REPRESENTATIONS OF N AS A SUM OF DISTINCT ELEMENTS FROM SPECIAL SEQUENCES

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1. INTRODUCTION

Let $\{a_k\}$ denote a sequence of natural numbers which satisfies the difference equation $a_{k+2} = a_{k+1} + a_k$ for $k = 1,2,\cdots$. It is easy to prove by induction that $a_1 + a_2 + \cdots + a_n = a_{n+2} - a_2$ for $n = 1,2,\cdots$; we use this fact in defining

(1) $$P(x) = \prod_{k=1}^{\infty}\left(1 + x^{a_k}\right) = \sum_{k=0}^{\infty} A(k)\,x^k$$

and

(2) $$P_n(x) = \prod_{k=1}^{n}\left(1 + x^{a_k}\right) = \sum_{k=0}^{a_{n+2}-a_2} A_n(k)\,x^k \quad .$$

It follows from these definitions that $A(k)$ enumerates the number of representations

(3) $$a_{i_1} + a_{i_2} + \cdots + a_{i_j} = k \qquad \text{with} \qquad 0 < i_1 < \cdots < i_j \ ,$$

and that $A_n(k)$ enumerates the number of these representations with $i_j \le n$.

Hoggatt and Basin [9] found recurrence formulae satisfied by $\{A_n(k)\}$ and $\{A(k)\}$ when $\{a_n\}$ is the Fibonacci sequence; in Section 2 we give generalizations of these results.

Hoggatt and King [10] defined a complete sequence of natural numbers $\{a_n\}$ as one for which $A(n) > 0$ for $n = 1,2,\cdots$ and found that (i) $\{F_n\}$ is complete, (ii) $\{F_n\}$ with any term deleted is complete, and (iii) $\{F_n\}$ with any two terms deleted is not complete. Brown [1] gave a simple necessary and sufficient condition for completeness of an arbitrary sequence of natural numbers and showed that the Fibonacci sequence is characterized by properties (ii) and (iii) already mentioned. Zeckendorf [13] showed that if F_1 is deleted from the Fibonacci sequence, then the resulting sequence has the property that every natural number has exactly one representation as a sum of elements from this sequence whose subscripts differ by at least two. Brown [2] has given an exposition of this paper and Daykin [4] showed that the Fibonacci sequence is the only sequence with the properties mentioned in Zeckendorf's Theorem. More on the subject of Zeckendorf's Theorem can be found in another excellent paper by Brown [3]. Ferns [5], Lafer [11], and Lafer and Long [12] have discussed various aspects of the problem of representing numbers as sums of Fibonacci numbers. Graham [6] has investigated completeness properties of $\{F_n + (-1)^n\}$ and proved that every sufficiently large number is a sum of distinct elements of this sequence even after any finite subset has been deleted.

In Section 3 we take up the problem of determining the magnitude of $A(n)$ when $\{a_n\}$ is the Fibonacci sequence; in this case we write $A(n) = R(n)$. Hoggatt [7] proposed that it be shown that $R(F_{2n} - 1) = n$ and that $R(N) > n$ if $N > F_{2n} - 1$. We will show that

$$R(F_n - 1) = \left[\frac{n+1}{2}\right] ,$$

and that $F_n \le N \le F_{n+1} - 1$ implies

$$\left[\frac{n+2}{2}\right] \le R(N) \le 2\,F_{(n+1)/2}$$

if n is odd and

$$\left[\frac{n+2}{2}\right] \le R(N) \le F_{(n+4)/2}$$

if n is even.

In Section 4 we investigate the number of representations of k as a sum of distinct Fibonacci numbers, writing $a_n = F_{n+1}$ and $T(n)$ for $A(n)$ in this case. The behavior of the function $T(n)$ is somewhat different from that of

R(n) of Section 3. For example, we show that there exist infinitely many n for which $T(n) = k$ for a fixed k, and in particular we find the solution sets for each of the equations $T(n) = 1$, $T(n) = 2$, $T(n) = 3$. By definition $T(n) \leq R(n)$ so that $T(N) \leq n - 1$ if $N \leq F_{n+1} - 1$. We show that

$$T(F_{n+1}) = \left[\frac{n+1}{2}\right]$$

and $T(F_{n+1} + 1) = [n/2]$ for $n = 3, 4, \cdots$.

Hoggatt [8] proposed that one show that M(n), the number of natural numbers less than n which cannot be expressed as a sum of distinct Lucas numbers $L_n (L_1 = 1,\ L_2 = 3,\ L_{n+2} = L_{n+1} + L_n)$ has the property $M(L_n) = F_{n-1}$; also, he asked for a formula for M(n). In Section 5, we give a solution to the same question involving any incomplete sequence satisfying $a_{n+2} = a_{n+1} + a_n$ with $a_1 < a_2 < \cdots$. In a paper now in preparation we have shown that the only complete sequences of natural numbers which satisfy the Fibonacci recurrence are those with initial terms (i) $a_1 = a_2 = 1$, (ii) $a_1 = 1$, $a_2 = 2$, or (iii) $a_1 = 2$, $a_2 = 1$.

2. RECURRENCE RELATIONS

See Section 1 for definitions and notation.

Lemma 1. $A_n(k) = A_n(a_{n+2} - a_2 - k)$ for $k = 0, 1, \cdots, n$.

Proof. Using the product notation for P_n we see

$$x^{a_{n+2}-a_2} P_n\left(\frac{1}{x}\right) = P_n(x) \ . \tag{4}$$

The symmetric property of A_n now follows on equating coefficients of the powers of x in (4).

Lemma 2.

(a) $A_{n+1}(k) = A_n(k)$ if $0 \leq k \leq a_{n+1} - 1$.

(b) $A_{n+1}(k) = A_n(k) + A_n(k - a_{n+1})$ if $a_{n+1} \leq k \leq a_{n+2} - a_2$.

(c) $A_{n+1}(k) = A_n(k - a_{n+1})$ if $a_{n+2} - a_2 + 1 \leq k \leq a_{n+3} - a_2$.

Proof. Each of these statements is obtained by equating coefficients of x^k in the identity

(5) $$P_{n+1}(x) = \left(1 + x^{a_{n+1}}\right) P_n(x) \; .$$

Lemma 3.

(a) $A_{n+1}(k) = A(k)$ if $0 \leq k \leq a_{n+2} - 1$.

(b) $A_{n+1}(k) = A(a_{n+2} - a_2 - k) + A(k - a_{n+1})$ if $a_{n+1} \leq k \leq a_{n+2} - a_2$.

(c) $A_{n+1}(k) = A(a_{n+3} - a_2 - k)$ if $a_{n+2} - a_2 + 1 \leq k \leq a_{n+3} - a_2$.

Proof. (a) This follows by induction on part (a) of Lemma 2.

(c) Using Lemma 1 we have $A_{n+1}(k) = A_{n+1}(a_{n+3} - a_2 - k)$ and assuming $a_{n+2} - a_2 + 1 \leq k \leq a_{n+3} - a_2$ we have $0 \leq a_{n+3} - a_2 - k \leq a_{n+1} - 1$, so that we can apply (a) of this lemma to get $A_{n+1}(a_{n+3} - a_2 - k) = A(a_{n+3} - a_2 - k)$ for k in the range under consideration and this is (c).

(b) Statement (b) of Lemma 2 asserts $A_{n+1}(k) = A_n(k) + A_n(k - a_{n+1})$ for $A_{n+1} \leq k \leq a_{n+2} - a_2$; but by (c) of this lemma we have $A_n(k) = A(a_{n+2} - a_2 - k)$ for k in the range under consideration. Also, if $a_{n+1} \leq k \leq a_{n+2} - a_2$ we have $0 \leq k - a_{n+1} - a_2$, so by (a) of this lemma we have $A_n(k - a_{n+1}) = A(k - a_{n+1})$. Combining these results gives part b.

Lemma 4.

(a) $A(k) = A(a_{n+2} - a_2 - k) + A(k - a_{n+1})$ if $a_{n+1} \leq k \leq a_{n+2} - a_2$ and $n = 2, 3, \cdots$.

(b) $A(k) = A(a_{n+3} - a_2 - k)$ if $a_{n+2} - a_2 + 1 \leq k < a_{n+2} - 1$ and $n = 2, 3, \cdots$.

(c) $A(a_{n+2} - a_2 + k) = A(a_n - a_2 + k)$ if $1 \leq k \leq a_2 - 1$.

(Note that in (b) and (c) the range of k is the empty set unless $a_2 \geq 2$.)

Proof.

(a) This is merely a combination of (a) and (b) in Lemma 3.

(b) If $a_{n+2} - a_2 + 1 \leq k \leq a_{n+2} - 1$, then $a_{n+1} - a_2 + 1 \leq a_{n+3} - a_2 - k \leq a_{n+1} - 1$, so that by (a) of Lemma 3, $A(a_{n+3} - a_2 - k) = A_{n+1}(a_{n+3} - a_2 - k)$. By Lemma 1, $A_{n+1}(a_{n+3} - a_2 - k) = A_{n+1}(k)$ and using (a) of Lemma 3 again we see that $A_{n+1}(k) = A(k)$ for k in the proposed range.

(c) Writing $k = a_{n+2} - a_2 + j$ with $1 \leq j \leq a_2 - 1$ in (b) we get

(6) $$A(a_{n+2} - a_2 + j) = A(a_{n+3} - a_2 - a_{n+2} + a_2 - j) = A(a_{n+1} - j) \; ;$$

but $a_{n+1} - j = a_{n+1} - a_2 + (a_2 - j)$ where $1 \leq a_2 - j \leq a_2 - 1$ so that we can use (6) to obtain

$$A(a_{n+1} - j) = A(a_{n+1} - a_2 + (a_2 - j)) = A(a_n - (a_2 - j)) \quad . \tag{7}$$

Combining (6) and (7) we obtain (c).

Lemma 5. $A(a_{n+1} + j) = A(a_{n+2} - a_2 - j)$ for $0 \leq j \leq a_n - a_2$ and $n = 2, 3, \cdots$.

Proof. For j in the range under consideration we have $a_{n+1} \leq a_{n+1} + j \leq a_{n+2} - a_2$ so that by (a) of Lemma 4 we have

$$\begin{aligned} A(a_{n+1} + j) &= A(a_{n+2} - a_2 - a_{n+1} - j) + A(a_{n+1} + j - a_{n+1}) \\ &= A(a_n - a_2 - j) + A(j) \quad . \end{aligned} \tag{8}$$

But we also have $a_{n+1} \leq a_{n+2} - a_2 - j \leq a_{n+2} - a_2$ for the assumed range of j, so that we can apply Lemma 4 again to write

$$\begin{aligned} A(a_{n+2} - a_2 - j) &= A(a_{n+2} - a_2 - a_{n+2} + a_2 + j) + A(a_{n+2} - a_2 - j \\ &\quad - a_{n+1}) = A(j) + A(a_n - a_2 - j) \quad . \end{aligned} \tag{9}$$

Since the right members of (8) and (9) are the same, so are the left members.

Using Lemmas 4 and 5 it is not hard to calculate $A(k)$ for a given sequence $\{a_n\}$. Of particular interest to us are the cases when $\{a_n\}$ is the Fibonacci sequence, the Fibonacci sequence with the first term deleted, and the Lucas sequence; we write $A(k) = R(k)$, $T(k)$ and $S(k)$ respectively in these cases. A table is provided for each of these functions in order to illustrate some of our results.

3. SOME PROPERTIES OF R(k)

In light of Lemma 4, it is natural to consider the behavior of $R(k)$ in the intervals $[F_n, F_{n+1} - 1]$; thus, as a matter of convenience we write

$$I_n = \{R(F_n), R(F_n + 1), \cdots, R(F_{n+1} - 1)\} \tag{10}$$

and note that Lemma 4 implies

(11) $I_{n+1} = \{R(0) + R(F_n - 1),\ R(1) + R(F_n - 2),\ \cdots,\ R(F_n - 1) + R(0)\}.$

As we mentioned in the introduction, Hoggatt has proposed that one prove $R(F_{2n} - 1) = n$ and that $R(k) > R(F_{2n} - 1)$ if $k > F_{2n} - 1$. This problem has led us to prove a result involving special values of $R(k)$ and to find the maximum and minimum of $R(k)$ in I_n.

Theorem 1.

(a) $R(F_n) = \left[\frac{n+2}{2}\right]$ for $n > 1$,

(b) $R(F_n - 1) = \left[\frac{n+1}{2}\right]$ for $n > 0$,

(c) $R(F_n - 2) = n - 2$ for $n > 2$,

(d) $R(F_n - 3) = n - 3$ for $n > 4$.

Proof. We prove only (b) (the other proofs are analogous) which implies the first part of Hoggatt's proposal. First, we observe that (b) is true for small values of n by consulting Table 1. Next, suppose

$$R(F_t - 1) = \left[\frac{t+1}{2}\right] \text{ for } t = n \text{ and } n+1$$

and take $k = F_{n+2} - 1$ in (a) of Lemma 4 to obtain

$$R(F_{n+2} - 1) = R(0) + R(F_n - 1) = 1 + \left[\frac{n+1}{2}\right] = \left[\frac{n+3}{2}\right] . \tag{12}$$

Thus, the assertion follows by induction on n.

Theorem 2.

$$R(F_n) = \left[\frac{n+2}{2}\right]$$

is a minimum of $R(k)$ in I_n.

Proof. We can verify the theorem for small values of n by inspection in Table 1. Suppose the theorem holds for all $n \leq N - 1$. We know by Theorem 1 that

$$R(F_n) = \left[\frac{n+2}{2}\right]$$

so that we are assuming

(13) $\left[\frac{n+2}{2}\right] = R(F_n) \leq R(k)$ for $F_n \leq k \leq F_{n+1} - 1$ and $n = 1, 2, \cdots, N - 1$.

Now suppose $F_N \leq k \leq F_{N+1} - 1$ and write $n = N - 1$ in (a) of Lemma 4 to obtain

(14) $$R(k) = R(F_{N+1} - 1 - k) + R(k - F_N) ;$$

but $F_N \leq k \leq F_{N+1} - 1$ implies $0 \leq F_{N+1} - 1 - k \leq F_{N-1} - 1$ and $0 \leq k - F_N \leq F_{N-1} - 1$. Suppose

(15) $$F_t \leq F_{N+1} - 1 - k \leq F_{t+1} - 1 ,$$

where of course $F_{t+1} - 1 \leq F_{N-1} - 1$ or $0 \leq t \leq N - 2$ (we are taking $F_0 = 0$). Now

(16) $$F_N - F_{t+1} \leq k + F_N - F_{N-1} \leq F_N - F_t - 1 ,$$

but with $0 \leq t \leq N - 2$ we must have $F_{N-2} \leq F_N - F_{t+1}$ and $F_N - F_t - 1 \leq F_{N-1} - 1$ so that evidently

(17) $$F_{N-2} \leq k - F_N \leq F_{N-1} - 1 .$$

Using (16) and (17) along with (13) we have

(18) $$\left[\frac{N}{2}\right] \leq R(k - F_N)$$

and

$$1 \le \left[\frac{t+2}{2}\right] \le R(F_{N+1} - 1 - k) \text{ since } t \ge 0 . \tag{19}$$

Combining (18) and (19) in (14) gives

$$R(k) \ge \left[\frac{N}{2}\right] + 1 = \left[\frac{N+2}{2}\right] \tag{20}$$

for $F_N \le k \le F_{N+1} - 1$. Hence the theorem follows by induction on N.

Corollary. $R(k) > R(F_{2n} - 1) = n$ if $k > F_{2n} - 1$.

Proof. We know from Theorem 2 that the minimum value of $R(k)$ in I_{2n} and I_{2n+1} is $n+1$ in each of them; hence the minimum of $R(k)$ in $I_{2n} \cup I_{2n+1}$ is $n+1$. Thus, every value of $R(k)$ in $I_{2n+2} \cup I_{2n+3}$ is at least $n+2$ so that we can conclude by induction on n that $R(k) > R(F_{2n} - 1)$ if $k > F_{2n} - 1$.

Theorem 3. The maximum of $R(k)$ in I_{2n} is F_{n+2} and the maximum of $R(k)$ in I_{2n+1} is $2F_{n+1}$ for $n = 1, 2, \cdots$; also,

$$F_3 \le 2F_2 < F_4 < 2F_3 < \cdots < F_{n+2} < 2F_{n+1} < F_{n+3} < \cdots \quad \text{for } n = 2, 3, \cdots . \tag{21}$$

Proof. The result in (21) follows by a simple induction.

The results concerning the maximum values of $R(k)$ in I_{2n} and I_{2n+1} can be verified for small n by using Table 1. Suppose these results hold for all $n \le N$; then we have by (a) of Lemma 4,

$$R(F_{n+1} + t) = R(F_n - t - 1) + R(t) \quad \text{for } 0 \le t \le F_n - 1 . \tag{22}$$

Also, we know by (b) of Lemma 4 that $R(k)$ is symmetric in I_{n+1}, so it is enough to consider the values of only the first half of the elements of I_{n+1} in order to determine the maximum elements. More than the first half of the elements of I_{n+1} are contained in the sets

$$\{R(F_{n+1} + t) \mid t = 0, 1, \cdots, F_{n-1} - 1\} \text{ and } \{R(F_{n+1} + t) \mid t = F_{n-1}, \cdots, F_n - 1\}. \tag{23}$$

Consider first the maximum of the first of the two sets in (23); evidently,

$$
\begin{aligned}
\max_{0\le t\le F_{n-1}-1} R(F_{n+1}+t) &= \max_{0\le t\le F_{n-1}-1} \{R(F_n - t - 1) + R(t)\} \\
&\le \max_{0\le t\le F_{n-1}-1} R(F_n - t - 1) + \max_{0\le t\le F_{n-1}-1} R(t) \qquad (24)\\
&= 2 \max I_{n-2} .
\end{aligned}
$$

Next, we have for the second set in (23)

$$
\begin{aligned}
&\max_{F_{n-1}\le t\le F_n-1} R(F_{n+1}+t) \\
&\qquad \le \max_{F_{n-1}\le t\le F_n-1} R(F_n - t - 1) + \max_{F_{n-1}\le t\le F_n-1} R(t) \qquad (25)\\
&\qquad = \max I_{n-3} + \max I_{n-1} .
\end{aligned}
$$

Together (24) and (25) imply

$$\max I_{n+1} \le \max \{\max I_{n-1} + \max I_{n-3},\ 2 \max I_{n-2}\} . \tag{26}$$

Writing $n = 2N + 1$ in (26) and applying the induction hypothesis we have

$$\max I_{2N+2} \le \max \{F_{N+3},\ 4F_N\} = F_{N+3} ; \tag{27}$$

similarly, $n = 2N + 2$ in (26) gives

$$\max I_{2N+3} \le \max \{2F_{N+2},\ 2F_{N+2}\} = 2F_{N+2} . \tag{28}$$

In order to finish the proof of Theorem 3 we need to show that $F_{N+3} \in I_{2N+2}$ and $2F_{N+2} \in I_{2N+3}$.

Since $0 \le F_{2N} + t \le F_{2N+3} - 1$ for $t = 0, 1, \cdots, F_{2N-1} - 1$, we can use (22) and (*b*) of Lemma 4 to find

$$R(F_{2N+3} + F_{2N} + t) = R(F_{2N+1} - t - 1) + R(F_{2N} + t) = 2R(F_{2N} + t) , \tag{29}$$

for $t = 0, 1, \cdots, F_{2N-1} - 1$. From this we gather that all of the elements of I_{2N} multiplied by 2 occur in I_{2N+3}; hence, twice the maximum in I_{2N} is in I_{2N+3} and this is precisely $2F_{N+2}$.

It is not so obvious that $F_{N+3} \in I_{2N+2}$; to prove this we let λ_n denote an integer such that $R(F_{2N} + \lambda_n) = F_{n+2}$ for $n \le N$. We will also include in our induction hypothesis that an admissible value of λ_{n+1} for $n < N$ is given by $\lambda_{n+1} = F_{2n-1} - \lambda_n - 1$. Now consider

$$
\begin{aligned}
&R(F_{2N+2} + F_{2N-1} - \lambda_N - 1) \\
&\qquad = R(F_{2N+1} - F_{2N-1} + \lambda_N) + R(F_{2N-1} - \lambda_N - 1) \\
&\qquad = R(F_{2N} + \lambda_N) + R(F_{2N-2} + \lambda_{N-1}) \\
&\qquad = F_{N+2} + F_{N+1} = F_{N+3} .
\end{aligned} \tag{30}
$$

The second equality in (30) follows from (22). It is now clear that an admissible value for λ_{N+1} is $F_{2N-1} - \lambda_N - 1$ and that $F_{N+3} \in I_{2N+2}$. This completes the proof of Theorem 3.

4. T(n), THE NUMBER OF REPRESENTATIONS OF n AS A SUM OF DISTINCT FIBONACCI NUMBERS

For the moment we are taking $a_n = F_{n+1}$ in the lemmas of Section 2 and write $A(k) = T(k)$ in this case. The following theorem can be proved in the same way we proved Theorem 1, so we leave out the proof.

Theorem 4.

(a) $T(F_{n+1}) = \left[\frac{n+1}{2}\right]$ if $n = 1, 2, \cdots$.

(b) $T(F_{n+1} + 1) = \left[\frac{n}{2}\right]$ if $n = 3, 4, \cdots$.

Theorem 5.

(a) $T(N) = 1$ if and only if $N = F_{n+1} - 1$ for $n = 1, 2, \cdots$.

(b) $T(N) = 2$ if and only if $N = F_{n+3} + F_n - 1$ or $F_{n+4} - F_n - 1$ for $n = 1, 2, \cdots$.

(c) $T(N) > 0$ if $N \ge 0$.

(d) $T(N) = 3$ if and only if $N = F_{n+5} + F_n - 1,\ F_{n+5} + F_{n+1} - 1,\ F_{n+6} - F_n - 1,\ F_{n+6} - F_{n+1} - 1$ for $n = 1, 2, \cdots$.

Proof. (a) and (c): We can check Table 2 to see that $T(F_{n+1} - 1) = 1$ if $n = 1, 2, 3, 4$. Suppose $T(F_{n+1} - 1) = 1$ for all n less than $N > 4$. Then by (c) of Lemma 4 we have $T(F_N - F_3 + 1) = T(F_N - 1) = T(F_{N-3} - 1)$ which is 1 by assumption. Next, the table shows that the only values of $N < F_5$ for which $T(N) = 1$ are $N = F_2 - 1, F_3 - 1, F_4 - 1$ and $F_5 - 1$. Suppose for all $4 \leq n < N$, where $N > 5$, that $F_n \leq k < F_{n+1} - 1$ implies $T(k) > 1$. Then by (a) of Lemma 4 we have for $F_N \leq k < F_{N+1} - 1$, $T(k) = T(F_{N+1} - F_3 - k) + T(k - F_N) \geq 2$. This completes the proofs of both (a) and (c).

(b) By Lemma 5, we have $T(F_{n+3} + F_n - 1) = T(F_{n+4} - F_n - F_3 + 1)$, and since $F_{n+3} \leq F_{n+3} + F_n - 1 \leq F_{n+4} - F_3$ we can apply (a) of Lemma 4 to get $T(F_{n+3} + F_n - 1) = T(F_{n+4} - F_3 - F_{n+3} - F_n + 1) + T(F_{n+3} + F_n - 1 - F_{n+3})$ $= T(F_{n+1} - 1) + T(F_n - 1)$. By (a) of this lemma, the last sum is 2. To prove the "only if" part of (c), we use induction with (a) of Lemma 4 just as in the proof of the "only if" part of (a).

(d) The proof can be given using induction and (a) of Lemma 4 just as (a) and (b) were proved.

Theorem 6. For every natural number k there exist infinitely many N such that N has exactly k representations as a sum of distinct Fibonacci numbers, in fact,

$$T(F_{n+k+2} + 2F_{n+2} - 1) = k \quad \text{for } n = 1, 2, \cdots \text{ and } k = 4, 5, \cdots . \tag{31}$$

Proof. The theorem is true for $k = 1, 2, 3$, by (a), (b), and (d) of Theorem 5. We will verify the theorem for $k = 4$ and leave the verification for $k = 5$ as an exercise.

Since $F_{n+6} \leq F_{n+6} + 2F_{n+2} - 1 \leq F_{n+7} - F_3$ we can apply (a) of Lemma 4 to obtain

$$\begin{aligned} T(F_{n+6} + 2F_{n+2} - 1) &= T(F_{n+7} - F_3 - F_{n+6} - 2F_{n+2} + 1) \\ &\quad + T(F_{n+6} + 2F_{n+2} - 1 - F_{n+6}) \\ &= T(F_{n+5} - 2F_{n+2} - 1) + T(2F_{n+2} - 1) ; \end{aligned} \tag{32}$$

however, $2F_{n+2} = F_{n+2} + F_{n+1} + F_n = F_{n+3} + F_n$ so that

$$T(2F_{n+2} - 1) = T(F_{n+3} + F_n - 1) = 2, \tag{33}$$

$$T(F_{n+5} - 2F_{n+2} - 1) = T(F_{n+4} - F_n - 1) = 2 \tag{34}$$

by (b) of Theorem 5; combining (33) and (34) in the last member of (32) gives the desired result.

Now suppose (31) holds for all $k < N$ where $N > 5$. Since $F_{n+N+2} \le F_{n+N+2} + 2F_{n+2} - 1 \le F_{n+N+3} - F_3$, we can use (a) of Lemma 4 to obtain

$$\begin{aligned} T(F_{n+N+2} + 2F_{n+2} - 1) &= T(F_{n+N+3} - F_3 - F_{n+N+2} - 2F_{n+2} + 1) \\ &\quad + T(F_{n+N+2} + 2F_{n+2} - 1 - F_{n+N+2}) \\ &= T(F_{n+N+1} - 2F_{n+2} - 1) + T(2F_{n+2} - 1). \end{aligned} \tag{35}$$

Since $0 < 2F_{n+2} - 1 \le F_{n+N+1} - F_3$ we can use Lemma 5 to write

$$\begin{aligned} T(F_{n+N+1} - F_3 - 2F_{n+2} + 1) &= T(F_{n+N+1} - 2F_{n+2} - 1) \\ &= T(F_{n+N} + 2F_{n+2} - 1) \;; \end{aligned} \tag{36}$$

but, this last quantity is $n - 2$ by assumption and recalling (33) we see that the sum in the last member of (35) is $(N - 2) + 2 = N$. This concludes the proof.

5. INCOMPLETE SEQUENCES

In what follows, $N(n)$ denotes the number of non-negative integers $k \le n$ for which $A(k) = 0$.

<u>Lemma 6</u>. Let $0 < v_1 < v_2 < \cdots$ denote the sequence of numbers k for which $A(k) = 0$ and suppose $v_{t+1}, v_{t+2}, \cdots, v_{t+s}$ is a complete listing of the v's between a_n and $a_n + k + j \le a_{n+1}$ for $n \ge 2$; then $v_{t+j} = a_n + v_j$ for $j = 1, 2, \cdots, s$ and v_s is the largest v not exceeding $k + 1$.

<u>Proof.</u> The lemma can be verified for $n = 2$ and 3 by determining $A(k)$ for $0 \le k \le a_{4-1}$ using $P_4(x)$, since by (a) of Lemma 3 we have $A(k) = A_4(k)$ for k in the supposed range.

Suppose for some $N \ge 3$ that the v_i's between a_n and a_{n+1} are given by $a_n + v_1, a_n + v_2, \cdots, a_n + v_\ell$ where v_ℓ is the largest v_i not exceeding a_{n-1} and $n \le N$. We will show that this implies the v_i between a_N and $a_N + k < a_{N+1}$ are given by $a_N + v_1, a_N + v_2, \cdots, a_N + v_s$ where v_s is the largest v_i not exceeding $k + 1$.

Case 1. Let $a_N < a_N + v_j \le a_{N+1} - a_2$, then by (a) of Lemma 4 we have

$$
\begin{aligned}
A(a_N + v_j) &= A(a_{N+1} - a_2 - a_N - v_j) + A(a_N + v_j - a_N) \\
&= A(a_{N-1} - a_2 - v_j) + A(v_j) \qquad (37) \\
&= A(a_{N-1} - a_2 - v_j) \;;
\end{aligned}
$$

but for $a_N + v_j$ in the range being considered we have $0 \le v_j \le a_{N-1} - a_2$ so that by Lemma 5

$$A(a_{N-1} - a_2 - v_j) = A(a_{N-2} + v_j) \tag{38}$$

and the right member is zero by assumption, so that $A(a_N + v_j) = 0$ is a consequence.

Now suppose there is a t not a v_i such that $a_N \le a_N + t \le a_{N+1} - a_2$ and $A(a_N + t) = 0$; then by (a) of Lemma 4 we would have

$$A(a_N + t) = A(a_{N-1} - a_2 - t) + A(t) \quad . \tag{39}$$

But this is a contradiction since $A(t) \neq 0$ (t is not a v_i) and we assumed $A(a_N + t) = 0$.

Thus $a_N + v_1, a_N + v_2, \cdots, a_N + v_s \le a_N + k \le a_{N+1} - a_2$ is a complete listing of the v_j between a_N and $a_N + k \le a_{N+1} - a_2$.

Case 2. Let $a_{N+1} - a_2 < a_N + v_j \le a_{N+1}$, then by (c) of Lemma 4 we have

$$A(a_N + v_j) = A(a_{N-2} + v_j) \tag{40}$$

which is zero by assumption. If we suppose there is a t such that t is not a v_i and $a_{N+1} - a_2 < a_N + t < a_{N+1}$ implies $A(a_N + t) = 0$, we obtain a contradiction since $A(a_N + t) = A(a_{N-2} + t) = 0$ would imply t is a v_i.

Thus, $a_N + v_j, \cdots, a_N + v_\ell$, with v_j the smallest v_i not less than a_{N-1}, comprises a complete listing of the v_i between $a_{N+1} - a_2$ and a_{N+1}.

Taken together, the results proved in Cases 1 and 2 imply Lemma 6 by induction.

Corollary. If $A(k) > 0$ for $k \le a_2$, then $\{a_n\}$ is complete; this is equivalent to saying $(a_1, a_2) = (2,1), (1,2)$ or $(1,1)$.

Proof. This follows from Lemma 6 and induction. Also, note that if $\{a_n\}$ is not complete, then there exist infinitely many k such that $A(k) = 0$.

Lemma 7.

(a) $N(a_n + k) = N(a_n) + N(k)$ if $0 \le k \le a_{n-1}$ and $n = 2, 3, 4, \cdots$.

(b) $N(k) = k$ if $0 \le k < a_1$.

(c) $N(k) = k - 1$ if $a_1 \le k < a_2$.

(d) $N(k) = k - 2$ if $a_2 \le k \le a_3$.

(e) $N(a_n - 1) = N(a_n)$ if $n = 1, 2, \cdots$.

Proof. (a) Suppose $n > 2$, then by Lemma 6, the v_i such that $a_n < v_i \le a_n + k$ with $0 \le k \le a_{n-1}$ are given by $a_n + v_1, a_n + v_2, \cdots, a_n + v_j$, where v_j is the smallest v_i not exceeding k. Hence there are $N(k)$ v_i in the supposed range. By definition the number of $v_i \le a_n$ is $N(a_n)$ so $N(a_n + k) = N(a_n) + N(k)$.

(b) (c) (d) follow from the fact that $A(k) \neq 0$ with $k < a_3$ only if $k = 0, a_1, a_2$.

(e) Since a_n is never a v_i, $N(a_n - 1) = N(a_n)$.

Lemma 8. $N(a_1) = a_1 - 1$, $N(a_2) = a_2 - 2$, $N(a_3) = a_3 - 3$ and $N(a_{n+1}) = N(a_n) + N(a_{n-1})$ if $n = 3, 4, \cdots$.

Proof. $N(a_1) = N(a_1 - 1) = a_1 - 1$ by (e) and (b) of Lemma 7 respectively; the second and third statements follow by (e) and (c) and (e) and (d) of the same lemma respectively. The last statement follows by writing $k = a_{n-1}$ in (a) of Lemma 7.

Lemma 9. $N(a_n) = a_n - F_{n+1}$ if $n = 1, 2, \cdots$ and F_n denotes the n^{th} Fibonacci number.

Proof. The statement is clearly true for $n = 1, 2, 3$ and can be seen by the first part of Lemma 8. If we suppose the statement true for all $n \le k (k \ge 3)$ we can write

$$\begin{aligned} N(a_{k+1}) &= N(a_k) + N(a_{k-1}) = a_k - F_{k+1} + a_{k-1} - F_k \\ &= a_{k+1} - F_{k+2} \end{aligned} \tag{41}$$

by the last part of Lemma 8; so Lemma 9 follows by induction on k.

Lemma 10. Every natural number can be written in the form

$$n = a_{k_1} + a_{k_2} + \cdots + a_{k_i} + t \tag{42}$$

with $k_j + 1 > k_{j+1}$ and $0 \leq t < a_2$.

Proof. The lemma is trivially true for all $n \leq a_2$. Every natural number between a_2 and a_3 can be written $a_2 + t$ with $t < a_1$; $n = a_3$ is of the form (42).

Suppose (42) holds for all $n < N$, and let a_k denote the largest a_i not exceeding N and consider $N - a_k$. We must have $N = a_k < a_{k-1}$, since $N < a_k \geq a_{k-1}$ implies $N \geq a_{k+1}$ which contradicts the maximal property of a_k. It follows that $N - a_k < N$ can be represented in the form (42) with $k + 1 > k_1$; hence, $N = a_k + a_{k_1} + \cdots + a_{k_i} + t$ is also of the form (42).

Theorem 7. Let n be a number represented as in (42). Then

$$N(n) = \begin{cases} n - \{F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_i+1}\} & \text{if } 0 \leq t \leq a_1 \\ n - \{1 + F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_i+1}\} & \text{if } a_1 \leq t \leq a_2 \end{cases} \tag{43}$$

Proof. Since $a_{k_2} + \cdots + a_{k_i} + t < a_{k_1} - 1$ we can apply Lemma 7 to obtain

$$N(n) = N(a_{k_1}) + N(a_{k_2} + \cdots + a_{k_i} + t) \ ; \tag{44}$$

applying Lemma 7 repeatedly in (44) we get

$$N(n) = N(a_{k_1}) + N(a_{k_2}) + \cdots + N(a_{k_i} + t) \ . \tag{45}$$

Now if $a_{k_i} = a_2$, $0 \leq t \leq a_1$, since if $t \geq a_1$ we would have $a_{k_i} = a_3$ and we can write

$$N(a_{k_i} + t) = N(a_{k_i}) + N(t) \ ; \tag{46}$$

but if $a_{k_i} = a_1$, we would have $a_1 + t < a_2$ and we reselect t as $a_1 + t_1$; also, we can conclude from this that $a_{k_i} - 1 \geq a_3$ so (46) still holds in this case. Thus (45) can be written in the form

$$N(n) = N(a_{k_1}) + N(a_{k_2}) + \cdots + N(a_{k_i}) + N(t) \quad . \tag{47}$$

Applying Lemma 9 to the $N(a_{k_i})$ in the right member of (47) we get

$$\begin{aligned} N(n) &= a_{k_1} - F_{k_1+1} + a_{k_2} - F_{k_2+1} + \cdots + a_{k_i} - F_{k_i+1} + N(t) \\ &= a_{k_1} + a_{k_2} + \cdots + a_{k_i} + N(t) - \{F_{k_1+1} + \cdots + F_{k_i+1}\} \quad ; \end{aligned} \tag{48}$$

but if $t < a_1$, $N(t) = t$ and $a_{k_1} + \cdots + a_{k_i} + t = n$ by assumption so that the first part of (43) is true. If $a_1 \leq t < a_2$, $N(t) = t - 1$ and we see that the second part of (43) is also true. This completes the proof of Theorem 6.

Hoggatt (Problem H-53, Fibonacci Quarterly) has proposed that one show that $M(n)$, the number of natural numbers less than n which cannot be expressed as a sum of distinct Lucas numbers $L_n (L_1 = 1,\ L_2 = 3,\ L_{n+2} = L_{n+1} + L_n$ has the property

$$M(L_n) = F_{n-1} \quad ; \tag{49}$$

also, he asked for a formula for $M(n)$.

The Lucas sequence can be used in place of $\{a_n\}$ in all of our lemmas and theorems. In particular, Lemma 9 tells us $M(L_n) = N(L_n) = L_n - F_{n+1}$; it is a trivial matter to show $L_n - F_{n+1} = F_{n-1}$ by induction so (49) is proved. Writing $a_{k_i} = L_{k_i}$ in (42) and Theorem 7 gives a formula for $M(n)$ for all natural numbers n.

Table 1
$R(k)$ for $0 \leq k < 144$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
R(n)	1	2	2	3	3	3	4	3	4	5	4	5	4	4	6	5	6	6	5	6

n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
R(n)	4	5	7	6	8	7	6	8	6	7	8	6	7	5	5	8	7	9	9	8

Table 1 (Cont'd)

n	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
R(n)	10	7	8	10	8	10	8	7	10	8	9	9	7	8	5	6	9	8	11	10

n	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
R(n)	9	12	9	11	13	10	12	9	8	12	10	12	12	10	12	8	9	12	10	13

n	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99
R(n)	11	9	12	9	10	11	8	9	6	6	10	9	12	12	11	14	10	12	15	12

n	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115
R(n)	15	12	11	16	13	15	15	12	14	9	10	14	12	16	14	12

n	116	117	118	119	120	121	122	123	124	125	126	127	128	129	130	131
R(n)	16	12	14	16	12	14	10	9	14	12	15	15	13	16	11	12

n	132	133	134	135	136	137	138	139	140	141	142	143
R(n)	15	12	15	12	10	14	11	12	12	9	10	6

Table 2
T(k) for $0 \leq k \leq 55$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
T(n)	1	1	1	2	1	2	2	1	3	2	2	3	1	3	3	2	4	2	3	3

n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
T(n)	1	4	3	3	5	2	4	4	2	5	3	3	4	1	4	4	3	6	3	5

n	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55
T(n)	5	2	6	4	4	6	2	5	5	3	6	3	4	4	1	5

Table 3
S(k) for $0 \le k \le 68$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
S(n)	1	1	0	1	2	1	0	2	2	0	1	3	2	0	2	3	1	0	3	3

n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
S(n)	0	2	4	2	0	3	3	0	1	4	3	0	3	5	2	0	4	4	0	2

n	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
S(n)	5	3	0	3	4	1	0	4	4	0	3	6	3	0	5	5	0	2	6	4

n	60	61	62	63	64	65	66	67	68	69
S(n)	0	4	6	2	0	5	5	0	3	

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ON A GENERALIZATION OF MULTINOMIAL COEFFICIENTS FOR FIBONACCI SEQUENCES

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Let $m = n_1 + n_2 + \cdots + n_k$ be a partition of m into $k \geq 2$ positive integral parts and let $F_0 = 0$, $F_1 = 1, \cdots, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. This is known as the Fibonacci sequence. A multinomial coefficient for the Fibonacci sequence is defined to be the quotient

$$[m; n_1, n_2, \cdots, n_k] = \prod_{j=1}^{m} F_j \Big/ \prod_{j=1}^{n_1} F_j \prod_{j=1}^{n_2} F_j \cdots \prod_{j=1}^{n_k} F_j .$$

It is the purpose of this paper to show that such quotients are integer valued. In order to do this we first establish a representation of F_m in terms of a linear combination of the F_{n_i}. This result is of some interest in itself since it contains many of the classic formulae for Fibonacci sequences.

<u>Theorem 1</u>: Let $F_0 = 0$, $F_1 = 1, \cdots, F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, and let $m = n_1 + n_2 + \cdots + n_k$ be a partition of m into positive integral parts. Then

$$F_m = \sum_{i=1}^{k} G_i P_i F_{n_i}$$

where $G_1 = 1$, $G_i = F_{n_1+n_2+\cdots n_{i-1} - 1}$, $1 < i \leq k$; and $P_i = \prod_{j=i+1}^{k} F_{n_j+1}$, $1 \leq i < k$, $P_k = 1$.

For the proof of the theorem we require the following Lemmas:

<u>Lemma 1</u>: If $n_1 + n_2 + \cdots + n_k$ and $n_1' + n_2' + \cdots + n_k'$ are partitions of m into $k \geq 2$ positive integral parts where the parts $n_1', n_2', \cdots, n_k'$ are a permutation of the parts $n_1, n_2, \cdots, n_k$, then

$$\sum_{i=1}^{k} G_i P_i F_{n_i} = \sum_{i=1}^{k} G_i' P_i' F_{n_i}'$$

where

$$G_i = F_{n_1+n_2+\cdots+n_{i-1}-1}, \quad 1 < i \le k, \quad G_1 = 1;$$
$$P_i = \prod_{j=i+1}^{k} F_{n_j+1}, \quad 1 \le i < k, \quad P_k = 1;$$
$$G_i' = F_{n_1'+n_2'+\cdots+n_{i-1}'-1}, \quad 1 < i \le k, \quad G_1' = 1;$$
$$P_i' = \prod_{j=i+1}^{k} F_{n_j'+1}, \quad 1 \le i < k, \quad P_k' = 1 .$$

Proof: Since any permutation of the parts $n_1, n_2, \cdots, n_k$ can be obtained by successive transpositions of adjacent parts it suffices to show the conclusion for the case $n_{s+1} = n_s'$ and $n_s = n_{s+1}'$, $n_i = n_i'$ for $i \ne s, s+1$. From the definition of G_i and G_i' we have $G_i = G_i'$ for $1 \le i \le s$ and $s+2 \le i \le k$, $G_{s+1} = F_{n_1+n_2+\cdots+n_{s-1}+n_s-1}$, $G_{s+1}' = F_{n_1+n_2+\cdots+n_{s-1}+n_{s+1}-1}$. We also have $P_i = P_i'$ for $1 \le i \le s-1$ and $s+1 \le i \le k$, $P_s = F_{n_{s+1}+1}P_{s+1}$, and $P_s' = F_{n_s+1}P_{s+1}$. Thus every term in the unprimed sum equals the corresponding term in the primed sum except for the terms where $i = s$ and $i = s+1$. Considering just these terms, we must show that $G_sP_sF_{n_s} + G_{s+1}P_{s+1}F_{n_{s+1}} = G_s'P_s'F_{n_s'} + G_{s+1}'P_{s+1}'F_{n_{s+1}'}$.

$$\begin{aligned}
G_sP_sF_{n_s} + G_{s+1}P_{s+1}F_{n_{s+1}} &= G_sF_{n_{s+1}+1}P_{s+1}F_{n_s} \\
&\qquad + F_{n_1+n_2+\cdots n_{s-1}+n_s-1}P_{s+1}F_{n_{s+1}} \\
&= G_sF_{n_{s+1}+1}F_{n_s} \\
&\qquad + (F_{n_s}F_{n_1+n_2+\cdots n_{s-1}} + G_sF_{n_s-1})F_{n_{s+1}} \\
&= F_{n_s}F_{n_{s+1}}F_{n_1+n_2+\cdots n_{s-1}} \\
&\qquad + G_s(F_{n_{s+1}+1}F_{n_s} + F_{n_{s+1}}F_{n_s-1}) \\
&= F_{n_s}F_{n_{s+1}}F_{n_1+n_2+\cdots n_{s-1}} \\
&\qquad + G_sF_{n_s+n_{s+1}} \\
&= F_{n_{s+1}}F_{n_s}F_{n_1+n_2+\cdots n_{s-1}} \\
&\qquad + G_s(F_{n_s+1}F_{n_{s+1}} + F_{n_s}F_{n_{s+1}-1})
\end{aligned}$$

$$= G_s F_{n_s+1} F_{n_{s+1}}$$

$$+ (F_{n_{s+1}} F_{n_1+n_2+\cdots n_{s-1}} + G_s F_{n_{s+1}-1}) F_{n_s}$$

$$= G_s F_{n_s+1} F_{n_{s+1}}$$

$$+ F_{n_1+n_2+\cdots+n_{s-1}+n_{s+1}-1} F_{n_s}$$

$$= G_s F_{n_s+1} P_{s+1} F_{n_{s+1}}$$

$$+ F_{n_1+n_2+\cdots+n_{s-1}+n_{s+1}-1} P_{s+1} F_{n_s}$$

$$= G'_s P'_s F_{n'_s} + G'_{s+1} P'_{s+1} F_{n'_{s+1}} \quad .$$

where we have used repeatedly the classical formula $F_{m+n} = F_{m+1}F_n + F_{n-1}F_m$.

Lemma 2: If $n_1 + n_2 + \cdots + n_k$ is a partition of m into $k \geq 2$ positive integral parts with at least one part (say n_s) greater than 1, then

$$\sum_{i=1}^{k} G_i P_i F_{n_i} = \sum_{i=1}^{k} G'_i P'_i F_{n'_i}$$

where $n_i = n'_i$ for $i \neq s, r$; $n_s - 1 = n'_s$, $n_r + 1 = n'_r$, $s \neq r$, and G_i, P_i, G'_i and P'_i are all defined as in Lemma 1.

Proof: In view of Lemma 1 we can assume that $n_1 > 1$ and show the result for the partitions $n_1 + n_2 + \cdots + n_k$ and $n'_1 + n'_2 + \cdots + n'_k$ where $n'_1 = n_1 - 1$, $n'_2 = n_2 + 1$, $n'_i = n_i$ for $3 \leq i \leq k$. For this choice, $G_i = G'_i$ for $i = 1$ and $3 \leq i \leq k$, $G_2 = F_{n_1-1}$, $G'_2 = F_{n_1-2}$, and $P_i = P'_i$ for $1 < i \leq k$.

Here every term in the unprimed sum equals the corresponding term in the primed sum except for $i = 1, 2$. Considering only these terms,

$$G_1P_1F_{n_1} + G_2P_2F_{n_2} = (F_{n_1}) \prod_{j=2}^{k} F_{n_j+1} + (F_{n_1-1}F_{n_2}) \prod_{j=3}^{k} F_{n_j+1}$$

$$= (F_{n_1}F_{n_2+1} + F_{n_1-1}F_{n_2}) \prod_{j=3}^{k} F_{n_j+1}$$

$$= (F_{n_1+n_2}) \prod_{j=3}^{k} F_{n_j+1}$$

$$= (F_{n_2+2} F_{n_1-1} + F_{n_2+1} F_{n_1-2}) \prod_{j=3}^{k} F_{n_j+1}$$

$$= (F_{n_1-2}) \prod_{j=2}^{k} F_{n_j+1} + (F_{n_2+2} F_{n_1-1}) \prod_{j=3}^{k} F_{n_j+1}$$

$$= G_1' P_1' F_{n_1'} + G_2' P_2' F_{n_2'}$$

which completes the proof.

We now proceed to the proof of the theorem. When $m = k$ we have $n_i = 1$, $G_1 = 1$, $G_i = F_{i-2}$ for $2 \le i \le k$, $P_i = 1$ for $1 \le i \le k$ and

$$\sum_{i=1}^{k=m} G_i P_i F_{n_i} = F_1 + \sum_{i=0}^{m-2} F_i = F_1 + (F_m - 1) = F_m$$

by a well-known result for the Fibonacci sequence. When $m = k + 1$, all the parts are 1 except one part which is 2. By Lemma 1 we can assume that $n_k = 2$. For this we have $G_i = F_{i-2}$ for $1 < i \le k$, $G_1 = 1$, $P_i = F_2^{k-i-1} F_3$ for $1 \le i < k$, $P_k = 1$. Thus

$$\sum_{i=1}^{k} G_i P_i F_{n_i} = F_3 F_{k-1} + F_{k-2} = F_{k-1} + (F_{k-1} + F_{k-2}) = F_{k+1} \quad .$$

Now assume $m \ge k + 2$ and let $m = n_1 + n_2 + \cdots + n_k$ with $n_1 \le n_2 \le \cdots \le n_k$. There are two cases, $n_k \ge 3$ or $n_k \ge 2$ and $n_{k-1} \ge 2$. By applying Lemma 2 we can reduce the second case to the first. Thus we need only consider $n_1 \le n_2 \le \cdots \le n_k$ with $n_k \ge 3$. We assume that the result is valid for the partitions

$$m - 1 = n_1' + n_2' + \cdots + n_k' \quad \text{where} \quad n_i' = n_i, \; 1 \le i < k, \; n_k' = n_k - 1$$

$$m - 2 = n_1'' + n_2'' + \cdots + n_k'' \quad \text{where} \quad n_i'' = n_i, \; 1 \le i < k, \; n_k'' = n_k - 2$$

and show it holds for the partition

$$m = n_1 + n_2 + \cdots + n_k \ .$$

We have $G_i = G_i' = G_i''$ for $1 \le i \le k$ and

$$P_i = (F_{n_k+1}) \prod_{j=i+1}^{k-1} F_{n_j+1} \quad ,$$

$$P_i' = (F_{n_k}) \prod_{j=i+1}^{k-1} F_{n_j+1} \quad ,$$

$$P_i'' = (F_{n_k-1}) \prod_{j=i+1}^{k-1} F_{n_j+1}$$

for $1 \le i < k$, $P_k = P_k' = P_k'' = 1$. Hence,

$$F_m = F_{m-1} + F_{m-2} = \sum_{i=1}^{k} G_i'P_i'F_{n_i'} + \sum_{i=1}^{k} G_i''P_i''F_{n_i''}$$

$$= \sum_{i=1}^{k-1} G_i\left(\sum_{j=i+1}^{k-1} F_{n_j+1}\right)(F_{n_k} + F_{n_k-1})\, F_{n_i}$$

$$+ G_kP_k(F_{n_k-1} + F_{n_k-2})$$

$$= \sum_{i=1}^{k} G_iP_iF_{n_i} \quad ,$$

which is the desired result.

Utilizing the result of Theorem 1 we prove the following theorem:

Theorem 2: Let m and r be integers, $m \ge r \ge 2$, and let $n_1 + n_2 + \cdots + n_k$ be a partition of m into positive integral parts.

Then $[m;n_1,n_2,\cdots,n_r]$ is an integer.

Proof: If $m = 2$, then $r = 2$, and the only admissible partition has $n_1 = n_2 = 1$. Clearly $[2;1,1]$ is an integer. Now let $m > 2$ and assume that for every partition of $m - 1$ into positive integers where $m - 1 \geq s \geq 2$ we have that $[m - 1; n_1, n_2, \cdots, n_s]$ is an integer. If $r = m$, then each $n_i = 1$, $1 \leq i \leq r$, and $[m;n_1,n_2,\cdots,n_r]$ is an integer. If $2 \leq r < m$ then $m - 1 \geq r \geq 2$, and by the induction hypothesis $[m - 1;n_1 - 1, n_2,\cdots,n_r]$, $[m - 1;n_1,n_2 - 1,\cdots,n_r]$, $\cdots$, $[m - 1; n_1, n_2,\cdots,n_r - 1]$ are all integers.

Now

$$[m;n_1,n_2,\cdots,n_r] = \sum_{i=1}^{k} G_i P_i [m - 1; n_1, \cdots, n_{i-1}, \cdots n_r]$$

where we have used Theorem 1 to write F_m as a linear combination of the F_{n_i}, $1 \leq i \leq r$. The right-hand side is an integer since all the terms are integers. This completes the proof of the theorem.

Editorial Comment: The special multinomial coefficient where $k = 2$, that is, for $m = n_1 + n_2$,

$$\prod_{j=1}^{m} F_j \Big/ \prod_{j=1}^{n_1} F_j \prod_{j=1}^{n_2} F_j \quad ,$$

has been given the fitting name, "Fibonomial coefficient." Fibonomial coefficients appeared in this Quarterly in advanced problem H-4, proposed by T. Brennan and solved by J. L. Brown, Oct., 1963, p. 49, and in Brennan's paper, "Fibonacci Powers and Pascal's Triangle in a Matrix," April and October, 1964. Also, a proof of the theorem of this paper for the case $k = 2$ appears in D. Jarden's Recurring Sequences, p. 45.

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ON THE DETERMINATION OF THE ZEROS OF THE FIBONACCI SEQUENCE

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In his article [1], Brother U. Alfred has given a table of periods and zeros of the Fibonacci Sequence for primes in the range $2,000 < p < 3,000$. The range $p < 2,000$ has been investigated by D. D. Wall [2]. The present author has studied the extended range $p < 5,000$ by computer, and has found that approximately 68% of the primes have zeros which are maximal or half maximal, i.e., $Z(F,p) = p + 1$, $p - 1$, $(p + 1)/2$ or $(p - 1)/2$.

It would seem profitable, then, to seek a formula which gives the values of $Z(F,p)$ for some of these "time-consuming" primes. If these can be taken care of this way, the average time per prime would decrease since there are large primes with surprisingly small periods.

We have succeeded in producing a formula for two sets of primes. A table of zeros of the Fibonacci Sequence for primes in the range $3,000 < p < 10,000$ discovered by these formulas is included at the end of this paper. It is not known whether these formulae apply to more than a finite set of primes. See [3] for some discussion on this point.

To develop the ideas in a somewhat more general context, we introduce the Primary Numbers F_n defined by the recurrence relation:

$$F_{n+2} = aF_{n+1} + bF_n \ ; \ F_0 = 0, \quad F_1 = 1 \quad ,$$

where a and b are integral. F_n may be given explicitly in the Binet form;

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad , \tag{1}$$

where α and β are the (assumed distinct) roots of the quadratic equation $x^2 - ax - b = 0$. In a like manner, we may define the Secondary Numbers which play the same role as the well known Lucas Numbers do to the Fibonacci Numbers. Thus the Secondary Numbers L_n are defined by the recurrence relation:

$$L_{n+2} = aL_{n+1} + bL_n \; ; \; L_0 = 2, \; L_1 = a \; .$$

L_n may also be given explicitly in the Binet form as:

$$L_n = \alpha^n + \beta^n \tag{2}$$

The following three properties of the Primary Sequences may easily be established by induction, or by using formula (1).

1) $F_r = -(-b)^r F_{-r}$
2) If $(a,b) = 1$, then $(F_n, b) = 1$
3) If $(a,b) = 1$, then $(F_n, F_{n+1}) = 1$.

Using formula (1), it is a simple algebraic exercise to prove the next result.

Lemma 1. $F_m = F_{i+1}F_{m-i} + bF_iF_{m-i-1}$

Proof: Since α and β are the roots of $x^2 - ax - b = 0$, we have $\alpha\beta = -b$.

$$\begin{aligned}
\text{R.H.S.} &= ((\alpha^{i+1} - \beta^{i+1})(\alpha^{m-i} - \beta^{m-i}) - \alpha\beta\,(\alpha^i - \beta^i)(\alpha^{m-i-1} - \beta^{m-i-1}))/(\alpha - \beta)^2 \\
&= (\alpha^{m+1} - \alpha^{i+1}\cdot\beta^{m-i} - \alpha^{m-i}\cdot\beta^{i+1} + \beta^{m+1} - \beta\alpha^m + \alpha^{i+1}\cdot\beta^{m-i} + \alpha^{m-i} \\
&\qquad \cdot\beta^{i+1} - \alpha\beta^m)/(\alpha - \beta)^2 \\
&= (\alpha^{m+1} + \beta^{m+1} - \beta\alpha^m - \alpha\beta^m)/(\alpha - \beta)^2 \\
&= (\alpha - \beta)(\alpha^m - \beta^m)/(\alpha - \beta)^2 = (\alpha^m - \beta^m)/(\alpha - \beta) = \text{L.H.S.}
\end{aligned}$$

Making use of properties 1) and 3) and Lemma 1, we may prove the following Theorem which tells us that the factors of Primary Sequences occur in similar patterns to those encountered in the Fibonacci Sequence itself.

Theorem 1. Let $(a,b) = 1$. Chose a prime p and an integer j such that p^j exactly divides F_d* $(d > 0)$, but no Primary Number with smaller subscript. Then p^j divides F_n (not necessarily exactly) if and only if $n = dt$ for some integer t. Or: $F_d | F_n$ iff $n = dt$ for some integer t.

Proof. Suppose that $n = dt$. We prove by induction on t that p^j divides F_n. $t = 1$. p^j divides F_d.

Assume true for $t = t$, $t \geq 1$.

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*i.e., $p^j | F_d$ but $p^{j+1} \nmid F_d$.

Putting $m = d(t+1)$ and $i = d$ in Lemma 1, we have the identity:

$$F_{d(t+1)} = F_{d+1}F_{dt} + bF_dF_{dt-1}$$

p^j divides F_d and F_{dt}, so by (1), divides $F_{d(t+1)}$.

Conversely, suppose that p^j divides F_n, where $n = dt + r$ for some r satisfying $0 < r < d$. We seek a contradiction, forcing r to equal 0.

Putting $m = dt$ and $i = -r$ in Lemma 1, we have the identity:

$$F_{dt} = F_{-r+1}F_{dt+r} + bF_{-r}F_{dt+r-1} \quad .$$

Multiplying through by $-(-b)^{r-1}$ and using the fact that $F_r = -(-b)^rF_{-r}$, we have:

$$-(-b)^{r-1}F_{dt} = F_{r-1}F_{dt+r} - F_rF_{dt+r-1} \quad .$$

Since p^j divides both F_{dt} and F_{dt+r} it divides F_rF_{dt+r-1}. However, if $(a, b) = 1$, consecutive Primary Numbers are co-prime, and so p does not divide F_{dt+r-1}. Thus p^j divides F_r which is a contradiction.

Another result which we will need is contained in the next Theorem. This result is a direct generalization of the well-known result applied to Fibonacci Numbers. The proof follows precisely the one given by Hardy and Wright in [4], and so need not be repeated here.

Theorem 2. Let $k = a^2 + 4b \neq 0$ and p be a prime such that $p \nmid 2b$, then p divides F_{p-1}, F_p or F_{p+1} according as the Legendre Symbol (k/p) is $+1$, 0 or -1.

Proof. Let the roots of the quadratic equation $x^2 - ax - b = 0$ be:

$$\alpha = (a + \sqrt{a^2 + 4b})/2 \qquad \text{and} \qquad \beta = (a - \sqrt{a^2 + 4b})/2 \quad .$$

Hence

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{(a + \sqrt{k})^n - (a - \sqrt{k})^n}{2^n\sqrt{k}}$$

Case 1. $(k/p) = +1$

$$2^{p-2}F_{p-1} = ((a + \sqrt{k})^{p-1} - (a - \sqrt{k})^{p-1})/(2\sqrt{k})$$

$$= \left(\sum_{r=0}^{p-1} \binom{p-1}{r} a^{p-r-1}(\sqrt{k})^r - \sum_{r=0}^{p-1} \binom{p-1}{r} a^{p-r-1}(-\sqrt{k})^r\right) \Bigg/ (2\sqrt{k})$$

$$= \left(\sum_{\substack{r \text{ odd} \\ 1 \le r \le p-2}} \binom{p-1}{r} a^{p-r-1}(\sqrt{k})^r\right) \Bigg/ (\sqrt{k})$$

$$= \sum_{s=0}^{(p-3)/2} \binom{p-1}{2s+1} a^{p-2s-2}k^s$$

since $\binom{p-1}{2s+1} \equiv -1 \pmod p$ for $s = 0, 1, \cdots, (p-3)/2$, we find that

$$2^{p-2}F_{p-1} \equiv - \sum_{s=0}^{(p-3)/2} a^{p-2s-2}k^s \pmod p \quad .$$

Summing this geometric progression, we have:

$$2^p bF_{p-1} \equiv a^p - ak^{(p-1)/2} \pmod p \quad .$$

Making use of Euler's Criterion $k^{(p-1)/2} \equiv (k/p) \pmod p$ for the quadratic character of $k \pmod p$, assuming that $p \nmid 2b$, $(k/p) = +1$ and knowing that $a^p \equiv a \pmod p$, we have:

$$F_{p-1} \equiv 0 \pmod p \quad .$$

Case 2. $(k/p) = 0$

$$2^{p-1}F_p = ((a + \sqrt{k})^p - (a - \sqrt{k})^p)/(2\sqrt{k})$$

$$= \left(\sum_{r=0}^{p} \binom{p}{r} a^{p-r}(\sqrt{k})^r - \sum_{r=0}^{p} \binom{p}{r} a^{p-r}(-\sqrt{k})^r\right) \Bigg/ (2\sqrt{k})$$

$$= \left(\sum_{\substack{r \text{ odd} \\ 1 \le r \le p}} \binom{p}{r} a^{p-r}(\sqrt{k})^r\right) \Bigg/ (\sqrt{k}) = \sum_{s=0}^{(p-1)/2} \binom{p}{2s+1} a^{p-2s-1}k^s \quad .$$

p divides each Binomial Coefficient except the last and so:

$$2^{p-1}F_p \equiv k^{(p-1)/2} \pmod{p} \quad .$$

Since $p \nmid 2b$ and $(k/p) = 0$, we have

$$F_p \equiv 0 \pmod{p} \quad .$$

<u>Case 3.</u> $(k/p) = -1$

$$\begin{aligned}
2^p F_{p+1} &= ((a + \sqrt{k})^{p+1} - (a - \sqrt{k})^{p+1})/(2\sqrt{k}) \\
&= \left(\sum_{r=0}^{p+1} \binom{p+1}{r} a^{p-r+1}(\sqrt{k})^r - \sum_{r=0}^{p+1} \binom{p+1}{r} a^{p-r+1}(-\sqrt{k})^r \right) \Bigg/ (2\sqrt{k}) \\
&= \left(\sum_{\substack{r \text{ odd} \\ 1 \le r \le p}} \binom{p+1}{r} a^{p-r+1}(\sqrt{k})^r \right) \Bigg/ \sqrt{k} \\
&= \sum_{s=0}^{(p-1)/2} \binom{p+1}{2s+1} a^{p-2s} k^s \quad .
\end{aligned}$$

All the Binomial Coefficients except the first and last are divisible by p and so:

$$2^p F_{p+1} \equiv a^p + ak^{(p-1)/2} \pmod{p} \quad .$$

Since $p \nmid 2b$, $(k/p) = -1$ and $a^p \equiv a \pmod{p}$, we have:

$$F_{p+1} \equiv 0 \pmod{p} \quad .$$

Yet another well-known result which can be extended to the Primary Sequences is given in Lemma 2. A proof may be constructed on the model provided by Glenn Michael in [5], and is a simple exercise for the reader.

Lemma 2. If $(a,b) = 1$ and c,d are positive integers, then $(F_c, F_d) = |F_{(c,d)}|$.

Proof. Let $e = (c,d)$ and $D = (F_c, F_d)$. $e|c$ and $e|d$ hence by Theorem 1, $F_e|F_c$ and $F_e|F_d$. Thus $F_e|D$.

There exist integers x and y (given by the Euclidean Algorithm) such that $e = xc + yd$. Suppose without loss of generality that $x > 0$ and $y \le 0$. Using Lemma 1 with $m = xc$ and $i = e$ we have:

$$F_{xc} = F_{e-1}F_{-yd} + bF_eF_{-yd-1} \quad .$$

$D|F_c$ and F_d and so by Theorem 1, $D|F_{xc}$ and F_{-yd}. Thus $D|bF_eF_{-yd-1}$, but by property 2), $(D,b) = 1$, and by property 3), $(D, F_{-yd-1}) = 1$. Thus $D|F_e$. This, together with $F_e|D$ gives the result.

Lemma 3.

$$F_{2n-1} - F_{n-1}L_n = (-b)^{n-1}$$

Proof.

$$\begin{aligned}
\text{L. H. S.} &= (\alpha^{2n-1} - \beta^{2n-1} - (\alpha^{n-1} - \beta^{n-1})(\alpha^n + \beta^n))/(\alpha - \beta) \\
&= (\alpha^{2n-1} - \beta^{2n-1} - \alpha^{2n-1} - \alpha^{n-1}\beta^n + \beta^{n-1}\alpha^n + \beta^{2n-1})/(\alpha - \beta) \\
&= (-\alpha^{n-1}\beta^n + \beta^{n-1}\alpha^n)/(\alpha - \beta) \\
&= (\alpha - \beta)(\alpha\beta)^{n-1}/(\alpha - \beta) = (\alpha\beta)^{n-1} = (-b)^{n-1} = \text{R.H.S.}
\end{aligned}$$

MAIN RESULTS

We shall divide the main results of this paper into 6 parts — four Lemmas in which the essential ideas are proven, a Theorem utilizing these ideas and a Corollary applying them in particular to the Fibonacci Numbers. It will be implicitly understood that from now on, $(a,b) = 1$ and $p \nmid 2abk$.

Lemma 4. If $(-b/p) = (k/p) = +1$ (Legendre Symbols), then $p|F_{(p-1)/2}$.

Proof. Using Lemma 3 with $n = (p+1)/2$ gives

$$F_p - F_{\frac{p-1}{2}} L_{\frac{p+1}{2}} = (-b)^{(p-1)/2} \quad .$$

In the proof of Theorem 2 we find that

$$2^{p-1}F_p \equiv F_p \equiv (k/p) \pmod{p} \quad .$$

Thus:

$$(k/p) - F_{\frac{p-1}{2}} L_{\frac{p+1}{2}} \equiv (-b/p) \pmod{p} \tag{3}$$

Putting $(-b/p) = (k/p) = +1$ we have:

$$F_{\frac{p-1}{2}} L_{\frac{p+1}{2}} \equiv 0 \pmod{p} \quad .$$

Suppose, now, that p divides $L_{(p+1)/2}$. Since $L_{(p+1)/2} = F_{p+1}/F_{(p+1)/2}$, p divides F_{p+1}. Theorem 2 tells us that p divides F_{p-1} since $(k/p) = +1$. Applying Lemma 2, we see that p divides $F_{(p-1,p+1)}$ which is F_2.

But $F_2 = a$ and so we have a contradiction.

Lemma 5. If $(-b/p) = (k/p) = -1$, then $p \nmid F_{(p+1)/2}$.

Proof. Using (3) with $(-b/p) = (k/p) = -1$ we have:

$$F_{\frac{p-1}{2}} L_{\frac{p+1}{2}} \equiv 0 \pmod{p} \quad .$$

Suppose that $p \mid F_{(p-1)/2}$. Therefore $p \mid F_{p-1}$. By Theorem 2, $p \mid F_{p+1}$, and so as before, we find that $p \mid F_2 = a$ a contradiction. Hence $p \mid L_{(p+1)/2}$.

Since $L_n = aF_n + 2bF_{n-1}$, any prime divisor common to F_n and L_n must divide 2b by property 3). These primes are excluded, and so $p \nmid F_{(p+1)/2}$ as asserted.

Lemma 6. If $(-b/p) = +1$, $(k/p) = -1$, then $p \mid F_{(p+1)/2}$.

Proof. Putting $(-b/p) = +1$ and $(k/p) = -1$ in (3) we have:

$$F_{\frac{p-1}{2}} L_{\frac{p+1}{2}} \equiv -2 \pmod{p} \quad .$$

Thus $p \nmid L_{(p+1)/2}$ since $p \neq 2$. Suppose, to the contrary, that $p \nmid F_{(p+1)/2}$. By Theorem 2, $p \mid F_{p+1}$, and so $p \mid F_{p+1}/F_{(p+1)/2} = L_{(p+1)/2}$ a contradiction.

Lemma 7. If $(-b/p) = -1$ and $(k/p) = +1$, then $p \nmid F_{(p-1)/2}$.

Proof. Similarly we have:

$$F_{\frac{p-1}{2}} \; L_{\frac{p+1}{2}} \equiv +2 \pmod{p} \quad .$$

Clearly

$$p \nmid F_{(p-1)/2} \; .$$

To distinguish from the Fibonacci case, we shall employ the terminology $Z(F;a,b;p)$ for the first non-trivial zero (mod p) of the Primary Sequence with parameters a and b. Thus $Z(F;1,1;p) = Z(F,p)$ following the notation used by Brother U. Alfred in [1]. Similar remarks apply to $Z(L;a,b;p)$.

Main Theorem.

1) If r is a prime and $p = 2r + 1$ is a prime such that $(-b/p) = (k/p) = +1$, then $Z(F;a,b;p) = r$.

2) If s is a prime and $p = 2s - 1$ is a prime such that $(-b/p) = (k/p) = -1$, then $Z(F;a,b;p) = p + 1$.

3) If s is a prime and $p = 2s - 1$ is a prime such that $(-b/p) = +1$, and $(k/p) = -1$, then $Z(F;a,b;p) = s$.

4) If r is a prime and $p = 2r + 1$ is a prime such that $(-b/p) = -1$, and $(k/p) = +1$, then $Z(F;a,b;p) = p - 1$.

Proof of the Main Theorem.

1) Since $(k/p) = +1$, we see from Theorems 1 and 2 that $p \mid F_d$, where d is a divisor of $p - 1 = 2r$. The only divisors of 2r are 1, 2, r and 2r since r is prime. Clearly $p \nmid F_1 = 1$ and by assumption $p \nmid F_2 = a$. Lemma 4 tells us that $p \mid F_r$ and so $Z(F;a,b;p) = r$.

2) Since $(k/p) = -1$, $p \mid F_d$, where $d \mid p + 1 = 2s$. The divisors of 2s are 1, 2, s and 2s. $p \nmid F_1$ and $p \nmid F_2$. Lemma 5 then tells us that $p \nmid F_s$ and so p must divide $F_{2s} = F_{p+1}$, i.e., $Z(F;a,b;p) = p + 1$.

3) Since $(k/p) = -1$, $p \mid F_d$, where $d \mid p + 1 = 2s$. Thus d must be 1, 2, s or 2s because of the primality of s. $p \nmid F_1$ and $p \nmid F_2$. Lemma 6 tells us that $p \mid F_s$ and so $Z(F;a,b;p) = s$.

4) Since $(k/p) = +1$, $p \mid F_d$ where $d \mid p - 1 = 2r$. Again d must be one of: $1, 2, r$ or $2r$ since r is prime. $p \nmid F_1$ and $p \nmid F_2$. Lemma 7 tells us that $p \nmid F_r$ and so p must divide $F_{2r} = F_{p-1}$. Hence $Z(F;a,b;p) = p - 1$.

Specializing the above results to the case of the Fibonacci Sequence ($F_{n+2} = F_{n+1} + F_n$; $F_0 = 0$, $F_1 = 1$) by choosing $a = b = 1$ and hence $k = 5$, we find that parts 1) and 2) of the Main Theorem are now vacuous. Indeed, 1) requires p to be of the form $20k + 1$ or 9, and thus r to be of the form $10k + 0$ or 4 which cannot be prime; 2) requires p to be of the form $20k + 3$ or 7, and thus s to be of the form $10k + 2$ or 4 giving only the prime 2; 3) requires p to be of the form $20k + 13$ or 17 requiring s to be of the form $10k + 7$ or 9 which may now be prime and 4) requires p to be of the form $20k + 11$ or 19 and thus r to be of the form $10k + 5$ or 9 giving primes 5 and $10k + 9$. Thus we have established the following result:

Corollary. Employing the symbol $Z(F, p)$ to denote the first non-trivial zero (mod p) among the Fibonacci Sequence ($F_{n+2} = F_{n+1} + F_n$; $F_0 = 0$, $F_1 = 1$) we have:

1) $s = 2$ and $p = 2s - 1 = 3$ are both prime, and so $Z(F, 3) = 4$.

2) If $s \equiv 7$ or $9 \pmod{10}$ and $p = 2s - 1$ are both prime, then $Z(F,p) = s$.

3) $r = 5$ and $p = 2r + 1 = 11$ are both prime, and so $Z(F, 11) = 10$.

4) If $r \equiv 9 \pmod{10}$ and $p = 2r + 1$ are both prime, then $Z(F,p) = p - 1$.

It would be interesting to discover other sets of primes which have determinable periods and zeros. One such set is the set of Mersenne primes $M_p = 2^p - 1$, where p is a prime of the form $4t + 3$. Since $(-1/M_p) = (5/M_p) = -1$, Lemma 5 tells us that $M_p \nmid F_{2^{4t+2}}$ and so $M_p \nmid F_{2^g}$ for $0 \le g < 4t + 2$, otherwise we could obtain a contradiction from Theorem 1. However, Theorem 2 tells us that $M_p \mid F_{2^p}$, and so $Z(F, M_p) = 2^p$.

A definite formula for $Z(F, p)$ is not to be expected for the same reason that one would not expect to find a formula for the exponent to which a given integer c belongs modulo p. However, some problems, such as that of classifying the set of primes for which $Z(F,p)$ is even (the set of divisors of the Lucas Numbers ($p \neq 2$)) may have partial or complete solutions, and so we leave the reader to investigate them.

TABLE OF ZEROS

p	Z(F,p)	p	Z(F,p)	p	Z(F,p)
3119	3118	5399	5398	7393	3697
3217	1609	5413	2707	7417	3709
3253	1627	5437	2719	7477	3739
3313	1657	5639	5638	7537	3769
3517	1759	5879	5878	7559	7558
3733	1867	5939	5938	7753	3877
3779	3778	6037	3019	7933	3967
4057	2029	6073	3037	8039	8038
4079	4078	6133	3067	8053	4027
4139	4138	6217	3109	8317	4159
4177	2089	6337	3169	8353	4177
4259	4258	6373	3187	8677	4339
4273	2137	6599	6598	8699	8698
4357	2179	6637	3319	8713	4357
4679	4678	6659	6658	8819	8818
4799	4798	6719	6718	8893	4447
4919	4918	6779	6778	9013	4507
4933	2467	6899	6898	9133	4567
5077	2539	6997	3499	9277	4639
5099	5098	7057	3529	9817	4909
5113	2557	7079	7078	9839	9838
5233	2617	7213	3607	9973	4987

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Continued from p. 306

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SOME BINOMIAL COEFFICIENT IDENTITIES*

L. CARLITZ, Duke University, Durham, No. Carolina

1.

Put

$$H(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{i+j}{j}\binom{m-i+j}{j}\binom{i+n-j}{n-j}\binom{m+n-i-j}{n-j}.$$

The formula

$$(1) \qquad H(m,n) - H(m-1,n) - H(m,n-1) = \binom{m+n}{m}^2$$

was proposed as a problem by Paul Brock in the SIAM Review [1]; the published solution by David Slepian established the identity by means of contour integration. Another proof was subsequently given by R. M. Baer and the proposer [2].

The writer [3] gave a proof of (1) and of some related formulas by means of generating functions. The proof of (1) in particular depended on the expansion

$$(2) \qquad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\binom{i+j}{j}\binom{j+k}{k}\binom{k+\ell}{\ell}\binom{\ell+i}{i}u^i v^j w^k x^\ell$$

$$= \left\{[(1-v)(1-x) - w + u(1-w)]^2 - 4u(1-v-w)(1-w-x)\right\}^{-(1/2)}$$

If we take $u = w$, $v = x$ we get

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$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} H(m,n)u^m v^n = (1 - u - v)^{-1}(1 - 2u - 2v + u^2 - 2uv + v^2)^{-(1/2)}, \tag{3}$$

which implies (1). We now take $u = -w$, $v = -x$. Then the left member of (2) becomes

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} (-1)^{i-j}\binom{i+j}{j}\binom{j+k}{k}\binom{k+\ell}{\ell}\binom{\ell+i}{i} w^{i+k} x^{j+\ell}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \overline{H}(m,n)\, w^m x^n ,$$

where

$$\overline{H}(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} (-1)^{i+j}\binom{i+j}{j}\binom{m-i+j}{j}\binom{i+n-j}{n-j}\binom{m+n-i-j}{n-j}.$$

The right member of (2) becomes

$$\left\{\left[(1-u)^2 - x^2\right]^2 + 4w(1-u+x)(1-u-x)\right\}^{-\frac{1}{2}} = (1 - 2w^2 - 2x^2 + w^4 - 2w^2x^2 + x^4)^{-\frac{1}{2}}$$

It is proved in [3] that

$$(1 - 2w - 2x + w^2 - 2wx + x^2)^{-(1/2)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{m}^2 w^m x^n .$$

We therefore get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \overline{H}(m,n)\, w^m x^n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{m}^2 w^{2m} x^{2n} , \tag{4}$$

so that $\overline{H}(m,n) = 0$ if either m or n is odd, while

$$\overline{H}(2m,2n) = \binom{m+n}{m}^2 . \tag{5}$$

2.

If in (2) we take $u = v$, $w = x$, it is proved in [3] that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} J(m,n)v^m x^n = (1-2v)^{-(1/2)}(1-2x)^{-(1/2)}(1-2v-2x)^{-(1/2)} , \tag{6}$$

where

$$J(m,n) = \sum_{i=0}^{m}\sum_{k=0}^{n}\binom{m}{i}\binom{n}{k}\binom{m-i+k}{k}\binom{i+n-k}{i} .$$

Since

$$(1-2v)^{-(1/2)}(1-2x)^{-(1/2)}(1-2v-2x)^{-(1/2)} = (1-2v)^{-1}(1-2x)^{-1}\left\{1 - \frac{4vx}{(1-2v)(1-2x)}\right\}^{-(1/2)}$$

$$= \sum_{r=0}^{\infty}\binom{2r}{r}\frac{v^r x^r}{(1-2v)^{r+1}(1-2x)^{r+1}}$$

$$= \sum_{r=0}^{\infty}\binom{2r}{r} v^r x^r \sum_{m=0}^{\infty}\binom{m+r}{r}(2v)^m \sum_{n=0}^{\infty}\binom{n+r}{r}(2x)^n$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 2^{m+n} v^m x^n \sum_{r=0}^{\min(m,n)} 2^{-2r}\binom{2r}{r}\binom{m}{r}\binom{n}{r} .$$

so that

$$J(m,n) = 2^{m+n} \sum_{r=0}^{\min(m,n)} 2^{-2r} \binom{2r}{r}\binom{m}{r}\binom{n}{r}$$

$$= 2^{m+n} {}_3F_2\left[\begin{matrix} 1/2, & -m, & -n \\ & 1, & 1 \end{matrix}\right]$$

in the usual notation for generalized hypergeometric function. This may be compared with [3, (4.3)].

We now take $u = -v$, $w = -x$ in (2). Then the left member of (2) becomes

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \bar{J}(m,n) v^m x^n \quad ,$$

where

$$\bar{J}(m,n) = \sum_{i=0}^{m} \sum_{k=0}^{n} (-1)^{i+k} \binom{m}{i}\binom{n}{k}\binom{m-i+k}{k}\binom{i+n-k}{i}$$

As for the right member of (2) we get

$$\left\{(1 - 2v)^2 + 4v(1 - v + x)\right\}^{-(1/2)} = (1 + 4vx)^{-(1/2)} \quad ,$$

so that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \bar{J}(m,n) v^m x^n = (1 + 4vx)^{-(1/2)}$$

Since

$$(1 + 4vx)^{-(1/2)} = \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} v^n x^n \quad ,$$

it follows that

$$\bar{J}(m,n) = (-1)^n \binom{m+n}{m} \delta_{mn} \quad . \tag{9}$$

It follows from (7) that

$$\bar{J}(m,n) = (-1)^n \sum_{i=0}^{m} \sum_{k=0}^{n} (-1)^{i+k} \binom{m}{i}\binom{n}{k}\binom{i+k}{k}\binom{m+n-i-k}{n-k}$$

$$= (-1)^n \sum_{i=0}^{m} \sum_{k=0}^{n} (-1)^{i+k} \binom{m}{i}^2 \binom{n}{k}^2 \frac{(i+k)!\,(m+n-i-k)}{m!\,n!}$$

Thus (9) may be replaced by

$$\sum_{i=0}^{m} \sum_{k=0}^{n} (-1)^{i+k} \frac{\binom{m}{i}^2 \binom{n}{k}^2}{\binom{m+n}{i+k}} = \delta_{mn} \quad . \tag{10}$$

3.

The left member of (3) is equal to

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \binom{i+j}{j}\binom{j+k}{k}\binom{k+\ell}{\ell}\binom{\ell+i}{i} u^{i+k} v^{j+\ell}$$

$$= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} u^{i+k} \sum_{j=0}^{\infty} \binom{i+j}{j}\binom{k+j}{j} v^j \sum_{\ell=0}^{\infty} \binom{i+\ell}{\ell}\binom{k+\ell}{\ell} v^\ell$$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} u^{i+k} \left\{ \sum_{j=0}^{\infty} \frac{(i+1)_j (k+1)_j}{j!\,j!} v^j \right\}^2$$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} u^{i+k} \left\{ F(i+1,\ k+1;\ 1;\ v) \right\}^2 \quad ,$$

where $F(i+1),\ k+1;\ 1;\ v)$ is the hypergeometric function. If we put

$$G_m(v) = \sum_{k=0}^{m} \left\{ F(m-k+1,\ k+1;\ 1;\ v) \right\}^2$$

then (3) becomes

$$\text{(11)} \qquad \sum_{n=0}^{\infty} u^m G_m(v) = (1-u-v)^{-1} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{m+n}{m}^2 u^m v^n \quad .$$

Multiplying by $1-u-v$ and comparing coefficients of u^m we get

$$\text{(12)} \quad (1-v)G_m(v) - G_{m-1}(v) = \sum_{n=0}^{\infty} \binom{m+n}{m}^2 v^n = F(m+1,\ m+1;1;v) \quad .$$

This identity is evidently equivalent to (1).

In a similar manner, it follows from (4) that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} (-1)^i u^{i+k} F(i+1,\ k+1;\ 1;\ v)\, F(i+1,\ k+1;\ 1;\ -v)$$

$$= \sum_{m=0}^{\infty} u^{2m} \sum_{n=0}^{\infty} \binom{m+n}{m}^2 v^{2n} \quad ,$$

which yields the identity

$$(13)\quad \sum_{i=0}^{2m} (-1)^i F(i+1, 2m-i+1, 1; v) F(i+1, 2m-i+1; 1; -v) = \sum_{n=0} \binom{m+n}{n}^2 v^{2n}.$$

The identities corresponding to (7) and (9) seem less interesting.

4.

With a little manipulation the right member of (2) reduces to

$$\{(1 - u - v - w - x - uw - vx)^2 - 4uvwx\}^{-(1/2)}$$

We have therefore

$$(14)\quad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \binom{i+j}{j}\binom{j+k}{k}\binom{k+\ell}{\ell}\binom{\ell+i}{i} u^i v^j w^k x^\ell$$
$$= \{(1 - u - v - w - x + uw + vx)^2 - 4uvwx\}^{-(1/2)} .$$

Note that the right side is unchanged by the permutation (uvwx) and also by each of the transpositions (uw) and (vx) and therefore by the permutations of a group of order eight. The same symmetries are evident from the left member.

It may be of interest to remark that in the case of three variables we have the expansion

$$(15)\quad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{i+j}{j}\binom{j+k}{k}\binom{k+i}{i} u^i v^j w^k$$
$$= \{(1 - u - v - w)^2 - 4uvw)\}^{-(1/2)} .$$

Each side is plainly symmetric in u, v, w. As a special case of (15) we may mention $v = \epsilon u$, $w = \epsilon^2 u$, where ϵ, ϵ^2 are the primitive cube roots of unity.

The right member reduces to $(1 - 4u^3)^{-(1/2)}$ and therefore

$$\sum_{i+j+k=3n} \binom{i+j}{j}\binom{j+k}{k}\binom{k+i}{i} \epsilon^{j+2k} = \binom{2n}{n}$$

while

$$\sum_{i+j+k=n} \binom{i+j}{j}\binom{j+k}{k}\binom{k+i}{i} \epsilon^{j+2k} = 0 \quad (3 \nmid n).$$

If we expand the right member of (15) and compare coefficients we get

$$\sum_r \binom{2r}{r} \frac{(i+j+k-2r)!}{r!\,(i-r)!\,(j-r)!\,(k-r)!} = \binom{i+j}{j}\binom{j+k}{k}\binom{k+i}{i},$$

which can also be written in the form

$$\sum_r \frac{\binom{i}{r}\binom{j}{r}\binom{k}{r}}{\binom{i+j+k}{2r}} = \frac{(i+j)!\,(j+k)!\,(k+i)!}{i!\,j!\,k!\,(i+j+k)!} \tag{16}$$

5.

In the case of six variables a good deal of computation is required. Making use of 3, (5.1) we can show that

$$\sum_{i_1,\cdots,i_6=0}^{\infty} \binom{i_1+i_2}{i_2}\binom{i_2+i_3}{i_3}\binom{i_3+i_4}{i_4}\binom{i_4+i_5}{i_5}\binom{i_5+i_6}{i_6}\binom{i_6+i_1}{i_1} \cdot u_1^{i_1}u_2^{i_2}u_3^{i_3}u_4^{i_4}u_5^{i_5}u_6^{i_6} \tag{17}$$

$$= \left\{[1-u_1-u_2-u_3-u_4-u_5-u_6+u_1u_3+u_1u_4+u_1u_5+u_2u_4+u_2u_5+u_2u_6+u_3u_5+u_3u_6+u_4u_6-u_1u_3u_5-u_2u_4u_6]^2 - 4u_1u_2u_3u_4u_5u_6\right\}^{-\frac{1}{2}}$$

On the right of (17) the bilinear terms satisfy the following rule: in the cycle (123456) adjacent subscripts are not allowed; thus, for example u_1u_2 and u_1u_6 do not appear.

If we take $u_1 = u_4$, $u_2 = u_5$, $u_3 = u_6$, the right member of (7) reduces to

$$\left\{[1 - 2u_1 - 2u_2 - 2u_3 + (u_1 + u_2 + u_3)^2 - 2u_1u_2u_3]^2 - 4u_1^2u_2^2u_3^2\right\}^{-(1/2)}$$

$$=\{[(1 - u_1 - u_2 - u_3)^{-1}[(1 - u_1 - u_2 - u_3)^2 - 4u_1u_2u_3]^{-(1/2)}$$

in agreement with [3, (5.2)].

For five variables we find that

$$\sum_{i_1,\cdots,i_5=0} \binom{i_1+i_2}{i_2}\binom{i_2+i_3}{i_3}\binom{i_3+i_4}{i_4}\binom{i_4+i_5}{i_5}\binom{i_5+i_1}{i_1} u_1^{i_1}u_2^{i_2}u_3^{i_3}u_4^{i_4}u_5^{i_5} \tag{18}$$

$$=\left\{[1 - u_1 - u_2 - u_3 - u_4 - u_5 + u_1u_3 + u_1u_4 + u_2u_4 + u_2u_5 + u_3u_5]^2 - 4u_1u_2u_3u_4u_5\right\}^{-(1/2)}$$

The bilinear terms on the right are determined exactly as in (17); in the cycle (12345) adjacent subscripts are not allowed.

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3. L. Carlitz, "A Binomial Identity Arising from a Sorting Problem," SIAM Review, Vol. 6 (1964), pp. 20-30.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by V. E. HOGGATT, JR., San Jose State College, San Jose, Calif.

Send all communications concerning Advanced Problems and Solutions to Raymond Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within three months after publication of the problems.

H-93 ***Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*** (Corrected)

that

$$F_n = \prod_{k=1}^{\overline{n-1}} \left(3 + 2\cos\frac{2k\pi}{n}\right)$$

$$L_n = \prod_{k=1}^{\overline{n-2}} \left(3 + 2\cos\frac{(2k+1)\pi}{n}\right) ,$$

where $\bar{n}$ is the greatest integer in $n/2$.

H-96 ***Proposed by Maxey Brooke, Sweeny, Texas, and V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.***

Suppose a female rabbit produces F_n (L_n) female rabbits at the n^{th} time point and her female offspring follow the same birth sequence, then show that the new arrivals, C_n, (, D_n,) at the n^{th} time point satisfies

$$C_{n+2} = 2C_{n+1} + C_n \qquad C_1 = 1 \qquad C_2 = 2$$

and

$$D_{n+1} = 3D_n + (-1)^n \qquad D_1 = 1 \ .$$

H-97 ***Proposed by L. Carlitz, Duke University, Durham, N.C.***

Show

(a) $$\sum_{k=0}^{n} \binom{n}{k}^2 L_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} L_{n-k}$$

(b) $$\sum_{k=0}^{n} \binom{n}{k}^2 F_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} F_{n-k} \ .$$

H-98 **Proposed by George Ledin, Jr., San Francisco, Calif.**

If the sequence of integers is designated as J, the ring identity as I, and the quasi-inverse of J as F, then $(I - J)(I - F) = I$ should be satisfied. For further information see R. G. Buschman, "Quasi Inverses of Sequences," American Mathematical Monthly, Vol. 73, No. 4, III (1966) p. 134.

Find the quasi-inverse sequence of the integers (negative, positive, and zero).

H-99 **Proposed by Charles R. Wall, Harker Heights, Texas.**

Using the notation of H-63 (April 1965 FQJ, p. 116), show that if $\alpha = (1 + \sqrt{5})/2$,

$$\prod_{n=1}^{m} \sqrt{5}F_n \alpha^{-n} = 1 + \sum_{n=1}^{m} (-1)^{n(n-1)/2} F(n,m) \alpha^{-n(m+1)}$$

$$\prod_{n=1}^{m} L_n \alpha^{-n} = 1 + \sum_{n=1}^{m} (-1)^{n(n+1)/2} F(n,m) \alpha^{-n(m+1)} ,$$

where

$$F(n,m) = \frac{F_m F_{m-1} \cdots F_{m-n+1}}{F_1 F_2 \cdots F_n} .$$

H-100 **Proposed by D.W. Robinson, Brigham Young Univ., Provo, Utah.**

Let N be an integer such that $F_n \le N < F_{n+1}$, $n \ge 1$. Find the maximum number of Fibonacci numbers required to represent N as an Algebraic Sum of these numbers.

H-101 **Proposed by Harlan Umansky, Cliffside Park, N. J., and Malcolm Tallman, Brooklyn, N.Y.**

Let a, b, c, d be any four consecutive generalized Fibonacci numbers (say $H_1 = p$ and $H_2 = q$ and $H_{n+2} = H_{n+1} + H_n$, $n \ge 1$), then show

$$(cd - ab)^2 = (ad)^2 + (2bc)^2$$

Let $A = L_k L_{k+3}$, $B = 2L_{k+1}L_{k+2}$, and $C = L_{2k+2} + L_{2k+4}$. Then show

$$A^2 + B^2 = C^2 .$$

H-102 **Proposed by J. Arkin, Suffern, N. Y.**

Find a closed expression for A_m in the following recurrence relation.

$$\left[\frac{m}{2}\right] + \overset{\circ}{1} = A_m - A_{m-3} - A_{m-4} - A_{m-5} + A_{m-7} + A_{m-8} + A_{m-9} - A_{m-12},$$

where $m = 0, 1, 2, \cdots$ and the first thirteen values of A_0 through A_{12} are

1, 1, 2, 3, 5, 7, 10, 13, 18, 23, 30, 37, and 47, and $[x]$ is the greatest integer contained in x.

SOLUTIONS

EULER AND FIBONACCI

H-54 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

If F_n is the n^{th} Fibonacci number, then show that

$$\phi(F_n) \equiv 0 \pmod 4, \qquad n > 4$$

where $\phi(n)$ is Euler's function.

Solution by John L. Brown Jr., Penn. State Univ., State College, Pa.

It is well known that $m|n$ implies $F_m|F_n$. Further, $F_n = 3 \pmod 4$ implies $n = 6k - 2$ for $k = 1, 2, 3, \cdots$. Therefore, if F_n is a prime and $n > 4$, F_n must be of the form $4s + 1$ with s a positive integer. [Otherwise, $F_n, n > 4)$ would be of the form $4r + 3$ and hence $n = 6k - 2$ with $k \geq 2$ implying that $F_{3k-1}|F_n$, contrary to assumption.] Since $\phi(p) = p - 1$ for any prime p, it is therefore clear that F_n prime with $n > 4$ implies $\phi(F_n) \equiv \phi(4s + 1) \equiv (4s + 1) - 1 \equiv 4s \equiv 0 \pmod 4$.

Now, for any integer n,

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right),$$

where $p_1, p_2, \cdots, p_k$ are the distinct prime divisors of n. Therefore, for $n = ab$ with a and b integers,

$$\phi(ab) = ab\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right),$$

where $p_1, p_2, \cdots, p_k$ are the distinct prime divisors of ab. Since the distinct prime divisors of either a or b separately are included among those of ab, it is obvious that either $\phi(a) \equiv 0 \pmod 4$ or $\phi(b) \equiv 0 \pmod 4$ necessarily implies $\phi(ab) \equiv 0 \pmod 4$.

We shall now prove by an induction on n that $\phi(F_n) \equiv 0 \pmod 4$ for $n > 4$. First, the result is easily verified for $n = 5, 6, \cdots, 10$. Assume as an induction hypothesis that it has been proved for all $n \leq t$ where t is an integer ≥ 10. Then, if F_{t+1} is prime, we have $\phi(F_{t+1}) \equiv 0 \pmod 4$ by the result of the first paragraph. Otherwise, we distinguish 2 cases. If $t + 1$ is composite,

$t+1$ may be written as $t+1 = m_1 m_2$ where m_1 and m_2 are integers both >1, and at least one of them, say m_1 for definiteness, is >5 and $<t$. Then $F_{m_1} \mid F_{t+1}$, so that $F_{t+1} = F_{m_1} \cdot q$. Since $\phi(F_{m_1}) \equiv 0 \pmod 4$ by the induction hypothesis, we conclude from the remarks of the second paragraph that $\phi(F_{t+1}) \equiv \phi(F_{m_1} \cdot q) \equiv 0 \pmod 4$ as required.

In the alternative case where $t+1$ is prime, we note that F_{t+1} is odd (otherwise $t+1$ would be divisible by 3) and composite. Hence F_{t+1} has only <u>odd</u> prime factors, say $p_1, p_2, \cdots, p_k$, and

$$\phi(F_{t+1}) = F_{t+1}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$

Since $k \geq 2$, it is clear that

$$\phi(F_{t+1}) = \frac{F_{t+1}}{p_1 p_2 \cdots p_k}(p_1 - 1)(p_2 - 1)\cdots(p_k - 1)$$

is divisible by 4. Thus in all cases, $\phi(F_{t+1}) \equiv 0 \pmod 4$ and the proof is completed by mathematical induction.

Also solved by the proposer.

H-56 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Show

$$\sum_{n=1}^{\infty} \frac{F_k^n}{F_n F_{n+2} \cdots F_{n+k} F_{n+k+1}} = \frac{(F_k / F_{k+1})}{\prod_{i=1}^{k+1} F_i}, \quad k \geq 1 .$$

Solution by the Proposer.

1. $$\frac{1}{F_n F_{n+1} F_{n+2}} - \frac{1}{F_{n+1} F_{n+2} F_{n+3}} = \frac{F_{n+3} - F_n}{F_n F_{n+1} F_{n+2} F_{n+3}} = \frac{2}{F_n F_{n+2} F_{n+3}},$$

so that

$$\sum_{1}^{\infty} \frac{1}{F_n F_{n+2} F_{n+3} F_{n+4}} = \frac{1}{2}\sum_{1}^{\infty}\left(\frac{1}{F_n F_{n+1} F_{n+2} F_{n+3}} - \frac{1}{F_{n+1} F_{n+2} F_{n+3}}\right) = \frac{1}{4}.$$

2. $$\frac{1}{F_n F_{n+1} F_{n+2} F_{n+3}} - \frac{1}{F_{n+1} F_{n+2} F_{n+3} F_{n+4}} = \frac{F_{n+4} - 2F_n}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}}$$

$$= \frac{3}{F_n F_{n+2} F_{n+3} F_{n+4}} ,$$

so that

$$3 \sum_1^\infty \frac{2^n}{F_n F_{n+2} F_{n+3} F_{n+4}} = \sum_1^\infty \left(\frac{2^n}{F_n F_{n+1} F_{n+2} F_{n+3}} - \frac{2^{n+1}}{F_{n+1} F_{n+2} F_{n+3} F_{n+4}} \right)$$

$$= \frac{2}{F_1 F_2 F_3 F_4} = \frac{1}{3} .$$

3. $$\frac{1}{F_n F_{n+1} \cdots F_{n+k}} - \frac{F_k}{F_{n+1} F_{n+2} \cdots F_{n+k+1}} = \frac{F_{n+k+1} - F_k F_n}{F_n F_{n+1} \cdots F_{n+k+1}}$$

$$= \frac{F_{k+1}}{F_n F_{n+2} F_{n+3} \cdots F_{n+k+1}} ,$$

so that

$$F_{k+1} \sum_{n=1}^\infty \frac{F_k^n}{F_n F_{n+2} F_{n+3} \cdots F_{n+k+1}} = \frac{F_k}{F_3 F_4 \cdots F_{k+1}}$$

which is equivalent to the stated result.

ONE MOMENT, PLEASE

H-57 ***Proposed by George Ledin, Jr., San Francisco, Calif.***

If F_n is the n^{th} Fibonacci number, define

$$G_n = \left(\sum_{k=1}^n kF_k \right) \Big/ \left(\sum_{k=1}^n F_k \right)$$

and show

(i) $$\lim_{n \to \infty} (G_{n+1} - G_n) = 1$$

(ii) $$\lim_{n \to \infty} (G_{n+1} / G_n) = 1 .$$

Generalize.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

(i) Let H_n be the generalized Fibonacci numbers defined by $H_1 = p$, $H_2 = p + q$, $H_n = H_{n-1} + H_{n-2}$. We may show by induction

$$R_n = \sum_{k=1}^{n} kH_k = (n+1)H_{n+2} - H_{n+4} + H_3 \quad ;$$

$$S_n = \sum_{k=1}^{n} H_k = H_{n+2} - H_2$$

(The first is problem B-40, Fibonacci Quarterly, Vol. 2, No. 2, p. 154.) Let $G_n = R_n/S_n$. Then

$$L = \lim_{n\to\infty} (G_{n+1} - G_n)$$

$$= \lim_{n\to\infty}\left[\frac{(n+2)H_{n+3} - H_{n+5} + H_3}{H_{n+3} - H_2} - \frac{(n+1)H_{n+2} = H_{n+4} + H_3}{H_{n+2} - H_2}\right]$$

and so by dividing we get

$$\text{(1)} \qquad L = \lim_{n\to\infty}\left[\frac{n+2 - H_{n+5}/H_{n+3} + H_3/H_{n+3}}{1 - H_2/H_{n+3}} - \frac{n+1 - H_{n+4}/H_{n+2} + H_3/H_{n+2}}{1 - H_2/H_{n+2}}\right].$$

Horadam, "A Generalized Fibonacci Sequence," American Mathematical Monthly, Vol. 68 (1961), pp. 455-459, has shown

$$\text{(2)} \qquad \lim_{n\to\infty} H_{n+k}/H_n = a^k, \quad a = (1+\sqrt{5})/2$$

and it is easy to prove

$$\text{(3)} \qquad \lim_{n\to\infty} c/F_{n+r} = 0, \quad c, r \text{ constants,}$$

so that

$$L = \lim_{n\to\infty} (n + 2 - H_{n+5}/H_{n+3} - n - 1 + H_{n+4}/H_{n+2})$$
$$= 1 - a^2 + a^2 = 1.$$

(ii) By dividing the numerator and denominator of the main fractions in (1) by n, forming their quotient, applying (2), (3), and the easily proved $\lim 1/nF_{n+r} = 0$, r constant, we find $\lim (G_{n+1}/G_n) = 1$.

Putting $p = 1$, $q = 0$ in the above gives the desired results of the problem.

Also solved by John L. Brown Jr., Penn. State Univ., State College, Pa., and the Proposer.

COMPOSITIONS ANYONE?

H-58 ***Proposed by John L. Brown Jr., Penn. State Univ., State College, Pa.***

Evaluate, as a function of n and k, the sum

$$\sum_{i_1+i_2+\cdots+i_{k+1}=n} F_{2i_1+2}F_{2i_k+2}\cdots F_{2i_k+2}F_{2i_{k+1}+2} ,$$

where $i_1, i_2, i_3, \cdots, i_{k+1}$ constitute an ordered set of indices which take on the values of all permutations of all sets of $k+1$ non-negative integers whose sum is n.

Solution by David Zeitlin, Minneapolis, Minn.

If $V(n,k)$ is the desired sum, then

$$\sum_{n=0}^{\infty} V(n,k)x^n = \left[\sum_{n=0}^{\infty} F_{2n+2}x^n\right]^{k+1} = \frac{1}{(1 - 3x + x^2)^{k+1}} .$$

Since the generating function of the Gegenbauer (or ultraspherical) polynomial, $C_n^{(a)}(u)$, is

$$\sum_{n=0}^{\infty} C_n^{(a)}(u)x^n = \frac{1}{(1 - 2ux + x^2)^a} ,$$

we see that

$$V(n,k) = C_n^{(k+1)}\left(\frac{3}{2}\right)$$

where

$$C_n^{(a)}(\mu) = \frac{1}{\Gamma(a)} \sum_{m=0}^{[n/2]} (-1)^m \frac{\Gamma(a + n - m)}{m!\,(n - 2m)!} (2\mu)^{n-2m} .$$

Thus

$$V(n,k) = \sum_{m=0}^{[n/2]} (-1)^m \binom{k + n - m}{k} \binom{n - m}{m} 3^{n-2m} .$$

Also solved by the proposer.

* * * * *

FIBONACCI ON EGYPTIAN FRACTIONS

M. Dunton and R.E. Grimm, Sacramento State College, Sacramento, Calif.

When J. J. Sylvester wrote in [1] of his method for expressing any rational number between 0 and 1 as a sum of a finite number of unitary fractions, he was not aware that the method was known to Leonardo Pisano in 1202. B. M. Stewart in [2] has a chapter on Egyptian fractions, but makes no mention of Fibonacci. It is hoped that the following translation will make it possible for Fibonacci to speak for himself. In preparing the text which is to be incorporated in a complete, new edition of the Liber Abacci, we have used pp. 77–83 of Baldassare Boncompagni's Scritti di Leonardo Pisano, Vol. 1 (Rome, 1857). We have consulted microfilms of the manuscripts listed in the bibliography in attempting to correct apparent errors. At the same time, we have tried to reproduce Fibonacci's style as closely as seems consistent with readability.

* * *

In this chapter, we explain how to break up rational numbers into unitary fractions so that you may be able to distinguish more intelligently about fractions having any denominator, when considering what part or parts of a unit integer they are.

This work is divided into 7 sections, the first of which is when the larger number, which is below the bar, is divided by the smaller. The rule for this category is that you divide the larger by the smaller and you will have the part which the smaller is of the larger. E.g., we want to know what part 3/12 is to a unit integer. If 12 is divided by 3, the result is 4, for which you say 1/4, which is 3/12 of a unit integer. In the same way 4/20 is 1/5 of a unit integer; 5/100 is 1/20, since 100 divided by 5 gives 20. You should reason in the same way about similar cases.

Now this category is divided into 3 parts, the first of which is simple, the second compound; the third is called compound reversed. The simple is the one I just mentioned. The compound is when the simple has to do with the parts of a second number, as in the case of

$$\frac{\frac{2}{4}}{9}^{*},$$

since 2/4, which belongs to the first simple category, is divided by 9; wherefore,

$$\frac{\frac{2}{4}}{9}$$

becomes

$$\frac{\frac{1}{2}}{9} = \frac{1}{18},$$

while

$$\frac{\frac{2}{6}}{9}$$

becomes

$$\frac{\frac{1}{3}}{9}$$

and

$$\frac{\frac{3}{9}}{10}$$

becomes

$$\frac{\frac{1}{3}}{10},$$

since 3/9 reduced is 1/3, which compounded with 1/10 produces

$$\frac{\frac{1}{3}}{10};$$

and you should reason in this same way about similar cases. The first category compound reversed** is

$$\frac{3/5}{9},$$

*This is a 20th-century interpretation of what Leonardo writes as 20/49. We have consistently made such conversions in this translation.

**Notice his awareness of the commutative property!

this being

$$\frac{\frac{3}{9}}{5}$$

which equals

$$\frac{\frac{1}{3}}{5} \quad ;$$

reason about

$$\frac{\frac{4}{7}}{8}$$

in the same way; it reverses to

$$\frac{\frac{4}{8}}{7} = \frac{\frac{1}{2}}{7} \text{ and } \frac{\frac{5}{9}}{10}$$

becomes

$$\frac{\frac{5}{10}}{9} = \frac{\frac{1}{2}}{9} \quad .$$

The second category is when the larger number is not divisible by the smaller, but from the smaller can be made such parts that the larger is divided by any of the parts themselves. The rule for this category is that you make parts of the smaller by which the larger can be divided; and let the larger be divided by each one of these parts, and you will have unitary fractions which will be the smaller of the larger. E.g., we want to separate 5/6 into unitary fractions. Since 6 is not divisible by 5, 5/6 cannot be of the first category; but since 5 can be partitioned into 3 and 2, by each of which the larger, namely 6, is divisible, 5/6 is proved to belong to the second category. Hence, when 6 is divided by 3 and 2, the result is 2 and 3; for the 2, one takes 1/2, and for 3, take 1/3. Therefore 5/6 = 1/3 + 1/2 of the unit integer; or otherwise, by separating 5/6 into 3/6 and 2/6, each will be one of those two fractions: 3/6, belonging to the first category, equals 1/2, and 2/6 equals 1/3 of one; and 5/6 equals 1/3 + 1/2, as we said above. Likewise, if you resolve 7/8 into 4/8, 2/8 and 1/8, you will have 1/2 for 4/8 and 1/4 for 2/8 and 1/8 for 1/8, i.e., for

7/8 you will have 1/8 + 1/4 + 1/2. Now this second category has in like manner a compound part and a compound reversed part.

$$\frac{\frac{3}{4}}{10}$$

belongs to the compound part, since 3/4 according to the second category is 1/4 + 1/2; wherefore for

$$\frac{\frac{3}{4}}{10}$$

one gets the compounds

$$\frac{\frac{1}{2}}{10} \text{ and } \frac{\frac{1}{4}}{10} ,$$

i.e., 1/20 and 1/40. Likewise,

$$\frac{\frac{5}{8}}{9} \text{ becomes } \frac{\frac{1}{2}}{9} \text{ and } \frac{\frac{1}{8}}{9} ,$$

since 5/8 = 1/2 + 1/8; but for

$$\frac{\frac{5}{8}}{10} ,$$

since it belongs to the first category reversed, you will not resolve into

$$\frac{\frac{1}{2}}{10} \text{ and } \frac{\frac{1}{8}}{10} ,$$

since by the first category it should be reversed into

$$\frac{\frac{5}{10}}{8} = \frac{\frac{1}{2}}{8} ;$$

and this will take place because of the affinity which 5, which is above the 8, has for 10.* Now regarding the compound reversed part of this category, an

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*Recall that in his notation it reads $\frac{5\ \ 0}{8\ 10}$.

example is

$$\frac{\frac{3}{5}}{10}\text{, which is reversed into } \frac{\frac{3}{10}}{5} = \frac{\frac{1}{5}}{5} + \frac{\frac{1}{10}}{5}\ ,$$

i.e., 1/25 + 1/50; because 3/10 is reduced in simple form into 1/5 + 1/10, therefore

$$\frac{\frac{3}{10}}{5}$$

compounded will be resolved into

$$\frac{\frac{1}{5}}{5} \quad \text{and} \quad \frac{\frac{1}{5}}{10}\ .$$

In like manner

$$\frac{\frac{5}{7}}{8} \text{ gives } \frac{\frac{5}{8}}{7} = \frac{\frac{1}{2}}{7} + \frac{\frac{1}{8}}{7}\ ;$$

and you should reason in this way in similar cases. But since we recognize that the parts of the first and second category are necessary before the others in computations, we take pains to show at present in certain tables the breakdowns of the parts of certain numbers, which you should be eager to learn by heart in order better to understand what we mean in this section.*

The third category is when one more than the larger number is divisible by the smaller. The rule for this category is that you divide the number which is more than the larger by the smaller, and the quantity resulting from the division will be such part of the unit integer as is the smaller of the larger, plus the same part of the part which one is of the smaller number. E.g., we want to make unitary fractions of 2/11, which is in this category since one more than 11, i.e., 12, is divisible by 2, which is above the bar; since 6 is produced by this division, the results are 1/6 plus 1/6 of 1/11, i.e.,

$$\frac{\frac{1}{6}}{11}$$

as the unitary fractions of 2/11. In the same way, for 3/11 you will have 1/4 and

*To conserve space we have omitted his table of breakdowns for all rational numbers between 0 and 1 having denominators 6, 8, 12, 20, 24, 60, or 100.

$$\frac{\frac{1}{4}}{11}, \qquad \text{i.e.,} \qquad \frac{1}{44} + \frac{1}{4} \quad .$$

And for 4/11 you will have

$$\frac{1}{3} + \frac{\frac{1}{3}}{11}, \quad \text{i.e.,} \quad \frac{1}{33} + \frac{1}{3} \quad ;$$

and for 6/11 you will have

$$\frac{1}{2} + \frac{\frac{1}{2}}{11}, \quad \text{i.e.,} \quad \frac{1}{22} + \frac{1}{2} \quad ;$$

and for 5/19 you will have

$$\frac{\frac{1}{4}}{19}, \quad \text{i.e.,} \quad \frac{1}{76} + \frac{1}{4} \quad ,$$

since 5 which is above 19 is 1/4 of 20, which is 1 + 19. The third category as well is also compounded twice, as in the case of

$$\frac{\frac{2}{3}}{7} \text{ which is } \frac{\frac{1}{2}}{7} + \frac{\frac{1}{6}}{7}, \text{ since } \frac{2}{3} \text{ is } \frac{1}{6} + \frac{1}{2} \; ;$$

likewise

$$\frac{\frac{4}{7}}{9} \text{ is } \frac{\frac{1}{2}}{9} + \frac{\frac{1}{14}}{9}, \text{ since } \frac{4}{7} \text{ is } \frac{1}{14} + \frac{1}{2} \quad .$$

Andthis same category is also reversed, as

$$\frac{\frac{3}{7}}{11} \text{ or } \frac{\frac{3}{7}}{8} \; ; \text{ for } \frac{\frac{3}{7}}{11} \text{ is } \frac{\frac{3}{11}}{7} \quad .$$

The 3/11 by the third category is 1/44 + 1/4; wherefore

$\frac{\frac{3}{11}}{7}$ is $\frac{\frac{1}{4}}{7} + \frac{\frac{1}{44}}{7}$. Likewise $\frac{\frac{3}{7}}{8}$ is reversed to $\frac{\frac{3}{8}}{7}$, which belongs to two compounded categories, namely the second and third. According to the second compounded category

$$\frac{3/8}{7} \text{ is } \frac{1}{7}\left(\frac{1}{8}+\frac{1}{4}\right) = \frac{1/4}{7}+\frac{1/8}{7} \quad ;^{*}$$

also, according to the third category compounded,

$$\frac{3/8}{7} \text{ is } \frac{1}{7}\left(\frac{1}{24}+\frac{1}{3}\right) ,$$

since 3/8 results in 1/24 + 1/3; and you should reason in the same way about similar cases.

There are times when from this same category two parts can be made from the smaller number above the bar, by either of which one more than the larger is divided without remainder, as in 8/11 and 9/11.** For two parts can be made from 8/11, namely 6/11 and 2/11; whence for 6/11 we have, according to this reasoning, two unitary fractions, namely 1/22 + 1/2, and for 2/11 we have 1/66 + 1/6. Therefore for 8/11 we have 1/66 + 1/22 + 1/6 + 1/2. Likewise, for 9/11, through its being resolved into 6/11 and 3/11, we have

$$\frac{1}{44}+\frac{1}{22}+\frac{1}{4}+\frac{1}{2} \quad ;$$

and for 10/11 we have

$$\frac{1}{33}+\frac{1}{22}+\frac{1}{3}+\frac{1}{2} \quad ,$$

since the 10 above the 11 is 1/3 + 1/2 of 12, which is one more than the 11 under the bar.

The fourth category is when the larger is prime and one more than the larger is divided by one less than the smaller, as 5/11 and 7/11. The rule for this category is that you subtract one from the smaller, from which you will make one unitary fraction such as the number will be which is under the bar;

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*Note the distributive law! His notation is $\dfrac{1\ \ 1\ \ 0}{8\ \ 4\ \ 7}$.

**It appears that he is trying to generalize. If so, he must have recognized a flaw; for in the next section he uses a different approach and then you will have as remainders parts of the third category, as in the case when you subtract 1/11 from 5/11, with the remainder being 4/11. For this, by the third category, you will have unitary fractions 1/33 + 1/3; with the above-written 1/11 added on, the result is

$$\frac{1}{33} + \frac{1}{11} + \frac{1}{3} .$$

In the same way for 7/11 you will have

$$\frac{1}{22} + \frac{1}{11} + \frac{1}{2} ;$$

and for 3/7 you will have

$$\frac{1}{28} + \frac{1}{7} + \frac{1}{4} ;$$

and for 6/19 you will have

$$\frac{1}{76} + \frac{1}{19} + \frac{1}{4} ;$$

and for 7/29 you will have

$$\frac{1\frac{1}{5}}{29} + \frac{1}{5} , \quad \text{i.e.,} \quad \frac{1}{145} + \frac{1}{29} + \frac{1}{5} .$$

The fifth category is when the larger number is even, and one more than the larger is divisible by two less than the smaller. The rule for this category is that you subtract 2 from the smaller number, and this 2 will belong to the first category, but the remainder will belong to the third, as in 11/26 from which if you subtract 2/26, which equals 1/13 according to the rule of the first category, the remainder is 9/26, which equals

$$\frac{1/3}{26} + \frac{1}{3} , \quad \text{e.g.,} \quad \frac{1}{78} + \frac{1}{3} .$$

To this add 1/13; the result will be

$$\frac{1}{78} + \frac{1}{13} + \frac{1}{3}$$

as the unitary fractions of 11/26; and in the same way for 11/62 you will have

$$\frac{1/7}{62} + \frac{1}{31} + \frac{1}{7} \quad .$$

The sixth category is when the larger number is divisible by 3, and one more than the larger is divisible by 3 less than the smaller, as in 17/27. The rule for this is that from the parts themselves you will subtract three parts, i. e., that you will subtract 3 from the lesser. These three parts will belong to the first category, the rest to the third, as when from 17/27 you subtract 3/27 (1/9 according to the first category of our topic); and to 14/27, which by the third category is 1/54 + 1/2, adding the 1/9 written above, you get

$$\frac{1}{54} + \frac{1}{9} + \frac{1}{2}$$

as the parts of 17/27. In the same way for 20/33 you will have

$$\frac{1}{66} + \frac{1}{11} + \frac{1}{2} \quad .$$

The seventh category is when none of the above-described categories occurs. The rule for this category is very useful, since through it the parts of certain above-described categories, viz., the second, fourth, fifth and sixth categories, are occasionally found better than through the rules of those categories themselves. Hence the parts of the four categories themselves are always to be derived through this seventh rule, so that you can more precisely discover the neater parts, either through the rules of the categories themselves or through the present rule. The rule for this category is that you divide the larger number by the smaller; and when the division itself is not even, look to see between what two numbers that division falls. Then take the larger part,* and subtract it, and keep the remainder. If this will be from one of the above-described categories, then you will take the larger part of the remainder itself; and you will do this until the parts of one of the above-described categories remain, or until you have all unitary fractions. E. g., we want to break down 4/13 into unitary fractions. Now 13 divided by 4 falls between 3 and 4; wherefore 4/13 of the unit integer is less than 1/3 and more than 1/4 of it; wherefore

*By larger part, he evidently means the unitary fraction with the larger denominator. In the example this would be 1/4.

we perceive that 1/4 is the largest unitary fraction that can be taken from 4/13. For 13/13 makes the unit integer; wherefore one fourth of 13/13, i. e.,

$$\frac{13}{4 \times 13}$$

is 1/4 of the unit integer; wherefore subtract 13/(4 x 13) from 4/13, and the remainder will be 3/(4 x 13), which by the second category is

$$\frac{1}{13}\left(\frac{1}{4} + \frac{1}{2}\right), \text{ i. e., } \frac{1}{52} + \frac{1}{26} .$$

Or since 3/(4 x 13) is 3/52, which by the rule for the second category is in like manner 1/52 + 1/26, we thus have for 4/13 three unitary fractions, namely

$$\frac{1}{52} + \frac{1}{26} + \frac{1}{4} .$$

Otherwise, you can find the parts of 3/52 by this seventh rule, namely by dividing by 3; the result is 17, and more; wherefore 1/18 is the largest part which is in 3/52. Consequently, 52 divided by 18 gives $2 + \frac{8}{9}$; if this is taken from 3, the remainder is

$$\frac{1}{9 \times 52} \text{ or } \frac{1}{468} .$$

Therefore for 3/52 we have

$$\frac{1}{468} + \frac{1}{18} ;$$

wherefore for 4/13 we have

$$\frac{1}{468} + \frac{1}{18} + \frac{1}{4} .$$

Likewise, you will make unitary fractions of 9/61 in the following way. Divide 61 by 9, and the result will be 6, and more; wherefore you will have 1/7 as the largest unitary fraction of 9/61. And so you will divide 61 by 7, with the result $8 + \frac{5}{7}$, which are sixty-first parts of one; subtract this from 9/61, and the remainder will be

$$\frac{2}{7 \times 61}\text{, i.e., } \frac{2}{427}\text{; this } \frac{2}{427} \text{ is } \frac{1}{214} \text{ and } \frac{1}{214 \times 427} ,$$

according to the third category; therefore 9/61 is

$$\frac{1}{214 \times 427} + \frac{1}{214} + \frac{1}{7} .$$

Since 2/(7 x 61) by the compound third category results in

$$\frac{1}{4 \times 61} \quad \text{and} \quad \frac{1}{28 \times 61} ,$$

consequently for 9/61, the result is

$$\frac{1}{1708} + \frac{1}{244} + \frac{1}{7} .^{*}$$

Likewise, we want to demonstrate this same method concerning 17/29. If 29 is divided by 17, we obtain 1, and more; wherefore we perceive that 17/29 is more than half of the unit integer; and it is to be noted that 3/3, 4/4, 5/5, or 6/6 makes the unit integer; in like manner 29/29 makes the unit integer. If we take half of it, i.e., $14\frac{1}{2}/29$, and take this from 17/29, the remainder will be $2\frac{1}{2}/29$, i.e., 5/58; wherefore 17/29 is 5/58 + 1/2. Of this 5/58 one must make unitary fractions, namely by this same category; wherefore divide 58 by 5, and the result will be 11, and more. Whence one perceives that 1/12 is the largest unitary fraction that is in 5/58; whence one should take 1/12 from 58/58, i.e., from the integer, with the result $4\frac{5}{6}/58$, which is less than 5/58 by 1/(6 x 58), i.e., 1/348; and so you will have for 17/29 three unitary fractions, namely

$$\frac{1}{348} + \frac{1}{12} + \frac{1}{2} .$$

*By compound third category he means factor out 1/61 from 2/(7 x 61) and apply the third category method to 2/7. This gives 2/7 = 1/28 + 1/4, and so 2/(7 x 61) = 1/61 (1/28 + 1/4) = 1/(61 x 28) + 1/(61 x 4).

Now there is in similar cases a certain other universal rule, namely, that you find a number which has in it many divisors like 12, 24, 36, 48, 60, or any other number which is greater than half the number under the bar, or less than double it, so that for the above-written 17/29 we take 24, which is more than half of 29; and therefore multiply the 17 which is above the bar by 24, and it will be 408; divide this by 29 and by 24, with the result

$$\frac{14\frac{2}{29}}{24} \quad .$$

Then see what part 14 is of 24; the result is

$$\frac{1}{4} + \frac{1}{3} \text{ or } \frac{1}{12} + \frac{1}{2} \qquad ;$$

keep these as parts of 17/29; and again see what part the 2 which is above the 29 is of 24; the result is 1/12 of it, for which you will have (1/12)/29 among the parts of 17/29. Since 2/29 of 1/24 = 2/24 of 1/29, which is namely 1/348, therefore for 17/29 you will have

$$\frac{1}{348} + \frac{1}{4} + \frac{1}{3}, \text{ or } \frac{1}{348} + \frac{1}{12} + \frac{1}{2} \quad ,$$

as we found out above.

Likewise, if you multiply the 17 which is above the 29 by 36, just as you multiplied it by 24, and divide in a similar way by 29 and by 36, the result is

$$\frac{21\frac{3}{29}}{36}, \text{ the 21 being } \quad \frac{1}{4} + \frac{1}{3} \quad \text{or} \quad \frac{1}{12} + \frac{1}{2} \text{ of } 36;$$

and 3, which is above 29, is 1/12 of 36; and from the 3 which is above 29, the result will be

$$\frac{\frac{1}{12}}{29}, \text{ i.e., } \frac{1}{348} \quad ;$$

and so you will have again as the unitary fractions of 17/29,

$$\frac{1}{348} + \frac{1}{4} + \frac{1}{3}, \quad \text{or} \quad \frac{1}{348} + \frac{1}{12} + \frac{1}{2} \quad .$$

And if you want to know why we multiplied by 24 that 17 which is above the 29, and divided the product by 29, you should know that of 17/29 we made twenty fourths because 24 is a number composed of many numbers, whence its parts derive from the first and second category. For 17/29 has been found, as previously said, to be

$$\frac{14\,\frac{2}{29}}{24}\text{, of which } \frac{14}{24} \quad ,$$

which is at the head of the bar,* is expressed by the second category as

$$\frac{1}{4} + \frac{1}{3} \text{ or } \frac{1}{12} + \frac{1}{2} \quad ;$$

and for the remainder $\frac{2}{29}/24$, it is expressed by the first category reversed as

$$\frac{\frac{2}{24}}{29}\text{ , i. e., } \frac{\frac{1}{12}}{29} \quad .$$

Likewise, when you have multiplied 17 by 36 and divided by 29, then you have made thirty sixths of 17/29. For 29/29 is equal to 36/36; wherefore the ratio which 29 is to 36 will also hold true for 17 to the fourth number; wherefore we multiplied the third number, namely 17, by the second, namely 36, and divided the product by the first, because when the four numbers are proportional, the multiplication of the second by the third is equal to the multiplication of the first by the fourth, as has been shown by Euclid.

Likewise, if you want to break down 19/53 into unitary fractions, although it belongs to the fourth category, when one more than 53 is divided by one less than 19, whence for 19/53 you will have

$$\frac{1}{159} + \frac{1}{53} + \frac{1}{3} \quad ,$$

let us next show in what way it should be done by the seventh category. Now 53 divided by 19 falls between 2 and 3; wherefore we have 1/3 as the largest unitary

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*In his notation it reads $\frac{2}{29}\,\frac{14}{24}$. Since he reads numbers from right to left, $\frac{14}{24}$ is at the beginning or head of the fraction.

fraction which can be taken from 19/53; and subtract a third of 53, namely $17\frac{2}{3}$, from 19, with the remainder being $1\frac{1}{3}$, i. e., $1\frac{1}{3}/53$; therefore the unitary fractions of 19/53 are

$$\frac{1}{159} + \frac{1}{53} + \frac{1}{3} \quad ,$$

as we discovered through the rule of the fourth category.

Now by this rule one cannot so easily make the unitary fractions of 20/53. Hence you will find them through another rule, viz., by multiplying 20 by some number which has many divisors, as we have said previously. Now if 20 is multiplied by 48 and divided by 53 and then by 48, the result is

$$\frac{18\frac{6}{53}}{48}\ , \quad \text{the 18 being } \frac{1}{8} + \frac{1}{4} \text{ of 48,} \quad \text{or} \quad \frac{1}{24} + \frac{1}{3} \quad ;$$

and the 6 which is above the 53 is 1/8 of 48; wherefore it will be $\frac{1}{8}/53$, since the 6 is above the 53; therefore for the unitary fractions of 20/53 you have

$$\frac{\frac{1}{8}}{53} + \frac{1}{24} + \frac{1}{3} \quad ;$$

and you should strive to operate in this way in all similar cases; and when you cannot have through any of the above-mentioned rules convenient unitary fractions of any similar cases, you should strive to find them through one of the others; and one should note that there are many fractions which should be adapted before they are broken down into unitary fractions, namely when the larger number is not divided by the smaller and has in turn some common divisor, as in the case of 6/9, each of which numbers is exactly divided by 3; wherefore you will divide each of them by 3, with the resulting 2 above the bar and 3 below it, i. e., 2/3, which belongs to the third category, when one more than 3 is divisible by 2, whereby they are 1/6 + 1/2. It is the same in the case of 6/8, each of which numbers is divisible by 2, whence they are reduced to 3/4, equal to 1/4 + 1/2 by the second category; and you should reason in this way about similar cases. And if several fractions are under one bar, they should be reduced to one fraction beneath the bar, as in the case of $3\frac{1}{2}/8$, which is 7/16. And they are reduced as follows: 3, which is above the 8, is multiplied by 2, and 1 is added; we put it down, and thus we have 7; and you will multiply 2 by 8, which is under the bar, and get 16; this 16 we put under the bar, and above it we put the 7.

Likewise, $\dfrac{4\,\frac{3\frac{2}{3}}{5}}{9}$* is $\dfrac{71}{135}$,

which is found according to the above-written method, namely by multiplying the 4 above the 9 by 5 and adding 3; this product is multiplied by 3 and 2 is added; and so we have 71 above the bar; and from multiplying 3 by 5 and that product by 9 we have 135 below the bar; this 71/135 according to the seventh rule is broken down into

$$\frac{1}{270} + \frac{1}{45} + \frac{1}{2} \quad .$$

And it should be noted that when by the seventh rule you take the largest part, which the smaller number will be of the larger, and leave the unitary fractions that remain, the result is less than elegant. You will leave that larger part and work with the other following part, which is less than it; so that if the larger part is 1/5, you will work with 1/6; and if it is 1/7, you will work with 1/8. E.g., in 4/49 the largest part is 1/13; when this is taken from 4/49, the remainder is

$\dfrac{\frac{3}{13}}{49}$, namely $\dfrac{3}{637}$, which by the fourth category is $\dfrac{\frac{1}{319}}{637} + \dfrac{1}{319}$;

therefore, for 4/49 we have

$$\frac{\frac{1}{319}}{637} + \frac{1}{319} + \frac{1}{13} \quad ,$$

which is less than elegant; wherefore you will abandon 1/13 and work with 1/14; when this is taken from 4/49, the remainder is

$\dfrac{\frac{1}{2}}{49}$, i.e., $\dfrac{1}{98}$; and so for $\dfrac{4}{49}$ we have $\dfrac{1}{98} + \dfrac{1}{14}$,

---

*In his notation $\dfrac{2\ 3\ 4}{3\ 5\ 9}$.

which are more elegant fractions than those previously found. They are found also in another way, namely by dividing the 4 above the 49 by a divisor of 49, with the result $\frac{4}{7}/7$, which by the third category compounded is

$$\frac{1}{7}\left(\frac{1}{14} + \frac{1}{2}\right); \quad \text{for } \frac{\frac{1}{2}}{7} \quad \text{is} \quad \frac{1}{14}, \text{ and } \frac{\frac{1}{14}}{7} \quad \text{is} \quad \frac{1}{98} \quad ;$$

and thus for 4/49 we have in like manner 1/98 + 1/14.

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CORRECTION

In "An Almost Linear Recurrence," by Donald E. Knuth, April 1966 Fibonacci Quarterly, p. 123, replace c_n by $\ln c_n$ in Eq. 13.

In "On the Quadratic Character of the Fibonacci Root," by Emma Lehner, April 1966 Fibonacci Quarterly, please make the following corrections:

p. 136, line -9: This line should end in θ.

p. 136, line -5: Replace 5 by $\theta\sqrt{5}$.

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BINOMIAL SUMS OF FIBONACCI POWERS

JOHN WESSNER, Melbourne, Florida

At the impetus of Professor Hoggatt, a general solution was obtained for summations of the form

$$\sum_{k=0}^{n} \binom{n}{k} (\pm 1)^k F_k^b \quad ,$$

where $b = 2m$. Some suggestions will be made for attacking the problem for odd b, and it is hoped that a complete solution will be forthcoming.

Let us first consider the case where $b = 4p$. If $\alpha = 1/2(1 + \sqrt{5})$, $\beta = 1/2\ (1 - \sqrt{5})$, $F_k = (\alpha^k + \beta^k)/(\alpha - \beta)$, and $L_k = \alpha^k + \beta^k$, then

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} F_k^{4p} &= \sum_{k=0}^{n} \binom{n}{k} 5^{-2p} (\alpha^k - \beta^k)^{4p} \\
&= \sum_{k=0}^{n} \binom{n}{k} 5^{-2p} \sum_{t=0}^{4p} \binom{4p}{t} (-1)^t (\alpha^{4p-t} \beta^t)^k \\
&= 5^{-2p} \sum_{k=0}^{n} \binom{n}{k} \left\{ \sum_{j=0}^{2p-1} \binom{4p}{j} (-1)^{j(k+1)} \left[(\alpha^{4p-2j})^k + (\beta^{4p-2j})^k \right] \right. \\
&\qquad \left. + \binom{4p}{2p} (-1)^{2p(k+1)} \right\} \\
&= 5^{-2p} \sum_{j=0}^{2p-1} \binom{4p}{j} (-1)^j \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^{jk} L_{(4p-2j)k} \right\} + 5^{-2p} \binom{4p}{2p} 2^n ,
\end{aligned}$$

where we have made use of the fact that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \ .$$

Hence, we have reduced the problem to one involving Lucas numbers and no powers. We must digress to obtain the required Lucas formulas.

Consider

$$L(n,1,q) = \sum_{k=0}^{n} \binom{n}{k} L_{qk} = \sum_{k=0}^{n} \binom{n}{k} (\alpha^{qk} + \beta^{qk})$$
$$= (1 + \alpha^q)^n + (1 + \beta^q)^n .$$

If $q = 2g$, then

$$L(n,1,2g) = \left\{(\alpha\beta)^g(-1)^g + \alpha^{2g}\right\}^n + \left\{(\alpha\beta)^g(-1)^g + \beta^{2g}\right\}^n$$
$$= \alpha^{gn}\left\{\alpha^g + (-1)^g\beta^g\right\}^n + \beta^{gn}\left\{(-1)^g\alpha^g + \beta^g\right\}^n .$$

The manipulation of this depends upon the parity of g. Let $g = 2r$, or $q = 4r$:

$$L(n,1,4r) = (\alpha^{2rn} + \beta^{2rn})(\alpha^{2r} + \beta^{2r})^n = L_{2rn}L_{2r}^n .$$

If, instead, $q = 4r + 2$, then for odd values of n we obtain

$$L(n,1,4r+2) = (\alpha^{(2r+1)n} + (-1)^n\beta^{(2r+1)n})(\alpha^{2r+1} - \beta^{2r+1})^n$$
$$= 5^{\frac{1}{2}(n+1)}F_{(2r+1)n}F_{2r+1}^n .$$

For even values of n,

$$L(n,1,4r+2) = 5^{\frac{1}{2}n}L_{(2r+1)n}F_{2r+1}^n .$$

By the same methods we obtain

$$L(n,-1,4r+2) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{(4r+2)k} = (-1)^n L_{(2r+1)n}L_{2r+1}^n ,$$

and

$$L(n,-1,4r) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{4rk} = 5^{\frac{1}{2}n} L_{2rn} F_{2r}^n$$

for even n, and

$$L(n,-1,4r) = -5^{\frac{1}{2}(n+1)} F_{2rn} F_{2r}^n$$

if n is odd.

Now, let us return to the original problem.

$$\sum_{k=0}^{n} \binom{n}{k} F_k^{4p} = 5^{-2p} \binom{4p}{2p} 2^n$$

$$+ \; 5^{-2p} \sum_{j=0}^{2p-1} \binom{4p}{j} (-1)^j \sum_{k=0}^{n} \binom{n}{k} (-1)^{jk} L_{(4p-2j)k}$$

$$= 5^{-2p} \left\{ 2^n \binom{4p}{2p} + \sum_{i=0}^{p-1} \binom{4p}{2i} L(n,1,4[p-i]) - \sum_{i=1}^{p} \binom{4p}{2i-1} L(n,-1,4[p-i]+2) \right\}$$

$$= 5^{-2p} \left\{ 2^n \binom{4p}{2p} + \sum_{i=0}^{p-1} \binom{4p}{2i} L_{2(p-i)n} L_{2(p-i)}^n \right.$$

$$\left. - (-1)^n \sum_{i=1}^{n} \binom{4p}{2i-1} L_{2(p-i)n+n} L_{2(p-i)+1}^n \right\} .$$

Similarly, if $b = 4p + 2$,

$$\sum_{k=0}^{n} \binom{n}{k} F_k^{4p+2} = 5^{-(2p+1)} \sum_{k=0}^{n} \binom{n}{k} \left\{ \sum_{j=0}^{2p} \binom{4p+2}{j} (-1)^{j(k+1)} [(\alpha^{4p-2j+2})^k \right.$$

$$\left. + \; (\beta^{4p-2j+2})^k] + \binom{4p+2}{2p+1} (-1)^{(k+1)(2p+1)} \right\}$$

$$= 5^{-(2p+1)} \sum_{j=0}^{2p} \binom{4p+2}{j} (-1)^j \sum_{k=0}^{n} \binom{n}{k} (-1)^{jk} L_{(4p-2j+2)k}$$

$$+ 5^{-(2p+1)} \binom{4p+2}{2p+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1}$$

$$= 5^{-(2p+1)} \left\{ \sum_{i=0}^{p} \binom{4p+2}{2i} L(n,1,4[p-i]+2) - \sum_{i=0}^{p-1} \binom{4p+2}{2i+1} L(n,-1,4[p-i]) \right\}.$$

For summations involving alternating signs the same method of analysis gives similar results.

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k^{4p} = 5^{-2p} \left\{ \sum_{i=0}^{p-1} \binom{4p}{2i} L(n,-1,4[p-i]) - \sum_{i=1}^{p} \binom{4p}{2i-1} L(n,1,4[p-i]+2) \right\},$$

and

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k^{4p+2} = 5^{-(2p+1)} \left\{ \sum_{i=0}^{p} \binom{4p+2}{2i} L(n,-1,4[p-i]+2) - \sum_{i=0}^{p-1} \binom{4p+2}{2i+1} L(n,1,4[p-i]) - \binom{4p+2}{2p+1} 2^n \right\}.$$

Finally, a word regarding summations for odd powers of F_k. For powers $b \geqq 5$, the problem still reduces to a binomial sum involving Lucas numbers whose subscripts are in an arithmetic progression, but the expressions $(1+\alpha^q)$ and $(1+\beta^q)$ cannot be reduced in a manner similar to that used for even q. Surely they are reducible, and it is hoped that the expression obtained above may be extended to odd powers.

★ ★ ★ ★ ★

A NOTE ON A THEOREM OF JACOBI

JOSEPH ARKIN, Suffern, New York

It has been shown that the sequence, L_n, or $1, 3, 4, 7, 11, 18, 29, 47, \cdots$ is defined by

$$(1) \qquad L_n = u^n + (-u)^{-n} \qquad (u = (1 + \sqrt{5})/2) \quad ,$$

where L_n is the n^{th} Lucas number.

<u>Theorem 1</u>. If $L_{2^{n-1}} = c_n$ $(n = 2, 3, 4, \cdots)$, p is a prime $4m + 3$, $M_p = 2^p - 1$ and

$$F(x) = \prod_{n=2}^{\infty}\left(1 - x^{2^{n-1}}\right)\left(1 + 3x^{2^{n-1}-1} + x^{2^n-2}\right) = 1 + \sum_{n=2}^{\infty} c_n x^{2^{n-1}} \quad ;$$

Then, M_p is a prime if and only if $c_p \equiv 0 \pmod{M_p}$.

<u>Proof</u>. Using the famous identity of Jacobi from Hardy and Wright [1, p. 282],

$$(2) \quad \prod_{n=1}^{\infty}((1 - x^{2n})(1 + x^{2n-1}z)(1 + x^{2n-1}z^{-1})) = 1 + \sum_{n=1}^{\infty} x^{n^2}(z^n + z^{-n})$$

we put $z + z^{-1} = 3$ $(z = (3 + \sqrt{5})/2)$, and combining z with u in (1) we have $u^2 = z$, so that $L_{2n} = z^n + z^{-n}$, which leads to

$$(3) \quad \prod_{n=1}^{\infty}(1 - x^{2n})(1 + 3x^{2n-1} + x^{4n-2}) = 1 + \sum_{n=1}^{\infty} L_{2n} x^{n2} \quad .$$

Next, in Theorem 1 we put $L_{2^{n-1}} = c_n$ and replace n with 2^{n-2}, where it is evident the resulting equation is identical to $F(x)$.

We complete the proof of Theorem 1 with the following theorem of Lucas appearing in [2, p. 397]:

... If $4m + 3$ is prime, $P = 2^{4m+3} - 1$ is prime if the first term of the series $3, 7, 47, \cdots$, defined by $r_{n+1} = r_n^2 - 2$, which is divisible by P is of rank $4m + 2$; but P is composite if no one of the first $4m + 2$ terms is divisible by P... .

<u>Corollary</u>. If $[x]$ denotes the greatest integer contained in x and $n!/(n - r)!r! = \binom{n}{r}$, then

$$z^n + z^{-n} = n \sum_{r=0}^{[n/2]} (-1)^r (n - r)^{-1} \binom{n - r}{r} b^{n-2r} \quad .$$

The proof of the Corollary is obtained by elementary means if we put

$$z^n + z^{-n} = ((b + \sqrt{b^2 - 4})/2)^n + ((b + \sqrt{b^2 - 4})/2)^{-n}$$

and then add the right side of the equation.

In conclusion, although there are many special cases to the Corollary, the one obtained by setting $b = 0$ may be worth mentioning.

Let

$p(n)$ = the number of unrestricted partitions of an integer n,

$p_m(n)$ = the number of partitions of n into parts not exceeding m, where $p(0) = p_m(0) = 1$.

We then have the following:

<u>Theorem 2.</u> If

$$F_m(x) = \prod_{n=1}^{m} (1 - x^n)^{-1} = 1 + \sum_{n=1}^{\infty} p_m(n)x^n$$

and

$$\prod_{n=1}^{\infty} (1 - x^n)^{-1} = 1 + \sum_{n=1}^{\infty} p(n)x^n \quad ,$$

then

$$p(2u) \equiv p_2(u-2) + p_4(u-8) + \cdots + p_{2r}(u-2r^2) + \cdots \pmod 2$$

and

$$p(2u+1) \equiv p_1(u) + p_3(u-4) + \cdots + p_{2r+1}(u-2r^2-2r) + \cdots \pmod 2.$$

Proof. Putting $z + z^{-1} = 0$, then $z = i$ $(i^2 = -1)$, so that $z^{2n} + z^{-2n} = 2(-1)^n$. Then applying these results, together with our replacing x with x^2 in Eqs. (2) and (3), leads to

$$\prod_{n=1}^{\infty}(1-x^n)(1+x^{2n-1}) = 1 + 2\sum_{r=1}^{\infty}(-1)^r x^{2r^2}$$

and it is evident that

$$\prod_{n=1}^{\infty}(1+x^{2n-1}) \equiv \sum_{n=0}^{\infty} p(n)x^n \pmod 2 . \tag{4}$$

According to Hardy and Wright [1, p. 281], Euler proved that

$$\prod_{n=1}^{\infty}(1+x^{2n-1}) = 1 + \sum_{r=1}^{\infty} x^{r^2} F_r(x^2) ,$$

and combining this result with $F_m(x)$ in Theorem 2 and with Eq. (4), we complete the proof of Theorem 2.

ACKNOWLEDGMENT

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COMPOSITION OF RECURSIVE FORMULAE

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The purpose of this article is to consider some properties of composite functions, such as F_{F_n}, L_{F_n}, etc. For ease of notation, F(n) and L(n) will represent the usual Fibonacci and Lucas sequences, respectively. The following notations will also be adopted:

$$\begin{aligned} g(n) &\equiv L\{F(n)\} \\ h(n) &\equiv F\{F(n)\} \\ f(n) &\equiv F\{L(n)\} \\ k(n) &\equiv L\{L(n)\} \end{aligned}$$

Part I. Recursive Relations for g(n), h(n), f(n), k(n).

Although hybrid relations for the above were sought, only partial success was achieved.

(1) $$5h(n)h(n+1) = g(n+2) = (-1)^{F(n)}g(n-1)$$

Proof. In this and subsequent parts, considerable use was made of the well known identities,

(a) $$\sqrt{5}F(n) = \alpha^n - \beta^n$$

(b) $$L(n) = \alpha^n + \beta^n$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2} .$$

In (a) replace n by F(n) and obtain

$$\begin{aligned} \sqrt{5}h(n) &= \alpha F(n) - \beta F(n) \\ \sqrt{5}h(n+1) &= \alpha F(n+1) - \beta F(n+1) \end{aligned}$$

Multiplying these and observing

$$\begin{aligned} F(n+2) &= F(n) + F(n+1) \\ F(n+1) &= F(n) + F(n-1) \\ \alpha\beta &= -1 \end{aligned}$$

the result follows.

(2) $$\frac{g(n)g(n+1) - g(n+2)}{g(n-1)} = (-1)^{F(n)}$$

Proof. In (b) replace n by F(n) and then as above the result follows.

(3) $$g(n)g(n + 1) + 5h(n)h(n + 1) = 2g(n + 2)$$

Proof. Combine (1) and (2).

(4) $$5f(n)f(n + 1) = k(n + 2) - (-1)^{L(n)}k(n - 1)$$

Proof. In (a) replace n by L(n) and as above the result follows.

(5) $$\frac{k(n)k(n + 1) - k(n + 2)}{k(n - 1)} = (-1)^{L(n)}$$

Proof. In (b) replace n by L(n) and as in (2) the result follows upon rearrangement.

(6) $$k(n)k(n + 1) + 5f(n)f(n + 1) = 2k(n + 2)$$

Proof. Combine (4) and (5).

(7) $$g(n + 1)g(n - 1) = k(n) + (-1)^{F(n-1)}g(n) .$$

Proof. In (b) replace n by F(n) and observe that

$$L(n) = F(n + 1) + F(n - 1).$$

The result then follows.

(8) $$5h(n - 1)h(n + 1) = k(n) = (-1)^{F(n-1)}g(n).$$

Proof. In (a) replace n by F(n - 1) and F(n + 1) and multiply.

(9) $$g(n + 1)g(n - 1) + 5h(n - 1)h(n + 1) = 2k(n)$$

Proof. Combine (7) and (8).

(10) $$g(n + 1)g(n - 1) - 5h(n - 1)h(n + 1) = 2(-1)^{F(n-1)}g(n).$$

Proof. As above, combining (7) and (8) yields the result.

By combining the above relationships, many others may be obtained. Note that (2) and (5) are hybrid relations, whereas the others are mongrel. It would be of interest to obtain hybrid relations for h(n) and f(n). In the light of the above results one can scarcely help setting up the correspondences

$$h(n) \longleftrightarrow f(n)$$

$$g(n) \longleftrightarrow k(n).$$

Part II. Explicit Relations for $g(n)$, $h(n)$, $f(n)$, $k(n)$.

Using (a) and (b) one immediately obtains the rather clumsy formulae

$$g(n) = \alpha^{\frac{1}{\sqrt{5}}(\alpha^n - \beta^n)} + \beta^{\frac{1}{\sqrt{5}}(\alpha^n - \beta^n)}$$

$$h(n) = \frac{1}{\sqrt{5}}\left\{\alpha^{\frac{1}{\sqrt{5}}(\alpha^n - \beta^n)} - \beta^{\frac{1}{\sqrt{5}}(\alpha^n - \beta^n)}\right\}$$

$$f(n) = \frac{1}{\sqrt{5}}\left\{\alpha^{(\alpha^n + \beta^n)} - \beta^{(\alpha^n + \beta^n)}\right\}$$

$$k(n) = \alpha^{(\alpha^n + \beta^n)} + \beta^{(\alpha^n + \beta^n)}$$

Part III. Generating Functions for $g(n)$, $h(n)$, $f(n)$, $k(n)$.

The original goal of obtaining generating functions of the type

$$F(x) = \sum_0^\infty h(k)x^k ,$$

was not achieved.

The methods used by Riordan [1] did not appear to offer much help in this direction. However, ugly mixed types could be obtained as the following example illustrates:

$$\frac{x \ln \alpha}{1 - x^2} = \sum_0^\infty \ln\left\{\frac{\sqrt{5}h(i) + g(i)}{2}\right\} x^i$$

Proof.

$$\alpha^{F(n)} = \frac{\sqrt{5}h(n) + g(n)}{2}$$

Also, since

$$\frac{x}{1 - x - x^2} = \sum_0^\infty F_i x^i ,$$

the result follows.

Part IV. Upper Bounds for $h(n)$, $g(n)$, $f(n)$, $k(n)$.

Part II yields explicit values for $h(n)$, etc., but the relations are awkward to work with and more workable bounds were sought.

(1) $$h(n) \le 2^{2^{n-2}-2} \quad \text{for} \quad n \ge 3$$

Proof. $h(n)$, $g(n)$, $f(n)$, $k(n)$ are increasing functions of n, and it is well known that

$$F(n) \le 2^{n-2} \quad \text{for} \quad n \ge 3.$$

Hence

$$h(n) = F\{F(n)\} \le F(2^{n-2}) \le 2^{2^{n-2}-2}$$

(2) $$f(n) \leq 2^{5(2)^{n-3}-2} \quad \text{for} \quad n \geq 3$$

<u>Proof.</u>

$$L(n) = F(n-1) + F(n+1) \leq 2^{n-3} + 2^{n-1} = 5(2)^{n-3}$$

The result follows as above.

(3) $$k(n) \leq 5(2)^{5(2)^{n-3}-3} \quad \text{for} \quad n \geq 3$$

<u>Proof.</u> The result follows as above.

(4) $$g(n) \leq 5(2)^{2^{n-2}-3} \quad \text{for} \quad n \geq 3$$

<u>Proof.</u> As above.

The above inequalities may be replaced by strict inequalities for $n > 3$, since

$$F(n) < 2^{n-2} \qquad \text{for} \quad n > 3$$

Proposal for Future Investigations

To reiterate, the two most interesting avenues for future work are the development of generating functions for $h(n)$, $g(n)$, $f(n)$, $k(n)$, and hybrid recursive relations for $h(n)$ and $f(n)$.

REFERENCE

1. John Riordan, "Generating Functions for Powers of Fibonacci Numbers," <u>Duke Math. J.</u>, 29 (1962), pp. 5-12.

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NOTE ON THE EUCLIDEAN ALGORITHM

R. L. DUNCAN, Lock Haven State College, Lock Haven, Pa.

Let $\mu_1 = 1$; $\mu_2 = 2$ and $\mu_n = \mu_{n-1} + \mu_{n-2}$ $(n \geq 3)$ be the Fibonacci numbers. Then the Euclidean Algorithm for finding (μ_{n+1}, μ_n) is

$$\begin{aligned}
\mu_{n+1} &= \mu_n + \mu_{n-1} \\
\mu_n &= \mu_{n-1} + \mu_{n-2} \\
&\vdots \\
\mu_3 &= \mu_2 + \mu_1 \\
\mu_2 &= 2\mu_1
\end{aligned}$$

and the required number of divisions is n.

Let

$$\xi = \frac{1 + \sqrt{5}}{2} \quad .$$

Then $\xi > \mu_1$, and, since ξ is a root of the equation $\xi^2 = \xi + 1$, we have $\xi^2 > \mu_2$. Also, $\xi^3 = \xi^2 + \xi > \mu_2 + \mu_1 = \mu_3$ and, in general, $\xi^n > \mu_n$.

Now let p be the number of digits in μ_n. Then $\mu_n \geq 10^{p-1}$ and, by the preceding result, $10^{p-1} < \xi^n$ or

$$n > \frac{p - 1}{\log \xi} \quad .$$

Hence

$$n > \frac{p}{\log \xi} - 5 \quad \text{since} \quad \log \xi > \frac{1}{5}$$

In the proof of Lamé's Theorem [1] it is shown that

$$n < \frac{p}{\log \xi} + 1 \quad ,$$

where here n is the number of divisions in the Euclidean Algorithm for any two numbers and p is the number of digits in the smaller number. Thus we see that the upper bound for the number of divisions in the Euclidean Algorithm given by Lamé's Theorem is virtually the best possible.

REFERENCE

1. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill, 1939, p. 43.

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All subscription correspondence should be addressed to Brother U. Alfred, St. Mary's College, Calif. All checks ($4.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Manuscripts intended for publication in the Quarterly should be sent to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, Calif. All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscripts sent to the editors.

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MORE FIBONACCI IDENTITIES

M. N. S. SWAMY, Nova Scotia Technical College, Halifax, Canada

In an earlier article [1] the author has discussed in detail the properties of a set of polynomials $B_n(x)$ and $b_n(x)$. It has been shown that [2],

$$B_n(x) = \left[\left\{(x + 2 + \sqrt{x^2 + 4x})/2\right\}^{n+1} - \left\{(x + 2 - \sqrt{(x^2 + 4x)}/2\right\}^{n+1}\right] \Big/ \sqrt{(x^2 + 4x)}$$

Putting $x = 1$ and simplifying we can show that

$$B_n(1) = F_{2n+2} \tag{1a}$$

where F_n is the n^{th} Fibonacci number.

Hence,

$$b_n(1) = B_n(1) - B_{n-1}(1) = F_{2n+2} - F_{2n} = F_{2n+1} \tag{1b}$$

We shall now use (1) and the properties of B_n and b_n to establish some interesting Fibonacci identities:

It has been shown that [1],

$$\begin{vmatrix} B_m & B_n \\ b_m & b_n \end{vmatrix} = \begin{vmatrix} B_{m-r} & B_{n-r} \\ b_{m-r-1} & b_{n-r-1} \end{vmatrix} \tag{2}$$

and

$$\begin{vmatrix} B_m & B_n \\ B_{m-1} & B_{n-1} \end{vmatrix} = \begin{vmatrix} B_{m-r} & B_{n-r} \\ B_{m-r-1} & B_{n-r-1} \end{vmatrix} \tag{3}$$

From (1) and (2) we can establish that

$$\begin{vmatrix} F_m & F_n \\ F_{m-1} & F_{n-1} \end{vmatrix} = \begin{vmatrix} F_{m-2r} & F_{n-2r} \\ F_{m-2r-1} & F_{n-2r-1} \end{vmatrix} \tag{4}$$

and from (1) and (3) that

$$\begin{vmatrix} F_m & F_n \\ F_{m-2} & F_{n-2} \end{vmatrix} = \begin{vmatrix} F_{m-2r} & F_{n-2r} \\ F_{m-2r-2} & F_{n-2r-2} \end{vmatrix} \tag{5}$$

Using equations (33)-(37) of [1] we may deduce that [3],

$$\begin{aligned} F_2 + F_6 + \cdots + F_{4n-2} &= F_{2n}^2 \\ F_1 + F_5 + \cdots + F_{4n-3} &= F_{2n-1}F_{2n} \\ F_3 + F_7 + \cdots + F_{4n-1} &= F_{2n}F_{2n+1} \\ F_4 + F_8 + \cdots + F_{4n} &= F_{2n}F_{2n+2} \end{aligned} \tag{6}$$

From (33)-(37) of [1] we may establish the identities

$$(x^2 + 4x)\sum_0^n B_r^2 = B_{2n+2} - (2n + 3)$$

$$(x^2 + 4x)\sum_0^n B_rB_{r+1} = B_{2n+3} - (n + 2)(x + 2)$$

$$(x^2 + 4x)\sum_0^n b_rB_r = b_{2n+2} + (n + 1)x - 1$$

and

$$(x^2 + 4x)\sum_0^n b_r^2 = B_{2n+1} + 2(n + 1)$$

From the above identities and (1) we can deduce that

$$5(F_2F_4 + F_4F_6 + \cdots + F_{2n-2}F_{2n}) = F_{4n} - 3n \tag{7}$$

$$5(F_1F_2 + F_3F_4 + \cdots + F_{2n-1}F_{2n}) = F_{4n+1} + (n - 1) \tag{8}$$

(9) $$5(F_1^2 + F_3^2 + \cdots + F_{2n-1}^2) = F_{4n} + 2n$$

and

(10) $$5(F_2^2 + F_4^2 + \cdots + F_{2n}^2) = F_{4n+2} - (2n + 1)$$

Combining the identities of (9) and (10) we get

(11) $$5(F_1^2 + F_2^2 + \cdots + F_n^2) = F_{2n+2} + F_{2n} + (-1)^{n+1}$$

Also, we have the well-known identity,

(12) $$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

Hence from (11) and (12) we get

(13) $$F_{2n+2} + F_{2n} - 5F_{n+1}F_n = (-1)^n$$

From (14) and (31) of reference [1] we have the results,

(14) $$B_r^2 - B_{r+1}B_{r-1} = 1$$

and

(15) $$b_r B_r - b_{r+1}B_{r-1} = 1$$

Therefore, $(B_r / b_{r+1}) - (B_{r-1} / b_r) = 1/(b_r b_{r+1})$. Hence,

$$(1/b_{n-1} b_n) + (1/b_{n-2} b_{n-1}) + \cdots + (1/b_1 b_2) = (B_{n-1} / b_n) - (B_0 / b_1)$$

Since $B_0 = b_0 = 1$, we may write this result as,

$$\sum_1^n (1/b_r b_{r-1}) = (B_{n-1} / b_n)$$

Therefore

$$(B_n/b_n) = 1 + (B_{n-1}/b_n) = 1 + \sum_{1}^{n} (1/b_r b_{r-1}) \tag{16}$$

Similarly starting with (14) we can establish that

$$(b_n/B_n) = 1 - \sum_{1}^{n} (1/B_r B_{r-1}) \tag{17}$$

Combining the identities (16) and (17) we have,

$$\left\{1 + \sum_{1}^{n} (1/b_r b_{r-1})\right\}\left\{1 - \sum_{1}^{n} (1/B_r B_{r-1})\right\} = 1 \tag{18}$$

Substituting (1) in (18) we derive an interesting result that

$$\left\{1 + \sum_{1}^{n} \frac{1}{F_{2n-1} F_{2n+1}}\right\}\left\{1 - \sum_{1}^{n} \frac{1}{F_{2n} F_{2n+2}}\right\} = 1 \tag{19}$$

Many other interesting Fibonacci identities may be established using the properties of B_n and b_n, and it is left to the reader to develop these identities.

REFERENCES

1. M. N. S. Swamy, "Properties of the Polynomials Defined by Morgan-Voyce," Fibonacci Quarterly, Vol. 4, No. 1, pp. 73-81.
2. S. L. Basin, "The Appearance of Fibonacci Numbers and the Q Matrix in Electrical Network Theory," Mathematics Magazine, Vol. 36, March-April 1963, pp. 84-97.
3. K. Siler, "Fibonacci Summations," Fibonacci Quarterly, Vol. 1, October 1963, pp. 67-69.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A. P. HILLMAN, University of New Mexico, Albuquerque, New Mex.

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-100 ***Proposed by J. A. H. Hunter, Toronto, Canada.***

Let $u_{n+2} = u_{n+1} + u_n - 1$, with $u_1 = 1$ and $u_2 = 3$. Find the general solution for u_n.

B-101 ***Proposed by Thomas P. Dence, Bowling Green State Univ., Bowling Green, Ohio.***

Let $x_{i,n}$ be defined by $x_{1,n} = 1$, $x_{2,n} = n$, and $x_{i+2,n} = x_{i+1,n} + x_{i,n}$. Express $x_{i,n}$ as a function of F_n and n.

B-102 ***Proposed by Gerald L Alexanderson, University of Santa Clara, Santa Clara, Calif.***

The Pell sequence $1, 2, 5, 12, 29, \cdots$ is defined by $P_1 = 1$, $P_2 = 2$ and $P_{n+2} = 2P_{n+1} + P_n$. Let $(P_{n+1} + iP_n)^2 = x_n + iy_n$, with x_n and y_n real and let $z_n = |x_n + iy_n|$. Prove that the numbers x_n, y_n, and z_n are the lengths of the sides of a right triangle and that x_n and y_n are consecutive integers for every positive integer n. Are there any other positive integral solutions of $x^2 + (x \pm 1)^2 = z^2$ than $(x, z) = (x_n, z_n)$?

B-103 ***Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.***

Let

$$a_n = \sum_{d|n} F_d \qquad (n \geq 1),$$

where the sum is over all divisors d of n. Prove that $\{a_n\}$ is a strictly increasing sequence. Also show that

$$\sum_{n=1}^{\infty} \frac{F_n x^n}{1-x^n} = \sum_{n=1}^{\infty} a_n x^n .$$

B-104 ***Proposed by H. H. Ferns, Victoria, British Columbia.***

Show that

$$\sum_{n=1}^{\infty} \frac{F_{2n+1}}{L_n L_{n+1} L_{n+2}} = \frac{1}{3} ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively.

B-105 ***Proposed by Phil Mana, Univ. of New Mexico, Albuquerque, New Mex.***

Let g_n be the number of finite sequences $c_1, c_2, \cdots, c_n$ with $c_1 = 1$, each c_i in $\{0,1\}$, (c_i, c_{i+1}) never $(0,0)$, and (c_i, c_{i+1}, c_{i+2}) never $(0,1,0)$. Prove that for every integer $s > 1$ there is an integer t with $t \le s^3 - 3$ and g_t an integral multiple of s.

SOLUTIONS

AN INTEGER VALUED FUNCTION

B-82 ***Proposed by Nanci Smith, Univ. of New Mexico, Albuquerque, New Mex.***

Describe a function $g(n)$ having the table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	⋯
g(n)	0	1	1	2	1	2	2	3	1	2	2	3	2	⋯

Solution by the Proposer.

The function g(n) is the number of one's (i.e., sum of the "digits") in the binary representation of n.

Also solved by Joseph D.E. Konhauser and Jeremy C. Pond.

A RECURSION RELATION FOR SQUARES

B-83 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.*

Show that $F_n^2 + F_{n+4}^2 = F_{n+1}^2 + F_{n+3}^2 + 4F_{n+2}^2$.

Solution by James E. Desmond, Fort Lauderdale, Florida.

From Basin and Hoggatt ("A Primer on the Fibonacci Sequence—Part I," Fibonacci Quarterly, Vol. 1, No. 1, p. 66) we have that $F_{n+1}^2 + F_n^2 = F_{2n+1}$ and $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$. So,

$$
\begin{aligned}
&F_n^2 + F_{n+4}^2 - F_{n+1}^2 - F_{n+3}^2 - 4F_{n+2}^2 \\
&\quad = -(F_{n+2}^2 - F_n^2) + (F_{n+4}^2 - F_{n+2}^2) - (F_{n+2}^2 + F_{n+1}^2) - (F_{n+3}^2 + F_{n+2}^2) \\
&\quad = -F_{2(n+1)} + F_{2(n+3)} - F_{2(n+1)+1} - F_{2(n+2)+1} \\
&\quad = -F_{2n+2} + F_{2n+6} - F_{2n+3} - F_{2n+5} \\
&\quad = F_{2n+4} - F_{2n+4} \\
&\quad = 0 \quad .
\end{aligned}
$$

Also solved by Anne E. Bentley, Clyde A. Bridger, Thomas P. Dence, Joseph D.E. Konhauser, Karen S. Laskowski, Douglas Lind, Pat Miller, John W. Milsom, F. D. Parker, Jeremy C. Pond, Toni Ann Viggiani, Howard L. Walton, David Zeitlin, and the proposer.

TERM-BY-TERM SUMS

B-84 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.*

The Fibonacci polynomials are defined by $f_1(x) = 1$, $f_2(x) = x$, $f_{n+1}(x) = xf_n(x) + F_{n-1}(x)$, $n > 1$. If $z_r = f_r(x) + f_r(y)$, show that z_r satisfies

$$z_{n+4} - (x + y)z_{n+3} + (xy - 2)z_{n+2} + (x + y) + z_n = 0 \quad .$$

Solution by David Zeitlin, Minneapolis, Minn.

In B-65 (see Vol. 3, No. 4, 1965, p. 325), it was shown that if u_n and v_n are sequences which satisfy $u_{n+2} + au_{n+1} + bu_n = 0$ and $v_{n+2} + cv_{n+1} + dv_n = 0$, where a, b, c and d are constants, then $z_n = u_n + v_n$ satisfies $z_{n+4} + pz_{n+3} + qz_{n+2} + rz_{n+1} + sz_n = 0$, where $(E^2 + aE + b)(E^2 + cE + d) \equiv E^4 + pE^3 + qE^2 + rE + s$. Since $p = a + c$, $q = ac + b + d$, $r = bc + ad$, and $s = bd$, the desired result is obtained by setting $a = -x$, $b = -1$, $c = -y$, and $d = -1$.

Also solved by Clyde A. Bridger, James E. Desmond, Joseph D.E. Konhauser, Karen S. Laskowski, Douglas Lind, John W. Milsom, Jeremy C. Pond, Howard L. Walton, and the proposer.

SUMS OF SQUARES

B-85 ***Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.***

Find compact expressions for:

(a) $$F_2^2 + F_4^2 + F_6^2 + \cdots + F_{2n}^2$$

(b) $$F_1^2 + F_3^2 + F_5^2 + \cdots + F_{2n-1}^2 \quad .$$

Solution by David Zeitlin, Minneapolis, Minn.

Using mathematical induction, one may show that for $m = 0, 1, \cdots$,

$$5 \sum_{k=0}^{n} F_{2k+m}^2 = F_{4n+2m+2} + 2n(-1)^{m+1} + 6F_m^2 - F_{m+2}^2$$

Thus, for $m = 0$ and $m = 1$, we obtain, respectively,

(a) $$\sum_{k=0}^{n} F_{2k}^2 = (F_{4n+2} - 2n - 1)/5 \; ,$$

(b) $$\sum_{k=0}^{n-1} F_{2k+1}^2 = (F_{4n} + 2n)/5$$

Also solved by Joseph D.E. Konhauser, Jeremy C. Pond, Clyde Bridger, James E. Desmond, M.N.S. Swamy, and the Proposer.

RECURSION FOR CUBES

B-86 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

(Corrected version.) Show that the squares of every third Fibonacci number satisfy

$$y_{n+3} - 17y_{n+2} - 17y_{n+1} + y_n = 0 .$$

Solution by David Zeitlin, Minneapolis, Minn.

Since $F_{3n+m}^2 = C_1a^{6n} + C_2b^{6n} + C_3(-1)^n$, $m,n = 0,1,\cdots$, where a and b are roots of $x^2 - x - 1 = 0$, the difference equation, noting that $a^6 + b^6 = L_6 = 18$ is

$$\begin{aligned}(E + 1)(E - a^6)(E - b^6)y_n & \\ &\equiv (E + 1)(E^2 - 18E + 1)y_n \\ &\equiv (E^3 - 17E^2 - 17E + 1)y_n \\ &\equiv y_{n+3} - 17y_{n+2} - 17y_{n+1} + y_n = 0 .\end{aligned}$$

Also solved by James E. Desmond, Joseph D.E. Konhauser, Douglas Lind, Jeremy C. Pond, C.B.A. Peck, and the proposer.

A SPECIAL CASE OF AN IDENTITY

B-87 *Proposed by A.P. Hillman, University of New Mexico, Albuquerque, New Mex.*

Prove the identity in $x_0, x_1, \cdots, x_n$:

$$\sum_{k=0}^{n}\left[\frac{(-1)^{n-k}}{k!\,(n-k)!}\prod_{j=0}^{n}(x_j + k)\right] = \binom{n+1}{2} + \sum_{j=0}^{n} x_j \quad .$$

Solution by the Proposer.

If the difference equation (1), of "Generalized Binomial Coefficients," by Roseanna F. Torretto and J. Allen Fuchs, Fibonacci Quarterly, Vol. 2, No. 4, Dec. 1964, pp. 296-302, is chosen to be $y_{n+2} - 2y_{n+1} + y_n = 0$, then U_n becomes n, $\left[\begin{matrix} m \\ j \end{matrix}\right]$ becomes the binomial coefficient $\binom{m}{j}$, and the identity of this problem results from formula (5) of that paper upon division of both sides by $n!$.

Also solved by Joseph D.E. Konhauser, Douglas Lind, and David Zeitlin.

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