# The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION


## TABLE OF CONTENTS




VOLUME 40 MAY 2002 NUMBER 2

## PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

## EDITORIAL POLICY

**THE FIBONACCI QUARTERLY** seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

## SUBMITTING AN ARTICLE

Articles should be submitted using the format of articles in any current issues of **THE FIBONACCI QUARTERLY.** They should be typewritten or reproduced typewritten copies, that are clearly readable, double spaced with wide margins and on only one side of the paper. The full name and address of the author must appear at the beginning of the paper directly under the title. Illustrations should be carefully drawn in India ink on separate sheets of bond paper or vellum, approximately twice the size they are to appear in print. Since the Fibonacci Association has adopted $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \geq 2$ and $L_1=1$, $L_2=3$, $L_{n+1} = L_n + L_{n-1}$, $n \geq 2$ as the standard definitions for The Fibonacci and Lucas sequences, these definitions ***should not*** be a part of future papers. However, the notations ***must*** be used. One to three ***complete*** A.M.S. classification numbers ***must*** be given directly after references or on the bottom of the last page. **Papers not satisfying all of these criteria will be returned.** See the new worldwide web page at:

http://www.sdstate.edu/~wcsc/http/fibhome.html

for additional instructions.

Three copies of the manuscript should be submitted to: **CURTIS COOPER, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, CENTRAL MISSOURI STATE UNIVERSITY, WARRENSBURG, MO 64093-5045.**

Authors are encouraged to keep a copy of their manuscripts for their own files as protection against loss. The editor will give immediate acknowledgment of all manuscripts received.

The journal will now accept articles via electronic services. However, electronic manuscripts must be submitted using the typesetting mathematical wordprocessor AMS-TeX. Submitting manuscripts using AMS-TeX will speed up the refereeing process. AMS-TeX can be downloaded from the internet via the homepage of the American Mathematical Society.

## SUBSCRIPTIONS, ADDRESS CHANGE, AND REPRINT INFORMATION

Address all subscription correspondence, including notification of address change, to: **PATTY SOLSAA, SUBSCRIPTIONS MANAGER, THE FIBONACCI ASSOCIATION, P.O. BOX 320, AURORA, SD 57002-0320. E-mail: solsaap@itctel.com.**

Requests for reprint permission should be directed to the editor. However, general permission is granted to members of The Fibonacci Association for noncommercial reproduction of a limited quantity of individual articles (in whole or in part) provided complete reference is made to the source.

Annual domestic Fibonacci Association membership dues, which include a subscription to **THE FIBONACCI QUARTERLY**, are $40 for Regular Membership, $50 for Library, $50 for Sustaining Membership, and $80 for Institutional Membership; foreign rates, which are based on international mailing rates, are somewhat higher than domestic rates; please write for details. **THE FIBONACCI QUARTERLY** is published each February, May, August and November.

All back issues of **THE FIBONACCI QUARTERLY** are available in microfilm or hard copy format from **ProQuest INFORMATION & LEARNING, 300 NORTH ZEEB ROAD, P.O. BOX 1346, ANN ARBOR, MI 48106-1346.** Reprints can also be purchased from ProQuest at the same address.

# The Fibonacci Quarterly

***Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)***
***and Br. Alfred Brousseau (1907-1988)***

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION*
*DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES*



Fibonacci Association Web Page Address: **http://www.MSCS.dal.ca/Fibonacci/**

# MORGAN-VOYCE CONVOLUTIONS


**A. F. Horadam**

The University of New England, Armidale, Australia 2351



*(Submitted December 1999-Final Revision August 2000)*


## 1. GENERATING FUNCTIONS

### Morgan-Voyce Generators

Much has been written lately about the four Morgan-Voyce polynomials $B_n(x)$, $b_n(x)$, $C_n(x)$, and $c_n(x)$. Basic properties of these polynomials are developed in [2], which contains appropriate reference material.

The main purpose of this paper is to investigate the simplest features of the convolutions of the Morgan-Voyce polynomials and their corresponding numbers occurring when $x = 1$. Our Morgan-Voyce polynomials are defined [2] in terms of generating functions thus:

$$\sum_{n=1}^{\infty} B_n(x)y^{n-1} = [1-\overline{(2+x)y-y^2}]^{-1} = g, \; B_0(x) = 0, \tag{1.1}$$

$$\sum_{n=0}^{\infty} C_n(x)y^n = [2-(2+x)y]g, \tag{1.2}$$

$$\sum_{n=1}^{\infty} b_{n-1}(x)y^{n-1} = [1-(1+x)y]g, \tag{1.3}$$

$$\sum_{n=0}^{\infty} c_n(x)y^n = [-1+(3+x)y]g, \tag{1.4}$$

where, in (1.1)-(1.4), the functional notation $g(x, y) \equiv g$ has been dropped in the interest of simplicity. So, $g$ (1.1) may be said to be the "single parent" progenitor of the family (1.2)-(1.4)!

Partial differentiation with respect to $x$ (Section 5), which is a second feature of this paper, provides us with deeper insights into the essential nature of the polynomials. Two related papers could indeed have evolved from this paper but it is thought more desirable to preserve unity and cohesiveness.

### Motivation

Stimuli for pursuing this investigation are:

(i) in mountaineering language, "it [the challenge] is there!" and

(ii) it increases our knowledge of convolution analysis beyond that already established for other well-known polynomials.

### Initial Conditions

All the convolution number sequences displayed in (2.2a), (2.3a); (3.2a), (3.3a); (4.2a), (4.3a); (5.2a), (5.3a) for $B_n^{(k)}(x), C_n^{(k)}(x), b_n^{(k)}(x), c_n^{(k)}(x)$, respectively, when $k = 1, 2$, have been checked against those obtainable from the general formulas in [1] which were determined by means of Cauchy products. This signifies that our generating function definitions must, when $x = 1$, produce exactly the same two initial numbers of each sequence as are specified in [1].

## 2. CONVOLUTIONS FOR $B_n(x)$

**Definitions**

The $k^{th}$ *convolution polynomials* $B_n^{(k)}(x)$ of $B_n(x)$ are defined by

$$\sum_{n=1}^{\infty} B_n^{(k)}(x)y^{n-1} = g^{k+1}, \; B_0^{(k)}(x)=0, \tag{2.1}$$

$$= \left(\sum_{n=1}^{\infty} B_n(x)y^{n-1}\right)^{k+1}, \tag{2.1a}$$

so that $B_n^{(0)}(x) \equiv B_n(x)$.

Correspondingly, the $k^{th}$ *convolution numbers* $B_n^{(k)}(1) \equiv B_n^{(k)}$ arise in the special case when $x=1$.

**Examples**

$k = 1$

$$\begin{aligned} &B_1^{(1)}(x)=1,\; B_2^{(1)}(x)=4+2x,\; B_3^{(1)}(x)=10+12x+3x^2, \\ &B_4^{(1)}(x)=20+42x+24x^2+4x^3,\; B_5^{(1)}(x)=35+112x+108x^2+40x^3+5x^4, \dots \end{aligned} \tag{2.2}$$

$k = 2$

$$\begin{aligned} &B_1^{(2)}(x)=1,\; B_2^{(2)}(x)=6+3x,\; B_3^{(2)}(x)=21+24x+6x^2, \\ &B_4^{(2)}(x)=56+108x+60x^2+10x^3,\; B_5^{(2)}(x)=126+360x+330x^2+120x^3+15x^4, \dots \end{aligned} \tag{2.3}$$

**Special Cases**

$$\{B_n^{(1)}\}_1^{\infty} = 1, 6, 25, 90, 300, \dots \tag{2.2a}$$

$$\{B_n^{(2)}\}_1^{\infty} = 1, 9, 51, 234, 961, \dots \tag{2.3a}$$

Larger values of $k$ and $n$ clearly involve cumbersome expressions which do not excite our interest.

**Recurrence Relations**

Immediately from (1.1) and (2.1) we deduce that

$$B_n^{(k)}(x) = B_n^{(k+1)}(x) - (2+x)B_{n-1}^{(k+1)}(x) + B_{n-2}^{(k+1)}(x) \tag{2.4}$$

with the simplest instance ($k=0$) being

$$B_n(x) = B_n^{(1)}(x) - (2+x)B_{n-1}^{(1)}(x) + B_{n-2}^{(1)}(x). \tag{2.4a}$$

Partial differentiation with respect to $y$ in (2.1) and comparison of coefficients of $y^{n-2}$ leads to

$$(n-1)B_n^{(k)}(x) = (k+1)\{(2+x)B_{n-1}^{(k+1)}(x) - 2B_{n-2}^{(k+1)}(x)\}. \tag{2.5}$$

Amalgamating (2.4) and (2.5) and replacing $k$ by $k-1$, we obtain the reduction

$$(n-1)B_n^{(k)}(x) = (n+k-1)(2+x)B_{n-1}^{(k)}(x) - (n+2k-1)B_{n-2}^{(k)}(x). \tag{2.6}$$

Recurrence (2.6) enables us to consolidate a table for $B_n^{(k)}(x)$, given $x=1$, from two previously known successive values. Substitution of $k=0$ reduces (2.6) to the defining recurrence for $B_n(x)$. Furthermore, $k=0$ in (2.5) produces the simple link ($n \to n+1$)

$$nB_{n+1}(x) = (2+x)B_n^{(1)}(x) - 2B_{n-1}^{(1)}(x). \tag{2.5a}$$

Further partial differentiation, but this time with respect to $x$, will be investigated for all the Morgan-Voyce polynomials separately in Section 5.

## 3. CONVOLUTIONS FOR $C_n(x)$

Coming now to $C_n(x)$ we find ourselves enmeshed in more complicated algebra than that for $B_n(x)$, by virtue of the definition (1.2).

**Definitions**

The $k^{th}$ *convolution polynomials* $C_n^{(k)}(x)$ of $C_n(x)$ are defined by

$$\sum_{n=0}^{\infty} C_n^{(k)}(x)y^n = [2-(2+x)y]^{k+1}g^{k+1} \tag{3.1}$$

$$= \left(\sum_{n=0}^{\infty} C_n(x)y^n\right)^{k+1}, \tag{3.1a}$$

so that $C_n^{(0)}(x) \equiv C_n(x)$.

Correspondingly, the $k^{th}$ *convolution numbers* $C_n^{(k)}(1) \equiv C_n^{(k)}$ arise when $x = 1$.

**Examples**

$\boldsymbol{k=1}$

$$\begin{aligned} &C_0^{(1)}(x) = 4,\ C_1^{(1)}(x) = 4(2+x),\ C_2^{(1)}(x) = 12+20x+5x^2, \\ &C_3^{(1)}(x) = 16+56x+36x^2+6x^3,\ C_4^{(1)}(x) = 20+120x+142x^2+56x^3+7x^4, \dots \end{aligned} \tag{3.2}$$

$\boldsymbol{k=2}$

$$\begin{aligned} &C_0^{(2)}(x) = 8,\ C_1^{(2)}(x) = 12(2+x),\ C_2^{(2)}(x) = 48+72x+18x^2, \\ &C_3^{(2)}(x) = 80+240x+150x^2+25x^3,\ C_4^{(2)}(x) = 120+600x+678x^2+264x^3+33x^4, \dots \end{aligned} \tag{3.3}$$

**Special Cases**

$$\{C_n^{(1)}\}_0^{\infty} = 4, 12, 37, 114, 345, \dots \tag{3.2a}$$

$$\{C_n^{(2)}\}_0^{\infty} = 8, 36, 138, 495, 1695, \dots \tag{3.3a}$$

**Recurrence Relations**

Taken together, (2.1) and (3.1) give rise, when $k = 1$, to

$$C_{n-1}^{(1)}(x) = 4B_n^{(1)}(x) - 4(2+x)B_{n-1}^{(1)}(x) + (2+x)^2 B_{n-2}^{(1)}(x). \tag{3.4}$$

Differentiate (1.2) partially with respect to $y$ and equate coefficients of $y^{n-1}$. After simplification, the algebra reduces to

$$nC_n(x) = (2+x)B_n^{(1)}(x) - 4B_{n-1}^{(1)}(x) + (2+x)B_{n-2}^{(1)}(x). \tag{3.5}$$

Uniting (3.4) and (3.5), we establish, on tidying up, that

$$n(2+x)C_n(x) = (4+x)xB_n^{(1)}(x) + C_{n-1}^{(1)}(x). \tag{3.6}$$

Multiply numerator and denominator of (3.1), when $k = 0$, by $g$ (1.1). Simplification then shows, by (2.1), that

$$C_{n-1}(x) = 2B_n^{(1)}(x) - 3(2+x)B_{n-1}^{(1)}(x) + (6+4x+x^2)B_{n-2}^{(1)}(x) - (2+x)B_{n-3}^{(1)}(x). \tag{3.7}$$

Extending (3.5) to $k=2$, we quickly get

$$C_{n-1}^{(2)}(x)=8B_n^{(2)}(x)-12(2+x)B_{n-1}^{(2)}+6(2+x)^2B_{n-2}^{(2)}(x)-(2+x)^3B_{n-3}^{(2)}(x). \tag{3.8}$$

Beyond this, the formulas become even less algebraically attractive. Enchantment and time are lacking to pursue this unproductive activity.

## 4. CONVOLUTIONS FOR $b_n(x)$

### Definitions

The $k^{th}$ *convolution polynomials* $b_n^{(k)}(x)$ of $b_n(x)$ are defined by

$$\sum_{n=1}^{\infty} b_{n-1}^{(k)}(x)y^{n-1}=\{1-(1+x)y\}^{k+1}g^{k+1} \quad (\text{so } b_0^{(k)}(x)=1) \tag{4.1}$$

$$=\left(\sum_{n=1}^{\infty} b_{n-1}(x)y^{n-1}\right)^{k+1}. \tag{4.1a}$$

In particular, when $x=1$, the $k^{th}$ *convolution numbers* $b_n^{(k)}(1)\equiv b_n^{(k)}$ emerge.

### Examples

$k=1$

$$\begin{aligned}&b_1^{(1)}(x)=2,\ b_2^{(1)}(x)=3+2x,\ b_3^{(1)}(x)=4+8x+2x^2,\\ &b_4^{(1)}(x)=5+20x+13x^2+2x^3,\ldots.\end{aligned} \tag{4.2}$$

$k=2$

$$\begin{aligned}&b_1^{(2)}(x)=3,\ b_2^{(2)}(x)=6+3x,\ b_3^{(2)}(x)=10+15x+3x^2,\\ &b_4^{(2)}(x)=15+45x+24x^2+3x^3,\ldots.\end{aligned} \tag{4.3}$$

### Special Cases

$$\{b_n^{(1)}\}_0^{\infty}=1,2,5,14,40,\ldots. \tag{4.2a}$$

$$\{b_n^{(2)}\}_0^{\infty}=1,3,9,28,87,\ldots. \tag{4.3a}$$

### Recurrence Relations

Put $k=1$ in (4.1). Then we immediately construct the recurrence

$$b_n^{(1)}(x)=B_{n+1}^{(1)}(x)-2(1+x)B_n^{(1)}(x)+(1+x)^2B_{n-1}^{(1)}(x). \tag{4.4}$$

Partially differentiate (4.1) with respect to $y$. Then

$$nb_n(x)=B_n^{(1)}(x)-2B_{n-1}^{(1)}(x)+(1+x)B_{n-2}^{(1)}(x). \tag{4.5}$$

Together, with suitable adjustment, (4.4) and (4.5) produce

$$nb_n(x)=b_{n-1}^{(1)}(x)+2xB_{n-1}^{(1)}(x)-(x+x^2)B_{n-2}^{(1)}(x). \tag{4.6}$$

Next, let us multiply numerator and denominator of (4.1), when $k=1$, by $g$ (1.1). Upon the requisite algebraic manipulation with application of $b_n^{(2)}(x)$ given by (4.1), when $k=2$, namely,

$$b_n^{(2)}(x)=B_{n+1}^{(2)}(x)-3(1+x)B_n^{(2)}(x)+3(1+x)^2B_{n-1}^{(2)}(x)-(1+x)^3B_{n-2}^{(2)}(x), \tag{4.7}$$

it transpires that

$$b_n^{(2)}(x)=b_n^{(1)}(x)+B_n^{(2)}(x)-(3+2x)B_{n-1}^{(2)}(x)+(3+4x+x^2)B_{n-2}^{(2)}(x)-(1+x)^2B_{n-3}^{(2)}(x). \tag{4.8}$$

***Caveat!*** Anticipating (5.1) we might have been tempted to use the formula $b_n(x) = B_n(x) - B_{n-1}(x)$ [2, (2.13) : $x = 1$] to derive the valid generating function $\sum_{n=1}^{\infty} b_n(x)y^{n-1} = (1-y)g$. However, the difficulty here for convolutions is that the first element defined is $b_1(x) = 1$. What we need is $b_0(x) = 1$ to be covered by the definition. Consequently, we must abide by (4.1).

## 5. CONVOLUTIONS FOR $c_n(x)$

### Definitions

Care must be taken when we come to deal with the convolutions of the last of our four Morgan-Voyce polynomials. Our problem with $c_n(x)$ as defined in (1.4) is that $c_0(x) = -1$. But we do not want negative numbers as part of convolutions. So we begin the sequence for $c_n(x)$ with $c_1(x) = 1$.

Recalling [2, (3.7)] that $c_n(x) = B_n(x) + B_{n-1}(x)$, we define the $k^{th}$ *convolution polynomials* $c_n^{(k)}(x)$ of $c_n(x)$ to be given by $(n \geq 1)$

$$\sum_{n=1}^{\infty} c_n^{(k)}(x)y^{n-1} = (1+y)^{k+1}g^{k+1}. \tag{5.1}$$

Substitution of $x = 1$ engenders the $k^{th}$ *convolution numbers* $c_n^{(k)}(1) \equiv c_n^{(k)}$.

### Examples

$\boldsymbol{k = 1}$

$$\begin{aligned} &c_1^{(1)}(x) = 1,\ c_2^{(1)}(x) = 6+2x,\ c_3^{(1)}(x) = 19+16x+3x^2, \\ &c_4^{(1)} = 44+68x+30x^2+4x^3,\ c_5^{(1)} = 85+208x+159x^2+48x^3+5x^4, \ldots . \end{aligned} \tag{5.2}$$

$\boldsymbol{k = 2}$

$$\begin{aligned} &c_1^{(2)}(x) = 1,\ c_2^{(2)}(x) = 9+3x,\ c_3^{(2)}(x) = 42+33x+6x^2, \\ &c_4^{(2)}(x) = 138+189x+78x^2+10x^3,\ c_5^{(2)}(x) = 363+759x+528x^2+150x^3+15x^4, \ldots . \end{aligned} \tag{5.3}$$

### Special Cases

$$\{c_n^{(1)}\}_1^{\infty} = 1, 8, 38, 146, 505, \ldots . \tag{5.2a}$$

$$\{c_n^{(2)}\}_1^{\infty} = 1, 12, 81, 415, 1815, \ldots . \tag{5.3a}$$

### Recurrence Relations

From (5.1) and (1.1) we have automatically

$$c_n^{(1)}(x) = B_n^{(1)}(x) + 2B_{n-1}^{(1)}(x) + B_{n-2}^{(1)}(x). \tag{5.4}$$

Partial differentiation in (5.1) with respect to $y$, in conjunction with (1.1), and $n \to n+1$, produces

$$nc_{n+1}(x) = (3+x)B_n^{(1)}(x) - 2B_{n-1}^{(1)}(x) - B_{n-2}^{(1)}(x). \tag{5.5}$$

Joining (5.4) and (5.5) ensures the neat nexus

$$c_n^{(1)}(x) = (4+x)B_n^{(1)}(x) - nc_{n+1}(x). \tag{5.6}$$

Next, taking $k = 0$, multiply numerator and denominator in (5.1) by $g$. Organizing the resulting material and applying (1.1) then establishes the result:

$$c_{n+1}(x) = B_{n+1}^{(1)}(x) - (1+x)[B_n^{(1)}(x) + B_{n-1}^{(1)}(x)] + B_{n-2}^{(1)}(x). \tag{5.7}$$

## 6. PARTIAL DIFFERENTIATION

In this section, partial differentiation is performed **only with respect to $x$.**

**Notation**

Successive orders of partial differentiation (first, second, third, ..., $k^{\text{th}}$) will be represented by superscript primes $'$, $''$, $'''$, ..., $k$ primes, where the unbracketed superscript $k$ is to be clearly distinguished from the bracketed $k^{\text{th}}$ convolution order symbol superscript $(k)$. Thus, we will have

$$B_n'(x)=\frac{\partial B_n(x)}{\partial x},\ B_n''(x)=\frac{\partial^2 B_n(x)}{\partial x^2},\dots,B_n^k(x)=\frac{\partial^k B_n(x)}{\partial x^k}.$$

Likewise for $C_n(x)$, $b_n(x)$, and $c_n(x)$.

**I. $B_n^k(x)$:** Equate appropriate coefficients using (1.1) in

$$\sum_{n=1}^{\infty} B_n'(x)y^{n-1}=yg^2=\sum_{m=0}^{\infty} B_m^{(1)}(x)y^m$$

unfolding the nice result

$$B_n'(x)=B_{n-1}^{(1)}(x). \tag{6.1}$$

Repetition of the process gives

$$B_n''(x)=2B_{n-2}^{(2)}(x). \tag{6.1a}$$

Generally,

$$B_n^k(x)=k!\,B_{n-k}^{(k)}(x). \tag{6.1b}$$

Temporarily revert to $B_n^{(2)}(x)$. Then we may write

$$\sum_{n=1}^{\infty} B_n^{(2)}(x)y^{n-1}=[\{1-(2+x)y+y^2\}+\{(2+x)y-1\}]g^3=\sum_{n=1}^{\infty} B_n^{(1)}(x)y^{n-1}+\{(2+x)y-1\}g^3,$$

whence

$$B_n^{(2)}(x)=B_n^{(1)}(x)+(2+x)B_{n-1}^{(2)}(x)-B_{n-2}^{(2)}(x). \tag{6.2}$$

Accordingly, (6.1a) and (6.2) conjoined give

$$B_n^{(1)}(x)=B_n^{(2)}(x)-(2+x)B_{n-1}^{(2)}(x)+B_{n-2}^{(2)}(x) \tag{6.3}$$

which is (2.4) when $k=1$.

Two pleasant theorems now conclude this subsection.

***Theorem 1:*** $B_{n+2}''(x)-B_n''(x)=(n+1)B_n^{(1)}(x)$.

***Proof:***

$$\begin{aligned}B_{n+2}''(x)-B_n''(x)&=2B_n^{(2)}(x)-2B_{n-2}^{(2)}(x)\quad \text{by (2.12)}\\&=2B_n^{(1)}(x)+2\{(2+x)B_{n-1}^{(2)}(x)-B_{n-2}^{(2)}(x)\}-2B_{n-2}^{(2)}(x)\quad\text{by (2.7)}\\&=2B_n^{(1)}(x)+2\{(2+x)B_{n-1}^{(2)}(x)-2B_{n-2}^{(2)}(x)\}\\&=2B_n^{(1)}(x)+(n-1)B_n^{(1)}(x)\quad\text{by (2.8)}\\&=(n+1)B_n^{(1)}(x).\end{aligned}$$

***Corollary 1:*** $\sum_{n=2}^{m} nB_{n-1}^{(1)}(x)=B_m''(x)+B_{m+1}''(x)$.

More generally,

***Theorem 2:*** $B_{n+2}^k(x) - B_n^k(x) = (k-1)!(n+1)B_{n+2-k}^{(k-1)}(x)$.

In particular, we have $B'_{n+2}(x) - B'_n(x) = (n+1)B_{n+1}(x)$ $(= C'_{n+1}(x))$ as we expect from [2, (3.9), (3.24)].

**II. $C_n^k(x)$:** Consider the relation $C_n(x) = B_{n+1}(x) - B_{n-1}(x)$ [2, (3.9)]. Redefine (1.2) in this context to assert

$$\sum_{n=1}^{\infty} C_n(x)y^{n+1} = (y-y^3)g - y, \quad C_0(x) = 2. \tag{6.4}$$

Elementary processes then, with [2, (3.24)] produce

$$C'_n(x) = B_n^{(1)}(x) - B_{n-2}^{(1)}(x) = nB_n(x), \tag{6.5}$$

$$C''_n(x) = 2(B_{n-1}^{(2)}(x) - B_{n-3}^{(2)}(x)) = nB_{n-1}^{(1)}(x) = nB'_n(x), \tag{6.6}$$

culminating in

$$C_n^k(x) = k!(B_{n-k+1}^{(k)}(x) - B_{n-k-1}^{(k)}(x)) = n(k-1)!\,B_{n-k+1}^{(k-1)}(x), \tag{6.7}$$

whence

$$\sum_{n=k+1}^{m} C_n^k(x) = k!(B_{m-k+1}^{(k)}(x) + B_{m-k}^{(k)}(x) - 1). \tag{6.8}$$

In particular ($k = 1$),

$$\sum_{n=1}^{m} C'_n = B_m^{(1)}(x) + B_{m-1}^{(1)}(x). \tag{6.8a}$$

Analogously to Theorem 2 there is

$$C_{n+2}^k(x) - C_n^k(x) = nC_{n+1}^{k-1}(x) + 2B_{n+2}^{k-1}(x), \tag{6.9}$$

which can be expressed in convolution form. Proof of the assertion (6.9) is left to the reader.

**III. $b_n^k(x)$:** Convolutions of $b_n(x)$ do not appear in this section (see the *Caveat* in Section 4), so we may, on making use of [2, (2.13)], choose the definition

$$\sum_{n=1}^{\infty} b_n(x)y^{n-1} = (1-y)g, \quad b_0(x) = 1. \tag{6.10}$$

Then, by (1.1),

$$b'_n(x) = B_{n-1}^{(1)}(x) - B_{n-2}^{(1)}(x) = B'_n(x) - B'_{n-1}(x), \tag{6.11}$$

$$b''_n(x) = 2(B_{n-2}^{(2)}(x) - B_{n-3}^{(2)}(x)) = B''_n(x) - B''_{n-1}(x). \tag{6.12}$$

Eventually, and generally,

$$b_n^k(x) = k!(B_{n-k}^{(k)}(x) - B_{n-k-1}^{(k)}(x)) = B_n^k(x) - B_{n-1}^k(x). \tag{6.13}$$

Summation discloses that

$$\sum_{n=1}^{m} b'_n(x) = B_{m-1}^{(1)}(x) = B'_m(x) \tag{6.14}$$

while

$$\sum_{n=k+1}^{m} b_n^k(x) = k!\,B_{m-k}^{(k)}(x) = B_m^k(x). \tag{6.15}$$

From $nB_n(x) = C_n'(x) = b_n'(x) + b_{n+1}'(x)$, we may deduce after a little rearrangement that

$$\sum_{n=1}^{m} (-1)^{m+n} nB_n(x) = b_{m+1}'(x) \tag{6.16}$$

which can be generalized to $b_{m+1}^k(x)$.

**IV.** $c_n^k(x)$**:** Appealing to [2, (3.7)], we take

$$\sum_{n=0}^{\infty} c_n(x)y^n = (1+y)g. \tag{6.17}$$

Following the procedure in III, we rapidly reach the general situation:

$$c_n^k(x) = k!(B_{n-k}^{(k)}(x) + B_{n-k-1}^{(k)}(x)) = B_n^k(x) + B_{n-1}^k(x). \tag{6.18}$$

From $nB_n^k(x) = C_n^{k+1}(x) = c_{n+1}^{k+1}(x) - c_n^{k+1}(x)$ (see [2, (3.11)]), it then transpires that

$$\sum_{n=1}^{m} nB_n^k(k) = c_{m+1}^{k+1}(x). \tag{6.19}$$

Suppose $k = 1$ in (6.18). Addition then reveals that

$$\sum_{n=2}^{m} (-1)^n c_n'(x) = (-1)^m B_{m-1}^{(1)}(x) = (-1)^m B_m'(x), \tag{6.20}$$

whence, by (6.16),

$$\sum_{n=2}^{m} (-1)^n c_n'(x) = (-1)^m \sum_{n=2}^{m} b_n'(x). \tag{6.21}$$

## 7. CONCLUSION

Undertaking a thorough investigation of the latent features of the mixed foursome of Morgan-Voyce polynomials is a task of rather Herculean proportions, but no doubt somewhat more satisfying than cleansing the Augean stables. One challenge confronting us is an examination of the rising and falling diagonal polynomials associated with the Morgan-Voyce polynomials. For a related study of this kind of project, the recent paper [3], containing many references, is strongly suggested.

## ACKNOWLEDGMENT

I should like to thank the anonymous referee very much for the spirit in which the suggestions for improvement were offered.

## REFERENCES

1. V. E. Hoggatt, Jr., & G. E. Bergum. "Generalized Convolution Arrays." *The Fibonacci Quarterly* **13.3** (1975):193-98.
2. A. F. Horadam. "New Aspects of Morgan-Voyce Polynomials." *Applications of Fibonacci Numbers* **7:**161-76. Ed. G. E. Bergum, et al. Dordrecht: Kluwer, 1998.
3. M. N. S. Swamy. "Generalized Fibonacci and Lucas Polynomials and Their Associated Diagonal Polynomials." *The Fibonacci Quarterly* **37.3** (1999):213-22.

AMS Classification Number: 11B39

❖❖❖

# SOLVING NONHOMOGENEOUS RECURRENCE RELATIONS OF ORDER $r$ BY MATRIX METHODS

**Bouazza El Wahbi**
Département de mathématiques et informatique, Faculté des Sciences,
Université Abdelmalek Essaadi, B.P. 2121, Tétouan, Morocco

**Mehdi Mouline**
Département de mathématiques et informatique, Faculté des Sciences,
Université Mohammed V-Agdal, B.P. 1014, Rabat, Morocco

**Mustapha Rachidi**
Département de mathématiques et informatique, Faculté des Sciences,
Université Mohammed V-Agdal, B.P. 1014, Rabat, Morocco

(*Submitted December 1999-Final Revision July 2001*)

## 1. INTRODUCTION

Let $a_0, \ldots, a_{r-1}$ ($r \geq 2$, $a_{r-1} \neq 0$) be some real or complex numbers. Let $\{C_n\}_{n\geq 0}$ be a sequence of $\mathbb{C}$ (or $\mathbb{R}$). Sometimes, for reasons of convenience, we consider $\{C_n\}_{n\geq 0}$ under its equivalent form as a function $C: \mathbb{N} \to \mathbb{C}$ (or $\mathbb{R}$). And when no possible confusion can arise, we write $C(n)$ rather than $C_n$ and, similarly, in case of an indexed family of functions $C_j: \mathbb{N} \to \mathbb{C}$, we use $C_j(n)$ instead of $C_{j,n}$. Let $\{T_n\}_{n\geq 0}$ be the sequence defined by the following nonhomogeneous recurrence relation of order $r$,

$$T_{n+1} = a_0 T_n + a_1 T_{n-1} + \cdots + a_{r-1} T_{n-r+1} + C_{n+1} \quad \text{for } n \geq r-1, \tag{1}$$

where $T_0, \ldots, T_{r-1}$ are given initial values (or conditions). In the sequel, we refer to such sequence $\{T_n\}_{n\geq 0}$ as the solution of "recurrence relation (1)." If the function $C$ satisfies

$$C_n = \sum_{j=0}^{d} \beta_j C_{j,n}$$

for some finite sequence of functions $C_0, \ldots, C_d: \mathbb{N} \to \mathbb{C}$, the solution $\{T_n\}_{n\geq 0}$ may be expressed as

$$T_n = \sum_{j=0}^{d} \beta_j T_{j,n},$$

where $\{T_{j,n}\}_{n\geq 0}$ is the solution of (1) with $C_n = C_j(n)$. Solutions of (1) have been studied in the case in which $C$ equals a polynomial or a factorial polynomial (see, e.g., [1]-[4], [7], [9], [12]).

The purpose of this paper is to study a matrix formulation of (1), which extends those considered for (1) in [6], [10], and [11], when $C(n) = 0$. This allows us to provide a method for solving equation (1) for a general $C: \mathbb{N} \to \mathbb{C}$. Our expression for general solutions of (1) extends those obtained in [1] for $r \geq 2$. If the nonhomogeneous part equals a polynomial or a factorial polynomial, our general solution allows us to recover a well-known particular solution—Asveld's polynomials and factorial polynomials (see [2], [3], [9]).

This paper is organized as follows. In Section 2 we study an $r \times r$ matrix associated to (1), in connection with $r$-generalized Fibonacci sequences. In Section 3 we use a matrix formulation with an aim toward solving (1) for arbitrary $C:\mathbb{N}\to\mathbb{C}$. Section 4 is devoted to the study and discussion of our general solution in the polynomial and factorial polynomial cases. Section 5 consists of some final remarks.

## 2. MATRICES ASSOCIATED TO $r$-GENERALIZED FIBONACCI SEQUENCES

From the $r$-generalized Fibonacci sequence $V_{n+1} = a_0 V_n + \cdots + a_{r-1} V_{n-r+1}$ for $n \geq 0$, as studied by Andrade and Pethe [1], we take $r$ copies, indexed by $s$ $(0 \leq s \leq r-1)$:

$$V_{n+1}^{(s)} = a_0 V_n^{(s)} + \cdots + a_{r-1} V_{n-r+1}^{(s)} \quad \text{for } n \geq 0. \tag{2}$$

We provide these $r$ copies with mutually different sets of initial conditions, that is, $V_{-j}^{(s)} = \delta_{s,j}$ $(0 \leq j \leq r-1,\ 0 \leq s \leq r-1)$, where $\delta_{s,j}$ is the Kronecker symbol. Consider the following $r \times r$ matrix:

$$A = \begin{pmatrix} a_0 & a_1 & & \cdots & a_{r-1} \\ 1 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

Expression (3) shows that the columns and arrows of $A$ are indexed from 0 to $r-1$. The usual matrix indexing form $A = (\alpha_{i,j})_{1 \leq i,j \leq r}$ of (3) is given as follows: $\alpha_{1j} = a_{j-1}$ for every $1 \leq j \leq r$, and $\alpha_{ij} = \delta_{i,i-1}$ for every $2 \leq i \leq r$, $1 \leq j \leq r$.

The matrix (3) has been considered for $r$-generalized Fibonacci sequences in [6], [10], [11].

A straightforward computation allows us to establish that the matrix $A$ is related to the $r$-generalized Fibonacci sequences (2) as follows.

***Proposition 2.1:*** Let $A$ be the matrix defined by (3). Then, for every $n \geq 0$, we have

$$A^n = (a_{is}^n)_{0 \leq i,\, s \leq r-1}$$

where

$$a_{is}^n = V_{n-i}^{(s)}. \tag{4}$$

***Remark 2.1:*** Due to the initial conditions $V_{-j}^{(s)} = \delta_{sj}$ $(0 \leq j \leq r-1,\ 0 \leq s \leq r-1)$, we have indeed that $A^0$ equals the $r \times r$-identity matrix.

## 3. SOLVING (1) BY MATRIX METHODS

Consider $X_n = {}^t(T_n, \ldots, T_{n-r+1})$ and $D_n = {}^t(C_n, 0, \ldots, 0)$ for $n \geq r-1$, where ${}^tZ$ denotes the transpose of $Z$. We can easily verify that (1) is equivalent to the following matrix equation:

$$X_{n+1} = AX_n + D_{n+1}, \quad n \geq r-1, \tag{5}$$

where $A$ is the matrix (3). From (5), we derive that

$$X_n = A^{n-r+1} X_{r-1} + \sum_{k=r}^{n} A^{n-k} D_k, \quad n \geq r. \tag{6}$$

Let $R_n = \sum_{k=r}^{n} A^{n-k} D_k$. Then we can verify that $R_{n+1} = AR_n + D_{n+1}$. From expressions (4), (5), and (6), we derive the following result.

***Theorem 3.1:*** Let $\{T_n\}_{n\geq 0}$ be the solution of (1) whose initial conditions are $T_0, \ldots, T_{r-1}$. Then, for $n \geq 0$, we have

$$T_n = \sum_{s=0}^{r-1} V_{n-r+1}^{(s)} T_{r-s-1} + \sum_{k=r}^{n} V_{n-k}^{(0)} C_k. \tag{7}$$

Because of (2), the sequence $\{U_n\}_{n\geq 0}$ defined by $U_n = \sum_{s=0}^{r-1} V_{n-r+1}^{(s)} T_{r-s-1}$ is a solution of the homogeneous part of (1). Thus, the sequence $\{W_n^{\langle ps\rangle}\}_{n\geq 0}$, where

$$W_n^{\langle ps\rangle} = \sum_{k=r}^{n} V_{n-k}^{(0)} C_k = -\sum_{s=0}^{r-1} V_{n-r+1}^{(s)} T_{r-s-1} + T_n$$

is a particular solution of (1) that satisfies $W_n^{\langle ps\rangle} = 0$ for $n = 0, 1, \ldots, r-1$. We call $\{W_n^{\langle ps\rangle}\}_{n\geq 0}$ the *fundamental particular solution of (1)*. Hence, (6) and Theorem 3.1 allow us to formulate the following result.

***Theorem 3.2:*** Let $\{T_n\}_{n\geq 0}$ be a solution of (1). Then, for $n \geq 0$, we have

$$T_n = T_n^{\langle hs\rangle} + W_n^{\langle ps\rangle} = T_n^{\langle hs\rangle} - \sum_{s=0}^{r-1} V_{n-r+1}^{(s)} T_{r-s-1}^{\langle ps\rangle} + T_n^{\langle ps\rangle}, \tag{8}$$

where $\{W_n^{\langle ps\rangle}\}_{n\geq 0}$ is the fundamental particular solution of (1), $\{T_n^{\langle hs\rangle}\}_{n\geq 0}$ is a solution of the homogeneous part of (1) with initial conditions $T_0, \ldots, T_{r-1}$, and $\{T_n^{\langle ps\rangle}\}_{n\geq 0}$ is a particular solution of (1) with initial conditions $T_0^{\langle ps\rangle}, \ldots, T_{r-1}^{\langle ps\rangle}$.

Expression (8) extends the one established in [1], with the aid of Binet's formula in the polynomial case.

## 4. POLYNOMIAL AND FACTORIAL POLYNOMIAL CASES

### 4.1 Elementary Polynomial Solutions and Asveld's Polynomials

For $C(n) = n^j$ $(0 \leq j \leq d)$, the fundamental particular solution $\{W_{j,n}^{\langle ps\rangle}\}_{n\geq 0}$, called the *elementary fundamental particular solution*, is

$$W_{j,n}^{\langle ps\rangle} = \sum_{q=r}^{n} q^j V_{n-q}^{(0)} \quad \text{for } n \geq r.$$

Let $\{f_n\}_{n\geq r}$ be the sequence of $C^\infty$-functions defined on $\mathbb{R}$ as follows:

$$f_n(x) = \sum_{q=r}^{n} V_{n-q}^{(0)} \exp(qx). \tag{9}$$

For each function $f_n$, the $j^{\text{th}}$ derivative is

$$f_n^{(j)}(x) = \sum_{q=r}^{n} q^j V_{n-q}^{(0)} \exp(px).$$

Expressions (2) and (9) imply that $\{f_n^{(j)}\}_{n\geq r}$ satisfies the following nonhomogeneous recurrence relation of order $r$,

$$f_{n+1}^{(j)}(x) = \sum_{i=0}^{r-1} a_i f_{n-i}^{(j)}(x) + (n+1)^j \exp[(n+1)x]. \tag{10}$$

For reasons of simplicity, we suppose that $\{V_n^{(0)}\}_{n\geq -r+1}$ has simple characteristic roots. Thus, Binet's formula takes the form $V_n^{(0)} = \sum_{i=0}^{r-1} \alpha_i \lambda_i^n$. We have to distinguish the following exhaustive cases:

1. $\lambda_i \neq 1$ for every $i$ $(0 \leq i \leq r-1)$.
2. There exists $d$ $(0 \leq d \leq r-1)$ such that $\lambda_d = 1$.

In the sequel, we suppose (without loss of generality) that $\lambda_0 = 1$.

When $\lambda_i \neq 1$ for all $i$ $(0 \leq i \leq r-1)$, we consider

$$H_{1,n}(x) = g_1(x)e^{(n+1)x}, \ K_{1,n}(x) = \sum_{i=0}^{r-1} v_i(x)\lambda_i^{n-r+1}, \tag{11}$$

where

$$g_1(x) = \sum_{i=0}^{r-1} \frac{\alpha_i}{e^x - \lambda_i}, \quad v_i(x) = \frac{\alpha_i e^{rx}}{\lambda_i - e^x}.$$

And if $\lambda_0 = 1$, we set

$$G_n(x) = \alpha_0 \sum_{p=r}^{n} e^{px}, \ H_{2,n}(x) = g_2(x)e^{(n+1)x}, \ K_{2,n}(x) = \sum_{i=1}^{r-1} v_i(x)\lambda_i^{n-r+1}, \tag{12}$$

where

$$g_2(x) = \sum_{i=1}^{r-1} \frac{\alpha_i}{e^x - \lambda_i}.$$

We set $S_n(x) = H_{1,n}(x)$ if $\lambda_i \neq 1$ for all $i$ $(0 \leq i \leq r-1)$ and $S_n(x) = G_n(x) + H_{2,n}(x)$ if $\lambda_0 = 1$.

Because the $\lambda_i$'s are characteristic roots, we have

$$K_{p,n+1}^{(j)}(x) = \sum_{i=0}^{r-1} a_i K_{p,n-i}^{(j)}(x) \ (p = 1, 2).$$

Then, from (10), we derive that for $j \geq 0$ we have

$$S_{n+1}^{(j)}(x) = \sum_{i=0}^{r-1} a_i S_{n-i}^{(j)}(x) + (n+1)^j \exp[(n+1)x]. \tag{13}$$

As a consequence, we have the following lemma.

***Lemma 4.1:***

***(a)*** The elementary fundamental particular solution $\{W_{j,n}^{\langle ps\rangle}\}_{n\geq 0}$ of (1) is given by $W_{j,n}^{\langle ps\rangle} = f_n^{(j)}(0)$. More precisely, we have $W_{j,n}^{\langle ps\rangle} = H_{1,n}^{(j)}(0) + K_{1,n}^{(j)}(0)$ if $\lambda_i \neq 1$ for all $i$ $(0 \leq i \leq r-1)$, where $H_{1,n}(x)$ and $K_{1,n}(x)$ are given by (11), and $W_{j,n}^{\langle ps\rangle} = G_n^{(j)}(0) + H_{2,n}^{(j)}(0) + K_{2,n}^{(j)}(0)$ if $\lambda_0 = 1$, where $G_n(x)$, $H_{2,n}(x)$, and $K_{2,n}(x)$ are given by (12).

***(b)*** For $j \geq 0$, the sequence $\{S_n^{(j)}(0)\}_{n\geq 0}$ is a particular solution of (1) for $C(n) = n^j$.

By Leibnitz's formula, we have

$$H_{p,n}^{(j)}(x) = \sum_{i=0}^{j} \left\{ \sum_{k=i}^{j} \binom{k}{j} \binom{i}{k} g_p^{(j-k)}(x) \right\} n^i e^{(n+1)x} \text{ for } j \geq 0,$$

where $p = 1, 2$. If $\lambda_0 = 1$ is a characteristic root, then we have

$$G_n^{(j)}(0)=\alpha_0\sum_{p=r}^{n}p^j=\alpha_0\sum_{p=0}^{n-r}(n-p)^j.$$

It is known that $\sum_{p=0}^{n}p^j=Q_j(n)$, where $Q_j(n)$ is a polynomial of degree $j+1$. Thus, Lemma 4.1 and (13) allow us to derive the following result.

***Theorem 4.2:*** Let $\{T_n\}_{n\geq 0}$ be a solution of (1) with $C(n)=n^j$. Then the elementary polynomial solution $\{P_j(n)\}_{n\geq 0}$ of (1) is given by $P_j(n)=S_n^j(0)$. More precisely, if $\lambda_i\neq 1$ for all $i$ $(0\leq i\leq r-1)$, we have

$$P_j(n)=\sum_{k=0}^{j}\left\{\sum_{i=k}^{j}\binom{i}{j}\binom{k}{i}g_1^{(j-i)}(0)\right\}n^k, \tag{14}$$

and if $\lambda_0=1$ we have

$$P_j(n)=\alpha_0\sum_{k=0}^{j+1}\mu_k(n-r)^k+\sum_{k=0}^{j}\left\{\sum_{i=k}^{j}\binom{i}{j}\binom{k}{i}g_2^{(j-i)}(0)\right\}n^k. \tag{15}$$

If $\lambda_0=1$, the polynomial (15) may be written as $P_j(n)=\alpha_0 n^{j+1}+\sum_{k=0}^{j}\nu_{j,k}n^k$, where $\nu_{j,k}$ are constants (real or complex numbers).

Theorem 4.2 shows that particular polynomial solutions $P_j(n)$ $(0\leq j\leq d)$ defined by (14)-(15) are the well-known Asveld's polynomials studied in [2], [4], [9], and [12]. Our method of obtaining $P_j(n)$ $(0\leq j\leq d)$ is different. For their computation, we use the classic result on $\sum_{j=0}^{n}p^j=Q_j(n)$ and the $j^{\text{th}}$ derivative of $H_{p,n}(x)$ $(p=1,2)$ given by (11)-(12). The derivative of $H_{p,n}(x)$ $(p=1,2)$ can be derived from the following property.

***Proposition 4.3:*** Let $u(x)=\frac{1}{e^x-\lambda}$ with $\lambda\neq 0,1$ and $x\neq\ln(\lambda)$ if $\lambda>0$. Then we have

$$u^{(k)}(x)=\frac{T_k(e^x)}{(e^x-\lambda)^{k+1}},$$

where $T_{k+1}=X(X-\lambda)\frac{dT_k}{dX}-(k+1)XT_k$ for $k\geq 0$.

### 4.2 Elementary Factorial Polynomial Solutions and Asveld's Polynomials

For $C(n)=n^{(j)}$, the elementary fundamental particular solution $\{\widetilde{W}_{j,n}^{\langle ps\rangle}\}_{n\geq 0}$ is

$$\widetilde{W}_{j,n}^{\langle ps\rangle}=\sum_{p=r}^{n}p^{(j)}V_{n-p}^{(0)}\quad\text{for all } n\geq r.$$

Instead of (9), let $\{\tilde{f}_n\}_{n\geq r}$ be the sequence of $C^\infty$-functions on $\mathbb{R}^*=\mathbb{R}-\{0\}$ defined as follows:

$$\tilde{f}_n(x)=(-1)^j\sum_{k=r}^{n}V_{n-k}^{(0)}x^{-k+j-1}. \tag{16}$$

The $q^{\text{th}}$ $(q\geq 0)$ derivative of $h_{j,k}(x)=x^{-k+j-1}$ $(x\neq 0)$ is $h_{j,k}^{(q)}(x)=(-1)^q(k-j+q)^{(q)}x^{-k+j-q-1}$. Hence, the $j^{\text{th}}$ derivative of $\tilde{f}_n$ is

$$\tilde{f}_n^{(j)}(x)=\sum_{k=r}^{n}k^{(j)}V_{n-k}^{(0)}x^{-k-1}.$$

From (2), we derive that $\{\tilde{f}_n\}_{n\geq r}$ defined by (16) satisfies

$$\tilde{f}_{n+1}^{(j)}(x)=\sum_{i=0}^{r-1}a_i\tilde{f}_{n-i}^{(j)}(x)+(n+1)^{(j)}x^{-n-2}. \tag{17}$$

As in Subsection 4.1, we suppose that $\{V_n^{(0)}\}_{n\geq -r+1}$ has simple characteristic roots. We also consider the following two exhaustive cases: (a) $\lambda_i\neq 1$ for every $i$ $(0\leq i\leq r-1)$; (b) There exists $d$ $(0\leq d\leq r-1)$ such that $\lambda_d=1$. As in Subsection 4.1, we suppose in the second case that $\lambda_0=1$. The case in which $\lambda_d=1$ for some $d\neq 0$ can be derived easily.

When $\lambda_i\neq 1$ for all $i$ $(0\leq i\leq r-1)$, we set

$$\tilde{H}_{1,n}(x)=\tilde{g}_1(x)h_{j,n}(x),\ \tilde{K}_{1,n}(x)=\sum_{0\leq i\leq r-1}\tilde{v}_i(x)\lambda_i^{n-r+1}, \tag{18}$$

where

$$\tilde{g}_1(x)=(-1)^j\sum_{i=0}^{r-1}\frac{\alpha_i}{1-x\lambda_i},\ \tilde{v}_i(x)=(-1)^j\frac{\alpha_i x^{j-r}}{\lambda_i x-1}.$$

If $\lambda_0=1$, we set

$$\tilde{G}_n(x)=(-1)^j\alpha_0\sum_{k=r}^{n}h_{j,k}(x),\ \tilde{H}_{2,n}(x)=\tilde{g}_2(x)h_{j,n}(x),\ \tilde{K}_{2,n}(x)=\sum_{i=1}^{r-1}\tilde{v}_i(x)\lambda_i^{n-r+1}, \tag{19}$$

where

$$\tilde{g}_2(x)=(-1)^j\sum_{i=1}^{r-1}\frac{\alpha_i}{1-x\lambda_i}.$$

Because the $\lambda_i$'s are characteristic roots, we have

$$\tilde{K}_{p,n+1}^{(j)}(x)=\sum_{i=0}^{r-1}\alpha_i\tilde{K}_{p,n-i}^{(j)}(x)\ \ (p=1,2).$$

Then from (17) we derive that, for all $j\geq 0$, we have

$$\tilde{S}_{n+1}^{(j)}(x)=\sum_{i=0}^{r-1}a_i\tilde{S}_{n-i}^{(j)}(x)+(n+1)^{(j)}x^{-n-2}, \tag{20}$$

where $\tilde{S}_n(x)=\tilde{H}_{1,n}(x)$ if $\lambda_i\neq 1$ for all $i$ $(0\leq i\leq r-1)$ and $\tilde{S}_n(x)=\tilde{G}_n(x)+\tilde{H}_{2,n}(x)$ if $\lambda_0=1$.

Therefore, we have the analog of Lemma 4.1 as follows.

***Lemma 4.4***

***(a)*** The elementary fundamental particular solution $\{\tilde{W}_{j,n}^{\langle ps\rangle}\}_{n\geq 0}$ of (1) is given by $\tilde{W}_{j,n}^{\langle ps\rangle}=\tilde{f}_n^{(j)}(1)$. More precisely, we have $\tilde{W}_{j,n}^{\langle ps\rangle}=\tilde{H}_{1,n}^{(j)}(1)+\tilde{K}_{1,n}^{(j)}(1)$ if $\lambda_i\neq 1$ for all $i$ $(0\leq i\leq r-1)$, where $\tilde{H}_{1,n}(x)$ and $\tilde{K}_{1,n}(x)$ are given by (18), and $\tilde{W}_{j,n}^{\langle ps\rangle}=\tilde{G}_n^{(j)}(1)+\tilde{H}_{2,n}^{(j)}(1)+\tilde{K}_{2,n}^{(j)}(1)$ if $\lambda_0=1$, where $\tilde{G}_n(x)$, $\tilde{H}_{2,n}(x)$, and $\tilde{K}_{2,n}(x)$ are given by (19).

***(b)*** For $j\geq 0$, the sequence $\{\tilde{S}_n^{(j)}(1)\}_{n\geq 0}$ is a particular solution of (1) for $C_n=n^{(j)}$.

By Leibnitz's formula, we have

$$\tilde{H}_{p,n}^{(j)}(x)=\sum_{k=0}^{j}\binom{k}{j}g_p^{(j-k)}(x)h_{j,n}^{(k)}(x)\ \ (p=1,2).$$

Thus,

$$\widetilde{H}_{p,n}^{(j)}(x)=\sum_{k=0}^{j}(-1)^k\binom{k}{j}g_p^{(j-k)}(x)(n-j+k)^{(k)}x^{-n+j-k-1}\quad(p=1,2).$$

Consider the following "binomial theorem for factorial polynomials," which is designated by Asveld [3] as Lemma 1:

$$(x+y)^{(k)}=\sum_{i=0}^{k}\binom{i}{k}x^{(i)}y^{(k-i)}.$$

Then we have

$$\widetilde{H}_{p,n}^{(j)}(1)=\sum_{i=0}^{j}\left(\sum_{k=i}^{j}(-1)^k\binom{k}{j}\binom{i}{k}g_p^{(j-k)}(1)(k-j)^{(k-i)}\right)n^{(i)}\quad(p=1,2).$$

Hence, $\widetilde{H}_{p,n}(1)$ $(p=1,2)$ is a factorial polynomial. If $\lambda_0=1$, we have

$$\widetilde{G}_n^{(j)}(1)=\alpha_0\sum_{k=0}^{n-r}(n-k)^{(j)}.$$

Next, we establish that $\widetilde{G}_n^{(j)}(1)$ is a factorial polynomial.

***Lemma 4.5:*** For $j\geq 0$, we have

$$\sum_{k=0}^{n}k^{(j)}=\sum_{k=0}^{j+1}\beta_{j,k}n^{(k)},$$

where $\beta_{j,k}$ are constants (real or complex numbers).

***Proof:*** Consider Stirling numbers of the first kind $s(t,j)$ and Stirling numbers of the second kind $S(t,j)$, which are defined by

$$x^{(j)}=\sum_{t=0}^{j}s(t,j)x^t\quad\text{and}\quad x^i=\sum_{t=0}^{i}S(t,i)x^{(t)}.$$

By successive applications of the two preceding formulas and the following classic result,

$$\sum_{k=0}^{n}k^t=\sum_{i=0}^{t+1}a_{i,t}n^i,$$

we derive that

$$\sum_{k=0}^{n}k^{(j)}=\sum_{q=0}^{j+1}\beta_{j,q}n^{(q)},$$

where

$$\beta_{j,q}=\sum_{i=q}^{j}\sum_{i=0}^{t+1}a_{i,t}s(t,j)S(q,j).\ \square$$

Now, using Lemma 4.4, we derive the following result.

***Theorem 4.6:*** Let $\{T_n\}_{n\geq 0}$ be a solution of (1) with $C(n)=n^{(j)}$. Then the elementary factorial polynomial solution $\{\widetilde{P}_j(n)\}_{n\geq 0}$ of (1) is given by $\widetilde{P}_j(n)=\widetilde{S}_n^{(j)}(1)$. More precisely, if $\lambda_i\neq 1$ for all $i$ $(0\leq i\leq r-1)$, we have

$$\widetilde{P}_j(n) = \sum_{i=0}^{j}\left(\sum_{k=i}^{j}(-1)^k\binom{k}{j}\binom{i}{k}\widetilde{g}_1^{(j-k)}(1)(k-j)^{(k-i)}\right)n^{(i)}. \tag{21}$$

And if $\lambda_0 = 1$, we have

$$\widetilde{P}_j(n) = (-1)^j\alpha_0\sum_{k=0}^{j+1}\gamma_{j,k}n^{(k)} + \sum_{i=1}^{j}\left(\sum_{k=i}^{j}(-1)^k\binom{k}{j}\binom{i}{k}\widetilde{g}_2^{(j-k)}(1)(k-j)^{(k-i)}\right)n^{(i)}, \tag{22}$$

where $\gamma_{j,k}$ are constants (real or complex numbers).

The particular factorial polynomial solutions $\widetilde{P}_j(n)$ $(0 \le j \le d)$ defined by (21)-(22) are the well-known Asveld factorial polynomials studied in [4] and [7]. Our method for obtaining $\widetilde{P}_j(n)$ $(0 \le j \le d)$ is different from Asveld's. For their computation, we use Lemma 4.5 and the $j^{\text{th}}$ derivative of $\widetilde{H}_{n,p}(x)$ $(p = 1, 2)$ as defined by (18)-(19).

### 4.3 Polynomial and Factorial Polynomial Solutions for $\lambda_0 = 1$ of Multiplicity $m \ge 1$

Suppose that $\lambda_i \ne 1$ for all $i$ $(0 \le i \le r-1)$. Then (14) and (21) imply, respectively, that the Asveld polynomials $P_j(n)$ $(0 \le j \le d)$ are of degree $j$ and the Asveld factorial polynomials $\widetilde{P}_j(n)$ $(0 \le j \le d)$ are of degree $j$. Meanwhile, for $\lambda_0 = 1$, (15) and (22) show that $P_j(n)$ and $\widetilde{P}_j(n)$ $(0 \le j \le d)$ may be of degree $j+1$. More generally, an extension of Theorems 4.2 and 4.6 may be derived by the same method using, respectively,

$$G_n(x) = \sum_{i=0}^{m-1}\sum_{k=r}^{n}\alpha_{0,i}(n-k)^i e^{kx}$$

instead of $G_n(x)$ and

$$\widetilde{G}_n(x) = (-1)^j\sum_{i=0}^{m-1}\alpha_{0,i}\sum_{k=r}^{n}(n-k)^i x^{-k+j-1}$$

instead of $\widetilde{G}_n(x)$ of (19).

More precisely, we have the following result.

***Theorem 4.7:*** Let $\{T_n\}_{n\ge0}$ be a solution of (1) and suppose that $\lambda_0 = 1$ has multiplicity $m \ge 1$, and the other characteristic roots $\lambda_1, \ldots, \lambda_s$ (where $s = r-m-1$) are simple.

***(a)*** For $C(n) = n^j$, the elementary polynomial solution $\{P_j(n)\}_{n\ge0}$ of (1) is given by

$$P_j(n) = \sum_{k=0}^{j+m}\nu_{j,k}n^k + \sum_{k=0}^{j}\left\{\sum_{i=k}^{j}\binom{i}{j}\binom{k}{i}g_2^{(j-i)}(0)\right\}n^k,$$

where $\nu_{j,k}$ are constants (real or complex numbers) and

$$g_2(x) = \sum_{i=1}^{s}\frac{\alpha_i}{e^x - \lambda_i}.$$

***(b)*** For $C(n) = n^{(j)}$, the elementary factorial polynomial solution $\{\widetilde{P}_j(n)\}_{n\ge0}$ of (1) is given by

$$\widetilde{P}_j(n) = \sum_{k=0}^{j+m}\nu_{j,k}n^{(k)} + \sum_{k=0}^{j}\left\{\sum_{i=k}^{j}\binom{i}{j}\binom{k}{i}\widetilde{g}_2^{(j-i)}(1)\right\}n^{(k)},$$

where $\nu_{j,k}$ are constants (real or complex numbers) and

$$\widetilde{g}_2(x) = (-1)^j \sum_{i=1}^{s} \frac{\alpha_i}{1 - x\lambda_i}.$$

Theorem 4.7 shows that $P_j(n)$ and $\widetilde{P}_j(n)$ may be of degree $j+m$, where $m$ is the multiplicity of $\lambda_0 = 1$.

### 4.4 Solutions of (1) for General $\{C_n\}_{n\geq 0}$

In the general situation, polynomial and factorial polynomial solutions of (1) are as follows.

***Proposition 4.8:*** Let $\{T_n\}_{n\geq 0}$ be a solution of (1) and suppose that the characteristic roots $\lambda_0, \ldots, \lambda_{r-1}$ are simple. Then:

***(a)*** For $C(n) = \sum_{j=0}^{d} \beta_j n^j$, the particular fundamental polynomial solution $\{P(n)\}_{n\geq 0}$ of (1) is given by $P(n) = \sum_{j=0}^{d} \beta_j S_n^{(j)}(0)$. More precisely, $P(n) = \sum_{j=0}^{d} \beta_j P_j(n)$, where $P_j(n)$ is given by (14) if $\lambda_i \neq 1$ for all $i$ $(0 \leq i \leq r-1)$ and (15) if $\lambda_0 = 1$.

***(b)*** For $C(n) = \sum_{j=0}^{d} \beta_j n^{(j)}$, the particular fundamental factorial polynomial solution $\{\widetilde{P}(n)\}_{n\geq 0}$ of (1) is given by $\widetilde{P}(n) = \sum_{j=0}^{d} \beta_j \widetilde{S}_n^{(j)}(1)$. More precisely, $\widetilde{P}(n) = \sum_{j=0}^{d} \beta_j \widetilde{P}_j(n)$, where $\widetilde{P}_j(n)$ is given by (21) if $\lambda_i \neq 1$ for all $i$ $(0 \leq i \leq r-1)$ and by (22) if $\lambda_0 = 1$.

From Lemma 4.1 and Theorem 4.2, we derive that in the polynomial case the elementary fundamental particular solutions of (1) are

$$W_{j,n}^{\langle ps\rangle} = P_j(n) + \sum_{i=0}^{r-1} v_i^{(j)}(0)\lambda_i^{n-r+1}$$

if $\lambda_i \neq 1$ for all $i$ $(0 \leq i \leq r-1)$, where $P_j(n)$ is given by (14) and

$$v_i(x) = \frac{\alpha_i e^{rx}}{\lambda_i - e^x}$$

And if $\lambda_0 = 1$, we have

$$W_{j,n}^{\langle ps\rangle} = P_j(n) + \sum_{i=0}^{r-1} u_i^{(j)}(0)\lambda_i^{n-r+1},$$

where $P_j(n)$ is given by (15) above. For $C(n) = \sum_{j=0}^{d} \beta_j n^j$, the fundamental particular solution $\{W_n^{\langle ps\rangle}\}_{n\geq 0}$ is given by

$$W_n^{\langle ps\rangle} = \sum_{j=0}^{d} \beta_j W_{j,n}^{\langle ps\rangle}.$$

In the same manner, Lemma 4.4 and Theorem 4.6 imply that, for the factorial polynomial case, elementary fundamental particular solutions are

$$\widetilde{W}_{j,n}^{\langle ps\rangle} = \widetilde{P}_j(n) + \sum_{i=0}^{r-1} \widetilde{v}_i^{(j)}(1)\lambda_i^{n-r+1}$$

if $\lambda_i \neq 1$ for all $i$ $(0 \leq i \leq r-1)$, where $\widetilde{P}_j(n)$ is given by (21) above, and

$$\widetilde{v}_i(x) = (-1)^j \frac{\alpha_i x^{j-r}}{\lambda_i x - 1}.$$

And if $\lambda_0 = 1$, we have

$$\widetilde{W}_{j,n}^{\langle ps\rangle} = \widetilde{P}_j(n) + \sum_{i=0}^{r-1} \widetilde{v}_i^{(j)}(1)\lambda_i^{n-r+1},$$

where $\widetilde{P}_j(n)$ is given by (22) above. For $C(n) = \sum_{j=0}^{d} \beta_j n^{(j)}$, the fundamental particular solution $\{\widetilde{W}_n^{\langle ps\rangle}\}_{n\geq 0}$ of (1) may be expressed as

$$\widetilde{W}_n^{\langle ps\rangle} = \sum_{j=0}^{d} \beta_j \widetilde{W}_{j,n}^{\langle ps\rangle}.$$

More precisely, Lemmas 4.1 and 4.4, Theorems 4.2 and 4.6, and Proposition 4.8 imply

***Proposition 4.9:*** Let $\{T_n\}_{n\geq 0}$ be a solution of (1) and suppose that the characteristic roots $\lambda_0, \ldots, \lambda_{r-1}$ are simple. Then

***(a)*** For $C(n) = \sum_{j=0}^{d} \beta_j n^j$, the fundamental particular solution $\{W_n^{\langle ps\rangle}\}_{n\geq 0}$ of (1) is

$$W_n^{\langle ps\rangle} = \sum_{j=0}^{d} \beta_j P_j(n) + \sum_{i=0}^{r-1} \left( \sum_{j=0}^{d} \beta_j v_i^{(j)}(0) \right) \lambda_i^{n-r+1}$$

if $\lambda_i \neq 1$ for all $i$ $(0 \leq i \leq r-1)$, where

$$v_i(x) = \frac{\alpha_i e^{rx}}{e^x - \lambda_i}$$

and $P_j(n)$ is given by (14). And if $\lambda_0 = 1$, we have

$$W_n^{\langle ps\rangle} = \sum_{j=0}^{d} \beta_j P_j(n) + \sum_{i=1}^{r-1} \left( \sum_{j=0}^{d} \beta_j v_i^{(j)}(0) \right) \lambda_i^{n-r+1},$$

where $P_j(n)$ is given by (15).

***(b)*** For $C(n) = \sum_{j=0}^{d} \beta_j n^{(j)}$, the fundamental particular solution $\{\widetilde{W}_n^{\langle ps\rangle}\}_{n\geq 0}$ of (1) is

$$\widetilde{W}_n^{\langle ps\rangle} = \sum_{j=0}^{d} \beta_j \widetilde{P}_j(n) + \sum_{i=0}^{r-1} \left( \sum_{j=0}^{d} \beta_j \widetilde{v}_i^{(j)}(1) \right) \lambda_i^{n-r+1}$$

if $\lambda_i \neq 1$ for all $i$ $(0 \leq i \leq r-1)$, where

$$\widetilde{v}_i(x) = (-1)^j \frac{\alpha_i x^{j-r}}{\lambda_i x - 1}$$

and $\widetilde{P}_j(n)$ is given by (21). And if $\lambda_0 = 1$, we have

$$\widetilde{W}_n^{\langle ps\rangle} = \sum_{j=0}^{d} \beta_j \widetilde{P}_j(n) + \sum_{i=1}^{r-1} \left( \sum_{j=0}^{d} \beta_j \widetilde{v}_i^{(j)}(1) \right) \lambda_i^{n-r+1},$$

where $\widetilde{P}_j(n)$ is given by (22).

## 5. CONCLUDING REMARKS

***Remark 5.1: Relation with Genocchi and Bernoulli Numbers.*** In the $j^{\text{th}}$ derivative of $H_{p,n}(x)$ $(p = 1, 2)$ given by (11)-(12) appears the $k^{\text{th}}$ $(0 \leq k \leq j)$ derivative of functions $u_i(x) = \frac{\alpha_i}{e^x - \lambda_i}$. Let $u(x) = \frac{\alpha}{e^x - \lambda}$, where $\lambda < 0$, then

$$u(x) = \nu\frac{1}{e^{x+\beta}+1} = \frac{2\nu}{x+\beta}v(x+\beta),$$

where $\nu = -\frac{\alpha}{\lambda}$, $\beta = -\ln(-\lambda)$, and $v(t) = \frac{2t}{e^t+1}$. The Genocchi numbers $G_n$ $(n \geq 0)$ are defined by

$$\sum_{n=0}^{+\infty} G_n \frac{t^n}{n!} = v(t)$$

(see [5] and [8]). So, because $G_0 = 0$, we have

$$u(x) = \frac{1}{2\nu}\sum_{n=0}^{+\infty} G_{n+1}\frac{(x+\beta)^n}{n!} = \frac{1}{2\nu}\sum_{n=0}^{+\infty}\left(\sum_{k=n}^{+\infty}\frac{G_{n+1}}{(n-k)!(k+1)}\beta^{k-n}\right)\frac{x^n}{n!}.$$

Particularly, for $\lambda = -1$, we have

$$u(x) = \frac{1}{2\alpha}\sum_{n=0}^{+\infty} G_{n+1}\frac{x^n}{n!}.$$

If $\lambda_0 = 1$ is a simple characteristic root, we may take, for any $x \neq 0$, $G_n(x) = \alpha_0 h_n(x) w(x)$, where $h_n(x) = \frac{e^{(n-r+1)x}-1}{x}$ and $w(x) = \frac{x}{e^x - 1}$. Expansion series of these two functions are

$$h_n(x) = \sum_{k=0}^{+\infty}\frac{(n-r+1)^k}{k+1}\frac{x^k}{k!}, \quad w(x) = \sum_{k=0}^{+\infty} B_k \frac{x^k}{k!},$$

where $B_k$ are the Bernoulli numbers (see, e.g., [5] and [8]). Then Leibnitz's formula

$$G_n^{(k)}(x) = \alpha_0 \sum_{i=0}^{k}\binom{i}{k} h_n^{(i)} u^{(k-i)}(x)$$

implies that

$$G_n^{(k)}(0) = \alpha_0 \sum_{i=0}^{k}\binom{i}{k}\frac{(n-r+1)^i}{i+1}B_{k-i}.$$

Hence, Asveld's polynomials $P_j(n)$ $(0 \leq j \leq d)$ depend on the Genocchi and Bernoulli numbers when $\lambda < 0$ or $\lambda_0 = 1$.

***Remark 5.2: Degree of $P_j(n)$ and $\widetilde{P}_j(n)$.*** Theorems 4.2, 4.6, and 4.7 show that Asveld's polynomials $P_j(n)$ and factorial polynomials $\widetilde{P}_j(n)$ $(0 \leq j \leq d)$ are of degree $j+m$, where $m$ is the multiplicity of $\lambda_0 = 1$. This property is established by the two last authors using an alternative method for solving (1), which is the subject of another paper.

***Remark 5.3: The Case of Multiplicities $\geq 1$.*** In Section 4 we considered that the characteristic roots are simple except for Theorem 4.7, where $\lambda_0 = 1$ is supposed of multiplicity $m \geq 1$. The problem is to derive the particular polynomial or factorial polynomial solutions of (1) using the method of Section 3 when the characteristic roots $\lambda_0, \ldots, \lambda_p$ $(p \leq r-1)$ are of arbitrary multiplicities $m_0, \ldots, m_p$.

## ACKNOWLEDGMENT

The authors would like to express their sincere gratitude to the referee for several useful and valuable suggestions that improved the presentation of this paper. The third author would like to thank Professor A. Horadam for his encouragement and for sending him some of his papers. He also thanks Professor C. Cooper for some useful remarks.

## REFERENCES

1. A. Andrade & S. P. Pethe. "On the $r^{\text{th}}$-Order Nonhomogeneous Recurrence Relation and Some Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **30.3** (1992):256-62.
2. P. R. J. Asveld. "A Family of Fibonacci-Like Sequences." *The Fibonacci Quarterly* **25.1** (1987):81-83.
3. P. R. J. Asveld. "Another Family of Fibonacci-Like Sequences." *The Fibonacci Quarterly* **25.4** (1987):361-64.
4. P. R. J. Asveld. "Fibonacci-Like Differential Equation with a Polynomial Nonhomogeneous Part." *The Fibonacci Quarterly* **27.2** (1989):303-09.
5. L. Comtet. *Advanced Combinatorics*. Dordrecht: Reidel, 1974.
6. F. Dubeau, W. Mota, M. Rachidi, & O. Saeki. "On Weighted $r$-Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **35.2** (1997):137-47.
7. A. F. Horadam. "Falling Factorial Polynomials of Generalized Fibonacci Type." In *Applications of Fibonacci Numbers*, pp. 139-53. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1990.
8. A. F. Horadam. "Genocchi Polynomials." In *Applications of Fibonacci Numbers*, pp. 145-66. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1991.
9. A. F. Horadam & A. G. Shannon. "Asveld's Polynomials $p_j(n)$." In *Applications of Fibonacci Numbers*, pp. 163-76. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1988.
10. D. Kalman. "Generalized Fibonacci Sequences by Matrix Methods." *The Fiboacci Quarterly* **20.1** (1982):73-76.
11. M. Mouline & M. Rachidi. "Suites de Fibonacci généralisées et chaines de Markov." *Real Academia de Ciencias Exatas, Fisicas y Naturales de Madrid* **89.1-2** (1995):61-77.
12. J. C. Turner. "Note on a Family of Fibonacci-Like Sequences." *The Fibonacci Quarterly* **27.3** (1989):229-32.

AMS Classification Numbers: 40A05, 40A25, 45M05

❖❖❖

# AN EXTENSION OF AN OLD PROBLEM OF DIOPHANTUS AND EULER–II

**Andrej Dujella**
Dept. of Math., University of Zagreb, Bijenicka cesta 30, 10000 Zagreb, Croatia
*(Submitted December 1999-Final Revision May 2000)*

Diophantus found three rationals $\frac{3}{10}, \frac{21}{5}, \frac{7}{10}$ with the property that the product of any two of them increased by the sum of those two gives a perfect square (see [5], pp. 85-86, 215-217), and Euler found four rationals $\frac{65}{224}, \frac{9}{224}, \frac{9}{56}, \frac{5}{2}$ with the same property (see [4], pp. 518-519).

We will call a set $\{x_1, x_2, \ldots, x_m\}$ of $m$ rationals such that $x_i x_j + x_i + x_j$ is a perfect square for all $1 \le i < j \le m$ a *Eulerian m-tuple*.

In [8], we found the Eulerian quintuple

$$\left\{-\frac{27}{40}, \frac{17}{8}, \frac{27}{10}, 9, \frac{493}{40}\right\}. \tag{1}$$

This example leads us to the following questions: Is there any Eulerian quintuple consisting of positive rationals (this would be more in the style of Diophantus)? Are there infinitely many such quintuples? In the present paper we give affirmative answers to both questions.

We mention that it is not known whether there exists any Eulerian quadruple consisting of integers. In [3]. [10], and [12], it was proved that some particular Eulerian triples cannot be extended to an integer quadruple; in [7], it was proved that the Eulerian pair $\{0, 1\}$ cannot be extended to an integer quadruple.

Let $q$ be a rational number. A set $\{a_1, a_2, \ldots, a_m\}$ of $m$ nonzero rationals is called a *Diophantine m-tuple with the property* $D(q)$ if $a_i a_j + q$ is a perfect square for all $1 \le i < j \le m$ (see [6]). It is clear that $\{x_1, x_2, \ldots, x_m\}$ is a Eulerian $m$-tuple iff $\{x_1+1, x_2+1, \ldots, x_m+1\}$ is a Diophantine $m$-tuple with the property $D(-1)$.

In [8], we proved that the set

$$\begin{aligned}\big\{\tfrac{1}{3}(x^2+6x-18)(-x^2+2x+2),\ \tfrac{1}{3}x^2(x+5)(-x+3),\ (x-2)(5x+6),\\ \tfrac{1}{3}(x^2+4x-6)(-x^2+4x+6),\ 4x^2\big\}\end{aligned} \tag{2}$$

has the property $D(\frac{16}{9}x^2(x^2-x-3)(x^2+2x-12))$. From (2) for $x = \frac{5}{2}$, we obtain the Eulerian quintuple (1).

Consider the quartic curve

$$Q:\ y^2 = -(x^2-x-3)(x^2+2x-12).$$

We have a rational point $\left(\frac{5}{2}, \frac{3}{4}\right)$ on $Q$. Using the construction from [1], we find that, with the substitution

$$x = \frac{63s+10t+2619}{18s+4t+2403},\quad y = \frac{24s^3-6777s^2-12t^2-34749t+54898479}{(18s+4t+2403)^2}, \tag{3}$$

$Q$ is birationally equivalent to the elliptic curve

$$\begin{aligned}E:\ t^2 &= s^3-18981s-1001700\\ &= (s-159)(s+75)(s+84).\end{aligned}$$

Using the program package SIMATH (see [14]), we obtain the following information about curve $E$: $E(\mathbf{Q})_{\text{tors}} \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, $E(\mathbf{Q})_{\text{tors}} = \{\mathcal{O}, A = (159, 0),\ B = (-75, 0),\ C = (-84, 0)\}$, rank $E(\mathbf{Q}) = 1$, $E(\mathbf{Q})/E(\mathbf{Q})_{\text{tors}} = \langle P \rangle$, where $P = (2103, -96228)$. The author is grateful to the referee for the observation that the minimal equation for $E$ is $v^2 = u^3 - u^2 - 234u - 1296$. This is curve 1248E1 in John Cremona's online tables, which confirm that the rank of $E$ is equal to 1.

As a direct consequence of the fact that rank $E(\mathbf{Q}) = 1$, we conclude that there are infinitely many rational points on $Q$. By (2), we obtain infinitely many Diophantine quintuples with the property $D(-\frac{16}{9}x^2y^2)$, and multiplying elements of these quintuples by $\frac{3}{4xy}$ we obtain quintuples with the property $D(-1)$. Therefore, we have proved

***Theorem 1:*** There exist infinitely many Diophantine quintuples with the property $D(-1)$.

***Corollary 1:*** There exist infinitely many Eulerian quintuples.

The next question is: Which points $(s, t)$ on $E(\mathbf{Q})$ induce Eulerian quintuples with positive elements or, equivalently, Diophantine quintuples with the property $D(-1)$ whose elements are $> 1$?

Therefore, we would like to find the points $(x, y)$ on $Q$ such that the five rationals

$$\frac{(x^2+6x-18)(-x^2+2x+2)-4xy}{4xy},\ \frac{x(x+5)(-x+3)-4y}{4y},\ \frac{3(x-2)(5x+6)-4xy}{4xy}$$
$$\frac{(x^2+4x-6)(-x^2+4x+6)-4xy}{4xy},\ \text{and}\ \frac{3x-y}{y}$$

are all positive. Let us denote these five expressions by $R_1(x, y), \ldots, R_5(x, y)$. First of all, from $(x^2-x-3)(x^2+2x-12) = -y^2 \le 0$, it follows that

$$x \in \left[-1-\sqrt{13},\ \frac{1-\sqrt{13}}{2}\right] \cup \left[\frac{1+\sqrt{13}}{2},\ -1+\sqrt{13}\right]. \tag{4}$$

Here,

$$-1-\sqrt{13} \approx -4.605551275464,\ \frac{1-\sqrt{13}}{2} \approx -1.302775637732,$$
$$\frac{1+\sqrt{13}}{2} \approx 2.302775637732,\ -1+\sqrt{13} \approx 2.605551275464.$$

Set $\alpha = \frac{1+\sqrt{13}}{2}$ and $\beta = \frac{-1+\sqrt{13}}{2}$. Then condition (4) may be written in the form

$$x \in [-2\alpha, -\beta] \cup [\alpha, 2\beta].$$

Assume first that $y > 0$. Then we find (using MATHEMATICA) that $R_1(x, y) > 0$ if and only if $x \in \langle \alpha, x^{(1)} \rangle \cup \langle x^{(2)}, 2\beta \rangle$, where

$$x^{(1)} \approx 2.306300513595,\ x^{(2)} \approx 2.601569034318;$$

$R_2(x, y) > 0$ iff $x \in \langle \alpha, 2\beta \rangle$; $R_3(x, y) > 0$ iff $x \in \langle \alpha, 2\beta \rangle$; $R_4(x, y) > 0$ iff $x \in \langle \alpha, 2\beta \rangle$; $R_5(x, y) > 0$ iff $x \in \langle \alpha, 2\beta \rangle$.

Assume now that $y < 0$. Then we find that $R_1(x, y) > 0$ iff $x \in \langle -2\alpha, -\beta \rangle$; $R_2(x, y) > 0$ iff $x \in \langle 2\alpha, x^{(3)} \rangle \cup \langle -3, -\beta \rangle$, where

$$x^{(3)} \approx -4.482360405707;$$

$R_3(x,y) > 0$ iff $x \in \langle -2\alpha, -2\rangle \cup \langle x^{(4)}, -\beta\rangle$, where

$$x^{(4)} \approx -1.338580448007;$$

$R_4(x,y) > 0$ iff $x \in \langle -2\alpha, -\beta\rangle$; $R_5(x,y) > 0$ iff $x \in \langle -2\alpha, -3\rangle \cup \langle -2, -\beta\rangle$.

Summarizing these computations, we may write that $R_i(x,y) > 0$ for $i = 1, \ldots, 5$ iff

$$x \in \langle \alpha, x^{(1)}\rangle \cup \langle x^{(2)}, 2\beta\rangle,\ y > 0 \ \text{ or } \ x \in \langle -2\alpha, x^{(3)}\rangle \cup \langle x^{(4)}, -\beta\rangle,\ y < 0. \tag{5}$$

We can see also that we have only three possibilities for the signs of $R_i(x,y)$, $i = 1, \ldots, 5$. Namely, we may have zero, one, or five negative numbers among them. This is not surprising. Indeed, it is a consequence of the following simple fact.

***Proposition 1:*** There does not exist a Eulerian triple $\{x_1, x_2, x_3\}$ such that $x_1 > 0$, $x_2 < 0$, and $x_3 < 0$.

***Proof:*** Let $y_2 = -x_2$ and $y_3 = -x_3$. Since $-x_1y_2 - y_2 + x_1 \geq 0$, we have $y_2 < 1$ and, similarly, $y_3 < 1$. On the other hand, $y_2y_3 - y_2 - y_3 \geq 0$ implies $y_2y_3 \geq 4$, a contradiction. □

Now we may determine the points on $E$ such that the corresponding points $(x,y)$ on $Q$ satisfy (5). Using (3), we obtain that these points are

$$s \in \langle s^{(1)}, s^{(2)}\rangle \cup \langle s^{(3)}, s^{(4)}\rangle,\ t > 0 \ \text{ or } \ s \in \langle s^{(5)}, s^{(6)}\rangle \cup \langle s^{(7)}, s^{(8)}\rangle,\ t < 0, \tag{6}$$

where

$$\begin{aligned} s^{(1)} &\approx -79.224984709848, & s^{(2)} &\approx -76.849933010661,\\ s^{(3)} &\approx 458.63743164323, & s^{(4)} &\approx 937.53800125946,\\ s^{(5)} &\approx -82.093984103146, & s^{(6)} &\approx -79.690329099008,\\ s^{(7)} &\approx 232.03689724592, & s^{(8)} &\approx 348.76934786866. \end{aligned}$$

Our final task is to determine rational points on $E$ which satisfy (6). We know that rational points on $E$ have the form $X = T + mP$, where $T \in \{\mathcal{O}, A, B, C\}$ and $m \in \mathbf{Z}$.

We may parameterize elliptic curve $E$ by the Weierstrass function

$$s = \wp(z), \quad t = \tfrac{1}{2}\wp'(z).$$

We will denote the parameter $z$ corresponding to the point $X = (s,t)$ by $\omega(X)$. The Weierstrass $\wp$-function is periodic, with complex and real periods given by

$$\omega_1 = i\int_{-\infty}^{-84} \frac{ds}{\sqrt{1001700 + 18981s - s^3}} \approx 0.391753653118i,$$

$$\omega_2 = \int_{159}^{+\infty} \frac{ds}{\sqrt{s^3 - 18981s - 1001700}} \approx 0.203439216566$$

(see [11], pp. 22-29). We have $\omega(A) = \frac{\omega_2}{2}$, $\omega(B) = \frac{\omega_2}{2} + \frac{\omega_1}{2}i$, $\omega(C) = \frac{\omega_1}{2}i$. Using PARI [2], we find that $\omega(P) = \sigma$, where

$$\sigma \approx 0.0218157627564.$$

Also using PARI, we find that condition (6) is equivalent to

$$\omega(X) \in \left\langle \gamma^{(1)}, \delta^{(1)}\right\rangle \cup \left\langle \gamma^{(1)} + \frac{\omega_2}{2}, \delta^{(1)} + \frac{\omega_2}{2}\right\rangle \tag{7}$$

or

$$\omega(X)-\frac{\omega_1}{2}i \in \langle \gamma^{(2)}, \delta^{(2)}\rangle \cup \left\langle \gamma^{(2)}+\frac{\omega_2}{2}, \delta^{(2)}+\frac{\omega_2}{2}\right\rangle, \tag{8}$$

where

$$\gamma^{(1)} \approx 0.0545490289958, \qquad \delta^{(1)} \approx 0.0689863420434,$$
$$\gamma^{(2)} \approx 0.0525347833467, \qquad \delta^{(2)} \approx 0.0710005876925.$$

Note that points $X$ and $A+X$ induce the same quintuple. Namely, if $X$ induces the point $(x, y)$ on $Q$, then $A+X$ induces the point $(\frac{6}{x}, \frac{6y}{x^2})$, and the only effect of these changes on $R_i$'s is the permutation of $R_2(x, y)$ and $R_3(x, y)$. Therefore, it suffices to consider the points of the form $mP$ and $C+mP$.

The point $X = mP$ satisfies condition (7) iff $m\sigma \bmod \frac{\omega_2}{2} \in \langle \gamma^{(1)}, \delta^{(1)}\rangle$ or, equivalently,

$$m\cdot\left(\frac{2\sigma}{\omega_2}\right) \bmod 1 \in \left\langle \frac{2\gamma^{(1)}}{\omega_2}, \frac{2\delta^{(1)}}{\omega_2}\right\rangle. \tag{9}$$

Analogously, the point $X = C+mP$ satisfies condition (8) iff

$$m\cdot\left(\frac{2\sigma}{\omega_2}\right) \bmod 1 \in \left\langle \frac{2\gamma^{(2)}}{\omega_2}, \frac{2\delta^{(2)}}{\omega_2}\right\rangle. \tag{10}$$

Assume that $\frac{2\sigma}{\omega_2} = \frac{k}{l} \in \mathbf{Q}$. Then $\omega(2lP) = 0$, which means that $P$ is a torsion point, a contradiction. Therefore, $\frac{2\sigma}{\omega_2}$ is an irrational number and we may apply Bohl-Sierpinski-Weyl theorem (see [13], pp. 24-27), which implies that the sequence $\{m\cdot(\frac{2\sigma}{\omega_2}) \bmod 1\}$ is dense in [0, 1].

Therefore, there are infinitely many integers $m$ that satisfy condition (9), resp. (10), and then the corresponding points $mP$, $C+mP$ on $E(\mathbf{Q})$ satisfy conditions (7), resp. (8).

Hence, we have proved

***Theorem 2:*** There exist infinitely many Eulerian quintuples consisting of positive rationals.

***Example 1:*** Condition (9) can be approximated by

$$m\cdot 0.214469590718 \bmod 1 \in \langle 0.536286571189, 0.678201019526\rangle,$$

and condition (10) by

$$m\cdot 0.214469590718 \bmod 1 \in \langle 0.51646663051, 0.698002960205\rangle.$$

It is easy to find "small solutions" of (9):

$$m \in M_1 = \{\ldots, -100, -95, -86, -81, -72, -67, -58, -53, -44, -39, -30, -25, -16, -11, -2, 3, 12, 17, 26, 31, 40, 45, 54, 59, 68, 73, 82, 87, 96, \ldots\},$$

and of (10):

$$m \in M_2 = \{\ldots, -100, -95, -90, -86, -81, -72, -67, -58, -53, -44, -39, -30, -25, -16, -11, -2, 3, 12, 17, 26, 31, 40, 45, 54, 59, 68, 73, 82, 87, 91, 96, \ldots\}.$$

Note that, for $i = 1, 2$, $m \in M_i$ holds if and only if $1-m \in M_i$. Namely, the points $mP$ and $A+(1-m)P$ induce the same point on $Q$. This fact explains why $\gamma^{(1)}+\delta^{(1)} = \gamma^{(2)}+\delta^{(2)} = \sigma+\frac{\omega_2}{2}$.

Note also that, for "many" elements $m$ of the set $M_i$, $i=1,2$, $m+28 \in M_i$ holds. This happens because $28\sigma$ is close to $3\omega_2$.

The Eulerian quintuples induced by the points $-2P$ and $C-2P$ are listed in the following table:

| point on $E$ | Eulerian quintuple |
|---|---|
| $-2P$ | $\left\{\frac{1225373882407176816090280933127280538 1}{13356284738726537361337339615814680856}, \frac{402280625581345978468093983 33}{20273776660492527125756260 72}, \frac{904102036076757756322317387 35}{264016552841465436885252699 8}, \frac{14592496608158331417199201827533275885 89}{13356284738726537361337339615814680856}, \frac{1646347887706876161 5}{200378051669604563}\right\}$ |
| $C-2P$ | $\left\{\frac{24384004810826647895250908584025016017}{12260187519716576269892403630624702 20}, \frac{1117453457253188077607784537 3}{1225575724730803312553801852}, \frac{2004087612633081351104639 18}{2004504853296123500054560 55}, \frac{287670780013453292618651769213853277 7}{122601875197165762698924036306247022 0}, \frac{1329253988561517422}{200378051669604563}\right\}$ |

***Remark 1:*** In the same manner as in the proof of Theorem 2, we can prove that there are infinitely many Eulerian quintuples consisting of negative rationals, and infinitely many Eulerian quintuples consisting of one negative and four rationals.

***Remark 2:*** In [9], we asked the following question: For which nonzero rationals $q$ do there exist infinitely many rational Diophantine quintuples with the property $D(q)$? It is clear that it suffices to consider square-free integers $q$. It was already known to Euler that there exist infinitely many rational Diophantine quintuples with the property $D(1)$ (see [4], p. 517). In [9], we gave an affirmative answer to the above question for $q=-3$, and Theorem 1 solves the case $q=-1$. In our forthcoming paper, we will give an affirmative answer to the above question for a large class of rationals $q$, including 114 integers in the range $-100 \le 2 \le 100$.

## REFERENCES


1. A. O. L. Atkin & F. Morain. "Finding Suitable Curves for the Elliptic Curve Method of Factorization." *Math. Comp.* **60** (1993):399-405.
2. C. Batut, D. Bernardi, H. Cohen, & M. Olivier. GP/PARI, Université Bordeux I, 1994.
3. E. Brown. "Sets in Which $xy+k$ Is Always a Square." *Math. Comp.* **45** (1985):613-20.
4. L. E. Dickson. *History of the Theory of Numbers.* Vol. 2. New York: Chelsea, 1966.
5. Diophantus of Alexandria. *Arithmetics and the Book of Polygonal Numbers.* Ed. I. G. Bashmakova. Moscow: Nauka, 1974. (In Russian.)

6. A. Dujella. "On Diophantine Quintuples." *Acta Arith.* **81** (1997):69-79.
7. A. Dujella. "Complete Solution of a Family of Simultaneous Pellian Equations." *Acta Math. Inform. Univ. Ostraviensis* **6** (1998):59-67.
8. A. Dujella. "An Extension of an Old Problem of Diophantus and Euler." *The Fibonacci Quarterly* **37.4** (1999):312-14.
9. A. Dujella. "A Note on Diophantine Quintuples." *Algebraic Number Theory and Diophantine Analysis*, pp. 123-27. Ed. F. Halter-Koch and R. F. Tuchy. Berlin: Walter de Gruyter, 2000.
10. K. S. Kedlaya. "Solving Constrained Pell Equations." *Math. Comp.* **67** (1998):833-42.
11. N. Koblitz. *Introduction to Elliptic Curves and Modular Forms.* New York: Springer-Verlag, 1993.
12. S. P. Mohanty & A. M. S. Ramasamy. "The Simultaneous Diophantine Equations $5y^2 - 20 = x^2$ and $2y^2 + 1 = z^2$." *J. Number Theory* **18** (1984):356-59.
13. I. Niven. *Diophantine Approximations.* New York: John Wiley, 1963.
14. SIMATH Manual. Saarbrücken: Universität des Saarlandes, 1997.


AMS Classification Numbers: 11G05, 11D09

❖❖❖

# THE MULTIPLE SUM ON THE GENERALIZED LUCAS SEQUENCES

**Ping-an He and Zhizheng Zhang**
Institute of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P.R. China
(*Submitted January 2000-Final Revision September 2000*)

The computation of the multiple sum on a linear recurrence sequence is an interesting question. Many fine results have been given. This paper will establish a computational formula for the multiple sum on the generalized Lucas sequence. A new method will be used and some congruence relations will be given.

We define a linear recurrence sequence $W_n = W_n(a, b; p, q)$, $n = 0, 1, \ldots$, as

$$\begin{aligned} &W_n = pW_{n-1} - qW_{n-2} \quad (n \geq 2) \\ &W_0 = a,\ W_1 = b. \end{aligned}$$

We consider the sequence

$$\begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p; p, q). \end{cases}$$

Then $U_n$ and $V_n$ are called the generalized Fibonacci sequence and the generalized Lucas sequence. Their Binet formulas are, respectively,

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}.$$

In [2], W. Zhang gave a computational formula involving the multiple sum on the generalized Fibonacci sequence when $U_0 = 0$.

In this paper, we shall use another method (formal power) to establish a computational formula for the multiple sum on the generalized Lucas sequence, i.e.,

$$\sum_{a_1 + a_2 + \cdots + a_k = n} V_{a_1} V_{a_2} \cdots V_{a_k},$$

where the summation is taken over all $n$-tuples with positive coordinates $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$.

The generating function of the Generalized Lucas sequence $\{V_n\}_0^\infty$ is

$$H(x) = \sum_{n \geq 0} V_n x^n = \frac{2 - px}{1 - px + qx^2}.$$

Let

$$H_k(x) = \left( \frac{2 - px}{1 - px + qx^2} \right)^k = \sum_{n \geq 0} V_n^{(k)} x^n.$$

Obviously, $V_n^{(1)} = V_n$. Then

$$\sum_{a_1 + a_2 + \cdots + a_m = n} V_{a_1}^{(k_1)} V_{a_2}^{(k_2)} \cdots V_{a_m}^{(k_m)} = V_n^{(k_1 + k_2 + \cdots + k_m)}.$$

If we take $k_1 = k_2 = \cdots = k_m = 1$, we obtain the following lemma.

***Lemma:*** $\sum_{a_1+a_2+\cdots+a_m=n} V_{a_1}V_{a_2}\cdots V_{a_m} = V_n^{(m)}$.

***Theorem 1:*** Let $V_n^{(k)}$ be defined as above. Then

$$V_n^{(k+1)} = \frac{1}{k(p^2-4q)}\{4(n+2)V_{n+2}^{(k)} - 2p(2n+k+2)V_{n+1}^{(k)} + p^2(n+k)V_n^{(k)}\}.$$

***Proof:*** We note the following equalities:

$$\begin{aligned}
\frac{d}{dx}(H_k(x)) &= \frac{d}{dx}\left(\frac{2-px}{1-px+qx^2}\right)^k \\
&= k\left(\frac{2-px}{1-px+qx^2}\right)^{k-1}\frac{-p(1-px+qx^2)+(2-px)(p-2qx)}{(1-px+qx^2)^2} \\
&= k\left(\frac{2-px}{1-px+qx^2}\right)^{k-1}\frac{-p+p^2x-pqx^2+2p-4qx-p^2x+2pqx^2}{(1-px+qx^2)^2} \\
&= k\left(\frac{2-px}{1-px+qx^2}\right)^{k-1}\frac{p-4qx+pqx^2}{(1-px+qx^2)^2} \\
&= k\left(\frac{2-px}{1-px+qx^2}\right)^{k-1}\frac{p(1-px+qx^2)+(p^2-4q)x}{(1-px+qx^2)^2} \\
&= \frac{kp}{2-px}\left(\frac{2-px}{1-px+qx^2}\right)^k + \frac{k(p^2-4q)x}{(2-px)^2}\left(\frac{2-px}{1-px+qx^2}\right)^{k+1}.
\end{aligned}$$

Thus,

$$kx(p^2-4q)\left(\frac{2-px}{1-px+qx^2}\right)^{k+1} = (2-px)^2\frac{d}{dx}\left(\frac{2-px}{1-px+qx^2}\right)^k - kp(2-px)\left(\frac{2-px}{1-px+qx^2}\right)^k.$$

So

$$kx(p^2-4q)\sum_{n\geq 0}V_n^{(k+1)}x^n = (4-4px+p^2x^2)\sum_{n\geq 0}nV_n^{(k)}x^{n-1} - kp(2-px)\sum_{n\geq 0}V_n^{(k)}x^n.$$

Comparing coefficients on both sides of the equation, we have

$$\begin{aligned}
k(p^2-4q)V_{n-1}^{(k+1)} &= 4(n+1)V_{n+1}^{(k)} - 4npV_n^{(k)} + p^2(n-1)V_{n-1}^{(k)} - 2kpV_n^{(k)} + kp^2V_{n-1}^{(k)} \\
&= 4(n+1)V_{n+1}^{(k)} - 2p(2n+k)V_n^{(k)} + p^2(n+k-1)V_{n-1}^{(k)}.
\end{aligned}$$

This completes the proof. □

Taking $k = 1, 2, 3, 4$ in the lemma and using Theorem 1, we obtain the following results.

***Theorem 2:*** Let $(V_n)$ be defined as above. We have the following identities:

***(a)*** $\sum_{a+b=n} V_aV_b = \frac{1}{p^2-4q}\{2pV_{n+1} + [(n+1)(p^2-4q)-4q]V_n\}$.

***(b)*** $\displaystyle\sum_{a+b+c=n} V_aV_bV_c = \frac{n+2}{2(p^2-4q)}\{6pV_{n+1}+[(n+1)(p^2-4q)-12q]V_n\}.$

***(c)*** $$\sum_{a+b+c+d=n} V_aV_bV_cV_d = \frac{1}{3!(p^2-4q)}\{12[(n+3)_2(p^2-4q)+(n+1)(p^2-4q)-2q]V_{n+2} + [(n+3)_3(p^2-4q)^2-12q(n+3)_2(p^2-4q)+24q^2]V_n\}.$$

***(d)*** $$\sum_{a+b+c+d+e=n} V_aV_bV_cV_dV_e = \frac{1}{4!(p^2-4q)^2}\{48(n+1)(n+4)V_{n+4}-4[12q(n+2)(n^2+10n+23) - 2(n+4)_3(p^2-4q)-3p^2(n+4)(n^2+6n+7)+6nq]V_{n+2} + (n+4)[48q^2(n+3)^2+(n+3)_3(p^2-4q)^2-8q(n+3)_2(p^2-4q) -12p^2q(n+3)_2+24q^2]V_n\}.$$

Here, $(n)_k = n(n-1)(n-2)\cdots(n-k+1)$.

***Theorem 3:*** Under the conditions of Theorem 2, we have the following:

***(a)*** $2V_{n+2}-2qV_n \equiv 0 \pmod{p^2-4q}$.

***(b)*** $12[(n+3)_2(p^2-4q)+(n+1)(p^2-4q)-2q]V_{n+2} + [(n+3)_3(p^2-4q)^2-12q(n+3)_2(p^2-4q)+24q^2]V_n \equiv 0 \pmod{3!(p^2-4q)^2}$.

***(c)*** $(n+4)[48q^2(n+3)^2+(n+3)_3(p^2-4q)^2-8q(n+3)_2(p^2-4q)-12p^2q(n+3)_2+24q^2]V_n - 4[12q(n+2)(n^2+10n+23)-2(n+4)_3(p^2-4q)-3p^2(n+4)(n^2+6n+7)+6nq]V_{n+2} + 48(n+1)(n+4)V_{n+4} \equiv 0 \pmod{4!(p^2-4q)^2}$.

***Proof:*** Use Theorem 2(a), (c), (d).

Taking $p=-q=1$, $V_n=L_n$ is the Lucas sequence, i.e., $L_0=2$, $L_1=1$, $L_2=3$, $L_3=4$, ... . Thus, from Theorem 2, we obtain Corollaries 1 and 2.

***Corollary 1:*** Let $(L_n)$ be the Lucas sequence. Then we have the following:

***(a)*** $\displaystyle\sum_{a+b=n} L_aL_b = \frac{1}{5}\{2L_{n+1}+(5n+9)L_n\}.$

***(b)*** $\displaystyle\sum_{a+b+c=n} L_aL_bL_c = \frac{n+2}{10}\{6L_{n+1}+(5n+17)L_n\}.$

***(c)*** $\displaystyle\sum_{a+b+c+d=n} L_aL_bL_cL_d = \frac{1}{150}\{12(5n^2+30n+37)L_{n+1}+(25n^3+270n^2+935n+978)L_n\}.$

***(d)*** $$\sum_{a+b+c+d+e=n} L_aL_bL_cL_dL_e = \frac{1}{600}\{48(n+1)(n+4)L_{n+4}+4(25n^3+264n^2+875n+876)L_{n+2}\} + \frac{1}{600}[(n+3)(n+4)(25n^2+175n+298)+24(n+4)]L_n.$$

***Corollary 2:*** Let $(L_n)$ be the Lucas sequence. Then we have the following congruences:

***(a)*** $L_{n+2}+L_n \equiv 0 \pmod 5$.

***(b)*** $12(5n^2+30n+37)L_{n+1}+(25n^3+120n^2+35n+78)L_n \equiv 0 \pmod{150}$.

*(c)* $48(n+1)(n+4)L_{n+4}+4(25n^3+264n^2+875n+876)L_{n+2}$
$+[(n+3)(n+4)(25n^2+175n+298)+24(n+4)]L_n \equiv 0 \pmod{600}$.

First, we gave the multiple sum on the generalized Lucas sequence. Then, we discussed the multiple sum on the even generalized Lucas sequence. Now

$$\sum_{n\geq 0} V_{2n}x^{2n}=\frac{1}{2}\left\{\frac{2-px}{1-px+qx^2}+\frac{2+px}{1+px+qx^2}\right\}=\frac{2-(p^2-2q)x^2}{1-(p^2-2q)x^2+q^2x^4}.$$

We use methods similar to those employed above. Let

$$\sum_{n\geq 0} R_{2n}^{(k)}x^n=\left(\frac{2-(p^2-2q)x}{1-(p^2-2q)x+q^2x^2}\right)^k.$$

Obviously, $R_{2n}^{(1)}=V_{2n}$,

$$\sum_{a_1+a_2+\cdots+a_m=n} V_{2a_1}V_{2a_2}\cdots V_{2a_m}=R_{2n}^{(m)},$$

$$R_{2n}^{(k+1)}=\frac{1}{kp^2(p^2-4q)}\{4(n+2)R_{2n+4}^{(k)}-2(p^2-2q)(2n+k+2)R_{2n+2}^{(k)}+(p^2-2q)^2(n+k)R_{2n}^{(k)}\}.$$

Hence, we have the following theorems.

***Theorem 4:*** $\displaystyle\sum_{a+b=n} V_{2a}V_{2b}=\frac{1}{p^2(p^2-4q)}\{2(p^2-2q)V_{2n+2}+[(n+1)p^2(p^2-4q)-4q^2]V_{2n}\}$,

$$\sum_{a+b+c=n} V_{2a}V_{2b}V_{2c}=\frac{1}{2p^2(p^2-4q)}\{6(p^2-2q)V_{2n+2}+[(n+1)p^2(p^2-4q)-12q^2]V_{2n}\}.$$

***Theorem 5:*** $2(p^2-2q)V_{2n+2}-4q^2V_{2n}\equiv 0 \pmod{p^2(p^2-4q)}$.

$$6(p^2-2q)V_{2n+2}+\{p^2(n+1)(p^2-4q)-12q^2]V_{2n}\equiv 0 \pmod{2p^2(p^2-4q)}.$$

## REFERENCES


1. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.2** (1965):161-76.
2. Wenpeng Zhang. "Some Identities Involving the Fibonacci Numbers." *The Fibonacci Quarterly* **35.3** (1997):225-29.


AMS Classification Numbers: 11B37, 11B39, 06B10

❖❖❖

# ON THE SUMMATION OF GENERALIZED ARITHMETIC-GEOMETRIC TRIGONOMETRIC SERIES


**Ziqing Xie**

Department of Mathematics, Hunan Normal University, Changsha 410081, China



*(Submitted January 2000-Final Revision June 2000)*


## 1. INTRODUCTION

For any real or complex number $\beta$, we denote

$$(t\,|\,\beta)_p = \prod_{j=0}^{p-1}(t - j\beta),$$

where $p$ is a positive integer, with $(t\,|\,\beta)_0 = 1$, and call it the generalized falling factorial with increment $\beta$. In particular, we write $(t\,|\,1)_p = (t)_p$ and $(t\,|\,0)_p = t^p$. It is known that the Dickson polynomial in $t$ of degree $p$ with real parameter $\alpha$ is defined as

$$D_p(t, \alpha) = \sum_{i=0}^{[p/2]} \frac{p}{p-i}\binom{p-i}{i}(-\alpha)^i t^{p-2i} \tag{1.1}$$

with $D_0(t, \alpha) = 2$ (cf. [4]). Evidently $D_p(t, 0) = t^p$.

In this paper, we find closed summation formulas for the series

$$S_1^{(1)}(n) = \sum_{k=0}^{n}(k+\lambda\,|\,\beta)_p r^k \cos k\theta, \quad S_1^{(1)}(\infty) = \sum_{k=0}^{\infty}(k+\lambda\,|\,\beta)_p r^k \cos k\theta, \tag{1.2}$$

$$S_1^{(2)}(n) = \sum_{k=0}^{n}(k+\lambda\,|\,\beta)_p r^k \sin k\theta, \quad S_1^{(2)}(\infty) = \sum_{k=0}^{\infty}(k+\lambda\,|\,\beta)_p r^k \sin k\theta, \tag{1.3}$$

$$S_2^{(1)}(n) = \sum_{k=\alpha}^{n+\alpha} D_p(k, \alpha) r^k \cos k\theta, \quad S_2^{(1)}(\infty) = \sum_{k=\alpha}^{\infty} D_p(k, \alpha) r^k \cos k\theta, \tag{1.4}$$

$$S_2^{(2)}(n) = \sum_{k=\alpha}^{n+\alpha} D_p(k, \alpha) r^k \sin k\theta, \quad S_2^{(2)}(\infty) = \sum_{k=\alpha}^{\infty} D_p(k, \alpha) r^k \sin k\theta, \tag{1.5}$$

where $\alpha$ is any given integer, $\lambda$ and $\beta$ are real numbers, and $|r| < 1$ for $S_i^{(j)}(\infty)$, $i, j = 1, 2$.

In [2], L. C. Hsu and P. J. S. Shiue have obtained closed summation formulas for the series

$$S_1(n) = \sum_{k=0}^{n}(k+\lambda\,|\,\beta)_p x^k, \; S_1(\infty) = \sum_{k=0}^{\infty}(k+\lambda\,|\,\beta)_p x^k, \tag{1.6}$$

$$S_2(n) = \sum_{k=\alpha}^{n+\alpha} D_p(k, \alpha) x^k, \quad S_2(\infty) = \sum_{k=\alpha}^{\infty} D_p(k, \alpha) x^k, \tag{1.7}$$

where $\alpha$ is any given integer, $\lambda$ and $\beta$ are real or complex numbers, and $|x| < 1$ for both $S_1(\infty)$ and $S_2(\infty)$. The results of this paper are based on the conclusions above.

## 2. MAIN RESULTS

We first define the rank as follows.

***Definition 2.1:*** The number of the summation symbols $\sum$ appearing in the right-hand side of a closed summation formula is called the *rank* of this summation formula.

Recall Howard's degenerate weighted Stirling numbers $S(p, j, \lambda|\beta)$ $(0 \le j \le p)$ can be defined by the basis transformation relation

$$(t+\lambda|\beta)_p = \sum_{j=0}^{p} j!S(p, j, \lambda|\beta)\binom{t}{j}. \tag{2.1}$$

Indeed, by applying the forward difference operator $\Delta$ defined by $\Delta f(x) = f(x+1) - f(x)$ and $\Delta^j = \Delta\Delta^{j-1}$ $(j \ge 2)$, and using the Newton interpolation formula to the LHS of (2.1), we see that the numbers $j!S(p, j, \lambda|\beta)$ in the RHS of (2.1) may be written as (cf. [2])

$$j!S(p, j, \lambda|\beta) = \Delta^j(t+\lambda|\beta)_{p|t=0} = \sum_{m=0}^{j}(-1)^{j-m}\binom{j}{m}(m+\lambda|\beta)_p. \tag{2.2}$$

Equation (2.2) shows the rank of $S(p, j, \lambda|\beta)$ is 1.

On the other hand, a kind of generalized Stirling numbers, called Dickson Stirling numbers, can be introduced by the relations (cf. [1], [2])

$$D_p(t, \alpha) = \sum_{j=0}^{p} S(p, j, \alpha)(t-\alpha)_j \quad (p = 1, 2, \ldots). \tag{2.3}$$

Of course, these relations may be rewritten as follows:

$$D_p(t+\alpha, \alpha) = \sum_{j=0}^{p} S(p, j, \alpha)(t)_j \quad (p = 1, 2, \ldots). \tag{2.4}$$

In fact, similar to the expression of $S(p, j, \lambda|\beta)$, the Dickson-Stirling numbers have the finite difference expression

$$S(p, j, \alpha) = \frac{1}{j!}\Delta^j D_p(t, \alpha)|_{t=\alpha}$$

and its rank is 1.

In the following, we first list the main results of L. C. Hsu and P. J. S. Shiue (cf. [2]) which are important to our conclusions. Denote

$$\phi(x, n, j) = \frac{1}{1-x}\left[\left(\frac{x}{1-x}\right)^j - x^{n+1}\sum_{r=0}^{j}\binom{n+1}{j-r}\left(\frac{x}{1-x}\right)^r\right]. \tag{2.5}$$

***Lemma 2.1:*** For $x \ne 1$, we have the summation formula

$$\sum_{k=0}^{n}(k+\lambda|\beta)_p x^k = \sum_{j=0}^{p} j!S(p, j, \lambda|\beta)\phi(x, n, j), \tag{2.6}$$

where $\phi(x, n, j)$ is given by (2.5).

***Lemma 2.2:*** For $|x| < 1$, we have the summation formula

$$\sum_{k=0}^{\infty}(k+\lambda|\beta)_p x^k = \sum_{j=0}^{p}\frac{j!S(p,j,\lambda|\beta)x^j}{(1-x)^{j+1}}. \tag{2.7}$$

***Lemma 2.3:*** For any given integer $\alpha$, we have the summation formula

$$\sum_{k=\alpha}^{n+\alpha}D_p(k,\alpha)x^k = x^{\alpha}\sum_{j=0}^{p}j!S(p,j,\alpha)\phi(x,n,j), \tag{2.8}$$

where the Dickson-Stirling numbers are defined by (2.3) and $\phi(x,n,j)$ is given by (2.5).

***Lemma 2.4:*** For $|x|<1$ and any given integer $\alpha$, we have the formula

$$\sum_{k=\alpha}^{\infty}D_p(k,\alpha)x^k = x^{\alpha}\sum_{j=0}^{p}\frac{j!S(p,j,\alpha)x^j}{(1-x)^{j+1}}. \tag{2.9}$$

The proofs of Lemmas 2.1 through 2.4 can be seen in [2].

In these lemmas, the constants $\lambda, \beta$ are real or complex numbers, $\alpha$ is an integer. From now on, unless specified, we assume $\lambda, \beta$ are real parameters, $\alpha$ is an integer.

We first recall the famous Chebyshev polynomial $T_n(x)$ defined as follows:

$$T_n(x) = \cos(n\arccos x), \quad x\in[-1,1].$$

It is known that $T_n(x)$ satisfies the recurrence relations $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ with $T_0(x)=1$, $T_1(x)=x$. For simplicity, denote

$$\begin{aligned}
\cos t_1 &= \frac{\cos\theta - r}{\sqrt{1+r^2-2r\cos\theta}},\\
\sin t_1 &= \frac{\sin\theta}{\sqrt{1+r^2-2r\cos\theta}},\\
T_l &= T_l\left(\frac{\cos\theta-r}{\sqrt{1+r^2-2r\cos\theta}}\right),
\end{aligned} \tag{2.10}$$

where $T_l(x)$ is the Chebyshev polynomial of degree $l$.

***Theorem 2.1:*** Assume $0\le\theta\le 2\pi$, $r\ge 0$. If $r\ne 1$ or $\theta\ne 0, 2\pi$, we have the summation formulas

$$\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k\cos k\theta = \sum_{j=0}^{p}j!S(p,j,\lambda|\beta)\phi_1^{(1)}(r,\theta,n,j), \tag{2.11}$$

$$\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k\sin k\theta = \sum_{j=0}^{p}j!S(p,j,\lambda|\beta)\phi_1^{(2)}(r,\theta,n,j), \tag{2.12}$$

where

$$\phi_1^{(1)}(r,\theta,n,j) = r^j\left[\frac{(1-r\cos\theta)T_j}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}} - \frac{r\sin^2\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right]$$

$$-r^{n+1}\sum_{m=0}^{j}\binom{n+1}{j-m}r^m\left\{\frac{T_m[\cos(n+1)\theta - r\cos n\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+2}{2}}} - \frac{\sin\theta\sin(n+1)\theta - r\sin\theta\sin n\theta}{(1+r^2-2r\cos\theta)^{\frac{m+3}{2}}}\sum_{l=0}^{m-1}T_lT_1^{m-1-l}\right\},$$

$$\phi_1^{(2)}(r,\theta,n,j) = r^j\left[\frac{r\sin\theta T_j}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}} + \frac{\sin\theta(1-r\cos\theta)}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1} T_l T_1^{j-1-l}\right]$$

$$-r^{n+1}\sum_{m=0}^{j}\binom{n+1}{j-m} r^m\left\{\frac{T_m[\sin(n+1)\theta - r\sin n\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+2}{2}}} + \frac{\sin\theta\cos(n+1)\theta - r\sin\theta\cos n\theta}{(1+r^2-2r\cos\theta)^{\frac{m+3}{2}}}\sum_{l=0}^{m-1} T_l T_1^{m-1-l}\right\}.$$

Equations (2.11) and (2.12) imply that the summation formulas of $\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k\cos k\theta$ and $\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k\sin k\theta$ have the same rank 5.

***Proof:*** In (2.6), set $x = re^{1\theta}$, then

$$\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k\cos k\theta = \sum_{j=0}^{p} j!S(p,j,\lambda|\beta)\,\mathrm{Re}\,\phi(re^{i\theta},n,j), \tag{2.13}$$

$$\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k\sin k\theta = \sum_{j=0}^{p} j!S(p,j,\lambda|\beta)\,\mathrm{Im}\,\phi(re^{i\theta},n,j). \tag{2.14}$$

We first obtain

$$\begin{aligned}\frac{x}{1-x} &= \frac{re^{i\theta}}{1-re^{i\theta}} = \frac{r(\cos\theta + i\sin\theta)}{1-r\cos\theta - ir\sin\theta} = \frac{r(\cos\theta - r + i\sin\theta)}{1+r^2-2r\cos\theta}\\ &= \frac{r\sqrt{1+r^2-2r\cos\theta}}{1+r^2-2r\cos\theta}\left(\frac{\cos\theta - r}{\sqrt{1+r^2-2r\cos\theta}} + \frac{i\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}\right)\\ &= \frac{r}{\sqrt{1+r^2-2r\cos\theta}}(\cos t_1 + i\sin t_1) = \frac{re^{it}1}{\sqrt{1+r^2-2r\cos\theta}},\end{aligned} \tag{2.15}$$

where $\cos t_1$ and $\sin t_1$ are defined in (2.10), and

$$\begin{aligned}\frac{1}{1-x} &= \frac{1}{1-re^{i\theta}} = \frac{1}{\sqrt{1+r^2-2r\cos\theta}}\left(\frac{1-r\cos\theta}{\sqrt{1+r^2-2r\cos\theta}} + i\frac{r\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}\right)\\ &= \frac{e^{it}2}{\sqrt{1+r^2-2r\cos\theta}},\end{aligned} \tag{2.16}$$

where

$$\cos t_2 = \frac{1-r\cos\theta}{\sqrt{1+r^2-2r\cos\theta}},\quad \sin t_2 = \frac{r\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}. \tag{2.17}$$

Therefore,

$$\begin{aligned}&\phi(x,n,j) = \phi(re^{i\theta},n,j)\\ &= \frac{e^{it}2}{\sqrt{1+r^2-2r\cos\theta}}\left[\frac{r^j e^{jt}1^i}{(1+r^2-2r\cos\theta)^{\frac{j}{2}}} - r^{n+1}e^{i(n+1)\theta}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m e^{mt}1^i}{(1+r^2-2r\cos\theta)^{\frac{m}{2}}}\right]\\ &= \frac{r^j e^{(jt_1+t_2)i}}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}} - r^{n+1}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m e^{[mt_1+t_2+(n+1)\theta]i}}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}}.\end{aligned} \tag{2.18}$$

Equation (2.18) implies that

$$\operatorname{Re}\phi(re^{i\theta}, n, j) = \frac{r^j \cos(jt_1+t_2)}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}} - r^{n+1}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m\cos[mt_1+t_2+(n+1)\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}}, \tag{2.19}$$

$$\operatorname{Im}\phi(re^{i\theta}, n, j) = \frac{r^j \sin(jt_1+t_2)}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}} - r^{n+1}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m\sin[mt_1+t_2+(n+1)\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}}. \tag{2.20}$$

By the definition of Chebyshev polynomial and (2.10), we know $T_l = \cos(lt_1)$. Set $I_j = \sin jt_1$, then

$$\begin{aligned}
I_j &= \sin jt_1 = \sin(j-1)t_1\cos t_1 + \cos(j-1)t_1 \sin t_1 \\
&= I_{j-1}\cos t_1 + T_{j-1}\sin t_1 = (I_{j-2}\cos t_1 + T_{j-2}\sin t_1)\cos t_1 + T_{j-1}\sin t_1 \\
&= (I_{j-3}\cos t_1 + T_{j-3}\sin t_1)\cos^2 t_1 + T_{j-2}\sin t_1\cos t_1 + T_{j-1}\sin t_1 \\
&= I_{j-3}\cos^3 t_1 + \sin t_1(T_{j-3}\cos^2 t_1 + T_{j-2}\cos t_1 + T_{j-1}) \\
&\cdots \\
&= I_1\cos^{j-1} t_1 + \sin t_1(T_1\cos^{j-2} t_1 + T_2\cos^{j-3} t_1 + \cdots + T_{j-2}\cos t_1 + T_{j-1}) \\
&= \sin t_1(\cos^{j-1} t_1 + T_1\cos^{j-2} t_1 + T_2\cos^{j-3} t_1 + \cdots + T_{j-2}\cos t_1 + T_{j-1}) \\
&= \sin t_1 \sum_{l=0}^{j-1} T_l t_1^{j-1-l}.
\end{aligned} \tag{2.21}$$

From (2.21) and (2.17), it is easy to obtain that

$$\begin{aligned}
\cos(jt_1+t_2) &= \cos jt_1\cos t_2 - \sin jt_1 \sin t_2 = T_j\cos t_2 - \sin t_1\sin t_2\sum_{l=0}^{j-1}T_l t_1^{j-1-l} \\
&= \frac{(1-r\cos\theta)T_j}{\sqrt{1+r^2-2r\cos\theta}} - \frac{r\sin^2\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}.
\end{aligned} \tag{2.22}$$

Similarly,

$$\begin{aligned}
\sin(jt_1+t_2) &= \sin jt_1\cos t_2 + \cos jt_1\sin t_2 \\
&= \frac{\sin\theta(1-r\cos\theta)}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l} + \frac{rT_j\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}.
\end{aligned} \tag{2.23}$$

Hence,

$$\begin{aligned}
\cos(jt_1+t_2+\alpha\theta) &= \cos(jt_1+t_2)\cos\alpha\theta - \sin(jt_1+t_2)\sin\alpha\theta \\
&= \left[\frac{(1-r\cos\theta)T_j}{\sqrt{1+r^2-2r\cos\theta}} - \frac{r\sin^2\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right]\cos\alpha\theta \\
&\quad - \left[\frac{\sin\theta(1-r\cos\theta)}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l} + \frac{rT_j\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}\right]\sin\alpha\theta \\
&= \frac{T_j(\cos\alpha\theta - r\cos\theta\cos\alpha\theta - r\sin\theta\sin\alpha\theta)}{\sqrt{1+r^2-2r\cos\theta}} \\
&\quad - \frac{r\sin^2\theta\cos\alpha\theta + \sin\theta\sin\alpha\theta - r\sin\theta\cos\theta\sin\alpha\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l} \\
&= \frac{T_j[\cos\alpha\theta - r\cos(1-\alpha)\theta]}{\sqrt{1+r^2-2r\cos\theta}} - \frac{\sin\theta\sin\alpha\theta + 4\sin\theta\sin(1-\alpha)\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l},
\end{aligned} \tag{2.24}$$

$$\begin{aligned}\sin(jt_1+t_2+\alpha\theta) &= \sin(jt_1+t_2)\cos\alpha\theta+\cos(jt_1+t_2)\sin\alpha\theta\\ &= \left[\frac{\sin\theta(1-r\cos\theta)}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}+\frac{rT_j\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}\right]\cos\alpha\theta\\ &\quad+\left[\frac{(1-r\cos\theta)T_j}{\sqrt{1+r^2-2r\cos\theta}}-\frac{r\sin^2\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right]\sin\alpha\theta\\ &= \frac{T_j[\sin\alpha\theta+r\sin(1-\alpha)\theta]}{\sqrt{1+r^2-2r\cos\theta}}+\frac{\sin\theta\cos\alpha\theta-r\sin\theta\cos(1-\alpha)\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}.\end{aligned}\tag{2.25}$$

In (2.24) and (2.25), set $j=m$ and $\alpha=n+1$ to obtain

$$\begin{aligned}\cos[mt_1+t_2+(n+1)\theta] &= \frac{T_m[\cos(n+1)\theta-r\cos n\theta]}{\sqrt{1+r^2-2r\cos\theta}}\\ &\quad-\frac{\sin\theta\sin(n+1)\theta-r\sin\theta\sin n\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{m-1}T_lT_1^{m-1-l},\end{aligned}\tag{2.26}$$

$$\begin{aligned}\sin[mt_1+t_2+(n+1)\theta] &= \frac{T_m[\sin(n+1)\theta-r\sin n\theta]}{\sqrt{1+r^2-2r\cos\theta}}\\ &\quad+\frac{\sin\theta\cos(n+1)\theta-r\sin\theta\cos n\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{m-1}T_lT_1^{m-1-l}.\end{aligned}\tag{2.27}$$

From (2.13), (2.14), (2.19), (2.20), (2.22), (2.23), (2.26), and (2.27), we obtain (2.11) and (2.12) immediately.

In (2.11) and (2.12), set $n\to\infty$ to obtain the following conclusion.

***Theorem 2.2:*** If $r<1$ and $0\le\theta\le 2\pi$, then

$$\sum_{k=0}^{\infty}(k+\lambda|\beta)_p r^k\cos k\theta=\sum_{j=0}^{p}j!S(p,j,\lambda|\beta)\psi_1^{(1)}(r,\theta,j)\tag{2.28}$$

and

$$\sum_{k=0}^{\infty}(k+\lambda|\beta)_p r^k\sin k\theta=\sum_{j=0}^{p}j!S(p,j,\lambda|\beta)\psi_1^{(2)}(r,\theta,j),\tag{2.29}$$

where

$$\psi_1^{(1)}(r,\theta,j)=r^j\left[\frac{T_j(1-r\cos\theta)}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}}-\frac{r\sin^2\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right]$$

and

$$\psi_1^{(2)}(r,\theta,j)=r^j\left[\frac{rT_j\sin\theta}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}}+\frac{\sin\theta(1-r\cos\theta)}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right].$$

The rank of (2.28) and (2.29) is 3.

***Theorem 2.3:*** Assume $r\ge 0$, $0\le\theta\le 2\pi$. If $r\ne 1$ or $\theta\ne 0, 2\pi$ for any given integer $\alpha$, we have the summation formulas:

$$\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha) r^k \cos k\theta = \sum_{j=0}^{p} j!\,S(p,j,\alpha)\phi_2^{(1)}(\alpha, r, \theta, n, j), \tag{2.30}$$

$$\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha) r^k \sin k\theta = \sum_{j=0}^{p} j!\,S(p,j,\alpha)\phi_2^{(2)}(\alpha, r, \theta, n, j), \tag{2.31}$$

where

$$\begin{aligned}\phi_2^{(1)}(\alpha, r, \theta, n, j) = r^{j+\alpha}&\left[T_j \frac{\cos\alpha\theta - r\cos(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}} - \frac{\sin\theta\sin\alpha\theta + r\sin\theta\sin(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1} T_l T_1^{j-1-l}\right] \\ &- r^{n+1+\alpha}\sum_{m=0}^{j}\binom{n+1}{j-m} r^m \left[T_m \frac{\cos(n+1+\alpha)\theta - r\cos(n+\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{m+2}{2}}}\right. \\ &\left. - \frac{\sin\theta\sin(n+1+\alpha)\theta - r\sin\theta\sin(n+\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{m+3}{2}}}\sum_{l=0}^{m-1} T_l T_1^{m-1-l}\right],\end{aligned}$$

$$\begin{aligned}\phi_2^{(2)}(\alpha, r, \theta, n, j) = r^{j+\alpha}&\left[T_j \frac{\sin\alpha\theta - r\sin(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}} + \frac{\sin\theta\cos\alpha\theta - r\sin\theta\cos(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1} T_l T_1^{j-1-l}\right] \\ &- r^{n+1+\alpha}\sum_{m=0}^{j}\binom{n+1}{j-m} r^m \left[T_m \frac{\sin(n+1+\alpha)\theta - r\sin(n+\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{m+2}{2}}}\right. \\ &\left. + \frac{\sin\theta\cos(n+1+\alpha)\theta - r\sin\theta\cos(n+\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{m+3}{2}}}\sum_{l=0}^{m-1} T_l T_1^{m-1-l}\right].\end{aligned}$$

Equations (2.30) and (2.31) imply that the summation formulas of $\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha) r^k \cos k\theta$ and $\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha) r^k \sin k\theta$ have the same rank 5.

***Proof:*** In (2.8), set $x = re^{1\theta}$, then

$$\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha) r^k \cos k\theta = \sum_{j=0}^{p} j!\,S(p,j,\alpha)\,\mathrm{Re}[r^\alpha e^{i\alpha\theta}\phi(re^{i\theta}, n, j)], \tag{2.32}$$

$$\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha) r^k \sin k\theta = \sum_{j=0}^{p} j!\,S(p,j,\alpha)\,\mathrm{Im}[r^\alpha e^{i\alpha\theta}\phi(re^{i\theta}, n, j)]. \tag{2.33}$$

By (2.18), we have

$$\begin{aligned}x^\alpha\phi(x,n,j) &= r^\alpha e^{i\alpha\theta}\phi(re^{i\theta}) \\ &= \frac{r^{j+\alpha}e^{(jt_1+t_2+\alpha\theta)i}}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}} - r^{n+1+\alpha}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m e^{[mt_1+t_2+(n+1+\alpha)\theta]i}}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}}.\end{aligned} \tag{2.34}$$

This implies

$$\begin{aligned}\mathrm{Re}[r^\alpha e^{i\alpha\theta}\phi(re^{i\theta}, n, j)] &= \frac{r^{j+\alpha}\cos(jt_1+t_2+\alpha\theta)}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}} \\ &\quad - r^{n+1+\alpha}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m\cos[mt_1+t_2+(n+1+\alpha)\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}},\end{aligned} \tag{2.35}$$

$$\begin{aligned}\operatorname{Im}[r^\alpha e^{i\alpha\theta}\phi(re^{i\theta},n,j)] &= \frac{r^{j+\alpha}\sin(jt_1+t_2+\alpha\theta)}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}} \\ &\quad - r^{n+1+\alpha}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m\sin[mt_1+t_2+(n+1+\alpha)\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}}.\end{aligned} \tag{2.36}$$

By (2.32), (2.33), (2.35), (2.36), (2.24), and (2.25), we obtain (2.30) and (2.31).

In (2.30) and (2.31), set $n\to\infty$, then we easily obtain the following conclusion.

***Theorem 2.4:*** If $r<1$, $0\le\theta\le 2\pi$, then

$$\sum_{k=\alpha}^{\infty} D_p(k,\alpha)r^k\cos k\theta = \sum_{j=0}^{p} j!S(p,j,\alpha)\psi_2^{(1)}(\alpha,r,\theta,n,j), \tag{2.37}$$

$$\sum_{k=\alpha}^{\infty} D_p(k,\alpha)r^k\sin k\theta = \sum_{j=0}^{p} j!S(p,j,\alpha)\psi_2^{(2)}(\alpha,r,\theta,n,j), \tag{2.38}$$

where

$$\psi_2^{(1)}(\alpha,r,\theta,n,j) = r^{j+\alpha}\left[T_j\frac{\cos\alpha\theta - r\cos(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}} - \frac{\sin\theta\sin\alpha\theta + r\sin\theta\sin(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right],$$

$$\psi_2^{(2)}(\alpha,r,\theta,n,j) = r^{j+\alpha}\left[T_j\frac{\sin\alpha\theta - r\sin(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}} + \frac{\sin\theta\cos\alpha\theta - r\sin\theta\cos(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right].$$

These imply that the rank of (2.37) and (2.38) is 3.

## ACKNOWLEDGMENT

This work was inspired by a lecture of Professor L. C. Hsu during his visit to Hunan Normal University. I gratefully thank Professor Hsu for his suggestions.

## REFERENCES

1. L. C. Hsu, G. L. Mullen, & P. J. S. Shiue. "Dickson-Stirling Numbers." *Proc. Edin. Math. Soc.* **40** (1997):409-23.
2. L. C. Hsu & P. J. S. Shiue. "On Certain Summation Problems and Generalizations of Eulerian Polynomials and Numbers." *Discrete Math.* **204** (1999):237-47.
3. T. Lengyel. "On Some Properties of the Series $\sum_{k=0}^{\infty} k^n x^k$ and the Stirling Numbers of the Second Kind." *Discrete Math.* **150** (1996):281-92.
4. R. Lidl, G. L. Mullen, & G. Turnwald. "Dickson Polynomials, Pitman Mono." *Surveys in Pure Appl. Math.* **65**. Longman Sci. Tech. Essex, England, 1993.

AMS Classification Numbers: 40A25, 42A24

❖❖❖

# PASCAL DECOMPOSITIONS OF ARITHMETIC AND CONVOLUTION ARRAYS IN MATRICES

**Johann Leida and Yongzhi (Peter) Yang**
Mail #OSS 201, University of St. Thomas, 2115 Summit Avenue, St. Paul, MN 55105
E-mail: leida@math.wisc.edu; y9yang@stthomas.edu
*(Submitted January 2000-Final Revision July 2000)*

## 1. INTRODUCTION

Bicknell and Hoggatt [1]-[6] , [9] published several articles in the 1970s involving matrices made up of generalized arithmetic progressions and convolutions of sequences with first term one. We give a new proof of their result using a novel decomposition of such matrices and then extend their result to convolution matrices of sequences whose first term does not equal one. In the process, we gain an increased understanding of the underlying structures of such matrices. We also note that these results should be readily extensible to a class of matrices recently discussed by Ollerton and Shannon [11].

## 2. ARITHMETIC PROGRESSION MATRICES

In [5] and others, Bicknell and Hoggatt define an *arithmetic progression of* $r^{\text{th}}$ *order*, or $(AP)_r$, as any sequence of numbers whose $r^{\text{th}}$ row of differences is a nonzero constant while the $(r-1)^{\text{st}}$ is not. The constant number in the $r^{\text{th}}$ row is called the *constant of the progression*. The sequence itself is the zeroth row of differences, so a constant nonzero sequence is an $(AP)_0$. They then give the following theorem.

***Theorem 1 ("Eves' Theorem"):*** Let $A$ be an $n\times n$ matrix whose $i^{\text{th}}$ row $(i=1,2,\ldots,n)$ is composed of $n$ terms of an $(AP)_{i-1}$ with constant of progression $a_i$. Then $|A|$ must be equal to $\prod_{i=1}^{n} a_i$.

Bicknell and Hoggatt refer to this as Eves' Theorem after a letter they received from Howard Eves; however, very similar results may be found much earlier in Muir and Metzler (see [10], pp. 47-48 and Ch. XX). The first example of such a matrix given in both of these sources is the familiar rectangular form of Pascal's triangle,

$$T=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 1 & 3 & 6 & 10 & 15 & 21 & \cdots \\ 1 & 4 & 10 & 20 & 35 & 56 & \cdots \\ 1 & 5 & 15 & 35 & 70 & 126 & \cdots \\ 1 & 6 & 21 & 56 & 126 & 252 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1}$$

whose $i^{\text{th}}$ row $(i=1,2,\ldots)$ is an $(AP)_{i-1}$ with constant 1. According to the theorem, then, the determinant of any $n\times n$ submatrix of $T$ with one side on the left column of ones (or, by symmetry, its top row along the top row of ones) must equal $\prod_{i=1}^{n} a_i=\prod_{i=1}^{n} 1=1$.

An alternate approach involves the observation that

$$T=\begin{pmatrix}1&0&0&0&0&0&\cdots\\1&1&0&0&0&0&\cdots\\1&2&1&0&0&0&\cdots\\1&3&3&1&0&0&\cdots\\1&4&6&4&1&0&\cdots\\1&5&10&10&5&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\cdot\begin{pmatrix}1&1&1&1&1&1&\cdots\\0&1&2&3&4&5&\cdots\\0&0&1&3&6&10&\cdots\\0&0&0&1&4&10&\cdots\\0&0&0&0&1&5&\cdots\\0&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}; \tag{2}$$

that is, Pascal's triangle in rectangular form is equal to the matrix product of its lower triangular form with its upper triangular form.

From this decomposition, it is easy to see why the upper left corner determinants discussed above must equal one. Furthermore, it begs the question: can other arithmetic matrices be decomposed in a similar way?

The answer is yes. In fact, any matrix $A$ whose rows are arithmetic progressions satisfying the criteria of Eves' theorem may be decomposed similarly. We state this formally as

***Theorem 2 (Pascal Decomposition Theorem):*** Let any $n\times n$ matrix whose $i^{\text{th}}$ row is an $(AP)_{i-1}$ for $i=1,2,\dots,n$ be known as an arithmetic matrix. Then $A$ is an arithmetic matrix if and only if it may be rewritten as the product of an $n\times n$ lower triangular seed matrix $S$ with nonzero diagonal elements and the upper triangular matrix form of Pascal's triangle.

***Proof:*** We will first present a constructive proof that such a matrix decomposes and then deal with the reverse case. Let

$$A=\begin{pmatrix}\overline{A}_1\\\overline{A}_2\\\vdots\\\overline{A}_n\end{pmatrix}, \tag{3}$$

where $\overline{A}_i$ is the $i^{\text{th}}$ row of $A$, that is, $\overline{A}_i=(a_{i1},a_{i2},\dots,a_{in})$, and let $\overline{A}_i$ be an $(AP)_{i-1}$ as defined above. We write out the difference table of this $i^{\text{th}}$ row as in Sloane and Plouffe (see [12], p. 13), labeling the leading diagonal $\{b_{i1},b_{i2},\dots\}$:

$$\begin{array}{lcccccccc}
\overline{A}_i & b_{i1}=a_{i1} & a_{i2} & a_{i3} & a_{i4} & a_{i5} & a_{i6} & \cdots \\
\Delta\overline{A}_i & & b_{i2} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Delta^2\overline{A}_i & & & b_{i3} & \cdot & \cdot & \cdot & \cdot \\
\vdots & & & & \ddots & \cdot & \cdot & \cdot \\
\Delta^{i-1}\overline{A}_i & & & & b_{ii} & \cdot & \cdot & \\
\Delta^{i}\overline{A}_i & & & & & 0 & \cdot &
\end{array}, \tag{4}$$

where $\Delta^k\overline{A}_i$ denotes the $k^{\text{th}}$ row of $\overline{A}_i$'s differences; that is, the $j^{\text{th}}$ element of $\Delta\overline{A}_i$ is $\Delta a_{ij}=a_{i(j+1)}-a_{ij}$ and, in general, the $j^{\text{th}}$ element of $\Delta^k\overline{A}_i$ is $\Delta^k a_{ij}=\Delta^{k-1}a_{i(j+1)}-\Delta^{k-1}a_{ij}$.

Now, since $\overline{A}_i$ is an $(AP)_{i-1}$, its $(i-1)^{\text{th}}$ row of differences must be equal to the nonzero constant of the progression. In particular, $b_{ii}$ must equal the constant of the progression. Also, any elements below row $i$ (on the leading diagonal, all $b_{ij}$, $j>i$) must equal zero. From [12], we have the following relationships between the top row of our difference table and its leading diagonal:

$$a_{ik} = \sum_{j=1}^{k} \binom{k-1}{j-1} b_{ij} \quad \text{and} \quad b_{ik} = \sum_{j=1}^{k} (-1)^{k-j} \binom{k-1}{j-1} a_{ij}. \tag{5}$$

Substituting for the $a_{ik}$ that make up our matrix $A$, we have

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{1} \binom{0}{j-1} b_{1j} & \cdots & \sum_{j=1}^{n} \binom{n-1}{j-1} b_{1j} \\ \sum_{j=1}^{1} \binom{0}{j-1} b_{2j} & \cdots & \sum_{j=1}^{n} \binom{n-1}{j-1} b_{2j} \\ \vdots & & \vdots \\ \sum_{j=1}^{1} \binom{0}{j-1} b_{nj} & \cdots & \sum_{j=1}^{n} \binom{n-1}{j-1} b_{nj} \end{pmatrix}. \tag{6}$$

Now, since some of the $b_{ij}$ were shown to be zero above, we can reindex the sums and see that

$$A = \begin{pmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & \cdots & \binom{n-1}{0} \\ 0 & 1 & 2 & \cdots & \binom{n-1}{1} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n-1}{n-2} \\ 0 & 0 & 0 & \cdots & \binom{n-1}{n-1} \end{pmatrix}. \tag{7}$$

Therefore, $A$ can be written as the product of a lower triangular matrix $S$ and the $n \times n$ upper triangular Pascal matrix. Moreover, it is easy to see that $b_{ii} \neq 0$ for $(i = 1, 2, \dots, n)$ by the definition of the $(AP)_i$.

As for the reverse case, we notice that so long as the diagonal elements of $S$ are nonzero, the process outlined above can be run backwards. Hence, any matrix that is the product of a lower triangular seed matrix $S$ with nonzero diagonal elements and the upper triangular matrix form of the Pascal triangle must be an $(AP)$ matrix, and our theorem is proved. What's more, we now know the exact structure of the seed matrix $S$, and can calculate it from our original matrix $A$. We call this process the *Pascal decomposition* of $A$.

***Corollary:*** $|A| = \prod_{i=1}^{n} b_{ii}$.

As an example, we can apply our theorem to the numbers $M_{k,r}$ examined by Wong and Maddocks in [13]. These numbers, with properties somewhat similar to binomial coefficients, satisfy the recurrence relation

$$M_{k+1,r+1} = M_{k+1,r} + M_{k,r+1} + M_{k,r} \tag{8}$$

with initial conditions $M_{0,0} = M_{1,0} = M_{0,1} = 1$. If we write these numbers out in a matrix where $k$ is the row number and $r$ indicates the column, we have the following arithmetic matrix:

$$M=\begin{pmatrix}1&1&1&1&1&1&\cdots\\1&3&5&7&9&11&\cdots\\1&5&13&25&41&61&\cdots\\1&7&25&63&129&231&\cdots\\1&9&41&129&321&681&\cdots\\1&11&61&231&681&1683&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}. \tag{9}$$

To decompose $M$, we multiply it by the inverse of the upper triangular Pascal matrix. Equivalently, we could write out the difference tables for each row of $M$, but the inversion method is more succinct:

$$\begin{pmatrix}1&1&1&1&1&1&\cdots\\1&3&5&7&9&11&\cdots\\1&5&13&25&41&61&\cdots\\1&7&25&63&129&231&\cdots\\1&9&41&129&321&681&\cdots\\1&11&61&231&681&1683&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\cdot\begin{pmatrix}1&1&1&1&1&1&\cdots\\0&1&2&3&4&5&\cdots\\0&0&1&3&6&10&\cdots\\0&0&0&1&4&10&\cdots\\0&0&0&0&1&5&\cdots\\0&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}^{-1}$$
$$=\begin{pmatrix}1&0&0&0&0&0&\cdots\\1&2&0&0&0&0&\cdots\\1&4&4&0&0&0&\cdots\\1&6&12&8&0&0&\cdots\\1&8&24&32&16&0&\cdots\\1&10&40&80&80&32&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}, \tag{10}$$

which may be rewritten as

$$M=\begin{pmatrix}1&0&0&0&0&0&\cdots\\1&2&0&0&0&0&\cdots\\1&4&4&0&0&0&\cdots\\1&6&12&8&0&0&\cdots\\1&8&24&32&16&0&\cdots\\1&10&40&80&80&32&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\cdot\begin{pmatrix}1&1&1&1&1&1&\cdots\\0&1&2&3&4&5&\cdots\\0&0&1&3&6&10&\cdots\\0&0&0&1&4&10&\cdots\\0&0&0&0&1&5&\cdots\\0&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}. \tag{11}$$

From equation (11), it is easy to see that $|M|_{n\times n}=2^{n(n-1)/2}$, as predicted by Eves' theorem [just note that each row $i$ ($i=1,\dots,n$) has constant $2^{i-1}$].

Interestingly, symmetric matrices such as $M$ are subject to further decomposition using the lower triangular matrix form of Pascal's triangle; note that

$$M=\begin{pmatrix}1&0&0&0&0&0&\cdots\\1&1&0&0&0&0&\cdots\\1&2&1&0&0&0&\cdots\\1&3&3&1&0&0&\cdots\\1&4&6&4&1&0&\cdots\\1&5&10&10&5&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0&0&0&\cdots\\0&2&0&0&0&0&\cdots\\0&0&4&0&0&0&\cdots\\0&0&0&8&0&0&\cdots\\0&0&0&0&16&0&\cdots\\0&0&0&0&0&32&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\cdot\begin{pmatrix}1&1&1&1&1&1&\cdots\\0&1&2&3&4&5&\cdots\\0&0&1&3&6&10&\cdots\\0&0&0&1&4&10&\cdots\\0&0&0&0&1&5&\cdots\\0&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}. \tag{12}$$

This result is valid for symmetric arithmetic matrices in general, so we present another corollary.

***Corollary:*** Let $A$ be any symmetric matrix that also satisfies the conditions of Theorem 2. Then $A = P^T \cdot D \cdot P$, where $P$ is the upper triangular Pascal triangle matrix and $D$ is $diag(c_1, c_2, \ldots, c_n)$ with $c_i$ being the constant of the progression for the $i^{\text{th}}$ row of $A$ $(i = 1, 2, \ldots, n)$.

***Proof:*** By Theorem 2, $A = S_1 \cdot P$ and $A^T = P^T \cdot S_2$, where $S_1$ and $S_2$ are lower triangular and upper triangular matrices, respectively. Since $A$ is symmetric and $P$ is invertible, we can write $A = P^T \cdot L_1 \cdot P$ and $A^T = P^T \cdot L_2 \cdot P$, where $L_1$ is lower triangular and $L_2$ is upper triangular. Since $A = A^T$ by symmetry, $P^T \cdot L_1 \cdot P = P^T \cdot L_2 \cdot P$. Thus, $L_1 = L_2$, and since $L_1$ is lower triangular and $L_2$ is upper triangular, they must be a diagonal matrix, denoted by $diag(l_1, l_2, \ldots, l_n)$. Now, the diagonal elements of $P$ and $P^T$ are all one, and by the first corollary to Theorem 2 the determinant of the principal $(k \times k)$ submatrix of $A$ is equal to the product of the progression constants of its rows, $c_1 c_2 \ldots c_k$. This means that $c_1 c_2 \ldots c_k = l_1 l_2 \ldots l_k$ for $k = 1, 2, \ldots, n$. Therefore, by induction on $k$, the diagonal elements of $D = L_1 = L_2$ must equal the progression constants for $A$'s rows.

## 3. CONVOLUTION MATRICES FOR SEQUENCES WITH FIRST TERM ONE

The convolution matrices Bicknell and Hoggatt studied next provide further interesting examples of the decomposition technique, and they also lead to an interesting generalization. The *convolution* of two sequences $\{a_n\}$ and $\{b_n\}$ $(n = 0, 1, \ldots)$ is defined to be the sequence $\{c_n\}$ such that $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. The *convolution matrix* of a sequence is the matrix whose $i^{\text{th}}$ column is the $(i-1)^{\text{th}}$ convolution of the sequence with itself $(i = 1, 2, \ldots)$. The rectangular form of Pascal's triangle, for instance, is the convolution matrix for the sequence $\{1, 1, 1, \ldots\}$. Bicknell and Hoggatt did a detailed analysis of the convolutions of the Catalan numbers

$$\{C_n\} = \left\{\frac{1}{n+1}\binom{2n}{n}\right\} = \{1, 1, 2, 5, 14, \ldots\}$$

over the course of several papers; in [2] and [3], they present the following convolution matrix for this sequence:

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 2 & 5 & 9 & 14 & 20 & 27 & \cdots \\ 5 & 14 & 28 & 48 & 75 & 110 & \cdots \\ 14 & 42 & 90 & 165 & 275 & 429 & \cdots \\ 42 & 132 & 297 & 572 & 1001 & 1638 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{13}$$

Bicknell and Hoggatt showed in [3] that any convolution matrix for a sequence whose first term is one must be an arithmetic progression matrix with row constants all equal to one and, hence, must—by Eves' theorem—have determinant one. Nevertheless, examining the Pascal decompositions for these matrices is worthwhile since it reveals a detailed underlying structure not otherwise apparent.

Looking at the Pascal decomposition of $C$, we note that the seed matrix $S$ seems to have a close relationship to the even columns of $C$:

$$C=\begin{pmatrix}1&0&0&0&0&0&\cdots\\1&1&0&0&0&0&\cdots\\2&3&1&0&0&0&\cdots\\5&9&5&1&0&0&\cdots\\14&28&20&7&1&0&\cdots\\42&90&75&35&9&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\cdot\begin{pmatrix}1&1&1&1&1&1&\cdots\\0&1&2&3&4&5&\cdots\\0&0&1&3&6&10&\cdots\\0&0&0&1&4&10&\cdots\\0&0&0&0&1&5&\cdots\\0&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}. \tag{14}$$

What can account for this? To tease out the answer, we first examine convolution matrices in general. First, we note that any $n\times n$ convolution matrix $V$ of a sequence $\{v_n\}$ may be written in the form

$$V=(\overline{V},\ A\cdot\overline{V},\ A^2\cdot\overline{V},\dots,A^{n-1}\cdot\overline{V}), \tag{15}$$

where $\overline{V}$ is the first $n$ terms of $\{v_n\}$ and

$$A=\begin{pmatrix}v_0&0&0&\cdots&0\\v_1&v_0&0&\cdots&0\\v_2&v_1&v_0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\v_{n-1}&v_{n-2}&v_{n-3}&\cdots&v_0\end{pmatrix}. \tag{16}$$

If we set $v_0=1$, then from [3] we know that $V$ is a matrix satisfying Theorem 2 and must, therefore, have a Pascal decomposition, i.e., $V=S\cdot P$, where $S$ is a lower triangular seed and $P$ represents the upper triangular Pascal matrix. We can solve this for $S=V\cdot P^{-1}$; substituting for $V$ gives

$$S=(\overline{V},\ A\cdot\overline{V},\ A^2\cdot\overline{V},\dots,A^{n-1}\cdot\overline{V})\cdot P^{-1}. \tag{17}$$

Since the inverse of $P$ is clearly

$$P^{-1}=\begin{pmatrix}1&-1&1&-1&1&-1&\cdots\\0&1&-2&3&-4&5&\cdots\\0&0&1&-3&6&-10&\cdots\\0&0&0&1&-4&10&\cdots\\0&0&0&0&1&-5&\cdots\\0&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}, \tag{18}$$

we can rewrite $S$:

$$S=(\overline{V},\ (A-I)\cdot\overline{V},\ (A-I)^2\cdot\overline{V},\dots,(A-I)^{n-1}\cdot\overline{V}), \tag{19}$$

where $I$ is the identity matrix.

Thus, each column of $S$ is a successive convolution of $\{v_0,v_1,\dots,v_n\}$ with $\{0,v_1,v_2,\dots,v_n\}$; i.e., if

$$B=(A-I)=\begin{pmatrix}0&0&0&\cdots&0\\v_1&0&0&\cdots&0\\v_2&v_1&0&\cdots&0\\\vdots&\vdots&\ddots&\ddots&\vdots\\v_{n-1}&v_{n-2}&\cdots&v_1&0\end{pmatrix}. \tag{20}$$

then

$$S=(\overline{V},\ B\cdot\overline{V},\ B^2\cdot\overline{V},\dots,B^{n-1}\cdot\overline{V}). \tag{21}$$

We summarize this discussion in the following theorem.

***Theorem 3 (Weak Convolution Decomposition Theorem):*** If $V$ is a convolution matrix of a sequence $\{v_n\}$ with first term one, then $V = S \cdot P$ for some lower triangular matrix $S$ and the upper triangular Pascal triangle matrix $P$. Moreover, successive columns of $S$ are successive convolutions of the sequence $\{v_n\}$ with the sequence $\{0, v_1, v_2, v_3, \ldots\}$.

Returning to our Catalan convolution matrix example, we can reexamine its seed matrix in light of this theorem. As predicted, each column is a convolution of the sequences $\{1, 1, 2, 5, \ldots\}$ and $\{0, 1, 2, 5, \ldots\}$. Besides this, we can make our earlier conjecture about the relationship between the columns of the Catalan seed matrix, denoted by $S_C$, and the even columns of $C$ explicit: the $i^{\text{th}}$ column of $S_C$ is equal to the $(2i)^{\text{th}}$ column of $C$ shifted down $i$ places ($i = 0, 1, \ldots$).

Symbolically, we let $C = (\overline{C},\ A \cdot \overline{C},\ A^2 \cdot \overline{C}, \ldots, A^{n-1} \cdot \overline{C})$, where $\overline{C}$ is the column vector filled with the first $n$ Catalan numbers and

$$A = \begin{pmatrix} C_0 & 0 & 0 & \cdots & 0 \\ C_1 & C_0 & 0 & \cdots & 0 \\ C_2 & C_1 & C_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & C_{n-2} & C_{n-3} & \cdots & C_0 \end{pmatrix}. \tag{22}$$

Also, let $S_C = (\overline{C},\ B \cdot \overline{C},\ B^2 \cdot \overline{C}, \ldots, B^{n-1} \cdot \overline{C})$, where

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ C_1 & 0 & 0 & \cdots & 0 \\ C_2 & C_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ C_{n-1} & C_{n-2} & \cdots & C_1 & 0 \end{pmatrix}. \tag{23}$$

Then what we are trying to show is that $B^k \cdot \overline{C}$ is equal to $A^{2k} \cdot \overline{C}$ shifted down $k$ spots.

We first note that the Catalan numbers have the well-known recursive relation $\sum_{j=0}^{i} C_{i-j} C_j = C_{i+1}$ for $i = 0, 1, \ldots$. (See [8].)

By this relation, we have

$$A^2 = \begin{pmatrix} C_1 & 0 & 0 & \cdots & 0 \\ C_2 & C_1 & 0 & \cdots & 0 \\ C_3 & C_2 & C_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & C_{n-1} & C_{n-2} & \cdots & C_1 \end{pmatrix} \tag{24}$$

and

$$I_S \cdot A^2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ C_1 & 0 & 0 & \cdots & 0 \\ C_2 & C_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ C_{n-1} & C_{n-2} & \cdots & C_1 & 0 \end{pmatrix} = B, \tag{25}$$

where

$$I_S = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{26}$$

Since $I_S \cdot A^2 = B$, we can show by mathematical induction that

$$\begin{aligned} S_C &= (\overline{C},\ B\cdot\overline{C},\ B^2\cdot\overline{C}, \ldots, B^{n-1}\cdot\overline{C}) \\ &= (\overline{C}, (I_S\cdot A^2)\cdot\overline{C}, (I_S^2\cdot A^4)\cdot\overline{C}, \ldots, (I_S^{n-1}\cdot A^{2(n-1)})\overline{C}), \end{aligned} \tag{27}$$

thereby showing the desired relationship between the columns of $S_C$ and the even columns of $C$.

## 4. CONVOLUTION MATRICES OF SEQUENCES WITH FIRST TERM OTHER THAN ONE

We now have a very detailed understanding of the structure of any convolution matrix of a sequence whose first term is one. What happens, though, if the sequence's first term does not equal one? A good example is the following convolution matrix of the Lucas numbers {2, 1, 3, 4, 7, ...} (we use the standard definition and notation, but begin with $L_0 = 2$ instead of $L_1 = 1$):

$$L = \begin{pmatrix} 2 & 4 & 8 & 16 & 32 & 64 & \cdots \\ 1 & 4 & 12 & 32 & 80 & 192 & \cdots \\ 3 & 13 & 42 & 120 & 320 & 816 & \cdots \\ 4 & 22 & 85 & 280 & 840 & 2368 & \cdots \\ 7 & 45 & 195 & 705 & 2290 & 6924 & \cdots \\ 11 & 82 & 399 & 1588 & 5601 & 10204 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{28}$$

Eves' theorem has nothing to say in this case since the rows are no longer arithmetic progressions. However, if we multiply it twice by the inverse of the upper triangular Pascal triangle matrix, which we will again denote $P$, we obtain a seed matrix very like the ones encountered in our earlier work:

$$L\cdot(P^{-1})^2 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 7 & 2 & 0 & 0 & 0 & \cdots \\ 4 & 14 & 13 & 2 & 0 & 0 & \cdots \\ 7 & 31 & 43 & 19 & 2 & 0 & \cdots \\ 11 & 60 & 115 & 90 & 25 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{29}$$

In particular, each column of this matrix is equal to the convolution of the sequences {2, 1, 3, 4, ...} and {0, 1, 3, 4, ...}.

Note that this sequence had first term two, and that we multiplied the matrix by $P^{-1}$ twice. This was by no means coincidental; in fact, we may state this correlation as part of a general theorem.

***Theorem 4 (Strong Convolution Decomposition Theorem):*** Let $\{v_n\}$ be a sequence whose first term is a positive integer $v_0$, and let $V$ be the convolution matrix of that sequence. Then $V = S\cdot P^{v_0}$ for some lower triangular matrix $S$ and the upper triangular Pascal triangle matrix $P$. Moreover, successive columns of $S$ are successive convolutions of the sequence $\{v_n\}$ with the sequence $\{0, v_1, v_2, v_3, \ldots\}$.

***Proof:*** The proof is constructive. We first note that if $V$ is any convolution matrix of a sequence $\{v_n\}$ with first term $v_0$, then $V = (\overline{V},\ A\cdot\overline{V},\ A^2\cdot\overline{V}, \ldots, A^{n-1}\cdot\overline{V})$, where $\overline{V}$ is the column vector whose $i^{\text{th}}$ element is $v_i$ and

$$A=\begin{pmatrix} v_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ v_1 & v_0 & 0 & 0 & 0 & 0 & \cdots \\ v_2 & v_1 & v_0 & 0 & 0 & 0 & \cdots \\ v_3 & v_2 & v_1 & v_0 & 0 & 0 & \cdots \\ v_4 & v_3 & v_2 & v_1 & v_0 & 0 & \cdots \\ v_5 & v_4 & v_3 & v_2 & v_1 & v_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{30}$$

If we multiply the convolution matrix $v_0$ times by the inverse of the upper triangular Pascal triangle matrix $P$, we have

$$\begin{aligned} S &= V\cdot(P^{-1})^{v_0} \\ &= (\overline{V},\ A\cdot\overline{V},\ A^2\cdot\overline{V},\ldots,\ A^{n-1}\cdot\overline{V})\cdot(P^{-1})^{v_0} \\ &= (\overline{V},\ (A-I)\cdot\overline{V},\ (A-I)^2\cdot\overline{V},\ldots,(A-I)^{n-1}\cdot\overline{V})\cdot(P^{-1})^{v_0-1} \\ &= (\overline{V},\ (A-2I)\cdot\overline{V},\ (A-2I)^2\cdot\overline{V},\ldots,(A-2I)^{n-1}\cdot\overline{V})\cdot(P^{-1})^{v_0-2} \\ &\ \vdots \\ &= (\overline{V},\ (A-v_0I)\cdot\overline{V},\ (A-v_0I)^2\cdot\overline{V},\ldots,(A-v_0I)^{n-1}\cdot\overline{V}). \end{aligned} \tag{31}$$

Let a new matrix $B=A-v_0I$, i.e.,

$$B=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ v_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ v_2 & v_1 & 0 & 0 & 0 & 0 & \cdots \\ v_3 & v_2 & v_1 & 0 & 0 & 0 & \cdots \\ v_4 & v_3 & v_2 & v_1 & 0 & 0 & \cdots \\ v_5 & v_4 & v_3 & v_2 & v_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{32}$$

Then it is clear that $S$ is a convolution matrix since $S=(\overline{V},\ B\cdot\overline{V},\ B^2\cdot\overline{V},\ldots,\ B^{n-1}\cdot\overline{V})$. More specifically, successive columns of $S$ are successive convolutions of the sequence $\{v_0, v_1, v_2, \ldots\}$ with $\{0, v_1, v_2, \ldots\}$, as was to be shown.

***Corollary:*** For any convolution matrix $V$ satisfying the conditions of Theorem 4, $|V|=v_0^n v_1^{n(n-1)/2}$.

***Proof:*** By Theorem 4, $V=S\cdot P^{v_0}$. Now, $|P^{v_0}|=1^{v_0}=1$, so $|V|=|S|$. Since $S$ is lower triangular with diagonal elements $v_0, v_0v_1^1, v_0v_1^2,\ldots, v_0v_1^{n-1}$, $|S|=v_0^n v_1^{1+2+\cdots+(n-1)}$. Hence, $|V|=|S|=v_0^n v_1^{n(n-1)/2}$.

***Remark:*** The determinant of any convolution matrix is wholly determined by the first two elements of the sequence.

## 5. CONCLUSION AND FUTURE GOALS

Pascal decompositions allow easy calculation of determinants for arbitrary sized matrices, for once the sequence on the diagonal of the seed matrix is understood, it is a simple matter to calculate its product. What's more, this technique provides a visual tool to examine the structure of several flavors of matrices, such as the arithmetic and convolution matrices discussed above.

In a future paper, we hope to further generalize this technique and add to this list the recursion relation matrices studied by Ollerton and Shannon [11].

## ACKNOWLEDGMENT

The authors gratefully acknowledge the support of the Council on Undergraduate Research 1999 Summer Undergraduate Research Fellowship which funded this paper's development.

## REFERENCES

1. M. Bicknell. "A Primer for the Fibonacci Numbers: Part XIII. The Fibonacci Convolution Triangle, Pascal's Triangle, and Some Interesting Determinants." *The Fibonacci Quarterly* **11.5** (1973):511-16.
2. M. Bicknell & V. E. Hoggatt, Jr. "Catalan and Related Sequences Arising from Inverses of Pascal's Triangle Matrices." *The Fibonacci Quarterly* **14.5** (1976):395-405.
3. M. Bicknell & V. E. Hoggatt, Jr. "Pascal, Catalan, and General Sequence Convolution Arrays in a Matrix." *The Fibonacci Quarterly* **14.2** (1976):135-43.
4. M. Bicknell & V. E. Hoggatt, Jr. "Sequences of matrix Inverses from Pascal, Catalan, and Related Convolution Arrays." *The Fibonacci Quarterly* **14.3** (1976):224-32.
5. M. Bicknell & V. E. Hoggatt, Jr. "Special Determinants Found within Generalized Pascal Triangles." *The Fibonacci Quarterly* **11.5** (1973):469-79.
6. M. Bicknell & V. E. Hoggatt, Jr. "Unit Determinants in Generalized Pascal Triangles." *The Fibonacci Quarterly* **11.2** (1973):131-44.
7. B. A. Bondarenko. *Generalized Pascal Triangles and Pyramids*. Santa Clara, CA: The Fibonacci Association, 1993.
8. D. Callan. "A Combinatorial Interpretation of a Catalan Numbers Identity." *Math. Mag.* **72.4** (1999):295-98.
9. V. E. Hoggatt, Jr., & G. E. Bergum. "Generalized Convolution Arrays." *The Fibonacci Quarterly* **13.3** (1975):193-98.
10. T. Muir & W. H. Metzler. *A Treatise on the Theory of Determinants*. New York: Longmans, Green and Co., 1933.
11. R. L. Ollerton & A. G. Shannon. "Some Properties of Generalized Pascal Squares and Triangles." *The Fibonacci Quarterly* **36.2** (1998):98-109.
12. N. J. A. Sloane & S. Plouffe. *The Encyclopedia of Integer Sequences*. San Diego, CA: Academic Press, 1995.
13. C. K. Wong & T. W. Maddocks. "A Generalized Pascal's Triangle." *The Fibonacci Quarterly* **13.2** (1975):134-36.

AMS Classification Numbers: 15A23, 11B25, 11B65

# MATRIX POWERS OF COLUMN-JUSTIFIED PASCAL TRIANGLES AND FIBONACCI SEQUENCES

**Rhodes Peele and Pantelimon Stanica**
Department of Mathematics, Auburn University Montgomery, Montgomery, AL 36124
e-mail: peelhor@mindspring.com; stanpan@strudel.aum.edu
*(Submitted January 2000-Final Revision October 2000)*

## 1. MOTIVATION

It is known that if $L_n$, respectively $R_n$, are $n \times n$ matrices with the $(i, j)^{\text{th}}$ entry the binomial coefficient $\binom{i-1}{j-1}$, respectively $\binom{i-1}{n-j}$, then $L_n^2 \equiv I_n \pmod 2$, respectively $R_n^3 \equiv I_n \pmod 2$, where $I_n$ is the identity matrix of dimension $n > 1$ (see, e.g., Problem P10735 in the May 1999 issue of *Amer. Math. Monthly*).

The entries of $L_n$ form a left-justified Pascal triangle and the entries of $R_n$ result from taking the mirror-image of this triangle with respect to its first column.

The questions we ask are: Can this result be extended to other primes or, better yet, is it possible to find a closed form for the entries of powers of $L_n$ and $R_n$?

$L_n$ succumbs easily, as we shall see in our first result. $R_n$ in turn fights back, since closed forms for its powers are not found. However, we show a beautiful connection between matrices similar to $R_n$ and the Fibonacci numbers. If $n = 2$, the connection is easily seen, since

$$R_2^e = \begin{pmatrix} F_{e-1} & F_e \\ F_e & F_{e+1} \end{pmatrix}.$$

A simple consequence of our results is that the order of $L_n$ modulo a prime $p$ is $p$, and the order of $R_n$ modulo $p$ divides four times the entry point of the Fibonacci sequence modulo $p$.

## 2. HIGHER POWERS OF $L_n$ AND $R_n$

The first approach that comes to mind is to find a closed form for all entries of powers of $L_n$ and $R_n$. It is not difficult to obtain all the powers of $L_n$. Denoting the entries of the $e^{\text{th}}$ power of $L_n$ by $l_{i,j}^{(e)}$, we can prove

***Theorem 1:*** The entries of $L_n^e$ are

$$l_{i,j}^{(e)} = e^{i-j}\binom{i-1}{j-1}. \tag{1}$$

***Proof:*** We use induction on $e$. The result is certainly true for $e = 1$. Now, using induction and matrix multiplication,

$$l_{i,j}^{(e+1)} = \sum_{s=1}^{n}\binom{i-1}{s-1}e^{s-j}\binom{s-1}{j-1} \overset{[5],\text{p.}3}{=} \sum_{s=1}^{n} e^{s-j}\binom{i-1}{j-1}\binom{i-j}{i-s}$$
$$= \binom{i-1}{j-1}\sum_{k=0}^{i-j} e^{i-j-k}\binom{i-j}{k} = \binom{i-1}{j-1}(e+1)^{i-j}. \qquad \square$$

To prove a similar result for $R_n$ is no easy matter. In fact, except for a few lower-dimensional cases and a few of its rows/columns, simple closed forms for the entries of $R_n^e$ are not found.

In the sequel, we consider the tableau with entries $a_{ij}$, $i \geq 1$, $j \geq 0$, satisfying

$$a_{i,\,j-1} = a_{i-1,\,j-1} + a_{i-1,\,j}, \tag{2}$$

with boundary conditions $a_{1,\,n} = 1$, $a_{1,\,j} = 0$, $j \neq n$. We shall use the following consequences of the boundary conditions and recurrence (2): $a_{i,\,j} = 0$ for $i + j \leq n$, and $a_{i,\,n+1} = 0$, $1 \leq i \leq n$ [in fact, we use *only* these consequences and (2)]. The matrix $R$ will be defined as $(a_{i,\,j})_{i=1\ldots n,\,j=1\ldots n}$. We treat the second and third powers first, since it gives us the idea about the general case. To clear up the mysteries of some of the steps in our calculations, we will refer to matrix multiplication as *m.m.* and the boundary conditions as *b.c.*.

***Lemma 2:*** The entries of the matrix $R^2$ satisfy

$$b_{i,\,j+1} = b_{i-1,\,j+1} + 2b_{i-1,\,j} - b_{ij}, \; 2 \leq i \leq n, \; 1 \leq j \leq n-1, \tag{3}$$

and the entries of $R^3$ satisfy

$$c_{i+1,\,j} = 2c_{i,\,j} + 3c_{i,\,j-1} - 2c_{i+1,\,j-1}, \; 1 \leq i \leq n-1, \; 2 \leq j \leq n. \tag{4}$$

***Proof:*** Using matrix multiplication and (2), we obtain

$$\begin{aligned} b_{i,\,j+1} &\overset{m.m.}{=} \sum_{s=1}^{n} a_{i,\,s} a_{s,\,j+1} \overset{(2)}{=} \sum_{s=1}^{n} a_{i,\,s}(a_{s+1,\,j} - a_{s,\,j}) \\ &= \sum_{s=1}^{n} a_{i,\,s} a_{s+1,\,j} - \sum_{s=1}^{n} a_{i,\,s} a_{s,\,j} \overset{m.m.}{=} \sum_{s=1}^{n} a_{i,\,s} a_{s+1,\,j} - b_{i,\,j}. \end{aligned} \tag{5}$$

Therefore, denoting $S_{i,\,j} = \sum_{s=1}^{n} a_{i,\,s} a_{s+1,\,j}$, we obtain

$$b_{i,\,j+1} + b_{i,\,j} = S_{i,\,j}. \tag{6}$$

If $2 \leq i \leq n$ and $1 \leq j \leq n$,

$$\begin{aligned} S_{i,\,j} &= \sum_{s=1}^{n} (a_{i-1,\,s} + a_{i-1,\,s+1}) a_{s+1,\,j} \overset{t=s+1}{=} S_{i-1,\,j} + \sum_{t=2}^{n+1} a_{i-1,\,t} a_{t,\,j} \\ &\overset{m.m.}{=} S_{i-1,\,j} + b_{i-1,\,j} + a_{i-1,\,n+1} a_{n+1,\,j} - a_{i-1,\,1} a_{1,\,j} \overset{b.c.}{=} S_{i-1,\,j} + b_{i-1,\,j}. \end{aligned}$$

Using (6) in the previous recurrence, we obtain $b_{i,\,j+1} + b_{i,\,j} = b_{i-1,\,j+1} + b_{i-1,\,j} + b_{i-1,\,j}$, which gives us (3).

If the relations (3) are satisfied, we obtain, for $j \geq 2$,

$$\begin{aligned} c_{i,\,j} &\overset{m.m.}{=} \sum_{s=1}^{n} a_{i,\,s} b_{s,\,j} \overset{(3)}{=} \sum_{s=1}^{n} a_{i,\,s}(b_{s-1,\,j} + 2b_{s-1,\,j-1} - b_{s,\,j-1}) \\ &= \sum_{s=1}^{n} a_{i,\,s} b_{s-1,\,j} + 2\sum_{s=1}^{n} a_{i,\,s} b_{s-1,\,j-1} - c_{i,\,j-1} = T_{i,\,j} + 2T_{i,\,j-1} - c_{i,\,j-1}, \end{aligned}$$

where $T_{i,\,j} = \sum_{s=1}^{n} a_{i,\,s} b_{s-1,\,j}$. Furthermore, for $i \leq n-1$,

$$\begin{aligned} T_{i,\,j} &\overset{(2)}{=} \sum_{s=1}^{n} (a_{i+1,\,s-1} - a_{i,\,s-1}) b_{s-1,\,j} \overset{m.m.}{=} c_{i+1,\,j} + a_{i+1,\,0} b_{0,\,j} - a_{i+1,\,n} b_{n,\,j} - c_{i,\,j} - a_{i,\,0} b_{0,\,j} + a_{i,\,n} b_{n,\,j} \\ &\overset{(2)}{=} c_{i+1,\,j} - c_{i,\,j} + a_{i,\,1} b_{0,\,j} - a_{i,\,n+1} b_{n,\,j} \overset{b.c.}{=} c_{i+1,\,j} - c_{i,\,j}. \end{aligned}$$

Therefore,

$$c_{i,j} = T_{i,j} + 2T_{i,j-1} - c_{i,j-1} = c_{i+1,j} - c_{i,j} + 2c_{i+1,j-1} - 2c_{i,j-1} - c_{i,j-1},$$

which will produce the equations (4). □

***Corollary 3:*** The entries of the second and third power of $R$ can be expressed in terms of the entries of the previous row:

$$b_{i+1,j} = b_{i,j} - \sum_{k=1}^{j-1}(-1)^k b_{i,j-k}, \quad c_{i+1,j} = 2c_{i,j} + \sum_{k=1}^{j-1}(-1)^k 2^{k-1} c_{i,j-k}.$$

We have wondered if relations similar to (3) or (4) are true for higher powers of $R$. It turns out that

***Theorem 4:*** The entries $a_{i,j}^{(e)}$ of the $e^{\text{th}}$ power of $R$ satisfy the relation

$$F_{e-1} a_{i,j}^{(e)} = F_e a_{i-1,j}^{(e)} + F_{e+1} a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)},$$

where $F_e$ is the Fibonacci sequence.

***Proof:*** We show first that the entries of $R^e$ satisfy a relation of the form

$$\delta_e a_{i,j}^{(e)} = \alpha_e a_{i-1,j}^{(e)} + \beta_e a_{i-1,j-1}^{(e)} + \gamma_e a_{i,j-1}^{(e)} \tag{7}$$

and then will proceed to find these coefficients. From Lemma 2, we observe that $\delta_1 = 0$, $\alpha_1 = 1$, $\beta_1 = 1$, $\gamma_1 = -1$, $\delta_2 = 1$, $\alpha_2 = 1$, $\beta_2 = 2$, $\gamma_2 = -1$, and $\delta_3 = 1$, $\alpha_3 = 2$, $\beta_3 = 3$, $\gamma_3 = -2$. Now, the coefficients of $R^e$ satisfy, for $i, j \geq 2$,

$$\begin{aligned}\delta_{e-1} a_{i-1,j}^{(e)} &\overset{m.m.}{=} \sum_{s=1}^{n} \delta_{e-1} a_{i-1,s} a_{s,j}^{(e-1)} \overset{(7)}{=} \sum_{s=1}^{n} a_{i-1,s}\left(\alpha_{e-1} a_{s-1,j}^{(e-1)} + \beta_{e-1} a_{s-1,j-1}^{(e-1)} + \gamma_{e-1} a_{s,j-1}^{(e-1)}\right) \\ &\overset{m.m.}{=} \alpha_{e-1} U_{i-1,j} + \beta_{e-1} U_{i-1,j-1} + \gamma_{e-1} a_{i-1,j-1}^{(e)},\end{aligned} \tag{8}$$

where $U_{i,j} = \sum_{s=1}^{n} a_{i,s} a_{s-1,j}^{(e-1)}$. We evaluate, for $2 \leq i \leq n$,

$$\begin{aligned}U_{i-1,j} &\overset{(2)}{=} \sum_{s=1}^{n} (a_{i,s-1} - a_{i-1,s-1}) a_{s-1,j}^{(e-1)} \\ &\overset{m.m.}{=} a_{i,j}^{(e)} - a_{i-1,j}^{(e)} + a_{i,0} a_{0,j}^{(e-1)} - a_{i-1,0} a_{0,j}^{(e-1)} - a_{i,n} a_{n,j}^{(e-1)} + a_{i-1,n} a_{n,j}^{(e-1)} \\ &\overset{(2)}{=} a_{i,j}^{(e)} - a_{i-1,j}^{(e)} + a_{i-1,1} a_{0,j}^{(e-1)} - a_{i-1,n+1} a_{n,j}^{(e-1)} = a_{i,j}^{(e)} - a_{i-1,j}^{(e)},\end{aligned}$$

since $a_{i-1,1} = 0$, $i \leq n$, and $a_{i-1,n+1} = 0$. Thus,

$$\alpha_{e-1} a_{i,j}^{(e)} = (\delta_{e-1} + \alpha_{e-1}) a_{i-1,j}^{(e)} + (\beta_{e-1} - \gamma_{e-1}) a_{i-1,j-1}^{(e)} - \beta_{e-1} a_{i,j-1}^{(e)}.$$

Therefore, we obtain the following system of sequences:

$$\begin{aligned}\delta_e &= \alpha_{e-1}, \\ \alpha_e &= \alpha_{e-1} + \delta_{e-1}, \\ \beta_e &= \beta_{e-1} - \gamma_{e-1}, \\ \gamma_e &= -\beta_{e-1}.\end{aligned} \tag{9}$$

From this, we deduce $\delta_e = F_{e-1}$, $\alpha_e = F_e$, $\beta_e = F_{e+1}$, $\gamma_e = -F_e$, where $F_e$ is the Fibonacci sequence with $F_0 = 0$, $F_1 = 1$. □

Corollary 3 can be generalized, with a little more work and anticipating (10), to obtain the elements in the $(i+1)^{\text{th}}$ row of $R^e$, in terms of the elements in the previous row.

***Proposition 5:*** We have

$$F_{e-1}a_{i+1,j}^{(e)} = F_e a_{i,j}^{(e)} - \sum_{k=1}^{j-1}(-1)^{k+e}\frac{F_e^{k-1}}{F_{e-1}^k}a_{i,j-k}^{(e)}.$$

## 3. HIGHER POWERS OF $L_n$ AND $R_n$ MODULO A PRIME $p$

As before let $L_n$, respectively $R_n$, be defined as the matrices with entries $\binom{i-1}{j-1}$, respectively $\binom{i-1}{n-j}$. We use the notation "*matrix* $\equiv a \pmod p$" with the meaning "*matrix* $\equiv aI_n \pmod p$".

We ask the question of whether or not the order of $L_n$ and $R_n$ modulo a prime $p$ is finite. We can easily prove a result for $L_n$ using Theorem 1.

***Theorem 6:*** The order of $L_n$ $(n \ge 2)$ modulo $p$ is $p$.

***Proof:*** We have shown that the entries of $L_n^e$ are $l_{i,j}^{(e)} = e^{i-j}\binom{i-1}{j-1}$ for any integer $e$. Thus, the entries on the principal diagonal of $L_n^e$ are all 1. If $i \ne j$, then $p \mid l_{i,j}^{(p)}$. Assume there is an integer $e$ with $0 < e < p$ such that $p \mid l_{i,j}^{(e)}$ for all $i \ne j$. Take $i = 2$ and $j = 1$. Then $p \mid e$ is a contradiction. Therefore, the integer $p$ is the least integer $e > 0$ for which $p \mid l_{i,j}^{(e)}$ for all $i \ne j$, which proves our assertion. □

We can prove the finiteness of the order of $R_n$ modulo $p$ in a simple manner. By the Pigeonhole Principle, there exist $s < t$ such that $R_n^s \equiv R_n^t \pmod p$. Since $R_n$ is an invertible matrix (det $R_n = (-1)^n \not\equiv 0 \pmod p$), $R_n^{t-s} \equiv I_n \pmod p$. More precise results will be proved next. In order to do that, we need some known facts about the period of the Fibonacci sequence. It was shown that the period of the Fibonacci sequence modulo $m$ (not necessarily prime) is less than or equal to $6m$ (with equality holding for infinitely many values of $m$) (see P. Freyd, Problem E 3410, *Amer. Math. Monthly*, December 1990, with a solution provided in *ibid.*, March 1992). In the case of a prime, the result can be strengthened (see Theorem 7). The least integer $n \ne 0$ with the property $m \mid F_n$ is called the *entry point* modulo $m$.

In [1] and [7], the authors obtain (see also [6], Chs. VI-VII, for a more updated source)

***Theorem 7 (Bloom-Wall):*** Denote the period of the Fibonacci sequence modulo $p$ by $\mathcal{P}(p)$. Let $p$ be an odd prime with $p \ne 5$. If $p \equiv \pm 1 \pmod 5$, then the period $\mathcal{P}(p) \mid (p-1)$. If $p \equiv \pm 3 \pmod 5$, then the entry point $e \mid (p+1)$ and the period $\mathcal{P}(p) \mid 2(p+1)$.

***Remark 8:*** For $p = 2$, the entry point is 3 and the period is 3. In the case $p = 5$, the entry point is 5 and the period is 20.

***Theorem 9:*** If $e$ is the entry point modulo $p$ of $F_e$, then $R_{2k}^e \equiv (-1)^{(k+1)e}F_{e-1}I_{2k} \pmod p$ and $R_{2k+1}^e \equiv (-1)^{ke}I_{2k+1} \pmod p$. Moreover, $R_n^{4e} \equiv I_n \pmod p$.

***Proof:*** We prove by induction on $e$ that the elements in the first row and first column of $R_n^e$ are

$$a_{1,j}^{(e)} = \binom{n-1}{j-1} F_{e-1}^{n-j} F_e^{j-1} \quad \text{and} \quad a_{i,1}^{(e)} = F_{e-1}^{n-i} F_e^{i-1}. \tag{10}$$

First, we deal with the elements in the first row. The first equation is certainly true for $e=1$, if we define $0^0 = 1$. Now,

$$\begin{aligned} a_{1,j}^{(e+1)} &\overset{m.m.}{=} \sum_{s=1}^{n} a_{1,s}^{(e)} a_{s,j} = \sum_{s=1}^{n} F_{e-1}^{n-s} F_e^{s-1} \binom{n-1}{s-1}\binom{s-1}{n-j} \\ &= \sum_{s=1}^{n} F_{e-1}^{n-s} F_e^{s-1} \binom{n-1}{j-1}\binom{j-1}{n-s} = F_e^{n-1}\binom{n-1}{j-1}\sum_{s=1}^{n}\left(\frac{F_{e-1}}{F_e}\right)^{n-s}\binom{j-1}{n-s} \\ &= F_e^{n-1}\binom{n-1}{j-1}\left(1+\frac{F_{e-1}}{F_e}\right)^{j-1} = F_e^{n-j} F_{e+1}^{j-1}\binom{n-1}{j-1}. \end{aligned}$$

Again by induction, we prove the result for the elements in the first column. The case $e=1$ can be checked easily. Then

$$\begin{aligned} a_{i,1}^{(e+1)} &\overset{m.m.}{=} \sum_{s=1}^{n} a_{i,s} a_{s,1}^{(e)} = \sum_{s=1}^{n} \binom{i-1}{n-s} F_{e-1}^{n-s} F_e^{s-1} \\ &= F_e^{n-1}\sum_{s=1}^{n}\left(\frac{F_{e-1}}{F_e}\right)^{n-s}\binom{i-1}{n-s} = F_e^{n-1}\left(1+\frac{F_{e-1}}{F_e}\right)^{i-1} = F_e^{n-i} F_{e+1}^{i-1}. \end{aligned}$$

Let $e$ be the entry point modulo $p$ of the Fibonacci sequence. By Bloom-Wall's result, we have $e \le p+1$. Using Theorem 4, we obtain $F_{e-1} a_{i,j}^{(e)} \equiv F_e a_{i-1,j}^{(e)} + F_{e+1} a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)} \pmod{p}$. Thus,

$$F_{e-1} a_{i,j}^{(e)} \equiv F_{e+1} a_{i-1,j-1}^{(e)} \pmod{p}. \tag{11}$$

Since $F_{e-1} + F_e = F_{e+1}$, $p \mid F_e$, and $p \nmid F_{e-1}$, we obtain $F_{e-1} \equiv F_{e+1} \pmod{p}$ and

$$a_{i,j}^{(e)} \equiv a_{i-1,j-1}^{(e)} \pmod{p}. \tag{12}$$

We see from what was proved above that, modulo $p$, the elements in the first row and column of $R_n \pmod{p}$ are all zero, except for the one in the first position, which is $F_{e-1}^{n-1} \not\equiv 0 \pmod{p}$. Using (12), we get $R_n^e \equiv F_{e-1}^{n-1} I_n \pmod{p}$. Using Cassini's identity $F_{e-1}F_{e+1} - F_e^2 = (-1)^e$ (see [2], p. 292), we obtain $F_{e-1}^2 \equiv F_{e+1}^2 \equiv (-1)^e \pmod{p}$. If $n = 2k$, then

$$F_{e-1}^{n-1} = F_{e-1}^{2k-1} \equiv (F_{e-1}^2)^k F_{e-1}^{-1} \equiv (-1)^{ke} F_{e-1}^{-1} \equiv (-1)^{(k+1)e} F_{e-1} \pmod{p}.$$

If $n = 2k+1$, then $F_{e-1}^{n-1} = F_{e-1}^{2k} \equiv (F_{e-1}^2)^k \equiv (-1)^{ke} \pmod{p}$.

The previous two congruences replaced in $R_n^e \equiv F_{e-1}^{n-1} I_n \pmod{p}$, will give the first two assertions of our theorem.

It is well known (a very particular case of Matijasevich's lemma) that $F_{2e-1} = F_{e-1}^2 + F_e^2 \equiv F_{e-1}^2 \pmod{F_e^2}$, so $F_{2e-1}^2 \equiv 1 \pmod{p}$. Thus, since $F_m$ divides $F_{sm}$ for all $m$ and $s$ (in particular, for $s=2$, $m=e$), it follows that $F_{2e} \equiv 0 \pmod{p}$ and $R_n^{4e} = (R_n^{2e})^2 \equiv (F_{2e-1}^2)^{n-1} I_n \equiv I_n \pmod{p}$. □

***Remark 10:*** We remark here the fact that the bound $4e$ for the order of $R$ is tight. That can be seen by taking, for example, the prime 13, since the entry point for the Fibonacci sequence is 7, and the order of $R_{4k}$ is 28.

Using some elementary number theory, we can prove

***Theorem 11:*** If $p \mid F_{p-1}$, then $R_n^{p-1} \equiv I_n \pmod p$.

***Proof:*** We observe that, since $p \mid F_{p-1}$, we have $p \equiv \pm 1 \pmod 5$, otherwise, $p \equiv \pm 2 \pmod 5$, and by Bloom-Wall's theorem, the entry point $e$ divides $p+1$. Thus, $e \mid p-1$ and $e \mid p+1$. Therefore, $e$ must be 2. This is not possible because $F_2 = 1$, which is not divisible by any prime. So, $p \equiv \pm 1 \pmod 5$ and $F_p \equiv F_{p-2} \pmod p$. Thus, $R_n^{p-1} \equiv F_{p-2}^{n-1} I_n \equiv F_p^{n-1} I_n \pmod p$.

By the previous Bloom-Wall theorem, $\mathcal{P}(p) \mid (p-1)$; therefore, $F_{p-1} \equiv 0$, $F_p \equiv 1$, $F_{p+1} \equiv 1$, etc. Hence, $R_n^{p-1} \equiv F_p^{n-1} I_n \equiv I_n \pmod p$. □

Another interesting result is the following theorem.

***Theorem 12:*** If $p \mid F_{p+1}$, then $R_{2k+1}^{p+1} \equiv I_{2k+1} \pmod p$ and $R_{2k}^{p+1} \equiv -I_{2k} \pmod p$.

***Proof:*** Assume $p = 2$. The entry point of the Fibonacci sequence modulo 2 is $e = 3$. Since $F_2 = 1$, Theorem 9 shows the result in this case. Assume $p > 2$. We know that in this case we must have $p \equiv \pm 2 \pmod 5$. Using the known formula (see, e.g., [3], Theorem 180)

$$F_j = 2^{1-j}\left[\binom{j}{1} + 5\binom{j}{3} + 5^2\binom{j}{5} + \cdots\right],$$

taking $j = p$, and using Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod p$, we obtain

$$F_p \equiv 5^{(p-1)/2}\binom{p}{p} \equiv -1 \pmod p,$$

since, for the primes $\equiv \pm 2 \pmod 5$, 5 is a quadratic nonresidue.

When $n$ is odd, $R_n^{p+1} \equiv F_p^{n-1} I_n \equiv (F_p^2)^{\frac{n-1}{2}} I_n \equiv I_n \pmod p$. Consider the case of $n$ even. Since $F_p \equiv -1 \pmod p$, we have $R_n^{p+1} \equiv F_p^{n-1} I_n \equiv (-1)^{n-1} I_n \equiv -I_n \pmod p$. □

The proofs of the previous two theorems imply

***Corollary 13:*** If $p \equiv \pm 1 \pmod 5$ and $p-1$ is the entry point for the Fibonacci sequence modulo $p$, then the period is exactly $p-1$. If $p \equiv \pm 2 \pmod 5$ and $p+1$ is the entry point for the Fibonacci sequence modulo $p$, then the period is exactly $2(p+1)$.

***Corollary 14:*** The order of $R_n \pmod p$ is less than or equal to $2(p+1)$ and the bound is met.

***Proof:*** If $p \equiv \pm 1 \pmod 5$, then the order of $R_n \pmod p$ is $\le p-1$. If $p \equiv \pm 2 \pmod 5$, then $F_p \equiv -1 \pmod p$. Therefore, $R_n^{p+1} \equiv F_p^{n-1} I_n \equiv (-1)^{n-1} I_n \pmod p$. Thus, $R_n^{2p+2} \equiv I_n \pmod p$. The bound is met for all primes $p \equiv \pm 2 \pmod 5$ and all even integers $n$. □

## 4. FURTHER PROBLEMS AND RESULTS

The inverses of $R_n$ and $L_n$ are not difficult to find. We have

***Theorem 15:*** The inverse of

$$L_n = \left(\binom{i-1}{j-1}\right)_{1 \le i, j \le n} \quad \text{is} \quad L_n^{-1} = \left((-1)^{i+j}\binom{i-1}{j-1}\right)_{1 \le i, j \le n}.$$

The inverse of

$$R_n = \left(\binom{i-1}{n-j}\right)_{1\le i,\, j\le n} \quad \text{is} \quad R_n^{-1} = \left((-1)^{n+i+j+1}\binom{n-i}{j-1}\right)_{1\le i,\, j\le n}.$$

***Proof:*** We have

$$\sum_{s}^{n}(-1)^{i+s}\binom{i-1}{s-1}\binom{s-1}{j-1} \overset{[5],\text{p.}3}{=} \sum_{s}^{n}(-1)^{i+s}\binom{i-1}{j-1}\binom{i-j}{i-s} \overset{k=i-s}{=} \binom{i-1}{j-1}\sum_{k=0}^{i-j}(-1)^k\binom{i-j}{k},$$

which is 0, unless $i = j$, in which case it is 1. A similar analysis for $R_n$ will produce its inverse. □

Another approach to find a closed form for all entries of powers of $R_n$ would be to find all eigenvalues of $R_n$, and use the diagonalization of the matrix to find the entries of $R_n$. We found the following empirically and we state it as a conjecture.

***Conjecture 16:*** Denote $\phi = \frac{1+\sqrt{5}}{2}$, $\overline{\phi} = \frac{1-\sqrt{5}}{2}$. The eigenvalues of $R_n$ are:

***(a)*** $\{(-1)^{k+i}\phi^{2i-1}, (-1)^{k+i}\overline{\phi}^{2i-1}\}_{i=1,\dots,k}$ if $n = 2k$.

***(b)*** $\{(-1)^k\} \cup \{(-1)^{k+i}\phi^{2i}, (-1)^{k+i}\overline{\phi}^{2i}\}_{i=1,\dots,k}$ if $n = 2k+1$.

Another venue of research would be to study the matrices associated to other interesting sequences—Lucas, Pell, etc.—and we will approach this matter elsewhere.

*Note Added to Proof:* Recently, the above-mentioned conjecture was settled in the affirmative, independently, by P. Stanica and R. Peele, by D. Callan, and by H. Prodinger.

## ACKNOWLEDGMENT

The authors would like to thank the anonymous referee for his helpful comments, which improved significantly the presentation of the paper.

## REFERENCES

1. D. M. Bloom. "On Periodicity in Generalized Fibonacci Sequences." *Amer. Math. Monthly* (1965):856-61.
2. R. L. Graham, D. E. Knuth, & O. Patashnik. *Concrete Mathematics.* New York: Addison-Wesley, 1989.
3. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers.* Oxford: Oxford-Clarendon Press, 1960.
4. M. Petkovsek, H. Wilf, & D. Zeilberger. $A = B$. Wellesley, Mass.: A. K. Peters, 1997.
5. J. Riordan. *Combinatorial Identities.* New York: John Wiley & Sons, 1968.
6. S. Vajda. *Fibonacci & Lucas Numbers and the Golden Section: Theory and Applications.* New York: John Wiley & Sons, 1989.
7. D. D. Wall. "Fibonacci Sequences Modulo *m*." *Amer. Math. Monthly* (1960):525-32.

AMS Classification Numbers: 05A10, 11B39, 11B65, 11C20, 15A33

❖❖❖

# A NOTE ON THE DIVISIBILITY OF THE GENERALIZED LUCAS SEQUENCES

**Pingzhi Yuan**
Dept. of Math. and Mechanics, Central South University, Hunan Changsha, 410075, P.R. China
e-mail: yuanpz@csru.edu.cn
*(Submitted January 2000-Final Revision March 2001)*

In this paper we discuss the divisibility theory of the generalized Lucas sequences $U_n$ and $V_n$ which were defined by D. H. Lehmer [1] as follows:

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta), \tag{1}$$

$$V_n = \alpha^n + \beta^n, \quad V_0 = 2, \tag{2}$$

where $\alpha = (\sqrt{R} + \sqrt{\Delta})/2$, $\beta = (\sqrt{R} - \sqrt{\Delta})/2$ are the roots of $x^2 - R^{1/2}x + Q = 0$, $R$ and $Q$ are coprime integers, $R > 0$, the discriminant $\Delta = R - 4Q$, and $n \geq 0$ is an integer.

The main theorem of this paper is a complement of that of Lehmer [1], and this result is essential in the applications to exponential Diophantine equations, as we will show in another paper. Moreover, the main results of McDaniel [2] will be extended, and this can be deduced easily from the main theorem of this paper.

It is easy to see that $U_{2k+1}$ and $V_{2k}$ are rational integers and that $U_{2k}$ and $V_{2k+1}$ are integral multiples of $R^{1/2}$. Let $Z$ be the set of integers, $R^{1/2}Z = \{aR^{1/2} \mid a \in Z\}$. If we define the divisibility of the elements of the set $Z \cup R^{1/2}Z$ as follows: For any $A$, $B \in Z \cup R^{1/2}Z$, $A|B \Leftrightarrow B = A \cdot C$, and $C \in Z \cup R^{1/2}Z$, then most of the propositions below are well known (see, e.g., [3], Chapter 2). Proposition 1(e) was recently proved in [2]; however, as we will show, this proposition is not true for the most general definition of the generalized Lucas sequences as defined above.

***Proposition 1:*** Let $m$ and $n$ be arbitrary integers:

***(a)*** $V_n^2 - \Delta U_n^2 = 4Q^n$.

***(b)*** If $m|n$, then $U_m|U_n$; if $n/m$ is odd, then $V_m|V_n$.

***(c)*** $U_{2n} = U_n V_n$; $V_{2n} = V_n^2 - 2Q^n$.

***(d)*** If $d = \gcd(m, n)$, then $\gcd(U_m, U_n) = U_d$.

***(e)*** If $d = \gcd(m, n)$, then $\gcd(V_m, V_n) = V_d$ if $m/d$ and $n/d$ are odd, and 1, or 2, otherwise.

***(f)*** If $p$ is a prime and $\omega$ is the minimal positive integer with $p|U_\omega$ ([1] defined $\omega$ to be the appearance of $p$ in $U_n$), then for any positive integers $k$ and $\lambda$, we have $p^{\lambda+1}|U_{k\omega p^\lambda}$.

***(g)*** If an odd prime $p$, with $p \nmid R\Delta$, $\varepsilon = (\Delta R/p)$ is the Kronecker symbol, then $U_{p-\varepsilon} \equiv 0 \pmod{p}$.

For any prime $p$, $A \in Z \cup R^{1/2}Z$, $\text{ord}_p A$ is defined to be the rational number $s$ with $2s$ being an integer and $p^{2s} \| A^2$, denoted by $\text{ord}_p A = s$. We now have the following theorem.

***Theorem 1:*** If $p$, $q$ are odd primes and $s$, $t$ are positive integers with $p^s \| \Delta$, $q^t \| R$, then:

***(a)*** If $p^s > 3$, then $\text{ord}_p U_m = \text{ord}_p m$, $\text{ord}_p V_m = 0$.

***(b)*** For $q^t > 3$: if $m$ is odd, then $\text{ord}_q U_m = 0$, $\text{ord}_q V_m / V_1 = \text{ord}_q m$; if $m$ is even, then $\text{ord}_q V_m = 0$, $\text{ord}_q U_m = \text{ord}_q m + t/2$.

*(c)* Suppose $p^s = 3$ and $\lambda$ is an integer with $3^\lambda \| 3R+\Delta$, then $\mathrm{ord}_3 V_m = 0$, $\mathrm{ord}_3 U_{3m} = \lambda + \mathrm{ord}_3 m$; if $3 \nmid m$, then $\mathrm{ord}_3 U_m = 0$.
*(d)* Suppose now that $q^t = 3$ and $\mu$ is an integer with $3^\mu \| 3\Delta + R$. If $m$ is odd, then $\mathrm{ord}_3 U_m = 0$, $\mathrm{ord}_3 V_{3m}/V_1 = \mathrm{ord}_3 m + \mu$, and $\mathrm{ord}_3 V_m / V_1 = 0$ with $3 \nmid m$; if $m$ is even, then $\mathrm{ord}_3 V_m = 0$, $\mathrm{ord}_3 U_{3m} = \mathrm{ord}_3 m + \mu + 1/2$, and $\mathrm{ord}_3 U_m = 1/2$ with $3 \nmid m$.
*(e)* Let $2 \| R$: if $2 \nmid m$, then $\mathrm{ord}_2 U_m = \mathrm{ord}_2 V_m / V_1 = 0$ $(2 \nmid m)$; if $2 \| m$, then $\mathrm{ord}_2 V_m = \mathrm{ord}_2 V_2$ and $\mathrm{ord}_2 U_m = 1/2$; if $4 | m$, then $\mathrm{ord}_2 V_m = 1/2$ and $\mathrm{ord}_2 U_m = \mathrm{ord}_2 m - 1/2$.
*(f)* Let $4 | R$: if $m$ is odd, then $\mathrm{ord}_2 U_m = 0$ and $\mathrm{ord}_2 V_m = \mathrm{ord}_2 V_1$; if $m$ is even, then $\mathrm{ord}_2 U_m = \mathrm{ord}_2 m + \frac{1}{2}\mathrm{ord}_2 R - 1$ and $\mathrm{ord}_2 V_m = 1$.

***Proof:*** We divide the proof of the theorem into three parts:

**(I)** If $m$ is odd, subtracting the $m^{\text{th}}$ power of $2\beta = R^{1/2} - \Delta^{1/2}$ from the $m^{\text{th}}$ power of $2\alpha = R^{1/2} + \Delta^{1/2}$, we get

$$2^{m-1}U_m = \sum_{i=0}^{(m-1)/2} \binom{m}{2i+1} \Delta^i R^{(m-2i-1)/2} = mR^{(m-1)/2} + \sum_{i=1}^{(m-1)/2} \frac{m}{2i+1}\binom{m-1}{2i} \Delta^i R^{(m-2i-1)/2}. \tag{3}$$

Let $u$ be a positive integer with $p^u \| m$, $u > 0$, and notice that

$$\mathrm{ord}_p \frac{m}{2i+1}\Delta^i = si + u - \mathrm{ord}_p(2i+1) \ge si + u - \log_p(2i+1). \tag{4}$$

If $p^s \neq 3$, then $p^{si} > 2i+1$ for any $i \ge 1$, so from (4) we know that every term of the summation of (3) is a multiple of $p^{u+1}$; therefore, $\mathrm{ord}_p U_m = \mathrm{ord}_p m = u$. This result together with Proposition 1(a) and $(R, Q) = 1$ implies that $\mathrm{ord}_p V_m = 0$, i.e., Theorem 1(a) holds for odd $m$.

If $p^s = 3$, then $4U_3 = 3R + \Delta$, so from (3) we conclude that $3 | U_m$ when $3 | m$. Subtracting the $m^{\text{th}}$ power of $2\beta^3 = V_3 - \Delta^{1/2}U_3$ from the $m^{\text{th}}$ power of $2\alpha^3 = V_3 + \Delta^{1/2}U_3$, we get

$$2^{m-1}U_{3m}/U_3 = \sum_{i=0}^{(m-1)/2} \binom{m}{2i+1} (\Delta U_3^2)^i V_3^{m-2i-1}. \tag{5}$$

Similar to the above, we have $\mathrm{ord}_3 U_{3m}/U_3 = \mathrm{ord}_3 m$ and $\mathrm{ord}_3 V_m = 0$, i.e., Theorem 1(c) holds for odd $m$.

If $m$ is odd, from [1] and Proposition 1(a) we have

$$2^{m-1}V_m / V_1 = \sum_{i=0}^{(m-1)/2} \binom{m}{2i+1} R^i \Delta^{(m-2i-1)/2}, \tag{6}$$

$$R(V_m/V_1)^2 - \Delta U^2 = 4Q^m. \tag{7}$$

Symmetrically, from (6) and (7) we conclude that Theorem 1(b) and (d) hold for odd $m$.

**(II)** Now suppose that $m$ is even, then $U_2^2 = R$, so $R | U_m^2$ for any even $m$; therefore, $\mathrm{ord}_p V_m = 0 = \mathrm{ord}_q V_m$ by Proposition 1(a). Let $m = 2^a m_1$, $2 \nmid m_1$, $a \ge 1$, be an integer, and notice that by Proposition 1(c) we have

$$U_{2^a m_1} = U_{m_1} V_{m_1} V_{2m_1} \ldots V_{2^{a-1}m_1}. \tag{8}$$

Thus, $\mathrm{ord}_p U_m = \mathrm{ord}_p U_{m_1}$ and $\mathrm{ord}_q U_m = \mathrm{ord}_q V_{m_1}$, and from the above result of the odd number $m_1$ we know that Theorem 1(a)-(d) hold for even $m$.

**(III)** For Theorem 1(e), it is well-known that $\{U_m\}$ satisfies the following recurrence relation,

$$U_{m+2} = R^{1/2}U_{m+1} - QU_m, \quad U_0 = 0, \ U_1 = 1. \tag{9}$$

Since $(R, Q) = 1$ and $2 \| R$, we have $Q \equiv 1 \pmod 2$ and $\Delta = R - 4Q \equiv 2 \pmod 4$. Taking modulo 2 for the sequence (9), we obtain a sequence with a period 4,

$$U_m \equiv 0, 1, R^{1/2}, 1, 0, 1, R^{1/2}, 1, \ldots. \tag{10}$$

If $2 \nmid m$, then (10) implies that $\mathrm{ord}_2 U_m = 0$, and from $2 \| \Delta$ and $V_m^2 - \Delta U_m^2 = 4Q^m$ we have $\mathrm{ord}_2 V_m = 1/2$; if $4 | m$, then (10) implies that $\mathrm{ord}_2 U_m \geq 1$, and from $2 \| \Delta$ and $V_m^2 - \Delta U_m^2 = 4Q^m$ we have $\mathrm{ord}_2 V_m = 1$. Then from (8) we have

$$\mathrm{ord}_2 U_m = \mathrm{ord}_2 U_{m_1} + \mathrm{ord}_2 V_{m_1} + \sum_{i=1}^{a-1} \mathrm{ord}_2 V_{2^i m_1} = 0 + \frac{1}{2} + (a-1) = \mathrm{ord}_2 m - \frac{1}{2}.$$

If $2 \| m$, say, $m = 2m_1$, $2 \nmid m_1$, then $V_2 \equiv R - 2Q \equiv 0 \pmod 4$, and adding the $m^{\text{th}}$ powers of $2\alpha^2 = V_2 + (R\Delta)^{1/2}$ and $2\beta^2 = V_2 - (R\Delta)^{1/2}$, we get

$$2^{m_1 - 1} V_{2m_1} / V_2 = \sum_{i=0}^{(m_1-1)/2} \binom{m_1}{2i+1} V_2^{2i} (\Delta R)^{(m_1 - 2i - 1)/2} \tag{11}$$

and $\mathrm{ord}_2(V_2^{2i}(\Delta R)^{(m_1-2i-1)/2}) \geq m_1 - 1$, and the equality holds if and only if $i = 0$. Thus, by taking modulo $2^{m_1}$ for (11), we get $\mathrm{ord}_2 V_{2m_1}/V_2 = 0$, and from (8) we have $\mathrm{ord}_2 V_{2m_1} = \mathrm{ord}_2 V_{m_1} = 1/2$. Summing the above result we complete the proof of Theorem 1(e).

For Theorem 1(f), if $4 | R$, put $R = 4R_1$, then $\Delta = R - 4Q = 4\Delta_1$ and $Q$ is odd, so $2 | R_1 \Delta_1$, and if $m$ is odd,

$$U_m = \sum_{i=0}^{(m-1)/2} \binom{m}{2i+1} \Delta_1^i R_1^{(m-2i-1)/2} = mR_1^{(m-1)/2} + \sum_{i=1}^{(m-3)/2} \frac{m}{2i+1} \binom{m-1}{2i} \Delta_1^i R_1^{(m-2i-1)/2} + \Delta_1^{(m-1)/2}.$$

Therefore, $\mathrm{ord}_2 U_m = 0$. Similarly, $\mathrm{ord}_2 V_m = \mathrm{ord}_2 V_1$. If $m$ is even, then from (8) we have $2 | U_m$, and $V_m^2/4 - \Delta_1 U_m^2 = Q^m$ implies that $V_m/2$ is odd, i.e., $\mathrm{ord}_2 V_m = 1$. From the results for odd $m$ and again using (8) we have $\mathrm{ord}_2 U_m = \mathrm{ord}_2 m - 1 + \mathrm{ord}_2 V_1 = \mathrm{ord}_2 m + \frac{1}{2}\mathrm{ord}_2 R - 1$. This completes the proof of Theorem 1.

***Remark 1:*** Put $\alpha_1 = \alpha^m$, $\beta_1 = \beta^m$, $R_1 = \alpha_1 + \beta_1$, $\Delta_1 = (\alpha_1 - \beta_1)^2$, $U_n^{(1)} = (\alpha_1^n - \beta_1^n)/(\alpha_1 - \beta_1)$, and $V_n^{(1)} = \alpha_1^n + \beta_1^n$. Then we have $U_n^{(1)} = U_{mn}/U_m$, $V_n^{(1)} = V_{mn}$, and $\Delta_1 = \Delta U_m^2$. Applying Theorem 1 to $U_n^{(1)}$, $V_n^{(1)}$, we obtain the largest power of $q$ in $U_n$ or $V_n$ if $q | U_m$ or $q | V_m$.

Now let us remark that if $2 \nmid R$ then $2 \nmid \Delta$, since $U_n$ and $V_n$ satisfy recurrence relation (9) and the following one, respectively,

$$V_{n+2} = R^{1/2} V_{m+1} - QV_m, \quad V_0 = 2, \ V_1 = R^{1/2}. \tag{12}$$

Taking modulo 2, we have $2 \nmid U_m V_m$ when $m > 0$, and if $2 \nmid Q$ then $2 | U_m$ and $2 | V_m$ if and only if $3 | m$ and $3 | n$, respectively. Hence, from Remark 1 and the above discussion, we need only consider the case of $2 | R$ when we study the behavior of the 2-part of $U_m$ and $V_n$.

We will now prove the following corollary which is an extension of Proposition 1(e) above.

***Corollary:*** If $d = \gcd(m, n)$, then $\gcd(V_m, V_n) = V_d$ if $m/d$ and $n/d$ are odd, and 1, $\sqrt{2}$, or 2, otherwise.

***Proof:*** For $d = \gcd(m, n)$, we may suppose without loss of generality that $km = d + \ell n$, where $k$ and $\ell$ are positive integers. If $k$ is odd, notice that $V_m | V_{km}$ and $(U_m, V_m) | 2$ for any $m \geq 0$ and

$$2V_{km} = (\alpha^d - \beta^d)(\alpha^{\ell n} - \beta^{\ell n}) + V_d V_{\ell n} \tag{13}$$

and $V_n | V_{\ell n}$ if $\ell$ is odd, $V_n | U_{\ell n}$ if $\ell$ is even. Thus,

$$(V_m, V_n) | ((\alpha^d - \beta^d)(\alpha^{\ell n} - \beta^{\ell n}),\ V_d V_{\ell n}) | 8V_d. \tag{14}$$

If $k$ is even, then $\ell n$ is an odd multiple of $d$, and we see that

$$2(\alpha^{km} - \beta^{km}) / (\alpha^d - \beta^d) = V_d(\alpha^{\ell n} - \beta^{\ell n}) / (\alpha^d - \beta^d) + V_{\ell n}, \tag{15}$$

$V_m | 2(\alpha^{km} - \beta^{km}) / (\alpha^d - \beta^d)$, and $V_n | V_{\ell n}$, so

$$(V_m, V_n) | 2V_d. \tag{16}$$

Furthermore, for any prime divisor $p$ of $2V_d$ from Remark 1, applying Theorem 1 to $V_m$ and $V_n$ we obtain the desired results.

***Remark 2:*** Lehmer proved the following theorem.

***Theorem A (Lehmer [1], Theorem 1.6):*** If $2\alpha$ is a positive integer such that $q^\alpha$ is the highest power of a prime $q$ dividing $U_m$, and if $k$ is any integer not divisible by $q$, then for any integer $\lambda$, $U_{kmq^\lambda}$ is divisible by $q^{\alpha+\lambda}$, and if $q^\alpha \neq 2$, this is the highest power of $q$ dividing $U_{kmq^\lambda}$.

Comparing Theorem A with Theorem 1 of this paper, we can easily find out that: If $q^\alpha = 3$, $m = 2$, $3 \| R$, and $9 | 3\Delta + R$, and we put $\lambda = 1$ in Theorem A, then the last conclusion of Theorem A is incorrect. This is indispensable in its applications to exponential Diophantine equations, as will be shown in a future paper.

***Example:*** Let $R = 2$ and $\Delta = -1$, then we have

$$V_0 = 2,\ V_1 = \sqrt{2},\ V_2 = 4,\ V_3 = 5\sqrt{2},\ V_4 = 14,\ V_5 = 19\sqrt{2}, \ldots,$$

which means that $\gcd(V_4, V_5) = \sqrt{2}$.

## ACKNOWLEDGMENT

The author is very grateful to the anonymous referee from valuable suggestions.

## REFERENCES

1. D. H. Lehmer. "An Extended Theory of Lucas Functions." *Ann. Math.* **31** (1930):419-48.
2. W. L. McDaniel. "The g.c.d. in Lucas Sequences and Lehmer Number Sequences." *The Fibonacci Quarterly* **29.1** (1991):24-29.
3. P. Ribenboim. *The Book of Prime Number Records*. New York: Springer-Verlag, 1989.

AMS Classification Numbers: 11B37, 11A07

# AN INTERESTING PROPERTY OF A RECURRENCE RELATED TO THE FIBONACCI SEQUENCE

**M. N. Deshpande**
Institute of Science, Nagpur-440001, India
E-mail: dpratap@nagpur.dot.net

**Andrej Dujella**
Dept. of Mathematics, University of Zagreb, Bijenicka cesta 30, 10000 Zagreb, Croatia
E-mail: duje@math.hr
*(Submitted February 2000-Final Revision May 2000)*

## 1. INTRODUCTION

The sequence of Fibonacci numbers with even subscripts $(F_{2n})$ has one remarkable property. If we choose three successive elements of this sequence, then the product of any two of them increased by 1 is a perfect square. Indeed,

$$F_{2n}\cdot F_{2n+2}+1=F_{2n+1}^2,\quad F_{2n}\cdot F_{2n+4}+1=F_{2n+2}^2.$$

This property was studied and generalized by several authors (see references). Let us just mention that Hoggatt and Bergum [8] proved that the number $d=4F_{2n+1}F_{2n+2}F_{2n+3}$ has the property that $F_{2n}\cdot d+1$, $F_{2n+2}\cdot d+1$, and $F_{2n+4}\cdot d+1$ are perfect squares, and Dujella [7] proved that the positive integer $d$ with the above property is unique.

The purpose of this paper is to characterize linear binary recursive sequences which possess the similar property as the above property of Fibonacci numbers.

We will consider binary recursive sequences of the form

$$G_{n+1}=AG_n-G_{n-1}, \tag{1}$$

where $A$, $G_0$, and $G_1$ are integers. We call the sequence $(G_n)$ *nondegenerated* if $|G_0|+|G_1|>0$ and the quotient of the roots $\alpha,\beta\in\mathbf{C}$ of the characteristic equation of $G_n$,

$$x^2-Ax+1=0,$$

is not a root of unity. Let $D=A^2-4$, $C=G_1^2-AG_0G_1+G_0^2$. Then nondegeneracy implies that $|A|\geq 3$ and $C\neq 0$. Solving recurrence (1), we obtain

$$G_n=\frac{a\alpha^n-b\beta^n}{\alpha-\beta},$$

where $a=G_1-G_0\beta$, $b=G_1-G_0\alpha$.

***Definition 1:*** Let $k$ be an integer. A sequence $(G_n)$ is said to have the property $P(k)$ if both $G_nG_{n+1}+k$ and $G_nG_{n+2}+k$ are perfect squares for all $n\geq 0$.

With this notation, we may say that the sequence $(F_{2n})$ has the property $P(1)$.

Our main result is the following theorem.

***Theorem 1:*** Let $(G_n)$ be a nondegenerated binary recursive sequence given by (1). If $G_n$ has the property $P(k)$ for some $k\in\mathbf{Z}$, then $A=3$ and $k=G_0^2-3G_0G_1+G_1^2$.

***Remark 1:*** The sequences from Theorem 1 have the form

$$G_n = G_1F_{2n} - G_0F_{2n-2},$$

and for $G_0 = 0$ and $G_1 = 1$ we obtain exactly the sequence $(F_{2n})$. Note that the converse of Theorem 1 is also valid. This follows from the formula $(F_{2n})$ proved below, and the general fact that if $ab+k=r^2$ then $a(a+b-2r)+k=(a-r)^2$.

## 2. PROOF OF THEOREM 1

Assume that $k$ is an integer such that the sequence $(G_n)$ has the property $P(k)$. This implies that $G_nG_{n+2}+k$ is a perfect square for all $n \geq 0$. On the other hand,

$$\begin{aligned} G_nG_{n+2} &= \frac{a^2\alpha^{2n+2}+b^2\beta^{2n+2}-ab(\alpha\beta)^n(\alpha^2+\beta^2)}{(\alpha-\beta)^2} \\ &= \left(\frac{a\alpha^{n+1}-b\beta^{n+1}}{\alpha-\beta}\right)^2 - \frac{ab(\alpha\beta)^n(\alpha-\beta)^2}{(\alpha-\beta)^2} \\ &= G_{n+1}^2 - ab = G_{n+1}^2 - C. \end{aligned}$$

Hence, $G_{n+1}^2 - C + k$ is a perfect square for all $n \geq 0$. This implies that $k = C$.

Our problem is now reduced to find sequences such that $G_nG_{n+1}+C$ is a perfect square for all $n \geq 0$.

We have $G_{n+1}^2 - AG_nG_{n+1} + G_n^2 = C$ (see [9]). Denote $G_nG_{n+1}+C = G_n^2-(A-1)G_nG_{n+1}+G_{n+1}^2$ by $H_n$. It can be verified easily that the sequence $(H_n)$ satisfies the recurrence relation

$$H_{n+1} = (A^2-2)H_n - H_{n-1} - C(A^2-A-4).$$

Finally, put $S_n = (A^2-4)H_n - C(A^2-A-4)$. Then the sequence $(S_n)$ satisfies the homogeneous recurrence relation

$$S_{n+1} = (A^2-2)S_n - S_{n-1}.$$

Denote the polynomial $(A^2-4)x^2 - C(A^2-A-4)$ by $R(x)$. Then our condition implies that, for every $n \geq 0$, there exist $x \in \mathbf{Z}$ such that

$$S_n = R(x). \tag{2}$$

Therefore, equation (2) has infinitely many solutions.

Let $D_1 = (A^2-2)^2 - 4 = A^2(A^2-4)$ and $C_1 = S_1^2 - S_0S_2 = -(A^2-4)A^2C^2$ be the discriminant and the characteristic of the sequence $(S_n)$, respectively. Assume also that

$$S_n = \frac{a_1\alpha^{2n} - b_1\beta^{2n}}{\alpha^2-\beta^2} \quad \text{for some } a_1 \text{ and } b_1,$$

and put

$$T_n = a_1\alpha^{2n} + b_1\beta^{2n} \quad \text{for all } n \geq 0.$$

Then, since

$$T_n^2 = D_1S_n^2 + 4C_1 \quad \text{for all } n \geq 0,$$

and since the equation $S_n = R(x)$ has infinitely many integer solutions $(n, x)$, it follows that the equation

$$y^2 = D_1 R(x)^2 + 4C_1$$

has infinitely many integer solutions $(x, y)$. By a well-known theorem of Siegel [20], we get that the polynomial $F(X) = D_1 R(X)^2 + 4C_1$ has at most two simple roots. Since $F$ is of degree 4, it follows that $F$ must have a double root. Notice that

$$F'(X) = 2D_1 R(X) R'(X) = 4(A^2 - 4) D_1 R(X) X.$$

Certainly, $F$ and $R$ cannot have a common root because this would imply that $C_1 = 0$, which is impossible since $(G_n)$ is nondegenerated. Hence, $F(0) = 0$, which is equivalent to

$$A^2(A^2-4)[C(A^2-A-4)]^2 - 4A^2(A^2-4)C^2 = 0. \tag{3}$$

Formula (3) implies that $A^2 - A - 4 = \pm 2$.

$$A^2 - A - 4 = \pm 2.$$

If $A^2 - A - 4 = 2$, then $A = 3$ or $A = -2$, and if $A^2 - A - 4 = -2$, then $A = 2$ or $A = -1$. Since we assumed that the sequence $(G_n)$ is nondegenerated, i.e., $|A| \geq 3$, we conclude that $A = 3$. □

***Remark 2:*** In degenerate cases with $A = 0, \pm 1, \pm 2$, the sequence $(G_n)$ also may have property $P(k)$ for some $k \in \mathbb{Z}$. For example, for $A = 2$, the sequence $G_n = a$ has property $P(b^2 - a^2)$; for $A = 0$, the sequence $G_{2n} = 0$, $G_{4n+1} = 2ab$, $G_{4n+3} = -2ab$ has property $P((a^2 + b^2)^2)$; for $A = -1$, the sequence $G_{3n} = a$, $G_{3n+1} = b$, $G_{3n+2} = -a - b$ has property $P(a^2 + ab + b^2)$. Here, $a$ and $b$ are arbitrary integers.

## ACKNOWLEDGMENTS

In the first version of this paper, a theorem of Nemes and Pethö (see Theorem 3 in [17]) was used in the last part of the proof of Theorem 1. The authors would like to thank the anonymous referee for a detailed suggestion on how to avoid using the result of Nemes and Pethö by direct application of the theorem of Siegel.

## REFERENCES

1. J. Arkin, D. C. Arney, F. R. Giordano, R. A. Kolb, & G. E. Bergum. "An Extension of an Old Classical Diophantine Problem." *Applications of Fibonacci Numbers* **5**:45-48. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1993.
2. G. Berzsenyi. "Problems, Puzzles, Paradoxes." *Consortium* **25** (1988):5.
3. M. N. Deshpande & G. E. Bergum. "Interesting Arrays Associated with Fibonacci Sequences." *Applications of Fibonacci Numbers* **6**:85-92. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.
4. A. Dujella. "Generalization of a Problem of Diophantus." *Acta Arith.* **65** (1993):15-27.
5. A. Dujella. "Generalized Fibonacci Numbers and the Problem of Diophantus." *The Fibonacci Quarterly* **34.2** (1996):164-75.
6. A. Dujella. "A Problem of Diophantus and Pell Numbers." *Applications of Fibonacci Numbers* **7**:61-68. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.
7. A. Dujella. "A Proof of the Hoggatt-Bergum Conjecture." *Proc. Amer. Math. Soc.* **127** (1999):1999-2005.

8. V. E. Hoggatt, Jr., & G. E. Bergum. "A Problem of Fermat and the Fibonacci Sequence." *The Fibonacci Quarterly* **15.4** (1977):323-30.
9. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." *Duke Math. J.* **32** (1965):437-46.
10. A. F. Horadam. "Generalization of a Result of Morgado." *Portugaliae Math.* **44** (1987): 131-36.
11. B. W. Jones. "A Second Variation on a Problem of Diophantus and Davenport." *The Fibonacci Quarterly* **16.2** (1978):155-65.
12. C. Long & G. E. Bergum. "On a Problem of Diophantus." *Applications of Fibonacci Numbers* **2:**183-91. Ed. A. N. Philippou, A. F. Horadam, & G. E. Bergum. Dordrecht: Kluwer, 1988.
13. S. P. Mohanty & A. M. S. Ramasamy. "On $P_{r,k}$ Sequences." *The Fibonacci Quarterly* **23** (1985):36-44.
14. J. Morgado. "Generalization of a Result of Hoggatt and Bergum on Fibonacci Numbers." *Portugaliae Math.* **42** (1983-1984):441-45.
15. J. Morgado. "Note on a Shannon Theorem Concerning the Fibonacci Numbers and Diophantine Quadruples." *Portugaliae Math.* **48** (1991):429-39.
16. J. Morgado. "Note on the Chebyshev Polynomials and Applications to the Fibonacci Numbers." *Portugaliae Math.* **52** (1995):363-78.
17. I. Nemes & A. Pethö. "Polynomial Values in Linear Recurrences II." *J. Number Theory* **24** (1986):47-53.
18. A. M. S. Ramasamy. "A Remarkable Sequence." *Banyan Math. J.* **2** (1995):69-76.
19. A. G. Shannon. "Fibonacci Numbers and Diophantine Quadruples: Generalizations of Results of Morgado and Horadam." *Portugaliae Math.* **45** (1988):165-69.
20. C. L. Siegel. "The Integer Solutions of the Equation $y^2 = ax^n + bx^{n-1} + \cdots + k$." *J. London Math. Soc.* **1** (1926):66-68.
21. Gh. Udrea. "A Note on the Sequence $(W_n)_{n\geq 0}$ of A. F. Horadam." *Portugaliae Math.* **53** (1996):143-55.


AMS Classification Numbers: 11B39, 11B37, 11D61

❖❖❖

# THE BRAHMAGUPTA POLYNOMIALS IN TWO COMPLEX VARIABLES AND THEIR CONJUGATES

**R. Rangarajan**
Department of Mathematics, Mysore University, Manasagangotri, 570 006, India
E-mail: rajra63@yahoo.com

**H. S. Sudheer**
Department of Mathematics, Kasturbha College, Shimoga, Karnataka, India

(*Submitted February 2000-Final Revision August 2000*)

## 1. INTRODUCTION

The Brahmagupta matrix and polynomials in two real variables were first introduced by Suryanarayan [7]. Later they were extended to two complex variables [8]. There is yet another way to extend naturally from the real variables case to the complex variables case. This is done by using two complex variables with their conjugates. In this paper we will explore this way of generalizing the matrix and the polynomials. This method yields quite different results than the ones developed in [8].

We define the Brahmagupta matrix, see (1) below, involving two complex variables as well as their conjugates and show that it generates a class of homogeneous polynomials. The two complex variables $z$ and $w$ lie in two distinct complex planes. This space is denoted $\mathbb{C}\times\mathbb{C}$ or $\mathbb{C}^2$. A typical member of this space has the form $\zeta=(z,w)$. Following [8], the points in $\mathbb{C}^2$ can be identified naturally with the points of $\mathbb{R}^4$ by the scheme:

$$(z,w)\in\mathbb{C}^2\leftrightarrow(x+iy,u+iv)\leftrightarrow(x,y,u,v)\in\mathbb{R}^4.$$

The polynomials generated by the matrix contain some of the well-known real polynomials like Chebychev polynomials of the first and second kind and Morgan-Voyce polynomials, among others. Thus, the paper provides a unified approach to the study of Brahmagupta polynomials.

In this paper we study the Brahmagupta matrix and the Brahmagupta polynomials in two complex variables and their conjugates. This study is similar to those in [7] and [8] and provides a natural way to extend them from the real case to the complex case. The emerging polynomials have a unique feature, namely, their real and imaginary parts form only two polynomials instead of four, involving essentially two variables. However, they have to be studied in two different cases depending on the nature of the variables: (i) both real; (ii) one real and the other purely imaginary. It is interesting to note that in the former case the Brahmagupta matrix and Brahmagupta polynomials are particular cases of those given in [7]; in the latter case, they are special cases of those given in [8]. In fact, Section 2 is clearly different from [7] and [8]. Section 5 is intended to show that the extended class of polynomials contain many of the well-known polynomials.

## 2. BRAHMAGUPTA MATRIX WITH COMPLEX ENTRIES

Let $z=x+iy$ and $w=u+iv$ be two complex variables and let $\bar{z}=x-iy$ and $\bar{w}=u-iv$ be their conjugates. Let $t\neq 0$ be a fixed real number. Consider the matrix

$$B_J=B_J(z,w)=\begin{bmatrix} z & w \\ t\bar{w} & \bar{z}\end{bmatrix}=B(x,u)+JB(y,v), \tag{1}$$

where

$$B(\xi,\eta)=\begin{bmatrix}\xi & \eta\\ t\eta & \xi\end{bmatrix} \text{ and } J=\begin{bmatrix} i & 0\\ 0 & -i\end{bmatrix}.$$

Let $\beta=\det(B_J)=|z|^2-t|w|^2$. It is clear that, if $y=0$ and $v=0$, then (1) reduces to the real case [7]. Let $\mathbf{B_J}$ denote the set of all matrices of the form $B_J$. Define $\overline{B}_J=B_J(\overline{z},\overline{w})$. $\mathbf{B_J}$ shows that, if $B_k=B_J(z_k,w_k)$, then $\mathbf{B_J}$ satisfies the following properties:

$$B_1B_2\neq B_2B_1,\ \overline{B}_1\overline{B}_2=\overline{B}_1\overline{B}_2,\ B_J\overline{B}_J\neq\overline{B}_JB_J.$$

Thus, if the entries of $B_J$ are real, then $\mathbf{B_J}$ forms a commutative subgroup of $GL(2,R)$. But in the present case, $\mathbf{B_J}$ is a *noncommutative* subgroup of $GL(2,C)$.

Let $\beta=\det(B_J)\neq 0$. Set $\alpha^2=x^2-\beta$. Notice that $\alpha$ is real if $x^2-\beta>0$ and $\alpha$ is imaginary if $x^2-\beta<0$. The eigenvalues of $B_J$ are $\lambda_\pm=x\pm\alpha$, with corresponding eigenvectors $E_\pm=[\pm w,\ \alpha\mp iy]^T$, where $T$ denotes the transpose. Using the eigenrelations, $B_J$ can be diagonalized in the form

$$\begin{bmatrix} z & w\\ t\overline{w} & \overline{z}\end{bmatrix}=\frac{1}{2w\alpha}\begin{bmatrix} w & -w\\ \alpha-iy & \alpha+iy\end{bmatrix}\begin{bmatrix} x+\alpha & 0\\ 0 & x-\alpha\end{bmatrix}\begin{bmatrix}\alpha+iy & w\\ -\alpha+iy & w\end{bmatrix}. \tag{2}$$

Define

$$\begin{bmatrix} z & w\\ t\overline{w} & \overline{z}\end{bmatrix}^n=\begin{bmatrix} z_n & w_n\\ t\overline{w}_n & \overline{z}_n\end{bmatrix}.$$

Then, using the above eigenrelations, we find that

$$\begin{bmatrix} z & w\\ t\overline{w} & \overline{z}\end{bmatrix}^n=\frac{1}{2w\alpha}\begin{bmatrix} w & -w\\ \alpha-iy & \alpha+iy\end{bmatrix}\begin{bmatrix} (x+\alpha)^n & 0\\ 0 & (x-\alpha)^n\end{bmatrix}\begin{bmatrix}\alpha+iy & w\\ -\alpha+iy & w\end{bmatrix}.$$

From the above result, we derive the following Binet forms for $z_n$ and $w_n$:

$$z_n=\frac{1}{2\alpha}[(\alpha+iy)(x+\alpha)^n+(\alpha-iy)(x-\alpha)^n], \tag{3}$$

$$w_n=\frac{w}{2\alpha}[(x+\alpha)^n-(x-\alpha)^n]. \tag{4}$$

Let us consider the two cases: (a) $\alpha$ is real; (b) $\alpha$ is imaginary.

**Case (a).** For $\alpha$ real, we can separate the real and imaginary parts of $z_n=x_n+iy_n$ and $w_n=u_n+iv_n$ and obtain

$$\begin{aligned}
&(i) && x_n=\frac{1}{2}[(x+\alpha)^n+(x-\alpha)^n],\\
&(ii) && y_n=\frac{y}{2\alpha}[(x+\alpha)^n-(x-\alpha)^n],\\
&(iii) && u_n=\frac{u}{2\alpha}[(x+\alpha)^n-(x-\alpha)^n],\\
&(iv) && v_n=\frac{v}{2\alpha}[(x+\alpha)^n-(x-\alpha)^n].
\end{aligned} \tag{5}$$

Set

$$\frac{\alpha_n}{\alpha}=\frac{y_n}{y}=\frac{u_n}{u}=\frac{v_n}{v}. \tag{6}$$

From the above results, we see that instead of the four forms $x_n$, $y_n$, $u_n$, and $v_n$, there are essentially two forms to consider, namely,

$$x_n = \frac{1}{2}[(x+\alpha)^n + (x-\alpha)^n] \quad \text{and} \quad \alpha_n = \frac{1}{2}[(x+\alpha)^n - (x-\alpha)^n]. \tag{7}$$

We can generate $x_n$ and $\alpha_n$ by the matrix

$$A = A(x, \alpha) = \begin{bmatrix} x & \alpha \\ \alpha & x \end{bmatrix}.$$

**Case (b).** For $\alpha$ imaginary, let us write $\alpha = i\hat{\alpha}$. Following a similar procedure as for the real case, we find that

$$\frac{\hat{\alpha}_n}{\alpha} = \frac{\hat{y}_n}{y} = \frac{\hat{u}_n}{u} = \frac{\hat{v}_n}{v}, \tag{8}$$

where $\hat{y}_n$ is obtained by replacing $\alpha$ by $i\alpha$ in (5, ii) and similarly we define $\hat{x}_n$, $\hat{u}_n$, $\hat{v}_n$, and $\hat{\alpha}_n$. In relation (7), replacing $\alpha$ by $i\hat{\alpha}$, we find that

$$\hat{x}_n = \frac{1}{2}[(x+i\hat{\alpha})^n + (x-i\hat{\alpha})^n] \quad \text{and} \quad i\hat{\alpha}_n = \frac{1}{2}[(x+i\hat{\alpha})^n - (x-i\hat{\alpha})^n]. \tag{9}$$

From (7) and (9), we see that $x_n \pm \alpha_n = (x \pm \alpha)^n$ and $\hat{x}_n \pm i\hat{\alpha}_n = (x \pm i\hat{\alpha})^n$. Similarly, we can generate $\hat{x}_n$ and $\hat{\alpha}_n$ by the matrix

$$\hat{A} = \hat{A}(x, i\hat{\alpha}) = \begin{bmatrix} x & i\hat{\alpha} \\ i\hat{\alpha} & x \end{bmatrix}.$$

## 3. PROPERTIES OF $A$ AND $\hat{A}$

Notice that the determinant of $A$ as well as that of $\hat{A}$ is $x^2 - \alpha^2 \neq 0$. Since

$$A(x_1, \alpha_1)A(x_2, \alpha_2) = A(x_2, \alpha_2)A(x_1, \alpha_1),$$

the set of matrices of the form $A$ commute. Set

$$A_n = A^n = \begin{bmatrix} x & \alpha \\ \alpha & x \end{bmatrix}^n = \begin{bmatrix} x_n & \alpha_n \\ \alpha_n & x_n \end{bmatrix}.$$

The Binet forms of $A$ are given by (7). $x_n$ and $\alpha_n$ satisfy the following recurrence relations:

$$x_{n+1} = xx_n + \alpha\alpha_n; \quad \alpha_{n+1} = x\alpha_n + \alpha x_n. \tag{10}$$

From the recurrence relation (10), we derive the *three-term recurrence relations* satisfied by $x_n$ and $\alpha_n$:

$$x_{n+1} = 2xx_n - (x^2 - \alpha^2)x_{n-1}; \quad \alpha_{n+1} = 2x\alpha_n - (x^2 - \alpha^2)\alpha_{n-1}.$$

It is clear that, if $\alpha$ is imaginary, the three-term recurrence relation becomes

$$\hat{x}_{n+1} = 2x\hat{x}_n - (x^2 + \hat{\alpha}^2)\hat{x}_{n-1}; \quad \hat{\alpha}_{n+1} = 2x\hat{\alpha}_n - (\hat{x}^2 + \hat{\alpha}^2)\hat{\alpha}_{n-1}.$$

If $\xi_n = x_n + \alpha_n$ and $\eta_n = x_n - \alpha_n$, then $\xi^n = \xi_n$ and $\eta^n = \eta_n$.

From the above results we see that, for real $\alpha$,

$$e^A = e^x \begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix}.$$

To show this, we write $2x_k = \xi_k + \eta_k$ and $2\alpha_k = \xi_k - \eta_k$. Since

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad \text{and} \quad A_k = \begin{bmatrix} x_k & \alpha_k \\ \alpha_k & x_k \end{bmatrix},$$

we express $x_k$ and $\alpha_k$ in terms of $\xi$ and $\eta$ and obtain the desired results. Notice that $\det e^A = e^{2x}$.

On the other hand, if $\alpha$ is imaginary, we replace $\alpha$ by $i\hat{\alpha}$ and follow a similar reasoning to show that

$$e^{\hat{A}} = e^x \begin{bmatrix} \cos\hat{\alpha} & i\sin\hat{\alpha} \\ i\sin\hat{\alpha} & \cos\hat{\alpha} \end{bmatrix}.$$

In this case also, $\det e^{\hat{A}} = e^{2x}$.

$x_n$ and $\alpha_n$ can be extended to the negative integers also by defining $x_{-n} = x_n\beta^{-n}$ and $\alpha_{-n} = -\alpha_n\beta^{-n}$. Then we will have

$$A^{-n} = \begin{bmatrix} x & \alpha \\ \alpha & x \end{bmatrix}^{-n} = \begin{bmatrix} x_{-n} & \alpha_{-n} \\ \alpha_{-n} & x_{-n} \end{bmatrix} = A_{-n};$$

here we have used the property

$$\left(\begin{bmatrix} x & \alpha \\ \alpha & x \end{bmatrix}^{-1}\right)^n = \left(\frac{1}{\beta}\begin{bmatrix} x & -\alpha \\ -\alpha & x \end{bmatrix}\right)^n = \frac{1}{\beta^n}\begin{bmatrix} x_n & -\alpha_n \\ -\alpha_n & x_n \end{bmatrix}.$$

Notice that $A^0 = I$, the identity matrix. A similar result holds for $\hat{A}^{-n}$.

## 4. RECURRENCE RELATIONS

From the Binet forms (7) and (9), the reader may verify the following.

**Recurrence Relations:**

$$\begin{aligned}
&(i) && x_{m+n} = x_m x_n \pm \alpha_m\alpha_n, \\
&(ii) && \alpha_{m+n} = x_m\alpha_m + \alpha_m x_n, \\
&(iii) && \beta^{2n}x_{m-n} = x_m x_n \mp \alpha_m\alpha_n, \\
&(iv) && \beta^{2n}\alpha_{m-n} = x_n\alpha_m \mp x_m\alpha_n, \\
&(v) && x_{m+n} + \beta^{2n}x_{m-n} = 2x_m x_n, \\
&(vi) && \alpha_{m+n} + \beta^{2n}\alpha_{m-n} = 2x_n\alpha_m, \\
&(vii) && x_{m+n} - \beta^{2n}x_{m-n} = \pm 2\alpha_m\alpha_n, \\
&(viii) && \alpha_{m+n} - \beta^{2n}\alpha_{m-n} = 2x_m\alpha_n,
\end{aligned} \tag{11}$$

where the top sign is chosen if $\alpha$ is real; if $\alpha$ is imaginary, the bottom sign is chosen. Notice that (v) and (vi) are the generalizations of the three-term recurrence relations.

Let $\sum_{k=1}^{n} = \sum$. Again using the Binet forms, the reader may verify the following

$$\begin{aligned}
&(i) && \sum x_k = \frac{\beta^2 x_n - x_{n+1} + x - \beta^2}{\beta^2 - 2x + 1},\\
&(ii) && \sum \alpha_k = \frac{\beta^2 \alpha_n - \alpha_{n+1} + \alpha}{\beta^2 - 2x + 1},\\
&(iii) && \sum x_k^2 = \frac{\beta^2 x_{2n} - x_{2n+2} + x_2 - \beta^2}{2(\beta^2 - 2x_2 + 1)} + \frac{\beta^2(\beta^{2n} - 2)}{2(\beta^2 - 1)},\\
&(iv) && \sum \alpha_k^2 = \frac{\beta^2 \alpha_{2n} - \alpha_{2n+2} + \alpha_2 - \beta^2}{2(\beta^2 - 2\alpha_2 + 1)} + \frac{\beta^2(\beta^{2n} - 1)^2}{2(\beta^2 - 1)},\\
&(v) && 2\sum x_k x_{n+1-k} = nx_{n+1} + \frac{\beta^2 \alpha_n}{\alpha},\\
&(vi) && 2\sum \alpha_k \alpha_{n+1-k} = nx_{n+1} - \frac{\beta^2 \alpha_n}{\alpha},\\
&(vii) && 2\sum x_k \alpha_{n-k+1} = 2\sum \alpha_k x_{n-k+1} = n\alpha_{n+1}.
\end{aligned} \tag{12}$$

(12, v, vi, vii) are convolution formulas. For $\alpha$ imaginary, a set of similar formulas holds.

From the Binet forms for (7) we see that, for $\alpha > 0$, $x_n$ and $\alpha_n$ satisfy

**The Limiting Properties:**

$$\lim_{n\to\infty} \frac{x_n}{\alpha_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n}{x_{n-1}} = \lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n-1}} = x + \alpha.$$

**The Divisors of $x_{2n}$ and $\alpha_{2n}$:**

From (11 i) we see that, if $\alpha$ is imaginary, then $x + i\alpha$ and $x - i\alpha$ are factors of $x_{2n}$ for real $\alpha$.

From (11 ii) we see that $x_n$ and $\alpha_n$ are factors of $\alpha_{2n}$. The last statement can be generalized: If $r$ divides $s$, then $x_r$ and $\alpha_r$ are factors of $\alpha_s$.

## 5. BRAHMAGUPTA POLYNOMIALS

With the help of the binomial expansions for $x_n \pm \alpha_n = (x \pm \alpha)^n$, we find that

$$x_n = x^n + \binom{n}{2} x^{n-2}\alpha^2 + \binom{n}{4} x^{n-4}\alpha^4 + \cdots,$$
$$\alpha_n = nx^{n-1}\alpha + \binom{n}{3} x^{n-3}\alpha^3 + \binom{n}{5} x^{n-5}\alpha^5 + \cdots.$$

Similarly, expanding $\hat{x}_n \pm i\hat{\alpha}_n = (x \pm i\hat{\alpha})^n$, we obtain

$$\hat{x}_n = x^n - \binom{n}{2} x^{n-2}\hat{\alpha}^2 + \binom{n}{4} x^{n-4}\hat{\alpha}^4 - + \cdots,$$
$$\hat{\alpha}_n = nx^{n-1}\hat{\alpha} - \binom{n}{3} x^{n-3}\hat{\alpha}^3 + \binom{n}{5} x^{n-5}\hat{\alpha}^5 - + \cdots.$$

For $\alpha$ real, the first few polynomials of $x_n$ and $\alpha_n$ are:

$$x_0 = 1,\ x_1 = x,\ x_2 = x^2 + \alpha^2,\ x_3 = x^3 + 3x\alpha^2,\ x_4 = x^4 + 6x^2\alpha^2 + y^4, \ldots;$$
$$\alpha_0 = 0,\ \alpha_1 = \alpha, \alpha_2 = 2x\alpha,\ \alpha_3 = 3x^2\alpha + \alpha^3,\ \alpha_4 = 4x\alpha^3 + 4x^3\alpha, \ldots.$$

Similarly, for $\alpha$ imaginary, we can write the first few polynomials of $\hat{x}_n$ and $\hat{\alpha}_n$.

**Special Cases of the Polynomials:**

*a. Brahmagupta sequences*

If $x=2$ and $\alpha=\sqrt{3}$, the Binet forms reduce to

$$2x_n=(2+\sqrt{3})^n+(2-\sqrt{3})^n;\ 2\sqrt{3}u_n=(2+\sqrt{3})^n-(2-\sqrt{3})^n.$$

These sequences appear in obtaining Heron triangles with consecutive integer sides [1]; $2x_n$ denotes the middle side and $2u_n$ denotes the height of the triangle.

*b. Lucas and Fibonacci sequences $L_n$ and $F_n$*

In $B_J$ in (1), set $x=y=u=\frac{1}{2}$, $v=0$, and $t=6$. Then we get $\beta^2=\alpha^2-x^2=1$, $\alpha=\frac{\sqrt{5}}{2}$, and

$$2x_n=\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n=L_n,\ 2\sqrt{5}u_n=\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n=F_n,$$

and, in this case, we have

$$2B_J(L_n+iF_n,F_n)=\begin{bmatrix} L_n+iF_n & F_n \\ 6F_n & L_n-iF_n \end{bmatrix}.$$

*c. Pell sequences*

In $B_J$, if we set $x=y=u=1$, $v=0$, and $t=3$, we get $\beta=-1$, $\alpha=\sqrt{2}$, and $x_n$ and $u_n$ reduce to Pell sequences given by:

$$2x_n=(1+\sqrt{2})^n+(1-\sqrt{2})^n,\ 2\sqrt{2}u_n=(1+\sqrt{2})^n-(1-\sqrt{2})^n.$$

Also, $B_n$ becomes

$$B_J(x_n+iy_n,y_n)=\begin{bmatrix} x_n+iy_n & y_n \\ 3y_n & x_n-iy_n \end{bmatrix}.$$

*d. Brahmagupta polynomials*

If $v=0=y$, then $x_n$ and $y_n$ reduce to the Brahmagupta polynomials in the real case:

$$x_n=\frac{1}{2}[(x+y\sqrt{t})^n+(x-y\sqrt{t})^n],\ y_n=\frac{1}{2\sqrt{t}}[(x+y\sqrt{t})^n-(x-y\sqrt{t})^n].$$

The properties of these polynomials have been studied in [7].

*e. The Chebyshev polynomials*

Set $\beta=1$, $\alpha=\sqrt{x^2-1}$, and $u=1$, then

$$x_n=\frac{1}{2}[(x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n]=T_n(x),$$
$$u_n=\frac{u}{2\sqrt{x^2-1}}[(x+\sqrt{x^2-1})^n-(x-\sqrt{x^2-1})^n]=U_n(x).$$

The Chebyshev polynomials occur in many branches of mathematics like Interpolation Theory, Orthogonal Polynomials, Approximation Theory, Numerical Analysis, etc. [6].

*f. Polynomials similar to Chebyshev polynomials*

If we set $\beta=-1$ and $\alpha=\sqrt{x^2+1}$, we obtain polynomials similar to the Chebyshev polynomials:

$$x_n = \frac{1}{2}[(x+\sqrt{x^2+1})^n + (x-\sqrt{x^2-1})^n] = V_n(x),$$
$$\frac{\alpha_n}{\alpha} = \frac{1}{2\sqrt{x^2+1}}[(x+\sqrt{x^2+1})^n + (x-\sqrt{x^2-1})^n] = W_n(x).$$

*g. Morgan-Voyce polynomials*

If $x$ is replaced by $(x+2)/2$ and $\alpha$ by $\sqrt{x^2+4x}/2$ in the matrix $A$, then the det $A=1$. If, in addition, $u=1$, then

$$2x_n = \left(\frac{x+2+\sqrt{x^2+4x}}{2}\right)^n + \left(\frac{x+2-\sqrt{x^2+4x}}{2}\right)^n,$$
$$2\frac{u_n}{u}\sqrt{x^2+4x} = \left(\frac{x+2+\sqrt{x^2+4x}}{2}\right)^n - \left(\frac{x+2-\sqrt{x^2+4x}}{2}\right)^n = B_n,$$

where $B_n$ is the Morgan-Voyce polynomial [4], [9]. The three-term recurrence relation for these polynomials are $B_n = (2+x)B_{n-1} - B_{n-2}$. Morgan-Voyce polynomials are used in the analysis of ladder networks and electric line theory [4], [9].

*h. Catalan numbers*

If $x=1$ and $\alpha^2 = 1+4u$ in (5), we find that

$$2x_n = (1+\sqrt{1+4u})^n + (1-\sqrt{1+4u})^n,$$
$$2u_n = \frac{1}{\sqrt{1+4u}}[(1+\sqrt{1+4u})^n - (1-\sqrt{1+4u})^n].$$

Both $x_n$ and $u_n$ appear in the study of Catalan numbers [2].

Let $\alpha$ be real. Then we find, from (7),

$$\frac{\partial x_n}{\partial x} = \frac{\partial \alpha_n}{\partial \alpha} = n x_{n-1}, \quad \frac{\partial x_n}{\partial \alpha} = \frac{\partial \alpha_n}{\partial x} = n\alpha_{n-1}.$$

From the above relations, we infer that $x_n$ and $\alpha_n$ are the polynomial solutions of the wave equation:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \alpha^2}\right)X = 0.$$

On the other hand, let $\alpha$ be imaginary. Put $\alpha = i\hat{\alpha}$. Then we find, from (9),

$$\begin{aligned}\frac{\partial \hat{x}_n}{\partial x} &= \frac{\partial \hat{\alpha}_n}{\partial \hat{\alpha}} = n\hat{x}_{n-1},\\ \frac{\partial \hat{x}_n}{\partial \hat{\alpha}} &= \frac{-\partial \hat{\alpha}_n}{\partial x} = -n\hat{\alpha}_{n-1}.\end{aligned} \tag{13}$$

From these relations, we infer that $\hat{x}_n$ and $\hat{\alpha}_n$ are the polynomial solutions of the Laplace equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \hat{\alpha}^2}\right)X = 0.$$

## 6. GENERATING FUNCTIONS

We shall now show that the generating functions for $z_n$ and $w_n$ are:

$$(i)\ \sum_{n=0}^{\infty} z_n s^n = \frac{1-\bar{z}s}{1-2xs+\beta^2 s^2};\quad (ii)\ \sum_{n=0}^{\infty} w_n s^n = \frac{ws}{1-2xs+\beta^2 s^2}. \tag{14}$$

We shall assume that $s$ is real; then we can separate the real and imaginary parts on both sides to obtain the following generating functions for $x_n$ and $\alpha_n$:

$$(i)\ \sum_{n=0}^{\infty} x_n s^n = \frac{1-xs}{1-2xs+\beta s^2};\quad (ii)\ \sum_{n=0}^{\infty} \alpha_n s^n = \frac{\alpha s}{1-2xs+\beta s^2}. \tag{15}$$

To show (13), we use the standard result: For $\|B_J s\| < 1$, we have

$$\sum_{n=0}^{\infty} (B_J s)^n = (I - B_J s)^{-1}.$$

Now,

$$I - B_J s = \begin{bmatrix} 1-zs & -ws \\ -t\overline{w}s & 1-\bar{z}s \end{bmatrix},$$

$$\det(I - B_J s) = 1-(z+\bar{z})s + (|z|^2 - t|w|^2)s^2 = 1-2xs+\beta s^2,$$

and

$$(1-2xs+\beta s^2)\sum_{n=0}^{\infty}(B_J s)^n = \begin{bmatrix} 1-\bar{z}s & ws \\ t\overline{w}s & 1-zs \end{bmatrix}.$$

The claim (14) follows from the above result.

It is known that, if $F(s)$ and $L(s)$ are generating functions of $F_n$ and $L_n$, respectively, then $F(s) = e^{L(s)}$ [3]. This result can be generalized to the generating functions of $x_n$ and $\alpha_n$. Let

$$X(s) = \sum_{k=1}^{\infty} x_k s^k,\quad A(s) = \sum_{k=1}^{\infty} \frac{\alpha_k}{\alpha} s^{k-1},\quad \chi(s) = \sum_{k=1}^{\infty} x_k \frac{s^k}{k}.$$

Notice that $s\chi'(s) = X(s)$. Now, we state this result as the following theorem.

***Theorem:*** $e^{2\chi(s)} = A(s)$.

***Proof:*** Set $\xi = x+\alpha$ and $\eta = x-\alpha$. Then

$$\xi+\eta = 2x,\quad \xi\eta = x^2-\alpha^2 = \beta,\quad 2x_n = (\xi^n+\eta^n),\quad 2\alpha_n = (\xi^n - \eta^n).$$

Now consider

$$\begin{aligned}\chi(s) &= \sum_{k=1}^{\infty} x_k \frac{s^k}{k} \\ &= \frac{1}{2}\left[\xi s + \xi^2\frac{s^2}{2} + \cdots + \xi^n \frac{s^n}{n} + \cdots\right] + \frac{1}{2}\left[\eta s + \eta^2 \frac{s^2}{2} + \cdots + \eta^n \frac{s^n}{n} + \cdots\right] \\ &= -\frac{1}{2}[\ln(1-\xi s) + \ln(1-\eta s)] = -\frac{1}{2}\ln(1-2xs+\beta s^2),\end{aligned}$$

which implies $2\chi(s) = \ln A(s)$ or $e^{2\chi(s)} = A(s)$.

All the infinite series summation properties involving reciprocals of $x_n$ and $y_n$ developed in [7] can be extended to $x_n$ and $\alpha_n$ (or $\hat{\alpha}_n$). Since the arithmetic goes through without any changes, we do not list them here.

## ACKNOWLEDGMENT

The authors thank the anonymous referee for valuable suggestions that led to the improvement of the presentation.

## REFERENCES

1. R. A. Beauregard & E. R. Suryanarayan. "Brahmagupta Triangles." *College Math. J.* **29** (1998):13.
2. R. L. Graham, D. E. Knuth, & O. Patashnik. *Concrete Mathematics*, pp. 204, 344. New York: Addison-Wesley, 1988.
3. R. Honsberger. *Mathematical Gems*, p. 102. The MAA Publication, 1985.
4. A. F. Horadam. "New Aspects of Morgan-Voyce Polynomials." *Applications of Fibonacci Numbers* **7:**161. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.
5. J. M. Mahon & A. F. Horadam. "Infinite Series Summation Involving Reciprocals of Pell Polynomials. *Fibonacci Numbers and Their Applications*. Dordrecht: Reidel, 1986.
6. T. J. Rivlin. *Chebyshev Polynomials*. New York: Wiley, 1974.
7. E. R. Suryanarayan. "Brahmagupta Polynomials." *The Fibonacci Quarterly* **34.1** (1996):30.
8. E. R. Suryanarayan. "Brahmagupta Polynomials in Two Complex Variables." *The Fibonacci Quarterly* **36.1** (1998):34.
9. M. N. S. Swamy. "Further Properties of Morgan-Voyce Polynomials." *The Fibonacci Quarterly* **6.2** (1968):167.

AMS Classification Numbers: 01A32, 11B37, 11B39

# ON FAREY SERIES AND DEDEKIND SUMS

**Wenpeng Zhang and Yuan Yi**
Research Center for Basic Science, Xi'an Jiaotong University, Xi'an, Shaanxi, P.R. China
*(Submitted March 2000-Final Revision September 2001)*

## 1. INTRODUCTION

As usual, the Farey series $\mathcal{F}_n$ of order $n$ is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed $n$. Thus, $h/k$ belongs to $\mathcal{F}_n = \{\rho_0, \rho_1, \rho_2, \ldots, \rho_m\}$, where $m = \phi(1)+\phi(2)+\cdots+\phi(n)$, if $0 \le h \le k \le n$, $(h,k)=1$; the numbers 0 and 1 are included in the forms $\frac{0}{1}$ and $\frac{1}{1}$. For example, $\mathcal{F}_5$ is:

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

The many characteristic properties of $\mathcal{F}_n$ can be found in references [1] and [3]. In this paper, we shall study the distribution problems of Dedekind sums for Farey fractions, and obtain some interesting identities. For convenience, we first introduce the definition of the Dedekind sum $S(h,q)$. For a positive integer $q$ and an arbitrary integer $h$, we define

$$S(h,q) = \sum_{a=1}^{q} \left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x-[x]-\frac{1}{2} & \text{if } x \text{ is not an integer;} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

The various arithmetical properties of $S(h,k)$ were investigated by many authors. Perhaps the most famous property of $S(h,k)$ is the reciprocity formula (see [2], [4], and [6]):

$$S(h,q)+S(q,h) = \frac{h^2+q^2+1}{12hq} - \frac{1}{4} \tag{1}$$

for all $(h,q)=1$, $h>0$, $q>0$. Regarding Dedekind sums and uniform distribution, G. Myerson [5], Z. Zheng [10], and I. Vardi [7] have also obtained some meaningful results. But for any fraction $a_i/b_i$ belonging to the Farey series $\mathcal{F}_q$, the authors are not aware of the study of the properties of $S(a_i,b_i)$. The main purpose of this paper is to study the properties of $S(a_i,b_i)$ for $a_i/b_i$ belonging to the Farey series $\mathcal{F}_q$, and give an interesting identity. That is, we shall prove the following two main theorems.

***Theorem 1:*** Let $0 < a \le q$ be a positive integer with $(a,q)=1$. Then we have the identity

$$S(a,q) = \frac{1}{12}\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k}+\frac{b_k}{b_{k-1}}\right)+\frac{a}{12q}-\frac{n}{4},$$

where $n$ is the position of $\rho_n = a/q = a_n/b_n$ in the Farey series $\mathcal{F}_q$, $b_i$ $(0 \le i \le n)$ is the denominator of $\rho_i = \frac{a_i}{b_i}$ with $\frac{a_i}{b_i} \le \rho_n = \frac{a}{q}$ in the Farey series $\mathcal{F}_q$.

***Theorem 2:*** Let $p$ be a prime and let $a$ be a positive integer with $a < p$, then we have the identity

$$\sum_{\chi(-1)=-1} \chi(a)|L(1, \chi)|^2 = \frac{\pi^2(p-1)}{12p}\left[\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k}+\frac{b_k}{b_{k-1}}\right)+\frac{a}{p}-3n\right],$$

where $\chi$ is the Dirichlet character mod $p$ and $L(1, \chi)$ is the Dirichlet $L$-function corresponding character $\chi$.

For $a = 2$ and 3, from Theorem 2 and the properties of character, we immediately obtain the following two corollaries.

***Corollary 1:*** Let $p$ be a prime and $\chi$ be the Dirichlet odd character mod $p$. Then we have

$$\sum_{\chi(-1)=-1} \chi(2)|L(1, \chi)|^2 = \frac{\pi^2(p-1)^2(p-5)}{24p^2}.$$

***Corollary 2:*** Let $p$ be a prime and $\chi$ be the Dirichlet character modulo $p$. Then

$$\sum_{\chi(-1)=-1} \chi(3)|L(1, \chi)|^2 = \begin{cases} \frac{\pi^2}{36}\cdot\frac{(p-1)^2(p-10)}{p^2} & \text{if } p \equiv 1 \bmod 3; \\ \frac{\pi^2}{36}\cdot\frac{(p-1)(p-2)(p-5)}{p^2} & \text{if } p \equiv 2 \bmod 3. \end{cases}$$

It is clear that these two corollaries are an extension of Walum [8].

## 2. SOME LEMMAS

To complete the proof of Theorems 1 and 2, we need the following two lemmas.

***Lemma 1:*** If $h/k$ and $h'/k'$ are two successive terms in $\mathscr{F}_n$, then $kh' - hk' = 1$.

***Proof:*** See Theorem 5.5 of [1].

***Lemma 2:*** Let $k$ and $h$ be integers with $k \geq 3$ and $(h, k) = 1$. Then we have

$$S(h, k) = \frac{1}{\pi^2 k}\sum_{d|k}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h)|L(1, \chi)|^2$$

where $\phi(k)$ is Euler's function.

***Proof:*** See [9].

## 3. PROOF OF THE THEOREMS

In this section, we shall complete the proof of the theorems. First, we prove Theorem 1. We write the Farey fractions $\mathscr{F}_q$ as follows:

$$\frac{0}{1}, \frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}, \ldots, \frac{1}{1},$$

and suppose $\frac{a_n}{b_n} = \frac{a}{q}$.

For the successive terms $\frac{a_n}{b_n}$ and $\frac{a_{n-1}}{b_{n-1}}$, from Lemma 1 we know that

$$a_n b_{n-1} - b_n a_{n-1} = 1. \tag{2}$$

Using the properties of Dedekind sums and (2), we get

$$\begin{aligned} S(a_n, b_n) &= S(a_n b_{n-1}\overline{b_{n-1}}, b_n) \\ &= S(\overline{b_{n-1}}(1+a_{n-1}b_n), b_n) \\ &= S(\overline{b_{n-1}}, b_n) = S(b_{n-1}, b_n). \end{aligned} \tag{3}$$

Similarly, we also have

$$\begin{aligned} S(a_{n-1}, b_{n-1}) &= S(a_{n-1}b_n\overline{b_n}, b_{n-1}) \\ &= S((a_n b_{n-1} - 1)\,\overline{b_n},\, b_{n-1}) \\ &= S(-\overline{b_n}, b_{n-1}) = -S(b_n, b_{n-1}), \end{aligned} \tag{4}$$

where $\overline{b_n}$ denotes the solution $x$ of the congruence equation $xb_n \equiv 1 \pmod{b_{n-1}}$.

So, from (3), (4), and the reciprocity formula (1), we obtain

$$S(a_n, b_n) - S(a_{n-1}, b_{n-1}) = S(b_{n-1}, b_n) + S(b_n, b_{n-1}) = \frac{b_n^2 + b_{n-1}^2 + 1}{12 b_n b_{n-1}} - \frac{1}{4}. \tag{5}$$

Hence, by expression (5) and Lemma 1, we obtain

$$\begin{aligned} S(a_n, b_n) &= S(a_{n-1}, b_{n-1}) + \frac{1}{12}\left(\frac{b_{n-1}}{b_n} + \frac{b_n}{b_{n-1}}\right) + \frac{1}{12 b_n b_{n-1}} - \frac{1}{4} \\ &= S(a_{n-1}, b_{n-1}) + \frac{1}{12}\left(\frac{b_{n-1}}{b_n} + \frac{b_n}{b_{n-1}}\right) + \frac{1}{12}\left(\frac{a_n}{b_n} - \frac{a_{n-1}}{b_{n-1}}\right) - \frac{1}{4} \\ &\cdots \\ &= \frac{1}{12}\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k} + \frac{b_k}{b_{k-1}}\right) + \frac{a}{12q} - \frac{n}{4}. \end{aligned} \tag{6}$$

From (6) and the fact that $a_n / b_n = a / q$, we immediately have

$$S(a, q) = \frac{1}{12}\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k} + \frac{b_k}{b_{k-1}}\right) + \frac{a}{12q} - \frac{n}{4}.$$

This completes the proof of Theorem 1.

***Proof of Theorem 2:*** Using Lemma 2, we have

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(a)\,|L(1,\chi)|^2 = \frac{\pi^2(p-1)}{p} S(a, p). \tag{7}$$

Then from Theorem 1 and (7), we can easily obtain

$$\begin{aligned} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(a)\,|L(1,\chi)|^2 &= \frac{\pi^2(p-1)}{p}\left[\frac{1}{12}\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k} + \frac{b_k}{b_{k-1}}\right) + \frac{a}{12p} - \frac{n}{4}\right] \\ &= \frac{\pi^2(p-1)}{12p}\left[\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k} + \frac{b_k}{b_{k-1}}\right) + \frac{a}{p} - 3n\right]. \end{aligned}$$

This completes the proof of Theorem 2.

***Proof of the Corollaries:*** If $a=2$, then the position of $2/p$ in the Farey fractions $\mathcal{F}_p$ is $\frac{p+3}{2}$, so $n=\frac{p+3}{2}$. Thus, from Theorem 2, we have

$$\begin{aligned}\sum_{k=1}^{\frac{p+3}{2}}\left(\frac{b_{k-1}}{b_k}+\frac{b_k}{b_{k-1}}\right)&=\frac{1}{p}+\frac{p}{p-1}+\cdots+\frac{p-\frac{p-1}{2}+1}{p-\frac{p-1}{2}}+\frac{p-\frac{p-1}{2}}{p}\\&\quad+p+\frac{p-1}{p}+\frac{p-2}{p-1}+\cdots+\frac{p-\frac{p-1}{2}}{p-\frac{p-1}{2}+1}+\frac{p}{p-\frac{p-1}{2}}\\&=p+1+2\cdot\frac{p-3}{4}+\frac{2p-\frac{p-1}{2}+1}{p-\frac{p-1}{2}}+\frac{p-\frac{p-1}{2}}{p}.\end{aligned}\tag{8}$$

So, from (8) and Theorem 2, we have

$$\begin{aligned}\sum_{\chi(-1)=-1}\chi(2)|L(1,\chi)|^2&=\frac{\pi^2(p-1)}{12p}\left[\sum_{k=1}^{\frac{p+3}{2}}\left(\frac{b_{k-1}}{b_k}+\frac{b_k}{b_{k-1}}\right)+\frac{2}{p}-\frac{3(p+3)}{2}\right]\\&=\frac{\pi^2(p-1)}{12p}\left[p+1+2\cdot\frac{p-3}{4}+\frac{2p-\frac{p-1}{2}+1}{p-\frac{p-1}{2}}+\frac{p-\frac{p-1}{2}}{p}+\frac{2}{p}-\frac{3(p+3)}{2}\right]\\&=\frac{\pi^2(p-1)^2(p-5)}{24p^2}.\end{aligned}$$

This proves Corollary 1.

Using Theorem 2, or the reciprocity formula (1) and Lemma 2, we may immediately deduce

$$\sum_{\chi(-1)=-1}\chi(3)|L(1,\chi)|^2=\begin{cases}\frac{\pi^2}{36}\cdot\frac{(p-1)^2(p-10)}{p^2} & \text{if } p\equiv 1 \bmod 3;\\ \frac{\pi^2}{36}\cdot\frac{(p-1)(p-2)(p-5)}{p^2} & \text{if } p\equiv 2 \bmod 3.\end{cases}$$

This completes the proof of Corollary 2.

## ACKNOWLEDGMENTS

The authors express their gratitude to the anonymous referee for very helpful and detailed comments.

## REFERENCES

1. Tom M. Apostol. *Modular Functions and Dirichlet Series in Number Theory*. New York: Springer-Verlag, 1976.
2. L. Carlitz. "The Reciprocity Theorem for Dedekind Sums." *Pacific J. Math.* **3** (1953):523-27.
3. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford: Clarendon Press, 1960.
4. L. J. Mordell. "The Reciprocity Formula for Dedekind Sums." *Amer. J. Math.* **73** (1951): 593-98.
5. G. Myerson. "Dedekind Sums and Uniform Distribution." *J. Number Theory* **28** (1991): 1803-07.
6. H. Rademacher. "On the Transformation of $\log\eta(\tau)$." *J. Indian Math. Soc.* **19** (1955):25-30.

7. I. Vardi. "A Relation between Dedekind Sums and Kloosterman Sums." *Duke Math. J.* **55** (1987):189-97.
8. H. Walum. "An Exact Formula for an Average of $L$-Series." *Illinois J. Math.* **26** (1982):1-3.
9. Wenpeng Zhang. "On the Mean Values of Dedekind Sums." *Journal de theorie des nombres de Bordeaux* **8** (1996):429-42.
10. Z. Zheng. "Dedekind Sums and Uniform Distribution (mod 1)." *Acta Mathematica Sinica* **11** (1995):62-67.

AMS Classification Numbers: 11B37, 11B39, 11N37

❖❖❖

# WALKING INTO AN ABSOLUTE SUM


**Hans J. H. Tuenter**

Schulich School of Business, York University, Toronto, Ontario, Canada M3J 1P3

E-mail: htuenter@schulich.yorku.ca



*(Submitted March 2000-Final Revision September 2000)*


## 1. INTRODUCTION

Recently, it was asked by Paul Bruckman [1] to show that the sum

$$S_r(n)=\sum_{k=0}^{2n}\binom{2n}{k}|n-k|^r \tag{1}$$

evaluates to $n^2\binom{2n}{n}$ for $r=3$. In the published solution [16], it was also noted that $S_1(n)=n\binom{2n}{n}$, and, as a consequence, it was conjectured that $S_{2r+1}(n)$ equals the product of $\binom{2n}{n}$ and a monic polynomial of degree $r+1$.

We show this conjecture to be true, albeit with the modification of discarding the adjec-tival modifier "monic". In fact, we show that $S_{2r+1}(n)=P_r(n)n\binom{2n}{n}$ and $S_{2r}(n)=Q_r(n)2^{2n-r}$, where $P_r(n)$ and $Q_r(n)$ are both polynomials of degree $r$ with integer coefficients. We then investigate the relationship of these polynomials to the Dumont-Foata polynomials [6]. These are generalizations of the Gandhi polynomials, which find their origin in a representation of the Genocchi numbers, first conjectured by Gandhi [9]. Finally, we show that the sums $S_r(n)$ are essentially the moments of a random variate, measuring the absolute distance to the origin in a symmetric Bernoulli random walk, after $2n$ time steps.

## 2. DERIVATION

We note that the sum can be rewritten as

$$S_r(n)=2\sum_{k=0}^{n}\binom{2n}{n-k}k^r-\binom{2n}{n}\delta_{r0},$$

with $\delta_{r0}$ the Kronecker delta. Now consider, for $r\geq 1$,

$$n^2S_r(n)-S_{r+2}(n)=2\sum_{k=0}^{n-1}\binom{2n}{n-k}k^r(n^2-k^2)=4n(2n-1)\sum_{k=0}^{n-1}\binom{2n-2}{n-1-k}k^r,$$

leading directly to the recursion

$$S_{r+2}(n)=n^2S_r(n)-2n(2n-1)S_r(n-1). \tag{2}$$

For $r=0$, the derivation is slightly more elaborate because we need to keep track of the additional term, but leads to the same recursion so that (2) is valid for all nonnegative integers $r$. To start the recursion, we find the value $S_0(n)=2^{2n}$ by an application of the binomial theorem to (1). The value of $S_1(n)$ is easily obtained by breaking up the summand $k$ to create two sums:

$$S_1(n)=\sum_{k=0}^{n}\binom{2n}{n-k}[(n+k)-(n-k)]=2n\sum_{k=0}^{n}\binom{2n-1}{n-k}-2n\sum_{k=0}^{n-1}\binom{2n-1}{n-k-1},$$

and one sees that, after changing the range of summation of the second sum to start at $k=1$, all terms cancel out, with the exception of the summand $2n\binom{2n-1}{n}$. Rearranging terms gives the desired $S_1(n)=n\binom{2n}{n}$.

It is now clear that the structure of the sum depends upon the parity of $r$. Starting with the odd values, we simplify the recursion (2) by the substitution $S_{2r+1}(n)=P_r(n)n\binom{2n}{n}$ to give

$$P_{r+1}(n)=n^2[P_r(n)-P_r(n-1)]+nP_r(n-1), \tag{3}$$

with initial condition $P_0(n)=1$. An inductive argument now shows that $P_r(n)$ is a polynomial of degree $r$ with integer coefficients, and proves the modified conjecture. It is not difficult to show that $r!$ is the leading coefficient of $P_r(n)$, and, hence, that these polynomials are not monic. In fact, the only cases for which the leading coefficient is 1 are $r=0$ and $r=1$. The first few polynomials are now easily determined as:

$$\begin{aligned}
&P_0(n)=1,\\
&P_1(n)=n,\\
&P_2(n)=(2n-1)n,\\
&P_3(n)=(6n^2-8n+3)n,\\
&P_4(n)=(24n^3-60n^2+54n-17)n,\\
&P_5(n)=(120n^4-480n^3+762n^2-556n+155)n.
\end{aligned}$$

For the even sums, we substitute $S_{2r}(n)=Q_r(n)2^{2n-r}$ to give the recursion

$$Q_{r+1}(n)=2n^2[Q_r(n)-Q_r(n-1)]+nQ_r(n-1), \tag{4}$$

with initial condition $Q_0(n)=1$. This shows that $Q_r(n)$ is a polynomial of degree $r$ with integer coefficients. It is not difficult to establish that the leading coefficient is given by $(2r-1)\cdot(2r-3)\cdot\cdots 3\cdot 1=(2r)!/(2^r r!)$ and, hence, that these polynomials are also not monic. Applying the recursion gives the first few polynomials as

$$\begin{aligned}
&Q_0(n)=1,\\
&Q_1(n)=n,\\
&Q_2(n)=(3n-1)n,\\
&Q_3(n)=(15n^2-15n+4)n,\\
&Q_4(n)=(105n^3-210n^2+147n-34)n,\\
&Q_5(n)=(945n^4-3150n^3+4095n^2-2370n+496)n.
\end{aligned}$$

It is worth noting that, by evaluating $S_r(n)$ for particular values of $n$, one can derive various properties of [the coefficients of] the polynomials $P_r(n)$ and $Q_r(n)$. For instance, it is not difficult to show that the coefficients of $P_r(n)$ sum to unity, and those of $Q_r(n)$ to $2^{r-1}$ (for $r\geq 1$) by evaluating the sums for $n=1$. Indeed, one can derive the closed form solutions for $S_{2r}(n)$ and $S_{2r+1}(n)$ by solving a system of linear equations in $r$ unknowns, representing the coefficients of the corresponding polynomial.

In the constant of the polynomials $P_r(n)/n$, one recognizes the Genocchi numbers (see [4], [10]) named after the Italian mathematician Angelo Genocchi (1817-1889):

$$G_2=-1,\ G_4=1,\ G_6=-3,\ G_8=17,\ G_{10}=-155,\ G_{12}=2073,\ldots.$$

These numbers are defined through the exponential generating function

$$\frac{2t}{e^t+1}=t+\sum_{r\geq 1}G_{2r}\frac{t^{2r}}{(2r)!},$$

and are related to the Bernoulli numbers by $G_{2r}=2(1-2^{2r})B_{2r}$. The Genocchi numbers are listed as sequence A001469 in the on-line version of the encyclopedia of integer sequences [15], where additional references may be found. The constant of the polynomials $Q_r(n)/n$ matches the first terms of the sequence A002105 in [15], and is related to the tangent numbers. The connection to the Genocchi numbers will be explored further in the next section, where the polynomials $P_r(n)$ and $Q_r(n)$ are found to be related to special cases of the Dumont-Foata polynomials.

Another matter of interest is the leading coefficient of the polynomials, characterizing the behavior of the sums $S_r(n)$ for large values of $n$. For the even-indexed sums, this is easily established as

$$S_{2r}(n)\sim\frac{(2r)!}{2^{2r}r!}2^{2n}n^r, \tag{5}$$

and for the odd-indexed sums we can use Stirling's formula to give $\binom{2n}{n}\sim 2^{2n}/\sqrt{\pi n}$, so that

$$S_{2r+1}(n)\sim\frac{r!}{\sqrt{\pi}}2^{2n}n^{r+\frac{1}{2}}. \tag{6}$$

In these expressions, one recognizes the moments of a central chi-distribution (see, for instance, [12], pp. 420-21). That this is no coincidence will be shown in Section 4, where we establish the connec-tion between the sums $S_r(n)$ and the distance to the origin in a symmetric Bernoulli random walk.

## 3. DUMONT-FOATA POLYNOMIALS

In this section we show that the polynomials $P_r(n)$ and $Q_r(n)$ are related to special cases of the Dumont-Foata polynomials [6]. These are defined recursively by means of

$$F_{r+1}(x,y,z)=(x+z)(y+z)F_r(x,y,z+1)-z^2F_r(x,y,z), \tag{7}$$

with initial condition $F_1(x,y,z)=1$. Explicit expressions for these polynomials and their generating functions have been derived by Carlitz [3], but are too lengthy to display here.

The Dumont-Foata polynomials can be regarded as generalizations of the Gandhi polynomials (see, for instance [5], [17]), which are defined by the recursion

$$F_{r+1}(z)=(z+1)^2F_r(z+1)-z^2F_r(z), \tag{8}$$

with initial condition $F_1(z)=1$. The coefficients of the first few of these polynomials are shown in Table 1, and can also be found in sequence A036970 in [15]. The Gandhi polynomials arose from a conjecture by Gandhi [9] concerning a representation of the Genocchi numbers. Gandhi's conjec-ture that $F_r(0)=(-1)^rG_{2r}$ was proved by Carlitz [2] and also by Riordan and Stein [14]. Another polynomial that can be derived as a special case of the Dumont-Foata polynomials is obtained by the recursion

$$\widetilde{F}_{r+1}(z)=(2z+1)(z+1)\widetilde{F}_r(z+1)-2z^2\widetilde{F}_r(z), \tag{9}$$

with initial condition $\widetilde{F}_1(z)=1$. The coefficients of the first few of these polynomials are given in Table 2. Comparing these and the coefficients of the Gandhi polynomials to the coefficients of the polynomials $P_r(n)$ and $Q_r(n)$, the connection to the Dumont-Foata polynomials becomes evident. By substitution in (3) and (4), it is easily verified that

$$P_r(n) = (-1)^{r-1} nF_r(1, 1, -n) \text{ and } Q_r(n) = (-2)^{r-1} nF_r(\tfrac{1}{2}, 1, -n),$$

for $r \geq 1$. The occurrence of the Genocchi numbers in the expressions for $P_r(n)$ is seen to be a direct consequence of Gandhi's conjecture: $P_r'(0) = (-1)^{r-1} F_r(0) = -G_{2r}$. The occurrence of the Genocchi numbers in the expressions for $Q_r(n)$ is conjectured by the present author, in the form $\widetilde{F}_r(0) = (-2)^r G_{2r} / (2r)$, where $\widetilde{F}_r(z)$ are the polynomials defined by (9).

**TABLE 1. Coefficients of the Gandhi Polynomials, Arranged in Triangular Form**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | 1 |
| | | | | | | 2 | 1 |
| | | | | | 6 | 8 | 3 |
| | | | | 24 | 60 | 54 | 17 |
| | | | 120 | 480 | 762 | 556 | 155 |
| | | 720 | 4200 | 10248 | 12840 | 8146 | 2073 |
| | 5040 | 40320 | 139440 | 263040 | 282078 | 161424 | 38227 |
| 40320 | 423360 | 1965600 | 5170800 | 8240952 | 7886580 | 4163438 | 929569 |

**TABLE 2. Coefficients of the Polynomials $\widetilde{F}_r(z)$, Arranged in Triangular Form**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | 1 |
| | | | | | | 3 | 1 |
| | | | | | 15 | 15 | 4 |
| | | | | 105 | 210 | 147 | 34 |
| | | | 945 | 3150 | 4095 | 2370 | 496 |
| | | 10395 | 51975 | 107415 | 111705 | 56958 | 11056 |
| | 135135 | 945945 | 2837835 | 4579575 | 4114110 | 1911000 | 349504 |
| 2027025 | 18918900 | 77567490 | 178378200 | 244909665 | 197722980 | 85389132 | 14873104 |

## 4. SYMMETRIC BERNOULLI RANDOM WALKS

In a symmetric Bernoulli random walk, one considers the movements of a particle starting at time $t = 0$ at the origin. Its movements are determined by a chance mechanism, where a fair coin is flipped and the particle is moved one unit to the right if it is heads up, and one unit to the left if it is tails up. A more exhaustive description and in-depth study of random walks can be found in Feller [8] or Révész [13]. A more playful introduction to the topic is given in the monograph by Dynkin and Uspenskii [7]. A topic of interest is the position of the particle after $2n$ coin tosses: $Y_{2n} = X_1 + X_2 + \cdots + X_{2n}$, where $X_i$ is +1 or −1 depending upon whether or not the coin showed heads in the $i^{\text{th}}$ coin toss. Note that the $X_i$ are independent and identically distributed variates with mean 0 and variance 1. The probability distribution of the position of the particle after $2n$

moves can be derived from a simple combinatorial argument [see, e.g., [8], p. 75, or [13], p. 13) and is given by

$$\text{Prob}(Y_{2n} = 2k) = \binom{2n}{n-k} 2^{-2n},$$

where $k = -n, -n+1, \ldots, n$ and $n$ is a positive integer. The matter of interest in the context of this note is the distance to the origin $|Y_{2n}|$ at time $t = 2n$. Its moments are given by

$$\mathrm{E}|Y_{2n}|^r = \sum_{k=-n}^{n} \binom{2n}{n-k} 2^{-2n} |2k|^r,$$

and one sees that $\mathrm{E}|Y_{2n}|^r = 2^{r-2n} S_r(n)$, thus establishing the connection to the absolute sums from the introduction. The limit behavior of these sums now becomes clear. By the central limit theorem (see, e.g., [11], p. 18), one has that $Y_{2n}$, for sufficiently large $n$, follows a normal distribution with mean 0 and variance $2n$. This implies that, asymptotically, $|Y_{2n}|$ has a half-normal or central chi-distribution, so that

$$\mathrm{E}|Y_{2n}|^r \sim \frac{\Gamma[(r+1)/2]}{\Gamma(1/2)} 2^r n^{r/2}$$

(see, e.g., [12], pp. 420-21). This gives the asymptotic behavior of the sums as

$$S_r(n) = 2^{2n-r}\,\mathrm{E}|Y_{2n}|^r \sim \frac{\Gamma[(r+1)/2]}{\Gamma(1/2)} 2^{2n} n^{r/2},$$

and, upon expanding the gamma functions, one recovers the limit results (5) and (6).

## 5. DISCUSSION

One could possibly use the relation of the Gandhi polynomials to the sums $S_{2r+1}(n)$ to gain new insights on the former. In particular, one now has an expression to derive the function values of the Gandhi polynomials for negative integral arguments:

$$F_r(-n) = (-1)^{r-1} \frac{2}{n^2} \binom{2n}{n}^{-1} \sum_{k=1}^{n} \binom{2n}{n-k} k^{2r+1} \quad (n, r \geq 1).$$

For example, one easily obtains $F_r(-1) = (-1)^{r-1}$ and $F_r(-2) = (-1)^{r-1}(2^{2r-1}+1)/3$.

Likewise, one can use the relation of the moments of the absolute distance to the origin in a symmetric Bernoulli random walk and the sums $S_r(n)$ to express these moments in terms of the polynomi-als $P_r(n)$ and $Q_r(n)$:

$$\mathrm{E}|Y_{2n}|^{2r} = 2^r Q_r(n) \quad \text{and} \quad \mathrm{E}|Y_{2n}|^{2r+1} = \binom{2n}{n} 2^{2(r-n)+1} n P_r(n).$$

This equivalence can be used to establish the rate of convergence to the moments of the half-normal distribution.

Finally, it should be noted that one can also determine expressions for $S_{2r}(n)$ by means of the generating function

$$f_n(\varphi) = \sum_{k=0}^{2n} \binom{2n}{k} e^{(n-k)\varphi} = e^{n\varphi}[1+e^{-\varphi}]^{2n} = [e^{\varphi} + 2 + e^{-\varphi}]^n,$$

so that $S_{2r}(n) = f_n^{(2r)}(0)$. However, this approach covers only the even-indexed case, and does not give the same insights as the one we have followed here.

## ACKNOWLEDGMENT

I would like to thank the anonymous referee for drawing attention to the occurrence of the Genocchi numbers in the polynomials $P_r(n)$. This led to a further investigation and the characterization in terms of the Gandhi and Dumont-Foata polynomials.

## REFERENCES

1. Paul S. Bruckman. "Problem B-871." *The Fibonacci Quarterly* **37.1** (1999):85.
2. L. Carlitz. "A Conjecture Concerning Genocchi Numbers." *Det Kongelige Norske Videnskabers Selskabs Skrifter* **9** (1972):1-4.
3. L. Carlitz. "Explicit Formulas for the Dumont-Foata Polynomial." *Discrete Mathematics* **30.3** (1980):211-25.
4. Louis Comtet. *Advanced Combinatorics: The Art of Finite and Infinite Expansions*. Boston: D. Reidel, 1974.
5. Dominique Dumont. "Interpretations combinatoires des nombres de Genocchi." *Duke Math. J.* **41** (1974):305-18.
6. Dominique Dumont & Dominique Foata. "Une propriété de symétrie des nombres de Genocchi." *Bulletin de la Société Mathématique de France* **104.4** (1976):433-51.
7. E. B. Dynkin & V. A. Uspenskii. *Random Walks: Part Three of Mathematical Conversations*. Boston: D. C. Heath and Company, 1963.
8. William Feller. *An Introduction to Probability Theory and Its Applications*. 3rd ed. New York: John Wiley & Sons, 1950.
9. J. M. Gandhi. "A Conjectured Representation of Genocchi Numbers." *Amer. Math. Monthly* **77.5** (1970):505-06.
10. Ronald L. Graham, Donald E. Knuth, & Oren Patashnik. *Concrete Mathematics*. Reading, Mass.: Addison-Wesley, 1989.
11. Allan Gut. *Stopped Random Walks*. New York: Springer-Verlag, 1988.
12. Normal L. Johnson, Samuel Kotz, & N. Balakrishnan. *Continuous Univariate Distributions*. Vol. 1. New York: John Wiley & Sons, 1994.
13. Pál Révész. *Random Walk in Random and Non-Random Environments*. Singapore: World Scientific Pub. Co., 1990.
14. John Riordan & Paul R. Stein. "Proof of a Conjecture on Genocchi Numbers." *Discrete Mathematics* **5.4** (1973):381-88.
15. N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences*. Published electronically at http://www.research.att.com/~njas/sequences/.
16. Indulis Strazdins. "Solution to Problem B-871." *The Fibonacci Quarterly* **38.1** (2000):86-87.
17. Volker Strehl. "Alternating Permutations and Modified Ghandi-Polynomials." *Discrete Mathematics* **28** (1979):89-100.

AMS Classification Numbers: 11B65, 60G50, 44A60

❖❖❖

# ELEMENTARY PROBLEMS AND SOLUTIONS

## *Edited by*
## Russ Euler and Jawad Sadek

*Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.*

*If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.*

*Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2002. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".*

### BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

### PROBLEMS PROPOSED IN THIS ISSUE

**B-935** ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI***

Prove that

$$8\sin\left(\frac{F_3}{2}\right)\sin\left(\frac{F_9}{2}\right)\sin\left(\frac{F_{12}}{2}\right) < 1$$

where the arguments are measured in degrees.

**B-936** ***Proposed by José Luis Diaz & Juan José Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain***

Let $n$ be a nonnegative integer. Show that the equation

$$x^5 + F_{2n}x^4 + 2(F_{2n} - 2F_{n+1}^2)x^3 + 2F_{2n}(F_{2n} - 2F_{n+1}^2)x^2 + F_{2n}^2 x + F_{2n}^3 = 0$$

has only integer roots.

**B-937** ***Proposed by Paul S. Bruckman, Sacramento, CA***

Prove the following identities:

***(a)*** $(F_n)^2 + (F_{n+1})^2 + 4(F_{n+2})^2 = (F_{n+3})^2 + (L_{n+1})^2$;

***(b)*** $(L_n)^2 + (L_{n+1})^2 + 4(L_{n+2})^2 = (L_{n+3})^2 + (5F_{n+1})^2$.

**B-938** ***Proposed by Charles K. Cook, University of South Carolina at Sumter, Sumter, SC***

Find the smallest positive integer $k$ for which the given series converges and find its sum:

*(a)* $\sum_{n=1}^{\infty} \frac{nF_n}{k^n}$;

*(b)* $\sum_{n=1}^{\infty} \frac{nL_n}{k^n}$.

**B-939** ***Proposed by N. Gauthier, Royal Military College of Canada***

For $n \geq 0$ and $s$ arbitrary integers, with

$$f(l, m, n) \equiv f(l, m) = (-1)^{n-l}\binom{n}{l}\binom{n}{m},$$

prove the following identities:

*(a)* $2^n F_{n+s} = \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m, m) F_{l+s}$;

*(b)* $3 \cdot 2^{n-1} n F_{n+s+2} = \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m, m)[(l-2m)F_{l+s} + mF_{l+s-1}]$.

## SOLUTIONS

### A Relatively Prime Fibonacci Couple

**B-921** ***Proposed by the editors***
***(Vol. 39, no. 3, June-July 2001)***

Determine whether or not $F_{6n}-1$ and $F_{6n-3}+1$ are relatively prime for all $n \geq 1$.

***Solution by Russell Jay Hendel, Towson University, Baltimore, MD***

We go beyond the problem requirements by also providing explicit formulas for the relative primeness.

Recall that two integers $a$ and $b$ are relatively prime if and only if there exist integers $x$ and $y$ such that

$$ax + by = 1. \tag{1}$$

Accordingly, let

$$a = F_{6n} - 1, \quad a = F_{6n} - 1,$$
$$b = F_{6n-3} + 1. \, b = F_{6n-3} + 1.$$

The parallel processor algorithm of Hendel [2] motivates defining

$$x = F_{6n-5} - \{F_{6n-4} - F_{6n-10} - 4\} / 16,$$
$$y = \{F_{6n+3} + F_{6n+1} - F_{6n-3} - F_{6n-5} - 12\} / 16.$$

Using periodicity properties of the Fibonacci sequence modulo 16, it is straightforward to verify that $x$ and $y$ are in fact integers.

Using these definitions of $x$ and $y$, (1) can be proven for all $n$ by using the Verification Theorem of Dresel [1]. We need only check (1) for the first values of $n$ and this is easily done by hand calculator. For example, when $n = 3$, (1) yields the explicit identity $2583*211-611*892=1$.

**References**

1. L. A. G. Dresel. "Transformations of Fibonacci-Lucas Identities." In *Applications of Fibonacci Numbers* **5:**169-84. Ed. G. Bergum, et al. Dordrecht: Kluwer, 1993.
2. R. J. Hendel. "A Fibonacci Problem Classification Scheme Useful to Undergraduate Pedagogy." In *Applications of Fibonacci Numbers* **5:**289-304. Dordrecht: Kluwer, 1993.

***Also solved by Paul S. Bruckman, L. A. G. Dresel, Lake Superior State University Problem Group, H.-J. Seiffert, Gabriela & Pantelimon Stănică (jointly), and the proposers.***

## A Prime Search

**B-922** ***Proposed by Irving Kaplansky, Math. Sciences Research Institute, Berkeley, CA (Vol. 39, no. 3, June-July 2002)***

Determine all primes $p$ such that the Fibonacci numbers modulo $p$ yield all residues.

***Solution by Pantelimon Stănică, Auburn University, Montgomery, AL***

In *The Fibonacci Quarterly* **6.2** (1968):139-41 ("Fibonacci Sequence Modulo *m*"), A. P. Shah proved that if $p$ is a prime and $p \equiv 1, 9 \pmod{10}$ then the Fibonacci sequence does not form a complete residue modulo $p$.

In *The Fibonacci Quarterly* **8.3** (1970):000-00 ["Fibonacci Sequence Modulo a Prime $p \equiv 3$ (mod 4)"], G. Bruckner proved the same for the remaining cases if $p > 7$. Therefore, the Fibonacci sequence modulo $p$ yields all residues if and only if $p = 2, 3, 5, 7$ by an easy calculation and using the above references.

In *The Fibonacci Quarterly* **38.3** (2000):272-81 ("Complete and Reduced Residue Systems of Second-Order Recurrences Modulo *p*"), H.-C. Li proved that even the generalized Fibonacci sequence with parameters $(a, 1)$ does not form a complete residue system modulo $p > 5$.

*L. A. G. Dresel also referred to the G. Bruckner reference.*

***Also solved by P. Bruckman, L. A. G. Dresel, and the proposer.***

## The Fraction Continues

**B-923** ***Proposed by José Luis Diaz & Juan José Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain (Vol. 39, no. 3, June-July 2002)***

Let $\alpha_l$ be the $l^{\text{th}}$ convergent of the continued fractional expansion:

$$\alpha = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

Prove that

***(a)*** $\frac{1}{n}\sum_{k=0}^{n-1}\alpha_{l+k} \geq [F_n\alpha_l + F_{n-1}]^{1/n}$,

***(b)*** $\alpha_k^l = \sum_{j=0}^{k}\binom{k}{j}\frac{1}{\alpha_{l-1}^j}$ for all $k \in \mathbb{N}$.

***Solution by Paul S. Bruckman, Sacramento, CA***

The readers of this journal will readily recognize the following result:

$$\alpha_j = F_{j+1} / F_j$$

(for typographical clarity, the notation is modified).

***(a)*** Let

$$A(n, j) = 1/n\sum_{k=0}^{n-1}\alpha_{j+k}.$$

Note that $A(n, j)$ is the arithmetic average (A.M.) of the quantities $\alpha_j, \alpha_{j+1}, \ldots, \alpha_{j+n-1}$. By the A.M.-G.M. inequality,

$$A(n, j) \geq G(n, j) \equiv \left(\prod_{k=0}^{n-1}\alpha_{j+k}\right)^{1/n}.$$

Note that $G(n, j) = (F_{j+n} / F_j)^{1/n}$.

Also,

$$F_n\alpha_j + F_{n-1} = (F_nF_{j+1} + F_{n-1}F_j) / F_j = F_{j+n} / F_j.$$

Thus, $A(n, j) \geq (F_n\alpha_j + F_{n-1})^{1/n}$. Q.E.D.

***(b)*** Let

$$S(k, j) = \sum_{i=0}^{k} {}_kC_i(\alpha_{j-1})^{-i}.$$

Then

$$S(k, j) = (1 + 1/\alpha_{j-1})^k = (1 + F_{j-1} / F_j)^k$$
$$= \{(F_j + F_{j-1}) / F_j\}^k = (F_{j+1} / F_j)^k = (\alpha_j)^k.$$

This corrects the statement of this part of the problem.

***Also solved by H.-J. Seiffert (essentially the same as the featured solution) and the proposer.***

## A Generalization of a Lucas Numbers Identity

**A <u>B-924</u>** ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 39, no. 3, June-July 2001)***

For $n$ an arbitrary integer, the following identity is easily established for Lucas numbers:

$$L_{2n+2} + L_{2n-2} = 3L_{2n}. \tag{1}$$

Consider the Fibonacci and Lucas polynomials, $\{F_n(u)\}_{n=0}^{\infty}$ and $\{L_n(u)\}_{n=0}^{\infty}$, defined by

$$F_0(u) = 0,\ F_1(u) = 1,\ F_{n+2}(u) = uF_{n+1}(u) + F_n(u),$$

and

$$L_0(u) = 2,\ L_1(u) = u,\ L_{n+2}(u) = uL_{n+1}(u) + L_n(u),$$

respectively. The corresponding generalization of (1) is

$$L_{2n+1}(u) + L_{2n-2}(u) = (u^2+2)L_{2n}(u). \tag{2}$$

For $m$ a nonnegative integer, with the convention that a discrete sum with a negative upper limit is identically zero, prove the following generalization of (2):

$$\begin{aligned}&(n+1)^{2m}L_{2n+2}(u) + (n-1)^{2m}L_{2n-2}(u)\\ &= (u^2+2)\left[\sum_{l=0}^{m}\binom{2m}{2l}n^{2l}\right]L_{2n}(u) + u(u^2+4)\left[n\sum_{l=0}^{m-1}\binom{2m}{2l+1}n^{2l}\right]F_{2n}(u).\end{aligned} \tag{3}$$

Also prove the following companion identity:

$$\begin{aligned}&(n+1)^{2m+1}F_{2n+2}(u) + (n-1)^{2m+1}F_{2n-2}(u)\\ &= u\left[\sum_{l=0}^{m}\binom{2m+1}{2l}n^{2l}\right]L_{2n}(u) + (u^2+2)\left[n\sum_{l=0}^{m-1}\binom{2m+1}{2l+1}n^{2l+1}\right]F_{2n}(u).\end{aligned} \tag{4}$$

***Solution by H.-J. Seiffert, Berlin, Germany***

In (4), the upper index in the second sum on the right-hand side must be replaced by $m$.

It is known [see A. F. Horadam & Bro. J. M. Mahon, "Pell and Pell-Lucas Polynomials," *The Fibonacci Quarterly* **23.1** (1985):7-20, equations (3.23), (2.2), (2.1), and (3.22)] that

$$(u^2+2)L_{2n}(u) = L_{2n+2}(u) + L_{2n-2}(u), \tag{5}$$

$$(u^2+4)F_{2n}(u) = L_{2n-1}(u) + L_{2n+1}(u), \tag{6}$$

$$L_{2n}(u) = F_{2n-1}(u) + F_{2n+1}(u), \tag{7}$$

$$(u^2+2)F_{2n}(u) = F_{2n+2}(u) + F_{2n-2}(u); \tag{8}$$

note that (5) is the corrected version of (2).

$$\sum_{l=0}^{m}\binom{2m}{2l}n^{2l} = \frac{(n+1)^{2m}+(n-1)^{2m}}{2}, \tag{9}$$

$$\sum_{l=0}^{m-1}\binom{2m}{2l+1}n^{2l+1} = \frac{(n+1)^{2m}-(n-1)^{2m}}{2}, \tag{10}$$

$$\sum_{l=0}^{m}\binom{2m+1}{2l}n^{2l} = \frac{(n+1)^{2m+1}-(n-1)^{2m+1}}{2}, \tag{11}$$

$$\sum_{l=0}^{m}\binom{2m+1}{2l+1}n^{2l+1} = \frac{(n+1)^{2m+1}+(n-1)^{2m+1}}{2}. \tag{12}$$

***Proof of (3):*** In view of (5), (6), (9), and (10), we must show that

$$\begin{aligned}&(n+1)^{2m}L_{2n+2}(u) + (n-1)^{2m}L_{2n-2}(u)\\ &= \frac{(n+1)^{2m}+(n-1)^{2m}}{2}(L_{2n+2}(u) + L_{2n-2}(n)) + \frac{(n+1)^{2m}-(n-1)^{2m}}{2}(uL_{2n-1}(u) + uL_{2n+1}(u)),\end{aligned}$$

which is true because

$$L_{2n-2}(u)+uL_{2n-1}(u)+uL_{2n+1}(u)=L_{2n+2}(u)$$

and, equivalently,

$$L_{2n+2}(u)-uL_{2n-1}(u)-uL_{2n+1}(u)=L_{2n-2}(u).$$

***Proof of (4):*** This is easily verified by applying (7), (8), (11), and (12), and using

$$uF_{2n-1}(u)+uF_{2n+1}(u)+F_{2n-2}(u)=F_{2n+2}(u).$$

***Also solved by P. Bruckman and the proposer.***

## The Gandhi Polynomials

In response to Paul Bruckman's question, Reiner Martin sent the following remark:

In the August 2001 issue of *The Fibonacci Quarterly*, Paul Bruckman asks whether the polynomials $P(r,n)$ given by $P(1,n)=n$ and $P(r+1,n)=n^2(P(r,n)-P(r,n-1))$ are new to the literature.

Indeed, these polynomials (or, rather, a trivial variation thereof) are known as Gandhi polynomials. References are:

[1] D. Dumont, "Sur une conjecture de Gandhi concernant les nombres de Genocchi," *Discrete Mathematics* **1** (1972):321-27.

[2] D. Dumont, "Interpretations combinatoires des nombres de Genocchi," *Duke Math. Journal* **41** (1974):305-18.

Identifying these polynomials illustrates the usefulness of Sloane's On-Line Encyclopedia of Integer Sequences (http://www.research.att.com/~njas/sequences/). Entering the first few nonzero coefficients as $1,-1,2,3,-8,6,-17,54,-60,24$ into the database yields a hit (up to signs) with the sequence A036970 (triangle of coefficients of Gandhi polynomials), where the references can be found.

*We wish to belatedly acknowledge the solution to problem B-915 by Walther Janous. In fact, his solution gives a sharper inequality that will appear in a separate proposal.*

❖❖❖

# ADVANCED PROBLEMS AND SOLUTIONS

## *Edited by*
## Raymond E. Whitney

*Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.*

### PROBLEMS PROPOSED IN THIS ISSUE

**H-583** ***Proposed by N. Gauthier, Royal Military College of Canada***

**A Theorem on Generalized Fibonacci Convolutions**

This is a generalization of Problem **B-858** by W. Lang (*The Fibonacci Quarterly* **36.3**, 1998).

Let $n \geq 0$, $a$, $b$ be integers; also let $A$, $B$ be arbitrary yet known real numbers and consider the generalized Fibonacci sequence $\{G_n \equiv A\alpha^n + B\beta^n\}_{n=-\infty}^{\infty}$, where

$$\alpha = \frac{1}{2}[1+\sqrt{5}],\ \beta = \frac{1}{2}[1-\sqrt{5}].$$

For $m$ a nonnegative integer, prove the following generalized convolution theorem for the sequences $\{(a+n)^m\}_{n=-\infty}^{\infty}$ and $\{G_n\}_{n=-\infty}^{\infty}$,

$$\sum_{k=0}^{n}(a+k)^m G_{b-a-k} = \sum_{l=0}^{m} l![c_l^m(a)G_{b-a+l+1} - c_l^m(a+n+1)G_{b-a-n+1+l}],$$

where the set of coefficients $\{c_l^m(v);\ 0 \leq m;\ 0 \leq l \leq m;\ v = a \text{ or } a+n+1\}$ satisfies the following second-order linear recurrence relation

$$c_l^{m+1}(v) = (v+l)c_l^m(v) + c_{l-1}^m(v);\ c_{l=0}^{m=0}(v) = 1, c_{l=0}^{m=1}(v) = v, c_{l=1}^{m=1}(v) = 1$$

with the understanding that $c_{-1}^m(v) \equiv 0$ and that $c_{m+1}^m(v) \equiv 0$.

Prob. **B-858** follows as a special case if one sets $a = 0$, $m = 1$, $b = n$, and $A = -B = (\alpha - \beta)^{-1}$ in the above theorem. Indeed, one then gets that

$$G_n = F_n,\ c_0^1(0) = 0,\ c_1^1(0) = 1,\ c_0^1(n+1) = n+1, \text{ and } c_1^1(n+1) = 1$$

and the result follows directly.

**H-584** ***Proposed by Paul S. Bruckman, Sacramento, CA***

Prove the following identity:

$$\begin{aligned}&(F_{n+4} + L_{n+3})^5 + (F_n + L_{n+1})^5 + (2F_{n+1} + L_{n+2})^5 \\ &= (2F_{n+3} + L_{n+2})^5 + (F_{n+2})^5 + (5F_{n+2})^5 + 1920F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}.\end{aligned}$$

## SOLUTIONS

### Some Operator!

**H-571** ***Proposed by D. Tsedenbayar, Mongolian Pedagogical University, Warsaw, Poland (Vol. 39, no. 1, February 2001)***

Prove: If $(T_\alpha f)(t) = t^\alpha \int_0^t f(s)\,ds$, with $\alpha \in \mathbf{R}$, then

$$(T_\alpha^n f)(t) = \frac{t^\alpha}{(\alpha+1)^{(n-1)}(n-1)!}\int_0^t (t^{\alpha+1} - s^{\alpha+1})^{n-1} f(s)\,ds, \quad \text{for } \alpha \neq -1$$

and

$$(T_\alpha^n f)(t) = \frac{1}{t(n-1)!}\int_0^t \left(\ln\frac{t}{s}\right)^{n-1} f(s)\,ds, \quad \text{for } \alpha = -1.$$

***Remark:*** If $\alpha = -1$, then $T_{-1}$ is a Cesaro operator; if $\alpha = 0$, then $T_0$ is a Volterra operator.

***Solution by Paul S. Bruckman, Sacramento, CA***

Our proof is by induction on $n$. We let $S(\alpha)$ denote the set of positive integers $n$ such that the statements of the problem are true. Note that the statements of the problem are true for $n = 1$, since they reduce to the definitions of $(T_\alpha)(f(t))$. That is, $1 \in S(\alpha)$.

Suppose $n \in S(\alpha)$. Then $(T_\alpha)^{n+1}(f(t)) = (T_\alpha)(T_\alpha)^n(f(t))$.

If $\alpha \neq -1$,

$$\begin{aligned}(T_\alpha)^{n+1}(f(t)) &= (T_\alpha)\frac{t^\alpha}{(\alpha+1)^{n-1}(n-1)!}\int_0^t (t^{\alpha+1} - s^{\alpha+1})^{n-1} f(s)\,ds \\ &= \frac{t^\alpha}{(\alpha+1)^{n-1}(n-1)!}\int_0^t s^\alpha \int_0^s (s^{\alpha+1} - u^{\alpha+1})^{n-1} f(u)\,du\,ds \\ &= \frac{t^\alpha}{(\alpha+1)^{n-1}(n-1)!}\int_0^t f(u)\int_u^t s^\alpha (s^{\alpha+1} - u^{\alpha+1})^{n-1}\,ds\,du \\ &= \frac{t^\alpha}{(\alpha+1)^{n-1}(n-1)!}\int_0^t f(u)\left.\frac{(s^{\alpha+1} - u^{\alpha+1})^n}{(\alpha+1)n}\right]_u^t du \\ &= \frac{t^\alpha}{(\alpha+1)^n(n)!}\int_0^t f(s)(t^{\alpha+1} - s^{\alpha+1})^n\,ds,\end{aligned}$$

which is the statement of the first part of the problem ($\alpha \neq -1$) for $n+1$. That is, $n \in S(\alpha) \Rightarrow (n+1) \in S(\alpha)$ if $\alpha \neq -1$.

The second part of the problem (for $\alpha = -1$) is treated similarly. In this case,

$$\begin{aligned}(T_{-1})^{n+1}(f(t)) &= (T_{-1})(1/t(n-1)!)\int_0^t (\log t/s)^{n-1} f(s)\,ds \\ &= (1/t(n-1)!)\int_0^t 1/s \int_0^s (\log s/u)^{n-1} f(u)\,du\,ds \\ &= (1/t(n-1)!)\int_0^t f(u)\int_u^t 1/s(\log s/u)^{n-1}\,ds\,du\end{aligned}$$

$$= (1/t(n-1)!)\int_0^t f(u)/n\left[(\log s/u)^n\right]_u^t du$$

$$= (1/t(n)!)\int_0^t f(u)(\log t/u)^n du$$

$$= (1/t(n)!)\int_0^t f(s)(\log t/s)^n ds.$$

This is the statement of the problem for $\alpha = -1$, $n+1$. Therefore, $n \in S(-1) \Rightarrow (n+1) \in S(-1)$. We have shown that, for all real $\alpha$, $n \in S(\alpha) \Rightarrow (n+1) \in S(\alpha)$. The desired results follow by induction.

## Sum Problem

**H-572** ***Proposed by Paul S. Bruckman, Berkeley, CA***
***(Vol. 39, no. 2, May 2001)***

Prove the following, where $\varphi = \alpha^{-1}$:

$$\sum_{n=0}^{\infty}\{\varphi^{5n+1}/(5n+1)+\varphi^{5n+3}/(5n+2)-\varphi^{5n+4}/(5n+3)-\varphi^{5n+4}/(5n+4)\} = (\pi/25)(50-10\sqrt{5})^{1/2}.$$

***Solution by Kenneth B. Davenport, Frackville, PA***

Since, for $|x|<1$,

$$\frac{1}{1-x^5} = 1+x^5+x^{10}+x^{15}+\cdots = \sum_{n=0}^{\infty} x^{5n}, \tag{1}$$

we let, for $-1<x<1$,

$$A(x) = \int_0^{\varphi}\frac{1}{1-x^5}dx = \sum_{n=0}^{\infty}\frac{\varphi^{5n+1}}{(5n+1)}, \tag{2}$$

$$B(x) = \varphi\int_0^{\varphi}\frac{x}{1-x^5}dx = \sum_{n=0}^{\infty}\frac{\varphi^{5n+3}}{(5n+2)}, \tag{3}$$

$$C(x) = -\varphi\int_0^{\varphi}\frac{x^2}{1-x^5}dx = -\sum_{n=0}^{\infty}\frac{\varphi^{5n+4}}{(5n+3)}, \tag{4}$$

$$D(x) = -\int_0^{\varphi}\frac{x^3}{1-x^5}dx = \sum_{n=0}^{\infty}\frac{\varphi^{5n+4}}{(5n+4)}. \tag{5}$$

Making use of an integral expression:

$$\int\frac{x^m}{1-x^n}dx = -\frac{1}{n}\cos\frac{2(m+1)\pi}{n}\log\left(1-2x\cos\frac{2\pi}{n}+x^2\right)-\frac{1}{n}\cos\frac{4(m+1)\pi}{n}\log\left(1-2x\cos\frac{4\pi}{n}+x^2\right)$$
$$-\frac{1}{n}\cos\frac{6(m+1)\pi}{n}\log\left(1-2x\cos\frac{6\pi}{n}+x^2\right)-\cdots+\frac{2}{n}\sin\frac{2(m+1)\pi}{n}\arctan\frac{x\sin\frac{2\pi}{n}}{1-x\cos\frac{2\pi}{n}}$$
$$+\frac{2}{n}\sin\frac{4(m+1)\pi}{n}\arctan\frac{x\sin\frac{4\pi}{n}}{1-x\cos\frac{4\pi}{n}}+\frac{2}{n}\sin\frac{6(m+1)\pi}{n}\arctan\frac{x\sin\frac{6\pi}{n}}{1-x\cos\frac{6\pi}{n}}+\cdots-\frac{1}{n}\log(1-x).$$

From *Tables of Indefinite Integrals* by G. Petit Bois (Dover Publications, 1961), we derive the following.

**For $A(x)$:**

$$\begin{aligned}
&-\frac{1}{5}\cos\frac{2\pi}{5}\log\left(1-2x\cos\frac{2\pi}{5}+x^2\right)\\
&-\frac{1}{5}\cos\frac{4\pi}{5}\log\left(1-2x\cos\frac{4\pi}{5}+x^2\right)\\
&+\frac{2}{5}\sin\frac{2\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{2\pi}{5}}{1-x\cos\frac{2\pi}{5}}\right]\\
&+\frac{2}{5}\sin\frac{4\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{4\pi}{5}}{1-x\cos\frac{4\pi}{5}}\right].
\end{aligned}\tag{6}$$

**For $B(x)$:**

$$\begin{aligned}
&-\frac{1}{5}\cos\frac{4\pi}{5}\log\left(1-2x\cos\frac{2\pi}{5}+x^2\right)\\
&-\frac{1}{5}\cos\frac{8\pi}{5}\log\left(1-2x\cos\frac{4\pi}{5}+x^2\right)\\
&+\frac{2}{5}\sin\frac{4\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{2\pi}{5}}{1-x\cos\frac{2\pi}{5}}\right]\\
&+\frac{2}{5}\sin\frac{8\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{4\pi}{5}}{1-x\cos\frac{4\pi}{5}}\right].
\end{aligned}\tag{7}$$

**For $C(x)$:**

$$\begin{aligned}
&+\frac{1}{5}\cos\frac{6\pi}{5}\log\left(1-2x\cos\frac{2\pi}{5}+x^2\right)\\
&+\frac{1}{5}\cos\frac{12\pi}{5}\log\left(1-2x\cos\frac{4\pi}{5}+x^2\right)\\
&-\frac{2}{5}\sin\frac{6\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{2\pi}{5}}{1-x\cos\frac{2\pi}{5}}\right]\\
&-\frac{2}{5}\sin\frac{12\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{4\pi}{5}}{1-x\cos\frac{4\pi}{5}}\right].
\end{aligned}\tag{8}$$

**For $D(x)$:**

$$\begin{aligned}
&+\frac{1}{5}\cos\frac{8\pi}{5}\log\left(1-2x\cos\frac{2\pi}{5}+x^2\right)\\
&+\frac{1}{5}\cos\frac{16\pi}{5}\log\left(1-2x\cos\frac{4\pi}{5}+x^2\right)\\
&-\frac{2}{5}\sin\frac{8\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{2\pi}{5}}{1-x\cos\frac{2\pi}{5}}\right]\\
&-\frac{2}{5}\sin\frac{16\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{4\pi}{5}}{1-x\cos\frac{4\pi}{5}}\right].
\end{aligned}\tag{9}$$

And now, keeping in mind that (7) is multiplied by the factor $\varphi$, (8) by $-\varphi$, and (9) by $-1$, we observe that (6) and (9) when summed cancel the *logarithmic* parts due to sign and likewise (7) and (8) when summed will cancel the *logarithmic* parts. Thus, upon evaluating (6) and (9) as well as (7) and (8) between the bounds 0 and $\varphi$, one will then have:

**(6) + (9) =**

$$+\frac{2}{5}\sin\frac{2\pi}{5}\cdot\frac{\pi}{5}+\frac{2}{5}\sin\frac{4\pi}{5}\tan^{-1}\left[\frac{\varphi\sin\frac{4\pi}{5}}{1-\varphi\cos\frac{4\pi}{5}}\right]$$

$$-\frac{2}{5}\sin\frac{8\pi}{5}\cdot\frac{\pi}{5}-\frac{2}{5}\sin\frac{16\pi}{5}\tan^{-1}\left[\frac{\varphi\sin\frac{4\pi}{5}}{1-\varphi\cos\frac{4\pi}{5}}\right]; \tag{10}$$

**(7) + (8) =**

$$+\frac{2}{5}\sin\frac{4\pi}{5}\cdot\frac{\pi}{5}+\frac{2}{5}\sin\frac{8\pi}{5}\tan^{-1}\left[\frac{\varphi\sin\frac{4\pi}{5}}{1-\varphi\cos\frac{4\pi}{5}}\right]$$

$$-\frac{2}{5}\sin\frac{6\pi}{5}\cdot\frac{\pi}{5}-\frac{2}{5}\sin\frac{12\pi}{5}\tan^{-1}\left[\frac{\varphi\sin\frac{4\pi}{5}}{1-\varphi\cos\frac{4\pi}{5}}\right]. \tag{11}$$

And now, noting that

$$\left[\sin\frac{4\pi}{5}-\sin\frac{16\pi}{5}+\varphi\sin\frac{8\pi}{5}-\varphi\sin\frac{12\pi}{5}\right]=0,$$

we may simplify (10) and (11) to obtain

$$\frac{2\pi}{25}\left[\sin\frac{2\pi}{5}-\sin\frac{8\pi}{5}+\sin\frac{4\pi}{5}-\sin\frac{6\pi}{5}\right]. \tag{12}$$

Analytically, this reduces to the expression:

$$\frac{\pi}{25}(10+2\sqrt{5})^{1/2}+\left(\frac{3-\sqrt{5}}{2}\right)^{1/2}(10+2\sqrt{5})^{1/2}=\frac{\pi}{25}(10+2\sqrt{5})^{1/2}+(20-8\sqrt{5})^{1/2}. \tag{13}$$

And (13) is equivalent to

$$\frac{\pi}{25}(50-10\sqrt{5})^{1/2}.$$

***Also solved by F. Ovidiu, H.-J. Seiffert, and the proposer.***

## Fee Fi Fo Fum

**H-573** ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 39, no. 2, May 2001)***

"By definition, a magic matrix is a square matrix whose lines, columns, and two main diagonals all add up to the same sum. Consider a $3\times 3$ magic matrix $\Phi$ whose elements are the following combinations of the $n^{\text{th}}$ and $(n+1)^{\text{th}}$ Fibonacci numbers:

$$\Phi_{11}=3F_{n+1}+F_n;\quad \Phi_{12}=F_{n+1};\quad \Phi_{13}=2F_{n+1}+2F_n;$$
$$\Phi_{21}=F_{n+1}+2F_n;\quad \Phi_{22}=2F_{n+1}+F_n;\quad \Phi_{23}=3F_{n+1};$$
$$\Phi_{31}=2F_{n+1};\quad \Phi_{32}=3F_{n+1}+2F_n;\quad \Phi_{33}=F_{n+1}+F_n.$$

Find a closed-form expression for $\Phi^m$, where $m$ is a *positive* integer, and determine all the values of $m$ for which it too is a magic matrix."

***Solution by the proposer***

It is well known that the elements of a $3 \times 3$ magic matrix can generally be written in the form:

$$\begin{array}{lll} \Phi_{11} = a+b; & \Phi_{12} = a-(b+c); & \Phi_{13} = a+c; \\ \Phi_{21} = a-(b-c); & \Phi_{22} = a; & \Phi_{23} = a+(b-c); \\ \Phi_{31} = a-c; & \Phi_{32} = a+(b+c); & \Phi_{33} = a-b. \end{array}$$

In the present situation, $a = F_{n+3}$, $b = F_{n+1}$, $c = F_n$.

Now define three magic matrices, as follows:

$$A \equiv \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}; \quad B \equiv \frac{1}{\sqrt{3}}\begin{pmatrix} +1 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -1 \end{pmatrix}; \quad C \equiv \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -1 & +1 \\ +1 & 0 & -1 \\ -1 & +1 & 0 \end{pmatrix}.$$

Then $\Phi = \alpha A + \beta B + \gamma C$, where $\alpha \equiv 3F_{n+3}$, $B \equiv \sqrt{3}F_{n+1}$, $\gamma \equiv \sqrt{3}F_n$.

Next, for $m$ an integer, one can simply verify the following multiplication properties:

$$A^m = A, m > 0; \;\; AB = BA = AC = CA = N; \;\; BC = -CB;$$

$$B^2 = -C^2 = I - A; \;\; B^{2m} = B^2, m > 0; \;\; B^{2m+1} = B, m \geq 0.$$

$N$ and $I$ are the $3 \times 3$ null and identity matrices, respectively. Consequently,

$$\Phi^2 = \alpha^2 A + (\beta^2 - \gamma^2)B^2,$$

and since $A$, $B$ commute, with $AB = BA = N$, and

$$\beta^2 - \gamma^2 = (\beta - \gamma)(\beta + \gamma) = 3F_{n+2}F_{n-1},$$

we find that

$$\begin{aligned} \Phi^{2m} &= [\alpha^2 A + (\beta^2 - \gamma^2)B^2]^m = \sum_{k=0}^{m}\binom{m}{k}(\alpha^2 A)^k[(\beta^2 - \gamma^2)B^2]^{m-k} \\ &= \alpha^{2m}A + (\beta^2 - \gamma^2)^m B^2 \\ &= 3^m[3^m F_{n+3}^{2m} - F_{n+2}^m F_{n-1}^m]A + [3^m F_{n+2}^m F_{n-1}^m]I \end{aligned}$$

for $m$ a ***positive*** integer. Furthermore, for $m$ a ***nonnegative*** integer,

$$\begin{aligned} \Phi^{2m+1} &= (\alpha A + \beta B + \gamma C)[\alpha^{2m}A + [(\beta^2 - \gamma^2)^m B^2] \\ &= \alpha^{2m+1}A + (\beta^2 - \gamma^2)^m[\beta B + \gamma C] \\ &= 3^{2m+1}F_{n+3}^{2m+1}A + 3^{m+1/2}F_{n-1}^m F_{n+2}^m[F_{n+1}B + F_n C]. \end{aligned}$$

*Odd powers of the magic matrix $\Phi$ are always magic* as well, whereas even powers are only so if $\beta^2 = \gamma^2$. This completes the solution.

***Also solved by P. Bruckman.***

❖❖❖

## SUSTAINING MEMBERS

*H.L. Alder
G.L. Alexanderson
P. G. Anderson
S. Ando
R. Andre-Jeannin
A.T. Benjamin
J.G. Bergart
G. Bergum
*M. Bicknell-Johnson
W. Blumberg
H. Bottcher
P.S. Bruckman
D. Callan
G.D. Chakerian
H. Chen
C.K. Cook
C. Cooper
M.J. DeBruin
M.J. DeLeon
J. De Kerf
E. Deutsch
L.A.G. Dresel
U. Dudley
M. Elia
L.G. Ericksen, Jr.
D.R. Farmer
D.C. Fielder
C.T. Flynn
E. Frost
N. Gauthier
*H.W. Gould
P. Hagis, Jr.
V. Hanning
H. Harborth
*A.F. Horadam
Y. Horibe
F.T. Howard
R.J. Howell
R.E. Kennedy
C.H. Kimberling
Y.H.H. Kwong
J.C. Lagarias
J. Lahr
*C.T. Long
G. Lord
F. Luca
W.L. McDaniel
F.U. Mendizabal
M.G. Monzingo
J.F. Morrison
E. Nenzell
H. Niederhausen
S.A. Obaid
T.J. Osler
A. Prince
D. Redmond
C. Reiter
B.F. Robertson
B.M. Romanic
S. Sato
D.L. Schweizer
H.J Seiffert
A.G. Shannon
L.W. Shapiro
L. Somer
P. Spears
W.R. Spickerman
P.K. Stockmeyer
D.R. Stone
I. Strazdins
J. Suck
M.N.S. Swamy
*D. Thoro
J.C. Turner
N. Vallous
C. Vanden Eynden
T.P. Vaughan
J.N. Vitale
M.J. Wallace
J.E. Walton
W.A. Webb
V. Weber
R.E. Whitney

*Charter Members

## INSTITUTIONAL MEMBERS

BIBLIOTECA DEL SEMINARIO MATHEMATICO
*Padova, Italy*

CALIFORNIA STATE UNIVERSITY SACRAMENTO
*Sacramento, California*

FAHD PETRO MIN UNIVERSITY
*Dhahran, Saudi Arabia*

GONZAGA UNIVERSITY
*Spokane, Washington*

HOWELL ENGINEERING COMPANY
*Yucaipa, California*

ITHACA COLLEGE
*Ithaca, New York*

KLEPCO, INC.
*Reno, Nevada*

KOBENHAVNS UNIVERSITY
MATEMATISK INSTITUT
*Copenhagen, Denmark*

MISSOURI SOUTHERN STATE COLLEGE
*Joplin, Missouri*

SAN JOSE STATE UNIVERSITY
*San Jose, California*

SANTA CLARA UNIVERSITY
*Santa Clara, California*

UNIVERSITE DE MONTREAL
*Montreal, Quebec, Canada*

UNIVERSITY OF NEW ENGLAND
*Armidale, N.S.W. Australia*

WAKE FOREST UNIVERSITY
*Winston-Salem, North Carolina*

WASHINGTON STATE UNIVERSITY
*Pullman, Washington*

YESHIVA UNIVERSITY
*New York, New York*

# BOOKS AVAILABLE
## THROUGH THE FIBONACCI ASSOCIATION

*Introduction to Fibonacci Discovery* by Brother Alfred Brousseau, Fibonacci Association (FA), 1965. $18.00

*Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr. FA, 1972. $23.00

*A Primer for the Fibonacci Numbers.* Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972. $32.00

*Fibonacci's Problem Book,* Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974. $19.00

*The Theory of Simply Periodic Numerical Functions* by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969. $6.00

*Linear Recursion and Fibonacci Sequences* by Brother Alfred Brousseau. FA, 1971. $6.00

*Fibonacci and Related Number Theoretic Tables.* Edited by Brother Alfred Brousseau. FA, 1972. $30.00

*Number Theory Tables.* Edited by Brother Alfred Brousseau. FA, 1973. $39.00

*Tables of Fibonacci Entry Points, Part One.* Edited and annotated by Brother Alfred Brousseau. FA, 1965. $14.00

*Tables of Fibonacci Entry Points, Part Two.* Edited and annotated by Brother Alfred Brousseau. FA, 1965. $14.00

*A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume.* Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980. $38.00

*Applications of Fibonacci Numbers, Volumes 1-7.* Edited by G.E. Bergum, A.F. Horadam and A.N. Philippou. Contact Kluwer Academic Publishers for price.

*Applications of Fibonacci Numbers, Volume 8.* Edited by F.T. Howard. Contact Kluwer Academic Publishers for price.

*Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications* by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger. FA, 1993. $37.00

*Fibonacci Entry Points and Periods for Primes 100,003 through 415,993* by Daniel C. Fielder and Paul S. Bruckman. $20.00

**Shipping and handling charges will be $4.00 for each book in the United States and Canada. For Foreign orders, the shipping and handling charge will be $9.00 for each book.**

**Please write to the Fibonacci Association, P.O. Box 320, Aurora, S.D. 57002-0320, U.S.A., for more information.**