{p-1}\left(\frac{a}{q}\right) - B_{p-1} \equiv 2^{-1}p^{-1}q(U_p - 1) \pmod{p},$$

where U_p is a linear recurrence of order $[q/2]$ depending only on a, q, and the least positive residue of p modulo q. Their work extended a list of congruences given by E. Lehmer [9].

In this note we illustrate a means of finding congruence relations among Bernoulli and Euler polynomials evaluated at various rational numbers. We do this by considering the polynomials at values that have not been discussed previously. By applying (7)-(9), we build linear relationships among certain rational evaluations. Some recent results concerning the values of Bernoulli and Euler polynomials at rational points at rational points then enable us to obtain congruences based on the coefficients of these relations. Before proceeding with the derivation of the congruences, we shall present these results.

2. SOME RECENT RESULTS

The following result concerning Bernoulli polynomials was recently presented by G. Almkvist and A. Meurman in [1]. Other versions of the proof of this are given in [2], [3], and [13].

Theorem 2.1: Let $r, s \in \mathbb{Z}$, $s \neq 0$. Then $s^n(B_n(r/s) - B_n(0)) \in \mathbb{Z}$.

Since Euler polynomials satisfy many properties that are similar to those that Bernoulli polynomials satisfy, we would expect a result similar to Theorem 2.1 for Euler polynomials. In fact, we have such a result, presented in [4].

Theorem 2.2: Let $r, s \in \mathbb{Z}$, $s \neq 0$. Then $s^n(E_n(r/s) - (-1)^{rs}E_n(0)) \in \mathbb{Z}$.

Note that Theorems 2.1 and 2.2 will be the key components that enable us to derive the congruences that we intend to illustrate. They imply that, whenever k is a positive integer, for all $r, s \in \mathbb{Z}$, $s \neq 0$, the Bernoulli polynomials satisfy

$$ks^nB_n\left(\frac{r}{s}\right) \equiv ks^nB_n \pmod{k},$$

and for the Euler polynomials,

$$ks^nE_n\left(\frac{r}{s}\right) \equiv (-1)^{rs}ks^nE_n(0) \pmod{k}.$$

Note that this last congruence can be written in terms of B_n since we can also express $E_n(0)$ in such manner.

3. SOME EXAMPLES

The multiplication identities (7)-(9) provide a linear relationship among a set of values of particular Bernoulli and Euler polynomials at various rational numbers, these numbers also satisfying their own linear relationship. Varying the parameters t and q in (7)-(9) may provide several distinct linear relationships among these values. By a partial reduction of such a system, the coefficients of these values are modified so that, by applying Theorems 2.1 and 2.2, a congruence relationship can be obtained modulo one of these coefficients.

3.1 A Congruence Relating $B_n(2r/s)$ and $E_{n-1}(2r/s)$

This example gives a congruence relation, modulo a power of 2, between Bernoulli and Euler polynomials evaluated at the same rational number.

Theorem 3.1: Let $r, s \in \mathbb{Z}$ such that $(2r, s) = 1$. Then for positive integers n,

$$2B_n\left(\frac{2r}{s}\right) - nE_{n-1}\left(\frac{2r}{s}\right) \equiv 2^{n+1}B_n \pmod{2^{n+1}}.$$

Proof: Letting $q = 2$ and $t = r/s$ in (7) and (9) yields

$$B_n\left(\frac{2r}{s}\right) = 2^{n-1}B_n\left(\frac{r}{s}\right) + 2^{n-1}B_n\left(\frac{s+2r}{2s}\right),$$

$$-nE_{n-1}\left(\frac{2r}{s}\right) = 2^nB_n\left(\frac{r}{s}\right) - 2^nB_n\left(\frac{s+2r}{2s}\right).$$

Combining these two relations so as to eliminate $B_n((s+2r)/(2s))$, we obtain

$$2B_n\left(\frac{2r}{s}\right) - nE_{n-1}\left(\frac{2r}{s}\right) = 2^{n+1}B_n\left(\frac{r}{s}\right),$$

and thus, by Theorem 2.1,

$$2s^n B_n\left(\frac{2r}{s}\right) - ns^n E_{n-1}\left(\frac{2r}{s}\right) \equiv 2^{n+1} s^n B_n \pmod{2^{n+1}}.$$

This then yields the theorem, since $(2, s) = 1$. □

This result implies that, for odd $n \geq 3$, we have

$$2B_n\left(\frac{2r}{s}\right) \equiv nE_{n-1}\left(\frac{2r}{s}\right) \pmod{2^{n+1}},$$

since $B_n = 0$, and for all $n \geq 1$,

$$2B_n\left(\frac{2r}{s}\right) \equiv nE_{n-1}\left(\frac{2r}{s}\right) \pmod{2^n},$$

by the von Staudt-Clausen theorem.

3.2 Congruences for $B_n(t)$ and $E_{n-1}(t)$ at Multiples of $1/10$ for n Even

This additional example concerns the values of $B_n(t)$, at $1/10$ and $3/10$, and $E_{n-1}(t)$, at $1/5$ and $2/5$, for even $n \geq 2$.

Lemma 3.2: Let be an even positive integer. Then

$$10^n B_n\left(\frac{1}{10}\right) + 10^n B_n\left(\frac{3}{10}\right) = (2^{n-1} - 1)(5^n - 5)B_n, \tag{10}$$

$$n5^n E_{n-1}\left(\frac{1}{5}\right) - n5^n E_{n-1}\left(\frac{2}{5}\right) = -(2^n - 1)(5^n - 5)B_n, \tag{11}$$

$$(2^n + 1)10^n B_n\left(\frac{1}{10}\right) + (2^{n-1} + 1)n5^n E_{n-1}\left(\frac{1}{5}\right) = -2^{n-1}(5^n - 5)B_n, \tag{12}$$

$$2(2^n + 1)5^n B_n\left(\frac{1}{5}\right) + n5^n E_{n-1}\left(\frac{1}{5}\right) = -2^n(5^n - 5)B_n. \tag{13}$$

Proof: In view of (3), by letting $q = 2$ and $t = 1/5, 1/10$ in (7) and (9), we obtain

$$2^n \cdot 5^n B_n\left(\frac{1}{5}\right) - 2 \cdot 5^n B_n\left(\frac{2}{5}\right) + 10^n B_n\left(\frac{3}{10}\right) = 0, \tag{14}$$

$$2^n \cdot 5^n B_n\left(\frac{1}{5}\right) - 10^n B_n\left(\frac{3}{10}\right) + n5^n E_{n-1}\left(\frac{2}{5}\right) = 0, \tag{15}$$

$$-2 \cdot 5^n B_n\left(\frac{1}{5}\right) + 2^n \cdot 5^n B_n\left(\frac{2}{5}\right) + 10^n B_n\left(\frac{1}{10}\right) = 0, \tag{16}$$

$$-2^n \cdot 5^n B_n\left(\frac{2}{5}\right) + 10^n B_n\left(\frac{1}{10}\right) + n5^n E_{n-1}\left(\frac{1}{5}\right) = 0. \tag{17}$$

The case of $q = 5$ and $t = 0$ in (7), yields

$$5^n B_n\left(\frac{1}{5}\right) + 5^n B_n\left(\frac{2}{5}\right) = -\frac{1}{2}(5^n - 5)B_n. \tag{18}$$

Note that, by adding corresponding left-hand and right-hand sides of each equation, the combination $(14)+(16)-(2^n-2)(18)$ yields (10). Also, $(17)+2(2^n-1)(18)-(14)-(15)-(16)$ yields (11). From $2^{n-1}(16)+(2^{n-1}+1)(17)+2^n(18)$, we obtain (12), and $(17)+2^{n+1}(18)-(16)$ yields (13). □

Now, from (10)-(13), we can derive congruences related to each of the values $10^nB_n(1/10)$, $10^nB_n(3/10)$, $n5^nE_{n-1}(1/5)$, and $n5^nE_{n-1}(2/5)$. We shall first focus on those for the Euler polynomials.

Theorem 3.3: For n an even positive integer,

$$n5^nE_{n-1}\left(\frac{1}{5}\right)\equiv-(2^n(5^n-5)+5^n(2^{n+1}+2))B_n \pmod{2^{n+1}+2}, \tag{19}$$

$$n5^nE_{n-1}\left(\frac{2}{5}\right)\equiv-(5^n-5+5^n(2^{n+1}+2))B_n \pmod{2^{n+1}+2}. \tag{20}$$

Proof: If we use Theorem 2.1 to reduce (13) modulo $2^{n+1}+2$, we obtain (19). Now reduce (11) modulo $2^{n+1}+2$, utilizing (19) to represent $n5^nE_{n-1}(1/5)$, and we obtain (20). □

Corollary 3.4: Let n be an even positive integer, and let p be prime such that $p\,|\,(2^n+1)$. Then

$$n5^nE_{n-1}\left(\frac{1}{5}\right)\equiv-n5^nE_{n-1}\left(\frac{2}{5}\right)\equiv(5^n-5)B_n \pmod{p}.$$

Proof: If $p\,|\,(2^n+1)$, then $(p-1)\nmid n$ since, otherwise, $2^n+1\equiv 2 \pmod{p}$. Thus, by the von Staudt-Clausen theorem, p is not in the denominator of B_n, and so $5^n(2^{n+1}+2)B_n\equiv 0 \pmod{p}$. Therefore, (19) and (20) reduce to yield the result. □

Corollary 3.5: Let p be prime such that $p\equiv 5 \pmod 8$. Then

$$5^{(p-1)/2}E_{(p-3)/2}\left(\frac{1}{5}\right)\equiv-5^{(p-1)/2}E_{(p-3)/2}\left(\frac{2}{5}\right)\equiv-2(5^{(p-1)/2}-5)B_{(p-1)/2} \pmod{p}.$$

Proof: Note that $p\equiv 5 \pmod 8$ implies that $(p-1)/2$ is even and that $\left(\frac{2}{p}\right)=-1$, where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol corresponding to p. Euler's criterion states that $\left(\frac{2}{p}\right)=-1$ if and only if $2^{(p-1)/2}+1\equiv 0 \pmod{p}$. Therefore, by taking $n=(p-1)/2$, the result follows. □

Corollary 3.6: Let p be prime such that $p\equiv 13 \pmod{24}$. If there exist integers C and D for which $p=C^2+27D^2$, then

$$5^{(p-1)/6}E_{(p-7)/6}\left(\frac{1}{5}\right)\equiv-5^{(p-1)/6}E_{(p-7)/6}\left(\frac{2}{5}\right)\equiv-6(5^{(p-1)/6}-5)B_{(p-1)/6} \pmod{p}.$$

Proof: In [8], page 119, we see that there are integers C and D such that $p=C^2+27D^2$ if and only if 2 is a cubic residue modulo p. Now, 2 is a cubic residue modulo p if and only if $2^{(p-1)/3}\equiv 1 \pmod{p}$, and since $(p-1)/6$ must be an (even) integer, we can write

$$2^{(p-1)/3}-1=(2^{(p-1)/6}-1)(2^{(p-1)/6}+1),$$

where either $2^{(p-1)/6}\equiv-1 \pmod{p}$ or $2^{(p-1)/6}\equiv 1 \pmod{p}$.

If $2^{(p-1)/6}\equiv 1 \pmod{p}$, then $2^{(p-1)/2}\equiv 1 \pmod{p}$; thus, by Euler's criterion, $\left(\frac{2}{p}\right)=1$. However, $p\equiv 13 \pmod{24}$ implies that $\left(\frac{2}{p}\right)=-1$. Therefore, $2^{(p-1)/6}\equiv-1 \pmod{p}$, yielding the result. □

Now we consider congruences for $10^n B_n(1/10)$ and $10^n B_n(3/10)$.

Theorem 3.7: Let n be an even positive integer. Then

$$(2^n+1)10^n B_n\left(\frac{1}{10}\right) \equiv -(5^n(2^{2n}+2^n-2)+2^{n-1}(5^n-5))B_n \pmod{5n(2^{n-1}+1)}, \tag{21}$$

$$(2^n+1)10^n B_n\left(\frac{3}{10}\right) \equiv (5^n(2^{2n}+2^n-2)+(2^{2n-1}-1)(5^n-5))B_n \pmod{5n(2^{n-1}+1)}. \tag{22}$$

Proof: By Theorem 2.2, we can reduce (12) modulo $5n(2^{n-1}+1)$ to obtain

$$(2^n+1)10^n B_n\left(\frac{1}{10}\right) - (2^{n-1}+1)n5^n E_{n-1}(0) \equiv -2^{n-1}(5^n-5)B_n \pmod{5n(2^{n-1}+1)}.$$

Since $-nE_{n-1}(0) = (2^{n+1}-2)B_n$, this yields (21). Now multiply (10) through by 2^n+1 and reduce modulo $5n(2^{n-1}+1)$, utilizing (21) to represent $(2^n+1)10^n B_n(1/10)$, and we obtain (22). □

Corollary 3.8: Let p be prime, $p>3$, and let n be an even positive integer such that $p \mid (2^{n-1}+1)$. Then

$$10^n B_n\left(\frac{1}{10}\right) \equiv 10^n B_n\left(\frac{3}{10}\right) \equiv -(5^n-5)B_n \pmod{p}.$$

Proof: If $p \mid (2^n+2)$, then $(p-1) \nmid n$ since, otherwise, $2^n+2 \equiv 3 \pmod{p}$. Thus, p is not in the denominator of B_n. This implies that we can reduce (21) to the form

$$(2^n+1)10^n B_n\left(\frac{1}{10}\right) \equiv (5^n-5)B_n \pmod{p}.$$

Also, from Theorem 2.1,

$$\begin{aligned}(2^n+1)10^n B_n\left(\frac{1}{10}\right) &\equiv (2^n+2)10^n B_n - 10^n B_n\left(\frac{1}{10}\right) \pmod{2^n+2}\\ &\equiv -10^n B_n\left(\frac{1}{10}\right) \pmod{p}.\end{aligned}$$

Thus, we have the congruence for $10^n B_n(1/10)$. By incorporating this into the reduction of (10) modulo p, we can obtain the congruence for $10^n B_n(3/10)$. □

Corollary 3.9: Let p be prime, $p>3$, such that $p \equiv 3 \pmod 8$. Then

$$10^{(p+1)/2} B_{(p+1)/2}\left(\frac{1}{10}\right) \equiv 10^{(p+1)/2} B_{(p+1)/2}\left(\frac{3}{10}\right) \equiv -(5^{(p+1)/2}-5)B_{(p+1)/2} \pmod{p}.$$

Proof: If $p \equiv 3 \pmod 8$, then $(p+1)/2$ is even and $\left(\frac{2}{p}\right) = -1$. By Euler's criterion, we then have $2^{(p-1)/2}+1 \equiv 0 \pmod{p}$. The result follows by taking $n=(p+1)/2$. □

Corollary 3.10: Let p be prime such that $p \equiv 11$ or $19 \pmod{40}$. Then p divides the numerators of $B_{(p+1)/2}(1/10)$ and $B_{(p+1)/2}(3/10)$.

Proof: If $p \equiv 11$ or $19 \pmod{40}$, then $p \equiv 3 \pmod 8$ and $\left(\frac{5}{p}\right) = 1$. By Euler's criterion, $\left(\frac{5}{p}\right) = 1$ implies that $5^{(p+1)/2}-5 \equiv 0 \pmod{p}$. Since these conditions also imply that $p \nmid 10$, the result follows. □

Corollary 3.11: Let p be prime such that $p \equiv 19 \pmod{24}$. If there exist integers C and D for which $p = C^2 + 27D^2$, then

$$10^{(p+5)/6} B_{(p+5)/6}\left(\frac{1}{10}\right) \equiv 10^{(p+5)/6} B_{(p+5)/6}\left(\frac{3}{10}\right) \equiv -(5^{(p+5)/6} - 5) B_{(p+5)/6} \pmod{p}.$$

Proof: Recall that there are integers C and D such that $p = C^2 + 27D^2$ if and only if $2^{(p-1)/3} \equiv 1 \pmod{p}$. Since $(p-1)/6$ is an integer, this implies that either $2^{(p-1)/6} \equiv -1 \pmod{p}$ or $2^{(p-1)/6} \equiv 1 \pmod{p}$.

If $2^{(p-1)/6} \equiv 1 \pmod{p}$, then $2^{(p-1)/2} \equiv 1 \pmod{p}$; thus, $\left(\frac{2}{p}\right) = 1$. However, $p \equiv 19 \pmod{24}$ implies that $\left(\frac{2}{p}\right) = -1$. Therefore, $2^{(p-1)/6} \equiv -1 \pmod{p}$, and so $p \mid (2^{(p+5)/6} + 2)$. □

4. CONCLUSION

We have illustrated how some simple properties of Bernoulli and Euler polynomials can be utilized to construct congruences for certain rational evaluations of these polynomials. Congruences involving more terms can be easily obtained, but the difficulty to interpret their meaning increases with the number of terms involved. The examples given here are simple, but they are quite effective at illustrating how this method provides an opportunity to obtain previously unknown divisibility properties of rational values of Bernoulli and Euler polynomials.

REFERENCES

1. G. Almkvist & A. Meurman. "Values of Bernoulli Polynomials and Hurwitz's Zeta Function at Rational Points." *C. R. Math. Rep. Acad. Sci. Canada* **13** (1991):104-08.
2. K. Bartz & J. Rutkowski. "On the von Staudt-Clausen Theorem." *C. R. Math. Rep. Acad. Sci. Canada* **15** (1993):46-48.
3. F. Clarke & I. Sh. Slavutskii. "The Integrality of the Values of Bernoulli Polynomials and of Generalised Bernoulli Numbers." *Bull. London Math. Soc.* **29** (1997):22-24.
4. G. Fox. "Euler Polynomials at Rational Numbers." *C. R. Math. Rep. Acad. Sci. Canada* **21** (1999):87-90.
5. J. Glaisher. "The Bernoullian Function." *Quart. J. Pure Appl. Math.* **29** (1898):1-168.
6. J. Glaisher. "On a Set of Coefficients Analogous to the Eulerian Numbers." *Proc. Lond. Math. Soc.* **31** (1899):216-35.
7. A. Granville & Z.-W. Sun. "Values of Bernoulli Polynomials." *Pacific J. Math.* **172** (1996): 117-37.
8. K. Ireland & M. Rosen. *A Classical Introduction to Modern Number Theory*. New York: Springer-Verlag, 1982.
9. E. Lehmer. "On Congruences Involving Bernoulli Numbers and the Quotients of Fermat and Wilson." *Ann. of Math.* **39** (1938):350-60.
10. N. Nielsen. *Traité élémentaire des nombres de Bernoulli*. Paris, 1923.
11. N. E. Nörlund. *Vorlesungen über Differenzenrechnung*. Berlin: Springer-Verlag, 1924.
12. J. Raabe. "Zurückführung einiger Summen und bestimmten Integrale auf die Jacob-Bernoullische Function." *J. Reine Angew. Math.* **42** (1851):348-67.
13. B. Sury. "The Value of Bernoulli Polynomials at Rational Numbers." *Bull. London Math. Soc.* **25** (1993):327-29.

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ON THE SOLVABILITY OF A FAMILY OF DIOPHANTINE EQUATIONS

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1. INTRODUCTION

A frequently occurring problem in the theory of binary quadratic forms is to determine, for a given integer m, the existence of solutions to the Diophantine equation

$$f(x,y) := ax^2 + bxy + cy^2 = m,$$

having discriminant $\Delta = b^2 - 4ac$. In the case of a strictly positive nonsquare discriminant, it is well known that the occurrence of one solution to $f(x,y) = m$ implies the existence of infinitely many other solutions. Using this fact, one may attempt to investigate the solvability of the following family of binomial Diophantine equations,

$$\binom{x+1}{2} = d\binom{y+1}{2}, \tag{1}$$

where $d \in N \setminus \{0\}$, as they can be recast into a quadratic form by completion of the square to obtain the following Pell-like equation

$$X^2 - dY^2 = 1 - d, \tag{2}$$

where $X = 2x+1$ and $Y = 2y+1$. Indeed, as (1, 1) is a solution of (2), there will exist infinitely many solutions (X, Y) when $\Delta = 4d > 0$ and is nonsquare. Unfortunately, in order to relate this to the solvability of our family of Diophantine equations, we must demonstrate that within the solution set of (2) there exists an infinite subset of solutions (X, Y) for which both X and Y are odd integers. To address such a problem, it will be necessary here to exploit a group action on the solution set $\mathcal{S} := \{(x,y) \in Z^2 : f(x,y) = m\}$, which allows one to generate an infinite subset of elements in $\mathcal{S}$ from a given solution in $\mathcal{S}$. In the case of (2) for a nonsquare $d \in N \setminus \{0\}$, an infinite subset of odd solutions can be generated from (1, 1). Although the solvability of (1) has been proved using elementary arguments (see [1]), the approach taken here is more direct and can be applied to a wider class of Diophantine equations. To illustrate, the above method will be used to establish, for each $m \in Z \setminus \{0\}$, the existence of infinitely many integer solutions to the more general family of equations

$$x(x+m) = dy(y+m), \tag{3}$$

when $d \in N \setminus \{0\}$ is nonsquare. The subset of solutions generated from the above group action are often referred to within the literature as orbits since they are closed with respect to the group action. It is well known (see [2]) that the solution set $\mathcal{S}$, when nonempty, is equal to a finite union of distinct orbits, each generated from a unique solution in $\mathcal{S}$. Consequently, in addition to proving the solvability of (3), we shall derive an asymptotic formula for the maximum number of distinct orbits that are required to completely describe $\mathcal{S}$ in the case of (3) as $d \to \infty$ through nonsquare values. Despite the reliance in this paper on algebraic methods, it is possible to demonstrate the solvability of the original class of Diophantine equations, for $d = 2, 3$, by a more elementary argument than that used in [1]. This method, which has already been applied to the case $d = 2$ in connection with the study of Pythagorean triples (see [3]), will result here in an algorithm for generating all positive integer solutions for the case $d = 3$. As an interesting aside, we further provide what the authors believe to be an unknown characterization for the solutions of the "negative Pell equation" $X^2 - 2Y^2 = -1$ in terms of the set of square triangular numbers; this follows as a direct consequence of the analysis in [3].

2. MAIN RESULT

We begin in this section by introducing some well-known concepts and results from the theory of binary quadratic forms that will be required in describing the group action on the set $\mathcal{S} = \{(x, y) \in Z^2 : f(x, y) = m\}$. The background material that follows has been taken from [2], where quadratic forms are treated from the perspective of quadratic number fields and their rings of integers. In what follows, assume Δ is a positive, nonsquare integer with $\Delta \equiv 0 \pmod 4$.

Definition 2.1: Let $Q(\sqrt{\Delta})$ be the quadratic extension of Q obtained by adjoining $\sqrt{\Delta}$. Define conjugation σ and norm N as follows: For $x, y \in Q$ and $\alpha = x + y\sqrt{\Delta}$, set $\sigma(\alpha) = x - y\sqrt{\Delta}$ and $N(\alpha) = \alpha\sigma(\alpha) = x^2 - \Delta y^2 \in Q$.

Using the well-known fact that $\sigma : Q(\sqrt{\Delta}) \to Q(\sqrt{\Delta})$ is an automorphism, it is easily deduced that the norm map N is multiplicative. In the theory of binary quadratic forms, the Pell equation plays a central role. We now introduce this equation and briefly examine the algebraic structure of its solution set.

Definition 2.2: The Pell equation is given by $f_\Delta(x, y) = 1$, where f_Δ is an integral binary form as follows:

$$f_\Delta(x, y) = x^2 - \frac{\Delta}{4}y^2,$$

with discriminant Δ. The negative Pell equation is $f_\Delta(x, y) = -1$. One also defines $Pell^{\pm}(\Delta) = \{(x, y) \in Z^2 : f_\Delta(x, y) = \pm 1\}$ and $Pell(\Delta) = \{(x, y) \in Z^2 : f_\Delta(x, y) = 1\}$.

For the above values of the discriminant Δ, it is known that $Pell(\Delta)$ has infinitely many elements. More importantly, all solutions with positive x and y can be generated as a power of a minimal "fundamental" solution. These results can be deduced by analyzing the Pell equation from the context of the subring $\mathcal{O}_\Delta$ of $Q(\sqrt{\Delta})$ having the underlying set $\{x + y\rho_\Delta : x, y \in Z\}$, where $\rho_\Delta = \sqrt{\Delta / 4}$. We expand here a little on this analysis, which not only leads to the group structure of $Pell(\Delta)$ but will also help to effect the desired group action of $\mathcal{S}$.

As every ordered pair $(x, y) \in Pell^{\pm}(\Delta)$ can be uniquely represented as an element $x + y\rho_\Delta \in \mathcal{O}_\Delta$, one sees from the calculation $N(x + y\rho_\Delta) = f_\Delta(x, y)$ that solving the positive or negative Pell equation is equivalent to finding the elements in $\mathcal{O}_\Delta$ having norm equal to ± 1. However, from the multiplicativity of N, it is easily established that $N(\alpha) = \pm 1$ for $\alpha \in \mathcal{O}_\Delta$ if and only if α is a unit in the ring $\mathcal{O}_\Delta$. Consequently, if one denotes the group of units in $\mathcal{O}_\Delta$ by $\mathcal{O}_\Delta^\times$, then $\psi := x + y\rho_\Delta$ defines a bijection $\psi : Pell^{\pm}(\Delta) \to \mathcal{O}_\Delta^\times$. Hence, $Pell^{\pm}(\Delta)$ is a group, as it is in bijection with the commutative group $\mathcal{O}_\Delta^\times$. Moreover, by using ψ to map the group law from $\mathcal{O}_\Delta^\times$, it is easily seen that the product of solutions in $Pell^{\pm}(\Delta)$ is given by

$$(u,v)\cdot(U,V)=\left(uU+\frac{\Delta}{4}vV,uV+vU\right). \tag{4}$$

If we further define $\mathcal{O}_{\Delta,1}^\times=\{\alpha\in\mathcal{O}_\Delta^\times : N(\alpha)=1\}$ and $\mathcal{O}_{\Delta,+}^\times=\{\alpha\in\mathcal{O}_\Delta^\times:\alpha>0\}$, then restricting ψ to the subgroup $\mathcal{O}_{\Delta,1}^\times$ of $\mathcal{O}_\Delta^\times$ gives an isomorphism $Pell(\Delta)\cong\mathcal{O}_{\Delta,1}^\times$. The cyclic nature of the group $Pell(\Delta)$ can be deduced by first noting that $\mathcal{O}_{\Delta,+}^\times$ contains a minimal element ε over all elements in $\mathcal{O}_{\Delta,+}^\times$ that are greater than unity (see [2]). Using this fact, it can be easily shown, as $\mathcal{O}_{\Delta,+}^\times\subseteq(0,\infty)=\bigcup_{n\in Z}[\varepsilon^n,\varepsilon^{n+1})$, that any $\alpha\in\mathcal{O}_{\Delta,+}^\times$ is of the form $\alpha=\varepsilon^n$ for some $n\in Z$. Since any $\beta=\pm\varepsilon^n$ for some $n\in Z$. Thus, if one formally defines

$$\tau_\Delta=\begin{cases}\varepsilon & \text{if } N(\varepsilon)=1,\\ \varepsilon^2 & \text{if } N(\varepsilon)=-1,\end{cases}$$

then $\mathcal{O}_{\Delta,1}^\times=\{\pm\tau_\Delta^m: m\in Z\}\cong Pell(\Delta)$.

Remark 2.1: Note that, if $\alpha=\mathcal{O}_\Delta^\times\backslash\mathcal{O}_{\Delta,+}^\times$, then as $-1=-1+0\rho_\Delta\in\mathcal{O}_\Delta^\times$ and $-\alpha\in\mathcal{O}_{\Delta,+}^\times$, one must have $-\alpha=\pm\varepsilon^n$ or $\alpha=\mp\varepsilon^n$ for some $n\in Z$. Consequently, if $\varepsilon=a+b\rho_\Delta$, then from the bijection $\psi: Pell^{\pm}(\Delta)\to\mathcal{O}_\Delta^\times$ it is clear that $Pell^{\pm}(\Delta)=\{\pm(x_n,y_n):n\in Z\}$, where $x_n+y_n\rho_\Delta=(a+b\rho_\Delta)^n$. Similarly, the solutions in $Pell(\Delta)=\{\pm(x_n,y_n):n\in Z\}$ can be calculated from $x_n+y_n\rho_\Delta=\tau_\Delta^n$. While in the case in which $N(\varepsilon)=-1$ the solutions in $Pell^{-}(\Delta)$ are obtained from the subset $\{\pm(x_{2n+1},y_{2n+1}):n\in Z\}$ of $Pell^{\pm}(\Delta)$ as they are the only ordered pairs in $Pell^{\pm}(\Delta)$ for which $N(\psi)=-1$.

To help define the group action on $\mathcal{S}$, we proceed in a similar manner to the above, by first generalizing the construction of the ring $\mathcal{O}_\Delta$. The definition given below is motivated by the factorization

$$f(x,y)=\frac{1}{a}\left(xa+y\frac{b+\sqrt{\Delta}}{2}\right)\left(xa+y\frac{b-\sqrt{\Delta}}{2}\right).$$

Definition 2.3: The module M_f of an integral binary quadratic form $f(x,y)$, which has discriminant Δ, is the $\mathcal{O}_\Delta$ module having the underlying set $\{xa+y(b+\sqrt{\Delta})/2:x,y\in Z\}\subseteq Q(\sqrt{\Delta})$.

It is the closure of M_f under multiplication by elements in $\mathcal{O}_\Delta$ that most interests us here. The important calculation is $(u+v\rho_\Delta)(xa+y(b+\sqrt{\Delta})/2)=(x'a+y'(b+\sqrt{\Delta})/2)$, where

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}u-\frac{b}{2}v & -cv\\ av & u+\frac{b}{2}v\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}. \tag{5}$$

Equation (5) can be used to define an action of the group $\mathcal{O}_{\Delta,1}^\times$ on $\mathcal{S}$, which, given an $(x,y)\in\mathcal{S}$, one can generate an infinite set)or orbit) of solutions in $\mathcal{S}$ by repeated application of (5). To see this, first observe that any $(x,y)\in\mathcal{S}$ is uniquely represented as an element $ax+y(b+\sqrt{\Delta})/2\in M_f$. Now, as in the case of the Pell form, one can set $\psi(x,y):=xa+y(b+\sqrt{\Delta})/2$, from which it is immediate that $N(\psi(x,y))=af(x,y)$. Hence, ψ defines a bijection $\psi:\mathcal{S}\to\{\gamma\in M_f:N(\gamma)=am\}$. If we formally define the action of an element $\alpha\in\mathcal{O}_{\Delta,1}^\times$ on the set $\mathcal{S}$ by

$\alpha \cdot (x, y) := \psi^{-1}(\alpha\psi(x, y))$, then from the multiplicity of N it is clear that $N(\psi(\alpha \cdot (x, y))) = N(\alpha\psi(x, y)) = am$, consequently, $\alpha \cdot (x, y) \in \mathcal{S}$. As $\mathbb{O}_{\Delta,1}^{\times}$ is a cyclic group of infinite order, the set $\mathcal{S}$, when nonempty, will at least contain the infinite subset of solutions in the orbit given by $\mathbb{O}_{\Delta,1}^{\times} \cdot (x, y) = \{\pm\tau_{\Delta}^{n} \cdot (x, y), n \in Z\}$. Also, from the bijection $\psi(x, y)$, the elements in $\mathbb{O}_{\Delta,1}^{\times} \cdot (x, y)$ can be calculated explicitly by repeated application of (5). We now apply the above group action to establish the solvability of the general Diophantine equation in (3).

Theorem 2.1: Suppose $d > 1$ is a nonsquare integer and $m \in Z \setminus \{0\}$, then there exist infinitely many solutions to the Diophantine equation

$$x(x+m) = dy(y+m). \tag{6}$$

Proof: By completing the square, the Diophantine equation in question can be rewritten in the form

$$X^2 - dY^2 = m^2(1-d), \tag{7}$$

where $X = 2x+m$ and $Y = 2y+m$. When $m = 2s$, equation (7) can be reduced further to the quadratic form

$$Z^2 - dW^2 = s^2(1-d), \tag{8}$$

where $Z = x+s$ and $W = y+s$. Now, for the assumed values of d, equation (8) has the nonsquare discriminant $\Delta = 4d$ and so an infinite number of solutions can be generated from the orbit $\mathbb{O}_{\Delta,1}^{\times} \cdot (s, s) = \{(Z, W) = \pm\tau_{\Delta}^{n} \cdot (s, s) : n \in Z\}$. Hence, the original Diophantine equation in (6) will have at least the infinite subset of solutions given by $\{(x, y) = (Z-s, W-s) : (Z, W) \in \mathbb{O}_{\Delta,1}^{\times} \cdot (s, s)\}$. If m is odd, then the question of solvability of (6) is reduced to knowing whether there exist infinitely many odd solutions to (7). We now examine the orbit of solutions generated by the action of $\mathbb{O}_{\Delta,1}^{\times}$ on (m, m). If $\tau_{\Delta} = u + v\rho_{\Delta}$, then, by (5), the sequence of elements $\{\tau_{\Delta}^{n} \cdot (m, m)\}_{n=0}^{\infty}$ can be generated using

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} u & dv \\ v & u \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{9}$$

with $(x_0, y_0) = (m, m)$. We claim that, for all nonsquare $d > 1$, the sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ contains infinitely many odd ordered pairs. To demonstrate this by induction it will be convenient, since $u^2 = 1 + dv^2$, to deal with the following cases separately. For brevity, one need only attend to the inductive step in each case.

Case 1. $2 \nmid d$

In this instance, u and v will be of opposite parity. If, for some $n \geq 0$, it is assumed that (x_n, y_n) is an odd ordered pair, then by (9) both x_{n+1} and y_{n+1} are the sum of an odd and even number and so must be odd.

Case 2. $2 \mid d$

Now u will always be odd irrespective of the parity of v. If v is even, then the oddness of the ordered pair (x_n, y_n) follows by an analogous argument to the one above. For v odd, we shall establish that all the odd solutions are contained in the subsequence $\{(x_{2n}, y_{2n})\}_{n=0}^{\infty}$. Therefore, suppose x_{2n} and y_{2n} are odd for some $n \geq 0$, then by (9) x_{2n+1} is odd while y_{2n+1} is even.

However, by another application of (9), one finds that both x_{2n+2} and y_{2n+2} are the sum of an odd and an even integer and so must be odd. □

Corollary 2.1: Suppose $d > 1$ is a nonsquare integer and T_n denoted the n^{th} triangular number, then there exist infinitely many pairs of positive integers (m, n) such that $T_m = dT_n$. If d is a perfect square then, in general, only at most finitely many solutions (m, n) can be found while, in particular, non exist when $d = p^{2s}$ for p a prime.

Proof: The first statement follows from setting $m = 1$ in Theorem 2.1. Suppose now d is a perfect square, with $m = \sqrt{d}$. Clearly, the equation $f(X, Y) = 1 - d$, where $f(X, Y) = X^2 - dY^2$ can have only finitely many integer solutions due to the factorization $f(X, Y) = (X - mY)(X + mY)$. In the case of $d = p^{2s}$, consider equation (1) given here as $x(x+1) = dy(y+1)$. If one assumes to the contrary that a positive integer solution (x, y) exists, then $p^{2s} \mid x(x+1)$. However, this can only be true if either $p^{2s} \mid x$ or $p^{2s} \mid (x+1)$ as $(x, x+1) = 1$. Suppose $x = mp^{2s}$ for some fixed $m \in N \setminus \{0\}$, then y must be a root of the quadratic $0 = y^2 + y - (m^2 p^{2s} + m)$. However, as the discriminant of this equation satisfies the inequality

$$(2p^s m)^2 < 4p^{2s}m^2 + 4m + 1 < (2p^s m + 1)^2$$

and so cannot be a square, one deduces that $y \notin N$. A similar contradiction follows if $x + 1 = mp^{2s}$, as the discriminant of the resulting quadratic satisfies the inequality

$$(2p^s m - 1)^2 < 4p^{2s}m^2 - 4m + 1 < (2p^s m)^2. \ \square$$

Remark 2.2: One can use the above argument to compute an infinite subset of solutions to the Diophantine equation $x(x+m) = dy(y+m)$ for nonsquare d via (5). All that is required is the determination of the element $\tau_\Delta = u + v\rho_\Delta$, which will result upon finding the unit $\varepsilon \in \mathcal{O}^{\times}_{\Delta,+}$. This can be achieved by applying the following method taken from [2]. Consider the quadratic form $f_{4d}(x, y) = x^2 - dy^2$, which has a nonsquare discriminant $4d > 0$. If y is the smallest positive integer such that one of the $dy^2 + 1$ or $dy^2 - 1$ is a square and x is the positive integer root, then $\varepsilon = x + y\sqrt{d}$.

When determining the full solution set one will, of course, have to find all the distinct orbits that comprise $\mathcal{S}$. This can be achieved because a finite list containing the generators of each such orbit can be constructed using the following result (see [2]).

Proposition 2.1: Let $f(x, y)$ be an integral form with discriminant Δ and suppose $m \neq 0 \in Z$. If $\tau = \tau_\Delta$ is the smallest unit in $\mathcal{O}^{\times}_{\Delta,1}$ that is greater than unity, then:

(i) Every orbit of integral solutions of $f(x, y) = m$ contains a solution $(x, y) \in Z^2$ such that $0 \le y \le U$, where $U = |am\tau / \Delta|^{1/2}(1 - 1/\tau)$ if $am > 0$ and $U = |am\tau / \Delta|^{1/2}(1 + 1/\tau)$ if $am < 0$.

(ii) Two distinct solutions $(x_1, y_1) \neq (x_2, y_2) \in Z^2$ of the equation $f(x, y) = m$ such that $0 \le y_i \le U$ belong to the same orbit if and only if $y_1 = y_2 = 0$ or $y_1 = y_2 = U$.

Since every orbit of solutions to $f(x, y) = m$ contains an element in the finite set

$$\mathcal{S}' = \{(x, y) : 0 \le y \le U\},$$

any $(x, y) \in \mathcal{S}'$ can be listed and sorted into orbits using Proposition 2.1. The set $\mathcal{S}'$ which contains the representatives of the orbits can be viewed as a finite list from which the solutions in $\mathcal{S}$ can be generated from the group action. In the case of (6), it is of interest to estimate the maximum number of distinct orbits needed to describe $\mathcal{S}$ as $d \to \infty$. Using the following result, which is taken from [4], we can obtain an asymptotic bound for the maximum number of possible orbits for the Diophantine equation in (6).

Lemma 2.1: If $\tau_\Delta = u + \rho_\Delta v$ is the smallest unit in $\mathcal{O}_{\Delta,1}^\times$ that is greater than unity with $\Delta = 4d$, then $u \sim \sqrt{d}e^{\sqrt{d}+O(1)}$ as $d \to \infty$ through nonsquare values.

Theorem 2.2: For a given $m \in Z \setminus \{0\}$, the maximum number of distinct orbits for the equation $x(x+m) = dy(y+m)$ is given by $[M(d)]$, where

$$M(d) \sim \frac{|k|d^{1/4}}{\sqrt{2}}e^{\frac{\sqrt{d}}{2}+O(1)}$$

as $d \to \infty$ through nonsquare values where $k = m/2$ for m even and $k = m$ for k odd.

Proof: We first note that no proper orbit can be generated from the solution $(0, 0)$. Thus, from Proposition 2.1, the maximum number of distinct orbits is equal to the total number of positive integers less than or equal to U, that is, $[U]$. Now when $m = 2s$ and d is nonsquare, the solutions of the Diophantine equation in (2) arise directly from the orbits of $Z^2 - dW^2 = s^2(1-d)$ via a translation of these orbits by subtraction of the ordered pair (s, s). Consequently, we have by Lemma 2.1 that

$$\begin{aligned} M(d) &= \left|\frac{s^2(1-d)}{4d}\right|^{1/2}\left|\tau_\Delta\left(1+\frac{1}{t_\Delta}\right)^2\right|^{1/2} = \frac{|s|}{2}\left|\frac{1-d}{d}\right|^{1/2}|\tau_\Delta + 2 + \sigma(\tau_\Delta)|^{1/2} \\ &= \frac{|s|}{\sqrt{2}}\left|\frac{1-d}{d}\right|^{1/2}|u+1|^{1/2} \sim \frac{|s|}{\sqrt{2}}(\sqrt{d}e^{\sqrt{d}+O(1)})^{1/2} \end{aligned}$$

as $d \to \infty$. When m is odd, the solutions are derived from the odd solutions in the orbits of $X^2 - dY^2 = m^2(1-d)$. Thus, if in each of these orbits there exists an infinite subset of odd solutions, then the maximum number of distinct orbits is again $[U]$ and the asymptotic bound will result as in the above by replacing s by m. □

3. AN ELEMENTARY APPROACH

In contrast to the algebraic methods used previously, we present in this section an alternate technique for demonstrating the solvability of (1) for the cases $d = 2, 3$. Although of interest on its own, the elementary approach employed here has the advantage of allowing one to deduce a characterization for the solutions of the negative Pell equation in terms of square triangular numbers. We first observe that, if $0 < x \le y$, then $x(x+1) < dy(y+1)$, while, if $x \ge dy > 0$, then $x(x+1) > dy(y+1)$. Consequently, for an arbitrary $y \in N \setminus \{0\}$, the only integer values which x may possibly assume in order that $x(x+1) = dy(y+1)$ are those in which $y < x < dy$. So, if (x, y) is a solution, then there must exist a fixed $t \in N \setminus \{0\}$ such that $x = y + t$ and $(y+t)(y+t+1) = dy(y+1)$. With the introduction of the parameter t, one can then solve for y in terms of t and so the question of solvability is necessarily reduced to knowing whether the discriminant of the associated quadratic is a square for infinitely many t. We now apply this method of the case $d = 3$. The following technical lemma will be required to establish the necessary condition for the existence of integer solutions.

Lemma 3.1: Suppose (v, u) is a positive integer solution of the Pell equation $v^2 - 3u^2 = 1$. Then $(v, u) = (2, 1)^n$, for some $n \geq 1$, where the product of solutions is taken in the sense of (4).

Proof: Applying the method in Remark 2.2, one deduces for $\Delta = 12$ that the element $\varepsilon \in \mathbb{O}_{\Delta,+}^{\times}$ is given by $\varepsilon = 2+\sqrt{3}$. As $N(\varepsilon) = 1$, we have $\tau_{12} = \varepsilon$ and so $Pell(\Delta) = \{\pm(x_n, y_n)$: $n \in Z\}$, where $(x_n + y_n\sqrt{3}) = (2+\sqrt{3})^n$. Thus, the positive solutions are given by (x_n, y_n), where $n \in N \setminus \{0\}$. Now, since $(x_{n+1}, y_{n+1}\sqrt{3}) = (x_n + y_n\sqrt{3})(2+\sqrt{3})$, one sees that $x_{n+1} = 2x_n + 3y_n$ and $y_{n+1} = x_n + 2y_n$. Consequently, via the product formula for solutions in (4) we have $(x_{n+1}, y_{n+1}) = (x_n, y_n)\cdot(2, 1)$, from which it is deduced that $(x_n, y_n) = (2, 1)^n$ as $(x_1, y_1) = (2, 1)$. □

Theorem 3.1: There are infinitely many positive integer solutions to (1) in the case $d = 3$. Moreover, all such solutions (x, y) are given by

$$\left(\frac{3u_n + v_n - 1}{2}, \frac{u_n + v_n - 1}{2}\right),$$

where the ordered pair (v_n, u_n) is generated recursively using

$$\begin{pmatrix} v_{i+1} \\ u_{i+1} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_i \\ u_i \end{pmatrix}, \tag{10}$$

with $(v_1, u_1) = (2, 1)$.

Proof: We first prove existence. Suppose (x, y) satisfies the Diophantine equation, then by the above method there must exist a fixed $t > 0$ such that $x = y + t$. Substituting this expression into the Diophantine equation and simplifying yields the quadratic $0 = 2y^2 + 2(1-t)y - (t^2 + t)$. Remembering that y is assumed positive, one finds upon solving this equation that

$$y = \frac{t - 1 + \sqrt{3t^2 + 1}}{2}. \tag{11}$$

However, from Lemma 3.1, there are infinitely many $s, t \in N$ such that $3t^2 + 1 = s^2$; moreover, by a simple parity argument, the numerator in (11) can be shown to be an even integer for all such t. Consequently, there are infinitely many integers (x, y) that satisfy $s(s+1) = 3y(y+1)$, all of which may be determined via (11). It is now a simple task to construct the accompanying algorithm. Set $t = v_n$ and $s = u_n$ as in Lemma 3.1, then, clearly, from (11) we have

$$y = \frac{v_n - 1 + \sqrt{u_n^2}}{2} = \frac{v_n + u_n - 1}{2}, \quad x = v_n + y = \frac{3v_n + u_n - 1}{2}.$$

Finally, as $(v_{n+1}, u_{n+1}) = (2, 1)\cdot(2, 1)^n = (2, 1)\cdot(v_n, u_n)$, one deduces from the product formula in (4) the recurrence relation of (10). □

For larger values of d, the above method cannot be applied due to the increased difficulty in verifying the existence of infinitely many t for which the discriminant of $(y+t)(y+t+1) = dy(y+1)$ is a square. To conclude, we examine an application of our elementary method to uncover a curious connection between the solutions of the Diophantine equation $X^2 - 2Y^2 = -1$ and the sequence of square triangular numbers. Following the above analysis it is easily seen, in the case $d = 2$, that for (x, y) to be a positive integer solution of $x(x+1) = 2y(y+1)$ there must exist a $t > 0$ such that $x = y + t$ with

$$y = \frac{2t - 1 + \sqrt{8t^2 + 1}}{2}.$$

Since y is an integer, $8t^2 + 1$ must be an odd perfect square. Consequently, we require $8t^2 + 1 = (2m+1)^2$ for some $m \in N \setminus \{0\}$, so that t and m satisfy $2t^2 = m(m+1)$. Thus, by denoting T_n as the square root of the n^{th} square triangular number, of which there are infinitely many, one deduces for some $n \in N \setminus \{0\}$ that

$$x = \frac{4T_n - 1 + \sqrt{8T_n^2 + 1}}{2}, \quad y = \frac{2T_n - 1 + \sqrt{8T_n^2 + 1}}{2}. \tag{12}$$

Using these relations, one can deduce the following characterization.

Theorem 3.2: All positive integer solutions (X, Y) of the negative Pell equation $X^2 - 2Y^2 = -1$ are of the form

$$(4T_n + \sqrt{8T_n^2 + 1},\ 2T_n + \sqrt{8T_n^2 + 1}),$$

where T_n denotes the positive square root of the n^{th} square triangular number.

Proof: Recall that the negative Pell equation $X^2 - 2Y^2 = -1$, where $X = 2x + 1$, $Y = 2y + 1$, can be derived by completing the square on $x(x+1) = 2y(y+1)$. The result will follow from (12) if one can show that all the positive solutions (X, Y) consist only of odd integers. To establish this, we first observe from Remark 2.2 that, for $\Delta = 8$, the element $\varepsilon \in \mathbb{O}_{\Delta,+}^{\times}$ is given by $\varepsilon = 1 + \sqrt{2}$. Therefore, $Pell^{\pm}(8) = \{\pm(x_n, y_n) : n \in Z\}$, where $x_n + y_n\sqrt{2} = (1+\sqrt{2})^n$. However, since $N(1+\sqrt{2}) = -1$, the positive solutions of the negative Pell equation must be given by $(X_n, Y_n) = (x_{2n+1}, y_{2n+1})$, where $n \in N$. Moreover, since $x_{2n+3} + y_{2n+3}\sqrt{2} = (1+\sqrt{2})^2(x_{2n+1} + y_{2n+1}\sqrt{2})$, one can see that the solutions (X_n, Y_n) satisfy the recurrence relation

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}, \text{ with } (X_0, Y_0) = (1, 1).$$

The desired conclusion follows now by a simple inductive argument. □

REFERENCES

1. E. Becker, R. Robson, G. Schrange. "Variationen über ein diophantisches Thema." *El. Math.* **50** (1995):45-64.
2. D. E. Flath. *Introduction to Number Theory*. New York: Wiley & Sons, 1988.
3. M. A. Nyblom. "A Note on the Set of Almost Isosceles Right-Angled Triangles." *The Fibonacci Quarterly* **36.4** (1998):319-22.
4. A. Peretti. "The Diophantine Equation $t^2 - Dy^2 = 1$." *Bull. Number Theory* **13** (1989):61-68.

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ON PALINDROMIC SEQUENCES FROM IRRATIONAL NUMBERS

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INTRODUCTION

A *palindrome* is a finite sequence $(x_1, x_2, \dots, x_n)$ of numbers satisfying

$$(x_1, x_2, \dots, x_n) = (x_n, x_{n-1}, \dots, x_1).$$

Let $\Delta_n = \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor$ for some positive irrational α, and $n = 1, 2, \dots$. In [2], Kimberling shows that there are infinitely many palindromes $(\Delta_1, \dots, \Delta_l)$ in the infinite Δ-sequence (or the *characteristic* word of the *Beatty sequence*). For example, for $\alpha = (1+\sqrt{5})/2$, the Δ-sequence begins $1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, \dots$. So $(\Delta_1, \dots, \Delta_l)$ is a palindrome for

$$l \in \{1, 3, 8, 21, 55, 144, 377, 987, \dots\},$$

and $(\Delta_2, \dots, \Delta_{l-1})$ is a palindrome for

$$l \in \{3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \dots\}.$$

(The examples in [2] only partly match this observation.) In [1] Droubay proves that, if $\alpha = (1+\sqrt{5})/2$, the number of palindromes of length n is exactly 1 if n is even, and 2 if n is odd (see also [3], e.g.). Then, how can we describe all the palindromes in the Δ-sequence? This paper gives an answer to this question.

MAIN RESULTS

As usual, we denote the continued fraction expansion of α by $\alpha = [a_0; a_1, a_2, \dots]$. Then its n^{th} (*total*) convergent $p_n / q_n = [a_0; a_1, \dots, a_n]$ is given by the recurrence relations

$$p_n = a_n p_{n-1} + p_{n-2} \quad (n = 0, 1, \dots), \quad p_{-2} = 0, \quad p_{-1} = 1,$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad (n = 0, 1, \dots), \quad q_{-2} = 1, \quad q_{-1} = 0.$$

Define the n^{th} *intermediate* (or *partial*) *convergents* by $p_{n,r}/q_{n,r}$ $(r = 0, 1, 2, \dots, a_n - 1)$, where $p_{n,r} = rp_{n+1} + p_n$ and $q_{n,r} = rq_{n+1} + q_n$ ([3], cf. [5]). So, $p_{n, a_{n+2}} = p_{n+2}$ and $q_{n, a_{n+2}} = q_{n+2}$.

We define the fractional part of x by $\{x\} = x - \lfloor x \rfloor$.

Lemma 1: Let l and m be integers satisfying $l \ge 2m-1$. Then $(\Delta_m, \Delta_{m+1}, \dots, \Delta_{l-m+1})$ is a palindrome if and only if $\{k\alpha\} + \{(l-k)\alpha\}$ is invariant of k for $k = m-1, m, \dots, \lfloor (l+1)/2 \rfloor$.

Proof: By definition, $(\Delta_m, \Delta_{m+1}, \dots, \Delta_{l-m+1})$ is a palindrome if and only if, for $k = m-1$, m, $\dots$, $\lfloor (l+1)/2 \rfloor$,

$$\lfloor (k+1)\alpha \rfloor + \lfloor (l-k-1)\alpha \rfloor = \lfloor k\alpha \rfloor + \lfloor (l-k)\alpha \rfloor,$$

or

$$\{(k+1)\alpha\} + \{(l-k-1)\alpha\} = \{k\alpha\} + \{(l-k)\alpha\}.$$

Of course, this also holds for $k = \lfloor (l+1)/2 \rfloor + 1$, $\lfloor (l+1)/2 \rfloor + 2, \dots, l-m$.

Lemma 2 (cf. Theorem 1, [2]): Let q be an integer with $q > q_1$. There are integers n and r with $n = 0, 1, \ldots$ and $r = 1, 2, \ldots, a_{n+2}$ such that $q = q_{n,r}$ if and only if, for $k = 1, 2, \ldots, q-1$, the sum $\{k\alpha\} + \{(q-k)\alpha\}$ is invariant of *k*, that is,

$$\{k\alpha\} + \{(q-k)\alpha\} = \begin{cases} \{q\alpha\} + 1 & \text{if } n \text{ is even,} \\ \{q\alpha\} & \text{if } n \text{ is odd.} \end{cases}$$

Sublemma (Theorem 3.3, [5]): Let $q = 1, 2, \ldots, N-1$. If $q_{n,r-1} < N \le q_{n,r}$ $(2 \le r \le a_{n+2},\ n \ge 0)$, then

$$\{q_{n,r-1}\alpha\} \le \{q\alpha\} \le \{q_{n+1}\alpha\} \quad \text{if } n \text{ is even,}$$
$$\{q_{n+1}\alpha\} \le \{q\alpha\} \le \{q_{n,r-1}\alpha\} \quad \text{if } n \text{ is odd.}$$

If $q_{n+1} < N \le q_{n,1}$ $(n \ge 0)$, then

$$\{q_n\alpha\} \le \{q\alpha\} \le \{q_{n+1}\alpha\} \quad \text{if } n \text{ is even,}$$
$$\{q_{n+1}\alpha\} \le \{q\alpha\} \le \{q_n\alpha\} \quad \text{if } n \text{ is odd.}$$

If $N \le q_1$, then $\{\alpha\} < \{2\alpha\} < \cdots < \{(N-1)\alpha\}$.

Proof of Lemma 2: If $q = q_{n,r}$ for some integers n and r, then by the Sublemma for $k = 1, 2, \ldots, q-1$,

$$\{k\alpha\} > \{q\alpha\} \quad \text{if } n \text{ is even,}$$
$$\{k\alpha\} < \{q\alpha\} \quad \text{if } n \text{ is odd.}$$

Thus, for $k = 1, 2, \ldots, q-1$,

$$\{k\alpha\} + \{(q-k)\alpha\} > \{q\alpha\} \quad \text{if } n \text{ is even,}$$
$$\{k\alpha\} + \{(q-k)\alpha\} < \{q\alpha\} + 1 \quad \text{if } n \text{ is odd.}$$

Therefore, for $k = 1, 2, \ldots, q-1$,

$$\{k\alpha\} + \{(q-k)\alpha\} = \begin{cases} \{q\alpha\} + 1 & \text{if } n \text{ is even,} \\ \{q\alpha\} & \text{if } n \text{ is odd.} \end{cases}$$

Because $\{k\alpha\} + \{(q-k)\alpha\}$ takes only the values $\{q\alpha\}$ or $\{q\alpha\} + 1$, the sum $\{k\alpha\} + \{(q-k)\alpha\}$ is invariant of *k*.

On the other hand, if $q \ne q_{n,r}$ for some integers n and r, then there exist integers k' and k'' with $k' \ne k''$ and $0 < k',\ k'' < q$ such that $\{k'\alpha\} < \{q\alpha\} < \{k''\alpha\}$. Hence,

$$\{k'\alpha\} + \{(q-k')\alpha\} < \{q\alpha\} + 1 \quad \text{and} \quad \{k''\alpha\} + \{(q-k'')\alpha\} > \{q\alpha\}.$$

Since $\{k\alpha\} + \{(q-k)\alpha\}$ takes only the values $\{q\alpha\}$ or $\{q\alpha\} + 1$, the sum is not invariant of *k* for $k = 1, 2, \ldots, q-1,$.

When $m = 2$, we have the first main theorem by using Lemmas 1 and 2.

Theorem 1: Let the continued fraction expansion of an irrational α be

$$\alpha = [a_0; a_1, a_2, \ldots, a_n, \ldots].$$

Then $(\Delta_2, \ldots, \Delta_{l-1})$ is a palindrome only for

$$l \in \{\underbrace{1, 2, \ldots, q_1}_{a_1}, \underbrace{q_1+1, 2q_1+1, \ldots, q_2}_{a_2}, \underbrace{q_2+q_1, 2q_2+q_1, \ldots, q_3}_{a_3}, \ldots,$$
$$\underbrace{q_{n-1}+q_{n-2}, 2q_{n-1}+q_{n-2}, \ldots, q_n}_{a_n}, \ldots\} - \{1, 2\}.$$

Proof: Since $1/(a_1+1) < \{\alpha\} = [0; a_1, a_2, \ldots] < 1/a_1$, we have, for $a_1 \geq 2$,

$$\Delta_2 + \cdots + \Delta_{q_1} = \lfloor a_1\alpha \rfloor - \lfloor \alpha \rfloor = a_1\lfloor \alpha \rfloor - \lfloor \alpha \rfloor = (a_1-1)\lfloor \alpha \rfloor,$$

yielding $\Delta_2 = \cdots = \Delta_{q_1} = \lfloor \alpha \rfloor$ because $\Delta_n = \lfloor \alpha \rfloor$ or $\lfloor \alpha \rfloor + 1$. Hence, $(\Delta_2, \ldots, \Delta_{l-1})$ is a palindrome for $l = 3, 4, \ldots, q_1+1$. For $a_1 = 1$, it is trivial that $l = 3$.

Set $n = 0, 1, 2, \ldots$. By Lemma 2 for $k = 1, 2, \ldots, q_{n,r} - 2$ $(r = 1, 2, \ldots, a_{n+2})$,

$$\{(k+1)\alpha\} + \{(q_{n,r} - (k+1))\alpha\} = \{k\alpha\} + \{(q_{n,r} - k)\alpha\}.$$

Thus, by Lemma 1, $(\Delta_2, \ldots, \Delta_{l-1})$ is a palindrome for $l = q_{n,r}$ $(r = 1, 2, \ldots, a_{n+2})$. Lemma 2 also shows that there is no other possibility for l.

Example 1: Let $\alpha = e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots]$. Then the denominators of its convergents are

$$(q_1, q_2, q_3, \ldots, q_{10}, \ldots) = (1, 3, 4, 7, 32, 39, 71, 465, 536, 1001, \ldots).$$

Hence, $(\Delta_2, \ldots, \Delta_{l-1})$ is a palindrome for

$$l \in \{\underbrace{1}_{1}, \underbrace{2, 3}_{2}, \underbrace{4}_{1}, \underbrace{7}_{1}, \underbrace{11, 18, 25, 32}_{4}, \underbrace{39}_{1},$$
$$\underbrace{71}_{1}, \underbrace{110, 181, 252, 323, 394, 465}_{6}, \underbrace{536}_{1}, \underbrace{1001}_{1}, \ldots\} - \{1, 2\}$$
$$= \{3, 4, 7, 11, 18, 25, 32, 39, 71, 110, 181, 252, 323, 394, 465, 536, 1001, \ldots\}.$$

In fact, Δ begins with $2, \check{3}, \hat{3}, 2, 3, \hat{3}, 3, 2, 3, \hat{3}, 2, 3, 3, 3, 2, 3, \hat{3}, 2, 3, 3, 3, 2, 3, \hat{3}, 2, 3, 3, 3, 2, 3, \hat{3}, 2, \ldots$. One can see the palindromes between ˇ and ^ (included).

Next, we put $m = 1$ to obtain the following result.

Theorem 2: $(\Delta_1, \ldots, \Delta_l)$ is a palindrome only for

$$l \in \{\underbrace{1, 2, \ldots, q_1}_{a_1}, \underbrace{q_2+q_1, 2q_2+q_1, \ldots, q_3}_{a_3}, \underbrace{q_4+q_3, 2q_4+q_3, \ldots, q_5}_{a_5}, \ldots,$$
$$\underbrace{q_{2n}+q_{2n-1}, 2q_{2n}+q_{2n-1}, \ldots, q_{2n+1}}_{a_{2n+1}}, \dot{s}\}.$$

Proof: Since $\Delta_1 = \Delta_2 = \cdots = \Delta_{q_1} = \lfloor \alpha \rfloor$, $(\Delta_1, \ldots, \Delta_l)$ is a palindrome for $l = 1, 2, \ldots, q_1$. Set $n = 0, 1, 2, \ldots$. By Lemma 2 for $k = 2, 3, \ldots, q_{n,r} - 1$ $(r = 1, 2, \ldots, a_{n+2})$,

$$\{k\alpha\} + \{(q_{n,r} - k)\alpha\} = \{(k-1)\alpha\} + \{(q_{n,r} - k + 1)\alpha\}.$$

And for $k = 1$, $\{\alpha\} + \{(q_{n,r} - 1)\alpha\} = \{q_{n,r}\alpha\}$ is true only when n is odd. Therefore, $(\Delta_2, \ldots, \Delta_{l-1})$ is a palindrome for $l = q_{2n-1,r}$ $(r = 1, 2, \ldots, a_{2n+1};\ n = 1, 2, \ldots)$. By Lemma 2, all the possibilities for l appear here.

MORE PALINDROMES

There are infinitely many palindromes that do not start from Δ_1 or Δ_2 in the Δ-sequence. In other words, for any integer m, there exist infinitely many integers l with $l \geq 2m-1$ such that

$$(\Delta_m, \Delta_{m+1}, \ldots, \Delta_{l-m+1})$$

is palindromic. Defining $\Delta_0 = \lfloor 0\alpha \rfloor - \lfloor -\alpha \rfloor$, we have the following theorem.

Theorem 3: $(\Delta_0, \Delta_1, \ldots, \Delta_{l+1})$ is a palindrome only for

$$l \in \{q_1, \underbrace{q_2+q_1, 2q_2+q_1, \ldots, q_3}_{a_3}, \underbrace{q_4+q_3, 2q_4+q_3, \ldots, q_5}_{a_5}, \ldots,$$

$$\underbrace{q_{2n}+q_{2n-1}, 2q_{2n}+q_{2n-1}, \ldots, q_{2n+1}}_{a_{2n+1}}, \ldots\}.$$

Proof: Since $\Delta_0 = -\lfloor -\alpha \rfloor = \lfloor \alpha \rfloor + 1 = \Delta_{q_1+1}$ and $\Delta_1 = \Delta_2 = \cdots = \Delta_{q_1} = \lfloor \alpha \rfloor$, $(\Delta_0, \ldots, \Delta_{l+1})$ is a palindrome for $l = q_1$. By Lemma 2,

$$\{(k-1)\alpha\} + \{(q_{n,r}-k+1)\alpha\} = \{(k-2)\alpha\} + \{(q_{n,r}-k+2)\alpha\}$$

holds for $k = 3, 4, \ldots, q_{n,r}-1$. For $k=2$, $\{\alpha\} + \{(q_{n,r}-1)\alpha\} = \{q_{n,r}\alpha\}$ is true only when n is odd.

Consider the case $k=1$. When n is odd,

$$\{q_{n,r}\alpha\} + \{\alpha\} = q_{n,r}\alpha - \lfloor q_{n,r}\alpha \rfloor + \{\alpha\}$$

$$= 1 + \{\alpha\} - (p_{n,r} - q_{n,r}\alpha) > 1 + \frac{1}{a_1+1} - \frac{1}{q_{n+1}} \geq 1.$$

Therefore, $\{q_{n,r}\alpha\} + \{\alpha\} = \{(q_{n,r}+1)\alpha\} + 1$ or $\{q_{n,r}\alpha\} = \{-\alpha\} + \{(q_{n,r}+1)\alpha\}$. Of course, there are no other possibilities for l.

Next, we shall consider the cases where $m \geq 3$. From Theorem 1, we immediately obtain the following.

Corollary: For $m = 3, 4, \ldots, (\Delta_m, \Delta_{m+1}, \ldots, \Delta_{l-m+1})$ is a palindrome for

$$l \in \{\underbrace{1, 2, \ldots, q_1}_{a_1}, \underbrace{q_1+1, 2q_1+1, \ldots, q_2}_{a_2}, \underbrace{q_2+q_1, 2q_2+q_1, \ldots, q_3}_{a_3}, \ldots,$$

$$\underbrace{q_{n-1}+q_{n-2}, 2q_{n-1}+q_{n-2}, \ldots, q_n}_{a_n}, \ldots\}$$

with $l \geq 2m-1$.

However, this does not necessarily show all the palindromes. If $\{k\alpha\} + \{(l-k)\alpha\}$ is invariant of k just for $k = m-1, m, \ldots, \lfloor (l+1)/2 \rfloor$, $(\Delta_m, \Delta_{m+1}, \ldots, \Delta_{l-m+1})$ already becomes a palindrome. For example, when $m=3$, all the palindromes are described as follows.

Theorem 4: $(\Delta_3, \Delta_4, \ldots, \Delta_{l-2})$ is a palindrome only for

$$l \in \{\underbrace{1, 2, \ldots, q_1}_{a_1}, \underbrace{q_1+1, 2q_1+1, \ldots, q_2}_{a_2}, \underbrace{q_2+q_1, 2q_2+q_1, \ldots, q_3}_{a_3}, \ldots,$$

$$\underbrace{q_{n-1}+q_{n-2}, 2q_{n-1}+q_{n-2}, \ldots, q_n}_{a_n}, \ldots\}$$

with $l \geq 5$, or

$$l = q_1 + 2 \text{ if } a_1 \geq 3; \quad l = q_2 + 2 \text{ if } a_1 = 1 \text{ and } a_2 \leq 2.$$

Proof: Let n be even. By Lemma 2, if $\{\alpha\} + \{(q-1)\alpha\} = \{q\alpha\}$ and, for $k = 2, 3, \dots, q-2$, $\{k\alpha\} + \{(q-k)\alpha\} = \{q\alpha\} + 1$, then $(\Delta_3, \Delta_4, \dots, \Delta_{q-2})$ is a palindrome. Therefore, $\{\alpha\} < \{q\alpha\}$ or $\{(q-1)\alpha\} < \{q\alpha\}$, and $\{k\alpha\} > \{q\alpha\}$ $(k = 2, 3, \dots, q-2)$.

If $q < q_1$, this is clearly impossible.

If $q_{n+1} < q < q_{n,1}$, then, by the Sublemma, $\{q_n\alpha\} < \{q\alpha\} < \{q_{n+1}\alpha\}$. So, $q_n = 1$ or $q_n = q-1$. But $q_n = 1$ is impossible because $q \geq 5$. The case $q = q_n + 1$ does not satisfy $q > q_{n+1}$.

If $q_{n,r-1} < q < q_{n,r}$ for some integers n and $r \geq 2$, then, by the Sublemma, $\{q_{n,r-1}\alpha\} < \{q\alpha\} < \{q_{n,r}\alpha\}$. So, $q_{n,r-1} = 1$ or $q_{n,r-1} = q-1$. But $q_{n,r-1} = 1$ is impossible because $q \geq 5$. Suppose that $q_{n,r-1} = q-1$. Since

$$\{q_{n,r-1}\alpha\} < \{((r-2)q_{n+1} + q_n)\alpha\} < \{(q_{n,r-1}+1)\alpha\} = \{q\alpha\},$$

we must have $(r-2)q_{n+1} + q_n = 1$, yielding $r = 2$. Hence, $n = 0$. Similarly, we have $n = 1$ and $a_1 = 1$ when n is odd. Therefore, $q = q_{0,1} + 1 = q_1 + 2$ if $a_1 \geq 3$; $q = q_{1,1} + 1 = q_2 + 2$ if $a_1 = 1$ and $a_2 \geq 2$.

But it is not so easy to describe all the palindromes for general $m \geq 3$. It is convenient to use the following Lemma to find the extra palindromes in addition to those appearing in the Corollary.

Lemma 3: Let $q \neq q_{n,r}$ for any integers n and r. Suppose that the sequence $\{\alpha\}$, $\{2\alpha\}$, ..., $\{q\alpha\}$ is sorted as

$$\{u_1\alpha\} < \{u_2\alpha\} < \cdots < \{u_k\alpha\} < \{q\alpha\} < \{u_{k+1}\alpha\} < \cdots < \{u_{q-1}\alpha\},$$

where $\{u_1, u_2, \dots, u_k, u_{k+1}, \dots, u_{q-1}\} = \{1, 2, \dots, q-1\}$. Put

$$M = \max_{i \leq j \leq k} \min(u_j, q-u_j) \quad \text{and} \quad M' = \max_{k+1 \leq j \leq q-1} \min(u_j, q-u_j).$$

If $q \geq 2M+3$, then $(\Delta_m, \dots, \Delta_{q-m+1})$ is palindromic with $m = M+2, M+3, \dots, \lfloor (q+1)/2 \rfloor$.

If $q \geq 2M'+3$, then $(\Delta_m, \dots, \Delta_{q-m+1})$ is palindromic with $m = M'+2, M'+3, \dots, \lfloor (q+1)/2 \rfloor$.

Remark: The conditions $q \geq 2M+3$ and $q \geq 2M'+3$ do not hold simultaneously. For, either $M = q/2$ or $M' = q/2$ when q is even; either $M = (q-1)/2$ or $M' = (q-1)/2$ when q is odd. It is possible that both conditions fail for some q's.

Proof: First of all, notice that $\{k\alpha\}$ and $\{(q-k)\alpha\}$ lie on the same side of $\{q\alpha\}$. If $\{k\alpha\} < \{q\alpha\} < \{(q-k)\alpha\}$, then $\{q\alpha\} < \{k\alpha\} + \{(q-k)\alpha\} < \{q\alpha\} + 1$, yielding a contradiction because $\{k\alpha\} + \{(q-k)\alpha\}$ must be either $\{q\alpha\}$ or $\{q\alpha\} + 1$. Now, since $\{M\alpha\} < \{q\alpha\} < \{k\alpha\}$ $(k = M+1, M+2, \dots, \lfloor (q+1)/2 \rfloor$, we have

$$\{M\alpha\} + \{(q-M)\alpha\} < \{q\alpha\} + 1 \quad \text{and}$$
$$\{k\alpha\} + \{(q-k)\alpha\} > \{q\alpha\} \quad (k = M+1, M+2, \dots, \lfloor (q+1)/2 \rfloor),$$

yielding

$$\{M\alpha\} + \{(q-M)\alpha\} = \{q\alpha\} \quad \text{and}$$
$$\{k\alpha\} + \{(q-k)\alpha\} = \{q\alpha\} + 1 \quad (k = M+1, M+2, \dots, \lfloor (q+1)/2 \rfloor).$$

Together with Lemma 1 we have the desired result. The proof for M' is similar and is omitted here.

Example 2: Let $\alpha = (\sqrt{29}+5)/2 = [5; 5, 5, 5, \ldots]$. Then the sequence $\{\alpha\}$, $\{2\alpha\}$, ..., $\{483\alpha\}$ is sorted as

$$\begin{aligned}\{431\alpha\} &< \{296\alpha\} < \{161\alpha\} < \{26\alpha\} < \{457\alpha\} < \{322\alpha\}\\ &< \{187\alpha\} < \{52\alpha\} < \{483\alpha\} < \underbrace{\cdots\cdots\cdots}_{\text{all the others}} < \{462\alpha\}\\ &< \{327\alpha\} < \{192\alpha\} < \{57\alpha\} < \{353\alpha\} < \{218\alpha\} < \{83\alpha\}\\ &< \{379\alpha\} < \{244\alpha\} < \{109\alpha\} < \{405\alpha\} < \{270\alpha\} < 135\alpha\}.\end{aligned}$$

When $q = 483$, $M = \max(52, 187, 161, 26) = 187$, and $q \geq 2M+3$. By Lemma 3, $(\Delta_m, \Delta_{m+1}, \Delta_{q-m+1})$ is palindromic for $q = 483$ with $m = 189, 190, \ldots, 242$ only. Of course, $M' = 241$ does not satisfy the condition $q \geq 2M'+3$.

When $q = 462$, $M = \max(135, 192, 57, 109, 218, 83) = 218$, and $q \geq 2M+3$. By Lemma 3, $(\Delta_m, \Delta_{m+1}, \Delta_{q-m+1})$ is palindromic only for $q = 462$ with $m = 220, 221, \ldots, 231$.

HOW TO FIND M OR M' IN LEMMA 3

Lemma 3 shows that once M or M' is given for an arbitrary positive integer q with $q \neq q_{n,r}$, all the palindromes $(\Delta_m, \ldots, \Delta_{q-m+1})$ can be discovered without omission. It is, however, tiresome to sort the sequence $\{\alpha\}, \{2\alpha\}, \ldots, \{q\alpha\}$ as seen in Example 2. In fact, M or M' can be determined without any real sorting.

Consider the general integer q with $q \neq q_{n,i}$ for arbitrary integers n and i. For example, put $q = rq_{n+1} + jq_n$ $(r = 1, 2, \ldots, a_{n+2};\ j = 2, 3, \ldots, a_{n+1})$. Then, since

$$\begin{aligned}&\{(rq_{n+1}+q_n)\alpha\} < \cdots < \{(q_{n+1}+q_n)\alpha\} < \{q_n\alpha\}\\ &< \{(rq_{n+1}+2q_n)\alpha\} < \cdots < \{(q_{n+1}+2q_n)\alpha\} < \{2q_n\alpha\} < \cdots\\ &< \{(rq_{n+1}+jq_n)\alpha\} < \cdots < \{(q_{n+1}+jq_n)\alpha\} < \{jq_n\alpha\} < \cdots\end{aligned}$$

when n is even (the order is reversed, and M' replaces M, when n is odd; cf. [5]). M in Lemma 3 can be determined by

$$M = \begin{cases}(r-1)q_{n+1}/2 + (j-1)q_n & \text{if } r \text{ is odd,}\\ (rq_{n+1} + (j-1)q_n)/2 & \text{if } r \text{ is even and } j \text{ is odd,}\\ (rq_{n+1} + jq_n)/2 & \text{if } r \text{ is even and } j \text{ is even.}\end{cases}$$

The condition in Lemma 3, $q \geq 2M+3$, is satisfied if $q_{n+1} \geq (j-2)q_n + 3$ (r: odd); $q_n \geq 3$ (r: even, j: odd). But this condition is never satisfied if r is even and j is even.

Similarly, for $q = rq_{n+1} + jq_n - iq_{n-1}$ $(r = 1, 2, \ldots, a_{n-2};\ j = 2, 3, \ldots, a_{n+1};\ i = 1, 2, \ldots, a_n)$, we have

$$M = \begin{cases}(rq_{n+1} + jq_n - (i+1)q_{n-1})/2 & \text{if } r\text{: odd, } j + a_{n+1} \equiv 0 \pmod 2,\ i\text{: even;}\\ (rq_{n+1} + jq_n - iq_{n-1})/2 & \text{if } r\text{: odd, } j + a_{n+1} \equiv 0 \pmod 2,\ i\text{: odd;}\\ (rq_{n+1} + (j-1)q_n - q_{n-1})/2 & \text{if } r\text{: odd, } j + a_{n+1} \equiv 1 \pmod 2;\\ (rq_{n+1} + (j-1)q_n)/2 & \text{if } r\text{: even, } j\text{: odd;}\\ (rq_{n+1} + jq_n - iq_{n-1})/2 & \text{if } r\text{: even, } j\text{: even, } i\text{: even;}\\ (rq_{n+1} + jq_n - (i+1)q_{n-1})/2 & \text{if } r\text{: even, } j\text{: even, } i\text{: odd.}\end{cases}$$

And the condition $q \geq 2M+3$ is satisfied when

$$\begin{cases} q_{n-1} \geq 3 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: even;} \\ \text{never satisfied} & \text{if } r\text{: odd, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: odd;} \\ q_n \geq (i-1)q_{n-1}+3 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 1 \pmod 2; \\ q_n \geq iq_{n-1}+3 & \text{if } r\text{: even, } j\text{: odd;} \\ \text{never satisfied} & \text{if } r\text{: even, } j\text{: even, } i\text{: even;} \\ q_{n-1} \geq 3 & \text{if } r\text{: even, } j\text{: even, } i\text{: odd.} \end{cases}$$

Next, put $q = rq_{n+1} - jq_n$ $(r = 1, 2, \ldots, a_{n+2}+1;\ j = 0, 1, \ldots, a_{n+1})$. Then M in Lemma 3 can be determined by

$$M = \begin{cases} (r-1)q_{n+1}/2 & \text{if } r \text{ is odd,} \\ (rq_{n+1}-(j+1)q_n)/2 & \text{if } r \text{ is even and } j \text{ is odd,} \\ (rq_{n+1}-jq_n)/2 & \text{if } r \text{ is even and } j \text{ is even,} \end{cases}$$

because

$$\begin{aligned} &\{q_{n+1}\alpha\} < \{2q_{n+1}\alpha\} < \cdots < \{rq_{n+1}\alpha\} \\ &< \{(q_{n+1}-q_n)\alpha\} < \{(2q_{n+1}-q_n)\alpha\} < \cdots < \{(rq_{n+1}-q_n)\alpha\} \\ &< \{(q_{n+1}-2q_n)\alpha\} < \{(2q_{n+1}-2q_n)\alpha\} < \cdots < \{(rq_{n+1}-2q_n)\alpha\} < \cdots \\ &< \{(q_{n+1}-jq_n)\alpha\} < \{(2q_{n+1}-jq_n)\alpha\} < \cdots < \{(rq_{n+1}-jq_n)\alpha\} < \cdots \end{aligned}$$

when n is odd (the order is reversed, and M' replaces M, when n is even).

The condition $q \geq 2M+3$ is satisfied when

$$\begin{cases} q_{n+1} \geq jq_n+3 & \text{if } r \text{ is odd,} \\ q_n \geq 3 & \text{if } r \text{ is even and } j \text{ is odd,} \\ \text{never satisfied} & \text{if } r \text{ is even and } j \text{ is even.} \end{cases}$$

Similarly, for $q = rq_{n+1} - jq_n + iq_{n-1}$ $(r = 1, 2, \ldots, a_{n+2}+1;\ j = 0, 1, \ldots, a_{n+1};\ i = 0, 1, \ldots, a_n)$, we have

$$M = \begin{cases} (rq_{n+1}-jq_n+(i-1)q_{n-1})/2 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: even;} \\ (rq_{n+1}-jq_n+iq_{n-1})/2 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: odd;} \\ (rq_{n+1}-(j+1)q_n+(2i-1)q_{n-1})/2 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 1 \pmod 2; \\ (rq_{n+1}-(j+1)q_n+2iq_{n-1})/2 & \text{if } r\text{: even, } j\text{: odd;} \\ (rq_{n+1}-jq_n+iq_{n-1})/2 & \text{if } r\text{: even, } j\text{: even, } i\text{: even;} \\ (rq_{n+1}-jq_n+(i-1)q_{n-1})/2 & \text{if } r\text{: even, } j\text{: even, } i\text{: odd.} \end{cases}$$

The condition $q \geq 2M+3$ is satisfied when

$$\begin{cases} q_{n-1} \geq 3 & \text{if } r\text{: even, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: even;} \\ \text{never satisfied} & \text{if } r\text{: even, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: odd;} \\ q_n \geq (i-1)q_{n-1}+3 & \text{if } r\text{: even, } j+a_{n+1} \equiv 1 \pmod 2; \\ q_n \geq iq_{n-1}+3 & \text{if } r\text{: odd, } j\text{: odd;} \\ \text{never satisfied} & \text{if } r\text{: odd, } j\text{: even, } i\text{: even;} \\ q_{n-1} \geq 3 & \text{if } r\text{: odd, } j\text{: even, } i\text{: odd.} \end{cases}$$

Generally speaking, M (or M') in Lemma 3 can be determined as follows.

Lemma 4: If $M = (\cdots - uq_{N+1} + (s-1)q_N)/2$ for $q = \cdots - uq_{N+1} + sq_N$, then $M = (\cdots - uq_{N+1} + (s-1)q_N)/2$ for $q = \cdots - uq_{N+1} + sq_N - tq_{N-1}$ $(t = 1, 2, \ldots, a_N)$.

If $M = (\cdots - uq_{N+1} + sq_N)/2$ for $q = \cdots - uq_{N+1} + sq_N$, then

$$M=\begin{cases}(\cdots-uq_{N+1}+sq_N-tq_{N-1})/2 & \text{if } t \text{ is even,}\\ (\cdots-uq_{N+1}+sq_N-(t+1)q_{N-1})/2 & \text{if } t \text{ is odd,}\end{cases}$$

for $q=\cdots-uq_{N+1}+sq_N-tq_{N-1}$.

If $M=(\cdots-(u+1)q_{N+1}+(2s-1)q_N)/2$ for $q=\cdots-uq_{N+1}+sq_N$, then

$$M=\begin{cases}(\cdots-uq_{N+1}+sq_N-tq_{N-1})/2 & \text{if } s+a_{N+1}\equiv 1 \pmod 2 \text{ and } t \text{ is odd,}\\ (\cdots-uq_{N+1}+sq_N-(t+1)q_{N-1})/2 & \text{if } s+a_{N+1}\equiv 1 \pmod 2 \text{ and } t \text{ is even,}\\ (\cdots-uq_{N+1}+(s-1)q_N-q_{N-1})/2 & \text{if } s+a_{N+1}\equiv 0 \pmod 2,\end{cases}$$

for $q=\cdots-uq_{N+1}+sq_N-tq_{N-1}$.

If $M=(\cdots-(u+1)q_{N+1}+2sq_N)/2$ for $q=\cdots-uq_{N+1}+sq_N$, then

$$M=\begin{cases}(\cdots-uq_{N+1}+sq_N-tq_{N-1})/2 & \text{if } s+a_{N+1}\equiv 0 \pmod 2 \text{ and } t \text{ is odd,}\\ (\cdots-uq_{N+1}+sq_N-(t+1)q_{N-1})/2 & \text{if } s+a_{N+1}\equiv 0 \pmod 2 \text{ and } t \text{ is even,}\\ (\cdots-uq_{N+1}+(s-1)q_N-q_{N-1})/2 & \text{if } s+a_{N+1}\equiv 1 \pmod 2,\end{cases}$$

for $q=\cdots-uq_{N+1}+sq_N-tq_{N-1}$.

Lemma 4': If $M=(\cdots+uq_{N+1}-(s+1)q_N)/2$ for $q=\cdots+uq_{N+1}-sq_N$, then $M=(\cdots+uq_{N+1}-(s+1)q_N+2tq_{N-1})/2$ for $q=\cdots+uq_{N+1}-sq_N+tq_{N-1}$ $(t=1,2,\ldots,a_N)$.

If $M=(\cdots+uq_{N+1}-sq_N)/2$ for $q=\cdots+uq_{N+1}-sq_N$, then

$$M=\begin{cases}(\cdots+uq_{N+1}-sq_N+tq_{N-1})/2 & \text{if } t \text{ is even,}\\ (\cdots+uq_{N+1}-sq_N+(t-1)q_{N-1})/2 & \text{if } t \text{ is odd,}\end{cases}$$

for $q=\cdots+uq_{N+1}-sq_N+tq_{N-1}$.

If $M=(\cdots+(u-1)q_{N+1}-q_N)/2$ for $q=\cdots+uq_{N+1}-sq_N$, then

$$M=\begin{cases}(\cdots+uq_{N+1}-sq_N+tq_{N-1})/2 & \text{if } s+a_{N+1}\equiv 1 \pmod 2 \text{ and } t \text{ is odd,}\\ (\cdots+uq_{N+1}-sq_N+(t-1)q_{N-1})/2 & \text{if } s+a_{N+1}\equiv 1 \pmod 2 \text{ and } t \text{ is even,}\\ (\cdots+uq_{N+1}-(s+1)q_N+(2t-1)q_{N-1})/2 & \text{if } s+a_{N+1}\equiv 0 \pmod 2,\end{cases}$$

for $q=\cdots+uq_{N+1}-sq_N+tq_{N-1}$.

If $M=(\cdots+(u-1)q_{N+1})/2$ for $q=\cdots+uq_{N+1}-sq_N$, then

$$M=\begin{cases}(\cdots+uq_{N+1}-sq_N+tq_{N-1})/2 & \text{if } s+a_{N+1}\equiv 0 \pmod 2 \text{ and } t \text{ is odd,}\\ (\cdots+uq_{N+1}-sq_N+(t-1)q_{N-1})/2 & \text{if } s+a_{N+1}\equiv 0 \pmod 2 \text{ and } t \text{ is even,}\\ (\cdots+uq_{N+1}-(s+1)q_N+(2t-1)q_{N-1})/2 & \text{if } s+a_{N+1}\equiv 1 \pmod 2,\end{cases}$$

for$q=\cdots+uq_{N+1}-sq_N+tq_{N-1}$.

Example 3: There is a reason for our providing two alternative expressions for each integer q. For instance, let

$$\alpha=\frac{\sqrt{29}+5}{2}=[5;5,5,5,\ldots].$$

For $q=3q_2-q_1+4q_0=3\cdot 26-5+4\cdot 1=77$, we have $M=(3q_2-q_1+3q_0)/2=38$, not satisfying $q\geq 2M+3$. However, for $q=2q_2+5q_1=77$, we obtain $M'=(2q_2+4q_1)/2=36$, satisfying $q\geq 2M'+3$ and leading to the conclusion that $(\Delta_m,\ldots,\Delta_{q-m+1})$ is palindromic for $q=77$ with $m=38$ and 39.

SUMMARY

When $q = q_{n,r}$, the palindromic sequences $(\Delta_m, \ldots, \Delta_{q-m+1})$ can be found by Theorem 1, 2, 3, 4, or the Corollary. When $q \neq q_{n,r}$, all the other palindromes can be discovered by Lemma 3 with Lemma 4 and Lemma 4$'$.

REFERENCES

1. X. Droubay. "Palindromes in the Fibonacci Word." *Inform. Process. Lett.* **55** (1995):217-221.
2. C. Kimberling. "Palindromic Sequences from Irrational Numbers." *The Fibonacci Quarterly* **36.2** (1998):171-73.
3. S. Lang. *Introduction to Diophantine Approximations*. New expanded ed. New York: Springer-Verlag, 1995.
4. A. de Luca & F. Mignosi. "Some Combinatorial Properties of Sturmian Word." *Theoret. Comp. Sci.* **136**:361-85.
5. T. van Ravenstein. "The Three Gap Theorem (Steinhaus Conjecture)." *J. Austral. Math. Soc.* (Ser. A) **45** (1988):360-70.

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THE FIRST 330 TERMS OF SEQUENCE A013583

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1. INTRODUCTION

Let $R(N)$ be the number of representations of the natural number N as the sum of distinct Fibonacci numbers. The values of $R(N)$ are well recognized as the coefficient of x^N in the infinite product $\prod_{i=2}^{\infty}(1+x^{F_i})=(1+x)(1+x^2)(1+x^3)(1+x^5)\cdots=$

$$1+\mathbf{1x^1}+x^2+\mathbf{2x^3}+x^4+2x^5+2x^6+x^7+\mathbf{3x^8}+2x^9+2x^{10}+\cdots. \tag{1}$$

Combinatorially, each term $R(N)x^N$ counts the $R(N)$ partitions of N into distinct Fibonacci numbers. Some of the recursion properties of this sequence are investigated in [2]. The difficulties in producing this sequence are more computational than analytic in that usual generation methods quickly consume computer resources.

Our major interest is in the related sequence 1, 3, 8, 16, 24, 37, ..., $A_{n,\dots}$ whose n^{th} term is the least N such that $n=R(N)$, emphasized in boldface in (1) above. The general term of this sequence (see [8]), designated A013583 in Sloane's' on-line database of sequences, is still unknown. The 330 terms found in this note almost triple the 112 terms reported by Shallit [8]. Carlitz [3, 4], Klarner [7], and Hoggatt [4, 6], among others, have studied the representation of integers as sums of Fibonacci numbers and particularly Zeckendorf representations. The Zeckendorf representation of a natural number N uses only positive-subscripted, distinct, and non-consecutive Fibonacci numbers and is unique. We have used the Zeckendorf representation of N to write $R(N)$ in [1] and [2].

2. A PEEK AT A013583 FIRST

Let us begin by listing the terms of A013583 that we have computed. We will note very quickly why this sequence is so intractable. Table 1 lists 46 complete rows with 10 entries per row. The first 33 rows have no missing sequence terms; hence, 330 complete sequence terms. The first missing entry appears in the 34^{th} row as the yet unknown 331^{st} term. While there are necessarily missing terms in at least some of the remaining rows, there are also many useful calculated sequence terms. Our computer output concluded with a partial 47^{th} row with 5 unknown entries followed by the 446^{th} sequence term, 229971.

3. SOME OBSERVED AND COMPUTATIONAL PROPERTIES OF $\prod_{i=2}^{\infty}(1+x^{F_i})$

When $\prod_{i=2}^{\infty}(1+x^{F_i})$ is expanded, the terms are partitioned according to sets of palindromically arranged, successive $R(N)$ coefficients. For this reason, we refer to it as the *palindromic sequence*. The first few terms are given in (2) below.

TABLE 1. Terms of Sequence A013583 (Index 1 through 330 complete)

1	3	8	16	24	37	58	63	97	105
152	160	168	249	257	270	406	401	435	448
440	647	1011	673	723	715	1066	1058	1050	1092
1160	1147	1694	1155	1710	1702	2647	1846	1765	1854
2736	1867	2757	2744	2841	2990	2752	2854	2985	3019
4511	3032	6967	4456	3024	4477	4616	4451	7349	4629
7218	4917	4621	4854	4904	7179	7166	4896	7200	7247
7310	7213	7831	8187	7488	7205	11614	7480	7815	7857
7925	11593	18154	7912	11813	11682	11653	11619	7920	11669
11724	12669	12106	11661	12656	12093	18151	12648	18795	12792
19154	12101	20358	12711	12800	19099	20756	18761	18850	12813
18905	18871	46913	19557	19138	18858	19476	31134	20502	19565
20701	30579	18866	20832	21018	19578	47434	20463	20696	20777
20730	30414	30689	30359	30977	20743	47418	30503	47507	30702
30529	30969	30422	20735	30511	33176	30694	34684	47795	31676
53712	30524	49104	49201	33705	31689	47523	33108	49405	33286
49502	33574	49159	49112	31681	50091	49358	33278	33616	33561
50002	49489	53683	49366	49120	49222	49408	33553	49497	49434
49387	49667	53534	53670	53589	50107	54178	49400	50989	53615
54555	51222	56152	54521	51272	53581	124519	79607	49392	53856
79481	81141	79874	51264	79463	86241	53573	53848	54327	54225
124506	54293	81078	87927	80073	79476	80366	79853	82856	54280
80971	80086	131203	79942	82513	124433	124378	79913	124522	81073
79646	79879	54288	79984	79929	129221	82840	80361	129292	82882
125132	87694	82950	129538	86694	79921	86749	87131	131897	87681
129551	82937	128614	124417	130999	86686	128593	87673	142699	88016
129242	87817	128703	128831	129224	130004	82945	128606	129546	129402
129347	87126	87736	131177	87825	129216	130910	201246	133499	130012
142877	129326	128598	131190	134049	128873	129208	87838	129352	129250
201306	140539	129318	130025	140154	146927	140243	202466	142882	134185
131182	140298	142238	129305	140264	133494	142848	141861	216776	142780
140531	134104	141895	201314	134193	140251	142094	209286	208414	140958
217174	129313	211980	142225	142411	208244	209058	208037	209252	134206
______	212192	209668	209087	140259	140971	141856	142089	______	142170
208524	209396	142123	209634	209074	227408	______	209121	______	208079
212323	208511	______	209299	212268	142136	211985	209676	______	209409
227107	209210	217119	210396	227395	226777	______	227345	209236	227010
______	212260	______	208519	217436	209257	142128	209401	______	209218
______	231102	______	216881	210388	237867	217114	230958	______	217195
______	226968	217148	______	226929	______	______	209231	227112	228094
______	______	230416	______	230123	______	______	217161	______	______
______	______	______	226942	______	228107	______	230136	______	229704
______	______	______	229992	______	______	______	______	______	______
______	217153	______	______	______	______	______	______	______	228099
229696	230034	______	______	229979	______	______	______	______	______
______	______	______	______	______	______	______	______	______	______
______	______	______	______	______	229971				

$$\begin{aligned}&[1x^1]+x^2+[2x^3]+x^4+[2x^5+2x^6]+x^7+[3x^8+2x^9+2x^{10}+3x^{11}]+x^{12}\\&+[3x^{13}+3x^{14}+2x^{15}+4x^{16}+2x^{17}+3x^{18}+3x^{19}]+x^{20}+[4x^{21}+3x^{22}+3x^{23}\\&+5x^{24}+2x^{25}+4x^{26}+4x^{27}+2x^{28}+5x^{29}+3x^{30}+3x^{31}+4x^{32}]+x^{33}+[4x^{34}\\&+4x^{35}+3x^{36}+6x^{37}+3x^{38}+5x^{39}+5x^{40}+2x^{41}+6x^{42}+4x^{43}+4x^{44}+6x^{45}\\&+2x^{46}+5x^{47}+5x^{48}+3x^{49}+6x^{50}+3x^{51}+4x^{52}+4x^{53}]+x^{54}+[5x^{55}+\cdots\end{aligned} \tag{2}$$

Throughout this paper, we use the *floor* symbol $\lfloor x \rfloor$ to denote the **greatest** integer $\le x$ and the *ceiling* symbol $\lceil x \rceil$ to denote the **least** integer $\ge x$.

Square brackets identify coefficient palindromes. Palindromic sections share *external* boundaries of the form $1x^N$, $N = F_n - 1$, consistent with $R(F_n - 1) = 1$ given in [3]. For data-handling and computation, we omitted the overlapping terms with unit coefficients and partitioned the expansion into palindromic sections which we call k-sections. The first term of a k-section is $\lfloor (k+2)/2 \rfloor x^N$, $N = F_{k+2}$, and the last term is $\lfloor (k+2)/2 \rfloor x^N$, $N = F_{k+3} - 2$. In (2), observe coefficients [3 2 2 3] (for $k = 4$) starting with x^8.

Since the second half of a k-section adds no new coefficients but merely repeats those of the first half in reverse order, we use $\frac{1}{2}k$-sections. If the number of terms is odd, we include the center term, which becomes the last term of the $\frac{1}{2}k$-section. The coefficient of the last term of the $\frac{1}{2}k$-section is always a power of 2.

The value of these central coefficients can be established using identity (3), which can be proved using mathematical induction.

$$\sum_{i=1}^{p} F_{3i+1} = F_{3p+1} + F_{3p-2} + \cdots + F_7 + F_4 = (F_{3p+3} - 2)/2. \tag{3}$$

Thus,

$$R\left(\sum_{i=1}^{p} F_{3i+1}\right) = 2^{p-1} R(F_4) = 2^p = R((F_{3p+3} - 2)/2) \tag{4}$$

by repeatedly applying $R(F_{n+3} + K) = 2R(K)$, $F_n \le K < F_{n+1}$, and $R(F_4) = 2$ from [2].

Take $k = 3p - 1$. The powers of x on the left and right internal boundaries of the k-section become F_{3p+1} and $F_{3p+2} - 2$, and the central term has exponent $(F_{3p+1} + F_{3p+2} - 2)/2 = (F_{3p+3} - 2)/2$, which is an integer since $2 \mid F_{3p}$, and the coefficient is 2^p by (3) and (4).

Next, take $k = 3p$. The central pair of terms have exponents of x given by $(F_{3p+4} - 2 - 1)/2$ and $(F_{3p+4} - 2 + 1)/2$, which are integers since F_{3p+4} is odd. We can establish the values of A and B below by mathematical induction:

$$F_{3p+2} + F_{3p-1} + \cdots + F_8 + F_5 = (F_{3p+4} - 3)/2 = A, \tag{5}$$

$$F_{3p+2} + F_{3p-1} + \cdots + F_8 + F_5 + F_2 = (F_{3p+4} - 1)/2 = B. \tag{6}$$

By again applying $R(F_{n+3} + K) = 2R(K)$ and $R(F_4) = 2$ to (5) and (6),

$$R(A) = 2^{p-1} R(F_5) = 2^{p-1}(2) = 2^p, \tag{7}$$

$$R(B) = 2^p R(F_2) = 2^p(1) = 2^p, \tag{8}$$

so that $R(A) = R(B) = 2^p$.

In the same way, when $k = 3p + 1$, the two central terms have equal coefficients given by 2^p. This establishes $2^{\lfloor \frac{k+1}{3} \rfloor}$ as the coefficient of the right boundary of a $\frac{1}{2}k$-section for all k.

Also from [2], we can apply $R(F_n) = [n/2] = R(F_{n+1} - 2)$ to the first and last terms of the bracketed palindromic sequences, and $R(F_n - 1) = 1$ explains the overlapping external boundaries of the k-sections.

The only practical way available at present to find the j^{th} term of A013583 is to search for the *first* appearance of j as a coefficient in the palindromic sequence and to record the corresponding exponent of x as the j^{th} term in A013583. Table 2 lists numerical properties of k-sections useful for setting up and checking our computational procedures.

TABLE 2. Numerical Parameters of Palindrome Sequence ($1 \le k \le 26$)

1	2	3	4	5	6	7	8	9	10	11
1		1	2	1	2	1	2	2	0	0
2		2	3	2	3	2	3	4	1	1*
3	4	2	5	2	5	2	6	7	2	1
4	7	3	8	2	9	3	11	12	4	2
5	12	3	13	4	16	3	19	20	7	4*
6	20	4	21	4	26	4	32	33	12	6
7	33	4	34	4	43	4	53	54	20	10
8	54	5	55	8	71	5	87	88	33	17*
9	88	5	89	8	115	5	142	143	54	27
10	143	6	144	8	187	6	231	232	88	44
11	232	6	233	16	304	6	375	376	143	72*
12	376	7	377	16	492	7	608	609	232	166
13	609	7	610	16	797	7	985	986	376	188
14	986	8	987	32	1291	8	1595	1596	609	305*
15	1596	8	1597	32	2089	8	2582	2583	986	493
16	2583	9	2584	32	3381	9	4179	4180	1596	798
17	4180	9	4181	64	5472	9	6763	6764	2583	1292*
18	6764	10	6765	64	8854	10	10944	10945	4180	2090
19	10945	10	10946	64	14327	10	17709	17710	6764	3382
20	17710	11	17711	128	23183	11	28655	28656	10945	5973*
21	28656	11	28657	128	37511	11	46366	46367	17710	8855
22	46367	12	46368	128	60695	12	75023	75024	28656	14328
23	75024	12	75025	256	98208	12	121391	121392	46367	23184*
24	121392	13	121393	256	158904	13	196416	196417	75024	37512
25	196417	13	196418	256	257113	13	317809	317810	121392	60696
26	317810	14	317811	512	416019	14	514227	514228	196417	98209

1	Value of k.	k
2	Power of x of left <u>external</u> boundary of k- or $\frac{1}{2}k$-sections.	$F_{k+2} - 1$
3	Integer coefficient of left <u>interior</u> boundary of k- or $\frac{1}{2}k$-sections.	$\lfloor \frac{k+2}{2} \rfloor$
4	Power of x of left interior boundary of k- or $\frac{1}{2}k$-sections.	F_{k+2}
5	Integer coefficient of right boundary of $\frac{1}{2}k$-section.	$2^{\lfloor \frac{k+1}{3} \rfloor}$
6	Power of x of right interior boundary of $\frac{1}{2}k$-section.	$\lfloor \frac{F_{k+4}-2}{2} \rfloor$
7	Integer coefficient of right interior boundary of k-section.	$\lfloor \frac{k+2}{2} \rfloor$
8	Power of x of right interior boundary of k-section.	$F_{k+3} - 2$
9	Power of x of right exterior boundary of k-section.	$F_{k+3} - 1$
10	Number of terms in k-section.	$F_{k+1} - 1$
11	Number of terms in $\frac{1}{2}k$-section. When 10 is odd, * indicates $\frac{1}{2}k$-section ends with unique center term of the k-section.	$\lceil \frac{F_{k+1}-1}{2} \rceil$

4. SUCCESSIVELY BETTER WAYS OF GETTING DATA FROM $\prod_{i=2}^{\infty}(1+x^{F_i})$

For small k-sections we inspected each successive printout by hand to select the first occurrence of each coefficient value. We found the first 112 terms in computing for $k \leq 18$.

For further reduction in data handling we described the entries of $\frac{1}{2}k$-sections as {coefficient, power of x} pairs. Since only the unique {coefficient, smallest power of x for that coefficient} pairs from each $\frac{1}{2}k$-section qualify as potential pairs for A013583, we eliminated all pairs with duplicate "coefficient" portions except that pair with the least power of x. At the same time, the surviving pairs per $\frac{1}{2}k$-section emerge sorted by increasing coefficient size.

As an example, each line of (9) contains $\frac{1}{2}k$-section data, reduced and sorted as suggested. By suppressing the pairs that do not qualify, the highlighted {coefficient, powers of x} pairs for A013583 are immediately evident.

$$
\begin{aligned}
&\{\mathbf{\{1,1\}}\},\ k=1\\
&\{\mathbf{\{2,3\}}\},\ k=2\\
&\{\{2,5\}\},\ k=3\\
&\{\{2,9\},\ \mathbf{\{3,8\}}\},\ k=4\\
&\{\{2,15\},\{3,13\},\ \mathbf{\{4,16\}}\},\ k=5\\
&\{\{2,25\},\{3,22\},\{4,21\},\ \mathbf{\{5,24\}}\},\ k=6\\
&\{\{2,41\},\{3,36\},\{4,34\},\{5,39\},\ \mathbf{\{6,37\}}\},\ k=7\\
&\{\{2,67\},\{3,59\},\{4,56\},\{5,55\},\{6,60\},\ \mathbf{\{7,58\},\{8,63\}}\},\ k=8\\
&\{\{2,109\},\{3,96\},\{4,91\},\{5,89\},\{6,98\},\{7,94\},\{8,92\},\ \mathbf{\{9,97\},\{10,105\}}\},\ k=9
\end{aligned}
\tag{9}
$$

However, we were at the memory limit of our personal computer. We had to find new Fibonacci approaches to continue. When we found a way to let the *indices* of the Fibonacci numbers guide the computations in place of the Fibonacci numbers themselves, we had a fresh start with tremendously reduced computational requirements. The interaction between the Fibonacci numbers and their integer indices here is not the same as the divisibility properties noted in the many past studies of Fibonacci entry points and their periods. We needed formulas developed in [2] relating $R(N)$ to the Zeckendorf representation of N. By looking deeper into the structure of Fibonacci indices, we removed a core of redundancy to speed up and shorten our calculations and developed an improved way of assembling data and discarding duplicate data. We proceeded to calculate the remainder of the 330 terms of A013583 that you see in this paper. Even with our best available computation techniques, described below, we found size and time requirements to be impracticable for calculations beyond $k = 25$.

5. EXPLORING NEW WAYS TO FIND COEFFICIENTS OF k-SECTIONS

Since the combinatorial interpretation of the coefficients of the palindromic sequence is the number of partitions of the power of x into distinct Fibonacci members, we explore that point of view. We will use results of selected numerical examples to imply a general case. In the partial expansion of $\prod_{i=2}^{\infty}(1+x^{F_i})$ in (2), we observe the term $4x^{43}$, which tells us there are 4 partitions of 43 into distinct Fibonacci numbers. As is well known, 43 has the unique Zeckendorf representation, $43 = F_9 + F_6 + F_2 = 34 + 8 + 1$, where we rule out adjacent Fibonacci indices. As additional visual information, $F_9 = 34$ is the power of x of the left boundary of the k-section to which 43 belongs, and $F_{k+2} = F_9 = 34$ is the only Fibonacci power of x in its k-section, thus, $k = 7$. In general, we can represent any power in a k-section by its Zeckendorf representation which starts with F_{k+2}.

In [5], Fielder developed new *Mathematica*-oriented algorithms and programs for calculating and tabulating Zeckendorf representations and calculated the first 12,000 representations. We imbedded the algorithms in our work where needed. Reference [5] and *Mathematica* programs are available from Daniel C. Fielder. The indices in the Zeckendorf representation of an integer N give formulas for finding $R(N)$, as reported in [2]. We next describe how the indices are applied to our computer programs.

We noted earlier that the power of x of the first term of a k-section is not only a predictable Fibonacci number, but is the only power of x in that k-section which is a Fibonacci number. Because of the Fibonacci recursion, $F_{n+2} = F_{n+1} + F_n$, it is very easy to partition any Fibonacci number into distinct Fibonacci members. As an example, we represent the partition of $F_9 = 34$ as successive triangular arrays in (10):

$$\begin{array}{rrrr} & & & 34 \\ & & 13 & 21 \\ & 5 & 8 & 21 \\ 2 & 3 & 8 & 21 \end{array} \qquad \begin{array}{cccc} & & & F_9 \\ & & F_7 & F_8 \\ & F_5 & F_6 & F_8 \\ F_3 & F_4 & F_6 & F_8 \end{array} \qquad \begin{array}{cccc} & & & 9 \\ & & 7 & 8 \\ & 5 & 6 & 8 \\ 3 & 4 & 6 & 8 \end{array} \tag{10}$$

The first array consists of the partition integers, the second consists of the Fibonacci number symbols with subscripts, and the third consists of Fibonacci *indices* only.

The enumeration of sequence subscripts for powers in general involves interaction among the restricted partitions of the several Fibonacci numbers used in the Zeckendorf representation. Computations controlled by the indices of the right array have advantages of symmetry. For example, the left-descending diagonal will always consist of all the consecutive odd or even integers starting with the index of the Fibonacci number to be partitioned. (Recall that we do not admit a 1 index.) Once the diagonal of odd (or even) indices is in place, the remaining column lower entries are all one less than their diagonal entry. The number of restricted partitions is the floor of half the largest index. In the example, $\lfloor \frac{9}{2} \rfloor = 4$ partitions. If the power of x were the single F_{k+2}, the number of partitions and, thereby, the coefficient would be $\lfloor \frac{k+2}{2} \rfloor$.

When we consider our example $43 = F_9 + F_6 + F_2$, we represent the individual partitions as three triangles of Fibonacci indices with the Zeckendorf Fibonacci indices as apexes. (The order from low to high is a computational preference.)

$$2 \qquad \begin{array}{ccc} & & 6 \\ & 4 & 5 \\ 2 & 3 & 5 \end{array} \qquad \begin{array}{cccc} & & & 9 \\ & & 7 & 8 \\ & 5 & 6 & 8 \\ 3 & 4 & 6 & 8 \end{array} \tag{11}$$

By distributing each set of rows over the others, 12 sets of indices are found as the *Mathematica* string:

$$\begin{array}{l} \{9,6,2\}, \{9,4,5,2\}, \{9,2,3,5,2\}, \{7,8,6,2\}, \{7,8,4,5,2\}, \{7,8,2,3,5,2\}, \\ \{5,6,8,6,2\}, \{5,6,8,4,5,2\}, \{5,6,8,2,3,5,2\}, \{3,4,6,8,6,2\}, \\ \{3,4,6,8,4,5,2\}, \{3,4,6,8,2,3,5,2\}. \end{array} \tag{12}$$

Each set of indices of (12) evokes a partition of 43 into Fibonacci numbers having those indices. There are $1\times 3\times 4 = 12$ such partitions. Thus, we can use Zeckendorf representations both to count and to name partitions consisting solely of nonzero Fibonacci numbers. The results in (12) also suggest a very simple way to find the coefficients of the expansion $\prod_{i=2}^{\infty}[1/(1-x^{F_i})]$.

Our first computational improvement over direct expansion of (1) is given by our *Mathematica* program 10229601.ma. This program accepts 43, the power of x, and returns the coefficient 4 by using the equivalent of (11) to find (12) and then discarding sets with duplicate indices. The 4 sets of indices counted by 10229601.ma in the example are:

$$\{2,6,9\}, \{2,4,5,9\}, \{2,6,7,8\}, \{2,4,5,7,8\} \tag{13}$$

By using 10229601.ma in a loop, selected ranges of powers can be probed for power-coefficient pairs.

As the size of the powers increased, however, even 10229601.ma could not match the demands on it. This is because the distribution of indices in 10229601.ma takes place over all triangles, and memory is not released to be used again until the *end* of the computations. As an improvement, we distributed the index integers over the first two triangles on the right and eliminated sets with repeated integers. We applied this result to the next triangle alone, make the reductions, and repeated the process over the remaining triangles one at a time. The memory and time savings were substantial. In spite of the new computational advantages, the distribution was still over all of each triangle. With full triangle distribution, however, it is possible that there may be partitions with arbitrary length runs of repeating index integers. Since we want to count partitions with *no* repeating members, producing partitions through full distribution is not an optimum strategy.

Our next improvement restricted repeating members to a fixed and predictable limit per partition. We retained our earlier size order of the index triangles and eliminated enough lower rows so that the smallest member of a higher-order triangle is either equal to or just greater than the largest (or apex) member of its immediate lower-order neighbor. For illustration, we show the set of *partial* index triangles obtained from suitable modification of (11):

$$\begin{array}{ccccccccc} 2 & & & & 6 & & & & 9 \\ & & & 4 & 5 & & & 7 & 8 \\ & & 2 & 3 & 5 & & & & \end{array} \tag{14}$$

Now, when distribution is made over all partial triangles, triple or higher repeats of individual integers cannot occur. The only possibility of a repeated integer lies between the least integer of a triangle and the greatest integer of its immediate left neighbor. This means that when repeats occur, there is at most one pair of those integers per partition. In fact, if each Zeckendorf representation index differed from the preceding index by an *odd* integer, there would be **no** repeating partition members, and the distribution operation on the partial triangles would immediately yield the integer indices of the desired set of Fibonacci partitions.

As proved in [2], $R(N)$ can be written immediately by repeatedly applying the formulas:

$$R(F_{n+2k+1}+K) = (k+1)R(K),\ F_n \le K < F_{n+1}, \tag{15}$$

$$R(F_{n+2k}+K) = kR(K) + R(F_{n+1}-K-2). \tag{16}$$

In our example, the distribution and reduction process yield integer sets $\{2,6\}$, $\{2,4,5\}$ from the first two reduced triangles. The process continues to the third partial triangle and produces the final $\{2,6,9\}, \{2,4,5,9\}, \{2,6,7,8\}, \{2,4,5,7,8\}$. Our program 10229601.ma incorporates the concept of partial triangles along with previous improvements. It was the first sufficiently robust program for handling k values of 24 and especially 25, necessary to obtain coefficients from the palindromic sequence to complete the last of the 330 terms of A013583. Next we study the 330 terms from Table 1.

6. THE 330 PAIRS $\{n, A_n\}$ SORTED BY INTERVALS

Returning to Table 1 which lists $\{n, A_n\}$ and also gives all known values of $A_n < F_{28}$, we sort the data by intervals as given in our k-sections. In Table 3 we select all $\{n, A_n\}$ such that $F_m \le A_n < F_{m+1}$ and sort by increasing index values. For consistency with the terminology of [1] and [2], we take $m = k + 2$.

TABLE 3. Indices n for $\{n, A_n\}$ Sorted by Intervals
$F_m \le A_n < F_{m+1}$, $16 \le m \le 27$

$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	Missing values for n (partial list)
16	987	8	23	32	34	1	33
17	1597	7	33	36	42	2	37, 41
18	2584	12	37	50	55	3	51, 53, 54
19	4181	11	51	52	68	5	53, 59, 61, 66, 67
20	6765	19	53	76	89	7	77, 82, 83, 85-88
21	10946	19	77	82	110	9	83, 97, 99, 101, 103, 106-109
22	17711	28	83	112	144	15	113, 118, 122, 127, 132-135, 137-143
23	28657	27	118	112	178	22	113, 127, 137, 139, 149, 151, 153, 154, 157, 159, 161, 163-164, 166-167, 171-177
24	46368	50	113	196	233	27	197, 198, 201-203, 205, 206, 211, 213-219 221-232
25	75025	43	198	196	288	39	197, 211, 223, 226, 227, 229, 236, 239, 241, 244, 249, 251, 253-255, 257, 259, 261, 263-266, 268-271, 274, 276-287
26	121393	76	197	277	377	52	278, 291, 298, 309, 314, 318, 319, 321, 323, 326-329, 331-334, 339, 341, 342, 344-355, 357-376
27	196418	72	278	330	466	69	331, 339, 347, 349, 353, 359, 367, 371, 373, 379, 381, 383, 389, 391, 394, 396, 397, 401, 402, 404, 406, 407, 409-413, 415, 417, 419, 421-423, 425-431, 433-439, 443, 444, 446-465

Column descriptions:

$\boxed{1}$ Value of m which defines the interval.

$\boxed{2}$ F_m

$\boxed{3}$ Number of pairs of $\{n, A_n\}$ in interval.

$\boxed{4}$ Smallest index n appearing in interval.

$\boxed{5}$ Every index n less than or equal to this number has appeared by interval's end.

$\boxed{6}$ Largest index n appearing in interval.

$\boxed{7}$ Number of missing indices less than the largest n in the interval.

Notice that the largest index n in each interval is a Fibonacci number or twice a Fibonacci number. If $m=2p$, the largest index is $n=F_{p+1}$; if $m=2p+1$, $n=2F_p$.

In every k-section that we computed beyond $k=12$, some indices were missing and appear for the first time in a later k-section. However, the "missing values" appear in an orderly way. The indices $n=F_{p+1}-1$ and $n=2F_p-1$ always are missing values for the respective intervals $m=2p$ and $m=2p-1$. (We note in passing that $n=112$ was the highest index available before the disclosures of this paper, and $m=20$ is complete for n through 112.)

Putting this all together, the first appearance of $n=F_{p+1}$ is for $A_n=F_{p+1}^2-1$ in the interval $F_{2p}<A_n<F_{2p+1}$, and the list of indices is complete for $n\le F_p$. The first appearance of $n=2F_p$ is for $A_n=F_{p+3}F_p-1+(-1)^{p+3}$ in the interval $F_{2p+1}\le A_n<F_{2p+2}$, and the indices are complete for $n\le 2F_{p-2}$. The first appearances of F_k and $2F_k$ are discussed in [1].

We notice that, if n is the largest index which appears for A_n in the interval $F_m\le A_n<F_{m+1}$, then the indices $n-1$, $n-2$, $n-3, \ldots, n-(F_{\lceil\frac{m}{2}\rceil-5}-1)$ are missing values.

The values for A_n are not a strictly increasing sequence if sorted by index, as can be seen from Table 1. However, if $F_p<n\le F_{p+1}$, then $F_{2p-1}<A_n<F_{2p+4}$. If n is prime, then $F_{2p}<A_n<F_{2p+1}$ or $F_{2p+2}<A_n<F_{2p+3}$. In fact, if n is prime, the Fibonacci numbers used in the Zeckendorf representation of A_n are all even subscripted.

We found palindromic subsequences and fractal-like recursions in tables of $\{n, A_n\}$. We developed many formulas relating $R(N)$ and the Zeckendorf representation of N, but we still cannot describe a general term for $\{n, A_n\}$. The formulas we developed and the programming data we generated each extended our knowledge while suggesting new approaches. Theory and application worked hand-in-glove throughout this entire project.

7. POSTSCRIPT AND AFTERMATH

After all the 330 consecutive terms and many other nonconsecutive terms of A013583 were calculated and recorded, and much of the paper completed, we stumbled onto a very simple *Mathematica*-implemented algorithm which uses the combinatorial principle of Inclusion-Exclusion to find the coefficients of $\prod_{i=2}^{\infty}(1+x^{F_i})$ for powers of x. While too late to help us gather data for the 330 terms, it provides a reassuring check on the work already completed, and should prove an invaluable aid in our continuing assault on sequence A013583. The *Mathematica* algorithm implementation is many times faster than that used for getting the 330 terms of A013583. Would you believe that a preliminary trial program with the new algorithm verified the coefficient of

$$x^{96153171424047206983355738695915460604026 3}$$

as 147573952589676412928 in 2.62 seconds on a 133-Mhz PowerMac 7200 running *Mathematica*, version 2.2? Table 2 verifies this result because the power of x is that of column $\boxed{6}$ for $k=200$, while the coefficient matches the known value in column $\boxed{5}$ for $k=200$ in Table 2. Our paper describing the algorithm and its application has been reviewed and accepted for presentation at, and inclusion in, the proceedings of SOCO'99, Genoa, Italy, June 1-4, 1999.

A short paper outlining the Fibonacci and Zeckendorf algorithms of [5] has been accepted for presentation at the Southeastern MAA annual regional meeting in Memphis, TN, March 12-13, 1999.

We are also very optimistic about the ongoing development of an algorithm, hopefully with *Mathematica* implementation, which will generate terms of A013583 directly. Preliminary results have been most encouraging. The algorithm is based on ideas gathered from this note and reference [2].

REFERENCES

1. M. Bicknell-Johnson. "The Smallest Positive Integer Having F_k Representations as Sums of Distinct Fibonacci Numbers." Presented at The Eighth International Conference on Fibonacci Numbers and Their Applications, Rochester, NY, 1998. In *Applications of Fibonacci Numbers* **8**:47-52. Ed. F. T. Howard. Dordrecht: Kluwer Academic Publishers, 1999.
2. M. Bicknell-Johnson & D. C. Fielder. "The Number of Representations of N Using Distinct Fibonacci Numbers, Counted by Recursive Formulas." *The Fibonacci Quarterly* **37.1** (1999): 47-60.
3. L. Carlitz. "Fibonacci Representations." *The Fibonacci Quarterly* **6.4** (1968):193-220.
4. L. Carlitz, R. Scoville, & V. E. Hoggatt, Jr. "Fibonacci Representations." *The Fibonacci Quarterly* **10.1** (1972):1-28.
5. D. C. Fielder. "CERL Memorandum Report DCF05/22/97, On Finding the Largest Fibonacci Number in a Positive Integer with Extensions to Zeckendorf Sequences." Computer Engineering Research Laboratory, School of Electrical and Computer Engineering, Georgia Institute of Technology, June 24, 1997.
6. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969. Rpt. The Fibonacci Association, Santa Clara, CA, 1979.
7. D. A. Klarner. "Partitions of N into Distinct Fibonacci Numbers." *The Fibonacci Quarterly* **6.4** (1968):235-44.
8. J. Shallit. Private correspondence and posting of terms on N. J. A. Sloane's "On-Line Encyclopedia of Integer Sequences," June 1996: http://www.research.att.com/~njas/sequences/

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2001. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-911 ***Proposed by M. N. Deshpande, Institute of Science, Nagpur, India***

Determine whether $L_n + 2(-1)^m L_{n-2m-1}$ is divisible by 5 for all positive integers m and n.

B-912 ***Proposed by the editor***

Express $F_{n+(n \bmod 2)} \cdot L_{n+1-(n \bmod 2)}$ as a sum of Fibonacci numbers.

B-913 ***Proposed by Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA***

Fix an integer $k \geq 1$. The Fibonacci numbers satisfy an "accelerated" recurrence of the form

$$F_{n2^k} = \alpha_k F_{(n-1)2^k} - F_{(n-2)2^k} \quad (n = 2, 3, \ldots)$$

with $F_0 = 0$ and F_{2^k} to start the recurrence. For example, when $k = 1$, we have

$$F_{2n} = 3F_{2(n-1)} - F_{2(n-2)} \quad (n = 2, 3, \ldots; F_0 = 0; F_2 = 1).$$

a. Find the constant α_k by identifying it as a certain member of a sequence that is known by readers of these pages.

b. Generalize this result by similarly identifying the constant β_m for which the accelerated recurrence

$$F_{mn+h} = \beta_m F_{m(n-1)+h} + (-1)^{m+1} F_{m(n-2)+h},$$

with appropriate initial conditions, holds.

B-914 ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain***

Let $n \geq 2$ be an integer. Prove that

$$\prod_{k=2}^{n}\left\{\sum_{j=1}^{k}\frac{1}{(F_{k+2}-F_j-1)^2}\right\} \geq \frac{1}{F_2F_{n+1}}\left(\frac{n}{F_3F_4\ldots F_n}\right)^2.$$

B-915 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN***

If $|x| \leq 1$, prove that

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{i} i2^{-j-1}F_jx^{i-1}\right| < n^3.$$

SOLUTIONS

An Exponential Equation with Fibonacci Base

B-894 ***Proposed by the editor***
(Vol. 38, no. 1, February 2000)

Solve for x: $F_{110}^x + 442F_{115}^x + 13F_{119}^x = 221F_{114}^x + 255F_{117}^x$.

Solution by Paul S. Bruckman, Berkeley CA

The desired value of x must clearly be rational, if it exists at all. We may deduce the appropriate value of x by an approximation technique. If we replace F_{110} by u, say, then it is *approximately* true that $F_{114} = u\alpha^4$, $F_{115} = u\alpha^5$, $F_{117} = u\alpha^7$, and $F_{119} = u\alpha^9$. Therefore, the desired equation is replaced by the following approximate equation:

$$13\alpha^{9x} - 255\alpha^{7x} + 442\alpha^{5x} - 221\alpha^{4x} + 1 \approx 0. \tag{1}$$

Let $G(x)$ represent the expression in the left member of (1). It is easily found that $G(0) = -20$. We may also compute the following approximate values: $G(1) \approx -3027.513$, $G(2) \approx -95869.589$, $G(3) = 0$ (exactly), $G(4) \approx 2.58992{*}10^8$. Clearly, $G(x)$ increases without bound for all $x \geq 4$, since the term $13\alpha^{9x}$ dominates $G(x)$. Therefore, it appears that $x = 3$ is the unique desired solution; however, this must be verified in the exact (original) equation.

The cubes of the Fibonacci numbers have a characteristic polynomial of degree 4; if $P_3(z)$ is this polynomial, it is easily seen that

$$\begin{aligned} P_3(x) &= (z-\alpha^3)(z-\alpha^2\beta)(z-\alpha\beta^2)(z-\beta^3) = (z^2-4z-1)(z^2+z-1) \\ &= (z^2-1)^2 - 3z(z^2-1) - 4z^2 = z^4 - 6z^2 + 1 - 3z^3 + 3z \end{aligned}$$

or

$$P_3(z) = z^4 - 3z^3 - 6z^2 + 3z + 1. \tag{2}$$

That is, we have the following recurrence relation for the cubes of the Fibonacci numbers, valid for all n:

$$(F_{n+4})^3 - 3(F_{n+3})^3 - 6(F_{n+2})^3 + 3(F_{n+1})^3 + (F_n)^3 = 0. \tag{3}$$

Using (3), we need to verify the following relation:

$$13(F_{119})^3 - 255(F_{117})^3 + 442(F_{115})^3 - 221(F_{114})^3 + (F_{110})^3 = 0. \tag{4}$$

From (3), we gather that

$$(F_{110})^3 = -(F_{114})^3 + 3(F_{113})^3 + 6(F_{112})^3 - 3(F_{111})^3.$$

Also,

$$(F_{119})^3 = 3(F_{118})^3 + 6(F_{117})^3 - 3(F_{116})^3 - (F_{115})^3;$$

thus, if S represents the expression in the left member of (4), we obtain (after some simplification):

$$S = 39(F_{118})^3 - 177(F_{117})^3 - 39(F_{116})^3 + 429(F_{115})^3 - 222(F_{114})^3 + 3(F_{113})^3 + 6(F_{112})^3 - 3(F_{111})^3.$$

Also,

$$(F_{111})^3 = -(F_{115})^3 + 3(F_{114})^3 + 6(F_{113})^3 - 3(F_{112})^3$$

and

$$(F_{118})^3 = 3(F_{117})^3 + 6(F_{116})^3 - 3(F_{115})^3 - (F_{114})^3.$$

After further simplification, we obtain:

$$S = -60(F_{117})^3 + 195(F_{116})^3 + 315(F_{115})^3 - 270(F_{114})^3 - 15(F_{113})^3 + 15(F_{112})^3.$$

Finally, we make the substitutions:

$$(F_{112})^3 = -(F_{116})^3 + 3(F_{115})^3 + 6(F_{114})^3 - 3(F_{113})^3$$

and

$$(F_{117})^3 = 3(F_{116})^3 + 6(F_{115})^3 - 3(F_{114})^3 - (F_{115})^3.$$

Then, after further simplification, we obtain $S = 0$, identically. This establishes (4) and shows that $x = 3$ is the unique solution to the problem.

Brian D. Beasley noted that the solution $x = 3$ works for n, $n+5$, $n+9$, $n+4$, $n+7$ in place of 110, 115, 119, 114, and 117, respectively.

Also solved by Brian D. Beasley, Indulis Strazdins, and the proposer.

A Recurrence for F_{n^2}

B-895 ***Proposed by Indulis Strazdins, Riga Technical University, Latvia (Vol. 38, no. 2, May 2000)***

Find a recurrence for F_{n^2}.

Solution by Paul S. Bruckman, Berkeley CA

For brevity, write $Q_n \equiv F_{n^2}$. We may easily demonstrate that the following recurrence relation is satisfied:

$$Q_{n+1} - Q_{n-1} = F_{2n}L_{n^2+1}. \tag{1}$$

We may verify (1), using the identity,

$$F_u L_v = F_{v+u} - (-1)^u F_{v-u}, \tag{2}$$

by setting $u = 2n$, $v = n^2 + 1$. Also, setting $u = 4n$, $v = n^2 + 4$ yields

$$Q_{n+2} - Q_{n-2} = F_{4n}L_{n^2+4}. \tag{3}$$

Now we note (or easily verify) the following identities:

$$L_{m+4} = 7L_{m+1} - 10F_m \tag{4}$$

and

$$F_{4n} = F_{2n}L_{2n}. \tag{5}$$

Then, setting $m = n^2$ in (4) we obtain, from (1) and (3):

$$Q_{n+2} - Q_{n-2} - 7L_{2n}(Q_{n+1} - Q_{n-1}) + 10F_{4n}Q_n = 0. \tag{6}$$

Since $Q_{-n} = Q_n$, we see that (6) is valid for all integral n. We may also express (6) in the asymmetric form:

$$Q_{n+4} = 7L_{2n+4}Q_{n+3} - 10F_{4n+8}Q_{n+2} - 7L_{2n+4}Q_{n+1} + Q_n. \tag{7}$$

It does not appear that we can obtain a linear recurrence for the Q_n's that contains constant coefficients.

H.-J. Seiffert gave the formula

$$G_{n+2} = \frac{(F_{2n+1}L_{2n+3} - 1)G_{n+1} + F_{2n+3}G_n}{F_{n_{2n+1}}},$$

where $G_n = F_{n^2}$, *and L. A. G. Dresel gave the formula*

$$T_{n+1} = \frac{1}{2}(5S_n + T_n)L_{2n} - T_{n-1},$$

where $S_n = F_{n^2}$ *and* $T_n = L_{n^2}$. *He also noted that the factor* L_{2n} *occurring in the above formula can be obtained from the formula* $L_{2(n+1)} = 3L_{2n} - L_{2(n-1)}$.

Also solved by L. A. G. Dresel, H.-J. Seiffert, and the proposer.

An Independent Constant Fibonacci Sum

B-896 ***Proposed by Andrew Cusumano, Great Neck, NY***
(Vol. 38, no. 2, May 2000)

Find an integer k such that the expression $F_n^4 + 2F_n^3F_{n+1} + kF_n^2F_{n+1}^2 - 2F_nF_{n+1}^3 + F_{n+1}^4$ is a constant independent of n.

Solution by Kee-Wai Lau, Hong Kong, China

We show that, for $k = 1$, the given expression equals 1. This amounts to proving that

$$F_n^4 + 2F_n^3F_{n+1} - F_n^2F_{n+1}^2 - 2F_nF_{n+1}^3 + F_{n+1}^4 - 1 = 0. \tag{1}$$

In fact, the left-hand side of (1) equals

$$\begin{aligned}&F_n(F_n + F_{n+1})(F_n - F_{n+1})(F_{n+1} + (F_{n+1} + F_n)) + (F_{n+1}^2 + 1)(F_{n+1}^2 - 1)\\ &= -F_nF_{n+2}F_{n-1}F_{n+3} + (F_{n+1}^2 + 1)(F_{n+1}^2 - 1).\end{aligned}$$

Hence, to prove (1), it suffices to show that

$$F_{n+1}^2 - F_nF_{n+2} = (-1)^n \tag{2}$$

and

$$F_{n+1}^2 - F_{n-1}F_{n+3} = (-1)^{n-1}. \tag{3}$$

However, both (2) and (3) can be established readily by using the relation

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \text{ where } \alpha > 0,\ \alpha + \beta = 1, \text{ and } \alpha\beta = -1.$$

This completes the solution.

R. J. Hendel noted that the result follows from the "verification theorem" in L. A. G. Dresel's article "Transformation of Fibonacci-Lucas Identities" in Application of Fibonacci Numbers **5**, *ed. G. E. Bergum et al. (Dordrecht: Kluwer, 1993), after one trivially verifies that, for $k=-1$, the expression evaluates to 1 for $n=-2$ to $n=2$.*

Also solved by P. Bruckman, C. Cook, L. A. G. Dresel, R. J. Hendel, H. Kwong, K. Lewis, H.-J. Seiffert, and J. Sellers.

An Initial Value Problem

B-897 ***Proposed by Brian D. Beasley, Presbyterian College, Clinton, SC (Vol. 38, no. 2, May 2000)***

Define $\langle a_n \rangle$ by $a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n$ for $n \geq 0$ with initial conditions $a_0 = 4$, $a_1 = 2$, and $a_2 = 10$. Express a_n in terms of Fibonacci and/or Lucas numbers.

Solution by Richard André-Jeannin, Cosnes et Romain, France

The characteristic polynomial of the proposed recurrence can be easily factorized:

$$X^3 - 2X^2 - 2X + 1 = (X - \alpha^2)(X - \beta^2)(X + 1).$$

From this, we see that there exist constants A, B, and C such that $a_n = A\alpha^{2n} + B\beta^{2n} + C(-1)^n$. Considering the initial conditions, we obtain the linear system:

$$\begin{cases} A + B + C = 4, \\ A\alpha^2 + B\beta^2 - C = 2, \\ A\alpha^4 + B\beta^4 + C = 10. \end{cases}$$

After some calculations, we get $A = B = \frac{6}{5}$ and $C = \frac{8}{5}$; thus,

$$\begin{aligned} a_n &= \frac{6(\alpha^{2n} + \beta^{2n}) + 8(-1)^n}{5} = \frac{6L_{2n} + 8(-1)^n}{5} \\ &= L_{2n} + 2(-1)^n + \frac{L_{2n} - 2(-1)^n}{5} = L_n^2 + F_n^2. \end{aligned}$$

Also solved by P. Bruckman, J. Cigler, C. Cook, K. Davenport, L. A. G. Dresel, H. Kwong, K. Lewis, D. Redmond, M. Rose, H.-J. Seiffert, J. Sellers, I. Strazdins, and the proposer.

Some Fibonacci Sum

B-898 ***Proposed by Alexandru Lupaş, Sibiu, Romania (Vol. 38, no. 2, May 2000)***

Evaluate

$$\sum_{k=0}^{s} (-1)^{(n-1)(s-k)} \binom{2s+1}{s-k} F_{n(2k+1)}.$$

Solution by L. A. G. Dresel, Reading, England

Denoting the given expression by $E(s,n)$, consider the $2s+2$ terms of the expansion of $(\alpha^n-\beta^n)^{2s+1}$. The two central terms combine to give

$$\binom{2s+1}{s}(-\alpha^n\beta^n)^s(\alpha^n-\beta^n)=\binom{2s+1}{s}(-1)^{(n-1)s}(\sqrt{5})F_n.$$

Then proceeding outward and combining pairs of terms equidistant from the center, we obtain $(\alpha^n-\beta^n)^{2s+1}=(\sqrt{5})E(s,n)$. Therefore, since $(\alpha^n-\beta^n)=(\sqrt{5})F_n$, we have $E(s,n)=5^s(F_n)^{2s+1}$.

Note: The result corresponds to equation (80) in *Fibonacci & Lucas Numbers, and the Golden Section,* by S. Vajda (Chichester: Ellis Horwood Ltd., 1989).

Don Redmond obtained a similar result for a Lucas analog to the problem. He showed that

$$\sum_{k=0}^{s}(-1)^{n(s-k)}\binom{2s+1}{s-k}L_{n(2k+1)}=L_n^{2s+1}.$$

Also solved by P. Bruckman, J. Cigler, D. Redmond, H.-J. Seiffert, I. Strazdins, and the proposer.

On a Bruckman Conjecture

A Comment by N. Gauthier, Canada

In the Feb. 2000 issue of this quarterly, Dr. Rabinowitz published the solution to Elementary Problem B-871 by Paul S. Bruckman ["Absolute Sums," *The Fibonacci Quarterly* **38.1** (2000):86-87]. In a footnote to Indulis Strazdin's solution, Dr. Rabinowitz then commented that "*Bruckman noted that*

$$\sum_{k=0}^{2n}\binom{2n}{k}|n-k|=n\binom{2n}{n}$$

and conjectured that

$$\sum_{k=0}^{2n}\binom{2n}{k}|n-k|^{2r-1}=P_r(n)\binom{2n}{n}$$

for some monic polynomial $P_r(n)$ of degree r." Here, $n\ge 0$ and $r\ge 1$ are integers.

Professor Bruckman's conjecture about the general form of the latter sum seems to be correct based on evidence I collected with MAPLE V$^{(R)}$, but contrary to the conjecture, $P_r(n)$ is generally not a monic polynomial in n. From the evidence collected, the leading coefficient of $P_r(n)$ appears to be equal to $(r-1)!$; for $r=1$ and $r=2$, one then finds the leading terms to be n and n^2, respectively, in agreement with the values presented in the solution of problem B-871. But, for $r\ge 3$, the leading term of $P_r(n)$ is $(r-1)!n^r$.

The Bruckman polynonmials $P_r(n)$ were obtained for $1\le r\le 20$, using MAPLE V$^{(R)}$, by noting that

$$P_r(n)=\frac{1}{\binom{2n}{n}}\sum_{k=0}^{2n}\binom{2n}{k}|n-k|^{2r-1}=\frac{2}{\binom{2n}{n}}\sum_{k=0}^{n-1}\binom{2n}{k}(n-k)^{2r-1}$$

and by asking MAPLE for $P_r(n)$. For easy reference, here are the values of $P_r(n)$ for $r=3$, 4, 5, and 6:

$P_3(n)=2n^3-n^2$; $\quad P_4(n)=6n^4-8n^3+3n^2$;

$P_5(n)=24n^5-60n^4+54n^3-17n^2$; $\quad P_6(n)=120n^6-480n^5+762n^4-556n^3+155n^2$.

(I will gladly provide the polynomials for $7\le r\le 20$ on request for interested readers who might not have access to MAPLE.)

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-570 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Show that, for all positive integers n:

(a)
$$5^{n-1}F_{2n-1}=\sum_{\substack{k=0\\ 5\nmid 2n-k-1}}^{2n-1}(-1)^k\binom{4n-2}{k};$$

(b)
$$5^{n-1}L_{2n}=\sum_{\substack{k=0\\ 5\nmid 2n-k}}^{2n}(-1)^{k+1}\binom{4n}{k}.$$

Two closely related identities were given in H-518.

H-571 ***Proposed by D. Tsedenbayar, Mongolian Pedagogical University, Warsaw, Poland***

Prove: If $(T_\alpha f)(t)=t^\alpha\int_0^t f(s)\,ds$ with $\alpha\in\mathbf{R}$, then

$$(T_\alpha^n f)(t)=\frac{t^\alpha}{(\alpha+1)^{(n-1)}(n-1)!}\int_0^t(t^{\alpha+1}-s^{\alpha+1})^{n-1}f(s)\,ds,\ \text{ for } \alpha\neq -1,$$

and

$$(T_\alpha^n f)(t)=\frac{1}{t(n-1)!}\int_0^t\left(\ln\frac{t}{s}\right)^{n-1}f(s)\,ds,\ \text{ for } \alpha=-1.$$

Remark: If $\alpha=-1$, then T_{-1} is a Cesaro operator; if $\alpha=0$, then T_0 is a Volterra operator.

A Correction:

H-568 ***Proposed by N. Gauthier, Royal Military College of Canada, Kingston, Ontario***

The following was inspired by Paul S. Bruckman's Problem B-871 (Vol. 37, no. 1, February 1999; solved Vol. 38, no. 1, February 2000).

"For integers $n, m\geq 1$, prove or disprove that

$$f_m(n)\equiv\frac{1}{\binom{2n}{n}^2}\sum_{k=0}^{2n}\binom{2n}{k}^2|n-k|^{2m-1}$$

is the ratio of two polynomials with integer coefficients,

$$f_m(n) = P_m(n) / Q_m(n),$$

where $P_m(n)$ is of degree $\lfloor\frac{3m}{2}\rfloor$ in n and $Q_m(n)$ is of degree $\lfloor\frac{m}{2}\rfloor$; determine $P_m(n)$ and $Q_m(n)$ for $1 \le m \le 5$."

SOLUTIONS

A Piece of Pi

H-558 ***Proposed by Paul S. Bruckman, Berkeley, CA (Vol. 37, no. 4, November 1999)***

Prove the following:

$$\pi = \sum_{n=0}^{\infty}(-1)^n\{6\varepsilon_{10n+1} - 6\varepsilon_{10n+3} - 4\varepsilon_{10n+5} - 6\varepsilon_{10+7} + 6\varepsilon_{10n+9}\}, \text{ where } \varepsilon_m = \alpha^{-m}/m. \qquad (*)$$

Solution by the proposer

Looking at the form of the series expression, it is evidently composed of decisections of the logarithmic series. We begin with the definitions:

$$F_r(z) \equiv \sum_{n=0}^{\infty} z^{10n+r}/(10n+r), \quad r = 1, 2, \ldots, 10, \quad |z| < 1. \qquad (1)$$

Note that $F_r(0) = 0$ and $F_r'(z) = \sum_{n=0}^{\infty} z^{10n+r-1} = z^{r-1}/(1-z^{10})$. Letting $\theta = \exp(i\pi/5)$ (a tenth root of unity), we find, using residue theory (or otherwise), that $F_r'(z) = 1/10\sum_{k=1}^{10}\theta^{-k(r-1)}(1-x\theta^k)^{-1}$; then, by integration,

$$F_r(z) = -1/10\sum_{k=1}^{10}\theta^{-kr}\log(1-x\theta^k). \qquad (2)$$

The following transformation is implemented, valid for all complex $z = re^{i\phi}$:

$$\operatorname{Log} z = \log r + i\phi. \qquad (3)$$

Here, "Log" designates the "principal" logarithm, with $-\pi \le \phi \le \pi$; $r = |z|$, $\phi = \operatorname{Arg} z$. We also note that $2\cos(\pi/5) = \alpha$, $2\cos(2\pi/5) = -\beta = 1/\alpha$, and we let s_j denote $\sin(j\pi/5)$, $j = 1, 2$. We readily find that $2s_1 = (\sqrt{5}/\alpha)^{1/2}$ and $2s_2 = (\alpha\sqrt{5})^{1/2} = 2\alpha s_1$. After a trite but straightforward computation, we obtain the following expressions:

$$F_1(z) = \alpha A(x,\alpha) + \beta A(x,\beta) + B(x) + s_1C(x) + s_2D(x), \qquad (4)$$

where

$$A(x,c) = 1/20\log\{(1+cx+x^2)/(1-cx+x^2)\}, \quad B(x) = 1/10\log\{(1+x)/(1-x)\},$$
$$C(x) = 1/5\tan^{-1}\{2xs_1/(1-x^2)\}, \quad D(x) = 1/5\tan^{-1}\{2xs_2/(1-x^2)\};$$

$$F_2(x) = \alpha P(x,\alpha) + \beta P(x,\beta) + Q(x) + s_1U(x) + s_2V(x), \qquad (5)$$

where

$$P(x,c) = 1/20\log(1+cx^2+x^4), \quad Q(x) = -1/10\log(1-x^2),$$
$$U(x) = 1/5\tan^{-1}\{2x^2s_1/(2+\alpha x^2)\}, \quad V(x) = 1/5\tan^{-1}\{2x^2s_2/(2+\beta x^2)\};$$

$$F_3(z) = \beta A(x,\alpha) + \alpha A(x,\beta) + B(x) + s_2C(x) - s_1D(x); \qquad (6)$$

$$F_4(x) = \beta P(x,\alpha) + \alpha P(x,\beta) + Q(x) - s_2U(x) + s_1V(x); \qquad (7)$$

$$F_5(z) = -2A(x,\alpha) - 2A(x,\beta) + B(x) = 1/10\log\{(1+x^5)/(1-x^5)\}; \tag{8}$$
$$F_6(x) = \beta P(x,\alpha) + \alpha P(x,\beta) + Q(x) + s_2U(x) - s_1V(x); \tag{9}$$
$$F_7(z) = \beta A(x,\alpha) + \alpha A(x,\beta) + B(x) - s_2C(x) + s_1D(x); \tag{10}$$
$$F_8(x) = \alpha P(x,\alpha) + \beta P(x,\beta) + Q(x) - s_1U(x) - s_2V(x); \tag{11}$$
$$F_9(z) = \alpha A(x,\alpha) + \beta A(x,\beta) + B(x) - s_1C(x) - s_2D(x); \tag{12}$$
$$F_{10}(x) = -2P(x,\alpha) - 2P(x,\beta) + Q(x) = -1/10\log(1-x^{10}). \tag{13}$$

Next, we note the following: $F_1(x)+F_3(x)+F_7(x)+F_9(x) = 2A(x,\alpha)+2A(x,\beta)+4B(x)$. Then, using (8):

$$\begin{aligned}G(x) &\equiv 6\{F_1(x)+F_3(x)+F_7(x)+F_9(x)\} - 4F_5(x)\\
&= 12A(x,\alpha)+12A(x,\beta)+24B(x)+8A(x,\alpha)+8A(x,\beta)-4B(x)\\
&= 20\{A(x,\alpha)+A(x,\beta)+B(x)\}\\
&= \log\{\{(1+\alpha x+x^2)(1+\beta x+x^2)(1+x)^2\}/\{(1-\alpha x+x^2)(1-\beta x+x^2)(1-x)^2\}\}\\
&= \log\{\{(1+x+x^2+x^3+x^4)(1+x)^2\}/\{(1-x+x^2+x^3-x^4)/(1-x)^2\}\}\\
&= \log\{(1-x^5)(1+x)^3/[(1-x^5)(1-x)^3]\}.\end{aligned}$$

Thus, $G(ix) = -\log\{(1+ix^5)/(1-ix^5)\} + 3\log\{(1+ix)/(1-ix)\}$, i.e.,

$$6\{F_1(ix)+F_3(ix)+F_7(ix)+F_9(ix)\} - 4F_5(ix) = -2i\tan^{-1}x^5 + 6i\tan^{-1}x. \tag{14}$$

The left side of (14), employing the series definitions, becomes

$$i\sum_{n=0}^{\infty}(-1)^n\{6\varepsilon_{10n+1}(x) - 6\varepsilon_{10n+3}(x) - 4\varepsilon_{10n+5}(x) + 6\varepsilon_{10n+7}(x) - 6\varepsilon_{10n+9}(x)\},$$

where $\varepsilon_m(x) = x^m/m$. We see that, in order to prove the desired identity (*), it suffices to show:

$$-\tan^{-1}(\alpha^{-5}) + 3\tan^{-1}(\alpha^{-1}) = \pi/2. \tag{15}$$

If $\varphi = \tan^{-1}(\alpha^{-1})$, then

$$\tan(3\varphi) = (3\tan\varphi - \tan^3\varphi)/(1-3\tan^2\varphi) = (3\alpha^2-1)/(\alpha^3-3\alpha) = (3\alpha+2)/(1-\alpha) = -\alpha^5.$$

Thus, $3\varphi = \pi - \tan^{-1}(\alpha^5) = \pi - (\pi/2 - \tan^{-1}(\alpha^{-5}))$, which is (15). Q.E.D.

Also solved by R. Martin and H.-J. Seiffert

SUM Formulae

H-559 ***Proposed by N. Gauthier, Royal Military College of Canada***
(Vol. 38, no. 1, February 2000)

Let n and q be nonnegative integers and show that:

a. $$S_n(q) := \sum_{k=1}^{n}\frac{1}{2\cos(2\pi k/n) + (-1)^{q+1}L_{2q}} = \frac{(-1)^{q+1}nL_{qn}}{5F_{2q}F_{qn}};$$

b. $$s_n(q) := \sum_{k=1}^{n}\frac{1}{0.8\sin^2(2\pi k/n) + F_{2q}^2} = \begin{cases}\dfrac{nL_{2qn}}{F_{2q}L_{2q}F_{qn}L_{qn}}, & n \text{ odd},\\[2ex] \dfrac{nL_{qn}}{F_{2q}L_{2q}F_{qn}}, & n \text{ even}.\end{cases}$$

L_n and F_n are Lucas and Fibonacci numbers.

Solution by H.-J. Seiffert, Berlin, Germany

Let n be a positive integer. Differentiating the known identity [see I.S. Gradshteyn & I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed., p. 41, eqn. 1.394 (New York: Acad. Press, 1994)]

$$\prod_{k=1}^{n}(x^2+1-2x\cos(2\pi k/n))=(x^n-1)^2$$

logarithmically gives

$$\sum_{k=1}^{n}\frac{2x-2\cos(2\pi k/n)}{x^2+1-2x\cos(2\pi k/n)}=\frac{2nx^{n-1}}{x^n-1}.$$

Multiplying by x and then subtracting n from both sides of the resulting equation yields

$$\sum_{k=1}^{n}\frac{x^2-1}{x^2+1-2x\cos(2\pi k/n)}=n\frac{x^n+1}{x^n-1}.$$

It now easily follows that

$$\sum_{k=1}^{n}\frac{1}{2\cos(2\pi k/n)-x-1/x}=\frac{nx(x^n+1)}{(1-x^2)(x^n-1)},\tag{1}$$

valid for all real numbers x such that $x\neq 0$ and $x\neq 1$.

Taking $x=(\alpha/\beta)^q$, $q\in Z$, and $q\neq 0$, and using the known Binet forms of the Fibonacci and the Lucas numbers, we easily obtain the desired equation of the first part.

Replacing x by $-x$ in (1) and subtracting the so obtained identity from (1) gives

$$\sum_{k=1}^{n}\frac{2x+2/x}{4\cos^2(2\pi k/n)-(x+1/x)^2}=\frac{nx}{1-x^2}\left(\frac{x^n+1}{x^n-1}+\frac{(-x)^n+1}{(-x)^n-1}\right).$$

Since $\cos^2(2\pi k/n)=1-\sin^2(2\pi k/n)$, after some simple manipulations, we find that

$$T_n(x):=\sum_{k=1}^{n}\frac{1}{0.8\sin^2(2\pi k/n)+0.2(x-1/x)^2}=\frac{nx^2}{0.4(x^4-1)}\left(\frac{x^n+1}{x^n-1}+\frac{(-x)^n+1}{(-x)^n-1}\right),$$

valid for all real numbers x such that $x\neq 0$ and $x\neq 1$. Hence:

$$T_n(x)=\frac{nx^2(x^{2n}+1)}{0.2(x^4-1)(x^{2n}-1)}\ \text{ if } n \text{ is odd;}\quad T_n(x)=\frac{nx^2(x^n+1)}{0.2(x^4-1)(x^n-1)}\ \text{ if } n \text{ is even.}$$

Taking $x=(-\alpha/\beta)^q$, $q\in Z$, and $q\neq 0$, one easily deduces the requested equations of the second part. Q.E.D.

Also solved by P. Bruckman and the proposer.

A Complex Problem

H-560 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 38, no. 1, February 2000)

Define the sequences of Fibonacci and Lucas polynomials by

$$F_0(x)=0,\ F_1(x)=1,\ F_{n+1}(x)=xF_n(x)+F_{n-1}(x),\ n\in N,$$

and

$$L_0(x)=2,\ L_1(x)=x,\ L_{n+1}(x)=xL_n(x)+L_{n-1}(x),\ n\in N,$$

respectively. Show that, for all complex numbers x and all positive integers n,

$$\sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} x^k F_{3k}(x) = F_{2n}(x) + (-x)^n F_n(x)$$

and

$$\sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} x^k L_{3k}(x) = L_{2n}(x) + (-x)^n L_n(x).$$

Solution by Paul S. Bruckman, Berkeley, CA

We begin with the following well-known explicit expressions for $F_n(x)$ and $L_n(x)$, namely,

$$F_n(x) = (\alpha^n - \beta^n)/(\alpha - \beta), \;\; L_n(x) = \alpha^n + \beta^n, \;\; n = 0, 1, 2, \dots, \tag{1}$$

where

$$\alpha = \alpha(x) = (x+\theta)/2, \;\; \beta = \beta(x) = (x-\theta)/2, \tag{2}$$

$$\theta = \theta(x) = (x^2+4)^{1/2} = \alpha - \beta. \tag{3}$$

Next, we make the following definitions:

$$G_n(y) = \sum_{k=0}^{[n/2]} n/(n-k) \cdot {}_{n-k}C_k \cdot y^k, \tag{4}$$

where ${}_rC_s$ is the combinatorial symbol commonly known as "r choose s," i.e., $\binom{r}{s}$.

Then, if $U_n(x)$ and $V_n(x)$ denote the first and second sum expressions, respectively, given in the statement of the problem, we obtain

$$U_n(x) = \theta^{-1} \sum_{k=0}^{[n/2]} n/(n-k) \cdot {}_{n-k}C_k \cdot x^k(\alpha^{3k} - \beta^{3k}), \text{ or}$$

$$U_n(x) = \theta^{-1}(G_n(\alpha^3 x) - G_n(\beta^3 x)). \tag{5}$$

Similarly,

$$V_n(x) = G_n(\alpha^3 x) + G_n(\beta^3 x), \tag{6}$$

where we also make the following definitions: $U_0(x) = 0$, $V_0(x) = 2$.

Next, we form the following generating functions:

$$R(z, x) = \sum_{n=0}^{\infty} U_n(x) z^n, \;\; S(z, x) = \sum_{n=0}^{\infty} V_n(x) z^n, \tag{7}$$

$$T(z, y) = \sum_{n=0}^{\infty} G_n(y) z^n. \tag{8}$$

We see that

$$\begin{aligned} R(z, x) &= \theta^{-1}\{T(z, x\alpha^3) - T(z, x\beta^3)\}, \\ S(z, x) &= T(z, x\alpha^3) + T(z, x\beta^3). \end{aligned} \tag{9}$$

We obtain a closed form expression for $T(z, y)$ as follows:

$$\begin{aligned} T(z, y) &= \sum_{n,\, k=0}^{\infty} (n+2k)/(n+k)_{n+k}C_k z^{n+2k} y^k \\ &= \sum_{n,\, k=0}^{\infty} {}_{n+k}C_k z^n (z^2 y)^k + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} {}_{n+k-1}C_{k-1} z^n (z^2 y)^k \end{aligned}$$

$$= (1+z^2y)\sum_{n,\,k=0}^{\infty} {}_{n+k}C_k z^n (z^2y)^k = (1+z^2y)\sum_{n,\,k=0}^{\infty} {}_{-n-1}C_k z^n(-z^2y)^k$$
$$= (1+z^2y)\sum_{n=0}^{\infty} z^n(1-z^2y)^{-n-1} = (1+z^2y)(1-z^2y)^{-1}\{1-z/(1-z^2y)\}^{-1},$$

or

$$T(z,y) = (1+z^2y)/(1-z-z^2y). \tag{10}$$

Then

$$T(z,\alpha^3x) = (1+z^2\alpha^3x)/(1-z-z^2\alpha^3x); \tag{11}$$
$$T(z,\beta^3x) = (1+z^2\beta^3x)/(1-z-z^2\beta^3x). \tag{12}$$

We now note that $(1-z\alpha^2)(1+z\alpha x) = 1+z\alpha(x-\alpha)-z^2\alpha^3x = 1+z\alpha\beta-z^2\alpha^3x = 1-z-z^2\alpha^3x$. Similarly, we find that $(1-z\beta^2)(1+z\beta x) = 1-z-z^2\beta^3x$. We may also verify the following:

$$T(z,\alpha^3z) = -1+(1-z\alpha^2)^{-1}+(1+z\alpha x)^{-1}; \tag{13}$$
$$T(z,\beta^3z) = -1+(1-z\beta^2)^{-1}+(1+z\beta x)^{-1}. \tag{14}$$

Then, by expansion in (13) and (14):

$$T(z,\alpha^3x) = 1+\sum_{n=1}^{\infty}(\alpha^{2n}+(-x)^n\alpha^n)z^n;$$
$$T(z,\beta^3x) = 1+\sum_{n=1}^{\infty}(\beta^{2n}+(-x)^n\beta^n)z^n.$$

Now, using (9), we see that $R(z,x) = \theta^{-1}\sum_{n=0}^{\infty} z^n\{\alpha^{2n}-\beta^{2n}+(-x)^n(\alpha^n-\beta^n)\}$, or

$$R(z,x) = \sum_{n=0}^{\infty} z^n\{F_{2n}+(-x)^nF_n\}. \tag{15}$$

Likewise, $S(z,x) = 2+\sum_{n=1}^{\infty} z^n\{\alpha^{2n}-\beta^{2n}+(-x)^n(\alpha^n+\beta^n)\}$, or

$$S(z,x) = 2+\sum_{n=1}^{\infty} z^n\{L_{2n}+(-x)^nL_n\}. \tag{16}$$

Comparing the coefficients of z^n in (15) and (16) with those in (7) yields the desired results:

$$U_n(x) = F_{2n}+(-x)^nF_n,\ V_n(x) = L_{2n}+(-x)^nL_n,\ n=1,2,\ldots \text{ Q.E.D.} \tag{17}$$

Note: The Fibonacci polynomials are defined provided $x^2+4\neq 0$, i.e., $x\neq\pm 2i$. However, we may extend the definition of these polynomials to such exceptional values using continuity, i.e., by defining $F_n(2i) = ni^{n-1}$, $F_n(-2i) = n(-i)^{n-1}$. We also obtain $L_n(2i) = 2i^n$, $L_n(-2i) = 2(-i)^n$. With such definitions, we find that the results of the problem are indeed true for all complex x, including these exceptional values.

Also solved by A. J. Stam and the proposer.

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A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972. $32.00

Fibonacci's Problem Book, Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974. $19.00

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