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SOME CONVERGENT RECURSIVE SEQUENCES, HOMEOMORPHIC IDENTITIES, AND INDUCTIVELY DEFINED COMPLEMENTARY SEQUENCES

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I. Consider the following recursive formula for generating sequences, where λ is a given positive number:

(1.1) $$x_{n+1} = x_{n-1} - \lambda x_n .$$

It is well known that any sequence so generated is of the form

(1.2) $$x_n = C_1R_1^n + C_2R_2^n ,$$

where C_1 and C_2 are constants, and R_1 and R_2 are the roots of

(1.3) $$x^2 = 1 - \lambda x .$$

Let R_1 be the positive root which is less than one, and let R_2 be the root that is less than minus one. Then this sequence converges if and only if $C_2 = 0$; and when it does converge, it converges to zero. So given two positive numbers x_{-1} and x_0, this sequence converges if and only if $x_0/x_{-1} = R_1$. Furthermore, if λ is an integer, then R_1 is irrational.

It should be noted that sequences generated by a recursive formula such as (1.1) may be continued in the opposite direction. If $\lambda = 1$, then the recursive formula obtained for going in the opposite direction is the formula used for generating Fibonacci numbers.

Let P be a homeomorphism of $[0,\infty)$ onto itself. In other words, $P(0) = 0$ and P is a continuous, unbounded, strictly increasing function of non-negative real numbers. Then the question arises as to the convergence of the sequence starting from two positive numbers x_{-1} and x_0 and generated by the formula

(1.4) $$x_{n+1} = x_{n-1} - P(x_n) .$$

Although the properties to be discussed will depend on the nature of P on $[0,\infty)$ only, to facilitate the discussion it will be assumed that P is a homeomorphism of the entire real line. But for this exception, the word homeomorphism as used in this paper will refer to a homeomorphism of $[0, \infty)$ onto itself. Given two homeomorphisms h and g, their sum and products are defined as those homeomorphisms respectively satisfying

$$(1.5) \qquad \begin{aligned} (h + g)(t) &= h(t) + g(t) \\ (hg)(t) &= h[g(t)] \quad \text{and} \\ (gh)(t) &= g[h(t)] \quad \text{for all } t \geq 0. \end{aligned}$$

The inverse h^{-1} of h is that homeomorphism such that

$$(1.6) \qquad hh^{-1} = h^{-1}h = I ,$$

where I is the identity homeomorphism. The relation $h < g$ is defined to mean that $h(t) < g(t)$ for all $t > 0$. Similarly, $h \leq g$ means that $h(t) \leq g(t)$ for all $t > 0$. Note that $h < g$ if and only if $h^{-1} > g^{-1}$ and that $h \leq g$ if and only if $h^{-1} \geq g^{-1}$.

Given two homeomorphisms h and g, define $h \cup g$ as that homeomorphism such that $(h \cup g)(t)$ is the largest of the two numbers $h(t)$ and $g(t)$ for all $t \geq 0$. Define $h \cap g$ as that homeomorphism such that

$$(1.7) \qquad h \cap g + h \cup g = h + g \ .$$

In other words, $h \cap g$ is the minimum of g and h. Note that for any h and g,

$$(1.8) \qquad h \cap g = g \cap h \leq h \leq h \cup g = g \cup h \ .$$

Also note that for any homeomorphism h,

$$(1.9) \qquad h \cap h^{-1} \leq I \leq h \cup h^{-1} \ .$$

The remainder of this first section of this paper is devoted to proving the following five interrelated theorems:

<u>Theorem 1</u>: There exists a unique homeomorphism h such that

$$(1.10) \qquad h = P + h^{-1} \ .$$

Theorem 2: Let h and g be two homeomorphisms such that

$$h = P + g^{-1} \quad \text{and} \tag{1.11}$$

$$g = P + h^{-1} \quad .$$

Then $h = g =$ the homeomorphism of Theorem 1.

Theorem 3: Let g_1 be any homeomorphism. Then the sequence of homeomorphisms $\{g_n\}$ defined inductively by

$$g_{n+1} = P + g_n^{-1} \tag{1.12}$$

converges uniformly on every bounded subset of $[0, \infty)$ to the homeomorphism h of Theorem 1.

Theorem 4: The sequence generated by (1.4) from two positive numbers x_{-1} and x_0 converges if and only if $x_{-1} = h(x_0)$, where h is the homeomorphism of Theorem 1. Also, whenever this sequence converges, it converges to zero.

If $h(x_0) > x_{-1}$, then all of the elements of the sequence with even subscripts are positive, but all but a finite number of the elements with odd subscripts are negative.

If $h(x_0) < x_{-1}$, then all of the odd subscripted elements are positive and all but a finite number of the even subscripted elements are negative.

Theorem 5: If P maps integers into integers, then the h of Theorem 1 will not map any positive integer into an integer.

Proofs: Let h_1 be a homeomorphism such that $h_1 \leq P$. By induction, for n a positive integer, define

$$h_{n+1} = P + h_n^{-1} \quad . \tag{1.13}$$

Then $n > 1$ implies that

$$h_1 \leq P < P + h_{n-1}^{-1} = h_n \quad . \tag{1.14}$$

By induction on m, if $0 < m < n$ and m is even, then

(1.15) $$h_m = P + h_{m-1}^{-1} > P + h_{n-1}^{-1} = h_n$$

and similarly, if $m \geq 3$ is odd, then

(1.16) $$h_m = P + h_{m-1}^{-1} < P + h_{n-1}^{-1} = h_n \; .$$

So the increasing sequence of homeomorphisms with odd subscripts

(1.17) $$h_1 < h_3 < h_5 < h_7 < h_9 < \cdots$$

is bounded from above by the decreasing sequence

(1.18) $$h_2 > h_4 > h_6 > h_8 > h_{10} > \cdots \; .$$

Also, the decreasing sequence

(1.19) $$h_3^{-1} > h_5^{-1} > h_7^{-1} > h_9^{-1} > \cdots$$

is bounded from below by the increasing sequence

(1.20) $$h_2^{-1} < h_4^{-1} < h_6^{-1} < h_8^{-1} < \cdots \; .$$

Therefore, these four sequences must be pointwise convergent.

Let us now prove that the homeomorphisms h_n^{-1} for $n > 1$ are uniformly equicontinuous on every bounded subset of $[0, \infty)$. For any $r > 0$, $[0, r]$ is compact and so for any $\epsilon > 0$, there exists a $\delta > 0$ such that $0 \leq P(t_1) < P(t_2) \leq r$ and $P(t_2) - P(t_1) < \delta$ imply that $t_2 < t_1 + \epsilon$.

Let t_1, t_2 and $n > 1$ be such that

(1.21) $$0 \leq h_n(t_1) < h_n(t_2) \leq r \quad \text{and}$$

(1.22) $$h_n(t_2) - h_n(t_1) < \delta.$$

Then

(1.23) $$0 \leq P(t_1) < P(t_2) < P(t_2) + h_{n-1}^{-1}(t_2) = h_n(t_2) \leq r \; .$$

Also

$$P(t_2) - P(t_1) < P(t_2) - P(t_1) + h_{n-1}^{-1}(t_2) - h_{n-1}^{-1}(t_1) \tag{1.24}$$

$$= h_n(t_2) - h_n(t_1) < \delta .$$

Therefore $t_1 < t_2 < t_1 + \epsilon$. Consequently the h_n^{-1} are uniformly equicontinuous on any set $[0, r]$.

Since the pointwise convergent sequences (1.19) and (1.20) are uniformly equicontinuous on every bounded set $[0, r]$, they converge uniformly to continuous functions. However, sequences (1.17) and (1.18) are related to (1.19) and (1.20) by (1.13). Therefore, these sequences also converge uniformly to continuous functions. But if a series of homeomorphisms and their inverses both converge to continuous functions, then these continuous functions must be homeomorphisms. Therefore, there exist homeomorphisms h and g such that h, g, h^{-1} and g^{-1} are the limits of (1.17), (1.18), (1.19), and (1.20) respectively. Also, (1.13) implies (1.11).

Let h and g be any pair of homeomorphisms which satisfy (1.11). That at least one such pair exists has just been proven. Let x_0 be a positive real number and let $x_{-1} = h(x_0)$. Then let the sequence

$$x_{-1},\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ \cdots \tag{1.25}$$

be defined inductively by (1.4). Then it will be shown by induction that for $n \geq 0$

$$x_{2n} = (h^{-1}g^{-1})^n\,(x_0) \quad \text{and} \tag{1.26}$$

$$x_{2n-1} = h(h^{-1}g^{-1})^n\,(x_0) \quad . \tag{1.27}$$

Equations (1.26) and (1.27) are obviously satisfied for $n = 0$ since $(h^{-1}g^{-1})^0$ is defined as I. If they are true for a given n, then

$$(1.28) \qquad \begin{aligned} x_{2(n+1)-1} &= x_{2n-1} - P(x_{2n}) \\ &= \left[h(h^{-1}g^{-1})^n - P(h^{-1}g^{-1})^n\right](x_0) \\ &= (h-P)(h^{-1}g^{-1})^n (x_0) \\ &= g^{-1}(h^{-1}g^{-1})^n (x_0) \\ &= h(h^{-1}g^{-1})^{n+1} (x_0) \ . \end{aligned}$$

and also

$$(1.29) \qquad \begin{aligned} x_{2(n+1)} &= x_{2n} - P\,(x_{2(n+1)-1}) \\ &= (h^{-1}g^{-1})^n (x_0) - Ph(h^{-1}g^{-1})^{n+1} (x_0) \\ &= (g-P)h(h^{-1}g^{-1})^{n+1} (x_0) \\ &= (h^{-1}g^{-1})^{n+1} (x_0) \ . \end{aligned}$$

However, (1.11) implies that

$$(1.30) \qquad h^{-1}g^{-1} < h^{-1}(P + g^{-1}) = h^{-1}h = I \ .$$

Therefore, $(h^{-1}g^{-1})^n (x_0)$ converges to zero as n tends to infinity. Consequently, (1.25) also converges to zero.

Let x_0, y_0, x_{-1} and y_{-1} be any positive numbers such that $x_0 = y_0$ and $y_{-1} - x_{-1} = \varepsilon > 0$. Define the sequence $\{x_n\}$ inductively by (1.4) and likewise the sequence $\{y_n\}$ inductively by

$$(1.31) \qquad y_{n+1} = y_{n-1} - P(y_n) \quad .$$

Equations (1.4) and (1.31) yield

$$(1.32) \qquad y_1 - x_1 = y_{-1} - P(y_0) - x_{-1} + P(x_0) = y_{-1} - x_{-1} = \varepsilon \ .$$

Then by induction, for $n > 0$,

(1.33)
$$y_{2n} - x_{2n}$$
$$= y_{2n-2} - x_{2n-2} - P(y_{2n-1}) + P(x_{2n-1})$$
$$< y_{2n-2} - x_{2n-2} \leq 0$$

and

(1.34)
$$y_{2n+1} - x_{2n+1}$$
$$= y_{2n-1} - x_{2n-1} - P(y_{2n}) + P(x_{2n})$$
$$> y_{2n-1} - x_{2n-1} \geq \varepsilon \cdot$$

If $x_{-1} = h(x_0)$, then the y_{2n} terms decrease to less than

(1.35)
$$\lim_{n\to\infty} x_{2n} = 0$$

but the y_{2n+1} terms are bounded above

(1.36)
$$\lim_{n\to\infty} x_{2n+1} + \varepsilon = \varepsilon .$$

Conversely, if $y_{-1} = h(y_0)$, then the x_{2n} terms stay above zero but the x_{2n+1} terms decrease below $-\varepsilon$.

In view of the symmetric roles of h and g in (1.11), it may be similarly shown that the sequence defined by (1.4) converges if and only if $x_{-1} = g(x_0)$. Since this is true for any $x_0 > 0$, it follows that whenever h and g satisfy (1.11) they are the same. Therefore, except for the uniqueness of h, Theorems 1 and 2 have been proven. But the uniqueness of h is also similarly proven since it may likewise be shown that if

(1.37)
$$\hat{h} = P + \hat{h}^{-1}$$

for some homeomorphism $\hat{h}$, then (1.25) converges if and only if $x_{-1} = \hat{h}(x_0)$.

Let g_1 be any homeomorphism for which Theorem 3 is to be tested, and choose

(1.38)
$$h_1 = g_1 \cap g_2 \cap P .$$

Then $h_1 \leq g_1$ implies by induction that for $n > 0$,

$$(1.39) \qquad g_{2n} = P + g_{2n-1}^{-1} \leq P + h_{2n-1}^{-1} = h_{2n}$$

and

$$(1.40) \qquad g_{2n+1} = P + g_{2n}^{-1} \geq P + h_{2n}^{-1} = h_{2n+1} .$$

Therefore, the sequence $\{g_{2n+1}\}$ is bounded from below by $\{h_{2n+1}\}$ and $\{g_{2n}\}$ is bounded from above by $\{h_{2n}\}$. However, $h_1 \leq g_2$ similarly implies that $\{g_{2n+1}\}$ is bounded from above by $\{h_{2n}\}$ and $\{g_{2n}\}$ is bounded from below by $\{h_{2n-1}\}$. $h_1 \leq P$ implies that both $\{h_{2n+1}\}$ and $\{h_{2n}\}$ converge uniformly to h on every bounded subset of $[0, \infty)$. Therefore, $\{g_n\}$ also converges uniformly to h on every bounded subset of $[0, \infty)$. Identity (1.12) then implies that $\{g_n^{-1}\}$ also converges uniformly on every such bounded subset.

To prove Theorem 5, note that if P maps integers into integers and that if x_0 and x_{-1} are positive integers such that $h(x_0) = x_{-1}$, then the sequence defined inductively by (1.4) must consist of integers. But

$$(1.41) \qquad x_n = (h^{-1})^n (x_0) \quad \text{for } n \geq 0 .$$

implies a slow convergence of $\{x_n\}$ which contradicts the assertion that the elements of the sequence are integers.

II. Sequences defined by

$$(2.1) \qquad x_{n+1} = P(x_n) - x_{n-1}$$

are considered in this section of the paper. The homeomorphic identity

$$(2.2) \qquad h + h^{-1} = P$$

associated with (2.1) is also discussed here. In order to establish theorems concerning the unique convergence of sequences generated by (2.1) and concerning the existence of solutions to (2.2), additional properties of P will need to be assumed.

Lemma 1: Let h and g be any two homeomorphisms. Then

$$(h \cup g)^{-1} = h^{-1} \cap g^{-1} \tag{2.3}$$

and

$$(h \cap g)^{-1} = h^{-1} \cup g^{-1} \quad . \tag{2.4}$$

Proof: To prove (2.3), it is sufficient to show that $(h \cup g)(x) = y$ implies $(h^{-1} \cap g^{-1})(y) = x$. Whenever

$$g(x) \leqq h(x) = y \quad , \tag{2.5}$$

then

$$h^{-1}(y) = x = g^{-1}g(x) \leqq g^{-1}h(x) = g^{-1}(y) \tag{2.6}$$

and so

$$(h^{-1} \cap g^{-1})(y) = h^{-1}(y) = x \quad . \tag{2.7}$$

Similarly, whenever

$$h(x) \leqq g(x) = y \quad , \tag{2.8}$$

then (2.6) and (2.7) follow when h is replaced by g and g is replaced by h. Hence, (2.3) has been proved.

Replacing h by h^{-1} and g by g^{-1} and applying (2.3) proves (2.4).

Lemma 2: Let h and g be two homeomorphisms such that

$$h + h^{-1} = g + g^{-1} \quad . \tag{2.9}$$

Then

$$h \cup g + (h \cup g)^{-1} = h + h^{-1} \tag{2.10}$$

and

$$h \cap g + (h \cap g)^{-1} = h + h^{-1} \tag{2.11}$$

Proof: The hypothesis (2.9) implies that for every x, $(h \cup g)(x) = h(x)$ if and only if $(h^{-1} \cap g^{-1})(x) = h^{-1}(x)$. Therefore,

$$
\begin{aligned}
(2.12) \qquad & h + h^{-1} \\
&= (h + h^{-1}) \cap (g + g^{-1}) \\
&\le h \cup g + h^{-1} \cap g^{-1} \\
&\le (h + h^{-1}) \cup (g + g^{-1}) \\
&= h + h^{-1} \quad .
\end{aligned}
$$

Since the middle term of (2.12) equals $h + h^{-1}$, application of Lemma 1 to it yields (2.10). If h is replaced by h^{-1} and g by g^{-1}, then (2.9) remains invariant. Therefore, if these substitutions are applied to (2.10), the result is also valid. But in view of Lemma 1, this is equivalent to (2.11).

Lemma 3: Let h and g be any two homeomorphisms such that $h \geq g \geq I$. Then for any $x > 0$,

$$
(2.13) \qquad \int_0^x \left[h(t) + h^{-1}(t)\right] dt \geq \int_0^x \left[g(t) + g^{-1}(t)\right] dt
$$

and (2.13) becomes an equality if and only if

$$
(2.14) \qquad g(t) = h(t) \quad \text{for all} \quad h^{-1}(x) \leq t \leq x \quad .
$$

Proof: The set of all points (s, t) such that

$$
\begin{aligned}
(2.15) \qquad & 0 \leq s \leq x \ , \\
& h^{-1}(s) \leq t \leq g^{-1}(s)
\end{aligned}
$$

is the same as the set such that

$$
\begin{aligned}
(2.16) \qquad & 0 \leq s \leq x, \\
& g(t) \leq s \leq h(t) \quad \text{and} \\
& 0 \leq t \leq g^{-1}(x)
\end{aligned}
$$

Therefore (see Figures 1 and 2)

$$(2.17)\qquad \int_0^x \left[h(t) + h^{-1}(t)\right] dt - \int_0^x \left[g(t) + g^{-1}(t)\right] dt$$

$$= \int_0^x \left[h(t) - g(t)\right] dt - \int_0^x \left[g^{-1}(s) - h^{-1}(s)\right] ds$$

$$= \int_0^x \left[h(t) - g(t)\right] dt - \int_0^{g^{-1}(x)} \left[\min(x, h(t)) - g(t)\right] dt$$

$$= \int_{h^{-1}(x)}^{g^{-1}(x)} \left[h(t) - x\right] dt + \int_{g^{-1}(x)}^{x} \left[h(t) - g(t)\right] dt \geq 0$$

with equality if and only if $h^{-1}(x) = g^{-1}(x)$ and $h(t) = g(t)$ for $g^{-1}(x) \leq t \leq x$. But these last two conditions together are equivalent to (2.14).

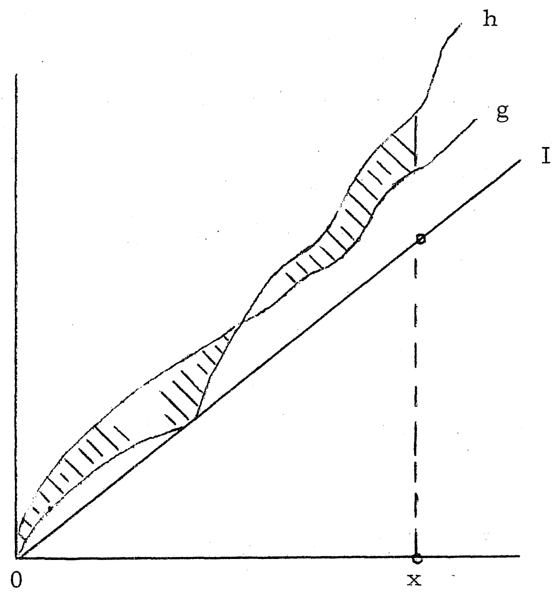

Figure 1: $\int_0^x \left[h(t) - g(t)\right] dt$ for two typical homeomorphisms h and g such that $h \geq g \geq I$ and $h(x) > g(x)$.

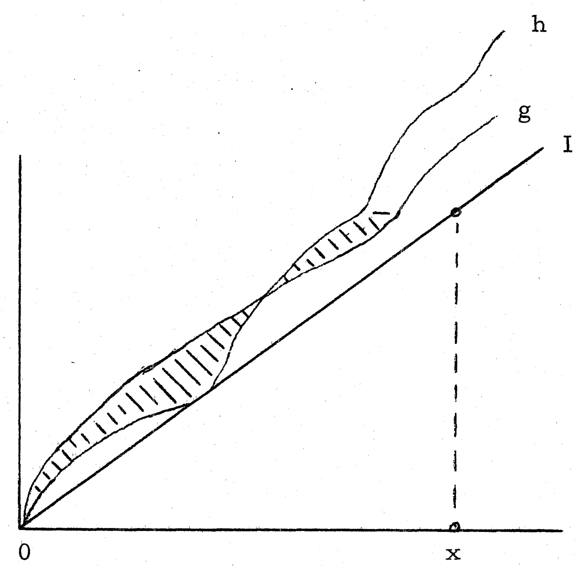

Figure 2: $\int_0^x \left[g^{-1}(t) - h^{-1}(t)\right] dt$ for two typical homeomorphisms h and g such that $h \geq g \geq I$ and $h(x) > g(x)$.

Theorem 6: Let h and g be any homeomorphisms. Then

$$h + h^{-1} = g + g^{-1} \tag{2.9}$$

if, and only if,

$$g(x) = \text{either } h(x) \text{ or } h^{-1}(x) \text{ for all } x \geq 0 \quad . \tag{2.18}$$

Proof: Define

$$f_1 = (h \cup h^{-1}) \cup (g \cup g^{-1}) \quad \text{and} \tag{2.19}$$

$$f_2 = (h \cup h^{-1}) \cap (g \cup g^{-1}) \quad .$$

Then (1.9) implies that

$$f_1 \geq f_2 \geq I \quad . \tag{2.20}$$

Whenever (2.9) holds, Lemma 2 may be applied four times to yield

$$f_1 + f_1^{-1} = f_2 + f_2^{-1} = h + h^{-1} \quad . \tag{2.21}$$

But integrating (2.21) and applying Lemma 3 to f_1 and f_2 proves that $f_1 = f_2$. Therefore

$$h \cup h^{-1} = g \cup g^{-1} \quad . \tag{2.22}$$

Now Lemma 1 may be applied to obtain

$$h \cap h^{-1} = (h \cup h^{-1})^{-1} = (g \cup g^{-1})^{-1} = g \cap g^{-1} \quad . \tag{2.23}$$

But (2.22) and (2.23) together imply (2.18).

To prove the converse, note that (2.18) implies that $h(x) = x$ if and only if $g(x) = x$. Therefore, the set $\{x \mid h(x) \neq x\}$ is the same set as $\{x \mid g(x) \neq x\}$. Since g and h are both homeomorphisms, each component of this set is mapped homeomorphically onto itself by h and also by g. Furthermore, neither $h-I$ nor $g-I$ changes sign on any such component. So (2.18) implies that, on each component, either $g = h$ and $g^{-1} = h^{-1}$ or else $g = h^{-1}$ and $g^{-1} = h$. Therefore, (2.9) holds on each such component. But (2.9) also holds wherever $h(x) = g(x) = x$.

Corollary: Given any homeomorphism h, there exists one and only one homeomorphism g such that $g \geq I$ and (2.9) $h + h^{-1} = g + g^{-1}$.

Theorem 7: Let h be any homeomorphism. Then for each $x \geq 0$,

$$\int_0^x \left[h(t) + h^{-1}(t)\right] dt \geq x^2 \tag{2.24}$$

and (2.24) becomes an equality if and only if

$$h(x) = x \quad . \tag{2.25}$$

Proof: Lemma 2 implies that

$$h \cup h^{-1} + (h \cup h^{-1})^{-1} = h + h^{-1} \quad . \tag{2.26}$$

Inequality (1.9) implies that the two homeomorphisms $h \cup h^{-1}$ and I satisfy the conditions of Lemma 3. Therefore, (2.24) is established, and (2.24) becomes an equality if and only if

$$(2.27) \qquad (h \cup h^{-1})(t) = t \quad \text{for all} \quad (h \cup h)^{-1}(x) \le t \le x \; .$$

But (2.27) is equivalent to (2.25).

Definition: Given two functions f and g, let either $f \uparrow g$ or $g \downarrow f$ be defined to mean that

$$(2.28) \qquad f(t_2) - f(t_1) \ge g(t_2) - g(t_1)$$

for all t_1 and t_2 on the domains of f and g such that $t_2 > t_1$. Note that if f and g both have continuous derivatives, then this is equivalent to

$$(2.29) \qquad \frac{d}{dt} f(t) \ge \frac{d}{dt} g(t) \quad \text{for all } t \; .$$

Remark: Let α be a positive real number and let f be a continuous function of $[0, \infty)$. Then $f(0) = 0$ and $f \uparrow \alpha I$ if and only if f is a homeomorphism and $f^{-1} \downarrow \alpha^{-1} I$.

Theorem 8: Let $\alpha \ge 2$ and $P \uparrow \alpha I$. Define the sequence of homeomorphisms $\{h_n\}$ inductively by $h_1 = I$ and

$$(2.30) \qquad h_{n+1} = P - h_n^{-1} \quad n > 1 \; .$$

Then the sequence $\{h_n\}$ converges to a homeomorphism h such that

$$(2.2) \qquad h + h^{-1} = P$$

and

$$(2.31) \qquad h \uparrow \left\{ \frac{\alpha + \sqrt{\alpha^2 + 4}}{2} \right\} I \; .$$

Furthermore, the convergence is uniform on every bounded subset of $[0, \infty)$.

Proof: Define by induction, $r_1 = 1$ and $r_{n+1} = \alpha - r_n^{-1}$. Then $h_1 \uparrow r_1 I$ and by induction,

$$h_n^{-1} \downarrow r_n^{-1}\, I \tag{2.32}$$

and

$$h_{n+1} = P - h_n^{-1} \uparrow \alpha\, I - r_n^{-1}\, I = r_{n+1}\, I \tag{2.33}$$

and the h_n are all homeomorphisms.

Also,

$$h_2 = P - I \geq I = h_1 \tag{2.34}$$

and so by induction,

$$h_{n+2} = P - h_{n+1}^{-1} \geq P - h_n^{-1} = h_{n+1} \quad . \tag{2.35}$$

Since for each $x \geq 0$, $h_n(x)$ is a monotonic non-decreasing sequence of numbers bounded above by $P(x)$, the sequence $\{h_n\}$ is pointwise convergent. Since the r_n are increasing, (2.32) implies that the h^{-1} are uniformly equicontinuous on every bounded subset of $[0, \infty)$. But this combined with (2.30) implies that the h_n are also uniformly equicontinuous on each such subset. Therefore, the sequence $\{h_n\}$ converges uniformly on every bounded subset of $[0, \infty)$ to some homeomorphism h.

Since $\{r_n\}$ is increasing but bounded by α, it must converge to some number r such that $1 < r \leq \alpha$. By continuity, $r = \alpha - r^{-1}$. Therefore, $h \uparrow r\, I$ which is the same as (2.31).

Theorem 9: In addition to the hypothesis to Theorem 8, let $P \downarrow \beta I$ where β is some real number $\geq \alpha$.

Then

$$h \downarrow \left(\frac{\beta + \sqrt{\beta^2 + 4}}{2} \right) I \ , \tag{2.36}$$

where h is the homeomorphism to which the sequence of homeomorphisms of Theorem 8 converge.

Proof: Define $v_1 = 1$ and by induction

$$v_{n+1} = \beta - v_n^{-1} \quad . \tag{2.37}$$

Then by induction

$$h_{n+1} \downarrow P - h_n^{-1} \downarrow \beta I - v_n^{-1} I = v_{n+1} I \ . \tag{2.38}$$

Therefore, $h \downarrow v\, I$, where

$$v = \lim_{n \to \infty} v_n = \left(\frac{\beta + \sqrt{\beta^2 + 4}}{2} \right) . \tag{2.39}$$

Corollary: Let $P = \alpha I$. Then

$$h = \left(\frac{\alpha + \sqrt{\alpha^2 + 4}}{2} \right) I \ . \tag{2.40}$$

Lemma 4: Let

$$h + h^{-1} = P \tag{2.2}$$

and let x_0 and x_{-1} be two positive real numbers such that

$$x_0 < h(x_0) = x_{-1} \tag{2.41}$$

Then the sequence $\{x_n\}$ defined inductively by

$$x_{n+1} = P(x_n) - x_{n-1} \tag{2.1}$$

will converge monotonically to y, where y is the largest real number such that

$$h(y) = y < x_0 \ . \tag{2.42}$$

However, for no $n > 0$ is $x_n = y$.

Proof: For $n = -1$ or 0, we have that $x_n = h^{-n}(x_0)$, where h^0 is defined as I. Therefore, by induction, for $n > 0$,

$$\begin{aligned} x_{n+1} &= P(x_n) - x_{n-1} \\ &= h(x_n) + h^{-1}(x_n) - x_{n-1} \\ &= h^{-n+1}(x_0) + h^{-n-1}(x_0) - h^{-n+1}(x_0) \\ &= h^{-n-1}(x_0) \ . \end{aligned} \tag{2.43}$$

Since $h^{-1}(x_0) < x_0$, the sequence $\{x_n\}$ must converge to y as described.

Theorem 10: Let h be any homeomorphism such that $h + h^{-1}$ maps positive integers into integers. Then h will never map any positive integer p into an integer unless $h(p) = p$.

Proof: If the theorem is false, then there exist positive integers p and q such that

$$h \cup h^{-1}(p) = q > p \quad . \tag{2.44}$$

Lemma 2 implies that

$$h \cup h^{-1} + (h \cup h^{-1})^{-1} = h + h^{-1} \quad . \tag{2.45}$$

Define x_0 as p and x_{-1} as q and define the sequence $\{x_n\}$ inductively by

$$x_{n+1} = h(x_n) + h^{-1}(x_n) - x_{n-1} \quad . \tag{2.46}$$

Then applying Lemma 4 to $h \cup h^{-1}$, one may see that $\{x_n\}$ must slowly converge as described in the lemma. However, this contradicts the fact, which may be easily verified by induction, that the sequence $\{x_n\}$ consists of integers.

Theorem 11: Let $P \uparrow 2I$ and $P > 2I$ on $(0,\infty)$. Let x_0 and x_{-1} be two positive real numbers.

Then a necessary and sufficient condition that the sequence defined inductively by

$$x_{n+1} = P(x_n) - x_{n-1} \tag{2.1}$$

converges is that $h(x_0) = x_{-1}$, where h is the unique homeomorphism such that $h \geq I$ and

$$h + h^{-1} = P \quad . \tag{2.2}$$

The sequence will contain a non-positive element if and only if $h(x_0) < x_{-1}$. Also, $x_{n+1} > x_n$ for some element if and only if $h(x_0) > x_{-1}$, and this holds if and only if

(2.47)
$$\lim_{n\to\infty} x_n = +\infty .$$

Proof: The existence and uniqueness of h is given by Theorem 8 and the corollary to Theorem 6. $P > 2I$ and (2.2) imply that $h > I$. If $x_{-1} = h(x_0)$, then Lemma 4 implies that x_n converges monotonically to zero.

Let x_0, y_0, x_{-1} and y_{-1} be positive numbers such that $x_0 = y_0$ and $y_{-1} - x_{-1} = \varepsilon > 0$. Define $\{x_n\}$ inductively by (2.1) and likewise $\{y_n\}$ by

(2.48)
$$y_{n+1} = P(y_n) - y_{n-1} .$$

If $n = -1$, then

(2.49)
$$x_n - y_n \geq n\,\varepsilon$$

and

(2.50)
$$(x_{n+1} - y_{n+1}) - (x_n - y_n) \geq \varepsilon .$$

Therefore, by induction, for $n \geq 0$,

(2.50)
$$\begin{aligned}&(x_{n+1} - y_{n+1}) - (x_n - y_n)\\ &= P(x_n) - P(y_n) - (x_n - y_n) - (x_{n-1} - y_{n-1})\\ &\geq (x_n - y_n) - (x_{n-1} - y_{n-1}) \geq \varepsilon\end{aligned}$$

and

$$\begin{aligned}&x_n - y_n\\ &= (x_n - y_n) - (x_{n-1} - y_{n-1}) + (x_{n-1} - y_{n-1})\\ &\geq \varepsilon + (x_{n-1} - y_{n-1})\\ &\geq \varepsilon + (n-1)\,\varepsilon = n\,\varepsilon .\end{aligned}$$

If $y_{-1} = h(y_0)$, then (2.49) implies that x_n is always positive for $n \geq 0$ and it converges to infinity. If $x_{-1} = h(x_0)$, then (2.49) implies that y_n is monotonic and will attain negative values.

III. Let a and b be monotonic increasing mappings of positive integers into positive integers. Then the sequences $\{a(n)\}$ and $\{b(n)\}$ are said to be complementary if and only if each positive integer is represented in one and only one of these sequences.

Given a real number r, define $[r]$ as the integer part of r, namely $[r]$ is that integer such that

$$[r] \le r < [r] + 1 . \tag{3.1}$$

Define $[r]^*$ as that integer such that

$$[r]^* < r \le [r]^* + 1 , \tag{3.2}$$

or equivalently

$$[r]^* = -1 - [-r] . \tag{3.3}$$

A result of S. Beatty, see Reference [1], is essentially that given a positive irrational number x, then the sequences $[(1 + x)\, n]$ and $[(1 + x^{-1})\, n]$ are complementary. This result has since turned up many times in the literature, often in the form that if α and β are two positive irrational numbers such that $\alpha^{-1} + \beta^{-1} = 1$, then the two sequences $[\alpha n]$ and $[\beta n]$ are complementary.

A generalization of this result by Lambek and Moser states that the sequences $\{a(n)\}$ and $\{b(n)\}$ are complementary if and only if for each pair of positive integers m and n, either $a(m) - m < n$ or else $b(n) - n < m$ but never both. This result combined with Lemma 5 may also be used to prove Theorem 12 instead of the proof given.

<u>Lemma 5</u>: Let f and g be homeomorphisms such that

$$f^{-1} + g^{-1} = I . \tag{3.4}$$

Then $f - I$ and $g - I$ are homeomorphisms and

$$(f - I)\,(g - I) = I . \tag{3.5}$$

Conversely, let h be any homeomorphism. Then

$$(I + h)^{-1} + (I + h^{-1})^{-1} = I . \tag{3.6}$$

Proof:

$$f - I = (I - f^{-1})\, f = g^{-1} f \quad \text{and} \tag{3.7}$$

$$g - I = (I - g^{-1})\, g = f^{-1} g \quad .$$

But $g^{-1} f$ and $f^{-1} g$ are homeomorphisms which are inverses of each other.

$$(I + h)^{-1} + (I + h^{-1})^{-1} \tag{3.8}$$

$$= (I + h)^{-1} + (I + h^{-1})^{-1} (h + I)\, h^{-1} h\, (I + h)^{-1}$$

$$= (I + h)^{-1} + (I + h^{-1})^{-1} (I + h^{-1})\, h\, (I + h)^{-1}$$

$$= (I + h)^{-1} + h\, (I + h)^{-1}$$

$$= (I + h)\, (I + h)^{-1} = I \quad .$$

Theorem 12: Let f and g be two homeomorphisms such that

$$f^{-1} + g^{-1} = I \quad . \tag{3.4}$$

Then the two sequences $\{[f(n)]\}$ and $\{[g(n)]^*\}$ are complementary.

Proof: Given a non-negative integer m, let n_1 be the number of elements of $\{[f(n)]\}$ which are less than or equal to m, and let n_2 be the number of such elements of $\{[g(n)]^*.\}$ Then

$$f(n_1) < m + 1 \le f(n_1 + 1) \quad \text{and} \tag{3.9}$$

$$g(n_2) \le m + 1 < g(n_2 + 1) \quad .$$

Applying f^{-1} and g^{-1} to (3.9) yields

$$n_1 = f^{-1} f(n_1) < f^{-1}(m + 1) \le f^{-1} f(n_1 + 1) = n_1 + 1 \tag{3.10}$$

$$n_2 = g^{-1} g(n_2) \le g^{-1}(m + 1) < g^{-1} g(n_2 + 1) = n_2 + 1 \quad .$$

Adding the two parts of (3.10) together yields

$$n_1 + n_2 < m + 1 < n_1 + n_2 + 2 \quad . \tag{3.11}$$

Since n_1, n_2 and m are all integers, it follows that

$$n_1 + n_2 = m \quad . \tag{3.12}$$

Therefore, each positive integer is represented once and only once by the sequences, but only positive integers are represented.

Corollary: Let h be any homeomorphism. Then the sequences $n + [h^{-1}(n)]^*$ and $n + [h(n)]$ are complementary.

Proof: Apply Lemma 5 to the theorem.

The analysis of Wythoff's game (see Reference [4]) involves complementary sequences $\{a(n)\}$ and $\{b(n)\}$ such that

$$b(n) = a(n) + n \quad . \tag{3.13}$$

In a later paper, a generalization of Wythoff's game will be given for which the analysis will involve complementary sequences such that

$$b(n) = a(n) + (k + 1)\, n, \tag{3.14}$$

where k is some non-negative integer which defines the game. Beatty's result is easily used to show that the complementary sequences satisfying (3.14) are

$$a(n) = \left[\left(\frac{1-k+\sqrt{(k+1)^2+4}}{2}\right) n\right] \quad \text{and} \tag{3.15}$$

$$b(n) = \left[\left(\frac{3+k+\sqrt{(k+1)^2+4}}{2}\right) n\right]$$

Theorem 13 may be thought of as a generalization of this result.

Theorem 13: Let P map integers into integers. Let the sequences be defined inductively as follows:

$$a(1) = 1 \tag{3.16}$$

$$b(n) = a(n) + P(n) \qquad n > 0 \tag{3.17}$$

$$a(n+1) = \text{smallest integer not represented by either } a(i) \text{ or } b(i) \text{ for some } i \le n \quad . \tag{3.18}$$

Then

$$a(n) = n + [h^{-1}(n)] \quad \text{and} \tag{3.19}$$

$$b(n) = n + [h(n)] ,$$

where h is the unique homeomorphism such that

$$h = P + h^{-1} . \tag{1.10}$$

Proof: Theorem 5 implies that

$$[h(n)]^* = [h(n)] . \tag{3.20}$$

So the corollary to Theorem 12 shows that the sequences defined by (3.19) are complementary. Since (1.10) implies that $h > h^{-1}$, (3.18) and (3.16), which is a special case of (3.18), are satisfied. Equation (3.17) follows from (1.10) and (3.19) and the fact that the $P(n)$ are integers.

In Reference [4] is presented the following result: Let k be an integer greater than 4. Then the sequences defined by

$$a(n) = \left[\left(\frac{k-\sqrt{k^2-4k}}{2}\right)n\right] \quad \text{and} \tag{3.21}$$

$$b(n) = \left[\left(\frac{k+\sqrt{k^2-4k}}{2}\right)n\right]$$

are the sequences such that for n any positive integer,

$$a(n) + b(n) = nk-1 \tag{3.22}$$

and such that $a(n)$ is the smallest positive integer not represented by any $a(i)$ or $b(i)$ with $i < n$.

The following theorem and its corollary may be thought of as generalizations of this result since they imply it with the help of the corollary to Theorem 9.

Theorem 14: Let P map integers into integers. Let there exist a homeomorphism h satisfying

$$h + h^{-1} = P . \tag{2.2}$$

Let the sequences $\{a(n)\}$ and $\{b(n)\}$ be defined inductively as follows: $a(n)$ is the smallest positive integer not represented by earlier elements of a and b, and

$$b(n) = P(n) + 2n-1-a(n) \quad . \tag{3.23}$$

Then no positive integer will be represented twice by the two sequences and for each $n > 0$,

$$a(n) = n + [h^{-1}(n)]^* \quad \text{and} \tag{3.24}$$

$$b(n) = n + [h(n)] \quad ,$$

where h is the unique homeomorphism such that $h \geq I$ and (2.2) is valid.

Proof: The sequences defined by (3.24) are complementary by the corollary to Theorem 12. Since h satisfies (2.2), the sequences defined by (3.24) satisfy (3.23). Finally, monotonicity of the sequences, their being complementary and the fact that $h^{-1} \leq h$ imply that $a(n)$ is the first such integer not previously represented.

Corollary: Let $P(n) \neq 2n$ for any integer $n > 0$. Then

$$a(n) = n + [h^{-1}(n)] \quad . \tag{3.25}$$

Proof: Theorem 10 implies that $[h^{-1}(n)]^* = [h^{-1}(n)]$.

Theorem 15: Let $\{a(n)\}$ and $\{b(n)\}$ be the sequences of Theorem 13. Let x_0 and x_{-1} be any two positive integers. Let the sequence $\{x_n\}$ be inductively defined by

$$x_{n+1} = x_{n-1} - P(x_n) \tag{1.4}$$

Then the first element of this sequence of integers to be non-positive will have an even subscript if and only if

$$x_0 \leq a(x_{-1}) - x_{-1} \tag{3.26}$$

which in turn is equivalent to

$$x_{-1} > b(x_0) - x_0 \quad . \tag{3.27}$$

Proof: Theorem 5 implies that $h(x_0) \neq x_{-1}$. Hence (3.19) implies that (3.26) and (3.27) are both equivalent to $x_{-1} > h(x_0)$. The proof is now completed by applying Theorem 4.

This theorem may also be proven by the results of Lambek and Moser (Reference [6]).

Theorem 16: Let P map integers into integers and $P \uparrow 2I$ and $P > 2I$ on $[0, \infty)$ and let $a(n)$ and $b(n)$ be the sequences of Theorem 14. Let x_0 and x_{-1} be any two positive integers. Let the sequence x_n be inductively defined by

(2.1) $$x_{n+1} = P(x_n) - x_{n-1} \quad .$$

Then the following four statements are logically equivalent:

(3.26) $$x_0 \leq a(x_{-1}) - x_1$$

(3.27) $$x_{-1} > b(x_0) - x_0$$

(3.28) $$\{x_n\} \text{ contains a non-positive element}$$

$$\{x_n\} \text{ is monotonic decreasing}$$

Proof: Theorem 10 implies that $h(x_0) \neq x_{-1}$. Hence (3.24) implies that (3.26) and (3.27) are both equivalent to $x_{-1} > h(x_0)$. The proof is now completed by application of Theorem 11.

IV. In this section, representations are sought for homeomorphisms and corresponding complementary sequences associated with P's such that

(4.1) $$P(n) = 2\alpha n + 2\beta$$

for n a positive integer. The numbers 2α and 2β are assumed to be integer constants. The requirement that P be a homeomorphism leads to the conditions that $\alpha > 0$ and

(4.2) $$\alpha + \beta = \frac{1}{2} P(1) > 0.$$

Example 1: For this example, let the function F be defined as

(4.3) $$F(x) = (\sqrt{\alpha^2+1} - \alpha)\, x - \beta + (\sqrt{\alpha^2+1} - 1)\, \beta/\alpha \quad .$$

The inverse of this function is

(4.4) $$F^{-1}(x) = (\sqrt{\alpha^2+1} + \alpha)\, x + \beta + (\sqrt{\alpha^2+1} - 1)\,\beta/\alpha \ .$$

Define

(4.5) $$h^{-1}(x) = xF(1) \qquad 0 \le x \le 1$$
$$h^{-1}(x) = F(x) \qquad x \ge 1$$

For h^{-1} to be a homeomorphism, it is necessary that $F(1) > 0$. With some algebraic manipulation, it is readily seen that this requirement is equivalent to

(4.6) $$(\alpha + 2\beta)(1-\beta) > \alpha\beta \quad \text{or}$$
$$(\alpha + \beta)(1-2\beta) > -\beta \ .$$

By utilizing (4.2), it is seen that (4.6) is satisfied if and only if $\beta \le 1/2$. Condition (4.2) and the requirement that $\alpha > 0$ imply that $h^{-1}(1) < 1$. Therefore,

(4.7) $$h(x) = F^{-1}(x) \qquad x \ge 1$$

and so for n a positive integer,

(4.8) $$h(n) = 2\alpha n + 2\beta + h^{-1}(n) \ .$$

So Theorems 5 and 13 give that the sequences

(4.9) $$a(n) = n + [F(n)] \quad \text{and}$$
$$b(n) = n + [F^{-1}(n)]$$

are complementary and satisfy

(4.10) $$b(n) = 2\alpha n + 2\beta + a(n) > a(n)$$

unless $\beta \ge 1$. In the case where $\beta \ge 1$, other representations are needed. Setting $\beta = 0$ and $\alpha = (k+1)/2$ yields (3.15).

For the next two examples, a homeomorphism $h \ge I$ is sought such that for n a positive integer

(4.11) $$h(n) + h^{-1}(n) = 2\alpha n + 2\beta \ .$$

However, in some cases of Example 3, a homeomorphism is found that only generates the complementary sequences that would be generated by a homeomorphism satisfying (4.11). In these cases, a representation of these sequences is obtained.

Theorem 7 implies that $\alpha \geq 1$. Since the sum of two unequal positive integers is at least three, Theorem 14 implies that

$$(4.12) \qquad \alpha + \beta = ([h(1)] + [h^{-1}(1)]^* + 1)/2 \geq 1$$

Example 2: For this example, let $\beta \leq 0$ and let the function F be defined as

$$(4.13) \qquad F(x) = (\alpha \sqrt{\alpha^2 - 1})\, x + \beta - \beta \sqrt{\alpha^2 - 1}/(\alpha - 1) \;.$$

If $\alpha = 1$, then $\beta = 0$ and let the last term of (4.13), which would be indeterminate, be assumed to vanish. The inverse of this function is

$$(4.14) \qquad F^{-1}(x) = (\alpha + \sqrt{\alpha^2 - 1})x + \beta + \beta \sqrt{\alpha^2 - 1}/(\alpha - 1) \;.$$

Let h^{-1} be defined according to (4.5) with this F being used instead of the F of Example 1. The conditions on α and β imply that

$$(4.15) \qquad 0 < F(x) \leq x \qquad x \geq 1$$

Therefore, for $x \geq 1$, $h(x) = F^{-1}(x)$ and so (4.11) is satisfied for any positive integer n.

If $\alpha > 1$ and $\beta = 0$, then application of the corollary to Theorem 14 yields Ky Fan's result summarized by (3.21) and (3.22). If $\alpha = 1$ and $\beta = 0$, then $h = I$ and the resulting complementary sequences are represented by

$$(4.16) \qquad \begin{aligned} a(n) &= 2n - 1 \quad \text{and} \\ b(n) &= 2n \;. \end{aligned}$$

Example 3: For this example, let $\beta \geq 1/2$ and let the function F be defined for $x \geq 1$ as

$$(4.17) \qquad \begin{aligned} F(x) &= \alpha x + \beta - \sqrt{(\alpha x + \beta)^2 - (x - \beta)^2 - \varepsilon} \\ &= \alpha x + \beta - \sqrt{(\alpha + 1)(\alpha x^2 - x^2 + 2\beta x) - \varepsilon} \end{aligned}$$

where ε is a constant to be appropriately chosen. The inverse to this function is often two-valued. Considering only the largest of these two values yields

(4.18) $$F^{-1}(x) = \alpha x + \beta + \sqrt{(\alpha x+\beta)^2 - (x-\beta)^2 - \varepsilon}$$

It may be shown that $dF(x)/dx > 0$ if and only if

(4.19) $$(x+\beta)\left\{(\alpha-1)x + \beta\alpha + \beta\right\} > \alpha^2\varepsilon/(\alpha+1) \ .$$

Furthermore, if $\varepsilon = 0$, then $F(x)$ is positive for all $x > \beta$. So for the case where $\beta = \frac{1}{2}$, set $\varepsilon = 0$ and define h^{-1} with this F according to (4.5).

For this case, $0 < F(1) < 1$. Therefore

(4.20) $$h(x) = F^{-1}(x) \qquad x \geq 1 \ .$$

Application of Theorem 14 and its corollary imply that the sequences defined by

(4.21) $$a(n) = \left[(\alpha+1)n + \frac{1}{2} - \sqrt{(\alpha^2-1)n^2 + (\alpha+1)n}\right] \qquad \text{and}$$

$$b(n) = \left[(\alpha+1)n + \frac{1}{2} + \sqrt{(\alpha^2-1)n^2 + (\alpha+1)n}\right]$$

are complementary and that

(4.22) $$a(n) < b(n) = 2(\alpha+1)n - a(n) \ .$$

In the paper that will generalize Wythoff's game, related games will be presented whose analysis utilizes these two complementary sequences.

For $\beta = 1$, choose $\varepsilon > 0$ but sufficiently small that

(4.23) $$0 < F(1) < F(2) < 1$$

and that (4.19) is satisfied for $x \geq 2$.

Define

(4.24) $$\begin{aligned} h^{-1}(x) &= xF(x) & 0 \leq x \leq 1 \\ h^{-1}(x) &= (2-x)F(1) + (x-1)F(2) & 1 \leq x \leq 2 \\ h^{-1}(x) &= F(x) & x > 2 \ . \end{aligned}$$

Then $h(x) = F^{-1}(x)$ for $x \geq 1$. Consequently, h will satisfy (4.11) for this case.

If $\beta \geq 1\frac{1}{2}$, then $F(1) > 0$ implies that $\mathcal{E} < (\beta-1)^2$. It may be shown that this in turn implies that $F(1) > F(2\beta-1)$. Consequently, there does not exist any homeomorphism which equals F for positive integers. However, for certain cases, homeomorphisms will be defined such that

$$[h^{-1}(n)]^* = [F(n)]^* = [F(n)] \quad \text{and} \tag{4.25}$$

$$h(n) = F^{-1}(n).$$

So in these cases, the sequences defined by (4.9) are complementary and satisfy

$$a(n) < b(n) = 2(\alpha+1)n + 2\beta - 1 - a(n) \ . \tag{4.26}$$

If $2\beta \geq 3$ is odd, then the requirement that $F(\beta \pm \frac{1}{2})$ be positive implies that $\mathcal{E} > -1/4$. If $2\beta \geq 4$ is even, then positivity of $F(\beta)$ implies that $\mathcal{E} > 0$. For the sequences defined by (4.9) to be monotonic and complementary, it is necessary that

$$[F(1)] = a(1) - 1 = 0 \ . \tag{4.27}$$

The requirement that $F(1) < 1$ is equivalent to

$$2(\alpha+1) - (\beta-2)^2 > \mathcal{E} \ . \tag{4.28}$$

If 2β is odd, then the left side of (4.28) equals 3/4 modulo one, but if 2β is even, the left side is an integer. Therefore, a necessary condition for the attainment of the present objectives is that

$$2(\alpha+1) > (\beta-2)^2 \tag{4.29}$$

This condition will also turn out to be sufficient. Furthermore, to attain these objectives when (4.29) is valid, it is sufficient that

$$0 < \mathcal{E} < \frac{1}{4} \ . \tag{4.30}$$

Condition (4.30) implies that (4.19) is satisfied whenever $x \geq \beta + \frac{1}{2}$. Also, (4.30) may be shown to imply that

$$F(\beta+1) < 1 \ . \tag{4.31}$$

Define

(4.32) $$h^{-1}(x) = xF([\beta+1])/[\beta+1] \qquad 0 \le x \le [\beta+1]$$

$$h^{-1}(x) = F(x) \qquad x \ge [\beta+1] \;.$$

Then h^{-1} is a homeomorphism, and since

(4.33) $$h^{-1}([\beta+1]) < 1 \;,$$

it follows that

(4.34) $$h(x) = F^{-1}(x) \qquad x \ge 1$$

Condition (4.30) implies that $F(n)$ can never be a multiple of $\frac{1}{2}$ for any integer n. For any $1 \le x \le [\beta+1]$, both $h^{-1}(x)$ and $F(x)$ are between zero and one. Therefore, (4.25) is satisfied for all positive integers as desired.

V. The purpose of this section is to generalize the results of the first section. Whereas the proof of Theorem 17 uses ideas not found in the first section, the remainder of this section utilizes mostly straightforward generalizations of the techniques of Section I plus applications of Theorem 17. In this section, μ always refers to a positive real number and μ^{-1} to its reciprocal.

Theorem 17: Let $\mu \le 1$ and let h and g be two homeomorphisms such that

(5.1) $$h + \mu g^{-1} = g + \mu h^{-1} \;.$$

Let $h(x) \ne g(x)$ for some $x > 0$. Then $h(t) > t$ for all $t \ge x$.

Proof: Given any point $t > 0$, if $h(t) > g(t)$, then

(5.2) $$h^{-1}h(t) = g^{-1}g(t) < g^{-1}h(t)$$

which by (5.1) implies that

(5.3) $$hh(t) > gh(t)$$

Similarly, $h(t) < g(t)$ implies (5.3) with the inequality reversed.

If the theorem is false, then for some point x, $h(x) \neq g(x)$ and either

$$h(x) \leq x \tag{5.4}$$

or else

$$x < h(x) < x_0 = h(x_0) \tag{5.5}$$

for some point x_0. Define x_1 as either x or $h(x)$, whichever satisfies

$$h(x_1) < g(x_1) \quad . \tag{5.6}$$

For $n > 1$, define x_n as $h^{n-1}(x_1)$. Then in case of (5.4), the sequence $\{x_n\}$ is monotonic non-increasing and bounded below by zero. In case of (5.5), $\{x_n\}$ is monotonic increasing and bounded above by x_0.

The first paragraph of this proof implies that for $n > 0$,

$$h(x_{2n-1}) < g(x_{2n-1}) \quad \text{and} \tag{5.7}$$

$$h(x_{2n}) > g(x_{2n}) \quad .$$

Let (y_n, z_n) be the component of $\{t \mid h(t) \neq g(t)\}$ which contains x_n. Then the first paragraph of this proof implies that the open intervals (y_n, z_n) are all disjoint and that for $n > 0$

$$y_{n+1} = h(y_n) = g(y_n) \quad \text{and} \tag{5.8}$$

$$z_{n+1} = h(z_n) = g(z_n) \quad .$$

When (5.4) holds, then

$$z_{n+1} \leq y_n < x_n < z_n \quad , \tag{5.9}$$

and when (5.5) holds, then

$$y_n < x_n < z_n \leq y_{n+1} < x_0 \quad . \tag{5.10}$$

Integration by parts and (5.8) imply that

$$(5.11)\qquad (-\mu)^n \int_{y_{n+1}}^{z_{n+1}} \left\{ g(t) - h(t) \right\} dt$$

$$= -(-\mu)^n \int_{y_{n+1}}^{z_{n+1}} \left\{ t dg(t) - t dh(t) \right\}$$

which replacing $g(t)$ by u and $h(t)$ by v

$$= -(-\mu)^n \left\{ \int_{y_n}^{z_n} g^{-1}(u) du - \int_{y_n}^{z_n} h^{-1}(v) dv \right\}$$

$$= (-\mu)^{n-1} \int_{y_n}^{z_n} \left\{ g(t) - h(t) \right\} dt$$

which by induction on n

$$= \int_{y_1}^{z_1} \left\{ g(t) - h(t) \right\} dt > 0 \ .$$

In case of (5.4), define x_0 as z_1. Then for either case,

$$(5.12)\qquad \int_0^{x_0} \left\{ h \cup g - h \cap g \right\} (t)\ dt$$

$$\geq \sum_{n=1}^{\infty} \int_{y_n}^{z_n} \left\{ h \cup g - h \cap g \right\} (t)\ dt$$

$$= \sum_{n=1}^{\infty} \mu^{1-n} \int_{y_1}^{z_1} \left\{ g(t) - h(t) \right\} dt = \infty$$

which is impossible.

Corollary: Let $\mu \geq 1$ and let h and g satisfy (5.1). Then $h(x) \neq g(x)$ implies that $h(t) < t$ for all $t \geq x$.

Proof: Replace h by h^{-1}, g by g^{-1} and μ by μ^{-1} and apply the theorem.

Theorem 18: Let h and g be two-homeomorphisms such that

$$h + g^{-1} = g + h^{-1} \quad . \tag{5.13}$$

Then $h = g$.

Proof: Use Theorem 17 and its corollary.

Theorem 19: Let h_1 be a homeomorphism such that $h_1 \leq P$. Define by induction for $n > 0$,

$$h_{n+1} = P + \mu h_n^{-1} \quad . \tag{5.14}$$

Then on each bounded subset of $[0, \infty)$, $\{h_{2n-1}\}$ and $\{h_{2n}\}$ converge uniformly to homeomorphisms h and g respectively. Furthermore, $h \leq g$ and

$$h = P + \mu g^{-1} \quad \text{and} \tag{5.15}$$

$$g = P + \mu h^{-1} \quad .$$

Proof: The arguments of (1.13) through (1.24) and the next paragraph remain unchanged except that h_{n-1}^{-1} is replaced by μh_{n-1}^{-1} and h_{m-1}^{-1} is replaced by μh_{m-1}^{-1}.

Theorem 20: Let $\mu \leq 1$ and x_0 be a positive number. Then there cannot exist more than one positive number x_{-1} such that the sequence x_n defined inductively by

$$x_{n+1} = \mu^{-1} x_{n-1} - \mu^{-1} P(x_n) \tag{5.16}$$

converges.

Proof: Let y_0, x_{-1}, ε and y_{-1} be positive numbers such that $y_0 = x_0$ and $y_{-1} - x_{-1} = \varepsilon$.

Define $\{y_n\}$ inductively by

$$y_{n+1} = \mu^{-1} y_{n-1} - \mu^{-1} P(y_n) \tag{5.17}$$

Then analogous to (1.32), (1.33) and (1.34) are the similarly obtained results

$$y_1 - x_1 = \mu^{-1} \varepsilon \tag{5.18}$$

and

$$y_{2n} - x_{2n} < \mu^{-1}(y_{2n-2} - x_{2n-2}) \leq 0 \tag{5.19}$$

and

$$y_{2n+1} - x_{2n+1} > \mu^{-1}(y_{2n-1} - x_{2n-1}) \geq \mu^{-n-1} \varepsilon \geq \varepsilon . \tag{5.20}$$

So at most one of the sequences may converge.

Theorem 21: Let h and g be two homeomorphisms such that

$$h = P + \mu h^{-1} \tag{5.21}$$

and

$$g = P + \mu g^{-1} . \tag{5.22}$$

Then $h = g$.

Proof: Identities (5.21) and (5.22) imply (5.1). If $\mu \geq 1$, then (5.21) implies that $h > h^{-1}$. Therefore, $h > I$ and the corollary to Theorem 17 finishes the proof for $\mu \geq 1$.

If $\mu < 1$, then for any $x > 0$, either $h(x) \leq x$ or $g(x) \leq x$ or else $h(x) > x$ and $g(x) > x$. In the first two cases Theorem 17 implies that $h(x) = g(x)$. In the last case, the sequences $\{x_n\} = \{h^{-n}(x)\}$ and $\{y_n\} = \{g^{-n}(x)\}$ are both convergent. But these sequences satisfy (5.16) and (5.17). Since $x_0 = y_0 = x$, Theorem 20 implies that

$$h(x) = x_{-1} = y_{-1} = g(x) . \tag{5.23}$$

Theorem 22: Let $\mu \leq 1$ and $P + I > I$. Let g_1 be any homeomorphism. Then the sequence of homeomorphisms defined inductively by

(5.24) $$g_{n+1} = P + \mu g_n^{-1}$$

converges uniformly on every bounded subset of $[0, \infty)$ to a unique homeomorphism h such that

(5.21) $$h = P + \mu h^{-1} .$$

Proof: Choose

(5.25) $$h_1 = g_1 \cap g_2(\mu I) \cap P .$$

Then Theorem 19 is applicable. For any $x > 0$, $g(x) > x$ for if not, $h(x) \leq g(x) \leq x$ which implies that $h^{-1}(x) \geq x$ and hence that

(5.26) $$g(x) = P(x) + \mu h^{-1}(x) \geq P(x) + \mu x > x .$$

Therefore,

(5.27) $$hg = (P + \mu g^{-1})g = Pg + \mu I > P + \mu I > I .$$

But $hg > I$ implies that $h > g^{-1}$ which in turn implies that $gh > I$.

For any point $x_0 > 0$, define x_{-1} as $h(x_0)$. Let the sequence $\{x_n\}$ be defined inductively by (5.16). Then arguments analogous to (1.28) and (1.29) imply (1.26) and (1.27). Since

(5.28) $$h^{-1}g^{-1} = (gh)^{-1} < I ,$$

the sequence $\{x_n\}$ converges to zero. By similar arguments, if x_{-1} is defined as $g(x_0)$, the sequence $\{x_n\}$ will still converge. Theorem 20 therefore implies that $g(x_0) = h(x_0)$. Since $h = g$, the convergence for g_1, insert μ into the proper positions of (1.39) and (1.40), and continue the argument of the paragraph containing (1.39) and (1.40). Uniqueness of h is obtained from Theorem 21.

Corollary: Let h and g be two homeomorphisms such that

(5.15) $$h = P + \mu g^{-1} \quad \text{and}$$

$$g = P + \mu h^{-1} .$$

Let $\mu \leq 1$ and $P + \mu I > I$. Then $h = g$.

Theorem 23: Let $\mu \leq 1$ and $P + \mu I > I$. Then a sequence generated by (5.16) will converge if and only if $x_{-1} = h(x_0)$ where h is the homeomorphism of Theorem 22. Furthermore, if it does converge, it will converge to zero.

If $h(x_0) > x_{-1}$, then all of the evenly subscripted elements of the sequence are positive, but all but a finite number of the elements with odd subscripts are negative.

If $h(x_0) < x_{-1}$, then all of the odd subscripted elements are positive and all but a finite number of the even subscripted elements are negative.

Proof: $P + \mu I > I$ and (5.21) imply that $h > I$. If $h(x_0) = x_{-1}$, then $\left\{ x_n \right\} = \left\{ h^{-n}(x_0) \right\}$ and the sequence converges to zero. When $h(x_0) \neq x_{-1}$, then (5.19) and (5.20) may replace (1.33) and (1.34) in order to continue the arguments of the paragraphs containing (1.33) through (1.36).

Theorem 24: Let μ^{-1} be an integer, let P map integers into integer multiples of μ and let $P + \mu I > I$. Then the h satisfying

$$h = P + \mu h^{-1} \tag{5.21}$$

will never map a positive integer into an integer.

Proof: Use the proof presented in the last paragraph of Section I.

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*Courtesy of H. W. Gould

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EXTENSIONS OF RECURRENCE RELATIONS

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The purpose of this article is to investigate analytic extensions of F_n and L_n to the complex plane. We shall begin by considering a particular extension. Later we will consider alternate extensions. We begin with the following notation

$$\alpha = (1+\sqrt{5})/2 \quad \text{and} \quad \beta = (1-\sqrt{5})/2 \quad .$$

Since $\beta < 0$ we adopt the convention $\beta = e^{i\pi}(-\beta)$.
With these conventions, we shall make the following definitions:

The Fibonacci Function, $F(z) = 1/\sqrt{5}\ (\alpha^z - \beta^z)$

The Lucas Function, $L(z) = \alpha^z + \beta^z$.

Note that $F(n) = F_n$ and $L(n) = L_n$, where n denotes an integer.

I Periodic Properties of $F(z)$ and $L(z)$

Theorem 1. α^z is periodic with period $2\pi i/\ln\alpha = p_\alpha$.

Proof. $\alpha^{z+p_\alpha} = \alpha^z e^{2\pi i} = \alpha^z$.

Theorem 2. β^z is periodic with period $2\pi/(\ln^2\alpha + \pi^2)(\pi - i\ln\alpha) = p_\beta$.

Proof. Since $-\ln\alpha = \ln(-\beta)$, we have

$$\beta^{z+p_\beta} = \beta^z e^{2\pi i} = \beta^z \quad .$$

Theorem 3. $F(z)$ and $L(z)$ are not periodic.

Proof. Deny! Assume $F(z)$ has period ω. $F(0) = 0 = F(\omega)$ implies $\alpha^\omega = \beta^\omega$.
Thus $F(z+\omega) = 1/\sqrt{5}\ \alpha^\omega(\alpha^z - \beta^z)$.

Hence $\alpha^\omega = 1$, so $\mathrm{Re}(\omega) = 0$. Then $\beta^\omega \neq 1$ unless $\omega = 0$.
The proof for $L(z)$ is similar.

II Zeroes of $F(z)$ and $L(z)$.

Theorem 4. The zeroes of $F(z)$ are

$$4k\pi\ln\alpha/(4\ln^2\alpha + \pi^2)(-\pi/2\ln\alpha + i) \quad .$$

Proof. Note that this theorem implies the only real zero of

$F(z)$ is 0.

$F(z) = 0$ implies $(\alpha/\beta)^z = 1 = e^{2k\pi i}$, k an integer.

Setting $z = x + iy$ and collecting real and imaginary parts and equating, the result follows.

The moduli of the zeroes are $|2k|\pi/\sqrt{4\ln^2\alpha + \pi^2}$.

Theorem 5. The zeroes of $L(z)$ are

$2(2k+1)\ln\alpha/(4\ln^2\alpha + \pi^2)(-\pi/2\ln\alpha + i) = z_k$,

where k is an integer.

Proof. Note that this theorem implies $L(z)$ has no real zeroes. Write $-1 = e^{(2k+1)\pi i}$ and proceed as above.

The moduli of the zeroes are $|2k{+}1|\pi/\sqrt{4\ln^2\alpha + \pi^2}$.

Observe that all of the zeroes of $L(z)$ and $F(z)$ are on the ray $\theta = \operatorname{Arctan}((-2\ln\alpha)/\pi) \sim -20^\circ$.

III <u>Behavior of $F(z)$ and $L(z)$ on the real axis.</u>

Theorem 6. On the real axis, the only real values of $F(z)$ and $L(z)$ are at $z = n$ (an integer), that is, F_n and L_n.

Proof. Since $y = 0$, $\alpha^z = \alpha^x$, $\beta^z = e^{-x\ln\alpha + \pi x i}$;

$\operatorname{Im} F(z) = \operatorname{Im} L(z) = 0$ yields

$-1/\sqrt{5}\, e^{-x\ln\alpha} \sin \pi x = 0$ or

$e^{-x\ln\alpha} \sin \pi x = 0$.

Hence $x = k$, k an integer.

(It is not too difficult to show that the only lattice points with real images for $F(z)$ are on the real axis.)

IV <u>Identities Satisfied by $F(z)$ and $L(z)$.</u>

Many of the identities of F_n and L_n carry over to $F(z)$ and $L(z)$. We shall list a few of them. They are easy to verify.

a. $F(z+2) = F(z+1) + F(z)$

b. $L(z+2) = L(z+1) + L(z)$

c. $F(z+1)F(z-1) - F^2(z) = e^{\pi i z}$

d. $L^2(z) - 5F^2(z) = 4e^{\pi i z}$

e. $F(-z) = -e^{\pi iz} F(z)$

f. $F(z)L(z) = F(2z)$

g. $F(z+w) = F(z-1)F(w) + F(z)F(w+1)$

h. $F(3z) = F^3(z+1) + F^3(z) - F^3(z-1)$

i. $\lim\limits_{x \to \infty} F(x)/F(x+1) =$

$\lim\limits_{y \to \infty} F(iy)/F(iy+1) = -\beta$

In general, $(-1)^n$ in an identity for F_n and L_n carries over to $e^{\pi iz}$. The identities which do not carry over to $F(z)$ and $L(z)$ are those which only make sense for integral argument. That is, those which involve binomial coefficients, etc.

V Analytic Properties of F(z) and L(z).

Note that our convention for β implies $\ln\beta = \pi i + \ln(-\beta)$. It is thus immediate that $F(z)$ and $L(z)$ are holomorphic in the plane (entire functions).

From the Taylor formula, we have for any finite z,

$$F(z) = 1/\sqrt{5} \sum_{k=0}^{\infty} \left\{ \left[(\ln^k \alpha)\alpha^w - (\ln^k \beta)\beta^w \right] /k! \right\} (z-w)^k \quad \text{and}$$

$$L(z) = \sum_{k=0}^{\infty} \left\{ \left[(\ln^k \alpha)\alpha^w + (\ln^k \beta)\beta^w \right] /k! \right\} (z-w)^k .$$

Note the results when these are used with $w = 0$ and $z = n$ or with $w = n-1$ and $z = n$.

$$F_n = 1/\sqrt{5} \sum_{k=0}^{\infty} \left[(\ln^k \alpha)\alpha^{n-1} - (\ln^k \beta)\beta^{n-1} \right] /k! .$$

This is, I believe, a new representation for F_n. The Hadamard Factorization theorem can be used to express $L(z)$ as a canonical product. As in theorem 5, let z_k represent a zero of $L(z)$. Renumber z_k as follows:

$$k = -1,\ 0,\ -2,\ 1,\ -3,\ 2,\ \ldots$$
$$n = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$$

Now $|z_n| \leq |z_{n+1}|$ and $|z_n| = 0(n)$. It is easy to see that $L(z)$ is of order and genus 1 and we have

$$L(z) = e^{cz} \prod_{n=1}^{\infty} (1 - z/z_n)\, e^{z/z_n}, \quad \text{where}$$

$$c = - \sum_{n=1}^{\infty} \left[\ln(1 - 1/z_n) + 1/z_n\right] .$$

We shall now discuss exceptional values of $F(z)$ and $L(z)$. Since $F(z)$ and $L(z)$ are entire functions with essential singularities at ∞, by Picard's theorem, they must take on every value, except possibly one, and infinite number of times.

$$\lim_{x\to\infty} L(x-ix) = \lim_{x\to\infty} F(x-ix) = 0$$

Thus 0 is an asymptotic value for $F(z)$ and $L(z)$.

$$\lim_{x\to\infty} L(x) = \lim_{x\to\infty} F(x) = \infty \text{ and } \infty$$

is an asymptotic value for $F(z)$ and $L(z)$.

Ahlfors has shown that entire functions of order P have at most 2P asymptotic values [1]. Further, if an integral function has z as an exceptional value, then z is an asymptotic value [2]. Now 0 is not an exceptional value for $F(z)$ or $L(z)$; Part II. Hence $F(z)$ and $L(z)$ have no finite exceptional values.

Thus the Fibonacci Prime Conjecture is trivial in the complex plane; that is, there are an infinite number of Fibonacci images which are distinct primes. It is conceivable that a knowledge of the distribution of prime images might yield a resolution of this conjecture, although this problem is probably more difficult than the conjecture itself. Poisson's formulae for real and imaginary parts of $F(z)$ might be useful, but the integrals are horrible Fresnel type integrals [3].

A characterization of the point set corresponding to $\operatorname{Im} F(z) = 0$ should present an interesting problem. Graphs of $\{z \mid \operatorname{Re}F(z) = 0\}$,

$\{z \mid \mathrm{Im}\,F(z) = 0\}$, $\{z \mid |F(z)| = M\}$ in some neighborhood of the origin should yield interesting diagrams.

VI Alternate Extensions.

There are an infinite number of extensions of F_n and L_n to entire functions in the complex plane. If the functional equation

$$G(z+2) = G(z+1) + G(z);\quad G(0) = 0,\ G(1) = 1\ ,$$

is used as a starting point, it appears that very little can be established. However it is possible to obtain extensions which are real at every point of the real axis. Consider, for example,

$$F_1(z) = 1/\sqrt{5}\left[\alpha^z - \sin\left(\frac{2z+1}{2}\pi\right)(-\beta)^z\right]\ .$$

Note that $F_1(z)$ satisfies the relation,

$$F_1(z+1)F_1(z-1) - F_1^2(z) = \sin(2z+1)\pi/2\ .$$

$F_1(z)$ is an entire function and has zeroes on the negative real axis and $F_1(n) = F_n$, n an integer.

Another type of extension is,

$$F_2(z) = e^{2\pi i z}F(z) + \sin\pi z\ .$$

Practically none of the above theorems hold for arbitrary extensions. The following construction seems to indicate that F_n could be extended to a periodic entire function in the complex plane. Consider the rectangle, R, in the complex plane bounded by

$$(1, 0),\ (1, 1),\ (-1, 1),\ (-1, 0)\ .$$

Select a function, $F_3(z)$, subject to the following conditions:

a. $F_3(0) = 0$

b. $F_3(-1+iy) + F_3(iy) = F_3(1+iy);\ y \in [0, 1]$

c. $F_3(x) = F_3(x+i);\ x \in [-1, 1]$

d. $F_3(-1) = F_3(1) = 1$

e. $F_3(z)$, analytic on R .

Extend $F_3(z)$ vertically by periodicity and horizontally by the functional equation, $F_3(z+2) = F_3(z+1) + F_3(z)$. The extension would be an entire function with period i and $F_3(n) = F_n$, n an integer.

REMARKS

Selection of a proper extension for $F(n)$ should, via the machinery of Analytic Function theory, put a powerful wrench on the Fibonacci Prime Conjecture.

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CORRECTIONS

"Binomial Coefficients, the Bracket Function, and Compositions with Relatively Prime Summands" by H. W. Gould, Fibonacci Quarterly, 2(1964), pp. 241-260.

Page 241. The second paragraph should begin: "Indeed this result is equivalent to the identical congruence $(1 - x)^p \equiv 1 - x^p \pmod p$ . . . "

Page 245. In Theorem 3 it is necessary to require $a_i > 0$.

Page 257. Line after relation (48), replace "out" by "our".

Page 251. Line 9 from bottom, for "as" read "an".

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SOME ORTHOGONAL POLYNOMIALS RELATED TO FIBONACCI NUMBERS

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1. We consider polynomials $f_n(x)$ such that

$$f_{n+2}(x) = (x+2n+p+1)f_{n+1}(x) - (n^2+pn+q)f_n(x) \qquad (n = 0, 1, 2, \ldots), \tag{1}$$

where

$$f_0(x) = 0, \; f_1(x) = 1 \ . \tag{2}$$

It follows at once that $f_n(x)$ is a polynomial in x of degree $n-1$ for $n \geq 1$. The parameters p, q are arbitrary but we shall assume that

$$p^2 - 4q \neq 0 \ . \tag{3}$$

Let α, β denote the roots of the equation

$$x^2 - px + q = 0 \ . \tag{4}$$

In view of (3), the roots α, β are distinct and

$$\alpha+\beta = p, \quad \alpha\beta = q \ . \tag{5}$$

We shall construct a generating function for $f_n(x)$:

$$F(t) = F(t, x) = \sum_{n=0}^{\infty} f_n(x)t^n/n! \tag{6}$$

It is easily verified that (1), (2) and (6) imply

$$(1-t)^2F''(t) - [(x+(p+1)(1-t)]F'(t) + qF(t) = 0 \ , \tag{7}$$

where the primes indicate differentiation with respect to t.

It is convenient to define an operator.

$$\Delta = (1-t)^2D^2 - (p+1)D + q \quad (D = d/dt) \ . \tag{8}$$

Thus (7) becomes

$$\Delta F(t) = xF'(t) \ . \tag{9}$$

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Consider

$$\Delta(1-t)^{-\alpha-k} = \left\{(\alpha+k)(\alpha+k+1)-(p+1)(\alpha+k) + q\right\}(1-t)^{-\alpha-k} .$$

Making use of (4) we find that this reduces to

$$(10) \qquad \Delta(1-t)^{-\alpha-k} = k(2\alpha-p+k)(1-t)^{-\alpha-k} .$$

Thus, if we put

$$(11) \qquad \Phi(t, \alpha) = \sum_{k=0}^{\infty} \frac{(\alpha)_k x^k}{k!(2\alpha-p+1)_k} (1-t)^{-\alpha-k} ,$$

where

$$(\alpha)_k = \alpha(\alpha+1) \ldots (\alpha+k-1) ,$$

we get

$$\Delta \Phi(t, \alpha) = \sum_{k=0}^{\infty} \frac{(\alpha)_{k+1} x^{k+1}}{k!(2\alpha-p+1)_k} (1-t)^{-\alpha-k-1} .$$

We have therefore

$$(12) \qquad \Delta \Phi(t, \alpha) = x \Phi'(t, \alpha)$$

and in exactly the same way

$$(13) \qquad \Delta \Phi(t, \beta) = x \Phi'(t, \beta) .$$

It follows from (11) that

$$\Phi(t, \alpha) = \sum_{k=0}^{\infty} \frac{(\alpha)_k x^k}{k!(2\alpha-p+1)_k} \sum_{n=0}^{\infty} \frac{(\alpha+k)_n}{n!} t^n$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(\alpha)_{n+k} x^k}{k!(2\alpha-p+1)_k} .$$

If we put

$$(14) \qquad \phi_n(x, \alpha) = \sum_{k=0}^{\infty} \frac{(\alpha)_{n+k} x^k}{k!(2\alpha-p+1)_k} ,$$

then we have

$$\Phi(t, \alpha) = \sum_{n=0}^{\infty} \phi_n(x, \alpha)t^n/n! \quad . \tag{15}$$

Note that (14) implies

$$\phi_n(x, \alpha) = (\alpha)_n \cdot {}_1F_1(\alpha+n; 2\alpha-p+1; x) \quad , \tag{16}$$

where ${}_1F_1$ denotes a hypergeometric function in the usual notation.

2. If we make use of (12) and (15) we find without much difficulty that $\phi_n(x, \alpha)$ satisfies the recurrence

$$\phi_{n+2}(x, \alpha) = (x+2n+p+1)\phi_{n+1}(x, \alpha) - (n^2+pn+q)\phi_n(x, \alpha) \quad (n \geq 0) \quad . \tag{17}$$

Clearly $\phi_n(x, \beta)$ satisfies the same recurrence. Thus any linear combination

$$\psi_n(x) = A\phi_n(x, \alpha) + B\phi_n(x, \beta) \quad ,$$

where A, B, are independent of n but may depend on x, α, β, will also satisfy (17).

We choose A, B so that

$$\psi_0(x) = 0, \quad \psi_1(x) = 1 \quad . \tag{18}$$

This requires

$$AC = \phi_0(x, \beta), \qquad BC = -\phi_0(x, \alpha) \quad ,$$

where

$$C = \phi_1(x, \alpha)\phi_0(x, \beta) - \phi_1(x, \beta)\phi_0(x, \alpha) \quad . \tag{19}$$

It is clear by comparison of (17) and (18) with (1) and (2) that

$$\psi_n(x) = f_n(x) \qquad (n = 0, 1, 2, \ldots) \quad .$$

We have therefore

$$f_n(x) = \frac{\phi_n(x, \alpha)\phi_0(x, \beta) - \phi_n(x, \beta)\phi_0(x, \alpha)}{C} \tag{20}$$

with C defined by (19).

Thus by (6) and (15)

$$F(t) = C^{-1}\left\{\Phi(t,\alpha)\phi_0(x,\beta) - \Phi(t,\beta)\phi_0(x,\alpha)\right\}, \tag{21}$$

so that we have obtained a generating function for $f_n(x)$.

3. In addition to the polynomial $f_n(x)$ we may construct a second solution $g_n(x)$ of (1) such that

$$g_0(x) = 1, \quad g_1(x) = x+p+1 . \tag{22}$$

Thus $g_n(x)$ is a polynomial in x of degree n. By exactly the same method we have used above, we find that

$$g_n(x) = -2\,\frac{\phi_n(x,\alpha)\phi_1(x,\beta) - \phi_n(x,\beta)\phi_1(x,\alpha)}{C} + (x+p)f_n(x) . \tag{23}$$

If we put

$$G(t) = G(t,x) = \sum_{n=0}^{\infty} g_n(x)\, t^n/n! \tag{24}$$

it follows that

$$G(t) = -2\,\frac{\Phi(t,\alpha)\phi_1(x,\beta) - \Phi(t,\beta)\phi_1(x,\alpha)}{C} + \frac{x+p}{C}\left(\Phi(t,\alpha)\phi_0(x,\beta) - \Phi(t,\beta)\phi_0(x,\alpha)\right) . \tag{25}$$

If the coefficient n^2+pn+q occurring in (1) is positive for all $n \geq 0$ then by a known result [1] we can assert that the polynomials $g_n(x)$ are orthogonal on the real line with respect to some weight function. The same remark applies to the $f_n(x)$. It would be of interest to explicitly determine these weight functions.

4. We have assumed in the above discussion that α and β are distinct. When α and β are equal we may replace (1) by

$$f_{n+2}(x) = (x+2n+2\alpha+1)f_{n+1}(x) - (n+\alpha)^2 f_n(\alpha) \qquad (n = 0, 1, 2, \ldots) . \tag{26}$$

We now put

$$\phi_n(x) = \sum_{k=0}^{\infty} \frac{(a)_{n+k}}{k!\,k!} x^k , \tag{27}$$

$$\Phi(t) = \sum_{n=0}^{\infty} \phi_n(x) t^n/n! = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{k!\,k!} (1-t)^{-a-k} . \tag{28}$$

It is easily verified that

$$\phi_{n+2}(x) = (x+2n+2a+1)\phi_{n+1}(x) - (n+a)^2 \phi_n(x) \quad (n = 0, 1, 2, \ldots) \tag{29}$$

and that

$$\Delta \Phi(t) = x\Phi'(t) . \tag{30}$$

As a second solution of (26) we take

$$\psi_n(x) = \sum_{k=0}^{\infty} \frac{(a)_{n+k}}{k!\,k!} (\sigma_{n+k}(a) - 2\sigma_k) x^k , \tag{31}$$

where

$$\sigma_k(a) = \frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+k-1} , \tag{32}$$

$$\sigma_k = \sigma_k(1)\ 1\ 1 + \frac{1}{2} + \ldots + \frac{1}{k} .$$

We omit the proof that $\psi_n(x)$ does indeed satisfy (26).

It is convenient to put

$$\Psi(t) = \sum_{n=0}^{\infty} \psi_n(x) t^n/n! . \tag{33}$$

It can be verified that $\Psi(t)$ also satisfies (30).

If we now put

$$f_n(x) = \frac{\phi_n(x)\psi_0(x) - \phi_0(x)\psi_n(x)}{\phi_1(x)\psi_0(x) - \phi_0(x)\psi_1(x)} \quad (n = 0, 1, 2, \ldots) , \tag{34}$$

then we have

$$f_0(x) = 0, \quad f_1(x) = 1 \quad . \tag{35}$$

Thus $f_n(x)$ is a polynomial of degree $n-1$ in x for $n \geq 1$ and is the unique solution of (26) that satisfies (35).

Similarly if we put

$$g_n(x) = 2\,\frac{\phi_1(x)\psi_n(x) - \psi_1(x)\phi_n(x)}{\phi_1(x)\psi_0(x) - \phi_0(x)\psi_0(x)} + (x+2\alpha+1)f_n(x) \tag{36}$$

then

$$g_0(x) = 2, \quad g_1(x) = x + 2\alpha+1 \quad . \tag{37}$$

Thus $g_n(x)$ is a polynomial of degree n in x and is the unique solution of (26) that satisfies (37).

Explicit formulas for the generating functions $\Phi(t)$ and $\Psi(t)$ can now be stated without any difficulty.

Here again it would be of interest to explicitly determine the weight functions connected with $\{f_n(x)\}$ and $\{g_n(x)\}$, respectively.

REFERENCE

1. J. Favard, Sur les polynomies de Tchebicheff, Comptes rendus de l' Academie des Sciences, Paris, vol. 200 (1935), pp. 2052-2053.

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A NOTE ON ORTHOGONAL POLYNOMIALS

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1. INTRODUCTION

The recurrence relation for orthogonal polynomials $q_n(x)$ (leading coefficient one) associated with the density function $f(x)$ over the interval $[a, b]$ is derived explicitly in terms of the moments of $f(x)$. Further, an alternative proof is given of the theorem that if $f(x)$ is symmetrical about $x = 0$, then the polynomials $q_n(x)$ are even or odd functions according as n is even or odd.

2. RESULTS

Let $f(x)$ denote the density of a distribution function $F(x)$ with infinitely many points of increase in the finite or infinite interval a, b, and let the moments

$$m_r = \int_a^b x^r f(x)dx$$

exist for $r = 0, 1, 2, \ldots$.

It is well known, see Szegö [1], that there exists a sequence of polynomials $p_0(x), p_1(x), p_2(x), \ldots$ uniquely determined by the following conditions:

(a) $p_n(x)$ is a polynomial of precise degree n in which the coefficient of x^n is positive.

(b) the system $p_n(x)$ is orthonormal, that is

$$\int_a^b p_m(x)p_n(x)f(x)dx = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

If, on the other hand, $F(x)$ has only N points of increase, then the $p_n(x)$ exist and are uniquely determined for $n = 0, 1, \ldots, N-1$; and if $F(x)$ has only finitely many finite moments, say m_{2k} or

m_{2k+1} exist, then the $p_n(x)$ exist and are uniquely determined for $n = 0, 1, \ldots, k$.

The polynomials $p_n(x)$ satisfying conditions (a) and (b) above are of the form

$$p_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} m_0 & m_1 & m_2 \cdots\cdots & m_n \\ m_1 & m_2 & m_3 \cdots\cdots & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ m_{n-1} & m_n & m_{n+1} \cdots & m_{2n-1} \\ 1 & x & x^2 \cdots\cdots & x^n \end{vmatrix}, \tag{1}$$

where

$$D_n = \begin{vmatrix} m_0 & m_1 & m_2 \cdots\cdots & m_n \\ m_1 & m_2 & m_3 \cdots\cdots & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ m_{n-1} & m_n & m_{n+1} \cdots & m_{2n-1} \\ m_n & m_{n+1} & m_{n+2} \cdots & m_{2n} \end{vmatrix}$$

and where the leading coefficients of $p_n(x)$ are

$$\sqrt{\frac{D_{n-1}}{D_n}}\ .$$

If we now define the polynomials $q_n(x)$ as

$$q_n(x) = \sqrt{\frac{D_n}{D_{n-1}}}\ p_n(x)\ , \tag{2}$$

then the $q_n(x)$ are orthogonal polynomials whose leading coefficients are always one.

According to Szegö[1], the following relation holds for any three consecutive orthonormal polynomials:

$$(3) \qquad p_n(x) = (A_n x + B_n)p_{n-1}(x) - C_n p_{n-2}(x), \quad n = 2, 3, \ldots$$

where A_n, B_n and C_n are constants, $A_n > 0$ and $C_n > 0$. If the highest coefficient of $p_n(x)$ is denoted by k_n, then

$$A_n = \frac{k_n}{k_{n-1}}, \quad C_n = \frac{A_n}{A_{n-1}} = \frac{k_n k_{n-2}}{k_{n-1}^2}$$

Since $k_n = \sqrt{\frac{D_{n-1}}{D_n}}$, we shall have

$$A_n = \sqrt{\frac{D_{n-1}}{D_n D_{n-2}}}, \quad C_n = \sqrt{\frac{D_{n-3}}{D_n}} \left(\frac{D_{n-1}}{D_{n-2}}\right)^{3/2}$$

The relation (3) then becomes

$$(4) \qquad p_n(x) = \left(\frac{D_{n-1}}{\sqrt{D_n D_{n-2}}} x + B_n\right) p_{n-1}(x) - \sqrt{\frac{D_{n-3}}{D_n}} \left(\frac{D_{n-1}}{D_{n-2}}\right)^{3/2} p_{n-2}(x)$$

Multiplying both sides of (4) by

$$\sqrt{\frac{D_n}{D_{n-1}}}$$

and using (2), we get

$$(5) \qquad q_n(x) = \left(x + \frac{\sqrt{D_n D_{n-2}}}{D_{n-1}} B_n \right) q_{n-1}(x) - \frac{D_{n-1} D_{n-3}}{D_{n-2}^2} q_{n-2}(x)$$

To find B_n, let us suppose that k'_n is the coefficient of x^{n-1} in $p_n(x)$, while k_n is the coefficient of x^n in $p_n(x)$. By equating the coefficients of x^{n-1} on both sides of (3), we get

$$k'_n = A_n k'_{n-1} + B_n k_{n-1}$$

which gives

$$(6) \qquad B_n = \frac{k'_n}{k_{n-1}} - A_n \frac{k'_{n-1}}{k_{n-1}}$$

But

$$A_n = \frac{k_n}{k_{n-1}} ,$$

so that (6) can be written as

$$B_n = \frac{k_n}{k_{n-1}} \left[\frac{k'_n}{k_n} - \frac{k'_{n-1}}{k_{n-1}} \right] \tag{7}$$

Let D_n^* denote the determinant obtained by deleting the (n+1)th row and the nth column of D_n. Then

$$k'_n = - \frac{D_n^*}{\sqrt{D_n D_{n-1}}}$$

Substituting for k_n, k'_n and k'_{n-1} in (7), we get

$$B_n = \frac{D_{n-1}}{\sqrt{D_n D_{n-2}}} \left[- \frac{D_n^*}{D_{n-1}} + \frac{D_{n-1}^*}{D_{n-2}} \right] \tag{8}$$

Using the value of B_n given by (8) in (5), we obtain

$$q_n(x) = \left(x - \frac{D_n^*}{D_{n-1}} + \frac{D_{n-1}^*}{D_{n-2}} \right) q_{n-1}(x) - \frac{D_{n-1} D_{n-3}}{D_{n-2}^2} q_{n-2}(x) . \tag{9}$$

Thus (9) gives the recurrence relation for orthogonal polynomials associated with the density function $f(x)$ explicitly in terms of the moments of $f(x)$. The recurrence relation (9) is valid also for $n = 1$ if we set $D_0^* = 0$, $D_{-1} = 1$ and $D_{-2} = 0$.

If the density function $f(x)$ is symmetrical about $x = 0$, that is, if $f(-x) = f(x)$ and $a = -b$, then

$$m_1 = m_3 = \ldots\ldots = m_{2r+1} = \ldots\ldots = 0$$

If the odd order moments are all zero, we shall prove below in Theorem 1 that D_n^* vanishes for $n = 1, 2, \ldots$ which will imply that $B_n = 0$ for $n = 1, 2, \ldots$.

We shall also prove below in Theorem 2 that, in this case, the polynomials $q_n(x)$ are even or odd function according as n is even or odd.

The recurrence relation for orthogonal polynomials associated with the symmetrical density function $f(x)$ is then obtained as

$$q_n(x) = xq_{n-1}(x) - \frac{D_{n-1}D_{n-3}}{D_{n-2}^2} q_{n-2}(x) \tag{10}$$

In particular, for $n = 0, 1, 2, 3,$ and 4, the orthogonal polynomials associated with the symmetrical density function $f(x)$ are obtained as follows:

$$q_0(x) = 1$$

$$q_1(x) = x$$

$$q_2(x) = x^2 - m_2$$

$$q_3(x) = x^3 - \frac{m_4}{m_2} x$$

$$q_4(x) = x^4 - \frac{m_6 - m_2 m_4}{m_4 - m_2^2} x^2 + \frac{m_2 m_6 - m_4^2}{m_4 - m_2^2}$$

We now prove the following two Theorems:

Theorem 1. Let $D = \left[d_{ij} \right]$ be an $(n \times n)$ matrix where $d_{ij} = 0$ for $i+j$ odd, and d_{ij} is arbitrary for $i+j$ even. Let D^* be an $(n-1)x(n-1)$ matrix obtained by deleting the uth row and the vth column of D such that $u+v$ is odd. Then the determinant of D^* is zero.

Proof. To prove the theorem we consider two cases: (1) n even and (2) n odd.

Case 1 $n = 2k$ (even)

Let us assume that we get D^* by deleting an odd row and an even column. Then by shifting rows and columns of D^*, we obtain

$$D^{**} = \begin{bmatrix} A_1 & A_2 \\ & \\ A_3 & A_4 \end{bmatrix}$$

where D^{**} is a matrix of $(2k-1)x(2k-1)$ elements and

$A_1 = k \times k$ matrix with zero elements

$A_2 = k \times (k-1)$ matrix with arbitrary elements

$A_3 = (k-1) \times k$ matrix with arbitrary elements

$A_4 = (k-1) \times (k-1)$ matrix with zero elements.

If we now take the Laplace expansion of D^{**} by $(k \times k)$ minors, then it can be easily seen that the determinant of D^{**} is zero, which will imply that the determinant of D^* is zero.

The result also follows if we take D^* by deleting an even row and an odd column.

Case 2 $n = 2k+1$ (odd)

In this case, we obtain

$$D^{**} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

where D^{**} is a matrix of $(2k \times 2k)$ elements and

$B_1 = k \times (k+1)$ matrix with zero elements

$B_2 = k \times (k-1)$ matrix with arbitrary elements

$B_3 = k \times (k+1)$ matrix with arbitrary elements

$B_4 = k \times (k-1)$ matrix with zero elements.

If we take the Laplace expansion of D^{**} by $(k \times k)$ minors, then we shall have the determinant of D^{**} equal to zero, which will imply that the determinant of D^* is zero.

Theorem 2 Let $q_n(x)$, defined by (2), be the orthogonal polynomials associated with the density function $f(x)$ symmetrical about $x = 0$. Then the polynomials $q_n(x)$ are even or odd functions according as n is even or odd.

Proof If the density function $f(x)$ is symmetrical about $x = 0$, then all the odd order moments are zero, that is

$$m_1 = m_3 = \dots = m_{2r+1} = \dots = 0$$

The proof of the theorem follows immediately by expanding $p_n(x)$, defined by (1), in terms of the last row of the determinant and making use of the result of Theorem 1.

REFERENCE

1. Szegö, G., Orthogonal Polynomials, rev. ed., Amer. Math. Soc. Colloquium Publications, Vol. 23, New York, 1959.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by Verner E. Hoggatt, Jr.
San Jose State College, San Jose, California

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-78 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

(i) Show $$\frac{x^{n-1}}{(1-x)^n} = \sum_{m=0}^{\infty} \binom{m}{n-1} x^m, \quad (n \geq 1)$$

where $\binom{m}{n}$ are the binomial coefficients. See "Diagonals of Pascal's Triangle", D. C. Duncan, pg 1115, AMM, Dec. 1965.

(ii) Show $$\frac{x}{1-x-x^2} = \sum_{m=0}^{\infty} \begin{bmatrix} m \\ 1 \end{bmatrix} x^m$$

$$\frac{x^2}{1-2x-2x^2+x^3} = \sum_{m=0}^{\infty} \begin{bmatrix} m \\ 2 \end{bmatrix} x^m ,$$

$$\frac{x^3}{1-3x-6x^2+3x^3+x^4} = \sum_{m=0}^{\infty} \begin{bmatrix} m \\ 3 \end{bmatrix} x^m ,$$

where $\begin{bmatrix} m \\ n \end{bmatrix}$ are the Fibonomial coefficients as in H-63, April 1965, FQJ and H-72 of Dec. 1965, FQJ.

The generalization is:

Let $$f(x) = \sum_{h=0}^{k} (-1)^{h(h+1)/2} \begin{bmatrix} k \\ h \end{bmatrix} x^h ,$$

then show

$$\frac{x^{k-1}}{f(x)} = \sum_{m=0}^{\infty} \left[{m \atop k-1}\right] x^m, \qquad (k \geq 1) .$$

H-79 *Proposed by J.A.H. Hunter, Toronto, Ontario, Canada*

Show

$$F_{n+1}^4 + F_n^4 + F_{n-1}^4 = 2\left[2F_n^2 + (-1)^n\right]^2$$

H-80 *Proposed by J.A.H. Hunter, Toronto, Ontario, Canada and Max Rumney, London, England*

Show

$$\sum_{r=0}^{n} \binom{n}{r} F_{r+2}^2 = \sum_{r=0}^{n} \binom{n-1}{r} F_{2r+5}^2 .$$

H-81 *Proposed by Vassili Daiev, Sea Cliff, N.Y.*

Find the nth term of the sequence

1, 1, 3, 1, 5, 3, 7, 1, 9, 5, 11, 3, 13, 7, 15, 1, 17, 9, 19, 5, . . .

H-82 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

If $f_0(x) \equiv 0$ and $f_1(x) \equiv 1$, $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$ then show

$$\mathrm{Tan}^{-1}\frac{1}{x} = \sum_{n=1}^{\infty} \mathrm{Tan}^{-1}\left(\frac{x}{f_{2n+1}(x)}\right) .$$

H-83 *Proposed by Mrs. William Squire, Morgantown, West Va.*

Show

$$\sum_{t=1}^{\left[\frac{m+1}{2}\right]} (-1)^{t-1} \binom{m-t}{t-1} 3^{m+1-2t} = F_{2m} ,$$

where $[x]$ is the greatest integer function.

SOLUTIONS

THE LOST IS FOUND

H-42 *Proposed by J.D.E. Konhauser, State College, Pa.*

A set of nine integers having the property that no two pairs have the same sum is the set consisting of the nine consecutive Fibonacci numbers, 1, 2, 3, 5, 8, 13, 21, 34, 55 with total sum 142. Starting with 1, and annexing at each step the smallest positive integer which produces a set with the stated property yields the set 1, 2, 3, 5, 8, 13, 21, 30, 39 with sum 122. Is this the best result? Can a set with lower total sum be found?

Solution by Frank Urbanija, Student, St. Joseph High School, Cleveland, Ohio

The following solution is submitted:

1, 2, 3, 5, 7, 15, 20, 25, 41,

the sum of which is 121.

Editorial Comment: This is the missing solution and the best received to date.

A BETTER PROBLEM SOLUTION

H-74 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Let f(n) denote the number of positive Fibonacci numbers not greater than a specified integer n. Show that for $n > 1$

$$f(n) = [K \ln(n\sqrt{5} + \tfrac{1}{2}]\quad ,$$

where $[x]$ denotes greatest integer not exceeding x, and K is a constant nearly equal to 2.078086943. (See H. W. Gould's Non-Fibonacci Numbers, Oct. 1965, FQJ).

Comments by John D. Cloud, Manhattan Beach, California

Problem H-74 in the Dec. 1965 issue of the Fibonacci Quarterly is not new. I proposed the same problem in the Nov. 1963 issue of the American Mathematical Monthly (Vol. 70, No. 9, p. 1005, prob. E1636). A more precise solution than the one indicated by Mr. Lind in the FQ appears on p. 798 of the AMM for Sept. 1964 (Vol. 71, No. 7).

The solution by William D. Jackson alluded to shows:

The number of Fibonacci numbers not greater than N is the greatest integer less than

$$\frac{\log\left[(N + \frac{1}{2})\sqrt{5}\right]}{\log\left(\frac{1+\sqrt{5}}{2}\right)} - 1 \quad .$$

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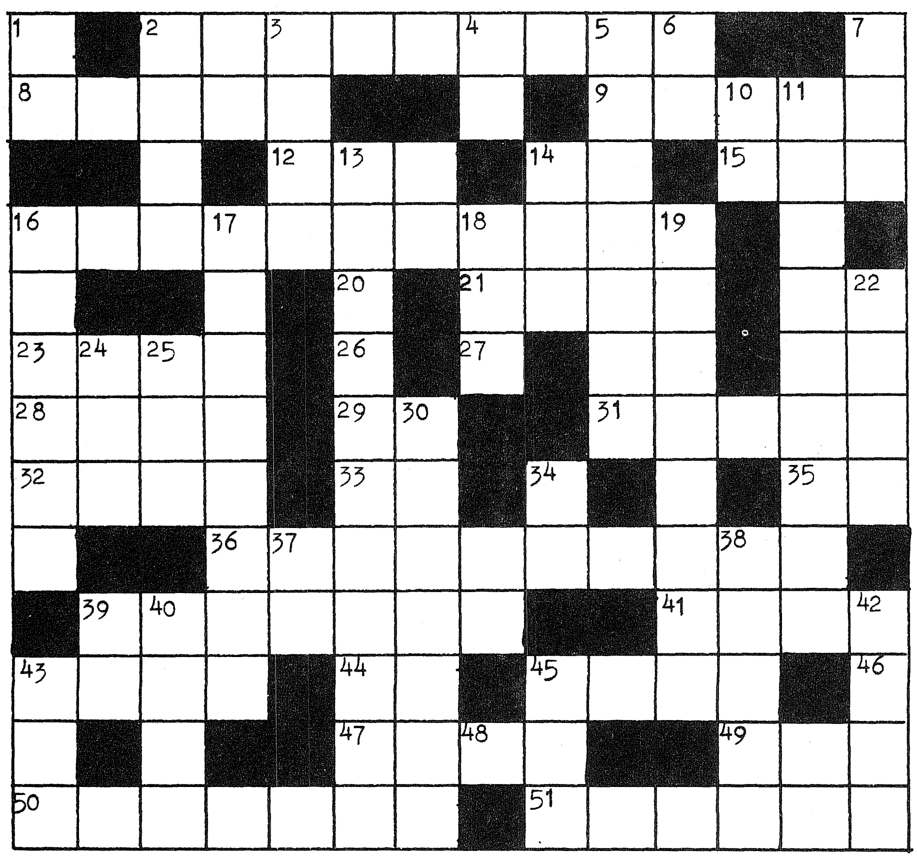
A FIBONACCI CROSSWORD PUZZLE

H.W. Gould
West Virginia University



ACROSS

2 A kind of number discovered by 17 Down of 22 Down while studying 50 Across.
8 Series (French).
9 A river in Kansas and Missouri.
12 Smallest number into which each of two numbers will divide.
14 The smallest natural number (French).
15 One more than a perfect square.
16 Occurrence of Fibonacci numbers in nature.
20 $\lim_{n \to \infty} (1 + 1/n)^n$.
21 Device sometimes used to generate random numbers.
23 Combining form of "Chinese".
26 Letter used to denote a famous number sequence.
27 Sign of differentiation.
28 Brother of Abel.
29 Iron.
31 The numbers 1, 3, 4, 7, 11, ... are ______ numbers.
32 Girl's name.
33 Measure of intelligence (they say).
35 Chemical element.
36 $F_{n+1} = F_n + F_{n-1}$ is an example.
39 Roman emperor.
41 $(e^x + e^{-x})/2$.
43 Amateur.
44 Latin connective.
45 Utilizes.
46 Square root of minus one.
47 If n is a natural number, then $n+1$ is the ______ natural number.
49 Bitter Vetch.
50 Studied in 1202 AD by 2 Across.
51 Throws out.

DOWN

1 The first degree.
2 Battle.
3 Famous mathematician who wrote novels under the nom de plume John Taine.
4 Definite article (Arabic).
5 Cone-shaped.
6 A form of the copula.
7 Base of the decimal system.
10 Preposition.
11 The function $x(1 - x - x^2)^{-1}$ "______" the Fibonacci numbers.
13 In fact, a Fibonacci number is the ______ of x^n in the power series expansion of $x(1 - x - x^2)^{-1}$.
16 Discovered a famous triangle named after him. Known earlier to the Chinese.
17 Another name of 2 Across.
18 One of the simplest ways to combine numbers.
19 The numbers 1, 1, 2, 3, 5, 8, ... form a ______.
22 A town in Italy.
24 An adjectival suffix.
25 A famous game whose theory is based on the binary system.
30 What one usually does with 13 Down in order to discover a relation.
34 City in ancient Sumer.
37 Chemical element discovered in Ytterby, Sweden.
38 Abbreviation for a trigonometrical function; counterpart of 41 Across.
39 Man's nickname.
40 A person who might live near 40°E. Long., 22°N. Lat.
42 To make a sharp sibilant sound.
43 Black substance obtained from coal.
45 An Indian of the Shoshonean tribes.
48 The Unknown Quantity.

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CORRECTIONS

"A Variant of Pascal's Triangle" by H. W. Gould, FQJ Vol. 3, Dec. 1965:

Page 265, line 6, for (6.7) read (6.6).

Page 268, relation (7.14): Left-hand member should read

$$K_{n+1}K_{n+b} - K_nK_{n+b+1}$$

Page 268, line 5 up from bottom: Right-hand member should read

$$(-1)^n(K_aK_b - K_0K_{a+b})$$

and line 3 up from bottom: Right-hand member should read

$$(-1)^n(K_{a-1}K_b - K_0K_{a-1+b})$$

Page 269, line 1: Right-hand member should read

$$(-1)^n(K_{a+1}K_b - K_0K_{a+1+b})$$

and line at bottom of page 269: In right-hand member for K_{n+a-b} read K_{n+a+b}.

"Power Identities for Sequences Defined by $W_{n+2} = dW_{n+1} - cW_n$" by David Zeitlin, FQJ Vol. 3, No. 4, 1965

Page 245. Line 2 of (3.1) should read

$$c^{rn_o + (mq(4r-q-1)/2)} \binom{2r}{r} (4c-d^2)^{q-r}$$

Page 246. The factor, c^{2mqr}, was omitted from the first line of the page.

Page 247. The factor, β^{mn}, was omitted from the last line (on the right hand side) at the bottom of the page.

Page 250. In line (1) of (3.16), the exponent of c should read $(r-1)n_0 + i_{2r-1}$.

Page 250. In line (2) of (4.3), $(-5)^q$ should read $(-5)^{q-r}$.

Page 251. In (4.8), $+4$ should read -4.

Page 254. The second line in (5.9) begins with a $+$ sign.

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ON THE INTEGER SOLUTION OF THE EQUATION $5x^2 \pm 6x + 1 = y^2$ AND SOME RELATED OBSERVATIONS

Edgar I. Emerson
Boulder, Colorado

The integer solution of the equation

$$(1) \qquad 5x^2 \pm 6x + 1 = y^2$$

is interesting because of the Fibonacci and Lucas relationships that appear.

One method of solving the problem involves the solution of the Pythagorean, (Py), equation

$$(2) \qquad X^2 + Y^2 = Z^2 ,$$

where $X = 2ab$, $Y = a^2 - b^2$, $Z = a^2 + b^2$, and $a > b$. Since no other restrictions are placed on a and b this solution of (2) is not necessarily primitive.

When $4x^2$ is added to both sides of (1) we obtain

$$(3) \qquad 9x^2 \pm 6x + 1 = y^2 + 4x^2 \qquad \text{or}$$

$$(4) \qquad (3x \pm 1)^2 = y^2 + (2x)^2 . \qquad \text{Now let}$$

$$(5a) \qquad 3x \pm 1 = Z = a^2 + b^2 ,$$

$$(5b) \qquad y = Y = a^2 - b^2 \qquad \text{and}$$

$$(5c) \qquad 2x = X = 2ab \qquad \text{or}$$

$$(5d) \qquad x = ab.$$

Substituting this value of x in (5a) we get

$$3ab \pm 1 = a^2 + b^2 \qquad \text{or}$$

$$(6) \qquad a^2 - 3ab + (b^2 \pm 1) = 0 .$$

Solving this equation for $a > b$ we have

$$(7) \qquad a = \frac{3b + \sqrt{9b^2 - 4(b^2 \pm 1)}}{2} = \frac{3b + \sqrt{5b^2 \pm 4}}{2}$$

If the values of b are such that $5b^2 \pm 4 = \square$ then $3b + \sqrt{5b^2 \pm 4}$ is always even and therefore a is always integral. Changing equation (7) to

$$2a = 3b + \sqrt{5b^2 \pm 4} \tag{8}$$

we prepare Table I by filling in the column under b with the Fibonacci numbers, F, and the column beneath the radical sign with the Lucas numbers, L. The rest of the table is then calculated.

Table I Showing Fibonacci and Lucas Relationships Involved in the Solution of

$$5x^2 \pm 6x + 1 = y^2$$

n,	a,	b;	$2a = 3b + \sqrt{5b^2 \pm 4}$;	$x = ab$;	$y = a^2 - b^2$
0,	1,	0;	2 = 0 + 2;	0;	1
1,	2,	1;	4 = 3 + 1;	2;	3
2,	3,	1;	6 = 3 + 3;	3;	8
3,	5,	2;	10 = 6 + 4;	10;	21
4,	8,	3;	16 = 9 + 7;	24;	55
5,	13,	5;	26 = 15 + 11;	65;	144
6,	21,	8;	42 = 24 + 18;	168;	377
-	-	-	- - -	-	-
n,	F_{n+2},	F_n;	$2F_{n+2} = 3F_n + L_n$;	$F_{n+1}^2 - (-1)^n = F_n F_{n+2}$;	$F_{n+2}^2 - F_n^2 = F_{2n+2}$

$$= (L_{n+1}^2 - F_{n+1}^2)/4; = L_{n+1} F_{n+1}$$

Note that $x_n + x_{n+1} = F_{2n+3} = F_{n+1}^2 + F_{n+2}^2$

The solution to equation (1) is

$$x_n = F_{n+1}^2 - (-1)^n = F_n F_{n+2} = 2(x_{n-1} + x_{n-2}) - x_{n-3} \tag{9}$$

$$y_n = F_{n+2}^2 - F_n^2 = F_{2n+2} = L_{n+1} F_{n+1} = 3y_{n-1} - y_{n-2} \;. \tag{10}$$

From (9) we have the interesting recurrent equation

$$x_n = 2(x_{n-1} + x_{n-2}) - x_{n-3} \;,$$

which can be expressed with Fibonacci terms as:

(11) $$F_{n+1}^2-(-1)^n = 2\left[F_n^2-(-1)^{n-1}+F_{n-1}^2-(-1)^{n-2}\right] - \left[F_{n-3}^2-(-1)^{n-3}\right] .$$

The (-1) terms disappear so that

(12) $$F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \qquad \text{or}$$

$$F_{n+1}^2 - F_{n-1}^2 = (F_n^2 - F_{n-2}^2) + (F_n^2 + F_{n-1}^2) \qquad \text{or}$$

$$F_{2n} = F_{2n-2} + F_{2n-1} \qquad \text{or}$$

$$F_{2n} = F_{2n}$$

and thus we have proved (11) and (12). Equation (12) can be written as

$$2(F_n^2+F_{n-1}^2) = F_{n+1}^2+F_{n-2}^2 \qquad \text{or}$$

(13) $$2F_{2n-1} = F_{n+1}^2+F_{n-2}^2 \quad ,$$

an interesting Fibonacci identity. Another interesting Fibonacci identity turns up when the appropriate F and L terms are substituted in equation (8), $2a = 3b + \sqrt{5b^2 \pm 4}$. We have

(14) $$2F_{n+2} = 3F_n+L_n \quad .$$

This identity is proved by adding $3F_n$ to each side of the identity $F_{n-1}+F_{n+1} = L_n$ as follows:

$$F_{n-1}+F_{n+1} = L_n$$

$$F_n+F_n+F_n = 3F_n$$

$$F_n+(F_n+F_{n-1})+(F_{n+1}+F_n) = 3F_n+L_n$$

$$(F_n+F_{n+1})+F_{n+2} = 3F_n+L_n$$

$$F_{n+2}+F_{n+2} = 3F_n+L_n$$

$$2F_{n+2} = 3F_n+L_n$$

Equation (1) can be written as

$$5x_n^2 + 6(-1)^n x_n + 1 = y_n^2 \tag{14}$$

and when the appropriate F terms are substituted, this equation becomes

$$5F_n^2F_{n+2}^2 + 6(-1)^nF_nF_{n+2} + 1 = F_{2n+2}^2 \tag{15}$$

which equation is equivalent to

$$5F_{n-1}^2F_{n+1}^2 - 6(-1)^nF_{n-1}F_{n+1} + 1 = F_{2n}^2 \quad \text{or} \tag{16}$$

$$5[F_n^2+(-1)^n]^2 - 6(-1)^n[F_n^2+(-1)^n] + 1 = F_{2n}^2 = L_n^2F_n^2 \quad . \tag{17}$$

When the indicated operations are performed we have successively

$$5[F_n^4+2(-1)^nF_n^2+(-1)^{2n}] - 6(-1)^n[F_n^2+(-1)^n] + 1 = L_n^2F_n^2$$

$$5F_n^4 + 10(-1)^nF_n^2 + 5 - 6(-1)^nF_n^2 - 6(-1)^{2n} + 1 = L_n^2F_n^2$$

$$5F_n^4 + 4(-1)^nF_n^2 = L_n^2F_n^2$$

$$5F_n^2 + 4(-1)^n = L_n^2$$

and thus we have proved the identities (16) and (17).

Now we examine the solution of equation (2), $X^2 + Y^2 = Z^2$, where F and L terms are used for (a) and (b). For this purpose we first prepare Table II where the a's and b's are transferred from Table I. The rest of Table II is then calculated.

The solution of $X^2 + Y^2 = Z^2$ is

$$X = 2ab = 2F_nF_{n+2} \tag{18a}$$

$$Y = a^2-b^2 = F_{n+2}^2-F_n^2 = F_{2n+2}$$

$$Z = a^2+b^2 = F_{n+2}^2+F_n^2 = 3F_{n+1}^2-2(-1)^n \quad .$$

Table II Showing Fibonacci and Lucas Relationships Involved in the Solution of $X^2+Y^2 = Z^2$

$X = 2ab$, $Y = a^2-b^2$, $Z = a^2+b^2$, $a = F_{n+2}$, and $b = F_n$.

n,	a,	b,	2ab=X,	$a^2-b^2=Y$,	$a^2+b^2=Z$,	a-b,	a+b
0,	1	0	1	1	1	1	1
1,	2	1	4	3	5	1	3
2,	3	1	6	8	10	2	4
3,	5	2	20	21	29	3	7
4,	8	3	48	55	73	5	11
5,	13	5	130	144	194	8	18
6,	21	8	336	377	505	13	29
-	-	-	-	-	-	-	-
n,	F_{n+2},	F_n,	$2F_nF_{n+2}$,	$F_n^2-F_{n+2}^2$,	$F_n^2+F_{n+2}^2$,	F_{n+1},	L_{n+1}

$2[F_{n+1}^2-(-1)^n]$, F_{2n+2}, $(L_{n+1}^2+F_{n+1}^2)/2$,

$(L_{n+1}^2-F_{n+1}^2)/2$, $L_{n+1}F_{n+1}$, $3F_{n+1}^2-2(-1)^n$

The identity (18c), $F_{n+2}^2+F_n^2 = 3F_{n+1}^2 - 2(-1)^n$, is equivalent to

(19) $$F_{n+1}^2+F_{n-1}^2 = 3F_n^2+2(-1)^n \qquad \text{but}$$

$$F_{n+1}F_{n-1} = F_n^2+(-1)^n \qquad \text{and}$$

$$2F_{n+1}F_{n-1} = 2F_n^2+2(-1)^n \qquad \text{and therefore}$$

$$F_{n+1}^2+F_{n-1}^2 = 2F_{n+1}F_{n-1} + F_n^2 \qquad \text{or}$$

$$F_{n+1}^2-2F_{n+1}F_{n-1}+F_{n-1}^2 = F_n^2 \qquad \text{or}$$

$$(F_{n+1}-F_{n-1})^2 = F_n^2 \qquad \text{or}$$

$$F_{n+1}-F_{n-1} = F_n \qquad \text{or}$$

$$F_{n+1} = F_n+F_{n-1}$$

and thus we have proved the Fibonacci identity for Z in (18c).

An equivalent equation for $X^2 + Y^2 = Z^2$ is the following Fibonacci identity:

$$4\left[F_{n+1}^2-(-1)^n\right]^2 + F_{2n+2}^2 = \left[3F_{n+1}^2-2(-1)^n\right]^2 \quad . \tag{20}$$

When the indicated operations are performed and the terms are collected this equation becomes

$$5F_{n+1}^4-4(-1)^nF_{n+1}^2 = F_{2n+2}^2 \qquad \text{or}$$

$$5F_n^4+4(-1)^nF_n^2 = F_{2n}^2 = L_n^2F_n^2 \qquad \text{or}$$

$$5F_n^2+4(-1)^n = L_n^2$$

and thus we have proved the Fibonacci identity expressed by equation (20).

The following equations represent further observations.

$$Z+X = a^2+2ab+b^2 = (a+b)^2 = (F_{n+2}+F_n)^2 = L_{n+1}^2 \qquad \text{and} \tag{21}$$

$$Z-X = a^2-2ab+b^2 = (a-b)^2 = (F_{n+2}-F_n)^2 = F_{n+1}^2 \quad . \tag{22}$$

Adding these equations and dividing by 2 we have

$$Z = (L_{n+1}^2+F_{n+1}^2)/2 \tag{23}$$

and subtracting the equations and dividing by 2 we get

$$X = (L_{n+1}^2-F_{n+1}^2)/2 \tag{24}$$

and multiplying (21) by (22) we obtain

$$Z^2-X^2 = L_{n+1}^2F_{n+1}^2 = Y^2 \qquad \text{or} \tag{25}$$

$$Y = L_{n+1}F_{n+1} \quad . \tag{26}$$

The area, A, of the Py triangle is

$$A = ab(a^2-b^2) \quad . \tag{27}$$

In general, the module, $ab(a^2-b^2)$, is divisible by 6, consequently when the appropriate F and/or L terms are substituted in the module the resulting expression must likewise be divisible by 6. Thus the following expressions are all divisible by 6:

$$F_nF_{n+2}(F_{n+2}^2-F_n^2) \; ; \quad [F_{n+1}^2-(-1)^n][F_{n+2}^2-F_n^2] \; ;$$

$$F_nF_{n+1}F_{n+2}L_{n+1} \; ; \quad F_nF_{n+2}F_{2n+2} \; ;$$

$$[F_{n+1}^2-(-1)^n]F_{2n+2} \; ; \quad [F_{n+1}^2-(-1)^n]F_{n+1}L_{n+1} \; ,$$

and

$$(L_{n+1}^2-F_{n+1}^2)L_{n+1}F_{n+1} \qquad \text{or}$$

$$(L_{n+1}^2-F_{n+1}^2)F_{2n+2} \qquad \text{or}$$

$$(L_{n+1}^2-F_{n+1}^2)(F_{n+2}^2-F_n^2)$$

are all divisible by 24 since $ab = (L_{n+1}^2-F_{n+1}^2)/4$.

In the foregoing considerations the values of a and b were restricted by equation (1) to $a = F_{n+2}$, $b = F_n$. If now, in the solution of a Py triangle, we substitute for a and b any arbitrary F and/or L terms then Fibonacci and/or Lucas number identities are easily produced in infinite variety and divisibility expressions are easily produced and proved.

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STAR GEOMETRY

Pythagoras, Fibonacci and Beard

R.S. Beard

The diagonal of the pentagon is made the unit of length in the upper left circle of the accompanying drawings. K is the ratio of the side of the pentagon to its diagonal.

Line 4.3 is the short side of triangle 1-4-3 and the long side of triangle 3-4-6.

Line 4.6 is the short side of triangle 3-4-6 and the long side of triangle 4-6-7.

Therefore the sides of these three similar isoseeles triangles are respectively 1 and K, K and K^2, K^2 and K^3. Lines 1-6 and 3-6 have the same length, K, as they are the equal sides of the isosceles triangle 1-3-6. Diagonal 1-4 of unit length is thus divided into segments K and K^2, that is $K^2 + K = 1$.

It follows from this equation that

$$K = \frac{1}{2}\sqrt{5} - \frac{1}{2} = 0.618034$$

and that $K^n = K^{n+1} + K^{n+2}$.

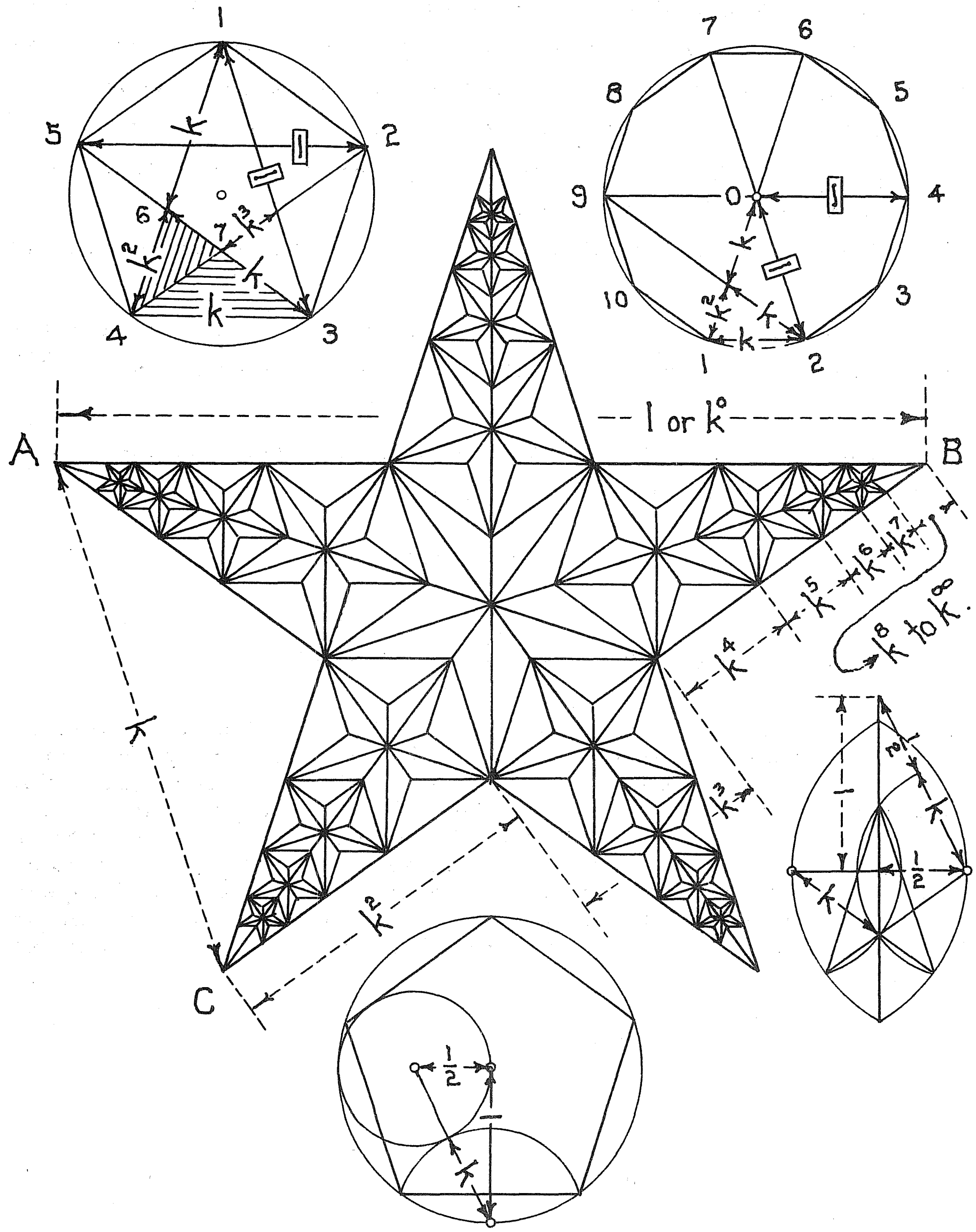
Since each power of K is the sum of its next two higher powers, these powers form a Fibonacci series when arranged in their descending order.

The radius is made the unit of length in the upper right circle. Similar triangles divide radius O-1 into segments K and K^2. This construction demonstrates that each side of a regular decagon has the golden section ratio to the radius of its circumscribing circle.

The right triangle in the lower circle has sides of 1, 1/2 and $1/2\sqrt{5}$. The dimension K with the value of $1/2\sqrt{5}-1/2$ is the length of the side of the inscribed decagon.

The leaf shaped figure shows one simple way to construct a five pointed star of a given width.

The Fibonacci star of stars is proportioned in the ten successive powers of the golden section from K^0 to K^9.


A
B
C
1 or k^0
k
k^2
k^3
k^4
k^5
k^6
k^7
k^8 to k^∞.
$\frac{1}{2}$

AB is made the unit of length or K^0.

AC, the side of the bounding pentagon has the length of K.

The side of each ray of the master star is K^2 long.

The bounding pentagon of the central star has sides of K^3 length.

Since one diagonal of each of the smaller stars is a side of the bounding pentagon of the next larger star all of the corresponding dimensions of these successive stars are in golden section ratio.

The bounding pentagons of the successive stars have sides of K^4, K^5, K^6 and K^7 respectively.

The rays of the smallest stars have K^8 edges and base widths of K^9.

This figure can be used to demonstrate that any power of the golden section K^n, is the sum of all of its higher powers from K^{n+2} to K^∞.

The lines connecting the tips of the central star and the centers of the five next smaller stars form a decagon.

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PROPERTIES OF THE POLYNOMIALS DEFINED BY MORGAN-VOYCE

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1. Introduction

In dealing with electrical ladder networks, A. M. Morgan-Voyce defined a set of polynomials by:

(1) $$b_n(x) = x\ B_{n-1}(x) + b_{n-1}(x) \qquad (n \geq 1)$$

(2) $$B_n(x) = (x+1)\ B_{n-1}(x) + b_{n-1}(x) \qquad (n \geq 1)$$

with,

(3) $$b_0(x) = B_0(x) = 1$$

These polynomials b_n and B_n have a number of very fascinating and interesting properties, and is the subject matter of this article. A few properties of these have been studied by Basin.

From (1) and (2) we see that

(4) $$b_n = B_n - B_{n-1}$$

(5) and, $$x\ B_n = b_{n+1} - b_n$$

Substituting (4) in (1) we have that B_n satisfies the difference equation,

$$B_n(x) = (x+2)\ B_{n-1}(x) - B_{n-2}(x) \qquad (n \geq 2)$$

with

(6) $$B_0(x) = 1, \text{ and } B_1(x) = x + 2$$

From (1) and (2) it can easily be derived that $b_n(x)$ also satisfies the same difference equation, namely,

$$b_n(x) = (x+2)\ b_{n-1}(x) - b_{n-2}(x) \qquad (n \geq 2)$$

with

(7) $$b_0(x) = 1, \text{ and } b_1(x) = x + 1$$

The difference equation (6) may be expressed as the continuant,

$$
(8) \qquad B_n(x) = \begin{vmatrix} x+2 & 1 & 0 & . & . & .0 \\ 1 & x+2 & 1 & 0 & . & 0 \\ 0 & 1 & x+2 & 1 & . & \\ . & . & . & . & . & \\ . & . & . & . & . & 0 \\ . & . & . & . & . & 1 \\ 0 & . & . & 0 & 1 & x+2 \end{vmatrix}_n \quad (n \geq 1)
$$

and hence we may study the properties of B_n by using those of the continuants. We shall list below only such of those properties of $B_n(x)$ which we will use in studying $b_n(x)$ and in deriving relations between the polynomials $b_n(x)$ and $B_n(x)$:

(9) $$B_{m+n} = B_m \; B_n - B_{m-1} \; B_{n-1}$$

(10) $$B_{2n} = B_n^2 - B_{n-1}^2$$

(11) $$B_{2n-1} = B_{n-1} \; (B_n - B_{n-2})$$

(12) $$(x + 2) \; B_{2n-1} = B_n^2 - B_{n-2}^2$$

(13) $$B_n \; B_{r-h+1} = B_r \; B_{n-h+1} - B_{h-2} \; B_{n-r-1}$$

(14) $$B_{n-1} \; B_{n+1} - B_n^2 = -1$$

(15) $$\frac{d}{dx} \; B_n(x) = \sum_0^{n-1} (B_r \cdot B_{n-1-r})$$

2. Relations between $b_n(x)$ and $B_n(x)$, and properties of $b_n(x)$:

From (5) and (7),

(16) $$x \; B_n = (x + 1) \; b_n - b_{n-1}$$

Also we have,

(17) $$B_{n+1} - B_{n-1} = b_{n+1} + b_n$$

From (4) and (5),

(18) $$b_{n+1} - b_{n-1} = x \quad (B_n + B_{n-1})$$

By successively substituting 0, 1, 2, ... for n in (5) and adding we have,

(19) $$x \sum_0^n B_r = b_{n+1} - 1$$

Similarly from (4) we may deduce that,

(20) $$\sum_0^n b_r = B_n$$

Now,

$$b_{m+n} = B_{m+n} - B_{m+n-1} = (B_m B_n - B_{m-1} B_{n-1}) - (B_m B_{n-1} - B_{m-1} B_{n-2})$$

$$= B_m \quad (B_n - B_{n-1}) - (B_{n-1} - B_{n-2}) \quad B_{m-1}$$

Hence,

(21a) $$b_{m+n} = B_m b_n - B_{m-1} b_{n-1}$$

Interchanging m and n we have,

(21b) $$b_{m+n} = b_m B_n - b_{m-1} B_{n-1}$$

Hence,

(22) $$b_m B_n - B_m b_n = b_{m-1} B_{n-1} - B_{m-1} b_{n-1}$$

We will see later that this is a particular case of the more general relationship (29).

Putting $m = n$ in (21),

(23) $$b_{2n} = b_n B_n - b_{n-1} B_{n-1}$$

Putting $m = n+1$ in (21),

(24a) $$b_{2n+1} = b_{n+1}B_n - b_nB_{n-1}$$

(24b) $$= B_{n+1}b_n - B_nb_{n-1}$$

From (7) we have

$$(x + 2)\, b_{2n+1} = b_{2n+2} + b_{2n}$$

$$= b_{n+1}B_{n+1} - b_nB_n + b_nB_n - b_{n-1}B_{n-1}$$

Hence,

(24c) $$(x + 2)\, b_{2n+1} = b_{n+1}B_{n+1} - b_{n-1}B_{n-1}$$

Also from (12),

$$(x + 2)\, B_{2n+1} = B_{n+1}^2 - B_{n-1}^2$$

Hence,

$$(x + 2)(B_{2n+1} - b_{2n+1}) = B_{n+1}(B_{n+1} - b_{n+1}) - B_{n-1}(B_{n-1} - b_{n-1})$$

Hence,

(25) $$(x + 2)\, B_{2n} = B_{n+1}B_n - B_{n-1}B_{n-2}$$

From (23) and (24) we deduce that,

(26) $$b_{2n} - b_{2n-1} = b_n^2 - b_{n-1}^2$$

Subtracting (12) from (25),

$$(x + 2)(B_{2n} - B_{2n-1}) = B_n(B_{n+1} - B_n) - B_{n-2}(B_{n-1} - B_{n-2})$$

Hence,

(27a) $$(x + 2)\, b_{2n} = B_nb_{n+1} - B_{n-2}b_{n-1}$$

(27b) $$= b_nB_{n+1} - b_{n-2}B_{n-1}$$

We will now derive a relationship between the polynomials $b_n(x)$ and $B_n(x)$, corresponding to the relation (13) for B_n:

Consider the expression,

$$b_{n-h+1}B_r - B_{h-2}b_{n-r-1}$$

$$= (B_{n-h+1} - B_{n-h})B_r - (B_{n-r-1} - B_{n-r-2})B_{h-2}$$

$$= (B_{n-h+1}B_r - B_{n-r-1}B_{h-2}) - (B_{n-h}B_r - B_{n-r-2}B_{h-2})$$

$$= B_nB_{r-h+1} - B_{n-1}B_{r-h+1} \qquad \text{from (13)}$$

$$= (B_n - B_{n-1})B_{r-h+1} = b_nB_{r-h+1}$$

Hence,

(28a) $$b_nB_{r-h+1} = b_{n-h+1}B_r - B_{h-2}b_{n-r-1}$$

Similarly,

(28b) $$b_nB_{r-h+1} = B_{n-h+1}b_r - b_{h-2}B_{n-r-1}$$

Hence from (28a) and (28b) we get the relation,

$$B_rb_{n-h+1} - B_{h-2}b_{n-r-1} = b_rB_{n-h+1} - b_{h-2}B_{n-r-1}$$

Changing r to m, h-2 to m-4, and n to m+n+1-r in the above relation,

(29a) $$B_mb_n - B_{m-r}b_{n-r} = b_mB_n - b_{m-r}B_{n-r}$$

Using the relation (4) in (29a) we derive the corresponding relation for $B_n(x)$ as,

(30a) $$B_mB_{n-1} - B_{m-r}B_{n-r-1} = B_nB_{m-1} - B_{n-r}B_{m-r-1}$$

These relations may be written neatly in the form of determinants:

(29b) $$\begin{vmatrix} B_m & B_n \\ b_m & b_n \end{vmatrix} = \begin{vmatrix} B_{m-r} & B_{n-r} \\ b_{m-r} & b_{n-r} \end{vmatrix}$$

and

(30b) $$\begin{vmatrix} B_m & B_{m-1} \\ B_n & B_{n-1} \end{vmatrix} = \begin{vmatrix} B_{m-r} & B_{m-1-r} \\ B_{n-r} & B_{n-1-r} \end{vmatrix}$$

Now putting $h = 2$, and $n = r+1$ in equation (28) we get,

(31) $$b_rB_r - b_{r+1}B_{r-1} = 1$$

Putting $m = n-1$, and $r = n-1$ in (29b) we get,

$$B_n b_{n-1} - b_n B_{n-1} = 1 \tag{32}$$

From (31) and (32) we see that $b_n(x)$ is prime to $b_{n-1}(x)$, $B_n(x)$ and $B_{n-1}(x)$ for integral values of x. Also, for integral values of x, $B_n(x)$ is prime to $B_{n-1}(x)$, $b_n(x)$ and $b_{n+1}(x)$.

By successively substituting 1, 2, 3, ... for n in (10) and adding, we have

$$\sum_1^n B_{2r} = B_n^2 - B_0^2 = B_n^2 - B_0$$

Hence,

$$\sum_0^n B_{2r} = B_n^2 \tag{33}$$

Similarly, using (11), (23), (24) and (26) we derive:

$$\sum_0^{n-1} B_{2r+1} = B_n B_{n-1} \tag{34}$$

$$\sum_0^n B_{2r} = b_n B_n \tag{35}$$

$$\sum_0^{n-1} b_{2r+1} = b_n B_{n-1} \tag{36}$$

$$\sum_0^{2n} (-1)^r b_r = b_n^2 \tag{37}$$

Let us now find an expression for the derivative of $b_n(x)$:

$$b_n'(x) = B_n' - B_{n-1}' = \sum_0^{n-1} B_r B_{n-1-r} - \sum_0^{n-2} B_r B_{n-2-r}$$

$$= B_{n-1}B_0 + \sum_0^{n-2} B_r(B_{n-r-1} - B_{n-r-2}) = B_{n-1}b_0 + \sum_0^{n-2} B_r b_{n-r-1}$$

Hence,

(38) $$b_n(x) = \sum_{0}^{n-1} B_r b_{n-1-r}$$

3. Explicit polynomial expressions for $B_n(x)$ and $b_n(x)$:

We can establish by induction that,

$$B_n(x) = \sum_{k=0}^{n} (c_n^k x^k) \quad ,$$

where,

(39) $$c_n^k = \binom{n+k+1}{n-k} \quad .$$

Now

(39) $$b_n(x) = B_n(x) - B_{n-1}(x) = \sum_{0}^{n} \left[\binom{n+k+1}{n-k} - \binom{n+k}{n-k-1} \right] x^k$$

$$= \sum_{0}^{n} \binom{n+k}{n-k} x^k \quad .$$

Therefore we have

$$b_n(x) = \sum_{k=0}^{n} (d_n^k x^k)$$

where,

(40) $$d_n^k = \binom{n+k}{n-k} \quad .$$

The equations (39) and (40 are explicit polynomial expressions for b_n and B_n, and show that they are of degree n.

We shall now derive a formula for

$$\int B_n(x) \, dx \quad :$$

From (39),

$$\int B_n(x)\,dx = \sum_0^n \left(c_n^k x^{k+1}/k+1\right) + c$$

Now the coefficient of x^{k+1} for the expression $B_{n+1} - B_{n-1}$ is,

$$c_{n+1}^{k+1} - c_{n-1}^{k+1} = \binom{n+k+3}{n-k} - \binom{n+k+1}{n-k-2} = (n+1)\,c_n^k / (k+1)$$

$$= (n+1)(\text{coefficient of } x^{k+1} \text{ in } \int B_n(x)\,dx\,.)$$

Hence,

$$\int B_n(x)\,dx = \frac{B_{n+1} - B_{n-1}}{n+1} + c \quad . \tag{41}$$

It may also be established that over the interval (-4, 0), $B_n(x)$ and $b_n(x)$ are orthogonal polynomials with respect to the weight functions $\sqrt{4-(x+2)^2}$ and $\sqrt{(x+4)/-x}$ respectively.

It may also be seen from (6) that,

$$B_n(x) = S_n(x+2) \tag{42a}$$

and hence,

$$b_n(x) = S_n(x+2) - S_{n-1}(x+2) \quad , \tag{42b}$$

where $S_n(x)$ is the Chebyshev polynomial.

4. Conclusions:

The article deals with the properties of a set of polynomials $b_n(x)$ and $B_n(x)$ defined by (1), (2) and (3). Even though they are related to the Chebyshev polynomials, the author believes that $B_n(x)$ and $b_n(x)$ are of use in the study of ladder networks and hence deserve a study of this nature.

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PERFECT NUMBER "ENDINGS"

J.A.H. Hunter
Toronto, Ontario, Canada

Helen A. Merrill, in her book Mathematical Excursions, (Dover Publications, Inc.; New York) outlined the main features of Perfect Numbers. After stating that 6, 28, and 396 are Perfect Numbers, she continued:

> "The next Perfect Number is 8128, and the next contains eight digits. All these Perfect Numbers end in 6 or 28, but no one knows whether this is true of all such numbers."

The 1st edition of that book was published in 1933. It is possible that an elementary proof for the noted "endings" has been published somewhere since then, but if so it has escaped my notice.

Accordingly I venture to lay out the necessary but quite elementary proof that all even Perfect Numbers end in 6 or 28; the non-existence of any odd Perfect Number has not yet been proven.

Every even Perfect Number is known to be of general form: $2^{n-1}(2^n - 1)$.

The mod 100, by actual calculation for successive values of n $(n > 1)$, we see that $2^{n-1}(2^n - 1)$ has successive values 6, 28, 0, 6; this sequence of 4 values being repeated for all cases up to $n = 22$. Still to mod 100, each of 2^{n-1} - $2^n - 1$ has a period of 20, repeating the remainders. Hence, the "6, 28, 0, 6" sequence of endings for the product $2^{n-1}(2^n - 1)$ must continue for all values of n.

It will be noted, the proof being trivial, that zero endings occur only when $n = 4k$.

Now, the actual particular form for all even Perfect Numbers requires n to be prime. Hence, with $n = 4k$, we can have no Perfect Number.

So, for all even Perfect Numbers we must have 6 or 28 as "ending."

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A DIVISIBILITY PROPERTY OF FIBONACCI NUMBERS

Lenard Weinstein
Massachusetts Institute of Technology

The following is an interesting theorem concerning the first $2n$ Fibonacci numbers:

Theorem

Give any set of $n + 1$ Fibonacci numbers selected from the set $F_1, F_2, \ldots, F_{2n}$, it is always possible to choose two elements from the $n + 1$ Fibonacci numbers such that one divides the other.

This theorem will be proved using the following two theorems:

Theorem 1:

Give any set of $n + 1$ integers selected from the set $1, 2, \ldots, 2n$, it is always possible to choose two elements from the $n + 1$ integers such that one divides the other.

Proof:

We shall use induction. The theorem is trivially true for the case $n = 1$. Assume it true for $n = k$. If $n = k+1$, we must prove that any set of $k+2$ integers selected from the set $1, 2, \ldots, 2(k+1)$ contains two elements such that one divides the other. If the $k+2$ integers are contained in the set $1, 2, \ldots, 2k$, then by the inductive hypothesis, the theorem is true. Similarly, if $k+1$ of the integers are contained in the set $1, 2, \ldots, 2k$, and the other integer is either $2k+1$ or $2(k+1)$, the same reasoning applies as above. If only k of the integers belong to the set $1, 2, \ldots, 2k$, then the other two integers must be the numbers $2k+1$ and $2(k+1)$. Consider any set of k numbers from the set $1, 2, \ldots, 2k$ and the integer $k+1$. By the inductive hypothesis, we can find two numbers from this set such that one number divides the other. Since $k+1$ does not divide any number but itself, in the set $1, 2, \ldots, 2k$, at best it is divisible by another element of the set. Any number that divides $k+1$, though, divides $2(k+1)$. Thus, any set of k elements chosen from the set $\{1, 2, \ldots, 2k\}$ and the element $2(k+1)$, contains two numbers such that one number divides the other. Therefore, the set of any k integers from the set 1, 2, ..., 2k , plus the numbers 2k+1 and 2(k+1), also contains two elements such that one divides the other.

Theorem 2:

If F_n is the n^{th} Fibonacci number ($F_1 = 1$, $F_2 = 1$), then $F_{(a,b)} = (F_a, F_b)$ [1] . ((a, b) is the greatest common divisor of the integers a and b.) This is a widely known theorem, easily proved.

It follows that if any set of n+1 Fibonacci numbers F_{a_1}, F_{a_2}, ... $F_{a_{n+1}}$, is chosen from the set $F_1, F_2, \ldots, F_{2n}$, there exist two elements of the n+1 Fibonacci numbers such that one number divides the other. For consider the numbers $a_1, a_2, \ldots, a_{n+1}$. By theorem 1, we can find two numbers a_j, a_k from these n+1 integers such that $a_j | a_k$. Thus $(a_j, a_k) = a_j$. It follows that $F_{(a_j, a_k)} = F_{a_j} = (F_{a_j}, F_{a_k})$ by theorem 2. This means $F_{a_j} | F_{a_k}$, and the theorem is proved.

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A LETTER TO THE EDITOR

Enrico T. Federighi and Ronald G. Roll
The Johns Hopkins University
Silver Spring, Maryland

A careful study of the Tables of Fibonacci Entry Points has led us to make some observations regarding the factors of Fibonacci numbers.

It is readily seen that $Z(p)$ divides $p-1$ whenever $p \equiv \pm 1 \pmod{10}$ and divides $p+1$ whenever $p \equiv \pm 3 \pmod{10}$.

Our problem is to determine, if possible, the primes p for which $Z(p)$ divides $p(k) = (p \pm 1)/k$ for $k > 1$. We conjecture the following solutions.

$k=2$: $Z(p)$ divides $p(2)$ if and only if $p=4n+1$.

$k=3$: $Z(p)$ divides $p(3)$ if and only if $p=x^2 + 135y^2$ or $p = 5x^2 + 27y^2$.

$k=4$: $Z(p)$ divides $p(4)$ if and only if $p=x^2 + 80y^2$ or $p = 5x^2 + 16y^2$.

The rule is more complicated but is still reasonable if k is a small prime. Let k be such a prime and define $\beta = (k-1)/2$.

Let $p(k,\ell) = 2 \cos (2\pi\ell/k)$ and $\gamma_{ij} = g^{i+j-2}$, where g is a primitive root of k. Also let $\epsilon_{kj} = 2 - p(k, g^{j-1})$. Define

$$C(k, x, y) = \prod_{j=1}^{\beta} \left\{ \left[\sum_{i=1}^{\beta} a_i p(k, \gamma_{ij}) \right]^2 x^2 + \epsilon_{kj} \left[\sum_{i=1}^{\beta} b_i p(k, \gamma_{ij}) \right]^2 y^2 \right\}$$

$$= \sum_{r=0}^{\beta} c_{2r} x^{2\beta-2r} y^{2r} .$$

$k > 4$: $Z(p)$ divides $p(k)$ if and only if $mp = C(k, \sqrt{5}, 1)$ or $mp = C(k, 1, \sqrt{5})$ where $c_{2\beta} \equiv 0 \pmod{k^3}$. The nature of m is uncertain but it will usually be unity or a power of 2^{β}. If $5^n \equiv 1 \pmod{k}$ for $n < \beta$, m may also contain an even power of 5.

To illustrate how the formulations look in practice, a table of all primes under 1000 with $k \geq 3$ is listed. It is assumed that $x=y=1$. Type 1 means that

$$mp = \sum_{r=0}^{\beta} c_{2r} 5^r$$

and type 2 means that

$$mp = \sum_{r=0}^{\beta} c_{2r} 5^{\beta-r} .$$

p	k	type	m	c_0	c_2	c_4	c_6
47	3	1	1	4	27		
61	4	1	1	9	16		
89	4	2	1	9	16		
	8	2	1	1	12	4	
107	3	1	1	16	27		
109	4	1	1	9	64		
113	3	1	1	1	108		
139	3	2	1	4	27		
149	4	1	1	1	144		
151	3	2	1	16	27		
199	3	2	1	64	27		
	9	2	64	361	585	243	27
211	5	2	16	1	50	125	
233	3	1	1	25	108		
	9	1	64	1	153	2187	27
263	3	1	1	4	243		
269	4	1	1	25	144		
281	5	2	16	121	250	125	
307	7	1	64	1	581	931	343
331	3	2	1	196	27		
347	3	1	1	64	27		
353	3	1	1	49	108		
389	4	1	1	49	144		
401	4	2	1	81	64		
421	4	1	1	81	16		
	5	2	16	361	650	125	

p	k	type	m	c_0	c_2	c_4	c_6	c_8	c_{10}	c_{12}
461	5	2	16	1	850	125				
521	4	2	1	441	16					
	5	2	16	961	850	125				
541	3	2	1	1	108					
557	3	1	1	25	432					
	9	1	64	1	153	4779	7803			
563	3	1	1	64	243					
619	3	2	1	484	27					
661	3	2	1	121	108					
	4	1	1	81	256					
677	3	1	1	49	432					
691	5	2	16	1681	1250	125				
701	4	1	1	25	576					
709	3	2	1	169	108					
743	3	1	1	100	243					
761	4	2	1	441	64					
	8	2	1	1	96	256				
769	4	2	1	49	144					
	8	1	1	289	76	4				
797	7	1	64	169	917	1323	343			
809	4	2	1	729	16					
811	3	2	1	676	27					
821	4	1	1	49	576					
829	3	2	1	289	108					
	4	1	1	9	784					
859	11	1	1024	1	319	3146	9438	9317	1331	
881	5	2	16	3721	1450	125				
911	13	1	102400	1	182	10595	222404	1595191	3405350	1373125
919	3	2	1	784	27					
	9	2	64	1	153	2187	27			
953	3	1	1	169	108					
	9	1	64	361	585	243	27			
967	11	1	32768	1	1551	105754	642510	286165	1331	
977	3	1	1	1	972					
991	5	2	16	3481	1850	125				

Comments on Table:

1) As can be seen whenever k is an odd prime c_{k-1} is divisible by k^3.

2) If k is an odd prime $\sum b_i \equiv 0 \pmod{k}$ will ensure that c_{k-1} be divisible by k^3.

3) If one sums the coefficients in the table without first multiplying by powers of five one obtains k-th power residues of two.

Some of this has undoubtedly been observed before and even probably proved but we have no idea how much.

We have enjoyed playing around with these concepts and actually suspect much more than we have indicated here. If anyone is interested in pursuing this further, we shall be glad to hear from him.

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LETTER TO THE EDITOR

Lenard Weinstein
Massachusetts Institute of Technology

Conjecture 2., made by Mr. Thoro, on page 186 of the October issue, follows immediately from the following theorem found on page 126 of W. J. Leveque, Topics in Number Theory, Vol. I:

Definition

A representation of a positive integer n as a sum of two squares, say $n = x^2 + y^2$ is termed proper if $(x, y) = 1$.

Theorem

If p is a prime of the form $4k + 3$ and $p | n$, then n has no proper representation.

Since $F_{2n+1} = F_n^2 + F_{n+1}^2$, and $(F_n, F_{n+1}) = 1$, F_{2n+1} always has a proper representation. Therefore, by the above theorem, no prime of the form $4k + 3$ can divide F_{2n+1}.

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A LOGARITHMIC FORMULA FOR FIBONACCI NUMBERS

Gerard R. Deily
United States Department of Defense
Washington, D.C.

If the logarithm of the Fibonacci number F_n is plotted against n, it can be seen that for large n the graph is a straight line. Thus one might expect that Fibonacci numbers could be computed from a formula of the form

$$\log F_n = m n + b \ , \tag{1}$$

where m is the slope of the line and b its intersection with the vertical axis. That this is so can easily be shown by manipulating Binet's formula

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right] \tag{2}$$

into the following form:

$$F_n = \frac{1}{5} \left(\frac{1 + \sqrt{5}}{2} \right)^n \left[1 - \left(\frac{1 - \sqrt{5}}{1 + \sqrt{5}} \right)^n \right] \tag{3}$$

For large n, the second term within the bracket becomes negligible, and hence (3) becomes

$$F_n \ \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n \tag{4}$$

Taking logarithms then gives

$$\log F_n \quad n\left[\log (1 + \sqrt{5}) - \log 2 \right] - \frac{1}{2} \log 5 \tag{5}$$

which is of the form (1). If base 10 is used, the characteristic of the logarithm computed in (5) then gives the order of magnitude of F_n. This knowledge is useful in determining required sizes of registers when setting up Fibonacci problems for computation.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. Hillman
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within two months of publication.

B-82 *Proposed by Nanci Smith, University of New Mexico, Albuquerque, N.M.*

Describe a function $g(n)$ having the table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$g(n)$	0	1	1	2	1	2	2	3	1	2	2	3	2	...

B-83 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada*

Show that $F_n^2 + F_{n+4}^2 = F_{n+1}^2 + F_{n+3}^2 + 4F_{n+2}^2$.

B-84 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada*

The Fibonacci polynomials are defined by $f_1(x) = 1$, $f_2(x) = x$,

$$f_{n+1}(x) = xf_n(x) + f_{n-1}(x), \qquad n > 1 \ .$$

If $z_r = f_r(x) + f_r(y)$, show that z_r satisfies

$$z_{n+4} - (x+y)z_{n+3} + (xy-2)z_{n+2} + (x+y)z_{n+1} + z_n = 0 \ .$$

B-85 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Find compact expressions for:

(a) $$F_2^2 + F_4^2 + F_6^2 + \ldots + F_{2n}^2$$

(b) $$F_1^2 + F_3^2 + F_5^2 + \ldots + F_{2n-1}^2 \ .$$

B-86 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that the squares of every third Fibonacci number satisfy

$$y_{n+3} - 17y_{n+2} + 17y_{n+1} - y_n = 0 \quad .$$

B-87 *Proposed by A.P. Hillman, University of New Mexico, Albuquerque, N.M.*

Prove the identity in

$$\sum_{k=0}^{n} \left[\frac{(-1)^{n-k}}{k!\,(n-k)!} \prod_{j=0}^{n} (x_j + k)\right] = \binom{n+1}{2} + \sum_{j=0}^{n} x_j \quad .$$

SOLUTIONS

AN N-TUPLE INTEGRAL

B-70 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Denote x^a by ex(a). Show that the following expression, containing n integrals,

$$\int_0^1 \mathrm{ex}\left(\int_0^1 \mathrm{ex}\left(\int_0^1 \mathrm{ex}(\ldots \int_0^1 \mathrm{ex}\left(\int_0^1 x\,dx\right)dx\right)\ldots dx\right)dx\right)dx$$

equals F_{n+1}/F_{n+2}, where F_n is the n-th Fibonacci number.

Solution by John Wessner, Melbourne, Florida

Let I_n denote the n-th such integral. Then

$$I_1 = \int_0^1 x\,dx = 1/2 \quad .$$

Let us assume that $I_{n-1} = F_n/F_{n+1}$, in which case

$$I_n = \int_0^1 x^{F_n/F_{n+1}} dx = \left\{(F_n/F_{n+1}) + 1\right\}^{-1}$$

$$= \left\{(F_n + F_{n+1})/F_{n+1}\right\}^{-1} = F_{n+2}/F_{n+1} \quad ,$$

which was to be shown.

Also solved by R.J. Hursey, Jr; M.N.S. Swamy; Howard L. Walton; David Zeitlin; and the proposer

B-71 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Find $a^{-2} + a^{-3} + a^{-4} + \ldots$, where $a = (1 - 5)/2$.

Solution by John W. Milsom, Slippery Rock State College, Slippery Rock, Penna.

If $S = a^{-2} + a^{-3} + a^{-4} + \ldots$, then $a^2 S = 1 + a^{-1} + a^{-2} + \ldots$.

Subtracting the first equation from the second,

$$a^2 S - S = 1 + a^{-1}$$

$$S = (1 + a^{-1})/(a^2 - 1)$$

$$S = 1/[a(a - 1)] \ .$$

Using $a = (1 + \sqrt{5})/2$, we find that $S = 1$. If you use $a = (1 + 5)/2 = 6/2 = 3$, as the problem reads, the result is $S = 1/6$.

Also solved by R.J. Hursey, Jr; Sidney Kravitz; M.N.S. Swamy; C.W. Trigg; Howard L. Walton; John Wessner; David Zeitlin; and the proposer.

ADDING RABBITS?

B-72 *Proposed by J.A.H. Hunter, Toronto, Canada*

Each distinct letter in this simple alphametic stands for a particular and different digit. We all know how rabbits link up with the Fibonacci series, so now evaluate our RABBITS.

```
RABBITS
   BEAR
RABBITS
     AS
A SERIES
```

Solution by Charles W. Trigg, San Diego, California

By the first column from the left, $0 < R < 5$. By the seventh column, $2S + R = 10k$, so $S \neq 0$, and R is even. That is, $R = 2$ or 4.

By the fourth column, $3B + 1 = R$, so B is odd.

With these and the obvious relations from the other columns we can proceed to establish the values of the letters in the order given in the table below:

R	B	A	S	E	T	I
2	7	4	9	6	3	3
					8	2
4	1	9	8	2	6	5

Since the first two sets contain duplicate digits, the third set is the unique solution. Thus

$$\begin{array}{r} 4911568 \\ 1294 \\ 4911568 \\ 98 \\ \hline 9824528 \end{array}$$

That is, RABBITS = 4911568, which just goes to show what 2 rabbits can do.

Also solved by Murray Berg; Rudolph W. Castown; Sidney Kravitz; John W. Milsom; Azriel Rosenfeld; and the proposer.

DOUBLE SUMS

B-73 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Prove that

$$\sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k}\binom{k+r-j-1}{j} = 1 + \sum_{m=0}^{2n+r-2} \sum_{p=0}^{m}\binom{m-p-1}{p} ,$$

where $\binom{n}{r} = 0$ for $n < r$.

Solution by David Zeitlin, Minneapolis, Minnesota

The given identity is valid only for $r \le n + 1$. Since

$$F_{n-1} = \sum_{j=0}^{[n/2]} \binom{n-j}{j}, \quad \sum_{k=0}^{n} F_k = F_{n+2} - 1, \text{ and } \sum_{k=0}^{n} \binom{n}{k} F_{k+m} = F_{2n+m} ,$$

we have

$$1 + \sum_{m=0}^{2n+r-2} \sum_{p=0}^{m} \binom{m-p-1}{p} = 1 + \sum_{m=0}^{2n+r-2} F_m = F_{2n+r} ;$$

while for $r \leq n + 1$, we have

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{n} \binom{k+r-j-1}{j} = \sum_{k=0}^{n} \binom{n}{k} F_{k+r} = F_{2n+r} .$$

Also solved by the proposer.

FIBONACCI POLYNOMIALS

B-74 *Proposed by M.N.S. Swamy, University of Saskatchewan, Regina, Canada*

The Fibonacci polynomial $f_n(x)$ is defined by $f_1 = 1$, $f_2 = x$, and $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$ for $n > 2$. Show the following:

(a) $$x \sum_{r=1}^{n} f_r(x) = f_{n+1} + f_n - 1 .$$

(b) $$f_{m+n+1} = f_{m+1} f_{n+1} + f_m f_n .$$

(c) $$f_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1} ,$$

where $[k]$ is the greatest integer not exceeding k. Hence show that the n-th Fibonacci number

$$F_n = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} .$$

Solution by David Zeitlin, Minneapolis, Minnesota

(a) Assuming the relation to be true for $n = n$, we have

$$\begin{aligned} x \sum_{r=1}^{n+1} f_r(x) &= xf_{n+1} + (f_{n+1} + f_n - 1) \\ &= f_{n+2} + f_{n+1} - 1 , \end{aligned}$$

and the result now follows by mathematical induction.

(b) Using formula (6) in my paper, "On summation formulas for Fibonacci and Lucas numbers," this Quarterly, vol. 2, 1964, No. 2, p. 105, we have (since $f_0 = 0$)

$$\text{(1)} \qquad \frac{f_{m+1} + f_m \cdot y}{1 - xy - y^2} = \sum_{n=0}^{\infty} f_{m+n+1}\, y^n ,$$

$$\text{(2)} \qquad \frac{f_{m+1}}{1 - xy - y^2} = \sum_{n=0}^{\infty} f_{m+1}\, f_{n+1}\, y^n ,$$

$$\text{(3)} \qquad \frac{f_m \cdot y}{1 - xy - y^2} = \sum_{n=0}^{\infty} f_m f_n\, y^n .$$

Since (1) = (2) + (3), the result follows by equating coefficients of y^n.

(c) We note that

$$\frac{y}{1-xy-y^2} = \sum_{n=0}^{\infty} f_n(x) y^n$$

and recall that

$$\frac{1}{1-2tz+z^2} = \sum_{n=0}^{\infty} U_n(t) z^n ,$$

where $U_n(t)$ is the Chebyshev polynomial of the second kind defined by

$$\text{(4)} \qquad U_n(t) = \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} (2t)^{n-2j} .$$

with $i^2 = -1$, we see that for $z = iy$ and $t = x/2i$, we have

$$\frac{1}{1-xy-y^2} = \sum_{n=0}^{\infty} i^n\, U_n\left(\frac{x}{2i}\right) y^n ,$$

and thus $f_{n+1}(x) = i^n\, U_n(x/2i)$, the desired result, using (4).

Since $F_n = f_n(1)$, we obtain

$$F_n = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} .$$

Also solved by the proposer.

DERIVATIVES OF FIBONACCI POLYNOMIALS

B-75 *Proposed by M.N.S. Swamy, University of Saskatchewan, Regina, Canada*

Let $f_n(x)$ be as defined in B-74. Show that the derivative

$$f'_n(x) = \sum_{r=1}^{n-1} f_r(x)\, f_{n-r}(x) \quad \text{for } n > 1.$$

Solution by David Zeitlin, Minneapolis, Minnesota

If we differentiate with respect to x the identity

$$\frac{y}{1-xy-y^2} = \sum_{n=0}^{\infty} f_n(x) y^n ,$$

we obtain

$$\sum_{n=0}^{\infty} f'_n(x) y^n = \left(\frac{y}{1-xy-y^2}\right)^2 = \left(\sum_{n=0}^{\infty} f_n(x) y^n\right)^2$$

$$= \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} f_r(x) f_{n-r}(x)\right] y^n .$$

If we equate coefficients of y^n, we obtain

$$f'_n(x) = \sum_{r=0}^{n} f_r(x) f_{n-r}(x)$$

$$= \sum_{r=1}^{n-1} f_r(x) f_{n-r}(x) \qquad (\text{since } f_0(x) = 0) .$$

Also solved by Lawrence D. Gould and the proposer.

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