FIBONACCI SPACES

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INTRODUCTION

The Fibonacci sequence $\{F_j\} = 0, 1, 1, 2, \ldots$, with $F_j + F_{j+1} = F_{j+2}$, $j \geq 0$, may be regarded as one element of a certain <u>space</u> of sequences associated with the quadratic polynomial $f(x) = -1 - x + x^2$, from which its remarkable properties derive. In the following pages, we present first, in modern algebraic terminology, an exposition of those parts of the general theory of such spaces of linear recurring sequences which form a background for this point of view. The spaces arising from quadratic polynomials are then considered in this setting, with some applications to number theory, in particular to various tests for primality of the Mersenne and Fermat numbers.

It is hoped that the paper may thus serve as an introduction and source of reference for these aspects of the subject. [1]**

1. THE SPACE OF A POLYNOMIAL

Let $f(x) = -a_0 - a_1 x - \ldots - a_{n-1}x^{n-1} + x^n = (x - r_1)\ldots(x - r_n)$ be an arbitrary monic polynomial of degree n in $Z[x]$, i.e., with coefficients a_j in the domain Z of rational integers, its (complex) zeros being therefore algebraic integers. With $f(x)$ we associate the set $C(f)$ of all sequences $S = \{s_0, s_1, \ldots\}$ with components s_j in the complex field [2] C, in which $s_0, \ldots, s_{n-1}$ are arbitrary but having

$$\text{(R)} \qquad a_0 s_j + a_1 s_{j+1} + \ldots + a_{n-1} s_{j+n-1} = s_{j+n}$$

for all $j \geq 0$. Clearly, $C(f)$ is a vector space of order n over C. An obvious basis consists of the integral sequences [3]** (i.e., with components in Z):

$$U_0 = \{u_{0j}\} = \{1, \ldots, 0, a_0, \ldots\}, \ldots, U_{n-1} = \{u_{n-1,j}\} = \{0, \ldots, 1, a_{n-1}, \ldots$$

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**Refer to footnotes at end of article.

of $C(f)$, in terms of which every sequence S of the space may be expressed uniquely, in the form

$$S = s_0U_0 + \ldots + s_{n-1}U_{n-1}, \; s_j \in C \quad .$$

A "Geometric" sequence $\{1, r, r^2, \ldots\}$ with $r \in C$ is in $C(f)$ if and only if $f(r) = 0$. Thus

$$R_i = \{1, r_i, r_i^2, \ldots\}, \qquad i = 1, \ldots, n$$

constitute the geometric sequences of $C(f)$. Every element

$$c_1R_1 + \ldots + c_nR_n, \; c_i \in C \quad ,$$

of the space they span is therefore in $C(f)$, in particular the integral sequence $V = R_1 + \ldots + R_n = \{v_j\} = \{n, a_{n-1}, \ldots\}$, with $v_j = r_1^j + \ldots + r_n^j$, $j \geq 0$. Being in $C(f)$, its components v_{j+n} satisfy the relations (R); these, together with the less obvious recursions

$$(a_{n-m})m + a_{n-m+1}v_1 + \ldots + a_{n-1}v_{m-1} = v_m, \; 1 \leqq m < n$$

are "Newton's formulas".

The geometric sequences R_i also form a basis for $C(f)$ over C if and only if the zeros of $f(x)$ are distinct. For, the matrix $R = [r_i^j]$, $i = 1, \ldots, n$; $j = 0, \ldots, n-1$, has the Vandermonde determinant $\pi_{k > i}(r_k - r_i)$. When the latter is not zero, the inverse matrix $R^{-1} = [r_{ij}]$ exists [4] (over C) with $I = R^{-1}R$, from which it follows that

(B) $\qquad U_i = r_{i1}R_1 + \ldots + r_{in}R_n, \quad i = 0, \ldots, n-1$

By means of these equations, which may be regarded as the "Binet formulas" for the general case, every sequence

$$S = \{s_j\} = \{s_0U_0 + s_{n-1}U_{n-1}\} \text{ of } C(f)$$

is expressible uniquely in the form

$$S = c_1R_1 + \ldots + c_nR_n, \; c_i \in C$$

with $s_j = c_1r_1^j + \ldots + c_nr_n^j$, $j \geq 0$, when the r_i are distinct. This underlying structure of the sequences of C(f), particularly of the integral sequences, is one of the most curious features of the subject.

For example, if all zeros of f(x) are h-th roots of unity, it is clear that every $S = \Sigma c_i R_i$ is pure periodic, with period [5]** dividing h. When $a_0 \neq 0$ and the r_i are distinct, the existence of a sequence S, with all the $c_i \neq 0$, of period k, implies that all r_i are k-th roots of unity. For, $a_0 \neq 0$ insures, via (R), that S is pure periodic, and we should then have a linear system $\Sigma r_i^j (r_i^k - 1) c_i = 0$, $j = 0, \ldots, n-1$, with determinant det $R \neq 0$. Thus $(r_i^k - 1)c_i = 0$ for each i, and either $c_i = 0$ or r_i is a k-th root of unity. Consequently, if f(x) has distinct zeros which are roots of unity, and h is the least positive integer for which all $r_i^h = 1$, then every $S = \Sigma c_i R_i$ of C(f) with all $c_i \neq 0$ is pure periodic of period k equal to h. For, by the first remark, $k|h$, and by the second, $h \leqq k$, hence, every period $k = h$.

Note: The following generalization of the "geometric basis" theorem was suggested by the referee: If r is a zero of multiplicity m for f(x), then r is a zero of the first m-1 derivatives of f(x). From this one may show that the m sequences

$$s_j^{(h)} = \binom{j}{h} r^j; \quad j \geqq 0, \ h = 0, 1, \ldots, m-1 \ ,$$

are in C(f), where $\binom{j}{h} = 1$ for $h = 0$, and $\binom{j}{h} = (j)(j-1)\ldots(j-h+1)/h!$ for $h \geqq 1$. Moreover, if f(x) has the <u>distinct</u> zeros $r_1, \ldots, r_k$, $1 \leqq k \leqq n$, of multiplicities $m_1, \ldots, m_k$ respectively, then the set of n sequences consisting of k subsets such as that above, one for each zero r_i, are linearly independent, and so form a basis for C(f) in the general case, provided only that zero, if it occurs among the r_i, has multiplicity 1. The linear independence follows from the nonsingularity of the "confluent Vandermonde matrix." Cf. ref. [1].

2. LIMIT THEOREMS

If $\lim s_{j+1}/s_j = r$ exists for a terminally non-zero sequence S of C(f), then r must be one of the zeros of f(x). For, the recursion (R) implies $a_0 + a_1(s_{j+1}/s_j) + \ldots + a_{n-1}(s_{j+n-1}/s_j) = s_{j+n}/s_j$. Since $s_{j+2}/s_j = (s_{j+2}/s_{j+1})(s_{j+1}/s_j)$, and so on, we have $f(r) = 0$ in the limit. As a partial answer to the questions arising here we include

Theorem 1. (A). If one zero r_1 of $f(x)$ exceeds all others in absolute value, then $S = \{s_j\} = \Sigma c_i R_i$ has $\lim s_j/r_1^j = c_1$. Hence if $c_1 \neq 0$, S is terminally non-zero, and $\lim s_{j+1}/s_j = r_1$.

(B). If $|r_i| < 1$ for all $i \geqq 2$, the sequence S has $\lim (s_j - c_1 r_1^j) = 0$. Hence, if S is an integral sequence, s_j is the closest integer to $c_1 r_1^j$ for large j, and, if $a_0 \neq 0$, no other integral sequence $\Sigma c_i R_i$ has the same c_1.

Proof. (A). $s_j/r_1^j = c_1 + c_2(r_2/r_1)^j + \ldots + c_n(r_n/r_1)^j \to c_1$.

If $c_1 \neq 0$, then $s_j \neq 0$, $j \geqq J$, and so $s_{j+1}/s_j =$

$$r_1(s_{j+1}/r_1^{j+1})/(s_j/r_1^j) \to r_1 c_1/c_1 = r_1 \quad .$$

(B). $s_j - c_1 r_1^j = c_2 r_2^j + \ldots + c_n r_n^j \to 0$. Two integral sequences with the same c_1 are therefore terminally identical, and hence identical, if $a_0 \neq 0$. Indeed, it is clear from (R) that two sequences are equal if they agree on any n consecutive corresponding indices.

3. INTEGRAL SEQUENCES

The integral sequences $F = \{f_j\}$ of $C(f)$ form a module $Z(f)$, with integral basis $U_0, \ldots, U_{n-1}$, every such sequence being uniquely expressible, with integral coefficients, in the form

$$F = f_0 U_0 + \ldots + f_{n-1} U_{n-1}, \quad f_j \in Z \quad .$$

The sequences of $Z(f)$ with $f_0 = 0$ form a sub-module of $Z(f)$, and have remarkable divisibility properties, as we shall see.

The sequences of $Z(f)$, considered modulo m, form a finite module $Z_m(f)$ of exactly m^n sequences with components in the modular ring $Z_m = Z/Zm$, the first n arbitrary, the rest governed by the recursion (R) mod m.

Suppose now that $m \geqq 2$ is an integer prime to a_0, and let $F = \{f_j\}$ be a sequence of $Z_m(f)$. It is clear from (R) that F is pure periodic if it is periodic at all, hence that F is periodic if and only if its n-tuples

$$(f_0, \ldots, f_{n-1}) \text{ and } (f_k, \ldots, f_{k+n-1})$$

are identical for some positive k, the least such k being its period. Since $F = 0$, of period $k = 1$, is the only sequence of $Z_m(f)$ containing the zero n-tuple, a non-zero sequence F can contain at most m^{n-1} different n-tuples, and so must be pure periodic of period $k \leqq m^n - 1$.

Moreover, F has period $k = m^n - 1$ if and only if its first $m^n - 1$ n-tuples

$$(f_0, \ldots, f_{n-1}), (f_1, \ldots, f_n), \ldots, (f_{m^n-2}, \ldots, f_{m^n+n-3})$$

are all distinct. In such a case, each of the $m^n - 1$ non-zero sequences of $Z_m(f)$ is a terminal sub-sequence of F, and so also has this period. The situation cannot arise for a composite modulus $m = ab$, $a \geqq 2$, $b \geqq 2$. For, such an F contains the n-tuples $(1, 0, \ldots, 0)$ and $(b, 0, \ldots, 0)$, hence aF is in the space and contains the n-tuples $(a, 0, \ldots, 0) \neq (0, 0, \ldots, 0)$, which is impossible.

The maximum period $m^n - 1$ <u>is</u> attainable in case of a prime modulus, which may be seen from the theory of Galois fields [7]. Let p be a prime in Z, and suppose $f(x) = -a_0 - a_1 x - \ldots - a_{n-1}x^{n-1} + x^n$ is an irreducible polynomial in $Z_p[x]$. Such an $f(x)$ exists for every p and n. The sequences of $Z_p(f)$ may then be regarded not only as the integral sequences, mod p, of the space $C(f)$, but also in a quite different light. For there exists a field $C^* \supset Z_p$, the "root-field" of $f(x)$ (abstractly, the Galois field $GFp^n)$) of exactly p^n elements, uniquely expressible in the form

$$s = f_0 + f_1 r + \ldots + f_{n-1} r^{n-1}, \quad f_j \in Z_p$$

in terms of which we may write

$$f(x) = (x - r)(x - r^p) \ldots (x - r^{p^{n-1}})$$

with n distinct zeros in C^*.

Following the plan of § 1, we find that the set $C^*(f)$ of all sequences $S = \{s_j\}$, with $s_j \in C^*$ governed by the recursion (R) in C^*, is a vector space of order n over C^*, consisting of exactly $(p^n)^n$ sequences, and containing the original module $Z_p(f)$ of p^n sequences $F = \{f_j\}$, $f_j \in Z_p$.

The zeros of $f(x)$ being distinct in C^*, the geometric sequences R_i which they generate form a basis for $C^*(f)$, in terms of which every sequence of the space may be uniquely expressed:

$$S = c_1 R_1 + \ldots + c_n R_n, \quad c_j \in C^*$$

with components $s_j = c_1 r_1^{\,j} + \ldots + c_n r_n^{\,j}$, $j \geqq 0$, where the r_i are now the zeros of $f(x)$ displayed above.

This sub-structure of the sequences S reveals their periodic character. For, the multiplicative group G of C^* is cyclic, with a generator s of period $p^n - 1$. If h is the period of r in G, then $h \mid p^n - 1$, and the element r^{p^i} has period $h/(h, p^i) = h$, which is therefore the period of <u>every</u> zero of $f(x)$. An obvious modification of the argument at the end of § 1 shows that every non-zero sequence S of $C^*(f)$ has period h.

The element r itself need not have period $p^n - 1$; however, some element $s \neq 0$ of C^* does generate the full group G, and its minimal polynomial mod p is irreducible of degree n in $Z_p[x]$. Hence there exists, for every p and n, an irreducible $f(x)$ of degree n in $Z_p[x]$ for which the zero r in C^* generates G, and every sequence $S \neq 0$ of the corresponding space $C^*(f)$ has period $p^n - 1$. We summarize these results in more conventional terms in

Theorem 2. If $m \geqq 2$ is an integer prime to a_0, then, modulo m, every integral sequence $F = \{f_j\}$ of $C(f)$ is pure periodic of period $k \leqq m^n - 1$. Hence, if $m \mid f_0$, then also $m \mid f_k, f_{2k}, f_{3k}, \ldots$. If m is composite, every period is less than $m^n - 1$. If p is a prime, and $f(x)$ is irreducible [6]** mod p, all integral sequences F of $C(f)$ are pure periodic with the same period h, where $h \mid p^n - 1$. For every p and n, there exists an $f(x)$ such that $h = p^n - 1$.

Example 1. For $f(x) = -2-x+x^3$ mod 3,

$$V = \{\dot{0}0202122102220010121120111\dot{0}0, \ldots\},$$

the period [7] being $h = 26$.

If an integer $m \geqq 1$ divides some f_j, $j \geqq 1$, of an integral sequence F, the least positive such j is called the rank of m in F.

Corollary 1. If F is an integral sequence of $C(f)$, with $f_0 = 0$, every positive integer m prime to a_0 has a rank $r \leqq k \leqq m^n - 1$ in F.

4. THE SPACE OF A QUADRATIC

For $f(x) = -a_0 - a_1 x + x^2 = (x-r_1)(x-r_2)$, the associated space $C(f)$ of all sequences $S = \{s_j\}$, $s_j \in C$, satisfying

$$\text{(R)} \qquad a_0 s_j + a_1 s_{j+1} = s_{j+2}, \quad j \geqq 0$$

has the basis $U_0 = \{1, 0, a_0, \ldots\}$, $U_1 = \{0, 1, a_1, \ldots\}$, and contains the sub-space of vectors $c_1R_1 + c_2R_2$, where $R_i = \{1, r_i, r_i^2, \ldots\}$, $i = 1, 2$ are its geometric sequences. The particular sequence

$$V = R_1 + R_2 = \{v_j\} = \{2, a_1, 2a_0 + a_1^2, \ldots\}$$

consisting of the integers $v_j = r_1^j + r_2^j$, $j \geqq 0$, is of special importance.

The geometric sequences R_i form a basis for $C(f)$ if and only if $r_1 \neq r_2$, in which case the matrix R^{-1} of § 1 is

$$\frac{1}{r_1 - r_2} \begin{bmatrix} -r_2 & r_1 \\ 1 & -1 \end{bmatrix}$$

The corresponding "Binet formulas" are accordingly

$$\text{(B)} \qquad \begin{aligned} U_0 &= (-r_2R_1 + r_1R_2)/(r_1 - r_2) \quad , \\ U_1 &= (R_1 - R_2)/(r_1 - r_2) \quad , \end{aligned}$$

or explicitly,

$$u_{0j} = (-r_2r_1^j + r_1r_2^j)/(r_1 - r_2), \quad u_{1j} = (r_1^j - r_2^j)/(r_1 - r_2)$$

The relation $u_{0j} = a_0 u_{1,j-1}$ is here manifest.

We know from § 2 that, if $|r_1| > |r_2|$, $\lim u_{1j}/r_1^j = 1/(r_1 - r_2)$, $\lim v_j/r_1^j = 1$, and $\lim u_{1,j+1}/u_{1j} = \lim v_{j+1}/v_j = r_1$.

Again, if $|r_2| < 1$, then, for example, $\lim (v_j - r_1^j) = 0$ and v_j is the nearest integer to r_1^j, $j \geqq J$; moreover if $a_0 \neq 0$, V is the only integral sequence $R_1 + c_2R_2$.

The integral sequences $F = \{f_j\}$ of $C(f)$ form a module, with integral basis U_0, U_1, and, modulo an integer m prime to a_0, are all pure periodic with periods less than m^2. For a prime modulus p, if $f(x)$ is irreducible mod p, these periods are equal, and divide $p^2 - 1$, and there exists an $f(x)$ such that every period is exactly $p^2 - 1$.

The sub-module of sequences $F = f_1U_1$ (i.e., with $f_0 = 0$) has the single basic sequence U_1, which is hereafter denoted simply by $U = \{u_j\} = \{0, 1, a_1, \ldots\}$. For $f(x) = -1-x+x^2$, it is of course the Fibonacci sequence.

Every integer m prime to a_0 has a rank $r \leqq k < m^2$ in U, where k is the period of U mod m; indeed $m | u_k, u_{2k}, \ldots$; similar statements may be made for every $F = f_1U$, and $F = f_0U_0$.

It is interesting that every sequence of $C(f)$ is expressible in terms of U alone. For example, from $V = 2U_0 + a_1U$ follows

$$v_j = a_0u_{j-1} + u_{j+1}, \quad j \geqq 1 .$$

Example 2. For

$$f(x) = 1 - 2x + x^2 = (x - 1)^2, \quad U = 0, 1, 2, 3, 4, \ldots \quad V = 2, 2, 2, 2, 2, \ldots .$$

Example 3. For

$$f(x) = 2 - 3x + x^2 = (x-2)(x-1), \quad u_j = 2^j - 1, \quad v_j = 2^j + 1$$

satisfy the simple recursions $u_{j+1} = 2u_j + 1$, $v_{j+1} = 2v_j - 1$. Note that $u_p = 2^p - 1$, $v_{2^k} = 2^{2^k} + 1$. The sequence U mod u_j has period j.

Example 4. For

$$f(x) = -2 - x + x^2 = (x - 2)(x + 1), \quad u_p = (2^p + 1)/3, \quad v_p = 2^p - 1 ,$$

for odd p, and $v_{2^k} = 2^{2^k} + 1$, $k \geqq 1$.

5. THE SEQUENCE U

Even for the general quadratic [8], the sequence

$$U = \{0, 1, a_1, a_0 + a_1^2, \ldots\}$$

has some remarkable properties, which flow from the

Lemma 1. For all

$$j \geqq 0,\ t \geqq 0,\ a_0 u_j u_t + u_{j+1} u_{t+1} = u_{j+t+1} \quad .$$

The statement is easily proved, for fixed $t \geqq 0$, by induction from j, $j+1$ to $j+2$, being obvious for $j = 0, 1$. The induction step reads

$$a_0 u_{j+2} u_t + u_{j+3} u_{t+1} = a_0(a_0 u_j + a_1 u_{j+1})u_t + (a_0 u_{j+1} + a_1 u_{j+2})u_{t+1} =$$
$$a_0 u_{j+t+1} + a_1 u_{j+t+2} = u_{j+t+3} \quad .$$

From this follows

Theorem 3. The correspondence $j \to u_j$ preserves divisibility i.e., $j|k$ implies $u_j|u_k$, or $u_j|u_j, u_{2j}, u_{3j}, \ldots$.

Since $u_0 = 0$, $u_1 = 1$, the final statement is trivial for $j = 0, 1$. Fixing $j \geqq 2$, we prove $u_j|u_{jq}$ by induction on $q \geqq 1$. This is trivial for $q = 1$. Fix $q \geqq 1$, and assume $u_j|u_{jq}$. Setting $t = jq - 1$ ($\geqq 1$) in the Lemma shows that $u_j|u_{j(q+1)}$.

Lemma 2. If a prime p divides any two consecutive u_k, then p divides a_0.

If $p|u_{j+1}, u_{j+2}$ but not a_0, then from (R) follows $p|u_j, u_{j+1}$ and ultimately $p|u_0 = 0$, $u_1 = 1$, which is false.

Theorem 4. Let m be a positive integer prime to a_0. Then, modulo m, U is pure periodic of period $k < m^2$, and $m|u_0, u_k, u_{2k}, \ldots$. Thus m has a rank $r \leqq k$ in U. Moreover, $m|u_0, u_r, u_{2r}, \ldots$ and no other u_j, i.e., $m|u_\ell$ if and only if $r|\ell$. In particular, $r|k$.

We have only to prove $m|u_\ell$ implies $r|\ell$, which is obvious for $\ell = 0$. For $\ell \geqq 1$, we have $r \leqq \ell$, by the minimality of r. Write $\ell = rq + j$, $0 \leqq j < r$, $q \geqq 1$. For $t = rq - 1$ ($\geqq 0$), Lemma 1 reads

$$a_0 u_j u_{rq-1} + u_{j+1} u_{rq} = u_\ell \quad .$$

Since $m|u_\ell, u_r, u_{rq}$ (Th. 3), we have $m|a_0 u_j u_{rq-1}$. Now m is prime to a_0, and hence also to u_{rq-1}, since a prime common to this and m is also in u_{rq}, contradicting Lemma 2. It follows that $m \mid u_j$, where $0 \leqq j < r$. Hence $j = 0$ and $\ell = rq$. We turn to the special case of

Lemma 3. If $(a_0, a_1) = 1$, the sequence U has the properties:

(a) $(a_0, u_j) = 1$, for all $j \geqq 1$,

(b) $(u_j, u_{j+1}) = 1$, for all $j \geqq 0$,

(c) $(u_j, u_k) = (u_k, u_{j+k})$ for all $j, k \geqq 0$.

Proof. (a) Induction on $j \geqq 1$. For $j = 1$, trivial. Assuming $(a_0, u_{j+1}) = 1$ for fixed $j \geqq 0$, we see from (R) that $(a_0, u_{j+2}) = 1$ also. For, a prime common to these divides $a_1 u_{j+1}$ and hence u_{j+1}, since $(a_0, a_1) = 1$, violating the induction hypothesis.

(b) is clear from Lemma 2 and (a).

(c) is trivial when j or k is zero.

For $j, k \geqq 1$, we have from Lemma 1, $a_0 u_j u_{k-1} + u_{j+1} u_k = u_{j+k}$. Clearly $(u_j, u_k) \mid u_k, u_{j+k}$ and hence $(u_j, u_k) \mid (u_k, u_{j+k})$. Conversely,

$$(u_k, u_{j+k}) \mid a_0 u_j u_{k-1} \quad .$$

The former is prime to a_0 by (a) and to u_{k-1} by (b). Thus it divides u_j, u_k, and so it divides (u_j, u_k).

Note: It is clear from (a) that the only integers m dividing components u_j of U are prime to a_0.

Theorem 5(A). If $\{g_j\}$ is an arbitrary sequence of integers with $g_0 = 0$, then the correspondence $j \to g_j$ preserves g.c.d.'s, that is, $(g_\ell, g_k) = g_{(\ell, k)}$ if and only if $(g_j, g_k) = (g_k, g_{j+k})$ for all $j, k \geqq 0$.

(B) In particular, the sequence U has this property whenever $(a_0, a_1) = 1$.

Proof: (A) The necessity is obvious. In proving the sufficiency we may suppose $\ell \geqq k$. The conclusion is clear for $k = 0$, since $g_0 = 0$. If $\ell \geqq k > 0$ and $k \mid \ell = qk$, we have

$$g_{(\ell, k)} = g_k = (g_k, g_k) = (g_k, g_{2k}) = \ldots = (g_k, g_\ell) \quad .$$

In the remaining case, $\ell > k > 0$, $k \nmid \ell$, we write $\ell = kq + j$, $0 < j < k < \ell$, and obtain $(g_j, g_k) = (g_k, g_{j+k}) = \ldots = (g_k, g_\ell)$. It is then clear how the Euclidean algorithm, proceeding from this relation through a sequence of similar steps and terminating in an equation such as $L = KQ + J$, $0 < J < K < L$, with $(\ell, k) = J \mid K$, leads to the conclusion $(g_\ell, g_k) = \ldots = (g_L, g_K) = (g_K, g_J) = g_J$.

(B) The application of (A) to U is now clear from Lemma 3(c).

Note: The non-trivial part of Th. 4 follows elegantly from Th. 5(B) when $(a_0, a_1) = 1$. For, if r is the rank of m, and $m \mid u_\ell$, then

$$m \mid (u_r, u_\ell) = u_{(r,\ell)}, \quad r \leqq (r, \ell) \mid r, \quad \text{and} \quad r = (r, \ell) \mid \ell \ .$$

Corollary 2. If $r_1 \neq r_2$ are relatively prime integers the correspondence $j \rightarrow u_j = (r_1{}^j - r_2{}^j)/(r_1 - r_2)$ is g.c.d.-preserving.

For, $f(x) = (x - r_1)(x - r_2) = -a_0 - a_1 x + x^2$ has $a_1 = r_1 + r_2$ and $a_0 = -r_1 r_2$ relatively prime.

Note: It is clear that the set of all g.c.d.-preserving functions $g(j)$ on the integers is a closed associative system (semi-group) with identity under the composition $f(g(j))$. Theorem 5(A), suggested by the referee, characterizes these functions. The sequences U resulting from quadratics with $(a_0, a_1) = 1$ are non-trivial functions of the kind. As a "trivial" example consider $g(2j - 1) = 1$, $g(2j) = 2j$. Although "well-known" we include the seldom fully stated

Corollary 3. For integers $r_1 \neq r_2$ with $(r_1, r_2) = 1$ and difference $d = r_1 - r_2$, let $u_j = (r_1{}^j - r_2{}^j)/(r_1 - r_2)$, $j \geqq 1$.

(a) $(d, u_j) = (d, j)$

(b) A prime $p \mid u_p$ if and only if $p \mid d$. Such a prime p has rank p in U.

(c) If $p \nmid d$, then $p \nmid u_p$ and $(d, u_p) = 1$.

(d) If $p \mid d$, then $p \mid u_p$, $(d, u_p) = p$, and if p is odd, $p^2 \nmid u_p$.

(e) Every prime factor $q \neq p$ of u_p is of form $1 + hp$.

(f) If $r_1 > r_2 > 0$, then for every odd prime p there exists a prime $q = q(p)$ dividing u_p, of form $1 + hp$, and $q(p)$ is one-one.

Proof. (a) is trivial for $j = 1$, and follows for $j \geqq 2$ from

$$u_j = ((d+r_2)^j - r_2^j)/d = (d^{j-2} + C^j_1 d^{j-3} r_2 + \ldots + C^j_{j-2} r_2^{j-2})d + jr_2^{j-1} \quad (a^*)$$

since $(d, r_2) = 1$.

(b) From (a^*), $u_p \equiv d^{p-1} \bmod p$. The statement about the rank of p follows from (a).

(c) follows from (b) and (a).

(d) follows from (b), (a), and the congruence $u_p \equiv pr_2^{p-1} \bmod p^2$ implied by (a^*) for a prime $p \geqq 3$.

(e) If $q | u_p$, $q \neq p$, then $r_1^p \equiv r_2^p \bmod q$, $r_1 \not\equiv r_2 \bmod q$, since (d, u_p) is 1 or p; and $q \nmid r_1, r_2$. We present two proofs: (1) Letting $r_2 r_2' \equiv 1 \bmod q$, we have $(r_1 r_2')^p \equiv 1$, $r_1 r_2' \not\equiv 1 \bmod q$ implies p = period $(r_1 r_2' \bmod q) | \Phi(q) = q - 1 = hp$. (2) $r_1^{q-1} \equiv 1 \equiv r_2^{q-1} \bmod q$, $q | (u_{q-1}, u_p) = u_{(q-1, p)}$, by Cor. 2. Hence $p | (q - 1) = hp$, otherwise $(q - 1, p) = 1$, $q | u_1 = 1$.

(f) Since $u_p = r_1^{p-1} + \ldots + r_2^{p-1} > pr_2^{p-1} \geqq p$, it follows that there exists a prime $q = q(p)$ of form $1 + hp \geqq 7$ dividing u_p for p odd, and by Cor. 2, this function is one-one. Of course the construction is valid for every pair r_1, r_2 covered by the corollary, the simplest being 2, 1 with $u_p = 2^p - 1$. It is not known whether an infinity of primes $1 + 2p$ exist.

Corollary 4. If

$(a_0, a_1) = 1$ and $a_0, a_1 \geqq 1$, then $u_0 = 0 < u_1 = 1 \leqq u_2 = a_1 < u_3 < u_4 \ldots$,

and

(a) j composite implies u_j composite, (b) $u_j | u_k$ implies $j | k$, except in the single case $a_1 = 1$. If $a_1 = 1$, (a) is false only when

$j = 4$ and u_4 is prime, while (b) is false only when $j = 2$ and k is odd.

Proof. It is clear from (R) that u_j is increasing as stated.

(a) Let $j = hi$, $h \geqq 2$, $i \geqq 2$. By Th. 3, $u_h, u_k | u_j$, where $u_h, u_i < u_j$, since $2 \leqq h$, $i < j$. If u_h or u_i exceeds 1, u_j is composite. Suppose both are 1. Then clearly $h = i = 2$, and $1 = u_h = u_i = u_2 = a_1$, $j = hi = 4$, $u_j = u_4$, and (a) follows with its proviso.

(b) If $u_j | u_k$, then $u_j = (u_j, u_k) = u_{(j,k)}$ (Th. 5), where $(j, k) \leqq j$. If equality holds, $j | k$; if inequality, we must have $(j, k) = 1$, $j = 2$, $1 = u_1 = u_2 = a_1$, k odd, and (b) follows.

6. THE LUCASIAN SEQUENCES U, V

The integral sequences

$$U = \{u_j\} = \{0, 1, a_1, \ldots\} \quad \text{and} \quad V = \{v_j\} = \{2, a_1, 2a_0 + a_1^2, \ldots\} \quad ,$$

with

$$u_j = (r_1^j - r_2^j)/(r_1 - r_2), \quad v_j = r_1^j + r_2^j \quad ,$$

of the space $C(f)$ associated with the quadratic

$$f(x) = -a_0 - a_1 x + x^2 = (x - r_1)(x - r_2) \quad ,$$

where

$$r_1 = \frac{1}{2}(a_1 + Q^{\frac{1}{2}}), \quad r_2 = \frac{1}{2}(a_1 - Q^{\frac{1}{2}}), \quad Q = a_1^2 + 4a_0,$$

$$r_1 + r_2 = a_1, \quad r_1 r_2 = -a_0, \quad r_1 - r_2 = Q^{\frac{1}{2}}, \quad a_0 Q \neq 0 \quad ,$$

have curious interrelations, which have been exploited by Lucas [4; p. 223], and Lehmer [5], (in even more general form) in the design of various "tests for primality". We present here some old and new aspects of this.

The following relations are easily verified using the above formulas for u_j, v_j:

(1) $u_j v_k + u_k v_j = 2u_{j+k}$ (1a) $u_j v_j = u_{2j}$

(2) $u_j v_k - u_k v_j = -2(-a_0)^j u_{k-j}$ $(k \geqq j)$

(3) $v_j v_k + Qu_j u_k = 2v_{j+k}$ (3a) $v_j^2 + Qu_j^2 = 2v_{2j}$

(4) $v_j v_k - Qu_j u_k = 2(-a_0)^j v_{k-j}$ $(k \geqq j)$

(4a) $v_j^2 - Qu_j^2 = 4(-a_0)^j$

(5) $v_{2j} = v_j^2 - 2(-a_0)^j$

(6) $u_1 = 1, \ v_1 = a_1$

(7) $u_p \equiv Q^{(p-1)/2} \bmod p$, for every odd prime p.

For example, we compute

$$2^p Q^{\frac{1}{2}} u_p = (a_1 + Q^{\frac{1}{2}})^p - (a_1 - Q^{\frac{1}{2}})^p = \sum_0^p (1-(-1)^i) C^p_i a_1^{p-i} Q^{i/2},$$

so

$$2^{p-1} u_p = \sum_{\substack{0 \\ i \text{ odd}}}^{p} C^p_i a_1^{p-i} Q^{(i-1)/2}, \quad \text{from which}$$

$$u_p \equiv Q^{(p-1)/2} \bmod p.$$

A prime is said to be <u>regular</u> (relative to $f(x)$) in case $p \nmid 2a_0 Q$. We know from Th. 4 such a prime has rank $r = r(p) \mid k(p) \leqq p^2 - 1$ in U, where $k(p)$ is the period of U mod p, and $p \mid u_0, u_r, u_{2r}, \ldots$ and no other u_j. More remarkably, we see now, from (2), (1), (6), and (7), that

$$a_1 - Q^{(p-1)/2} a_1 \equiv 2a_0 u_{p-1} \quad \text{and} \quad a_1 + Q^{(p-1)/2} a_1 \equiv 2u_{p+1} \bmod p.$$

Lemma 4. If p is a regular prime, then $p \mid u_{p-1}$ if $(Q/p) = 1$, where $p \mid u_{p+1}$ if $(Q/p) = -1$ (Legendre symbol!), so that always $r(p) \leqq p + 1$ in U.

Lemma 5. If p is a regular prime, then $p \mid v_{2^k}$ if and only if $r(p) = 2^{k+1}$ in U.

Proof. If $p \mid v_{2^k}$ then $p \mid u_{2^{k+1}}$ by (1a), but $p \nmid u_{2^k}$ by (4a), hence $r(p) = 2^{k+1}$. Conversely, if $p \mid u_{2^{k+1}}$ but not u_{2^k}, then $p \mid v_{2^k}$ by (1a).

These are the basic lemmas. For computational reasons, we note that the sequence $\{v_{2^k}\}$, with $v_2 = 2a_0 + a_1^2$, $v_{2^{k+1}} = (v_{2^k})^2 - 2a_0^{2^k}$, $k \geqq 1$ (cf. [5]) is related to the auxiliary sequence $\{S_k\}$ defined by

$$S_1 = 2 + (a_1^2/a_0),\ S_{k+1} = S_k^2 - 2,\ k \geqq 1 \quad ,$$

via the simple equation $v_{2^k} = a_0^{2^{k-1}} S_k$, $k \geqq 1$. Thus, whenever $a_0 \mid a_1^2$, $\{S_k\}$ is an integral sequence, and a regular prime p divides v_{2^k} if and only if $p \mid S_k$.

We may state one of Lehmer's results as

Theorem 6. Let $M = 2^q - 1$, where q is an odd prime, and suppose a_0, a_1 are integers with the properties

(a) If p is a prime divisor of M, then $p \nmid 2a_0Q$ where $Q = a_1^2 + 4a_0$.

(b) M prime implies $(Q/M) = -1$ and $(a_0/M) = 1$.

Then M is prime if and only if $M \mid v_{(M+1)/2}$, (equivalently S_{q-1}, if $a_0 \mid a_1^2$).

Proof. If prime $p \mid M \mid v_{(M+1)/2}$, p is regular by (a); hence, in U, $r(p) = M+1 \leq p+1$ by Lemmas 5, 4. Thus $p \mid M \leqq p$, and $M = p$, prime. Conversely, if M is prime, it is regular by (a), and from (b), (3), (6), and (7) follows $a_1^2 - Q \equiv 2v_{M+1}$, or $v_{M+1} \equiv -2a_0$ mod M. Then from (5) and (b), $-2a_0 \equiv (v_{(M+1)/2})^2 - 2a_0^{(M+1)/2}$, and M divides the stated v_j.

Example 5. For $a_0 = a_1 = 2$, $Q = 12$, (a) and (b) are satisfied. For, only the primes 2 and 3 divide $2a_0Q$, and $M \equiv 1$ mod 2 and $M \equiv 1$ mod 3. Moreover, if M is a prime $(Q/M) = (3/M) = -1$, and

$(a_0/M) = (2/M) = 1$, since $M \equiv -1 \mod 8$. Since $a_0 | a_1^2$, the sequence $\{S_k\}$ is integral with $S_1 = 4$, and M is prime if and only if $M | S_{q-1}$.

In the same fashion one may prove

Theorem 7. Let $F = 2^{2^t} + 1$, $t \geqq 1$, and suppose a_0, a_1 are integers with the properties

(a) If p is a prime divisor of F, then $p \nmid 2a_0Q$.

(b) F prime implies $(Q/F) = 1$, $(a_0/F) = -1$.

Then F is prime if and only if $F | v_{(F-1)/2}$, (equivalently S_{2^t-1}, if $a_0 | a_1^2$).

Proof. Suppose prime $p | F | v_{(F-1)/2}$. Then p is regular and in U has $r(p) = F - 1 \leqq p + 1$, so $p | F \leqq p + 2$. Clearly $F = p$, otherwise $2p \leqq F \leqq p + 2$, $p \leqq 2$. If F is prime, it is regular by (a), and from (b), (6), (7), (4), $a_1^2 - Q \equiv -2a_0v_{F-1} \mod F$ or $v_{F-1} \equiv 2 \mod F$. From (5) and (b), $2 \equiv (v_{(F-1)/2})^2 + 2 \mod F$ and the theorem follows.

Example 6. For $a_0 = a_1 = 3$, $Q = 21$, only the primes 2, 3, and 7 divide $2a_0Q$, whereas $F \equiv 1 \mod 2$, $F \equiv 2 \mod 3$; as for 7, note that either $2^t = 1 + 3h$, and then $F \equiv 3 \mod 7$, or $2^t = 2 + 3h$, and then $F \equiv 5 \mod 7$. Hence (a) holds. If F is prime, we have $(3/F) = (F/3) = (2/3) = -1$, and also $(7/F) = (F/7) = -1$, since $(3/7) = -1 = (5/7)$. Hence $(Q/F) = (3/F)(7/F) = 1$, and $(a_0/F) = (3/F) = -1$, as required. The auxiliary S_k are integers, with $S_1 = 5$.

Note: The tests indicated in Th. 7 have no computational advantage over the orthodox N and S condition $3^{(F-1)/2} \equiv -1 \mod F$. Indeed, the latter is a special case of Th. 7, with $a_0 = 3$, $a_1 = 2$. For the latest computational results see the relevant articles in Math. Comp. 18 (Jan. 1964) and Scientific American (Nov. 1964, p. 12). The least undecided Fermat number is F_{17}.

7. THE SPACE $Z_p(f)$

Let p be a prime of Z, and $f(x) = -a_0 - a_1x + x^2 \in Z_p[x]$. The regularity condition $p \nmid 2a_0Q$, which we here assume, insures that p is odd, and the zeros r_i of $f(x)$ in its root field are non-zero and distinct.

Since $f(x) = (x-2'a_1)^2 - (2')^2Q$, where $2'$ is the inverse of 2 mod p, we see that this root field is Z_p if $(Q/p) = 1$, or the Galois field $C^* \cong GF(p^2)$ of § 3 if $(Q/p) = -1$.

(I.) If $(Q/p) = 1$, there is a $b \in Z$ such that $b^2 \equiv Q \bmod p$, and $f(x)$ is reducible in $Z_p[x]$. Indeed, in Z_p, $f(x) = (x - 2'(a_1+b))(x - 2'(a_1 - b))$ has the distinct zeros indicated. The space $Z_p(f)$ itself has basis R_1, R_2 over Z_p, each of its sequences being expressible in the form $s_j = c_1r_1^j + c_2r_2^j$, $c_i, r_i \in Z_p$. Since $r_i^{p-1} \equiv 1$ mod p, every S is pure periodic of period dividing $p-1$. By an argument now familiar, if h is the least exponent for which both $r_i^h \equiv 1$, every sequence with both $c_i \neq 0$ (e.g., U_0, U, and V) has period h.

This case is of special interest when $p = 2^{2^t} + 1$ is a prime. The conditions $(Q/p) = 1$, $(a_0/p) = -1$ then hold if and only if $f(x)$ has zeros in Z_p with (say) r_2 a quadratic residue, and r_1 a non-residue. In such a case, r_1 has period $p-1$, and r_2 may have any period $m \mid \frac{1}{2}(p-1)$, in Z_p. From the above expression of s_j in terms of the r_i it is clear that one sequence $(S = 0)$ has period 1, exactly $p-1$ (those with $c_1 = 0$, $c_2 \neq 0$) have period m, and the $(p-1)p$ remaining sequences (with $c_1 \neq 0$, c_2 arbitrary) have period $p-1$.

(II.) If $(Q/p) = -1$, $f(x)$ is irreducible in $Z_p[x]$, with zeros r, r^p in C^*, where, in the cyclic group G of order $p^2 - 1$, r and r^p have a period $h \mid p^2 - 1$. The geometric sequences R_i are now in $C^*(f)$ and form a basis for the latter space over C^*. All sequences $S \neq 0$ of $C^*(f)$, in particular those of $Z_p(f)$, have period h. Since $(r)^{p+1} - (r^p)^{p+1} = r^{p+1}(1 - r^{p^2-1}) = 0$, $p \mid u_{p+1}$.

For every $p \geqq 3$, there exists an irreducible quadratic (relative to which p is necessarily regular) for which r has period p^2-1. Every sequence of the corresponding space has period p^2-1. Since $p \mid u_{p+1}$, and every pair $(0, 1), \ldots, (0, p-1)$, indeed every pair $\neq (0, 0)$, must appear exactly once as an adjacent pair in the sequence $\{u_0, \ldots, u_{p^2-1}\}$ mod p, it is clear that p+1 is itself the rank of p in U, and the above sequence consists of the terminal element and p-1 blocks of p+1 elements each. Moreover, each block arises from the first by renaming its elements, since each is the beginning of a sequence of the space which is a multiple of U itself.

Such a sequence thus provides a solution of very special type for the $m = p$, $n = 2$ problem (Cf. footnote [7]). Lehmer's quadratic (Ex. 5) $f(x) = -2-2x+x^2$ mod $p = 2^3 - 1 = 7$ has a root r of period $7^2 - 1 = 48$, and mod 7 we find

$U = \{01262216\ 05323352\ 04131143\ 06515561\ 02454425\ 03646634\ 0\ldots\}$.

FOOTNOTES

1. Most of the ideas presented may be found elsewhere, sometimes in less general form. See for example references [3, 6, 10].

2. A parallel version is obtained if C is everywhere replaced by the "root field" of f(x) over the rational field, or by the abstract root field of f(x) mod p. See [7].

3. Although linearly independent, one may note, among others, the relation $u_{oj} = a_o u_{n-1, j-1}$, $j \geqq 1$.

4. Explicitly, for $i = 0, 1, \ldots, n-1$; $j = 1, \ldots, n$,

$$r_{ij} = (-1)^i \sigma(n-1-i;j) / \prod_{k \neq j} (r_k - r_j)$$

where the σ denotes the elementary symmetric function of degree n-1-i in the n-1 roots r_k other than r_j. (Here $\sigma \equiv 1$ when $n-1-i = 0$). Cf. ref. [8].

5. Period always means minimal period, while pure periodic means that periodicity obtains from the beginning of the sequence.

6. The root field for a reducible f mod p exists, but the periodicity properties are more complicated.

7. The method indicated (with suitable insertion of a zero) provides

an algebraic construction of a sequence of integers mod m, of length $m^n + n - 1$, containing no repeated n-tuple, in the case of prime m. The existence of such sequences for arbitrary m, n is a well-known corollary of a theorem on graphs [2, 9]. (Remark of referee.)

8. For the Fibonacci case, see [10], on which the present section is modelled.

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RELATIVELY PRIME SEQUENCE SOLUTIONS OF NON-LINEAR DIFFERENCE EQUATIONS

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In the following we present a one-parameter family of first order non-linear difference equations which are shown to possess solutions which are sequences of (pair-wise) relatively prime numbers. In particular, we show that the sequences

(A) $F_k = 2^{2^k} + 1$

(B) $G_k^{\pm} = 2 \cosh \left[2^k \cosh^{-1} \frac{n_0}{2} \right] + \frac{1 \pm 3}{2}$,

where n_0 is an odd or even integer according as the sign in (B) is plus or minus, respectively, consist of relatively prime numbers by virtue of being solutions of such difference equations. Sequence (A) is the famous sequence of so-called Fermat numbers originally conjectured by Fermat to consist only of prime numbers (later disproved by Euler). The present method provides a new proof of the relatively primacy of these numbers (see [1], p. 14). Sequence (B) is apparently new and is the only other important solution which has been obtained in closed form.

These specific results are based on the following simple theorem:

Theorem: If the sequence of integers $\{u_k\}$ satisfies the difference equation

$$u_{k+1} = u_k^2 - bu_k + b \tag{1}$$

where $k = 0, 1, 2, \ldots$ and b is integral, then the distinct elements u_{k_1}, u_{k_2} have no common divisor except possibly for the divisors of b.

Proof. We may write equation (1) in the form

$$u_k - b = u_{k-1} (u_{k-1} - b) \quad ,$$

which by iteration may be expressed as (Continued on page 152.)

AN ALMOST LINEAR RECURRENCE

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A general linear recurrence with constant coefficients has the form

$$u_0 = a_1,\ u_1 = a_2, \ldots, u_{r-1} = a_r \quad ;$$

$$u_n = b_1 u_{n-1} + b_2 u_{n-2} + \ldots + b_r u_{n-r}, \quad n \geq r \quad .$$

The Fibonacci sequence is the simplest non-trivial case. Consider, however, the following sequence:

(1)
$$\phi_0 = 1 \quad ;$$

$$\phi_n = \phi_{n-1} + \phi_{[n/2]}, \quad n > 0 \quad .$$

In this case, successive terms are formed from the previous one by adding the term "halfway back" in the sequence. This recurrence, which may be considered as a new kind of generalization of the Fibonacci sequence, has a number of interesting properties which we will examine here.

The sequence begins 1, 2, 4, 6, 10, 14, 20, 26, 36, ... It is easy to see that all terms except the first are even, and furthermore ϕ_n is divisible by 4 if and only if $n \equiv 2^{2k-1} \pmod{2^{2k}}$ for some $k \geq 1$. We leave it to the reader to discover further arithmetic properties of the sequence.

The sequence ϕ_n has an interesting combinatorial interpretation: ϕ_n is precisely the number of partitions of the number $2n$ into powers of 2. For example, $6 = 4 + 2 = 4 + 1 + 1 = 2 + 2 + 2 = 2 + 2 + 1 + 1 = 2 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1$, and $\phi_3 = 6$. To verify this interpretation, let $P(m)$ be the number of partitions of m

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into powers of 2. If $2n = a_1 + a_2 + \ldots + a_k$, where $a_1 \geqq a_2 \geq \cdots \geqq a_k$ and each a_i is a power of 2, there are two cases: (i) $a_k = 1$; then $a_1 + \ldots + a_{k-1}$ is a partition of $2n-1$; (ii) $a_k > 1$; then $a_1/2 + a_2/2 + \ldots + a_k/2$ is a partition of n. Conversely, all partitions of $2n$ are obtained from partitions of $2n-1$ and n in this way, so $P(2n) = P(2n-1) + P(n)$. We also find $P(2n+1) = P(2n)$ by a similar argument; here only case (i) can arise since $2n + 1$ is an odd number. These recurrence relations for P, together with $P(1) = 1$ and $P(2) = 2$, establish the fact that $\phi_n = P(2n)$.

The same sequence also arises in other ways; the author first noticed it in connection with the solution of the recurrence relation

(1a)
$$M(0) = 0$$
$$M(n) = n + \min_{0 \leqq k < n} (2M(k)+M(n-1-k))$$

for which it can be shown that $M(n) - M(n-1) = m$ if $\phi_m \leq 2n < \phi_{m+1}$, and

$$M(\tfrac{1}{2}\phi_n - 1) = \frac{n-1}{2}\phi_n - \left[\tfrac{1}{4}\phi_{2n-1}\right] .$$

Recurrences such as (1a) occur in the study of dynamic programming problems, and they will be the subject of another paper.

Let us begin our analysis of ϕ_n by noticing some of its most elementary properties. By applying the rule (1) repeatedly, we find

(2)
$$\phi_{2n+1} = 2(\phi_0 + \ldots + \phi_n) .$$

Another immediate consequence of (1) is

(3)
$$\phi_{2n}^2 - \phi_{2n+1}\phi_{2n-1} = \phi_n^2 .$$

The sequence ϕ_n grows fairly rapidly; for example,

$$\phi_{500} = 1981471878$$
$$\phi_{10000} = 2.14 \times 10^{20} .$$

In fact, we now show that ϕ_n grows more rapidly than any power of n:

<u>Theorem 1</u>. <u>For any power</u> k, <u>there is an integer</u> N_k <u>such that</u> $\phi_n > n^k$ <u>for all</u> $n \geq N_k$.

<u>Proof</u>: Let N be such that $(2^{k+1}+1) \geq (2 + \frac{1}{N})^{k+1}$, and let

$$\alpha = \min_{N \leq n \leq 2N} (\phi_n/n^{k+1}) .$$

Then by induction $\phi_n \geq \alpha n^{k+1}$ for all $n \geq N$, since this is true for $N \leq n \leq 2N$, and if $n > 2N$

$$\phi_n = \phi_{n-1} + \phi_{[n/2]} \geq \alpha(n-1)^{k+1} + [n/2]^{k+1})$$

$$\geq \alpha((n-1)^{k+1} + (\frac{n-1}{2})^{k+1}) = \alpha(1 + \frac{1}{2^{k+1}})(n-1)^{k+1} \geq \alpha(1 + \frac{1}{2N})^{k+1}(n-1)^{k+1}$$

$$\geq \alpha(1 + \frac{1}{n-1})^{k+1}(n-1)^{k+1} = \alpha n^{k+1} .$$

If we choose $N_k \geq 1/\alpha$ and $N_k \geq N$, the proof is complete.

We now consider the generating function for ϕ_n. Let

$$F(x) = \phi_0 + \phi_1 x + \phi_2 x^2 + \phi_3 x^3 + \ldots \quad . \tag{4}$$

Notice that

$$(1+x)(F(x^2) = \phi_0 + \phi_0 x + \phi_1 x^2 + \phi_1 x^3 + \phi_2 x^4 + \phi_2 x^5 + \ldots$$

$$= \phi_0 + (\phi_1 - \phi_0)x + (\phi_2 - \phi_1)x^2 + (\phi_3 - \phi_2)x^3 + (\phi_4 - \phi_3)x^4 + \ldots$$

$$= (1-x)F(x) \quad ;$$

thus

$$F(x) = \frac{1+x}{1-x} F(x^2) = \frac{(1+x)(1+x^2)}{(1-x)(1-x^2)} F(x^4) = \ldots \quad .$$

We have therefore

$$F(x) = \frac{(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots}{(1-x)(1-x^2)(1-x^4)(1-x^8)\cdots} = \frac{1}{(1-x)^2(1-x^2)(1-x^4)(1-x^8)\cdots} \tag{5}$$

From this form of the generating function, we see that $F(x)$ converges for $|x| < 1$. (As a function of the complex variable z, $F(z)$ has the unit circle as a natural boundary.) It follows that

$$\limsup \sqrt[n]{\phi_n} = 1 \quad ,$$

i.e. the sequence ϕ_n grows more slowly than α^n for any constant $\alpha > 1$. This is in marked contrast to linear recurrences such as the Fibonacci numbers.

In the remainder of this paper we will determine the true rate of growth of the sequence ϕ_n; it will be proved by elementary methods that

$$\ln \phi_n \sim \frac{1}{\ln 4} (\ln n)^2 \quad ,$$

i.e.

$$\phi_n = e^{\frac{1}{\ln 4}(\ln n)^2 + o((\ln n)^2)} \quad . \tag{6}$$

The techniques are similar to others which have been used for determining the order of magnitude of the partition function (see [2]).

We start by observing that

$$\ln F(x) = -\ln(1-x) + \sum_{k=0}^{\infty} (-\ln(1-x^{2^k}))$$

$$= \sum_{r=1}^{\infty} \frac{x^r}{r} + \sum_{k=0}^{\infty} \sum_{r=1}^{\infty} \frac{x^{2^k r}}{r}$$

and hence by differentiation

$$\frac{F'(x)}{F(x)} = \sum_{r=1}^{\infty} x^{r-1} + \sum_{k=0}^{\infty} \sum_{r=1}^{\infty} 2^k x^{2^k r}$$

$$= 2 + 4x + 2x^2 + 8x^3 + 2x^4 + 4x^5 + \ldots + \theta_k x^{k-1} + \ldots$$

where θ_k is twice the highest power of 2 dividing k. Therefore

$$\frac{F'(x)}{F(x)} = (1-x)(2+6x+8x^2+16x^3+18x^4+22x^5+\ldots+\psi_k x^{k-1}+\ldots)$$

where if

$$k = 2^{a_1} +\ldots+ 2^{a_r}, \; a_1 > a_2 > \ldots > a_r \geq 0,$$

the coefficient of x^{k-1} in the power series on the righthand side is

$$\psi_k = \theta_1 + \theta_2 + \ldots + \theta_k = a_1 2^{a_1} +\ldots+ a_r 2^{a_r} + 2k \; .$$

(The reader will find the verification of this latter formula an interesting exercise in the use of the binary system.) We can estimate the magnitude of ψ_k as follows:

$$\psi_k \geqq a_1 k + 2k - (2^{a_1-1} + 2 \cdot 2^{a_1-2} + \ldots + a_1)$$

$$= (a_1+2)k - 2^{a_1+1} + a_1 + 2 \geq (1+\log_2 k)k - 2k \quad ;$$

hence

$$k \log_2 k - k \leqq \psi_k \leqq k \log_2 k + 2k \quad . \tag{7}$$

This estimate and the monotonicity of Φ_n are the only facts about $F(x)$ which are used in the derivation below.

Let
$$G(x) = e^{\frac{1}{\ln 4}(\ln(1-x))^2} \; .$$

Then

$$\frac{G'(x)}{G(x)} = \frac{-\log(1-x)}{\ln 2\,(1-x)} = (1-x)\left(\frac{1}{\ln 2}x + \frac{5}{2\ln 2}x^2 + \frac{13}{3\ln 2}x^3 + \frac{77}{12\ln 2}x^4+\ldots\right).$$

Since the derivative of $-\log(1-x)/(1-x)$ is $(1-\log(1-x))/(1-x)^2$, we find that the coefficient of x^{k-1} in the power series on the right is

$$\chi_k = \frac{k}{\ln 2}(h_k - 1) \quad , \tag{8}$$

where

(9) $$h_k = 1 + \frac{1}{2} + \dots + \frac{1}{k} \quad .$$

Since $h_k = \ln k + 0(1)$, we have therefore established the equations

(10) $$\frac{F'(x)}{F(x)} = (1-x)\sum_{k=1}^{\infty} \psi_k x^{k-1} \,, \quad \frac{G'(x)}{G(x)} = (1-x)\sum_{k=1}^{\infty} \chi_k x^{k-1} \quad ,$$

and

(11) $$\psi_k = \chi_k + 0(k) \quad .$$

This suggests a possible relation between the coefficients of $F(x)$ and those of $G(x)$. Note that if

$$\frac{F'(x)}{F(x)} = (1-x)f(x) \quad ,$$

then

$$F(x) = \exp \int_0^x (1-t)f(t)dt \quad .$$

Therefore the following lemma shows how relations (10) and (11) might be applied to our problem:

<u>Lemma</u> 1. <u>Let</u>

$$A(x) = \exp \int_0^x (1-t)a(t)dt \quad ,$$

$$B(x) = \exp \int_0^x (1-t)b(t)dt \quad ,$$

<u>where</u>

$$A(x) = \quad A_k x^k, \; a(x) = \sum a_k x^{k-1}, \; B(x) = \sum B_k x^k, \; b(x) = \sum b_k x^{k-1} \; .$$

<u>Assume the coefficients of</u> $A(x)$ <u>and of</u> $b(x)$ <u>are non-negative and non-decreasing. Then if</u> $a_k \le b_k$ <u>for all</u> k, $A_k \le B_k$; <u>if</u> $a_k \ge b_k$ <u>for all</u> k, $A_k \ge B_k$.

Proof: $A_0 = B_0 = 1$. Assume $a_k \le b_k$ for all k, and $A_k \le B_k$ for $0 \leq k < n$. Then since $A'(x) = (1-x)a(x)A(x)$, we have

$$\begin{aligned} nA_n &= a_n A_o + a_{n-1}(A_1 - A_o) + \ldots + a_1(A_{n-1} - A_{n-2}) \\ &\le b_n A_o + b_{n-1}(A_1 - A_o) + \ldots + b_1(A_{n-1} - A_{n-2}) \\ &= A_o(B_n - b_{n-1}) + A_1(b_{n-1} - b_{n-2}) + \ldots + A_{n-1}b_1 \\ &\le B_o(b_n - b_{n-1}) + B_1(b_{n-1} - b_{n-2}) + \ldots + B_{n-1}b_1 = nB_n \quad . \end{aligned}$$

Essentially the same argument works if $a_k \geq b_k$ for all k.

The problem is now one of estimating the coefficients of

$$G(x) = e^{\frac{1}{\ln 4}\ln^2(1-x)} \quad .$$

Theorem 2. If

$$e^{a \ln^2(1-x)} = \sum c_n x^n \quad , \tag{12}$$

we have

$$c_n = a \ln^2 n + 0((\ln n)(\ln \ln n)) \quad . \tag{13}$$

Proof: First we show that

$$\ln^m(1-x) = \sum_{n=m}^{\infty} \frac{m}{n} H_{m,n} x^n \quad , \tag{14}$$

where

$$H_{m,n} = \sum \frac{1}{a_1 \cdots a_{m-1}}$$

summed over all integers $a_1, \ldots, a_{m-1}$ such that $1 \leq a_i < n$, and the a_i are distinct. This follows inductively, since the derivative of (14) is

$$\frac{\ln^{m-1}(1-x)}{(x-1)} = \sum_{n=m}^{\infty} H_{m,n} x^{n-1} \quad ,$$

and we have

(15) $$H_{m,n} = H_{m,n-1} + \frac{m-1}{n-1} H_{m-1,n-1} .$$

Turning to equation (12), we have

(16) $$\sum_{n=0}^{\infty} c_n x^n = \sum_{m=0}^{\infty} \frac{a^m \ln^{2m}(1-x)}{m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^m}{m!} \binom{2m}{n} H_{2m,n} \quad x^n .$$

(We define $H_{m,n} = 0$ if $m > n$, so the parenthesized summation is actually a finite sum for any fixed value of n.)

Our theorem relies on the estimates

(17) $$(h_{n-1} - h_{m-1})^{m-1} \leq H_{m,n} \leq h_{n-1}^{m-1}, \quad \text{if } m \leq n .$$

The righthand inequality is obvious, since this is the sum

$$\sum \frac{1}{a_1 \cdots a_{m-1}}$$

without the restriction that the a's are distinct. On the other hand, given any term of

$$(h_{n-1} - h_{m-1})^{m-1} = \sum_{m \leq a_1 < n} \frac{1}{a_1 \cdots a_{m-1}} ,$$

we form a term

$$\frac{1}{b_1 \cdots b_{m-1}}$$

belonging to $H_{m,n}$, where $b_k = a_k - r$ if a_k is the r-th largest of $\{a_1, \ldots, a_{m-1}\}$. Thus, we decrease the largest element by 1, the second largest by 2, and so on; in case of ties, an arbitrary order is taken. No two terms

$$\frac{1}{a_1 \cdots a_{m-1}}$$

map into the same

$$\frac{1}{b_1 \cdots b_{m-1}}, \quad \text{and} \quad \frac{1}{a_1 \cdots a_{m-1}} \le \frac{1}{b_1 \cdots b_{m-1}},$$

so the lefthand side of (17) is established.

Putting the righthand side of (17) into (16), we obtain

$$(18) \qquad c_n = \frac{2}{n} \sum_{m=0}^{\infty} \frac{a^m}{(m-1)!} H_{2m,n} \le \frac{2ah_{n-1}}{n} \sum_{m=1}^{\infty} \frac{a^{m-1} h_{n-1}^{2m-2}}{(m-1)!} = \frac{2a}{n} e^{ah_{n-1}^2}$$

On the other hand,

$$(19) \qquad c_n > \frac{2}{n} \frac{a^m}{(m-1)!} H_{2m,n}$$

for any particular value of m. We choose m to be approximately $ah_{n-1}^2 + 1$, assuming n is large. Then we evaluate the logarithm of the term on the right, using Stirling's approximation and the left hand side of (17), and discarding terms of order less than $(\ln n)(\ln \ln n)$:

$$\ln c_n > \ln\left(\frac{2a}{n} \frac{a^{m-1}}{(m-1)!} (h_{n-1} - h_{2m-1})^{2m-1}\right)$$

$$= ah_{n-1}^2 \ln a + 2ah_{n-1}^2 \ln(h_{n-1} - h_{2m-1}) - ah_{n-1}^2 (\ln(ah_{n-1}^2) - 1) + 0(\ln n)$$

$$= a h_{n-1}^2 + 2a h_{n-1}^2 \ln\left(1 - \frac{h_{2m-1}}{h_{n-1}}\right) + 0(\ln n)$$

$$= a h_{n-1}^2 - 2a h_{n-1} h_{2m-1} + 0(\ln n) \quad .$$

This together with (18) establishes theorem 2.

Theorem 3. Let c_n be as in theorem 2. Then

$$\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = 1 \quad .$$

Proof: Since $H_{m,n+1} \ge H_{m,n}$, we have

$$\frac{c_{n+1}}{c_n} \ge \frac{n}{n+1}$$

by (16).

We also observe that $H_{m,n} \le h_{n-1}H_{m-1,n}$ and hence by (15)

$$H_{m,n+1} \le H_{m,n} + \frac{m-1}{n} h_{n-1}H_{m-2,n} ;$$

thus

$$c_{n+1} \le \sum_{m=1}^{\infty} \frac{a^m}{m!} \left(\frac{2m}{n+1}\right)H_{2m,n} + \frac{2a}{(n+1)} h_{n-1} \sum_{m=2}^{\infty} \frac{a^{m-1}}{(m-1)!}\left(\frac{2m-1}{2m-2}\right)\left(\frac{2(m-1)}{n}\right)H_{2(m-1),n}$$

$$\le \frac{n}{n+1}c_n + \frac{3ah_{n-1}}{n+1} c_n .$$

Corollary 3. If $P(x)$ is any polynomial, and if

$$\sum C_n x^n = e^{a \ln^2(1-x) + P(x)} ,$$

then

$$\ln C_n = \ln c_n + O(1) .$$

Proof: If $e^{P(x)} = a_0 + a_1x + a_2x^2 + \ldots$, we have

$$\frac{C_n}{c_n} = \frac{a_0c_n + a_1c_{n-1} + \ldots + a_nc_0}{c_n} \to e^{P(1)} .$$

Theorem 4. $\ln \phi_n \sim \frac{1}{\ln 4}(\ln n)^2$.

Proof: Let $\epsilon > 0$ be given. By (11), we can find N so that when $n > N$, $(1-\epsilon)\chi_k < \psi_k < (1+\epsilon)\chi_k$. Apply lemma 1 with $A(x) = F(x)$,

$$b(x) = \psi_1 + \psi_2 x + \ldots + \psi_N x^{n-1} + \sum_{k=N+1}^{\infty} (1+\epsilon)\chi_k x^{k-1} .$$

We find $\phi_n \le C_n$ where, by Corollary 3,

$$\ln C_n \sim \left(\frac{1+\epsilon}{\ln 4}\right)\ln^2 n .$$

Then apply lemma 1 with

$$A(x) = F(x), \quad b(x) = \sum_{k=N+1}^{\infty} (1-\epsilon)\chi_k x^{k-1} .$$

This gives us $\phi_n \geq C'_n$ where

$$\ln C'_n \sim \left(\frac{1-\epsilon}{\ln 4}\right)\ln^2 n \quad .$$

Therefore

$$\left| \frac{\ln \phi_n}{(\ln n)^2} - \frac{1}{\ln 4} \right|$$

is arbitrarily small when n is large enough.

Of course, the estimate we have derived in this theorem is very crude as far as the actual value of ϕ_n is concerned. Empirical tests based on the exact values of ϕ_n for $n \leq 10000$ reveal excellent agreement with the following formula:

$$(20) \qquad \ln \phi_n \approx \frac{\ln n}{\ln 4}\,(\ln n - 2(\ln \ln n) + 1) + \ln n - .843 \quad .$$

The error is less than .05 for $n \geq 10$; it reaches a low of about $-.05$ when n is near 50, then increases to approximately .032 when n is near 5000, and it slowly decreases after that. Thus we can use (20) to calculate

$$(21) \qquad \phi_n \approx .472 n^{1.721}\left(\frac{\sqrt{n}}{\ln n}\right)^{\log_2 n}$$

with an error of at most 5% when $10 < n \leq 10000$. Although formula (20) gives very good accuracy, it should be remembered that only the first term of the expansion has been verified, and the comparatively small values of $\ln \ln n$ for the range of n considered makes it possible that (20) is not the true asymptotic result. On the assumption that the true formula is a relatively "simple" one, however, equation (20) gives striking agreement. A similar situation exists in the study of the partition function; the methods used here can be applied with ease to that problem, to give

$$\log\, p(n) \sim \pi\sqrt{\frac{2}{3}n} \quad ;$$

the actual asymptotic formula for $p(n)$ itself is

$$p(n) = \left(\frac{1}{4\sqrt{3}} - \frac{1}{4\pi\sqrt{2}(n - \frac{1}{24})}\right) \frac{e^{\pi\sqrt{\frac{2}{3}n - \frac{1}{36}}}}{(n - \frac{1}{24})} + 0(e^{A\sqrt{n}}) ,$$

where $A < \pi\sqrt{\frac{2}{3}}$;

$$p(n) \sim \frac{1}{4\sqrt{3}\,n}\, e^{\pi\sqrt{\frac{2}{3}n}} .$$

It is doubtful that it would have been guessed empirically in either of these forms. For an account of this and a bibliography, see [1] .

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SOME DETERMINANTS CONTAINING POWERS OF FIBONACCI NUMBERS

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1. F. D. Parker (Problem H-46, this Quarterly, Vol. 2 (1964), p. 303) has proposed the evaluation of the determinant

$$|F^4_{n+i+j-2}| \quad (i,\ j = 1,\ 2,\ 3,\ 4,\ 5,)\ .$$

This suggests the more general problem of evaluating

$$D_k = D_{k,n} = |F^k_{n+r+s}| \quad (r\ ,\ s = 0,\ 1,\ \ldots,\ k)\ .$$

We shall show that

$$D_k = (-1)^{\frac{1}{2}k(k+1)(n+1)} \prod_{j=0}^{k} \binom{k}{j} \cdot (F_1^k F_2^{k-1} \ldots F_k)^2\ . \tag{1}$$

For example

$$D_1 = (-1)^{n+1},\ D_2 = (-1)^{n+1}2,\ D_3 = 36\ ,$$

$$D_4 = 13824.$$

To prove (1) we consider the quadratic form

$$Q = \sum_{r,\, s=0}^{k} F^k_{n+r+s}\, u_r\, u_s\ .$$

Since

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \left(\alpha = \frac{1+\sqrt{5}}{2}\ ,\ \beta = \frac{1-\sqrt{5}}{2}\right)\ ,$$

we have

$$(\alpha-\beta)^k Q = \sum_{r,\, s=0}^{k} u_r u_s \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \alpha^{(n+r+s)j} \beta^{(n-r-s)(k-j)}$$

$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \alpha^{nj} \beta^{n(k-j)} \sum_{r,\, s=0}^{k} \alpha^{(r+s)j} \beta^{(r+s)(k-j)} u_r u_s$$

$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \alpha^{nj} \beta^{n(k-j)} \left(\sum_{r=0}^{k} \alpha^{rj} \beta^{r(k-j)} u_r \right)^2 .$$

If we put

$$v_j = \sum_{r=0}^{k} (\alpha^j \beta^{k-j})^r u_r , \tag{2}$$

it is clear that

$$(\alpha-\beta)^k Q = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \alpha^{nj} \beta^{n(k-j)} v_j^2 . \tag{3}$$

Thus by means of the linear transformation (2), we have reduced Q to diagonal form. If Δ denotes the determinant of the linear transformation (2), it follows from (3) that

$$D_k = (-1)^{\frac{1}{2}k(k+1)} \prod_{j=0}^{k} \binom{k}{j} \cdot (\alpha-\beta)^{-k(k+1)} (\alpha\beta)^{\frac{1}{2}nk(k+1)} \Delta^2 . \tag{4}$$

Now

$$\Delta = \left| (\alpha^r \beta^{k-r})^s \right| \quad (r,\ s = 0, 1, \ldots, k) .$$

Since this is a Vandermonde determinant we get

$$\Delta = \prod_{0 \le r < s \le k} (\alpha^s \beta^{k-s} - \alpha^r \beta^{k-r})$$

$$= \prod_{0 \le r < s \le k} \alpha^r \beta^{k-s} (\alpha-\beta) F_{s-r}$$

$$= (\alpha-\beta)^{\frac{1}{2}k(k+1)} \prod_{r=0}^{k-1} \prod_{s=0}^{k-r} \alpha^r \beta^{k-r-s} F_s$$

$$= (\alpha-\beta)^{\frac{1}{2}k(k+1)} \prod_{r=0}^{k-1} \alpha^{r(k-r)} \beta^{\frac{1}{2}(k-r)(k-r-1)} \prod_{s=1}^{k-r} F_s$$

$$= (\alpha-\beta)^{\frac{1}{2}k(k+1)} (\alpha-\beta)^{\frac{1}{6}k(k+1)(k-1)} F_1^k F_2^{k-1} \cdots F_k \quad .$$

Therefore (4) becomes

$$D_k = (-1)^{\frac{1}{2}k(k+1)(n+1)} \prod_{j=0}^{k} \binom{k}{j} \cdot (F_1^k F_2^{k-1} \cdots F_k)^2 \quad .$$

This completes the proof of (1).

2. As for the determinant

$$D_k(L) = |L^k_{n+r+s}| \ (r, \ s = 0, 1, \ldots, k) \quad ,$$

consideration of the quadratic form

$$\sum_{r,\,s=0}^{k} L^k_{n+r+s} u_r u_s$$

$$= \sum_{j=0}^{k} \binom{k}{j} \alpha^{nj}\beta^{n(k-j)} \sum_{r,\,s=0}^{k} \alpha^{(r+s)j} \beta^{(r+s)(k-j)} u_r u_s$$

$$= \sum_{j=0}^{k} \binom{k}{j} \alpha^{nj} \beta^{n(k-j)} \left(\sum_{r=0}^{k} \alpha^{rj} \beta^{r(k-j)} u_r \right)^2$$

yields

$$D_k(L) = \prod_{j=0}^{k} \binom{k}{j} \alpha^{nj} \beta^{n(k-j)} \ . \ \Delta^2$$

$$= (-1)^{\frac{1}{2}nk(k+1)} \prod_{j=0}^{k} \binom{k}{j} \cdot (\alpha-\beta)^{k(k+1)} (F_1^k F_2^{k-1} \cdots F_k)^2 \quad .$$

It follows that

$$(5) \qquad D_k(L) = (-1)^{\frac{1}{2}nk(k+1)} 5^{\frac{1}{2}k(k+1)} \prod_{j=0}^{k} \binom{k}{j} \cdot (F_1^k F_2^{k-1} \cdots F_k)^2 .$$

3. Formulas (1) and (5) can be generalized in an obvious way. Consider the sequence $\{W_n\}$ defined by

$$W_{n+1} = p\, W_n - q\, W_{n-1} \qquad (n \geq 1) ,$$

where W_0, W_1 are assigned. Put

$$D_k(W) = |W_{n+r+s}^k| \qquad (r, s = 0, 1, \ldots, k) .$$

If $p^2 - 4q \neq 0$ and

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} , \qquad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}$$

then $W_n = A\alpha^n + B\beta^n$, where

$$A = \frac{W_1 - \beta W_0}{\alpha - \beta} , \quad B = \frac{\alpha W_0 - W_1}{\alpha - \beta} , \quad AB = -\frac{W_1^2 - pW_0W_1 + qW_0^2}{p^2 - 4q} .$$

We find that

$$(6) \qquad D_k(W) = (-1)^{\frac{1}{2}k(k+1)} q^{\frac{1}{2}nk(k+1) + \frac{1}{3}k(k+1)(k-1)} \prod_{j=0}^{k} \binom{k}{j}$$

$$\cdot (W_1^2 - pW_0W_1 + qW_0^2)^{\frac{1}{2}k(k+1)} (U_1^k U_2^{k-1} \cdots U_k)^2 ,$$

where

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} .$$

Indeed (6) holds also when $p^2 - 4q = 0$, provided we now take $U_n = n(p/2)^{n-1}$. This can be proved directly in the following way. We have

$$W_n = \left(\frac{p}{2}\right)^n (A + Bn) ,$$

where $A = W_0$, $pB = 2W_1 - pW_0$. Then

$$D_k(W) = \left|\left(\frac{p}{2}\right)^{(n+r+s)k} (A + B(n+r+s))^k\right| \quad (r,\ s=0,\ 1,\ \ldots,\ k)$$

$$= \left(\frac{p}{2}\right)^{nk(k+1)+k^2(k+1)} \left|(A + B(n+r+s))^k\right| \quad .$$

We recall that for a determinant of the type

$$|u_{n+r+s}| \qquad (r,\ s=0,\ 1,\ \ldots,\ k)$$

we have

$$|u_{n+r+s}| = |\Delta^{r+s} u_n| \qquad (r,\ s=0,\ 1,\ \ldots,\ k) \quad ,$$

where Δ is the usual finite difference operator. (See for example [1, p. 103].) In the present instance $u_n = (A+Bn)^k$, so that

$$\Delta^{r+s} u_n = 0 \qquad (r + s > k), \qquad \Delta^k u_n = k!\ B^k \quad .$$

It follows that

$$\left|(A+B(n+r+s))^k\right| = (-1)^{\frac{1}{2}k(k+1)} (k!)^{k+1} B^{k(k+1)}$$

and therefore

$$(7) \quad D_k(W) = (-1)^{\frac{1}{2}k(k+1)} \left(\frac{p}{2}\right)^{nk(k+1)+k(k+1)(k-1)} (k!)^{k+1} \left(W_1 - \frac{p}{2} W_0\right)^{k(k+1)} \quad .$$

On the other hand (6) becomes

$$D_k(W) = (-1)^{\frac{1}{2}k(k+1)} \left(\frac{p}{2}\right)^{nk(k+1)+\frac{2}{3}k(k+1)(k-1)} \prod_{j=0}^{k} \binom{k}{j}$$

$$\cdot \left(W_1 - \frac{p}{2} W_0\right)^{k(k+1)} \prod_{j=1}^{k} j^{2(k-j+1)} \left(\frac{p}{2}\right)^{2(j-1)(k-j+1)}$$

$$= (-1)^{\frac{1}{2}k(k+1)} \left(\frac{p}{2}\right)^{nk(k+1)+k(k+1)(k-1)} \left(W_1 - \frac{p}{2} W_0\right)^{k(k+1)} \cdot \prod_{j=1}^{k} \binom{k}{j} j^{2(k-j+1)}$$

Since

$$\prod_{j=1}^{k} \binom{k}{j} j^{2(k-j+1)} = \prod_{j=1}^{k} \binom{k}{j} \left(\frac{j!}{(j-1)!}\right)^{2(k-j+1)} = \prod_{j=0}^{k} \binom{k}{j} (j!)^2$$

$$= \prod_{j=0}^{k} \frac{k!\ j!}{(k-j)!} = (k!)^{k+1} ,$$

it is clear that (6) and (7) are in agreement.

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ON THE QUADRATIC CHARACTER OF THE FIBONACCI ROOT

Emma Lehmer
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Let $\theta = (1+\sqrt{5})/2$ be a root of the quadratic equation

$$x^2 - x - 1 = 0 \quad .$$

The n-th term of the Fibonacci sequence can be given by

$$F_n = \frac{\theta^{2n} - (-1)^n}{\sqrt{5}\,\theta^n} \tag{1}$$

Hence for any prime $p \neq 5$ we can state the criterion:

$$F_n \equiv 0 \pmod{p} \ \underline{\text{if}}\ \underline{\text{and}}\ \underline{\text{only}}\ \underline{\text{if}}\ \ \theta^{2n} \equiv (-1)^n \pmod{p} \quad . \tag{2}$$

If we define $\varepsilon = \pm 1$ in terms of the Legendre symbol as

$$\varepsilon = \left(\frac{5}{p}\right) \equiv 5^{\frac{p-1}{2}} \pmod{p} \qquad p \neq 5 \tag{3}$$

then a special case of Lucas' theorem 1 states

$$F_{p-\varepsilon} \equiv 0 \pmod{p} \tag{4}$$

while a special case of a theorem of Lehmer 2 gives

$$F_{\frac{p-\varepsilon}{2}} \equiv 0 \pmod{p} \quad \text{if and only if} \quad p = 4m+1 \quad . \tag{5}$$

Both (4) and (5) follow immediately from the criterion (2) and the easily verifiable congruence

$$\theta^{p-\varepsilon} \equiv \varepsilon \pmod{p} \tag{6}$$

It is the purpose of this note to give a criterion for the quadratic character of θ and to apply it to find the condition for the divisibility of

$$F_{\frac{p-\varepsilon}{4}}$$

by p.

In the first place, if p divides

$$F_{\frac{p-\varepsilon}{4}},$$

then it must also divide

$$F_{\frac{p-\varepsilon}{2}}$$

and therefore by (5) we have $p = 4m+1$, but since $(p - 6)/4$ must be an integer, $\varepsilon = 1$, so that $p = 20m+1, 9$. The quadratic character of θ for such primes is contained in the following lemma.

Lemma

$$(7) \qquad \theta^{\frac{p-1}{2}} = \left(\frac{\theta}{p}\right) = \begin{cases} \left(\frac{\sqrt{5}}{p}\right) & \text{if } p = 10m+1 \\ -\left(\frac{\sqrt{5}}{p}\right) & \text{if } p = 10m-1 \end{cases}$$

Proof. Let α be a primitive fifth root of unity so that $\alpha^5 = 1$, while $\alpha \neq 1$, then it is well known that

(8) α <u>is an integer modulo</u> p <u>if and only if</u> $p = 10m+1$.

It is also clear that we can write $\Theta = -(\alpha + \alpha^{-1})$ so that $\alpha^2 + \alpha + 1 = 0$ and hence

$$(9) \qquad \alpha = \frac{-\Theta \pm \sqrt{\Theta^2 - 4}}{2}$$

Considering (9) as a congruence modulo p and remembering that is an integer for the primes under consideration we see that α will be an integer modulo p only when $\Theta^2 - 4$ is a quadratic residue. But

$$(10) \qquad \theta^2 - 4 = (\theta - 2)(\theta + 2) = -(\theta - 1)^2\, \theta\sqrt{5} \quad .$$

Hence α is an integer modulo p if and only if 5 is a quadratic residue of p. Hence by (8) we obtain

$$(11) \qquad \left(\frac{\theta\sqrt{5}}{p}\right) = \begin{cases} 1 & \text{if } p = 10m+1 \\ -1 & \text{if } p = 10m-1 \end{cases}$$

from which (7) and the lemma follow at once.

But the quadratic character of $\sqrt{5}$ is the same as the quartic character of 5 and this has been expressed in terms of the quadratic partition

$$p = a^2 + b^2 , \quad a \equiv 1 \pmod 4) \tag{12}$$

as follows [3] :

5 is a quartic residue of $p = 4m+1$ *if and only if* $b \equiv 0 \pmod 5$

5 is a quadratic, but not a quartic residue of p *is and only if* $a = 0 \pmod 5$.

Hence our lemma leads to the following theorem.

Theorem 1. Let $p = a^2 + b^2$ *with* $a \equiv 1 \pmod 4$, *then*

$$\theta^{\frac{p-1}{2}} = \left(\frac{\theta}{p}\right) = \begin{cases} 1 \text{ if } p=20m+1,\ b\equiv 0 \pmod 5 \text{ or } p=20m+9,\ a=0 \pmod 5 \\ -1 \text{ if } p=20m+1,\ a\equiv 0 \pmod 5 \text{ or } p=20m+9,\ b\equiv 0 \pmod 5 \end{cases}$$

Combining Theorem 1 with condition (2) for $n = (p-1)/4$ we obtain

Theorem 2. Let $p = a^2 + b^2$ *with* $a \equiv 1 \pmod 4$, *then*

$$F_{\frac{p-1}{4}} \equiv 0 \pmod p \quad \text{if and only if}$$

$$\text{either } p = 40m+1, 29 \quad \text{and} \quad b \equiv 0 \pmod 5)$$

$$\text{or } p = 40m+9, 21 \quad \text{and} \quad a \equiv 0 \pmod 5)$$

The primes $p < 1000$ satisfying theorem 2 are

$p = 61, 89, 109, 149, 269, 389, 401, 421, 521, 661, 701, 761, 769, 809, 821, 829$

The primes $p < 1000$ for which θ is a quadratic residue are

$p = 29, 89, 101, 181, 229, 349, 401, 461, 509, 521, 541, 709, 761, 769, 809, 941$

$\theta = 6, 10, 23, 14, 82, 144, 112, 22, 122, 100, 173, 171, 92, 339, 343, 228$

For some of these primes the following theorem holds.

Theorem 3. If θ *is a quadratic, but not a higher power residue of a prime* $p = 10n\pm 1$, *then all the quadratic residues of* p *can be generated by addition as follows:*

$$r_0 = 1, \ r_1 = \theta \pmod p), \ r_{n+1} = r_n + r_{n-1} \pmod p)$$

This follows at once from the fact that $\theta^n + \theta^{n+1} = \theta^n(\theta + 1) = \theta^{n+2}$. For example for $p = 29$, $\theta = 6$, all the quadratic residues are

$$1, 6, 7, 13, 20, 4, 24, 28, 23, 22, 16, 9, 25, 5$$

after which the sequence repeats modulo p.

Further results along the lines of Theorems 1 and 2 are as follows.

Marguerite Dunton conjectured and the author proved for $p=30m+1$ that θ is a cubic residue of p, and hence that p divides

$$F_{\frac{p-1}{3}},$$

if and only if p is represented by the form

$$p = s^2 + 135\, t^2$$

The proof uses cyclotomic numbers of order 15 and is too long to give here. Such primes < 1000 are 139, 151, 199, 331, 541, 619, 661, 709, 811, 829 and 919. The author 4 has shown that θ is a quintic residue of p and hence that

$$F_{\frac{p-1}{5}}$$

is divisible by p if and only if p is an "artiad" of Lloyd Tanner [5]. The artiads < 1000 are 211, 281, 421, 461, 521, 691, 881 and 991.

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ON EVALUATING CERTAIN COEFFICIENTS

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The coefficients to be discussed are those involved when expressing the general term of certain sequences, defined by difference equations, in terms of the roots of the related characteristic equation.

Case I: If the characteristic equation

$$(1) \qquad a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \cdots + a_{m-1}x + a_m = 0, \; a_0 = 1 \quad ,$$

has no multiple root then

$$u_n = \sum_{k=1}^{m} C_k x_k^{n+1} \;, \quad u_n = 0, 1, 2, \ldots,$$

where x_k, $k = 1, 2, \ldots, m$, is a root of (1). If the boundary conditions are given by $u_0 = u_1 = \cdots = u_{m-1} = 1$ then

$$(2) \qquad C_k = \frac{\begin{vmatrix} x_1 x_2 & \cdots & x_{k-1} & 1 & x_{k+1} & \cdots & x_m \\ x_1^2 x_2^2 & \cdots & x_{k-1}^2 & 1 & x_{k+1}^2 & \cdots & x_m^2 \\ \cdot\;\cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot\;\cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot\;\cdot & & \cdot & \cdot & \cdot & & \cdot \\ x_1^m x_2^m & & x_{k-1}^m & 1 & x_{k+1}^m & \cdots & x_m^m \end{vmatrix}}{\begin{vmatrix} x_1 x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_m \\ x_1^2 x_2^2 & \cdots & x_{k-1}^2 & x_k^2 & x_{k+1}^2 & \cdots & x_m^2 \\ \cdot\;\cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot\;\cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot\;\cdot & & \cdot & \cdot & \cdot & & \cdot \\ x_1^m x_2^m \cdots & & x_{k-1}^m & x_k^m & x_{k+1}^m & \cdots & x_m^m \end{vmatrix}} = \frac{N}{D}$$

Expanding the determinants and dividing common factors from the numerator and denominator gives

(3) $$N = (-1)^{k-1} \prod_{\substack{i=1 \\ i \neq k}}^{m} (x_i - 1)$$

(4) $$D = (-1)^{k-1} x_k \prod_{\substack{i=1 \\ i \neq k}}^{m} (x_i - x_k)$$

Since

$$f(x) = \sum_{i=0}^{m} a_i x^{m-i} = \prod_{i=1}^{m} (x - x_i), \; a_0 = 1 \quad ,$$

$$f(1) = \prod_{i=1}^{m} (1 - x_i) \quad \text{and} \quad f'(x_k) = \prod_{\substack{i=1 \\ i \neq k}}^{m} (x_k - x_i) \quad ,$$

Using these identities, (3) becomes

$$N = \frac{(-1)^{m+k} f(1)}{(1 - x_k)} \quad , \qquad \text{if } x_k \neq 1$$

and (4) can be written

$$D = (-1)^{m+k} x_k f'(x_k)$$

Substituting these in (2) gives

$$C_k = 1, \qquad x_k = 1$$

$$C_k = \frac{f(1)}{x_k(1 - x_k)f'(x_k)} \quad , \qquad x_k \neq 1 \quad .$$

Parker [4] investigated the general term of a recursive sequence and gives a method for determining these coefficients but does not give the general formula.

For the Fibonacci sequence the characteristic equation is

$$x^2 - x - 1 = 0 \quad \text{and} \quad u_0 = u_1 = 1 \quad .$$

Therefore

$$C_k = \frac{(-1)(-1)}{x_k(x_k - 1)(2x_k - 1)} = \frac{1}{2x_k - 1}$$

Some characteristic equations obtained in generalizations of the Fibonacci sequence and the values of C_k for each follow.

The characteristic equation in the generalization by Dickinson [1] is $x^c - x^a - 1 = 0$, a, c integers. Since

$$x_k f'(x_k) = cx_k^c - ax_k^a = c(x_k^a + 1) - ax_k^a = (c - a)x_k^a + c \quad ,$$

$$C_k = \frac{1}{(x_k - 1)\left[(c - a)x_k^a + c\right]}$$

for the sequence in which $u_0 = u_1 = \ldots = u_{c-1} = 1$.

In the generalization by Harris and Styles [2] the characteristic equation is $x^p(x - 1)^q - 1 = 0$, p, q integers, $p \geq 0$, $q \geq 1$ and $u_0 = u_1 = \ldots = u_{p+q-1} = 1$.

$$C_k = \frac{1}{(p + q)x_{k-p}}$$

as was shown in [2] without this formula.

Miles [3] used the characteristic equation

$$x^k - x^{k-1} - \ldots - x - 1 = 0, \quad k \text{ integral} \geq 2 \quad .$$

For the sequence in which the initial conditions are given by

$$u_0 = u_1 = \ldots = u_{k-1} = 1 \quad ,$$

$$C_j = \frac{k - 1}{2x_j^k - (k + 1)} \quad , \qquad j = 1, 2, \ldots, k \quad .$$

Raab [5] used the characteristic equation

$$x^{r+1} - ax^r - b = 0, \text{ a, b real}, \quad r \text{ integral} \geq 1 \quad .$$

For the sequence in whichthe initial conditions are given by

$$u_0 = u_1 = \ldots = u_r = 1 \quad ,$$

$$C_k = \frac{b + a - 1}{(a - 1)x_k^{r+1} + b\left[(r + 1)x_k - r\right]}$$

The boundary conditions can be generalized slightly. If

$$u_0 = pr,\ u_1 = pr^2,\ u_2 = pr^3,\ \ldots,\ u_n = pr^{n+1}\ ,$$

$$C_k = \frac{pf(r)}{x_k(1 - x_k)f'(x_k)}$$

Case II: If the characteristic equation (1) has a root of multiplicity 2 then

$$u_n = (C_1 + 2C_2)x_2^{n+1} + \sum_{k=3}^{m} C_k x_k^{n+1}\ ,\quad n = 0, 1, 2, \ldots$$

and x_2 is the repeated root of (1). If the boundary conditions are given by $u_0 = u_1 = \ldots = u_{m-1} = 1$, then,

$$(5)\qquad C_1 = \frac{\begin{vmatrix} 1 & x_2 & x_3 & \cdots & x_m \\ 1 & 2x_2^2 & x_3^2 & \cdots & x_m^2 \\ 1 & 3x_2^3 & x_3^3 & \cdots & x_m^3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & mx_2^m & x_3^m & \cdots & x_m^m \end{vmatrix}}{\begin{vmatrix} x_2 & x_2 & x_3 & \cdots & x_m \\ x_2^2 & 2x_2^2 & x_3^2 & \cdots & x_m^2 \\ x_2^3 & 3x_2^3 & x_3^3 & \cdots & x_m^3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_2^m & mx_2^m & x_3^m & \cdots & x_m^m \end{vmatrix}} = \frac{N_1}{D}$$

Expanding the determinants gives

$$(6)\quad N_1 = \prod_{i=2}^{m} x_i \prod_{i=3}^{m} (x_i - 1) \prod_{i=3}^{m-1} (x_m - x_i) \prod_{i=3}^{m-2} (x_{m-1} - x_i) \cdots \prod_{i=3}^{4} (x_5 - x_i)$$

$$(x_4 - x_3) \left\{ (-1)^{\frac{(m-2)(m-3)}{2}} \left[(2x_2 - 1) \prod_{i=3}^{m} (x_i - x_2) - x_2(x_2 - 1) \right.\right.$$

(the sum of all possible factors $(x_i - x_2)$, $i = 3, 4, \ldots, m$, taken $m-3$ at a time$\left.\left.\right]\right\}$

Since

$$f''(x_2) = 2 \prod_{i=3}^{m} (x_2 - x_i) \text{ and } f'''(x_2) = 6$$

(the sum of all possible products of the factors $(x_2 - x_i)$, $i = 3, 4, \ldots, m$, taken $m-3$ at a time), the quantity in the braces in (6) can be expressed

$$(-1)^{\frac{(m-1)(m-2)}{2}} \left[\frac{(2x_2 - 1)f''(x_2)}{2} + x_2(x_2 - 1)\frac{f'''(x_2)}{6} \right]$$

Therefore,

$$N_1 = (-1)^{\frac{(m-1)(m-2)}{2}} \prod_{i=2}^{m} x_i \prod_{i=3}^{m} (x_i - 1) \prod_{i=3}^{m-1} (x_m - x_i)$$

$$\prod_{i=3}^{m-2} (x_{m-1} - x_i) \cdots \prod_{i=3}^{4} (x_5 - x_i) . \ (x_4 - x_3) \left[(2x_2 - 1)\frac{f''(x_2)}{2} \right.$$

$$\left. + x_2(x_2 - 1) \frac{f'''(x_2)}{6} \right]$$

Expanding the determinant in the denominator gives

$$(8)\qquad D = x_2^2 \prod_{i=2}^{m} x_i \prod_{i=3}^{m} (x_i - x_2)^2 \prod_{i=4}^{m} (x_i - x_3) \prod_{i=5}^{m} (x_i - x_4) \cdots$$

$$\prod_{i=m-1}^{m} (x_i - x_{m-2}) \prod_{i=m}^{m} (x_i - x_{m-1})$$

Substituting (7) and (8) in (5) and simplifying gives

$$C_1 = \frac{\prod_{i=3}^{m} (1 - x_i)}{x_2^2 \prod_{i=3}^{m} (x_2 - x_i)^2} \left[\frac{2x_2 - 1)f''(x_2)}{2} + \frac{x_2(x_2 - 1)f'''(x_2)}{6} \right]$$

If $x_2 = 1$, $C_1 = 1$. If $x_2 \neq 1$,

$$C_1 = \frac{4(1-x_2)^2 \prod_{i=3}^{m} (1 - x_i)}{x_2^2 \left[f''(x_2)\right]^2 (1-x_2)^2} \left[\frac{(2x_2 - 1)f''(x_2)}{2} + \frac{x_2(x_2 - 1)f'''(x_2)}{6} \right]$$

Therefore,

$$C_1 = \frac{2f(1)}{x_2(x_2 - 1)f''(x_2)} \left[\frac{1}{x_2} + \frac{1}{x_2 - 1} + \frac{f'''(x_2)}{3f''(x_2)} \right], \quad x_2 \neq 1 .$$

To determine C_2 the numerator in (5) is replaced by

$$\begin{vmatrix} x_2 & 1 & x_3 & \cdots & x_m \\ x_2^2 & 1 & x_3^2 & \cdots & x_m^2 \\ x_2^3 & 1 & x_3^3 & & x_m^3 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ x_2^m & 1 & x_3^m & \cdots & x_m^m \end{vmatrix} = N_2$$

Evaluating gives

$$(9) \qquad N_2 = (-1)^m \prod_{i=2}^{m} x_i \prod_{i=2}^{m} (1 - x_i) \prod_{i=3}^{m} (x_i - x_2) \prod_{i=4}^{m} (x_1 - x_3) \cdots$$

$$\prod_{i=m-1}^{m} (x_i - x_{m-2}) \prod_{i=m}^{m} (x_i - x_{m-1})$$

Dividing (9) by (8) and simplifying gives

$$C_2 = \frac{(-1)^m \prod_{i=2}^{m} (1-x_i)}{x_2^2 \prod_{i=3}^{m} (x_i - x_2)} = \frac{2 \prod_{i=2}^{m} (1-x_i)}{x_2^2 f''(x_2)}$$

For $x_2 = 1$, $C_2 = 0$. For $x_2 \neq 1$,

$$C_2 = \frac{2f(1)}{x_2^2 f''(x_2)(1-x_2)} .$$

To determine C_k, $k = 3, 4, \ldots, m$, the numerator in (5) is replaced by

$$\begin{vmatrix} x_2 & x_2 & x_3 & \cdots & x_{k-1} & 1 & x_{k+1} & \cdots & x_m \\ x_2^2 & 2x_2^2 & x_3^2 & \cdots & x_{k-1}^2 & 1 & x_{k+1}^2 & \cdots & x_m^2 \\ x_2^3 & 3x_2^3 & x_3^3 & \cdots & x_{k-1}^3 & 1 & x_{k+1}^3 & \cdots & x_m^3 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ x_2^m & mx_2^m & x_3^m & \cdots & x_{k-1}^m & 1 & x_{k+1}^m & \cdots & x_m^m \end{vmatrix} = N_k$$

Evaluating N_k yields

$$(11)\quad N_k = (-1)^{k-1} x_2^3 \prod_{\substack{i=3\\i\neq k}}^{m} x_i \prod_{\substack{i=2\\i\neq k}}^{m} (1-x_i) \prod_{\substack{i=2\\i\neq k}}^{m-1} (x_m-x_i) \prod_{\substack{i=2\\i=k}}^{m-2} (x_{m-1}-x_i)\cdots$$

$$\prod_{\substack{i=2\\i\neq k}}^{k+1} (x_{k+2}-x_i) \prod_{i=2}^{k-1} (x_{k+1}-x_i) \cdots \prod_{i=2}^{3} (x_4-x_i) \prod_{i=2}^{2} (x_3-x_i)$$

$$(1-x_2) \prod_{\substack{i=3\\i\neq k}}^{m} (x_i-x_2)$$

Substituting (11) and (8) in (10) leads to

$$C_k = \frac{(-1)^{k+m}(1-x_2)^2 \prod_{\substack{i=3\\i\neq k}}^{m} (1-x_i)}{x_k(x_k-x_2) \prod_{i=2}^{k-1} (x_k-x_i) \prod_{i=k+1}^{m} (x_i-x_k)}$$

$$C_k = \frac{(-1)^{k+m}(1-x_2)^2 \prod_{i=3}^{m} (1-x_i)}{x_k(1-x_k) \prod_{i=2}^{k-1} (x_k-x_i)(-1)^{m-k} \prod_{i=k+1}^{m} (x_k-x_i)}, \quad x_k \neq 1$$

There $C_k = 1$, $x_k = 1$.

$$C_k = \frac{f(1)}{x_k(1-x_k)f'(x_k)} \quad \text{if } x_k \neq 1 \ .$$

Summary: If the roots of

$$f(x) = \sum_{i=0}^{m} a_i x^{m-1} = 0$$

are not repeated and $u_0 = u_1 = u_2 = \ldots = u_{m-1} = 1$,

$$C_k = \begin{cases} 1, & x_k = 1 \\ \dfrac{f(1)}{x(1-x_k)f'(x_k)}, & x_k \neq 1 \end{cases}$$

If $f(x)$ has a double root, $x_1 = x_2$, and $u_0 = u_1 = \cdots = u_{m-1} = 1$,

$$C_1 = \begin{cases} 1, & x_2 = 1 \\ \dfrac{2f(1)}{x_2(x_2-1)f''(x_2)} \left[\dfrac{1}{x_2} + \dfrac{1}{x_2 - 1} + \dfrac{f'''(x_2)}{3f''(x_2)} \right], & x_2 \neq 1 \end{cases}$$

$$C_2 = \begin{cases} 0, & x_2 = 1 \\ \dfrac{2f(1)}{x_2^2(1-x_2)f''(x_2)}, & x_2 \neq 1 \end{cases}$$

$$C_k = \begin{cases} 1, & x_k = 1 \\ \dfrac{f(1)}{x_k f'(x_k)(1-x_k)}, & x_k \neq 1 \end{cases}, \quad k = 3, 4, \ldots, m$$

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by Verner E. Hoggatt, Jr.
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Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-84 *Proposed by Oswald Wyler, Carnegie Institute of Technology*

For any positive integer m, there is an integer $k(m)$ with the property that m divides a Fibonacci number F_n if and only if $k(m)$ divides n. For a prime number p, it is known that $k(p)$ divides $p - 1$ if $p \equiv \pm 1 \pmod 5$, and that $k(p)$ divides $p + 1$ if $p \equiv \pm 2 \pmod 5$. Using a table of Fibonacci numbers, one finds that $F_{k(p)}$ is a multiple of p, but not of p^2, for small prime numbers p, or in other words that $k(p) < k(p^2)$ for small prime numbers p. It does not seem to be known whether this is the case for all prime numbers p or not.

H-85 *Proposed by H.W. Gould, West Virginia University, Morgantown, West Va.*

Let

$$D_n = f_n x^n - [f_n x^n],$$

where

$$F_{n+1} = f_n + f_{n-1} \text{ with } f_0 = f_1 = 1, \quad x = (1 + \sqrt{5})/2 \quad ,$$

and $[z]$ = greatest integer $\leq z$ (so that $z - [z]$ = fractional part of z). Prove (or disprove) the existence of the limits

$$\lim_{n \to \infty} D_{2n} = 0.27\ldots = A$$

and

$$\lim_{n \to \infty} D_{2n+1} = 0.72\ldots = B$$

with $A + B = 1$. Generalize to case of $u_{n+1} = pu_n + qu_{n-1}$, where p and q are real and u_0 and u_1 are given.

H-86 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

Let p and q be integers such that $p + q \geq 1$, $q \geq 0$ show that if

$x^p(x-1)^q - 1 = 0$ has roots $r_1, r_2, \ldots, r_{p+q}$

and

$(x-1)^{p+q} - x^p = 0$ has roots $s_1, s_2, \ldots, s_{p+q}$

then

$s_i^{p+q} = r_i^q$ for $i = 1, 2, 3, \ldots, p+q$.

H-87 *Proposed by Monte Boisen, Jr., San Jose State College, San Jose, Calif.*

Show that, if

$$u_0 = u_2 = u_3 = \ldots = u_{n-1} = 1 \quad \text{and}$$

$$u_k = u_{k-1} + u_{k-2} + \ldots + u_{k-n} \quad k \geq n ,$$

then

$$\frac{1 - x^2 - 2x^3 - \ldots - (n-2)x^{n-1}}{1 - x - x^2 - \ldots - x^n} = \sum_{k=0}^{\infty} u_k x^k .$$

H-88 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

$$\sum_{k=0}^{n} F_{4mk} \binom{n}{k} = L_{2k}^{m} F_{2km} ,$$

where F_m and L_m are the mth Fibonacci and mth Lucas numbers, respectively.

H-80 *Proposed by J.A.H. Hunter, Toronto, Ontario, Canada and Max Rumney, London, England – Corrected*

Show

$$\sum_{r=0}^{n} \binom{n}{r} F_{r+2}^2 = \sum_{r=0}^{n} \binom{n-1}{r} F_{2r+5} \quad .$$

SOLUTIONS

THE FINAL WORD

H-42 The corrected list is:

1, 2, 3, 5, 9, 15, 20, 25, 41 .

J. D. Konhauser first noted the typing error.

Solution to the Crossword Puzzle by H.W. Gould, West Virginia University

B		F	I	B	O	N	A	C	C	I			T
S	E	R	I	E			L		O	S	A	G	E
		A		L	C	M		U	N		T	E	N
P	H	Y	L	L	O	T	A	X	I	S		N	
A			E		E		D	I	C	E		E	P
S	I	N	O		F		D		A	Q		R	I
C	A	I	N		F	E			L	U	C	A	S
A	L	M	A		I	Q		U		E		T	A
L			R	E	C	U	R	R	E	N	C	E	
	H	A	D	R	I	A	N			C	O	S	H
T	Y	R	O		E	T		U	S	E	S		I
A		A			N	E	X	T			E	R	S
R	A	B	B	I	T	S		E	V	I	C	T	S

XXXXXXXXXXXXXXX

TERMINAL DIGIT COINCIDENCES BETWEEN FIBONACCI NUMBERS AND THEIR INDICES

Gerard R. Deily
U.S. Department of Defense, Washington, D.C.

In [1] the Editor of this journal proposed essentially the following question: What Fibonacci numbers of index less than 10,000 have terminal digits coincident with the index? This note answers that question by supplying a computer-generated table of such coincidences.

The computer program was written in the ALGOL 60 international algorithmic language [2] and is given in the Appendix. For those readers not familiar with the ALGOL language, the operations performed by the program are basically as follows: With starting values given, all Fibonacci numbers with indices 1 to 9999 are computed modulo 10,000. Fibonacci numbers F_1 through F_9 are then reduced mod 10 and compared with their respective indices, F_{10} through F_{99} are reduced mod 100 and likewise compared, and similar reductions and comparisons are performed for F_{100} through F_{999} and for F_{1000} through F_{9999}. If the comparison yields an answer YES, the index for which this occurs is marked with an asterist. A 100 x 100 table, shown listed as tables I and II, is then printed out with row coordinates in hundreds and column coordinates in units, and with asterisks in the locations where coincidences occur. Hence, an asterisk in row zero column 61 indicates that F_{61} has 61 as its last two digits, and similarly an asterisk in row 4 column 85 indicates that F_{485} has 485 as its last three digits. Note the regularity of patterns in Tables I and II; these might prove to be an interesting subject for further investigation.

A digest of the results reported herein is given in Table III. These results were checked through F_{505} by inspecting a table [3] of Fibonacci numbers and by running several versions of the basic program.

REFERENCES

1. Brother U. Alfred, "Exploring Fibonacci Numbers with a Calculator," Fibonacci Quarterly, Vol. 2, No. 2 (April 1964), p. 138.

2. P. Naur et al, "Revised Report on the Algorithmic Language ALGOL 60", Communications of the ACM, Vol. 6, No. 1 (January 1963), pp. 1-17.

3. S. L. Basin and V. E. Hoggatt, Jr., "The First 571 Fibonacci Numbers", Recreational Mathematics, No. 11 (October 1962), pp. 19-30. (Continued on page 153.)

(Continued from page 116.)

$$u_k - b = u_{k-1}u_{k-2} \cdots u_{k-(k-1)} \left[u_{k-(k-1)} - b \right] \quad .$$

Hence

$$u_{k_1} = (u_0 - b) \prod_{i=0}^{i=k_1-1} u_i + b \quad .$$

Choose $k_2 < k_1$ without loss of generality and the conclusion is apparent.

We now consider equation (1) in several special cases. If $b = 0$ the equation is easily solved but in this case the theorem holds trivially. Let $b = 2$. Then we have

$$u_{k+1} = u_k^2 - 2u_k + 2 \quad ,$$

which can be written in the form

$$u_{k+1} - 1 = (u_k - 1)^2 \quad .$$

The solution of this equation is clearly

$$u_k = A^{2^k} + 1 \quad . \tag{2}$$

Hence the sequence with elements $A^{2^k} + 1$, for integral A, is relatively prime except possibly for the common divisor 2. The exception is obviously removed when A is an even integer. When $A = 2$, we have the Fermat numbers mentioned previously.

(Continued on page 164.)

APPENDIX

ALGOL-60 Program for Producing Fibonacci Terminal Digit Concidences

```
begin
    procedure hollerith (name, representation);
    real name;
    string representation;
        name: = representation;
    integer procedure mod (x, m);
    value x, m;
    integer x, m;
        mod: = x - m × (x ÷ m);
    integer i, j, k;
    real blank, star;
    integer array F[-1:9999] ;
    real array cell[0:9999] ;
    hollerith (blank, ' ');
    hollerith (star, '*');
    F[-1] : = 1;
    F[0] : = 0;
    cell[0] : = star;
    for k: = 1 step 1 until 4 do
    begin
        integer finish, modulus;
        modulus: = 10 ↑ k;
        finish: = modulus - 1;
        for i: = modulus / 10 step 1 until finish do
        begin
            F[i]    : = mod (F[i-1] + F[i-2] , 10000);
            cell[i] : = if mod (F[i] , modulus) = i
                        then star
                        else blank;
        end calculations for k-th order of magnitude;
    end setup of coincidence table;
    write (for i: = 0 step 1 until 99 do
           for j: = 0 step 1 until 99 do cell[100×i + j] );
end Fibonacci terminal digit coincidence program;
```

TABLE I PAGE 5 OF 7

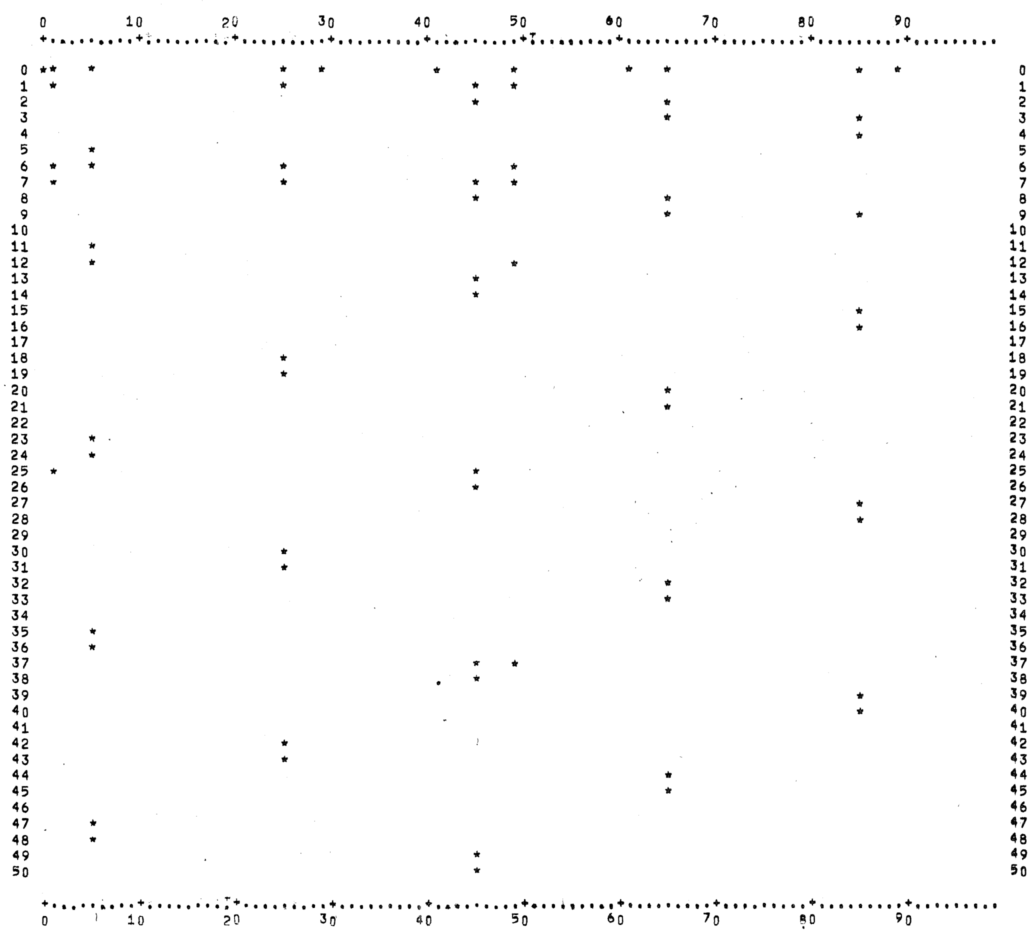

TABLE II PAGE 6 OF 7

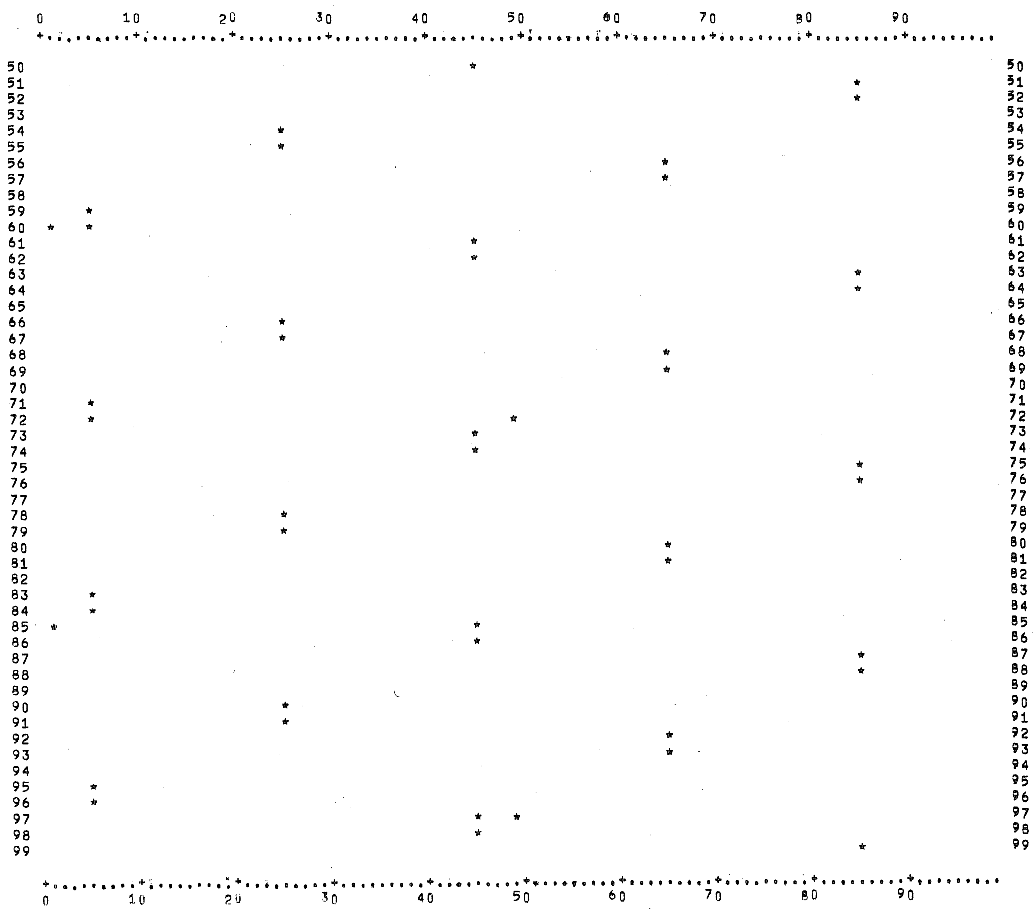

TABLE III

INDICES COINCIDENT WITH FIBONACCI NUMBER TERMINAL DIGITS

1, 5,
25, 29,
41, 49,
61, 65,
85, 89
101, 125, 145, 149,
245, 265,
365, 385,
485, 505,
601, 605, 625, 649,
701, 725, 745, 749,
845, 865,
965, 985,

1105,		1205,	1249,	1345,		1445,	1585,	1685,	1825,	1925,	2065,	21
2305,		2405,	2501,	2545,		2645,	2785,	2885,	3025,	3125,	3265,	33
3505,		3605,		3745,	3749,	3845,	3985,	4085,	4225,	4325,	4465,	45
4705,		4805,		4945,		5045,	5185,	5285,	5425,	5525,	5665,	57
5905,	6001,	6005,		6145,		6245,	6385,	6485,	6625,	6725,	6865,	69
7105,		7205,	7249,	7345,		7445,	7585,	7685,	7825,	7925,	8065,	81
8305,		8405,	8501,	8545,		8645,	8785,	8885,	9025,	9125,	9265,	93
9505,		9605,		9745,	9749,	9845,	9985,	others exceed 10000.				

xxxxxxxxxxxxxxx

Also solved by Douglas Lind and Donald Howells.

PHI, THE GOLDEN RATIO (to 4599 Decimal Places), AND FIBONACCI NUMBERS

Murray Berg
Oakland, Calif.

The golden ratio can be obtained by the division of a line segment into two parts such that the ratio of the total line to the larger segment is the same as the larger segment to the smaller segment. This ratio was given the name, phi, by the U.S. mathematician, Mark Barr, over 50 years ago [1].

If B were the larger segment and A the smaller segment, then

$$\frac{A + B}{B} = \frac{B}{A} \tag{1}$$

Letting A = unity, we have

$$\frac{1 + B}{B} = B$$

and the positive root, B is the number, phi:

$$B = \frac{1 + \sqrt{5}}{2}$$

If we let B equal unity, then the result would be one less than phi. Phi is the only number whose reciprocal is obtained by subtracting one from itself.

Phi is an irrational number, and its determination depends on the calculation of $\sqrt{5}$. It can also be calculated from the Fibonacci series [1] and [2].

The Fibonacci numbers are an infinite series, where each element in the series is the sum of the preceding two elements, and the first two are equal to 1.

Thus $$F_1 = F_2 = 1$$

and for $i > 2$ $$F_i = F_{i-1} + F_{i-2}\,,$$

where F_i = the ith number in the Fibonacci series.

A program was written for the IBM 1401 which produced Fibonacci numbers up to 4000 digits in length. A pair of consecutive numbers were printed out at each 100 digit level.

F_{476} is the first 100 digit Fibonacci number. For each succeeding 100 digit level, the F number increased by 478 and 479 alternately, i.e., F_{954} is the first Fibonacci number with 200 digits, F_{1433} has 300, F_{1911} has 400, up to F_{19137} with 4000 digits. Table 1 lists the Fibonacci numbers F_{11003} and F_{11004}.

It is evident from equation (1) that phi is the ratio between two consecutive members of the Fibonacci series. Theoretically, we can obtain the ratio to any number of decimal places by performing one division with Fibonacci numbers that are far out in the series. However, this must have practical limits.

A separate program performed such a division for F_{11003} and F_{11004}, both 2300 digits in length. The division instruction took 21 minutes. This was repeated with the numerator and denominator reversed. The ratios coincided to 4598 decimal places.

Empirical observation of members early in the series show that the true ratio lies somewhere between the ratios obtained by the two divisions. In fact, it is approximately one fourth the distance between the first two non-coinciding digits of the two divisions. These two digits for the first division were 42 and for the second division, 10. The true first digit of the two would be 3. This is the 4599th digit of phi. Table 2 lists the fractional digits of phi.

REFERENCES

1. Martin Gardner, Mathematical Games, Scientific American, Vol. 201, No. 2, Aug. 1959, pp. 128-134.
2. N. Vorob'ev, Fibonacci Numbers, Translated from the Russian by Halina Moss, Blaisdell Publishing Co., 1961.

xxxxxxxxxxxxxxx

TABLE 1

F11003

1385226108	1389849146	3318369579	2494356958	4648374158	2397008279	8631194478	0123270814	9606278507	1751399903
2651224797	6473891243	0891846980	3039604879	0046956080	4905717207	6604928573	5218136522	1731643799	6629392564
7173511856	2146914169	6708707974	6699151188	2442466984	5710547919	6199746044	7472341072	4003782106	0553357908
7574996945	6924869639	6917120736	5268922184	1896192623	7606833193	0271352823	3209213511	6654782175	9454851487
2496093268	9753082994	9214939556	6355848406	4062988440	6134620903	7912031476	6964248790	8726924515	7016612608
0051225211	5428814745	3016761256	7294777233	1289541117	0885655450	2312356840	1372262062	2298301027	7269258591
2012890852	4806673466	9663354501	1142199481	3650636786	5229287829	8848188137	8172683771	0267970914	5115676896
2723799650	8596935856	5423569649	1078336886	5232864678	6569734815	6337166059	1819627031	1521792082	0236762110
5572088355	6215509544	0202887462	8129578616	9581239454	0372956238	2134273674	4384466075	5128521668	0497974568
7660404686	1519119027	4619727906	1507794036	8906861517	2622180521	1023706129	5731454302	7936180979	1895837832
2764124938	6088403251	4049695388	4991667711	8842003238	3436710506	9902097057	1865017517	9126821677	1994293318
2740869994	0851091246	9478056732	4448753439	5432510823	1301627655	8708508729	2151315800	9109439700	1412268114
8090725106	3521862288	6871088487	7752208419	8640236523	8079861118	4773187478	6521109077	2668319024	3505313833
5635374951	5618951083	0648797240	0490175133	8680223685	3297035838	1198854328	8383937835	9669940528	7162356283
5632860204	6956657639	5257855778	2371306204	3737166938	2038668754	7674835212	3478098396	4289148123	8919252426
4129306207	6694621619	2343575456	2042187238	4602706582	1109668539	8459488464	3113389750	2861398637	8833783107
0887842148	5647170984	3084772401	9888001330	8095527704	2088481463	9095526048	6913994736	0451421700	6160580357
4139509331	5149971174	6116928633	0190593393	4778407869	5760226150	4212756331	3772061206	7071231484	0297335718
8165137824	6242619136	9332914897	6535760412	9776981220	2863991636	1855595685	2283544770	1435962831	8080741054
3977633239	4708095587	4875953506	4381158531	0839843746	7700728139	7778864638	1922731227	0284538002	6509070757
6975227979	1208062350	8359916939	5154135119	9519556612	8740630214	0954042220	3240718052	7657932922	9545448966
0339811947	9365812919	9616056923	2347309682	3963617027	2634743224	7003593412	0651665497	7760111150	9982857292
5460935494	6299557873	1660491087	1345701086	7269788440	1871245899	2914722272	8848008668	9496842560	1752488377

TABLE 1 (Continued)

F11004

2241342925	0726149417	8768445221	7640750624	9224305335	3277323564	4619416117	2235392377	3005663265	5745110745
2509731507	7828619535	7415711132	5130674465	4508593892	5043085035	6940122848	8191759950	7301724135	8171825183
0928585522	0324203872	2540352820	1125292205	7271034243	4836044721	0363252053	1266365415	0010467211	5995273570
0519690070	8154592872	8421330888	7876866356	2200570692	4022736279	1533037075	6075911934	2980240833	5229508679
4835803161	5626140930	3642851474	0824316080	8489859296	8152543605	7710729043	4702322192	0394506576	7556471880
1852815052	8910088959	0065297343	9636830863	1040299947	0813959319	2799580310	1107921801	9909982891	8337347082
0729628005	3193120486	2310564502	2262602234	2833899932	2885933136	1942355534	9314355005	4676741907	4830542906
9507437916	1485588617	1102677540	5378988772	6054590766	7167333068	9901230463	3515170420	2771791077	4707282953
9807088368	5168935663	7998120065	4173341196	6919087513	2306681796	1109461965	4626972652	4844032105	7089563389
2558962811	8372532811	5690534216	0287770603	6864679470	1130003741	4686266305	5064980434	5989905718	8508082738
9223333007	6393902074	7344803185	0112496308	7426219214	3034580269	9566190926	5696653468	2397757935	0645559192
8527203580	1513269923	9354011433	8718644225	3582205650	1943318605	8535551141	2072772788	9582224424	1176699658
4223655959	0237488568	5519687985	3969117752	3387338406	7085151201	3165688111	6727373403	7025157771	5171781686
3583422096	2143997905	2819057484	3271191933	0627247913	5974124740	7459818257	2713876742	4295849216	4540420708
5664623931	7181718163	5574390609	3286721969	0540640885	3679865367	6770799571	2239359060	4474264222	5386481817
1129799983	1819808009	7828847818	6912050400	6977241945	5576981463	4364542393	3025531789	7895313665	2905867204
8280069287	1985281490	6219182278	2098843186	3500754266	8667392361	4481010338	5608921945	4604373723	2083981697
0743459356	3245433618	0826392315	5182193383	9849278009	0521701221	6729698803	2212727621	4583381448	8919561483
2472352138	0294211905	8572350989	5990695907	6354573336	3605187057	7977696355	4843478023	8250297146	4130520029
7263043600	5077368545	7178988583	5679066442	2174890839	9413215948	1841222396	8058820056	7544797041	7480859537
1468275379	5884519860	4036505692	7219638895	4773751421	4006846575	5214682501	8676764452	2065126375	9990118794
3738874031	9405122694	2526076260	6900250347	6468505567	2906807306	8337622732	0580472090	7584467309	6730487626
8214395238	0405570837	5413169662	4660432562	1471942062	3375517429	9699405764	8550102519	4559914918	5436210128

TABLE 2

								.61803398	8749894848
2045868343	6563811772	0309179805	7628621354	4862270526	0462818902	4497072072	0418939113	7484754088	0753868917
5212663386	2223536931	7931800607	6672635443	3389086595	9395829056	3832266131	9928290267	8806752087	6689250171
1696207032	2210432162	6954862629	6313614438	1497587012	2034080588	7954454749	2461856953	6486444924	1044320771
3449470495	6584678850	9874339442	2125448770	6647809158	8460749988	7124007652	1705751797	8834166256	2494075890
6970400028	1210427621	7711177780	5315317141	0117046665	9914669798	7317613560	0670874807	1013179523	6894275219
4843530567	8300228785	6997829778	3478458782	2891109762	5003026961	5617002504	6433824377	6486102838	3126833037
2429267526	3116533924	7316711121	1588186385	1331620384	0052221657	9128667529	4654906811	3171599343	2359734949
8509040947	6213222981	0172610705	9611645629	9098162905	5520852479	0352406020	1727997471	7534277759	2778625619
4320827505	1312181562	8551222480	9394712341	4517022373	5805772786	1600868838	2952304592	6478780178	8992199027
0776903895	3219681986	1514378031	4997411069	2608867429	6226757560	5231727775	2035361393	6210767389	3764556060
6059216589	4667595519	0040055590	8950229530	9423124823	5521221241	5444006470	3405657347	9766397239	4949946584
4948730231	5417676893	7521030687	3788034417	0093954409	6279558986	7872320951	2426893557	3097045095	9568440175
5519881921	8020640529	0551893494	7592600734	8522821010	8819464454	4222318891	3192946896	2200230144	3770269923
0078030852	6118075451	9288770502	1096842493	6271359251	8760777884	6658361502	3891349333	3122310533	9232136243
1926372891	0670503399	2822652635	5620902979	8642472759	7725655086	1548754357	4826471814	1451270006	0238901620
7773224499	4353088999	0950168032	8112194320	4819643876	7586331479	8571911397	8153978074	7615077221	1750826945
8639320456	5209896985	5567814106	9683728840	5874610337	8105444390	9436835835	8138113116	8993855576	9754841491
4453415091	2954070050	1947754861	6307542264	1729394680	3673198058	6183391832	8599130396	0720144559	5044977921
2076124785	6459161608	3705949878	6006970189	4098864007	6443617093	3417270919	1433650137	1576601148	0381430626
2380514321	1734815100	5590134561	0118007905	0638142152	7093085880	9287570345	0507808145	4588199063	3612982798
1411745339	2731208092	8972792221	3298064294	6878242748	7401745055	4067787570	8323731097	5915117762	9784432847
4790817651	8097787268	4161176325	0386121129	1436834376	7023503711	1633072586	9883258710	3363222381	0980901211
0198991768	4149175123	3134015273	3843837234	5009347860	4979294599	1582201258	1045982309	2552872124	1370436149

TABLE 2 (Continued)

1020547185	5496118087	6426576511	0605458814	7560443178	4798584539	7312863016	2544876114	8520217064	4041116607
6695059775	7832570395	1108782308	2710647893	9021115691	0392768384	5386333321	5658296597	7310343603	2322545743
6372041244	0640888267	3758433953	6795931232	2134373209	9574988946	9956564736	0072959998	3912881031	9742631251
7971414320	1231127955	1894778172	6914158911	7799195648	1255800184	5506563295	2859859100	0908621802	9775637892
5999164994	6428193022	2935523466	7475932695	1654214021	0913630181	9472270789	0122087287	3617073486	4999815625
5472811373	4798716569	5274890081	4438405327	4837813782	4669174442	2963491470	8157007352	5457070897	7267546934
3822619546	8615331209	5335792380	1460927351	0210119190	2183606750	9730895752	8957746814	2295433943	8549315533
9630380729	1691758461	0146099505	5064803679	3041472365	7203986007	3550760902	3173125016	1320484358	3648177048
4818109916	0244252327	1672190189	3345963786	0878752870	1739359303	0133590112	3710239171	2659047026	3494028307
6687674363	8651327106	2803231740	6931733448	2343564531	8505813531	0854973335	0759966778	7124490583	6367541328
9086240632	4563953572	1252426117	0278028656	0432349428	3730172557	4405837278	2679960317	3936401328	7627701243
6798311446	4369476705	3127249241	0471670013	8247831286	5650649343	4180390041	0178053395	0587724586	6557552293
9158239708	4177298337	2823115256	9260929959	4224000056	0626678674	3579239724	5408481765	1973436265	2689448885
5272027477	8747335983	5367277614	0759171205	1326934483	7529916499	8093602461	7844267572	7767900191	9190703805
2204612324	8239132610	4327191684	5123060236	2789354543	2461769975	7536890417	6365025478	5138246314	6583363833
7602357789	9267298863	2161858395	9036399818	3845827644	9124598093	7043055559	6137973432	6134830494	9496868108
9535696348	2817812886	2536460842	0339465381	9441945714	2666823718	3949183237	0908574850	2665680398	9744066210
5360306400	2608171126	6599541993	6873160945	7228881092	0778822772	0363668448	1532561728	4117690979	2666655223
8468831137	1852991921	6319052015	6863122282	0715599876	4684235520	5928537175	7807656050	3677313097	5191223973
8872246825	8057159744	5740484298	7807352215	9842667662	5780770620	1943040054	2550158312	5030175340	9411719101
9298903844	7250332988	0245014367	9684416947	9595453045	9103138116	2187045679	9786636617	4605957000	3445970113
5251813460	0656553520	3478881174	1499412748	2641521355	6776394039	0710387088	1823380680	48	

POWERS OF THE GOLDEN SECTION

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The Golden Section is the positive root k of the equation $x^2 + x = 1$,

$$k = (\sqrt{5} - 1)/2 = 0.618034.$$

The negative root, $(-\sqrt{5} - 1)/2$, is $-1/k$, the negative reciprocal of k. From the above equation,

$$k^n = k^{n+1} + k^{n+2} = k^{n-2} - k^{n-1},$$

That is, any power of k is the sum of the next two higher powers or the difference between the next two lower powers.

If the powers of k are listed serially as in the box tabulation with the accompanying diagram, the ascending and descending ratios of the successive powers are k and $1/k$ respectively.

Each power of k can be expressed in terms of its first power and a Fibonacci number as indicated in the right column of the box tabulation. Starting with the successive powers k^0 and k^1, this column can be completed by repeated application of the k^n formula. It is evident from this tabulation and the k^n formula that the powers of k form a Fibonacci series which can be separated into two component Fibonacci series.

Continuing further, all power of k can be expressed in terms of any other power and a number as shown on the accompanying transformation tables. The box tabulation on the diagram can be used to determine the values given in the transformation table. For example, the value $k = \frac{1}{8}k^6 - \frac{5}{8}$ is obtained from $k^6 = 5 - 8k$. This value for k, coupled with the value $k^0 = 0 + \frac{8}{8}$, can be used to determine all of the values listed in the vertical k^6 column with the aid of the k^n formulas.

It is interesting to note the recurrence of the Fibonacci sequence in the numerators in the vertical columns and in the denominators in the horizontal columns. Both the Fibonacci and Lucas series appear in the k^0 vertical column. Complex expressions involving various powers of k can be very much simplified by reference to these tables.

REFERENCE

Robert S. Beard, "The Golden Section and Fibonacci Numbers", Scripta Mathematica, Vol. 16, Mar. - June, 1950 pp. 116-119.

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(Tables and Chart are on pages 165, 166 and 167.)

(Continued from page 152.)

In general we may transform equation (1) by writing

(3) $$u_k = F(V_k) \quad .$$

Suppose that furthermore we require that F satisfy the functional equation

(4) $$F^2(s) - bF(s) + b = F(2s) \quad .$$

Then our equation becomes $F(V_{k+1}) = F(2V_k)$, a solution of which is given by $V_k = A2^k$. Hence we have

(5) $$u_k = F(A2^k) \quad .$$

We now consider the functional equation (4). Let

$$F(s) - \frac{b}{2} = 2L(s) \quad .$$

Then equation (4) becomes

(6) $$L^2(s) = \frac{1}{2}\left[\left(\frac{b^2}{8} - \frac{b}{4}\right) + L(2s)\right] \quad .$$

(Continued on page 169.)

TRANSFORMATION OF POWERS OF K. (Equivalents in Other Powers.) RSB.

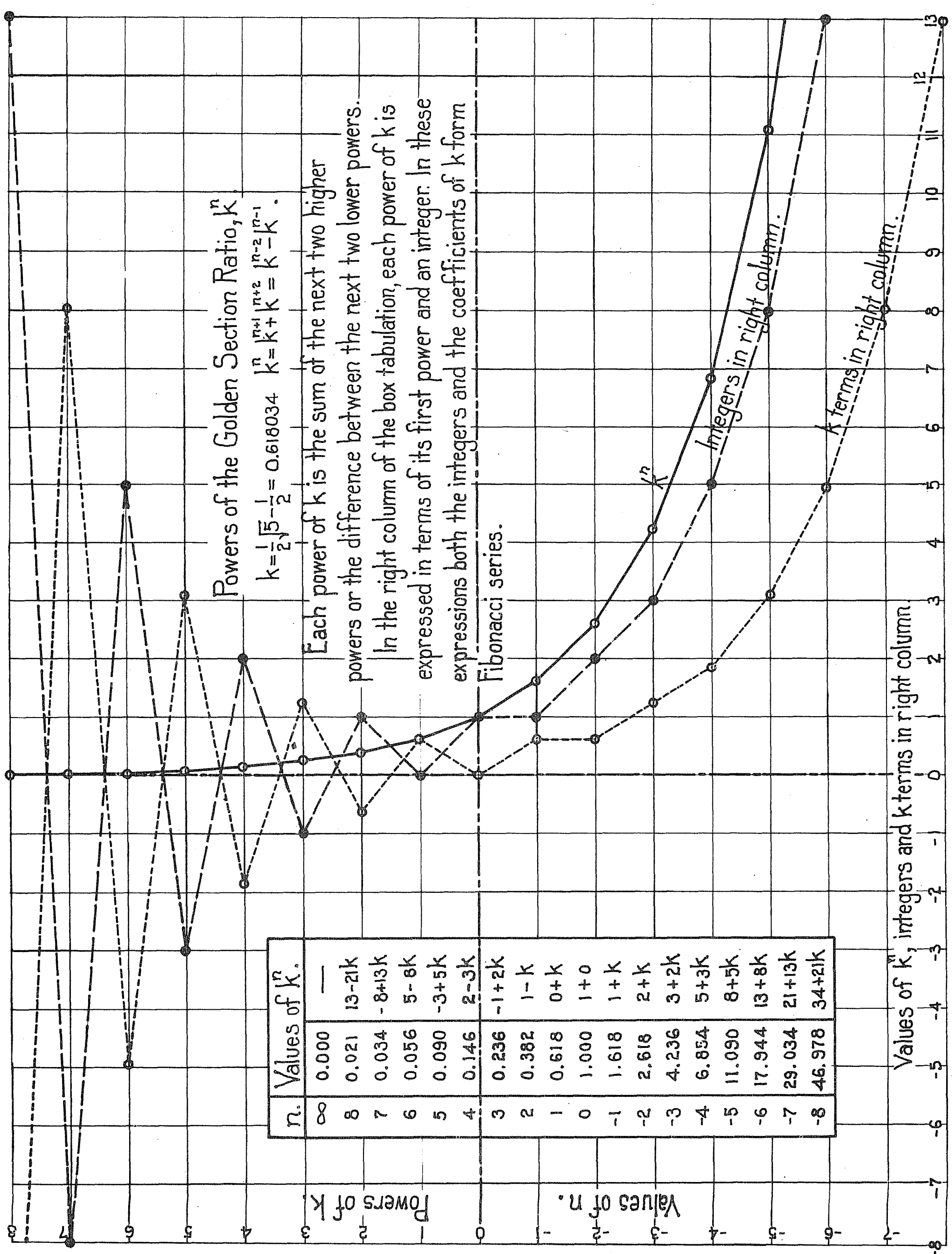
Powers of K.	K^0	K^{-1}	K^{-2}	K^{-3}	K^{-4}	K^{-5}	K^{-6}	K^{-7}	K^{-8}	K^{-9}	K^{-10}	Powers of K.
-10	$\frac{55}{2}\sqrt{5}+\frac{123}{2}$	$55k^{-1}+34$	$55k^{-2}-21$	$\frac{55}{2}k^{-3}+\frac{13}{2}$	$\frac{55}{3}k^{-4}-\frac{8}{3}$	$\frac{55}{5}k^{-5}+\frac{5}{5}$	$\frac{55}{8}k^{-6}-\frac{3}{8}$	$\frac{55}{13}k^{-7}+\frac{2}{13}$	$\frac{55}{21}k^{-8}-\frac{1}{21}$	$\frac{55}{34}k^{-9}+\frac{1}{34}$	$\frac{55}{55}k^{-10}+0$	-10
-9	$\frac{34}{2}\sqrt{5}+\frac{76}{2}$	$34k^{-1}+21$	$34k^{-2}-13$	$\frac{34}{2}k^{-3}+\frac{8}{2}$	$\frac{34}{3}k^{-4}-\frac{5}{3}$	$\frac{34}{5}k^{-5}+\frac{3}{5}$	$\frac{34}{8}k^{-6}-\frac{2}{8}$	$\frac{34}{13}k^{-7}+\frac{1}{13}$	$\frac{34}{21}k^{-8}-\frac{1}{21}$	$\frac{34}{34}k^{-9}+0$	$\frac{34}{55}k^{-10}-\frac{1}{55}$	-9
-8	$\frac{21}{2}\sqrt{5}+\frac{47}{2}$	$21k^{-1}+13$	$21k^{-2}-8$	$\frac{21}{2}k^{-3}+\frac{5}{2}$	$\frac{21}{3}k^{-4}-\frac{3}{3}$	$\frac{21}{5}k^{-5}+\frac{2}{5}$	$\frac{21}{8}k^{-6}-\frac{1}{8}$	$\frac{21}{13}k^{-7}+\frac{1}{13}$	$\frac{21}{21}k^{-8}+0$	$\frac{21}{34}k^{-9}+\frac{1}{34}$	$\frac{21}{55}k^{-10}+\frac{1}{55}$	-8
-7	$\frac{13}{2}\sqrt{5}+\frac{29}{2}$	$13k^{-1}+8$	$13k^{-2}-5$	$\frac{13}{2}k^{-3}+\frac{3}{2}$	$\frac{13}{3}k^{-4}-\frac{2}{3}$	$\frac{13}{5}k^{-5}+\frac{1}{5}$	$\frac{13}{8}k^{-6}-\frac{1}{8}$	$\frac{13}{13}k^{-7}+0$	$\frac{13}{21}k^{-8}-\frac{1}{21}$	$\frac{13}{34}k^{-9}-\frac{1}{34}$	$\frac{13}{55}k^{-10}-\frac{2}{55}$	-7
-6	$\frac{8}{2}\sqrt{5}+\frac{18}{2}$	$8k^{-1}+5$	$8k^{-2}-3$	$\frac{8}{2}k^{-3}+\frac{2}{2}$	$\frac{8}{3}k^{-4}-\frac{1}{3}$	$\frac{8}{5}k^{-5}+\frac{1}{5}$	$\frac{8}{8}k^{-6}+0$	$\frac{8}{13}k^{-7}+\frac{1}{13}$	$\frac{8}{21}k^{-8}+\frac{1}{21}$	$\frac{8}{34}k^{-9}+\frac{2}{34}$	$\frac{8}{55}k^{-10}+\frac{3}{55}$	-6
-5	$\frac{5}{2}\sqrt{5}+\frac{11}{2}$	$5k^{-1}+3$	$5k^{-2}-2$	$\frac{5}{2}k^{-3}+\frac{1}{2}$	$\frac{5}{3}k^{-4}-\frac{1}{3}$	$\frac{5}{5}k^{-5}+0$	$\frac{5}{8}k^{-6}-\frac{1}{8}$	$\frac{5}{13}k^{-7}-\frac{1}{13}$	$\frac{5}{21}k^{-8}-\frac{2}{21}$	$\frac{5}{34}k^{-9}-\frac{3}{34}$	$\frac{5}{55}k^{-10}-\frac{5}{55}$	-5
-4	$\frac{3}{2}\sqrt{5}+\frac{7}{2}$	$3k^{-1}+2$	$3k^{-2}-1$	$\frac{3}{2}k^{-3}+\frac{1}{2}$	$\frac{3}{3}k^{-4}+0$	$\frac{3}{5}k^{-5}+\frac{1}{5}$	$\frac{3}{8}k^{-6}+\frac{1}{8}$	$\frac{3}{13}k^{-7}+\frac{2}{13}$	$\frac{3}{21}k^{-8}+\frac{3}{21}$	$\frac{3}{34}k^{-9}+\frac{5}{34}$	$\frac{3}{55}k^{-10}+\frac{8}{55}$	-4
-3	$\frac{2}{2}\sqrt{5}+\frac{4}{2}$	$2k^{-1}+1$	$2k^{-2}-1$	$\frac{2}{2}k^{-3}+0$	$\frac{2}{3}k^{-4}-\frac{1}{3}$	$\frac{2}{5}k^{-5}-\frac{1}{5}$	$\frac{2}{8}k^{-6}-\frac{2}{8}$	$\frac{2}{13}k^{-7}-\frac{3}{13}$	$\frac{2}{21}k^{-8}-\frac{5}{21}$	$\frac{2}{34}k^{-9}-\frac{8}{34}$	$\frac{2}{55}k^{-10}-\frac{13}{55}$	-3
-2	$\frac{1}{2}\sqrt{5}+\frac{3}{2}$	$k^{-1}+1$	$k^{-2}+0$	$\frac{1}{2}k^{-3}+\frac{1}{2}$	$\frac{1}{3}k^{-4}+\frac{1}{3}$	$\frac{1}{5}k^{-5}+\frac{2}{5}$	$\frac{1}{8}k^{-6}+\frac{3}{8}$	$\frac{1}{13}k^{-7}+\frac{5}{13}$	$\frac{1}{21}k^{-8}+\frac{8}{21}$	$\frac{1}{34}k^{-9}+\frac{13}{34}$	$\frac{1}{55}k^{-10}+\frac{21}{55}$	-2
-1	$\frac{1}{2}\sqrt{5}+\frac{1}{2}$	$k^{-1}+0$	$k^{-2}-1$	$\frac{1}{2}k^{-3}-\frac{1}{2}$	$\frac{1}{3}k^{-4}-\frac{2}{3}$	$\frac{1}{5}k^{-5}-\frac{3}{5}$	$\frac{1}{8}k^{-6}-\frac{5}{8}$	$\frac{1}{13}k^{-7}-\frac{8}{13}$	$\frac{1}{21}k^{-8}-\frac{13}{21}$	$\frac{1}{34}k^{-9}-\frac{21}{34}$	$\frac{1}{55}k^{-10}-\frac{34}{55}$	-1
0	$0+1$	$0+1$	$0+1$	$0+\frac{2}{2}$	$0+\frac{3}{3}$	$0+\frac{5}{5}$	$0+\frac{8}{8}$	$0+\frac{13}{13}$	$0+\frac{21}{21}$	$0+\frac{34}{34}$	$0+\frac{55}{55}$	0
1	$\frac{1}{2}\sqrt{5}-\frac{1}{2}$	$k^{-1}-1$	$k^{-2}-2$	$\frac{1}{2}k^{-3}-\frac{3}{2}$	$\frac{1}{3}k^{-4}-\frac{5}{3}$	$\frac{1}{5}k^{-5}-\frac{8}{5}$	$\frac{1}{8}k^{-6}-\frac{13}{8}$	$\frac{1}{13}k^{-7}-\frac{21}{13}$	$\frac{1}{21}k^{-8}-\frac{34}{21}$	$\frac{1}{34}k^{-9}-\frac{55}{34}$	$\frac{1}{55}k^{-10}-\frac{89}{55}$	1
2	$-\frac{1}{2}\sqrt{5}+\frac{3}{2}$	$-k^{-1}+2$	$-k^{-2}+3$	$-\frac{1}{2}k^{-3}+\frac{5}{2}$	$-\frac{1}{3}k^{-4}+\frac{8}{3}$	$-\frac{1}{5}k^{-5}+\frac{13}{5}$	$-\frac{1}{8}k^{-6}+\frac{21}{8}$	$-\frac{1}{13}k^{-7}+\frac{34}{13}$	$-\frac{1}{21}k^{-8}+\frac{55}{21}$	$-\frac{1}{34}k^{-9}+\frac{89}{34}$	$-\frac{1}{55}k^{-10}+\frac{144}{55}$	2
3	$\frac{2}{2}\sqrt{5}-\frac{4}{2}$	$2k^{-1}-3$	$2k^{-2}-5$	$\frac{2}{2}k^{-3}-\frac{8}{2}$	$\frac{2}{3}k^{-4}-\frac{13}{3}$	$\frac{2}{5}k^{-5}-\frac{21}{5}$	$\frac{2}{3}k^{-6}-\frac{34}{8}$	$\frac{2}{13}k^{-7}-\frac{55}{13}$	$\frac{2}{21}k^{-8}-\frac{89}{21}$	$\frac{2}{34}k^{-9}-\frac{144}{34}$	$\frac{2}{55}k^{-10}-\frac{233}{55}$	3
4	$-\frac{3}{2}\sqrt{5}+\frac{7}{2}$	$-3k^{-1}+5$	$-3k^{-2}+8$	$-\frac{3}{2}k^{-3}+\frac{13}{2}$	$-\frac{3}{3}k^{-4}+\frac{21}{3}$	$-\frac{3}{5}k^{-5}+\frac{34}{5}$	$-\frac{3}{8}k^{-6}+\frac{55}{8}$	$-\frac{3}{13}k^{-7}+\frac{89}{13}$	$-\frac{3}{21}k^{-8}+\frac{144}{21}$	$-\frac{3}{34}k^{-9}+\frac{233}{34}$	$-\frac{3}{55}k^{-10}+\frac{377}{55}$	4
5	$\frac{5}{2}\sqrt{5}-\frac{11}{2}$	$5k^{-1}-8$	$5k^{-2}-13$	$\frac{5}{2}k^{-3}-\frac{21}{2}$	$\frac{5}{3}k^{-4}-\frac{34}{3}$	$\frac{5}{5}k^{-5}-\frac{55}{5}$	$\frac{5}{8}k^{-6}-\frac{89}{8}$	$\frac{5}{13}k^{-7}-\frac{144}{13}$	$\frac{5}{21}k^{-8}-\frac{233}{21}$	$\frac{5}{34}k^{-9}-\frac{377}{34}$	$\frac{5}{55}k^{-10}-\frac{610}{55}$	5
6	$-\frac{8}{2}\sqrt{5}+\frac{18}{2}$	$-8k^{-1}+13$	$-8k^{-2}+21$	$-\frac{8}{2}k^{-3}+\frac{34}{2}$	$-\frac{8}{3}k^{-4}+\frac{55}{3}$	$-\frac{8}{5}k^{-5}+\frac{89}{5}$	$-\frac{8}{8}k^{-6}+\frac{144}{8}$	$-\frac{8}{13}k^{-7}+\frac{233}{13}$	$-\frac{8}{21}k^{-8}+\frac{377}{21}$	$-\frac{8}{34}k^{-9}+\frac{610}{34}$	$-\frac{8}{55}k^{-10}+\frac{987}{55}$	6
7	$\frac{13}{2}\sqrt{5}-\frac{29}{2}$	$13k^{-1}-21$	$13k^{-2}-34$	$\frac{13}{2}k^{-3}-\frac{55}{2}$	$\frac{13}{3}k^{-4}-\frac{89}{3}$	$\frac{13}{5}k^{-5}-\frac{144}{5}$	$\frac{13}{8}k^{-6}-\frac{233}{8}$	$\frac{13}{13}k^{-7}-\frac{377}{13}$	$\frac{13}{21}k^{-8}-\frac{610}{21}$	$\frac{13}{34}k^{-9}-\frac{987}{34}$	$\frac{13}{55}k^{-10}-\frac{1597}{55}$	7
8	$-\frac{21}{2}\sqrt{5}+\frac{47}{2}$	$-21k^{-1}+34$	$-21k^{-2}+55$	$-\frac{21}{2}k^{-3}+\frac{89}{2}$	$-\frac{21}{3}k^{-4}+\frac{144}{3}$	$-\frac{21}{5}k^{-5}+\frac{233}{5}$	$-\frac{21}{8}k^{-6}+\frac{377}{8}$	$-\frac{21}{13}k^{-7}+\frac{610}{13}$	$-\frac{21}{21}k^{-8}+\frac{987}{21}$	$-\frac{21}{34}k^{-9}+\frac{1597}{34}$	$-\frac{21}{55}k^{-10}+\frac{2584}{55}$	8
9	$\frac{34}{2}\sqrt{5}-\frac{76}{2}$	$34k^{-1}-55$	$34k^{-2}-89$	$\frac{34}{2}k^{-3}-\frac{144}{2}$	$\frac{34}{3}k^{-4}-\frac{233}{3}$	$\frac{34}{5}k^{-5}-\frac{377}{5}$	$\frac{34}{8}k^{-6}-\frac{610}{8}$	$\frac{34}{13}k^{-7}-\frac{987}{13}$	$\frac{34}{21}k^{-8}-\frac{1597}{21}$	$\frac{34}{34}k^{-9}-\frac{2584}{34}$	$\frac{34}{55}k^{-10}-\frac{4181}{55}$	9
10	$-\frac{55}{2}\sqrt{5}+\frac{123}{2}$	$-55k^{-1}+34$	$-55k^{-2}+144$	$-\frac{55}{2}k^{-3}+\frac{233}{2}$	$-\frac{55}{3}k^{-4}+\frac{377}{3}$	$-\frac{55}{5}k^{-5}+\frac{610}{5}$	$-\frac{55}{8}k^{-6}+\frac{987}{8}$	$-\frac{55}{13}k^{-7}+\frac{1597}{13}$	$-\frac{55}{21}k^{-8}+\frac{2584}{21}$	$-\frac{55}{34}k^{-9}+\frac{4181}{34}$	$-\frac{55}{55}k^{-10}+\frac{6765}{55}$	10

TRANSFORMATION OF POWERS OF K. (Equivalents in Other Powers.)

RSB.

Powers of K.	K^0	K	K^2	K^3	K^4	K^5
-10	122.991870	$55k+89$	$-55k^2+144$	$\frac{55}{2}k^3+\frac{233}{2}$	$-\frac{55}{3}k^4+\frac{377}{3}$	$\frac{55}{5}k^5+\frac{610}{5}$
-9	76.013156	$34k+55$	$-34k^2+89$	$\frac{34}{2}k^3+\frac{144}{2}$	$-\frac{34}{3}k^4+\frac{233}{3}$	$\frac{34}{5}k^5+\frac{377}{5}$
-8	46.978714	$21k+34$	$-21k^2+55$	$\frac{21}{2}k^3+\frac{89}{2}$	$-\frac{21}{3}k^4+\frac{144}{3}$	$\frac{21}{5}k^5+\frac{233}{5}$
-7	29.034442	$13k+21$	$-13k^2+34$	$\frac{13}{2}k^3+\frac{55}{2}$	$-\frac{13}{3}k^4+\frac{89}{3}$	$\frac{13}{5}k^5+\frac{144}{5}$
-6	17.944272	$8k+13$	$-8k^2+21$	$\frac{8}{2}k^3+\frac{34}{2}$	$-\frac{8}{3}k^4+\frac{55}{3}$	$\frac{8}{5}k^5+\frac{89}{5}$
-5	11.090170	$5k+8$	$-5k^2+13$	$\frac{5}{2}k^3+\frac{21}{2}$	$-\frac{5}{3}k^4+\frac{34}{3}$	$\frac{5}{5}k^5+\frac{55}{5}$
-4	6.854102	$3k+5$	$-3k^2+8$	$\frac{3}{2}k^3+\frac{13}{2}$	$-\frac{3}{3}k^4+\frac{21}{3}$	$\frac{3}{5}k^5+\frac{34}{5}$
-3	4.236068	$2k+3$	$-2k^2+5$	$\frac{2}{2}k^3+\frac{8}{2}$	$-\frac{2}{3}k^4+\frac{13}{3}$	$\frac{2}{5}k^5+\frac{21}{5}$
-2	2.618034	$k+2$	$-k^2+3$	$\frac{1}{2}k^3+\frac{5}{2}$	$-\frac{1}{3}k^4+\frac{8}{3}$	$\frac{1}{5}k^5+\frac{13}{5}$
-1	1.618034	$k+1$	$-k^2+2$	$\frac{1}{2}k^3+\frac{3}{2}$	$-\frac{1}{3}k^4+\frac{5}{3}$	$\frac{1}{5}k^5+\frac{8}{5}$
0	1.000000	$0+1$	$0+1$	$0+\frac{2}{2}$	$0+\frac{3}{3}$	$0+\frac{5}{5}$
1	0.618034	$k+0$	$-k^2+1$	$\frac{1}{2}k^3+\frac{1}{2}$	$-\frac{1}{3}k^4+\frac{2}{3}$	$\frac{1}{5}k^5+\frac{3}{5}$
2	0.381966	$-k+1$	k^2+0	$-\frac{1}{2}k^3+\frac{1}{2}$	$\frac{1}{3}k^4+\frac{1}{3}$	$-\frac{1}{5}k^5+\frac{2}{5}$
3	0.236068	$2k-1$	$-2k^2+1$	$\frac{2}{2}k^3+0$	$-\frac{2}{3}k^4+\frac{1}{3}$	$\frac{2}{5}k^5+\frac{1}{5}$
4	0.145898	$-3k+2$	$3k^2-1$	$-\frac{3}{2}k^3+\frac{1}{2}$	$\frac{3}{3}k^4+0$	$-\frac{3}{5}k^5+\frac{1}{5}$
5	0.090170	$5k-3$	$-5k^2+2$	$\frac{5}{2}k^3-\frac{1}{2}$	$-\frac{5}{3}k^4+\frac{1}{3}$	$\frac{5}{5}k^5+0$
6	0.055728	$-8k+5$	$8k^2-3$	$-\frac{8}{2}k^3+\frac{2}{2}$	$\frac{8}{3}k^4-\frac{1}{3}$	$-\frac{8}{5}k^5+\frac{1}{5}$
7	0.034442	$13k-8$	$-13k^2+5$	$\frac{13}{2}k^3-\frac{3}{2}$	$-\frac{13}{3}k^4+\frac{2}{3}$	$\frac{13}{5}k^5-\frac{1}{5}$
8	0.021286	$-21k+13$	$21k^2-8$	$-\frac{21}{2}k^3+\frac{5}{2}$	$\frac{21}{3}k^4-\frac{3}{3}$	$-\frac{21}{5}k^5+\frac{2}{5}$
9	0.013156	$34k-21$	$-34k^2+13$	$\frac{34}{2}k^3-\frac{8}{2}$	$-\frac{34}{3}k^4+\frac{5}{3}$	$\frac{34}{5}k^5-\frac{3}{5}$
10	0.008130	$-55k+34$	$55k^2-21$	$-\frac{55}{2}k^3+\frac{13}{2}$	$\frac{55}{3}k^4-\frac{8}{3}$	$-\frac{55}{5}k^5+\frac{5}{5}$

K^6	K^7	K^8	K^9	K^{10}	Powers of K.
$-\frac{55}{8}k^6+\frac{987}{8}$	$\frac{55}{13}k^7+\frac{1597}{13}$	$-\frac{55}{21}k^8+\frac{2584}{21}$	$\frac{55}{34}k^9+\frac{4181}{34}$	$-\frac{55}{55}k^{10}+\frac{6765}{55}$	-10
$-\frac{34}{8}k^6+\frac{610}{8}$	$\frac{34}{13}k^7+\frac{987}{13}$	$-\frac{34}{21}k^8+\frac{1597}{21}$	$\frac{34}{34}k^9+\frac{2584}{34}$	$-\frac{34}{55}k^{10}+\frac{4181}{55}$	-9
$-\frac{21}{8}k^6+\frac{377}{8}$	$\frac{21}{13}k^7+\frac{610}{13}$	$-\frac{21}{21}k^8+\frac{987}{21}$	$\frac{21}{34}k^9+\frac{1597}{34}$	$-\frac{21}{55}k^{10}+\frac{2584}{55}$	-8
$-\frac{13}{8}k^6+\frac{233}{8}$	$\frac{13}{13}k^7+\frac{377}{13}$	$-\frac{13}{21}k^8+\frac{610}{21}$	$\frac{13}{34}k^9+\frac{987}{34}$	$-\frac{13}{55}k^{10}+\frac{1597}{55}$	-7
$-\frac{8}{8}k^6+\frac{144}{8}$	$\frac{8}{13}k^7+\frac{233}{13}$	$-\frac{8}{21}k^8+\frac{377}{21}$	$\frac{8}{34}k^9+\frac{610}{34}$	$-\frac{8}{55}k^{10}+\frac{987}{55}$	-6
$-\frac{5}{8}k^6+\frac{89}{8}$	$\frac{5}{13}k^7+\frac{144}{13}$	$-\frac{5}{21}k^8+\frac{233}{21}$	$\frac{5}{34}k^9+\frac{377}{34}$	$-\frac{5}{55}k^{10}+\frac{610}{55}$	-5
$-\frac{3}{8}k^6+\frac{55}{8}$	$\frac{3}{13}k^7+\frac{89}{13}$	$-\frac{3}{21}k^8+\frac{144}{21}$	$\frac{3}{34}k^9+\frac{233}{34}$	$-\frac{3}{55}k^{10}+\frac{377}{55}$	-4
$-\frac{2}{8}k^6+\frac{34}{8}$	$\frac{2}{13}k^7+\frac{55}{13}$	$-\frac{2}{21}k^8+\frac{89}{21}$	$\frac{2}{34}k^9+\frac{144}{34}$	$-\frac{2}{55}k^{10}+\frac{233}{55}$	-3
$-\frac{1}{8}k^6+\frac{21}{8}$	$\frac{1}{13}k^7+\frac{34}{13}$	$-\frac{1}{21}k^8+\frac{55}{21}$	$\frac{1}{34}k^9+\frac{89}{34}$	$-\frac{1}{55}k^{10}+\frac{144}{55}$	-2
$-\frac{1}{8}k^6+\frac{13}{8}$	$\frac{1}{13}k^7+\frac{21}{13}$	$-\frac{1}{21}k^8+\frac{34}{21}$	$\frac{1}{34}k^9+\frac{55}{34}$	$-\frac{1}{55}k^{10}+\frac{89}{55}$	-1
$0+\frac{8}{8}$	$0+\frac{13}{13}$	$0+\frac{21}{21}$	$0+\frac{34}{34}$	$0+\frac{55}{55}$	0
$-\frac{1}{8}k^6+\frac{5}{8}$	$\frac{1}{13}k^7+\frac{8}{13}$	$-\frac{1}{21}k^8+\frac{13}{21}$	$\frac{1}{34}k^9+\frac{21}{34}$	$-\frac{1}{55}k^{10}+\frac{34}{55}$	1
$\frac{1}{8}k^6+\frac{3}{8}$	$-\frac{1}{13}k^7+\frac{5}{13}$	$\frac{1}{21}k^8+\frac{8}{21}$	$-\frac{1}{34}k^9+\frac{13}{34}$	$\frac{1}{55}k^{10}+\frac{21}{55}$	2
$-\frac{2}{8}k^6+\frac{2}{8}$	$\frac{2}{13}k^7+\frac{3}{13}$	$-\frac{2}{21}k^8+\frac{5}{21}$	$\frac{2}{34}k^9+\frac{8}{34}$	$-\frac{2}{55}k^{10}+\frac{13}{55}$	3
$\frac{3}{8}k^6+\frac{1}{8}$	$-\frac{3}{13}k^7+\frac{2}{13}$	$\frac{3}{21}k^8+\frac{3}{21}$	$-\frac{3}{34}k^9+\frac{5}{34}$	$\frac{3}{55}k^{10}+\frac{8}{55}$	4
$-\frac{5}{8}k^6+\frac{1}{8}$	$\frac{5}{13}k^7+\frac{1}{13}$	$-\frac{5}{21}k^8+\frac{2}{21}$	$\frac{5}{34}k^9+\frac{3}{34}$	$-\frac{5}{55}k^{10}+\frac{5}{55}$	5
$\frac{8}{8}k^6+0$	$-\frac{8}{13}k^7+\frac{1}{13}$	$\frac{8}{21}k^8+\frac{1}{21}$	$-\frac{8}{34}k^9+\frac{2}{34}$	$\frac{8}{55}k^{10}+\frac{3}{55}$	6
$-\frac{13}{8}k^6+\frac{1}{8}$	$\frac{13}{13}k^7+0$	$-\frac{13}{21}k^8+\frac{1}{21}$	$\frac{13}{34}k^9+\frac{1}{34}$	$-\frac{13}{55}k^{10}+\frac{2}{55}$	7
$\frac{21}{8}k^6-\frac{1}{8}$	$-\frac{21}{13}k^7+\frac{1}{13}$	$\frac{21}{21}k^8+0$	$-\frac{21}{34}k^9+\frac{1}{34}$	$\frac{21}{55}k^{10}+\frac{1}{55}$	8
$-\frac{34}{8}k^6+\frac{2}{8}$	$\frac{34}{13}k^7-\frac{1}{13}$	$-\frac{34}{21}k^8+\frac{1}{21}$	$\frac{34}{34}k^9+0$	$-\frac{34}{55}k^{10}+\frac{1}{55}$	9
$\frac{55}{8}k^6-\frac{3}{8}$	$-\frac{55}{13}k^7+\frac{2}{13}$	$\frac{55}{21}k^8-\frac{1}{21}$	$-\frac{55}{34}k^9+\frac{1}{34}$	$\frac{55}{55}k^{10}+0$	10



n.	Values of k^n.	
∞	0.000	—
8	0.021	13-21k
7	0.034	-8+13k
6	0.056	5-8k
5	0.090	-3+5k
4	0.146	2-3k
3	0.236	-1+2k
2	0.382	1-k
1	0.618	0+k
0	1.000	1+0
-1	1.618	1+k
-2	2.618	2+k
-3	4.236	3+2k
-4	6.854	5+3k
-5	11.090	8+5k
-6	17.944	13+8k
-7	29.034	21+13k
-8	46.978	34+21k

FIBONACCI AND EUCLID

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For many centuries number addicts have had much fun with the numbers generated by the Pythagorean identity

$$(2mn)^2 + (m^2-n^2)^2 = (m^2+n^2)^2$$

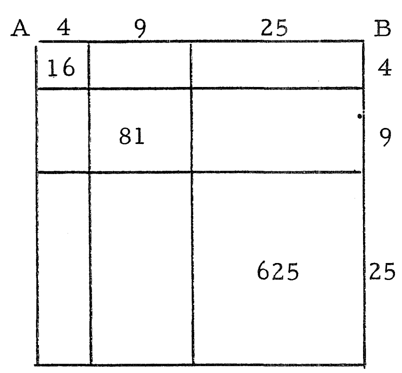
Candido's identity viz.

$$\left[m^2+n^2+(m+n)^2\right]^2 = 2\left[m^4+n^4+(m+n)^4\right]$$

may have its moments too, especially for Fibonacci fans. It is easily verified by expanding both sides and obviously holds for all integral values of m and n. Therefore it holds when m and n are consecutive Fibonacci or Lucas numbers. It has its application to geometry as illustrated by the following problem.

> "Divide a given straight line into three segments so that the square on the whole line is twice the sum of the squares on the three segments."

Solution: Construct a square on line AB. Let AB (and the remaining sides) be divided into three segments which are measured by the second powers of any three consecutive Fibonacci numbers, e.g., 2^2, 3^2, 5^2 or 4, 9, 25.



Then the cross lines joining the points of division divide the square into nine rectangles — three are squares and six are non-squares. Thus we see that the great square is twice the sum of the three smaller squares, i.e.,

$$38^2 = 2\left[16 + 81 + 625\right] = 1444.$$

Furthermore the six non-squares are equal in pairs and therefore represent the six faces of a cuboid. The area and diagonal of the large square are equal respectively to twice the area and diagonal of the cuboid. This identity can be applied to the areas of circles and spheres on the line and its segments or to any similar polygons drawn on the same.

Problemists may find many applications to geometry, e.g., "From the corners of an equilateral triangle cut off similar triangles so that half the area remains."

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(Continued from page 164.)

Now if
$$\frac{b^2}{8} - \frac{b}{4} = 1 \quad (b = 1 \pm 3) \quad ,$$

equation (6) has the solutions

$$L(s) = \begin{cases} \cos s \\ \cosh s \end{cases} .$$

Therefore

$$(7) \qquad F(s) = 2 \begin{Bmatrix} \cos s \\ \cosh s \end{Bmatrix} + \frac{b}{2} = 2 \begin{Bmatrix} \cos s \\ \cosh s \end{Bmatrix} + \frac{1 \pm 3}{2} .$$

Equation (5) now gives

$$(8) \qquad u_k = 2 \cosh A2^k + \frac{1 \pm 3}{2}$$

for the hyperbolic cosine alternative in equation (6). Now if A is chosen to be Arccosh $\frac{n_0}{2}$ where n_0 is an odd integer, it is easily shown that the first term of the right number of equation (8) is always an odd integer and hence that u_k is not divisible by 2 with the choice of the positive sign. A similar result holds for n_0 even with the negative sign. Therefore by the theorem, the sequence (b) is relatively prime. The cosine alternative in equation (7) leads to a bounded sequence of integers and therefore is not very interesting.

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SUMS OF n-th POWERS OF ROOTS OF A GIVEN QUADRATIC EQUATION

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The quadratic equation whose roots are the sum or difference of the n-th powers of the roots of a given quadratic equation is of interest to the algorithmatist both for the number patterns to be seen and for the implied connection between quadratic equations and the factorability and primeness of numbers. In this paper, an elementary approach which points out relationships which might otherwise be missed is used to derive general expressions for the subject equation and to show how Lucas and Fibonacci numbers arise as special cases.

Sums of like powers of roots of a given equation have been studied in detail. For a few examples, in this Quarterly S. L. Basin [1] has used a development of Waring's formula [2] for sums of like powers of roots to write a generating function for Lucas numbers and to write the kth Lucas number as a sum of binomial coefficients. R. G. Buschman [3] has used a linear combination of like powers of roots of a quadratic to study the difference equation $u_{n+1} = au_n + bu_{n-1}$, arriving at a result which expresses u_n in terms of a and b. Paul F. Byrd [4], [5] has studied the coefficients in expansions of analytic functions in polynomials associated with Fibonacci numbers. A number of the results in the foregoing references appear as special cases of those of the present paper.

A. THE SUMS OF THE n-th POWERS

Given the primitive equation $f(x) = x^2 - px - q = 0$, with roots r_1 and r_2, the first problem considered here is to write the equation

$$f(x)^n = x^2 - (r_1^n + r_2^n)x - (r_1 r_2)^n = 0 \quad ,$$

with roots r_1^n and r_2^n.

Solving the primitive equation for r_1 and r_2 by the quadratic formula,

$$r_1 = (p + \sqrt{p^2 + 4q})/2 , \qquad r_2 = (p - \sqrt{p^2 + 4q})/2 \quad .$$

Since r_1 and r_2 satisfy the equation $(x - r_1)(x - r_2) = 0$,

$$x^2 - (r_1 + r_2)x + r_1 r_2 = x^2 - px - q = 0$$

identically, so $r_1 + r_2 = p$ and $r_1 r_2 = -q$. Next, write the equation $f(x)^2$, having roots r_1^2 and r_2^2. Squaring roots,

$$r_1^2 = \left[(p + \sqrt{p^2 + 4q})/2\right]^2 = (2p^2 + 4q + 2p\sqrt{p^2 + 4q})/4 ,$$

$$r_2^2 = \left[(p - \sqrt{p^2 + 4q})/2\right]^2 = (2p^2 + 4q - 2p\sqrt{p^2 + 4q})/4 .$$

Adding, $r_1^2 + r_2^2 = p^2 + 2q$. The desired equation is then

$$f(x)^2 = x^2 - (p^2 + 2q)x + q^2 = 0 .$$

Similarly, the coefficient of x for the equation $f(x)^3 = 0$, by multiplying and adding, is $r_1^3 + r_2^3 = p^3 + 3pq$. Continuing in this manner, the coefficients for x for the equations whose roots are higher powers of r_1 and r_2 can be tabulated as follows:

The sequence for $r_1^n + r_2^n$ ends when, for a given n, the last term in the sequence becomes equal to zero. The penultimate term assumes the form $2q^{n/2}$ when n is even and $npq^{(n-1)/2}$ when n is odd.

Writing the numerical coefficients from Table 1 in the form of an array of m columns and n rows and letting $C(m, n)$ denote the coefficient in the mth column and nth row leads to Table 2. It is obvious that the iterative operation in Table 2 to generate additional numerical coefficients is

$$C(m, n) = C(m-1, n-2) + C(m, n-1) ,$$

which suggests a linear combination of binomial coefficients. By direct expansion,

$$\frac{n(n-m)(n-m-1)\ldots(n-2m+3)}{(m-1)!} = 2\binom{n+1-m}{m-1} - \binom{n-m}{m-1} .$$

TABLE 1: $r_1^n + r_2^n$

$$r_1 + r_2 = p$$

$$r_1^2 + r_2^2 = p^2 + 2q$$

$$r_1^3 + r_2^3 = p^3 + 3pq$$

$$r_1^4 + r_2^4 = p^4 + 4p^2q + 2q^2$$

$$r_1^5 + r_2^5 = p^5 + 5p^3q + 5pq^2$$

$$r_1^6 + r_2^6 = p^6 + 6p^4q + 9p^2q^2 + 2q^3$$

$$r_1^7 + r_2^7 = p^7 + 7p^5q + 14p^3q^2 + 7pq^3$$

$$r_1^8 + r_2^8 = p^8 + 8p^6q + 20p^4q^2 + 16p^2q^3 + 2q^4$$

$$\cdots \qquad\qquad \cdots$$

$$r_1^n + r_2^n = p^n + np^{n-2}q + \frac{n(n-3)}{2!}p^{n-4}q^2 + \frac{n(n-4)(n-5)}{3!}p^{n-6}q^3$$

$$+ \cdots + \frac{n(n-m)(n-m-1)\cdots(n-2m+3)}{(m-1)!}p^{n-2m+2}q^{m-1}$$

TABLE 2: $C(m, n)$

n \ m	1	2	3	4	5	$\Sigma C(m, n)$
1	1	0				1
2	1	2				3
3	1	3	0			4
4	1	4	2			7
5	1	5	5	0		11
6	1	6	9	2		18
7	1	7	14	7	0	29
8	1	8	20	16	2	47

From the above tables,

$$\text{(A. 1)} \qquad r_1^n + r_2^n = \sum_{i=0}^{[n/2]} \left[2\binom{n-i}{i} - \binom{n-i-1}{i} \right] p^{n-2i} q^i \quad ,$$

where $[x]$ is the greatest integer less than or equal to x, and $\binom{m}{n}$ is the binomial coefficient

$$\frac{m!}{(m-n)!\ n!}$$

Proof: By algebra,

$$\begin{aligned} r_1^{k+1} + r_2^{k+1} &= (r_1^{k+1} + r_2^{k+1} + r_1 r_2^k + r_1^k r_2) - (r_1 r_2^k + r_1^k r_2) \\ &= (r_1^k + r_2^k)(r_1 + r_2) - r_1 r_2 (r_1^{k-1} + r_2^{k-1}) \\ &= p(r_1^k + r_2^k) + q(r_1^{k-1} + r_2^{k-1}) \quad . \end{aligned}$$

From Table 1, (A. 1) holds for $n = 1, 2, \ldots, 8$. To prove (A. 1) by mathematical induction, assume that (A. 1) holds when $n = k$ and $n = k-1$. Using the result just given and the inductive hypothesis,

$$\begin{aligned} r_1^{k+1} + r_2^{k+1} = p &\sum_{i=0}^{[k/2]} \left[2\binom{k-i}{i} - \binom{k-i-1}{i} \right] p^{k-2i} q^i \\ &+ q \sum_{i=0}^{[(k-1)/2]} \left[2\binom{k-1-i}{i} - \binom{k-i-2}{i} \right] p^{k-1-2i} q^i \quad . \end{aligned}$$

Multiplying as indicated and using the index substitution $i-1$ for i in the second series yields

$$\begin{aligned} r_1^{k+1} + r_2^{k+1} = &\sum_{i=0}^{[k/2]} \left[2\binom{k-i}{i} - \binom{k-i-1}{i} \right] p^{k+1-2i} q^i \\ &+ \sum_{i=0}^{[(k+1)/2]} \left[2\binom{k-i}{i-1} - \binom{k-i-1}{i-1} \right] p^{k+1-2i} q^i \quad . \end{aligned}$$

Since the recursion formula for binomial coefficients is

$$\binom{m}{n} + \binom{m}{n+1} = \binom{m+1}{n+1} \quad ,$$

combining the series above will yield

$$\left[2\binom{k+1-i}{i} - \binom{k-i}{i}\right]$$

for the coefficient of $p^{k+1-2i}q^i$. Considering the series for k even and for k odd leads to

$$r_1^{k+1} + r_2^{k+1} = \sum_{i=0}^{[(k+1)/2]} \left[2\binom{k+1-i}{i} - \binom{k-i}{i}\right] p^{k+1-2i}q^i \quad ,$$

so that (A.1) follows by mathematical induction. An expression equivalent to (A.1) was given by Basin in [1].

Using formula (A.1), we can write the desired equation

$$f(x)^n = x^2 - (r_1^n + r_2^n)x + (-1)^n q^n = 0 \quad .$$

Example: Given the equation $x^2 - 5x + 6 = 0$, write the equation whose roots are the fourth powers of the roots of the given equation, without solving the given equation. In the given equation, $p = 5$, $q = 6$. From Table 1 or formula (A.1),

$$r_1^4 + r_2^4 = (5)^4 + 4(5)^2(-6) + 2(-6)^2 = 97 \quad ,$$

so $f(x)^4 = x^2 - 97x + 6^4 = 0$. As a check, by factoring $x^2 - 5x + 6 = 0$, $r_1 = 2$ and $r_2 = 3$, giving us $f(x)^4 = (x - 2^4)(x - 3^4) = x^2 - 97x + 6^4$.

Returning to Table 2, the summation of the coefficients by rows yields the series 1, 3, 4, 7, 11, 18, 29, 47, ..., the successive Lucas numbers defined by $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$. The general equation for $r_1^n + r_2^n$ reduces to the numerical values of $C(m, n)$ of Table 2 for $p = q = 1$ in the primitive equation $x^2 - px - q = 0$, which becomes $x^2 - x - 1 = 0$ with roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. This yields an expression for the nth Lucas number, since $L_n = \alpha^n + \beta^n$;

(A.2) $$L_n = 1 + n + n(N-3)/2! + n(n-4)(n-5)/3! + n(n-5)(n-6)(n-7)/4!$$

$$= \sum_{i=0}^{[n/2]} \left[2\binom{n-i}{i} - \binom{n-i-1}{i}\right] .$$

For example, $L_6 = 1 + 6 + 6(3)/2 + 6(2)(1)/6 = 18$.

Also, the equation whose roots are the nth roots of $x^2 - x - 1 = 0$ is $x^2 - L_n x + (-1)^n = 0$. (Essentially, equation (A.2), as well as (A.3), (B.2), (B.3), appears in [3] by Buschman.)

An alternate expression for L_n is obtained by expressing $r_1^n + r_2^n$ in terms of p and Δ, where $\Delta = \sqrt{p^2 + rq}$. Now,

$$r_1^n = [(p + \Delta)/2]^n = 2^{-n}(p^n + np^{n-1}\Delta + \frac{n(n-1)}{2!} p^{n-2}\Delta^2 + \frac{n(n-1)(n-2)}{3!} p^{n-3}\Delta^3 + \ldots)$$

$$r_2^n = [(p - \Delta)/2]^h = 2^{-n}(p^n - np^{n-1}\Delta + \frac{n(n-1)}{2!} p^{n-2}\Delta^2 + \frac{n(n-1)(n-2)}{3!} p^{n-3}\Delta^3 + \ldots)$$

which, for $p = q = 1$, $\Delta = \sqrt{5}$, implies, on adding,

(A.3) $$L_n = 2^{1-n}(1 - 5n(n-1)/2! + 5^2 n(n-1)(n-2)(n-3)/4! + \ldots)$$

$$= \frac{1}{2^{n-1}} \sum_{i=0}^{[n/2]} 5^i \binom{n}{2i} .$$

B. THE DIFFERENCES OF THE nth POWERS

Now let us turn to the equation whose roots are r_1^n and $-r_2^n$, where r_1 and r_2 are the roots of the primitive equation $x^2 - px - q = 0$. Since

$$r_1 - r_2 = (p + \sqrt{p^2 + rq})/2 - (p - \sqrt{p^2 + 4q})/2 = \sqrt{p^2 + 4q} = \Delta \quad ,$$

the equation whose roots are r_1 and $-r_2$ is $x^2 - \sqrt{p^2 + rq}\, x - r_1 r_2 = 0$, and $f(x)_\Delta = x^2 - \Delta x + q = 0$. Similarly,

$$r_1^2 - r_2^2 = [(p + \sqrt{p^2 + 4q})/2]^2 - [(p - \sqrt{p^2 + 4q})/2]^2 = p\Delta \quad .$$

Continuing in this manner, and assembling results, we have

TABLE 3: $r_1^n - r_2^n$

n	$r_1^n - r_2^n$
1	Δ
2	$p\Delta$
3	$(p^2 + q)\Delta$
4	$(p^3 + 2pq)\Delta$
5	$(p^4 + 3p^2q + q^2)\Delta$
6	$(p^5 + 4p^3q + 3pq^2)\Delta$
7	$(p^6 + 5p^4q + 6p^2q^2 + q^3)\Delta$
...	...
n	$(p^{n-1} + (n-2)p^{n-3}q + \frac{(n-3)(n-4)}{2!} p^{n-5}q^2 + \ldots + \frac{(n-m-1)(n-m-2)\ldots(n-2m)}{m!} p^{n-2m-1}q^m)\Delta$

The sequence for $r_1^n - r_2^n$ as given in Table 3 ends, for a given n, on arriving at term zero.

Writing the numerical coefficients from Table 3 in the form of an array of m columns and n rows, and letting $C_\Delta(m, n)$ denote the coefficient in the mth column and nth row,

TABLE 4: $C_\Delta(m, n)$

n \ m	1	2	3	4	5	6	$\Sigma C_\Delta(m, n)$
1	1	0					1
2	1	0					1
3	1	1	0				2
4	1	2	0				3
5	1	3	1	0			5
6	1	4	3	0			8
7	1	5	6	1	0		13
8	1	6	10	4	0		21
9	1	7	15	10	1	0	34

It is obvious that the iterative process to generate additional numerical coefficients rests on the relation

$$C_\Delta(m,n) = C_\Delta(m-1,\ n-2) + C_\Delta(m,\ n-1) \quad ,$$

the same as for Table 2. The coefficients generated are also the coefficients appearing in the Fibonacci polynomials defined by

$$f_{n+1}(x) = xf_n(x) + f_{n-1}(x),\quad f_1(x) = 1,\quad f_2(x) = x \quad .$$

(For further references to Fibonacci polynomials, see Byrd, [4], p. 17.)

Tables 3 and 4 lead to

$$\text{(B.1)} \qquad r_1^n - r_2^n = \Delta \sum_{j=0}^{[n/2]} \binom{n-j-1}{j} p^{n-j-1}q^j \quad ,$$

which is proved similarly to (A.1)

The equation whose roots are r_1^n and $-r_2^n$, where r_1 and r_2 are the roots of $x^2 - px - q = 0$, is $f(x)^n_\Delta = x^2 - (r_1^n - r_2^n)x + q^n = 0$. We use (B.1) to solve the following: Given the equation $x^2 - 10x + 21 = 0$ with unsolved roots r_1 and r_2, write the quadratic equation whose roots are r_1^3 and $-r_2^3$. From the given equation, $p = 10$, $q = -21$, and $\Delta = \sqrt{10^2 - 4(21)} = 4$. Using (B.1) or Table 3,

$$r_1^3 - r_2^3 = (p^2 + q) \quad = (10^2 - 21)4 = 316 \quad .$$

Hence $f(x)^3_\Delta = x^2 - 316x + (-21)^3 = 0$. As a check, the roots of the problem equation are $r_1 = 7$, $r_2 = 3$, so $f(x)^3_\Delta = (x - 7^3)(x + 3^3) = x^2 - 316x - 21^3 = 0$.

The summation of terms in the successive rows of Table 4 is seen to yield the sequence of Fibonacci numbers, defined by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. This result can be demonstrated by substituting $p = q = 1$ in the general expression (B.1) and dividing by $\Delta = \sqrt{5}$, for the left hand member becomes the Binet form, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$. In general,

(B.2) $$F_n = 1 + (n-2) + (n-3)(n-4)/2! + (n-4)(n-5)(n-6)/3! + \dots$$
$$= \sum_{j=0}^{[n/2]} \binom{n-j-1}{j} ,$$

the sum of terms in rising diagonals of Pascal's triangle. Also, the quadratic equation whose roots are α^n and $-\beta^n$, nth powers of the roots of the quadratic equation $x^2 - x - 1 = 0$ with roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$, is $x^2 - \sqrt{5}\ F_n x - 1 = 0$.

Expressing $r_1^n - r_2^n$ in terms of only p and Δ and assembling terms as we did for $r_1^n + r_2^n$ in (A.3), we arrive at an alternate expression for the nth Fibonacci number,

(B.3) $$F_n = 2^{1-n}(n + 5n(n-1)(n-2)/3! + 5^2 n(n-1)(n-2)(n-3)(n-4)/5! + \dots)$$
$$= 2^{1-n} \sum_{i=0}^{[(n-1)/2]} 5^i \binom{n}{2i+1} .$$

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DETERMINANTS INVOLVING Kth POWERS FROM SECOND ORDER SEQUENCES

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INTRODUCTION

Let a_n be a sequence of complex numbers satisfying the difference equation

(1) $$a_{n+2} = \alpha\, a_{n+1} - \beta\, a_n \quad \text{for} \quad n = 0, 1, \ldots ,$$

where α and β are fixed complex numbers, for such a sequence we define

(2) $$A_k(a_n) = \begin{vmatrix} a_n^k & a_{n+1}^k & \cdots & a_{n+k}^k \\ a_{n+1}^k & a_{n+2}^k & \cdots & a_{n+k+1}^k \\ \vdots & \vdots & & \vdots \\ a_{n+k}^k & a_{n+k+1}^k & \cdots & a_{n+2k}^k \end{vmatrix} \quad \text{for } n = 0, 1, \ldots .$$

It is the purpose of this note to prove

(3) $$A_k(a_n) = \beta^{nk(k+1)/2} A_k(a_0) ,$$

and give examples of the result.

A DIFFERENCE EQUATION FOR $[a_n^k]$:

Let $(1 - \theta_1 x)(1 - \theta_2 x) = 1 - \alpha x + \beta x^2$, so that $\theta_1 + \theta_2 = \alpha$ and $\theta_1 \theta_2 = \beta$, and assume $\theta_1 \neq \theta_2$. Carlitz [3] has proved that

(4) $$p_k(x) / q_k(x) = \sum_{n=0}^{\infty} a_n^k x^n$$

where

(5) $$q_k(x) = \prod_{i=0}^{k} (1 - \theta_1^i \theta_2^{k-i} x)$$

and $p_k(x)$ is a polynomial of degree less than the degree of $q_k(x)$. Letting

(6) $$q_k(x) = 1 - \sum_{i=1}^{k} \alpha_{k+1-i}(k) x^i$$

(the constants $\alpha_j(k)$ are polynomials symmetric in θ_1 and θ_2 determined by (5)) we see after multiplying through (4) with $q_k(x)$ (as given in (6)) and equating coefficients of x^n in the right and left members that

(7) $$a_{n+k+1}^k = \alpha_{k+1}(k) a_{n+k}^k + \alpha_k(k) a_{n+k-1}^k + \dots + \alpha_1(k) a_n^k$$

for $n = 0, 1, \dots$. We also know from (5) and (6) that

(8) $$-\alpha_1(k) = (-1)^{k+1} (\theta_1 \theta_2)^{1+2+\dots+k} \qquad \text{or}$$

$$\alpha_1(k) = (-1)^{k+2} \beta^{k(k+1)/2}$$

Now let k be a fixed natural number and consider for $n \geq 0$,

(9) $$(-1)^k (-1)^{k+2} \beta^{k(k+1)/2} A_k(a_n) = (-1)^k \alpha_1(k) A_k(a_n)$$

$$= \begin{vmatrix} a_{n+1}^k & a_{n+2}^k & \dots & a_{n+k}^k & \alpha_1(k) a_n^k \\ a_{n+2}^k & a_{n+3}^k & \dots & a_{n+k+1}^k & \alpha_1(k) a_{n+1}^k \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n+k+1}^k & a_{n+k+2}^k & \dots & a_{n+2k}^k & \alpha_1(k) a_{n+k}^k \end{vmatrix} = A_k(a_{n+1}) \; .$$

The second equality in (9) follows since we have interchanged k columns and multiplied the last column by $\alpha_1(k)$. The last equality follows since appropriate multiples of the first columns can be added to the last column to make it the last column of $A_k(a_{n+1})$; the "appropriate multiples" are the $a_j(k)$ given in (7).

Thus, we have shown

$$(-1)^k(-1)^{k+2}\beta^{k(k+1)/2}A_k(a_n) = \beta^{k(k+1)/2}A_k(a_n) = A_k(a_{n+1}) \quad ,$$

so that (3) can be proved by induction on n.

As a corollary to (3) we note that if $\{a_n\}$ satisfies (1), then $\{a_{qn+p}\}$, where q and p are non-negative integers, is a second order sequence as well; in fact,

$$(10) \qquad a_{q(n+2)+p} = (\theta_1^q + \theta_2^q)\, a_{q(n+1)+p} - \beta^q a_{qn+p}$$

for $n = 0, 1, \ldots$ Hence we can rewrite (3) to obtain

$$(11) \qquad A_k(a_{qn+p}) = \beta^{qnk(k+1)/2} A_k(a_p) \quad .$$

EXAMPLES INVOLVING THE FIBONACCI SEQUENCES

When a_n is the Fibonacci sequence $\{F_n\} = \{0, 1, 1, 2, \ldots\}$, $\beta = -1$ in (3) so that we have

$$(12) \qquad \begin{vmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{vmatrix} = (-1)^n \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (-1)^{n+1} \quad ,$$

$$(13) \qquad \begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ F_{n+2}^2 & F_{n+3}^2 & F_{n+4}^2 \end{vmatrix} = (-1)^{3n} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 4 \\ 1 & 4 & 9 \end{vmatrix} = (-1)^{n+1}\, 2 \quad ,$$

$$(14) \qquad \begin{vmatrix} F_n^3 & F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 \\ F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 \\ F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 & F_{n+5}^3 \\ F_{n+3}^3 & F_{n+4}^3 & F_{n+5}^3 & F_{n+6}^3 \end{vmatrix} = (-1)^{6n} \begin{vmatrix} 0 & 1 & 1 & 8 \\ 1 & 1 & 8 & 27 \\ 1 & 8 & 27 & 125 \\ 8 & 27 & 125 & 512 \end{vmatrix} = 36$$

The result in (12) is well known, Brother Alfred proposed (13) as a problem in the very first issue of the Fibonacci Quarterly [1], and Erbacker, Fuchs and Parker proposed (14) in a later issue [5].

If we redefine $a_0 = F_1$, $a_1 = F_2$, ... we have $\{a_n\} = \{u_n\}$ in the standard notation; fixing $q = 2$ and $p = 1$ in (11) we obtain for $k = 1$ and 2,

(15) $$A_1(u_{2n+1}) = A_1(u_1) = -1 ,$$

(16) $$A_2(u_{2n+1}) = A_2(u_1) = -18 ,$$

respectively; on the other hand if we fix $q = 2$ and $p = 0$ in (11) we have for $k = 1$ and 2,

(17) $$A_1(u_{2n}) = A_1(u_0) = 1 ,$$

(18) $$A_2(u_{2n}) = A_2(u_0) = 18 ,$$

respectively. Together (16)and (18) imply

(19) $$\begin{vmatrix} u_n^2 & u_{n+2}^2 & u_{n+4}^2 \\ u_{n+2}^2 & u_{n+4}^2 & u_{n+6}^2 \\ u_{n+4}^2 & u_{n+6}^2 & u_{n+8}^2 \end{vmatrix} = (-1)^{n+1} \, 18$$

which has also been proposed as a problem by Brother Alfred [2].

AN EXAMPLE INVOLVING A SEQUENCE OF POLYNOMIALS

Lorch and Moser [8] proposed that one prove

(20) $$\begin{vmatrix} v_n & v_{n+1} \\ v_{n+1} & v_{n+2} \end{vmatrix} = x \quad \text{for } n = 0, 1, 2, \ldots$$

where $v_0 = 1$ and

(21) $$v_n = \sum_{v=0}^{n} \binom{n+v}{n-v} x^v \quad \text{for } n = 1, 2, \ldots .$$

In proving (2), Carlitz [4] proved

$$v_{n+2} = (x+2)\, v_{n+1} - v_n \quad \text{for } n = 0, 1, 2, \ldots \quad ; \tag{22}$$

hence, we can prove (20) and obtain generalizations by using (3). For $k = 1$ and 2 we have respectively,

$$A_1(v_n) = A_1(v_0) = x \quad , \tag{23}$$

$$A_2(v_n) = A_2(v_0) = 2x^3(x+2)^2 \quad . \tag{24}$$

A second generalization of this problem was also given by Gould [6].

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A GENERALIZED LANGFORD PROBLEM

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Let $n > 1$ be an integer and consider the integers $1, 2, 3, \ldots, n$. The sequence $a_1, a_2, a_3, \ldots, a_{2n}$ is said to be a perfect sequence for n if each of the integers $1, 2, 3, \ldots, n$ occurs in the sequence exactly twice and the integer i is separated in the sequence by exactly i entries. For example, 1 7 1 2 6 4 2 5 3 7 4 6 3 5 is a perfect sequence for 7. C. D. Langford [2] posed the problem of determining all n having a perfect sequence. It was shown by C. J. Priday [3] and Roy O. Davies [1] that n has a perfect sequence if, and only if, n is of the form $4m - 1$ or $4m$. For $n = 3$, 3 1 2 1 3 2 is the only perfect sequence except for the same sequence in reverse order and for $n = 4$, 4 1 3 1 2 4 3 2 is the only perfect sequence except for the same sequence in reverse order. According to Davies there are 25 perfect sequences for 7. He stated the problem, as yet unsolved, of finding a function giving the number of perfect sequences for n of the form $4m - 1$ or $4m$.

In this note we define a generalized perfect s-sequence for the integer $n > 1$ to be a sequence of length sn in which each of the integers $1, 2, 3, \ldots, n$ occurs exactly s times and between any two occurences of the integer i there are i entries. Thus, a perfect sequence for n is a generalized perfect 2-sequence.

The authors are unable to discover an n for which there is a generalized perfect s-sequence for $s > 2$ and pose as a problem the determination of all s and n for which there are generalized perfect s-sequences for n.

The following partial result is given in case $s = 3$. The method of proof becomes tedious for large n but could be settled for any given n on a machine.

Theorem. <u>There is no generalized perfect 3-sequence for</u> $n = 2, 3, 4, 5, 6$.

Proof. The case $n = 2$ is trivial.

Consider the case $n = 3$. Assume that there is a generalized perfect 3-sequence for $n = 3$. Beginning with the first 3 in the sequence we must have $3, a_1, a_2, a_3, 3$. There are 9 elements in the sequence including another 3 hence the entire sequence must be of the form

$$3, a_1, a_2, a_3, 3, b_1, b_2, b_3, 3 \quad .$$

The first occurence of 2 in the sequence is at a_2 or a_3 hence either $a_2 = b_1 = 2$ or $a_3 = b_2 = 2$ but in neither case is there room for another 2 and so there is no generalized perfect 3-sequence for $n = 3$.

Now, let $n = 4$. If the desired sequence is possible, beginning with the first 4 in the sequence we have $4, a_1, a_2, a_3, a_4, 4, b_1, b_2, b_3, b_4, 4$. Because of the positions of the 4's, a_1, a_2, a_4, b_1, b_4 are not 3 hence $a_3 = b_2 = 3$ and the sequence either begins or ends with a 3. Consider the case

$$4, a_1, a_2, 3, a_4, 4, b_1, 3, b_3, b_4, 4, 3 \quad .$$

the alternate case is similar. Because of the spaces already occupied by the 3's and 4's it is not possible to put the 2's in the sequence and so $n = 4$ is impossible.

In case $n = 5$, we must have the subsequence

$$5, a_1, a_2, a_3, a_4, a_5, 5, b_1, b_2, b_3, b_4, b_5, 5$$

in the proposed sequence. It is obvious that a_1, a_2, a_5 cannot be a 4. If $a_3 = 4$, the sequence is

(1) $$4, c_1, 5, a_1, a_2, 4, a_4, a_5, 5, b_1, 4, b_3, b_4, b_5, 5$$

or has

(2) $$5, a_1, a_2, 4, a_4, a_5, 5, b_1, 4, b_3, b_4, b_5, 5, 4$$

as a subsequence. If $a_4 = 4$, there is a subsequence

(3) $$4, 5, a_1, a_2, a_3, 4, a_5, 5, b_1, b_2, 4, b_4, b_5, 5$$

or the entire sequence is

(4) $$5, a_1, a_2, a_3, 4, a_5, 5, b_1, b_2, 4, b_4, b_5, 5, d_1, 4$$

For sequence (1), it is clear that one must have $a_1 = a_5 = b_3 = 3$ hence $a_4 = b_1 = b_4 = 2$ but this is impossible since one cannot have $c_1 = a_2 = b_5 = 1$. The argument for sequence (2) is the same.

For the sequence (3), the only possible choices for 3 make $a_3 = b_1 = b_5 = 3$. This done, the only choices for 2 make $a_2 = a_5 = b_2 = 2$ but this requires that $a_1 = b_4 = 1$ which is impossible. The argument for the sequence (4) is the same and it is seen that the case $n = 5$ cannot occur.

The case $n = 6$ is treated similarly. The details are numerous and will be omitted.

The authors are indebted to the referee for the following theorem.

Theorem. There is no generalized perfect s-sequence for $n < s$.

Proof. There are s terms equal to n, and between each of the $s-1$ pairs of adjacent n's is an interval of length n. The total length, $s + n(s-1)$, must not be greater than sn, which implies $n \geq s$.

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A LOWER BOUND FOR MAXIMUM ZERO-ONE DETERMINANTS

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What is the largest possible determinant of order n, if zero and one are the only entries allowed? This question, last posed by Harary [1], seems difficult.

Williamson [2] obtained the values 1, 1, 2, 3, 5, 9 for $n = 1, 2, 3, 4, 5, 6$; and proved the general problem equivalent to a similar Hadamard question with allowed entries 1 and -1.

Cohn [3] derived an asymptotic lower bound,

$$(n+1)^{(\frac{1}{2} - \epsilon)(n+1)}/2^n ,$$

where ϵ is any positive number. The upper bound

$$(n+1)^{(n+1)/2}/2^n$$

follows from Hadamard's inequality [4], applied to the 1, -1 version of the problem.

Clements and Lindström [5] have announced the lower bound

$$(n+1)^K/2^n ,$$

where $K = (n+1)(1-(\log 4/3)/\log(n+1))/2$, and the logarithms are base two.

In this note, I show that the Fibonacci sequence 1, 1, 2, 3, 5, 8, ... is a lower bound for the sequences of maximum zero-one determinants. Also, I compare this bound with the Clements-Lindström bound.

Theorem: The maximum zero-one determinant of order n is at least as large as the n^{th} Fibonacci number.

This is proved by exhibiting zero-one matrices whose determinants are the Fibonacci numbers.

Let $a(n)$ be the row vector with n entries which are alternately one and zero, starting with one. Consider the n^{th} order matrix

$$F(n) = \begin{pmatrix} a(n) & & & & & \\ 1 & a(n-1) & & & & \\ 0 & 1 & \cdot & & & \\ \cdot & 0 & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ \cdot & \cdot & & & a(2) & \\ 0 & 0 & & & 1 & a(1) \end{pmatrix}$$

For example,

$$F(1) = (1), \;\; F(2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ and } \;\; F(3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Notice that det F(1) = 1, det F(2) = 1, and det F(3) = 2 .

Expanding det F(n) by the first column gives det F(n) =

$$\det\begin{pmatrix} a(n-1) & & & & \\ 1 & \cdot & & & \\ 0 & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & a(2) & \\ 0 & & & 1 & a(1) \end{pmatrix} - \det\begin{pmatrix} 0 & a(n-2) & & & \\ 1 & a(n-2) & & & \\ 0 & 1 & a(n-3) & & \\ \cdot & 0 & 1 & \cdot & \\ \cdot & \vdots & \cdot & a(2) & \\ 0 & 0 & 0 & 1 & a(1) \end{pmatrix}$$

$$= \det F(n-1) - (-1)\det\begin{pmatrix} a(n-2) & & & & & \\ 1 & a(n-3) & & & & \\ 0 & 1 & \cdot & & & \\ \cdot & 0 & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ \cdot & \cdot & & & a(2) & \\ 0 & 0 & & & 1 & a(1) \end{pmatrix}$$

= det F(n-1) + det F(n-2). Therefore, the sequence det F(n) is the Fibonacci sequence.

To compare this bound with the Clements-Lindström bound, examine the following table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
det F(n)	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$(n+1)^K/2^n$	.8	.9	1.1	1.7	2.8	5.2	10.1	21.1	46.3	107.2	259.5	654.9	1,717.7	4,669	13,122

If n is greater than 8, the Clements-Lindström bound is better.

For special n, still better bounds can be found. One of Cohn's inequalities [3] becomes, for zero-one determinants,

$$M(mn-1) \geq 2^{(m-1)(n-1)} [M(m-1)]^n [M(n-1)]^m \quad ,$$

where M(i) is the maximum determinant of order i. If mn-1 = 14 and mn-1 = 15, then

$$M(14) \geq 6,912, \quad \text{and}$$
$$M(15) \geq 131,072 \quad .$$

The numbers in the table above were bought from Diane K. Middents for 2 palindromes.

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6. Don Walters, American Mathematical Monthly, June-July 1949 Problem E-834.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. Hillman
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-88 *Proposed by John Wessner, Melbourne, Florida*

Let $L_0, L_2, L_4, L_6, \ldots$ be the Lucas numbers $2, 3, 7, 18, \ldots$. Show that

$$L_{2k} \equiv 2(-1)^k \pmod 5 .$$

B-89 *Proposed by Robert S. Seamons, Yakima Valley College, Yakima, Washington*

Let F_n and L_n be the n-th Fibonacci and n-th Lucas number, respectively. Let $[x]$ be the greatest integer function. Show that $L_{2m} = 1 + [\sqrt{5}\, F_{2m}]$ for all positive integers m.

B-90 *Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico*

Let $b_1, b_2, \ldots$ be the sequence $3, 7, 47, \ldots$ with recurrence relation $b_{n+1} = b_n^2 - 2$. Show that the roots of

$$x^2 - 2b_n x + 4 = 0$$

are expressible in the form $c + d\sqrt{5}$, where c and d are integers.

B-91 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia*

If F_n is the n-th Fibonacci number, show that

$$\sum_{j=1}^{\infty} (1/F_j)$$

converges while

$$\sum_{j=3}^{\infty} (1/\ln F_j)$$

diverges.

B-92 *Proposed by J.L. Brown, Jr., The Pennsylvania State University*

Let (x, y) denote the g.c.d. of positive integers x and y. Show that $(F_m, F_n) = (F_m, F_{m+n}) = (F_n, F_{m+n})$ for all positive integers m and n.

B-93 *Proposed by Martin Pettet, Toronto, Ontario, Canada*

Show that if n is a positive prime, $L_n \equiv 1 \pmod n$. Is the converse true?

SOLUTIONS

The deadlines for submitting solutions to the Elementary Problems Section have proved to be unrealistic and are being changed as of this issue. We take this opportunity to list some errata and some of the solvers whose solutions were received after copy for this section went into production.

ERRATA

B-33 The solution printed on page 235, Vol. 2, No. 3, was submitted by Charles R. Wall, Texas Christian University, Ft. Worth, Texas. This problem was also solved by John H. Halton, B. Litvack, and the proposer.

B-39 In the note after the solution to B-39 on page 327, Vol. 2, No. 4, the reference to the inequality $a^{n-1} < F_n < a^n$ for $n > 1$ in An Introduction to the Theory of Numbers, by Niven and Zuckerman, should state that their initial conditions on the F_n are $F_1 = 0$ and $F_2 = 1$.

B-57 On the next to last line of the solution on page 160, Vol. 3, No. 2, the second "=" sign should obviously be a ">" sign.

SOLUTIONS RECEIVED AFTER PRODUCTION DATE FOR PROBLEM SECTION

Problem	Solvers
B-17	Ken Siler
B-19	F. D. Parker
B-20	S. L. Basin
B-22	Ken Siler
B-24	Sr. Mary De Sales McNabb, Charles R. Wall
B-25	Douglas Lind, Charles R. Wall
B-26	Gurdial Singh
B-29	Douglas Lind
B-30	Douglas Lind
B-35	Denis Hanson
B-38	Brian Scott
B-44	J. L. Brown, Jr.
B-45	J. L. Brown, Jr.
B-46	Clyde A. Bridger; C. A. Church, Jr.; Kenneth E. Newcomer; Charles R. Wall
B-47	J. L. Brown, Jr.; Charles R. Wall
B-48	J. L. Brown, Jr.; Douglas Lind; Charles R. Wall
B-49	Douglas Lind, Charles R. Wall
B-50	J. L. Brown, Jr.; C. B. A. Peck; Charles R. Wall
B-52	M. N. S. Swamy, Charles R. Wall
B-53	M. N. S. Swamy, Charles R. Wall
B-54	M. N. S. Swamy, Charles R. Wall
B-55	Howard L. Walton
B-70	Dermott A. Breault; James E. Desmond; John E. Homer, Jr.; George Ledin, Jr.; C. B. A. Peck; Dean B. Priest
B-71	Dermott A. Breault; J. L. Brown, Jr.; John E. Homer, Jr; George Ledin, Jr.; C. B. A. Peck; Dean B. Priest
B-72	James E. Desmond; George Ledin, Jr.
B-74	Douglas Lind
B-75	J. L. Brown, Jr.; Douglas Lind.

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Corrected

B-86 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that the squares of every third Fibonacci number satisfy

$$y_{n+3} - 17y_{n+2} - 17y_{n+1} + y_n = 0 \quad .$$