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SETS IN WHICH THE PRODUCT OF ANY K ELEMENTS INCREASED BY t IS A k^{th}-POWER

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Let t be an integer. A P_t-set of size n is a set $A = \{x_1, x_2, \ldots, x_n\}$ of distinct positive integers such that $x_i x_j + t$ is a square of an integer whenever $i \neq j$. These P_t-sets are said to verify Diophantus' property. In fact, Diophantus was the first to note that the product of any two elements of the set $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ increased by 1 is a square of a rational number. We now introduce a more general definition.

Definition 1: Let $k > 1$ be a positive integer, and let t be an integer. A $P_t^{(k)}$-set of size n is a set $A = \{x_1, x_2, \ldots, x_n\}$ of distinct positive integers such that $\prod_{i \in I} x_i + t$ is a k^{th}-power of an integer for each $I \subset \{1, 2, \ldots, n\}$ where $\operatorname{card}(I) = k$.

A $P_t^{(k)}$-set A is said to be extendible if there exists an integer $a \notin A$ such that $A \cup \{a\}$ is a $P_t^{(k)}$-set. When $k = 2$, these sets are exactly the P_t-sets. The problem of extending P_t-sets is very old and dates back to the time of Diophantus (see Dickson [5], vol. II). The first famous result in this area is due to Baker and Davenport [3], who showed that the P_1-set $\{1, 3, 8, 120\}$ is nonextendible by using Diophantine approximation. Several others have recently made efforts to characterize the P_t-sets (see references). However, nothing is known about the $P_t^{(k)}$-sets when $k \geq 3$.

The purpose of this paper is to exhibit a $P_t^{(3)}$-set of size 4, and to show (Theorem 1) that this set is nonextendible. We also prove (Theorem 2) that the $P_{-8}^{(4)}$-set $\{1, 2, 3, 4\}$ and the $P_1^{(4)}$-set $\{1, 2, 5, 8\}$ are nonextendible. In Theorem 3 we show that any $P_t^{(k)}$-set is finite.

Example of a $P_t^{(3)}$-set: The set $\{1, 3, 4, 7\}$ is a $P_{-20}^{(3)}$-set of size 4.

Theorem 1: The $P_{-20}^{(3)}$-set $\{1, 3, 4, 7\}$ is nonextendible.

Proof: Suppose there exists an integer a such that $\{1, 3, 4, 7, a\}$ is a $P_{-20}^{(3)}$-set. Then the following system of equations

$$\begin{cases} 3a - 20 = u^3, \\ 21a - 20 = v^3, \\ 12a - 20 = w^3, \end{cases} \tag{1}$$

has an integral solution $(u, v, w) \in \mathbb{N}^3$. One can derive more equations in the system (1) but this is not necessary for our proof. The system (1) yields

$$u^3 + v^3 = 2w^3 \text{ with } (u, v, w) \in \mathbb{N}^3. \tag{2}$$

However, it is well known from the work of Euler and Lagrange (see Dickson [5], vol. II, pp. 572-73) that all solutions of equation (2) in positive integers are given by $u=v=w$, which is impossible in the system (1). $\square$

It would be interesting to know if there exists any $P_t^{(k)}$-set of size $n>k\geq 4$. For $n=k$, the problem is easy. In fact, there are two strategies for finding a $P_t^{(k)}$-set of size k.

(1) Fix any k positive integers $a_1, a_2, \ldots, a_k$. Let A be an integer and $t=A^k-\prod_{i=1}^k a_i$. Then the set $\{a_1, a_2, \ldots, a_k\}$ is a $P_t^{(k)}$-set of size k. For example, let $k=4$, $a_1=1$, $a_2=2$, $a_3=3$, $a_4=4$, and $A=2$. Then $t=-8$ and $\{1,2,3,4\}$ is a $P_{-8}^{(4)}$-set of size 4.

(2) Fix any t, and choose any integer A such that there exist k different factors $a_1, a_2, \ldots, a_k$ nonnecessary primes and $A^k-t=\prod_{i=1}^k a_i$. Then the set $\{a_1, a_2, \ldots, a_k\}$ is a $P_t^{(k)}$-set of size k. For example, let $k=4$, $t=1$, and $A=2$. Then $A^4-t=80=1\cdot 2\cdot 5\cdot 8$ and $\{1,2,5,8\}$ is a $P_1^{(4)}$-set of size 4.

Theorem 2:

(a) The $P_{-8}^{(4)}$-set $\{1,2,3,4\}$ is nonextendible.

(b) The $P_1^{(4)}$-set $\{1,2,5,8\}$ is nonextendible.

Proof:

(a) Suppose there exists an integer a such that $\{1,2,3,4,a\}$ is a $P_{-8}^{(4)}$-set. Then the following system of equations

$$\begin{cases} 6a-8=x^4, \\ 8a-8=y^4, \\ 12a-8=z^4, \\ 24a-8=w^4, \end{cases} \tag{3}$$

has an integral solution $(x,y,z,w)\in\mathbb{N}^4$. A congruence mod 16 shows that this is impossible.

(b) Suppose there exists an integer a such that $\{1,2,5,8,a\}$ is a $P_1^{(4)}$-set. Then the following system of equations

$$\begin{cases} 10a+1=x^4, \\ 16a+1=y^4, \\ 40a+1=z^4, \\ 80a+1=w^4, \end{cases} \tag{4}$$

has an integral solution $(x,y,z,w)\in(\mathbb{N}^*)^4$. The system (4) yields

$$w^4+1=2z^4 \quad \text{with } (z,w)\in(\mathbb{N}^*)^2. \tag{5}$$

But it is well known (see [13], pp. 17-18) that all solutions of (5) are given by $w=z=1$, and this gives $a=0$. $\square$

Theorem 3: Any $P_t^{(k)}$-set is finite.

Proof: Let $\{a_1, \ldots, a_k, a_{k+1}, N\}$ be a $P_t^{(k)}$-set. Let $a=a_1a_2\ldots a_ka_{k+1}$,

$$\alpha=\frac{a}{a_1a_2}, \quad \beta=\frac{a}{a_1a_3}, \quad \text{and} \quad \gamma=\frac{a}{a_2a_3}.$$

Then there exist integers x, y, and z such that

$$\alpha N + t = x^k, \quad \beta N + t = y^k, \quad \text{and} \quad \gamma N + t = z^k.$$

Hence, we obtain a superelliptic curve

$$(\alpha N + t)(\beta N + t)(\gamma N + t) = w^k$$

(for $k = 2, 3$, this is an elliptic curve), and from Theorems 6.1 and 6.2 in [15] it follows that $N \le C$ for some computable number C depending only on k, α, β, γ, and t. $\square$

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DIVERGENT RATS SEQUENCES

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1. INTRODUCTION

In 1990 John Conway invented a digital game called RATS [1]. RATS is an acronym for Reverse, Add, Then Sort. A game of RATS produces a sequence of positive integers. Each positive integer in the sequence has its digits arranged in nondecreasing order. To play a "game" of RATS, we take a positive integer whose digits are arranged in nondecreasing order, Reverse the digits, Add the reversed digits to the number, delete the zero digits in the sum, and Then Sort the remaining digits of the sum in nondecreasing order. The resulting number is the next number in the sequence.

For example, if we begin a game of RATS with 3, assuming base 10, the RATS sequence is

3, 6, 12, 33, 66, 123, 444, 888, 1677, 3489, 12333, 44556,
111, 222, 444, 888, 1677, 3489, 12333, 44556, ...,

which exhibits a cycle of length 8 and least member 111.

In [5], Curt McMullen gave a list of base 10 RATS cycles he had discovered. Computer searches were done by Curtis Cooper and Robert E. Kennedy (see [2] and [3]) to find more base 10 cycles. A list of these cycles and the search techniques used can be found in the [2] and [3].

We also have sequences that diverge. The most fundamental one, in base 10, starts with 1.

1, 2, 4, 8, 16, 77, 145, 668, 1345, 6677, 13444, 55778, 133345, 666677, 1333444,
5567777, 12333445, 66666677, 133333444, 556667777, 1233334444, 5566667777, ...

Notice that in each successive number, the number of 3's and 6's both increase by 1. This is the mark of a divergent sequence. This sequence, in particular, is known as Conway's Divergent Sequence.

Due to the size and repetitive nature of the digits in the RATS game, we will use superscripts to denote repeated digits in a number. For example, 11122223344444, will be represented as $1^3 2^4 3^2 4^5$. Using this notation, we can give the closed form of Conway's Divergent Sequence.

Conway's Divergent Sequence: Let $m \geq 2$. Then

$$1\,2\,3^m 4^4,\ 5^2\,6^m\,7^4$$

is a length 2 divergent sequence in base 10. Here, length 2 means that the sequence "comes back to itself" after the second iteration.

This paper will emphasize divergent sequences for bases other than 10. Some preliminary work has already been done for bases 19, 37, and 50 by McMullen [5]. In bases larger than 10, the digits bigger than 10 will be denoted with parentheses around them.

Lemma 1: Let $m \geq 19$. Then

$$1\,2\,3^3\,4^4\,5^{12}\,6^m\,7^{40},\ 8^2\,9^2\,(10)^6\,(11)^8\,(12)^{m+5}\,(13)^{38}$$

is a length 2 divergent sequence in base 19.

Lemma 2: Let $m \geq 1257$. Then

$$1\,2\,3^3\,4^4\,5^{12}\,6^{16}\,7^{48}\,8^{64}\,9^{192}\,(10)^{256}\,(11)^{768}\,(12)^{m}\,(13)^{2616},$$
$$(14)^2\,(15)^2\,(16)^6\,(17)^8\,(18)^{24}\,(19)^{32}\,(20)^{96}\,(21)^{128}\,(22)^{384}\,(23)^{512}\,(24)^{m+285}\,(25)^{2502}$$

is a length 2 divergent sequence in base 37.

Cooper and Gentges [4] found a divergent sequence in base 55.

Lemma 3: Let $m \geq 80099$. Then

$$1\,2\,3^3\,4^4\,5^{12}\,6^{16}\,7^{48}\,8^{64}\,9^{192}\,(10)^{256}\,(11)^{768}\,(12)^{1024}\,(13)^{3072}$$
$$(14)^{4096}\,(15)^{12288}\,(16)^{16384}\,(17)^{49152}\,(18)^{m}\,(19)^{167480},$$
$$(20)^2\,(21)^2\,(22)^6\,(23)^8\,(24)^{24}\,(25)^{32}\,(26)^{96}\,(27)^{128}\,(28)^{384}\,(29)^{512}$$
$$(30)^{1536}\,(31)^{2048}\,(32)^{6144}\,(33)^{8192}\,(34)^{24576}\,(35)^{32768}\,(36)^{m+18205}\,(37)^{160198}$$

is a length 2 divergent sequence in base 55.

Finally, Cooper and Gentges [4] found a closed form for the family of divergent sequences in bases $18n+1$, $n \geq 1$.

Theorem 4: Let m be a large positive integer and let $18n+1$ be the base, for $n \geq 1$. Then

$$1\,2\,3^3\,4^4\,5^{12}\,6^{16}\,7^{48}\,8^{64}\ldots(6n)^{m}(6n+1)^{(23\cdot 64^n-32)/36},$$
$$(6n+2)^2(6n+3)^2(6n+4)^6(6n+5)^8(6n+6)^{24}(6n+7)^{32}(6n+8)^{96}(6n+9)^{128}$$
$$\ldots(12n)^{m+(5\cdot 64^n+40)/72}(12n+1)^{(22\cdot 64^n-40)/36}$$

is a length 2 divergent sequence.

All of these divergent sequences have been of length 2. This paper will examine divergent RATS sequences of length $t \geq 2$. First, we will show other divergent RATS sequences of length 2. Next, we will show explicit divergent RATS sequences of lengths 3, 4, 5, and 6. In addition, we will prove that there are arbitrarily long divergent RATS sequences.

2. DIVERGENT RATS SEQUENCES OF LENGTH 2

Divergent sequences consisting of two elements were found for bases 28, 46, and 64. This led to finding a closed form for divergent sequences of length 2 in base $18n+10$, where $n \geq 1$.

Lemma 5: Let $m \geq 191$. Then

$$1\,2\,3^3\,4^4\,5^{12}\,6^{16}\,7^{48}\,8^{64}\,9^{m}\,(10)^{312},$$
$$(11)^2\,(12)^2\,(13)^6\,(14)^8\,(15)^{24}\,(16)^{32}\,(17)^{96}\,(18)^{m-35}\,(19)^{326}$$

is a length 2 divergent sequence in base 28.

The interested reader can obtain the proof from the authors.

By finding similar patterns for bases 46 and 64, we were led to the following closed form for a length 2 divergent RATS sequence in base $18n+10$, where $n \in \mathbb{Z}^+$.

Theorem 6: Let m be a large positive integer. Then

$$1\,2\,3^3\,4^4\,5^{12}\,6^{16}\ldots(6n+1)^{(3\cdot 64^n)/4}(6n+2)^{64^n}(6n+3)^m(6n+4)^{(44\cdot 64^n-8)/9},$$
$$(6n+5)^2(6n+6)^2(6n+7)^6(6n+8)^8(6n+9)^{24}(6n+10)^{32}$$
$$\ldots(12n+6)^{m-(5\cdot 64^n-5)/9}(12n+7)^{(46\cdot 64^n-10)/9}$$

is a length 2 divergent sequence in base $18n+10$, $n\geq 1$.

Again, the interested reader can obtain the proof from the authors.

3. DIVERGENT RATS SEQUENCES OF LENGTH 3, 4, 5, AND 6

In this section, we set out to find divergent RATS sequences of length 3 and longer in different bases. McMullen [5] has discovered the following divergent sequence in base 50. The interested reader can obtain the proof from the authors.

Lemma 7: Let $m\geq 55$. Then

$$1\,3\,4^7\,6^8\,7^m\,8^{40},\,9^2\,(11)^2\,(12)^{14}\,(14)^{m-7}\,(15)^{46},$$
$$(24)^4\,(26)^4\,(27)^{28}\,(28)^{m-35}\,(29)^{56}$$

is a length 3 divergent sequence in base 50.

Using the same proof technique as above, the following two lemmas can be proved.

Lemma 8: Let $m\geq 2591$. Then

$$1\,3\,4^7\,6^8\,7^{56}\,9^{64}\,(10)^{448}\,(12)^{512}\,(13)^{3584}\,(14)^m\,(15)^{7272},$$
$$(16)^2\,(18)^2\,(19)^{14}\,(21)^{16}\,(22)^{112}\,(24)^{128}\,(25)^{896}\,(27)^{1024}\,(28)^{m+4577}\,(29)^{5182},$$
$$(45)^4\,(47)^4\,(48)^{28}\,(50)^{32}\,(51)^{224}\,(53)^{256}\,(54)^{1792}\,(56)^{m+3637}\,(57)^{5976}$$

is a length 3 divergent sequence in base 99.

Lemma 9: Let $m\geq 797131$. Then

$$1\,3\,4^7\,6^8\,7^{56}\,9^{64}\,(10)^{448}\,(12)^{512}$$
$$(13)^{3584}\,(15)^{4096}\,(16)^{28672}\,(18)^{32768}\,(19)^{229376}\,(21)^m\,(22)^{765032},$$
$$(23)^2\,(25)^2\,(26)^{14}\,(28)^{16}\,(29)^{112}\,(31)^{128}\,(32)^{896}\,(34)^{1024}$$
$$(35)^{7168}\,(37)^{8192}\,(38)^{57344}\,(40)^{65536}\,(41)^{458752}\,(42)^{m-465439}\,(43)^{930878},$$
$$(66)^4\,(68)^4\,(69)^{28}\,(71)^{32}\,(72)^{224}\,(74)^{256}\,(75)^{1792}\,(77)^{2048}$$
$$(78)^{14336}\,(80)^{16384}\,(81)^{114688}\,(83)^{131072}\,(84)^{m+120373}\,(85)^{663384}$$

is a length 3 divergent sequence in base 148.

Searching next for a base with a length 4 divergent sequence, we found the following number in base 226. The proof of this lemma is similar to the previous proofs.

Lemma 10: Let $m \geq 8683$. Then

$$1\,4\,5^{15}\,8^{16}\,9^{240}\,(12)^{256}\,(13)^{3840}\,(15)^{m}\,(16)^{10176},$$
$$(17)^{2}\,(20)^{2}\,(21)^{30}\,(24)^{32}\,(25)^{480}\,(28)^{512}\,(29)^{7680}\,(30)^{m-5807}\,(31)^{11614},$$
$$(48)^{4}\,(51)^{4}\,(52)^{60}\,(55)^{64}\,(56)^{960}\,(59)^{1024}\,(60)^{m+6677}\,(61)^{5752},$$
$$(109)^{8}\,(112)^{8}\,(113)^{120}\,(116)^{128}\,(117)^{1920}\,(120)^{m+5089}\,(121)^{7272}$$

is a length 4 divergent sequence in base 226.

Searching next for a base with a length 5 divergent sequence we found the following number in base 962.

Lemma 11: Let $m \geq 1040187391$. Then

$$1\,5\,6^{31}\,(10)^{32}\,(11)^{992}\,(15)^{1024}\,(16)^{31744}\,(20)^{32768}$$
$$(21)^{1015808}\,(25)^{1048576}\,(26)^{32505856}\,(30)^{33554432}\,(31)^{m}\,(32)^{213407584},$$
$$(33)^{2}\,(37)^{2}\,(38)^{62}\,(42)^{64}\,(43)^{1984}\,(47)^{2048}\,(48)^{63488}\,(52)^{65536}$$
$$(53)^{2031616}\,(57)^{2097152}\,(58)^{65011712}\,(62)^{m-780107455}\,(63)^{290432638},$$
$$(96)^{4}\,(100)^{4}\,(101)^{124}\,(105)^{128}\,(106)^{3968}\,(110)^{4096}\,(111)^{126976}\,(115)^{131072}$$
$$(116)^{4063232}\,(120)^{4194304}\,(121)^{130023424}\,(124)^{m-299266427}\,(125)^{442317944},$$
$$(221)^{8}\,(225)^{8}\,(226)^{248}\,(230)^{256}\,(231)^{7936}\,(235)^{8192}\,(236)^{253952}\,(240)^{262144}$$
$$(241)^{8126464}\,(245)^{8388608}\,(246)^{260046848}\,(248)^{m-603037029}\,(249)^{607541224},$$
$$(470)^{16}\,(474)^{16}\,(475)^{496}\,(479)^{512}\,(480)^{15872}\,(484)^{16384}\,(485)^{507904}\,(489)^{524288}$$
$$(490)^{16252928}\,(494)^{16777216}\,(495)^{520093696}\,(496)^{566516434}\,(497)^{m-933483599}\,(498)^{660893120}$$

is a length 5 divergent sequence in base 962.

Lemma 12: Let m be a large positive integer. Then

$$1\,6\,7^{63}\,(12)^{64}\,(13)^{4032}\,(18)^{4096}\,(19)^{258048}\,(24)^{262144}$$
$$(25)^{16515072}\,(30)^{16777216}\,(31)^{1056964608}\,(36)^{1073741824}\,(37)^{67645734912}$$
$$(42)^{68719476736}\,(43)^{4329327034368}\,(48)^{4398046511104}\,(49)^{277076930199552}$$
$$(54)^{281474976710656}\,(55)^{17732923532771328}\,(60)^{18014398509481984}$$
$$(61)^{1134907106097364992}\,(63)^{m}\,(64)^{2472288970952097408}$$

is one of the six numbers in a length 6 divergent sequence in base 3970.

4. ARBITRARILY LONG DIVERGENT RATS SEQUENCES

The arbitrarily long divergent RATS sequences will follow the patterns of the divergent RATS sequences we have seen in the previous chapters. We will come as close to explicitly constructing divergent RATS sequences as possible. We will state the base of operation, the form of the smallest element in the divergent RATS sequence, and the exponents (repetition factors) of each of the elements in the smallest element in the divergent sequence. The only part that will be left to the imagination is one of the exponents, which is a component solution to a particular linear system.

The use of primes and pseudoprimes in the following came about by doing extensive trials on numerous different lengths and noticing a pattern for all lengths that were prime or pseudoprime. This pattern is best described as follows: Divergent sequences of length t, where t is prime or pseudoprime, have three consecutive integers at the end of the first term of the sequence. After one iteration of RATS is complete, there are only two consecutive integers at the end and the prior number is $t-1$ smaller than the first consecutive integer. This "gap" shrinks by one at each successive iteration until in the $(t+1)^{\text{st}}$ iteration; the gap closes and we return to three consecutive integers at the end. Then this pattern begins again. Other patterns may exist for lengths that are not prime or pseudoprime.

Let t be a prime or a pseudoprime in base 2. A pseudoprime in base 2 [7] is a composite number n such that

$$2^n \equiv 2 \pmod{n}.$$

The smallest pseudoprime in base 2 (psp) is 341. Since t is either a prime or a psp, it follows that

$$2^t \equiv 2 \pmod{t}.$$

The preceding considerations lead to Theorem 13.

Theorem 13: Let m be a large positive integer and t be a prime or a pseudoprime in base 2. Let

$$\begin{aligned} a_1 &= 1+1+(2^t-1)+2^t+\cdots+(2^{2^t-2-t}-2^{2^t-2-2t})+(2^{2^t-2-t}) \\ a_2 &= 2(a_1-2^{2^t-2-t}) \\ a_3 &= 2a_2 \\ &\vdots \\ a_t &= 2a_{t-1} \end{aligned}$$

and let

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_t \end{pmatrix}$$

be the solution to

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & -1 & \cdots & 0 \\ 0 & 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_t \end{pmatrix} = \begin{pmatrix} 2a_1 \\ 2a_2 \\ 2a_3 \\ \vdots \\ 2a_t \end{pmatrix}.$$

Then the number

$$1t(t+1)^{2^t-1}(2t)^{2^t}\cdots(2^t-1-t)^{2^{2^t-2-t}-2^{2^t-2-2t}}(2t-2)^{2^{2^t-2-t}}(2^t-1)^m(2^t)^{x_1}$$

is the start of a divergent RATS sequence of length t in base $(2^t-1)^2+1$.

5. QUESTIONS

As we continue to study the game of RATS, we are led to many questions.

First of all, in the first section, the exponent on the next to last number in the sequence for the length 2 divergent sequence in base 19 is 19, the exponent on the next to last number in the sequence for the length 2 divergent sequence in base 37 is 1257, and the exponent on the next to last number in the sequence for the length 2 divergent sequence in base 55 was 80099. Is there a pattern to these exponents? Can an explicit formula be found for these exponents? What about these exponents for divergent sequences of other lengths or in other bases?

Combining Theorem 6 and Lemma 12 gives the following result. That is, in base 3970, there are two different divergent RATS sequences of different lengths, one of length 2 and one of length 6. Can we find other bases with two or more divergent RATS sequences of different lengths?

Theorem 13 proved the existence of length t divergent RATS sequences in base $(2^t-1)^2+1$, where t is a prime or psp. What about the case when t is not a prime and is not a psp?

John Conway has a simple sounding yet tremendously hard conjecture based on his RATS game in base 10. So far, every positive integer with digits in nondecreasing order (up to 15 digits) which starts a RATS game either cycles or enters Conway's Divergent RATS Sequence. Conway conjectures that this is true for every positive integer. This is still an open problem.

Finally we conjecture that there is only one divergent sequence of length t for each base.

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DERIVED SEQUENCES, THE TRIBONACCI RECURRENCE AND CUBIC FORMS

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1. INTRODUCTION

Integer representations by forms are sources of a series of very interesting Diophantine equations. For instance, the cubic form $x^3+y^3+z^3$ represents 1 and 2 in an infinite number of ways, whereas only two representations $(1,1,1)$ and $(4,4,-5)$ are known for the number 3 and it is unknown whether there are other representations. Number 4 has no representations, as was proved using congruential arguments. But, in general, we do not know a definitive criterion for testing if a number is representable as a sum of three cubes, nor a method for finding such a representation (see [12]). For the quadratic forms, the situation is different; Gauss developed his theory of quadratic forms [7] and solved the related integer representation problem. In particular, quadratic forms, computed as first derived sequences using the Simson determinant, are invariant for some second-order linear recurrent sequence. Therefore, each representable integer admits an infinite number of representations deduced from the recurrent sequence (see [17], [6]).

In this paper we discuss integer representations by a cubic form associated with a third-order recurrence known as the Tribonacci recurrence [9]. The technique of derived sequences is used to define an invariant cubic form, computed as second derived sequences [5] of a third-order linear recurrent sequence. Therefore, an infinite number of representations is produced whenever a representation exists. Before stating the problem, let us briefly review the properties of a third-order linear recurrent sequence $\{T_0, T_1, T_2, \dots\}$ defined by the recurrence

$$T_{n+3} = pT_{n+2}+qT_{n+1}+rT_n, \quad p,q,r \in \mathbb{Z}, \tag{1}$$

over $\mathbb{Z}$, the ring of rational integers, with initial integer values $T_2=c$, $T_1=b$, and $T_0=a$. The characteristic polynomial of recurrence (1) is $\Lambda(x)=x^3-px^2-qx-r$, and expressions that allow us to directly compute T_n are

$$T_n = \begin{cases} A\alpha^n+B\beta^n+C\gamma^n, \\ (A+Bn)\alpha^n+C\gamma^n, \\ (A+Bn+Cn^2)\alpha^n, \end{cases} \tag{2}$$

according to whether $\Lambda(x)$ has three simple roots α,β,γ, a double root $\alpha=\beta$ and γ, or a triple root $\alpha=\beta=\gamma$. The second derived sequence $T_n^{(2)}$ (see [5], Vol. I, p. 410) of a third-order recurrent sequence is defined by

$$T_n^{(2)} = \begin{vmatrix} T_n & T_{n+1} & T_{n+2} \\ T_{n+1} & T_{n+2} & T_{n+3} \\ T_{n+2} & T_{n+3} & T_{n+4} \end{vmatrix}.$$

The development of this determinant, using the recurrence (1) to eliminate T_{n+3} and T_{n+4}, yields a cubic form in three variables, T_n, T_{n+1}, and T_{n+2}:

$$\begin{aligned}-T_n^3r^2-(pq+r)T_{n+1}^3-T_{n+2}^3+(3r-pq)T_nT_{n+1}T_{n+2}-2qrT_n^2T_{n+1}-prT_n^2T_{n+2}\\ -(pr+q^2)T_nT_{n+1}^2+qT_nT_{n+2}^2+(q-p^2)T_{n+1}^2T_{n+2}+2pT_{n+1}T_{n+2}^2.\end{aligned} \tag{3}$$

Whereas a closed-form expression for the sequence $T_n^{(2)}$ as a function of n is obtained using (2) before expanding the determinant:

$$T_n^{(2)}=\begin{cases}-r^nABC[(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)]^2,\\ -r^nB^2C[(\alpha-\gamma)^2\alpha]^2,\\ -r^n8C^3\alpha^6,\end{cases} \tag{4}$$

according to the three situations of simple, double, or triple roots.

From expression (4) we can conclude that, whatever may be the root multiplicity of $\Lambda(x)$, $T_n^{(2)}$ satisfies the first-order recurrence $T_n^{(2)}=rT_{n-1}^{(2)}$. From the same equation (4) we can see that the cubic form (3) is an invariant for the sequence T_n if and only if $r=1$. Rewriting expression (3) with $r=1$, and substituting the variables x, y, and z for T_n, T_{n+1}, and T_{n+2}, respectively, we obtain the invariant cubic form $\mathscr{C}(x,y,z)$:

$$-x^3-y^3(pq+1)-z^3+xyz(3-pq)-2qx^2y-px^2z-(p+q^2)xy^2+qxz^2+(q-p^2)zy^2+2pyz^2. \tag{5}$$

The integer representation problem consists of finding all triples $(x_0,y_0,z_0)\in\mathbb{Z}^3$ that are solutions of the Diophantine equation $\mathscr{C}(x_0,y_0,z_0)=m\ \forall m\in\mathbb{Z}$. If a triple (x_0,y_0,z_0) exists, it is called a representation of m and $\mathscr{C}(x,y,z)$ is said to represent m. Otherwise, it is said that $\mathscr{C}(x,y,z)$ does not represent m. From the invariance of $\mathscr{C}(x,y,z)$ it is evident that, using a triple (x_0,y_0,z_0) as an initial condition in the recurrence (1), we get an infinite number of representations for m.

When $p=q=r=1$, expression (1) is known as the Tribonacci recurrence, and the sequences with initial conditions $T_2=1$, $T_1=1$, $T_0=0$, and $T_2=3$, $T_1=1$ $T_0=3$ are called the Tribonacci sequence K_n and the generalized Lucas sequence S_n, respectively (see [9]).

In this paper we investigate the properties of the Tribonacci cubic form $\mathscr{T}(x,y,z)$, which we define as the opposite of the expression obtained by setting $p=q=1$ in (5), namely,

$$\mathscr{T}(x,y,z)=x^3+2y^3+z^3-2xyz+2yx^2+zx^2+2xy^2-xz^2-2yz^2. \tag{6}$$

The related representation problem is fully solved. With this aim, the paper is organized as follows: Section 2 collects the main properties of the Tribonacci recurrence and a related ring that we will call the *Tribonacci ring*; Section 3 studies the Tribonacci cubic form; Section 4 solves the integer representation problem for the Tribonacci cubic form; Section 5 comments on related and open problems.

2. THE TRIBONACCI RECURRENCE AND CUBIC RINGS

Let $\Theta(x)=x^3-x^2-x-1$ denote the characteristic polynomial, called *Tribonacci polynomial*, of the Tribonacci recurrence. The polynomial $\Theta(x)$ is irreducible over the rational field $\mathbb{Q}$ with Galois group $\mathscr{G}_\Theta$ isomorphic to $\mathscr{S}_3$, the symmetric group on three elements. Let us denote by τ the real root and with τ_1 and τ_2 the complex conjugate roots of $\Theta(x)$. These roots, expressed by means of the Tartaglia-Lagrange formulas, are

$$\begin{cases} \tau = \frac{1}{3}\left[1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}\right], \\ \tau_1 = \frac{1}{3}\left[1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}\right], \\ \tau_2 = \frac{1}{3}\left[1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}\right], \end{cases}$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity.

Let $\mathbb{Q}[\tau]$ denote a cubic field generated by the real root τ of the Tribonacci polynomial. The field $\mathbb{Q}[\omega, \tau]$ of complete reducibility for $\Theta(x)$ is obtained from $\mathbb{Q}[\tau]$ by the adjunction of ω. Using (2) and the roots of the Tribonacci polynomial, we get the following explicit expressions for the Tribonacci and the generalized Lucas sequence, respectively:

$$\begin{cases} K_n = \dfrac{\tau^{n+1}}{(\tau-\tau_1)(\tau-\tau_2)} + \dfrac{\tau_1^{n+1}}{(\tau_1-\tau)(\tau_1-\tau_2)} + \dfrac{\tau_2^{n+1}}{(\tau_2-\tau)(\tau_2-\tau_1)}, \\ S_n = \tau^n + \tau_1^n + \tau_2^n. \end{cases}$$

Notice that S_n is the n-power symmetric function of τ, τ_1, and τ_2.

In a ring, the solution of the representation problem for decomposable forms is based on ring factoring properties [1]. In the case of the Tribonacci cubic form, the solution depends on the integral ring of $\mathbb{Q}[\tau]$ that we will call the *Tribonacci ring*.

2.1 The Tribonacci Ring

In this section we will prove two main properties of the integral ring of $\mathbb{Q}[\tau]$:

i) the ring of integers in $\mathbb{Q}[\tau]$ is $\mathbb{Z}[\tau]$, the extension of $\mathbb{Z}$ by the adjunction of τ;

ii) $\mathbb{Z}[\tau]$ is a principal ideal domain (PID).

Let us recall that the norm in $\mathbb{Q}[\tau]$ is defined as

$$\begin{aligned} Nr(a+b\tau+c\tau^2) &= (a+b\tau+c\tau^2)(a+b\tau_1+c\tau_1^2)(a+b\tau_2+c\tau_2^2) \\ &= a^3+b^3+c^3-4abc-c^2(b+a)+b^2(c-a)+a^2(3c+b), \end{aligned} \tag{7}$$

where the product expansion turns out to be an irreducible cubic form ([1], p. 80) over $\mathbb{Z}$.

A direct calculation shows that the basis discriminant is $D[1, \tau, \tau^2] = -44$. Since the norm of τ is unity, i.e., $Nr(\tau) = \tau \cdot \tau_1 \cdot \tau_2 = 1$, then τ is a unit, and the triple $\{1, \tau, \tau^2\}$, hereinafter called *polynomial basis*, is an integral basis for $\mathbb{Q}[\tau]$.

Theorem 1: The triple $\{1, \tau, \tau^2\}$ is an integral basis for the integer ring of $\mathbb{Q}[\tau]$.

Proof: Let us consider an integral basis $\{\omega_1, \omega_2, \omega_3\}$ for the ring of integers in $\mathbb{Q}[\tau]$. Since τ is an integer, then a 3×3 matrix $C = (c_{hj})$ with $c_{hj} \in \mathbb{Z}$ exists such that

$$\begin{cases} 1 = c_{11}\omega_1 + c_{12}\omega_2 + c_{13}\omega_3, \\ \tau = c_{21}\omega_1 + c_{22}\omega_2 + c_{23}\omega_3, \\ \tau^2 = c_{31}\omega_1 + c_{32}\omega_2 + c_{33}\omega_3. \end{cases} \tag{8}$$

The determinant $\det(C)$ is an integer, and (see [14], p. 67) we have

$$D[1, \tau, \tau^2] = \det(C)^2 D[\omega_1, \omega_2, \omega_3] = -2^2 \cdot 11;$$

hence, possible values for $\det(C)$ are ± 1 and ± 2. Let us first observe that it is sufficient to consider positive values for $\det(C)$, since negative values only correspond to a different ordering of the basis elements. Let $C_{jh} \in \mathbb{Z}$ denote the cofactor of c_{hj}. If $\det(C) = 1$, then, inverting (8), we have

$$\begin{cases} \omega_1 = C_{11} \cdot 1 + C_{12}\tau + C_{13}\tau^2, \\ \omega_2 = C_{21} \cdot 1 + C_{22}\tau + C_{23}\tau^2, \\ \omega_3 = C_{31} \cdot 1 + C_{32}\tau + C_{33}\tau^2, \end{cases} \tag{9}$$

and $\{1, \tau, \tau^2\}$ is evidently an integral basis for $\mathbb{Q}[\tau]$.

If $\det(C) = 2$, then, inverting (8), we obtain

$$\begin{cases} \omega_1 = \dfrac{C_{11} \cdot 1 + C_{12}\tau + C_{13}\tau^2}{2}, \\ \omega_2 = \dfrac{C_{21} \cdot 1 + C_{22}\tau + C_{23}\tau^2}{2}, \\ \omega_3 = \dfrac{C_{31} \cdot 1 + C_{32}\tau + C_{33}\tau^2}{2}, \end{cases} \tag{10}$$

where at least one of the coefficients C_{jh} is an odd number, otherwise $\det(C^{-1})$ is an integer, contrary to the assumption that $\det(C^{-1}) = 1/2$.

To demonstrate that $\det(C) = 2$ is not possible, let us compute the norm $Nr(\omega_j) = \frac{1}{8}\Omega$, where Ω is the cubic form

$$C_{j1}^3 + C_{j2}^3 + C_{j3}^3 - 4C_{j1}C_{j2}C_{j3} - C_{j3}^2(C_{j2} + C_{j1}) + C_{j2}^2(C_{j3} - C_{j1}) + C_{j1}^2(3C_{j3} + C_{j2}).$$

Since $Nr(\omega_j)$ is an integer, Ω must be a multiple of 8, which will turn out to be impossible unless all C_{jh} are even, a case already excluded. In fact, taking the congruence modulo 2 of the expression between square brackets, we find the condition $C_{j1} + C_{j2} + C_{j3} = 0$, $j = 1, 2, 3$, where, for at least one j, one addend is even and two addends are odd. For instance, let $C_{j1} = 2a$, $C_{j2} = 2b+1$, and $C_{j3} = 2c+1$ be substituted in the above bracketed expression, then, taking the congruence modulo 4 of the resulting expression, we have

$$(2b+1)^3 + (2c+1)^3 - (2c+1)^2(2b+1+2a) + (2b+1)^2(2c+1-2a) = 2 \mod 4.$$

This result shows that Ω is twice an odd number; hence, it cannot be a multiple of 8. The other two cases, $C_{j1} = 2a+1, C_{j2} = 2b, C_{j3} = 2c+1$ and $C_{j1} = 2a+1, C_{j2} = 2b+1, C_{j3} = 2c$, respectively, yield the same conclusion. Therefore, $\det(C) = 2$ is not possible. Thus, the ring of integers in $\mathbb{Q}[\tau]$ is the integral extension $\mathbb{Z}[\tau]$, and any integer in $\mathbb{Q}[\tau]$ is of the form $a + b\tau + c\tau^2$, with $a, b, c \in \mathbb{Z}$. $\square$

It follows from the above proposition that the field discriminant is $D_{\mathbb{Q}[\tau]/\mathbb{Q}} = -44$. Moreover, we have already observed that $Nr(\tau) = 1$. Since $\Theta(x)$ has one real root and two complex roots, Dirichlet's theorem ([1], p. 112) allows us to conclude that the multiplicative group $\mathbb{U}[\tau]$ of the units in $\mathbb{Z}[\tau]$ is an Abelian group generated by -1 and τ. In other terms, $\mathbb{U}[\tau] = \mathbb{C}_2 \times \mathbb{C}_\infty$, where $\mathbb{C}_2 = \{1, -1\}$ and $\mathbb{C}_\infty$ is a cyclic group of infinite order isomorphic to the additive group of $\mathbb{Z}$.

We end this section by showing that $\mathbb{Z}[\tau]$ is a principal ideal domain (PID), consequently, $\mathbb{Z}[\tau]$ is a unique factorization domain. For this purpose, let us recall three basic concepts:

- In $\mathbb{Q}[\tau]$, a notion of ideal equivalence is introduced by defining two ideals $\mathcal{V}_1$ and $\mathcal{V}_2$ as equivalent if a rational number $u \in \mathbb{Q}$ exists such that $\mathcal{V}_1 = u\mathcal{V}_2$.
- Given a nonzero ideal $\mathcal{B}$ of a ring $\mathbb{R}$, we call the number card $(\mathbb{R}/\mathcal{B})$ (the cardinality of the quotient $\mathbb{R}/\mathcal{B}$) the norm of $\mathcal{B}$ and denote it by $Nr(\mathcal{B})$ (see [15], p. 52).
- $\mathbb{Z}[\tau]$ is the only ideal with norm 1 in $\mathbb{Q}[\tau]$.

Theorem 2: The ring $\mathbb{Z}[\tau]$ is a principal ideal domain.

Proof: The ring $\mathbb{Z}[\tau]$ is a Dedekind ring; hence, it is integrally closed and, by Corollary 1 on page 58 of [15], every ideal class contains an integral ideal $\mathcal{B}$ such that

$$Nr(\mathcal{B}) \le \left(\frac{4}{\pi}\right)\frac{3!}{3^3}|-44|^{1/2} \le 1.877 < 2,$$

and this means that $\mathbb{Z}[\tau]$ belongs to every ideal class, so that in $\mathbb{Z}[\tau]$ every ideal is principal (see [1], p. 231). □

Remark 1: Dedekind gave a pictorial description of an ideal number in his masterful survey paper on the theory of algebraic integers [4]. Ideal numbers were introduced by Dirichlet and Kummer in order to recover unique factorization in algebraic number fields. A "true" ideal number "*is never defined in its own right, but only as a divisor of actual number* ω" ([4], p. 94) in the ring. If the unique factorization holds in the ring of integers of an algebraic number field, then no ideal number exists and all ideals belong to a single equivalence class. Whereas, if unique factorization does not hold true, then a "true" ideal number of Kummer occurs. "True" ideal numbers produce different classes of ideals in the algebraic number field, and these classes cannot contain the integral ring as an element.

3. THE TRIBONACCI CUBIC FORM

We defined the Tribonacci cubic form $\mathcal{T}(x, y, z)$ as having a positive x^3 coefficient. This assumption is not restrictive, since a sign change of the three variables corresponds to a sign change of the form, that is,

$$\mathcal{T}(-x, -y, -z) = -\mathcal{T}(x, y, z). \tag{11}$$

The transformation (11) belongs to a set of variable substitutions that specify the equivalence of forms. This concept, together with the notions of reducibility and decomposability of forms, is elemental to classify $\mathcal{T}(x, y, z)$. Let us recall, for the sake of reference, their definitions from [1].

Definition 1: Let all cubic forms considered have coefficients in $\mathbb{Z}$:

i) Two cubic forms $\mathcal{C}'(y_1, y_2, y_3)$ and $\mathcal{C}(x_1, x_2, x_3)$ are called *equivalent* if there is a nonsingular linear change of variables which takes one form to the other. The transformation is characterized by a matrix with integer entries and its determinant is ± 1.

ii) A cubic form is said to be *irreducible* over $\mathbb{Q}$ if it cannot be written as a product of a linear form and a quadratic form with coefficients in $\mathbb{Q}$.

iii) A cubic form is said to be *decomposable* over $\mathbb{Q}$ if it can be written as a product of linear forms with coefficients in some finite algebraic extension of $\mathbb{Q}$; it is called nondecomposable otherwise.

The above definition has its nearly direct consequence in the following facts:

1. Equivalence of forms is an equivalence relation.
2. The variable transformation is invertible, and the matrix of the inverse transformation has integer coefficients.
3. Equivalent forms represent the same set of integers.
4. Any cubic form is always equivalent to a form with at most eight nonzero coefficients.
5. Reducible cubic forms with integer coefficients are reducible over $\mathbb{Z}$.
6. Cubic forms with integer coefficients, which are decomposable in a finite algebraic extension $\mathfrak{Q}$ of $\mathbb{Q}$, are decomposable in the integral ring of $\mathfrak{Q}$.

Proposition 1: The Tribonacci cubic form $\mathcal{T}(x,y,z)$ is irreducible in $\mathbb{Q}$, decomposable in $\mathbb{Q}[\tau,\omega]$ and equivalent to a form $\mathcal{C}(t,u,v)$ with eight nonzero coefficients.

Proof: The equivalent form $\mathcal{C}(t,u,v)$ with eight nonzero coefficients,

$$u^3+4v^3-7t^3-2tuv+9ut^2+2vt^2-5tu^2-4uv^2, \tag{12}$$

is obtained performing on $\mathcal{T}(x,y,z)$ the variable substitution

$$\begin{cases} x=-t+v, \\ y=v, \\ z=-2t+v+u. \end{cases}$$

Irreducibility is easily proved by setting $y=0$ and $z=-1$ to obtain the irreducible polynomial $\mathcal{T}(x,0,-1)=x^3-x^2-x-1$ in a single variable.

Decomposability is proved by factoring $\mathcal{T}(x,y,z)$ into a linear and a quadratic form over the real field $\mathbb{Q}[\tau]$. The full decomposition into three linear forms over $\mathbb{Q}[\omega,\tau]$ is obtained by taking the conjugate of the linear factor under the Galois group $\mathcal{G}_\Theta$.

Let us consider the real decomposition

$$\mathcal{T}(x,y,z)=(x+ay+bz)(x^2+cy^2+dz^2+exy+fyz+gxz); \tag{13}$$

therefore, we obtain the following system of nine equations in seven variables from the comparison of the coefficients in their expanded version with the coefficients of equal monomials in (6):

$$\begin{cases} ac=2 & \Rightarrow & c=2/a=a^2-2a+2, \\ bd=1 & \Rightarrow & d=b^2-b-1, \\ ag+be+f=-2 & \Rightarrow & f=-2-a-2b+2ab, \\ e+a=2 & \Rightarrow & e=2-a, \\ c+ae=2 & \Rightarrow & a^3-2a^2+2a-2=0, \\ af+bc=0 & \Rightarrow & f=-b(a^2-a), \\ ad+bf=-2 & \Rightarrow & f=-2(b^2-b-1)-a(2b-b^2), \\ d+bg=-1 & \Rightarrow & b^3-b^2-b-1=0, \\ b+g=1 & \Rightarrow & g=1-b. \end{cases}$$

The system is compatible. Moreover, the factorization (13) takes place in $\mathbb{Q}[\tau]$ because we can express both b and a in terms of τ. Actually, we have $b=\tau$ and, from the birational substitution $b=1/(a-1)$ that transforms the equation $\Theta(b)=b^3-b^2-b-1=0$ into the equation $a^3-2a^2+2a-2=0$, we get $a=(1+b)/b=(1+\tau)/\tau$. The coefficients of the decomposition (13) are explicitly

$$a=\frac{\tau+1}{\tau}=(\tau-1)\tau,\ b=\tau,\ c=2\frac{\tau}{\tau+1}=-\tau^2+2\tau+1,\ d=\tau^2-\tau-1,$$
$$e=\frac{\tau-1}{\tau}=-\tau^2+\tau+2,\ f=-\tau^2-1,\ g=1-\tau.$$

Over the complex extension $\mathbb{Q}[\omega,\tau]$, $\mathcal{T}(x,y,z)$ decomposes into three linear factors as

$$\mathcal{T}(x,y,z)=(\tau x+(1+\tau)y+\tau^2z)(\tau_1x+(1+\tau_1)y+\tau_1^2z)(\tau_2x+(1+\tau_2)y+\tau_2^2z). \tag{14}$$

It is evident that the Tribonacci cubic is a norm in $\mathbb{Q}[\tau]$ with respect to the integral basis $\{\tau, 1+\tau, \tau^2\}$, hereinafter called *Tribonacci basis.*

Remark 2: Along with a cubic form, it is interesting to consider the cubic curve in a complex projective plane with homogeneous coordinates x, y, and z, defined by the equation $\mathcal{C}(x,y,z)=0$ (see [11]). Whenever the cubic curve has a singular point, a translation of the singular point into the origin usually yields to a simpler expression of cubic form. However, the curve $\mathcal{T}(x,y,z)=0$ has no singular point, in fact it is a degenerate curve which is the product of three straight lines that are not concurrent in a single point. Since the Tribonacci cubic form cannot be simplified using this artifice, it is likely that the reduced form (12) with eight coefficients is the simplest one possible.

4. THE TRIBONACCI CUBIC AND REPRESENTATION OF INTEGERS

The Tribonacci cubic gives infinitely many representations for $m=-1$ and $m=-44$ by Tribonacci and generalized Lucas sequences, respectively, and no representation for $m=3$. As a consequence of (14), the representation problem for the Tribonacci cubic can be completely solved, since $\mathbb{Z}[\tau]$ is a PID. In particular, it is evident that rational primes, which are still primes in $\mathbb{Z}[\tau]$, are not represented by the Tribonacci cubic. The following theorem fully characterizes the rational primes of $\mathbb{Z}[\tau]$.

Theorem 3: A prime p in $\mathbb{Z}$ is also prime in $\mathbb{Z}[\tau]$ if and only if $\Theta(x)$ is irreducible over $\mathbb{Z}_p$.

Proof: First, let us assume that p is a rational prime in $\mathbb{Z}[\tau]$, then the set of residues $\mathbb{Z}[\tau]_p$ modulo p is a field isomorphic to $GF(p^3)$ with basis $\{1,\tau,\tau^2\}$, so $\Theta(x)$ is irreducible over $\mathbb{Z}_p$.

Second, let us assume that $\Theta(x)$ is irreducible over $\mathbb{Z}_p$; therefore, the Galois field $GF(p^3)$ is generated by a root of $\Theta(x)$. If we assume that p factors properly in $\mathbb{Z}[\tau]$, then we have a decomposition

$$\begin{aligned}p=(a_0+a_1\tau+a_2\tau^2)\cdot[&(a_0^2-a_1^2+a_0a_1+3a_0a_2-2a_1a_2)\\&+(-a_1^2+2a_2^2-a_0a_1-a_1a_2)\tau+(a_1^2-a_2^2-a_0a_2+a_1a_2)\tau^2],\end{aligned}$$

where $a_0,a_1,a_2\in\mathbb{Z}$. Taking the congruence modulo p, we see that $(\tilde{a}_0+\tilde{a}_1\tau+\tilde{a}_2\tau^2)$, where $\tilde{a}_0,\tilde{a}_1,\tilde{a}_2\in\mathbb{Z}_p$ is a zero divisor in $GF(p^3)$, a contradiction. Thus, p is a prime in $\mathbb{Z}[\tau]$. □

Remark 3: Theorem 3 is a reformulation adapted to our cubic field of the well-known fact that rational Gaussian primes are primes $p\,(=4k+3)$ for which -1 is a quadratic nonresidue; in other words, the polynomial x^2+1 is irreducible over $\mathbb{Z}_p$. Whereas, for primes $p=4k+1$, since -1 is a quadratic residue, the polynomial x^2+1 is reducible over $\mathbb{Z}_p$. Hence, p factors over $\mathbb{Z}[i]$, i.e., $p=x^2+y^2$ has a solution in rational integers x and y (see [4]).

Remark 4: It is easy to check whether an integral polynomial $m(x)$ is irreducible over $GF(p)$ by computing the greatest common civisor (GCD), via the Euclidean algorithm, with $x^{p-1}-1$. If the GCD is 1, then $m(x)$ is irreducible, otherwise we get the product of its linear irreducible factors, possibly $m(x)$ itself.

It follows from Theorem 3 that 3, 5, 23, 31, 37, ... are primes in $\mathbb{Z}[\tau]$; therefore, they are not represented by the Tribonacci cubic. In the next table, we summarize the representation of the rational primes up to 29 that factor in $\mathbb{Z}[\tau]$. The prime factors are explicitly written in the polynomial basis, whereas the representing triples are given in the Tribonacci basis, which is useful to initiate the Tribonacci recurrence generating an infinite number of representations. The rational primes 2 and 11 are factors of the discriminant D; hence, they are the only primes that ramify in $\mathbb{Z}[\tau]$ (see [15], Theorem 1, p. 58). In particular, they are the only rational primes divisible by a square of a prime in $\mathbb{Z}[\tau]$.

$$\begin{array}{rcll}
2 &=& (1+\tau)(1-\tau)^2 & \Rightarrow (0,1,0), \\
2^s &=& (1+\tau)^s(1-\tau)^{2s} & \Rightarrow (a,b,c), \\
11 &=& (3+4\tau+4\tau^2)(3-2\tau)^2 & \Rightarrow (1,3,4), \\
7 &=& (1+2\tau)(-1-6\tau+4\tau^2) & \Rightarrow (1,1,0), \\
13 &=& (3+\tau-\tau^2)(4-\tau+2\tau^2) & \Rightarrow (-2,3,-1), \\
17 &=& (-1+2\tau^2)(-5+8\tau-2\tau^2) & \Rightarrow (1,-1,2), \\
19 &=& (-2-\tau+2\tau^2)(-3+7\tau-1\tau^2) & \Rightarrow (1,-2,2), \\
29 &=& (2+3\tau)(1-15\tau+9\tau^2) & \Rightarrow (1,2,0).
\end{array}$$

Finally, we give conditions that are necessary and sufficient to represent an integer M by the Tribonacci form.

Theorem 4: A rational integer $M \in \mathbb{Z}$ is represented by the Tribonacci form if and only if its prime decomposition is

$$M = \pm 2^a 11^b \prod_{j=1}^{s} p_j^{m_j} \prod_{h=1}^{r} q_h^{3n_h},$$

where p_j are distinct rational primes that factor in $\mathbb{Z}[\tau]$, and q_h are distinct rational primes in $\mathbb{Z}[\tau]$ with $a, b \geq 0$.

Proof: From the norm product property, it follows that a representation for M is obtained as a product of the prime power factor representations. Therefore, the conclusion stems from the following facts:

a) $\mathbb{Z}[\tau]$ is a PID;

b) let (a_j, b_j, c_j) be a representation of p_j, then $N(a_j\tau + b_j(\tau+1) + c_j\tau^2) = p_j$;

c) any cube $q_h^{3n_h}$ is represented as $(q_h^{n_h}, 0, 0)$, given that 1 is represented as $(1, 0, 0)$;

d) neither $q_h^{3n_h+1}$ nor $q_h^{3n_h+2}$ is represented, because q_h is not represented, and they are not cube-powers. □

5. REMARKS AND CONCLUSIONS

In this paper we have introduced the Tribonacci cubic form $\mathcal{T}(x, y, z)$ and solved the related representation problem. To this end, we have described the Tribonacci ring, namely, the ring of integers in the real cubic field $\mathbb{Q}[\tau]$ containing the real root τ of the Tribonacci polynomial $x^3 - x^2 - x - 1$. In particular, we have computed the integral basis, the discriminant, the group of the units, and we have shown that the integral ring $\mathbb{Z}[\tau]$ is a principal ideal ring.

The integer representation problem for cubic forms, in general, has unpredictable features, unlike the one for binary quadratic forms. For instance, the equation $z(x^2+y^2)=m$ has a finite number $N(m)$ of solutions for every m, depending on the factorization of m, a solution being the triple $(1, 0, m)$. In the Introduction we recalled that the cubic $x^3+y^3+z^3$ despite numerous attempts still remains unsolved (see [12]). Unfortunately, this cubic is neither a Tribonacci cubic nor does it seem to be equivalent to any invariant cubic of a third-order recurrence. Therefore, in this context, cubic forms like the Tribonacci cubic, yielding none or infinitely many representations for every integer, have a rather regular behavior.

In conclusion, although Gauss began the study of integer representations by cubic forms, the theory is far from complete, unlike the theory of binary quadratic forms, but this is a challenging source of beautiful problems.

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MORGAN-VOYCE POLYNOMIAL DERIVATIVE SEQUENCES

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1. INTRODUCTION

The aim of this paper is to study the main properties of the derivatives $B_n^{(1)}(x)$ and $C_n^{(1)}(x)$ of the *Morgan-Voyce polynomials* $B_n(x)$ and $C_n(x)$ (e.g., see [8]) described in the next section. Here x is an indeterminate and the bracketed superscript symbolizes the first derivative with respect to x. As done in previous papers, we shall confine ourselves to considering the case $x = 1$. For notational convenience, the terms $B_n^{(1)}(1)$ and $C_n^{(1)}(1)$ will be denoted by R_n and S_n, respectively.

Various papers have dealt with this kind of polynomial pairs. For example, the polynomial pairs (Fibonacci, Lucas), (Pell, Pell-Lucas), and (Jacobsthal, Jacobsthal-Lucas) have been studied in [1], [2], [3], [4], [9], and [10].

The paper is set out as follows. After recalling some background material on the Morgan-Voyce polynomials, we show first some basic properties of the numbers R_n and S_n the most interesting of which are, perhaps, expressions for sums and differences involving subscript sums and differences (see Section 3.3). In Section 4, we evaluate certain finite sums involving R_n and S_n. We conclude the paper with some properties of divisibility and the primality of R_n and S_n.

1.1 Some Useful Results for Fibonacci and Lucas Numbers F_n, L_n

Binet forms are

$$F_n = (a^n - b^n)/\sqrt{5}, \tag{1.1}$$

$$L_n = a^n + b^n, \tag{1.2}$$

where a and b are the roots of the equation $t^2 - t - 1 = 0$, i.e.,

$$a = (1+\sqrt{5})/2,\ \ b = (1-\sqrt{5})/2 \ \ (\text{so } a+b=1,\ ab=-1,\ a-b=\sqrt{5}). \tag{1.3}$$

From (1.1)-(1.3), it follows readily that

$$F_{n+2p} + F_{n-2p} = F_n L_{2p}, \tag{1.4}$$

$$F_{n+2p} - F_{n-2p} = L_n F_{2p}, \tag{1.5}$$

$$L_{n+2p} + L_{n-2p} = L_n L_{2p}, \tag{1.6}$$

$$L_{n+2p} - L_{n-2p} = 5F_n F_{2p}. \tag{1.7}$$

Some relationships among Fibonacci, Lucas, and Morgan-Voyce polynomials that are applicable to the development of our theme include

$$xB_n(x^2) = F_{2n}(x), \tag{1.8}$$

$$C_n(x^2) = L_{2n}(x). \tag{1.9}$$

These occur as (4.1) and (4.3) of [8]. Substituting $x = 1$ in this pair of relationships yields

$$B_n = F_{2n}, \tag{1.8a}$$

$$C_n = L_{2n}, \tag{1.9a}$$

where $B_n := B_n(1), \ldots$.

Background information on the Fibonacci and Lucas numbers may be found in [11].

2. BACKGROUND MATERIAL

Consider the polynomial sequence $\{X_n(x)\}$ defined by the recurrence

$$X_n(x) = (x+2)X_{n-1}(x) - X_{n-2}(x) \quad (n \geq 2) \tag{2.1}$$

with initial conditions

$$X_0(x) = a_0, \;\; X_1(x) = a_1 \quad (a_0, a_1 \text{ integers}). \tag{2.2}$$

Special cases for the Morgan-Voyce polynomials $B_n(x)$ and $C_n(x)$ are:

$$\begin{cases} (a_0, a_1) = (0, 1) & \text{if } X_n(x) \equiv B_n(x), \\ (a_0, a_1) = (2, 2+x) & \text{if } X_n(x) \equiv C_n(x). \end{cases} \tag{2.3}$$

It has to be pointed out that, in the very special case $x = 0$, we have

$$B_n(0) = n \quad \text{and} \quad C_n(0) = 2 \;\forall n. \tag{2.4}$$

Combinatorial expressions for the above polynomials are

$$B_n(x) = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} x^k \quad (n \geq 0) \quad [8, (2.20)], \tag{2.5}$$

$$C_n(x) = \sum_{k=0}^{n-1} \frac{2n}{n-k} \binom{n+k-1}{2k} x^k + x^n \quad (n \geq 1) \quad [8, (3.22)]. \tag{2.6}$$

Observe that, if we assume that $0^0 = 1$ (see [5] for some considerations on this assumption), then (2.5) and (2.6) hold also for $x = 0$ [cf. (2.4)].

Binet forms are

$$B_n(x) = (\alpha^n - \beta^n)/\Delta, \tag{2.7}$$

$$C_n(x) = \alpha^n + \beta^n, \tag{2.8}$$

where the roots $\alpha := \alpha(x)$, $\beta := \beta(x)$ of the characteristic equation $t^2 - (x+2)t + 1 = 0$ are

$$\alpha = (x+2+\Delta)/2, \;\; \beta = (x+2-\Delta)/2 \tag{2.9}$$

so that

$$\alpha + \beta = x+2, \;\; \alpha\beta = 1, \;\; \alpha - \beta = \Delta := \Delta(x) = \sqrt{x(x+4)}. \tag{2.10}$$

Clearly, (1.3) contrasted with (2.9) and (2.10) together reveals that $a^2 = \alpha(1)$, $b^2 = \beta(1)$.

Notice that

$$\alpha^{(1)} := \frac{d\alpha(x)}{dx} = \frac{\alpha}{\Delta}, \;\; \beta^{(1)} := \frac{d\beta(x)}{dx} = -\frac{\beta}{\Delta}, \tag{2.11}$$

leading to

$$(\alpha^n)^{(1)} = \frac{n\alpha^n}{\Delta},\ (\beta^n)^{(1)} = -\frac{n\beta^n}{\Delta}, \tag{2.12}$$

and

$$\Delta^{(1)} := \frac{d\Delta(x)}{dx} = \frac{x+2}{\Delta(x)}. \tag{2.13}$$

3. SOME ELEMENTARY PROPERTIES OF R_n AND S_n

3.1 Basics

From (2.5) and (2.6) we immediately obtain the derivatives

$$B_n^{(1)}(x) := \frac{d}{dx}B_n(x) = \sum_{k=0}^{n-1} k\binom{n+k}{2k+1}x^{k-1} \quad (n \ge 0), \tag{3.1}$$

$$C_n^{(1)}(x) := \frac{d}{dx}C_n(x) = \sum_{k=0}^{n-1} \frac{2nk}{n-k}\binom{n+k-1}{2k}x^{k-1} + nx^{n-1} \quad (n \ge 1). \tag{3.2}$$

For example, $B_4^{(1)}(x) = 3x^2 + 12x + 10$ and $C_4^{(1)}(x) = 4x^3 + 24x^2 + 40x + 16$.

When $x = 1$, the following table can be constructed $[B_n^{(1)}(1) := R_n,\ C_n^{(1)}(1) := S_n]$ from (3.1) and (3.2).

TABLE 1. Values of R_n and S_n for $0 \le n \le 10$

n	0	1	2	3	4	5	6	7	8	9	10
R_n	0	0	1	6	25	90	300	954	2939	8850	26195
S_n	0	1	6	24	84	275	864	2639	7896	23256	67650

Observe that the value of R_0 can be obtained by letting $x = 1$ in (3.1) with the assumption that a sum vanishes whenever the upper range indicator is less than the lower one. The value of S_0 comes from the fact that the initial condition $C_0(x) = 2$ is independent of x.

Using the Binet forms (2.7) and (2.8) with (2.12) and (2.13), we deduce that

$$B_n^{(1)}(x) = [nC_n(x) - (x+2)B_n(x)]/\Delta^2, \tag{3.3}$$

$$C_n^{(1)}(x) = nB_n(x) \quad \text{as in [8, (3.24)]}, \tag{3.4}$$

whence

$$R_n := B_n^{(1)}(1) = (nL_{2n} - 3F_{2n})/5 \quad \text{[by (1.8a) and (1.9a)]}, \tag{3.5}$$

$$S_n := C_n(1) = nF_{2n} \quad \text{[by (1.10a)]}, \tag{3.6}$$

results which are of subsequent application.

3.2 Negative Subscripts

Direct differentiation of $B_{-n}(x) = -B_n(x)$, $C_{-n}(x) = C_n(x)$ [8, (5.1), (5.2)] yields

$$R_{-n} = -R_n, \tag{3.7}$$

$$S_{-n} = S_n. \tag{3.8}$$

3.3 Sums and Differences Involving Subscript Sums and Differences

Routine algebraic computation applied to standard Fibonacci and Lucas number knowledge [see (1.4)-(1.7)] with (3.5) produces the identities

$$R_{n+p}+R_{n-p}=L_{2p}R_n+F_{2n}S_p, \tag{3.9}$$

$$R_{n+p}-R_{n-p}=L_{2p}R_p+F_{2p}S_n, \tag{3.10}$$

with special cases

($p=1$):

$$R_{n+1}+R_{n-1}=3R_n+F_{2n}, \tag{3.11}$$

$$R_{n+1}-R_{n-1}=nF_{2n}=S_n \quad \text{by (3.6)}. \tag{3.12}$$

($p=n$):

$$R_{2n}=L_{2n}R_n+nF_{2n}^2=L_{2n}R_n+F_{2n}S_n. \tag{3.13}$$

Furthermore, with (3.6),

$$S_{n+p}+S_{n-p}=L_{2p}S_n+L_{2n}S_p, \tag{3.14}$$

$$S_{n+p}-S_{n-p}=nL_{2n}F_{2p}+pF_{2n}L_{2p}, \tag{3.15}$$

whence

$$S_{2n}=2L_{2n}S_n. \tag{3.16}$$

4. EVALUATION OF SOME FINITE SUMS FOR R_n AND S_n

As a calculational aid in the ensuing investigations, we need the following identities [1], [7] which are valid for arbitrary y:

$$\sum_{r=0}^{k} ry^r=[ky^{k+2}-(k+1)y^{k+1}+y]/(y-1)^2, \tag{4.1}$$

$$\sum_{r=0}^{k}\binom{k}{r}y^r=(y+1)^k, \tag{4.2}$$

$$\sum_{r=0}^{k}\binom{k}{r}ry^r=ky(y+1)^{k-1}. \tag{4.3}$$

Proposition 1:

$$\sum_{r=0}^{k} R_r=1+(kL_{2k+1}-L_{2k}-3F_{2k+1})/5. \tag{4.4}$$

Proposition 2:

$$\sum_{r=0}^{k} S_r=kF_{2k+1}-F_{2k}. \tag{4.5}$$

Proof of Proposition 1: Taking (3.5) into account, rewrite the left-hand side of (4.4) as

$$\begin{aligned}
&\frac{1}{5}\left[\sum_{r=0}^{k} rL_{2r}-3\sum_{r=0}^{k} F_{2r}\right]\\
&=\frac{1}{5}\left[\sum_{r=0}^{k} r(a^{2r}+b^{2r})-\frac{3}{\sqrt{5}}\sum_{r=0}^{k}(a^{2r}-b^{2r})\right] \quad \text{by (1.1), (1.2)}\\
&=[kL_{2k+2}-(k+1)L_{2k}+2-3(F_{2k+1}-1)]/5 \quad \text{by (4.1) with } y=a^2,b^2\\
&=(kL_{2k+1}-L_{2k}-3F_{2k+1}+5)/5. \ \square
\end{aligned}$$

Identity (4.5) can be proved in a way similar to that for (4.4).

Proposition 3:

$$\sum_{r=0}^{k}\binom{k}{r}R_r=\begin{cases}5^{(k-2)/2}(kF_{k+1}-3F_k) & (k\text{ even}),\\ 5^{(k-3)/2}(kL_{k+1}-3L_k) & (k\text{ odd}).\end{cases} \tag{4.6}$$

Proposition 4:

$$\sum_{r=0}^{k}\binom{k}{r}S_r=\begin{cases}5^{(k-2)/2}kL_{k+1} & (k\text{ even}),\\ 5^{(k-1)/2}kF_{k+1} & (k\text{ odd}).\end{cases} \tag{4.7}$$

To prove Propositions 3 and 4, we need the following lemmas.

Lemma 1:

$$\sum_{r=0}^{k}\binom{k}{r}F_{2r}=\begin{cases}5^{k/2}F_k & (k\text{ even}),\\ 5^{(k-1)/2}L_k & (k\text{ odd}).\end{cases} \tag{4.8}$$

Lemma 2:

$$\sum_{r=0}^{k}\binom{k}{r}rL_{2r+m}=\begin{cases}5^{k/2}kF_{k+m+1} & (k\text{ even}),\\ 5^{(k-1)/2}kL_{k+m+1} & (k\text{ odd}).\end{cases} \tag{4.9}$$

Lemma 3:

$$\sum_{r=0}^{k}\binom{k}{r}rF_{2r+m}=\begin{cases}5^{(k-2)/2}kL_{k+m+1} & (k\text{ even}),\\ 5^{(k-1)/2}kF_{k+m+1} & (k\text{ odd}).\end{cases} \tag{4.10}$$

To prove these three lemmas, use (1.1)-(1.2) along with (4.2)-(4.3) while recalling the key relationships $a^2+1=a\sqrt{5}$ and $b^2+1=-b\sqrt{5}$ deduced from (1.3).

Proof of Proposition 3 (a sketch): From (3.5), rewrite the left-hand side of (4.6) as

$$\frac{1}{5}\sum_{r=0}^{k}\binom{k}{r}(rL_{2r}-3F_{2r}),$$

whence the right-hand side of (4.6) can be obtained after some algebraic enterprises involving (4.8), and (4.9) with $m=0$. □

With the aid of Lemma 3, Proposition 4 can be proved in a similar way.

5. SOME DIVISIBILITY PROPERTIES OF R_n AND S_n

In this section, the divisibility of R_n and S_n by the first three primes is investigated. To save space, only Proposition 7 is proved in detail. A glimpse to the primality of the integers under study is caught at the end of the section.

Proposition 5: ***(i)*** R_n is odd iff $n=2(3k\pm1)$, while ***(ii)*** S_n is odd iff $n=6k\pm1$.

Proposition 6: ***(i)*** R_n is divisible by 3 iff either $n=3k$ or $n=6k\pm1$, while ***(ii)*** S_n is divisible by 3 iff either $n=2k$ or $n=3k$.

Corollary to Propositions 5 and 6: Both R_n and S_n are divisible by 3 iff they are even.

Proposition 7: ***(i)*** R_n is divisible by 5 iff either $n=5k$ or $n=5k\pm1$, while ***(ii)*** S_n is divisible by 5 iff $n=5k$.

Proof: The proof of (ii) is trivial as it is based on (3.6) and the well-known fact that F_n is divisible by 5 iff n is. As for (i), from (ii) and (3.12) we can say that

$$R_{n+1} \equiv R_{n-1} \pmod 5 \Leftrightarrow n \equiv 0 \pmod 5. \tag{5.1}$$

Further, from the recurrence $R_n = 3R_{n-1} - R_{n-2} + F_{2n-2}$ [that is readily obtained by calculating at $x = 1$ the first derivative with respect to x of both sides of (2.1) with $X \equiv B$, and using (1.8a)], and from the conditions on n for F_n to be divisible by 5, we have that

$$R_{n+1} \equiv 3R_n - R_{n-1} \pmod 5 \Leftrightarrow n \equiv 0 \pmod 5. \tag{5.2}$$

From (5.1)-(5.2) we can write $2R_{n+1} \equiv 3R_n \pmod 5 \Rightarrow n \equiv 0 \pmod 5$, that is,

$$R_{n+1} \equiv -R_n \pmod 5 \Rightarrow n \equiv 0 \pmod 5. \tag{5.3}$$

From (5.3) and (5.1), it remains to prove that

$$n \equiv 0 \pmod 5 \Rightarrow R_n \equiv 0 \pmod 5. \tag{5.4}$$

Put $n = 5k$ in (3.5) thus getting

$$R_{5k} = kL_{10k} - 3F_{10k}/5. \tag{5.5}$$

On using (2.4)-(2.4′) of [6], we can express $F_{10k}/5$ (for k even) as

$$\sum_{r=1}^{k/2}(F_{20r-17} + F_{20r-14} + F_{20r-4} + F_{20r-7} + F_{20r-9} + F_{20r-11}) \tag{5.6}$$

and (for k odd)

$$11 + \sum_{r=1}^{(k-1)/2}(F_{20r-7} + F_{20r-4} + F_{20r+6} + F_{20r+3} + F_{20r+1} + F_{20r-1}). \tag{5.7}$$

For $r = 1$, expression (5.6) is congruent to 3 modulo 5. Since the repetition period of the Fibonacci sequence reduced modulo 5 is 20, the congruence above holds for all $r \le k/2$. It follows that $F_{10k}/5 \equiv 3k/2 \pmod 5$ if k is even. Analogously, it can be seen from (5.7) that $F_{10k}/5 \equiv k \pmod 5$ if k is odd. Summarizing, we found that

$$3F_{10k}/5 \equiv \begin{cases} 2k \pmod 5 & (k \text{ even}), \\ 3k \pmod 5 & (k \text{ odd}). \end{cases} \tag{5.8}$$

Finally, the inspection of the sequence $\{kL_{10k}\}$ reduced modulo 5 shows that

$$kL_{10k} \equiv \begin{cases} 2k \pmod 5 & (k \text{ even}), \\ 3k \pmod 5 & (k \text{ odd}). \end{cases} \tag{5.9}$$

Identity (5.5) along with congruences (5.8) and (5.9) prove (5.4) and the proposition. □

5.1 On the Primality of R_n and S_n

Since $S_n \equiv 0 \pmod n$ for $n \ge 1$ [see (3.6)], these integers cannot be prime. From Propositions 5-7, we see that a necessary condition for R_n to be a prime is that $n \equiv 2$, or 8, or 22, or 28 (mod 60). By using the function "nextprime" of the software package DERIVE, we found only two prime R_n for $n \le 248$, namely,

$$R_8 = 2939 \quad \text{and} \quad R_{68} = 352{,}536{,}175{,}722{,}757{,}107{,}150{,}131{,}558{,}879.$$

6. CONCLUSIONS

What has been presented in the preceding theory provides us with some feeling for the flow of ideas emanating from the initial sources.

Future directions of related research studies could lead to the investigation of partial derivative aspects of the Morgan-Voyce polynomials and, perhaps more importantly, to the *integration sequences* associated with these mathematically fertile polynomials.

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ON THE FREQUENCY OF OCCURRENCE OF α^i IN THE α-EXPANSIONS OF THE POSITIVE INTEGERS

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1. INTRODUCTION

Most students are familiar with representations of integers using various integral bases. In [1], George Bergman introduced a system using the irrational base $\alpha = \frac{1+\sqrt{5}}{2}$. The number α is of course the well-known golden ratio,* often defined as the limit of the sequence $\{F_n / F_{n-1}\}$, where F_n is the n^{th} Fibonacci number. Under this system, we can represent any natural number n (uniquely) as the sum of nonconsecutive powers of α. This means that, for any natural number n, there exists a unique sequence $\{e_i\}$, where $e_i \in \{0, 1\}$ for all i, such that $n = \sum_{i=-\infty}^{\infty} e_i \alpha^i$ and $e_i e_{i+1} = 0$ for each i. The *α-expansion of n* is $\ldots e_{-2}e_{-1}\underline{e_0}e_1e_2\ldots$, where we adopt the convention of underlining the zero$^{\text{th}}$ coordinate and omitting leading and trailing zeros when convenient. For example, $5 = \alpha^{-4} + \alpha^{-1} + \alpha^3$, so the base-$\alpha$ representation of 5 is $1001\underline{0}001$. Table 1 shows the α-expansions of the first 30 natural numbers. Table 2 shows the base 2 representations.

If we look down any column of the base 2 representations, it is easy to detect the patterns, which involve strings of 0s and 1s of equal length, so that the ratio of 1s to 0s is almost 1. The situation for other positive integral bases is analogous. In contrast, the columns in the α-expansions also exhibit patterns, but these are not so easy to detect or describe. The purpose of this paper is to explore some of these patterns. For each positive integer n, let $Ratio_i(n)$ be the ratio of the numbers $k \le n$ that do have α^i in their α-expansions to those that do not. In other words, $Ratio_i(n)$ is the ratio of 1s to 0s in the i^{th} column (i.e., the column corresponding to α^i) of the α-expansions of the integers 1 through n.

Hart and Sanchis showed in [6] that $Ratio_0(n) \to \alpha^{-2}$ as $n \to \infty$, thus proving Conjecture 1 from [2], as well as answering a question posed by Bergman in [1]. In this paper, we generalize the techniques used in [6] to derive the behavior of $Ratio_i(n)$ for all other values of i. It should come as no surprise that α-expansions are closely related to the Fibonacci sequence. Indeed, any natural number n can be expressed uniquely as the sum of Fibonacci numbers F_k (here $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$). This is the well-known Zeckendorf decomposition of n. Grabner et al. ([3], [4]) showed that, for $m \ge \log_m k$, the Zeckendorf decomposition of kF_m can be produced by replacing each α^i in the α-expansion of k with F_{m+i}. Thus, our results also provide information about the occurrence of F_{k+i} in the Zeckendorf decomposition of kF_k.

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* In [5] and [6], the symbol β was used for this quantity; we have decided to change to the more commonly used α.

TABLE 1. α-Expansions of the Integers 1-30

n																
1									$\underline{1}$							
2							1	0	$\underline{0}$	1						
3							1	0	$\underline{0}$	0	1					
4							1	0	$\underline{1}$	0	1					
5					1	0	0	1	$\underline{0}$	0	0	1				
6					1	0	0	0	$\underline{0}$	1	0	1				
7					1	0	0	0	$\underline{0}$	0	0	0	1			
8					1	0	0	0	$\underline{1}$	0	0	0	1			
9					1	0	1	0	$\underline{0}$	1	0	0	1			
10					1	0	1	0	$\underline{0}$	0	1	0	1			
11					1	0	1	0	$\underline{1}$	0	1	0	1			
12			1	0	0	1	0	1	$\underline{0}$	0	0	0	0	1		
13			1	0	0	1	0	0	$\underline{0}$	1	0	0	0	1		
14			1	0	0	1	0	0	$\underline{0}$	0	1	0	0	1		
15			1	0	0	1	0	0	$\underline{1}$	0	1	0	0	1		
16			1	0	0	0	0	1	$\underline{0}$	0	0	1	0	1		
17			1	0	0	0	0	0	$\underline{0}$	1	0	1	0	1		
18			1	0	0	0	0	0	$\underline{0}$	0	0	0	0	0	1	
19			1	0	0	0	0	0	$\underline{1}$	0	0	0	0	0	1	
20			1	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	1	
21			1	0	0	0	1	0	$\underline{0}$	0	1	0	0	0	1	
22			1	0	0	0	1	0	$\underline{1}$	0	1	0	0	0	1	
23			1	0	1	0	0	1	$\underline{0}$	0	0	1	0	0	1	
24			1	0	1	0	0	0	$\underline{0}$	1	0	1	0	0	1	
25			1	0	1	0	0	0	$\underline{0}$	0	0	0	1	0	1	
26			1	0	1	0	0	0	$\underline{1}$	0	0	0	1	0	1	
27			1	0	1	0	1	0	$\underline{0}$	1	0	0	1	0	1	
28			1	0	1	0	1	0	$\underline{0}$	0	1	0	1	0	1	
29			1	0	1	0	1	0	$\underline{1}$	0	1	0	1	0	1	
30	1	0	0	1	0	1	0	1	$\underline{0}$	0	0	0	0	0	0	1

TABLE 2. Base 2 Expansions

n					
1	0	0	0	0	1
2	0	0	0	1	0
3	0	0	0	1	1
4	0	0	1	0	0
5	0	0	1	0	1
6	0	0	1	1	0
7	0	0	1	1	1
8	0	1	0	0	0
9	0	1	0	0	1
10	0	1	0	1	0
11	0	1	0	1	1
12	0	1	1	0	0
13	0	1	1	0	1
14	0	1	1	1	0
15	0	1	1	1	1
16	1	0	0	0	0
17	1	0	0	0	1
18	1	0	0	1	0
19	1	0	0	1	1
20	1	0	1	0	0
21	1	0	1	0	1
22	1	0	1	1	0
23	1	0	1	1	1
24	1	1	0	0	0
25	1	1	0	0	1
26	1	1	0	1	0
27	1	1	0	1	1
28	1	1	1	0	0
29	1	1	1	0	1
30	1	1	1	1	0

Definition 1.1: For each integer r, define R_r as follows:

a. $R_0 = \alpha^{-2}$,

b. $R_r = \dfrac{L_{r-1}-1}{\alpha L_r + 1}$ for odd $r > 0$,

c. $R_r = \dfrac{L_{r-1}+1}{\alpha L_r - 1}$ for even $r > 0$,

d. $R_r = \dfrac{L_{-r}}{\alpha L_{-r+1}}$ for $r < 0$,

where $L_0 = 2$, $L_1 = 1$, and $L_{i+1} = L_i + L_{i-1}$ are the Lucas numbers for $i \geq 0$. Alternatively, $L_i = F_{i-1} + F_{i+1}$. It was shown in [6] that $\lim_{n\to\infty} Ratio_0(n) = R_0$. In this paper we show that, for each $r \neq 0$, $\lim_{n\to\infty} Ratio_r(n) = R_r$. Note also the interesting fact that, as $r \to \infty$ and as $r \to -\infty$, R_r approaches the limit R_0. Our strategy is to first establish some recursive patterns along each column (these are established in Lemma 3.5) which will allow us to obtain precise expressions for $Ratio_i(n_k)$ and $Ratio_i(m_k)$, where $\{n_k\}$ and $\{m_k\}$ are two subsequences of the natural numbers. The limits of these subsequences can then easily be obtained from known limit results about Fibonacci and Lucas numbers. We then show that, as $n \to \infty$, members of the full sequence $Ratio_i(n)$ must be caught between these two subsequences and hence the full sequence converges. Our proofs use only combinatorial and algorithmic techniques and do not require any specialized number theory background.

2. DEFINITIONS AND PRELIMINARIES

We use definitions and notation similar to those used in [5] and [6]. In particular, $\ell(n)$ denotes the absolute value of the smallest power of α in the α-expansion of n, and $u(n)$ denotes the largest such power. The following is a restatement of Theorem 1 from [4] in terms of the α-expansion.

Theorem 2.1 (Grabner, Nemes, Petho, Tichy): For $k \geq 1$, we have $\ell(n) = u(n) = 2k$ whenever $L_{2k} \leq n \leq L_{2k+1}$, and we have $\ell(n) = 2k+2$ and $u(n) = 2k+1$ whenever $L_{2k+1} < n < L_{2k+2}$.

The following definitions are from [6].

Definition 2.2: We define V to be the infinite dimensional vector space over $\mathbb{Z}$ given by $V \equiv \{(\ldots, v_{-1}, \underline{v_0}, v_1, v_2, \ldots) : v_i \in \mathbb{Z}\ \forall i$, with at most finitely many v_i nonzero$\}$. For clarity, we underline the zero$^{\text{th}}$ coordinate.

Definition 2.3: Define $\hat{V}$ to be the subset of V consisting of all vectors whose entries are in the set $\{0, 1\}$ and which have no two consecutive ones. We will call the elements of $\hat{V}$ totally reduced vectors. When convenient, we omit trailing and leading zeros, so for example,

$$(\ldots, 0, \ldots, 0, 0, 1, \underline{0}, 1, 0, 1, 0, 0, \ldots, 0, \ldots) = (1, \underline{0}, 1, 0, 1).$$

As in [5], we represent α-expansions by vectors in $\hat{V}$, where a one in the j^{th} coordinate represents α^j.

Definition 2.4: We define the function $\alpha : \mathbb{N} \to \hat{V}$ so that, when the α-expansion of n is $\sum_{i=-\infty}^{\infty} e_i\alpha^i$, $\alpha(n)$ is the vector in $\hat{V}$ with $v_i = e_i$.

It follows from Theorem 2.1 that, if $L_{2k} \leq n \leq L_{2k+1}$, we can write

$$n = \alpha^{-2k} + \sum_{i=-2k+2}^{2k-2} e_i\alpha^i + \alpha^{2k} \text{ so that } \alpha(n) = (1, 0, e_{-2k+2}, e_{-2k+3}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-2}, 0, 1) \tag{1}$$

and, if $L_{2k+1} < n < L_{2k+2}$, then we can write

$$n = \alpha^{-2k-2} + \sum_{i=-2k}^{2k-1} e_i\alpha^i + \alpha^{2k+1} \text{ so that } \alpha(n) = (1, 0, e_{-2k}, e_{-2k+1}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-1}, 0, 1). \tag{2}$$

Definition 2.5: The function $\sigma : V \to \mathbb{N}$ is defined as follows: $\sigma((\ldots, v_{-1}, v_0, v_1, \ldots)) = \sum_{i=-\infty}^{\infty} v_i\alpha^i$.

Thus $\sigma(\alpha(n)) = n$ for all natural numbers n. (Note that the definition of σ in [5] is in terms of Fibonacci numbers, and is not equivalent to the one given here. Specifically, the two functions are only guaranteed to be equal when applied to $\alpha(n)$ where $n \in \mathbb{N}$.) The following definitions are generalizations of definitions in [6]. (The definitions in [6] correspond to the case $i = 0$.)

Definition 2.6: We say that n has property $\mathcal{P}_i$ if α^i appears in the α-expansion of n.

Definition 2.7: For natural numbers n, m:

a. $Ones_i(n, m] = |\{k \in N : n < k \leq m,\ k \text{ has property } \mathcal{P}_i\}|$;

b. $Zeros_i(n, m] = |\{k \in N : n < k \leq m,\ k \text{ does not have property } \mathcal{P}_i\}|$;

c. $Ratio_i(n, m] = \frac{Ones_i(n,m]}{Zeros_i(n,m]}$.

By abuse of notation, we also define $Ratio_i(n) = Ratio_i(0, n]$. We will call a finite sequence of 0s and 1s a *pattern*. We use patterns to describe values of $[\alpha(n)]_i$ for fixed i and a sequence of consecutive natural numbers n. Recall that $[\alpha(n)]_i = 1$ if α^i occurs in the α-expansion of n, and that $[\alpha(n)]_i = 0$ otherwise. Thus, for $n \leq m$, we can define the pattern

$$Pat_i(n, m] = [\alpha(n+1)]_i[\alpha(n+2)]_i \cdots [\alpha(m)]_i.$$

Patterns can be concatenated. We will denote the concatenation operation with the operator +, but will omit it when convenient. So, for example, for $n \leq m \leq p$,

$$Pat_i(n, p] = Pat_i(n, m] + Pat_i(m, p] = Pat_i(n, m]Pat_i(m, p].$$

In addition, we use the notation P/n to denote the *prefix* of a pattern P obtained by deleting the rightmost n digits. So, for example, $11001/2 = 110$. By abuse of notation, if P is a pattern, we define $Ones(P)$ and $Zeros(P)$ to be the number of 1s and the number of 0s, respectively, appearing in the pattern P. We also define $Ratio(P) = Ones(P)/Zeros(P)$. We will be using the following known facts about Fibonacci and Lucas numbers: For any $h > 0$, the sequence F_{2n+h}/F_{2n} is decreasing, the sequence F_{2n+1+h}/F_{2n+1} is increasing, the sequence L_{2n+h}/L_{2n} is increasing, the sequence L_{2n+1+h}/L_{2n+1} is decreasing, and

$$\lim_{n\to\infty}(F_{n+h}/F_n) = \alpha^h, \tag{3}$$

$$\lim_{n\to\infty}(L_{n+h}/L_n) = \alpha^h, \tag{4}$$

$$F_{n+h}L_{n+k} - F_nL_{n+h+k} = (-1)^n F_h L_k, \tag{5}$$

$$F_{n+h}F_{n+k} - F_nF_{n+h+k} = (-1)^n F_h F_k, \tag{6}$$

$$\sum_{i=0}^{h} F_i = F_{h+2} - 1, \tag{7}$$

$$\sum_{i=0}^{h} F_{k+2i} = F_{k+2h+1} - F_{k-1}, \tag{8}$$

$$\alpha^k + \alpha^{k+2} = \alpha L_{k+1} + L_k. \tag{9}$$

Formulas (5) and (6) are from [7], page 177, (19b and 20a). The following Lemma will be used repeatedly.

Lemma 2.8: Let $a, b, c, d \in \mathbb{N}$, and $x, y \in \mathbb{R}$. If $\frac{a}{b} \leq x$ and $\frac{c}{d} \leq y$, then $\frac{a+c}{b+d} \leq \max\{x, y\}$. When each $\leq$ is replaced by $\geq$, the result holds with *max* replaced by *min*.

3. SOME USEFUL RESULTS

In the sequence of α-expansions of the natural numbers, the Lucas numbers play a special role. First, note that

$$\alpha(L_{2k}) = 10^{2k-1}\underline{0}0^{2k-1}1 \quad \text{and} \quad \alpha(L_{2k+1}) = (10)^k\underline{1}(01)^k.$$

(Readers may derive these formulas themselves, or refer to [2].) In Table 1, compare the expansions found between $L_4 = 7$ and $L_5 = 11$ with those found between $L_6 = 18$ and $2L_5 = 22$. The two sequences of expansions are identical if we restrict our attention to powers of α between α^{-3} and α^3. Similar observations can be made, for large enough k, by comparing the expansions of the numbers found between L_{2k-2} and L_{2k-1}, and those between L_{2k} and $2L_{2k-1}$: the expansions are identical for those powers of α between α^{-k} and α^k. It can be proved that this is always the case, using an algorithmic technique presented in [5]. In fact, a full recursive pattern in the sequence of α-expansions can be established. This was shown in [6], and we merely restate the relevant results here. Note that, for $n \geq 4$, $L_n < 2L_{n-1} < L_{n-2} + L_n < L_{n+1}$. Thus, we can partition the α-expansions between L_n and L_{n+1} into three segments: the first from L_n to $2L_{n-1}$, the second from $2L_{n-1}$ to $L_{n-2} + L_n$, and the third from $L_{n-2} + L_n$ to L_{n+1}. As partly indicated above (for even n), the sequence of α-expansions between L_n and $2L_{n-1}$ is similar to that between L_{n-2} and L_{n-1}. In addition, the sequence of α-expansions between $2L_{n-1}$ and $L_{n-2} + L_n$ is similar to that between L_{n-3} and L_{n-2}, and the sequence of α-expansions between $L_{n-2} + L_n$ and L_{n+1} is again similar to that between L_{n-2} and L_{n-1}. The exact ways in which the sequences are similar (or dissimilar) vary for each of the three segments, and also vary depending on whether n is even or odd. The full result is expressed in the following propositions, and was proved in Lemma 3.8 of [6].

Proposition 3.1: Let $k \geq 2$. If $0 < m < L_{2k-2}$ and $\alpha(L_{2k-1} + m) = (1, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-3}, 0, 1)$, then:

a. $e_{-(2k-2)} = 0$.

b. $\alpha(L_{2k+1} + m) = (1, 0, 0, 1, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-3}, 0, 0, 0, 1)$.

c. $\alpha(L_{2k-1} + L_{2k+1} + m) = (1, 0, 0, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-3}, 0, 1, 0, 1)$.

d. $\alpha(2L_{2k+1} + m) = (1, 0, 1, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-3}, 0, 1, 0, 0, 1)$.

Proposition 3.2: Let $k \geq 2$. If $0 \leq m \leq L_{2k-1}$ and $\alpha(L_{2k} + m) = (1, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-2}, 0, 1)$, then:

a. $\alpha(L_{2k+2} + m) = (1, 0, 0, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-2}, 0, 0, 0, 1)$.

b. $\alpha(L_{2k} + L_{2k+2} + m) = (1, 0, 1, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-2}, 0, 1, 0, 1)$.

c. $\alpha(2L_{2k+2} + m) = (1, 0, 0, 1, 0, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-2}, 0, 1, 0, 0, 1)$.

From Propositions 3.1 and 3.2, the following may be deduced.

Corollary 3.3: Let $k \geq 2$.

a. For $L_{2k+1} < n < 2L_{2k}$, $\alpha(n)$ begins with 10010 and ends in 0001.

b. For $L_{2k-1} + L_{2k+1} < n < L_{2k+2}$, $\alpha(n)$ begins with 10000 and ends in 0101.

c. For $2L_{2k+1} = L_{2k-1} + L_{2k+2} < n < L_{2k} + L_{2k+2}$, $\alpha(n)$ begins with 10100 and ends in 01001.

d. For $L_{2k+2} \leq n \leq 2L_{2k+1}$, $\alpha(n)$ begins with 1000 and ends in 0001.

e. For $L_{2k} + L_{2k+2} \leq n \leq L_{2k+3}$, $\alpha(n)$ begins with 1010 and ends in 0101.

f. For $2L_{2k+2} = L_{2k} + L_{2k+3} \leq n \leq L_{2k+1} + L_{2k+3}$, $\alpha(n)$ begins with 100100 and ends in 01001.

Definition 3.4: For $k \geq 1$, $P_r^k = Pat_r(L_{k-1}, L_k]$.

From the above propositions, recursive formulas for P_r^k easily follow.

Lemma 3.5: Let $k \geq 2$.

a. If $-2k+3 \leq r \leq 2k-2$, then $P_r^{2k+2} = P_r^{2k} P_r^{2k-1} P_r^{2k}$.

b. If $-2k+4 \leq r \leq 2k-3$, then $P_r^{2k+1} = P_r^{2k-1} P_r^{2k-2} P_r^{2k-1}$.

Proof: Fix k and r as above and define the following maps:

$$f_1 : [L_{2k-1}+1, L_{2k}-1] \to [L_{2k+1}+1, 2L_{2k}-1], \qquad f_1(x) = x + L_{2k};$$
$$f_2 : [L_{2k-2}, L_{2k-1}] \to [2L_{2k}, L_{2k-1}+L_{2k+1}], \qquad f_2(x) = x + L_{2k+1};$$
$$f_3 : [L_{2k-1}+1, L_{2k}-1] \to [L_{2k-1}+L_{2k+1}+1, L_{2k+2}-1], \quad f_3(x) = x + L_{2k+1}.$$

Clearly, these maps are one-to-one and onto. Moreover, by Propositions 3.1 and 3.2, $[\alpha(x)]_r = [\alpha(f_i(x))]_r$ for any x in the domain of f_i, if $-2k+3 \leq r \leq 2k-2$. It follows that

$$Pat_r(L_{2k-1}, L_{2k}-1] = Pat_r(L_{2k+1}, 2L_{2k}-1],$$
$$Pat_r(L_{2k-2}-1, L_{2k-1}] = Pat_r(2L_{2k}-1, L_{2k-1}+L_{2k+1}],$$
$$Pat_r(L_{2k-1}, L_{2k}-1] = Pat_r(L_{2k-1}+L_{2k+1}, L_{2k+2}-1].$$

Then

$$\begin{aligned} &Pat_r(L_{2k-1}, L_{2k}-1] + Pat_r(L_{2k-2}-1, L_{2k-1}] + Pat_r(L_{2k-1}, L_{2k}-1] \\ &= Pat_r(L_{2k+1}, 2L_{2k}-1] + Pat_r(2L_{2k}-1, L_{2k-1}+L_{2k+1}] + Pat_r(L_{2k-1}+L_{2k+1}, L_{2k+2}-1]. \end{aligned}$$

Using the fact that $[\alpha(L_{2k})]_r = 0$ for every k, this simplifies to $P_r^{2k} P_r^{2k-1} P_r^{2k} = P_r^{2k+2}$. Similarly, we define the following maps:

$$g_1 : [L_{2k-2}, L_{2k-1}] \to [L_{2k}, 2L_{2k-1}], \qquad g_1(x) = x + L_{2k-1};$$
$$g_2 : [L_{2k-3}+1, L_{2k-2}-1] \to [2L_{2k-1}+1, L_{2k-2}+L_{2k}-1], \quad g_2(x) = x + L_{2k};$$
$$g_3 : [L_{2k-2}, L_{2k-1}] \to [L_{2k-2}+L_{2k}, L_{2k+1}], \qquad g_3(x) = x + L_{2k}.$$

Again by Propositions 3.1 and 3.2, these maps are bijections which leave the r^{th} term of the α-expansion of x fixed for $-2k+4 \leq r \leq 2k-3$. So, by concatenating the domains and ranges as above, we again obtain $P_r^{2k-1} P_r^{2k-2} P_r^{2k-1} = P_r^{2k+1}$. □

4. SOME SPECIAL SUBSEQUENCES OF *Ratio*$_r(n)$

Here we show that, for each r, there exist two subsequences of *Ratio*$_r(n)$ that converge to R_r. These subsequences are related to the odd and even Lucas numbers. One is increasing and the other is decreasing. In Section 5 we show that the sequence *Ratio*$_r(n)$ is trapped between these two monotone convergent subsequences, and therefore *Ratio*$_r(n)$ must also converge to R_r.

4.1 Positive Powers of α

We consider even and odd powers separately. For even powers, let $r = 2l$ where $l \geq 1$; for odd powers, let $r = 2l+1$ where $l \geq 0$. Using the recursive formulas derived in the previous section, it is straightforward to obtain closed formulas for *Ones*(P_r^k).

Lemma 4.1: For $k \ge 2$,

$$Ones(P_{2l}^k) = \begin{cases} 0, & k < 2l, \\ 1, & k = 2l, \\ L_{2l-1}, & k = 2l+1, \\ (L_{2l-1}+1)F_{k-2l-2}, & k \ge 2l+2; \end{cases} \qquad Ones(P_{2l+1}^k) = \begin{cases} 0, & k \le 2l+1, \\ L_{2l}-1, & k = 2l+2, \\ (L_{2l}-1)F_{k-2l-3}, & k \ge 2l+3. \end{cases}$$

Proof: The proof is by induction on k. The base cases are somewhat numerous but straightforward. We use Theorem 2.1 and Corollary 3.3 to compute the entries of the middle two columns of the following table, then compute the last column by simple counting.

	$L_{k-1} < n < L_k$	$n = L_k$	$Ones(P^k)$
$k < 2l$	$u(n) = k-1$ $[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 0$	$u(n) = k-1$ or k $[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l}^k) = 0$ $Ones(P_{2l+1}^k) = 0$
$k = 2l$	$u(n) = 2l-1$ $[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 0$	$u(n) = 2l$ $[\alpha(n)]_{2l} = 1$ $[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l}^k) = 1$ $Ones(P_{2l+1}^k) = 0$
$k = 2l+1$	$u(n) = 2l$ $[\alpha(n)]_{2l} = 1$ $[\alpha(n)]_{2l+1} = 0$	$u(n) = 2l$ $[\alpha(n)]_{2l} = 1$ $[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l}^k) = L_k - L_{k-1} = L_{2l-1}$ $Ones(P_{2l+1}^k) = 0$
$k = 2l+2$	$u(n) = 2l+1$ $[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 1$	$u(n) = 2l+2$ $[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l}^k) = 0$ $Ones(P_{2l+1}^k) = L_k - L_{k-1} - 1 = L_{2l-1}$

If $k = 2l+3$ and $L_{2l+2} < n \le L_{2l+3}$, then $u(n) = 2l+2$ again by Theorem 2.1. Corollary 3.3 again helps us to complete the following table:

$L_{2l+2} < n \le 2L_{2l+1}$	$2L_{2l+1} < n < L_{2l} + L_{2l+2}$	$L_{2l} + L_{2l+2} \le n \le L_{2l+3}$	$Ones(P^{2l+3})$
$[\alpha(n)]_{2l} = 0$	$[\alpha(n)]_{2l} = 0$	$[\alpha(n)]_{2l} = 1$	$Ones(P_{2l}^{2l+3}) = L_{2l+3} - (L_{2l} + L_{2l+2}) + 1 = L_{2l-1} + 1$
$[\alpha(n)]_{2l+1} = 0$	$[\alpha(n)]_{2l+1} = 0$	$[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l+1}^{2l+3}) = 0$

If $k = 2l+4$, then $u(n) = 2l+3$ for $L_{2l+3} < n < L_{2l+4}$ and $u(n) = 2l+4$ for $n = L_{2l+4}$. We again invoke Corollary 3.3 to complete the table:

$L_{2l+3} < n < 2L_{2l+2}$	$2L_{2l+2} \le n \le L_{2l+1} + L_{2l+3}$	$L_{2l+1} + L_{2l+3} < n < L_{2l+4}$	$n = L_{2l+4}$	$Ones(P^{2l+4})$
$[\alpha(n)]_{2l} = 0$	$[\alpha(n)]_{2l} = 1$	$[\alpha(n)]_{2l} = 0$	$[\alpha(n)]_{2l} = 0$	$Ones(P_{2l}^{2l+4})$ $= L_{2l+1} + L_{2l+3} - 2L_{2l+2} + 1 = L_{2l-1} + 1$
$[\alpha(n)]_{2l+1} = 0$	$[\alpha(n)]_{2l+1} = 0$	$[\alpha(n)]_{2l+1} = 1$	$[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l+1}^{2l+4})$ $= L_{2l+4} - L_{2l+1} - L_{2l+3} - 1 = L_{2l} - 1$

For the inductive step, assume that $k \ge 2l+5$. By Lemma 3.5,

$$\begin{aligned} Ones(P_{2l}^k) &= 2Ones(P_{2l}^{k-2}) + Ones(P_{2l}^{k-3}) \\ &= 2(L_{2l-1}+1)F_{k-2l-4} + (L_{2l-1}+1)F_{k-2l-5} = (L_{2l-1}+1)F_{k-2l-2}. \end{aligned}$$

Similarly,

$$\begin{aligned} Ones(P_{2l+1}^k) &= 2Ones(P_{2l+1}^{k-2}) + Ones(P_{2l+1}^{k-3}) \\ &= 2(L_{2l}-1)F_{k-2l-5} + (L_{2l}-1)F_{k-2l-6} = (L_{2l}-1)F_{k-2l-3}. \quad \square \end{aligned}$$

Now that we have formulas for $Ones(P_r^k)$ for positive r, closed formulas for $Ratio_r(L_n)$ for all Lucas numbers L_n may be obtained by straightforward calculations. However, the subsequences of the $Ratio_r(m)$ sequence in which we are interested happen to occur not at the Lucas numbers themselves but at points close to the Lucas numbers. Specifically, we will show that (for positive odd powers of α) the values $Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1)$ form a decreasing subsequence at which local maxima occur; and that the values $Ratio_{2l+1}(L_{2k}-L_{2l})$ form an increasing subsequence at which local minima occur. Similar subsequences occur for even positive powers of α, and for negative powers. To obtain formulas for these ratios, we need to first nail down the patterns occurring between these points and the Lucas numbers that they are close to. This is done in the following Lemma for positive powers of α. The proof, which is omitted, uses induction on k combined with results from Propositions 3.1 and 3.2, as well as Corollary 3.3.

Lemma 4.2: If $k \geq l+1$, then:

a. $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+1}] = 0^{L_{2l+1}+1}$.

b. $Pat_{2l+1}(L_{2k+1}-L_{2l+2}, L_{2k+1}-L_{2l+1}-1] = 1^{L_{2l}-1}$.

c. $Pat_{2l+1}(L_{2k}-L_{2l}, L_{2k}] = 1^{L_{2l}-1}0$.

d. $Pat_{2l+1}(L_{2k+2}-L_{2l+3}-1, L_{2k+2}-L_{2l}] = 0^{2L_{2l+1}+1}$.

e. $Pat_{2l}(L_{2k}-L_{2l}, L_{2k}] = 0^{L_{2l}}$.

f. $Pat_{2l}(L_{2k-1}-L_{2l-1}-1, L_{2k-1}] = 1^{L_{2l-1}+1}$.

The main results of this section are given in Theorems 4.3 and 4.4.

Theorem 4.3: For $l \geq 0$ and large enough k:

a. $Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1)$ decreases to R_{2l+1} as k increases.

b. $Ratio_{2l+1}(L_{2k}-L_{2l})$ increases to R_{2l+1} as k increases.

Proof: By Lemmas 4.1 and 4.2, if $k \geq l+1$, then using Formula (7),

$$Ones_{2l+1}(0, L_{2k+1}-L_{2l+1}-1] = Ones_{2l+1}(0, L_{2k+1}] = \sum_{j=1}^{2k+1} Ones(P_{2l+1}^j)$$
$$= L_{2l}-1+\sum_{j=2l+3}^{2k+1}(L_{2l}-1)F_{j-2l-3} = L_{2l}-1+(L_{2l}-1)(F_{2k-2l}-1) = (L_{2l}-1)F_{2k-2l}.$$

It follows that

$$Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1) = \frac{(L_{2l}-1)F_{2k-2l}}{L_{2k+1}-L_{2l+1}-1-(L_{2l}-1)F_{2k-2l}} = \frac{L_{2l}-1}{\frac{L_{2k+1}-L_{2l+1}-1}{F_{2k-2l}}-(L_{2l}-1)}.$$

Part (a) follows from the fact that $(L_{2k+1}-L_{2l+1}-1)/F_{2k-2l}$ is increasing for large enough k (which can be deduced from Formula (5)) and has limit $\alpha^{2l+2}+\alpha^{2l} = \alpha L_{2l+1}+L_{2l}$ (from Formulas (3) and (9)). Similarly, if $k \geq l+2$, then

$$Ones_{2l+1}(0, L_{2k}-L_{2l}] = Ones_{2l+1}(0, L_{2k}]-Ones_{2l+1}(L_{2k}-L_{2l}, L_{2k}]$$
$$= \sum_{j=1}^{2k} Ones(P_{2l+1}^j)-(L_{2l}-1) = (L_{2l}-1)\sum_{j=2l+3}^{2k} F_{j-2l-3} = (L_{2l}-1)(F_{2k-2l-1}-1).$$

It follows that

$$Ratio_{2l+1}(L_{2k}-L_{2l})=\frac{(L_{2l}-1)(F_{2k-2l-1}-1)}{L_{2k}-L_{2l}-(L_{2l}-1)(F_{2k-2l-1}-1)}=\frac{L_{2l}-1}{\dfrac{L_{2k}-L_{2l}}{F_{2k-2l-1}-1}-(L_{2l}-1)}.$$

Part (b) follows from the fact that $\frac{L_{2k}-L_{2l}}{F_{2k-2l-1}-1}$ is decreasing (for large enough k, again by Formula (5)) and has limit $\alpha^{2l+2}+\alpha^{2l}$. □

Theorem 4.4: For $l \geq 1$ and large enough k:

a. $Ratio_{2l}(L_{2k}-L_{2l})$ decreases to R_{2l} as k increases.

b. $Ratio_{2l}(L_{2k+1}-L_{2l-1}-1)$ increases to R_{2l} as k increases.

The proof of this theorem is omitted as it is similar to the proof of Theorem 4.3.

4.2 Negative Powers of α

We state here the results for the subsequences of $Ratio_r(n)$ where $r<0$. The proofs are completely analogous to those from the previous subsection.

Lemma 4.5: For $k \geq 1$:

$$Ones(P^k_{-2l})=\begin{cases}0, & k<2l,\\ L_{2l-2}, & k=2l,\\ L_{2l-1}, & k=2l+1,\\ 0, & k=2l+2,\\ L_{2l}(F_{k-2l-3}+1), & k \text{ odd}, \\ & k\geq 2l+3,\\ L_{2l}(F_{k-2l-3}-1), & k \text{ even},\\ & k\geq 2l+3;\end{cases}\qquad Ones(P^k_{-(2l+1)})=\begin{cases}0, & k\leq 2l+3,\\ L_{2l+1}(F_{k-2l-4}-1), & k \text{ odd},\\ & k\geq 2l+4,\\ L_{2l+1}(F_{k-2l-4}+1), & k \text{ even},\\ & k\geq 2l+4.\end{cases}$$

Lemma 4.6: If $k \geq l+1$, then:

a. $Pat_{-(2l+1)}(L_{2k}-L_{2l}, L_{2k}]=0^{L_{2l}}$.

b. $Pat_{-(2l+1)}(L_{2k+2}-L_{2l+2}, L_{2k+2}]=1^{L_{2l+1}}0^{L_{2l}}$.

c. $Pat_{-(2l+1)}(L_{2k+3}-2L_{2l+2}, L_{2k+3}]=0^{2L_{2l+2}}$.

d. $Pat_{-2l}(L_{2k}, L_{2k}+L_{2l-1}]=0^{L_{2l-1}}$.

e. $Pat_{-2l}(L_{2k+1}-L_{2l}, L_{2k+1}]=1^{L_{2l}}$.

f. $Pat_{-2l}(L_{2k+2}-L_{2l+2}, L_{2k+2}]=0^{L_{2l+2}}$.

Theorem 4.7: Let l be a nonnegative integer. For large enough k:

a. If $l\geq 0$, then $Ratio_{-(2l+1)}(L_{2k}-L_{2l})$ decreases to $R_{-(2l+1)}$ as k increases.

b. If $l\geq 0$, then $Ratio_{-(2l+1)}(L_{2k+1})$ increases to $R_{-(2l+1)}$ as k increases.

c. If $l\geq 1$, then $Ratio_{-2l}(L_{2k+1})$ decreases to $R_{-(2l)}$ as k increases.

d. If $l\geq 1$, then $Ratio_{-2l}(L_{2k}+L_{2l-1})$ increases to $R_{-(2l)}$ as k increases.

5. THE MAIN RESULT

In this section we show that, as $n \to \infty$, members of the full sequence $Ratio_r(n)$ must be caught between the two subsequences examined in the previous section. In order to do this, we bound the ratios of prefixes of patterns originating at members of the subsequences, and then use Lemma 2.8.

5.1 The Case $r > 0$ Odd

We start by examining in more detail the patterns appearing between Lucas numbers. The base cases are taken care of in the following corollary.

Corollary 5.1: For $l \geq 0$:

a. $Pat_{2l+1}(L_{2l+2}, L_{2l+3}] = 0^{L_{2l+1}}$.

b. $Pat_{2l+1}(L_{2l+3}, L_{2l+4}] = 0^{L_{2l+1}}1^{L_{2l}-1}0$.

c. $Pat_{2l+1}(L_{2l+4}, L_{2l+5}] = 0^{L_{2l+1}}1^{L_{2l}-1}0^{L_{2l+1}+1}$.

d. $Pat_{2l+1}(L_{2l+5}, L_{2l+6}] = 0^{L_{2l+1}}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0$.

e. $Pat_{2l+1}(L_{2l+6}, L_{2l+7}] = 0^{L_{2l+1}}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}$.

Proof: Parts (a)-(c) follow from Corollary 3.3 and Theorem 2.1. Parts (d) and (e) follow from (a)-(c) using Lemma 3.5. □

Lemma 5.2: For $k \geq l+2$:

a. $Ratio(P_{2l+1}^{2k-1}) \leq R_{2l+1}$.

b. $Ratio(P_{2l+1}^{2k-1}P_{2l+1}^{2k}) \leq R_{2l+1}$.

c. $Ratio(P_{2l+1}^{2k}) \geq R_{2l+1}$.

d. $Ratio(P_{2l+1}^{2k}P_{2l+1}^{2k+1}) \geq R_{2l+1}$.

Proof: By Lemma 4.1, if $k \geq l+2$, then

$$Ratio(P_{2l+1}^{2k-1}) = \frac{(L_{2l}-1)F_{2k-2l-4}}{L_{2k-3}-(L_{2l}-1)F_{2k-2l-4}} = \frac{L_{2l}-1}{\frac{L_{2k-3}}{F_{2k-2l-4}}-(L_{2l}-1)}.$$

Since $\frac{L_{2k-3}}{F_{2k-2l-4}}$ is decreasing with limit $\alpha^{2l+2}+\alpha^{2l}$,

$$Ratio(P_{2l+1}^{2k-1}) \leq \frac{L_{2l}-1}{\alpha^{2l+2}+\alpha^{2l}-L_{2l}+1} = R_{2l+1},$$

which proves (a). We also have

$$\begin{aligned} Ones(P_{2l+1}^{2k-1}P_{2l+1}^{2k}) &= Ones(P_{2l+1}^{2k-1}) + Ones(P_{2l+1}^{2k}) \\ &= (L_{2l}-1)F_{2k-2l-4} + (L_{2l}-1)F_{2k-2l-3} = (L_{2l}-1)F_{2k-2l-2}, \end{aligned}$$

and hence, by part (a),

$$Ratio(P_{2l+1}^{2k-1}P_{2l+1}^{2k}) = \frac{(L_{2l}-1)F_{2k-2l-2}}{L_{2k-1}-(L_{2l}-1)F_{2k-2l-2}} = Ratio(P_{2l+1}^{2k+1}) \leq R_{2l+1}.$$

Similarly,

$$Ratio(P_{2l+1}^{2k}) = \frac{(L_{2l}-1)F_{2k-2l-3}}{L_{2k-2}-(L_{2l}-1)F_{2k-2l-3}} = \frac{L_{2l}-1}{\frac{L_{2k-2}}{F_{2k-2l-3}}-(L_{2l}-1)}.$$

Since $\frac{L_{2k-2}}{F_{2k-2l-3}}$ is increasing with limit $\alpha^{2l+2}+\alpha^{2l}$,

$$Ratio(P_{2l+1}^{2k}) \geq \frac{L_{2l}-1}{\alpha^{2l+2}+\alpha^{2l}-L_{2l}+1} = R_{2l+1},$$

which proves (c). For part (d), the reader may check that

$$Ratio(P_{2l+1}^{2k}P_{2l+1}^{2k+1}) = Ratio(P_{2l+1}^{2k+2}) \geq R_{2l+1}$$

using part (c). □

We intend to show that, for $L_{2k+1}-L_{2l+1}-1 < n < L_{2k+3}-L_{2l+1}-1$, $\limsup_{n\to\infty} Ratio_{2l+1}(n) \leq R_{2l+1}$. By Theorem 4.3 and Lemma 2.8, it is sufficient to show that, if P is any prefix of $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+3}-L_{2l+1}-2]$, then $Ratio(P) \leq R_{2l+1}$. However, this last statement is not true for the largest prefixes unless k is sufficiently large. We therefore first prove a partial result applicable to prefixes P which do not include the tail end of ones found in $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+3}-L_{2l+1}-2]$.

Lemma 5.3: For $l \geq 0$ and $k \geq l+1$, if P is any prefix of $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+3}-L_{2l+2}]$, then $Ratio(P) \leq R_{2l+1}$.

Proof: We use repeatedly the fact that, by Lemma 2.8, the pattern obtained by concatenating two patterns whose ratios are $\leq R_{2l+1}$ also has ratio $\leq R_{2l+1}$. If $k = l+1$, then by Lemma 4.2 and Corollary 5.1,

$$\begin{aligned} &Pat_{2l+1}(L_{2l+3}-L_{2l+1}-1, L_{2l+5}-L_{2l+2}] \\ &= Pat_{2l+1}(L_{2l+3}-L_{2l+1}-1, L_{2l+3}] + P_{2l+1}^{2l+4}P_{2l+1}^{2l+5} / L_{2l+2} \\ &= 0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}. \end{aligned}$$

The prefix yielding the highest ratio is $P = 0^{2L_{2l+1}+1}1^{L_{2l}-1}$ so that

$$Ratio(P) = \frac{L_{2l}-1}{2L_{2l+1}+1} \leq \frac{L_{2l}-1}{\alpha L_{2l+1}+1} = R_{2l+1}.$$

If $k = l+2$, the pattern in question is

$$\begin{aligned} &Pat_{2l+1}(L_{2l+5}-L_{2l+1}-1, L_{2l+7}-L_{2l+2}] \\ &= Pat_{2l+1}(L_{2l+5}-L_{2l+1}-1, L_{2l+5}] + P_{2l+1}^{2l+6}P_{2l+1}^{2l+7} / L_{2l+2} \\ &= 0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1} \end{aligned}$$

by Lemma 4.2 and Corollary 5.1. We need only consider the prefixes ending in $1^{L_{2l}-1}$, since these yield the highest ratios. We have:

a. $Ratio(0^{2L_{2l+1}+1}1^{L_{2l}-1}) = \dfrac{L_{2l}-1}{2L_{2l+1}+1} < \dfrac{L_{2l}-1}{\alpha L_{2l+1}+1}$.

b. $Ratio(0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}) = \dfrac{2L_{2l}-2}{4L_{2l+1}+2} = \dfrac{L_{2l}-1}{2L_{2l+1}+1} < \dfrac{L_{2l}-1}{\alpha L_{2l+1}+1}$.

c. $Ratio(0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}1^{L_{2l}-1}) = \dfrac{3L_{2l}-3}{5L_{2l+1}+3} = \dfrac{L_{2l}-1}{(5/3)L_{2l+1}+1} < \dfrac{L_{2l}-1}{\alpha L_{2l+1}+1}.$

d. $Ratio(0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}) = \dfrac{4L_{2l}-4}{7L_{2l+1}+4}$

$$= \frac{L_{2l}-1}{(7/4)L_{2l+1}+1} < \frac{L_{2l}-1}{\alpha L_{2l+1}+1}.$$

For the inductive step, assume $k \ge l+3$. By Lemmas 3.5 and 4.2,

$$Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+3}-L_{2l+2}]$$
$$= 0^{L_{2l+1}+1}P_{2l+1}^{2k+2}P_{2l+1}^{2k+3} / L_{2l+2} = 0^{L_{2l+1}+1}P_{2l+1}^{2k}P_{2l+1}^{2k-1}P_{2l+1}^{2k}P_{2l+1}^{2k+1}P_{2l+1}^{2k}P_{2l+1}^{2k+1} / L_{2l+2}.$$

Suppose P is a prefix of this pattern.

Case 1: If P is a prefix of $0^{L_{2l+1}+1}P_{2l+1}^{2k}$, the result follows by the induction hypothesis.

Case 2: $P = 0^{L_{2l+1}+1}P_{2l+1}^{2k}Q = 0^{L_{2l+1}+1}P_{2l+1}^{2k-2}P_{2l+1}^{2k-3}P_{2l+1}^{2k-2}Q$, where Q is a prefix of P_{2l+1}^{2k-1}/L_{2l+2}. By the induction hypothesis, $Ratio(0^{L_{2l+1}+1}P_{2l+1}^{2k-2}Q) \le R_{2l+1}$, and $Ratio(P_{2l+1}^{2k-2}P_{2l+1}^{2k-3}) \le R_{2l+1}$ by Lemma 5.2. Hence, $Ratio(P) \le R_{2l+1}$.

Case 3: $P = 0^{L_{2l+1}+1}P_{2l+1}^{2k}(P_{2l+1}^{2k-1} / L_{2l+2})Q$, where Q is a prefix of $Pat_{2l+1}(L_{2k-1}-L_{2l+2}, L_{2k-1}] = 1^{L_{2l}-1}0^{L_{2l+1}+1}$ by Lemma 4.2. The prefix yielding the largest ratio is

$$P = 0^{L_{2l+1}+1}P_{2l+1}^{2k}(P_{2l+1}^{2k-1} / L_{2l+2})1^{L_{2l}-1}.$$

But this is a permutation of $P' = P_{2l+1}^{2k}(P_{2l+1}^{2k-1} / L_{2l+2})1^{L_{2l}-1}0^{L_{2l+1}+1} = P_{2l+1}^{2k}P_{2l+1}^{2k-1}$, so that $Ratio(P) = Ratio(P') \le R_{2l+1}$ by Lemma 5.2.

Case 4: $P = 0^{L_{2l+1}+1}P_{2l+1}^{2k}P_{2l+1}^{2k-1}Q$, where Q is a prefix of $P_{2l+1}^{2k}P_{2l+1}^{2k+1} / L_{2l+2}$. By the induction hypothesis, $Ratio(0^{L_{2l+1}+1}Q) \le R_{2l+1}$. By Lemma 5.2, $Ratio(P_{2l+1}^{2k}P_{2l+1}^{2k-1}) \le R_{2l+1}$, so $Ratio(P) \le R_{2l+1}$.

Case 5: $P = 0^{L_{2l+1}+1}P_{2l+1}^{2k}P_{2l+1}^{2k-1}P_{2l+1}^{2k}(P_{2l+1}^{2k+1} / L_{2l+2})Q = 0^{L_{2l+1}+1}P_{2l+1}^{2k+2}(P_{2l+1}^{2k+1} / L_{2l+2})Q$, where Q is a prefix of $Pat_{2l+1}(L_{2k+1}-L_{2l+2}, L_{2k+1}] = 1^{L_{2l}-1}0^{L_{2l+1}+1}$. As in Case 4, the prefix yielding the highest ratio is

$$P = 0^{L_{2l+1}+1}P_{2l+1}^{2k+2}(P_{2l+1}^{2k+1} / L_{2l+2})1^{L_{2l}-1}$$

which is a permutation of $P' = P_{2l+1}^{2k+1}P_{2l+1}^{2k+2}$, so that $Ratio(P) = Ratio(P') \le R_{2l+1}$ by Lemma 5.2.

Case 6: $P = 0^{L_{2l+1}+1}P_{2l+1}^{2k+2}P_{2l+1}^{2k+1}Q$, where Q is a prefix of $P_{2l+1}^{2k}P_{2l+1}^{2k+1} / L_{2l+2}$. By the induction hypothesis, $Ratio(0^{L_{2l+1}+1}Q) \le R_{2l+1}$. By Lemma 5.2, $Ratio(P_{2l+1}^{2k+2}P_{2l+1}^{2k+1}) \le R_{2l+1}$; thus, $Ratio(P) \le R_{2l+1}$. □

The result for all prefixes can now be proved as follows.

Lemma 5.4: For $l \ge 0$, there exists an integer K_l such that, for $k \ge K_l$, if P is any prefix of $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+3}-L_{2l+1}-2]$, then $Ratio(P) \le R_{2l+1}$.

Proof: Note that, by Lemma 4.2,

$$Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+3}-L_{2l+1}-2] = Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+3}-L_{2l+2}] + 1^{L_{2l}-2},$$

so, in view of Lemma 5.3, we need only show that, for large enough k,

$$Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1,\, L_{2k+3}-L_{2l+1}-2] \le R_{2l+1}.$$

Now, by Lemmas 4.2 and 4.1,

$$\begin{aligned}&Ones_{2l+1}(L_{2k+1}-L_{2l+1}-1,\, L_{2k+3}-L_{2l+1}-2]\\ &= Ones_{2l+1}(L_{2k+1}-L_{2l+1}-1,\, L_{2k+1}]+Ones_{2l+1}(L_{2k+1},\, L_{2k+3}]\\ &\quad - Ones_{2l+1}(L_{2k+3}-L_{2l+1}-2,\, L_{2k+3}] = (L_{2l}-1)F_{2k-2l+1}-1.\end{aligned}$$

So

$$\begin{aligned}Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1,\, L_{2k+3}-L_{2l+1}-2] &= \frac{(L_{2l}-1)F_{2k-2l+1}-1}{(L_{2k+2}-1)-((L_{2l}-1)F_{2k-2l+1}-1)}\\ &= \frac{1}{\dfrac{L_{2k+2}-1}{(L_{2l}-1)F_{2k-2l+1}-1}} \le \frac{1}{\dfrac{\alpha^{2l}+\alpha^{2l+2}}{L_{2l}-1}-1}\end{aligned}$$

since $\frac{L_{2k+2}-1}{(L_{2l}-1)F_{2k-2l+1}-1}$ is decreasing for k larger than some K_l, by Formula (5), with limit $\frac{\alpha^{2l}+\alpha^{2l+2}}{L_{2l}-1}$. Now

$$\frac{1}{\dfrac{\alpha^{2l}+\alpha^{2l+2}}{L_{2l}-1}-1} = \frac{L_{2l}-1}{\alpha^{2l}+\alpha^{2l+2}-L_{2l}+1} = \frac{L_{2l}-1}{\alpha L_{2l+1}+1} = R_{2l+1}.$$

This proves the lemma. □

The next step consists of showing that, for $L_{2k}-L_{2l} < n < L_{2k+2}-L_{2l}$,

$$\liminf_{n\to\infty} Ratio_{2l+1}(n) \ge R_{2l+1}.$$

Again, it is sufficient to consider proper prefixes of $Pat_{2l+1}(L_{2k}-L_{2l},\, L_{2k+2}-L_{2l}]$. The results and proofs are analogous to the ones just presented. We present only the statements of the results.

Lemma 5.5: For $l \ge 0$ and $k \ge l+1$, if P is any prefix of $Pat_{2l+1}(L_{2k}-L_{2l},\, L_{2k+2}-L_{2l+3}-1]$, then $Ratio(P) \ge R_{2l+1}$.

Lemma 5.6: For $l \ge 0$, there exists an integer $\widetilde{K}_l$ such that, if $k \ge \widetilde{K}_l$ and P is any prefix of $Pat_{2l+1}(L_{2k}-L_{2l},\, L_{2k+2}-L_{2l}-1]$, then $Ratio(P) \ge R_{2l+1}$.

We can now state the final result for positive odd powers of α.

Theorem 5.7: For any $l \ge 0$, $\lim_{n\to\infty} Ratio_{2l+1}(n) = R_{2l+1}$.

Proof: If $L_{2k+1}-L_{2l+1}-1 < n \le L_{2k+3}-L_{2l+1}-2$, then

$$Pat_{2l+1}(0, n] = Pat_{2l+1}(0,\, L_{2k+1}-L_{2l+1}-1]+P,$$

where P is a prefix of $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1,\, L_{2k+3}-L_{2l+1}-2]$. If k is large enough, then by Lemma 5.4, $Ratio(P) \le R_{2l+1}$, and by Theorem 4.3, $Ratio(L_{2k+1}-L_{2l+1}-1)$ decreases to the limit R_{2l+1}. So

$$Ratio_{2l+1}(n) \le \max\{Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1),\, Ratio(P)\}.$$

Letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} Ratio_{2l+1}(n) \le R_{2l+1}.$$

Similarly, if $L_{2k}-L_{2l}<n\le L_{2k+2}-L_{2l}-1$, then

$$Pat_{2l+1}(0,n]=Pat_{2l+1}(0,L_{2k}-L_{2l}]+P,$$

where P is a prefix of $Pat_{2l+1}(L_{2k}-L_{2l},L_{2k+2}-L_{2l}-1]$. If k is large enough, then by Lemma 5.6, $Ratio(P)\ge R_{2l+1}$, and by Theorem 4.3, $Ratio(L_{2k}-L_{2l})$ increases to the limit R_{2l+1}. Therefore,

$$Ratio_{2l+1}(n)\ge\min\{Ratio_{2l+1}(L_{2k}-L_{2l}),Ratio(P)\}.$$

Now, letting $n\to\infty$, we obtain

$$\liminf_{n\to\infty}Ratio_{2l+1}(n)\ge R_{2l+1}.\ \square$$

5.2 Other Cases

We state the results for the cases $r>0$ even and $r<0$ without proof. The proofs are very similar to those in the previous subsection.

Lemma 5.8: For $l\ge1$, there exists an integer K_l such that, for $k\ge K_l$, if P is any prefix of $Pat_{2l}(L_{2k}-L_{2l},L_{2k+2}-L_{2l}-1]$, then $Ratio(P)\le R_{2l}$.

Lemma 5.9: For $l\ge1$, there exists an integer $\widetilde{K}_l$ such that, for $k\ge\widetilde{K}_l$, if P is any prefix of $Pat_{2l}(L_{2k+1}-L_{2l-1}-1,L_{2k+3}-L_{2l-1}-2]$, then $Ratio(P)\ge R_{2l}$.

Theorem 4.4 together with Lemmas 5.8 and 5.9 leads to the following theorem.

Theorem 5.10: For any $l\ge1$, $\lim_{n\to\infty}Ratio_{2l}(n)=R_{2l}$.

Lemma 5.11: For $l\ge0$, there exists an integer K_l such that, for $k\ge K_l$, if P is any prefix of $Pat_{-(2l+1)}(L_{2k}-L_{2l},L_{2k+2}-L_{2l}-1]$, then $Ratio(P)\le R_{-(2l+1)}$.

Lemma 5.12: For $l\ge0$, there exists an integer $\widetilde{K}_l$ such that, for $k\ge\widetilde{K}_l$, if P is any prefix of $Pat_{-(2l+1)}(L_{2k+1},L_{2k+3}-1]$, then $Ratio(P)\ge R_{-(2l+1)}$.

Theorem 4.7 together with Lemmas 5.11 and 5.12 leads to the following theorem.

Theorem 5.13: For any $l\ge0$, $\lim_{n\to\infty}Ratio_{-(2l+1)}(n)=R_{-(2l+1)}$.

Lemma 5.14: For $l\ge1$, there exists an integer K_l such that, for $k\ge K_l$, if P is any prefix of $Pat_{-(2l)}(L_{2k+1},L_{2k+3}-1]$, then $Ratio(P)\le R_{-(2l)}$.

Lemma 5.15: For $l\ge1$, there exists an integer $\widetilde{K}_l$ such that, for $k\ge\widetilde{K}_l$, if P is any prefix of $Pat_{-(2l)}(L_{2k}+L_{2l-1},L_{2k+2}+L_{2l-1}-1]$, then $Ratio(P)\ge R_{-(2l)}$.

Theorem 4.7 together with Lemmas 5.14 and 5.15 leads to the following theorem.

Theorem 5.16: For any $l\ge1$, $\lim_{n\to\infty}Ratio_{-(2l)}(n)=R_{-(2l)}$.

6. CONCLUSION

We have characterized the frequency of occurrence of α^i in the α-expansions of the positive integers, for both positive and negative powers of α, using a recursive pattern found in these expansions. These results complete the characterization of the frequency of occurrence of the powers of α in the α-expansions of the positive integers, which was started in [6]. Other characteristics, such as the frequency of occurrence of certain specific patterns in the expansions, might be capable of being derived using similar methods.

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SOME PROPERTIES OF PARTIAL DERIVATIVES OF GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

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1. INTRODUCTION

In [5], Hongquan Yu and Chuanguang Liang considered the partial derivative sequences of the bivariate Fibonacci polynomials $U_n(x, y)$ and the bivariate Lucas polynomials $V_n(x, y)$. Some properties involving second-order derivative sequences of the Fibonacci polynomials $U_n(x)$ and Lucas polynomials $V_n(x)$ are established in [1] and [2]. These results may be extended to the k^{th} derivative case (see [4]).

In this paper we shall consider the partial derivative sequences of the generalized bivariate Fibonacci polynomials $U_{n,m}(x, y)$ and the generalized bivariate Lucas polynomials $V_{n,m}(x, y)$. We shall use the notation $U_{n,m}$ and $V_{n,m}$ instead of $U_{n,m}(x, y)$ and $V_{n,m}(x, y)$, respectively. These polynomials are defined by

$$U_{n,m} = xU_{n-1,m} + yU_{n-m,m}, \quad n \geq m, \tag{1.1}$$

with $U_{0,m} = 0$, $U_{n,m} = x^{n-1}$, $n = 1, 2, \ldots, m-1$, and

$$V_{n,m} = xV_{n-1,m} + yV_{n-m,m}, \quad n \geq m, \tag{1.2}$$

with $V_{0,m} = 2$, $V_{n,m} = x^n$, $n = 1, 2, \ldots, m-1$.

For $p = 0$ and $q = -y$, the polynomials $U_{n,m}$ are the known polynomials $\phi_n(0, -y; x)$ [3].

From (1.1) and (1.2), we find some first members of the sequences $U_{n,m}$ and $V_{n,m}$, respectively. These polynomials are given in the following table.

TABLE 1

n	$U_{n,m}$	$V_{n,m}$
0	0	2
1	1	x
2	x	x^2
3	x^2	x^3
$\vdots$	$\vdots$	$\vdots$
$m-1$	x^{m-2}	x^{m-1}
m	x^{m-1}	$x^m + 2y$
$m+1$	$x^m + y$	$x^{m+1} + 3xy$
$\vdots$	$\vdots$	$\vdots$
$2m-1$	$x^{2m-2} + (m-1)x^{m-2}y$	$x^{2m-1} + (m+1)x^{m-1}y$
$2m$	$x^{2m-1} + mx^{m-1}y$	$x^{2m} + (m+2)x^m y + 2y^2$
$\vdots$	$\vdots$	$\vdots$

The partial derivatives of $U_{n,m}$ and $V_{n,m}$ are defined by

$$U_{n,m}^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} U_{n,m} \quad \text{and} \quad V_{n,m}^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} V_{n,m}, \quad k \geq 0, \ j \geq 0.$$

Also, we find that $U_{n,m}$ and $V_{n,m}$ possess the following generating functions:

$$F = (1 - xt - yt^m)^{-1} = \sum_{n=1}^{\infty} U_{n,m} t^{n-1} \tag{1.3}$$

and

$$G = (2 - xt^{m-1})(1 - xt - yt^m)^{-1} = \sum_{n=0}^{\infty} V_{n,m} t^n. \tag{1.4}$$

From (1.3) and (1.4), we get the following representations of $U_{n,m}$ and $V_{n,m}$, respectively:

$$U_{n,m} = \sum_{k=0}^{[(n-1)/m]} \binom{n-1-(m-1)k}{k} x^{n-1-mk} y^k \tag{1.5}$$

and

$$V_{n,m} = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} x^{n-mk} y^k. \tag{1.6}$$

If $m = 2$, then polynomials $U_{n,m}$ and $V_{n,m}$ are the known polynomials U_n and V_n ([5]), respectively.

From Table 1, using induction on n, we can prove that

$$V_{n,m} = U_{n+1,m} + yU_{n+1-m,m}, \quad n \geq m-1. \tag{1.7}$$

2. SOME PROPERTIES OF $U_{n,m}^{(k,j)}$ AND $V_{n,m}^{(k,j)}$

We shall consider the partial derivatives $U_{n,m}^{(k,j)}$ and $V_{n,m}^{(k,j)}$. Namely, we shall prove the following theorem.

Theorem 2.1: The polynomials $U_{n,m}^{(k,j)}$ and $V_{n,m}^{(k,j)}$ $(n \geq 0, k \geq 0, j \geq 0)$ satisfy the following identities:

$$V_{n,m}^{(k,j)} = U_{n+1,m}^{(k,j)} + jU_{n+1-m,m}^{(k,j-1)} + yU_{n+1-m,m}^{(k,j)}; \tag{2.1}$$

$$U_{n,m}^{(k,j)} = kU_{n-1,m}^{(k-1,j)} + xU_{n-1,m}^{(k,j)} + jU_{n-m,m}^{(k,j-1)} + yU_{n-m,m}^{(k,j)}; \tag{2.2}$$

$$V_{n,m}^{(k,j)} = kV_{n-1,m}^{(k-1,j)} + xV_{n-1,m}^{(k,j)} + jV_{n-m,m}^{(k,j-1)} + yV_{n-m,m}^{(k,j)}; \tag{2.3}$$

$$V_{n,m}^{(k,j)} = \sum_{i=j}^{[(n-k)/m]} (n-(m-2)i) \frac{(n-(m-1)i)!}{(i-j)!(n-k-mi)!} x^{n-k-mi} y^{i-j}. \tag{2.4}$$

Proof: Differentiating (1.7), (1.1), and (1.2), first k-times with respect to x, then j-times with respect to y, we get (2.1), (2.2), and (2.3), respectively.

Also, if we differentiate (1.6) with respect to x, then with respect to y, we get (2.4).

Remark 2.1: If $m = 2$, then identities (2.1)-(2.4) become identities (i)-(iv) in [5].

Theorem 2.2: Let $k \geq 0,\ j \geq 0$. Then, we have:

$$\sum_{i=0}^{n} U_{i,m}^{(k,j)} U_{n-i,m} = \frac{1}{k+j+1} U_{n,m}^{(k+1,j)}; \tag{2.5}$$

$$\sum_{i=0}^{n} V_{i,m}^{(k,0)} V_{n-i,m}^{(0,j)} = \left((k+j+1)\binom{k+j}{j}\right)^{-1} (2 - xt^{m-1})^2 (2t^{-1} - t^{m-3} + yt^{2m-3}) U_{n+1,m}^{(k+j,j)}; \tag{2.6}$$

$$\sum_{i=0}^{n} U_{i+1,m}^{(0,j-1)} V_{n-i,m}^{(0,k)} = \left((j+k)\binom{j+k-1}{j-1} t^m\right)^{-1} V_{n,m}^{(0,j+k)}; \tag{2.7}$$

$$\sum_{i=0}^{n} U_{i,m}^{(k,j)} U_{n-i,m}^{(l,p)} = \left((k+j+p+l+1)\binom{k+j+p+l}{k+j}\right)^{-1} U_{n,m}^{(k+l+1,j+p)}. \tag{2.8}$$

Proof: Differentiating (1.3) k-times with respect to x, then j-times with respect to y, we get

$$F^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} F = \frac{(k+j)!\, t^{k+mj}}{(1 - xt - yt^m)^{k+j+1}} = \sum_{n=1}^{\infty} U_{n,m}^{(k,j)} t^{n-1}. \tag{i}$$

From (i), we have

$$F^{(0,0)} F^{(k,j)} = \frac{(k+j)!\, t^{k+jm}}{(1 - xt - yt^m)^{k+j+2}} = \sum_{n=1}^{\infty} \sum_{i=0}^{n} U_{i,m}^{(k,j)} U_{n-i,m} t^{n-2}.$$

Hence, we conclude that

$$\begin{aligned}\sum_{i=0}^{n} U_{i,m}^{(k,j)} U_{n-i,m} &= \frac{(k+j)!\, t^{k+1+jm}}{(1 - xt - yt^m)^{k+j+2}} \\ &= \frac{(k+j+1)!\, t^{k+1+jm}}{(k+j+1)(1 - xt - yt^m)^{k+j+2}} = \frac{1}{k+j+1} U_{n,m}^{(k+1,j)}.\end{aligned}$$

By the last equalities, we get (2.5)

In a similar way, we can obtain (2.6), (2.7), and (2.8).

Corollary 2.1: If $k = l,\ j = p$, from (2.8) we get

$$\sum_{i=0}^{n} U_{i,m}^{(k,j)} U_{n-i,m}^{(k,j)} = \left((2k+2j+1)\binom{2k+2j}{k+j}\right)^{-1} U_{n,m}^{(2k+1,2j)}.$$

Furthermore, we are going to prove the following general result.

Theorem 2.3: Let $k \geq 0, j \geq 0, s \geq 0$. Then

$$\sum_{i_1 + i_2 + \cdots + i_s = n} U_{i_1,m}^{(k,j)} U_{i_2,m}^{(k,j)} \cdots U_{i_s,m}^{(k,j)} = \frac{((k+j)!)^s}{(sk+sj+s-1)!} U_{n,m}^{(sk+s-1,sj)}. \tag{2.10}$$

Proof: From (i), i.e.,

$$F^{(k,j)} = \frac{(k+j)!\, t^{k+mj}}{(1 - xt - yt^m)^{k+j+1}} = \sum_{n=1}^{\infty} U_{n,m}^{(k,j)} t^{n-1},$$

we find:

$$F^{(k,j)}F^{(k,j)}\dots F^{(k,j)} = \frac{((k+j)!)^s t^{sk+sjm}}{(1-xt-yt^m)^{sk+sj+s}}$$
$$= \sum_{n=1}^{\infty} \sum_{i_1+i_2+\cdots+i_s=n} U^{(k,j)}_{i_1,m}U^{(k,j)}_{i_2,m}\dots U^{(k,j)}_{i_s,m}t^{n-s}.$$

Hence, we get

$$\sum_{n=1}^{\infty} \sum_{i_1+i_2+\cdots+i_s=n} U^{(k,j)}_{i_1,m}U^{(k,j)}_{i_2,m}\dots U^{(k,j)}_{i_s,m}t^{n-1} = \frac{((k+j)!)^s t^{sk+s-1+sjm}}{(1-xt-yt^m)^{sk+sj+s}}$$
$$= \frac{((k+j)!)^s}{(sk+sj+s-1)!}U^{(sk+s-1,\, sj)}_{n,m}.$$

The equality (2.10) follows from the last equalities.

Remark 2.2: We can prove that

$$\frac{((k+j)!)^s}{(sk+sj+s-1)!} = \prod_{i=2}^{s}\left((i\alpha-1)\binom{i\alpha-2}{\alpha-1}\right)^{-1},$$

where $\alpha = k+j+1$. So (2.10) takes the following form,

$$\sum_{i_1+i_2+\cdots+i_s=n} U^{(k,j)}_{i_1,m}U^{(k,j)}_{i_2,m}\dots U^{(k,j)}_{i_s,m} = \prod_{i=2}^{s}\left((i\alpha-1)\binom{i\alpha-2}{\alpha-1}\right)^{-1} U^{(sk+s-1,\, sj)}_{n,m},$$

where $\alpha = k+j+1$.

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GENERALIZING BAILEY'S GENERALIZATION OF THE CATALAN NUMBERS

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INTRODUCTION

In a recent note in *Mathematics Magazine* [1], D. F. Bailey developed a formula for the number of sequences $a_1, a_2, \ldots, a_{n+r}$ comprised of n 1's and r −1's such that $\sum_{i=1}^{j} a_j \geq 0$ for each $j = 1, 2, \ldots, n+r$.

He denoted the number of such sequences as $\left\{ {n \atop r} \right\}$ and noted that

$$\left\{ {n \atop r} \right\} = \left\{ {n \atop r-1} \right\} + \left\{ {n-1 \atop r} \right\} \tag{1}$$

for $1 < r < n$ and

$$\left\{ {n \atop n} \right\} = \left\{ {n \atop n-1} \right\} \tag{2}$$

for each $n \geq 1$. From these facts, it is easy to build the following table of values for $\left\{ {n \atop r} \right\}$.

TABLE 1. Values for $\left\{ {n \atop r} \right\}$

$n \backslash r$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	2						
3	1	3	5	5					
4	1	4	9	14	14				
5	1	5	14	28	42	42			
6	1	6	20	48	90	132	132		
7	1	7	27	75	165	297	429	429	
8	1	8	35	110	275	572	1001	1430	1430

Indeed, Bailey obtains the identity

$$\left\{ {n \atop r} \right\} = \frac{(n+1-r)(n+2)(n+3)\ldots(n+r)}{r!}$$

whenever $n \geq r \geq 2$. Bailey closes the paper by noting that

$$\left\{ {n \atop n} \right\} = \frac{1}{n+1}\binom{2n}{n},$$

the n^{th} Catalan number C_n. This is not surprising, as one of the classical combinatorial interpretations of C_n is the number of sequences of n 1's and n −1's that satisfy the subsequence sum restriction mentioned above. In this context, Bailey provided a very nice generalization of the classical Catalan numbers.

In this paper we generalize Bailey's work by considering sequences comprised of n $(m-1)$'s and r -1's (where $m \geq 2$) with the sum of each subsequence of the first j terms nonnegative. We will denote the number of such sequences as $\left\{ {n \atop r} \right\}_{m-1}$.

It is clear that the recurrences similar to (1) and (2) are satisfied for general values of m. Namely,

$$\left\{ {n \atop r} \right\}_{m-1} = \left\{ {n-1 \atop r} \right\}_{m-1} + \left\{ {n \atop r-1} \right\}_{m-1} \tag{3}$$

for $1 < r < n$ and

$$\begin{aligned} \left\{ {n \atop (m-1)n} \right\}_{m-1} &= \left\{ {n \atop (m-1)n-1} \right\}_{m-1} = \left\{ {n \atop (m-1)n-2} \right\}_{m-1} \\ &= \cdots = \left\{ {n \atop (m-1)n-(m-1)} \right\}_{m-1} \end{aligned} \tag{4}$$

for each $n \geq 1$. The recurrence in (3) is seen in a straightforward manner. Take a sequence of n $(m-1)$'s and r -1's that is counted by $\left\{ {n \atop r} \right\}_{m-1}$. The last element in the sequence is either an $m-1$ or a -1. If it is an $m-1$, then the preceding subsequence is one of those counted by $\left\{ {n-1 \atop r} \right\}_{m-1}$. On the other hand, if the last element of our original sequence is a -1, then the preceding subsequence is one of those enumerated in $\left\{ {n \atop r-1} \right\}_{m-1}$.

To establish (4), consider a sequence of n $(m-1)$'s and $(m-1)n$ -1's counted by $\left\{ {n \atop (m-1)n} \right\}_{m-1}$. It must be the case that the last element in this sequence is -1. (If not, then one of the subsequence sums would have to be negative, which is contradictory.) Hence, the subsequence preceding this final -1 will also satisfy the property that all of its subsequence sums are positive. Therefore, this preceding subsequence will be enumerated by $\left\{ {n \atop (m-1)n-1} \right\}_{m-1}$. In more general terms, it is clear that the last $m-1$ elements of our sequence of n $(m-1)$'s and $n(m-1)n$ -1's must be -1. Thus, the same argument as that above can be used to prove the full set of equalities in (4).

We include here a table comparable to Table 1 above in the case of $m = 3$.

TABLE 2. Values for $\left\{ {n \atop r} \right\}_2$

$n \backslash r$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1	1						
2	1	2	3	3	3				
3	1	3	6	9	12	12	12		
4	1	4	10	19	31	43	55	55	55
5	1	5	15	34	65	108	163	218	273
6	1	6	21	55	120	228	391	609	882
7	1	7	28	83	203	431	822	1431	2313
8	1	8	36	119	322	753	1575	3006	5319

Our goal is to present various results involving $\left\{ {n \atop r} \right\}_{m-1}$, including an interpretation of the sequences counted by $\left\{ {n \atop r} \right\}_{m-1}$ which truly generalizes the proof that

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

which appears in [2]. In particular, we pay special attention to the case when $r=(m-1)n$ (the largest allowable value of r for fixed m) and prove that

$$\left\{ {n \atop (m-1)n} \right\}_{m-1} = \frac{1}{(m-1)n+1}\binom{mn}{n}. \tag{5}$$

This is a clear generalization of C_n. Indeed, these quantities also enjoy a rich history and background, and can be found in the works of Raney [6] and others. See [5] and [7] for additional discussion.

THE MAIN GENERALIZATION OF THE CATALAN NUMBERS

We now consider sequences of the form $a_1, a_2, \ldots, a_{n+r}$ containing n $(m-1)$'s and r -1's for a fixed value of m larger than 1. Note that the total number of such sequences (with no restrictions) is $\binom{n+r}{n}$. We wish to count, in a natural way, the number of such sequences that satisfy

$$a_1 + a_2 + \cdots + a_j \geq 0 \tag{6}$$

for each $j=1,2,\ldots,n+r$. We will do so by subtracting from $\binom{n+r}{n}$ the number of sequences $\left\langle {n \atop r} \right\rangle_{m-1}$ that violate (6) above for at least one value of j. (These sequences will be referred to affectionately as "bad.") This use of inclusion/exclusion is, in essence, the approach taken in [2] to prove the formula for C_n, although the proof in [2] is not directly generalizable.

For simplicity, we first focus our attention on the case $r=(m-1)n$. We pick up the more general case in the next section.

On our way to a closed form for $\left\{ {n \atop (m-1)n} \right\}_{m-1}$, we first state the following lemma.

Lemma 1:

$$\sum_{k=0}^{n} \frac{w}{w+dk}\binom{p-bk}{n-k}\binom{q+bk}{k}$$
$$=\binom{p+q}{n} + \sum_{k=1}^{n}\binom{p+q-k}{n-k}\frac{(wb-qd)(wb-(q-1)d)\cdots(wb-(q-k+1)d)}{(w+d)(w+2d)\cdots(w+kd)}$$

for all values of p, q, w, n, b and d for which the terms are defined.

Proof: This result is proved by H. Gould and J. Kaucky in [4]. □

Using Lemma 1, we can prove the following corollary.

Corollary 1: For all $m>2$,

$$\sum_{k=0}^{n-1} \frac{1}{(m-1)k+1}\binom{mk}{k}\binom{mn-mk-1}{n-k} = (m-1)\binom{mn}{n-1}.$$

Proof: We apply Lemma 1 with $b=m$, $p=mn-1$, $q=0$, $w=1$, and $d=m-1$. Then we have, following (see [4]),

$$\sum_{k=0}^{n} \frac{1}{1+(m-1)k}\binom{mn-1-mk}{n-k}\binom{mk}{k}$$
$$=\binom{mn-1}{n} + \sum_{k=1}^{n}\binom{mn-1-k}{n-k}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \binom{mn-1-k}{n-k} \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{-mn+k+1+n-k-1}{n-k} \quad \text{using (3.4) of [4]} \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{-(m-1)n}{n-k} \\
&= \sum_{j=n}^{0} (-1)^{j} \binom{-(m-1)n}{j} \quad \text{using } j=n-k \\
&= \sum_{j=0}^{n} (-1)^{j} \binom{-(m-1)n}{j} \\
&= \binom{n+(m-1)n}{n} \quad \text{using identity 1.49 of [3]} \\
&= \binom{mn}{n}.
\end{aligned}
$$

Thus, we know that

$$
\begin{aligned}
\sum_{k=0}^{n-1} \frac{1}{(m-1)k+1}\binom{mk}{k}\binom{m(n-k)-1}{n-k} &= \binom{mn}{n} - \frac{1}{(m-1)n+1}\binom{mn}{n}\binom{-1}{0} \\
&= \binom{mn}{n} - \frac{1}{(m-1)n+1}\binom{mn}{n}\cdot 1 \\
&= \frac{(m-1)n}{(m-1)n+1}\binom{mn}{n} = (m-1)\binom{mn}{n-1}. \quad \square
\end{aligned}
$$

We are now in position to state the main result for $\left\{ {n \atop (m-1)n} \right\}_{m-1}$.

Theorem 1: For all $n \geq 1$ and $m \geq 2$,

$$\left\{ {n \atop (m-1)n} \right\}_{m-1} = \frac{1}{(m-1)n+1}\binom{mn}{n}.$$

Proof: We will prove this theorem by induction on n. First, we recall that

$$\left\{ {n \atop (m-1)n} \right\}_{m-1} = \binom{n+(m-1)n}{n} - \left\langle {n \atop (m-1)n} \right\rangle_{m-1}$$

and focus on the "bad" sequences counted by $\left\langle {n \atop (m-1)n} \right\rangle_{m-1}$. Let $\mathbb{a} = \langle a_1, a_2, \ldots, a_{mn} \rangle$ be a bad sequence of n $(m-1)$'s and $(m-1)n$ -1's and let j be the first subscript for which the partial sum $S_j = \sum_{i=1}^{j} a_i < 0$. (The existence of j is guaranteed since $\mathbb{a}$ is a bad sequence.) Then $S_j = -1$ and $a_j = -1$ (by the minimality of j) so that $S_{j-1} = 0$. Therefore, $j \equiv 1 \pmod{m}$.

We now set $k = \frac{j-1}{m}$. Then there are k $(m-1)$'s in the partial sequence $\mathbb{a}_{j-1} = \langle a_1, \ldots, a_{j-1} \rangle$, so $\mathbb{a}_{j-1}$ is a "good" sequence with k $(m-1)$'s and $(m-1)k$ -1's by the minimality of k. Moreover, the subsequence

$$\hat{\mathbf{a}}_j = \langle a_{j+1}, a_{j+2}, \ldots, a_{mn} \rangle$$

is an arbitrary sequence of $(n-k)$ $(m-1)$'s and $((m-1)(n-k)-1)$ -1's. (Remember that $a_j = -1$.) Since there are $\left\{{k \atop (m-1)k}\right\}_{m-1}$ ways to choose a good sequence with k $(m-1)$'s and $(m-1)k$ -1's and $\binom{mn-mk-1}{n-k}$ ways to choose an arbitrary sequence of $(n-k)$ $(m-1)$'s and $((m-1)(n-k)-1)$ -1's, then the number of bad sequences with n $(m-1)$'s and $(m-1)n$ -1's with the first bad partial sequence having length $mk+1$ is $\left\{{k \atop (m-1)k}\right\}_{m-1}\binom{mn-mk-1}{n-k}$. Hence,

$$\left\langle {n \atop (m-1)n} \right\rangle_{m-1} = \sum_{k=0}^{n-1} \left\{ {k \atop (m-1)k} \right\}_{m-1} \binom{mn-mk-1}{n-k}. \tag{7}$$

A quick comment is in order regarding the index of summation in (7). Since $j \equiv 1 \bmod m$, the smallest possible value of j is 1, whence $k = 0$ is the smallest value of k (since $k = \frac{j-1}{m}$). Since the whole sequence has length mn, the largest value of j is $mn-m+1$; hence, the largest value of k is $\frac{mn-m+1-1}{m} = n-1$.

Now, by induction,

$$\left\{ {k \atop (m-1)k} \right\}_{m-1} = \frac{1}{(m-1)k+1}\binom{mk}{k},$$

so

$$\left\langle {n \atop (m-1)n} \right\rangle_{m-1} = \sum_{k=0}^{n-1} \frac{1}{(m-1)k+1}\binom{mk}{k}\binom{mn-mk-1}{n-k}.$$

Now Corollary 1 can be applied:

$$\begin{aligned}\left\langle {n \atop (m-1)n} \right\rangle_{m-1} &= \sum_{k=0}^{n-1} \frac{1}{(m-1)k+1}\binom{mk}{k}\binom{mn-mk-1}{n-k} \\ &= (m-1)\binom{mn}{n-1}.\end{aligned}$$

Our result is now in reach.

$$\begin{aligned}\left\{ {n \atop (m-1)n} \right\}_{m-1} &= \binom{n+(m-1)n}{n} - \left\langle {n \atop (m-1)n} \right\rangle_{m-1} \\ &= \binom{mn}{n} - (m-1)\binom{mn}{n-1} \\ &= \binom{mn}{n} - (m-1)\cdot\frac{n}{(mn-n+1)}\binom{mn}{n} \\ &= \left[1 - \frac{n(m-1)}{mn-n+1}\right]\binom{mn}{n} \\ &= \frac{1}{(m-1)n+1}\binom{mn}{n} \quad \text{after simplification.}\end{aligned}$$

Therefore, we see that

$$\left\{ {n \atop (m-1)n} \right\}_{m-1} = \frac{1}{(m-1)n+1}\binom{mn}{n}. \quad \square$$

As noted in the introductory section, the values $\frac{1}{(m-1)n+1}\binom{mn}{n}$ have appeared in the past. However, we are unaware of their interpretation as the number of sequences described above. Moreover, the proof technique utilized in Theorem 1 does not seem readily available in the literature.

THE GENERAL CASE

Our initial motivation in this study (in the spirit of Bailey) was to find a closed form for $\left\{ {n \atop r} \right\}_{m-1}$ for all r satisfying $1 \le r \le (m-1)n$. The completion of this task has proved elusive. However, we can generalize (7) above to determine $\left\langle {n \atop r} \right\rangle_{m-1}$. Then it is clear that

$$\left\{ {n \atop r} \right\}_{m-1} = \binom{n+r}{n} - \left\langle {n \atop r} \right\rangle_{m-1}.$$

We now look at the generalization of (7).

Theorem 2:

$$\left\langle {n \atop r} \right\rangle_{m-1} = \sum_{k=0}^{\lceil \frac{r}{m-1} \rceil - 1} \frac{1}{(m-1)k+1} \binom{mk}{k} \binom{n+r-mk-1}{n-k}. \tag{8}$$

Proof: The proof of this is essentially the same as that in Theorem 1. The major difference is that the index of summation must be modified. To determine the extreme values of k we analyze as before. Since $j \equiv 1 \bmod m$, the smallest possible value of j is 1, whence $k=0$ is the smallest value of k (since $k = \frac{j-1}{m}$). Now, any "good" subsequence of **a** has km terms and so will have $(km-k)$ -1's in it. But if $km-k \ge r$, there are no -1's left to make the sequence bad. That is, we may not have $k \ge \frac{r}{m-1}$, so the maximum value of k is $\lfloor \frac{r}{m-1} \rfloor$ if r is not an integer multiple of $m-1$, and $\frac{r}{m-1}-1$ if r is an integer multiple of $m-1$. A more efficient way of expressing the maximum value of k is as $\lceil \frac{r}{m-1} \rceil - 1$. □

Unfortunately, we have been unable to determine a closed formula for (8). However, we note that this is still a useful insight, at least in a computational sense. Indeed, if one wants to determine (for example) $\left\{ {100 \atop 44} \right\}_4$ with $m=5$, $n=100$, and $r=44$, then (8) provides a very feasible way to calculate $\left\langle {100 \atop 44} \right\rangle_4$, so that

$$\left\{ {100 \atop 44} \right\}_4 = \binom{144}{44} - \left\langle {100 \atop 44} \right\rangle_4.$$

In this example, the sum in (8) only contains $\lceil \frac{44}{4} \rceil$ or 11 terms, each of which is simply a weighted product of two binomial coefficients. This is much quicker than calculating $\left\{ {100 \atop 44} \right\}_4$ from the recurrences (3) and (4).

CONCLUDING THOUGHTS AND QUESTIONS

While we have not fully reached our initial goal, we are satisfied with the results obtained, especially since the approach seems quite novel. We now share two thoughts in closing.

First, we covet a closed formula for the sum in (8). It is unclear how to accomplish this task. Second, we note a fairly interesting residual result from Bailey's work. Bailey proved that

$$\left\{ {n \atop r} \right\}_1 = \frac{(n+1-r)(n+r)!}{(n+1)n!r!} = \frac{n+1-r}{n+1}\binom{n+r}{n}.$$

Thus, it is clear that

$$\left\langle {n \atop r} \right\rangle_1 = \binom{n+r}{r} - \frac{n+1-r}{n+1}\binom{n+r}{r}$$
$$= \left(1 - \frac{n+1-r}{n+1}\right)\binom{n+r}{r} = \frac{r}{n+1}\binom{n+r}{r}.$$

By (8) above, we then have

$$\sum_{k=0}^{r-1} \frac{1}{k+1}\binom{2k}{k}\binom{n+r-2k-1}{n-k} = \frac{r}{n+1}\binom{n+r}{r}.$$

The proof of this summation result does not appear to be within reach via known tools such as Lemma 1, and we have been unable to prove this identity directly. A direct combinatorial proof of this result would be nice to see. If found, such a proof might allow us to better see a closed formula for the sum in (8).

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ON IRRATIONAL VALUED SERIES INVOLVING GENERALIZED FIBONACCI NUMBERS II

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1. INTRODUCTION

In the predecessor to this paper (see [7]) a family of rational termed series having irrational sums was constructed. These series whose terms, for a fixed $k \in N \setminus \{0\}$, were formed from the reciprocal of the factorial-like product of generalized Fibonacci numbers $U_k U_{k+1} \ldots U_{k+n}$, in addition exhibited irrational limits when summed over arbitrary infinite subsequences of N, by replacing n with a strictly monotone increasing function $f: N \to N$. Owing to this factorial-like form, the argument employed in [7] was closely modeled on that of Euler's for establishing the irrationality of e. However, as a consequence of the approach taken, one needed to restrict attention to those sequences $\{U_n\}$, generated with respect to the relatively prime pair (P, Q) with $|Q| = 1$ and $|P| > 1$. In view of these results it was later conjectured in [7] whether other irrational valued series could be constructed having terms formed from the reciprocal of such products as $U_{f(n)} \ldots U_{f(n)+k}$, where again $f: N \to N \setminus \{0\}$ was a strictly monotone increasing function. In this paper we shall provide evidence to support the conjecture by examining two disparate cases, namely, when $f(\cdot)$ satisfies a linear and an exponential growth condition. To help establish the result in the later case, a sufficient condition for irrationality will be derived. This condition, which is similar but slightly more restrictive than that employed in [2] and [6], will be demonstrated, for interest's sake, by an alternate proof based on the following well-known criterion for irrationality (see [8]): If there exists a $\delta > 0$ and a nonconstant infinite sequence $\{p_n / q_n\}$ of rational approximations to θ, with $(p_n, q_n) = 1$, and such that, for all n sufficiently large,

$$\left| \theta - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{1+\delta}},$$

then θ is irrational. In addition, the above sufficiency condition will allow us to prove that the conjecture also holds for generalized Lucas sequences $\{V_n\}$, when $f(\cdot)$ has exponential growth. One notable feature of these results compared to those obtained in [7] is that they apply to a much wider family of sequences, namely, those which are generated with respect to the relatively prime pair (P, Q) with $|Q| > 1$ and $P > |Q+1|$. Unfortunately, in the linear case (i.e., when $f(n) = n$), we cannot achieve the same level of generality, as irrational sums can only be deduced for those series involving generalized Fibonacci numbers where $Q = 1$ and $P > 2$. This restriction is due to the fact that (for k even) the sum of the series in question is given as a linear expression in s over the rationals, where $s = \sum_{n=1}^{\infty} 1/U_n$ is at present only known to be irrational for $|Q| = 1$. In the final part of the paper, we return to the family of series first considered in [7] and, by applying the above criterion for irrationality, we extend the results there to encompass those series involving both the generalized Fibonacci and Lucas numbers where $Q \neq 0$ and $P > |Q+1|$.

2. SERIES WITH TERMS $(U_{f(n)} \dots U_{f(n)+k})^{-1}$

We will first address the case of $f(\cdot)$ satisfying an exponential growth condition. To help motivate the required sufficient condition for irrationality, let us consider the following result which was first stated (without proof) in [3] but later proved by Badea in [2].

Theorem 2.1: Let $\{a_n\}$, for $n \geq 1$, be a sequence of integers such that $a_{n+1} > a_n^2 - a_n + 1$ holds for all n. Then the sum of the series $\sum_{n=1}^{\infty} 1/a_n$ is an irrational number.

This criterion, which is based on a sufficient condition for irrationality of Brun [3], is best possible, in the sense that rational valued series can occur if the strict inequality is replaced by equality. It is clear from Theorem 2.1 that, if $a_1 > 1$ and $a_{n+1} \geq a_n^2$ for $n > 1$, then the series of reciprocals $\{1/a_n\}$ must also sum to an irrational number. Such a weaker version of the above criterion was proved indirectly by McDaniel in [6] via a descent method and later was used to establish the irrationality of $\sum_{n=1}^{\infty} 1/U_{f(n)}$, where $f: N \to N$ satisfied the inequality $f(n+1) \geq 2f(n)$. In a similar manner, by using the more restrictive condition of $\inf_{n \in N}\{a_{n+1}/a_n^2\} > 1$, we can now extend the results obtained in [6] to those series involving the reciprocal of such products as $a_n = U_{f(n)} \dots U_{f(n)+k}$. Although not essential, the advantage in using this alternate condition is that we can demonstrate irrationality via a direct proof, as opposed to the indirect arguments employed in [6]. To this end, consider now the following technical lemma.

Lemma 2.1: If $\sum_{n=1}^{\infty} 1/a_n$ is a series of rationals with $a_n \in N \setminus \{0\}$ and $\inf_{n \in N}\{a_{n+1}/a_n^2\} > 1$, then the series converges to an irrational sum.

Proof: From the above inequality, the series is clearly convergent. Denoting the sum of the series by θ, we examine the sequence of rational approximations p_n/q_n to θ generated from the n^{th} partial sums, expressed in reduced form. As $a_n > 0$, for $n \geq 1$, the terms p_n/q_n must be strictly monotone increasing and so the sequence is nonconstant. To prove the irrationality of θ it is sufficient, in view of the aforementioned criterion, to demonstrate that $|q_n\theta - p_n| = o(1)$ as $n \to \infty$. Since $(p_m, q_m) = 1$, the lowest common denominator of the m fractions in the set $\{1/a_n\}_{n=1}^{m}$ must be greater than or equal to q_m but, as $a_1 a_2 \dots a_m$ is one common denominator, we deduce that $q_m \leq a_1 \dots a_m$. Thus, again by the above inequality,

$$|q_m\theta - p_m| = \sum_{n=m+1}^{\infty} \frac{q_m}{a_n} \leq a_1 \dots a_m \sum_{n=m+1}^{\infty} \frac{1}{a_n} = b_m \sum_{n=m+1}^{\infty} \frac{a_{m+1}}{a_n}$$
$$< b_m\left(1 + \sum_{r=1}^{\infty} \frac{a_{m+r}}{a_{m+r+1}}\right) < b_m\left(1 + \sum_{r=1}^{\infty} \frac{1}{a_{m+r}}\right) < b_m(1+\theta),$$

where $b_m = (a_1 \dots a_m)/a_{m+1}$. The result will follow after showing that $b_m \to 0$ as $m \to \infty$. To this end, we consider $\log(1/b_m)$. Via assumption, there must exist a $\delta > 0$ such that $a_{n+1}/a_n^2 \geq (1+\delta)$ for all n, and so

$$\log(1/b_m) = \sum_{r=1}^{m} (\log a_{r+1} - \log a_r) + \log a_1 - \sum_{r=1}^{m} \log a_r$$
$$= \sum_{r=1}^{m} \log\left(\frac{a_{r+1}}{a_r^2}\right) + \log a_1 \geq m \log(1+\delta) \to \infty \text{ as } m \to \infty. \quad \square$$

In the case of $a_n = U_{f(n)} \dots U_{f(n)+k}$, the condition of the previous lemma can be satisfied when $f(\cdot)$ has exponential growth. We demonstrate this using the following well-known identities:

$$U_{2m} = U_m V_m, \quad U_{2m-1} = U_m^2 - QU_{m-1}^2, \quad V_m > U_m. \tag{1}$$

Theorem 2.2: Suppose $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to the relatively prime pair (P, Q) with $Q \neq 0$ and $P > |Q+1|$. If, for a given $k \in N$, the function $f: N \to N \setminus \{0\}$ has the property $f(n+1) > 2f(n) + 2k$ for all $n \geq 1$, then the series $\sum_{n=1}^{\infty} 1/a_n$ converges to an irrational sum, where $a_n = U_{f(n)} \dots U_{f(n)+k}$.

Proof: We first note that, for the prescribed values of P and Q, $\{U_n\}$ and $\{V_n\}$ are strictly monotone increasing sequences of positive integers. To demonstrate the irrationality of the series sum, it will suffice in view of Lemma 2.1 to show that $\inf_{n \in N}\{a_{n+1}/a_n^2\} > 1$. Now, since

$$\frac{a_{n+1}}{a_n^2} = \prod_{r=0}^{k} \frac{U_{f(n+1)+r}}{U_{f(n)+r}^2},$$

observe from the assumption on $f(\cdot)$ and the identities in (1) that, for $r = 0, 1, \dots, k$ and $n \in N$,

$$\begin{aligned}\frac{U_{f(n+1)+r}}{U_{f(n)+r}^2} &\geq \frac{U_{2f(n)+2k+r+1}}{U_{f(n)+r}^2} \geq \frac{U_{2f(n)+2r+1}}{U_{f(n)+r}^2} = \frac{PU_{f(n)+r}V_{f(n)+r} - QU_{2f(n)+2r-1}}{U_{f(n)+r}^2} \\ &> \frac{PU_{f(n)+r}^2 - QU_{2f(n)+2r-1}}{U_{f(n)+r}^2} = \frac{U_{f(n)+r}^2(P-Q) + Q^2U_{f(n)+r-1}^2}{U_{f(n)+r}^2}.\end{aligned}$$

Consequently, as $P - Q \geq 2$, one deduces from the previous inequality that $\inf_{n \in N}\{a_{n+1}/a_n^2\} > 2^{k+1} > 1$. □

Via a similar application of Lemma 2.1, one can prove the irrationality of the above series when U_n is replaced with the terms of a generalized Lucas sequence $\{V_n\}$.

Theorem 2.3: Suppose $\{V_n\}$ is a generalized Lucas sequence generated with respect to the relatively prime pair (P, Q) with $Q \neq 0$ and $P > |Q+1|$. If, for a given $k \in N$, the function $f: N \to N \setminus \{0\}$ has the property that $f(n+1) > 2f(n) + 2k$ for all $n \geq 1$, then the series $\sum_{n=1}^{\infty} 1/a_n$ converges to an irrational sum, where $a_n = V_{f(n)} \dots V_{f(n)+k}$.

Proof: For the prescribed values of P and Q, it is readily seen that $\lim_{n\to\infty} V_{2n+1}/V_n^2 = \alpha > 1$. Thus, for an $0 < \varepsilon < \alpha - 1$, there must exist an $N(\varepsilon) > 0$ such that $V_{2n+1}/V_n^2 > \alpha - \varepsilon > 1$, when $n > N(\varepsilon)$. Let $N' := \min\{s \in N : f(n) > N(\varepsilon) \text{ for all } n \geq s\}$ and consider the remainder of the series given here by $\sum_{n=N'}^{\infty} 1/a_n$. To demonstrate the irrationality of the above series, it will suffice to prove that $\inf_{n \geq N'}\{a_{n+1}/a_n^2\} > 1$. Now, for $n \geq N'$ and $r = 0, 1, \dots, k$, one clearly must have $f(n) + r > N(\varepsilon)$ and so, from the assumption on $f(\cdot)$, observe that

$$\frac{V_{f(n+1)+r}}{V_{f(n)+r}^2} \geq \frac{V_{2f(n)+2k+r+1}}{V_{f(n)+r}^2} \geq \frac{V_{2f(n)+2r+1}}{V_{f(n)+r}^2} > \alpha - \varepsilon.$$

Consequently, $\inf_{n \geq N'}\{a_{n+1}/a_n^2\} > (\alpha - \varepsilon)^{k+1} > 1$. □

Turning now to the case of $f(n)=n$, it is readily apparent that one cannot apply Lemma 2.1 to prove irrationality as $a_{n+1}/a_n^2=\prod_{r=0}^{k}(U_{n+1+r}/U_{n+r}^2)\to 0$ as $n\to\infty$ and so the infimum over the natural numbers of the associated sequence must be equal to zero. In spite of this, one can still reach the desired conclusion for the series in question by an application of two existing results within the literature. The first of these, which is due to André-Jeannin (see [1]), asserted that the series $\sum_{n=1}^{\infty}1/U_n$ sums to an irrational number when $\{U_n\}$ is generated with respect to the ordered pair (P,Q), where $|Q|=1$ and $P>2$. By then combining this with the well-known reduction formula of Carlitz for Fibonacci summations, we can write the sum of $\sum_{n=1}^{\infty}1/a_n$ (for any fixed $k\in N$) in terms of a linear expression in θ over the rationals, where θ is an irrational number to be determined. Thus, consider now the following Lemma which forms the basis of the reduction formula that shall be used directly.

Lemma 2.2: Suppose that the sequences $\{U_n\}$ and $\{V_n\}$ are generated with respect to the ordered pair $(P,1)$ with $P\neq 1,2$ and let α and β be the roots of $x^2-Px+1=0$. If we denote $\left\{{m \atop r}\right\}=(U)_m/(U)_r(U)_{m-r}$, where $(U)_m=U_1\ldots U_m$ and $(U)_0=1$, then

$$\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}x^j=\prod_{j=1}^{m}[1-V_{2j-1}x+x^2], \tag{2}$$

$$\sum_{j=0}^{2m+1}(-1)^j\left\{{2m+1 \atop j}\right\}\alpha^{j+1}x^j=(\alpha-\alpha^{m+2}\beta^{-m}x)\prod_{j=1}^{m}[1-V_{2j-1}x+x^2], \tag{3}$$

$$\frac{1}{U_nU_{n+1}\cdots U_{n+2m}}=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}\frac{1}{U_{n+j}}. \tag{4}$$

The above restriction on the value of P is required in order that the Binet formula for U_n is not indeterminant. For a proof of the above identities, interested readers should consult [4].

Theorem 2.4: Suppose the sequence $\{U_n\}$ is generated with respect to the ordered pair (P,Q), where $P>2$ and $Q=1$, then the series $\sum_{n=1}^{\infty}1/a_n$ sums to an irrational number where $a_n=U_n\ldots U_{n+k}$ and $k\in N$.

Proof: For the prescribed values of P and Q, all series under consideration are clearly convergent. Addressing the case in which k is even, observe from Lemma 2.2 when $x=1$ that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{a_n}&=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{\infty}\frac{1}{U_{n+j}}\\&=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{\infty}\frac{1}{U_n}-\frac{1}{(U)_{2m}}\sum_{j=1}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{j}\frac{1}{U_n}\\&=\frac{1}{(U)_{2m}}\sum_{j=1}^{m}[2-V_{2j-1}]\sum_{n=1}^{\infty}\frac{1}{U_n}-\frac{1}{(U)_{2m}}\sum_{j=1}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{j}\frac{1}{U_n}.\end{aligned}$$

Consequently, the sum of the series in question is of the form $a\theta+b$, where $a,b\in Q$ and θ is an irrational number. However, as $\{V_n\}$ is a monotone increasing sequence and $V_1=P>2$, one must have $a\neq 0$. Hence, the result is established for k even. Suppose now that $k=2m+1$, then as in [4] we multiply (4) by $1/U_{n+2m+1}$ and upon summing we find

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{a_n}&=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\begin{Bmatrix}2m\\ j\end{Bmatrix}\sum_{n=1}^{\infty}\frac{1}{U_{n+j}U_{n+2m+1}}\\ &=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\begin{Bmatrix}2m\\ j\end{Bmatrix}\sum_{n=1}^{j}\frac{1}{U_nU_{n+2m-j+1}}-\frac{1}{(U)_{2m}}\sum_{j=1}^{2m}(-1)^j\begin{Bmatrix}2m\\ j\end{Bmatrix}\sum_{n=1}^{\infty}\frac{1}{U_nU_{n+2m-j+1}}.\end{aligned}\tag{5}$$

Now, dividing both sides of the standard identity $U_{n+r}U_{n-1}-U_nU_{n+r-1}=-(\alpha\beta)^{n-1}U_r$ by the term U_nU_{n+r}, with $r=2m-j+1$, and summing to N terms, where $N>r$, observe that

$$\begin{aligned}U_{2m-j+1}\sum_{n=1}^{N}\frac{1}{U_nU_{n+2m-j+1}}&=\sum_{n=1}^{N}\frac{U_{n+2m-j}}{U_{n+2m-j+1}}-\sum_{n=1}^{N}\frac{U_{n-1}}{U_n}\\ &=\sum_{n=1}^{2m-j+1}\frac{U_{N+n-1}}{U_{N+n}}-\sum_{n=1}^{2m-j+1}\frac{U_{n-1}}{U_n}.\end{aligned}\tag{6}$$

By assumption, $|\alpha|>|\beta|$, and so $U_{N+n-1}/U_{N+n}\to 1/\alpha$ as $N\to\infty$. Thus, combining the limiting value of (6) with (5), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{a_n}&=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\begin{Bmatrix}2m\\ j\end{Bmatrix}\frac{1}{U_{2m-j+1}}\left(\frac{2m-j+1}{\alpha}-\sum_{n=1}^{2m-j+1}\frac{U_{n-1}}{U_n}\right)\\ &\quad-\frac{1}{(U)_{2m}}\sum_{j=1}^{2m}(-1)^j\begin{Bmatrix}2m\\ j\end{Bmatrix}\sum_{n=1}^{j}\frac{1}{U_nU_{n+2m-j+1}}\\ &=\frac{1}{\alpha}a'+b',\end{aligned}$$

where $a',b'\in Q$ and α^{-1} is an algebraic irrational. It remains only to show that the constant $a'\neq 0$. From the definition of the generalized binomial coefficient in Lemma 2.2, we see that

$$\begin{aligned}\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\begin{Bmatrix}2m\\ j\end{Bmatrix}\frac{2m-j+1}{U_{2m-j+1}}&=\frac{1}{(U)_{2m+1}}\sum_{j=0}^{2m}(-1)^j\begin{Bmatrix}2m\\ j\end{Bmatrix}\frac{U_{2m+1}}{U_{2m-j+1}}(2m-j+1)\\ &=\frac{1}{(U)_{2m+1}}\sum_{j=0}^{2m+1}(-1)^j\begin{Bmatrix}2m+1\\ j\end{Bmatrix}(2m-j+1).\end{aligned}$$

Thus, if we denote the polynomial function in (3) by $R(x)$, then

$$a'=\frac{2m+1}{(U)_{2m+1}}\alpha^{-1}R(\alpha^{-1})-\frac{\alpha^{-2}}{(U)_{2m+1}}\frac{dR}{dx}(\alpha^{-1}).$$

However, by Lemma 2.2, $R(x)$ contains the quadratic factor $1-V_1x+x^2=(1-\beta x)(1-\alpha x)$ and so $R(\alpha^{-1})=0$. Moreover,

$$\begin{aligned}\frac{dR}{dx}(\alpha^{-1})&=(1-V_1\alpha^{-1}+\alpha^{-2})\frac{d}{dx}\left\{(\alpha-\alpha^{m+2}\beta^{-m}x)\prod_{j=2}^{m}[1-V_{2j-1}x+x^2]\right\}(\alpha^{-1})\\ &\quad+(2\alpha^{-1}-V_1)(\alpha-\alpha^{m+1}\beta^{-m})\prod_{j=2}^{m}(1-V_{2j-1}\alpha^{-1}+\alpha^{-2})\\ &=(2\alpha^{-1}-V_1)(\alpha-\alpha^{m+1}\beta^{-m})\prod_{j=2}^{m}(1-V_{2j-1}\alpha^{-1}+\alpha^{-2}).\end{aligned}\tag{7}$$

Now, since $\alpha \neq \beta$, it is immediate that the first two factors in (7) must be nonzero, while as the quadratic factor $1-V_{2j-1}x+x^2$ for $j=2,\dots,m$ has the roots α^{j-1}/β^j, β^{j-1}/α^j of which neither is equal to α^{-1}, we can finally conclude that $R'(\alpha^{-1}) \neq 0$ and so $a' \neq 0$. □

Remark 2.1: The inequality $P>2$ in Theorem 2.4 cannot be weakened as the series in question will sum to a rational value when $(P,Q)=(2,1)$. To demonstrate this, we first consider as $U_n=n$ in the present case, the function $f(x)=(x^{n+k}(n-1)!)/(n+k)!$. Applying Taylor's theorem to $f(x)$ about the point $a=0$, observe that, for $x \geq 0$,

$$f(x)=\sum_{m=0}^{k}\frac{x^m f^{(m)}(0)}{m!}+\int_0^x \frac{(x-t)^k}{k!}t^{n-1}\,dt. \tag{8}$$

Now, as $f^m(0)=0$ for $m=0,\dots,k$, it is clear after setting $x=1$ in (8) and applying Lebesgue's Dominated Convergence Theorem that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(n+1)\cdots(n+k)}&=\sum_{n=1}^{\infty}\int_0^1\frac{(1-t)^k}{k!}t^{n-1}\,dt\\&=\int_0^1\sum_{n=1}^{\infty}\frac{(1-t)^k}{k!}t^{n-1}\,dt=\int_0^1\frac{(1-t)^{k-1}}{k!}dt=\frac{1}{kk!}.\end{aligned}$$

It still remains an open problem as to whether the series having terms of the above form continue to exhibit irrational sums when $f(n)$ is replaced by an arbitrary strictly monotone increasing integer valued function, such as a polynomial in n over the positive integers. Such a problem may be impossible to resolve, as it is difficult in general to predict the nature of a series sum. To illustrate this difficulty, we shall show that it is possible to construct a pair of infinite series having positive rational terms asymptotic to each other, with one summing to a rational number and the other to an irrational number. Consider $\sum_{n=1}^{\infty}1/a_n$, where a_n is generated from the recurrence relation $a_{n+1}=a_n^2-a_n+1$ with $a_1=2$. Then, as $1/(a_{n+1}-1)=1/(a_n-1)-1/a_n$, one deduces that $\sum_{n=1}^{N}1/a_n=1-(a_{N+1}-1)^{-1}$ and so the series converges to 1. However, if we define $b_n=a_n-1/n$, then clearly $1/b_n \sim 1/a_n$ as $n\to\infty$ with

$$\begin{aligned}b_{n+1}&=a_n^2-a_n+1-\frac{1}{n+1}\\&=(b_n^2-b_n+1)+\frac{2b_n}{n}+\frac{1}{n^2}-\frac{1}{n}-\frac{1}{n+1}>b_n^2-b_n+1,\end{aligned}$$

where the inequality follows from the fact that $2b_n/n+1/n^2-1/n-1/(n+1)>0$, which is easily deduced via the simple inequalities $2b_n>2$ and $2>1+n/(n+1)-1/n$. Thus, via Theorem 2.1, $\sum_{n=1}^{\infty}1/b_n$ will sum to an irrational number.

3. SERIES WITH TERMS $(U_kU_{k+1}\dots U_{k+f(n)})^{-1}$

In this section we shall again apply the criterion mentioned in the Introduction to establish irrational sums for the family of series considered in [7] but now involving the larger class of sequences $\{U_n\}$, $\{V_n\}$ generated with respect to the relatively prime pair (P,Q), where $Q\neq 0$ and $P>1$. As in the previous section, it will be convenient to first demonstrate irrationality for a general family of series having terms of the form $(x_kx_{k+1}\dots x_{k+f(n)})^{-1}$, where $\{x_n\}$ is an arbitrary strictly increasing sequence of positive integers. Although a similar result was established in [5] via an indirect argument, the version proved here is far stronger in comparison because we do not need to impose the restrictive divisibility assumption that for any $m \in N \setminus \{0\}$ there exists an n such that $m | x_1 x_2 \ldots x_n$. However, it should be noted that this condition, which was also used in [7], was the source for the restriction on the parameter Q that was needed to argue in a similar manner as in [5].

Theorem 3.1: Let $\{x_n\}$ be a strictly increasing sequence of positive integers and $g: N \to N \setminus \{0\}$ a strictly monotone increasing function. If, in addition, $\{b_n\}$ is a bounded sequence of nonzero integers, then $\sum_{n=1}^{\infty} b_n / a_n$ converges to an irrational number, where $a_n = x_1 x_2 \ldots x_{g(n)}$.

Proof: From the assumption, it is immediate that the series in question are absolutely convergent. Denoting the sum of the series by θ, we again consider the sequence of rational approximations p_n / q_n to θ generated from the n^{th} partial sums expressed in reduced form. As p_n / q_n are clearly nonconstant, the result will follow upon showing that $|q_n \theta - p_n| = o(1)$ as $n \to \infty$. Since $(p_n, q_n) = 1$, the lowest common denominator of the m fractions $\{b_n / a_n\}_{n=1}^{m}$ must be greater than or equal to q_m, but as $x_1 x_2 \ldots x_{g(m)}$ is one common denominator, we deduce that $q_m \le x_1 x_2 \ldots x_{g(m)}$. Thus, if $|b_n| \le M$ for all n, then

$$|q_m \theta - p_m| = q_m \left| \sum_{r=1}^{\infty} \frac{b_{m+r}}{a_{m+r}} \right| \le M \sum_{r=1}^{\infty} \frac{a_m}{a_{m+r}} = M \sum_{r=1}^{\infty} \frac{1}{a'_r}, \tag{9}$$

where $a'_r = x_{g(m)+1} \ldots x_{g(m+r)}$. Now, by the strict monotonicity of x_n, all $g(m+r) - g(m)$ terms in the definition of a'_r are greater than or equal to $x_{g(m)+1}$. Consequently, as $g(m+r) - g(m) \ge r$, one deduces $a'_r \ge x^r_{g(m)+1}$, and so

$$\sum_{r=1}^{\infty} \frac{1}{a'_r} \le \sum_{r=1}^{\infty} x^{-r}_{g(m)+1} = \frac{1}{x_{g(m)+1} - 1}. \tag{10}$$

Thus, by combining (9) with (10) together with the monotonicity of x_n and $g(\cdot)$, it is readily apparent that $|q_m \theta - p_m| \to 0$ as $m \to \infty$. □

Corollary 3.1: Suppose $\{U_n\}$ and $\{V_n\}$ are generated with respect to the relatively prime pair (P, Q), with $Q \ne 0$ and $P > |Q+1|$. If, in addition, $f: N \to N \setminus \{0\}$ is a strictly monotone increasing function and $\{b_n\}$ is a bounded sequence of nonzero integers, then $\sum_{n=1}^{\infty} b_n / a_n$ converges to an irrational number, where $a_n = U_k \ldots U_{k+f(n)}$ or $a_n = V_k \ldots V_{k+f(n)}$.

Proof: In Theorem 3.1, substitute x_n for either U_n or V_n, which are strictly monotone increasing sequences of positive integers. If $g(n) = f(n) + k$, then $\sum_{n=1}^{\infty} b_n / a_n$ sums to an irrational number. In the case in which $k > 1$, the result will follow upon multiplying the series by the product $x_1 \ldots x_{k-1}$. □

To conclude, we shall prove, as in [5], a companion result to Theorem 3.1 in which a class of irrational valued alternating series were constructed. Again one can dispense with the divisibility condition that was required in [5]; however, in its place we have imposed an order condition.

Theorem 3.2: Let $\{x_n\}$ and $\{y_n\}$ be two strictly increasing sequences of positive integers such that $y_n = o(x_n)$ as $n \to \infty$ and for all n sufficiently large,

$$\frac{y_{n+1}}{x_{n+1}} < y_n < x_n. \tag{11}$$

If, in addition, $g: N \to N \setminus \{0\}$ is a strictly monotone increasing function, then $\sum_{n=1}^{\infty}(-1)^n y_n / a_n$ converges to an irrational number where $a_n = x_1 x_2 \ldots x_{g(n)}$.

Proof: Using (11) and the fact that $g(n) \geq n$, observe that, for n sufficiently large,

$$\frac{y_{n+1}}{a_{n+1}}\frac{a_n}{y_n} = \frac{y_{n+1}}{y_n}\frac{1}{x_{g(n)+1}\cdots x_{g(n+1)}} \leq \frac{y_{n+1}}{y_n x_{g(n+1)}} \leq \frac{y_{n+1}}{y_n x_{n+1}} < 1$$

and

$$0 \leq \frac{y_n}{a_n} = (x_1 \ldots x_{g(n)-1})^{-1}\frac{y_n}{x_{g(n)}} \leq (x_1 \ldots x_{g(n)-1})^{-1}\frac{y_n}{x_n} < (x_1 \ldots x_{g(n)-1})^{-1} \to 0$$

as $n \to \infty$. So, by Leibniz' criterion, the alternating series converges. Denoting the sum of the series by θ, we again consider the sequence of rational approximations p_n / q_n to θ generated from the n^{th} partial sums expressed in reduced form. As p_n / q_n are clearly nonconstant, the result will follow upon showing that $|q_n\theta - p_n| = o(1)$ as $n \to \infty$. Since $(p_n, q_n) = 1$, the lowest common denominators of the m fractions $\{(-1)^n y_n / a_n\}_{n=1}^{m}$ must be greater than or equal to q_m, but since $x_1 x_2 \ldots x_{g(m)}$ is one common denominator, we deduce that $q_m \leq a_m$. Now

$$|q_m\theta - p_m| = q_m\left|\sum_{n=m+1}^{\infty}(-1)^n\frac{y_n}{a_n}\right| = q_m\left|\sum_{r=1}^{\infty}(-1)^{r+1}\frac{y_{m+r}}{a_{m+r}}\right| \leq a_m\left|\sum_{r=1}^{\infty}(-1)^{r+1}\frac{y_{m+r}}{a_{m+r}}\right|; \tag{12}$$

furthermore, by standard bounds from the theory of alternating series, we also have that

$$0 < \frac{y_{m+1}}{a_{m+1}} - \frac{y_{m+2}}{a_{m+2}} < \sum_{r=1}^{\infty}(-1)^{r+1}\frac{y_{m+r}}{a_{m+r}} < \frac{y_{m+1}}{a_{m+1}}.$$

Thus, we can obtain, from (12), the upper bound

$$|q_m\theta - p_m| \leq a_m\frac{y_{m+1}}{a_{m+1}} = \frac{y_{m+1}}{x_{g(m)+1}\cdots x_{g(m+1)}} \leq \frac{y_{m+1}}{x_{g(m+1)}} \leq \frac{y_{m+1}}{x_{m+1}}.$$

Hence, the result is established since, by assumption, $y_{m+1} / x_{m+1} \to 0$ as $m \to \infty$. □

As an application, we can now construct the following class of irrational valued series involving generalized Fibonacci numbers and the Euler totient function.

Corollary 3.2: Suppose $\{U_n\}$ is generated with respect to the relatively prime pair (P, Q) with $Q < 0$ and $P > 0$, then

$$\sum_{n=1}^{\infty}(-1)^n\frac{\varphi(n)}{U_1 U_2 \ldots U_n},$$

where $\varphi(n)$ is Euler's totient function, will converge to an irrational number.

Proof: In Theorem 3.2, substitute x_n for U_n, which is a strictly monotone increasing sequence of positive integers. If, in addition, we set $g(n) = n$ and $y_n = \varphi(n)$, then the irrationality of the series sum will follow if the inequality in (11) holds for n large and $\varphi(n) = o(U_n)$. To this end, we first note that, for the prescribed (P, Q) values, one must have $n = o(U_n)$, and since $\varphi(n) \leq n$ for all n, we deduce that $0 < \varphi(n)/U_n < n/U_n \to 0$ as $n \to \infty$. Consequently, for n sufficiently large,

$$\frac{\varphi(n+1)}{U_{n+1}} < 1 \leq \varphi(n) \leq n < U_n,$$

as required. □

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THE BINET FORMULA AND REPRESENTATIONS OF k-GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

We consider a generalization of Fibonacci sequence, which is called the k-generalized Fibonacci sequence for a positive integer $k \geq 2$. The *k-generalized Fibonacci sequence* $\{g_n^{(k)}\}$ is defined as

$$g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0, \; g_{k-1}^{(k)} = g_k^{(k)} = 1$$

and, for $n > k \geq 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}.$$

We call $g_n^{(k)}$ the n^{th} *k-generalized Fibonacci number*. For example, if $k = 2$, then $\{g_n^{(2)}\}$ is a Fibonacci sequence and, if $k = 5$, then $g_1^{(5)} = g_2^{(5)} = g_3^{(5)} = 0$, $g_4^{(5)} = g_5^{(5)} = 1$, and then the 5-generalized Fibonacci sequence is

$$0, 0, 0, 1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, \ldots.$$

Let I_{k-1} be the identity matrix of order $k-1$ and let E be a $1 \times (k-1)$ matrix whose entries are 1's. For any $k \geq 2$, the fundamental recurrence relation $g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}$ can be defined by the vector recurrence relation

$$\begin{bmatrix} g_{n+1}^{(k)} \\ g_{n+2}^{(k)} \\ \vdots \\ g_{n+k}^{(k)} \end{bmatrix} = Q_k \begin{bmatrix} g_{n}^{(k)} \\ g_{n+1}^{(k)} \\ \vdots \\ g_{n+k-1}^{(k)} \end{bmatrix},$$

where

$$Q_k = \begin{bmatrix} 0 & I_{k-1} \\ 1 & E \end{bmatrix}_{k \times k}. \tag{1}$$

The matrix Q_k is said to be a *k-generalized Fibonacci matrix*. In [4] and [5], we gave the relationships between the k-generalized Fibonacci sequences and their associated matrices.

In 1843, Binet found a formula giving F_n in terms of n. It is a very complicated-looking expression, and the formula is

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where α and β are eigenvalues of Q_2. In [6], Levesque gave a Binet formula for the Fibonacci sequence by using a generating function.

In this paper, we derive a generalized Binet formula for the k-generalized Fibonacci sequence by using the determinant and we give several combinatorial representations of k-generalized Fibonacci numbers.

2. GENERALIZED BINET FORMULA

Let $\{g_n^{(k)}\}$ be a k-generalized Fibonacci sequence. Throughout the paper we will use $g_n = g_{n+k-2}^{(k)}$, $n = 1, 2, \ldots$, and $G_k = (g_1, g_2, g_3, \ldots)$ for notational convenience.

For example, if $k = 2$, $G_2 = (1, 1, 2, 3, \ldots)$, and if $k \geq 3$, $G_k = (1, 1, 2, 4, \ldots)$. For G_k, $k \geq 2$, since $g_1 = g_2 = 1$, we can replace the matrix Q_k in (1) with

$$Q_k = \begin{bmatrix} 0 & g_1 & 0 & \cdots & 0 \\ 0 & 0 & g_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & g_1 \\ g_1 & g_1 & \cdots & g_1 & g_2 \end{bmatrix}.$$

Then we can find the following matrix in [3]:

$$Q_k^n = \begin{bmatrix} g_{n-(k-1)} & g_{1,2}^{\dagger} & g_{1,3}^{\dagger} & \cdots & g_{1,k-1}^{\dagger} & g_{n-(k-2)} \\ g_{n-(k-2)} & g_{2,2}^{\dagger} & g_{2,3}^{\dagger} & \cdots & g_{2,k-1}^{\dagger} & g_{n-(k-3)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ g_{n-1} & g_{k-1,2}^{\dagger} & g_{k-1,3}^{\dagger} & \cdots & g_{k-1,k-1}^{\dagger} & g_n \\ g_n & g_{k,2}^{\dagger} & g_{k,3}^{\dagger} & \cdots & g_{k,k-1}^{\dagger} & g_{n+1} \end{bmatrix}, \tag{2}$$

where

$$\begin{aligned} g_{i,2}^{\dagger} &= g_{n-(k-i)} + g_{n-(k-(i-1))} \\ g_{i,3}^{\dagger} &= g_{n-(k-i)} + g_{n-(k-(i-1))} + g_{n-(k-(i-2))} \\ &\vdots \\ g_{i,k-1}^{\dagger} &= g_{n-(k-i)} + g_{n-(k-(i-1))} + g_{n-(k-(i-2))} + \cdots + g_{n-(k-(i-(k-2)))}, \end{aligned}$$

$i = 1, 2, \ldots, k$. Since $Q_k^n Q_k^m = Q_k^{n+m}$, $g_{n+m} = (Q_k^{n+m})_{k,1}$; hence, we have the following theorem.

Theorem 2.1 (see [3]): Let $G_k = (g_1, g_2, \ldots)$. Then, for any positive integers n and m,

$$\begin{aligned} g_{n+m} = {} & g_n g_{m-(k-1)} + (g_n + g_{n-1}) g_{m-(k-2)} \\ & + (g_n + g_{n-1} + g_{n-2}) g_{m-(k-3)} + \cdots \\ & + (g_n + g_{n-1} + g_{n-2} + \cdots + g_{n-(k-2)}) g_{m-1} + g_{n+1} g_m. \end{aligned}$$

Note that $g_{n+m} = (Q_k^{n+m})_{k,1} = (Q_k^{n+m})_{k-1,k}$. Then we have the following corollary.

Corollary 2.2: Let $G_k = (g_1, g_2, \ldots)$. Then, for any positive integers n and m,

$$\begin{aligned} g_{n+m} = {} & g_{n-1} g_{m-(k-2)} + (g_{n-1} + g_{n-2}) g_{m-(k-3)} \\ & + (g_{n-1} + g_{n-2} + g_{n-3}) g_{m-(k-4)} + \cdots \\ & + (g_{n-1} + g_{n-2} + g_{n-3} + \cdots + g_{n-(k-1)}) g_m + g_n g_{m+1}. \end{aligned}$$

Now we are going to find the generalized Binet formula for the k-generalized Fibonacci sequence.

Lemma 2.3: Let $b_k = \frac{2^{k+1}}{k+1}\left(\frac{k}{k+1}\right)^k$. Then $b_k < b_{k+1}$ for $k \ge 2$.

Proof: Since $\frac{k+1}{k+2} > \frac{k}{k+1}$ and $k \ge 2$,

$$\left(\frac{k+1}{k+2}\right)^{k+1} > \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \frac{2^{k+2}}{k+2} \ge \frac{2^{k+1}}{k}.$$

Therefore,

$$b_{k+1} = \frac{2^{k+2}}{k+2}\left(\frac{k+1}{k+2}\right)^{k+1} > \frac{2^{k+1}}{k+1}\left(\frac{k}{k+1}\right)^k = b_k$$

for each positive integer k. □

Lemma 2.4: The equation $z^{k+1} - 2z^k + 1 = 0$ does not have multiple roots for $k \ge 2$.

Proof: Let $f(z) = z^k - z^{k-1} - \cdots - z - 1$ and let $g(z) = (z-1)f(z)$. Then $g(z) = z^{k+1} - 2z^k + 1$. So 1 is a root but not a multiple root of $g(z) = 0$, since $k \ge 2$ and $f(1) \ne 0$. Suppose that α is a multiple root of $g(z) = 0$. Note that $\alpha \ne 0$ and $\alpha \ne 1$. Since α is a multiple root, $g(z) = \alpha^{k+1} - 2\alpha^k + 1 = 0$ and $g'(\alpha) = (k+1)\alpha^k - 2k\alpha^{k-1} = \alpha^{k-1}((k+1)\alpha - 2k) = 0$. Thus, $\alpha = \frac{2k}{k+1}$, and hence

$$\begin{aligned}
0 &= -\alpha^{k+1} + 2\alpha^k - 1 = \alpha^k(2-\alpha) - 1 \\
&= \left(\frac{2k}{k+1}\right)^k\left(2 - \frac{2k}{k+1}\right) - 1 = \left(\frac{2k}{k+1}\right)^k\left(\frac{2k+2-2k}{k+1}\right) - 1 \\
&= \frac{2^{k+1}}{k+1}\left(\frac{k}{k+1}\right)^k - 1 = b_k - 1.
\end{aligned}$$

Since, by Lemma 2.3, $b_2 = (\frac{2}{3})^3 \times 2^2 = \frac{2^5}{3^3} > 1$ and $b_k < b_{k+1}$ for $k \ge 2$, $b_k \ne 1$, a contradiction.

Therefore, the equation $g(z) = 0$ does not have multiple roots. □

Let $f(\lambda)$ be the characteristic polynomial of the k-generalized Fibonacci matrix Q_k. Then $f(\lambda) = \lambda^k - \lambda^{k-1} - \cdots - \lambda - 1$, which is a well-known fact. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of Q_k. Then, by Lemma 2.4, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct. Let Λ be a $k \times k$ Vandermonde matrix as follows:

$$\Lambda = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{bmatrix}.$$

Set $V = \Lambda^T$. Let

$$\mathbf{d}_i = \begin{bmatrix} \lambda_1^{n+i-1} \\ \lambda_2^{n+i-1} \\ \vdots \\ \lambda_k^{n+i-1} \end{bmatrix}$$

and let $V_j^{(i)}$ be a $k \times k$ matrix obtained from V by replacing the j^{th} column of V by $\mathbf{d}_i$. Then we have the generalized Binet formula as the following theorem.

Theorem 2.5: Let $\{g_n^{(k)}\}$ be a k-generalized Fibonacci sequence. Then

$$g_n = \frac{\det(V_1^{(k)})}{\det(V)}, \tag{3}$$

where $g_n = g_{n+k-2}^{(k)}$.

Proof: Since the eigenvalues of Q_k are distinct, Q_k is diagonalizable. It is easy to show that $Q_k\Lambda = \Lambda D$, where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Since Λ is invertible, $\Lambda^{-1}Q_k\Lambda = D$. Thus, Q_k is similar to D. So we have $Q_k^n\Lambda = \Lambda D^n$. Let $Q_k^n = [q_{ij}]_{k\times k}$. Then we have the following linear system of equations:

$$\begin{aligned} q_{i1} + q_{i2}\lambda_1 + \cdots + q_{ik}\lambda_1^{k-1} &= \lambda_1^{n+i-1} \\ q_{i1} + q_{i2}\lambda_2 + \cdots + q_{ik}\lambda_2^{k-1} &= \lambda_2^{n+i-1} \\ \vdots \qquad\qquad & \quad \vdots \\ q_{i1} + q_{i2}\lambda_k + \cdots + q_{ik}\lambda_k^{k-1} &= \lambda_k^{n+i-1}. \end{aligned}$$

And, for each $j = 1, 2, \ldots, k$, we get

$$q_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}.$$

Therefore, by (2), we have the explicit form

$$q_{k1} = g_n = \frac{\det(V_1^{(k)})}{\det(V)}. \quad \square$$

We note that, if $k = 2$, then (3) is the Binet formula for the Fibonacci sequence.

3. COMBINATORIAL REPRESENTATIONS OF k-GENERALIZED FIBONACCI NUMBERS

In this section, we consider some combinatorial representations of $g_n = g_{n+k-2}^{(k)}$ for $k \geq 2$. Let S_k be a $k \times k$ (0, 1)-matrix as follows:

$$S_k = \begin{bmatrix} E & 1 \\ I_{k-1} & 0 \end{bmatrix}.$$

Then, by (2),

$$S_k^n = [s_{ij}] = \begin{bmatrix} g_{n+1} & g_{k,k-1}^{\dagger} & \cdots & g_{k,3}^{\dagger} & g_{k,2}^{\dagger} & g_n \\ g_n & g_{k-1,k-1}^{\dagger} & \cdots & g_{k-1,3}^{\dagger} & g_{k-1,2}^{\dagger} & g_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n-(k-3)} & g_{2,k-1}^{\dagger} & \cdots & g_{2,3}^{\dagger} & g_{2,2}^{\dagger} & g_{n-(k-2)} \\ g_{n-(k-2)} & g_{1,k-1}^{\dagger} & \cdots & g_{1,3}^{\dagger} & g_{1,2}^{\dagger} & g_{n-(k-1)} \end{bmatrix}. \tag{4}$$

In [1], we can find the following lemma.

Lemma 3.1 (see [1]):

$$s_{ij} = \sum_{(m_1,\ldots,m_k)} \frac{m_j + m_{j+1} + \cdots + m_k}{m_1 + \cdots + m_k} \times \binom{m_1 + \cdots + m_k}{m_1, \ldots, m_k},$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + km_k = n - i + j$ and defined to be 1 if $n = i - j$.

Corollary 3.2: Let $\{g_n^{(k)}\}$ be the k-generalized Fibonacci sequence. Then

$$g_n = \sum_{(m_1, \ldots, m_k)} \frac{m_k}{m_1 + \cdots + m_k} \times \binom{m_1 + \cdots + m_k}{m_1, \ldots, m_k},$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + km_k = n - 1 + k$.

Proof: From Lemma 3.1, if $i = 1$ and $j = k$, then the conclusion can be derived directly from (4). □

Let $A = [a_{ij}]$ be an $n \times n$ (0, 1)-matrix. The *permanent* of A is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where σ runs over all permutations of the set $\{1, 2, \ldots, n\}$. A matrix A is called *convertible* if there is an $n \times n$ $(1, -1)$-matrix H such that $\operatorname{per} A = \det(A \circ H)$, where $A \circ H$ denotes the Hadamard product of A and H. Such a matrix H is called a *converter* of A.

Let $\mathcal{F}^{(n,k)} = [f_{ij}] = T_n + B_n$, where $T_n = [t_{ij}]$ is the $n \times n$ $(0,1)$-matrix defined by $t_{ij} = 1$ if and only if $|i - j| \le 1$, and $B_n = [b_{ij}]$ is the $n \times n$ $(0, 1)$-matrix defined by $b_{ij} = 1$ if and only if $2 \le j - i \le k - 1$. In [4] and [5], the following theorem gave a representation of $g_n^{(k)}$.

Theorem 3.3 (see [4], [5]): Let $\{g_n^{(k)}\}$ be the k-generalized Fibonacci sequence. Then

$$g_n = \operatorname{per} \mathcal{F}^{(n-1,k)},$$

where $g_n = g_{n+k-2}^{(k)}$.

Let H be a $(1, -1)$-matrix of order $n - 1$, defined by

$$H = \begin{bmatrix} 1 & -1 & 1 & \cdots & 1 \\ 1 & 1 & -1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then the following theorem holds.

Theorem 3.4: Let $\{g_n^{(k)}\}$ be the k-generalized Fibonacci sequence. Then

$$g_n = \det(\mathcal{F}^{(n-1,k)} \circ H),$$

where $g_n = g_{n+k-2}^{(k)}$.

Proof: Since the matrix $\mathcal{F}^{(n-1,k)}$ is a convertible matrix with converter H, we have

$$\operatorname{per} \mathcal{F}^{(n-1,k)} = \det(\mathcal{F}^{(n-1,k)} \circ H)$$

and, by Theorem 3.3, the proof is complete. □

Now we consider the generating function of the k-generalized Fibonacci sequence. We can easily find the characteristic polynomial, $x^k - x^{k-1} - \cdots - x - 1$, of the k-Fibonacci matrix Q_k. It follows that all of the eigenvalues of Q_k satisfy $x^k = x^{k-1} + x^{k-2} + \cdots + x + 1$. And we can find the following fact in [5]:

$$\begin{aligned} x^n = {} & g_{n-k+2}x^{k-1} + (g_{n-k+1} + g_{n-k} + \cdots + g_{n-2k+3})x^{k-2} \\ & + (g_{n-k+1} + g_{n-k} + \cdots + g_{n-2k+4})x^{k-3} \\ & + \cdots + (g_{n-k+1} + g_{n-k})x + g_{n-k+1}. \end{aligned} \tag{5}$$

Let $G_k(x) = g_1 + g_2x + g_3x^2 + \cdots + g_{n+1}x^n + \cdots$. Then

$$G_k(x) - xG_k(x) - x^2G_k(x) - \cdots - x^kG_k(x) = (1 - x - x^2 - \cdots - x^k)G_k(x).$$

Using equation (5), we have $(1 - x - x^2 - \cdots - x^k)G_k(x) = g_1 = 1$. Thus,

$$G_k(x) = (1 - x - x^2 - \cdots - x^k)^{-1}$$

for $0 \le x + x^2 + \cdots + x^k < 1$.

Let $f_k(x) = x + x^2 + \cdots + x^k$. Then $0 \le f_k(x) < 1$ and we have the following lemma.

Lemma 3.5: For positive integers p and n, the coefficient of x^n in $(f_k(x))^p$ is

$$\sum_{l=0}^{p}(-1)^l\binom{p}{l}\binom{n-kl-1}{n-kl-p}, \ \frac{n}{k} \le p \le n.$$

Proof:

$$\begin{aligned} (f_k(x))^p &= (x + x^2 + \cdots + x^k)^p = x^p(1 + x + x^2 + \cdots + x^{k-1})^p \\ &= x^p\left(\frac{1-x^k}{1-x}\right)^p = x^p\left((1-x^k)\left(\frac{1}{1-x}\right)\right)^p \\ &= x^p\left(\left(\sum_{l=0}^{p}\binom{p}{l}(-1)^l x^{kl}\right)\left(\sum_{i=0}^{\infty}\binom{p+i-1}{i}x^i\right)\right). \end{aligned}$$

In the above equation, we consider the coefficient of x^n. Since the first term on the right is x^p, we have $kl + i = n - p$, that is, $i = n - kl - p$. If $l = q$ for any $q = 0, 1, \ldots, p$, then the second term on the right is

$$\left((-1)^q\binom{p}{q}\binom{n-kq-1}{n-kq-p}\right)x^{n-p}.$$

So the coefficient of x^n is

$$\sum_{l=0}^{p}(-1)^l\binom{p}{l}\binom{n-kl-1}{n-kl-p}, \ \frac{n}{k} \le p \le n. \ \square$$

Theorem 3.6: For positive integers p and n,

$$g_{n+1} = \sum_{\frac{n}{k} \le p \le n}\ \sum_{l=0}^{p}(-1)^l\binom{p}{l}\binom{n-kl-1}{n-kl-p}. \tag{6}$$

Proof: Since

$$G_k(x) = g_1 + g_2x + g_3x^2 + \cdots + g_{n+1}x^n + \cdots = \frac{1}{1 - x - x^2 - \cdots - x^k},$$

the coefficient of x^n is the $(n+k-1)^{\text{st}}$ Fibonacci number, that is, g_{n+1} in G_k. And

$$\begin{aligned} G_k(x) &= \frac{1}{1-x-x^2-\cdots-x^k} = \frac{1}{1-f_k(x)} \\ &= 1+f_k(x)+(f_k(x))^2+\cdots+(f_k(x))^n+\cdots \\ &= 1+f_k(x)+x^2\sum_{l=0}^{n}\binom{2}{l}(-1)^l x^{kl}\sum_{i=0}^{\infty}\binom{i+1}{i}x^i \\ &\quad +\cdots+x^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l x^{kl}\sum_{i=0}^{\infty}\binom{n+i-1}{i}x^i+\cdots. \end{aligned} \tag{7}$$

Since we need the coefficient of x^n, we only need the first $n+1$ terms on the right and the $(p+1)^{\text{st}}$ term in (7) such that

$$x^p\sum_{l=0}^{p}\binom{p}{l}(-1)^l x^{kl}\sum_{i=0}^{\infty}\binom{p+i-1}{i}x^i.$$

So $kl+i=n-p$, as we see in (6), and $\frac{n}{k}\le p\le n$. Thus, by Lemma 3.5, we have the theorem. □

From the above theorems, we have five representations for g_n, $g_n = g_{n+k-2}^{(k)}$. That is,

$$\begin{aligned} g_n &= \operatorname{per}\mathcal{F}^{(n-1,k)} = \det(\mathcal{F}^{(n-1,k)}\circ H) = \frac{\det(V_1^{(k)})}{\det(V)} \\ &= \sum_{\frac{n-1}{k}\le p\le n-1}\sum_{l=0}^{p}(-1)^l\binom{p}{l}\binom{n-kl-2}{n-kl-p-1} \\ &= \sum_{(m_1,\dots,m_k)}\frac{m_k}{m_1+\cdots+m_k}\times\binom{m_1+\cdots+m_k}{m_1,\dots,m_k}, \end{aligned}$$

where the summation is over nonnegative integers satisfying $m_1+2m_2+\cdots+km_k=n-1+k$.

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GENERALIZATIONS OF SOME IDENTITIES INVOLVING THE FIBONACCI NUMBERS

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The generalized Fibonacci and Lucas numbers are defined by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n \tag{1}$$

where $\alpha = \frac{p+\sqrt{p^2-4q}}{2}$, $\beta = \frac{p-\sqrt{p^2-4q}}{2}$, $p>0$, $q \neq 0$, and $p^2-4q>0$. It is obvious that $\{U_n\}$ and $\{V_n\}$ are the usual Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$ when $p=-q=1$. Recently, for the Fibonacci numbers, Zhang established the following identities in [2]:

$$\sum_{a+b=n} F_a F_b = \frac{1}{5}((n-1)F_n + 2nF_{n-1}), \; n \geq 1, \tag{2}$$

$$\sum_{a+b+c=n} F_a F_b F_c = \frac{1}{50}((5n^2-9n-2)F_{n-1} + (5n^2-3n-2)F_{n-2}), \; n \geq 2, \tag{3}$$

and when $n \geq 3$,

$$\sum_{a+b+c+d=n} F_a F_b F_c F_d = \frac{1}{150}((4n^3-12n^2-4n+12)F_{n-2} + (3n^3-6n^2-3n+6)F_{n-3}). \tag{4}$$

In this paper, we extend the above conclusions. We establish some identities related to $\{U_n\}$ and $\{V_n\}$. The equalities (2)-(4) emerge as special cases of our results.

Consider the generating function of $\{U_{nk}\}$: $G_k(x) = \sum_{n=0}^{\infty} U_{nk} x^n$, where k is a positive integer. Clearly, by (1) and the geometric formula,

$$G_k(x) = \frac{U_k x}{1 - V_k x + q^k x^2}, \quad |x| > \alpha^k.$$

Let $F_k(x) = \frac{G_k(x)}{x}$. Then

$$F_k(x) = \sum_{n=1}^{\infty} U_{nk} x^{n-1} = \frac{U_k}{1 - V_k x + q^k x^2}, \; |x| > \alpha^k. \tag{5}$$

For $F_k(x)$, we have the following lemma.

Lemma: If $F_k(x)$ is defined by (5), then $F_k(x)$ satisfies

$$F_k^2(x) = \frac{U_k}{V_k^2 - 4q^k}(F_k'(x)(V_k - 2q^k x) - 4q^k F_k(x)), \tag{6}$$

$$F_k^3(x) = \frac{U_k^2}{2(V_k^2-4q^k)^2}(F_k''(x)(V_k - 2q^k x)^2 - 14q^k F_k'(x)(V_k - 2q^k x) + 32q^{2k} F_k(x)), \tag{7}$$

and

$$F_k^4(x)=\frac{U_k^3}{6(V_k^2-4q^k)^3}(F_k'''(x)(V_k-2q^kx)^3-30q^kF_k''(x)(V_k-2q^kx)^2 \\ +228q^{2k}F_k'(x)(V_k-2q^kx)-384q^{3k}F_k(x)). \tag{8}$$

Proof: Noticing that

$$F_k'(x)=\frac{U_k(V_k-2q^kx)}{(1-V_kx+q^kx^2)^2}=\frac{(V_k-2q^kx)F_k(x)}{1-V_kx+q^kx^2},$$

we have

$$F_k'(x)(V_k-2q^kx)-4q^kF_k(x)=\frac{(V_k^2-4q^k)F_k(x)}{1-V_kx+q^kx^2}=\frac{V_k^2-4q^k}{U_k}F_k^2(x),$$

and hence (6) holds. Differentiating in (6), we get

$$2F_k(x)F_k'(x)=\frac{U_k}{V_k^2-4q^k}(F_k''(x)(V_k-2q^kx)-6q^kF_k'(x)).$$

Therefore,

$$2F_k(x)F_k'(x)(V_k-2q^kx)=\frac{U_k}{V_k^2-4q^k}(F_k''(x)(V_k-2q^kx)^2-6q^kF_k'(x)(V_k-2q^kx)).$$

Using (6), we have

$$2F_k(x)\left(\frac{V_k^2-4q^k}{U_k}F_k^2(x)+4q^kF_k(x)\right)=\frac{U_k}{V_k^2-4q^k}(F_k''(x)(V_k-2q^kx)^2-6q^kF_k'(x)(V_k-2q^kx)).$$

Using (6) again, we can prove that (7) holds. Similarly, differentiating in (6) and applying (6) and (7), we can obtain identity (8). □

From the above lemma, we have the main results of this paper.

Theorem: Suppose that k and n are positive integers. Then

$$\sum_{a+b=n}U_{ak}U_{bk}=\frac{U_k}{V_k^2-4q^k}((n-1)U_{nk}V_k-2q^knU_{(n-1)k}),\ n\geq 1, \tag{9}$$

$$\sum_{a+b+c=n}U_{ak}U_{bk}U_{ck}=\frac{U_k^2}{2(V_k^2-4q^k)^2}((n-1)(n-2)V_k^2U_{nk}-q^kV_k(4n^2-6n-4)U_{(n-1)k} \\ +(4n^2-28n+28(n-3)V_k+80)U_{(n-2)k}),\ n\geq 2, \tag{10}$$

and

$$\sum_{a+b+c+d=n}U_{ak}U_{bk}U_{ck}U_{dk}=\frac{U_k^3}{6(V_k^2-4q^k)^3}(V_k^3(n-1)(n-2)(n-3)U_{nk} \\ -6q^kV_k^2(n-2)(n-3)(n+1)U_{(n-1)k} \\ +12q^{2k}V_k(n-3)(n^2+n-1)U_{(n-2)k} \\ -8q^{3k}n(n^2-4)U_{(n-3)k}),\ n\geq 3. \tag{11}$$

Proof: To show that this theorem is valid, comparing the coefficients of x^{n-2}, x^{n-3}, and x^{n-4} on both sides of the Lemma, we have identities (9)-(11). □

Corollary. Assume that k and n are positive integers. Then

$$\sum_{a+b=n} F_{ak}F_{bk} = \frac{F_k}{L_k^2 - 4(-1)^k}((n-1)F_{nk}L_k - 2(-1)^k nF_{(n-1)k}),\ n \geq 1, \tag{12}$$

$$\begin{aligned}\sum_{a+b+c=n} F_{ak}F_{bk}F_{ck} = &\frac{F_k^2}{2(L_k^2 - 4(-1)^k)^2}((n-1)(n-2)L_k^2 F_{nk} - (-1)^k L_k(4n^2 - 6n - 4)F_{(n-1)k} \\ &+ (4n^2 - 28n + 28(n-3)L_k + 80)F_{(n-2)k}),\ n \geq 2,\end{aligned} \tag{13}$$

and

$$\begin{aligned}\sum_{a+b+c+d=n} F_{ak}F_{bk}F_{ck}F_{dk} = &\frac{F_k^3}{6(L_k^2 - 4(-1)^k)^3}((n-1)(n-2)(n-3)L_k^3 F_{nk} \\ &- 6(-1)^k(n-2)(n-3)(n+1)L_k^2 F_{(n-1)k} \\ &+ 12L_k(n-3)(n^2+n-1)F_{(n-2)k} - 8(-1)^k n(n^2-4)F_{(n-3)k}),\ n \geq 3.\end{aligned} \tag{14}$$

From the Corollary, it is very easy to obtain (2)-(4). If $k = 1$ in (14), then

$$\begin{aligned}\sum_{a+b+c+d=n} F_aF_bF_cF_d = &\frac{1}{750}((n-1)(n-2)(n-3)F_n + 6(n-2)(n-3)(n+1)F_{n-1} \\ &+ 12(n-3)(n^2+n-1)F_{n-2} + 8n(n^2-4)F_{n-3}).\end{aligned}$$

By using $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$, we can obtain (4). Similarly, from (12), (13), and $F_n = F_{n-1} + F_{n-2}$, we have identities (2) and (3). In addition, we can work out other sums from the Corollary. For example, when $k = 2$ and $q = -1$ in (13), we have

$$\sum_{a+b+c=n} F_{2a}F_{2b}F_{2c} = \frac{1}{50}(9(n-1)(n-2)F_{2n} - 3(4n^2 - 6n - 4)F_{2n-2} + (4n^2 + 56n - 172)F_{2n-4}).$$

Applying $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ again and again, we get

$$\sum_{a+b+c=n} F_{2a}F_{2b}F_{2c} = \frac{1}{50}((15n^2 - 63n + 66)F_{2n-3} + (10n^2 + 20n - 124)F_{2n-4}).$$

When $p = 2$ and $q = -1$ in (10), we obtain

$$\begin{aligned}\sum_{a+b+c=n} P_{ak}P_{bk}P_{ck} = &\frac{P_k^2}{2(Q_k^2 - 4(-1)^k)^2}((n-1)(n-2)Q_k^2 P_{nk} - (-1)^k Q_k(4n^2 - 6n - 4)P_{(n-1)k} \\ &+ (4n^2 - 28n + 28(n-3)Q_k + 80)P_{(n-2)k}),\ n \geq 2,\end{aligned}$$

where P_k and Q_k denote the k^{th} Pell and Pell-Lucas numbers (see [1]).

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APPROXIMATION OF ∞-GENERALIZED FIBONACCI SEQUENCES AND THEIR ASYMPTOTIC BINET FORMULA

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1. INTRODUCTION

The notion of an ∞-generalized Fibonacci sequence has been introduced and studied in [8], [9], and [11]. In fact, such a notion goes back to Euler. In his book [4], he discusses Bernoulli's method of using linear recurrences to approximate roots of (mainly polynomial) equations. At the very end, in Article 355 [4, p. 301], there is a brief example of the use of an ∞-generalized Fibonacci sequence for the approximation of a root of a power series equation.*

The class of sequences defined by linear recurrences of infinite order is an extension of the class of ordinary *r-generalized Fibonacci sequences* (r-GFS, for short) with r finite defined by linear recurrences of r^{th} order (for example, see [1], [2], [3], [6], [7], [10], etc.). More precisely, let $\{a_j\}_{j\geq 0}$ and $\{\alpha_{-j}\}_{j\geq 0}$ be two sequences of real or complex numbers, where $a_j \neq 0$ for some j. The former is called the *coefficient sequence* and the latter the *initial sequence*. The associated *∞-generalized Fibonacci sequence* (∞-GFS, for short) $\{V_n\}_{n\in \mathbf{Z}}$ is defined as follows:

$$V_n = \alpha_n \quad (n \leq 0), \tag{1.1}$$

$$V_n = \sum_{j=0}^{\infty} a_j V_{n-j-1} \quad (n \geq 1). \tag{1.2}$$

As is easily observed, the general terms V_n may not necessarily exist. In [8], a sufficient condition for the existence of the general terms has been given.

In this paper, we first give a necessary and sufficient condition for the existence of the general terms V_n $(n \geq 1)$ of an ∞-GFS (see Section 2). We will see that the condition in [8] satisfies our condition, but not vice versa. We then consider a process of approximating a given ∞-GFS by a sequence of r-GFS's, where $r < \infty$ varies (see Section 3). As is well known, there is a Binet-type

* The authors would like to thank the referee for kindly pointing out Euler's work.

formula for the general terms of an r-GFS (for example, see Theorem 1 in [3]). In Section 4, we use such a formula together with the approximation result in Section 3 to obtain an asymptotic Binet formula for an ∞-GFS. In Section 5, we study the asymptotic behavior of ∞-GFS's using the results in the previous sections. In Section 6, we concentrate on the case in which $a_j \geq 0$ and obtain some sharp results about the asymptotic behavior of ∞-GFS's. Finally, in Section 7, we give an explicit example of our main theorem of Section 6.

2. EXISTENCE OF GENERAL TERMS

Let $\{a_j\}_{j\geq 0}$ and $\{\alpha_{-j}\}_{j\geq 0}$ be as in Section 1 and $\{V_n\}_{n\in Z}$ be the associated ∞-GFS defined by (1.1) and (1.2). Equation (1.2) can be rewritten as follows:

$$V_n = \sum_{j=0}^{n-2} a_j V_{n-j-1} + \sum_{j=n-1}^{\infty} a_j V_{n-j-1} = \sum_{j=0}^{n-2} a_j V_{n-j-1} + \sum_{j=0}^{\infty} a_{j+n-1}\alpha_{-j}. \tag{2.1}$$

Then it is easy to see that we have the following necessary and sufficient condition for the existence of the general terms V_n $(n \geq 1)$.

Proposition 2.1: The general term V_n exists for all $n \geq 1$ if and only if the following condition (C_∞) is satisfied.

(C_∞): The series $\sum_{j=0}^{\infty} a_{j+n-1}\alpha_{-j}$ converges for all $n \geq 1$.

Condition (C_∞) is trivially satisfied in the case of an r-GFS with r finite, since $a_j = 0$ for all $j \geq r$.

Remark 2.2: As particular cases of Proposition 2.1, we can easily prove the following.

(a) If the series $\sum_{j=0}^{\infty} \alpha_{-j}$ converges absolutely and the sequence $\{a_j\}_{j\geq 0}$ is bounded, then V_n exists for all $n \geq 1$.

(b) If the series $\sum_{j=0}^{\infty} a_j$ converges absolutely and the sequence $\{\alpha_{-j}\}_{j\geq 0}$ is bounded, then V_n exists for all $n \geq 1$.

For another existence result, see Lemma 6.6. Compare Remark 2.2 with Section 2.1 in [11].

Now let us compare our condition (C_∞) with the sufficient condition considered in [8] for the existence of the general terms V_n $(n \geq 1)$. Let $h(z)$ be the power series defined by $h(z) = \sum_{j=0}^{\infty} a_j z^j$. The conditions considered in [8] are the following.

$(C1)$: The radius of convergence R of the power series $h(z)$ is positive.

$(C2)$: There exist $C > 0$ and $T > 0$ with $0 < T < R$ satisfying $|\alpha_{-j}| \leq CT^j$ for all $j \geq 0$.

It was established in [8] that, if conditions $(C1)$ and $(C2)$ are satisfied, then the general term V_n of the associated ∞-GFS exists for all $n \geq 1$.

It is easy to see that, if conditions $(C1)$ and $(C2)$ are satisfied, then (C_∞) is also satisfied. On the other hand, the examples $a_j = (j+1)^{-3}$, $\alpha_{-j} = j$, and $a_j = (j+1)^{-1}$, $\alpha_{-j} = (-1)^j$ both satisfy condition $(C1)$, but not $(C2)$, while (C_∞) is satisfied in both cases. Therefore, condition (C_∞) is strictly weaker than $(C1)$ and $(C2)$.

3. APPROXIMATION BY r-GFS's WITH r FINITE

Let $\{a_j\}_{j\geq 0}$ and $\{\alpha_{-j}\}_{j\geq 0}$ be sequences of complex numbers as before. For each $r \geq 1$, let $\{V_n^{(r)}\}_{n\geq -r+1}$ be the r-GFS defined as follows:

$$V_n^{(r)} = \alpha_n \quad (n = -r+1,\ -r+2, \ldots, 0), \tag{3.1}$$

$$V_n^{(r)} = \sum_{j=0}^{r-1} a_j V_{n-j-1}^{(r)} \quad (n \geq 1). \tag{3.2}$$

Note that here we allow the case where $a_{r-1} = 0$, while $a_{r-1} \neq 0$ is assumed in [3].

In this section, we prove the following approximation theorem.

Theorem 3.1: The general term V_n exists for all $n \geq 1$ if and only if the sequence $\{V_n^{(r)}\}_{r\geq 1}$ converges for all $n \geq 1$. Furthermore, in this case, for all $n \geq 1$, we have

$$V_n = \lim_{r\to\infty} V_n^{(r)}. \tag{3.3}$$

Proof: We prove, by induction on k, that the terms $V_1, \ldots, V_k$ exist if and only if, for all n with $1 \leq n \leq k$, the sequence $\{V_n^{(r)}\}_{r\geq 1}$ converges and (3.3) holds. When $k = 1$, we have

$$V_1 = \sum_{j=0}^{\infty} a_j \alpha_{-j} \quad \text{and} \quad V_1^{(r)} = \sum_{j=0}^{r-1} a_j \alpha_{-j}$$

for all $r \geq 1$. Thus, V_1 exists if and only if the sequence $\{V_1^{(r)}\}_{r\geq 1}$ converges. Furthermore, in this case, we have $V_1 = \lim_{r\to\infty} V_1^{(r)}$.

Now suppose $k \geq 2$ and that the induction hypothesis holds for $k-1$. For $r \geq k$, we have

$$V_k = \sum_{j=0}^{k-2} a_j V_{k-j-1} + \sum_{j=k-1}^{\infty} a_j \alpha_{k-j-1}$$

and

$$V_k^{(r)} = \sum_{j=0}^{k-2} a_j V_{k-j-1}^{(r)} + \sum_{j=k-1}^{r-1} a_j \alpha_{k-j-1}. \tag{3.4}$$

Then, by our induction hypothesis, we see that the sequence $\{V_n^{(r)}\}_{r\geq 1}$ converges for all n with $1 \leq n \leq k$ if and only if the terms $V_1, \ldots, V_k$ exist. Furthermore, in this case, using our induction hypothesis, we see that (3.3) holds for $n = k$ by sending $r \to \infty$ in (3.4). □

4. ASYMPTOTIC BINET FORMULA

Let $\{a_j\}_{j\geq 0}$ and $\{\alpha_{-j}\}_{j\geq 0}$ be sequences of complex numbers. For each $r \geq 1$, consider the polynomial $Q_r(z)$ defined by

$$Q_r(z) = 1 - \sum_{j=0}^{r-1} a_j z^{j+1}. \tag{4.1}$$

Note that the characteristic polynomial $P_r(z)$ of the r-GFS $\{V_n^{(r)}\}_{n\geq -r+1}$ defined by (3.1) and (3.2) is given by

$$P_r(z) = z^r Q_r(z^{-1}), \tag{4.2}$$

which is a polynomial of degree r. Let $\lambda_1^{(r)}, \ldots, \lambda_{u(r)}^{(r)}$ be the complex roots of $P_r(z)$, whose respective multiplicities are $m_1^{(r)}, \ldots, m_{u(r)}^{(r)}$. Note that $m_1^{(r)} + \cdots + m_{u(r)}^{(r)} = r$. The classical Binet-type formula for the r-GFS $\{V_n^{(r)}\}_{n \geq -r+1}$ is given by the following:

$$V_n^{(r)} = \sum_{k=1}^{u(r)} \sum_{j=0}^{m_r^{(k)}-1} \beta_{k,j}^{(r)} n^j (\lambda_k^{(r)})^n, \tag{4.3}$$

where the complex numbers $\beta_{k,j}^{(r)}$ are determined by the initial sequence $\{\alpha_{-j}\}_{0 \leq j \leq r-1}$ (e.g., see [5, Theorem 3.7]; [3, Theorem 1]).

Remark 4.1: In [5] and [3] it is assumed that $a_{r-1} \neq 0$. When this condition is not satisfied, the polynomial $Q_r(z)$ may not necessarily be of degree r. On the other hand, the characteristic polynomial $P_r(z)$ is always of degree r, which may have zero as a root of some multiplicity. Hence, the above Binet-type formula (4.3) holds even if $a_{r-1} = 0$.

By Proposition 2.1, Theorem 3.1, and (4.3), we have the following asymptotic Binet formula.

Theorem 4.2: If condition (C_∞) is satisfied, then we have, for all $n \geq 1$,

$$V_n = \lim_{r \to \infty} \sum_{k=1}^{u(r)} \sum_{j=0}^{m_k^{(r)}-1} \beta_{k,j}^{(r)} n^j (\lambda_k^{(r)})^n. \tag{4.4}$$

Compare the above results with Problem 4.5 in [8].

Example 4.3: Consider the ∞-GFS $\{V_n\}_{n \in \mathbb{Z}}$ associated with the coefficient sequence $a_j = -\gamma^{j+1}$ and the initial sequence $\alpha_{-j} = \delta_{0j}$ $(j \geq 0)$, where γ is a nonzero complex number, $\delta_{0j} = 0$ if $j \neq 0$, and $\delta_{00} = 1$. Note that condition (C_∞) is trivially satisfied. By a straightforward calculation, we see that

$$V_n = \begin{cases} 0 & (n \neq 0, 1), \\ 1 & (n = 0), \\ -\gamma & (n = 1). \end{cases} \tag{4.5}$$

On the other hand, we have $P_r(z) = z^r + \gamma z^{r-1} + \cdots + \gamma^{r-1} z + \gamma^r$. Thus, all the roots are simple and they are of the form $\lambda_k^{(r)} = \gamma \xi_{r+1}^k$ $(k = 1, 2, \ldots, r)$ for a primitive $(r+1)^{\text{st}}$ root ξ_{r+1} of unity. Then we have*

$$\sum_{k=1}^{r} \beta_{k,0}^{(r)} (\lambda_k^{(r)})^n = \delta_{0n} \quad (-r+1 \leq n \leq 0). \tag{4.6}$$

We multiply each of the equations of (4.6) by γ^{-n} and sum them up for $n = -r+1, \ldots, 0$. Then we obtain

$$\sum_{k=1}^{r} \beta_{k,0}^{(r)} (\lambda_k^{(r)})^{-r} = -\gamma^{-r}, \tag{4.7}$$

since

$$\sum_{n=-r+1}^{0} (\lambda_k^{(r)})^n \gamma^{-n} = -(\lambda_k^{(r)})^{-r} \gamma^r.$$

* Using (4.6), we can obtain explicit values of $\beta_{k,0}^{(r)}$, although we do not need them here.

By successively multiplying (4.6) and (4.7) by $\gamma^{r+1} = (\lambda_k^{(r)})^{r+1}$, we see that

$$V_n^{(r)} = \begin{cases} 0, & n \not\equiv 0, 1 \pmod{r+1}, \\ \gamma^n, & n \equiv 0 \pmod{r+1}, \\ -\gamma^n, & n \equiv 1 \pmod{r+1}, \end{cases} \tag{4.8}$$

by (4.3). Hence, we have $\lim_{r\to\infty} V_n^{(r)} = V_n$ in view of (4.5).

5. ASYMPTOTIC BEHAVIOR OF ∞-GFS's

Let $\{a_j\}_{j\geq 0}$ and $\{\alpha_{-j}\}_{j\geq 0}$ be sequences of complex numbers. For each $r \geq 1$, consider the characteristic polynomial $P_r(z)$ of the r-GFS $\{V_n^{(r)}\}_{n\geq -r+1}$ as in (4.2). Let $r_0 \geq 1$ be an integer such that $a_{r_0-1} \neq 0$ and let us assume that, for each $r \geq r_0$, there exists a nonzero *dominant root* q_r of $P_r(z)$ with *dominant multiplicity* 1 (for these terminologies, refer to Section 3 in [3]). In [3], it has been shown that $L_r = \lim_{n\to\infty} V_n^{(r)}/q_r^n$ exists and its explicit value has been obtained in terms of q_r together with the coefficient and the initial sequences.

Let us assume that the sequence $\{q_r\}_{r\geq r_0}$ converges to a nonzero complex number q. If one looks at Theorem 4.2, then it might seem easy to obtain a convergence result for the sequence $\{V_n/q^n\}_{n\geq 1}$. However, since equation (4.4) is given by the limit for $r\to\infty$, we have to be careful with the relationship between the convergence with respect to r and that with respect to n. For this reason, we need the following definition.

Definition 5.1: Let $\{x_n^{(r)}\}_{n\geq n_0, r\geq r_0}$ be a doubly-indexed sequence of real or complex numbers. We say that the sequences $\{x_n^{(r)}\}_{n\geq n_0}$ are *uniformly convergent* for $r \geq r_0$ if there exists a sequence $\{L_r\}_{r\geq r_0}$ of real or complex numbers such that, for every $\varepsilon > 0$, there exists an $N \geq n_0$ satisfying $|x_n^{(r)} - L_r| < \varepsilon$ for all $n \geq N$ and all $r \geq r_0$. It is easy to see that in this case, if the sequence $\{x_n^{(r)}\}_{r\geq r_0}$ converges to x_n for each $n \geq n_0$, and if $L = \lim_{r\to\infty} L_r$ exists, then $\lim_{n\to\infty} x_n$ exists and is equal to L.

Then, combining the results of [3], Theorem 3.1 of the present paper, and the above definition, we obtain the following (for an explicit example, see Section 7).

Theorem 5.2: Suppose that

(a) $P_r(z)$ has a nonzero dominant root q_r of dominant multiplicity 1 for each $r \geq r_0$,
(b) $q = \lim_{r\to\infty} q_r$ exists and is nonzero,
(c) the general term V_n exists for all $n \geq 1$,
(d) the sequences $\{x_n^{(r)}\}_{n\geq 0} = \{V_n^{(r)}/q_r^n\}_{n\geq 0}$ are uniformly convergent for $r \geq r_0$ with $L_r = \lim_{n\to\infty} V_n^{(r)}/q_r^n$, and
(e) $L = \lim_{r\to\infty} L_r$ exists.

Then the limit $\lim_{n\to\infty} V_n/q^n$ exists and is equal to L.

Proof: By Theorem 3.1 and our assumptions, we have $V_n/q^n = \lim_{r\to\infty} V_n^{(r)}/q_r^n$ for each $n \geq 1$. Then, by the observation given in Definition 5.1 together with our assumptions, we have $\lim_{n\to\infty} V_n/q^n = L$. □

Remark 5.3: As in the above theorem, let us assume (a)-(c) and, instead of (d) and (e), let us assume that $L = \lim_{n,r\to\infty} x_n^{(r)}$ exists, where we write $\lim_{n,r\to\infty} x_n^{(r)} = L$ if, for every $\varepsilon > 0$, there exists an $N \geq r_0$ such that $|x_n^{(r)} - L| < \varepsilon$ for all $n, r \geq N$. Then we have

$$L = \lim_{n\to\infty} \frac{V_n}{q^n} = \lim_{r\to\infty} L_r. \tag{5.1}$$

The following lemma is easy to prove.

Lemma 5.4: Let $\{y_n^{(r)}\}_{n\geq n_0, r\geq r_0}$ be a doubly-indexed sequence of real or complex numbers such that, for every $n \geq n_0$, $\lim_{r\to\infty} y_n^{(r)} = \gamma_n$ exists and $\lim_{n\to\infty} \gamma_n = \gamma$ exists. Then, for every $n \geq n_0$, there exists an $r(n) \geq r_0$ such that $r(n) < r(n+1)$ for all $n \geq n_0$ and that the sequence $\{y_n^{(r(n))}\}_{n\geq n_0}$ converges to γ.

Let us assume conditions (a)-(c) of Theorem 5.2 and, for $n \geq 1$ and $r \geq r_0$, set $y_n^{(r)} = V_n / q^n - V_n^{(r)} / q_r^n$. Then, for every $n \geq 1$, we have $\lim_{r\to\infty} y_n^{(r)} = \gamma_n = 0$. Then $\lim_{n\to\infty} \gamma_n = 0$ trivially exists. Thus, Lemma 5.4 implies that, for every $n \geq 1$, there exists an $r(n) \geq r_0$ such that $r(1) < r(2) < r(3) < \cdots$ and $\lim_{n\to\infty} y_n^{(r(n))} = 0$. Therefore, we have the following theorem.

Theorem 5.5: Suppose that

(*a*) $P_r(z)$ has a nonzero dominant root q_r of dominant multiplicity 1 for each $r \geq r_0$,

(*b*) $q = \lim_{r\to\infty} q_r$ exists and is nonzero, and

(*c*) the general term V_n exists for all $n \geq 1$.

Then $L = \lim_{n\to\infty} V_n / q^n$ exists if and only if $\lim_{n\to\infty} V_n^{(r(n))} / q_{r(n)}^n$ exists. Furthermore, in this case, we have

$$L = \lim_{n\to\infty} \frac{V_n}{q^n} = \lim_{n\to\infty} \frac{V_n^{(r(n))}}{q_{r(n)}^n}. \tag{5.2}$$

In (5.1) and (5.2), we did not give the limiting value L explicitly. In the following section, we determine the explicit value in the case where a_j are nonnegative real numbers.

6. THE CASE OF NONNEGATIVE COEFFICIENTS

In this section, we assume that all the coefficients a_j are nonnegative real numbers and consider the same problem as in the previous section. We use the same notations.

It is not difficult to see that, for each $r \geq r_0$, there always exists a unique real number $q_r > 0$ such that $P_r(q_r) = Q_r(q_r^{-1}) = 0$ (for example, see Lemma 2 in [2], Lemma 8 in [3], and Section 12 in [12]), where Q_r is the polynomial defined by (4.1). Set $p_r = q_r^{-1}$. Define the power series $Q(z)$ by $Q(z) = 1 - zh(z) = 1 - \sum_{j=0}^{\infty} a_j z^{j+1}$ and let R be the radius of convergence of $Q(z)$, which coincides with that of $h(z)$. The following will be proved later in this section.

Theorem 6.1: The sequence $\{q_r^{-1}\}_{r\geq r_0} = \{p_r\}_{r\geq r_0}$ always converges and the following conditions are equivalent:

(*a*) Condition $(C1)$ is satisfied (i.e., $R > 0$) and $\lim_{x\to R-0} Q(x) \leq 0$.

(*b*) The limiting value $l = \lim_{r\to\infty} p_r > 0$ and $Q(l) = 0$.

(*c*) There exists a unique positive real number p such that $Q(p) = 0$.

Furthermore, if (c) is satisfied, then we have $p = \lim_{r\to\infty} p_r$.

The main result of this section is the following theorem.

Theorem 6.2: Assume that one of the three conditions of Theorem 6.1 is satisfied. Suppose that $d_{r_1} = 1$ for some $r_1 \geq r_0$, $0 < p < R$, and

$$q^j |\alpha_{-j}| < K \quad (j \geq 0) \tag{6.1}$$

for some constant $K > 0$, where $d_{r_1} = \gcd\{j+1 : a_j > 0, 0 \leq j \leq r_1 - 1\}$ and $q = p^{-1}$. If the sequences $\{V_n^{(r)} / q_r^n\}_{n \geq 1}$ are uniformly convergent for $r \geq r_1$, then V_n exists for all n and we have

$$\lim_{n\to\infty} \frac{V_n}{q^n} = \frac{\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} a_k q^{j-k-1}\right)\alpha_{-j}}{\sum_{j=0}^{\infty}(j+1)a_j q^{-(j+1)}}. \tag{6.2}$$

Let us begin by proving Theorem 6.1.

Proof of Theorem 6.1: Suppose that $r_0 \leq r < r'$. Then we have $Q_{r'}(p_r) = -a_r p_r^{r+1} - \cdots - a_{r'-1} p_r^{r'} \leq 0$. Furthermore, we have $Q_{r'}(p_{r'}) = 0$. Since $Q_{r'}(x)$ is a decreasing function on $(0, \infty)$, we have $p_r \geq p_{r'}$; i.e., the sequence $\{p_r\}_{r \geq r_0}$ of positive real numbers is nonincreasing. Hence, it is convergent. In the following, we set $l = \lim_{r\to\infty} p_r \geq 0$.

For every $r \geq r_0$, we have $0 \leq l \leq p_r$. Since $Q_r(x)$ is a decreasing function on $(0, \infty)$, we have $0 \leq Q_r(l) \leq 1$. On the other hand, since $Q_{r'}(l) - Q_r(l) = -a_r l^{r+1} - \cdots - a_{r'-1} l^{r'} \leq 0$ for r, $r' \geq r_0$ with $r < r'$, we see that the sequence $\{Q_r(l)\}_{r \geq r_0}$ is nonincreasing. Thus, $\lim_{r\to\infty} Q_r(l)$ exists and is equal to $Q(l)$. Furthermore, we have

$$0 \leq Q(l) \leq 1. \tag{6.3}$$

(a) $\Rightarrow$ (b): First, note that since $Q(l)$ exists we have $0 \leq l \leq R$.

Suppose $0 \leq l < R$ and $Q(l) > 0$. Since $Q(x)$ is a continuous function on the interval $(-R, R)$, there exists a sufficiently small positive real number η such that $Q(x) > 0$ for all $x \in (l - \eta, l + \eta) \subset (-R, R)$. Since $l = \lim_{r\to\infty} p_r$, there exists an $r' \geq r_0$ such that $p_r \in [l, l + \eta)$ for all $r \geq r'$. Thus, $Q(p_r) > 0$ for all $r \geq r'$. However, since $Q(p_r) = -\sum_{j=r}^{\infty} a_j p_r^{j+1} \leq 0$, this is a contradiction. Therefore, we have $Q(l) = 0$.

If $l = R$, then we have $0 \leq Q(R) \leq 1$ by (6.3). Thus, we have $Q(R) = Q(l) = 0$, since $Q(R) = \lim_{x\to R-0} Q(x) \leq 0$ by our assumption.

Therefore, we have $Q(l) = 0$, and this implies that $l > 0$, since, if $l = 0$, we would have $Q(l) = 1 > 0$.

(b) $\Rightarrow$ (c): Setting $p = l$, we have $Q(p) = 0$. The uniqueness follows from the fact that $Q(x)$ is a strictly decreasing function.

(c) $\Rightarrow$ (a): Since $p > 0$ and $Q(p) = 0$, we see that $0 < p \leq R$, which implies condition $(C1)$. Furthermore, since $Q(x)$ is a decreasing function on $(0, R)$, we have $\lim_{x\to R-0} Q(x) \leq Q(p) = 0$. This completes the proof. □

Remark 6.3: When some a_j is not a nonnegative real number, there does not always exist a root p of $Q(z)$. For instance, in Example 4.3 of Section 4, we have $Q(z) = 1/(1 - \gamma z)$, which never takes the value zero inside the convergence range. Compare this observation with Problem 4.5 in [8].

Since q_r is a root of the characteristic polynomial P_r, we have

$$\frac{a_0}{q_r}+\frac{a_1}{q_r^2}+\cdots+\frac{a_{r-1}}{q_r^r}=1. \tag{6.4}$$

Combining this with Theorems 3, 5, and 9 of [3], we have the following lemma.

Lemma 6.4: For each $r \geq r_0$, we have:

(a) $L_r = \lim_{n\to\infty} V_n^{(r)}/q_r^n$ exists for any initial values $\{\alpha_{-j}\}_{0\le j\le r-1}$ and is nonzero for some initial values if and only if $d_r = 1$.

(b) If there exists an $r_1 \geq r_0$ such that $d_{r_1}=1$, then $L_r = \lim_{n\to\infty} V_n^{(r)}/q_r^n$ exists for all $r \geq r_1$. Furthermore, this limit is given by

$$L_r = \frac{\sum_{j=0}^{r-1}\left(\sum_{k=j}^{r-1} a_k q_r^{j-k-1}\right)\alpha_{-j}}{\sum_{j=0}^{r-1}(j+1)a_j q_r^{-(j+1)}}. \tag{6.5}$$

Lemma 6.5: Assume that one of the three conditions of Theorem 6.1 is satisfied. Suppose that $d_{r_1}=1$ for some $r_1 \geq r_0$, $0<p<R$, and (6.1) holds for some constant $K>0$. Then, for $L_r = \lim_{n\to\infty} V_n^{(r)}/q_r^n$ $(r \geq r_1)$, we have

$$\lim_{r\to\infty} L_r = \frac{\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} a_k q^{j-k-1}\right)\alpha_{-j}}{\sum_{j=0}^{\infty}(j+1)a_j q^{-(j+1)}} < +\infty. \tag{6.6}$$

Proof: Set $S_r(x)=\sum_{j=0}^{r-1}(j+1)a_j x^{j+1}$. Since $0<p=q^{-1}\le p_r = q_r^{-1}$ for all $r\geq r_0$, we have

$$S_r(q^{-1})=\sum_{j=0}^{r-1}(j+1)a_j q^{-(j+1)} \le \sum_{j=0}^{r-1}(j+1)a_j q_r^{-(j+1)} = S_r(q_r^{-1}) \tag{6.7}$$

for all $r \geq r_0$. On the other hand, consider the function S defined by

$$S(x)=\sum_{j=0}^{\infty}(j+1)a_j x^{j+1} = -xQ'(x). \tag{6.8}$$

Note that S is continuous on the interval $[0, R)$ and, hence, at $x=p=q^{-1}$ by our assumption. Thus, we have

$$\lim_{r\to\infty} S(q_r^{-1}) = S(q^{-1}) = \sum_{j=0}^{\infty}(j+1)a_j q^{-(j+1)} < +\infty. \tag{6.9}$$

Furthermore,

$$S_r(q_r^{-1})=\sum_{j=0}^{r-1}(j+1)a_j q_r^{-(j+1)} \le S(q_r^{-1}) \tag{6.10}$$

for all $r \geq r_0$. Thus, by (6.7) and (6.10), we have $S_r(q^{-1}) \leq S(q_r^{-1})$ for all sufficiently large r and, hence, using (6.9) we see that $\lim_{r\to\infty} S_r(q_r^{-1}) = S(q^{-1}) < +\infty$. In other words, the denominator of (6.5) converges to that of (6.6) as r tends to ∞. Note that this value is not zero.

Let B_r denote the numerator of (6.5); i.e.,

$$B_r = \sum_{j=0}^{r-1}\left(\sum_{k=j}^{r-1} a_k q_r^{-(k+1)}\right) q_r^j \alpha_{-j} = \sum_{k=0}^{r-1} a_k q_r^{-(k+1)}\left(\sum_{j=0}^{k} q_r^j \alpha_{-j}\right).$$

Furthermore, set

$$C_r = \sum_{k=0}^{r-1} a_k q^{-(k+1)}\left(\sum_{j=0}^{k} q^j \alpha_{-j}\right) \quad \text{and} \quad H_r = \sum_{k=0}^{r-1} a_k q^{-(k+1)}\left(\sum_{j=0}^{k} q_r^j \alpha_{-j}\right)$$

so that we have

$$|B_r - C_r| \leq |B_r - H_r| + |H_r - C_r|. \tag{6.11}$$

First, let us consider $D_r = |B_r - H_r|$. We have

$$D_r \leq \sum_{k=0}^{r-1} a_k q_r^{-(k+1)}\left|1 - \frac{q^{-(k+1)}}{q_r^{-(k+1)}}\right|\left(\sum_{j=0}^{k} q_r^j |\alpha_{-j}|\right). \tag{6.12}$$

It is easy to see that $|1 - q^{-(k+1)}/q_r^{-(k+1)}| = |1-(q_r/q)^{k+1}| \leq (k+1)(1-(q_r/q))$ for all $k \geq 0$, since $q_r \leq q$. Thus, $D_r \leq (1-q_r/q)\sum_{k=0}^{r-1}(k+1)a_k q_r^{-(k+1)}(\sum_{j=0}^{k} q_r^j|\alpha_{-j}|)$ by (6.12). Furthermore, since $q_r \leq q$, we have $q_r^j|\alpha_{-j}| \leq q^j|\alpha_{-j}| < K$ for all $j \geq 0$ by our assumption. Hence, we obtain $D_r \leq K(1-q_r/q)\sum_{k=0}^{r-1}(k+1)^2 a_k q_r^{-(k+1)}$. Consider the function T defined by $T(x) = \sum_{j=0}^{\infty}(j+1)^2 a_j x^{j+1}$, which is continuous on the interval $[0, R)$, since $T(x) = xS'(x)$, where S is the function defined by (6.8). Since $0 < q^{-1} < R$ by our assumption and $\lim_{r\to\infty} q_r = q$, there exists an $r_2 \geq r_0$ such that $0 < q^{-1} \leq q_r^{-1} < R$ for all $r \geq r_2$. As $q_r \leq q_{r'}$ whenever $r < r'$, we obtain

$$D_r \leq K\left(1-\frac{q_r}{q}\right)\sum_{k=0}^{r-1}(k+1)^2 a_k q_{r_2}^{-(k+1)} = KT(q_{r_2}^{-1})\left(1-\frac{q_r}{q}\right) = M_1\left(1-\frac{q_r}{q}\right) \tag{6.13}$$

for all $r \geq r_2$, where $M_1 = KT(q_{r_2}^{-1})$ is a positive constant.

For $E_r = |H_r - C_r|$, we have $E_r \leq \sum_{k=0}^{r-1} a_k q^{-(k+1)}(\sum_{j=0}^{k}|q_r^j - q^j||\alpha_{-j}|)$. Therefore,

$$\sum_{j=0}^{k}|q_r^j - q^j||\alpha_{-j}| = \sum_{j=0}^{k} q^j\left|1-\left(\frac{q_r}{q}\right)^j\right||\alpha_{-j}| \tag{6.14}$$

for every $k \geq 0$. Furthermore, since $0 < q_r \leq q$, we have $|1-(q_r/q)^j| \leq j(1-q_r/q)$. Hence, (6.1) together with (6.14) implies

$$\sum_{j=0}^{k}|q_r^j - q^j||\alpha_{-j}| \leq \left(1-\frac{q_r}{q}\right)\sum_{j=0}^{k} jq^j|\alpha_{-j}| \leq \frac{K}{2}(k+1)^2\left(1-\frac{q_r}{q}\right).$$

Then we have

$$E_r \leq \frac{K}{2}\left(1-\frac{q_r}{q}\right)\sum_{k=0}^{\infty}(k+1)^2 a_k q^{-(k+1)} = M_2\left(1-\frac{q_r}{q}\right), \tag{6.15}$$

where $M_2 = KT(q^{-1})/2$ is a positive constant.

By (6.11), (6.13), and (6.15), we have

$$|B_r - C_r| \le M\left(1 - \frac{q_r}{q}\right), \tag{6.16}$$

where $M = M_1 + M_2 > 0$. On the other hand, since

$$\sum_{k=0}^{r-1} a_k q^{-(k+1)} \left(\sum_{j=0}^{k} q^j |\alpha_{-j}| \right) \le K \sum_{k=0}^{r-1} (k+1) a_k q^{-(k+1)} \le KS(q^{-1}) < +\infty \tag{6.17}$$

by our assumptions, $\lim_{r\to\infty} C_r$ exists and is equal to

$$\sum_{k=0}^{\infty} a_k q^{-(k+1)} \left(\sum_{j=0}^{k} q^j \alpha_{-j} \right) = \sum_{j=0}^{\infty} \left(\sum_{k=j}^{\infty} a_k q^{j-k-1} \right) \alpha_{-j}, \tag{6.18}$$

since (6.17) shows that the above series converges absolutely. Thus, by (6.16) together with the fact that $q = \lim_{r\to\infty} q_r$, we see that $\lim_{r\to\infty} B_r$ exists and is equal to the value as in (6.18), which is nothing but the numerator of (6.6). □

Lemma 6.6: Assume that one of the three conditions of Theorem 6.1 is satisfied. Then (6.1) implies condition (C_∞).

Proof: By (6.1), for all $n \ge 1$, we have

$$\sum_{j=0}^{\infty} a_{j+n-1} |\alpha_{-j}| \le K \sum_{j=0}^{\infty} a_{j+n-1} q^{-j} = K q^{n-1} \sum_{j=0}^{\infty} a_{j+n-1} q^{-(j+n-1)} \le K q^n,$$

since we have $\sum_{j=0}^{\infty} a_j q^{-(j+1)} = 1$. Thus, condition (C_∞) is satisfied. □

Combining Theorem 5.2, Lemma 6.5, and Lemma 6.6, we obtain Theorem 6.2.

When $p = R$, we have a partial result as follows.

Proposition 6.7: Assume that one of the three conditions of Theorem 6.1 is satisfied, that $d_{r_1} = 1$ for some $r_1 \ge r_0$, that $\sum_{j=0}^{\infty} (j+1) a_j q^{-(j+1)} = +\infty$, and that the series $\sum_{j=0}^{\infty} q^j |\alpha_{-j}|$ converges. If the sequences $\{V_n^{(r)} / q_r^n\}_{n\ge1}$ are uniformly convergent for $r \ge r_1$, then V_n exists for all n and we have $\lim_{n\to\infty} V_n / q^n = 0$.

Note that the above condition implies that $p = R$ [see (6.9)].

Proof of Proposition 6.7: Since we have $q \ge q_r$, we see easily that the numerator B_r of (6.5) satisfies

$$|B_r| \le \sum_{j=0}^{r-1} q_r^j |\alpha_{-j}| \le \sum_{j=0}^{r-1} q^j |\alpha_{-j}| \le \sum_{j=0}^{\infty} q^j |\alpha_{-j}| < +\infty. \tag{6.19}$$

The result now follows from Theorem 5.2, (6.5), Lemma 6.6, and (6.19). □

Remark 6.8: Results similar to Theorem 6.2 and Proposition 6.7 were obtained in Theorem 3.2 of [11] by using the Markov chain method. See, also, Theorem 3.10 of [8].

Problem 6.9: We do not know if $d_\infty = \gcd\{i+1 : a_i > 0\} = 1$ ($\Leftrightarrow d_{r_1} = 1$ for some $r_1 \ge r_0$) implies that $L = \lim_{n\to\infty} V_n / q^n$ exists in general. Note that in some special cases $d_\infty = 1$ if and only if $\lim_{n\to\infty} V_n / q^n$ exists, as was shown in [11].

7. EXAMPLE

Let us give an explicit example of our main theorem of the previous section.

Fix a real number $\alpha^{-1} = \beta > 1$ and set $\alpha_r^{-1} = \beta_r = \beta^{1-(1/r!)}$ for $r \geq 1$. Consider the sequence of real polynomials $\{U_r(x)\}_{r\geq 1}$ defined inductively by

$$U_1(x) = 2x - 2\beta_1, \tag{7.1}$$

$$U_{r+1}(x) = xU_r(x) - \beta_{r+1}U_r(\beta_{r+1}) \quad (r \geq 1). \tag{7.2}$$

Therefore, we have $U_r(x) = 2x^r - a_0x^{r-1} - \cdots - a_{r-2}x - a_{r-1}$ for some strictly positive real numbers a_j $(j \geq 0)$. Note that β_r is the unique positive real root of $U_r(x)$. Set $W_r(x) = 2 - a_0x - \cdots - a_{r-2}x^{r-1} - a_{r-1}x^r = x^rU_r(x^{-1})$. Then we have $W_r(0) = 2$ and $W_r(\alpha_r) = 0$. Furthermore, we set $W(x) = 2 - \sum_{j=0}^{\infty} a_jx^{j+1}$.

Lemma 7.1: We have $W(\alpha) = 0$ and $0 < \alpha \leq R$, where R is the radius of convergence of W.

Proof: Since $W_r(\alpha_r) = 0$ and $a_j = \beta_{j+1}U_j(\beta_{j+1}) \leq 2\beta_{j+1}^{j+1} \leq 2\beta^{j+1} = 2\alpha^{-(j+1)}$, we get $W_r(\alpha) = W_r(\alpha) - W_r(\alpha_r) = a_0(\alpha_r - \alpha) + a_1(\alpha_r^2 - \alpha^2) + \cdots + a_{r-1}(\alpha_r^r - \alpha^r)$. Thus,

$$\begin{aligned} W_r(\alpha) &\leq 2(\alpha_r - \alpha)/\alpha + 2(\alpha_r^2 - \alpha^2)/\alpha^2 + \cdots + 2(\alpha_r^r - \alpha^r)/\alpha^r \\ &= 2(\beta^{1/r!} - 1) + 2(\beta^{2/r!} - 1) + \cdots + 2(\beta^{r/r!} - 1). \end{aligned}$$

Therefore, we have

$$W_r(\alpha) \leq 2r(\beta^{1/(r-1)!} - 1) = (2r/(r-1)!)(r-1)!(\beta^{1/(r-1)!} - 1) \to 0 \quad (r \to \infty).$$

Thus, $W(\alpha) = \lim_{r\to\infty} W_r(\alpha) = 0$. □

Set $Q_r(x) = W_r(x) - 1$ and $Q(x) = W(x) - 1$. Then, for each $r \geq 1$, there exists a unique positive real root p_r of Q_r. Furthermore, by Theorem 6.1, $p = \lim_{r\to\infty} p_r$ exists and $Q(p) = 0$. Set $q_r = p_r^{-1}$ and $q = p^{-1}$ and note that $0 < p < R$, where R coincides with the radius of convergence of Q.

Lemma 7.2:

$$\lim_{r\to\infty} \left| \frac{p_r^r}{p^r} - 1 \right| = 0. \tag{7.3}$$

Proof: Let us fix an $r \geq 1$ for the moment. The functions $W(x)$ and $W_r(x)$ defined on the intervals $[0, d)$ and $[0, \infty)$, respectively, are differentiable with strictly negative derivatives. Let us denote by $g : (0, 2] \to [0, d)$ and $g_r : (-\infty, 2] \to [0, \infty)$, respectively, their inverse functions. Then define the differentiable function $f : (0, 2] \to \mathbf{R}$ by $f(y) = g(y)^r - g_r(y)^r$. For $y \in (0, 2)$, set $x = g(y)$ and $x_r = g_r(y)$. Then we obtain $x_r \geq x > 0$ and

$$\begin{aligned} -\frac{W'(x)}{x^{r-1}} &= \frac{a_0}{x^{r-1}} + 2\frac{a_1}{x^{r-2}} + \cdots + (r-1)\frac{a_{r-2}}{x} + ra_{r-1} + (r+1)a_rx + \cdots \\ &\geq \frac{a_0}{x_r^{r-1}} + 2\frac{a_1}{x_r^{r-2}} + \cdots + (r-1)\frac{a_{r-2}}{x_r} + ra_{r-1} = -\frac{W_r'(x_r)}{x_r^{r-1}} > 0. \end{aligned} \tag{7.4}$$

Hence, by (7.4), we have $f'(y) = rx^{r-1}W'(x)^{-1} - rx_r^{r-1}W_r'(x)^{-1} \geq 0$. Thus, the function f is nondecreasing and we obtain $\alpha^r - \alpha_r^r = \lim_{y\to+0} f(y) \leq f(1) = p^r - p_r^r$. Therefore,

$$|p^r - p_r^r| = p_r^r - p^r \le |\alpha^r - \alpha_r^r|$$

for all $r \ge 1$. Then we have

$$\left|\frac{p_r^r}{p^r} - 1\right| \le \left(\frac{\alpha}{p}\right)^r \left|\frac{\alpha_r^r}{\alpha^r} - 1\right| = \left(\frac{\alpha}{p}\right)^r |\beta^{1/(r-1)!} - 1| = \left(\frac{\alpha}{p}\right)^r \frac{1}{(r-1)!} \frac{|\beta^{1/(r-1)!} - 1|}{1/(r-1)!}. \tag{7.5}$$

Since $\lim_{r\to\infty}(\alpha/p)^r/(r-1)! = 0$ and $\lim_{r\to\infty}|\beta^{1/(r-1)!} - 1|(r-1)! = \ln\beta$, equation (7.3) holds. □

Let $\{V_n\}_{n\in Z}$ be the ∞-GFS defined by $V_n = q^n$. Let us show that the conditions of Theorem 6.2 are satisfied for this sequence. Recall that we denoted $x_n^{(r)} = V_n^{(r)}/q_r^n$; see Theorem 5.2.

Lemma 7.3: The sequences $\{x_n^{(r)}\}_{n\ge1}$ are uniformly convergent for $r \ge 1$.

Proof: By Lemma 7.2, for a given $\varepsilon > 0$, there exists an $r_2 > 0$ such that $|p^r/p_r^r - 1| < \varepsilon/2$ for all $r \ge r_2$. Let us fix an r with $r \ge r_2$. Then, by (3.1), for every n with $-r+1 \le n \le 0$, we have

$$|x_n^{(r)} - 1| = \left|\frac{V_n^{(r)}}{q_r^n} - 1\right| = \left|\frac{q^n}{q_r^n} - 1\right| \le \left|\left(\frac{q}{q_r}\right)^{-r} - 1\right| = \left|\frac{p^r}{p_r^r} - 1\right| < \frac{\varepsilon}{2}. \tag{7.6}$$

Suppose $|x_k^{(r)} - 1| < \varepsilon/2$ for all k with $-r+1 \le k \le n$, where $n \ge 0$. Then, by (6.4) and the relation $x_{n+1}^{(r)} = (a_0/q_r)x_n^{(r)} + (a_1/q_r^2)x_{n-1}^{(r)} + \cdots + (a_{r-1}/q_r^r)x_{n-r+1}^{(r)}$, we have

$$|x_{n+1}^{(r)} - 1| = \left|\frac{a_0}{q_r}(x_n^{(r)} - 1)\right| + \left|\frac{a_1}{q_r^2}(x_{n-1}^{(r)} - 1)\right| + \cdots + \left|\frac{a_{r-1}}{q_r^r}(x_{n-r+1}^{(r)} - 1)\right| < \frac{\varepsilon}{2}. \tag{7.7}$$

Thus, by induction, we see that $|x_n^{(r)} - 1| < \varepsilon/2$ for all n, provided that $r \ge r_2$.

On the other hand, by Lemma 6.4, $L_r = \lim_{n\to\infty} x_n^{(r)}$ exists for all $r \ge 1$ and we can check that $\lim_{r\to\infty} L_r = 1$ by using (6.5). Hence, there exists an $r_3 \ge r_2$ such that $|L_r - 1| < \varepsilon/2$ for all $r \ge r_3$. Therefore, for all $r \ge r_3$ and all $n \ge 1$, we have $|x_n^{(r)} - L_r| \le |x_n^{(r)} - 1| + |1 - L_r| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since we have only a finite number of r's with $r_3 > r \ge 1$, there exists an N such that $|x_n^{(r)} - L_r| < \varepsilon$ for all $n \ge N$ and all r with $r_2 > r \ge 1$. Thus, we have proved that the sequences $\{x^{(r)}\}_{n\ge1}$ are uniformly convergent for $r \ge 1$. □

Therefore, we have shown that all the conditions in Theorem 6.2 are satisfied. On the other hand, we see easily that

$$\lim_{n\to\infty}\frac{V_n}{q^n} = \frac{\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} a_k q^{j-k-1}\right)q^{-j}}{\sum_{j=0}^{\infty}(j+1)a_j q^{-(j+1)}} = 1. \tag{7.8}$$

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Russ Euler and Jawad Sadek

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If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2001. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-916 ***Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria***

Determine the value of $\prod_{k=0}^{n} (L_{2\cdot 3^k} - 1)$.

B-917 ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain***

Find the following sums:

(a) $\displaystyle\sum_{n\geq 0} \frac{1+L_{n+1}}{L_n L_{n+2}}$, *(b)* $\displaystyle\sum_{n\geq 0} \frac{L_{n-1}L_{n+2}}{L_n^2 L_{n+1}^2}$,

where L_k is the k^{th} Lucas number.

B-918 ***Proposed by M. N. Deshpande, Institute of Science, Nagpur, India***

Let i and j be positive integers such that $1 \leq j \leq i$. Let

$$T(i, j) = F_j F_{i-j+1} + F_j F_{i-j+2} + F_{j+1} F_{i-j+1}.$$

Determine whether or not

$$\underset{j}{\text{maximum}}\ T(i, j) - \underset{j}{\text{minimum}}\ T(i, j)$$

is divisible by 2 for all $i \geq 3$.

B-919 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***

Solve the equation $L_nF_{n+1} = p^m(p^m - 1)$, where m and n are natural numbers and p is a prime number.

B-920 ***Proposed by N. Gauthier, Royal Military College of Canada***

Prove that

$$\sum_{n=1}^{\infty} \sin\left(\frac{p\pi}{2} \cdot \frac{F_{n-1}}{F_nF_{n+1}}\right) \cos\left(\frac{p\pi}{2} \cdot \frac{F_{n+2}}{F_nF_{n+1}}\right) = 0$$

for p an arbitrary integer.

SOLUTIONS

It's A Toss

B-899 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY (Vol. 38, no. 2, May 2000)***

In a sequence of coin tosses, a *single* is a term (H or T) that is not the same as any adjacent term. For example, in the sequence HHTHHHTH, the singles are the terms in positions 3, 7, and 8. Let $S(n, r)$ be the number of sequences of n coin tosses that contain exactly r singles. If $n \geq 0$ and p is a prime, find the value modulo p of $\frac{1}{2}S(n+p-1,\ p-1)$.

Solution by the proposer

Answer: $\frac{1}{2}S(n+p-1, p-1) \equiv \begin{cases} 0, & \text{if } p \nmid n, \\ F_{m-1}, & \text{if } n = mp, \end{cases} \pmod{p}.$

Proof (with a few omissions): For $n = 0$, both sides equal 1; for $n = 1$, both sides are zero. Assuming $n \geq 2$, the n nonsingles in the sequence must appear in k blocks of lengths ≥ 2, where $1 \leq k \leq n/2$. For fixed k, the number of ways to choose the corresponding sequence of k block lengths each ≥ 2 (with sum n) equals the number of ways to partition a string of length $n-k$ into k *nonempty* blocks, namely, $\binom{n-k-1}{k-1}$. Once the k block lengths are given, the sequence of tosses is determined by (1) our choice of which $p-1$ of the $k+p-1$ blocks (including the singles) are singles, and (2) whether the sequence begins with H or T. Hence,

$$\frac{1}{2}S(n+p-1, p-1) = \sum_{1 \leq k \leq n/2} \binom{k+p-1}{p-1}\binom{n-k-1}{k-1}. \tag{1}$$

In what follows, it will be convenient to use the notation $[a, b]$ for the product $a(a-1)(a-2)\cdots(b)$ when $a \geq b$.

If $p \nmid k$, the factor $\binom{k+p-1}{p-1} = \frac{[k+p-1,\, k+1]}{[p-1,\, 1]}$ in (1) is divisible by p, since p divides the numerator but not the denominator. If $p \mid k$ but $p \nmid n$ (say $k = jp$, $n-k-1 = qp-r$ where $2 \leq r \leq p$), then

$$\begin{aligned}\binom{n-k-1}{k-1} = \binom{qp-r}{jp-1} &= \frac{[(q-j)p-0, (q-j)p-(r-2)]}{[qp-1, qp-(r-1)]}\binom{qp-1}{jp-1} \\ &= \frac{(\text{multiple of } p)}{(\text{nonmultiple of } p)} \cdot (\text{integer}) = (\text{multiple of } p).\end{aligned}$$

Our conclusion follows in the case $p \nmid n$; for the case $n = mp$, we need consider only those summands in (1) for which $k = jp$. Thus, we must show that

$$\sum_{1\le j\le m/2} \binom{jp+p-1}{p-1}\binom{(m-j)p-1}{jp-1} \equiv F_{m-1} \pmod{p}. \tag{2}$$

Both sides of (2) are zero when $m=1$, so assume $m \ge 2$. First, note that $\binom{jp+p-1}{p-1} = \frac{[jp+p-1,\, jp+1]}{[p-1,\, 1]} \equiv 1$ (mod p) since corresponding factors of numerator and denominator are congruent (mod p) and are all nonzero (mod p). A more complicated argument of similar character (omitted here) shows that $\binom{(m-j)p-1}{jp-1} \equiv \binom{m-j-1}{j-1}$ (mod p). Hence, (2) reduces to

$$\binom{m-2}{0}+\binom{m-3}{1}+\cdots \equiv F_{m-1} \pmod{p}. \tag{3}$$

But the left side of (3) in fact *equals* F_{m-1} (well known and easily shown by induction). Q.E.D.

Also solved by Paul S. Bruckman and Kathleen Lewis.

Always Rational

B-900 *Proposed by Richard André-Jeannin, Cosnes et Romain, France (Vol. 38, no. 4, August 2000)*

Show that $\tan(2n\arctan(\alpha))$ is a rational number for every $n \ge 0$.

Solution by the proposer

We will use the well-known relation

$$\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x \cdot \tan y},$$

where the values x, y, $x+y$ are different from an odd multiple of $\frac{\pi}{2}$. Further, it is clear that $\tan(\arctan\alpha) = \alpha$.

Now we will prove the given statement by induction. Denote $D_n = \tan(2n \arctan \alpha)$ for every $n \ge 0$.

It is easy to see that $D_0 = 0$ and

$$D_1 = \tan(2 \arctan \alpha) = \frac{2\tan(\arctan \alpha)}{1-\tan^2(\arctan \alpha)} = \frac{2\alpha}{1-\alpha^2} = \frac{2\alpha}{1-(1+\alpha)} = -2.$$

Suppose that D_n is a rational number, then we will show that D_{n+1} is also a rational number. We can write

$$\begin{aligned} D_{n+1} &= \tan(2(n+1)\arctan \alpha) = \tan(2n \arctan \alpha + 2 \arctan \alpha) \\ &= \frac{\tan(2n \arctan \alpha)+\tan(2 \arctan \alpha)}{1-\tan(2n \arctan \alpha)\cdot\tan(2 \arctan \alpha)} = \frac{D_n + D_1}{1-D_n \cdot D_1} = \frac{D_n - 2}{1+2D_n}. \end{aligned}$$

But this means that D_{n+1} is also a rational number and the proof is finished. Moreover, we have found the recurrence for D_n.

It still remains to show that, for all natural n, the condition $D_n \ne -\frac{1}{2}$ holds. If n is even, then $n = 2k$, where k is a natural number. Then we have the relation

$$D_{2k} = \frac{2D_k}{1-D_k^2} = -\frac{1}{2},$$

which can be rewritten as a quadratic equation $D_k^2 - 4D_k - 1 = 0$. But its roots are irrational numbers $D_k = 2+\sqrt{5}$ and $D_k = 2-\sqrt{5}$. This contradicts the fact that D_k must be a rational number.

If n is odd, then $n = 4k+1$ or $n = 4k+3$, where k is a natural number. Similarly, we can show that, if $D_{4k+1} = -\frac{1}{2}$ or $D_{4k+3} = -\frac{1}{2}$, then D_k is an irrational number.

Also solved by Paul S. Bruckman, M. Deshpande, L. A. G. Dresel, Steve Edwards, Walther Janous, Harris Kwong, Kee-Wai Lau, Reiner Martin, Don Redmond, H.-J. Seiffert, and Indulis Strazdins.

Back to Euler

B-901 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France (Vol. 38, no. 4, August 2000)***

Let A_n be the sequence defined by $A_0 = 1$, $A_1 = 0$, $A_n = (n-1)(A_{n-1} + A_{n-2})$ for $n \geq 2$. Find

$$\lim_{n\to+\infty} \frac{A_n}{n!}.$$

Solution by Paul S. Bruckman, Berkeley, CA

This problem is an old one, and occurs in the study of the number of *derangements* of n objects. We may express this in the following manner: In how many ways can the ordered set of integers $\{1, 2, 3, \ldots, n\}$ be permuted so that, in the new arrangement, none of these integers lies in its natural order? The answer turns out to be A_n, and an interesting explicit expression for A_n may be derived, which follows:

$$A_n = n! \sum_{k=0}^{n} (-1)^k / k! \tag{1}$$

In the older literature, this expression is denoted $n!!$. Here, we simply verify that the expression for A_n in (1) satisfies the conditions of the problem. By a change in variable, the given recurrence relation becomes $A_{n+1} = n(A_n + A_{n-1})$. Note that the initial conditions are satisfied by (1) for $n = 0$ and 1. Now, assuming that (1) is true for n and $n-1$, we have

$$n(A_n + A_{n-1}) = n \cdot n! \sum_{k=0}^{n} (-1)^k / k! + n \cdot (n-1)! \sum_{k=0}^{n-1} (-1)^k / k! = (n+1) \cdot n! \sum_{k=0}^{n} (-1)^k / k! - n!(-1)^n / n!$$
$$= (n+1)! \sum_{k=0}^{n} (-1)^k / k! + (n+1)!(-1)^{n+1} / (n+1)! = (n+1)! \sum_{k=0}^{n+1} (-1)^k / k! = A_{n+1}.$$

Applying induction establishes (1). We then see that

$$\lim_{n\to\infty} A_n / n! = \sum_{k=0}^{\infty} (-1)^k / k! = e^{-1}.$$

The featured solution sums up all comments and solutions of the other solvers. Several solvers gave references as to where equality (1) can be found. Harris Kwong cited Combinatorial Mathematics *by H. J. Ryser, Kee-Wai Lau listed* The Encyclopedia of Integer Sequences, *by N. J. A. Sloane and Simon Plouffe, H.-J. Seiffert included* Discrete and Combinatorial Mathematics, *2nd Edition, Exercise 9 on page 402, and Indulis Strazdins (to whom we owe the title of this problem) mentioned* Introduction to Combinatorial Analysis *by J. Riordan, Chapter 3, and Exercise B-853. He also mentioned that the recursion can be traced back to Euler.*

Also solved by Michael S. Becker, M. Deshpande, L. A. G. Dresel, Walther Janous, Harris Kwong, Kee-Wai Lau, Reiner Martin, Helmut Prodinger, H.-J. Seiffert, Indulis Strazdins, and the proposer.

A Pell Polynomials Identity

B-902 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 38, no. 4, August 2000)

The Pell polynomials are defined by $P_0(x)=0$, $P_1(x)=1$, and $P_n(x)=2xP_{n-1}(x)+P_{n-2}(x)$ for $n\geq 2$. Show that, for all nonzero real numbers x and all positive integers n,

$$\sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}P_k(x)=x^{n-1}P_n(1/x).$$

Solution by Reiner Martin, New York, NY

It is well known (and can also easily be verified using induction) that

$$2\sqrt{x^2+1}\cdot P_n(x)=\left(x+\sqrt{x^2+1}\right)^n-\left(x-\sqrt{x^2+1}\right)^n.$$

Thus,

$$\begin{aligned}&2\sqrt{x^2+1}\cdot\sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}P_k(x)\\&=\sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}\left(x+\sqrt{x^2+1}\right)^k-\sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}\left(x-\sqrt{x^2+1}\right)^k\\&=\left(1+\sqrt{x^2+1}\right)^n-\left(1-\sqrt{x^2+1}\right)^n=2\sqrt{x^2+1}\cdot x^{n-1}P_n(1/x).\end{aligned}$$

Almost all other solvers used a similar method to prove the equality.

Also solved by Richard André-Jeannin, Paul S. Bruckman, Johan Cigler, L. A. G. Dresel, Walther Janous, Harris Kwong, Kee-Wai Lau, Helmut Prodinger, and the proposer.

An Old Generation Function

B-903 ***Proposed by the editor***
(Vol. 38, no. 4, August 2000)

Find a closed form for $\sum_{n=0}^{\infty}F_n^2x^n$.

Solution by Walther Janous, Innsbruck, Austria

Because of $F_n^2=\frac{1}{5}(\alpha^n-\beta^n)^2=\frac{1}{5}(\alpha^{2n}+\beta^{2n}-2(-1)^n)$, we infer

$$\begin{aligned}\sum_{n=0}^{\infty}F_n^2x^n&=\frac{1}{5}\sum_{n=0}^{\infty}((\alpha^2x)^n+(\beta^2x)^n-2(-x)^n)\\&=\frac{1}{5}\left(\frac{1}{1-\alpha^2x}+\frac{1}{1-\beta^2x}-\frac{2}{1+x}\right)=\frac{1}{5}\cdot\frac{-80x(x-1)}{16(x+1)(x^2-3x+1)}\\&=\frac{x(1-x)}{(x+1)(x^2-3x+1)}\quad\left[=-\frac{x^2-x}{x^3-2x^2-2x+1}\right].\end{aligned}$$

(The domain of convergence of this expression is $\{x/|x|<\frac{1}{\alpha^2}\}$.)

The problem is well known, as some solvers pointed out. Harris Kwong mentioned some references where the problem had been generalized to any power of F_n [see J. Riordan, "Generating Functions for Powers of Fibonacci Numbers," *Duke Math. J.* **29** (1962):5-12] *and even extended* *to greater generality* [see L. Carlitz, "Generating Functions for Powers of Certain Sequences of Numbers," *Duke Math J.* **29** (1962):521-38.] *Some solvers used Maple to produce solutions for up to the 10th power of F_n. Richard André-Jeannin mentioned an article by Verner E. Hoggatt, Jr.* ["Some Special Fibonacci and Lucas Generating Functions," *The Fibonacci Quarterly* **9.2** (1971):121-23] *and H.-J. Seiffert commented that the answer to this problem was given in the solution to B-452* [*The Fibonacci Quarterly* **20.3** (1982):280-81]. *Pentti Haukkanen also cited three additional references.*

Also solved by Richard André-Jeannin, Paul S. Bruckman, Charles K. Cook, M. Deshpande, L. A. G. Dresel, Pentti Haukkanen, Harris Kwong, Joe Lewis, Reiner Martin, Jalis Morrison, Helmut Prodinger, Maitland Rose, Don Redmond, H.-J. Seiffert, Pantelimon Stănică, Indulis Strazdins, and the proposer.

A Fibonacci-Lucas Equality

B-904 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France (Vol. 38, no. 4, August 2000)***

Find the positive integers n and m such that $F_n = L_m$.

Solution by Harris Kwong, Fredonia, NY

For $m \geq 3$, we have $F_{m+1} < L_m < F_{m+2}$, because

$$L_m = F_{m+1} + F_{m-1} \geq F_{m+1} + F_2 > F_{m+1} \quad \text{and} \quad L_m = F_{m+1} + F_{m-1} < F_{m+1} + F_m = F_{m+2}.$$

Therefore, $F_n = L_m$ only when $m < 3$. The only solutions are $(n, m) = (1, 1), (2, 1), (4, 2)$.

Also solved by Richard André-Jeannin, Brian D. Beasley, Paul S. Bruckman, L. A. G. Dresel, Walther Janous, Lake Superior Problem Solving Group, Reiner Martin, Ibrahim Al-Pasari, H.-J. Seiffert, Indulis Strazdins, and the proposer.

A Three-Term Sum

B-905 ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain (Vol. 38, no. 4, August 2000)***

Let n be a positive integer greater than or equal to 2. Determine

$$\frac{(F_n^2+1)F_{n+1}F_{n+2}}{(F_{n+1}-F_n)(F_{n+2}-F_n)} + \frac{F_n(F_{n+1}^2+1)F_{n+2}}{(F_n-F_{n+1})(F_{n+2}-F_{n+1})} + \frac{F_nF_{n+1}(F_{n+2}^2+1)}{(F_n-F_{n+2})(F_{n+1}-F_{n+2})}.$$

Solution by Maitland A. Rose, Sumter, SC

If we replace $(F_{n+2}-F_n)$ by F_{n+1}, $(F_{n+2}-F_{n+1})$ by F_n, and simplify, the expression becomes

$$\frac{(F_n^2+1)F_{n+2}}{(F_{n+1}-F_n)} - \frac{(F_{n+1}^2+1)F_{n+2}}{(F_{n+1}-F_n)} + (F_{n+2}^2+1) = -\frac{(F_{n+1}^2-F_n^2)F_{n+2}}{(F_{n+1}-F_n)} + (F_{n+2}^2+1)$$

$$= -(F_{n+1}+F_n)F_{n+2} + (F_{n+2}^2+1) = -F_{n+2}^2 + F_{n+2}^2 + 1 = 1.$$

Also solved by Richard André-Jeannin, Brian D. Beasley, Scott H. Brown, Paul S. Bruckman, Julie Clark, Charles K. Cook, M. Deshpande, L. A. G. Dresel, Walther Janous, Harris Kwong, Carl Libis, Reiner Martin, H.-J. Seiffert, James Sellers, Pantelimon Stănică, Indulis Strazdins, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-572 ***Proposed by Paul S. Bruckman, Berkeley, CA***

Prove the following, where $\varphi = \alpha^{-1}$:

$$\sum_{n=0}^{\infty}\{\varphi^{5n+1}/(5n+1)+\varphi^{5n+3}/(5n+2)-\varphi^{5n+4}/(5n+3)-\varphi^{5n+4}/(5n+4)\}=(\pi/25)(50-10\sqrt{5})^{1/2}.$$

H-573 ***Proposed by N. Gauthier, Royal Military College of Canada***

"By definition, a magic matrix is a square matrix whose lines, columns, and two main diagonals all add up to the same sum. Consider a 3×3 magic matrix Φ whose elements are the following combinations of the n^{th} and $(n+1)^{\text{th}}$ Fibonacci numbers:

$$\begin{array}{lll}\Phi_{11}=3F_{n+1}+F_n; & \Phi_{12}=F_{n+1}; & \Phi_{13}=2F_{n+1}+2F_n;\\ \Phi_{21}=F_{n+1}+2F_n; & \Phi_{22}=2F_{n+1}+F_n; & \Phi_{23}=3F_{n+1};\\ \Phi_{31}=2F_{n+1}; & \Phi_{32}=3F_{n+1}+2F_n; & \Phi_{33}=F_{n+1}+F_n.\end{array}$$

Find a closed-form expression for Φ^m, where m is a *positive* integer, and determine all the values of m for which it too is a magic matrix."

SOLUTIONS

Geometric?

H-561 ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada (Vol. 38, no. 2, May 2000)***

Let n be an integer and set $s_{n+1}=\alpha^n+\alpha^{n-1}\beta+\cdots+\alpha\beta^{n-1}+\beta^n$, where $\alpha+\beta=a$, $\alpha\beta=b$, with $a\neq 0$, $b\neq 0$ two arbitrary parameters. Then prove that:

(a) $s_p^r s_{qr+n}=\sum_{\ell=0}^{r}\binom{r}{\ell}b^{q(r-\ell)}s_q^\ell s_{p-q}^{r-\ell}s_{p\ell+n}$;

(b) $b^{pr}s_q^r s_n=\sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}s_p^\ell s_{q+p}^{r-\ell}s_{q\ell+pr+n}$;

(c) $s_{2p+q}^r s_{qr+n}=\sum_{\ell=0}^{r}\binom{r}{\ell}b^{(p+q)(r-\ell)}s_p^{r-\ell}s_{p+q}^\ell s_{(2p+q)\ell-pr+n}$;

where $r\geq 0$, n, $p(\neq 0)$, and $q(\neq 0,\pm p)$ are arbitrary integers.

Solution by Paul S. Bruckman, Berkeley, CA

Let the sums indicated in the right-hand members of parts (a), (b), and (c) be denoted by $\mathcal{A}=\mathcal{A}(p,q,r,n)$, $\mathcal{B}=\mathcal{B}(p,q,r,n)$, $\mathcal{C}=\mathcal{C}(p,q,r,n)$, respectively. We distinguish two possibilities: Case 1, where $a^2 \neq 4b$, i.e., $\alpha \neq \beta$; Case 2, where $a^2 = 4b$, i.e., $\alpha = \beta$. For Case 1, the Binet formula holds: $s_n = (\alpha^n - \beta^n)/(\alpha-\beta)$; for Case 2, we have: $s_n = n\alpha^{n-1}$, in either case for all integers n. We first deal with Case 1; we also make the substitution $\theta = \alpha - \beta$. The sums may be evaluated by application of the appropriate formula for s_n and the Binomial theorem. We will also employ some auxiliary results, given in the form of lemmas.

Lemma 1: $b^q s_{p-q} + \alpha^p s_q = \alpha^q s_p$; $b^q s_{p-q} + \beta^p s_q = \beta^q s_p$.

Proof:

$$b^q s_{p-q} + \alpha^p s_q = \{(\alpha\beta)^q(\alpha^{p-q} - \beta^{p-q}) + \alpha^p(\alpha^q - \beta^q)\}/\theta = \alpha^p(\alpha^q-\beta^q)/\theta = \alpha^q s_p;$$

also,

$$b^q s_{p-q} + \beta^p s_q = \{(\alpha\beta)^q(\alpha^{p-q} - \beta^{p-q}) + \beta^p(\alpha^q - \beta^q)\}/\theta = \beta^p(\alpha^q-\beta^q)/\theta = \beta^q s_p. \ \square$$

Lemma 2: $s_{p+q} - \alpha^q s_p = \beta^p s_q$; $s_{p+q} - \beta^q s_p = \alpha^p s_q$.

Proof: In Lemma 1, replace p by $p+q$, then divide by α^q (or β^q). $\square$

Lemma 3: $\beta^{p+q}s_p + \alpha^p s_{p+q} = \alpha^{p+q}s_p + \beta^p s_{p+q} = s_{2p+q}$.

Proof: Replace q by $p+q$ in Lemma 2. $\square$

We may now proceed to the proof of the problem, at least for Case 1.

(a)
$$\begin{aligned}\mathcal{A} &= \sum_{k=0}^{r} {}_rC_k b^{q(r-k)}(s_q)^k(s_{p-q})^{r-k}(\alpha^{pk+n} - \beta^{pk+n})/\theta \\ &= (\alpha^n/\theta)(b^q s_{p-q} + \alpha^p s_q)^r - (\beta^n/\theta)(b^q s_{p-q} + \beta^p s_q)^r \\ &= (\alpha^n/\theta)(\alpha^q s_p)^r - (\beta^n/\theta)(\beta^q s_p)^r \ \text{(using Lemma 1)} \\ &= (s_p)^r(\alpha^{qr+n} - \beta^{qr+n})/\theta = (s_p)^r s_{qr+n}. \ \square\end{aligned}$$

(b)
$$\begin{aligned}\mathcal{B} &= \sum_{k=0}^{r}(-1)^k {}_rC_k(s_p)^k(s_{p+q})^{r-k}(\alpha^{qk+pr+n} - \beta^{qk+pr+n})/\theta \\ &= (\alpha^{pr+n}/\theta)(s_{p+q} - \alpha^q s_p)^r - (\beta^{pr+n}/\theta)(s_{p+q} - \beta^q s_p)^r \\ &= \alpha^{pr+n}/\theta)\beta^{pr}(s_q)^r - (\beta^{pr+n}/\theta)\alpha^{pr}(s_q)^r \ \text{(using Lemma 2)} \\ &= b^{pr}(s_q)^r(\alpha^n - \beta^n)/\theta = b^{pr}(s_q)^r s_n. \ \square\end{aligned}$$

(c)
$$\begin{aligned}\mathcal{C} &= \sum_{k=0}^{r} {}_rC_k b^{(p+q)(r-k)}(s_p)^{r-k}(s_{p+q})^k(\alpha^{(2p+q)k-pr+n} - \beta^{(2p+q)k-pr+n})/\theta \\ &= (\alpha^{-pr+n}/\theta)(b^{p+q}s_p + \alpha^{2p+q}s_{p+q})^r - (\beta^{-pr+n}/\theta)(b^{p+q}s_p + \beta^{2p+q}s_{p+q})^r \\ &= (\alpha^{-pr+n+pr+qr}/\theta)(\beta^{p+q}s_p + \alpha^p s_{p+q})^r - (\beta^{-pr+n+pr+qr}/\theta)(\alpha^{p+q}s_p + \beta^p s_{p+q})^r;\end{aligned}$$

hence, using Lemma 3, $\mathcal{C} = (\alpha^{qr+n}/\theta)(s_{2p+q})^r - (\beta^{qr+n}/\theta)(s_{2p+q})^r = (s_{2p+q})^r s_{qr+n}$. $\square$

It remains to prove (a), (b), and (c) for Case 2. We first show that Lemmas 1, 2, and 3 are valid for this case as well; the modified versions of these lemmas are denoted by a "prime" punctuation.

Lemma 1': $b^q s_{p-q} + \alpha^p s_q = \alpha^q s_p$.

Proof: $b^q s_{p-q} + \alpha^p s_q = (p-q)\alpha^{2q}(\alpha^{p-q-1}) + q\alpha^p(\alpha^{q-1}) = p\alpha^{p+q-1} = p\alpha^{p-1}\alpha^q = \alpha^q s_p$. □

Lemma 2': $s_{p+q} - \alpha^q s_p = \alpha^p s_q$.

Proof: Replace p by $p+q$ in Lemma 1′ and divide by α^q. □

Lemma 3': $\alpha^{p+q} s_p + \alpha^p s_{p+q} = s_{2p+q}$.

Proof: Replace q by $p+q$ in Lemma 2′.

(a)'
$$
\begin{aligned}
\mathscr{A} &= \sum_{k=0}^{r} {}_rC_k\, b^{q(r-k)}(s_q)^k (s_{p-q})^{r-k}(pk+n)\alpha^{pk+n-1} \\
&= pr\sum_{k=1}^{r} {}_{r-1}C_{k-1} b^{q(r-k)}(s_q)^k(s_{p-q})^{r-k}\alpha^{pk+n-1} + n\sum_{k=0}^{r} {}_rC_k\, b^{q(r-k)}(s_q)^k(s_{p-q})^{r-k}\alpha^{pk+n-1} \\
&= pr\sum_{k=0}^{r-1} {}_{r-1}C_k\, b^{q(r-1-k)}(s_q)^{k+1}(s_{p-q})^{r-1-k}\alpha^{pk+p+n-1} + n\sum_{k=0}^{r} {}_rC_k\, b^{q(r-k)}(s_q)^k(s_{p-q})^{r-k}\alpha^{pk+n-1} \\
&= prs_q\alpha^{p+n-1}\{b^q s_{p-q} + \alpha^p s_q\}^{r-1} + n\alpha^{n-1}\{b^q s_{p-q} + \alpha^p s_q\}^r \\
&= pr\alpha^{p+n-1+q(r-1)} s_q (s_p)^{r-1} + n\alpha^{n-1+qr}(s_p)^r \quad \text{(using Lemma 1′)} \\
&= \alpha^{p+n-2+qr}(s_p)^{r-1}(pqr+np) = p\alpha^{p-1}(qr+n)\alpha^{qr+n-1}(s_p)^{r-1} = (s_p)^r s_{qr+n}. \quad \square
\end{aligned}
$$

(b)'
$$
\begin{aligned}
\mathscr{B} &= \sum_{k=0}^{r}(-1)^k\, {}_rC_k (s_p)^k (s_{p+q})^{r-k}(qk+pr+n)\alpha^{qk+pr+n-1} \\
&= -qrs_p\alpha^{q+pr+n-1}\sum_{k=0}^{r-1}(-1)^k\, {}_{r-1}C_k(s_p)^k(s_{p+q})^{r-1-k}\alpha^{qk} \\
&\quad + (pr+n)\alpha^{pr+n-1}\sum_{k=0}^{r}(-1)^k\, {}_rC_k(s_p)^k(s_{p+q})^{r-k}\alpha^{qk} \\
&= -qrs_p\alpha^{q+pr+n-1}(s_{p+q} - \alpha^q s_p)^{r-1} + (pr+n)\alpha^{pr+n-1}(s_{p+q} - \alpha^q s_p)^r \\
&= -qrs_p\alpha^{q+pr+n-1}(\alpha^p s_q)^{r-1} + (pr+n)\alpha^{pr+n-1}(\alpha^p s_q)^r \quad \text{(using Lemma 2′)} \\
&= \{-rs_q s_p\alpha^{pr+n} + (pr+n)s_q\alpha^{pr+n-1+p}\}(\alpha^p s_q)^{r-1} \\
&= \{-rs_p\alpha^{2pr+n-p} + (pr+n)\alpha^{2pr+n-1}\}(s_q)^r = \{-pr\alpha^{2pr+n-1} + (pr+n)\alpha^{2pr+n-1}\}(s_q)^r \\
&= n\alpha^{2pr+n-1}(s_q)^r = \alpha^{2pr}(s_q)^r s_n = b^{pr}(s_q)^r s_n. \quad \square
\end{aligned}
$$

(c)':
$$
\begin{aligned}
\mathscr{C} &= \sum_{k=0}^{r} {}_rC_k\, b^{(p+q)(r-k)}(s_p)^{r-k}(s_{p+q})^k\{(2p+q)k - pr + n\}\alpha^{(2p+q)k-pr+n-1} \\
&= (2p+q)r\alpha^{2p+q-pr+n-1}s_{p+q}\sum_{k=0}^{r-1} {}_{r-1}C_k\, b^{(p+q)(r-1-k)}(s_p)^{r-1-k}(s_{p+q})^k\alpha^{(2p+q)k} +
\end{aligned}
$$

$$
\begin{aligned}
&+(-pr+n)\alpha^{-pr+n-1}\sum_{k=0}^{r} {}_rC_k\, b^{(p+q)(r-k)}(s_p)^{r-k}(s_{p+q})^k\alpha^{(2p+q)k}\\
&=(2p+q)r\alpha^{2p+q-pr+n-1}s_{p+q}(b^{p+q}s_p+\alpha^{2p+q}s_{p+q})^{r-1}\\
&\quad+(-pr+n)\alpha^{-pr+n-1}(b^{p+q}s_p+\alpha^{2p+q}s_{p+q})^r\\
&=(2p+q)r\alpha^{2p+q-pr+n-1}\alpha^{(p+q)(r-1)}s_{p+q}(\alpha^{p+q}s_p+\alpha^p s_{p+q})^{r-1}\\
&\quad+(-pr+n)\alpha^{-pr+n-1}\alpha^{(p+q)r}(\alpha^{p+q}s_p+\alpha^p s_{p+q})^r\\
&=(2p+q)r\alpha^{p-1+qr+n}s_{p+q}(s_{2p+q})^{r-1}+(-pr+n)\alpha^{qr+n-1}(s_{2p+q})^r\\
&=r\alpha^{p-1+qr+n-2p-q+1}s_{p+q}(s_{2p+q})^r+(-pr+n)\alpha^{qr+n-1}(s_{2p+q})^r\\
&=r(p+q)\alpha^{-p+qr+n-q+p+q-1}(s_{2p+q})^r+(-pr+n)\alpha^{qr+n-1}(s_{2p+q})^r\\
&=r(p+q)\alpha^{qr+n-1}(s_{2p+q})^r+(-pr+n)\alpha^{qr+n-1}(s_{2p+q})^r\\
&=(qr+n)\alpha^{qr+n-1}(s_{2p+q})^r=(s_{2p+q})^r s_{qr+n}.\ \square
\end{aligned}
$$

This completes the proof of the problem.

Also solved by H. Kwong, H.-J. Seiffert, and the proposer.

Greatest Problem

H-562 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 38, no. 2, May 2000)-corrected***

Show that, for all nonnegative integers n,

$$F_{2n+1}=4^n-5\sum_{k=0}^{\left[\frac{n-2}{5}\right]}\binom{2n+1}{n-5k-2},$$

where $[\cdot]$ denotes the greatest integer function.

Solution by the proposer

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, and $F_{n+2}(x)=xF_{n+1}(x)+F_n(x)$, for $n\in N_0$ (natural numbers).

Let $A_n:=i^{n-1}F_n(i\alpha)$, $n\in N_0$, where $i=\sqrt{(-1)}$. Writing $n=5m+r$, where $m\in N_0$ and $r\in\{0,1,2,3,4\}$, a simple induction proof on m yields

$$A_n=\begin{cases}0 & \text{if } n\equiv 0 \pmod 5,\\ 1 & \text{if } n\equiv 1 \pmod 5,\\ -\alpha & \text{if } n\equiv 2 \pmod 5,\\ \alpha & \text{if } n\equiv 3 \pmod 5,\\ -1 & \text{if } n\equiv 4 \pmod 5.\end{cases} \tag{1}$$

From H-518 [identity (8)], we know that, for all complex numbers x and all nonnegative integers n,

$$\sum_{j=0}^{n}(-1)^{j-1}\binom{2n}{n-j}F_j(x)^2=\frac{4^n-(-x^2)^n}{4+x^2}.$$

With $x = i\alpha$, this gives

$$\sum_{j=0}^{n}\binom{2n}{n-j}A_n^2 = \frac{4^n - \alpha^{2n}}{4-\alpha^2}. \tag{2}$$

Since $1/(4-\alpha^2) = \alpha/\sqrt{5}$, we have

$$\frac{4^n - \alpha^{2n}}{4-\alpha^2} = \frac{1}{\sqrt{5}}(4^n\alpha - \alpha^{2n+1}),$$

or, by $\alpha^r = (L_r + \sqrt{5}F_r)/2$, $r \in Z$,

$$\frac{4^n - \alpha^{2n}}{4-\alpha^2} = \frac{\sqrt{5}}{10}(4^n - L_{2n+1}) + \frac{1}{2}(4^n - F_{2n+1}). \tag{3}$$

On the other hand, from (1), it follows that

$$\sum_{j=0}^{n}\binom{2n}{n-j}A_j^2 = \sum_{k=0}^{[\frac{n-1}{5}]}\binom{2n}{n-5k-1} + \alpha^2\sum_{k=0}^{[\frac{n-2}{5}]}\binom{2n}{n-5k-2}$$
$$+\alpha^2\sum_{k=0}^{[\frac{n-3}{5}]}\binom{2n}{n-5k-3} + \sum_{k=0}^{[\frac{n-4}{5}]}\binom{2n}{n-5k-4}.$$

Using $\alpha^2 = (3+\sqrt{5})/2$, we then obtain

$$\sum_{j=0}^{n}\binom{2n}{n-j}A_j^2 = \frac{\sqrt{5}}{2}\left(\sum_{k=0}^{[\frac{n-2}{5}]}\binom{2n}{n-5k-2} + \sum_{k=0}^{[\frac{n-3}{5}]}\binom{2n}{n-5k-3}\right) + q_n,$$

where q_n is a rational number. Since $\sqrt{5}$ is irrational, (2), (3), and the latter equation imply that

$$\sum_{k=0}^{[\frac{n-2}{5}]}\binom{2n}{n-5k-2} + \sum_{k=0}^{[\frac{n-3}{5}]}\binom{2n}{n-5k-3} = \frac{1}{5}(4^n - L_{2n+1}).$$

This proves the desired identity, because

$$\binom{2n}{n-5k-2} + \binom{2n}{n-5k-3} = \binom{2n+1}{n-5k-2},\quad 0 \le k \le \left[\frac{n-2}{5}\right],$$

where we set $\binom{2n}{j} = 0$ for $j < 0$.

Also solved by H. Kwong and P. Bruckman.

A Stirling Problem

<u>H-563</u> ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada (Vol. 38, no. 2, May 2000)***

Let $m > 0$, $n \ge 0$, $p \ne 0$, $q \ne -p, 0$, and s be integers and, for $1 \le k \le n$, let $(n)_k := n(n-1)\dots(n-k+1)$ and $S_m^{(k)}$ be a Stirling number of the second kind.

Prove the following identity for Fibonacci numbers:

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^m[F_p/F_{p+q}]^r F_{qr+s} = (-1)^{np}[F_q/F_{p+q}]^n\sum_{k=1}^{m}(-1)^{(p+1)k}(n)_k S_m^{(k)}[F_p/F_q]^k F_{(p+q)k-np+s}.$$

Solution by the proposer

For x an arbitrary variable, consider the binomial expansion identity

$$\sum_{r=0}^{n}\binom{n}{r}x^r = (1-x)^n,$$

and apply the operator $D := x\frac{d}{dx}$ to it m times $(m>0)$ to get

$$\sum_{r=0}^{n} r^m\binom{n}{r}x^r = D^m(1+x)^n.$$

By a well-known result,

$$D^m = \sum_{k=1}^{m} S_m^{(k)}x^k\frac{d^k}{dx^k},$$

where $\{S_m^k : 1 \le k \le m,\ 1 \le m\}$ are the Stirling numbers of the second kind. Consequently,

$$\sum_{r=0}^{n} r^m\binom{n}{r}x^r = \sum_{k=1}^{m} S_m^{(k)}x^k\frac{d^k}{dx^k}(1+x)^n = \sum_{k=1}^{m} S_m^{(k)}x^k(n)_k(1+x)^{n-k}. \qquad (*)$$

Next, for integers $p\ (\neq 0)$, $q\ (\neq 0, -p)$, solve the following for u and w:

$$1+u\alpha^q = w\alpha^{-p};\ \ 1+u\beta^q = w\beta^{-p}.$$

One readily finds

$$u = -[F_p / F_{p+q}];\ \ w = (-1)^p[F_q / F_{p+q}]. \qquad (**)$$

Inserting $x = u\alpha^q$, $1+x = w\alpha^{-p}$ in $(*)$ and multipying the resulting equation by a^s then gives

$$\sum_{r=0}^{n} r^m\binom{n}{r}u^r\alpha^{qr+s} = \sum_{k=1}^{m} S_m^{(k)}(n)_k u^k\alpha^{qk+s}w^{n-k}\alpha^{kp-np}.$$

Finally, replace α and β in this result, subtract from the above, and divide by $\sqrt{5}$ to get

$$\sum_{r=0}^{n} r^m\binom{n}{r}u^r F_{qr+s} = \sum_{k=1}^{m} S_m^{(k)}(n)_k u^k w^{n-k}F_{(p+q)k-np+s}.$$

Inserting the values for u and w from $(**)$ then establishes the result claimed in the problem statement.

The case $m = 0$ is readily dealt with, and one gets

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\left(\frac{F_p}{F_{p+q}}\right)^r F_{qr+s} = (-1)^{pn}\left(\frac{F_q}{F_{p+q}}\right)^n F_{-pn+s}.$$

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