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PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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THE FIBONACCI QUARTERLY seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

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In addition to the members of the Board of Directors and the members of the Editorial Board, the following mathematicians, engineers, and physicists have assisted *The Fibonacci Quarterly* by refereeing papers during the past year. Their special efforts are sincerely appreciated, and we apologize in advance for any names that inadvertently may have been overlooked or misspelled.

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A COMBINATORIAL PROOF OF A RECURSIVE RELATION OF THE MOTZKIN SEQUENCE BY LATTICE PATHS

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We consider those lattice paths in the Cartesian plane running from $(0, 0)$ that use the steps from $S = \{U = (1, 1)$ (an up-step), $L = (1, 0)$ (a level-step), $D = (1, -1)$ (a down-step)$\}$. Let $A(n, k)$ be the set of all lattice paths ending at the point (n, k) and let $M(n)$ be the set of lattice paths in $A(n, 0)$ that never go below the x-axis. Let $a(n, k) = |A(n, k)|$ and $m_n = |M(n)|$, where m_n is called the *Motzkin number*. Here, we shall give a combinatorial proof of the three-term recursion of the Motzkin sequence,

$$(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2},$$

and also that

$$3 - \frac{6}{n+2} < \frac{m_n}{m_{n-1}} < 3 - \frac{4}{n+2}, \quad \lim_{n\to\infty} \frac{m_n}{m_{n-1}} = 3.$$

The first few Motzkin numbers are $m_0 = 1, 1, 2, 4, 9, 21, 51\ldots$. Let $B(n, k)$ denote the set of lattice paths in $A(n, k)$ that do not attain their highest value (i.e., maximum second coordinate) until the last step. Note that the last step of the paths in $B(n, k)$ is U. Let $b_{n,k} = |B(n, k)|$, then some entries of the matrices $(a_{n,k})$ and $(b_{n,k})$ are as follows:

$$\begin{bmatrix} n/k & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\ 0 & & & & & 1 & & & & \\ 1 & & & & 1 & 1 & 1 & & & \\ 2 & & & 1 & 2 & 3 & 2 & 1 & & \\ 3 & & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\ 4 & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \end{bmatrix},$$

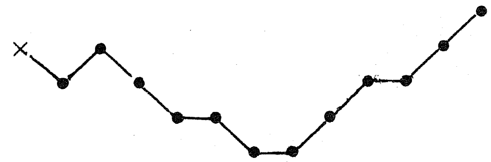
$$\begin{bmatrix} n/k & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 2 & 1 & 0 \\ 4 & 0 & 4 & 5 & 3 & 1 \end{bmatrix}.$$

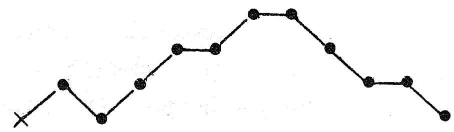
Lemma 1: There is a combinatorial proof for the equation $m_n = b_{n+1,1}$. See [1] and [3] for the cut and paste technique.

Proof: Let $P \in B(n+1, 1)$, remove the last step (U) and the reflection of the remaining is in $M(n)$. □

For example,

$$P = (DLDDUDLUULU)U \in B(12, 1) \to DLDDUDLUULU$$
$$\to ULUUDULDDLD = Q \in M(11),$$

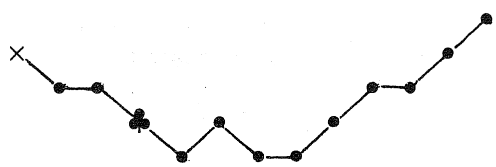
$P =$ 

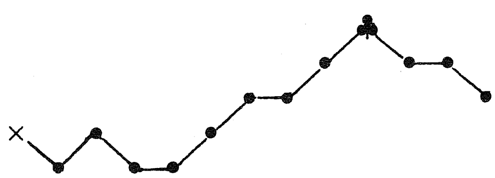
$Q =$ 

Theorem 2: There is a combinatorial proof for the equation $(n+1)b_{n+1,1} = a(n+1, 1)$. See also [5] for the proof and [1] and [3] for the cut and paste technique.

Proof: Let $S(n+1) = \{P^* : P \in B(n+1, 1),\ P^*$ with one marked vertex, which is one of the first $n+1$ vertices$\}$. Then $|S(n+1)| = (n+1)b_{n+1,1}$. Let $P^* \in S(n+1)$; this marked vertex partitions the path $P = FB$, where F is the front section and B is the back section. Then $Q = BF \in A(n+1, 1)$. Note that, graphically, the attached point is the leftmost highest point (the second coordinate) of Q. The converse starts with the leftmost highest point of Q in $A(n+1, 1)$ and reverse the above procedures. □

For example,

$\rightarrow P^* =$  $\in S(12)$,

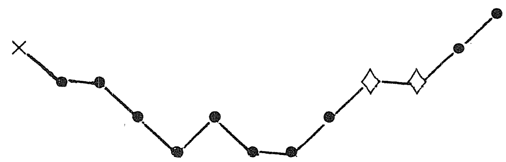
$\rightarrow Q =$  $\in A(12, 1)$.

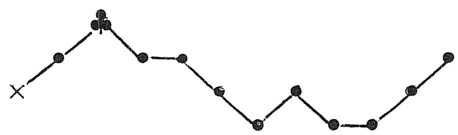
Proposition 3: The total number of L steps in $M(n)$ is the same as that in $B(n+1, 1)$ and is $a_{n,1}$.

Proof: From the proof of Lemma 1, the bijection between $M(n)$ and $B(n+1, 1)$ through reflection, it keeps the L steps. Hence, they have the same number of L steps.

Let $P = FLB \in B(n+1, 1)$ with L step. Then $Q = BF \in A(n, 1)$. Note that the joining point is the leftmost highest point in Q, since $P \in B(n+1, 1)$, by definition P reaches height 1 only at the end of the last step, the second coordinate of the L is less than or equal to 0; hence, any point in the subpath F from the initial point to L is lower or equal to the initial point and any point, before the terminal point, of the subpath B from L to the terminal point is of lower than the terminal point. This identification suggests the inverse mapping. □

For example,

$\rightarrow P = FLB =$  $\in B(12, 1)$,

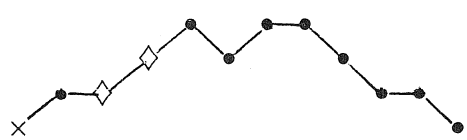
$Q = BF =$  $\in A(11, 1)$.

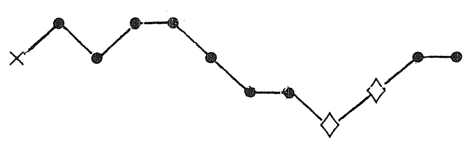
Proposition 4: There is a combinatorial proof for the equation

$$a_{n,0} = b_{n+1,1} + \frac{1}{2}(nb_{n+1,1} - a_{n,1}) = b_{n+1,1} + \frac{1}{2}(nb_{n+1,1} - nb_{n,1}).$$

Proof: Let $T(n) = \{P^e : P \in M(n),\ P^e \text{ is } P \text{ with an up-step marked}\}$. By Theorem 2 and Proposition 3, the number of level-steps among all paths in $M(n)$ is $a_{n,1} = nb_{n,1}$, and the total number of steps among all paths in $M(n)$ is $nm_n = nb_{n+1,1}$; hence, the total number of up-steps among all paths in $M(n)$ is $\frac{1}{2}(nb_{n+1,1} - nb_{n,1}) = |T(n)|$. Let $P^e = FUB \in T(n)$ with the U step marked, then $Q = BUF \in A(n, 0) - M(n)$ and the initial point of U in Q is the rightmost lowest point in Q. The inverse mapping starts with the rightmost lowest point. Note that $|M(n)| = m_n = b_{n+1,1}$. □

For example,

$P^e =$  $\in T(11)$,

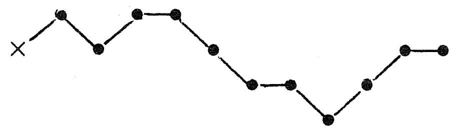
$Q =$  $\in A(11, 0)$.

Proposition 5: There is a combinatorial proof for the equation

$$a_{n,0} = a_{n-1,-1} + a_{n-1,0} + a_{n-1,1} = 2a_{n-1,1} + a_{n-1,0}$$
$$= 2(n-1)b_{n-1,1} + b_{n,1} + \frac{1}{2}((n-1)b_{n,1} - (n-1)b_{n-1,1}).$$

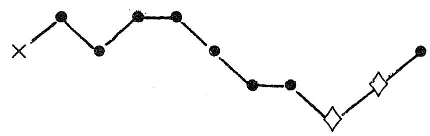
Proof: The first equality represents the partition of $A(n, 0)$ by the last step (U, L, or D), the second equality represents the symmetric property $a_{n-1,-1} = a_{n-1,1}$ and the last equality by Theorem 2 and Proposition 4. □

The following example shows the trail of one element for $n = 11$.



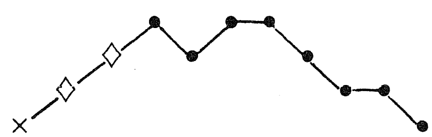
$\in A(11, 0)$.

Removing the last step, the second term of the first equality and the second term of the second equality,



$\in A(10, 0)$.

By Proposition 4, the second term of the third equality,



$\in T(10)$.

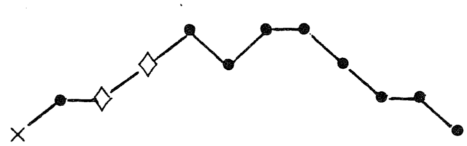
Theorem 6: There is a combinatorial proof for the equation

$$b_{n+1,1}+\frac{1}{2}(nb_{n+1,1}-nb_{n,1}) = 2(n-1)b_{n-1,1}+b_{n,1}+\left(\frac{1}{2}((n-1)b_{n,1}-(n-1)b_{n-1,1})\right).$$

Proof: The composition of the mappings in Proposition 4 and Proposition 5. □

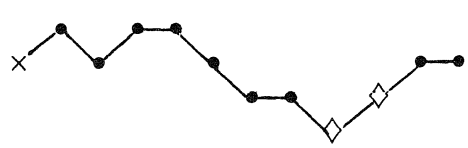
The following example shows the trail of one element for $n = 11$,

$P^e = FUB =$

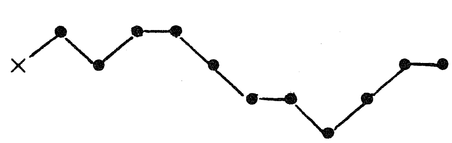


$\in T(11)$,

$Q = BUF =$

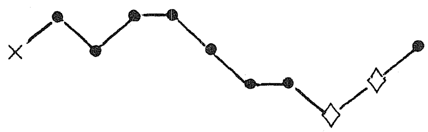


$\in A(11, 0)$,

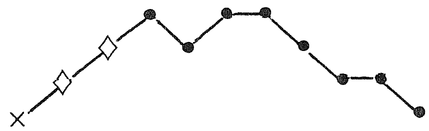


$\in A(11, 0)$.

Removing the last step,

 $\in A(10, 0)$.

By Proposition 4,

 $\in T(10)$.

The following result was proved in a combinatorial way in [2].

Theorem 7: $(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2}$.

Proof: By Theorem 6,

$$b_{n+1,1} + \frac{1}{2}(nb_{n+1,1} - nb_{n,1}) = 2(n-1)b_{n-1,1} + b_{n,1} + \left(\frac{1}{2}((n-1)b_{n,1} - (n-1)b_{n-1,1})\right).$$

By Lemma 1,

$$m_n + \frac{1}{2}(nm_n - nm_{n-1}) = 2(n-1)m_{n-2} + m_{n-1} + \left(\frac{1}{2}((n-2)m_{n-1} - (n-1)m_{n-2})\right).$$

Equivalently,

$$(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2}. \quad \square$$

Theorem 8: $3 - \frac{6}{n+2} < \frac{m_n}{m_{n-1}} < 3 - \frac{4}{n+2}$ for $n \geq 5$ and $\lim_{n\to\infty} \frac{m_n}{m_{n-1}} = 3$.

Proof: By Theorem 7, let

$$s_n := \frac{m_n}{m_{n-1}} = \frac{2n+1}{n+2} + \frac{3n-3}{n+2}\frac{m_{n-2}}{m_{n-1}} = \frac{2n+1}{n+2} + \frac{\frac{3n-3}{n+2}}{s_{n-1}},$$

$$a_n := \frac{2n+1}{n+2} = 2 - \frac{3}{n+2}, \quad b_n := \frac{3n-3}{n+2} = 3 - \frac{9}{n+2},$$

then

$$s_n = a_n + \frac{b_n}{s_{n-1}} \quad \text{and} \quad \frac{b_n}{s_n - a_n} = s_{n-1}.$$

If $s_{n-1} \leq 3$, then $\frac{b_n}{s_n - a_n} = s_{n-1} \leq 3$ and

$$s_n = a_n + \frac{b_n}{s_{n-1}} \geq 2 - \frac{3}{n+2} + \frac{3 - \frac{9}{n+2}}{3} = 3 - \frac{6}{n+2},$$

$$s_{n+1}=a_{n+1}+\frac{b_{n+1}}{s_n}\leq 2-\frac{3}{n+3}+\frac{3-\frac{9}{n+3}}{3-\frac{6}{n+2}}=2-\frac{3}{n+3}+\frac{\frac{3n}{n+3}}{\frac{3n}{n+2}}$$

$$=2-\frac{3}{n+3}+\frac{n+2}{n+3}=3-\frac{4}{n+3},$$

$$s_2=\frac{2}{1},\ s_3=\frac{4}{2},\ s_4=\frac{9}{4},\ s_5=\frac{21}{9},\ s_6=\frac{51}{21}<3.$$

By induction on both even and odd, we have the following:

$$3-\frac{6}{n+2}<\frac{m_n}{m_{n-1}}<3-\frac{4}{n+2},\quad \lim_{n\to\infty}\frac{m_n}{m_{n-1}}=3.\ \square$$

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ON THE RESOLUTION OF THE EQUATIONS $U_n = \binom{x}{3}$ AND $V_n = \binom{x}{3}$

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(*Submitted September 1999-Final Revision March 2000*)

1. INTRODUCTION

The purpose of the present paper is to prove that there are finitely many binomial coefficients of the form $\binom{x}{3}$ in certain binary recurrences, and give a simple method for the determination of these coefficients. We illustrate the method by the Fibonacci, the Lucas, and the Pell sequences. First, we transform both of the title equations into two elliptic equations and apply a theorem of Mordell [10], [11] to them. (Later, Siegel [16] generalized Mordell's result, and in 1968 Baker [1] gave its effective version.) After showing the finiteness, we use the program package SIMATH [15] which is a computer algebra system, especially useful for number theoretic purposes, and is able to find all the integer points on the corresponding elliptic curves. The algorithms of SIMATH are based on some deep results of Gebel, Pethö, and Zimmer [5].

Before going into detail, we present a short historical survey. Several authors have investigated the occurrence of special figurate numbers in the second-order linear recurrences. One such problem is, for example, to determine which Fibonacci numbers are square. Cohn [2], [3] and Wyler [18], applying elementary methods, proved independently that the only square Fibonacci numbers are $F_0 = 0$, $F_1 = F_2 = 1$, and $F_{12} = 144$. A similar result for the Lucas numbers was obtained by Cohn [4]: if $L_n = x^2$, then $n = 1$ or $n = 3$. London and Finkelstein [6] established full Fibonacci cubes. Pethö [12] gave a new proof of the theorem of London and Finkelstein, applying the Gel'fond-Baker method and computer investigations. Later Pethö found all the fifth-power Fibonacci numbers [14], and all the perfect powers in the Pell sequence [13].

Another special interest was to determine the triangular numbers $T_x = \frac{x(x+1)}{2}$ in certain recurrences. Hoggatt conjectured that there are only five triangular Fibonacci numbers. This problem was originally posed in 1963 by Tallman [17] in *The Fibonacci Quarterly*. In 1989 Ming [8] proved Hoggatt's conjecture by showing that the only Fibonacci numbers that are triangular are $F_0 = 0$, $F_1 = F_2 = 1$, $F_4 = 3$, $F_8 - 21$, and $F_{10} = 55$. Ming also proved in [9] that the only triangular Lucas numbers are $L_1 = 1$, $L_2 = 3$, and $L_{18} = 5778$. Moreover, the only triangular Pell number is $P_1 = 1$ (see McDaniel [7]).

Since the number T_{x-1} is equal to the binomial coefficient $\binom{x}{2}$, it is natural to ask whether the terms $\binom{x}{3}$ occur in binary recurrences or not. As we will see, the second-order linear recurrences, for instance, the Fibonacci, the Lucas, and the Pell sequences have few such terms.

Now we introduce some notation. Let the sequence $\{U_n\}_{n=0}^{\infty}$ be defined by the initial terms U_0, U_1, and by the recurrence relation

$$U_n = AU_{n-1} + BU_{n-2} \quad (n \geq 2), \tag{1}$$

where $U_0, U_1, A, B \in \mathbf{Z}$ with the conditions $|U_0| + |U_1| > 0$ and $AB \neq 0$. Moreover, let α and β be the roots of the polynomial

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$$p(x) = x^2 - Ax - B, \tag{2}$$

and we denote the discriminant $A^2 + 4B$ of $p(x)$ by D. Suppose $D \neq 0$ (i.e., $\alpha \neq \beta$). Throughout this paper we also assume that $U_0 = 0$ and $U_1 = 1$.

The sequence

$$V_n = AV_{n-1} + BV_{n-2} \quad (n \geq 2), \tag{3}$$

with the initial values $V_0 = 2$ and $V_1 = A$ is the associate sequence of U. The recurrences U and V satisfy the relation $V_n^2 - DU_n^2 = 4(-B)^n$.

Finally, it is even assumed that $|B| = 1$. Then

$$V_n^2 - DU_n^2 = 4(\pm 1)^n = \pm 4. \tag{4}$$

As usual, denote by F_n, L_n, and P_n the n^{th} term of the Fibonacci, the Lucas, and the Pell sequences, respectively.

The following theorems formulate precisely the new results.

Theorem 1: Both the equations $U_n = \binom{x}{3}$ and $V_n = \binom{x}{3}$ have only a finite number of solutions (n, x) in the integers $n \geq 0$ and $x \geq 3$.

Theorem 2: All the integer solutions of the equation

(i) $F_n = \binom{x}{3}$ are $(n, x) = (1, 3)$ and $(2, 3)$,

(ii) $L_n = \binom{x}{3}$ are $(n, x) = (1, 3)$ and $(3, 4)$,

(iii) $P_n = \binom{x}{3}$ is $(n, x) = (1, 3)$.

2. PROOF OF THEOREM 1

Let U and V be binary recurrences specified above. We distinguish two cases.

Case 1. First, we deal with the equation

$$U_n = \binom{x}{3} \tag{5}$$

in the integers n and x. Applying (4) together with $y = V_n$ and $x_1 = x - 1$, we have $U_n = \binom{x_1+1}{3}$ and

$$y^2 - D\left(\frac{x_1^3 - x_1}{6}\right)^2 = \pm 4. \tag{6}$$

Take the 36 times of the equation (6). Let $x_2 = x_1^2$ and $y_1 = 6y$, and using these new variables, from (6) we get

$$y_1^2 = Dx_2^3 - 2Dx_2^2 + Dx_2 \pm 144. \tag{7}$$

Multiplying by $3^6 D^2$ the equation (7) together with $k = 3^3 D y_1$ and $l = 3D(3x_2 - 2)$, it follows that

$$k^2 = l^3 - 27D^2 l + (54D^3 \pm 104976D^2). \tag{8}$$

By a theorem of Mordell [10], [11], it is sufficient to show that the polynomial $u(l) = l^3 - 27D^2 l + (54D^3 \pm 104976D^2)$ has three distinct roots. Suppose the polynomial $u(l)$ has a multiple root $\tilde{l}$. Then $\tilde{l}$ satisfies $u'(l) = 3l^2 - 27D^2 = 0$, i.e., $\tilde{l} = \pm 3D$. Since $u(3D) = \pm 104976D^2$, it follows that $D = 0$, which is impossible. Moreover, $u(-3D) = 108D^3 \pm 104976D^2$ implies $D = 0$ or $D = \pm 972$. But $D \neq 0$, and by $|B| = 1$ there are no integers A for which $D = A^2 + 4B = \pm 972$. Consequently, $u(l)$ has three distinct zeros.

Case 2. The second case consists of the examination of the Diophantine equation

$$V_n = \binom{x}{3} \tag{9}$$

in the integers n and x. Let $y = U_n$ and $x_1 = x - 1$. Applying the method step by step as above in Case 1, it leads to the elliptic equation

$$k^2 = l^3 - 27D^2 l + cD^3, \tag{10}$$

where $c = -104922$ if n is even and $c = 105030$ otherwise. The polynomial $v(l) = l^3 - 27D^2 l + cD^3$ also has three distinct roots because $v'(l) = 3l^2 - 27D^2$, $\tilde{l} = \pm 3D$, and $v(\pm 3D) = 0$ implies $D = 0$. Thus, the proof of Theorem 1 is complete. □

3. PROOF OF THEOREM 2

The corresponding elliptic curves of equations (8) and (10) are, in short, Weierstrass normal form, whence, for a given discriminant D, the theorem can be solved by SIMATH.

By (8) and (10), one can compute the coefficients of the elliptic curves in case of the Fibonacci, the Lucas, and the Pell sequences. The calculations are summarized in Table 1, as well as all the integer points belonging to them. Every binary recurrence leads to two elliptic equations because of the even and odd suffixes. For the Fibonacci and Lucas sequences, $D = 5$; for the Pell sequence and its associate sequence, $D = 8$.

TABLE 1

Equation	*Transformed equations*	*All the integer solutions* (l, k)
$F_n = \binom{x}{3}$	$k^2 = l^3 - 675l + 2631150$	$(15,1620), (-30,1620), (5199,374868), (735,19980), (150,2430), (-129,756)$
$F_n = \binom{x}{3}$	$k^2 = l^3 - 675l - 2617650$	$(150,810), (555,12960), (1014,32238), (195,2160), (451,9424), (4011,254016)$
$L_n = \binom{x}{3}$	$k^2 = l^3 - 675l - 13115250$	no solution
$L_n = \binom{x}{3}$	$k^2 = l^3 - 675l + 13128750$	$(375,8100), (-74,3574), (150,4050), (-201,2268), (2391,116964)$
$P_n = \binom{x}{3}$	$k^2 = l^3 - 1728l + 6746112$	$(-192,0), (24,2592), (-48,2592), (97,2737)$ $(312,6048), (564,13608), (5208,375840)$
$P_n = \binom{x}{3}$	$k^2 = l^3 - 1728l - 6690816$	$(240,2592), (609,14769)$

The last step is to calculate x and y from the solutions (l, k). By the proof of Theorem 1, it follows that $x = 1 + \sqrt{(l + 6D)/9D}$, $y = k/162D$ in the case of equation (5), and $y = k/162D^2$ in the case of the associate sequence. Except for some values x and y, they are not integers if $x \geq 3$. The exceptions provide all the solutions of equations (8) and (10). Then the proof of Theorem 2 is complete. □

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THE PROBABILITY THAT k POSITIVE INTEGERS ARE PAIRWISE RELATIVELY PRIME

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1. INTRODUCTION

In [3], Shonhiwa considered the function

$$G_k(n) = \sum_{\substack{1 \le a_1, \ldots, a_k \le n \\ (a_1, \ldots, a_k)=1}} 1,$$

where $k \ge 2$, $n \ge 1$, and asked: "What can be said about this function?" As a partial answer, he showed that

$$G_k(n) = \sum_{j=1}^{n} \sum_{d \mid j} \mu(d) \left[\frac{n}{d}\right]^{k-1},$$

where μ is the Möbius function (see [3], Theorem 4).

There is a more simple formula, namely,

$$G_k(n) = \sum_{j=1}^{n} \mu(j) \left[\frac{n}{j}\right]^{k}, \tag{1}$$

leading to the asymptotic result

$$G_k(n) = \frac{n^k}{\zeta(k)} + \begin{cases} O(n \log n), & \text{if } k = 2, \\ O(n^{k-1}), & \text{if } k \ge 3, \end{cases} \tag{2}$$

where ζ denotes, as usual, the Riemann zeta function. Formulas (1) and (2) are well known (see, e.g., [1]). It follows that

$$\lim_{n \to \infty} \frac{G_k(n)}{n^k} = \frac{1}{\zeta(k)},$$

i.e., the probability that k positive integers chosen at random are relatively prime is $\frac{1}{\zeta(k)}$.

For generalizations of this result, we refer to [2].

Remark 1: A short proof of (1) is as follows: Using the following property of the Möbius function,

$$G_k(n) = \sum_{1 \le a_1, \ldots, a_k \le n} \sum_{d \mid (a_1, \ldots, a_k)} \mu(d),$$

and denoting $a_j = db_j$, $1 \le j \le k$, we obtain

$$G_k(n) = \sum_{d=1}^{n} \mu(d) \sum_{1 \le b_1, \ldots, b_k \le n/d} 1 = \sum_{d=1}^{n} \mu(d) \left[\frac{n}{d}\right]^{k}.$$

In what follows, we investigate the question: What is the probability A_k that k positive integers are pairwise relatively prime?

For $k=2$ we have, of course, $A_2 = \frac{1}{\zeta(2)} = 0.607\ldots$ and for $k \geq 3$, $A_k < \frac{1}{\zeta(k)}$. Moreover, for large k, $\frac{1}{\zeta(k)}$ is nearly 1 and A_k seems to be nearly 0.

The next Theorem contains an asymptotic formula analogous to (2), giving the exact value of A_k.

2. MAIN RESULTS

Let $k, n, u \geq 1$ and let

$$P_k^{(u)}(n) = \sum_{\substack{1 \leq a_1, \ldots, a_k \leq n \\ (a_i, a_j)=1, i \neq j \\ (a_i, u)=1}} 1$$

be the number of k-tuples $\langle a_1, \ldots, a_k \rangle$ with $1 \leq a_1, \ldots, a_k \leq n$ such that $a_1, \ldots, a_k$ are pairwise relatively prime and each is prime to u.

Our main result is the following

Theorem: For a fixed $k \geq 1$, we have uniformly for n, $u \geq 1$,

$$P_k^{(u)}(n) = A_k f_k(u) n^k + O(\theta(u) n^{k-1} \log^{k-1} n), \tag{3}$$

where

$$A_k = \prod_p \left(1 - \frac{1}{p}\right)^{k-1} \left(1 + \frac{k-1}{p}\right),$$

$$f_k(u) = \prod_{p \mid u} \left(1 - \frac{k}{p+k-1}\right),$$

and $\theta(u)$ is the number of squarefree divisors of u.

Remark 2: Here $f_k(u)$ is a multiplicative function in u.

Corollary 1: The probability that k positive integers are pairwise relatively prime and each is prime to u is

$$\lim_{n \to \infty} \frac{P_k^{(u)}(n)}{n^k} = A_k f_k(u).$$

Corollary 2: $(u=1)$ The probability that k positive integers are pairwise relatively prime is

$$A_k = \prod_p \left(1 - \frac{1}{p}\right)^{k-1} \left(1 + \frac{k-1}{p}\right).$$

3. PROOF OF THE THEOREM

We need the following lemmas.

Lemma 1: For every $k, n, u \geq 1$,

$$P_{k+1}^{(u)}(n) = \sum_{\substack{j=1 \\ (j,u)=1}}^{n} P_k^{(ju)}(n).$$

Proof: From the definition of $P_k^{(u)}(n)$, we immediately have

$$P_{k+1}^{(u)}(n)=\sum_{\substack{a_{k+1}=1\\(a_{k+1},u)=1}}^{n}\sum_{\substack{1\le a_1,\dots,a_k\le n\\(a_i,a_j)=1,i\ne j\\(a_i,a_{k+1})=1\\(a_i,u)=1}}1=\sum_{\substack{a_{k+1}=1\\(a_{k+1},u)=1}}^{n}P_k^{(ua_{k+1})}(n)=\sum_{\substack{j=1\\(j,u)=1}}^{n}P_k^{(ju)}(n).$$

Lemma 2: For every $k,u\ge 1$,

$$f_k(u)=\sum_{d\mid u}\frac{\mu(d)k^{\omega(d)}}{\alpha_k(d)},$$

where

$$\alpha_k(u)=u\prod_{p\mid u}\left(1+\frac{k-1}{p}\right)$$

and $\omega(u)$ stands for the number of distinct prime factors of u.

Proof: By the multiplicativity of the involved functions, it is enough to verify for $n=p^a$ a prime power:

$$\sum_{d\mid p^a}\frac{\mu(d)k^{\omega(d)}}{\alpha_k(d)}=1-\frac{k}{p}\left(1+\frac{k-1}{p}\right)^{-1}=1-\frac{k}{p+k-1}=f_k(p^a).$$

Note that, for $k=2$, $\alpha_2(u)=\psi(u)$ is the Dedekind function.

Lemma 3: For $k\ge 1$, let $\tau_k(n)$ denote, as usual, the number of ordered k-tuples $\langle a_1,\dots,a_k\rangle$ of positive integers such that $n=a_1\cdot\cdots\cdot a_k$. Then

(a)
$$\sum_{n\le x}\frac{\tau_k(n)}{n}=O(\log^k x),\tag{4}$$

(b)
$$\sum_{n>x}\frac{\tau_k(n)}{n^2}=O\left(\frac{\log^{k-1}x}{x}\right).\tag{5}$$

Proof:

(a) Apply the familiar result $\sum_{n\le x}\tau_k(n)=O(x\log^{k-1}x)$ and partial summation.

(b) By induction on k. For $k=1$, $\tau_1(n)=1$, $n\ge 1$, and

$$\sum_{n\le x}\frac{1}{n^2}=\zeta(2)+O\left(\frac{1}{x}\right)\tag{6}$$

is well known. Suppose that

$$\sum_{n\le x}\frac{\tau_k(n)}{n^2}=\zeta(2)^k+O\left(\frac{\log^{k-1}x}{x}\right).$$

Then, from the identity $\tau_{k+1}(n)=\sum_{d\mid n}\tau_k(d)$, we obtain

$$\begin{aligned}\sum_{n\le x}\frac{\tau_{k+1}(n)}{n^2}&=\sum_{de\le x}\frac{\tau_k(e)}{d^2e^2}=\sum_{d\le x}\frac{1}{d^2}\sum_{e\le x/d}\frac{\tau_k(e)}{e^2}\\&=\sum_{d\le x}\frac{1}{d^2}\left(\zeta(2)^k+O\left(\left(\frac{x}{d}\right)^{-1}\log^{k-1}\frac{x}{d}\right)\right)\end{aligned}$$

$$= \zeta(2)^k \sum_{d \le x} \frac{1}{d^2} + O\left(\frac{\log^{k-1} x}{x} \sum_{d \le x} \frac{1}{d} \right)$$

$$= \zeta(2)^k \left(\zeta(2) + O\left(\frac{1}{x}\right) \right) + O\left(\frac{\log^{k-1} x}{x} \log x \right)$$

by (6), and we get the desired result (5).

Now, for the proof of the Theorem, we use induction on k. For $k = 1$, we have the Legendre function

$$P_1^{(u)}(n) = \sum_{\substack{1 \le a \le n \\ (a,u)=1}} 1 = \sum_{a=1}^{n} \sum_{d|(a,u)} \mu(d) = \sum_{a=1}^{n} \sum_{\substack{d|a \\ d|u}} \mu(d)$$

$$= \sum_{d|u} \mu(d) \sum_{1 \le j \le n/d} 1 = \sum_{d|u} \mu(d) \left[\frac{n}{d} \right] = \sum_{d|u} \mu(d) \left(\frac{n}{d} + O(1) \right)$$

$$= n \sum_{d|u} \frac{\mu(d)}{d} + O\left(\sum_{d|u} \mu^2(d) \right).$$

Hence,

$$P_1^{(u)}(n) = \sum_{\substack{a=1 \\ (a,u)=1}}^{n} 1 = n \frac{\phi(u)}{u} + O(\theta(u)) \tag{7}$$

and (3) is true for $k = 1$ with $A_1 = 1$, $f_1(u) = \frac{\phi(u)}{u}$, ϕ denoting the Euler function.

Suppose that (3) is valid for k and prove it for $k+1$. From Lemma 1, we obtain

$$\begin{aligned} P_{k+1}^{(u)}(n) &= \sum_{\substack{j=1 \\ (j,u)=1}}^{n} P_k^{(ju)}(n) = \sum_{\substack{j=1 \\ (j,u)=1}}^{n} (A_k f_k(ju) n^k + O(\theta(ju) n^{k-1} \log^{k-1} n)) \\ &= A_k f_k(u) n^k \sum_{\substack{j=1 \\ (j,u)=1}}^{n} f_k(j) + O\left(\theta(u) n^{k-1} \log^{k-1} n \sum_{j=1}^{n} \theta(j) \right). \end{aligned} \tag{8}$$

Here $\sum_{j=1}^{n} \theta(j) \le \sum_{j=1}^{n} \tau_2(j) = O(n \log n)$, where $\tau_2 = \tau$ is the divisor function.

Furthermore, by Lemma 2,

$$\sum_{\substack{j=1 \\ (j,u)=1}}^{n} f_k(j) = \sum_{\substack{de=j \le n \\ (j,u)=1}} \frac{\mu(d) k^{\omega(d)}}{\alpha_k(d)} = \sum_{\substack{d \le n \\ (d,u)=1}} \frac{\mu(d) k^{\omega(d)}}{\alpha_k(d)} \sum_{\substack{e \le n/d \\ (e,u)=1}} 1.$$

Using (7), we have

$$\begin{aligned} \sum_{\substack{j=1 \\ (j,u)=1}}^{n} f_k(j) &= \sum_{\substack{d \le n \\ (d,u)=1}} \frac{\mu(d) k^{\omega(d)}}{\alpha_k(d)} \left(\frac{n \phi(u)}{du} + O(\theta(u)) \right) \\ &= \frac{\phi(u)}{u} n \sum_{\substack{d \le n \\ (d,u)=1}} \frac{\mu(d) k^{\omega(d)}}{d \alpha_k(d)} + O\left(\theta(u) \sum_{d \le n} \frac{k^{\omega(d)}}{d} \right), \end{aligned} \tag{9}$$

since $\alpha_k(d) > d$.

Hence, the main term of (9) is

$$\frac{\phi(u)}{u}n\sum_{\substack{d=1\\(d,u)=1}}^{\infty}\frac{\mu(d)k^{\omega(d)}}{d\alpha_k(d)}=\frac{\phi(u)}{u}n\prod_{p\nmid u}\left(1-\frac{k}{p(p+k-1)}\right)$$

$$=n\prod_{p}\left(1-\frac{k}{p(p+k-1)}\right)\prod_{p\mid u}\left(1-\frac{1}{p}\right)\left(1-\frac{k}{p(p+k-1)}\right)^{-1},$$

and its O-terms are

$$O\left(n\sum_{d>n}\frac{k^{\omega(d)}}{d^2}\right)=O\left(n\sum_{d>n}\frac{\tau_k(d)}{d^2}\right)=O(\log^{k-1}n)$$

by Lemma 3(b) and

$$O\left(\theta(u)\sum_{d\le n}\frac{k^{\omega(d)}}{d}\right)=O\left(\theta(u)\sum_{d\le n}\frac{\tau_k(d)}{d}\right)=O(\theta(u)\log^k n)$$

from Lemma 3(a).

Substituting into (8), we get

$$P_{k+1}^{(u)}(n)=A_k\prod_{p}\left(1-\frac{k}{p(p+k-1)}\right)f_k(u)\prod_{p\mid u}\left(1-\frac{1}{p}\right)\left(1-\frac{k}{p(p+k-1)}\right)^{-1}n^{k+1}$$

$$+O(n^k\log^{k-1}n)+O(\theta(u)n^k\log^k n)=A_{k+1}f_{k+1}(u)n^{k+1}+O(\theta(u)n^k\log^k n)$$

by an easy computation, which shows that the formula is true for $k+1$ and the proof is complete.

4. APPROXIMATION OF THE CONSTANTS A_k

Using the arithmetic mean-geometric mean inequality we have, for every $k\ge 2$ and every prime p,

$$\left(1-\frac{1}{p}\right)^{k-1}\left(1+\frac{k-1}{p}\right)<\frac{1}{k^k}\left((k-1)\left(1-\frac{1}{p}\right)+\left(1+\frac{k-1}{p}\right)\right)^k=1,$$

and obtain the series of positive terms,

$$\sum_{p}\log\left(\left(1-\frac{1}{p}\right)^{-k+1}\left(1+\frac{k-1}{p}\right)^{-1}\right)=\sum_{n=1}^{\infty}\log\left(\left(1-\frac{1}{p_n}\right)^{-k+1}\left(1+\frac{k-1}{p_n}\right)^{-1}\right)=-\log A_k, \tag{10}$$

where p_n denotes the n^{th} prime.

Furthermore, the Bernoulli-inequality yields

$$\left(1-\frac{1}{p}\right)^{k-1}\ge 1-\frac{k-1}{p},$$

hence,

$$\left(1-\frac{1}{p}\right)^{k-1}\left(1+\frac{k-1}{p}\right)\ge 1-\left(\frac{k-1}{p}\right)^2$$

for every $k\ge 2$ and every prime p.

Therefore, the N^{th}-order error R_N of series (10) can be evaluated as follows. Taking $N > k-1$, we have $p_N > k-1$ and

$$R_N = \sum_{n=N+1}^{\infty} \log\left(\left(1-\frac{1}{p_n}\right)^{-k+1}\left(1+\frac{k-1}{p_n}\right)^{-1}\right) \le \sum_{n=N+1}^{\infty} \log\left(1-\left(\frac{k-1}{p_n}\right)^2\right)^{-1}$$
$$= \sum_{n=N+1}^{\infty} \log\left(1+\frac{(k-1)^2}{p_n^2-(k-1)^2}\right) < \sum_{n=N+1}^{\infty} \frac{(k-1)^2}{p_n^2-(k-1)^2}.$$

Now using that $p_n < 2n$, valid for $n \ge 5$, we have

$$R_N < \sum_{n=N+1}^{\infty} \frac{(k-1)^2}{4n^2-(k-1)^2} = \frac{k-1}{2}\sum_{n=N+1}^{\infty}\left(\frac{1}{2n-(k-1)}-\frac{1}{2n+(k-1)}\right)$$
$$= \frac{k-1}{2}\left(\frac{1}{2N-k+3}+\frac{1}{2N-k+5}+\cdots+\frac{1}{2N+k-1}\right) < \frac{(k-1)^2}{2(2N-k+3)}.$$

In order to obtain an approximation with r exact decimals, we use the condition

$$\frac{(k-1)^2}{2(2N-k+3)} \le \frac{1}{2}\cdot 10^{-r}$$

and have $N \ge \frac{1}{2}((k-1)^2\cdot 10^r + k - 3)$. Consequently, for such an N,

$$A_k \approx \prod_{n=1}^{N}\left(1-\frac{1}{p_n}\right)^{k-1}\left(1+\frac{k-1}{p_n}\right)$$

with r exact decimals.

Choosing $r = 3$ and doing the computations on a computer (I used MAPLE V), we obtain the following approximate values of the numbers A_k:

$$A_2 = 0.607\ldots,\ A_3 = 0.286\ldots,\ A_4 = 0.114\ldots,\ A_5 = 0.040\ldots,$$
$$A_6 = 0.013\ldots,\ A_7 = 0.004\ldots,\ A_8 = 0.001\ldots.$$

Furthermore, taking into account that the factors of the infinite product giving A_k are less than 1, we obtain

$$A_{10} < \prod_{n=1}^{20}\left(1-\frac{1}{p_n}\right)^9\left(1+\frac{9}{p_n}\right) < 10^{-4},\ A_{100} < \prod_{n=1}^{100}\left(1-\frac{1}{p_n}\right)^{99}\left(1+\frac{99}{p_n}\right) < 10^{-76}.$$

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SOME BINOMIAL CONVOLUTION FORMULAS

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0. INTRODUCTION

For a natural number ν and two sequences $\{A(k), B(k)\}_k$ of binomial coefficients, the following convolutions of Vandermonde type,

$$C(m,n,\nu) := \sum_k A(m+k\nu)B(n-k\nu),$$

will be investigated in this paper. When $\nu=2$, 3, 4, the convolutions will be nominated duplicate, triplicate, and quadruplicate, respectively. Thanks to the explicit solutions of the corresponding algebraic equations, we will establish the generating functions of binomial coefficients with running indices multiplicated accordingly. Then the formal power series method will be used to demonstrate several binomial convolution identities.

When $\nu=1$, we reproduce a pair of binomial identities and the related generating function relations, from which our argument will be developed. In this respect, there are two general convolution formulas due to Hagen and Rothe (cf. [9], §5.4),

$$\sum_{k=0}^{n} \frac{\alpha}{\alpha+k\beta}\binom{\alpha+k\beta}{k}\binom{\gamma-k\beta}{n-k}\frac{\gamma-n\beta}{\gamma-k\beta} = \frac{\alpha+\gamma-n\beta}{\alpha+\gamma}\binom{\alpha+\gamma}{n} \tag{0.1a}$$

and

$$\sum_{k=0}^{n} \frac{\alpha}{\alpha+k\beta}\binom{\alpha+k\beta}{k}\binom{\gamma-k\beta}{n-k} = \binom{\alpha+\gamma}{n}, \tag{0.1b}$$

which have been recovered by Gould [7] (see also [3], [6], and §4.5 in [10]) through manipulating the generating functions

$$\sum_{k=0}^{\infty} \frac{\alpha}{\alpha+k\beta}\binom{\alpha+k\beta}{k}\tau^k = \eta^\alpha \tag{0.2a}$$

and

$$\sum_{k=0}^{\infty} \frac{\alpha}{\alpha+k\beta}\binom{\alpha+k\beta}{k}\tau^k = \frac{\eta^{1+\alpha}}{\beta+\eta-\beta\eta}, \tag{0.2b}$$

where $\tau = (\eta-1)/\eta^\beta$. More binomial convolution formulas and the related hypergeometric identities may be found in [4] and [8].

For an indeterminate x and a complex sequence $\{T(k)\}_k$, the generating function is defined by the following formal power series:

$$f(x) = \sum_{k=0}^{\infty} T(k)x^k. \tag{0.3a}$$

Denote by $\omega_\nu = \exp(\frac{2\pi\sqrt{-1}}{\nu})$ the ν^{th} primitive root of unity. Then there exists a well-known formula to determine the generating function of the subsequence with running indices congruent to ι modulo ν,

$$\nu\sum_{k=0}^{\infty} T(k\nu+\iota)x^{k\nu+\iota} = \sum_{\kappa=0}^{\nu-1}\omega_\nu^{\kappa(\nu-\iota)} f(x\omega_\nu^\kappa), \tag{0.3b}$$

which will be used in this paper frequently without indication.

1. DUPLICATE CONVOLUTIONS

For $\beta = 1/2$, the functional equation between two variables τ and η in (0.2) becomes quadratic. The substitution of its solution $\eta(2\tau) \to U^2(\tau)$ leads the generating functions stated in (0.2a) and (0.2b) to the following lemma.

Lemma 1.1: For two indeterminates τ and U related by

$$2\tau = U - \frac{1}{U} \Leftrightarrow U = \tau + \sqrt{1+\tau^2}, \tag{1.1a}$$

we have functional equations

$$1 = U(\tau)\times U(-\tau), \tag{1.1b}$$

$$2\tau = U(\tau) - U(-\tau), \tag{1.1c}$$

$$2\sqrt{1+\tau^2} = U(\tau) + U(-\tau), \tag{1.1d}$$

$$1 + U^2(\tau) = \{U(\tau) + U(-\tau)\}U(\tau), \tag{1.1e}$$

$$1 + U^2(-\tau) = \{U(\tau) + U(-\tau)\}U(-\tau), \tag{1.1f}$$

and generating functions

$$\sum_{k=0}^{\infty} \frac{a}{a+k/2}\binom{a+k/2}{k}(2\tau)^k = U^{2a}(\tau) \tag{1.2a}$$

and

$$\sum_{k=0}^{\infty} \binom{a+k/2}{k}(2\tau)^k = \frac{2U^{1+2a}(\tau)}{U(\tau)+U(-\tau)}. \tag{1.2b}$$

Their combinations lead us immediately to the following proposition.

Proposition 1.2: With the complex function U defined in Lemma 1.1, we have generating functions on duplicated binomial coefficients:

$$U^{2a}(\tau) + U^{2a}(-\tau) = \sum_{k=0}^{\infty} \frac{2a}{a+k}\binom{a+k}{2k}(2\tau)^{2k}, \tag{1.3a}$$

$$U^{2a-1}(\tau) - U^{2a-1}(-\tau) = \sum_{k=0}^{\infty} \frac{2a-1}{a+k}\binom{a+k}{1+2k}(2\tau)^{2k+1}, \tag{1.3b}$$

$$\frac{U^{1+2a}(\tau) + U^{1+2a}(-\tau)}{U(\tau)+U(-\tau)} = \sum_{k=0}^{\infty} \binom{a+k}{2k}(2\tau)^{2k}, \tag{1.3c}$$

$$\frac{U^{2a}(\tau) - U^{2a}(-\tau)}{U(\tau)+U(-\tau)} = \sum_{k=0}^{\infty} \binom{a+k}{1+2k}(2\tau)^{2k+1}. \tag{1.3d}$$

Based on these relations, we are ready to establish binomial formulas on duplicate convolutions.

Theorem 1.3 (Duplicate convolution identities [5]):

$$\sum_k \frac{2a-m}{a+k}\binom{a+k}{m+2k}\binom{c-k}{n-2k} = \binom{a+c}{m+n} + (-1)^m \binom{c-a+m}{m+n}, \tag{1.4a}$$

$$\sum_k \frac{2a-m}{a+k}\binom{a+k}{m+2k}\binom{c-k}{n-2k}\frac{2c-n}{c-k} \tag{1.4b}$$

$$= \frac{2a+2c-m-n}{a+c}\binom{a+c}{m+n} + (-1)^m \frac{2c-2a+m-n}{c-a+m}\binom{c-a+m}{m+n}. \tag{1.4c}$$

Proof: By means of Lemma 1.1, manipulate generating functions

$$\{U^{2a}(\tau)+U^{2a}(-\tau)\} \times \frac{2U^{1+2c-n}(\tau)}{U(\tau)+U(-\tau)} = \frac{2U^{1+2a+2c-n}(\tau)}{U(\tau)+U(-\tau)} + \frac{2U^{1+2c-2a-n}(\tau)}{U(\tau)+U(-\tau)}$$

and

$$\{U^{2a-1}(\tau)-U^{2a-1}(-\tau)\} \times \frac{2U^{1+2c-n}(\tau)}{U(\tau)+U(-\tau)} = \frac{2U^{2a+2c-n}(\tau)}{U(\tau)+U(-\tau)} - \frac{2U^{2+2c-2a-n}(\tau)}{U(\tau)+U(-\tau)}.$$

According to Proposition 1.2, the coefficients of τ^n and τ^{1+n} in the formal power series expansions lead us, respectively, to the following convolution formulas,

$$\sum_k \frac{2a}{a+k}\binom{a+k}{2k}\binom{c-k}{n-2k} = \binom{a+c}{n} + \binom{c-a}{n} \tag{1.5a}$$

and

$$\sum_k \frac{2a-1}{a+k}\binom{a+k}{1+2k}\binom{c-k}{n-2k} = \binom{a+c}{1+n} - \binom{1+c-a}{1+n}, \tag{1.5b}$$

which have been discovered for the first time by Andrews-Burge (see [1], Eqs. 3.1-3.2), with the help of hypergeometric transformations in their work on plane partition enumerations and determinant evaluations.

Letting $\delta = 0,\ 1$ be the Kronecker delta, we can unify both formulas as a unilateral convolution identity,

$$\sum_k \frac{2a-\delta}{a+k}\binom{a+k}{\delta+2k}\binom{c-k}{n-2k} = \binom{a+c}{\delta+n} + (-1)^\delta \binom{\delta+c-a}{\delta+n},$$

which, in turn, is expressed under parameter replacements

$$\begin{aligned} &k \to k+p,\\ &a \to a-p,\\ &c \to c+p,\\ &n \to n+2p,\\ &\delta \to m-2p, \end{aligned}$$

as the first finite bilateral convolution formula stated in the theorem.

Again from Lemma 1.1 and Proposition 1.2, consider the generating functions

$$\{U^{2a}(\tau)+U^{2a}(-\tau)\} \times U^{2c-n}(\tau) = U^{2a+2c-n}(\tau) + U^{2c-2a-n}(\tau)$$

and

$$\{U^{2a-1}(\tau)-U^{2a-1}(-\tau)\} \times U^{2c-n}(\tau) = U^{2a+2c-n-1}(\tau) - U^{2c-2a-n+1}(\tau),$$

then the coefficients of τ^n and τ^{1+n} in the formal power series expansions result, respectively, in the following binomial convolution identities:

$$\sum_k \frac{2a}{a+k}\binom{a+k}{2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k} \tag{1.6a}$$

$$=\frac{2a+2c-n}{a+c}\binom{a+c}{n}+\frac{2c-2a-n}{c-a}\binom{c-a}{n}, \tag{1.6b}$$

$$\sum_k \frac{2a-1}{a+k}\binom{a+k}{1+2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k} \tag{1.6c}$$

$$=\frac{2a+2c-n-1}{a+c}\binom{a+c}{1+n}-\frac{2c-2a-n+1}{1+c-a}\binom{1+c-a}{1+n}. \tag{1.6d}$$

Their bilateralization derived exactly in the same way as in the proof of the first formula (1.4a) leads us to the second one (1.4b-1.4c). This completes the proof of Theorem 1.3. □

As a by-product, we present a pair of convolution formulas of Jensen type. From Lemma 1.1, it is trivial to have the formal power series

$$\frac{1}{U(\tau)+U(-\tau)}=\frac{1}{2\{U(\tau)-\tau\}}=\frac{1}{2}\sum_{k=0}^{\infty}\frac{\tau^k}{U^{1+k}(\tau)}.$$

By means of Proposition 1.2, we can establish the following expansions,

$$\frac{U^{1+2a}(\tau)+U^{1+2a}(-\tau)}{U(\tau)+U(-\tau)}\times\frac{2U^{1+2c}(\tau)}{U(\tau)+U(-\tau)}=\sum_{k=0}^{\infty}\frac{\tau^k U^{1+2a+2c-k}(\tau)+\tau^k U^{1+2a-2c+k}(-\tau)}{U(\tau)+U(-\tau)}$$

and

$$\frac{U^{2a}(\tau)-U^{2a}(-\tau)}{U(\tau)+U(-\tau)}\times\frac{2U^{1+2c}(\tau)}{U(\tau)+U(-\tau)}=\sum_{k=0}^{\infty}\frac{\tau^k U^{2a+2c-k}(\tau)-\tau^k U^{2c-2a-k}(\tau)}{U(\tau)+U(-\tau)},$$

whose coefficients of τ^n and τ^{1+n} lead us, respectively, to the Jensen convolutions

$$\sum_k\binom{a+k}{2k}\binom{c-k}{n-2k}=\sum_\iota\frac{1}{2^{1+\iota}}\left\{\binom{a+c-\iota}{n-\iota}+\binom{a-c+n}{n-\iota}(-1)^{n-\iota}\right\} \tag{1.7a}$$

and

$$\sum_k\binom{a+k}{1+2k}\binom{c-k}{n-2k}=\sum_\iota\frac{1}{2^{1+\iota}}\left\{\binom{a+c-\iota}{1+n-\iota}-\binom{a-c-\iota}{1+n-\iota}\right\}. \tag{1.7b}$$

Further formulas of Jensen type and binomial identities related to Theorem 1.3 as well as their applications to determinant evaluations can be found in [2] and [5].

2. TRIPLICATE CONVOLUTIONS

When $\beta=1/3$, the functional equation between two variables τ and η in (0.2) is cubic. The substitution of its solution $\eta(3\tau)\to V^3(\tau)$ can be used to reformulate the generating functions stated in (0.2a)-(0.2b) as follows.

Lemma 2.1: Denote the cubic root of unity by $\varepsilon=\exp(2\pi i/3)$. For two indeterminates τ and V related by

$$3\tau = V^2 - \frac{1}{V} \Leftrightarrow V = \Lambda(\tau) + \frac{\tau}{\Lambda(\tau)}, \tag{2.1a}$$

where

$$\Lambda(\tau) = \sqrt[3]{\{1+\sqrt{1-4t^3}\}/2}, \tag{2.1b}$$

we have the functional equations

$$1 = V(\tau)\times V(\tau\omega)\times V(\tau\omega^2), \tag{2.2a}$$

$$0 = V(\tau) + \omega V(\tau\omega) + \omega^2 V(\tau\omega^2), \tag{2.2b}$$

$$-3\tau = V(\tau\omega)V(\tau\omega^2) + \omega V(\tau)V(\tau\omega) + \omega^2 V(\tau)V(\tau\omega^2), \tag{2.2c}$$

and generating functions

$$\sum_{k=0}^{\infty} \frac{a}{a+k/3}\binom{a+k/3}{k}(3\tau)^k = V^{3a}(\tau), \tag{2.3a}$$

$$\sum_{k=0}^{\infty} \binom{a+k/3}{k}(3\tau)^k = \frac{3V^{3+3a}(\tau)}{1+2V^3(\tau)}. \tag{2.3b}$$

Their combinations yield the following generating functions on triplicated binomial coefficients.

Proposition 2.2: With complex function V defined as in Lemma 2.1, we have generating function relations:

$$V^{3a}(\tau) + V^{3a}(\tau\omega) + V^{3a}(\tau\omega^2) = \sum_{k=0}^{\infty} \frac{3a}{a+k}\binom{a+k}{3k}(3\tau)^{3k}, \tag{2.4a}$$

$$V^{3a}(\tau) + \omega^2 V^{3a}(\tau\omega) + \omega V^{3a}(\tau\omega^2) = \sum_{k=0}^{\infty} \frac{3a}{\frac{1}{3}+a+k}\binom{\frac{1}{3}+a+k}{1+3k}(3\tau)^{1+3k}, \tag{2.4b}$$

$$V^{3a}(\tau) + \omega V^{3a}(\tau\omega) + \omega^2 V^{3a}(\tau\omega^2) = \sum_{k=0}^{\infty} \frac{3a}{\frac{2}{3}+a+k}\binom{\frac{2}{3}+a+k}{2+3k}(3\tau)^{2+3k}. \tag{2.4c}$$

Theorem 2.3 (Triplicate convolution identities): Given two natural numbers m and n, define

$$\theta(m,n) = \omega^{m+2n} + \omega^{2m+n} = \begin{cases} +2, & m \equiv n \pmod 3, \\ -1, & m \not\equiv n \pmod 3. \end{cases} \tag{2.5}$$

Then there holds a binomial identity

$$\sum_k \frac{3a}{\frac{m}{3}+a+k}\binom{\frac{m}{3}+a+k}{m+3k}\frac{a}{\frac{n}{3}+a-k}\binom{\frac{n}{3}+a-k}{n-3k} \tag{2.6a}$$

$$= \frac{2a}{\frac{m+n}{3}+2a}\binom{\frac{m+n}{3}+2a}{m+n} + \frac{-a}{\frac{m+n}{3}-a}\binom{\frac{m+n}{3}-a}{m+n}\theta(m,n) \tag{2.6b}$$

and its reversal

$$\sum_k \frac{3c}{\frac{2m}{3}+c+2k}\binom{\frac{2m}{3}+c+2k}{m+3k}\frac{c}{\frac{2n}{3}+c-2k}\binom{\frac{2n}{3}+c-2k}{n-3k} \tag{2.7a}$$

$$= \frac{2c}{\frac{2m+2n}{3}+2c}\binom{\frac{2m+2n}{3}+2c}{m+n} + \frac{-c}{\frac{2m+2n}{3}-c}\binom{\frac{2m+2n}{3}-c}{m+n}\theta(m,n). \tag{2.7b}$$

Proof: By means of Lemma 2.1, manipulate generating functions:

$$V^{3a}(\tau)\times\{V^{3a}(\tau)+V^{3a}(\tau\omega)+V^{3a}(\tau\omega^2)\}=V^{6a}(\tau)+V^{-3a}(\tau\omega)+V^{-3a}(\tau\omega^2),$$
$$V^{3a}(\tau)\times\{V^{3a}(\tau)+\omega^2V^{3a}(\tau\omega)+\omega V^{3a}(\tau\omega^2)\}=V^{6a}(\tau)+\omega V^{-3a}(\tau\omega)+\omega^2V^{-3a}(\tau\omega^2),$$
$$V^{3a}(\tau)\times\{V^{3a}(\tau)+\omega V^{3a}(\tau\omega)+\omega^2V^{3a}(\tau\omega^2)\}=V^{6a}(\tau)+\omega^2V^{-3a}(\tau\omega)+\omega V^{-3a}(\tau\omega^2).$$

According to Proposition 2.2, the coefficients of $\tau^{n+\nu}$, $\nu=0,\ 1,\ 2,$ in the formal power series expansions lead us, respectively, to the following binomial convolutions,

$$\sum_k \frac{3a}{\frac{\nu}{3}+a+k}\binom{\frac{\nu}{3}+a+k}{\nu+3k}\frac{a}{\frac{n}{3}+a-k}\binom{\frac{n}{3}+a-k}{n-3k}$$
$$=\frac{2a}{\frac{n+\nu}{3}+2a}\binom{\frac{n+\nu}{3}+2a}{n+\nu}+\frac{-a}{\frac{n+\nu}{3}-a}\binom{\frac{n+\nu}{3}-a}{n+\nu}\theta(\nu,n),$$

which gives rise to the first finite bilateral convolution formula stated in the theorem under parameter replacements $k\to k+p$, $\nu\to m-3p$, and $n\to n+3p$. Rewriting every binomial coefficient in the first binomial identity through

$$\frac{\alpha}{\gamma}\binom{\gamma}{\ell}=(-1)^\ell\frac{-\alpha}{\ell-\gamma}\binom{\ell-\gamma}{\ell}$$

we immediately obtain the second one in the theorem. □

3. QUADRUPLICATE CONVOLUTIONS

For $\beta=1/4$, the functional equation between two variables τ and η in (0.2) becomes quartic. The substitution of its solution $\eta(4\tau)\to W^4(\tau)$ leads the generating functions stated in (0.2a)-(0.2b) to the following lemma.

Lemma 3.1: For two indeterminates τ and W related by

$$4\tau=W^3-\frac{1}{W}\Leftrightarrow W=\frac{\tau+\sqrt{\Omega^3(\tau)-\tau^2}}{\Omega},\tag{3.1a}$$

where

$$\Omega(\tau)=\sqrt{\phi^2(\tau)+\phi(\tau)\psi(\tau)+\psi^2(\tau)}\tag{3.1b}$$

and

$$\phi(\tau),\psi(\tau)=\sqrt[3]{\pm\tau^2+\sqrt{\tau^4+1/27}},\tag{3.1c}$$

we have the functional equations

$$\Omega(\tau)=\Omega(-\tau)=\Omega(i\tau)=\Omega(-i\tau),\tag{3.2a}$$
$$W(\tau)\times W(-\tau)\times W(i\tau)\times W(-i\tau)=1,\tag{3.2b}$$
$$\phi(\tau)\times\psi(\tau)=\tfrac{1}{3}\ \text{and}\ \phi^3(\tau)-\psi^3(\tau)=2\tau^2,\tag{3.2c}$$

and generating functions

$$\sum_{k=0}^{\infty}\frac{a}{a+k/4}\binom{a+k/4}{k}(4\tau)^k=W^{4a}(\tau),\tag{3.3a}$$

$$\sum_{k=0}^{\infty}\binom{a+k/4}{k}(4\tau)^k=\frac{4W^{4+4a}(\tau)}{1+3W^4(\tau)}. \tag{3.3b}$$

Their combinations bring about the following generating functions.

Proposition 3.2: With the complex function W as in Lemma 3.1, we have the following generating function relations:

$$\sum_{k=0}^{\infty}\frac{4c}{c+k}\binom{c+k}{4k}(4\tau)^{4k}=W^{4c}(\tau)+W^{4c}(-\tau) \tag{3.4a}$$

$$+W^{4c}(i\tau)+W^{4c}(-i\tau), \tag{3.4b}$$

$$\sum_{k=0}^{\infty}\frac{4c}{\frac{1}{4}+c+k}\binom{\frac{1}{4}+c+k}{1+4k}(4\tau)^{1+4k}=W^{4c}(\tau)-W^{4c}(-\tau) \tag{3.4c}$$

$$-iW^{4c}(i\tau)+iW^{4c}(-i\tau), \tag{3.4d}$$

$$\sum_{k=0}^{\infty}\frac{4c}{\frac{1}{2}+c+k}\binom{\frac{1}{2}+c+k}{2+4k}(4\tau)^{2+4k}=W^{4c}(\tau)+W^{4c}(-\tau) \tag{3.4e}$$

$$-W^{4c}(i\tau)-W^{4c}(-i\tau), \tag{3.4f}$$

$$\sum_{k=0}^{\infty}\frac{4c}{\frac{3}{4}+c+k}\binom{\frac{3}{4}+c+k}{3+4k}(4\tau)^{3+4k}=W^{4c}(\tau)-W^{4c}(-\tau) \tag{3.4g}$$

$$+iW^{4c}(i\tau)-iW^{4c}(-i\tau). \tag{3.4h}$$

Theorem 3.3 (Quadruplicate convolution identities): For two integers m and n, let

$$\varepsilon(m,n)=\begin{cases}+1, & m+n\not\equiv 0 \pmod 4,\\ -1, & m+n\equiv 0 \pmod 4.\end{cases} \tag{3.5}$$

Then, for $m\times n\not\equiv 1$ (mod 4), we have

$$\sum_k \frac{4c}{\frac{m}{4}+c+k}\binom{\frac{m}{4}+c-k}{m+4k}\frac{c}{\frac{n}{4}+c-k}\binom{\frac{n}{4}+c-k}{n-4k} \tag{3.6a}$$

$$+\sum_k \frac{-4c}{\frac{m}{4}-c+k}\binom{\frac{m}{4}-c+k}{m+4k}\frac{-c}{\frac{n}{4}-c-k}\binom{\frac{n}{4}-c-k}{n-4k}\varepsilon(m,n) \tag{3.6b}$$

$$=\frac{2c}{\frac{m+n}{4}+2c}\binom{\frac{m+n}{4}+2c}{m+n}+\frac{-2c}{\frac{m+n}{4}-2c}\binom{\frac{m+n}{4}-2c}{m+n}\varepsilon(m,n). \tag{3.6c}$$

Otherwise, for $m\times n\equiv 1$ (mod 4), there holds

$$\sum_k \frac{4c}{\frac{m}{4}+c+k}\binom{\frac{m}{4}+c-k}{m+4k}\frac{c}{\frac{n}{4}+c-k}\binom{\frac{n}{4}+c-k}{n-4k}\varepsilon(m,n) \tag{3.7a}$$

$$+\sum_k \frac{-4c}{\frac{2+m}{4}-c+k}\binom{\frac{m}{4}-c-k}{2+m+4k}\frac{-c}{\frac{n-2}{4}-c-k}\binom{\frac{n}{4}-c-k}{n-2-4k} \tag{3.7b}$$

$$=\frac{2c}{\frac{m+n}{4}+2c}\binom{\frac{m+n}{4}+2c}{m+n}\varepsilon(m,n)+\frac{-2c}{\frac{m+n}{4}-2c}\binom{\frac{m+n}{4}-2c}{m+n}. \tag{3.7c}$$

Proof: For $v = 0, 1, 2, 3$, define the binomial convolutions $\Diamond_v(n, c)$ by

$$\Diamond_v(n,c) = \sum_k \frac{4c}{\frac{v}{4}+c+k}\binom{\frac{v}{4}+c+k}{v+4k}\frac{c}{\frac{n}{4}+c-k}\binom{\frac{n}{4}+c-k}{n-4k}. \tag{3.8}$$

The proof of the theorem will be divided into four cases according to m (mod 4).

Case 1: $m \equiv 0 \pmod 4$. By means of Lemma 3.1 and (3.4a)-(3.4b), we may manipulate generating functions:

$$\begin{aligned}&-W^{8c}(\tau)+W^{4c}(\tau)\times\{W^{4c}(\tau)+W^{4c}(-\tau)+W^{4c}(i\tau)+W^{4c}(-i\tau)\}\\ &=W^{-4c}(i\tau)W^{-4c}(-i\tau)+W^{-4c}(-\tau)W^{-4c}(i\tau)+W^{-4c}(-\tau)W^{-4c}(-i\tau).\end{aligned}$$

The coefficients of τ^n in the formal power series expansions lead us to the following binomial convolutions,

$$\Delta_0(n,c) \overset{\text{def}}{=} \Diamond_0(n,c) - \frac{2c}{2c+\frac{n}{4}}\binom{2c+\frac{n}{4}}{n} \tag{3.9a}$$

$$= \sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{n-k}{4}-c}\binom{\frac{n-k}{4}-c}{n-k}\lambda_0(n,k), \tag{3.9b}$$

where $\lambda_0(n,k) = (-1)^n\{i^k + i^{3k} + i^{2k+3n}\}$, whose values are displayed in Table 1.

TABLE 1. Values of $\lambda_0(n, k)$

$k \backslash n$	0	1	2	3
0	3	$i-2$	1	$-2-i$
1	-1	$-i$	1	i
2	-1	$i+2$	-3	$2-i$
3	-1	$-i$	1	i

For $n \equiv 0 \pmod 2$, Table 1 suggests that we express (3.9b) as

$$\begin{aligned}\Delta_0(n,c) = {}&(-1)^{n/2}\sum_k \frac{-4c}{k-c}\binom{k-c}{4k}\frac{-c}{\frac{n}{4}-c-k}\binom{\frac{n}{4}-c-k}{n-4k}\\ &-(-1)^{n/2}\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{n-k}{4}-c}\binom{\frac{n-k}{4}-c}{n-k},\end{aligned}$$

which may be simplified, by means of (0.1a), to the following relation:

$$\Delta_0(n,c) = (-1)^{n/2}\Diamond_0(n,-c) - (-1)^{n/2}\frac{-2c}{\frac{n}{4}-2c}\binom{\frac{n}{4}-2c}{n} \tag{3.10a}$$

$$= (-1)^{n/2}\Delta_0(n,-c), \quad n \equiv 0 \pmod 2. \tag{3.10b}$$

While $n \equiv 1 \pmod 2$, it is easy to check from Table 1 that

$$\lambda_0(n,k) + \lambda_0(n,n-k) = -2(-1)^{\frac{k(n-k)}{2}}$$

Then the combination of (3.12b) and its reversal enables us to write

$$\Delta_0(n,c)=\sum_k \frac{c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{n-k}{4}-c}\binom{\frac{n-k}{4}-c}{n-k}(-1)^{\frac{k(n-k)}{2}}$$

$$=\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{n-k}{4}-c}\binom{\frac{n-k}{4}-c}{n-k}-4\sum_{k\equiv 0\,(\mathrm{mod}\,4)}\frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{n-k}{4}-c}\binom{\frac{n-k}{4}-c}{n-k}.$$

Applying (0.1a) to the penultimate sum, we get

$$\Delta_0(n,c) \;=-\Diamond_0(n,-c)+\frac{-2c}{\frac{n}{4}-2c}\binom{\frac{n}{4}-2c}{n} \tag{3.11a}$$

$$=-\Delta_0(n,-c),\quad n\equiv 1\ (\mathrm{mod}\,2). \tag{3.11b}$$

Both relations (3.10) and (3.11) may be stated as the single one $\Delta_0(n,c)+\Delta_0(n,-c)\varepsilon(0,n)=0$ which, in view of (3.9a), confirms the case $m\equiv 0$ (mod 4) of Theorem 3.3 with replacements $k\to k+p$ and $n\to n+4p$.

Case 2: $m\equiv 1$ (mod 4). In view of Lemma 3.1 and (3.4c)-(3.4d), we have the generating function relation:

$$-W^{8c}(\tau)+W^{4c}(\tau)\times\{W^{4c}(\tau)-W^{4c}(-\tau)-iW^{4c}(i\tau)+iW^{4c}(-i\tau)\}$$
$$=-W^{-4c}(i\tau)W^{-4c}(-i\tau)+iW^{-4c}(-\tau)W^{-4c}(i\tau)-iW^{-4c}(-\tau)W^{-4c}(-i\tau).$$

The coefficients of τ^{1+n} in their formal power series expansions leads us to the following binomial convolutions,

$$\Delta_1(n,c)\overset{\mathrm{def}}{=}\Diamond_1(n,c)-\frac{2c}{2c+\frac{1+n}{4}}\binom{2c+\frac{1+n}{4}}{1+n} \tag{3.12a}$$

$$=\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{1+n-k}{4}-c}\binom{\frac{1+n-k}{4}-c}{1+n-k}\lambda_1(n,k), \tag{3.12b}$$

where $\lambda_1(n,k)=(-1)^n\{i^{1+k}+i^{1+n+2k}+i^{3+3k}\}$, whose values are displayed in Table 2.

TABLE 2. Values of $\lambda_1(n,k)$

$k \backslash n$	0	1	2	3
0	i	1	$-i$	-1
1	$-2-i$	1	$i-2$	3
2	i	1	$-i$	-1
3	$2-i$	-3	$i+2$	-1

For $n\equiv 1$ (mod 2), Table 2 suggests that we rewrite (3.12b) as

$$\Delta_1(n,c)=(-1)^{\frac{n-1}{2}}\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{1+n-k}{4}-c}\binom{\frac{1+n-k}{4}-c}{1+n-k}$$
$$-(-1)^{\frac{n-1}{2}}\sum_k \frac{-4c}{\frac{3}{4}-c+k}\binom{\frac{3}{4}-c+k}{3+4k}\frac{-c}{\frac{n-2}{4}-c-k}\binom{\frac{n-2}{4}-c-k}{n-2-4k}$$

which may be simplified, by means of (0.1a), to the following relation:

$$\Delta_1(n,c) = (-1)^{\frac{n-1}{2}}\frac{-2c}{\frac{1+n}{4}-2c}\binom{\frac{1+n}{4}-2c}{1+n} \tag{3.13a}$$

$$-(-1)^{\frac{n-1}{2}}\Diamond_3(n-2,-c), \quad n\equiv 1 \pmod 2 \tag{3.13b}$$

$$=\Delta_1(n,-c), \quad n\equiv 3 \pmod 4, \tag{3.13c}$$

where the last line is derived by reversing the summation order.

When $n\equiv 0 \pmod 2$, it is easy to check from Table 2 that

$$\lambda_1(n,k)+\lambda_1(n,1+n-k) = -2(-1)^{\frac{(n-k)(k-1)}{2}}.$$

Then the combination of (3.12b) and its reversal enables us to express

$$\Delta_1(n,c) = \sum_k \frac{c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{1+n-k}{4}-c}\binom{\frac{1+n-k}{4}-c}{1+n-k}(-1)^{\frac{(n-k)(k-1)}{2}}$$

$$=\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{1+n-k}{4}-c}\binom{\frac{1+n-k}{4}-c}{1+n-k} - 4\sum_{k\equiv 1 \,(\mathrm{mod}\, 4)}\frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{1+n-k}{4}-c}\binom{\frac{1+n-k}{4}-c}{1+n-k}.$$

Applying (0.1a) to the penultimate sum, we get

$$\Delta_1(n,c) = -\Diamond_1(n,-c)+\frac{-2c}{\frac{1+n}{4}-2c}\binom{\frac{1+n}{4}-2c}{1+n} \tag{3.14a}$$

$$=-\Delta_1(n,-c), \quad n\equiv 0 \pmod 2. \tag{3.14b}$$

Both relations (3.13) and (3.14) may be restated as

$$\Delta_1(n,c)+\Delta_1(n,-c)\varepsilon(1,n)=0, \quad n\not\equiv 1 \pmod 4$$

$$\Diamond_1(n,c)+\Diamond_3(n-2,-c): \qquad n\equiv 1 \pmod 4$$

$$=\frac{2c}{\frac{1+n}{4}+2c}\binom{\frac{1+n}{4}+2c}{1+n}+\frac{-2c}{\frac{1+n}{4}-2c}\binom{\frac{1+n}{4}-2c}{1+n}$$

which, in view of (3.12a), confirms the case $m\equiv 1 \pmod 4$ of Theorem 3.3 with replacements $k\to k+p$ and $n\to n+4p$.

Case 3: $m\equiv 2 \pmod 4$. Using Lemma 3.1 and (3.4e)-(3.4f), perform the formal manipulation on generating functions:

$$-W^{8c}(\tau)+W^{4c}(\tau)\times\{W^{4c}(\tau)+W^{4c}(-\tau)-W^{4c}(i\tau)-W^{4c}(-i\tau)\}$$
$$=W^{-4c}(i\tau)W^{-4c}(-i\tau)-W^{-4c}(-\tau)W^{-4c}(i\tau)-W^{-4c}(-\tau)W^{-4c}(-i\tau).$$

The coefficients of τ^{2+n} in the formal power series expansions lead us to the following binomial convolutions,

$$\Delta_2(n,c) \stackrel{\mathrm{def}}{=} \Diamond_2(n,c)-\frac{2c}{2c+\frac{2+n}{4}}\binom{2c+\frac{2+n}{4}}{2+n} \tag{3.15a}$$

$$=\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{2+n-k}{4}-c}\binom{\frac{2+n-k}{4}-c}{2+n-k}\lambda_2(n,k), \tag{3.15b}$$

where

$$\lambda_2(n,k)=(-1)^n\{i^{2+k}+i^{2+3k}+i^{2+n+2k}\}$$

whose values are displayed in Table 3.

For $n \equiv 0 \pmod 2$, Table 3 suggests that we reformulate (3.15b) as

$$\Delta_2(n,c) = (-1)^{n/2}\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{2+n-k}{4}-c}\binom{\frac{2+n-k}{4}-c}{2+n-k}$$
$$-(-1)^{n/2}\sum_k \frac{-4c}{\frac{1}{2}-c+k}\binom{\frac{1}{2}-c+k}{2+4k}\frac{-c}{\frac{n}{4}-c-k}\binom{\frac{n}{4}-c-k}{n-4k},$$

which may be simplified, by means of (0.1a), to the following relation:

$$\Delta_2(n,c) = -(-1)^{n/2}\Diamond_2(n,-c) + (-1)^{n/2}\frac{-2c}{\frac{2+n}{4}-2c}\binom{\frac{2+n}{4}-2c}{2+n} \tag{3.16a}$$
$$= -(-1)^{n/2}\Delta_2(n,-c), \quad n \equiv 0 \pmod 2. \tag{3.16b}$$

TABLE 3. Values of $\lambda_2(n,k)$

$k \backslash n$	0	1	2	3
0	-3	$i+2$	-1	$2-i$
1	1	$-i$	-1	i
2	1	$i-2$	3	$-2-i$
3	1	$-i$	-1	i

While $n \equiv 1 \pmod 2$, it is easy to check from Table 3 above that

$$\lambda_2(n,k) + \lambda_2(n,2+n-k) = 2(-1)^{\frac{k(n+k)}{2}}.$$

Then the combination of (3.12b) and its reversal enables us to state

$$\Delta_2(n,c) = \sum_k \frac{c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{2+n-k}{4}-c}\binom{\frac{2+n-k}{4}-c}{2+n-k}(-1)^{\frac{k(n+k)}{2}}$$
$$= \sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{2+n-k}{4}-c}\binom{\frac{2+n-k}{4}-c}{2+n-k} - 4\sum_{k\equiv 2 \pmod 4}\frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{2+n-k}{4}-c}\binom{\frac{2+n-k}{4}-c}{2+n-k}.$$

Applying (0.1a) to the penultimate sum, we get

$$\Delta_2(n,c) = -\Diamond_2(n,-c) + \frac{-2c}{\frac{2+n}{4}-2c}\binom{\frac{2+n}{4}-2c}{2+n} \tag{3.17a}$$
$$= -\Delta_2(n,-c), \quad n \equiv 1 \pmod 2. \tag{3.17b}$$

Both relations (3.16) and (3.17) may be written as the single relation

$$\Delta_2(n,c) + \Delta_2(n,-c)\varepsilon(2,n) = 0,$$

which confirms, in view of (3.15a), the case $m \equiv 2 \pmod 4$ of Theorem 3.3 with replacements $k \to k+p$ and $n \to n+4p$.

Case 4: $m \equiv 3 \pmod 4$. Finally, from Lemma 3.1 and (3.4g)-(3.4h), we get the following functional equation:

$$-W^{8c}(\tau)+W^{4c}(\tau)\times\{W^{4c}(\tau)-W^{4c}(-\tau)+iW^{4c}(i\tau)-iW^{4c}(-i\tau)\}$$
$$=W^{-4c}(i\tau)W^{-4c}(-i\tau)-iW^{-4c}(-\tau)W^{-4c}(i\tau)+iW^{-4c}(-\tau)W^{-4c}(-i\tau).$$

The coefficients of τ^{3+n} in their formal power series expansions leads us to the following binomial convolutions,

$$\Delta_3(n,c)\stackrel{\text{def}}{=}\Diamond_3(n,c)-\frac{2c}{2c+\frac{3+n}{4}}\binom{2c+\frac{3+n}{4}}{3+n} \tag{3.18a}$$

$$=\sum_k\frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{3+n-k}{4}-c}\binom{\frac{3+n-k}{4}-c}{3+n-k}\lambda_3(n,k), \tag{3.18b}$$

where

$$\lambda_3(n,k)=(-1)^n\{i^{3+k}+i^{1+3k}+i^{3+n+2k}\}$$

whose values are displayed in Table 4.

TABLE 4. Values of $\lambda_3(n,k)$

$k \backslash n$	0	1	2	3
0	$-i$	-1	i	1
1	$i+2$	-1	$2-i$	-3
2	$-i$	-1	i	1
3	$i-2$	3	$-2-i$	1

For $n\equiv 1 \pmod 2$, Table 4 suggests that we rewrite (3.18b) as

$$\Delta_3(n,c)=(-1)^{\frac{n-1}{2}}\sum_k\frac{-4c}{\frac{1}{4}-c+k}\binom{\frac{1}{4}-c+k}{1+4k}\frac{-c}{\frac{2+n}{4}-c-k}\binom{\frac{2+n}{4}-c-k}{2+n-4k}$$

$$-(-1)^{\frac{n-1}{2}}\sum_k\frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{3+n-k}{4}-c}\binom{\frac{3+n-k}{4}-c}{3+n-k},$$

which may be simplified, by means of (0.1a), to the following relation,

$$\Delta_3(n,c)=(-1)^{\frac{n+1}{2}}\frac{-2c}{\frac{3+n}{4}-2c}\binom{\frac{3+n}{4}-2c}{3+n} \tag{3.19a}$$

$$+(-1)^{\frac{n-1}{2}}\Diamond_1(2+n,-c),\quad n\equiv 1 \pmod 2 \tag{3.19b}$$

$$=(-1)^{\frac{n-1}{2}}\Delta_3(n,-c),\quad n\equiv 1 \pmod 4, \tag{3.19c}$$

where the last line is derived by reversing the summation order.

While $n\equiv 0 \pmod 2$, it is easy to check from Table 4 that

$$\lambda_3(n,k)+\lambda_3(n,3+n-k)=-2(-1)^{\frac{(n-k)(k+1)}{2}}.$$

Then, the combination of (3.18b) and its reversal leads us to

$$\Delta_3(n,c)=\sum_k\frac{c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{3+n-k}{4}-c}\binom{\frac{3+n-k}{4}-c}{3+n-k}(-1)^{\frac{(n-k)(k+1)}{2}}$$

$$=\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{3+n-k}{4}-c}\binom{\frac{3+n-k}{4}-c}{3+n-k}-4\sum_{k\equiv 3\,(\mathrm{mod}\,4)}\frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{3+n-k}{4}-c}\binom{\frac{3+n-k}{4}-c}{3+n-k}.$$

Applying (0.1a) to the penultimate sum, we get

$$\Delta_3(n,c)=-\Diamond_3(n,-c)+\frac{-2c}{\frac{3+n}{4}-2c}\binom{\frac{3+n}{4}-2c}{3+n} \tag{3.20a}$$

$$=-\Delta_3(n,-c),\quad n\equiv 0\ (\mathrm{mod}\,2). \tag{3.20b}$$

Both relations (3.19) and (3.20) may be reproduced as

$$\Delta_3(n,c)+\Delta_3(n,-c)\varepsilon(3,n)=0,\quad n\not\equiv 3\ (\mathrm{mod}\,4),$$

$$\Diamond_3(n,c)+\Diamond_1(2+n,-c):\quad n\equiv 3\ (\mathrm{mod}\,4)$$

$$=\frac{2c}{\frac{3+n}{4}+2c}\binom{\frac{3+n}{4}+2c}{3+n}+\frac{-2c}{\frac{3+n}{4}-2c}\binom{\frac{3+n}{4}-2c}{3+n},$$

which confirms, in view of (3.18a), the case $m\equiv 3$ (mod 4) of Theorem 3.3 with replacements $k\to k+p$ and $n\to n+4p$.

Therefore, the proof of Theorem 3.3 is complete. □

Remark: During the 100th anniversary of Tricomi (October 1997, Rome), Richard Askey suggested that the author try another approach to the binomial identities stated in Theorem 1.3. This may be presented as follows:

Letting $\beta=1/2$ in (0.1b), we obtain

$$\sum\frac{a}{a+k/2}\binom{a+k/2}{k}\binom{c-k/2}{n-k}=\binom{a+c}{n}. \tag{3.21a}$$

By means of

$$\frac{a}{a+k/2}\binom{a+k/2}{k}=(-1)^k\frac{-a}{-a+k/2}\binom{-a+k/2}{k},$$

we can reformulate (3.21a) as

$$\sum_{k=0}^{n}(-1)^k\frac{a}{a+k/2}\binom{a+k/2}{k}\binom{c-k/2}{n-k}=\binom{c-a}{n}. \tag{3.21b}$$

Then identities (1.5a) and (1.5b) follow directly from the combinations of (3.21a) and (3.21b). Two other identities, stated in (1.6a)-(1.6b) and (1.6c)-(1.6d), may be derived similarly from (0.1a). The details are left to the reader.

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ON MODIFIED DICKSON POLYNOMIALS

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1. INTRODUCTION

The *modified Dickson polynomials*

$$Z_n(y,a) = \sum_{j=0}^{[n/2]} \frac{n}{n-j}\binom{n-j}{j}(-a)^j y^{[n/2]-j} \tag{1.1}$$

were defined and studied by P. Filipponi in the case $a = 1$ in [1], where several identities and congruences were established. In this note we generalize some of those theorems and present some new properties of these polynomials. One basic result is Proposition 2 in [1] which states that if p is an odd prime and k is an integer, then

$$Z_p(k,1) \equiv (k\,|\,p) \pmod{p}, \tag{1.2}$$

where $(k\,|\,p)$ is the Legendre symbol. The generalization is as follows.

Theorem 1: If p is an odd prime, a and k are integers, and m and r are positive integers, then

$$Z_{mp^r}(k,a) \equiv H_m(k)\cdot Z_{mp^{r-1}}(k,a) \pmod{p^r},$$

where

$$H_m(k) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ (k\,|\,p), & \text{if } m \text{ is odd.} \end{cases}$$

We will deduce this from a corresponding congruence for these polynomials in the polynomial ring $\mathbb{Z}[y,a]$, and present a few applications thereof in the next section. We give an analogous definition of modified Dickson polynomials of the second kind and give some identities, recurrences, and congruences for them in Section 3. We conclude by describing a compositeness test based on Theorem 1 in the last section.

2. CONGRUENCES FOR MODIFIED DICKSON POLYNOMIALS

The (usual) Dickson polynomials $D_n(x,a)$ are defined for $n > 0$ by

$$D_n(x,a) = \sum_{j=0}^{[n/2]} \frac{n}{n-j}\binom{n-j}{j}(-a)^j x^{n-2j} \tag{2.1}$$

(cf. [2]), with the convention that $D_0(x,a) = 2$. They may also be defined as the expansion coefficients of the rational differential form

$$\frac{dP}{P} = -\sum_{n=1}^{\infty} D_n(x,a)T^n \frac{dT}{T}, \tag{2.2}$$

where $P(T) = 1 - xT + aT^2$ (see [5], eq. (1.6)), and they satisfy the functional equation

$$D_n\left(u + \frac{a}{u}, a\right) = u^n + \left(\frac{a}{u}\right)^n. \tag{2.3}$$

By comparing (1.1) and (2.1), we see that as polynomials in y and a,

$$Z_n(y,a)=\begin{cases} D_n(y^{1/2},a), & \text{if } n \text{ is even,} \\ y^{-1/2}D_n(y^{1/2},a), & \text{if } n \text{ is odd} \end{cases} \tag{2.4}$$

(cf. [1], eq. (1.2)). We have the following congruence for the polynomials $Z_n(y,a)$.

Theorem 2: If p is an odd prime and m, r are positive integers, then the congruence

$$Z_{mp^r}(y,a)\equiv H_m\cdot Z_{mp^{r-1}}(y^p,a^p) \pmod{p^r\mathbb{Z}[y,a]}$$

holds in the polynomial ring $\mathbb{Z}[y,a]$, where

$$H_m=\begin{cases} 1, & \text{if } m \text{ is even,} \\ y^{(p-1)/2} & \text{if } m \text{ is odd.} \end{cases}$$

Proof: In Theorem 2 of [5] we showed that the congruence

$$D_{mp^r}(x,a)\equiv D_{mp^{r-1}}(x^p,a^p) \pmod{p^r\mathbb{Z}[y,a]} \tag{2.5}$$

holds in the polynomial ring $\mathbb{Z}[y,a]$. Replacing the indeterminate x with $y^{1/2}$ yields

$$D_{mp^r}(y^{1/2},a)\equiv D_{mp^{r-1}}(y^{p/2},a^p) \pmod{p^r\mathbb{Z}[y^{1/2},a]}, \tag{2.6}$$

where $y^{p/2}$ is defined to be $(y^{1/2})^p$. By (2.4), this gives the result for even m, since both sides of the congruence (2.6) lie in $\mathbb{Z}[y,a]$ in that case. For odd m, we divide both sides of (2.6) by $y^{1/2}$ to obtain the congruence

$$y^{-1/2}D_{mp^r}(y^{1/2},a)\equiv y^{(p-1)/2}\cdot(y^{-p/2}D_{mp^{r-1}}(y^{p/2},a^p)) \pmod{p^r\mathbb{Z}[y,a]}, \tag{2.7}$$

both sides of which now lie in $\mathbb{Z}[y,a]$. Comparison with (2.4) now gives the result for odd m.

Theorem 1 may be obtained directly from this as follows.

Proof of Theorem 1: Let a, k be integers and consider them as elements of the ring $\mathbb{Z}_p$ of p-adic integers. For an element u of $\mathbb{Z}_p$, the Teichmüller representative $\hat{u}$ of u is defined to be the unique solution to $x^p=x$ which is congruent to u modulo $p\mathbb{Z}_p$; it is also given by the p-adic limit $\hat{u}=\lim_{r\to\infty}u^{p^r}$. Observing that $\hat{a}^p=\hat{a}$, $\hat{k}^p=\hat{k}$, and $\hat{k}^{(p-1)/2}=(k\,|\,p)$, we evaluate the polynomial congruence of Theorem 2 at $y=k$, $a=\hat{a}$ to obtain

$$Z_{mp^r}(\hat{k},\hat{a})\equiv H_m(k)\cdot Z_{mp^{r-1}}(\hat{k},\hat{a}) \pmod{p^r\mathbb{Z}_p}, \tag{2.8}$$

where $H_m(k)$ is as defined in the statement of the theorem.

Now, from the second statement of Theorem 3 given in [5], applied with $i=1$, $n=1$, and $K=\mathbb{Q}_p(\sqrt{k})$, it follows that

$$D_{mp^r}(k^{1/2},a)\equiv D_{mp^r}(\hat{k}^{1/2},\hat{a}) \pmod{\pi p^r\mathfrak{O}_K} \tag{2.9}$$

for all r, where (π) is the maximal ideal in the ring of integers $\mathfrak{O}_K$ of the field K. For m even, comparison of (2.8) and (2.9) yields

$$D_{mp^r}(k^{1/2},a)\equiv D_{mp^{r-1}}(k^{1/2},a) \pmod{\pi p^{r-1}\mathfrak{O}_K}, \tag{2.10}$$

but both sides of this congruence are integers, so it must hold modulo $p^r\mathbb{Z}$. In this case the theorem then follows by comparison with (2.4).

If m is odd and $\hat{k} \neq 0$, multiplying both sides of (2.8) by $\hat{k}^{1/2}$ yields

$$D_{mp^r}(\hat{k}^{1/2}, \hat{a}) \equiv (k \mid p) \cdot D_{mp^{r-1}}(\hat{k}^{1/2}, \hat{a}) \pmod{p^r \mathfrak{O}_K}. \tag{2.11}$$

Comparison with (2.9) shows that

$$D_{mp^r}(k^{1/2}, a) \equiv (k \mid p) \cdot D_{mp^{r-1}}(k^{1/2}, a) \pmod{\pi p^{r-1} \mathfrak{O}_K}, \tag{2.12}$$

and then dividing by $k^{1/2}$ yields

$$Z_{mp^r}(k, a) \equiv (k \mid p) \cdot Z_{mp^{r-1}}(k, a) \pmod{\pi p^{r-1} \mathfrak{O}_K}, \tag{2.13}$$

but again both sides of this congruence are integers, so it holds modulo $p^r \mathbb{Z}$, proving the theorem in that case.

Finally, when $\hat{k} = 0$ and n is odd, we have the identity $Z_n(0, a) = (-a)^{(n-1)/2} \cdot n$ (cf. [1], eq. (2.7)), so that $Z_{mp^r}(0, a) \equiv 0 \pmod{p^r}$ when m is odd. Combining this with (2.9), we see that in this case we also have

$$Z_{mp^r}(k, a) \equiv H_m(k) \cdot Z_{mp^{r-1}}(k, a) \pmod{\pi p^{r-1} \mathfrak{O}_K}, \tag{2.14}$$

but again both sides are integers, proving the theorem.

Remarks: Perhaps the most interesting feature of these theorems is that while the "special element" H_m depends on y and on the parity of m, it does not depend on a. For example, taking $m = 1$, $r = 1$ in Theorem 1 and observing that $Z_1(y, a) = 1$ yields

$$Z_p(k, a) \equiv (k \mid p) \pmod{p}, \tag{2.15}$$

of which Filipponi's result (1.2) is a special case; indeed it is evident from (1.1) that $Z_p(k, a) \equiv k^{(p-1)/2} \pmod{p}$ for all a. In Section 4 below we propose a compositeness test based on (2.15).

One also obtains interesting congruences by combining Theorem 1 above with Filipponi's multiplication formula ([1], eq. (3.6)). For example, for n even the $h = 3$ case of Filipponi's result is the identity

$$Z_{3n} = Z_n^3 - 3Z_n \tag{2.16}$$

(cf. [1], eq. (3.5)), where $Z_n = Z_n(k, 1)$. Putting $n = m \cdot 3^r$ with m even, from Theorem 1.1 we obtain $Z_{3n} \equiv Z_n \pmod{3^{r+1}}$; combining this with (2.16) yields $Z_n(Z_n^2 - 4) \equiv 0 \pmod{3^{r+1}}$. It follows that, if n is even and divisible by 3^r, then Z_n is congruent to either -2, 0, or 2 modulo 3^{r+1}. A similar but slightly more complicated result holds for n odd. Many other such results may be obtained similarly.

We conclude this section with a generating form and recurrence for the $Z_n(y, a)$, which provides an efficient means for generating the sequence and for obtaining identities.

Theorem 3: For $n > 0$ the polynomials $Z_n(y, a)$ may be obtained as the expansion coefficients of the rational differential form

$$\sum_{n=1}^{\infty} Z_n(y, a) T^n \frac{dT}{T} = \frac{(1 - (2a - y)T - aT^2 - 2a^2T^3)\, dT}{1 + (2a - y)T^2 + a^2T^4}.$$

Consequently, the sequence $Z_n = Z_n(y, a)$ is given by the recurrence $Z_0 = 2$, $Z_1 = 1$, $Z_2 = y - 2a$, $Z_3 = y - 3a$, and $Z_{n+2} = (y - 2a)Z_n - a^2 Z_{n-2}$.

Proof: Use (2.4) to write the power series

$$\sum_{n=1}^{\infty} Z_n T^{n-1} = \frac{1}{2} y^{-1/2} \sum_{n=1}^{\infty} D_n(y^{1/2}, a)(T^{n-1} + (-T)^{n-1}) \\ + \frac{1}{2} \sum_{n=1}^{\infty} D_n(y^{1/2}, a)(T^{n-1} - (-T)^{n-1}) \tag{2.17}$$

as the sum of an even function of T and an odd function of T. Then from (2.2) we obtain

$$-\sum_{n=1}^{\infty} Z_n(y, a) T^n \frac{dT}{T} = \frac{1}{2}\left((1 + y^{-1/2}) \frac{dP(T)}{P(T)} + (1 - y^{-1/2}) \frac{dP(-T)}{P(-T)} \right), \tag{2.18}$$

where $P(T) = 1 - y^{1/2}T + aT^2$. Expanding and simplifying (2.18) yields the first statement of the theorem. The recurrence follows by multiplying both sides by $1 + (2a - y)T^2 + a^2T^4$ and equating coefficients of $T^n dT$.

3. MODIFIED DICKSON POLYNOMIALS OF THE SECOND KIND

The Dickson polynomials of the second kind $E_n(x, a)$ are defined for $n \geq 0$ by

$$E_n(x, a) = \sum_{j=0}^{[n/2]} \binom{n-j}{j} (-a)^j x^{n-2j} \tag{3.1}$$

(cf. [2]). They may also be defined as the expansion coefficients of the rational differential form

$$\frac{dT}{P(T)} = \sum_{n=0}^{\infty} E_n(x, a) T^n dT, \tag{3.2}$$

where $P(T) = 1 - xT + aT^2$ ([5], eq. (4.4)). By way of analogy with (1.1) we define the *modified Dickson polynomials of the second kind* $Y_n(y, a)$ by

$$Y_n(y, a) = \sum_{j=0}^{[n/2]} \binom{n-j}{j} (-a)^j y^{[n/2]-j}. \tag{3.3}$$

Comparison of (3.1) and (3.3) shows that as polynomials in y and a,

$$Y_n(y, a) = \begin{cases} E_n(y^{1/2}, a), & \text{if } n \text{ is even}, \\ y^{-1/2} E_n(y^{1/2}, a), & \text{if } n \text{ is odd}. \end{cases} \tag{3.4}$$

From this definition, we deduce the following generating form for the polynomials $Y_n(y, a)$.

Theorem 4: The polynomials $Y_n(y, a)$ may be obtained as the expansion coefficients of the rational differential form

$$\sum_{n=0}^{\infty} Y_n(y, a) T^n dT = \frac{(1 + T + aT^2) dT}{1 + (2a - y)T^2 + a^2T^4}.$$

Consequently, the sequence $Y_n = Y_n(y, a)$ is given by the recurrence $Y_0 = 1$, $Y_1 = 1$, $Y_2 = y - a$, $Y_3 = y - 2a$, and $Y_{n+2} = (y - 2a)Y_n - a^2 Y_{n-2}$.

Proof: Use (3.4) to write the power series $\sum_n Y_n(y,a)T^n$ as the sum of an even function of T and an odd function of T. Then from (3.2) we obtain

$$\sum_{n=0}^{\infty} Y_n(y,a)T^n dT = \frac{1}{2}\left((1+y^{-1/2})\frac{dT}{P(T)} + (1-y^{-1/2})\frac{dT}{P(-T)}\right), \tag{3.5}$$

where $P(T) = 1 - y^{1/2}T + aT^2$. Expanding and simplifying (3.5) yields the first statement of the theorem. The recurrence follows by multiplying both sides by $1+(2a-y)T^2+a^2T^4$ and equating coefficients of $T^n dT$.

The generating functions for Y_n and Z_n may be used directly to deduce several identities relating them to D_n and E_n, some of which we record here.

Theorem 5: In the polynomial ring $\mathbb{Z}[y,a]$ we have the identities

(i) $Y_{2m-1}(y,a) = E_{m-1}(y-2a, a^2)$ for $m > 0$,
(ii) $Z_{2m}(y,a) = D_m(y-2a, a^2)$ for $m \geq 0$,
(iii) $Y_{2m}(y,a) + Z_{2m+1}(y,a) = 2E_m(y-2a, a^2)$ for $m \geq 0$,
(iv) $Y_{2m}(y,a) - Z_{2m+1}(y,a) = 2aE_{m-1}(y-2a, a^2)$ for $m > 0$.

Proof: For (iii), use Theorems 3 and 4 and equation (3.2) to write

$$\begin{aligned}\sum_{n=0}^{\infty}(Y_n + Z_{n+1})T^n dT &= \frac{2\,dT}{1+(2a-y)T^2+a^2T^4} + (\text{odd function of } T)\, dT \\ &= 2\sum_{m=0}^{\infty} E_m(y-2a, a^2)T^{2m}dT + (\text{odd function of } T)\, dT.\end{aligned} \tag{3.6}$$

Equating coefficients of $T^{2m}dT$ gives the result. The other parts are obtained similarly.

Remarks: Replacing y with y^2 in (ii) and using (2.4) yields the $n=2$ case of the familiar composition formula

$$D_{mn}(y,a) = D_m(D_n(y,a), a^n) \tag{3.7}$$

for the usual Dickson polynomials. An analogous formula

$$E_{2m-1}(y,a) = y \cdot E_{m-1}(D_2(y,a), a^2) \tag{3.8}$$

is obtained in a like manner from (i). Similar composition formulas for E_{2m} and D_{2m+1} may be obtained by combining (iii) and (iv) and replacing y with y^2.

Another set of identities relating the polynomials Y_n and Z_n may be derived from the observation that the characteristic polynomial $1+(2a-y)T^2+a^2T^4$ is invariant under the transformation $a \mapsto -a$, $y \mapsto y-4a$, as follows.

Theorem 6: If m is a nonnegative integer we have, as identities in the polynomial ring $\mathbb{Z}[y,a]$,

(i) $Z_{2m+1}(y-4a, -a) = Y_{2m}(y,a)$,
(ii) $Z_{2m}(y-4a, -a) = Z_{2m}(y,a)$,
(iii) $Y_{2m}(y-4a, -a) = Z_{2m+1}(y,a)$,
(iv) $Y_{2m+1}(y-4a, -a) = Y_{2m+1}(y,a)$.

Proof: Using the generating form from Theorem 3, we compute

$$\begin{aligned}\sum_{n=1}^{\infty} Z_n(y-4a,-a)T^n \frac{dT}{T} &= \frac{(1-(2a-y)T+aT^2-2a^2T^3)\,dT}{1+(2a-y)T^2+a^2T^4} \\ &= \frac{(-(2a-y)T-2a^2T^3)\,dT}{1+(2a-y)T^2+a^2T^4}+\frac{(1+aT^2)\,dT}{1+(2a-y)T^2+a^2T^4}.\end{aligned} \tag{3.9}$$

Noting that the even part of this form agrees with the even part of the generating form for Y_n from Theorem 4, and the odd part of (3.9) agrees with the odd part of the generating form for Z_n from Theorem 3, gives results (i) and (ii). Repeating the argument starting from the generating form for Y_n from Theorem 4 gives (iii) and (iv).

Remark: Parts (i) and (iii) of Theorem 6 are equivalent.

Finally, we will use the results of Theorems 5 and 6 to give an analog of Theorem 1 for the values of the polynomials Y_n.

Theorem 7: If p is an odd prime, a and k are integers, and m and r are positive integers, then

$$Y_{mp^r-1}(k,a) \equiv G_m(k)\cdot Y_{mp^{r-1}-1}(k,a) \pmod{p^r},$$

where

$$G_m(k)=\begin{cases}(k(k-4a)\,|\,p), & \text{if } m \text{ is even,}\\ (k-4a\,|\,p), & \text{if } m \text{ is odd.}\end{cases}$$

Proof: First, suppose $m=2j$ is even. From Theorem 5(i), we have

$$Y_{mp^r-1}(k,a)=E_{jp^r-1}(k-2a,a^2)$$

for all $r\ge 0$. Using the congruence

$$E_{jp^r-1}(x,a) \equiv (x^2-4a\,|\,p)\cdot E_{jp^{r-1}-1}(x,a) \pmod{p^r} \tag{3.10}$$

(see [5], Cor. C; [4], Cor. 1(i)) with $x=k-2a$ and a replaced by a^2 yields the result for even m.

If m is odd, then mp^r-1 is even for all $r\ge 0$, and from Theorem 6(i) we have $Y_{mp^r-1}(y,a)=Z_{mp^r}(y-4a,-a)$. The result in this case then follows from the odd m case of Theorem 1.

Remarks: While it is possible to prove a polynomial congruence that holds modulo $p^r\mathbb{Z}[y,a]$ (analogous to Theorem 2) for the Y_n, the resulting congruence is rather inelegant due to the cumbersome "lifting of Frobenius" involved (cf. [5], Remark A.2, p. 43). However, the "mod p" case of this congruence may be stated rather simply: If p is an odd prime and m is a positive integer, then the congruence

$$Y_{mp-1}(y,a) \equiv G_m\cdot Y_{m-1}(y^p,a^p) \pmod{p\mathbb{Z}[y,a]} \tag{3.11}$$

holds in the polynomial ring $\mathbb{Z}[y,a]$, where

$$G_m=\begin{cases}(y(y-4a))^{(p-1)/2}, & \text{if } m \text{ is even,}\\ (y-4a)^{(p-1)/2}, & \text{if } m \text{ is odd.}\end{cases} \tag{3.12}$$

For m even, this follows from Theorem 5(i) above and from Theorem 5 in [5]; for m odd, it follows from Theorem 6(i) and from the odd m case of Theorem 2. In particular, the special case $m=1$ yields the congruence

$$Y_{p-1}(y,a) \equiv (y-4a)^{(p-1)/2} \pmod{p\mathbb{Z}[y,a]}, \tag{3.12}$$

and the case $m=2$ yields

$$Y_{2p-1}(y,a) \equiv (y(y-4a))^{(p-1)/2} \pmod{p\mathbb{Z}[y,a]}. \tag{3.13}$$

4. A COMPOSITENESS TEST

The congruence (2.15) furnishes a compositeness test which contains the usual Dickson polynomial test and the Solovay-Strassen test as special cases. If n is a prime then for all integers k and a we have

$$Z_n(k,a) \equiv (k\,|\,n) \pmod{n} \tag{4.1}$$

by (2.15), where $(k\,|\,n)$ now (and throughout this section) denotes the Jacobi symbol. If n is odd then in the special case in which $a=0$ the congruence (4.1) becomes

$$k^{(n-1)/2} \equiv (k\,|\,n) \pmod{n}, \tag{4.2}$$

which is the basis for the Solovay-Strassen test. On the other hand, suppose n is odd and k is a quadratic residue modulo n. Writing $k \equiv b^2 \pmod{n}$ and using (2.4), we have

$$bZ_n(k,a) \equiv bZ_n(b^2,a) = D_n(b,a) \pmod{n}, \tag{4.3}$$

whereas $(k\,|\,n)=1$. So, in the case where k is a quadratic residue modulo n, the congruence (4.1) is equivalent to the congruence

$$D_n(b,a) \equiv b \pmod{n}, \tag{4.4}$$

which is the basis of the usual Dickson polynomial compositeness test.

If n is a prime, it is clear that (4.4) is satisfied for all integers a and b from (2.5) with $m=r=1$ and $p=n$; and (4.2) is likewise satisfied for all integers k. However, if n is an odd composite number then there exist values of k with $(k,n)=1$ for which (4.2) holds; in this case, n is said to be a *Euler pseudoprime* to the base k. Furthermore, if n is an odd composite it may happen that (4.4) is satisfied for all integers b and a fixed integer a, in which case n is said to be a *strong Dickson pseudoprime* to the base a (cf. [2]). It is even possible that n may be a strong Dickson pseudoprime to every base; that is, (4.4) may hold for all integers a and b, although n is not prime.

It is quite easy to see that the compositeness test we propose based on the congruence (4.1) admits no "strong pseudoprimes" to any given base a; in fact, if n is not prime then for any a the congruence (4.1) fails at least half the time, as we now record.

Theorem 8: Let n be an odd composite integer, and let U_n denote the group of units in the ring $\mathbb{Z}/n\mathbb{Z}$. Then for any integer a, the congruence (4.1) fails for at least half the elements k of U_n.

Proof: First, suppose that n is a nonsquare and write $n=p^em$ with p prime, e odd, and $(m,p)=1$. Suppose that (4.1) holds for $k=b$. Using the Chinese remainder theorem, choose an integer c such that $c \equiv b \pmod{m}$ and $(c|p) = -(b|p)$. It then follows that $(c|n) = -(b|n)$ but $Z_n(c, a) \equiv Z_n(b, a) \pmod{m}$; hence (4.1) cannot hold for $k = c$. Using the isomorphism $U_n \cong U_m \times U_{p^e}$, we see that in fact half the integers c congruent to b modulo m have $(c|p) = -(b|p)$. Therefore, in any congruence class modulo m at most half the elements k can satisfy (4.1). The theorem then follows in this case.

Now suppose that n is a square and write $n = p^2 m$ with p prime. Since n is a square we have $(k|n) = 1$ for all integers k. Suppose then that (4.1) holds for $k = b$; then evaluating the polynomial congruence of Theorem 2 with $r = 2$ at $a = a$, $y = b$ yields

$$1 \equiv Z_{mp^2}(b, a) \equiv b^{(p-1)/2} Z_{mp}(b^p, a^p) \pmod{p^2}. \tag{4.5}$$

Now if c is any integer congruent to b modulo p, then $c^p \equiv b^p \pmod{p^2}$ and therefore $Z_{mp}(c^p, a^p) \equiv Z_{mp}(b^p, a^p) \pmod{p^2}$. However, if $c \equiv b \pmod{p}$, then $c^{(p-1)/2} \not\equiv b^{(p-1)/2} \pmod{p^2}$ unless $c \equiv b \pmod{p^2}$. Thus, if $c \equiv b \pmod{p}$ but $c \not\equiv b \pmod{p^2}$ then (4.1) cannot hold for $k = c$. Rewriting n as $n = p^e m'$ with e even and $(p, m') = 1$ and using the isomorphism $U_n \cong U_{m'} \times U_{p^e}$ shows that more than half the integers $c \in U_n$ which are congruent to b modulo p are not congruent to b modulo p^2. The theorem then follows in this case.

The test described here may be implemented in time commensurate with that required for other well-known tests. Using the identities

$$Z_{2n}(k, a) = \begin{cases} Z_n(k, a)^2 - 2a^n, & \text{if } n \text{ is even,} \\ kZ_n(k, a)^2 - 2a^n, & \text{if } n \text{ is odd,} \end{cases} \tag{4.6}$$

$$Z_{2n+1}(k, a) = Z_{n+1}(k, a) Z_n(k, a) - a^n, \tag{4.7}$$

and the recursion

$$Z_{n+1}(k, a) = \begin{cases} Z_n(k, a) - aZ_{n-1}(k, a), & \text{if } n \text{ is even,} \\ kZ_n(k, a) - aZ_{n-1}(k, a), & \text{if } n \text{ is odd,} \end{cases} \tag{4.8}$$

one may compute $Z_n(k, a)$ with $O(\log n)$ multiplications, as outlined in Lemma 2.5 of [2] for $D_n(k, a)$. The identities (4.6)-(4.8) were given in the case $a = 1$ in equations (3.2)-(3.4) of [1], and are proved for general a in the same manner.

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ON FIBONACCI AND PELL NUMBERS OF THE FORM kx^2 (Almost Every Term Has a $4r + 1$ Prime Factor)

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1. INTRODUCTION

In 1983 and 1984, Neville Robbins showed that neither the Fibonacci nor the Pell number sequence has terms of the form px^2 for prime $p \equiv 3 \pmod 4$, with one exception in each sequence [3], [4]. The main idea of Robbins' paper can be used to prove a stronger result, namely, that with a small number of exceptions, neither sequence has terms of the form kx^2 if k is an integer all of whose prime factors are congruent to 3 modulo 4. An interesting corollary is that, with 11 exceptions, every term of the Fibonacci sequence has a prime factor of the form $4r+1$ and, similarly, with 5 exceptions, for the Pell sequence.

The solutions of $F_n = x^2$ and $F_n = 2x^2$ were found by Cohn [1], and of $F_n = kx^2$, for certain values of $k > 2$, by Robbins [5]; of particular interest is Robbins' result that there are 15 values of k, $2 < k \le 1000$, for which solutions exist, and he gives these solutions. We refer the reader to [5].

2. SOME IDENTITIES AND RESULTS

We shall use the following identities and well-known facts relating the Fibonacci and Lucas numbers:

$$F_{2n} = F_n L_n, \tag{1}$$

$$\gcd(F_n, L_n) = 2 \text{ if } 3 \mid n \text{ and } 1 \text{ otherwise,} \tag{2}$$

$$F_{2n+1} = F_n^2 + F_{n+1}^2. \tag{3}$$

Let $S = \{3, 4, 6, 8, 16, 24, 32, 48\}$ and let $T = \{k' \mid k' > 1$ is square-free and each odd prime factor of k' is $\equiv 3 \pmod 4\}$. It may be noted that, in the following theorem, there is no loss of generality in assuming that k is square-free.

Theorem 1: If $n > 1$, then $F_n = kx^2$ for some square-free integer $k \ge 2$ whose odd prime factors are all $\equiv 3 \pmod 4$ iff $n \in S$.

Proof: The sufficiency:

$$\begin{array}{llll}
F_3 = 2 & (k=2) & F_{16} = 3\cdot 7\cdot 47 & (k=987)\\
F_4 = 3 & (k=3) & F_{24} = 2^5\cdot 3^2\cdot 7\cdot 23 & (k=322)\\
F_6 = 8 & (k=2) & F_{32} = 3\cdot 7\cdot 47\cdot 2207 & (k=2178309)\\
F_8 = 3\cdot 7 & (k=21) & F_{48} = 2^6\cdot 3^2\cdot 7\cdot 23\cdot 47\cdot 1103 & (k=8346401)
\end{array}$$

The necessity: Assume there exists at least one integer $n > 1$, $n \notin S$, such that F_n has the form $k'X^2$ for some $k' \in T$ and integer X. Then there exists a least such integer N; we let $F_N = kx^2$ for some $k \in T$ and integer x. Now N is not odd, since, by (3), if N is odd, then F_N is the sum of 2 squares and it is well known that the square-free part of the sum of 2 squares does not have a factor $\equiv 3 \pmod 4$. Let $N = 2m$. Then, by (1) and (2), $F_N = kx^2$ implies that there exist integers y and z, $x = yz$, such that either

(a) $F_m = y^2$ and $L_m = kz^2$, (c) $F_m = k_1y^2$ and $L_m = k_2z^2$, or

(b) $F_m = 2y^2$ and $L_m = 2kz^2$, (d) $F_m = 2k_1y^2$ and $L_m = 2k_2z^2$,

where $k_1k_2 = k$, $k_1 > 2$.

If (a), then by [1], $m = 1, 2,$ or 12. But then $N = 2$, which is not possible since $F_2 = 1$, or $N = 4$ or 24, contrary to our assumption that $N \notin S$.

If (b), then by [2], $m = 3$ or 6, but then $N = 6$, contrary to our assumption, or $N = 12$, but $F_{12} \neq kx^2$.

If (c), then, since $m < N$ and $k_1 \in T$, $m = 4, 6, 8, 16, 24, 32,$ or 48; that is, $N = 8$, 12, 16, 32, 48, 64, or 96. But 8, 16, 32, 48 are in S, $F_{12} \neq k_1x^2$, $4481 \mid F_{64}$, and $769 \mid F_{96}$ ($4481, 769 \equiv 1 \pmod 4$).

If (d), then either $2k_1 \in T$ or, if k_1 is even, $k_1 = 2k_3$ and $F_m = 2k_1y^2 = k_3(2y^2)$, with $k_3 \in T$; hence, the argument of (c) applies with k_1 replaced by $2k_1$ or k_3.

It follows that, if $n \notin S$, then $F_n \neq k'x^2$ for any $k' \in T$.

Since $F_n \neq kx^2$ implies $F_n \neq k$, we immediately have

Theorem 2: If $n \neq 0, 1, 2$ or an element of S, then F_n has at least one prime factor of the form $4r + 1$.

If P_n denotes the n^{th} Pell number, and R_n the n^{th} term of the "associated Pell sequence" ($R_0 = 2, R_1 = 1$), then, with one minor change, properties (1), (2), and (3) hold: $P_{2m} = P_mR_m$, $\gcd(P_m, R_m) = 2$ if m is even and 1 otherwise, and $P_{2m+1} = p_m^2 + p_{m+1}^2$.

We have the following results for Pell numbers. The proofs require the known facts that P_n is a square iff $n = 1$ or 7 and P_n is twice a square iff $n = 1$ (see [4]); since the proofs parallel those of Theorems 1 and 2, we omit them.

Theorem 3: If $n > 1$, then $P_n = kx^2$ for some square-free integer k whose odd prime factors are all $\equiv 3 \pmod 4$ iff $n = 2, 4,$ or 14.

Theorem 4: If $n \neq 0, 1, 2, 4,$ or 14, then P_n has at least one prime factor of the form $4r + 1$.

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THE $3x+1$ PROBLEM AND DIRECTED GRAPHS

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0. INTRODUCTION

Let **Z** denote the set of integers, **P** denote the positive integers, and **N** denote the nonnegative integers. Define the *Collatz mapping* $T: 2\mathbf{N}+1 \to 2\mathbf{N}+1$ by $T(x) = (3x+1)/2^j$, where $2^j \mid 3x+1$ but $2^{j+1} \nmid 3x+1$. The famous $3x+1$ *Conjecture*, or *Collatz Problem*, asserts that, for any $x \in 2\mathbf{N}+1$, there exists $k \in \mathbf{N}$ satisfying $T^k(x) = 1$, where T^k denotes k compositions of the function T. This paper's version of the Collatz mapping is also found in [4], whereas the most commonly used version is given in the comprehensive survey of Lagarias [6] and the research monograph of Wirsching [9]. It is obvious that our formulation of the $3x+1$ Conjecture is equivalent to those given in [6] and [9].

It is natural to study the $3x+1$ Conjecture in terms of the directed graph $G_{2\mathbf{N}+1}$ with vertices $2\mathbf{N}+1$ and directed edges from x to $T(x)$. A portion of this graph, known as the *Collatz graph* [6], is displayed in Figure 1. A slightly different version of the Collatz graph, which includes the positive even integers, is presented in [6], whereas $G_{2\mathbf{N}+1}$ excludes these with the purpose of making upcoming properties of certain vertices more transparent.

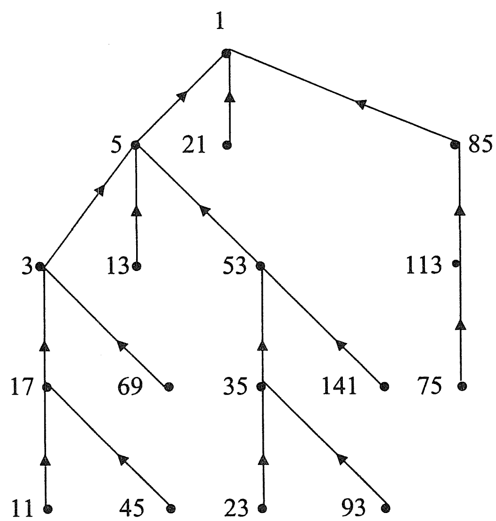

Figure 1. The Collatz Graph $G_{2\mathbf{N}+1}$ $(T^4(x) = 1, x < 150)$

A directed graph is said to be *weakly connected* if it is connected when viewed as an undirected graph, and we will call a pair of vertices weakly connected if they are connected by an undirected path. Using these graph-theoretical considerations, the $3x+1$ Conjecture can be restated as follows:

$3x+1$ Conjecture (1st form): The Collatz graph is weakly connected.

Our immediate goal is to identify a collection of vertices of $G_{2\mathbf{N}+1}$ which have a certain connectivity property (Section 1). We then use this result to analyze new directed graphs with vertex sets contained in $2\mathbf{N}+1$ for which weak connectivity also implies truth of the $3x+1$ Conjecture (Sections 2 and 3). Some conditions under which vertices of these new graphs are weakly connected are given. Certain numbers x satisfying the condition that $T^2(x)=1$ are discussed in Section 4. (A different characterization of some positive integers satisfying $T^k(x)=1$ can be found in [2].) In Section 4, we also prove the facts that cycles and divergent trajectories in our new graphs induce cycles and divergent trajectories in the original Collatz graph.

1. VERTICES WITH A SPECIAL CONNECTIVITY PROPERTY

To identify our vertex set, we need a few preliminaries. For $x \in 2\mathbf{N}+1$, the *total stopping time* of x, denoted $\sigma(x)$, is the least whole number k satisfying $T^k(x)=1$. (If no such k exists, set $\sigma(x)=\infty$.) Define the binary relation $\approx$ on $2\mathbf{N}+1$ as follows: $x \approx y$ if and only if there exists $k \in \mathbf{N}$ with $k \leq \min(\sigma(x), \sigma(y))$ satisfying $T^k(x)=T^k(y)$. Clearly, $\approx$ is an equivalence relation, hence each $x \in 2\mathbf{N}+1$ belongs to an equivalence class C_x. Observe that $\sigma(x)=\sigma(y)<\infty$ implies that $x \approx y$, and furthermore, the set $L_k=\{x \in 2\mathbf{N}+1 \mid \sigma(x)=k\}$ is an equivalence class under $\approx$.

Progress has been made recently in determining the density of positive integers x satisfying $\sigma(x)<\infty$. The strongest known result is in [3], where it is shown that, if $\pi(x)$ counts the number of integers n satisfying $|n|<x$ and $\sigma(n)<\infty$, then, for all sufficiently large x, $\pi(x) \geq x^{.81}$. Important groundwork for this result was provided by Krasikov [5], who used a scheme of difference inequalities to show that $\pi(x) \geq x^{3/7}$. A stochastic approach for analyzing total stopping times is presented in [7], and a thorough summary of currently known total stopping time results can be found in [9].

It also bears mentioning that, throughout the literature, there is a distinct difference between *stopping time* and *total stopping time*. The stopping time of x is defined to be the least positive integer k for which $T^k(x)<x$. The most important stopping time result is given in [8], where it is shown that the density of positive integers with finite stopping time is 1.

We are not ready to state and prove our first result, which can also be found in [1]. The proof reveals properties of certain vertices of the Collatz graph which are useful later; therefore, it is presented here.

Theorem 1: If $x>5$ is the smallest element in C_x, then there exists $n \in \mathbf{P}$ such that $T^n(x)=T^n(2x+1)$.

Proof: Let A_n denote the arithmetic progression $\{2^{n+2}m+2^n-1\}_{m=0}^{\infty}$, and let B_n denote the arithmetic progression $\{2^{n+2}m+2^{n+1}+2^n-1\}_{m=0}^{\infty}$. If we let $S_1=\bigcup_{n\in 2\mathbf{N}+1}(A_n)$, $S_2=\bigcup_{n\in 2\mathbf{P}}(B_n)$, $S_3=\bigcup_{n\in 2\mathbf{P}}(A_n)$, and $S_4=\bigcup_{n\in 2\mathbf{N}+1}(B_n)$, it is easy to verify that $\{S_1,S_2,S_3,S_4\}$ is a partition of $2\mathbf{N}+1$. We now show that $x \in S_3 \cup S_4$ is impossible. If $x \in S_3$, write $x=2^{n+2}m+2^n-1$, where n is even, and if $x \in S_4$ and $n=1$, choose y satisfying $4y+1=x$, else choose y satisfying $2y+1=x$. In all cases, a straightforward computation, taking parity of n into consideration when necessary, shows that $T^n(x)=T^n(y)$. Hence $y \approx x$ with $y<x$, contradicting the fact that x is smallest in its equivalence class. Therefore, $x \notin S_3 \cup S_4$, so $x \in S_1 \cup S_2$. If $x \in S_1$, write $x=2^{n+2}m+2^n-1$ with n odd, and if $x \in S_2$, write $x=2^{n+2}m+2^{n+1}+2^n-1$ with n even. Again applying the Collatz function n times and taking parity of n into account, we obtain $T^n(x)=T^n(2x+1)$. $\square$

Corollary 1 follows easily from Theorem 1.

Corollary 1: If x is the smallest element in L_k, then the vertices x and $2x+1$ of $G_{2\mathbf{N}+1}$ are weakly connected.

2. REDUCING THE VERTEX SET OF THE COLLATZ GRAPH

We now construct a new directed graph whose vertices are the smallest elements of the equivalence classes under $\approx$. The primary tool used is a mapping $\hat{T}$ induced by the Collatz mapping. The construction has the advantage of reducing the set of vertices of the Collatz graph, but the disadvantage of sacrificing some information about $T(x)$.

Let $M = \{x \in 2\mathbf{N}+1 | x \le y \text{ for all } y \in C_x\}$. For $S \subseteq \mathbf{P}$, define $\chi(S)$ to be the smallest element of S. Define $\hat{T}: M \to M$ by $\hat{T}(m) = \chi(C_{T(m)})$. Due to the fact that every vertex of the Collatz graph is weakly connected to some $m \in M$, the following statement is equivalent to the $3x+1$ Conjecture.

$3x+1$ Conjecture (2^{nd} form): The directed graph G_M with vertices M and directed edges from m to $\hat{T}(m)$ is weakly connected.

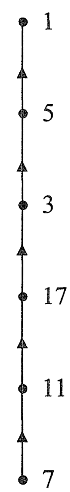

FIGURE 2. The Graph G_M ($\sigma(x) \le 5$)

A portion of G_M is displayed in Figure 2. The graph G_M, in effect, collapses the vertices of $G_{2\mathbf{N}+1}$ whose trajectories enter M, thereby reducing the set of vertices necessary to connect. Despite this reduction in the vertex set, it turns out that weak connectivity can be established for certain pairs of vertices of G_M, as shown in the next three theorems.

Theorem 2: Let $x \in M$ with $x \equiv 5 \pmod 6$, and define $T^{-1}(x)$ to be the smallest y in $2\mathbf{N}+1$ satisfying $T(y) = x$. Then x and $T^{-1}(x)$ are weakly connected vertices of G_M.

Proof: Letting $x = 6t+5$, it follows that $T^{-1}(x) = 4t+3$. We must show that $4t+3$ is a vertex of G_M. If $4t+3$ is not in M, then there exists $w < 4t+3$ with $w \approx 4t+3$, and using the definition of $\approx$, it follows that $T(w) \approx T(4t+3) = x$. Since $x \in M$, we obtain $x \le T(w)$, and this yields

$$6t+5 \le \frac{3w+1}{2^j}, \text{ where } j \ge 1.$$

Substituting the inequality $w < 4t+3$ yields

$$6t+5 < \frac{3(4t+3)+1}{2^j} = \frac{12t+10}{2^j} = 6t+5.$$

a contradiction. Hence, $x = 4t+3$ is a vertex of G_M. Finally, since

$$\hat{T}(T^{-1}(x)) = \chi(C_{T(T^{-1}(x))}) = \chi(C_x) = x,$$

we have x and $T^{-1}(x)$ weakly connected. □

Remark: If $x \in M$ with $x \equiv 1 \pmod 6$, then $T^{-1}(x)$ is not necessarily in M. For example, $379 = \chi(L_{19})$ and $283 = \chi(L_{20})$, but $T^{-1}(379) = 505$.

Theorem 3: Let $x \in M$ with $x \equiv 1 \pmod 8$, and let $y = \chi(C_{T(x)})$. Assume y is not a multiple of 3. Then $T(x) = y$, and x and y are weakly connected vertices of G_M.

Proof: Let $x = 8k+1$. If $T(x) \ne y$, then $y = \chi(C_{T(x)})$ implies that $y < T(x)$ and $y \approx T(x)$. Also, by hypothesis, y must be of the form $6t+1$ or $6t+5$. If $y = 6t+1$, then $y < T(x)$ gives $6t+1 < 6k+1$, hence $t < k$. Also, $y \approx T(x)$ implies $T^{-1}(y) \approx x$, where $T^{-1}(y)$ is the smallest inverse image of y under T. Therefore, $8t+1 \approx x$, and since $x \in M$, we must have $x \le 8t+1$. This yields $8k+1 \le 8t+1$, hence $k \le t$, a contradiction. If y is of the form $6t+5$, then $y < T(x)$ yields $6t+5 < 6k+1$, hence $t < k$. The condition $T^{-1}(y) \approx x$ yields $4t+3 \approx x$, hence $8k+1 \le 4t+3$. Substituting the inequality $t < k$ yields $8t+1 \le 4t+3$, again a contradiction. Therefore, $T(x) = y$ must hold. Since $\hat{T}(x) = \chi(C_{T(x)}) = y$, it follows that x and y are weakly connected vertices of G_M. □

Theorem 4: Let $x \in M$ with $x \equiv 25 \pmod{64}$, and let $y = \chi(C_{T(x)})$. Then $y = [3(x-1)]/8$, and the vertices x and $[3(x-1)]/8$ are weakly connected.

Proof: Let $x = 64k+25$. Simple computations show that $T(x) = 48k+19$ and that

$$T^2\left(\frac{T(x)-1}{2}\right) = T^2(T(x)).$$

Therefore, $[T(x)-1]/2 \approx T(x)$, hence $T(x) \ne y$. Also $x \equiv 1 \pmod 8$, so we can apply Theorem 3 to see that y must be a multiple of 3. Let $y = 3t$. By Theorem 1, we have $y \approx 2y+1 = 6t+1$. Since $T(8t+1) = 6t+1$, we have $T(8t+1) \approx y$, and using the fact that $y \approx T(x)$ along with the transitivity of $\approx$, we obtain $T(8t+1) \approx T(x)$. Using the definition of $\approx$, it follows that $8t+1 \approx x$, and since $x \in M$, we have $x \le 8t+1$. Furthermore, $y \approx T(x) \approx [T(x)-1]/2 = 24k+9$; thus, by the minimality of y, we see that $3t \le 24k+9$. From this inequality, we get $\frac{8}{3}(3t)+1 \le \frac{8}{3}(24k+9)+1$ which yields $8t+1 \le x$. Therefore, $x = 8t+1$, hence $y = [3(x-1)]/8$. It follows that $\hat{T}(x) = [3(x-1)]/8$ and that x and $[3(x-1)]/8$ are weakly connected. □

Observe that, if $x > 5$ is a vertex of G_M, then, by the proof of Theorem 1, it must be true that $x \in S_1 \cup S_2$. We can actually restrict the vertex set of G_M slightly further, according to the next theorem.

Theorem 5: If x is in the arithmetic progression $\{32m+17\}_{m=1}^{\infty}$, then x is not a vertex of G_M.

Proof: We will assume $x \in M$ and find a contradiction. Let $x = 32k+17$ with $k \geq 1$, and let $y = \chi(C_{T(x)})$. Since $T(x) = 24k+13$ and $T(24k+13) = T(6k+3)$, it follows that $T(x) \neq y$. Also, $x \equiv 1 \pmod 8$ and $x \in M$, so we can apply Theorem 3 to see that y must be a multiple of 3. Now, $y \approx 2y+1$ by Theorem 1 and $y \approx T(x)$ by the definition of y, hence $T(x) \approx 2y+1$. Since y is a multiple of 3, $T^{-1}(2y+1) = \frac{8}{3}y+1$, we have $x \approx \frac{8}{3}y+1$. This yields $x \leq \frac{8}{3}y+1$, and since $y \approx T(x) \approx 6k+3$, we have $y \leq 6k+3$. Combining these inequalities yields $x \leq 16k+9$, a contradiction. Therefore, $x \notin M$, and x is not a vertex of G_M. □

Remarks: A further systematic reduction of the vertex set beyond that of Theorem 5 would be of interest, as would further development of the weak connectivity results given in Theorems 1-4. It would also be interesting to state conditions which, when combined with the theorems in this section, would be sufficient to guarantee weak connectivity of G_M; in fact, $17 \in M$.

3. A DIFFERENT REDUCTION OF THE VERTEX SET

We now reduce the vertex set of the Collatz graph to a set properly containing M, and use this set to construct a new directed graph for which weak connectedness is equivalent to truth of the $3x+1$ Conjecture. First, we need some preliminaries to help describe our vertex set. If we let $f(x) = 4x+1$ and $g(x) = 2x+1$ and let M b defined as in Section 2, we have the following lemma.

Lemma 1: Let $x \in M$, $n \in \mathbf{N}$, and $\delta \in \{0, 1\}$. Then $f^n(x) \in C_x$ when $x > 1$, and $f^n g^\delta(x) \in C_x$ when $x > 5$.

Proof: When $x > 1$, a quick computation shows that $T(f(x)) = T(x)$, thus $T(f^n(x)) = T(x)$, and hence $f^n(x) \in C_x$ for $n \in \mathbf{N}$. When $x > 5$, we can apply Theorem 1 to obtain $g^\delta(x) \in C_x$, so $f^n g^\delta(x) \in C_x$. □

For $x > 5$, let $G_x = \{f^n g^\delta(x) | n \in \mathbf{N}, \delta \in \{0, 1\}\}$; for $x = 3$ and $x = 5$, let $G_x = \{f^n(x) | n \in N\}$. Note that G_x consists of a collection of vertices for which weak connectedness to x in the Collatz graph has been established. For convenience, set $G_1 = \{1\}$. Lemma 1 implies that $G_x \subseteq C_x$; therefore, it makes sense to study the vertices of C_x apart from G_x. We do so using the following inductive definition.

Definition: For $j \in N$, the j^{th} *exceptional number* in C_x is the smallest positive integer x_j satisfying $x_j \in C_x - \bigcup_{i=0}^{j} G_{x_{i-1}}$, where $G_{x_{-1}} = \emptyset$.

To clarify the previous definition, consider the example

$$G_{25} = \{25, 101, 405, \ldots\} \cup \{51, 205, 821, \ldots\}.$$

Since $\sigma(25) = 7$, it follows that $C_{25} = \{x \in \mathbf{P} | \sigma(x) = 7\}$. Direct computation shows that 217 is the smallest positive integer in $C_{25} - G_{25}$, hence 217 is the first exceptional number in C_7. Repeating the process, we compute

$$G_{217} = \{217, 869, 3477, \ldots\} \cup \{435, 1741, 6965, \ldots\},$$

and hence can verify that 433 is the smallest positive integer in $C_{25}-\{G_{25}\cup G_{217}\}$. Therefore, 433 is the second exceptional number in C_{25}. A table of exceptional numbers satisfying $\sigma(x)\leq 10$ and $j\leq 4$ is provided below.

TABLE 1. Exceptional Numbers ($\sigma(x)\leq 10, j\leq 4$)

$\sigma(x)$	x
0	1
1	5
2	3, 113, 7281, 466033, 29826161
3	17, 75, 1137, 2417, 4849
4	11, 201, 369, 401, 753
5	7, 241, 267, 497, 537
6	9, 81, 321, 331, 625
7	25, 49, 217, 433, 441
8	33, 65, 273, 289, 529
9	43, 89, 177, 385, 423
10	57, 59, 465, 473, 507

Let E denote the set of all exceptional numbers $2\mathbf{N}+1$. Using the methods of the proof of Theorem 4 of [1], it can be shown that, for $\sigma(x)>1$, $C_x-\bigcup_{i=0}^{j}G_{x_{i-1}}=\emptyset$ for all $j\in N$, hence, in this case, each $j\geq 0$ gives a distinct element of E. Furthermore, the following lemma gives a complete description of the set E.

Lemma 2: $S_1\cup S_2\cup\{3,5\}=E$.

Proof: The fact that $E\subseteq S_1\cup S_2\cup\{3,5\}$ is an immediate consequence of Lemmas 6 and 7 of [1] and Theorem 1. Therefore, we will show that $S_1\cup S_2\cup\{3,5\}\subseteq E$. If $x\in S_1\cup S_2\cup\{3,5\}$ with $x\leq 11$, numerical computation shows that $x\in E$, thus we will show that, if $x>11$, then $x\in S_1\cup S_2$ gives $x\in E$. If $x\notin E$, then $x=f^n g^{\delta}(y)$ for some $y\in E$. If $n\geq 1$, then $x\in 8m+5$, which is impossible, hence $n=0$. Therefore, $x=g^{\delta}(y)$ for some $y\in E$. If $\delta=0$, we obtain $x=y$, which contradicts the fact that $x\notin E$. Hence $\delta=1$; thus $x=2y+1$. Since $x\in S_1\cup S_2$, this yields $y\in S_3\cup S_4$ with $y>5$, which contradicts the fact that $y\in E\subseteq S_1\cup S_2\cup\{3,5\}$. Hence, our assumption that $x\notin E$ must be false. □

Remark: Using the equivalence classes defined in Section 1, the proofs of Theorems 3 and 4 of [1] can immediately be generalized to the case where $\sigma(x)\leq\infty$.

The primary purpose of Lemma 2 is to establish weak connectivity between certain vertices of a new directed graph (see Theorem 7). However, it is interesting to note that we can use Lemma 2 and the proof of Theorem 1 to immediately establish the following theorem, which is also given in [1].

Theorem 6: Let $x\in E$ with $x>5$. Then there exists $k\in\mathbf{N}$ such that $T^k(x)=T^k(2x+1)$.

We now use the sets G_x to construct a new partition of the positive odd integers. This partition will enable us to define a new directed graph.

Lemma 3: Let $\mathcal{P}=\{G_x\,|\,x\in E\}$. Then $\mathcal{P}$ is a partition of $2\mathbf{N}+1$.

Proof: Since

$$\bigcup_{x\in M} C_x = 2\mathbf{N}+1 \quad \text{and} \quad C_x = \bigcup_{x\in C_x\cap E} G_x,$$

it follows immediately that

$$\bigcup_{x\in E} G_x = 2\mathbf{N}+1.$$

It remains to show that, if x and y are in E, then $G_x \cap G_y = \emptyset$ when $G_x \neq G_y$. We will prove this by contradiction. If $z \in G_x \cap G_y$, then

$$z = f^{n_1} g^{\delta_1}(x) = f^{n_2} g^{\delta_2}(y),$$

where $n_1, n_2 \in N$, $\delta_1, \delta_2 \in \{0, 1\}$, $f(x) = 4x+1$, and $g(x) = 2x+1$. Without loss of generality, we can consider three cases: $\delta_1 = \delta_2 = 0$; $\delta_1 = 0$ and $\delta_2 = 1$; and $\delta_1 = \delta_2 = 1$.

In the first case, we have

$$f^{n_1}(x) = f^{n_2}(y),$$

and since we can assume $n_1 \leq n_2$ without loss of generality, we obtain $x = f^{n_2-n_1}(y)$. Assume $x \neq 5$, as the theorem follows trivially in this case. If $n_2 - n_1 = 0$, then $G_x = G_y$ is a contradiction, and if $n_2 - n_1 > 0$, we have x of the form $8m+5$ with $m > 1$ and $x \in E$, which contradicts Lemma 2. Hence, in any event, $\delta_1 = \delta_2 = 0$ is impossible.

In the second case, we have

$$f^{n_1}(x) = f^{n_2}(2y+1).$$

If $n_1 = n_2$, then $x = 2y+1$, hence $x \in G_y$. Since Theorem 6 implies that $x \in C_y$, we have contradicted the fact that $x \in E$. If $n_1 < n_2$, then $x = f^{n_2-n_1}(2y=1)$; therefore, x is of the form $8m+5$ with $m \geq 1$. Since $x \in E$, we have again contradicted Lemma 2. If $n_2 < n_1$, we obtain $2y+1 = f^{n_1-n_2}(x)$, which implies that $2y+1$ is of the form $8m+5$, contradicting the fact that y is odd. Hence, in any event, $\delta_1 = 0$ and $\delta_2 = 1$ is impossible.

Finally, the third case gives

$$f^{n_1}(2x+1) = f^{n_2}(2y+1).$$

Again, without loss of generality, assume $n_1 \leq n_2$. If $n_1 = n_2$, then $x = y$, hence $G_x = G_y$ is a contradiction. If $n_1 < n_2$, then $2x+1 = f^{n_2-n_1}(2y+1)$ implies that $2x+1$ is of the form $8m+5$. This forces x to be even, again a contradiction. Thus, $\delta_1 = \delta_2 = 1$ is also impossible, and hence our assumption that $G_x \cap G_y = \emptyset$ must be false. □

Using the partition $\mathcal{P}$, we define the equivalence relation $\sim$ as follows: $x \sim y$ if and only if x and y are in G_x for some $z \in E$. Denote by E_x the equivalence class under $\sim$ which contains x. For $e \in E$, define $\overline{T}: E \to E$ by $\overline{T}(e) = \chi(E_{T(e)})$. We now obtain another formulation of the $3x+1$ Conjecture.

$3x+1$ Conjecture (3rd form): The directed graph G_E with vertices E and directed edges from e to $\overline{T}(e)$ is weakly connected.

A portion of the directed graph G_E is displayed in Figure 3. The graph G_E collapses some vertices of $G_{2\mathbf{N}+1}$ whose trajectories enter E, while at the same time retaining enough vertices to permit establishing of substantial weak connectivity.

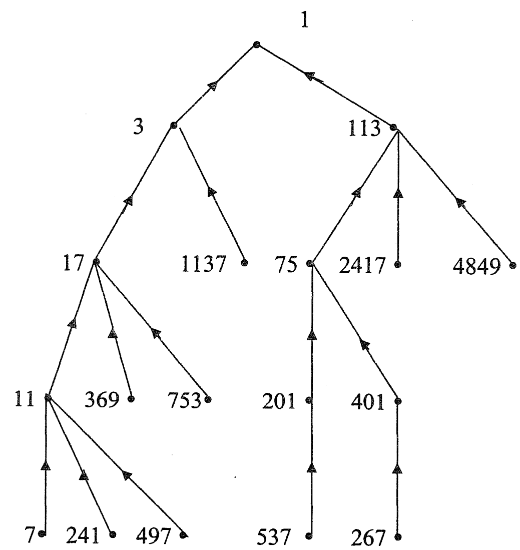

FIGURE 3. The Graph G_E $(\sigma(x) \le 5, j \le 4, x < 5000)$

Now let S_1 and S_2 be defined as in the proof of Theorem 1, and let $S = S_1 \cup S_2 - 1$. For x not a multiple of 3, let $T^{-1}(x)$ be the smallest y in $2\mathbf{N}+1$ satisfying $T(y) = x$, and define

$$T^{-1}(S) = \{T^{-1}(s) \mid s \in S - 3\mathbf{P}\}.$$

We then have the following results.

Lemma 4: $T^{-1}(S) \subseteq S$.

Proof: If $x \in S_1$, let $x = 2^{n+2}m + 2^n - 1$ with $n \in 2\mathbf{N}+1$. By considering congruences of m modulo 3, we see that x can be expressed in one of the following three forms:

$$x = 3 \cdot 2^{n+2}k + 2^n - 1;$$
$$x = 3 \cdot 2^{n+2}k + 2^{n+2} + 2^n - 1;$$
$$x = 3 \cdot 2^{n+2}k + 2^{n+3} + 2^n - 1.$$

If x is of the first form, then n odd yields $x \equiv 1 \pmod 6$. This gives

$$T^{-1}(x) = \frac{4x-1}{3}.$$

Hence $T^{-1}(x) \equiv 1 \pmod 8$, and therefore $T^{-1}(x) \in S$.

If x is of the second form, then n odd yields $x \equiv 0 \pmod 3$, hence $T^{-1}(x)$ does not exist.

If x is of the third form, then n odd yields $x \equiv 5 \pmod 6$. This gives

$$T^{-1}(x) = \frac{2x-1}{3} = 2^{n+3}k + 2^{n+2} + 2^{n+1} - 1,$$

hence $T^{-1}(x) \in S_2$. If $x \in S_2$, let $x = 2^{n+2}m + 2^{n+1} + 2^n - 1$ with $n \in 2\mathbf{P}$. Again considering congruences of m modulo 3, x can be expressed in one of the following three forms:

$$x = 3\cdot 2^{n+2}k + 2^{n+1} + 2^n - 1;$$
$$x = 3\cdot 2^{n+2}k + 2^{n+2} + 2^{n+1} + 2^n - 1;$$
$$x = 3\cdot 2^{n+2}k + 2^{n+3} + 2^{n+1} + 2^n - 1.$$

If x is of the first form, then $x \equiv 5 \pmod 6$. Therefore,

$$T^{-1}(x) = \frac{2x-1}{3} = 2^{n+3}k + 2^{n+1} - 1,$$

and hence $T^{-1}(x) \in S_1$.

If x is of the second form, then $x \equiv 0 \pmod 6$, and thus $T^{-1}(x)$ does not exist.

If x is of the third form, then $x \equiv 1 \pmod 6$ and, as before, $T^{-1}(x) \equiv 1 \pmod 8$, and thus is in S_1. Hence, in all cases, $T^{-1}(x) \in S$. □

Theorem 7: Let x be an element of E. Then the vertices x and $T^{-1}(x)$ of G_E are weakly connected.

Proof: We first show that $x \in E$ yields $T^{-1}(x) \in E$. We can assume without loss of generality that $x > 5$. Letting $x \in E$ and applying Lemma 2, we see that $x \in S_1 \cup S_2$. Applying Lemma 3 gives $T^{-1}(x) \in S_1 \cup S_2 - 1$, and again applying Lemma 2, we obtain $T^{-1}(x) \in E$. Finally, we get

$$\overline{T}(T^{-1}(x)) = \chi(E_{T(T^{-1}(x))}) = \chi(E_x) = x,$$

hence x and $T^{-1}(x)$ are weakly connected. □

4. TOTAL STOPPING TIMES OF CERTAIN EXCEPTIONAL NUMBERS AND CYCLES UNDER INDUCED MAPS

One possible approach to establishing weak connectedness of G_E is to characterize all $x \in E$ with a given finite total stopping time, and to apply T^{-1} repeatedly to those vertices. By Theorem 7, these inverse images would also be vertices of G_E, and perhaps would substantially "fill up" the set of all vertices of G_E. All $x \in E$ satisfying $\sigma(x) \le 2$ are described in Lemma 5 and Theorem 8.

Lemma 5: Let $x \in E$, and let $f(x) = 4x+1$. Then $\sigma(x) = 1$ if and only if $x = 5$.

Proof: It is well known that, for any $x \in 2\mathrm{N}+1$, $\sigma(x) = 1$ if and only if $x = \frac{1}{3}(4^{n+1}-1)$ for some $n \in P$ (see [4]). Since

$$\frac{1}{3}(4^{n+1}-1) = \sum_{i=0}^{n} 4^i = f^{n-1}(5)$$

and since $x \in E$, we must have $x = 5$. □

Lemma 6: Let

$$a_{m,n} = \frac{1}{3}\left(\sum_{i=0}^{3m} 4^{i+n} - 1\right) \quad \text{and} \quad b_{m,n} = \frac{1}{3}\left(2\sum_{i=0}^{3m-2} 4^{i+n-1} - 1\right).$$

Then $L_2 = \{a_{m,n} \mid m, n \in \mathbb{P}\} \cup \{b_{m,n} \mid m, n \in \mathbb{P}\}$.

Proof: The fact that $a_{m,n}$ and $b_{m,n}$ are in L_2 is easily verified by computation of $T^2(a_{m,n})$ and $T^2(b_{m,n})$. Thus, we need to show that $L_2 \subseteq \{a_{m,n} \mid m, n \in P\} \cup \{b_{m,n} \mid m, n \in P\}$. If $x \in L_2$, then $T(T(x))=1$, hence $T(x)=\frac{1}{3}(4^{k+1}-1)$ for some $k\in\mathbf{P}$. Since $T(x)=(3x+1)/2^j$ for some $j\in\mathbf{P}$, we obtain

$$\frac{3x+1}{2^j}=\frac{1}{3}(4^{k+1}-1)=\sum_{i=0}^{k}4^i,$$

hence $2^j\sum_{i=0}^{k}4^i\equiv 1 \pmod 3$. This yields $2^j(k+1)\equiv 1 \pmod 3$. Thus, if j is even, we have $k\equiv 0 \pmod 3$, and if j is odd, we have $k\equiv 1 \pmod 3$. In the first case, setting $j=2n$ and $k=3m$ gives $x=a_{m,n}$; in the second case, setting $j=2n-1$ and $k=3m-2$ gives $x=b_{m,n}$. □

If we let $x\in E$ with $\sigma(x)=2$, direct computation yields $E_x=\{3,113,7281,466033,\dots\}$. It is interesting to observe that the function $h(x)=64x+49$ generates all of E_x except for $x=3$, hence motivating our final lemma as well as Theorem 8.

Lemma 7: Let $x\in 2\mathbf{N}+1$, $g(x)=2x+1$, and $h(x)=64x+49$. Then $T^2(g(h^k(x)))=T^2(h^k(x))$ for all $k\in\mathbf{P}$.

Proof: We proceed by induction on k. When $k=1$, some simple computation shows that

$$T^2(g(h^k(x)))=T^2(h^k(x)).$$

Assuming the lemma is true for $k=j$, we show that the lemma holds for $k=j+1$. Since

$$T^2(g(h^{j+1}(x)))=T^2(g(h^j(h(x))))$$

and the induction hypothesis gives

$$T^2(g(h^j(h(x))))=T^2(h^j(h(x))),$$

we obtain

$$T^2(g(h^{j+1}(x)))=T^2(h^{j+1}(x)).$$

Hence, the case where $k=j+1$ holds true. □

Theorem 8: Let $x\in E$ with $x>5$ and let $h(x)=64x+49$. Then $\sigma(x)=2$ if and only if $x=h^n(1)$ for some $n\in\mathbf{P}$.

Proof: Assume $\sigma(x)=2$ and let $a_{m,n}$ and $b_{m,n}$ be defined as in Lemma 6. Using this lemma, we see that $x=a_{m,n}$ or $x=b_{m,n}$ for some $m,n\in\mathbf{P}$. If we let $f(x)=4x+1$, the relationships $a_{m,n+1}=f(a_{m,n})$ and $b_{m,n+1}=f(b_{m,n})$ are easily verified. Hence, using the fact that $x\in E$ in conjunction with Lemma 2, we see that $x=a_{m,1}$ or $x=b_{m,1}$. Now direct computation shows that $h(a_{m,1})=a_{m+1,1}$ for all $m\in\mathbf{P}$, so $a_{m+1,1}\equiv 1 \pmod 8$. Using Lemma 2 and verifying the case where $m=1$ independently, we obtain $a_{m,1}\in E$ for all $m\in\mathbf{P}$. Now let $g(x)=2x+1$. Since $T^2(a_{m,1})=T^2(b_{m+1,1})$ and $g(a_{m,1})=b_{m+1,1}$, we see that $b_{m+1,1}\in E$ only when $m=0$, hence when $b_{m+1,1}=3$. Since $x>5$, we conclude that $x=h^n(a_{m,1})$ for some $m\in\mathbf{P}$ and $n\in\mathbf{N}$. Using $h(a_{m,1})=a_{m+1,1}$ and the fact that $a_{1,1}=h(1)$, the result $x=h^n(1)$ for some $n\in\mathbf{P}$ follows.

We now show by induction that $\sigma(x)=2$ is a necessary condition for $x=h^n(1)$. For $n=1$, $\sigma(x)=2$ is easily verified. We assume that, for $x=h^n(1)\,x=h^k(1)$, we have $\sigma(x)=2$, and will show that $x=h^{k+1}(1)$ yields $\sigma(x)=2$. Direct computation shows that $T^2(h(x))=T^2(g(x))$, thus $T^2(h^{k+1}(x))=T^2(h(h^k(x)))=T^2(g(h^k(x)))$. Using Lemma 7, we obtain $T^2(h^{k+1}(x))=T^2(h^k(x))$. Finally, setting $x=1$ and invoking the induction hypothesis, we get $T^2(h^{k+1}(1))=1$; hence, for $x=h^{k+1}(1)$, we have $\sigma(x)=2$. □

Remark: A similar characterization for $x \in E$ satisfying $\sigma(x) = k$ when $k \geq 3$ would be of interest. In the case where $k = 3$, numerical computation suggests that $x = (h_1)^n(17)$ or $x = (h_2)^n(75)$, where $h_1(x) = 64x + 49$ and $h_2(x) = 32x + 17$. Furthermore, if we let $E_k = \{x \in E \mid \sigma(x) = k\}$, it can be conjectured that $E_k = \bigcup_{i=1}^{t_k} \{h_i^n(x_i) \mid n \in \mathbf{N}\}$ for some $t_k \in \mathbf{P}$ and $h_i = a_i x + b_i$. The behavior of t_k / k as $k \to \infty$ also merits further study.

We now demonstrate that a nontrivial cycle under T will induce a nontrivial cycle under the maps $\hat{T}$ and $\overline{T}$ (Theorems 9 and 10). Thus, to prove that nontrivial cycles do not exist under T, it is sufficient to prove that nontrivial cycles do not exist under either $\hat{T}$ or $\overline{T}$. Let $\approx$ and $\sim$ be the equivalence relations given in Sections 1 and 3, and let χ be defined as in Section 2. If we define $\hat{T} : 2\mathbf{N}+1 \to 2\mathbf{N}+$ by $\hat{T}(x) = \chi(C_{T(x)})$, we have the following lemmas.

Lemma 8: $\hat{T}^2(x) = \hat{T}(T(x))$ for all $x \in 2\mathbf{P}+1$.

Proof: Letting $y = \hat{T}(x)$ and $z = T(x)$, we have $y \approx z$, so $T(y) \approx T(z)$. Therefore, $C_{T(y)} = C_{T(z)}$, and thus $\chi(C_{T(y)}) = \chi(C_{T(z)})$. This gives $\hat{T}(y) = \hat{T}(z)$, and substituting for y and z gives the result. □

Lemma 9: $\hat{T}^{k+1}(x) = \hat{T}(T^k(x))$ for all $k \in \mathbf{P}$ and for all $x \in 2\mathbf{P}+1$ satisfying $\sigma(x) \geq k$.

Proof: We proceed by induction on k. The case in which $k = 1$ follows from Lemma 8. Assume that the lemma holds when $k = j$. Since

$$\hat{T}^{j+2}(x) = \hat{T}(\hat{T}^{j+1}(x)) = \hat{T}(\hat{T}(T^j(x))) = \hat{T}^2(T^j(x))$$

and since Lemma 8 gives

$$\hat{T}^2(T^j(x)) = \hat{T}(T(T^j(x))) = \hat{T}(T^{j+1}(x)),$$

the case when $k = j+1$ holds true. □

Theorem 9: If $T^k(x) = x$ for some $k \in \mathbf{P}$ and $x \in 2\mathbf{N}+1$, then there exists $y \in M$ satisfying $\hat{T}^k(y) = y$.

Proof: By Lemma 9, $\hat{T}^{k+1}(x) = \hat{T}(T^k(x))$, hence invoking the hypothesis of the theorem gives $\hat{T}^{k+1}(x) = \hat{T}(x)$, and setting $y = \hat{T}(x)$ gives the result. □

Lemma 10: Let $x, y \in E$ with $x \sim y$. Then $T(x) \sim T(y)$.

Proof: If $x \sim y$, then x and y are in G_z for some $z \in E$. Hence we can write $x = f^{n_1} g^{\delta_1}(z)$ and $y = f^{n_2} g^{\delta_2}(z)$, where $n_1, n_2 \in \mathbf{N}$, $\delta_1, \delta_2 \in \{0, 1\}$, $f(x) = 4x+1$, and $g(x) = 2x+1$. Applying Lemma 1, we see that $T(x) = T(g^{\delta_1}(x))$ and $T(y) = T(g^{\delta_2}(x))$. If $\delta_1 = \delta_2$, the result follows, so assume, without loss of generality, that $\delta_1 = 0$ and $\delta_2 = 1$. This yields $T(x) = T(z)$ and $T(y) = T(2z+1)$. If $z = 5$, the conclusion of the lemma is easily verified, so assume $z \neq 5$. Since $z \in E$, we can combine Lemma 2 with the proof of Theorem 1 to see that $T(z) = (3z+1)/2^j$ with $j = 1$ or $j = 2$. (The possibility of $j = 4$ is eliminated since $z \neq 5$.) Noting that $T(2z+1) = 3z+2$, we obtain $2^j T(z) + 1 = T(2z+1)$. When $j = 1$, this yields $g(T(x)) = T(y)$, and when $j = 2$, this yields $f(T(x)) = T(y)$; hence, in either case, $T(x) \sim T(y)$. □

Theorem 10: If $T^k(x) = x$ for some $k \in \mathbf{P}$, then there exists $e \in E$ satisfying $\overline{T}^k(e) = e$.

Proof: Using Lemma 10, the statements and proofs of Lemmas 8 and 9 hold with $\hat{T}$ replaced by $\overline{T}$, C_x replaced by E_x. and $\approx$ replaced by $\sim$. Hence, the result follows from a proof analogous to that of Theorem 9, with $\hat{T}$ replaced by $\overline{T}$ and M replaced by E. □

Finally, we will demonstrate that divergent trajectories under $\hat{T}$ and $\overline{T}$ will induce divergent trajectories under T.

Theorem 11: If $\{\hat{T}^k(x)\}_{k=1}^{\infty}$ is divergent, then $\{T^k(x)\}_{k=1}^{\infty}$ is divergent.

Proof: By Lemma 9, we obtain $\hat{T}^k(x) = \hat{T}(T^{k-1}(x))$, and by the definition of $\hat{T}$, we have $\hat{T}(T^{k-1}(x)) = \chi(C_{T^k(x)})$. Thus, $\hat{T}^k(x) = \chi(C_{T^k(x)})$, and hence $\hat{T}^k(x) \le T^k(x)$, from which the theorem immediately follows. □

Theorem 12: If $\{\overline{T}^k(x)\}_{k=1}^{\infty}$ is divergent, then $\{T^k(x)\}_{k=1}^{\infty}$ is divergent.

Proof: Since Lemma 9 holds with $\hat{T}$ replaced by $\overline{T}$, Theorem 12 follows from a proof analogous to that of Theorem 11, with $\hat{T}$ replaced by $\overline{T}$. □

Remarks: The results in this paper are primarily geared toward a constructive proof of the $3x+1$ Conjecture by establishing weak connectivity of G_M or G_E. It is interesting to note that, if $x \equiv 1$ (mod 32) and $f(x) = 8x+9$, then x and $f(x)$ are weakly connected in G_E. Furthermore, if x is in E, $x \equiv 3$ (mod 4), and $g(x) = 32x+17$, then x and $g(x)$ are weakly connected in G_E. Finally, if $x \equiv 1$ (mod 8) and $h(x) = 64x+49$, then x and $h(x)$ are weakly connected vertices of G_E. These results, coupled with Theorems 6 and 7, may be sufficient to establish weak connectivity of G_E. This appears to be a promising direction for future research.

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BOOK REVIEW

Fibonacci and Lucas Numbers with Applications, by Thomas Koshy (New York: Wiley-Interscience, 2001; ISBN: 0-47-39969-8)

This is a delightful book which should prove of great value not only to the professional mathematician but also to a great variety of other professionals like architects, biologists, neurophysiologists, physicists, and stock market analysts, to name but a few. The book is aimed at a broad student audience as well, and it contains a large selection of proposed problems which should make the book a valuable instrument for teaching a first course on number theory. Finally, the book should be of interest to Fibonacci enthusiasts and laypersons alike. This book has a very broad scope indeed!

Koshy begins by offering a lively and well-documented historical perspective on Leonardo Fibonacci and on his mathematical works; he also sketches the contributions made by Édouard Lucas. Next, the author takes the reader through some of the interesting occurrences of Fibonacci numbers in the biological, the chemical, the medical, the physical and the money market worlds. After having set this motivational stage in the first four chapters of the book, Koshy now adeptly moves on, in chapters 5 through 15, to some of the more elementary properties of Fibonacci and Lucas numbers: techniques for generating simple identities are presented and linear recurrence relations are discussed; links between the Fibonacci numbers and Pascal's triangle are established and explored, etc. Finally, the author proceeds to a presentation of more advanced subjects which involve Fibonacci and Lucas numbers, such as divisibility properties, generating functions, continued fractions, periodicity, weighted sums, matrices and the Q-matrix, Fibonacci and Lucas polynomials, Jacobsthal polynomials, Morgan-Voyce polynomials, Tribonacci polynomials, etc. The coverage is truly extensive.

The book is well written, well researched, and well organized. One should not overlook these features, particularly the last one, considering the astronomical amount of research and pedagogical literature that exists on Fibonacci numbers! The style is lively and precise. Difficult underlying concepts such as graphs, trees, etc. are all explained with great ease and confidence. I particularly enjoy the numerous theorems, identities, and results that are quoted throughout the book! I am also delighted by Koshy's tendency to provide anecdotal and biographical background on the authors he quotes. As a result of that tendency, the reader can catch a glimpse, albeit too brief at times, of the human face that lies behind this particular theorem or that particular technique. The entire book is liberally sprinkled with historical anecdotes and footnotes which show that the author has thoroughly researched his subject and which add color and vitality to the topic at hand.

I would venture to say that Koshy's book is the most comprehensive collection of results, theorems, and references regarding Fibonacci numbers and their applications to date. The timing and scope of the book make it a rather fitting tribute to the enduring impact of Leonardo Fibonacci's *Liber Abaci* since it was originally published in 1202, 800 years ago next year! I recommend Koshy's book without reservation to professional mathematicians who teach a course on number theory, to professional scientists and engineers, to students, and to the general amateurs and enthusiasts alike.

Reviewed by Napoleon Gauthier

AN ALTERNATE PROOF OF A THEOREM OF J. EWELL

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Let $t_r(n)$ denote the number of representations of n as a sum of r triangular numbers. In [1], J. Ewell derived a sextuple product identity, one of whose consequences is

Theorem 1: For each integer $n \geq 0$,

$$t_4(n) = \sigma(2n+1)$$

where σ denotes the arithmetical sum-of-divisors function.

In this note we present an alternate proof of Theorem 1.

Proof: Clearly, n is a sum of four triangular numbers if and only if $8n+4$ is a sum of the squares of four odd positive integers. Let $r_4(n)$ denote the number of representations of n as a sum of four squares, while $s_4(n)$ denotes the number of representations of n as the sum of four odd squares. An elementary argument shows that, if $8n+4$ is a sum of four squares, then these squares must all have the same parity. It is easily seen that

$$8n+4 = \sum_{i=1}^{4}(2b_i)^2$$

if and only if

$$2n+1 = \sum_{i=1}^{4} b_i^2.$$

Therefore, we have

$$\begin{aligned} s_4(8n+4) &= r_4(8n+4) - r_4(2n+1) = 8\left(\sum\{d : d \mid (8n+4), 4 \nmid d\} - \sum\{d : d \mid (2n+1)\}\right) \\ &= 8(\sigma(4n+2) - \sigma(2n+1)) = 16\sigma(2n+1), \end{aligned}$$

according to a well-known formula of Jacobi ([2], Theorem 386, p. 312). Therefore, the number of representations of $8n+4$ as the sum of the squares of four odd *positive* integers is

$$\frac{1}{16}s_4(8n+4) = \sigma(2n+1)$$

from which the conclusion follows.

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ON THE NUMBER OF PARTITIONS INTO AN EVEN AND ODD NUMBER OF PARTS

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INTRODUCTION

Let $q_i^e(n)$, $q_i^o(n)$ denote, respectively, the number of partitions into n evenly many, oddly many parts, with each part occurring at most i times. Let $\Delta_i(n) = q_i^e(n) - q_i^o(n)$. Let $\omega(j) = j(3j-1)/2$. It is well known that

$$\Delta_1(n) = \begin{cases} (-1)^n & \text{if } n = \omega(\pm j), \\ 0 & \text{otherwise.} \end{cases}$$

Formulas for $\Delta_i(n)$ were obtained by Hickerson [2] in the cases $i = 3$, i even; by Alder & Muwafi [1] in the cases $i = 5, 7$; by Hickerson [3] for i odd. In this note, we present a simpler formula for $\Delta_i(n)$, where i is odd, than that given in [3]. As a consequence, we obtain two apparently new recurrences concerning $q(n)$.

Remark: Note that, if f denotes any partition function, then we define $f(\alpha) = 0$ if α is not a nonnegative integer.

PRELIMINARIES

Definition 1: If $r \geq 2$, let $b_r(n)$ denote the number of r-regular partitions of n, i.e., the number of partitions of n into parts not divisible by r, or equivalently, the number of partitions of n such that each part occurs less than r times.

Let $x \in C$, $|x| < 1$. Then we have

$$\prod_{n\geq 1}(1-x^n) = 1 + \sum_{k\geq 1}(-1)^k(x^{\omega(k)} + x^{\omega(-k)}), \tag{1}$$

$$\sum_{n\geq 0} b_r(n)x^n = \prod_{n\geq 1}\frac{1-x^{rn}}{1-x^n}, \tag{2}$$

$$\sum_{n\geq 0} \Delta_i(n)x^n = \prod_{n\geq 1}\frac{1+(-1)^i x^{(i+1)n}}{1+x^n}, \tag{3}$$

$$\Delta_3(n) = \begin{cases} (-1)^n & \text{if } n = j(j+1)/2, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Theorem 1: If $r \geq 2$, then

$$\Delta_{2r-1}(n) = b_r\left(\frac{n}{2}\right) + \sum_{k\leq 1}(-1^k)\left\{\left(b_r\left(\frac{n-\omega(k)}{2}\right) + b_r\left(\frac{n-\omega(-k)}{2}\right)\right)\right\}.$$

Proof: Invoking (3), (2), and (1), we have

$$\begin{aligned}\sum_{n\le 0}\Delta_{2r-1}(n)x^n &= \prod_{n\ge 1}\frac{1-x^{2rn}}{1+x^n}\\ &= \prod_{n\ge 1}\frac{1-x^{2rn}}{1-x^{2n}}\prod_{n\ge 1}(1-x^n) = \left(\sum_{n\ge 0} b_r\left(\frac{n}{2}\right)x^n\right)\prod_{n\ge 1}(1-x^n)\\ &= \sum_{n\ge 0}\left(b_r\left(\frac{n}{2}\right)+\sum_{k\ge 1}(-1)^k\left\{\left(b_r\left(\frac{n-\omega(k)}{2}\right)+b_r\left(\frac{n-\omega(-k)}{2}\right)\right)\right\}\right)x^n.\end{aligned}$$

The conclusion now follows by matching coefficients of like powers of x.

Theorem 2:

(a) $$q(n)+\sum_{k\ge 1}(-1)^k\left\{q\left(n-\frac{\omega(k)}{2}\right)+q\left(n-\frac{\omega(-k)}{2}\right)\right\}=\begin{cases}1 & \text{if } n=j(j+1)/4,\\ 0 & \text{otherwise.}\end{cases}$$

(b) $$q(n)+\sum_{k\ge 2}(-1)^{k-1}\left\{q\left(n+\frac{1-\omega(k)}{2}\right)+q\left(n+\frac{1-\omega(-k)}{2}\right)\right\}=\begin{cases}1 & \text{if } n=j(j+3)/4,\\ 0 & \text{otherwise.}\end{cases}$$

Proof: Apply Theorem 1 with $r=2$, noting that $b_2(n)=q(n)$. This yields

$$q\left(\frac{n}{2}\right)+\sum_{k\ge 1}(-1)^k\left\{q\left(\frac{n-\omega(k)}{2}\right)+q\left(\frac{n-\omega(-k)}{2}\right)\right\}=\Delta_3(n).$$

If we invoke (4) and replace n by $2n$, we get (a); similarly, if we replace n by $2n+1$, we get (b).

Since it is easily seen that $2|\omega(k)$ iff $k\equiv 0,3 \pmod 4$, we may rewrite Theorem 2 in a fraction-free form as follows.

Theorem 2*:

(a) $$\begin{aligned}&q(n)-q(n-1)+\sum_{i\ge 1}(q(n-(4i-1)(3i-1))+q(n-(n-(4i+1)(3i+1)))\\ &\quad -\sum_{i\ge 1}(q(n-i(12i-1))+q(n-i(12i+1)))=\begin{cases}1 & \text{if } n=j(j+1)/4,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

(b) $$\begin{aligned}&q(n)+\sum_{i\ge 1}q(n-i(12i-5))+q(n-i(12i+5))-\sum_{i\ge 1}(q(n-(4i-3)(3i-1))\\ &\quad +q(n-(4i-1)(3i-2)))=\begin{cases}1 & \text{if } n=j(j+3)/4,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

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ON THE MORGAN-VOYCE POLYNOMIAL GENERALIZATION OF THE FIRST KIND

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In recent years, a number of papers appeared on the subject of generalization of the Morgan-Voyce (M-) polynomials (see, e.g., André-Jeannin [1]-[3] and Horadam [4]-[7]). The richness of results in these works prompted our investigation on this subject. We further generalized the M-polynomials in a particular way and obtained some new relations by means of the line-sequential formalism developed earlier (see, e.g., [8]-[10]). It was also shown that many known results were obtainable from these relations in a simple and systematic manner.

The recurrence relation of the M-polynomials is given by

$$-m_n + (2+x)m_{n+1} = m_{n+2},\ n \in \mathbf{Z}, \tag{1}$$

where m_n denotes the n^{th} term in the line-sequence; and $c=-1$ and $b=2+x$ are the parametric coefficients with x being the polynomial variable. The pair of basis, see (1.3a) and (1.4a) in [9], is given by, respectively,

$$M_{1,0}(-1, 2+x): \ldots, [1, 0], -1, -(2+x), -(3+4x+x^2), \ldots, \tag{2a}$$

$$M_{0,1}(-1, 2+x): \ldots, [0, 1], 2+x, 3+4x+x^2, 4+10x+6x^2+x^3, \ldots, \tag{2b}$$

which spans the two-dimensional M-vector space.

Let the n^{th} element of the line-sequence $M_{i,j}$ be denoted by $m_n[i, j]$, then by the definition of translation operation, (3.1) in [8], we have

$$Tm_n[1, 0] = m_{n+1}[1, 0];\ \ Tm_n[0, 1] = m_{n+1}[0, 1]. \tag{3a}$$

From (2a) and (2b), obviously the following translational relations hold:

$$TM_{1,0} = -M_{0,1},\ \ TM_{0,1} = M_{1,2+x}, \tag{3b}$$

where we have applied the rule of scalar multiplication in [8]. The first relation above also states the translational relation between the two basis line-sequences. In terms of the elements, it takes the form

$$m_{n+1}[1, 0] = -m_n[0, 1], \tag{3c}$$

in agreement with formula (1.2b) in [10]. Also, by the parity relations (1.3a) and (1.3b) in [10], we have, between the elements in the positive and negative branches of each of the two basis line-sequences, the following relations, respectively,

$$m_{-n}[1, 0] = -m_{n+2}[1, 0], \tag{4a}$$

$$m_{-n}[0, 1] = -m_n[0, 1]. \tag{4b}$$

The negative branches in (2a) and (2b) can be obtained by applying these relations, respectively.

Let the generating pair of a line-sequence be $[i, j]$, where $j = i + sx + r$, and $i, s, r \in \mathbf{Z}$, denote a set of parametric constants. This generating pair specifies a corresponding family of line-sequences lying in the M-space. We call this way of generalization adopted by André-Jeannin

[1] the generalization of the first kind, hence the title of this report. Later, André-Jeannin [2] also generalized the recurrence relation (1); thus, from the line-sequential point of view, generalized the M-space itself. We call this latter way of generalization the generalization of the second kind. In this report we shall concern ourselves with the former case only. The latter case will be discussed in a separate report.

Table 1 below gives the line-sequential conversion of those polynomial sequences treated in this report. The parametric coefficients in the Morgan-Voyce line-sequence are implicit in the designation of the letter M and henceforth omitted. There appears in the literature more than one set of conventions, we shall stick to those adopted in this table.

TABLE 1. Line-Sequences and Elements Conversion

Polynomials	Elements	Line-Sequences	References
	$m_n[i, i+sx+r]$	$M_{i,i+sx+r}$	(5a), (5b)
$B_n(x)$	$m_n[0, 1]$	$M_{0,1}$	[11]
$b_n(x)$	$m_n[1, 1]$	$M_{1,1}$	[11]
$P_n^{(r)}(x)$	$m_n[1, 1+x+r]$	$M_{1,1+x+r}$	[1]
$Q_n^{(r)}(x)$	$m_n[2, 2+x+r]$	$M_{2,2+x+r}$	[4]
$R_n^{(r,u)}(x)$	$m_n[u, u+x+r]$	$M_{u,u+x+r}$	[5]
$U_n(y)$	$m_n[0, 1]$	$M_{0,1}$	[11]

The line-sequence $M_{i,j}$ can be decomposed according to the rules of linear addition and scalar multiplication, see [8], as

$$M_{i,i+sx+r} = M_{1,1+sx+r} + (i-1)M_{1,1}. \tag{5a}$$

In terms of the elements, this becomes

$$m_n[i, i+sx+r] = m_n[1, 1+sx+r] + (i-1)m_n[1, 1]. \tag{5b}$$

Putting $i = u$, $s = 1$, and using the conversion in Table 1, we obtain

$$R_n^{(r,u)}(x) = P_n^{(r)}(x) + (u-1)b_n(x). \tag{5c}$$

This is Theorem 1 in [5] and, equivalently, Theorem 1 in [6]. See the Remark below for further explanation.

We may also decompose $M_{i,j}$ in the following manner:

$$M_{i,i+sx+r} = M_{i,2i+sx} + (r-i)M_{0,1}. \tag{6a}$$

Let $i = s = 1$, then we obtain

$$M_{1,1+x+r} = M_{1,2+x} + (r-1)M_{0,1}. \tag{6b}$$

In terms of the elements, applying (3a) and (3b), we find

$$m_n[1, 1+x+r] = m_{n+1}[0, 1] + (r-1)m_n[0, 1]. \tag{6c}$$

Applying conversions in Table 1, we obtain

$$P_n^{(r)}(x) = B_{n+1}(x) + (r-1)B_n(x) \tag{6d}$$

or, equivalently,

$$R_n^{(r,1)}(x) = U_{n+1}\left(\frac{2+x}{2}\right) + (r-1)U_n\left(\frac{2+x}{2}\right), \tag{6e}$$

which is (4.6) in [5].

If we decompose $M_{i,j}$ in the following manner,

$$M_{i,i+sx+r} = M_{1,2+sx} + (i-2+r)M_{0,1} + (i-1)M_{1,0}, \tag{7a}$$

and let $i = u$ and $s = 1$. Then, using (3b) and (3c), we obtain

$$M_{u,u+x+r} = TM_{0,1} + (u-2+r)M_{0,1} - (u-1)T^{-1}M_{0,1},$$

which, in terms of the elements, becomes

$$m_n[u, u+x+r] = m_{n+1}[0,1] + (u-2+r)m_n[0,1] - (u-1)m_{n-1}[0,1]. \tag{7b}$$

Using the conversions in Table 1, we obtain

$$R_n^{(r,u)}(x) = B_{n+1}(x) + (u-2+r)B_n(x) - (u-1)B_{n-1}(x)$$

or, equivalently,

$$R_n^{(r,u)}(x) = U_{n+1}\left(\frac{2+x}{2}\right) + (u-2+r)U_n\left(\frac{2+x}{2}\right) - (u-1)U_{n-1}\left(\frac{2+x}{2}\right). \tag{7c}$$

This is Theorem 2 in [5] (with a minor typographical correction). It is also valid for negative values of the index n (ref. Theorem 2 in [6]). See the Remark below.

We may also decompose $M_{i,j}$ in the following manner:

$$M_{i,j} = iM_{1,0} + jM_{0,1} \tag{8a}$$

$$= iM_{1,0} + iM_{0,1} + sxM_{0,1} + rM_{0,1}. \tag{8b}$$

Following André-Jeannin [1] and Horadam [4], we define

$$\mathbf{m}_n(i,s,r) = \sum_k \mathbf{m}_{n,k}(i,s,r)x^k, \quad k \geq 0. \tag{8c}$$

where the notation has been modified slightly for typographical convenience but is otherwise easily recognizable as compared to the relating symbols used in [1] and [4]. It is known (see [1]) that the coefficients of x in the basis line-sequence $M_{0,1}$ are generated by the combinatorial function $\binom{n+k}{2k+1}$; by the translational relation (3b), the coefficients in the complementary basis line-sequence $M_{1,0}$ are then generated by $-\binom{n+k-1}{2k+1}$. Substituting into (8b), using Pascal's theorem, we obtain the general coefficient formula:

$$\mathbf{m}_{n,k}(i,s,r) = (i-s)\binom{n+k-1}{2k} + s\binom{n+k}{2k} + r\binom{n+k}{2k+1}. \tag{9}$$

Repeated use of Pascal's theorem leads to relations for some special cases, following are some important examples.

Example 1: Let $i=1$ and $s=1$; we obtain formula (9) in [1]:

$$\mathbf{m}_{n,k}(1,1,r)=\binom{n+k}{2k}+r\binom{n+k}{2k+1}. \tag{10}$$

Example 2: Let $i=2$ and $s=1$; we obtain Theorem 1 in [4]:

$$\mathbf{m}_{n,k}(2,1,r)=\binom{n+k-1}{2k}+\binom{n+k}{2k}+r\binom{n+k}{2k+1}. \tag{11}$$

Example 3: Let $i=u$ and $s=1$; we obtain formula (2.12) in [5]:

$$\mathbf{m}_{n,k}(u,1,r)=(u-1)\binom{n+k-1}{2k}+\binom{n+k}{2k}+r\binom{n+k}{2k+1}. \tag{12}$$

Example 4: Applying the "negative whole" formula

$$\binom{-n}{r}=(-1)^r\binom{n+r-1}{r}$$

(which has its origin in the reflection symmetry of the Pascal array) to (9), we obtain the equivalent formula for $-n$:

$$\mathbf{m}_{-n,k}(i,s,r)=(i-s)\binom{n+k}{2k}+s\binom{n+k-1}{2k}-r\binom{n+k}{2k+1}. \tag{13}$$

Putting $i=u$ and $s=1$, we see that it reduces to

$$\mathbf{m}_{-n,k}(u,1,r)=(u-1)\binom{n+k}{2k}+\binom{n+k-1}{2k}-r\binom{n+k}{2k+1}, \tag{14}$$

which is equation (2.9) in [6]. And so forth.

It is easy to verify that

$$\binom{n+k-1}{2k}=2\binom{n+k-2}{2k}-\binom{n+k-3}{2k}+\binom{n+k-3}{2k-2}. \tag{15}$$

Using this identity, we obtain

$$\mathbf{m}_{n,k}(i,s,r)=2\mathbf{m}_{n-1,k}(i,s,r)-\mathbf{m}_{n-2,k}(i,s,r)+\mathbf{m}_{n-1,k-1}(i,s,r). \tag{16}$$

This reduces to (7) in [1] if $i=s=1$; to (2.10) in [4] if $i=2$ and $s=1$; and to (2.10) in [5] if $i=u$ and $s=1$.

Applying the "negative whole" formula to (15), we obtain

$$\binom{-n+k}{2k}=2\binom{-n+k+1}{2k}-\binom{-n+k+2}{2k}+\binom{-n+k}{2k-2}. \tag{17}$$

Using this identity, we obtain

$$\mathbf{m}_{-n,k}(i,s,r)=2\mathbf{m}_{-n-1,k}(i,s,r)-\mathbf{m}_{-n-2,k}(i,s,r)+\mathbf{m}_{-n-1,k-1}(i,s,r), \tag{18}$$

which reduces to (2.7) in [6] if $i=u$ and $s=1$.

Remark: Both identities (9) and (13) and identities (15) and (17) are valid for both positive and negative values of index n, a property intrinsic to the line-sequential formalism. Since (5a) is a line-sequential formulation, this means that equation (5c) is valid irrespective of the positivity or the negativity of the index n. Therefore, equation (5c) is equivalent to both Theorem 1 in [5] and Theorem 1 in [6]. Similarly, equation (7c) is equivalent to Theorem 2 in [5] and also equivalent to Theorem 2 in [6].

There are some special cases that are worth our attention.

Case 1. Let $i=s-r$ in (9). We then have

$$\mathbf{m}_{n,k}(s-r,s,r)=r\binom{n+k-1}{2k+1}+s\binom{n+k}{2k}=ra^{(1)}_{n-2,k}+sa^{(0)}_{n,k}. \tag{19}$$

This translates into the decomposition formula

$$M_{i,j}=(s-r)M_{1,0}+s(1+x)M_{0,1}. \tag{20}$$

The polynomial line-sequence is as follows:

$$\begin{gathered}M_{s-r,\,s(1+x)}(-1,2+x):\ldots,[s-r,s(1+x)],s+r+3sx+sx^2,\\ s+2r+(6s+r)x+5sx^2+sx^3,\ldots.\end{gathered} \tag{21}$$

Case 2. Let $s=r$ in (9). We then have

$$\mathbf{m}_{n,k}(i,r,r)=(i-r)\binom{n+k-1}{2k}+r\binom{n+k+1}{2k+1}=(i-r)a^{(0)}_{n-1,k}+ra^{(1)}_{n,k}. \tag{22}$$

This translates into the decomposition formula

$$M_{i,j}=iM_{1,0}+(i+r(1+x))M_{0,1}. \tag{23}$$

The polynomial line-sequence is as follows:

$$M_{i,(i+r(1+x))}(-1,1+2x):\ldots,[i,i+r(1+x)],i(1+x)+r(2+3x+x^2),\ldots. \tag{24}$$

Case 3 (a special case of Case 2). Let $i=0$ and $s=r$ in (9). Then we have

$$\mathbf{m}_{n,k}(0,r,r)=r\left(\binom{n+k-1}{2k+1}+\binom{n+k}{2k}\right)=r(a^{(1)}_{n-2,k}+a^{(0)}_{n,k}). \tag{25}$$

This translates into the decomposition formula, from (23),

$$M_{i,j}=r(1+x)M_{0,1}. \tag{26}$$

Hence, this reduces to a multiple of the second basis. The polynomial line-sequence is as follows:

$$M_{0,r(1+x)}(-1,1+2x):\ldots,[0,r(1+x)],r(2+3x+x^2),r(3+7x+5x^2+x^3),\ldots. \tag{27}$$

Case 4. Let $i=2r$ and $s=r$ in (9). We then have

$$\mathbf{m}_{n,k}(2r,r,r)=r\left(\binom{n+k-1}{2k}+\binom{n+k+1}{2k+1}\right)=r(b^{(0)}_{n,k}-a^{(0)}_{n,k}+a^{(1)}_{n,k}), \tag{28}$$

where $b^{(0)}_{n,k}$ is as given in Table 2 below. This translates into the decomposition formula

$$M_{2r,r(3+x)}=r(2M_{1,0}+(3+x)M_{0,1}). \tag{29}$$

The polynomial line-sequence is as follows:

$$M_{2r,\,r(3+x)}(-1, 1+2x): \ldots, [2r, r(3+x)], r+7rx+2rx^2, -2r+8rx+16rx^2+4rx^3, \ldots. \tag{30}$$

Note that, for $-n$, we also have, from (28),

$$\mathbf{m}_{-n,k}(2r, r, r) = r\left(\binom{n+k}{2k} + \binom{n+k-1}{2k} - \binom{n+k}{2k+1}\right) = r(2b_{n,k}^{(0)} - b_{n,k}^{(1)}), \tag{31}$$

where $b_{n,k}^{(1)}$ is as given in Table 2 below. And so forth.

Table 2 is a compilation of some conversion relations for convenience of reference.

TABLE 2. Conversion Relations

Relations	References
$\mathbf{m}_{n,k}(i, s, r) = (i-s)a_{n-1,k}^{(0)} + sa_{n,k}^{(0)} + ra_{n-1,k}^{(1)}$	[*see*(9)]
$\mathbf{m}_{n,k}(1, 1, 0) = \binom{n+k}{2k} = a_{n,k}^{(0)}$	[1]
$\mathbf{m}_{n,k}(1, 1, 1) = \binom{n+k+1}{2k+1} = a_{n,k}^{(1)}$	[1]
$\mathbf{m}_{n,k}(1, 1, r) = \binom{n+k}{2k} + r\binom{n+k}{2k+1} = a_{n,k}^{(r)}$	[1]
$\mathbf{m}_{n,k}(2, 1, 0) = \binom{n+k-1}{2k} + \binom{n+k}{2k} = b_{n,k}^{(0)}$	[4]
$\mathbf{m}_{n,k}(2, 1, 1) = \binom{n+k-1}{2k} + \binom{n+k}{2k} + \binom{n+k}{2k+1} = b_{n,k}^{(1)}$	[4]
$\mathbf{m}_{n,k}(2, 1, r) = \binom{n+k-1}{2k} + \binom{n+k}{2k} + r\binom{n+k}{2k+1} = b_{n,k}^{(r)}$	[4]

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❖❖❖

REPRESENTATION OF NUMBERS WITH NEGATIVE DIGITS AND MULTIPLICATION OF SMALL INTEGERS

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The usual way to multiply numbers in binary representation runs as follows: To compute $m \cdot n$, copy n to x. Multiply x by two. If the last digit of m is 1, then add n to x. Now delete the last digit of m. Repeat until $m = 1$, then $x = m \cdot n$.

Since multiplication by 2 needs almost no time, the running time of this algorithm depends only on the time to add two numbers and the number of 1's occurring in m. If n and m are both k-bit numbers, one needs almost always $\frac{1}{2}k^2$-bit operations.

In [1], Dimitrov and Donevsky used the Zeckendorf representation to construct a number system in which, on average, less nonvanishing digits are needed to represent a number. Thus, using this representation, multiplication becomes about 3.2% faster. In this note we will give another number system, which gives an algorithm to multiply n-digit numbers in expected time $\frac{3}{8}n^2 + 2n$, i.e., for large numbers this algorithm is 25% faster than the usual one.

Note that, for very large numbers, Karatsuba and Ofmann gave an algorithms with running time $O(n^{1.585})$, and Schoenhagen and Strasser gave one with running time $O(n \log n \log\log n)$ [2]; however, the constants implied by the O-notations are so large that these algorithms have no meaning for most computations. Thus, faster multiplication of small numbers might speed up many computations.

We will write integers as a string consisting of 1, 0, and -1, and interpret a string $\alpha_k \alpha_{k-1} \ldots \alpha_0$ as $\sum \alpha_i 2^i$. Our algorithm will make use of the following simple statement.

Proposition 1:

(a) Every integer n has a unique representation as above with the following additional requirements: there are no three consecutive 1's, no two consecutive nonvanishing digits are -1's, between a 1 and a -1 there are at least two 0's, and the first digit is 0 if and only if $n = 0$.

(b) The expected number of nonvanishing digits in the representation of an n-bit number is $\frac{3}{8}(n+3)$.

(c) The representation can be found by changing $\le \frac{3}{4}n$-bits on average.

Proof: First, we prove the uniqueness of this representation. Let n be the least number such that there are two different representations $\alpha_k \ldots \alpha_0$ and $\beta_l \ldots \beta_0$. If $k > l$, then

$$\alpha_k \ldots \alpha_0 - \beta_l \ldots \beta_0 \ge \underbrace{100-100\ldots}_{k+1 \text{ digits}} - \underbrace{110110\ldots}_{\le k \text{ digits}} = 1-1-1-1\ldots = 1.$$

Thus, $k = l$. Since, by the same computation, the leading digit of a positive digit is 1, deleting this digit together with the following 0's yields a counterexample of smaller absolute value, therefore inverting if necessary gives a smaller positive counterexample. However, we assumed n to be minimal.

To construct this representation, begin with the ordinary binary representation of n. Now, since $2^k + \cdots + 4 + 2 + 1 = 2^{k+1} - 1$, we have

$$\underbrace{11\ldots11}_{k\text{ digits}}=\underbrace{100\ldots00}_{k+1\text{ digits}}-1.$$

Thus, replacing every string of k consecutive 1's as above does not change the value of the string, and it is easily seen that the new representation fulfills all requirements, if we replace only blocks of length ≥ 3. During this replacement, we have to change $k+1$ digits for every block of length k. Since the expected value of the number of blocks of length k in an n-digit number is $n2^{-k-1}$, the expected value of replacements equals

$$\sum_{k=3}^{n} n(k+1)2^{-k-1} \leq \frac{n}{4}+\frac{n}{2}\sum_{k=3}^{\infty}\frac{k}{2^k}=\frac{3}{4}n.$$

In the resulting string, there is a single 1 for every substring 011 in the ordinary binary representation, two consecutive 1's for every substring 0110 and a 1 and a -1 for every block of length ≥ 3. Thus, to estimate the number of nonvanishing digits, we have to count the blocks in the ordinary binary representation. At every digit, a new block begins with probability 1/2, except the first one, where this is certain. If the last digit is 0, then there are as many 1-blocks as 0-blocks, otherwise, there is one 1-block more. Thus, the expected number of 1-blocks is $\frac{n+3}{4}$. Among these, there are $\frac{n+3}{8}$ blocks of length 1; thus, the total number of nonvanishing digits equals

$$2\frac{n+3}{4}-\frac{n+3}{8}=\frac{3}{8}(n+3).$$

Hence, all of our claims are proved. □

Now, adding and subtracting integers takes the same amount of time. Thus, to multiply two n-digit numbers using this modified binary system, we need $\frac{3}{8}(n+3)$ additions or subtractions on average. Each addition needs n bit operations, so this part of the multiplication algorithm needs $\frac{3}{8}n^2+\frac{9}{8}n$ steps. We also have to modify one of the two numbers to be multiplied, which takes $\frac{3}{4}n$ steps. Therefore, the total running time becomes

$$\frac{3}{8}n^2+\frac{15}{8}n<\frac{3}{8}n^2+2n$$

as claimed.

For $n>15$, we have $\frac{1}{2}n^2>\frac{3}{8}n^2+\frac{15}{8}n$. Thus, for numbers $>2^{15}=32768$, multiplying by using this number system is faster than the usual algorithm.

Note that things become even better if one has to do computations with the same number several times, since than one only has to convert the integers once. It is easily seen that in this case multiplication is always at least as fast as the standard algorithm.

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ON THE DISTRIBUTION OF TOTIENTS

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An integer n is called a totient if there is some integer x such that $\varphi(x)=n$, where φ is Euler's function. If this equation is not solvable, n is called a *nontotient*. In 1956, Schinzel [4] proved that, for any positive k, $2\cdot 7^k$ is a nontotient. In 1961, Ore (see [1]) proved that, for every α, there is some odd number k such that $2^\alpha\cdot k$ is a nontotient. In 1963, Selfridge [1] showed that the same is true with k restricted by $k\le 271129$. Recently, Mingzhi [3] proved that, for any given d, there are infinitely many primes p such that dp is a nontotient. In fact, his proof gives the existence of a, q with $(a,q)=1$ such that, for any sufficiently large prime $p\equiv a \pmod q$, dp is a nontotient. Thus, by the prime number theorem for arithmetic progressions, a positive percentage of all primes p has this property. However, here $q>d^{\tau(d)}$, where $\tau(n)$ denotes the number of divisors of n; thus, this percentage is quite small. In this note we will show this is true for almost all primes p. Further, we describe explicitly a large class of nontotients. We will prove the following theorems.

Theorem 1: There is an absolute constant c such that, for any integer d, the number of primes $p\le x$ such that dp is a totient is bounded above by $c\tau(d^2)\frac{x}{\log^2 x}$.

Here and in the sequel, the letter c denotes absolute positive constants and the letters p and q denote prime numbers only.

Theorem 2: Set $m=3\cdot 5\cdot 7\cdot 13\cdot 19\cdot 37\cdot 73$, $a=35274404$. If d is an integer relatively prime to m such that, for every prime divisor p of d, p is a quadratic residue (mod 73), and $q\equiv a \pmod m$ is a prime number such that $q-1\nmid d$, then dq is a nontotient.

It will be apparent from the proof that the value of a is by no means unique. Also, m may be subject to variation. We only use the fact that m has many prime divisors, and that the least common multiple of all $p-1$, where p ranges over the prime divisors of m, is very small. However, for other values of m, the computations would become extremely long.

The proof of Theorem 1 is based on the following theorem of Erdös [2].

Theorem 3: There is an absolute constant c such that, for any integer a, we have, for the number $N_a(x)$ of primes $p\le x$ such that $ap+1$ is also prime, the inequality

$$N_a(x)\le c\frac{x}{\log^2 x}\prod_{q|a}\left(1-\frac{1}{q}\right)^{-1}.$$

Proof of Theorem 1: Assume that p is some prime such that dp is a totient, say $dp=\varphi(n)$. Since φ is multiplicative and, for prime numbers q, we have $\varphi(q)=q-1$, there is either some prime divisor q of n such that $q\equiv 1 \pmod p$, say $q=ap+1$, or n is divisible by p^2. In the latter case we have $n=p^k m$, where $(p,m)=1$. Thus, we get $dp=(p-1)p^{k-1}\varphi(m)$ and $p-1\mid d$. So the number of such p is $\le\tau(d)$. In the first case, we have $n=qm$ with $(q,m)=1$; therefore, we get $dp=(q-1)\varphi(m)=ap\varphi(m)$. Especially, a is some divisor of d. We now fix some a, and count the number of primes $p\le x$ such that the equation $dp=\varphi(qm)$ is solvable when $q=ap+1$ is prime. This is at most the number of $p \le x$ such that $q = ap+1$ is prime, and by Theorem 2 this number is

$$\le c\frac{x}{\log^2 x}\prod_{q|a}\left(1-\frac{1}{q}\right)^{-1}.$$

Since a is a divisor of d, the total number of solutions is at most

$$c\frac{x}{\log^2 x}\sum_{a|d}\prod_{q|a}\left(1-\frac{1}{q}\right)^{-1}.$$

We have

$$\prod_{q|a}\left(1-\frac{1}{q}\right)^{-1} = \prod_{q|a}\left(1+\frac{1}{q}\right)\prod_{q|a}\left(1-\frac{1}{q^2}\right)^{-1}$$
$$< \frac{\pi^2}{6}\prod_{q|a}\left(1+\frac{1}{q}\right) \le c\sum_{t|a}\frac{1}{t}.$$

Hence, the sum above can be estimated as

$$\sum_{a|d}\prod_{q|a}\left(1-\frac{1}{q}\right)^{-1} \le c\sum_{a|d}\sum_{t|a}\frac{1}{t}.$$

The function $f(d) = \sum_{a|d}\sum_{t|a}\frac{1}{t}$ is multiplicative, since it is the Dirichlet convolution of multiplicative functions. For prime powers, we have

$$f(p^k) = \sum_{0\le l\le k}\sum_{0\le m\le l} p^{-m} = \sum_{0\le m\le k}(k-m+1)p^{-m} < 2k+1 = \tau(p^{2k}).$$

By multiplicativity, we get $f(n) \le \tau(n^2)$ for any n. Hence, the total number of primes $p \le x$ such that dp is totient is at most

$$c\frac{x}{\log^2 x}\tau(d^2) + \tau(d),$$

and by increasing c slightly, the second term may be neglected. This proves Theorem 1.

Proof of Theorem 2: Note that, if the equation $\varphi(x) = dq$ is solvable, either $q^2 | x$ or there is some prime $p \equiv 1 \pmod q$ such that $p-1 | dq$. In the first case, we have

$$q(q-1) = \varphi(q^2) \mid \varphi(x) = dq,$$

contradicting our first assumption on d. In the second case, number the prime divisors of m by r_j, $1 \le j \le 7$, and choose some primitive root π_j for each j. We may assume that p does not divide m; thus, the condition that p is prime implies $r_j \nmid p$. This is equivalent to $d'q \not\equiv 1 \pmod{r_j}$ for a certain divisor d' of d. Write

$$d' = \prod_{i=1}^{n} p_i^{x_i},$$

define a_{ij} by $\pi_j^{\alpha_{ij}} \equiv p_i \pmod{r_j}$, and define b_j by $\pi_j^{b_j} \equiv q \equiv a \pmod{r_j}$. Then the condition on p implies the system of incongruences

$$\sum_{i=1}^{n} \alpha_{ij}x_i \not\equiv -b_j \pmod{r_j - 1}.$$

Now, choosing π_j to be the least primitive root (mod r_j), i.e., $\pi_j = 2$ for $j \neq 3, 7$, $\pi_3 = 3$, and $\pi_7 = 5$, we obtain $b_1 = 1$, $b_2 = 2$, $b_3 = 4$, $b_4 = 8$, $b_5 = 12$, $b_6 = 24$, $b_7 = 0$. (Note that here we have much more freedom; we could choose different primitive roots, and we could use different values for the b_j, each resulting in different values for a.) To prove our claim, note first that by assumption all p_i are quadratic residues (mod 73), so all α_{i7} are even. Thus, since b_7 is even, too, we may divide the seventh incongruence by 2, obtaining an incongruence (mod 36). Further, for all j, we have $r_j - 1 \mid 36$, so every incongruence (mod $r_j - 1$) may be written as a set of incongruences (mod 36). Now, the solvability of the system is equivalent to the existence of some residue class (mod 36) that is not contained in one of the following seven sets, each defined by one of the seven incongruences:

$$\begin{aligned}
&\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35\},\\
&\{2, 6, 10, 14, 18, 22, 26, 30, 34\},\\
&\{4, 10, 14, 20, 26, 32\},\\
&\{8, 20, 32\}, \{12, 30\}, \{24\}, \{36\}.
\end{aligned}$$

By construction, the first four sets define residue classes (mod 12), and one easily checks that all but the class 0 are covered, whereas the last three sets contain the remaining class. Thus, our initial assumption on the solvability of the equation $\varphi(x) = dq$ was wrong, proving Theorem 2.

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REDUCTION FORMULAS FOR THE SUMMATION OF RECIPROCALS IN CERTAIN SECOND-ORDER RECURRING SEQUENCES

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1. INTRODUCTION

In [2], Brousseau considered sums of the form

$$S(k_1, k_2, \dots, k_m) = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k_1} F_{n+k_2} \cdots F_{n+k_m}} \tag{1.1}$$

and

$$T(k_1, k_2, \dots, k_m) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+k_1} F_{n+k_2} \cdots F_{n+k_m}}, \tag{1.2}$$

where the k_i are positive integers with $k_1 < k_2 < \cdots < k_m$. He stated that the sums in (1.1) and (1.2) could be written as

$$S(k_1, k_2, \dots, k_m) = r_1 + r_2 S(1, 2, \dots, m) \tag{1.3}$$

and

$$T(k_1, k_2, \dots, k_m) = r_3 + r_4 T(1, 2, \dots, m), \tag{1.4}$$

where r_1, r_2, r_3, and r_4 are rational numbers that depend upon $k_1, k_2, \dots, k_m$. He arrived at this conclusion after treating several cases involving small values of m.

Our aim in this paper is to prove Brousseau's claim by providing reduction formulas that accomplish this task. Recently, André-Jeannin [1] treated the case $m = 1$ by giving explicit expressions for the coefficients r_1, r_2, r_3, and r_4. Indeed, he worked with a generalization of the Fibonacci sequence, and we will do the same. In light of André-Jeannin's results, we consider only $m \geq 2$. We have found, for each of the sums (1.3) and (1.4), that two reduction formulas are needed for the case $m = 2$, and three are needed for $m \geq 3$. Consequently, we treat those cases separately.

Define the sequences $\{U_n\}$ and $\{W_n\}$ for all integers n by

$$\begin{cases} U_n = pU_{n-1} - qU_{n-2}, & U_0 = 0,\ U_1 = 1, \\ W_n = pW_{n-1} - qW_{n-2}, & W_0 = a,\ W_1 = b. \end{cases}$$

Here a, b, p, and q are assumed to be integers with $pq \neq 0$ and $\Delta = p^2 - 4q > 0$. Consequently, we can write down closed expressions for U_n and W_n (see [3]):

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.5}$$

where $\alpha = (p + \sqrt{\Delta})/2$, $\beta = (p - \sqrt{\Delta})/2$, $A = b - a\beta$, and $B = b - a\alpha$. Thus, $\{W_n\}$ generalizes $\{U_n\}$ which, in turn, generalizes $\{F_n\}$.

We note that

$$\alpha > 1 \text{ and } \alpha > |\beta| \text{ if } p > 0, \text{ while } \beta < -1 \text{ and } |\beta| > |\alpha| \text{ if } p < 0. \tag{1.6}$$

Consequently,

$$W_n \approx \frac{A}{\alpha - \beta}\alpha^n \text{ if } p > 0, \text{ and } W_n \approx \frac{-B}{\alpha - \beta}\beta^n \text{ if } p < 0. \tag{1.7}$$

Throughout the remainder of the paper, we take

$$S(k_1, k_2, \ldots, k_m) = \sum_{n=1}^{\infty} \frac{1}{W_n W_{n+k_1} W_{n+k_2} \cdots W_{n+k_m}} \tag{1.8}$$

and

$$T(k_1, k_2, \ldots, k_m) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{W_n W_{n+k_1} W_{n+k_2} \cdots W_{n+k_m}}, \tag{1.9}$$

where the k_i are positive integers as described earlier. From (1.6) it follows that $U_n \neq 0$ for $n \geq 1$. We shall suppose that $W_n \neq 0$ for $n \geq 1$. Then, by (1.6) and (1.7), use of the ratio test shows that the series in (1.8) and (1.9) are absolutely convergent.

We require the following identities:

$$U_m W_{n+1} - W_{n+m} = qU_{m-1}W_n, \tag{1.10}$$

$$U_{m-k+1}W_{n+k} - W_{n+m} = qU_{m-k}W_{n+k-1}, \tag{1.11}$$

$$U_m W_n + q^m U_d W_{n-m-d} = U_{m+d}W_{n-d}, \tag{1.12}$$

$$pW_{n+m} + q^2 U_{m-2}W_n = U_m W_{n+2}, \tag{1.13}$$

$$U_{m-l+1}W_{n+k} - U_{k-l+1}W_{n+m} = q^{k-l+1}U_{m-k}W_{n+l-1}. \tag{1.14}$$

Identity (1.11) follows from (1.10), which is essentially (3.14) in [3], where the initial values of $\{U_n\}$ are shifted. Identities (1.13) and (1.14) follow from (1.12), which occurs as (5.7) in [4].

2. THREE TERMS IN THE DENOMINATOR

Our results for the case in which the denominator consists of a product of three terms are contained in the following theorem.

Theorem 1: Let k_1 and k_2 be positive integers with $k_1 < k_2$. Then

$$S(k_1, k_2) = \frac{1}{qU_{k_2-k_1}}[U_{k_2-k_1+1}S(k_1-1, k_2) - S(k_1-1, k_1)] \quad \text{if } 1 < k_1, \tag{2.1}$$

$$S(1, k_2) = \frac{p}{U_{k_2}}S(1, 2) + \frac{q^2U_{k_2-2}}{U_{k_2}}\left[S(1, k_2-1) - \frac{1}{W_1W_2W_{k_2}}\right] \quad \text{if } 2 < k_2, \tag{2.2}$$

$$T(k_1, k_2) = \frac{1}{qU_{k_2-k_1}}[U_{k_2-k_1+1}T(k_1-1, k_2) - T(k_1-1, k_1)] \quad \text{if } 1 < k_1, \tag{2.3}$$

$$T(1, k_2) = \frac{p}{U_{k_2}}T(1, 2) + \frac{q^2U_{k_2-2}}{U_{k_2}}\left[\frac{1}{W_1W_2W_{k_2}} - T(1, k_2-1)\right] \quad \text{if } 2 < k_2. \tag{2.4}$$

Proof: With the use of (1.11), it follows that

$$\frac{qU_{k_2-k_1}}{W_nW_{n+k_1}W_{n+k_2}}=\frac{U_{k_2-k_1+1}}{W_nW_{n+k_1-1}W_{n+k_2}}-\frac{1}{W_nW_{n+k_1-1}W_{n+k_1}}, \tag{2.5}$$

and summing both sides we obtain (2.1). Likewise, to obtain (2.3), we first multiply (2.5) by $(-1)^{n-1}$ and sum both sides.

Next we have

$$\frac{U_{k_2}}{W_nW_{n+1}W_{n+k_2}}=\frac{p}{W_nW_{n+1}W_{n+2}}+\frac{q^2U_{k_2-2}}{W_{n+1}W_{n+2}W_{n+k_2}}, \tag{2.6}$$

which follows from (1.13). Now, if we sum both sides of (2.6) and note that

$$\sum_{n=1}^{\infty}\frac{1}{W_{n+1}W_{n+2}W_{n+k_2}}=S(1,k_2-1)-\frac{1}{W_1W_2W_{k_2}},$$

we obtain (2.2). Finally, to establish (2.4), we multiply (2.6) by $(-1)^{n-1}$, sum both sides, and note that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{W_{n+1}W_{n+2}W_{n+k_2}}=\frac{1}{W_1W_2W_{k_2}}-T(1,k_2-1).$$

This proves Theorem 1. □

It is instructive to work through some examples. Taking $W_n=F_n$ and using (2.1) and (2.2) repeatedly, we find that $S(3,6)=-\frac{269}{1920}+\frac{1}{4}S(1,2)$, and this agrees with the corresponding entry in Table III of [2]. Again, with $W_n=F_n$, we have $T(3,6)=-\frac{139}{1920}+\frac{1}{4}T(1,2)$.

3. MORE THAN THREE TERMS IN THE DENOMINATOR

Let $k_1, k_2, \ldots, k_m$ be positive integers and put $P(k_1,\ldots,k_m)=W_nW_{n+k_1}\ldots W_{n+k_m}$. With this notation, the work that follows will be more succinct. The main result of this section is contained in the theorem that follows, where we give only the reduction formulas for $S(k_1,k_2,\ldots,k_m)$. After the proof, we will indicate how the corresponding reduction formulas for $T(k_1,k_2,\ldots,k_m)$ can be obtained.

Theorem 2: For $m\ge 3$, let $k_1<k_2<\cdots<k_m$ be positive integers and set $k_0=0$. Then

$$\begin{aligned}S(k_1,k_2,\ldots,k_m)=&\frac{U_{k_m-k_j+1}}{q^{k_{m-1}-k_j+1}U_{k_m-k_{m-1}}}S(k_1,\ldots,k_{j-1},k_j-1,k_j,\ldots,k_{m-2},k_m)\\&-\frac{U_{k_{m-1}-k_j+1}}{q^{k_{m-1}-k_j+1}U_{k_m-k_{m-1}}}S(k_1,\ldots,k_{j-1},k_j-1,k_j,\ldots,k_{m-1})\\&\qquad\text{if } 1\le j\le m-2 \text{ and } k_{j-1}<k_j-1;\end{aligned} \tag{3.1}$$

$$\begin{aligned}S(k_1,\ldots,k_m)=&\frac{U_{k_m-k_{m-1}+1}}{qU_{k_m-k_{m-1}}}S(k_1,\ldots,k_{m-2},k_{m-1}-1,k_m)\\&-\frac{1}{qU_{k_m-k_{m-1}}}S(k_1,\ldots,k_{m-2},k_{m-1}-1,k_{m-1}) \text{ if } k_{m-2}<k_{m-1}-1;\end{aligned} \tag{3.2}$$

$$S(1,2,\ldots,m-1,k_m)=\frac{U_m}{U_{k_m}}S(1,2,\ldots,m)$$
$$+\frac{q^mU_{k_m-m}}{U_{k_m}}\left[S(1,2,\ldots,m-1,k_m-1)-\frac{1}{W_1W_2\ldots W_mW_{k_m}}\right]\quad\text{if } m<k_m. \tag{3.3}$$

Proof: With the use of (1.14), we see that

$$\frac{q^{k_{m-1}-k_j+1}U_{k_m-k_{m-1}}}{P(k_1,\ldots,k_m)}=\frac{U_{k_m-k_j+1}}{P(k_1,\ldots,k_{j-1},k_j-1,k_j,\ldots,k_{m-2},k_m)}-\frac{U_{k_{m-1}-k_j+1}}{P(k_1,\ldots,k_{j-1},k_j-1,k_j,\ldots,k_{m-1})},$$

and summing both sides we obtain (3.1).

Next we have

$$\frac{qU_{k_m-k_{m-1}}}{P(k_1,\ldots,k_m)}=\frac{U_{k_m-k_{m-1}+1}}{P(k_1,\ldots,k_{m-2},k_{m-1}-1,k_m)}-\frac{1}{P(k_1,\ldots,k_{m-2},k_{m-1}-1,k_{m-1})},$$

which can be proved with the use of (1.11). Summing both sides, we obtain (3.2).

Finally, with the aid of (1.12) we see that

$$\frac{U_{k_m}}{P(1,2,\ldots,m-1,k_m)}=\frac{U_m}{P(1,2,\ldots,m)}+\frac{q^mU_{k_m-m}}{W_{n+1}W_{n+2}\ldots W_{n+m}W_{n+k_m}}.$$

The reduction formula (3.3) follows if we sum both sides and observe that

$$\sum_{n=1}^{\infty}\frac{1}{W_{n+1}W_{n+2}\ldots W_{n+m}W_{n+k_m}}=S(1,2,\ldots,m-1,k_m-1)-\frac{1}{W_1W_2\ldots W_mW_{k_m}}.$$

This completes the proof of Theorem 2. □

As was the case in Theorem 1, the reduction formulas for T can be obtained from those for S. In (3.1) and (3.2), we simply replace S by T. In (3.3), we *first* replace the term in square brackets by

$$\frac{1}{W_1W_2\ldots W_mW_{k_m}}-S(1,2,\ldots,m-1,k_m-1)$$

and *then* replace S by T.

As an application of Theorem 2 we have, with $W_n=F_n$,

$$S(1,2,4,6,7)=-3S(1,2,3,4,7)+2S(1,2,3,4,6)\quad\text{by (3.1)}; \tag{3.4}$$

$$S(1,2,3,4,7)=\frac{1}{5070}+\frac{5}{13}S(1,2,3,4,5)-\frac{1}{13}S(1,2,3,4,6)\quad\text{by (3.3)}; \tag{3.5}$$

$$S(1,2,3,4,6)=\frac{1}{1920}+\frac{1}{2}S(1,2,3,4,5)\quad\text{by (3.3)}. \tag{3.6}$$

Together (3.4)-(3.6) imply that

$$S(1,2,4,6,7)=\frac{37}{64896}-\frac{1}{26}S(1,2,3,4,5).$$

4. CONCLUDING COMMENTS

Recently, Rabinowitz [5] considered the finite sums associated with (1.1) and (1.2). That is, he took the upper limit of summation to be N, and gave an algorithm for expressing the resulting sums in terms of

$$\sum_{n=1}^{N}\frac{1}{F_n},\ \sum_{n=1}^{N}\frac{(-1)^n}{F_n},\ \text{and}\ \sum_{n=1}^{N}\frac{1}{F_nF_{n+1}}.$$

In addition, he posed a number of interesting open questions.

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CONSECUTIVE BINOMIAL COEFFICIENTS IN PYTHAGOREAN TRIPLES AND SQUARES IN THE FIBONACCI SEQUENCE

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In this note, we find all triples consisting of consecutive binomial coefficients

$$\binom{n}{k}\ \binom{n}{k+1}\ \binom{n}{k+2} \tag{1}$$

forming Pythagorean triples. The result is

Theorem: If the three numbers listed at (1) above form a Pythagorean triple, then $n=62$ and $k=26$ or 34.

We first notice that it is enough to assume that $k+2 \le n/2$. Indeed, if $k \ge n/2$, then one can use the symmetry of the Pascal triangle to reduce the problem to the previous one, while the case in which $k < n/2$ but $k+2 > n/2$ is impossible because these conditions will lead to isosceles Pythagorean triangles which, as we all know, do not exist.

Proof: After performing the cancellations in the following equation,

$$\binom{n}{k}^2 + \binom{n}{k+1}^2 = \binom{n}{k+2}^2, \tag{2}$$

we get

$$(k+2)^2((k+1)^2+(n-k)^2) = (n-k)^2(n-k-1)^2. \tag{3}$$

We make the substitution $x := n-k$ and $y := k+1$. Equation (3) becomes

$$(y+1)^2(x^2+y^2) = x^2(x-1)^2. \tag{4}$$

Notice that equation (4) implies that x^2+y^2 is a square. Let $d := \gcd(x, y)$.

We distinguish two cases:

Case 1.

$$\begin{cases} x = 2duv, \\ y = d(u^2-v^2), \end{cases} \quad \text{where } \gcd(u, v) = 1 \text{ and } u \not\equiv v \pmod 2. \tag{5}$$

Combining formulas (5) and equation (4), we get

$$(d(u^2-v^2)+1)(u^2+v^2) = 2uv(2duv-1). \tag{6}$$

Since $\gcd(u^2+v^2, 2uv) = 1$, it follows from equation (6) that $(u^2+v^2) \mid (2duv-1)$. Hence,

$$\frac{2duv-1}{u^2+v^2} = \frac{d(u^2-v^2)+1}{2uv} = d_1, \tag{7}$$

where d_1 is an integer. One can rewrite the two equations (7) as

$$\begin{cases} d(2uv) - d_1(u^2+v^2) = 1, \\ d(u^2-v^2) - d_1(2uv) = -1. \end{cases} \tag{8}$$

One can now regard (8) as a linear system in two unknowns, namely, d and d_1. After solving it by using Kramer's rule, one gets

$$\begin{cases} d = \dfrac{-(u+v)^2}{u^4 - v^4 - 4u^2v^2}, \\ d_1 = \dfrac{-u^2+v^2-2uv}{u^4 - v^4 - 4u^2v^2}. \end{cases} \tag{9}$$

Let $\Delta = u^4 - v^4 - 4u^2v^2$ be the determinant of the coefficient matrix. We now show that $\Delta = \pm 1$. Indeed, notice that since $u \not\equiv v \pmod 2$, it follows that Δ is odd. Assume that $|\Delta| > 1$ and let p be an odd prime divisor of Δ. From the first formula (9) and the fact that d is an integer, we get that $p \mid (u+v)$. Since $p \mid \Delta = u^4 - v^4 - 4u^2v^2 = (u+v)(u-v)(u^2+v^2) - 4u^2v^2$, it follows that $p \mid uv$. But since $p \mid (u+v)$ also, we get that $p \mid \gcd(u, v)$, which is impossible. Hence,

$$u^4 - v^4 - 4u^2v^2 = \pm 1. \tag{10}$$

Notice that equation (10) can be rewritten as $(2(u^2-2v^2))^2 - 5(2v^2)^2 = \pm 4$. It is well known that all positive integer solutions of $X^2 - 5Y^2 = \pm 4$ are of the form $X = L_t$ and $Y = F_t$ for some positive integer t, where $(L_n)_{n\ge 0}$ and $(F_n)_{n\ge 0}$ are the Lucas and the Fibonacci sequence, respectively, given by $L_0 = 2$, $L_1 = 1$, $F_0 = 0$, $F_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ and $F_{n+2} = F_{n+1} + F_n$, respectively.* Now equation (11) implies that

$$\begin{cases} F_t = 2v^2, \\ L_t = \pm 2(u^2 - 2v^2). \end{cases} \tag{12}$$

It is known (see, e.g., [3]) that the only Fibonacci numbers which are twice times a square are $F_0 = 0$, $F_3 = 2$, and $F_6 = 8$. Hence, for our case, we get $t = 3$, $v = 1$, and $t = 6$, $v = 2$, respectively. In the first case, we get $u = 2$. From formula (9), we get $d = 9$, and then from formulas (5), we get $x = 36$ and $y = 27$. This gives the solution $n = 62$ and $k = 26$, and by the symmetry of the Pascal triangle, $k = 34$ as well. The case $t = 6$ and $v = 2$ does not lead to an integer solution for u.

Case 2.

$$\begin{cases} x = d(u^2 - v^2), \\ y = 2duv, \end{cases} \quad \text{where } \gcd(u, v) = 1 \text{ and } u \not\equiv v \pmod 2. \tag{13}$$

This case is very similar to the preceding one. With the notations (13), equation (4) becomes

$$(d(2uv)+1)(u^2+v^2) = (u^2-v^2)(d(u^2-v^2)-1). \tag{14}$$

Since $\gcd(u^2+v^2, u^2-v^2) = 1$, it follows that $(u^2+v^2) \mid (d(u^2-v^2)-1)$. Hence, equation (14) implies that

$$\frac{d(2uv)+1}{u^2-v^2} = \frac{d(u^2-v^2)-1}{u^2+v^2} = d_1, \tag{15}$$

where d_1 is an integer. One may now rewrite equation (15) as

* I could not find a reference for this fact.

$$\begin{cases} d(2uv) - d_1(u^2 - v^2) = -1, \\ d(u^2 - v^2) - d_1(u^2 + v^2) = 1. \end{cases} \tag{16}$$

Solving system (16) in terms of d and d_1 versus u and v, we get

$$\begin{cases} d = \dfrac{2u^2}{(u^2 - v^2)^2 - 2uv(u^2 + v^2)}, \\ d_1 = \dfrac{2uv + u^2 - v^2}{(u^2 - v^2)^2 - 2uv(u^2 + v^2)}. \end{cases} \tag{17}$$

One may again argue as in the preceding case that

$$(u^2 - v^2)^2 - 2uv(u^2 + v^2) = \pm 1. \tag{18}$$

Rewrite (18) as

$$(2(u^2 + v^2 - uv))^2 - 5(2uv)^2 = \pm 4. \tag{19}$$

Equation (19) implies that there exists $t > 0$ such that

$$\begin{cases} F_t = 2uv, \\ L_t = 2(u^2 + v^2 - uv). \end{cases} \tag{20}$$

Formulas (20) imply that

$$\frac{L_t - F_t}{2} = (u - v)^2. \tag{21}$$

Using the well-known fact that $L_t = F_t + 2F_{t-1}$ for all $t \geq 1$, it follows by formula (21) that

$$F_{t-1} = (u - v)^2. \tag{22}$$

It is well known (see [1] or [2]) that the only squares in the Fibonacci sequence are $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, and $F_{12} = 144$. Hence, by formula (22), we get that $t = 1, 2, 3, 13$. None of these values gives integer solutions u, v from the system of equations (20). The Theorem is therefore proved.

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SOLVING SOME GENERAL NONHOMOGENEOUS RECURRENCE RELATIONS OF ORDER *r* BY A LINEARIZATION METHOD AND AN APPLICATION TO POLYNOMIAL AND FACTORIAL POLYNOMIAL CASES

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1. INTRODUCTION

Let $a_0, \ldots, a_{r-1}$ $(r \geq 2, a_{r-1} \neq 0)$ be some fixed real (or complex) numbers and $\{C_n\}_{n\geq 0}$ be a sequence of real (or complex) numbers. Let $\{T_n\}_{n=0}^{+\infty}$ be the sequence defined by the following nonhomogeneous relation of order *r*,

$$T_{n+1} = a_0 T_n + a_1 T_{n-1} + \cdots + a_{r-1} T_{n-r+1} + C_n, \text{ for } n \geq r-1, \tag{1}$$

where $T_0, \ldots, T_{r-1}$ are specified by the initial conditions. In the sequel, we refer to these sequences as *sequences (1)*. The solutions $\{T_n\}_{n\geq 0}$ of (1) may be given as follows: $T_n = T_n^{(h)} + T_n^{(p)}$, where $\{T_n^{(h)}\}_{n\geq 0}$ is a solution of the homogeneous part of (1) and $\{T_n^{(p)}\}_{n\geq 0}$ is a particular solution of (1). If $C_n = \sum_{j=0}^{d} \beta_j C_n^j$, solutions $\{T_n\}_{n\geq 0}$ of (1) may be expressed as $T_n = \sum_{j=0}^{d} \beta_j T_n^j$, where $\{T_n^j\}_{n\geq 0}$ is a solution of (1) for $C_n = C_n^j$. Sequences (1) are studied in the case of C_n polynomial or factorial polynomial (see, e.g., [2], [3], [4], [5], [7], [12], and [8]).

The purpose of this paper is to study a linearization process of (1) when $C_n = V_n$, where $\{V_n\}_{n\geq 0}$ is an *m*-generalized Fibonacci sequence whose $V_0, \ldots, V_{m-1}$ are the initial terms and

$$V_{n+1} = b_0 V_n + \cdots + b_{m-1} V_{n-m+1}, \text{ for } n \geq m-1, \tag{2}$$

where $b_0, \ldots, b_{m-1}$ $(m \geq 2, b_{m-1} \neq 0)$ are given fixed real (or complex) numbers. This process permits the construction of a method for solving (1). In the polynomial and factorial polynomial cases, our linearization process allows us to express well-known particular solutions, particularly Asveld's polynomials and factorial polynomials, in another form. Examples and discussion are given.

This paper is organized as follows: In Section 2 we study a Linearization Process of (1). In Section 3 we apply this process to polynomial and factorial polynomial cases. Section 4 provides a concluding discussion.

2. LINEARIZATION PROCESS FOR SEQUENCES (1)

In this section we suppose $C_n = V_n$ with $\{V_n\}_{n\geq 0}$ defined by (2), where we set $m = s$ and $\sigma_2 = \{\mu_0, \ldots, \mu_t\}$ the set of its characteristic roots whose multiplicities are, respectively, $p_0, \ldots, p_t$.

Expression (1) implies that $V_{n+1} = T_{n+1} - \sum_{j=0}^{r-1} a_j T_{n-j}$ for any $n \geq r-1$. Let $n \geq r+s-1$, then for any j $(0 \leq j \leq s-1)$ we have $V_{n-j} = T_{n-j} - \sum_{k=0}^{r-1} a_k T_{n-j-k-1}$. Then from (2) we derive that

$$T_{n+1} = \sum_{j=0}^{r-1} a_j T_{n-j} + \sum_{j=0}^{s-1} b_j T_{n-j} - \sum_{j=0}^{s-1} \sum_{k=0}^{r-1} b_j a_k T_{n-j-k-1}. \tag{3}$$

Expression (3) shows that T_{n+1} $(n \geq r+s-1)$ is a linear recurrence relation of order $r+s$; more precisely, we have

$$T_{n+1} = (a_0 + b_0) T_n + \sum_{j=0}^{r_1-1} (a_j + b_j - c_j) T_{n-j} + \sum_{j=r_1}^{r_2-1} v_j T_{n-j} - \sum_{j=r_2}^{r+s-1} c_j T_{n-j},$$

where $c_j = \sum_{k+p=j;\, k \geq 1,\, p \geq 0} b_{k-1} a_p$ and $r_1 = \min(r, s)$, $r_2 = \max(r, s)$ with $v_j = a_j - c_j$ for $r > s$, $v_j = b_j - c_j$ for $r < s$, and $v_j = 0$ for $r = s$. Hence, we have the following result.

Theorem 2.1 (Linearization Process): Let $\{T_n\}_{n \geq 0}$ be a sequence (1) and $\{V_n\}_{n \geq 0}$ be a sequence (2), where $m = s$. Suppose $C_n = V_n$, then $\{T_n\}_{n \geq 0}$ is a sequence (2), where $m = r+s$. More precisely, $\{T_n\}_{n \geq 0}$ is a sequence (2) whose initial terms are $T_0, \ldots, T_{r+s-1}$ and whose characteristic polynomial is $p(x) = p_1(x) p_2(x)$, where $p_1(x) = x^r - \sum_{j=0}^{r-1} a_j x^{r-j-1}$ is the characteristic polynomial of the homogeneous part of (1) and $p_2(x) = x^s - \sum_{j=0}^{s-1} b_j x^{s-j-1}$ is the characteristic polynomial of (2).

Let $\sigma_1 = \{\lambda_0, \ldots, \lambda_q\}$ be the set of characteristic roots of the homogeneous part of (1) whose multiplicities are $n_0, \ldots, n_q$, respectively. Then $\sigma = \{v, p(v) = 0\} = \sigma_1 \cup \sigma_2$. Set $\sigma = \{v_0, \ldots, v_k\}$, where $v_i = \mu_i$ for $0 \leq i \leq t$ and $v_{i+t} = \lambda_{i-1}$ for $1 \leq i \leq k-t+1$. If $\sigma_1 \cap \sigma_2 = \emptyset$, we have $k = q + t + 1$, and if not, $k = q + t + 1 - u$, where u is the cardinal of $\sigma_1 \cap \sigma_2$. In the latter case, the Linearization Process shows that the multiplicity of $v_j \in \sigma_1 \cap \sigma_2$ is $m_j = n_j + p_j$, where n_j and p_j are multiplicities of v_j in $p_1(x)$ and $p_2(x)$, respectively. Therefore, we derive the Binet formula of $\{T_n\}_{n \geq 0}$ as

$$T_n = \sum_{j=0}^{t} R_j(n) v_j^n + \sum_{j=1}^{k-t+1} R_{j+t}(n) v_{j+t}^n \tag{4}$$

with $R_j(n) = \sum_{i=0}^{m_j-1} \beta_{ji} n^i$, where m_j is the multiplicity of v_j in $p(x) = p_1(x) p_2(x)$ and β_{ji} are constants derived as solution of a linear system of $r+s$ equations (see, e.g., [9] and [11]).

Because v_j for $t+1 \leq j \leq k$ satisfies $p_1(v_j) = 0$, we show that the sequence $\{T_n^{(h)}\}_{n \geq 0}$ defined by $T_n^{(h)} = \sum_{j=1}^{k-t+1} R_{j+t}(n) v_{j+t}^n$ is a solution of the homogeneous part of (1). Thus, we have the following result.

Theorem 2.2: Let $\{T_n\}_{n \geq 0}$ be a sequence (1) and $\{V_n\}_{n \geq 0}$ a sequence (2), where $m = s$. Suppose $C_n = V_n$, then the sequence $\{T_n^{(p)}\}_{n \geq 0}$, defined by

$$T_n^{(p)} = T_n - T_n^{(h)} = \sum_{j=0}^{t} R_j(n) v_j^n,$$

is a particular solution of (1).

Suppose $v_0 = \mu_0 = 1 \in \sigma_2$, then Binet's formula implies that $V_n = Q_0(n) + \sum_{j=1}^{t} Q_j(n) \mu_j^n$, where $Q_j(n)$ are polynomials in n of degree $p_j -$. Then a solution $\{T_n^{(p)}\}_{n \geq 0}$ of (1) may be expressed as follows: $T_n^{(p)} = T_n^1 + T_n^2$, where $\{T_n^1\}_{n\geq 0}$ and $\{T_n^2\}_{n\geq 0}$ are the solutions of (1) for, $C_n = Q_0(n)$ and $C_n = \sum_{j=1}^{t} Q_j(n)\mu_j^n$, respectively. We call $\{T_n^1\}_{n\geq 0}$ the *polynomial solutions of (1)*, corresponding to the polynomial part of C_n.

Corollary 2.1: Let $\{T_n\}_{n\geq 0}$ be a sequence (1) and $\{V_n\}_{n\geq 0}$ a sequence (2). Suppose $\nu_0 = \mu_0 = 1 \in \sigma_2$. Then the polynomial solution $\{T_n^1\}_{n\geq 0}$ of (1) is given by $T_n^1 = R_0(n)$, where $R_0(n) = \sum_{i=0}^{m_0-1} \beta_{0i} n^i$ is derived from the Binet formula (4) of the linearized expression (3) of $\{T_n\}_{n\geq 0}$. More precisely:

(a) If $1 \notin \sigma_1$, we have $T_n^1 = R_0(n)$ with $R_0(x)$ of degree $m_0 - 1$, where $m_0 = p_0$ is the multiplicity of $\mu_0 = 1$ in $p_2(x)$.

(b) If $1 \in \sigma_1$, we have $T_n^1 = R_0(n)$ with $R_0(x)$ of degree $m_0 - 1 = n_0 + p_0 - 1$, where n_0 and p_0 are multiplicities of $\lambda_0 = \mu_0 = 1$ in $p_1(x)$ and $p_2(x)$, respectively.

Corollary 2.1 shows that the polynomial solution $\{T_n^1\}_{n\geq 0}$ of (1) is nothing but the polynomial part of (4), corresponding to the solution of (1) for C_n, equal to the polynomial part in the Binet decomposition of $\{V_n\}_{n\geq 0}$.

Example 2.1: Let $\{T_n\}_{n\geq 0}$ be a sequence (1) whose initial terms are T_0, T_1, and $T_{n+1} = a_0 T_n + a_1 T_{n-1} + V_n$ for $n \geq 1$, where $\{V_n\}_{n\geq 0}$ is a sequence (2) with $m = s$. Then the Linearization Process implies that $\{T_n\}_{n\geq 0}$ is a sequence (2), where $m = s+2$, whose initial terms are $T_0, \ldots, T_{s+1}$ and whose coefficients are $u_0 = a_0 + b_0$, $u_1 = a_1 + b_1 - a_0 b_0$, $u_2 = b_1 - a_0 b_1 - a_1 b_0, \ldots, u_{s-1} = b_{s-1} - a_0 b_{s-2} - a_1 b_{s-3}$, $u_s = a_0 b_{s-1} - a_1 b_{s-2}$, and $u_{s+1} = -a_1 b_{s-1}$.

3. APPLICATIONS TO POLYNOMIAL AND FACTORIAL POLYNOMIAL CASES

3.1 Polynomial Case

In this subsection we consider $C_n = \sum_{j=0}^{d} \beta_j n^j$, where $n \in \mathbb{N}$. Let us first connect this case with the situation of Section 2. To this aim, we can show easily that if $\{V_n\}_{n\geq 0}$ is a sequence such that $V_n = \sum_{j=0}^{d} \beta_j n^j$, for $n \geq 0$, then $\{V_n\}_{n\geq 0}$ is a sequence (2) with $m = d+1$ whose initial terms are $V_0, \ldots, V_d$ and coefficients $b_j = (-1)^j \binom{d-j}{d+1}$, where $\binom{k}{n} = \frac{n!}{k!(n-k)!}$, are derived from its characteristic polynomial $p_2(x) = (x-1)^{d+1}$. Particularly, for $C_n = n^j$, we derive the following proposition from Corollary 2.1.

Proposition 3.1: Let $\{T_n\}_{n\geq 0}$ be a sequence (1) and let $C_n = n^j$. Then the polynomial solution $\{P_j(n)\}_{n\geq 0}$ of (1) is given by $P_j(n) = R_0(n)$, where $R_0(n) = \sum_{i=0}^{m_0-1} \beta_{0i} n^i$ is derived from the Binet formula (4) of the linearized expression (3) of $\{T_n\}_{n\geq 0}$. More precisely:

(a) If $1 \notin \sigma_1$, we have $P_j(n) = R_0(n)$ with $R_0(x)$ of degree $m_0 - 1 = j$.

(b) If $1 \in \sigma_1$, we have $P_j(n) = R_0(n)$ with $R_0(x)$ of degree $m_0 - 1 = n_0 + j$, where n_0 is the multiplicity of $\lambda_0 = 1$ in $p_1(x)$.

More generally, we have the following result.

Proposition 3.2: Let $\{T_n\}_{n\geq 0}$ be a sequence (1) and let $C_n = \sum_{j=0}^{d} \beta_j n^j$. Then the polynomial solution $\{P(n)\}_{n\geq 0}$ of (1) is $P(n) = R_0(n) = \sum_{j=0}^{d} \beta_j P_j(n)$, where $R_0(n) = \sum_{i=0}^{m_0-1} \beta_{0i} n^i$ is derived from the Binet formula (4) of the linearized expression (3) or $\{T_n\}_{n\geq 0}$. More precisely:

(a) If $1 \notin \sigma_1$, we have $P(n) = R_0(n)$ with $R_0(x)$ of degree d.

(b) If $1 \in \sigma_1$, we have $P(n) = R_0(n)$ with $R_0(x)$ of degree $m_0 - 1 = n_0 + d$, where n_0 is the multiplicity of $\lambda_0 = 1$ in $p_1(x)$.

Propositions 3.1 and 3.2 show that particular polynomial solutions $P_j(n)$ $(0 \le j \le d)$ are the well-known Asveld polynomials studied in [3], [5], [8], and [12]. Our method of obtaining $P_j(n)$ $(0 \le j \le d)$ is different. For their computation, we applied the *Linearization Process* of Section 2 to $\{T_n\}_{n\ge 0}$. Thus, the Binet formula (4) of the linearized expression (3) of (1) allows us to conclude that $P_j(n)$ can be considered as a polynomial part of (4). For $\lambda_0 = 1 \in \sigma_1$, we have $m_0 \ge j+2$, and Proposition 3.1 shows that $P_j(n)$ may be of degree $\ge j+1$ because the α_{0i} are not necessarily null for $j+1 \le i \le m_0 - 1$. This result has been verified by the authors with the aid of another method devised for solving equations (1) for a general C_n.

3.2 Factorial Polynomial Case

In this subsection, let $C_n = \sum_{j=0}^{d} \beta_j n^{(j)}$, where $n^{(j)} = n(n-1)\cdots(n-j+1)$. Note that $n^{(j)} = j!\binom{n}{j}$ for $j \ge 1$ and $n^{(0)} = 1$ $(0^{(0)} = 1)$. This case is related to the situation of Section 2 as follows. Consider Stirling numbers of the first kind $s(t, j)$ and Stirling numbers of the second kind, $S(t, j)$, which are defined by $x^{(j)} = \sum_{t=0}^{j} s(t, j) x^t$ and $x^i = \sum_{t=0}^{i} S(t, i) x^{(t)}$ (see, e.g., [1], [6], [7], and [10]). Hence, for any $j \ge 1$, we have $n^{(j)} = \sum_{t=0}^{j} s(t, j) n^t$. Therefore, $\{n^{(j)}\}_{n\ge 0}$ is a sequence (2), where $s = j+1$. We then derive the following proposition from Proposition 3.2.

Proposition 3.3: Let $\{T_n\}_{n\ge 0}$ be a sequence (1) and $C_n = n^{(j)}$. Then the factorial polynomial solution $\{\widetilde{P}_j(n)\}_{n\ge 0}$ of (1) is given by $\widetilde{P}_j(n) = \widetilde{R}_{0,j}(n)$, where $\widetilde{R}_{0,j}(n) = \sum_{t=0}^{j} s(j, t) P_t(n)$ with $P_t(n) = \sum_{i=0}^{t+m_0-1} \alpha_{0i}^t n^i$ $(0 \le t \le j)$ are solutions of the linearized expression (3) of $\{T_n\}_{n\ge 0}$ for $C_n = n^t$ $(0 \le t \le j)$. More precisely:

(a) If $1 \notin \sigma_1$, we have $\widetilde{P}_j(n) = \sum_{q=0}^{j} (\sum_{t=q}^{j} s(j, t) \gamma_{tq}) n^{(q)}$, where $\gamma_{tq} = \sum_{i=q}^{t} \alpha_{0i}^t S(i, q)$, with $S(i, q)$ the Stirling numbers of the second kind.

(b) If $1 \in \sigma_1$, we have $\widetilde{P}_j(n) = \sum_{q=0}^{j+m_0-1} (\sum_{t=q}^{j+m_0-1} s(j, t) \gamma_{tq}) n^{(q)}$, where $\gamma_{tq} = \sum_{i=q}^{t+m_0-1} \alpha_{0i}^t S(i, q)$, with $m_0 \ge 1$ the multiplicity of $\lambda_0 = 1$.

More generally, we have the following proposition.

Proposition 3.4: Let $\{T_n\}_{n\ge 0}$ be a sequence (1) and $C_n = \sum_{j=0}^{d} \beta_j n^{(j)}$. Then, the factorial polynomial solution $\{\widetilde{P}(n)\}_{n\ge 0}$ of (1) is given by $P(n) = R_0(n) = \sum_{j=0}^{d} \beta_j \widetilde{P}_j(n)$, where $\widetilde{P}_j(n)$ are factorial polynomial solutions of (1) for $C_n = n^{(j)}$ given by Proposition 3.3.

Propositions 3.3 and 3.4 show that particular factorial polynomial solutions $\widetilde{P}_j(n)$ $(0 \le j \le d)$ are the well-known Asveld factorial polynomials studied in [5] and [7]. Our method of obtaining $\widetilde{P}_j(n)$ $(0 \le j \le d)$ is different from those above. As for the polynomial case, if $1 \in \sigma_1$, we can show that $\widetilde{P}_j(n)$ $(0 \le j \le d)$ may be of *degree* $\ge j+1$. This result has also been verified by the authors using another method for solving (1) in the general case.

Example 3.1: Let $\{T_n\}_{n\ge 0}$ be a sequence (1) whose initial conditions are T_0, T_1, and $T_{n+1} = 3T_n - 2T_{n-1} + V_n$ for $n \ge 1$, where $V_n = n$. It is easy to see that $V_{n+1} = 2V_n - V_{n-1}$; therefore, the Linearization Process of Section 2 and Example 2.1 imply that $T_{n+1} = 5T_n - 9T_{n-1} + 7T_{n-2} - 2T_{n-3}$ for $n \ge 3$, where the initial conditions are T_0, T_1, $T_2 = 3T_1 - 2T_0 + 1$, and $T_3 = 7T_1 - 6T_0 + 5$. The characteristic polynomial of $\{T_n\}_{n\geq 0}$ is $p(x) = (x-1)^3(x-2)$. So the Binet formula of $\{T_n\}_{n\geq 0}$ is $T_n = P(n) + \eta 2^n$ for any $n \geq 0$, where $P(n) = an^2 + bn + c$. Also, the coefficients a, b, c, and η are a solution of the linear system of 4 equations, $(S): P(n) + \eta 2^n = T_n$, $n = 0, 1, 2, 3$. A straight computation allows us to verify that (S) is a Cramer system which owns a unique solution a, b, c, and η. In particular, we have $a = -\frac{1}{2}$. Hence, the polynomial solution $\{P(n)\}_{n\geq 0}$ of (1) is of degree 2.

4. CONCLUDING DISCUSSION AND EXAMPLE

4.1 Method of Substitution and Linearization Process

For $C_n = \sum_{j=0}^{d} \beta_j n^j$ (or $C_n = \sum_{j=0}^{d} \beta_j n^{(j)}$), the usual way for searching the particular polynomial (or factorial polynomial) solutions $\{P(n)\}_{n\geq 0}$ (or $\{\widetilde{P}(n)\}_{n\geq 0}$) of (1), and hence the Asveld polynomials (or factorial polynomials), is to consider them in the following form:

$$P(n) = \sum_{j=0}^{d} A_j n^j, \quad \widetilde{P}(n) = \sum_{j=0}^{d} A_j n^{(j)}. \tag{5}$$

Then the coefficients A_j $(0 \leq j \leq d)$ are computed from a series of equations that are obtained from the *substitution* of (5) in (1) (see, e.g., [3], [4], [5], [7], [8], and [12]).

The natural question is: How can we compare the Linearization Process of Section 2 and the method of substitution for searching particular solutions of (1) in polynomial and factorial polynomial cases? The Linearization Process of Section 2 shows that:

(a) If $\lambda_0 = 1 \notin \sigma_1$ [i.e., 1 is not a characteristic root of the homogeneous part of (1)], the Linearization Process shows that $\{P(n)\}_{n\geq 0}$ (or $\{\widetilde{P}(n)\}_{n\geq 0}$) is of the form (5). And the coefficients A_j $(0 \leq j \leq d)$ of (5) are obtained with the aid of the Binet formula applied directly to the linearized expression (3) of (1).

(b) If $\lambda_0 = 1 \in \sigma_1$ [i.e., 1 is a characteristic root of the homogeneous part of (1)], then these solutions may be of degree $\geq d$. More precisely, we have $P(n) = \sum_{j=0}^{d+n_0} A_j n^j$ and $\widetilde{P}(n) = \sum_{j=0}^{d+n_0} A_j n^{(j)}$, where n_0 is the multiplicity of $\lambda_0 = 1 \in \sigma_1$. If $P(n)$, or $\widetilde{P}(n)$, is of degree d, we must have $A_j = 0$ for $d+1 \leq j \leq d+n_0$. This means that we have some constraints on the coefficients $a_0, \ldots, a_{r-1}$, or on the initial terms $T_0, \ldots, T_{r-1}$.

The following simple example helps to make precise the difference between the Linearization Process and the method of substitution.

Example 4.1: Let $\{T_n\}_{n\geq 0}$ be a sequence (1) whose initial terms are T_0, T_1, and $T_{n+1} = a_0 T_n + a_1 T_{n-1} + V_n$ for $n \geq 1$, where $a_0 = 1 - \alpha$, $a_1 = \alpha$ with $\alpha \neq 1$, and $V_n = n$. Then we can see that $V_{n+1} = 2V_n - V_{n-1}$. Hence, the Linearization Process of Section 2 implies that

$$T_{n+1} = (3-\alpha)T_n + 3(\alpha - 1)T_{n-1} - (3\alpha - 1)T_{n-2} + \alpha T_{n-3} \text{ for } n \geq 3,$$

where initial terms are T_0, T_1, $T_2 = (1-\alpha)T_1 + \alpha T_0 + 1$, and $T_3 = (\alpha^2 - \alpha + 1)T_1 + \alpha(1-\alpha)T_0 + (3-\alpha)$. The characteristic polynomial of $\{T_n\}_{n\geq 0}$ is $p(x) = (x-1)^3(x+\alpha)$, and its Binet formula is $T_n = P(n) + \eta\lambda_1^n$ for $n \geq 0$, where $P(n) = an^2 + bn + c$ and $\lambda_1 = -\alpha$. The coefficients a, b, c, and η are derived from the following linear system 4 equations $(S): P(2) + \eta\lambda_1^j = T_j$, $j = 0, 1, 2$, and 3. A straight computation allows us to see that (S) is a Cramer system which owns a unique solution a, b, c, and η if $\Delta_\alpha = 2\alpha^3 + 6\alpha^2 + 6\alpha - 2 \neq 0$. In particular, we have $a = (\alpha+1)^2 / \Delta_\alpha$. Therefore, the polynomial solution $\{P(n)\}_{n\geq 0}$ of (1) is of degree 1 if $\alpha = -1$, and of degree 2 if not.

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AMS Classification Numbers: 40A05, 40A25, 45M05

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2002. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-930 ***Proposed by José Luis Díaz and Juan José Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain***

Let $n \geq 0$ be a nonnegative integer. Prove that $F_n^{L_n} L_n^{F_n} \leq (F_{n+1}^{F_{n+1}})^2$. When does equality occur?

B-931 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan***

Prove that $\gcd(L_n, F_{n+1}) = 1$ for all $n \geq 0$.

B-932 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan***

Prove that

A) $\dfrac{F_2 F_4 \ldots F_{2n}}{F_1 F_3 \ldots F_{2n+1}} < \dfrac{1}{\sqrt{F_{2n+1}}}$ for all $n \geq 1$

and

B) $\displaystyle\sum_{k=1}^{\infty} \frac{F_2 F_4 \ldots F_{2k}}{F_1 F_3 \ldots F_{2k+1}}$ converges.

B-933 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan***

Prove that $F_n^{F_{n+1}} > F_{n+1}^{F_n}$ for all $n \geq 4$.

B-934 ***Proposed by N. Gauthier, Royal Military College of Canada***

Prove that

$$2\sum_{n=1}^{m}\sin^2\left(\frac{\pi}{2}\frac{F_{m+1}}{F_mF_nF_{n+1}}\right)\sin\left(\pi\frac{F_nF_{m+1}}{F_mF_{n+1}}\right)=\sum_{n=1}^{m}(-1)^n\sin\left(\pi\frac{F_{m+1}}{F_mF_nF_{n+1}}\right)\cos\left(\pi\frac{F_nF_{m+1}}{F_{n+1}F_m}\right),$$

where m is a positive integer.

Note: *The Elementary Problems Editor, Dr. Russ Euler, is in need of more easy yet elegant and nonroutine problems. Due to space limitations, brevity is also important.*

SOLUTIONS

Subscript Is Power

B-916 ***Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria (Vol. 39, no. 2, May 2001)***

Determine the value of

$$\prod_{k=0}^{n}(L_{2\cdot 3^k}-1).$$

Solution 1 by H.-J. Seiffert, Berlin, Germany

In equation (2.6) of [1], it was shown that

$$\prod_{k=1}^{n}(L_{2\cdot 3^k}-1)=\frac{1}{2}F_{3^{n+1}}.$$

Multiplying by $L_2-1=2$ gives

$$\prod_{k=0}^{n}(L_{2\cdot 3^k}-1)=F_{3^{n+1}}.$$

Reference

1. P. Filipponi. "On the Fibonacci Numbers whose Subscript Is a Power." *The Fibonacci Quarterly* **34.3** (1996):271-76.

Solution 2 by Paul S. Bruckman, Sacramento, CA

For brevity, write $3^k=u$, $A_k\equiv F_u$. Note that $A_{k+1}/A_k=F_{3u}/F_u=(\alpha^{3u}-\beta^{3u})/(\alpha^u-\beta^u)=\alpha^{2u}+\alpha^u\beta^u+\beta^{3u}=L_{2u}-1$, since u is odd. Let P_n denote the given product. Then

$$P_n=\prod_{k=0}^{n}(A_{k+1}/A_k),$$

a telescoping product that reduces to A_{n+1}/A_0. Therefore, since $A_0=F_1=1$, it follows that $P_n=F_v$, where $v=3^{n+1}$.

Also solved by Brian Beasley, Kenneth Davenport, L. A. G. Dresel, Steve Edwards, Ovidiu Furdui, Russell Hendel, Walther Janous, Reiner Martin, Don Redmond, Maitland Rose, Jaroslav Seibert, and the proposer.

The solvers all used almost the same induction argument to show the result.

A Two Sum Problem

B-917 ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain (Vol. 39, no. 2, May 2001)***

Find the following sums:

$$(a)\ \sum_{n\geq 0}\frac{1+L_{n+1}}{L_nL_{n+2}}, \qquad (b)\ \sum_{n\geq 0}\frac{L_{n-1}L_{n+2}}{L_n^2L_{n+1}^2},$$

where L_k is the k^{th} Lucas number.

Solution by Reiner Martin, New York, NY

Note that

$$\sum_{n=0}^{m}\frac{1}{L_nL_{n+2}}=\sum_{n=0}^{m}\left(\frac{1}{L_nL_{n+1}}-\frac{1}{L_{n+1}L_{n+2}}\right)=\frac{1}{2}-\frac{1}{L_{m+1}L_{m+2}}$$

and

$$\sum_{n=0}^{m}\frac{L_{n+1}}{L_nL_{n+2}}=\sum_{n=0}^{m}\left(\frac{1}{L_n}-\frac{1}{L_{n+2}}\right)=\frac{3}{2}-\frac{1}{L_{m+1}}-\frac{1}{L_{m+2}}.$$

So we can evaluate (a) as

$$\sum_{n\geq 0}\frac{1+L_{n+1}}{L_nL_{n+2}}=\frac{1}{2}+\frac{3}{2}=2.$$

To find (b), observe that

$$\sum_{n=0}^{m}\frac{L_{n-1}L_{n+2}}{L_n^2L_{n+1}^2}=\sum_{n=0}^{m}\left(\frac{1}{L_n^2}-\frac{1}{L_{n+1}^2}\right)=\frac{1}{4}-\frac{1}{L_{m+1}^2},$$

which converges to 1/4.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Steve Edwards, Ovidiu Furdui, Russell Hendel, Walther Janous, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

Divisible or Not Divisible; That Is, by 2

B-918 ***Proposed by M. N. Deshpande, Institute of Science, Nagpur, India (Vol. 39, no. 2, May 2001)***

Let i and j be positive integers such that $1\leq j\leq i$. Let

$$T(i,j)=F_jF_{i-j+1}+F_jF_{i-j+2}+F_{j+1}F_{i-j+1}.$$

Determine whether or not

$$\underset{j}{\text{maximum}}\ T(i,j)-\underset{j}{\text{minimum}}\ T(i,j)$$

is divisible by 2 for all $i\geq 3$.

Solution by L. A. G. Dresel, Reading, England

Since $F_j+F_{j+1}=F_{j+2}$, we have $T(i,j)=F_{j+2}F_{i-j+1}+F_jF_{i-j+2}$, and therefore

$$T(i,j+1)=F_{j+3}F_{i-j}+F_{j+1}F_{i-j+1}\equiv F_jF_{i-j}+F_{j+1}F_{i-j+1}\ (\text{mod } 2),$$

since $F_j \equiv F_{j+3} \pmod 2$. Hence,

$$T(i,j)-T(i,j+1) \equiv (F_{j+2}-F_{j+1})F_{i-j+1}+F_j(F_{i-j+2}-F_{i-j}) = 2F_jF_{i-j+1} \equiv 0 \pmod 2.$$

It follows that, for a given i, $T(i,j)$ modulo 2 is independent of j, and therefore the difference between the given maximum and minimum is divisible by 2 for each i. We note that

$$T(i,j) \equiv T(i,1) \equiv F_{i+1} \pmod 2$$

for all j.

Also solved by Paul S. Bruckman, Walther Janous, Reiner Martin, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

A Prime Equation

B-919 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France (Vol. 39, no. 2, May 2001)***

Solve the equation $L_nF_{n+1} = p^m(p^m-1)$, where m and n are natural numbers and p is a prime number.

Solution by Jaroslav Seibert, Hradec Králové, The Czech Republic

First, we will prove that L_n and F_{n+1} are relatively prime numbers for each natural number n. Suppose there exists a prime q such that it divides the numbers L_n and F_{n+1} for some n. It is known that $L_nF_{n+1} = F_{2n+1}+(-1)^n = F_{n+1}L_{n+1}-F_nL_n+(-1)^n$ (see S. Vajda, *Fibonacci and Lucas Numbers & the Golden Section*, pp. 25, 36). Since the prime q divides $F_{n+1}L_{n+1}-F_nL_n$, it cannot divide F_nL_{n+1}, which is a contradiction. Further, it is easy to see that $F_{n+1} \le L_n \le 2F_{n+1}$, because $L_n = F_{n+1}+F_{n-1}$. To solve the given equation, it must be $F_{n+1} = p^m-1$ and $L_n = p^m$, which means $L_n - F_{n+1} = 1$. Then

$$\alpha^n+\beta^n-\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} = 1,$$

$$\alpha^{n+1}-\alpha^n\beta+\alpha\beta^n-\beta^{n+1}-\alpha^{n+1}+\beta^{n+1} = \alpha-\beta,$$

$$\alpha^{n-1}-\beta^{n-1} = \alpha-\beta,$$

$$\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta} = 1.$$

But $F_{n-1} = 1$ holds only for $n-1=1$, $n-1=2$, and $n-1=-1$ if we admit Fibonacci numbers with negative indices.

The given equation has only two solutions for natural numbers n, m. Concretely, if $n=2$, $m=1$, $p=3$, then $L_2F_3 = 3^1(3^1-1)$; if $n=3$, $m=2$, $p=2$, then $L_3F_4 = 2^2(2^2-1)$. In addition, if $n=0$, $m=1$, $p=2$, the equality $L_0F_1 = 2^1(2^1-1)$ also holds.

Florian Luca commented that the equation $L_nF_{n+1} = x(x-1)$, where n and x are nonnegative integers is a more general equation. He proved in one of his forthcoming papers that it has solutions at $n = 0$, 2, and 3 as well.

Also solved by Brian Beasley, Paul S. Bruckman, L. A. G. Dresel, Ovidiu Fordui, Walther Janous, Florian Luca, H.-J. Seiffert, and the proposer.

A Trigonometric Sum

B-920 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 39, no. 2, May 2001)***

Prove that

$$\sum_{n=1}^{\infty} \sin\left(\frac{p\pi}{2} \cdot \frac{F_{n-1}}{F_n F_{n+1}}\right) \cos\left(\frac{p\pi}{2} \cdot \frac{F_{n+2}}{F_n F_{n+1}}\right) = 0$$

for p an arbitrary integer.

Solution by Steve Edwards, Marietta, GA

$$\sum_{n=1}^{\infty} \sin\left(\frac{p\pi}{2} \cdot \frac{F_{n-1}}{F_n F_{n+1}}\right) \cos\left(\frac{p\pi}{2} \cdot \frac{F_{n+2}}{F_n F_{n+1}}\right) = \sum_{n=1}^{\infty} \sin\left(\frac{p\pi}{2} \cdot \frac{F_{n+1}-F_n}{F_n F_{n+1}}\right) \cos\left(\frac{p\pi}{2} \cdot \frac{F_{n+1}+F_n}{F_n F_{n+1}}\right)$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{p\pi}{2} \cdot \left(\frac{1}{F_n} - \frac{1}{F_{n+1}}\right)\right) \cos\left(\frac{p\pi}{2} \cdot \left(\frac{1}{F_n} + \frac{1}{F_{n+1}}\right)\right).$$

Now we can use the trig identity $\sin\frac{a-b}{2}\cos\frac{a+b}{2} = \frac{1}{2}(\sin a - \sin b)$ to get

$$\frac{1}{2}\sum_{n=1}^{\infty} \sin\frac{p\pi}{F_n} - \sin\frac{p\pi}{F_{n+1}}.$$

But this is a telescoping sum. Since $\frac{1}{F_n} \to 0$ as $n \to \infty$, only $\frac{1}{2}\sin\frac{p\pi}{F_1}$ survives, and this is 0 for any integer p.

Some solvers noted that the result is true for p any arbitrary complex number.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, L. A. G. Dresel, Ovidiu Fordui, Toufik Mansour, Reiner Martin, Maitland Rose, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

We wish to belatedly acknowledge the solution to Problem B-901 by Charles K. Cook, and the solution to Problem B-911 by Lake Superior State University Problem Solving Group.

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Announcement

TENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

June 24–June 28, 2002

Northern Arizona University, Flagstaff, Arizona

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CALL FOR PAPERS

The purpose of the conference is to bring together people from all branches of mathematics and science who are interested in Fibonacci numbers, their applications and generalizations, and other special number sequences. For the conference *Proceedings*, manuscripts that include new, unpublished results (or new proofs of known theorems) will be considered. A manuscript should contain an abstract on a separate page. For papers not intended for the *Proceedings*, authors may submit just an abstract, describing new work, published work or work in progress. Papers and abstracts, which should be submitted in duplicate to F. T. Howard at the address below, are due no later than May 1, 2002. Authors of accepted submissions will be allotted twenty minutes on the conference program. Questions about the conference may be directed to:

Professor F. T. Howard
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Box 7388 Reynolda Station
Winston-Salem, NC 27109 (U.S.A.)
e-mail: howard@mthcsc.wfu.edu

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-581 ***Proposed by José Luiz Díaz, Polytechnic University of Catalunya, Spain***

Let n be a positive integer. Prove that

(a) $F_n^{F_{n+1}} + F_{n+1}^{F_{n+2}} + F_{n+2}^{F_n} < F_n^{F_n} + F_{n+1}^{F_{n+1}} + F_{n+2}^{F_{n+2}}$,

(b) $F_n^{F_{n+1}} F_{n+1}^{F_{n+2}} F_{n+2}^{F_n} < F_n^{F_n} F_{n+1}^{F_{n+1}} F_{n+2}^{F_{n+2}}$.

H-582 ***Proposed by Ernst Herrmann, Siegburg, Germany***

a) Let A denote the set $\{2, 3, 5, 8, \ldots, F_{m+2}\}$ of m successive Fibonacci numbers, where $m \geq 4$. Prove that each real number x of the interval $I = [(F_{m+2} - 1)^{-1}, 1]$ has a series representation of the form

$$x = \sum_{i=1}^{\infty} \frac{1}{F_{k_1} F_{k_2} F_{k_3} \ldots F_{k_i}}, \tag{1}$$

where $F_{k_i} \in A$ for all $i \in N$.

b) It is impossible to change the assumption $m \geq 4$ into $m \geq 3$, that is, if $A = \{2, 3, 5\}$ and $I = [1/4, 1]$, then there are real numbers with no representation of the form (1), where $F_{k_i} \in A$. Find such a number.

SOLUTIONS

Inspiring

H-568 ***Proposed by N. Gauthier, Royal Military College of Canada, Kingston, Ontario (Vol. 38, no. 5, November 2000)***

The following was inspired by Paul. S. Bruckman's Problem B-871 in *The Fibonacci Quarterly* (proposed in Vol. 37, no. 1, February 1999; solved in Vol. 38, no. 1, February 2000).

"For integers $n, m \geq 1$, prove or disprove that

$$f_m(n) \equiv \frac{1}{\binom{2n}{n}^2} \sum_{k=0}^{2n} \binom{2n}{n}^2 |n-k|^{2m-1}$$

is the ratio of two polynomials with integer coefficients $f_m(n) = P_m(n)/Q_m(n)$, where $P_m(n)$ is of degree $\lfloor \frac{3m}{2} \rfloor$ in n and $Q_m(n)$ is of degree $\lfloor \frac{m}{2} \rfloor$; determine $P_m(n)$ and $Q_m(n)$ for $1 \le m \le 5$."

Solution by Paul S. Bruckman, Sacramento, CA

We let the combinatorial number read "a choose b" be denoted by the symbol ${}_aC_b$. After a bit of manipulation, we may express $f_m(n)$ as follows:

$$({}_{2n}C_n)^2 f_m(n) = 2\sum_{k=0}^{n} ({}_{2n}C_k)^2 (n-k)^{2m-1} = 2g_m(n), \text{ say.}$$

That is,

$$g_m(n) = \sum_{k=0}^{n} ({}_{2n}C_k)^2 (n-k)^{2m-1}. \tag{1}$$

For convenience, we make the following definitions:

$$A_r(n) = \sum_{k=0}^{n} ({}_{2n}C_k)^2 (n-k)^r; \tag{2}$$

$$B_r(n) = \sum_{k=0}^{n} ({}_{2n+1}C_k)^2 (n-k)^r; \tag{3}$$

$$U(n) = {}_{4n}C_{2n}; \; V(n) = ({}_{2n}C_n)^2. \tag{4}$$

Note that $g_m(n) = A_{2m-1}(n)$ and $f_m(n) = 2A_{2m-1}(n)/V(n)$. Also note that

$$U(n-1) = n(2n-1)U(n)/\{2(4n-1)(4n-3)\}, \; V(n-1) = n^2V(n)/\{4(2n-1)^2\}.$$

The following combinatorial identities are either directly found or easily derived from identities given in [1]; in some cases, their derivation is a bit lengthy, and is therefore abridged here:

$$A_0(n) = 1/2\{U(n)+V(n)\} \text{ (Identity (3.68) in [1]);} \tag{5}$$

$$B_0(n) = (4n+1)U(n)/(2n+1) \text{ (Identity (3.69) in [1]);} \tag{6}$$

$$A_2(n) = n^2U(n)/2(4n-1) \text{ (Identity (3.76) in [1] with } 2n \text{ replacing } n\text{);} \tag{7}$$

$$A_1(n) = nV(n)/4. \tag{8}$$

Proof of (8): The summand portion $(n-k)$ in $A_1(n)$ is equal to $(n^2-k^2+(n-k)^2)/2n$. Thus, after simplification,

$$\begin{aligned} 2nA_1(n) &= n^2A_0(n) - 4n^2B_0(n-1) + A_2(n) \\ &= n^2(U(n)+V(n))/2 - 2n^3U(n)/(4n-1) + n^2U(n)/(4n-1)4n^2, \end{aligned}$$

which reduces to (8). □

$$\sum_{k=0}^{n} ({}_{2n+1}C_k)^2(2n+1-2k)^2 = (2n+1)U(n) \text{ (Identity (3.76) in [1], with } 2n+1 \text{ replacing } n\text{).} \tag{9}$$

Now the summand portion $(n-k)$ in $B_1(n)$ may be written as

$$\{4n^2-1+(2n+1-2k)^2-4k^2\}/\{4(2n+1)\}.$$

It then follows that

$$4(2n+1)B_1(n) = (4n^2-1)B_0(n) + C(n) - 4(2n+1)^2(A_0(n) - V(n)),$$

where $C(n)$ is the expression given in the left member of (9). Then, after simplification, we obtain the following:

$$B_1(n) = (2n+1)V(n)/2 - (4n+1)U(n)/\{2(2n+1)\}. \tag{10}$$

Next, we note that

$$(n-k)^3 = (n-k)^2\{(2n-k)^2 - k^2\}/4n.$$

Then, using the above definitions, we see that

$$4nA_3(n) = 4n^2\{B_2(n-1) + 2B_1(n-1) + B_0(n-1)\} - 4n^2B_2(n-1);$$

hence,

$$A_3(n) = 2nB_1(n-1) + nB_0(n-1).$$

After further simplification, we obtain

$$A_3(n) = n^3V(n)/\{4(2n-1)\}. \tag{11}$$

From the definitions given in (1) and (2), along with the relation $f_m(n) = 2g_m(n)/V(n)$, and using the results of (8) and (11) we therefore have

$$f_1(n) = n/2, \quad f_2(n) = n^3/\{2(2n-1)\}. \tag{12}$$

We may prove Gauthier's conjecture by induction (on m). However, due to considerations of length, we can only outline the procedure and omit the details. The required tool for the proof is the following recurrence satisfied by the $f_m(n)$'s:

$$f_{m+2}(n) = 2n^2 f_{m+1}(n) - n^4 f_m(n) + n^4 f_m(n-1). \tag{13}$$

Proof of (13):

$$V(n)\{f_{m+2}(n) - 2n^2 f_{m+1}(n) + n^4 f_m(n)\}/2$$
$$= \sum_{k=0}^{n} ({}_{2n}C_k)^2 (n-k)^{2m-1}\{(n-k)^4 - 2n^2(n-k)^2 + n^4\}.$$

Note that the quantity in braces is equal to $\{(n-k)^2 - n^2\}^2 = k^2(2n-k)^2$; therefore, the last summation may be expressed as follows:

$$(2n)^2(2n-1)^2 \sum_{k=1}^{n} ({}_{2n-2}C_{k-1})^2 (n-k)^{2m-1}$$
$$= 4n^2(2n-1)^2 \sum_{k=0}^{n-1} ({}_{2n-2}C_k)^2 (n-1-k)^{2m-1}$$
$$= 4n^2(2n-1)^2 f_m(n-1)V(n-1)/2 = n^4 f_m(n-1)V(n)/2,$$

which reduces to (13). □

Instead of applying induction directly on (13), we transform this recurrence and apply it to a modified set of functions. Namely, we make the following transformation:

$$k_m(n) = f_m(n)T_m(n), \tag{14}$$

where

$$T_m(n) = 2^{r+1}(n-1/2)^{(r)} = 2(2n-1)(2n-3)\dots(2n-2r+1) \tag{15}$$

and

$$r = [m/2]. \tag{16}$$

Therefore, the $T_m(n)$'s are polynomials in n of degree r. By making the substitution indicated in (14) into the recurrence (13), we obtain our modified recurrence relation. It becomes more convenient to dichotomize this relation into the two cases $m = 2r$ and $m = 2r+1$:

$$k_{2r+2}(n) - 2n^2(2n-1-2r)k_{2r+1}(n) + n^4(2n-1-2r)k_{2r}(n) = n^4(2n-1)k_{2r}(n-1), \tag{17}$$

$$k_{2r+3}(n) - 2n^2k_{2r+2}(n) + n^4(2n-1-2r)k_{2r+1}(n) = n^4(2n-1)k_{2r+1}(n-1). \tag{18}$$

From (12) and the relation in (14), we obtain the initial values

$$k_1(n) = n,\ k_2(n) = n^3,\ k_3(n) = n^4,\ k_4(n) = 3n^6 - 5n^5 + n^4. \tag{19}$$

It follows (by an easy induction) from (17), (18), and (19) that the $k_m(n)$'s are polynomials in n with integer coefficients.

The following results are posited:

$$k_{2r}(n) = a_{2r}n^{3r} + R_{3r-1}(n),\ k_{2r-1}(n) = a_{2r-1}n^{3r-2} + R_{3r-3}(n); \tag{20}$$

$$a_{2r} = (2r-1)!/2^{r-1},\ a_{2r-1} = (2r-2)!/2^{r-1}. \tag{21}$$

In these formulas, the functions $R_M(n)$ are polynomials in n of degree M. In order to prove (20) and (21), we must first verify that they yield the correct values for $r = 1$ and $r = 2$. Using (19), we find that $a_1 = a_2 = a_3 = 1$, and $a_4 = 3$, thereby validating (20)-(21) for $r = 1$ and $r = 2$. If we apply the recurrence (17) to find $k_{2r+2}(n)$, expand each expression using the putative expressions in (20)-(21), and compare coefficients, we find that the coefficients of n^{3r+5} and n^{3r+4} vanish, while the coefficient of n^{3r+3} is found by the first formula in (21) with $r+1$ replacing r. This establishes the first half of the inductive step.

Then applying (18) to obtain $k_{2r+3}(n)$ and repeating the process, we find that the coefficients of n^{3r+6} and n^{3r+5} vanish, while the coefficient of n^{3r+4} is found by the second formula in (21) with $r+1$ replacing r. This establishes the second half of the inductive step. This is essentially equivalent to Gauthier's conjecture, with the added bonus of an expression for the leading term of $k_m(n)$.

Note that the degree of $k_{2r}(n)$ is $3r$, while the degree of $k_{2r-1}(n)$ is $3r-2$; this fact may be expressed concisely as follows: the degree of $k_m(n)$ is $[3m/2]$.

Having established (20)-(21), we may then revert to the original definitions. That is, we may express $f_m(n)$ as the ratio $P_m(n)/Q_m(n)$ of two polynomials with integer coefficients, where

$$P_m(n) = k_m(n) \text{ and } Q_m(n) = 2^r(n-1/2)^{(r)} = 2(2n-1)(2n-3)\dots(2n-2r+1),$$

with $r = [m/2]$. Thus, the degree of $P_m(n)$ is $[3m/2] = m+r$, while the degree of $Q_m(n)$ is r. This completes the demonstration of Gauthier's conjecture.

It only remains to fulfill the last part of the problem, namely, to display the functions $f_m(n)$ for $m = 1, 2, 3, 4, 5$. Since we already know that $f_m(n) = P_m(n)/Q_m(n)$, where

$$Q_m(n) = 2(2n-1)(2n-3)\dots(2n-2r+1),$$

it suffices to display the first few values of $P_m(n)$. As we have already shown,

$$P_1(n) = n,\ \ P_2(n) = n^3,\ \ P_3(n) = n^4,\ \ P_4(n) = n^4(3n^2-5n+1).$$

Continuing by means of (17) and (18), we find the following:

$$P_5(n) = n^4(6n^3-12n^2+6n-1),$$

$$P_6(n) = n^4(30n^5-150n^4+252n^3-185n^2+65n-9),$$

$$P_7(n) = n^4(90n^6-510n^5+1074n^4-1128n^3+650n^2-198n+25),$$

$$P_8(n) = n^4(630n^8-6300n^7+24990n^6-52200n^5 \\ +64506n^4-49356n^3+23111n^2-6087n+691),$$

etc.

By means of a little program names Derive, the author obtained the expanded expressions for $P_r(n)$ from $r=1$ to $r=15$. These are available upon request. It would be desirable to identify the "Gauthier polynomials" $P_r(n)$ with more familiar polynomials already appearing in the literature, whose properties may already be known.

Reference

1. H. W. Gould. *Combinatorial Identities.* Morgantown, W. Va., 1972.

A High Exponent

H-569 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 38, no. 5, November 2000)

Let $\tau(n)$ and $\sigma(n)$ denote, respectively, the number of divisors of the positive integer n and the sum of such divisors. Let $e_2(n)$ denote the highest exponent of 2 dividing n. Let p be any odd prime, and suppose $e_2(p+1)=h$. Prove the following for all odd positive integers a:

$$e_2(\sigma(p^a)) = e_2(\tau(p^a))+h-1. \tag{$*$}$$

Solution by H.-J. Seiffert, Berlin, Germany

If m and n are any positive integers, then

$$e_2(m)=0 \text{ if } m \text{ is odd},\ \ e_2(mn)=e_2(m)+e_2(n), \\ e_2(m+n)=\min(e_2(m),e_2(n)) \text{ if } e_2(m)\neq e_2(n). \tag{1}$$

Let h and q be positive integers such that q is odd and $2^hq-1>1$. We consider the positive integers

$$A_{k,m} := \sum_{j=0}^{2^k m-1}(2^hq-1)^j = \frac{(2^hq-1)^{2^k m}-1}{2^hq-2},$$

where k is any positive integer and m any odd positive integer. First, we prove that

$$e_2(A_{1,m}) = h \text{ for all odd } m\in N. \tag{2}$$

Since $A_{1,1}=2^hq$, this is true for $m=1$. Suppose that (2) holds for the odd positive integer m. Using the easily verified equation

$$A_{1,m+2} = (2^h q - 1)^4 A_{1,m} + 2^{h+1} q(2^{2h-1} q^2 - 2^h q + 1),$$

from (1) and the induction hypothesis, we obtain $e_2(A_{1,m+2}) = h$, so that (2) is established by induction. Next, we prove that, if $m \in N$ is odd, then

$$e_2(A_{k,m}) = k + h - 1 \text{ for all } k \in N. \tag{3}$$

By (2), this is true for $k = 1$. Suppose that (3) holds for $k \in N$. We have

$$A_{k+1,m} = A_{k,m}((2^h q - 1)^{2^k m} + 1) = 2A_{k,m}((2^{h-1} q - 1)A_{k,m} + 1),$$

so that, by (1) and the induction hypothesis, $e_2(A_{k+1,m}) = e_2(A_{k,m}) + 1 = k + h$. This completes the induction proof of (3).

Let p, a, and h be as in the proposal. Then there exist positive integers k, q, and m such that q and m are both odd, $p = 2^h q - 1 > 1$, and $a = 2^k m - 1$. Noting that

$$\sigma(p^a) = 1 + p + \cdots + p^a = \frac{p^{a+1} - 1}{p - 1} = A_{k,m}$$

and

$$\tau(p^a) = a + 1 = 2^k m,$$

we see that the requested equation $(*)$ is an immediate consequence of (3).

Also solved by L. A. G. Dresel, D. Iannucci, H. Kwong, R. Martin, J. Spilkes, and the proposer.

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Please send in proposals!!!

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