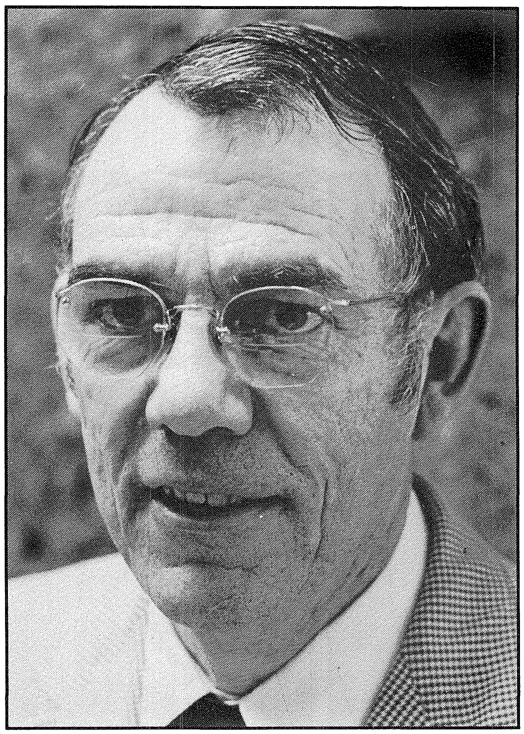
A Special Tribute to Calvin T. Long



Teacher • Researcher • Mentor
Leader • Friend

This edition of *The Fibonacci Quarterly* is being dedicated to Professor Calvin T. Long for his inspirations to his students, for his actions as a role model to his fellow Mathematicians, and for his extremely active participation in The Fibonacci Association as an author, referee and member of the Board of Directors.

Thank You!

A Short Biography of Calvin T. Long

Calvin T. Long was born in Rupert, Idaho, where he received his elementary and secondary education. Inspired by his teachers, he continued his education and was granted a B. S. degree from the University of Idaho in 1950, an M. S. degree from the University of Oregon in 1952 and a Ph.D. under the direction of Professor Ivan Niven, from the University of Oregon in 1955. After graduation, he spent one year as an analyst in Washington, D. C. working for the National Security Agency. In 1956, Cal accepted a position in the Mathematics Department at Washington State University (WSU) in Pullman, Washington, where he remained until his retirement in 1992.

It would be impossible to list all of Professor Long's accomplishments during his tenure at WSU so we will try to list only what we consider to be the most important ones. During the 1960s and early 1970s Cal served as director or associate director for several NSF-funded institutes for elementary, junior and high school mathematics teachers. This led to his deep interest in mathematics education. From 1970-78, he served as department chairman.

As a teacher, his students both at the graduate and undergraduate level respected him. He was a task-master but had a good sense of humor, sound scholarship and the ability to lead his students to their best efforts in an uncompromising way by insisting on excellence. For his efforts, Professor Long received the President's Faculty Excellence Award for Teaching in 1987 and was one of WSU's Case Award Nominees and Centennial Lecturers. He was also a visiting professor at three foreign and two American universities. During his career he directed 27 masters students and was the thesis advisor for five doctoral students.

As a researcher, he was the author or co-author of at least twenty-four grant proposals that funded programs or institutes related to mathematics education. He was the author of several books on number theory and mathematics education. He did extensive reviewing and refereeing of research papers and was an associate editor for two mathematics journals. He had more than 80-refereed publications and gave more than 150-invited lectures throughout the United States, Canada, Australia, New Zealand and Germany. He has also given at least 50 invited colloquium talks. Professor Long is a member of many honor societies, including Phi Beta Kappa. He is also an active member of many mathematical societies, including the Mathematical Association of America (MAA), the American Mathematical Society (AMS), the Fibonacci Association and the National Council of Teachers of Mathematics. He was elected Vice-Chairman, Chairman and Governor of the Northwest Section of the MAA. He served on numerous local, regional and national committees. For his dedication to his profession, he received the Certificate of Meritorious Service from the MAA in January of 1991.

As a mentor, he was always there for his fellow teachers as well as for his current and former students. As a leader, he was a state coordinator for the American High School Mathematics Examination, he was one of the organizers of the WSU Mathematics Honors Scholarship Competition program, and he was a consultant to the Washington State Superintendent of Public Instruction, to the State Department of Education and to the National Science Foundation.

Concerning The Fibonacci Association, Cal is a Charter Member. He served on the Board of Directors from July 6, 1983 to June 19, 1999 and he was the President for the last fifteen years. He was a strong supporter of the Fibonacci Research Conferences, attending most of them and presenting papers. Under his leadership, the organization became stronger and more unified.

On the unprofessional side, Cal is an avid fisherman and lover of the outdoors. It was not unusual to see him fly casting in the lakes and streams or walking the trails of the idyllic Idaho wilderness and sometimes you could even see him boating down the rapids of the Snake River. Cal also has a beautiful tenor voice, which he put to good use as a member of his church choir, a member of the Vandeleers, a well known University of Idaho choral group, a member of the Eugene Gleesmen, during his graduate years, a member of the Pullman/Moscow Chorale and a member of the Idaho-Washington Symphony Chorale. Cal was also a very dedicated husband whose strongest supporter was his wife Jean on whom he always knew he could count on because her support was always there. Finally, Cal was a devoted father to his two children, Tracy and Greg.

Cal, for all that you have done in so many ways for so many people, we say thank you. Enjoy retirement and know that you have made a difference to so many people who have crossed your path.

The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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THE BURGSTAHLER COINCIDENCE

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1. INTRODUCTION

Let $\tan x = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$. We know, of course, that $a_{2n} = 0$ for all n. Define a sequence A_n via

$$A_n = \sum_{k=0}^{n-1}\binom{n}{k}a_{k+1} = \sum_{2k<n}\binom{n}{2k}a_{2k+1}. \tag{1.1}$$

We have the following table:

n	1	2	3	4	5	6	7	8	9	10
A_n	1	1	2	3	5	8	$13\frac{2}{45}$	$21\frac{8}{45}$	$34\frac{167}{315}$	$56\frac{20}{63}$

At the 1999 MAA North Central Section Summer Seminar Sylvan Burgstahler posed the following question.

Question 1: Why is A_n approximately equal to a Fibonacci number?

In discussing this problem with the author, Dr. Burgstahler posed two more questions.

Question 2: If a_7 is changed from $\frac{17}{315}$ to $\frac{15}{315} = \frac{1}{21}$, then A_7 becomes 13 and A_8 becomes 21, but the new A_9 is $33\frac{314}{315}$ rather than 34. If we then change a_9 from $\frac{62}{2835}$ to $\frac{63}{2835} = \frac{1}{45}$, A_9 and A_{10} change to the appropriate Fibonacci numbers, but A_{11} remains incorrect. Does this pattern of obtaining two additional Fibonacci numbers for each correction persist?

More generally,

Question 3: Suppose that $f(x) = b_1x + b_3x^3 + b_5x^5 + \cdots$ is such that

$$F_n = \sum_{2k<n}\binom{n}{2k}b_{2k+1},$$

what can be said about the b's, and what can be said about $f(x)$?

In this paper we attempt to answer these questions. The first is straightforward, but the second and third are more interesting. The structure of this paper is as follows. In Section 2 we derive a formula for A_n that explains its proximity to the Fibonacci numbers. In Section 3 we recast this problem as a summation inversion problem to answer Question 2 and part of Question 3. We address the rest of Question 3 in Section 4. Throughout this paper we use the convention that $F_0 = 0$, $F_1 = 1$, α is the golden ratio,

$$\alpha = \frac{1+\sqrt{5}}{2} \quad\text{and}\quad \beta = \frac{1-\sqrt{5}}{2}.$$

We will make free use of the usual facts, e.g., $\alpha + \beta = 1$, $\alpha\beta = -1$.

2. A FORMULA FOR THE NUMBERS A_n

It is well known ([1], formula 4, p. 51) that the coefficients of $\tan x$ can be written explicitly in terms of Bernoulli numbers:

$$a_{2n-1} = (-1)^{n-1}\frac{2^{2n}(2^{2n}-1)}{(2n)!}B_{2n}, \tag{2.1}$$

where B_{2n} is the $2n^{\text{th}}$ Bernoulli number. The Bernoulli numbers are defined by the generating function ([1], formula 1, p. 35)

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \tag{2.2}$$

and have values 1, $-\frac{1}{2}$, $\frac{1}{6}$, 0, $-\frac{1}{30}$, 0, $\frac{1}{42}$, 0, $-\frac{1}{30}$, 0, $\frac{5}{66}$, ... They satisfy many identities including the recurrence ([1], formula 18, p. 38)

$$\sum_{k=0}^{n-1}\binom{n}{k}B_k = 0,$$

and series formulas

$$B_{2n} = (-1)^{n-1}\frac{(2n)!}{2^{2n-1}\pi^{2n}}\left(1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\cdots\right), \tag{2.3}$$

$$B_{2n} = (-1)^{n-1}\frac{2(2n)!}{(2^{2n}-1)\pi^{2n}}\left(1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\cdots\right). \tag{2.4}$$

These last two formulas can be found in most books of mathematical tables. Alternatively, (2.3) can be found in [1] (formula 22, p. 38) or in [2] (Vol. II, formula 2.60, p. 60). It is easy to derive (2.4) from (2.3).

Using (2.1) and (2.4) with (1.1), we have

$$\begin{aligned}
A_n &= \sum_{2k<n}\binom{n}{2k}a_{2k+1} = \sum_{2k<n}\binom{n}{2k}(-1)^k\frac{2^{2k+2}(2^{2k+1}-1)}{(2k+2)!}B_{2k+2} \\
&= \sum_{2k<n}\binom{n}{2k}(-1)^k\frac{2^{2k+2}(2^{2k+1}-1)}{(2k+2)!}(-1)^k\frac{2(2k+2)!}{(2^{2k+2}-1)\pi^{2k+2}}\left(1+\frac{1}{3^{2k+2}}+\frac{1}{5^{2k+2}}+\cdots\right) \\
&= \sum_{2k<n}\binom{n}{2k}\frac{2^{2k+3}}{\pi^{2k+2}}\left(1+\frac{1}{3^{2k+2}}+\frac{1}{5^{2k+2}}+\cdots\right),
\end{aligned}$$

so

$$A_n = \frac{8}{\pi^2}\sum_{2k<n}\sum_{j=0}^{\infty}\binom{n}{2k}\frac{2^{2k}}{(2j+1)^{2k+2}\pi^{2k}}. \tag{2.5}$$

Now consider the function

$$f_n(x) = \sum_{2k<n}\binom{n}{2k}x^{2k}. \tag{2.6}$$

It is easy to see that

$$f_n(x) = \frac{(1+x)^n+(1-x)^n-(1+(-1)^n)x^n}{2}. \tag{2.7}$$

Interchanging the order of summation in (2.5), we have

$$A_n = \frac{4}{\pi^2}\sum_{j=0}^{\infty}\frac{2}{(2j+1)^2}f_n\left(\frac{2}{(2j+1)\pi}\right),$$

or

$$A_n = \frac{4}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\left[\left(1+\frac{2}{(2j+1)\pi}\right)^n+\left(1-\frac{2}{(2j+1)\pi}\right)^n-(1+(-1)^n)\left(\frac{2}{(2j+1)\pi}\right)^n\right]. \qquad (2.8)$$

For example,

$$A_1 = \frac{4}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\left[\left(1+\frac{2}{(2j+1)\pi}\right)+\left(1-\frac{2}{(2j+1)\pi}\right)\right]$$
$$= \frac{8}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)=\frac{8}{\pi^2}\frac{\pi^2}{8}=1,$$

and

$$A_2 = \frac{4}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\left[\left(1+\frac{2}{(2j+1)\pi}\right)^2+\left(1-\frac{2}{(2j+1)\pi}\right)^2-\left(\frac{2}{(2j+1)\pi}\right)^2\right]$$
$$= \frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}=1.$$

With formula (2.8) for A_n, the main term is where $j=0$. This gives

$$A_n \cong \frac{4}{\pi^2}\left[\left(1+\frac{2}{\pi}\right)^n+\left(1-\frac{2}{\pi}\right)^n-(1+(-1)^n)\left(\frac{2}{\pi}\right)^n\right]\cong\frac{4}{\pi^2}\left(1+\frac{2}{\pi}\right)^n.$$

For example, letting

$$C_n = \frac{4}{\pi^2}\left(1+\frac{2}{\pi}\right)^n,$$

consider the expanded table

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55
A_n	1	1	2	3	5	8	13.04	21.18	34.53	56.32
C_n	.66	1.09	1.8	2.91	4.76	7.79	12.75	20.86	34.14	55.88

Finally,

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n-\beta^n)\cong\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \cong .447(1.618)^n,$$

whereas

$$A_n \cong \frac{4}{\pi^2}\left(1+\frac{2}{\pi}\right)^n \cong .405(1.637)^n.$$

Thus, $A_n/F_n \cong .906(1.0115)^n$. Hence, for small n, $A_n \cong F_n$, although the A's grow exponentially faster than F_n in the long run.

3. THE BURGSTAHLER PROBLEM AS AN INVERSION PROBLEM

The real problem considered in this paper is the following: Find the sequence b_{2n+1} given that

$$F_n = \sum_{2k<n} \binom{n}{2k} b_{2k+1}.$$

This can be cast as a sum inversion problem: Given a known sequence $\{a_n\}_{n=0}^{\infty}$, suppose a new sequence is defined by

$$a_n = \sum_k c_{n,k} b_k,$$

for some given set of constants $c_{n,k}$, what can be said about the b's in terms of the a's? It must be pointed out that such a sequence of b's need not always exist. For example, if we attempt to define a sequence b_{2n+1} by

$$F_n = \sum_{2k \le n} \binom{n}{2k} b_{2k+1},$$

we find that there is no solution: $F_1 = b_1$, $F_2 = b_1 + b_3$, $F_3 = b_1 + 3b_3$ is an inconsistent system of three equations and two unknowns. Similarly, if we attempt to solve the system

$$n = \sum_{2k<n} \binom{n}{2k} b_{2k+1}$$

rather than the given one, we obtain $1 = b_1$, $2 = b_1$, and again there is no solution. In order to even ask Question 3 in the Introduction, we need

$$F_n = \sum_{2k<n} \binom{n}{2k} b_{2k+1}$$

to define a consistent system. In fact, as we will see, a proof that this system is consistent will give an affirmative answer to Question 2.

Here is a standard technique (see [2], Vol. I, pp. 437, 438, or [3], formula 2.1.2, p. 28) for solving a class of inversion problems: Suppose that

$$a_n = \sum_k c_{n,k} b_n,$$

where $c_{n,k}$ depends on only $n-k$, say $c_{n,k} = c_{n-k}$. In this case a_n is a convolution of b_n and c_n. Thus, passing to generating functions, with

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,\ B(x) = \sum_{n=0}^{\infty} b_n x^n,\ C(x) = \sum_{n=0}^{\infty} c_n x^n,$$

we have $A(x) = B(x)C(x)$. Hence, $B(x) = C(x)^{-1}A(x)$.

We use this technique to solve the inversion problem

$$a_n = \sum_{k=0}^{n-1} \binom{n}{k} b_k. \tag{3.1}$$

This expression only makes sense for $n \ge 1$; we extend it by setting $a_0 = 0$. Dividing each side by $n!$ gives

$$\frac{a_n}{n!} = \sum_{k=0}^{n-1} \frac{1}{(n-k)!} \frac{b_k}{k!}.$$

Here, the $n-1$ in the upper limit introduces a complication. The c_n in the convolution is

$$c_n = \begin{cases} 0, & n = 0, \\ \frac{1}{n!}, & n \geq 1. \end{cases}$$

In this case, $C(x) = e^x - 1$. Using exponential generating functions for a_n and b_n,

$$A(x) = B(x)(e^x - 1),$$

so

$$B(x) = \frac{1}{e^x - 1} A(x). \tag{3.2}$$

Since $A(0) = 0$, we can write this as

$$B(x) = \frac{1}{e^x - 1} \frac{1}{x} A(x).$$

Thus, the b's will be a convolution of Bernoulli numbers with the a's. In particular, we have

Theorem 3.1: Suppose that sequences $\{a_n\}$ and $\{b_n\}$ are defined by

$$a_n = \sum_{k=0}^{n-1} \binom{n}{k} b_k.$$

Then

$$b_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} B_{n-k} a_{k+1} = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k a_{n-k+1}.$$

We next consider the specific case where $a_n = F_n$. In this case, $A(x)$, the exponential generating function for the Fibonacci numbers is

$$A(x) = \frac{1}{\sqrt{5}}(e^{\alpha x} - e^{\beta x}).$$

Thus, we have

$$B(x) = \frac{1}{e^x - 1} \frac{1}{\sqrt{5}}(e^{\alpha x} - e^{\beta x}) = \frac{1}{\sqrt{5}} \frac{\sinh(\sqrt{5}x/2)}{\sinh x/2}. \tag{3.3}$$

Since $B(x)$ is an even function, all the odd terms are zero.

Theorem 3.2: If the sequence $\{c_n\}$ is defined by

$$F_n = \sum_{k=0}^{n-1} \binom{n}{k} c_k,$$

then $c_{2n+1} = 0$ for all n. Consequently,

$$F_n = \sum_{2k<n} \binom{n}{2k} c_{2k}. \tag{3.4}$$

Moreover,

$$F_n = \sum_{2k<n} \binom{n}{2k} x_k \tag{3.5}$$

has as its unique solution, $x_n = c_{2n}$ for all n.

Proof: The remarks preceding the theorem show that $c_{2n+1} = 0$ for all n, which gives us (3.4). As a consequence, we know that the system in (3.5) has a solution of the form $x_n = c_{2n}$. That this is the only solution follows by induction on n, using

$$F_{2n+1} = \sum_{2k<2n+1} \binom{2n+1}{2k} x_k = \sum_{k=0}^{n} \binom{2n+1}{2k} x_k,$$

or

$$x_n = \frac{1}{2n+1}\left(F_{2n+1} - \sum_{k=0}^{n-1} \binom{2n+1}{2k} x_k\right).$$

Corollary 3.3: The two systems of equations

$$F_{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{2k} x_k \tag{3.6}$$

and

$$F_{2n+2} = \sum_{k=0}^{n} \binom{2n+2}{2k} x_k \tag{3.7}$$

each have the same solution $x_n = c_{2n}$ for all n.

Proof: Again, a solution $x_n = c_{2n}$ exists to each of these systems and, by induction, each has a unique solution.

Dr. Burgstahler's numbers b_{2n+1} are now just c_{2n} above. Combining Theorems 3.1 and 3.2, we have

Theorem 3.4: The system of equations

$$F_n = \sum_{2k<n} \binom{n}{2k} b_{2k+1}$$

is consistent and has a unique solution

$$b_{2n+1} = \frac{1}{2n+1} \sum_{k=0}^{2n} \binom{2n+1}{k} B_k F_{2n-k+1}.$$

We are now in a position to answer Dr. Burgstahler's second question: as coefficients in $\tan x$ are changed one by one to the b_{2n+1}, each change corrects two terms to Fibonacci numbers. This is because of Corollary 3.3, which indicates that both F_{2n+1} and F_{2n+2} can be expressed as sums involving $b_1, b_3, \ldots, b_{2n+1}$.

4. CONCLUDING REMARKS

We have not yet given a complete answer to Question 3. While we have given a formula for the terms of the sequence $\{b_{2n+1}\}$, we have not said anything about the function $f(x) = b_1 x + b_3 x^2 + b_5 x^5 + \cdots$.

Theorem 4.1: The power series $\sum_{n=0}^{\infty} b_{2n+1}x^{2n+1}$ has a radius of convergence of 0.

Sketch of Proof: Suppose, by way of contradiction, that this is not the case. That is, suppose that $\sum_{n=0}^{\infty} b_{2n+1}x^{2n+1}$ converges to a function $f(x)$, at least for $|x| < C$ for some constant $C > 0$. Then it may be shown that $f(x)$ satisfies the functional equation

$$f(x) = \frac{x^2}{1+x-x^2} + f\left(\frac{x}{1+x}\right), \tag{4.1}$$

in some neighborhood of the origin. Since $1+x-x^2 = 0$ at $x = \alpha$, $x = \beta$, this region must be a subset of the interval (β, α). However, given a function f satisfying (4.1), if $x = a$ is a pole for f, then so is $\frac{x}{1-x} = a$ or $x = \frac{a}{1-a}$. Iterating this, f has a pole at each of the values $x = \frac{a}{1-ka}$, if it has a pole at a. In particular, for $a = \beta$, this gives an increasing sequence of poles with 0 as its limit. As no convergent power series about the origin can have this property, we have a contradiction.

Thus, the first part of Question 3 was slightly naive—there was no guarantee that such a function $f(x)$ even existed; in fact, one does not. However, it was only by following the generating function approach above, and noting the problem of the poles that the author discovered this fact.

One may ask about an exponential generating function for the sequence $\{b_{2n+1}\}$ rather than the ordinary generating function, of course. As a consequence of formula (3.3), this exponential generating function is

$$\int_0^x \frac{t(e^{\alpha t} - e^{\beta t})}{\sqrt{5}(e^t - 1)}\,dt \quad \text{or} \quad \int_0^x \frac{t \sinh(\sqrt{5}t/2)}{\sqrt{5}\sinh t/2}\,dt.$$

The integral is needed to correct the index from c_{2n} to b_{2n+1}.

It is reasonable to ask when the system of equations

$$a_n = \sum_{2k<n} \binom{n}{2k} x_k \tag{4.2}$$

is consistent. We have the following result.

Theorem 4.2: The system in (4.2) is consistent if and only if the solution to the system

$$a_n = \sum_{k=0}^{n-1} \binom{n}{k} y_k \tag{4.3}$$

satisfies the condition $y_{2n+1} = 0$ for all n. In this case, the solution to (4.2) is given by $x_n = y_{2n}$ for all n.

Proof: If the solution to (4.3) satisfies the condition that $y_{2n+1} = 0$ for all n, then we obtain existence and uniqueness for solutions to (4.2) in exactly the same way as in Theorem 3.2. For the other direction, we assume that (4.2) has a solution and proceed in induction on n to show that in the solution to (4.3) all y_{2n+1} are 0 and that $y_{2n} = x_n$ for all n. To begin the induction, the equations $a_1 = x_0$ and $a_2 = x_0$ show that to be consistent, we need $a_1 = a_2$. In this case, $y_0 = a_1$, $y_0 + 2y_1 = a_2$ gives $y_1 = 0$. Moreover, since $x_0 = a_1$, we have that $x_0 = y_0$.

So, by way of induction, assume that, for $0 \le k \le n-1$, $y_{2k+1} = 0$ and $x_k = y_{2k}$. We have

$$a_{2n+1} = \sum_{2k<2n+1} \binom{2n+1}{2k} x_k = \sum_{k=0}^{n} \binom{2n+1}{2k} x_k,$$

and

$$a_{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{2k} y_k = \sum_{k=0}^{n} \binom{2n+1}{2k} y_{2k}.$$

Since $y_{2k} = x_k$ for all $k < n$, comparing these two expressions gives that $y_{2n} = x_n$. Now

$$a_{2n+2} = \sum_{2k<2n+2} \binom{2n+2}{2k} x_k = \sum_{k=0}^{n} \binom{2n+2}{2k} x_k$$

and

$$a_{2n+2} = \sum_{k=0}^{2n+1} \binom{2n+2}{k} y_k = \sum_{k=0}^{n} \binom{2n+2}{2k} y_{2k} + (2n+2) y_{2n+1}$$

force y_{2n+1} to be 0. This completes the proof.

We may now use generating function techniques to give more information.

Corollary 4.3: The system in (4.2) is consistent if and only if the exponential generating function $A(x)$ for $\{a_n\}$ satisfies the functional equation

$$A(x) = -e^x A(-x). \tag{4.4}$$

Proof: We may solve system (3.1) rather than (4.2). By formula (3.2), we have the relation

$$B(x) = \frac{1}{e^x - 1} A(x),$$

where $B(x)$ is the exponential generating function for the y_n. By the previous theorem, $B(x)$ must be an even function of x. Hence,

$$\frac{1}{e^{-x} - 1} A(-x) = \frac{1}{e^x - 1} A(x),$$

from which the functional equation follows.

The functional equation (4.4) does not place too heavy a restriction on sequences $\{a_n\}$. For example, if $f(x)$ is any odd function, then $\frac{2e^x}{e^x+1} f(x)$ will satisfy equation (4.4). We conclude with the following result.

Theorem 4.4: If the sequence $\{a_n\}$ satisfies a recurrence relation of the type $a_n = a_{n-1} + c a_{n-2}$, where c is an arbitrary constant and $a_0 = 0$, then system (4.2) is consistent.

Proof: The case where $c = 0$ is trivial; the solution to the recurrence relation being just the 0 sequence. Another special case is $c = \frac{-1}{4}$, in which case one may check that

$$a_n = n 2^{-n}, \quad A(x) = \frac{x}{2} e^{x/2}, \text{ and } B(x) = \frac{x}{2} \frac{e^{x/2}}{e^x - 1}.$$

In the cases where $c \neq 0$, $\frac{-1}{4}$, any solution satisfying $a_0 = 0$ will be of the form $a_n = C(u^n - v^n)$, where C is a constant, and $u + v = 1$ (u and v being the solutions to $x^2 - x - c = 0$). In this case, $A(x) = C(e^{ux} - e^{vx})$, so

$$-e^x A(-x) = -Ce^x(e^{-ux} - e^{-vx}) = C(e^{(1-v)x} - e^{1-u)x}) = C(e^{ux} - e^{vx}) = A(x),$$

so $A(x)$ satisfies the required functional equation, completing the proof.

As a very easy example, if $c = 2$, one may check that $a_n = 2^n - (-1)^n$ produces a consistent system for (4.2). In this case,

$$b_0 = 3 \quad \text{and} \quad b_n = \begin{cases} 0, & n \text{ odd}, \\ 2, & n \text{ even}, n > 0. \end{cases}$$

That this works can be independently checked using formulas (2.6) and (2.7).

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FACTORIZATIONS AND EIGENVALUES OF FIBONACCI AND SYMMETRIC FIBONACCI MATRICES

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1. INTRODUCTION

Matrix methods are a major tool in solving many problems stemming from linear recurrence relations. A matrix version of a linear recurrence relation on the Fibonacci sequence is well known as

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix}.$$

We let

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & F_1 \\ F_1 & F_2 \end{bmatrix},$$

then we can easily establish the following interesting property of Q by mathematical induction.

$$Q^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}.$$

From the equation $Q^{n+1}Q^n = Q^{2n+1}$, we get

$$\begin{bmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{bmatrix} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{2n+2} & F_{2n+1} \\ F_{2n+1} & F_{2n} \end{bmatrix},$$

which, upon tracing through the multiplication, yields an identity for each Fibonacci number on the right-hand side. For example, we have the elegant formula,

$$F_{n+1}^2 + F_n^2 = F_{2n+1}. \tag{1}$$

The sum of the squares of the first n Fibonacci numbers is almost as famous as the formula for the sum of the first n terms:

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}. \tag{2}$$

In particular, in [1], the authors gave several basic Fibonacci identities. For example,

$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{n-1}F_n = \frac{F_{2n-1} + F_nF_{n-1} - 1}{2}. \tag{3}$$

Now, we define a new matrix. The $n \times n$ *Fibonacci matrix* $\mathscr{F}_n = [f_{ij}]$ is defined as

$$\mathscr{F}_n = [f_{ij}] = \begin{cases} F_{i-j+1}, & i-j+1 \geq 0, \\ 0, & i-j+1 < 0. \end{cases}$$

For example,

$$\mathcal{F}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 \\ 5 & 3 & 2 & 1 & 1 \end{bmatrix},$$

and the first column of $\mathcal{F}_5$ is the vector $(1, 1, 2, 3, 5)^T$. Thus, several interesting facts can be found from the matrix $\mathcal{F}_n$.

The set of all n-square matrices is denoted by M_n. Any matrix $B \in M_n$ of the form $B = A^*A$, $A \in M_n$, may be written as $B = LL^*$, where $L \in M_n$ is a lower triangular matrix with nonnegative diagonal entries. This factorization is unique if A is nonsingular. This is called the *Cholesky factorization* of B. In particular, a matrix B is positive definite if and only if there exists a nonsingular lower triangular matrix $L \in M_n$ with positive diagonal entries such that $B = LL^*$. If B is a real matrix, L may be taken to be real.

A matrix $A \in M_n$ of the form

$$A = \begin{bmatrix} A_{11} & 0 & & \\ 0 & A_{22} & & 0 \\ & & \ddots & \\ & 0 & & A_{kk} \end{bmatrix}$$

in which $A_{ii} \in M_{n_i}$, $i = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} n_i = n$, is called *block diagonal*. Notationally, such a matrix is often indicated as $A = A_{11} \oplus A_{22} \oplus \cdots \oplus A_{kk}$ or, more briefly, $\oplus \sum_{i=1}^{k} A_{ii}$; this is called the *direct sum* of the matrices $A_{11}, \ldots, A_{kk}$.

2. FACTORIZATIONS

In [2], the authors gave the Cholesky factorization of the Pascal matrix. In this section we consider the construction and factorization of our Fibonacci matrix of order n by using the $(0, 1)$-matrix, where a matrix is said to be a $(0, 1)$-matrix if each of its entries is either 0 or 1.

Let I_n be the identity matrix of order n. Further, we define the $n \times n$ matrices S_n, $\overline{\mathcal{F}}_n$, and G_k by

$$S_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad S_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and $S_k = S_0 \oplus I_k$, $k = 1, 2, \ldots$, $\overline{\mathcal{F}}_n = [1] \oplus \mathcal{F}_{n-1}$, $G_1 = I_n$, $G_2 = I_{n-3} \oplus S_{-1}$, and, for $k \geq 3$, $G_k = I_{n-k} \oplus S_{k-3}$. Then we have the following lemma.

Lemma 2.1: $\overline{\mathcal{F}}_k S_{k-3} = \mathcal{F}_k$, $k \geq 3$.

Proof: For $k = 3$, we have $\overline{\mathcal{F}}_3 S_0 = \mathcal{F}_3$. Let $k > 3$. From the definition of the matrix product and the familiar Fibonacci sequence, the conclusion follows. □

From the definition of G_k, we know that $G_n = S_{n-3}$, $G_1 = I_n$, and $I_{n-3} \oplus S_{-1}$. The following theorem is an immediate consequence of Lemma 2.1.

Theorem 2.2: The Fibonacci matrix $\mathcal{F}_n$ can be factored by the G_k's as follows: $\mathcal{F}_n = G_1 G_2 \cdots G_n$. For example,

$$\mathcal{F}_5 = G_1G_2G_3G_4G_5 = I_5(I_2 \oplus S_{-1})(I_2 \oplus S_0)([1] \oplus S_1)S_2$$

$$= \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&1&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&1&1&0\\0&0&1&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&1&1&0&0\\0&1&0&1&0\\0&0&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}$$

$$= \begin{bmatrix} 1&0&0&0&0\\1&1&0&0&0\\2&1&1&0&0\\3&2&1&1&0\\5&3&2&1&1 \end{bmatrix}.$$

Now we consider another factorization of $\mathcal{F}_n$. The $n \times n$ matrix $C_n = [c_{ij}]$ is defined as

$$c_{ij} = \begin{cases} F_i, & j=1, \\ 1, & i=j, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{i.e., } C_n = \begin{bmatrix} F_1 & 0 & \cdots & 0 \\ F_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_n & 0 & \cdots & 1 \end{bmatrix}.$$

The next theorem follows by a simple calculation.

Theorem 2.3: For $n \ge 2$, $\mathcal{F}_n = C_n(I_1 \oplus C_{n-1})(I_2 \oplus C_{n-2}) \cdots (I_{n-2} \oplus C_2)$.

Also, we can easily find the inverse of the Fibonacci matrix $\mathcal{F}_n$. We know that

$$S_0^{-1} = \begin{bmatrix} 1&0&0\\-1&1&0\\-1&0&1 \end{bmatrix}, \quad S_{-1}^{-1} = \begin{bmatrix} 1&0&0\\0&1&0\\0&-1&1 \end{bmatrix}, \quad \text{and} \quad S_k^{-1} = S_0^{-1} \oplus I_k.$$

Define $H_k = G_k^{-1}$. Then

$$H_1 = G_1^{-1} = I_n, \quad H_2 = G_2^{-1} = I_{n-3} \oplus S_{-1}^{-1} = I_{n-2} \oplus \begin{bmatrix} 1&0\\-1&1 \end{bmatrix}, \quad \text{and} \quad H_n = S_{n-3}^{-1}.$$

Also, we know that

$$C_n^{-1} = \begin{bmatrix} F_1 & 0 & \cdots & 0 \\ -F_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -F_n & 0 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad (I_k \oplus C_{n-k})^{-1} = I_k \oplus C_{n-k}^{-1}.$$

So the following corollary holds.

Corollary 2.4: $\mathcal{F}_n^{-1} = G_n^{-1}G_{n-1}^{-1} \cdots G_2^{-1}G_1^{-1} = H_nH_{n-1} \cdots H_2H_1 = (I_{n-2} \oplus C_2)^{-1} \cdots (I_1 \oplus C_{n-1})^{-1}C_n^{-1}$.

From Corollary 2.4, we have

$$\mathcal{F}_n^{-1} = \begin{bmatrix} 1&0&0&0&\cdots&0\\-1&1&0&0&\cdots&0\\-1&-1&1&0&\cdots&0\\0&-1&-1&1&\cdots&0\\ \vdots&\vdots&\ddots&\ddots&\ddots&\vdots\\0&\cdots&0&-1&-1&1 \end{bmatrix}. \tag{4}$$

Now we define a *symmetric Fibonacci matrix* $\mathcal{Q}_n = [q_{ij}]$ as, for $i, j = 1, 2, \ldots, n$,

$$q_{ij} = q_{ji} = \begin{cases} \sum_{k=1}^{i} F_k^2, & i = j, \\ q_{i,j-2} + q_{i,j-1}, & i+1 \le j, \end{cases}$$

where $q_{1,0} = 0$. Then we have $q_{1j} = q_{j1} = F_j$ and $q_{2j} = q_{j2} = F_{j+1}$. For example,

$$\mathcal{Q}_{10} = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 \\ 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 \\ 2 & 3 & 6 & 9 & 15 & 24 & 39 & 63 & 102 & 165 \\ 3 & 5 & 9 & 15 & 24 & 39 & 63 & 102 & 165 & 267 \\ 5 & 8 & 15 & 24 & 40 & 64 & 104 & 168 & 272 & 440 \\ 8 & 13 & 24 & 39 & 64 & 104 & 168 & 272 & 440 & 712 \\ 13 & 21 & 39 & 63 & 104 & 168 & 273 & 441 & 714 & 1155 \\ 21 & 34 & 63 & 102 & 168 & 272 & 441 & 714 & 1155 & 1869 \\ 34 & 55 & 102 & 165 & 272 & 440 & 714 & 1155 & 1879 & 3025 \\ 55 & 89 & 165 & 267 & 440 & 712 & 1155 & 1869 & 3025 & 4895 \end{bmatrix}.$$

From the definition of $\mathcal{Q}_n$, we derive the following lemma.

Lemma 2.5: For $j \ge 3$, $q_{3j} = F_4(F_{j-3} + F_{j-2}F_3)$.

Proof: We know that $q_{3,3} = F_1^2 + F_2^2 + F_3^2 = F_3F_4$; hence, $q_{3,3} = F_4F_3 = F_4(F_0 + F_1F_3)$ for $F_0 = 0$. By induction, $q_{3j} = F_4(F_{j-3} + F_{j-2}F_3)$. □

We know that $q_{3,1} = q_{1,3} = F_3$ and $q_{3,2} = q_{2,3} = F_4$. Also we see that $q_{4,1} = q_{1,4}$, $q_{4,2} = q_{2,4}$, and $q_{4,3} = q_{3,4}$. By induction, we have the following lemma.

Lemma 2.6: For $j \ge 4$, $q_{4j} = F_4(F_{j-4} + F_{j-4}F_3 + F_{j-3}F_5)$.

From Lemmas 2.5 and 2.6, we know $q_{5,1}$, $q_{5,2}$, $q_{5,3}$, and $q_{5,4}$. From these facts and the definition of $\mathcal{Q}_n$, we have the following lemma.

Lemma 2.7: For $j \ge 5$, $q_{5j} = F_{j-5}F_4(1 + F_3 + F_5) + F_{j-4}F_5F_6$.

Proof: Since $q_{5,5} = F_5F_6$ we have, by induction, $q_{5j} = F_{j-5}F_4(1 + F_3 + F_5) + F_{j-4}F_5F_6$. □

From the definition of $\mathcal{Q}_n$ together with Lemmas 2.5, 2.6, and 2.7, we have the following lemma by induction on i.

Lemma 2.8: For $j \ge i \ge 6$,

$$q_{ij} = F_{j-i}F_4(1 + F_3 + F_5) + F_{j-i}F_5F_6 + F_{j-i}F_6F_7 + \cdots + F_{j-i}F_{i-1}F_i + F_{j-i+1}F_iF_{i+1}.$$

Now we have the following theorem.

Theorem 2.9: For $n \ge 1$ a positive integer, $H_nH_{n-1}\cdots H_2H_1\mathcal{Q}_n = \mathcal{F}_n^T$ and the Cholesky factorization of $\mathcal{Q}_n$ is given by $\mathcal{Q}_n = \mathcal{F}_n\mathcal{F}_n^T$.

Proof: By Corollary 2.4, $H_nH_{n-1}\cdots H_2H_1 = \mathcal{F}_n^{-1}$. So, if we have $\mathcal{F}_n^{-1}\mathcal{Q}_n = \mathcal{F}_n^T$, then the theorem holds.

Let $X = [x_{ij}] = \mathcal{F}_n^{-1}\mathcal{Q}_n$. Then, by (4), we have the following:

$$x_{ij} = \begin{cases} F_j, & \text{if } i = 1, \\ F_{j-1}, & \text{if } i = 2, \\ -q_{i-2,j} - q_{i-1,j} + q_{ij} & \text{otherwise.} \end{cases}$$

Now we consider the case $i \geq 3$. Since $\mathcal{Q}_n$ is a symmetric matrix, $-q_{i-2,j} - q_{i-1,j} + q_{ij} = -q_{j,i-2} - q_{j,i-1} + q_{ji}$. Hence, by the definition of $\mathcal{Q}_n$, $x_{ij} = 0$ for $j+1 \leq i$. So, we will prove that $-q_{i-2,j} - q_{i-1,j} + q_{ij} = F_{j-i+1}$ for $j \geq i$.

In the case in which $i \leq 5$, we have $x_{ij} = F_{j-i+1}$ by Lemmas 2.5, 2.4, and 2.7.

Now suppose that $j \geq i \geq 6$. Then, by Lemma 2.8, we have

$$\begin{aligned} x_{ij} &= -q_{i-2,j} - q_{i-1,j} + q_{ij} \\ &= (F_{j-i} - F_{j-i+1} - F_{j-i+2})F_4(1 + F_3 + F_5) + (F_{j-i} - F_{j-i+1} - F_{j-i+2})F_5F_6 \\ &\quad + \cdots + (F_{j-i} - F_{j-i+1} - F_{j-i+2})F_{i-3}F_{i-2} + (F_{j-i} - F_{j-i+1} - F_{j-i+3})F_{i-2}F_{i-1} \\ &\quad + (F_{j-i} - F_{j-i+2})F_{i-1}F_i + F_{j-i+1}F_iF_{i+1}. \end{aligned}$$

Since $F_{j-i} - F_{j-i+1} - F_{j-i+2} = -2F_{j-i+1}$, $F_{j-i} - F_{j-i+1} - F_{j-i+3} = -3F_{j-i+1}$, and $F_{j-i} - F_{j-i+2} = -F_{j-i+1}$, we have

$$x_{ij} = F_{j-i+1}[-2F_4 - 2(F_3F_4 + F_4F_5 + \cdots + F_{i-2}F_{i-1}) - F_{i-2}F_{i-1} - F_{i-1}F_i + F_iF_{i+1}].$$

Since $F_4 = 3$, using (3) we have

$$x_{ij} = \left[-6 - 2\left(\frac{F_{2(i-1)-1} + F_{i-1}F_{(i-1)-1} - 1}{2} - F_1F_2 - F_2F_3\right) - F_{i-2}F_{i-1} - F_{i-1}F_i + F_iF_{i+1}\right]F_{j-i+1}.$$

Since $F_{i+1} = F_i + F_{i-1}$ and by (1) we have

$$\begin{aligned} x_{ij} &= (1 - 2F_{i-1}F_{i-2} - F_{2i-3} - F_{i-1}F_i + F_iF_{i+1})F_{j-i+1} \\ &= (1 - 2F_{i-1}F_{i-2} - F_{2i-3} + F_i^2)F_{j-i+1} \\ &= (1 - F_{i-1}^2 - F_{i-2}^2 - 2F_{i-1}F_{i-2} + F_i^2)F_{j-i+1} \\ &= (1 - (F_{i-1} + F_{i-1})^2 + F_i^2)F_{j-i+1} \\ &= (1 - F_i^2 + F_i^2)F_{j-i+1} = F_{j-i+1}. \end{aligned}$$

Therefore, $\mathcal{F}_n^{-1}\mathcal{Q}_n = \mathcal{F}_n^T$, i.e., the Cholesky factorization of $\mathcal{Q}_n$ is given by $\mathcal{Q}_n = \mathcal{F}_n\mathcal{F}_n^T$. □

In particular, since $\mathcal{Q}_n^{-1} = (\mathcal{F}_n^T)^{-1}\mathcal{F}_n^{-1} = (\mathcal{F}_n^{-1})^T\mathcal{F}_n^{-1}$, we have

$$\mathcal{Q}_n^{-1} = \begin{bmatrix} 3 & 0 & -1 & 0 & \cdots & & & 0 \\ 0 & 3 & 0 & -1 & \cdots & & & 0 \\ -1 & 0 & 3 & 0 & \cdots & & & 0 \\ 0 & -1 & 0 & 3 & \ddots & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & -1 & 1 \end{bmatrix}. \tag{5}$$

From Theorem 2.9, we have the following corollary.

Corollary 2.10: If k is an odd number, then

$$F_nF_{n-k}+\cdots+F_{k+1}F_1=\begin{cases}F_nF_{n-(k-1)}-F_k & \text{if } n \text{ is odd,}\\ F_nF_{n-(k-1)} & \text{if } n \text{ is even.}\end{cases}$$

If k is an even number, then

$$F_nF_{n-k}+\cdots+F_{k+1}F_1=\begin{cases}F_nF_{n-(k-1)} & \text{if } n \text{ is odd,}\\ F_nF_{n-(k-1)}-F_k & \text{if } n \text{ is even.}\end{cases}$$

For the case when we multiply the i^{th} row of $\mathcal{F}_n$ and the i^{th} column of $\mathcal{F}_n$, we have the famous formula (2). Also, formula (2) is the case when $k=0$ in Corollary 2.10.

3. EIGENVALUES OF $\mathcal{Q}_n$

In this section, we consider the eigenvalues of $\mathcal{Q}_n$.

Let $\mathcal{D}=\{\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n : x_1\geq x_2\geq\cdots\geq x_n\}$. For $\mathbf{x},\mathbf{y}\in\mathcal{D}$, $\mathbf{x}\prec\mathbf{y}$ if $\sum_{i=1}^k x_i\leq\sum_{i=1}^k y_i$, $k=1,2,\ldots,n$ and if $k=n$, then the equality holds. When $\mathbf{x}\prec\mathbf{y}$, $\mathbf{x}$ is said to be *majorized* by $\mathbf{y}$, or $\mathbf{y}$ is said to *majorize* $\mathbf{x}$. The condition for majorization can be rewritten as follows: for $\mathbf{x},\mathbf{y}\in\mathcal{D}$, $\mathbf{x}\prec\mathbf{y}$ if $\sum_{i=0}^k x_{n-i}\geq\sum_{i=0}^k y_{n-i}$, $k=0,1,\ldots,n-2$, and if $k=n-1$, then equality holds.

The following is an interesting simple fact:

$$(\bar{x},\ldots,\bar{x})\prec(x_1,\ldots,x_n),\text{ where } \bar{x}=\frac{\sum_{n=1}^n x_i}{n}.$$

More interesting facts about majorizations can be found in [4].

An $n\times n$ matrix $P=[p_{ij}]$ is *doubly stochastic* if $p_{ij}\geq 0$ for $i,j=1,2,\ldots,n$, $\sum_{i=1}^n p_{ij}=1$, $j=1,2,\ldots,n$, and $\sum_{j=1}^n p_{ij}=1$, $i=1,2,\ldots,n$. In 1929, Hardy, Littlewood, and Polya proved that a necessary and sufficient condition that $\mathbf{x}\prec\mathbf{y}$ is that there exist a doubly stochastic matrix P such that $\mathbf{x}=\mathbf{y}P$.

We know both the eigenvalues and the main diagonal elements of a real symmetrix matrix are real numbers. The precise relationship between the main diagonal elements and the eigenvalues is given by the notion of majorization as follows: the vector of eigenvalues of a symmetrix matrix is majorized by the diagonal elements of the matrix.

Note that $\det\mathcal{F}_n=1$ and $\det\mathcal{Q}_n=1$. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues of $\mathcal{Q}_n$. Since $\mathcal{Q}_n=\mathcal{F}_n\mathcal{F}_n^T$ and $\sum_{i=1}^k F_i^2=F_{k+1}F_k$, the eigenvalues of $\mathcal{Q}_n$ are all positive and

$$(F_{n+1}F_n,F_nF_{n-1},\ldots,F_2F_1)\prec(\lambda_1,\lambda_2,\ldots,\lambda_n).$$

In [1], we find the interesting combinatorial property, $\sum_{i=0}^n\binom{n-i}{i}=F_{n+1}$. So we have the following corollaries.

Corollary 3.1: Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues of $\mathcal{Q}_n$. Then

$$\lambda_1+\lambda_2+\cdots+\lambda_n=\begin{cases}\left(\sum_{i=0}^n\binom{n-i}{i}\right)^2-1 & \text{if } n \text{ is odd,}\\ \left(\sum_{i=0}^n\binom{n-i}{i}\right)^2 & \text{if } n \text{ is even.}\end{cases}$$

Proof: Since $(F_{n+1}F_n, F_nF_{n-1}, \ldots, F_2F_1) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$, and from Corollary 2.10,

$$\lambda_1+\lambda_2+\cdots+\lambda_n = \begin{cases}(F_{n+1})^2-F_1 & \text{if } n \text{ is odd,}\\ (F_{n+1})^2 & \text{if n is even,}\end{cases} = \begin{cases}\left(\sum_{i=0}^n\binom{n-i}{i}\right)^2-1 & \text{if } n \text{ is odd,}\\ \left(\sum_{i=0}^n\binom{n-i}{i}\right)^2 & \text{if } n \text{ is even.}\end{cases} \quad \square$$

Corollary 3.2: If n is an odd number, then

$$n\lambda_n \le \left(\sum_{i=0}^n\binom{n-i}{i}\right)^2-1 \le n\lambda_1.$$

If n is an even number, then

$$n\lambda_n \le \left(\sum_{i=0}^n\binom{n-i}{i}\right)^2 \le n\lambda_1.$$

Proof: Let $s_n = \lambda_1+\lambda_2+\cdots+\lambda_n$. Since

$$\left(\frac{s_n}{n}, \frac{s_n}{n}, \ldots, \frac{s_n}{n}\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n),$$

we have $\lambda_n \le \frac{s_n}{n} \le \lambda_1$. Therefore, the proof is complete. $\square$

From equation (5), we have

$$(3, 3, \ldots, 3, 2, 1) \prec \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \ldots, \frac{1}{\lambda_1}\right). \tag{6}$$

Thus, there exists a doubly stochastic matrix $T=[t_{ij}]$ such that

$$(3, 3, \ldots, 3, 2, 1) = \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \ldots, \frac{1}{\lambda_1}\right)\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n}\\ t_{21} & t_{22} & \cdots & t_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ t_{n1} & t_{n2} & \cdots & t_{nn}\end{bmatrix}.$$

That is, we have $\frac{1}{\lambda_n}t_{1n}+\frac{1}{\lambda_{n-1}}t_{2n}+\cdots+\frac{1}{\lambda_1}t_{nn}=1$ and $t_{1n}+t_{2n}+\cdots+t_{nn}=1$.

Lemma 3.3: For each $i=1, 2, \ldots, n$, $t_{n-(i-1), n} \le \frac{\lambda_i}{n-1}$.

Proof: Suppose that $t_{n-(i-1), n} > \frac{\lambda_i}{n-1}$. Then

$$t_{1n}+t_{2n}+\cdots+t_{nn} > \frac{\lambda_1}{n-1}+\frac{\lambda_2}{n-1}+\cdots\frac{\lambda_n}{n-1} = \frac{1}{n-1}(\lambda_1+\lambda_2+\cdots+\lambda_n).$$

Since $t_{1n}+t_{2n}+\cdots+t_{nn}=1$ and $\sum_{i=1}^n\lambda_i \ge n$, this yields a contradiction, so $t_{n-(i-1), n} \le \frac{\lambda_i}{n-1}$. $\square$

From Lemma 3.3, we have $1-(n-1)\frac{1}{\lambda_i}t_{n-(i-1), n} \ge 0$. Let $\alpha = s_n-(n-1)$. Therefore, we have the following theorem.

Theorem 3.4: For $(\alpha, 1, 1, \ldots, 1) \in \mathfrak{D}$, $(\alpha, 1, 1, \ldots, 1) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Proof: A necessary and sufficient condition that $(\alpha, 1, 1, \ldots, 1) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is that there exist a doubly stochastic matrix P such that $(\alpha, 1, 1, \ldots, 1) = (\lambda_1, \lambda_2, \ldots, \lambda_n)P$.

We define an $n \times n$ matrix $P = [p_{ij}]$ as follows:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{12} \\ p_{21} & p_{22} & \cdots & p_{22} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{n2} \end{bmatrix},$$

where $p_{i2} = \frac{1}{\lambda_i} t_{n-(i-1),\, n}$ and $p_{i1} = 1-(n-1)p_{i2}$, $i = 1, 2, \ldots, n$. Since T is doubly stochastic and $\lambda_i > 0$, $p_{i2} \geq 0$, $i = 1, 2, \ldots, n$. By Lemma 3.3, $p_{i1} \geq 0$, $i = 1, 2, \ldots, n$. Then

$$p_{12} + p_{22} + \cdots + p_{n2} = \frac{t_{nn}}{\lambda_1} + \frac{t_{n-1,\, n}}{\lambda_2} + \cdots + \frac{t_{1n}}{\lambda_n} = 1,$$

$$p_{i1} + (n-1)p_{i2} = 1-(n-1)p_{i2} + (n-1)p_{i2} = 1,$$

and

$$\begin{aligned} p_{11} + p_{21} + \cdots + p_{n1} &= 1-(n-1)p_{12} + 1-(n-1)p_{22} + \cdots + 1-(n-1)p_{n2} \\ &= n - n(p_{12} + p_{22} + \cdots + p_{n2}) + p_{12} + p_{22} + \cdots + p_{n2} = 1. \end{aligned}$$

Thus, p is a doubly stochastic matrix. Furthermore,

$$\begin{aligned} \lambda_1 p_{12} + \lambda_2 p_{22} + \cdots + \lambda_n p_{n2} &= \lambda_1 \frac{t_{nn}}{\lambda_1} + \lambda_2 \frac{t_{n-1,\, n}}{\lambda_2} + \cdots + \lambda_n \frac{t_{1n}}{\lambda_n} \\ &= t_{nn} + t_{n-1,\, n} + \cdots + t_{1n} = 1 \end{aligned}$$

and

$$\begin{aligned} \lambda_1 p_{11} + \lambda_2 p_{21} + \cdots + \lambda_n p_{n1} &= \lambda_1(1-(n-1)p_{12}) + \cdots + \lambda_n(1-(n-1)p_{n2}) \\ &= \lambda_1 + \lambda_2 + \cdots + \lambda_n - (n-1)(\lambda_1 p_{12} + \lambda_2 p_{22} + \cdots + \lambda_n p_{n2}) \\ &= \lambda_1 + \lambda_2 + \cdots + \lambda_n - (n-1) = \alpha. \end{aligned}$$

Thus, $(\alpha, 1, 1, \ldots, 1) = (\lambda_1, \lambda_2, \ldots, \lambda_n)P$, so $(\alpha, 1, 1, \ldots, 1) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$. □

From equation (6), we have the following lemma.

Lemma 3.5: For $k = 2, 3, \ldots, n$, $\lambda_k \geq \frac{1}{3(k-1)}$.

Proof: From (6), for $k \geq 2$,

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_k} \leq 1 + 2 + 3 + \cdots + 3 = 3(k-1).$$

Thus,

$$\frac{1}{\lambda_k} \leq 3(k-1) - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_{k-1}}\right) \leq 3(k-1).$$

Therefore, for $k = 2, 3, \ldots, n$, $\lambda_k \geq \frac{1}{3(k-1)}$. □

Corollary 3.6: For $k = 1, 2, \ldots, n-2$, $\lambda_{n-k} \leq (k+1) - \frac{n-k}{3(n-1)}$. In particular, $\alpha \leq \lambda_1$ and $\frac{1}{3(k-1)} \leq \lambda_n \leq \frac{1}{3}$.

Proof: If $k = 1$, then $\lambda_n + \lambda_{n-1} \leq 2$. By Lemma 3.5, we have $\lambda_{n-1} \leq 2 - \frac{1}{3(n-1)}$. Hence, by induction on n, the proof is complete for $k = 1, 2, \ldots, n-2$. In particular, by Theorem 3.4 and (6), $\frac{1}{3(n-1)} \leq \lambda_n \leq \frac{1}{3}$. □

Since $\det \mathfrak{Q}_n = \lambda_1\lambda_2\ldots\lambda_n = 1$, $\lambda_2\lambda_3\ldots\lambda_n = \frac{1}{\lambda_1}$, we have $\lambda_1^{n-1} \geq \lambda_1\ldots\lambda_{n-1} = \frac{1}{\lambda_n}$. Thus,

$$\lambda_n \geq \left(\frac{1}{\lambda_1}\right)^{n-1}.$$

Therefore,

$$\left(\frac{1}{\lambda_1}\right)^{n-1} \leq \lambda_n \leq \frac{1}{3}.$$

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CONVOLUTIONS FOR JACOBSTHAL-TYPE POLYNOMIALS

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1. PRELIMINARIES

Object of the Paper

Basically, the purpose of this paper is to present data on convolution polynomials $J_n^{(k)}(x)$ and $j_n^{(k)}(x)$ for Jacobsthal and Jacobsthal-Lucas polynomials $J_n(x)$ and $j_n(x)$, respectively, and, more specifically, on the corresponding convolution numbers arising when $x = 1$.

Our information will roughly parallel and, therefore, should be compared with that offered for Pell and Pell-Lucas polynomials $P_n(x)$ and $Q_n(x)$, respectively, in [7] and [8] in particular.

Properties of $J_n(x)$ and $j_n(x)$ may be found in [5] and [6, p. 138]. Originally $J_n(x)$ was investigated by the Norwegian mathematician Jacobsthal [9]. For ease of reference, it is thought desirable to reproduce a few essential features of $J_n(x)$ and $j_n(x)$ in the next subsection.

Background articles of relevance on convolutions which could be consulted with benefit are [1], [2], and [3]. But observe that in [3] the x has to be replaced by $2x$ for our $J_n(x)$.

Convolution Arrays

Convolution numbers, symbolized by $J_n^{(k)}(1) \equiv J_n^{(k)}$ and $j_n^{(k)}(1) \equiv j_n^{(k)}$, where k represents the "order" of the convolution and n the sequence index, may be displayed in a *convolution array* (pattern). When $k = 0$, the ordinary Jacobsthal numbers $J_n^{(0)} \equiv J_n$ and the Jacobsthal-Lucas numbers $j_n^{(0)} \equiv j_n$ are generated.

Readers of [3, p. 401] will be aware that the n^{th}-order convolution sequence for $J_n^{(k)}$ appears there as columns of a matrix. As the convolution array for $j_n^{(k)}$ does not seem to have been previously recorded, we shall disclose its details in Table 2.

Mathematical Background

Definitions

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x), \quad J_0(x) = 0, \; J_1(x) = 1. \tag{1.1}$$

$$j_{n+2}(x) = j_{n+1}(x) + 2xj_n(x), \quad j_0(x) = 2, \; j_1(x) = 1. \tag{1.2}$$

For $0 \le n \le 10$, $J_n(x)$ and $j_n(x)$ are recorded in [6] in Tables 1 and 2, respectively, to which the reader is encouraged to refer.

Special Cases

$x = 1$: Jacobsthal numbers $J_n(1) = J_n$ and Jacobsthal-Lucas numbers $j_n(1) = j_n$.

$x = \frac{1}{2}$: $J_n(\frac{1}{2}) = F_n$, $j_n(\frac{1}{2}) = L_n$ (the n^{th} Fibonacci and Lucas numbers).

It follows that Tables 1 and 2 in [6] with (1.1) and (1.2) thus generate the number sequences

$$\{J_n(1)\} = 0, 1, 1, 3, 5, 11, 21, 43, \ldots, \tag{1.3}$$

$$\{j_n(1)\} = 2, 1, 5, 7, 17, 31, 65, 127, \ldots. \tag{1.4}$$

Binet Forms

From the characteristic equation $\lambda^2 - \lambda - 2x = 0$ for both (1.1) and (1.2), we deduce the roots

$$\alpha = \frac{1+\Delta}{2}, \quad \beta = \frac{1-\Delta}{2}, \tag{1.5}$$

so that

$$\alpha + \beta = 1, \ \alpha\beta = 2x, \ \alpha - \beta = \sqrt{1+8x} = \Delta. \tag{1.6}$$

Binet forms are then

$$J_n(x) = (\alpha^n - \beta^n)/\Delta, \tag{1.7}$$

$$j_n(x) = \alpha^n + \beta^n. \tag{1.8}$$

Generating Functions

$$\sum_{n=0}^{\infty} J_{n+1}(x)y^n = (1 - y - 2xy^2)^{-1}, \tag{1.9}$$

$$\sum_{n=0}^{\infty} j_{n+1}(x)y^n = (1+4xy)(1 - y - 2xy^2)^{-1}. \tag{1.10}$$

An immediate consequence of (1.9) and (1.10) is

$$j_n(x) = J_n(x) + 4xJ_{n-1}(x), \tag{1.11}$$

which is also quickly obtainable from (1.7) and (1.8).

Jacobsthal convolution polynomials $J_n^{(k)}(x)$ are defined [see (4.9) and (4.9a)] from (1.9) by

$$\sum_{n=0}^{\infty} J_{n+1}^{(k)}(x)y^n = (1 - y - 2xy^2)^{-(k+1)}. \tag{1.12}$$

The corresponding Jacobsthal-Lucas convolution polynomials $j_{n+1}^{(k)}(x)y^n$ are defined in (5.7) and (5.7a) by means of (1.10).

2. FIRST JACOBSTHAL CONVOLUTION POLYNOMIALS $J_n^{(1)}(x)$

Generating Function Definition

$$\sum_{n=0}^{\infty} J_{n+1}^{(1)}(x)y^n = (1 - y - 2xy^2)^{-2} \tag{2.1}$$

$$= \left(\sum_{r=0}^{\infty} J_{r+1}(x)y^r\right)^2 \text{ by (1.9).} \tag{2.1a}$$

Examples

$$\begin{aligned} &J_1^{(1)}(x) = 1, \ J_2^{(1)}(x) = 2, \ J_3^{(1)}(x) = 3+4x, \ J_4^{(1)}(x) = 4+12x, \ J_5^{(1)}(x) = 5+24x+12x^2, \\ &J_6^{(1)}(x) = 6+40x+48x^2, \ J_7^{(1)}(x) = 7+60x+120x^2+32x^3, \dots \end{aligned} \tag{2.2}$$

Special Case (First Jacobsthal Convolution Numbers: $x = 1$)

$$\{J_n^{(1)}(1)\} = 1, 2, 7, 16, 41, 94, 219, \dots. \tag{2.3}$$

Observe that this sequence of integers appears in the second column of the matrix in [3, p. 401].

Recurrence Relations

Immediately, from (1.9) and (2.1), we deduce the recurrence

$$J_{n+1}^{(1)}(x) - J_n^{(1)}(x) - 2xJ_{n-1}^{(1)}(x) = J_{n+1}(x). \tag{2.4}$$

By means of (2.4), the list of first convolution polynomials may be extended indefinitely.

Partial differentiation with respect to y of both sides of (1.9) along with the equating of the coefficients of y^{n-1} then yields, with (2.1),

$$nJ_{n+1}(x) = J_n^{(1)}(x) + 4xJ_{n-1}^{(1)}(x). \tag{2.5}$$

Combine (2.4) with (2.5) to obtain the recurrence

$$nJ_{n+1}^{(1)}(x) = (n+1)J_n^{(1)}(x) + 2x(n+2)J_{n-1}^{(1)}(x). \tag{2.6}$$

Eliminate $J_{n-1}^{(1)}(x)$ from (2.4) and (2.5). Then

$$(n+2)J_{n+1}(x) = 2J_{n+1}^{(1)}(x) - J_n^{(1)}(x). \tag{2.7}$$

Add (2.5) to (2.7), whence

$$(n+1)J_{n+1}(x) = J_{n+1}^{(1)}(x) + 2xJ_{n-1}^{(1)}(x). \tag{2.8}$$

Or, apply (2.9) below twice with reliance on (3.13), (3.12), and (1.2) in [6] and appeal to the (new) result, $j_{n+1}(x) + 4xj_n(x) = \Delta^2 J_{n+1}(x)$ obtained from Binet forms (1.7) and (1.8) above.

Other Main Properties

Next, we are able to derive the revealing connective relation

$$J_n^{(1)}(x) = \frac{nj_{n+1}(x) + 4xJ_n(x)}{\Delta^2}, \tag{2.9}$$

where Δ is given in (1.6). As a prelude to (2.9), we require the recursion

$$nj_{n+1}(x) = (1+4x)J_n^{(1)}(x) + 4xJ_{n-1}^{(1)}(x) + 8x^2 J_{n-2}^{(1)}(x). \tag{2.10}$$

Establishing (2.10) merely asks us to differentiate (1.10) partially with respect to y, and then perform appropriate algebraic interpretations involving (2.1). Corresponding coefficients of y^{n-1} are then equated.

Proofs of (2.9):

(a) **Induction.** The formula is verifiably valid for $n = 1, 2, 3, 4, 5$. Employing the induction method in conjunction with (2.4) leads us to the desired end.

(b) Alternatively (cf. [8, p. 61, (4.7)]), algebraic manipulation in (2.1) gives

$$\sum_{n=1}^{\infty} J_n^{(1)}(x) y^{n-1} = \frac{(1+4x+4xy+8x^2y^2) + 4x(1-y-2xy^2)}{(1+8x)(1-y-2xy^2)^2}$$

$$= \frac{1}{1+8x}\sum_{n=1}^{\infty}(nj_{n+1}(x) + 4xJ_n(x))y^{n-1} \quad \text{by (1.9), (1.10), (2.10).}$$

Compare coefficients of y^{n-1} and (2.9) ensues.

Observe that a Binet form may be deduced for $J_n^{(1)}(x)$ from (2.9) by means of (1.7) and (1.8). Worth noting in passing is that by combining (1.1) and [6, (3.12)] we may express the numerator of the right-hand side of (2.9) neatly as $(n+1)j_{n+1}(x)-J_{n+1}(x)$.

Explicit Combinatorial Form

Theorem 1:

$$J_n^{(1)}(x)=\sum_{r=0}^{\left[\frac{n-1}{2}\right]}\binom{n-r}{1}\binom{n-r-1}{r}(2x)^r \quad \text{(closed form).} \tag{2.11}$$

Proof (by induction): Using (2.2), we readily verify that the theorem is true for all $n=1$, 2, 3. Assume it is true for all $n\le N$, that is,

$$\text{Assumption: } J_N^{(1)}(x)=\sum_{r=0}^{\left[\frac{N-1}{2}\right]}\binom{N-r}{1}\binom{N-r-1}{r}(2x)^r. \tag{A}$$

Then the right-hand side of (2.6) becomes

$$N(J_N^{(1)}(x)+2xJ_{N-1}^{(1)}(x))+(J_N^{(1)}(x)+4xJ_{N-1}^{(1)}(x))$$

$$=N\sum_{r=0}^{\left[\frac{N}{2}\right]}(N-r)\binom{N-r}{r}(2x)^r+N\sum_{r=0}^{\left[\frac{N}{2}\right]}\binom{N-r}{r}(2x)^r \quad \text{from (A), on simplifying}$$

$$=N\sum_{r=0}^{\left[\frac{N}{2}\right]}(N-r+1)\binom{N-r}{r}(2x)^r \tag{B}$$

$$=NJ_{N+1}^{(1)}(x), \tag{C}$$

which must be the left-hand side of (2.6).

Consequently, (B) and (C) with (A) show that (2.11) is true for $n=N+1$ and thus for all n. Hence, Theorem 1 is completely demonstrated.

Remarks: Recourse is required in the proof to the use of

(i) N even, N odd considered separately (for convenience),

(ii) Pascal's Formula, and

(iii) the combinatorial result (readily computable)

$$(N-r)\binom{N-r-1}{r}+2(N-r)\binom{N-r-1}{r-1}=N\binom{N-r}{r}. \tag{2.11a}$$

Summation

From (2.4) and [6, (3.7)],

$$\sum_{r=1}^{n}J_r^{(1)}(x)=\frac{2xJ_{n+2}^{(1)}(x)-J_{n+4}(x)+1}{4x^2}. \tag{2.12}$$

Expanding the right-hand side of (2.1a), both sides having lower bound $n=1$, and equating coefficients, we arrive at

$$J_n^{(1)}(x) = \begin{cases} 2\sum_{r=1}^{[\frac{n}{2}]} J_r(x)J_{n-r+1}(x) & n \text{ even}, \\ 2\sum_{r=1}^{[\frac{n-1}{2}]} J_r(x)J_{n-r+1}(x) + J_{\frac{n+1}{2}}^2(x) & n \text{ odd}. \end{cases} \tag{2.13}$$

Differentiation and Convolutions

Let the prime ($'$) represent partial differentiation with respect to x. Differentiate both sides of (1.9) with respect to x. Compare this with (2.1). Then, on equating coefficients of y^{n-1}, we deduce the notably succinct connection

$$2J_{n-1}^{(1)}(x) = J'_{n+1}(x). \tag{2.14}$$

But $j'_n(x) = 2nJ_{n-1}(x)$ by [6, (3.21)]. Hence, the second derivative is

$$j''_n(x) = 4nJ_{n-3}^{(1)}(x). \tag{2.15}$$

3. FIRST JACOBSTHAL-LUCAS CONVOLUTION POLYNOMIALS $j_n^{(1)}(x)$

Generating Function Definition

$$\sum_{n=0}^{\infty} j_{n+1}^{(1)}(x)y^n = (1+4xy)^2(1-y-2xy^2)^{-2} \tag{3.1}$$

$$= \left(\sum_{r=0}^{\infty} j_{r+1}(x)y^r\right)^2 \quad \text{by (1.10)}. \tag{3.1a}$$

Examples:

$$\begin{aligned} &j_1^{(1)}(x) = 1,\ j_2^{(1)}(x) = 2+8x,\ j_3^{(1)}(x) = 3+20x+16x^2,\ j_4^{(1)}(x) = 4+36x+64x^2, \\ &j_5^{(1)}(x) = 5+56x+156x^2+64x^3,\ j_6^{(1)}(x) = 6+80x+304x^2+228x^3, \dots \end{aligned} \tag{3.2}$$

Special Case (First Jacobsthal-Lucas Convolution Numbers: $x = 1$)

$$\{j_1^{(1)}(1)\} = 1, 10, 39, 104, 281, 678, 1627, \dots \tag{3.3}$$

Recurrence Relations

Immediately, from (2.1) and (3.1), we have

$$j_n^{(1)}(x) = J_n^{(1)}(x) + 8xJ_{n-1}^{(1)}(x) + 16x^2J_{n-2}^{(1)}(x), \tag{3.4}$$

by means of which a list of convolution polynomials may be presented, in conjunction with (2.2), which may be checked against those already given in (3.2).

Combining (3.4) and (2.10), we deduce that

$$2nj_{n+1}(x) = j_n^{(1)}(x) + (1+8x)J_n^{(1)}(x) \quad (1+8x = \Delta^2). \tag{3.5}$$

Equations (2.9) and (3.5) generate the pleasing connection

$$j_n^{(1)}(x) = nj_{n+1}(x) - 4xJ_n(x), \tag{3.6}$$

which, with (1.11), may be cast in the form

$$(n-1)j_{n+1}(x) = j_n^{(1)}(x) - J_{n+1}(x). \tag{3.7}$$

Alternatively, (3.6) may be demonstrated in the following way.

$$\begin{aligned}\sum_{n=1}^{\infty} j_n^{(1)}(x)y^{n-1} &= (1+4xy)\cdot\frac{1+4xy}{(1-y-2xy^2)^2} \quad \text{by (3.1)}\\ &= (1+4xy)\sum_{n=1}^{\infty} nJ_{n+1}(x)y^{n-1} \quad \text{differentiating (1.9) w.r.t. } y\\ &= \sum_{n=1}^{\infty}(nJ_{n+1}(x)y^{n-1}+4x(n-1)J_n(x))y^{n-1},\end{aligned}$$

whence (3.6) emerges by (1.11).

Other Main Properties

Comparing the generating functions in (1.10) and (2.1), we calculate upon simplification that

$$j_n(x) = J_n^{(1)}(x)+(4x-1)J_{n-1}^{(1)}(x)-64xJ_{n-2}^{(1)}(x)-8x^2J_{n-3}(x). \tag{3.8}$$

Taken together, (2.9) and (3.6) produce

$$J_n^{(1)}(x)j_n^{(1)}(x) = \frac{n^2 j_{n+1}^2(x)-16x^2J_n^2(x)}{\Delta^2} \quad (\Delta^2 = 1+8x). \tag{3.9}$$

Equation (3.6), in conjunction with (1.7) and (1.8), allows us to display $j_n^{(1)}(x)$ in a Binet form.

Furthermore, (2.9) and (3.6) yield

$$\Delta^2 J_n^{(1)}(x)+j_n^{(1)}(x) = 2nj_{n+1}(x) \tag{3.10}$$

and

$$\Delta^2 J_n^{(1)}(x)-j_n^{(1)}(x) = 8xJ_n(x). \tag{3.11}$$

Lastly, we append a result which is left as an exercise for the curiosity of the reader:

$$(\Delta^2-1)j_n(x) = \Delta^2\{J_{n+1}^{(1)}(x)+2xJ_{n-1}^{(1)}(x)\}-\{j_{n+1}^{(1)}(x)+2xj_{n-1}^{(1)}(x)\}, \tag{3.12}$$

where $\Delta^2-1=8x$ by (1.6).

4. GENERAL JACOBSTHAL CONVOLUTION POLYNOMIALS $J_n^{(k)}(x)$ $(k>1)$

A. CASE $k=2$ (Second Jacobsthal Convolution Polynomials)

Generating Function Definition

$$\sum_{n=0}^{\infty} J_{n+1}^{(2)}(x)y^n = (1-y-2xy^2)^{-3} \tag{4.1}$$

$$= \left(\sum_{r=0}^{\infty} J_{r+1}(x)y^r\right)^3. \tag{4.1a}$$

Examples

$$\begin{aligned}&J_1^{(2)}(x)=1,\ J_2^{(2)}(x)=3,\ J_3^{(2)}(x)=6+6x,\ J_4^{(2)}(x)=10+24x,\ J_5^{(2)}(x)=15+60x+24x^2,\\ &J_6^{(2)}(x)=21+120x+120x^2,\ J_7^{(2)}(x)=28+210x+360x^2+80x^3,\ldots.\end{aligned} \tag{4.2}$$

Special Case (Second Jacobsthal Convolution Numbers: $x=1$)

$$\{J_n^{(2)}(1)\} = 1, 3, 12, 34, 99, 261, 678, \ldots. \tag{4.3}$$

Observe that this sequence of numbers occurs in the third column of the matrix array in [3, p. 401].

Recurrence Relations

Immediately, from (2.1) and (4.1) there comes

$$J_{n+1}^{(2)}(x) - J_n^{(2)}(x) - 2xJ_{n-1}^{(2)}(x) = J_{n+1}^{(1)}(x) \tag{4.4}$$

whereas (1.9) and (4.1) lead to

$$J_{n+1}^{(2)}(x) - 2J_n^{(2)}(x) + (1-4x)J_{n-1}^{(2)}(x) + 4xJ_{n-2}^{(2)}(x) + 4x^2J_{n-3}^{(2)}(x) = J_{n+1}(x). \tag{4.5}$$

Differentiate both sides of (2.1) partially with respect to y, then equate coefficients of y^{n-1} to obtain, by (4.1),

$$nJ_{n+1}^{(1)}(x) = 2(J_n^{(2)}(x) + 4xJ_{n-1}^{(2)}(x)). \tag{4.6}$$

Eliminate $J_{n+1}^{(1)}(x)$ from (4.4) and (4.6). Hence,

$$nJ_{n+1}^{(2)}(x) = (n+2)J_n^{(2)}(x) + 2x(n+4)J_{n-1}^{(2)}(x). \tag{4.7}$$

Next, eliminate $J_{n-1}^{(2)}(x)$ from (4.4) and (4.6). Accordingly,

$$(n+4)J_{n+1}^{(1)}(x) = 2(2J_{n+1}^{(2)}(x) - J_n^{(2)}(x)). \tag{4.8}$$

Not all results in Section 3 above ($k = 1$) extend readily to direct unit superscript increase on both sides of the equation [cf. (2.7), (4.8)].

B. CASE k General (k^{th} Jacobsthal Convolution Polynomials)

Generating Function Definition

$$\sum_{n=0}^{\infty} J_{n+1}^{(k)}(x)y^n = (1-y-2xy^2)^{-(k+1)} \tag{4.9}$$

$$= \left(\sum_{r=0}^{\infty} J_{r+1}(x)y^2\right)^{k+1} \quad \text{by (1.9).} \tag{4.9a}$$

Examples

$$\left.\begin{aligned} &J_1^{(k)}(x) = 1,\ J_2^{(k)}(x) = \binom{k+1}{1},\ J_3^{(k)}(x) = \binom{k+2}{2} + \binom{k+1}{1}2x, \\ &J_4^{(k)}(x) = \binom{k+3}{3} + \binom{k+2}{2}4x,\ J_5^{(k)}(x) = \binom{k+4}{4} + \binom{k+3}{3}\cdot 3\cdot 2x + \binom{k+2}{2}(2x)^2, \ldots \end{aligned}\right. \tag{4.10}$$

Special Case (k^{th} Jacobsthal Convolution Numbers: $x = 1$)

$$\{J_n^{(k)}(1)\} = 1,\ k+1,\ (k+1)\left(\frac{k+6}{2}\right),\ (k+1)(k+2)\left(\frac{k+15}{6}\right), \ldots. \tag{4.11}$$

Explicit Combinatorial Form

Theorem 2:

$$J_n^{(k)}(x) = \sum_{r=0}^{[\frac{n-1}{2}]} \binom{k+n-r-1}{k}\binom{n-r-1}{r}(2x)^r. \tag{4.12}$$

Proof: Constructing the proof parallels the procedures employed in Theorem 1, where $k = 1$. That is, apply (4.15), which will be proven independently below, and induction in tandem.

Remarks: Corresponding to the combinatorial identity (2.11a) for Theorem 1, we require in the proof of Theorem 2,

$$\begin{aligned} &k\left\{\binom{N+k-1-r}{k}\binom{N-r-1}{r}+2\binom{N+k-1-r}{k}\binom{N-r-1}{r-1}\right\} \\ &=N\binom{N+k-1-r}{k-1}\binom{N-r}{r}, \end{aligned} \tag{4.12a}$$

i.e., k is absorbed into the product and N emerges as a factor.

Finally, we have the sum

$$\begin{aligned} &N\left[\binom{N+k-1-r}{k}\binom{N-r}{r}+\binom{N+k-1-r}{k-1}\binom{N-r}{r}\right] \\ &=N\binom{N+k-r}{k}\binom{N-r}{r}. \end{aligned} \tag{4.12b}$$

Pascal's formula is needed in (4.12a) and (4.12b). The simplified form in (4.12b) relates to the expression for $J_{N+1}^{(k)}(x)$ in (4.12).

Knowledge of (4.12) now permits us to compute $J_n^{(k)}(x)$ for any k and n. In particular, $J_5^{(3)}(x) = 35 + 120x + 40x^2$. Refer also to (4.10).

Recurrence Relations

Appealing to (4.9) and (4.9) with $k-1$, we have the immediate consequence

$$J_{n+1}^{(k)}(x) - J_n^{(k)}(x) - 2xJ_{n-1}^{(k)}(x) = J_{n+1}^{(k-1)}(x). \tag{4.13}$$

Partially differentiate both sides of (4.9) with respect to y. Considering coefficients of y^{n-1} we then have, on replacing k by $k-1$,

$$nJ_{n+1}^{(k-1)}(x) = k(J_n^{(k)}(x) + 4xJ_{n-1}^{(k)}(x)). \tag{4.14}$$

Combine (4.13) and (4.14) to obtain the recurrence

$$nJ_{n+1}^{(k)}(x) = (n+k)J_n^{(k)}(x) + 2x(n+2k)J_{n-1}^{(k)}(x). \tag{4.15}$$

Furthermore, from (4.13) and (4.14), we arrive at

$$(n+2k)J_{n+1}^{(k-1)}(x) = k(2J_{n+1}^{(k)}(x) - J_n^{(k)}(x)). \tag{4.16}$$

Results when $k = 2$ may now be checked against those specialized in (4.1)-(4.8).

Convolution Array for $J_n^{(k)}$

In Table 1 below, we exhibit the simplest numbers occurring in the Jacobsthal array for the convolution numbers $J_n^{(k)}$.

Convolution numbers for $k = 1, 2$ and for small values of n are already publicized in (2.3), (4.3) and (3.3), (5.3). Applying the extremely useful formulas obtained (from the Cauchy convolutions of a sequence with itself) by induction in [1, pp. 193-94], where the initial conditions (1.1), (1.2) are known, we may develop the array for $J_n^{(k)}$ to our heart's desire. Or use Theorem 2 when

$x=1$. Systematic reduction to $n=1$ (boundary case) using (4.13) is a rewarding, if tedious, exercise. Reduction by (4.13) gives, for example, $J_4^{(2)}(x)=10+24x$ in conformity with (4.2).

TABLE 1. Convolution Array for $J_n^{(k)}$ $(n=1,2,\ldots,5)$

n/k	0	1	2	3	$\cdots$	k
1	1	1	1	1	$\cdots$	1
2	1	2	3	4	$\cdots$	$\binom{k+1}{1}$
3	3	7	12	18	$\cdots$	$\binom{k+2}{2}+2\binom{k+1}{1}$
4	5	16	34	60	$\cdots$	$\binom{k+3}{3}+4\binom{k+2}{2}$
5	11	41	99	195	$\cdots$	$\binom{k+4}{4}+6\binom{k+3}{3}+4\binom{k+2}{2}$

It should be noted that the formulas given in [1, pp. 193-94] relate to rows in the convolution array, whereas it is the columns that are generated in our approach, namely, one column for each convolution value of k.

Be aware that the notation in [1, pp. 193-94] is different, namely, we have the correspondence (subscripts in R_{nk} referring to rows and columns, respectively)

$$R_{nk} \Leftrightarrow J_n^{(k-1)}. \tag{4.17}$$

Formula (4.10) and [1, (1.6)] then both yield, for example, $R_{43}=J_4^{(2)}=34$ (Table 1).

Reverting briefly to [3, p. 401] we see that the abbreviated array for $J_n^{(k)}$ is exposed in matrix form in which the first, second, third, ... columns of the matrix B_2P are precisely our $J_n^{(0)}$, $J_n^{(1)}$, $J_n^{(2)}$, ..., respectively. *En passant*, we remark that the columns of the matrix A_2P are exactly the Pell convolution numbers $P_n^{(0)}$, $P_n^{(1)}$, $P_n^{(2)}$, ... examined in [8].

5. GENERAL JACOBSTHAL-LUCAS CONVOLUTION POLYNOMIALS $j_n^{(k)}(x)$ $(k>1)$

A. CASE $k=2$ (Second Jacobsthal-Lucas Convolution Polynomials)

Generating Function Definition

$$\sum_{n=0}^{\infty} j_{n+1}^{(2)}(x)y^n = (1+4xy)^3[1-y-2xy^2]^{-3} \tag{5.1}$$

$$= \left(\sum_{r=0}^{\infty} j_{r+1}(x)y^r\right)^3. \tag{5.1a}$$

Examples

$$\begin{gathered} j_1^{(2)}(x)=1,\ j_2^{(2)}(x)=3+12x,\ j_3^{(2)}(x)=6+42x+48x^2, \\ j_4^{(2)}(x)=10+96x+216x^2+64x^3,\ j_5^{(2)}(x)=15+180x+600x^2+480x^3,\ldots. \end{gathered} \tag{5.2}$$

Special Case (Second Jacobsthal-Lucas Convolution Numbers: $x=1$)

$$\{j_n^{(2)}(1)\}=1, 15, 96, 386, 1275, \ldots.$$

Recurrence Relations

Taken together, (1.10), (3.1), and (5.1) yield

$$\sum_{n=0}^{\infty} j_{n+1}^{(2)}(x)y^n = \left(\sum_{n=0}^{\infty} j_{n+1}(x)y^n\right)\left(\sum_{n=0}^{\infty} j_{n+1}^{(1)}(x)y^n\right). \tag{5.4}$$

Comparing coefficients of y^n, we deduce that

$$j_{n+1}^{(2)}(x) = \sum_{r=1}^{n+1} j_r(x) j_{n-r+2}^{(1)}(x). \tag{5.5}$$

Furthermore, from (4.1) and (5.1), we easily derive

$$j_n^{(2)}(x) = J_n^{(2)}(x) + 12xJ_{n-1}^{(2)}(x) + 48x^2 J_{n-2}^{(2)}(x) + 64x^3 J_{n-3}^{(2)}(x). \tag{5.6}$$

B. CASE *k* General (k^{th} Jacobsthal-Lucas Convolution Polynomials)

Generating Function Definition

$$\sum_{n=0}^{\infty} j_{n+1}^{(k)}(x)y^n = (1+4xy)^{k+1}[1-y-2xy^2]^{-(k+1)} \tag{5.7}$$

$$= \left(\sum_{r=0}^{\infty} j_{r+1}(x)y^r\right)^{k+1}. \tag{5.7a}$$

Examples

$$\begin{aligned} &j_1^{(k)}(x) = 1,\ j_2^{(k)}(x) = \binom{k+1}{1}(1+4x), \\ &j_3^{(k)}(x) = \binom{k+1}{2}16x^2 + 2x\binom{k+1}{1}\left\{2\binom{k+1}{1}+1\right\} + \binom{k+2}{2}, \ldots \end{aligned} \tag{5.8}$$

Special Case (k^{th} Jacobsthal-Lucas Convolution Numbers: $x = 1$)

$$\{j_n^{(k)}(1)\} = 1,\ 5\binom{k+1}{1},\ 16\binom{k+1}{2} + 2\binom{k+1}{1}\left\{2\binom{k+1}{1}+1\right\} + \binom{k+2}{2}, \ldots \tag{5.9}$$

Theorem 3:

$$j_n^{(k)}(x) = \sum_{r=0}^{k+1}\binom{k+1}{r}(4x)^r J_{n-r}^{(k)}(x), \tag{5.10}$$

where $J_{n-r}^{(k)}(x)$ are given in (4.12).

Proof: Expand $(1+4xy)^{k+1}$ in conjunction with (4.9) and (5.7) to produce

$$\begin{aligned} j_n^{(k)}(x) &= J_n^{(k)}(x) + \binom{k+1}{1}4xJ_{n-1}^{(k)}(x) + \binom{k+1}{2}(4x)^2 J_{n-2}^{(k)}(x) + \cdots \\ &\quad + \binom{k+1}{r}(4x)^r J_{n-r}^{(k)}(x) + \cdots + (4x)^{k+1} J_{n-k-1}^{(k)}(x). \end{aligned}$$

The theorem is thus demonstrated.

Armed with this knowledge (5.10), we may then appeal to (4.12) for the determination of the convolution polynomials $j_n^{(k)}(x)$ for any k and n. For example, application of (5.10) leads us to $j_5^{(2)}(x) = 15 + 180x + 600x^2 + 480x^3$, which confirms (5.2).

Convolution Array for $j_n^{(k)}$

A truncated array for $j_n^{(k)}$ is set out in Table 2.

TABLE 2. Convolution Array for $j_n^{(k)}$ ($n = 1, 2, \ldots, 5$)

n/k	0	1	2	3	$\cdots$	k
1	1	1	1	1	$\cdots$	1
2	5	10	15	20	$\cdots$	$5\binom{k+1}{1}$
3	7	39	96	178	$\cdots$	$16\binom{k+1}{2}+\binom{k+1}{1}\left\{4\binom{k+1}{1}+2\right\}+\binom{k+2}{2}$
4	17	104	386	488	$\cdots$	$\cdots$
5	31	281	1275	4163	$\cdots$	$\cdots$

As in (4.16), we have the correspondence of notation

$$R_{nk} \Leftrightarrow j_n^{(k-1)}, \tag{5.11}$$

where subscripts in R_{nk} refer to rows and columns, respectively, whence, for instance, $R_{32} = j_3^{(1)} = 39$ (Table 2).

Evidently, there is a law of diminishing returns evolving as we proceed to study the case for k general, and more so as we progress from $J_n^{(k)}(x)$ to $j_n^{(k)}(x)$. Perhaps we should follow a precept of Descartes and leave further discoveries for the pleasure of the assiduous investigator.

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VIETA POLYNOMIALS

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1. VIETA ARRAYS AND POLYNOMIALS

Vieta Arrays

Consider the combinatorial forms

$$B(n, j) = \binom{n-j-1}{j} \quad \left(0 \le j \le \left[\tfrac{n-1}{2}\right]\right) \tag{1.1}$$

and

$$b(n, j) = \frac{n}{n-j}\binom{n-j}{j} \quad \left(0 \le j \le \left[\tfrac{n}{2}\right]\right), \tag{1.2}$$

where $n(\ge 1)$ is the n^{th} row in an infinite left-adjusted triangular array. Then the entries in these arrays are as exhibited in Tables 1 and 2.

TABLE 1. Array for $B(n, j)$

1				
1				
1	1			
1	2			
1	3	1		
1	4	3		
1	5	6	1	
1	6	10	4	
1	7	15	10	1
1	8	21	20	5
⋮	⋮	⋮	⋮	⋮

TABLE 2. Array for $b(n, j)$

1					
1	2				
1	3				
1	4	2			
1	5	5			
1	6	9	2		
1	7	14	7		
1	8	20	16	2	
1	9	27	30	9	
1	10	35	50	25	2
⋮	⋮	⋮	⋮	⋮	⋮

In the notation and nomenclature of this paper, Table 1 will be called the *Vieta-Fibonacci array* and Table 2 the *Vieta-Lucas array*. The Table 2 array has already been displayed in [5] where its discovery is attributed to Vieta (or Viète, 1540-1603) [8].

Vieta Polynomials

From (1.1) and Table 1, we define the *Vieta-Fibonacci polynomials* $V_n(x)$ by

$$V_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n-k-1}{k} x^{n-2k-1}, \quad V_0(x) = 0. \tag{1.3}$$

From (1.3), we find:

$$\left.\begin{array}{l} V_1(x) = 1,\ V_2(x) = x,\ V_3(x) = x^2 - 1,\ V_4(x) = x^3 - 2x, \\ V_5(x) = x^4 - 3x^2 + 1,\ V_6(x) = x^5 - 4x^3 + 3x,\ V_7(x) = x^6 - 5x^4 + 6x^2 - 1, \ldots \end{array}\right\} \tag{1.4}$$

Equation (1.2) and Table 2 then invite the definition of the *Vieta-Lucas polynomials* $v_n(x)$ as

$$v_n(x)=\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\frac{n}{n-k}\binom{n-k}{k}x^{n-2k},\ v_0(x)=2. \tag{1.5}$$

From (1.5), we get:

$$\left.\begin{array}{l} v_1(x)=x,\ v_2(x)=x^2-2,\ v_3(x)=x^3-3x,\ v_4(x)=x^4-4x^2+2, \\ v_5(x)=x^5-5x^3+5x,\ v_6(x)=x^6-6x^4+9x^2-2,\dots \end{array}\right\} \tag{1.6}$$

Remark: Array Table 2 [8] and polynomials $v_n(x)$ were investigated in some detail in [5], while some fruitful pioneer work on $v_n(x)$ was accomplished in [3]. Array Table 1 and polynomials $V_n(x)$ were introduced in [6]. But see also [1, p. 14] and [4, pp. 312-13].

Recurrence Relations

Recursive definitions of the Vieta polynomials are

$$V_n(x)=xV_{n-1}(x)-V_{n-2}(x) \tag{1.7}$$

with

$$V_0(x)=0,\ V_1(x)=1, \tag{1.7a}$$

and

$$v_n(x)=xv_{n-1}(x)-v_{n-2}(x) \tag{1.8}$$

with

$$v_0(x)=2,\ v_1(x)=x. \tag{1.8a}$$

Characteristic Equation Roots

Both (1.7) and (1.8) have the characteristic equation

$$\lambda^2-\lambda x+1=0 \tag{1.9}$$

with roots

$$\alpha=\frac{x+\Delta}{2},\ \beta=\frac{x-\Delta}{2},\ \Delta=\sqrt{x^2-4} \tag{1.10}$$

so that

$$\alpha\beta=1,\ \alpha+\beta=x. \tag{1.11}$$

Purpose of this Paper

It is proposed

(i) to develop salient properties of $V_n(x)$ and $v_n(x)$, and

(ii) to explore the interplay of relationships among Vieta, Jacobsthal, and Morgan-Voyce polynomials (while observing the known connections with Fibonacci, Lucas, and Chebyshev polynomials).

2. VIETA-FIBONACCI POLYNOMIALS $V_n(x)$

Formulas (2.1) and (2.2) below flow from routine processes.

Binet Form

$$V_n(x)=\frac{\alpha^n-\beta^n}{\Delta}. \tag{2.1}$$

Generating Function

$$\sum_{n=1}^{\infty}V_n(x)y^{n-1}=[1-xy+y^2]^{-1}. \tag{2.2}$$

Simson's Formula

$$V_{n+1}(x)V_{n-1}(x) = V_n^2(x) = -1 \ \text{(by (2.1))]}. \tag{2.3}$$

Negative Subscript

$$V_{-n}(x) = -V_n(x) \ \text{(by (2.1))}. \tag{2.4}$$

Differentiation

$$\frac{dv_n(x)}{dx} = nV_n(x) \ \text{(by (2.1), (3.1))}. \tag{2.5}$$

A neat result:

$$V_n(x)V_{n-1}(-x) + V_n(-x)V_{n-1}(x) = 0 \ (n \geq 2). \tag{2.6}$$

Induction may be used to demonstrate (2.6); see [6].

3. VIETA-LUCAS POLYNOMIALS $v_n(x)$

Standard techniques reveal the following basic features of $v_n(x)$.

Binet Form

$$v_n(x) = \alpha^n + \beta^n. \tag{3.1}$$

Generating Function

$$\sum_{n=0}^{\infty} v_n(x)y^n = (2 - xy)[1 - xy + y^2]^{-1}. \tag{3.2}$$

Simson's Formula

$$v_{n+1}(x)v_{n-1}(x) - v_n^2(x) = \begin{cases} -1 & n \text{ odd}, \\ \Delta^2 & n \text{ even}. \end{cases} \tag{3.3}$$

Negative Subscript

$$v_{-n}(x) = v_n(x). \tag{3.4}$$

Miscellany

$$v_n(x)v_{n-1}(-x) + v_n(-x)v_{n-1}(x) = 0. \tag{3.5}$$

$$v_n^2(x) + v_{n-1}^2(x) - xv_n(x)v_{n-1}(x) = -\Delta^2. \tag{3.6}$$

$$v_n(x^2 - 2) - v_n^2(x) = -2. \tag{3.7}$$

Remarks:

(i) Results (3.3)-(3.7) may be determined by applying (3.1). To establish (3.5) by an alternative method, follow the approach used in [6] for the analogous equation for $V_n(x)$.

(ii) Both (3.6) and (3.7) occur, in effect, in [3].

(iii) There are no results for $V_n(x)$ corresponding to (3.6) and (3.7) for $v_n(x)$.

(iv) Observe that, for $v_n(x^2 - 2)$, the expressions corresponding to α, β, and Δ in (1.10) become $\alpha^* = \alpha^2$, $\beta^* = \beta^2$, $\Delta^* = x\Delta$.

Permutability

Theorem 1 (Jacobsthal [3]): $v_m(v_n(x)) = v_n(v_m(x)) = v_{mn}(x)$.

Proof: Adapting Jacobsthal's neat treatment of this elegant result, we notice the key nexus

$$v_n(x) = v_n\left(\alpha + \frac{1}{\alpha}\right) = \alpha^n + \alpha^{-n} \ \text{(by (1.11), (3.1))}. \tag{3.8}$$

whence

$$
\begin{aligned}
v_{mn}(x) &= \alpha^{nm} + \alpha^{-nm} \quad \text{(by (3.1))}\\
&= v_n(\alpha^m + \alpha^{-m}) \quad \text{(by (3.8))}\\
&= v_n(v_m(x)) \quad \text{(by (3.1))}\\
&= v_m(v_n(x)) \quad \text{also.}
\end{aligned}
$$

Remark: There is no result for $V_n(x)$ corresponding to Theorem 1 (Jacobsthal's theorem) for $v_n(x)$, i.e., the $V_n(x)$ are nonpermutable [cf. (9.3), (9.4)].

4. PROPERTIES OF $V_n(x)$, $v_n(x)$

Elementary methods, mostly involving Binet forms (2.1) and (3.1), disclose the following quintessential relations connecting $V_n(x)$ and $v_n(x)$.

$$V_n(x)v_n(x) = V_{2n}(x). \tag{4.1}$$

$$V_{n+1}(x) - V_{n-1}(x) = v_n(x). \tag{4.2}$$

$$v_{n+1}(x) - v_{n-1}(x) = \Delta^2 V_n(x). \tag{4.3}$$

$$v_n(x) = 2V_{n+1}(x) - xV_n(x). \tag{4.4}$$

$$\Delta^2 V_n(x) = 2v_{n+1}(x) - xv_n(x). \tag{4.5}$$

Notice that (4.4) is a direct consequence of the generating function definitions (2.2) and (3.2).

Summation

$$\Delta^2 \sum_{n=1}^{m} V_n(x) = v_{m+1}(x) + v_m(x) - x - 2 \quad \text{(by (4.3))}. \tag{4.6}$$

$$\sum_{n=1}^{m} v_n(x) = V_{m+1}(x) + V_m(x) - 1 \quad \text{(by (4.2))}. \tag{4.7}$$

Sums (Differences) of Products

$$V_m(x)v_n(x) + V_n(x)v_m(x) = 2V_{m+n}(x). \tag{4.8}$$

$$V_m(x)v_n(x) - V_n(x)v_m(x) = 2V_{m-n}(x). \tag{4.9}$$

$$v_m(x)v_n(x) + \Delta^2 V_m(x)V_n(x) = 2v_{m+n}(x). \tag{4.10}$$

$$v_m(x)v_n(x) - \Delta^2 V_m(x)V_n(x) = 2v_{m-n}(x). \tag{4.11}$$

Special cases $m = n$: In turn, the reductions are (4.1), $0 = 0$ (1.7a), and

$$v_n^2(x) + \Delta^2 V_n^2(x) = 2v_{2n}(x) \quad \text{(by (4.10))}, \tag{4.12}$$

$$v_n^2(x) - \Delta^2 V_n^2(x) = 4 \quad \text{(by (4.11))}. \tag{4.13}$$

Associated Sequences

Definitions: The k^{th} *associated sequences* $\{V_n^{(k)}(x)\}$ and $\{v_n^{(k)}(x)\}$ of $\{V_n(x)\}$ and $\{v_n(x)\}$ are defined by, respectively ($k \geq 1$),

$$V_n^{(k)}(x) = V_{n+1}^{(k-1)}(x) - V_{n-1}^{(k-1)}(x), \tag{4.14}$$

$$v_n^{(k)}(x) = v_{n+1}^{(k-1)}(x) - v_{n-1}^{(k-1)}(x), \tag{4.15}$$

where $V_n^{(0)}(x) = V_n(x)$ and $v_n^{(0)}(x) = v_n(x)$.

What are the ramifications of these ideas?

Immediately,

$$V_n^{(1)}(x) = v_n(x) \quad \text{(from (4.2))}, \tag{4.16}$$

$$v_n^{(1)}(x) = \Delta^2 V_n(x) \quad \text{(from (4.3))} \tag{4.17}$$

are the generic members of the *first associated sequences* $\{V_n^{(1)}(x)\}$ and $\{v_n^{(1)}(x)\}$.

Repeated application of the above formulas eventually reveals the succinct results:

$$V^{2m}(x) = v_n^{(2m-1)}(x) = \Delta^{2m} V_n(x), \tag{4.18}$$

$$V_n^{(2m+1)}(x) = v_n^{2m}(x) = \Delta^{2m} v_n(x). \tag{4.19}$$

5. THE ARGUMENT $-x^2$: VIETA AND MORGAN-VOYCE

Attractively simple formulas can be found to relate the Vieta polynomials to Morgan-Voyce polynomials having argument $-x^2$. Valuable space is preserved in this paper by asking the reader to consult [2] and [6] for the relevant combinatorial definitions of the Morgan-Voyce polynomials $B_n(x)$, $b_n(x)$, $C_n(x)$, and $c_n(x)$.

Alternative proofs are provided specifically to heighten insights into the structure of the polynomials. Equalities in some proofs require a reverse order of terms.

Theorem 2:

(a) $V_{2n}(x) = (-1)^{n-1} x B_n(-x^2)$.

(b) $V_{2n-1}(x) = (-1)^{n-1} b_n(-x^2)$.

(a)

Proof 1:

$$(-1)^{n-1} x B_n(-x^2) = \sum_{k=0}^{n-1} (-1)^{k+n-1} \binom{n+k}{2k+1} x^{2k+1} \quad \text{(by [6, (2.20)])}$$
$$= V_{2n}(x) \quad \text{(by (1.3))}.$$

Proof 2:

$$V_{2n}(x) = (-1)^{n-1} x[b_n(-x^2) + B_{n-1}(-x^2)] \quad \text{(by [6] adjusted)}$$
$$= (-1)^{n-1} x B_n(-x^2) \quad \text{(by [2, (2.13)])}.$$

(b)

Proof 1:

$$(-1)^{n-1} b_n(-x^2) = \sum_{k=0}^{n-1} (-1)^{k+n-1} \binom{n+k-1}{2k} x^{2k} \quad \text{(by [2, (2.21)])}$$
$$= V_{2n-1}(x) \quad \text{(by (1.13))}.$$

Proof 2:

$$V_{2n-1}(x) = (-1)^n (x^2 B_n(-x^2) - b_{n-1}(-x^2)) \quad \text{(by [6] adjusted)}$$
$$= (-1)^n (-b_n(-x^2)) \quad \text{(by [2, (2.15)])}$$
$$= (-1)^{n-1} b_n(-x^2).$$

Corollary 1: $V_{2n-1}(ix) = (-1)^{n-1} b_n(x^2)$ $(i^2 = -1)$.

Theorem 3:

(*a*) $v_{2n}(x) = (-1)^n C_n(-x^2)$.

(*b*) $v_{2n-1}(x) = (-1)^{n-1} x c_n(-x^2)$.

(a)

Proof:

$$\begin{aligned}(-1)^n C_n(-x^2) &= (-1)^n \left\{ \sum_{k=0}^{n-1} (-1)^k \frac{2n}{n-k} \binom{n-1+k}{n-1-k} x^{2k} + (-1)^n x^{2n} \right\} \quad \text{(by [6, (2.2)])} \\ &= v_{2n}(x) \quad \text{(by (1.5))} \\ &[= (-1)^n (C_{n-1}(-x^2) - x^2 c_n(-x^2)) \quad \text{(by (3.21)]).}\end{aligned}$$

(b)

Proof:

$$\begin{aligned}(-1)^{n-1} x c_n(-x^2) &= \sum_{k=1}^{n} (-1)^{k+n} \frac{2n-1}{2k-1} \binom{n+k-2}{n-k} x^{2k-1} \quad \text{(by [2, (3.23)])} \\ &= v_{2n-1}(x) \quad \text{(by (1.5))} \\ &[= (-1)^{n-1} x (C_{n-1}(-x^2) + c_{n-1}(-x^2)) \quad \text{(by [2, (3.11)]]).}\end{aligned}$$

Corollary 2: $v_{2n}(ix) = (-1)^n C_n(x^2)$ $(i^2 = -1)$.

6. THE ARGUMENT $-\frac{1}{x^2}$: VIETA AND JACOBSTHAL

Here, we discover connections between the Vieta and Jacobsthal polynomials.

Theorem 4:

(*a*) $V_n(x) = x^{n-1} J_n\left(-\frac{1}{x^2}\right)$.

(*b*) $v_n(x) = x^n j_n\left(-\frac{1}{x^2}\right)$ (by [6, (2.7)]).

(a)

Proof:

$$\begin{aligned}V_n(x) &= x^{n-1} \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-j-1}{j} \left(-\frac{1}{x^2}\right)^j \quad \text{(by (1.3))} \\ &= x^{n-1} J_n\left(-\frac{1}{x^2}\right) \quad \text{(by [6, (2.3)])} \\ &\left[\begin{array}{l} = x^{n-1}\left[J_{n-1}\left(-\frac{1}{x^2}\right) + \left(-\frac{1}{x^2}\right) J_{n-2}\left(-\frac{1}{x^2}\right)\right] \text{ by definition of } J_n(x) \\ = x^{n-1} J_{n-1}\left(-\frac{1}{x^2}\right) - x^{n-3} J_{n-2}\left(-\frac{1}{x^2}\right) \text{ as in [6] adjusted} \end{array}\right].\end{aligned}$$

(b)

Proof:

$$x^n j_n\left(-\frac{1}{x^2}\right) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n}{n-k}\binom{n-k}{k} x^{n-2k} \quad \text{(by [6, (2.6)])}$$
$$= v_n(x) \quad \text{(by (1.5) or [5, (1.9)])}$$
$$\left[\begin{array}{l} = x^n\left[j_{n-1}\left(-\frac{1}{x^2}\right) + \left(-\frac{1}{x^2}\right) j_{n-2}\left(-\frac{1}{x^2}\right)\right] \text{ by definition of } j_n(x) \\ = x^n j_{n-1}\left(-\frac{1}{x^2}\right) - x^{n-2} j_{n-2}\left(-\frac{1}{x^2}\right) \end{array}\right].$$

7. THE ARGUMENT $\frac{1}{x}$: JACOBSTHAL AND MORGAN-VOYCE

Next, we detect some attractive simple links between Jacobsthal and Morgan-Voyce polynomials involving reciprocal arguments x, $\frac{1}{x}$.

Theorem 5:

(a) $B_n(x) = x^{n-1} J_{2n}\left(\frac{1}{x}\right)$.

(b) $C_n(x) = x^n j_{2n}\left(\frac{1}{x}\right)$.

(a) This is stated and proved in [6, (2.8)].

(b)

Proof:

$$x^n j_{2n}\left(\frac{1}{x}\right) = \sum_{k=0}^{n} \frac{2n}{2n-k}\binom{2n-k}{k} x^{n-k} \quad \text{(by [6, (2.6)])}$$
$$= \sum_{k=0}^{n-1} \frac{2n}{2n-k}\binom{2n-k}{k} x^{n-k} + 2$$
$$= C_n(x) \quad \text{(by [6, (2.2)])}.$$

Upon making the transformation $x \to \frac{1}{x}$ in Theorem 5(a) and (b), we obtain their *Mutuality Properties* in Corollary 3(a) and (b).

Corollary 3 (Mutuality):

(a) $J_{2n}(x) = x^{n-1} B_n\left(\frac{1}{x}\right)$.

(b) $j_{2n}(x) = x^n C_n\left(\frac{1}{x}\right)$.

Combining Theorems 2(a) and 4(a), we get

$$x^{2n-1} J_{2n}\left(-\frac{1}{x^2}\right) = V_{2n}(x) = (-1)^{n-1} x B_n(-x^2)$$

leading to

$$B_n(-x^2) = (-x^2)^{n-1} J_{2n}\left(-\frac{1}{x^2}\right),$$

thus confirming Theorem 5(a) when $x \to -x^2$. Conclusions of a similar nature link $j_{2n}(-\frac{1}{x^2})$, $v_{2n}(x)$, and $b_n(-x^2)$ in Theorems 3(a), 4(b), and 5(b).

Theorem 6:

(a) $b_n(x) = x^{n-1} J_{2n-1}\left(\frac{1}{x}\right)$.

(b) $c_n(x) = x^{n-1} j_{2n-1}\left(\frac{1}{x}\right)$.

Proof: Similar to that for Theorem 5.

Corollary 4 (Mutuality):

(a) $J_{2n-1}(x) = x^{n-1} b_n\left(\frac{1}{x}\right)$.

(b) $j_{2n-1}(x) = x^{n-1} c_n\left(\frac{1}{x}\right)$.

8. ZEROS OF $V_n(x)$, $v_n(x)$

Known zeros of the Morgan-Voyce polynomials [2, (4.20)-(4.23)] may be employed to detect the zeros of the Vieta and the Jacobsthal polynomials. Some elementary trigonometry is required.

(a) $V_n(x) = 0$

By [2, (4.20)] and Theorem 2(a) with $x \to -x^2$, the $2n-1$ zeros of $V_{2n}(x)$ are 0 and the $2(n-1)$ zeros of $B_n(-x^2)$, namely ($r = 1, 2, \ldots, n-1$),

$$\begin{aligned} x &= \pm 2\sin\left(\frac{r}{n}\frac{\pi}{2}\right) = \pm 2\cos\left(\frac{n-r}{2n}\pi\right) \\ &= 2\cos\frac{r}{m}\pi \;\; (m = 2n, \text{ i.e., } m \text{ even}). \end{aligned} \tag{8.1}$$

Similarly, by [2, (4.21)] and Theorem 2(b) with $x \to -x^2$, the $2n-2$ zeros of $V_{2n-1}(x)$ are the $2(n-1)$ zeros of $b_n(-x^2)$, namely ($r = 1, 2, \ldots, n-1$),

$$\begin{aligned} x &= \pm 2\sin\left(\frac{2r-1}{2n-1}\frac{\pi}{2}\right) = \pm 2\cos\left(\frac{n-r}{2n-1}\pi\right) \\ &= 2\cos\frac{r}{m}\pi \;\; (m = 2n-1, \text{ i.e., } m \text{ odd}). \end{aligned} \tag{8.2}$$

Zeros $2\cos\frac{r}{m}\pi$ given in (8.1) and (8.2) are precisely those given in [7, (2.25)] for $y = -1$ (for $V_m(x)$) when m is even or odd. See also [7, (2.23)].

(b) $v_n(x)=0$

Invoking Theorems 3(a) and 3(b) next in conjunction with [2, (4.22), (4.23)] for $C_n(x)$ and $c_n(x)$ and making the transformation $x \to -x^2$, we discover the n zeros of $v_n(x)$ are $(r=1,\dots,n)$

$$x = 2\cos\left(\frac{2r-1}{2n}\pi\right)$$

which is in accord with [7, (2.26)]. See also [7, (2.24)].

Alternative approach to (a) and (b) above: Use the known roots for Chebyshev polynomials (9.3) and (9.4).

(c) Zeros of $J_n(x)$, $j_n(x)$

From Theorems 4(a), 4(b), it follows that the zeros of $J_n(x)$, $j_n(x)$ are given by $-\frac{1}{x^2} \to x$. This leads in (8.1)-(8.3) to the zeros of $J_n(x)$, $j_n(x)$ as

$$-\frac{1}{4\cos^2\frac{r\pi}{n}}, \quad -\frac{1}{4\cos^2\left(\frac{2r-1}{2n}\pi\right)},$$

that is, for

$$\underline{\underline{\text{(c)}}}\ J_n(x)=0:\quad x=-\frac{1}{4}\sec^2\frac{r\pi}{n}, \tag{8.4}$$

$$\underline{\underline{\text{(d)}}}\ j_n(x)=0:\quad x=-\frac{1}{4}\sec^2\left(\frac{2r-1}{2n}\pi\right). \tag{8.5}$$

These zero values concur with those given in [7, (2.28(, (2.29)] if we remember that $2x$ in the definitions for $J_n(x)$, $j_n(x)$ in [7] has to be replaced by x in this paper (as in [6]). Refer also to Corollaries 3(a) and 3(b).

9. MEDLEY

Lastly, we append some Vieta-related features of familiar polynomials.

Fibonacci and Lucas Polynomials $F_n(x)$, $L_n(x)$

$$V_n(ix) = i^{n-1}F_n(x) \quad (i^2=-1). \tag{9.1}$$

$$v_n(ix) = i^n L_n(x) \quad ([5]). \tag{9.2}$$

Chebyshev Polynomials $T_n(x)$, $U_n(x)$

$$V_n(x) = U_n\left(\frac{1}{2}x\right). \tag{9.3}$$

$$v_n(x) = 2T_n\left(\frac{1}{2}x\right) \quad ([3], [5]). \tag{9.4}$$

Suggested Topics for Further Development

1. **Irreducibility, divisibility:** Detailed analysis for $v_n(x)$ as in [5] is, for $V_n(x)$, left to the *aficionados* (having regard to Tables 1 and 2);
2. **Rising and falling diagonals** for Vieta polynomials (which has already been done for the Chebyshev polynomials and which has been almost completed for Vieta polynomials);

3. **Convolutions** for $V_n(x)$ and $v_n(x)$ (in which much progress has been achieved);
4. **Numerical values:** Consider various integer values of x in $V_n(x)$ and $v_n(x)$ to obtain sets of *Vieta numbers*. Some nice results ensue. Guidance may be sought in [2, pp. 172-73].

Conclusion

Apparently the $v_n(x)$ offer a slightly richer field of exploration than do the $V_n(x)$. However, many opportunities for discovery present themselves. Hopefully, this paper may whet the appetite of some readers to undertake further experiences.

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PENTAGONAL NUMBERS IN THE PELL SEQUENCE AND DIOPHANTINE EQUATIONS $2x^2 = y^2(3y-1)^2 \pm 2$

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1. INTRODUCTION

It is well known that a positive integer N is called a *pentagonal* (*generalized pentagonal*) *number* if $N = m(3m-1)/2$ for some integer $m > 0$ (for any integer m).

Ming Luo [1] has proved that 1 and 5 are the only pentagonal numbers in the *Fibonacci sequence* $\{F_n\}$. Later, he showed (in [2]) that 2, 1, and 7 are the only generalized pentagonal numbers in the *Lucas sequence* $\{L_n\}$. In [3] we have proved that 1 and 7 are the only generalized pentagonal numbers in the *associated Pell sequence* $\{Q_n\}$ defined by

$$Q_0 = Q_1 = 1 \text{ and } Q_{n+2} = 2Q_{n+1} + Q_n \text{ for } n \geq 0. \tag{1}$$

In this paper, we consider the *Pell sequence* $\{P_n\}$ defined by

$$P_0 = 0, P_1 = 1, \text{ and } P_{n+2} = 2P_{n+1} + P_n \text{ for } n \geq 0 \tag{2}$$

and prove that $P_{\pm 1}$, $P_{\pm 3}$, P_4, and P_6 are the only pentagonal numbers. Also we show that P_0, $P_{\pm 1}$, P_2, $P_{\pm 3}$, P_4, and P_6 are the only generalized pentagonal numbers. Further, we use this to solve the Diophantine equations of the title.

2. PRELIMINARY RESULTS

We have the following well-known properties of $\{P_n\}$ and $\{Q_n\}$: for all integers m and n,

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \text{ and } Q_n = \frac{\alpha^n + \beta^n}{2}, \text{ where } \alpha = 1+\sqrt{2} \text{ and } \beta = 1-\sqrt{2}, \tag{3}$$

$$P_{-n} = (-1)^{n+1}P_n \text{ and } Q_{-n} = (-1)^n Q_n, \tag{4}$$

$$Q_n^2 = 2P_n^2 + (-1)^n, \tag{5}$$

$$Q_{3n} = Q_n(Q_n^2 + 6P_n^2), \tag{6}$$

$$P_{m+n} = 2P_m Q_n - (-1)^n P_{m-n}. \tag{7}$$

If m is odd, then:

$$\left.\begin{array}{l} \text{(i) } Q_m^2 + 6P_m^2 \equiv 7 \pmod 8, \quad \text{(ii) } P_m \equiv 1 \pmod 4, \\ \text{(iii) } Q_m \equiv \pm 1 \pmod 4 \text{ according as } m \equiv \pm 1 \pmod 4. \end{array}\right\} \tag{8}$$

Lemma 1: If n, k, and t are integers, then $P_{n+2kt} \equiv (-1)^{t(k+1)} P_n \pmod{Q_k}$.

Proof: If $t = 0$, the lemma is trivial and it can be proved for $t > 0$ by using induction on t with (7). If $t < 0$, say $t = -m$, where $m > 0$, then by (4) we have

$$P_{n+2kt} = P_{n-2km} = P_{n+2(-k)m} \equiv (-1)^{t(-k+1)} P_n \pmod{Q_{-k}} \equiv (-1)^{t(k+1)} P_n \pmod{Q_k},$$

proving the lemma.

3. SOME LEMMAS

Since $N = m(3m-1)/2$ if and only if $24N+1=(6m-1)^2$, we have that N is generalized pentagonal if and only if $24N+1$ is the square of an integer congruent to 5 (mod 6). Therefore, in this section we identify those n for which $24P_n+1$ is a perfect square.

We begin with

Lemma 2: Suppose $n \equiv \pm 1 \pmod{2^2 \cdot 5}$. Then $24P_n+1$ is a perfect square if and only if $n = \pm 1$.

Proof: If $n = \pm 1$, then by (4) we have $24P_n + 1 = 24P_{\pm 1}+1 = 5^2$. Conversely, suppose $n = \pm 1 \pmod{2^2 \cdot 5}$ and $n \notin \{-1, 1\}$. Then n can be written as $n = 2\cdot 11^r \cdot 5m \pm 1$, where $r \geq 0$, $11 \nmid m$, and $2 | m$. Taking

$$k = \begin{cases} 5m & \text{if } m \equiv \pm 2 \text{ or } \pm 8 \pmod{22}, \\ m & \text{otherwise,} \end{cases}$$

we get that

$$k \equiv \pm 4, \pm 6, \text{ or } \pm 10 \pmod{22}, \text{ and } n = 2kg \pm 1, \text{ where } g \text{ is odd (in fact, } g = 11^r \cdot 5 \text{ or } 11^r). \tag{9}$$

Now, by Lemma 1, (9), and (4), we get

$$\begin{aligned} 24P_n + 1 = 24P_{2kg\pm 1} + 1 &\equiv 24(-1)^{g(k+1)}P_{\pm 1} + 1 \pmod{Q_k} \\ &\equiv 24(-1)+1 \pmod{Q_k} \equiv -23 \pmod{Q_k}. \end{aligned}$$

Therefore, the Jacobi symbol

$$\left(\frac{24P_n+1}{Q_k}\right) = \left(\frac{-23}{Q_k}\right) = \left(\frac{Q_k}{23}\right). \tag{10}$$

But modulo 23, the sequence $\{Q_n\}$ has period 22. That is, $Q_{n+22t} \equiv Q_n \pmod{23}$ for all integers $t \geq 0$. Thus, by (9) and (4), we get $Q_k \equiv Q_{\pm 4}$, $Q_{\pm 6}$, or $Q_{\pm 10} \pmod{23} \equiv 17$, 7, or 5 (mod 23), so that

$$\left(\frac{Q_k}{23}\right) = \left(\frac{17}{23}\right), \left(\frac{7}{23}\right), \text{ or } \left(\frac{5}{23}\right),$$

and in any case

$$\left(\frac{Q_k}{23}\right) = -1. \tag{11}$$

From (10) and (11), it follows that

$$\left(\frac{24P_n+1}{Q_k}\right) = -1 \text{ for } n \notin \{-1, 1\},$$

showing $24P_n+1$ is not a perfect square. Hence, the lemma.

Lemma 3: Suppose $n \equiv \pm 3 \pmod{2^4}$. Then $24P_n+1$ is a perfect square if and only if $n = \pm 3$.

Proof: If $n = \pm 3$, then by (4) we have $24P_n+1 = 24P_{\pm 3}+1 = 11^2$. Conversely, suppose $n = \pm 3 \pmod{2^4}$ and $n \notin \{-3, 3\}$. Then n can be written as $n = 2 \cdot 3^r \cdot k \pm 3$, where $r \geq 0$, $3 \nmid k$, and $8 | k$. And we get that

$$k \equiv \pm 8 \text{ or } \pm 16 \pmod{48} \text{ and } n = 2kg \pm 3, \text{ where } g = 3^r \text{ is odd and } k \text{ is even.} \tag{12}$$

Now, by Lemma 1, (12), and (4), we get

$$24P_n+1=24P_{2kg\pm 3}+1=24(-1)^{g(k+1)}P_{\pm 3}+1 \pmod{Q_k} \equiv -119 \pmod{Q_k}.$$

Hence, the Jacobi symbol

$$\left(\frac{24P_n+1}{Q_k}\right)=\left(\frac{-119}{Q_k}\right)=\left(\frac{Q_k}{119}\right). \tag{13}$$

But, modulo 119, the sequence $\{Q_n\}$ has period 48. Therefore, by (12) and (4), we get $Q_k \equiv Q_{\pm 8}$ or $Q_{\pm 16} \pmod{119} \equiv 101$ or $52 \pmod{119}$, and in any case,

$$\left(\frac{Q_k}{119}\right)=-1. \tag{14}$$

From (13) and (14), it follows that

$$\left(\frac{24P_n+1}{Q_k}\right)=-1 \text{ for } n \notin \{-3, 3\},$$

showing that $24P_n+1$ is not a perfect square. Hence the lemma.

Lemma 4: Suppose $n \equiv 4 \pmod{2^2 \cdot 5}$. Then $24P_n+1$ is a perfect square if and only if $n=4$.

Proof: If $n=4$, then $24P_n+1=24P_4+1=17^2$. Conversely, suppose $n \equiv 4 \pmod{2^2 \cdot 5}$ and $n \neq 4$. Then n can be written as $n=2\cdot 3^r \cdot 5m+4$, where $r \geq 0$, $2|m$, and $3\nmid m$. Taking

$$k=\begin{cases} m & \text{if } m \equiv \pm 10 \pmod{30}, \\ 5m & \text{otherwise,} \end{cases}$$

we get that

$$k \equiv \pm 10 \pmod{30} \text{ and } n=2kg+4, \text{ where } g \text{ is odd (in fact, } g=3^r \text{ or } 3^r\cdot 5). \tag{15}$$

Now, by Lemma 1 and (15), we get

$$24P_n+1=24P_{2kg+4}+1 \equiv 24(-1)^{g(k+1)}P_4+1 \pmod{Q_k} \equiv -287 \pmod{Q_k}.$$

Hence, the Jacobi symbol

$$\left(\frac{24P_n+1}{Q_k}\right)=\left(\frac{-287}{Q_k}\right)=\left(\frac{Q_k}{287}\right). \tag{16}$$

But, modulo 287, the sequence $\{Q_n\}$ has period 30. Therefore, by (15) and (4), we get $Q_k \equiv Q_{\pm 10} \pmod{287} \equiv 206 \pmod{287}$, so that

$$\left(\frac{Q_k}{287}\right)=\left(\frac{206}{287}\right)=-1. \tag{17}$$

From (16) and (17), it follows that

$$\left(\frac{24P_n+1}{Q_k}\right)=-1 \text{ for } n \neq 4,$$

showing that $24P_n+1$ is not a perfect square. Hence the lemma.

Lemma 5: Suppose $n \equiv 2 \pmod{2^2 \cdot 5 \cdot 7}$. Then $24P_n + 1$ is a perfect square if and only if $n = 2$.

Proof: If $n = 2$, then we have $24P_n + 1 = 24P_2 + 1 = 7^2$. Conversely, suppose $n \equiv 2$ (mod $2^2 \cdot 5 \cdot 7$) and $n \neq 2$. Then n can be written as $n = 2 \cdot 23^r \cdot 5 \cdot 7m + 2$, where $r \geq 0$, $23 \nmid m$, and $2|m$. Taking

$$k = \begin{cases} 7m & \text{if } m \equiv \pm 16 \pmod{46}, \\ 5m & \text{if } m = \pm 2, \pm 4, \pm 12, \pm 22 \pmod{46}, \\ m & \text{otherwise,} \end{cases}$$

we get that

$$\begin{aligned} &k \equiv \pm 6, \pm 8, \pm 10, \pm 14, \pm 18, \pm 20 \pmod{46} \text{ and } n = 2kg + 2, \text{ where } g \text{ is odd} \\ &(\text{in fact, } g = 23^r \cdot 5 \cdot 7,\ 23^r \cdot 7, \text{ or } 23^r \cdot 5). \end{aligned} \tag{18}$$

Now, by Lemma 1 and (18), we get

$$24P_n + 1 = 24P_{2kg+2} + 1 \equiv 24(-1)^{g(k+1)} P_2 + 1 \pmod{Q_k} \equiv -47 \pmod{Q_k}.$$

Hence, the Jacobi symbol

$$\left(\frac{24P_n + 1}{Q_k}\right) = \left(\frac{-47}{Q_k}\right) = \left(\frac{Q_k}{47}\right). \tag{19}$$

But, modulo 47, the sequence $\{Q_n\}$ has period 46. Therefore, by (18) and (4), we get $Q_k \equiv Q_{\pm 6}$, $Q_{\pm 8}$, $Q_{\pm 10}$, $Q_{\pm 14}$, $Q_{\pm 18}$, or $Q_{\pm 20}$ (mod 47) = 5, 13, 26, 33, 15, or 35 (mod 47), so that

$$\left(\frac{Q_k}{47}\right) = -1. \tag{20}$$

From (19) and (20), it follows that

$$\left(\frac{24P_n + 1}{Q_k}\right) = -1 \text{ for } n \neq 2,$$

showing $24P_n + 1$ is not a perfect square. Hence the lemma.

Lemma 6: Suppose $n \equiv 6 \pmod{2^2 \cdot 3 \cdot 5 \cdot 7}$. Then $24P_n + 1$ is a perfect square if and only if $n = 6$.

Proof: If $n = 6$, then we have $24P_n + 1 = 24P_6 + 1 = 41^2$. Conversely, suppose $n \equiv 6$ (mod $2^2 \cdot 3 \cdot 5 \cdot 7$) and $n \neq 6$. Then n can be written as $n = 2 \cdot 3^r \cdot 3 \cdot 5 \cdot 7m + 2$, where $r \geq 0$, $2|m$, and $3 \nmid m$, which implies that $m \equiv \pm 2 \pmod 6$. Taking

$$k = \begin{cases} 3 \cdot 5m & \text{if } m \equiv \pm 2, \pm 32, \pm 52, \pm 76, \pm 82, \pm 86, \pm 100, \pm 124, \\ & \qquad \pm 130, \pm 170, \pm 178, \text{or } \pm 188 \pmod{396}, \\ 7m & \text{if } m \equiv \pm 26, \pm 62, \text{ or } \pm 88 \pmod{396}, \\ 3m & \text{if } m \equiv \pm 4, \pm 10, \pm 14, \pm 20, \pm 22, \pm 28, \pm 40, \pm 58, \pm 74, \pm 98, \pm 104, \\ & \qquad \pm 110, \pm 116, \pm 136, \pm 146, \pm 148, \pm 172, \text{ or } \pm 196 \pmod{396}, \\ m & \text{otherwise,} \end{cases}$$

we get that

$$\begin{aligned} k \equiv\; & \pm 8, \pm 12, \pm 16, \pm 30, \pm 34, \pm 38, \pm 42, \pm 44, \pm 46, \pm 48, \pm 50, \pm 56, \pm 60, \\ & \pm 64, \pm 66, \pm 68, \pm 70, \pm 80, \pm 84, \pm 92, \pm 94, \pm 102, \pm 106, \pm 112, \pm 118, \\ & \pm 120, \pm 122, \pm 128, \pm 134, \pm 140, \pm 142, \pm 152, \pm 154, \pm 158, \pm 160, \pm 164, \\ & \pm 166, \pm 174, \pm 176, \pm 182, \pm 184, \pm 190, \pm 192, \pm 194, \pm 202, \pm 204, \pm 206, \\ & \pm 212, \pm 214, \pm 220, \pm 222, \pm 230, \pm 232, \pm 236, \pm 238, \pm 242, \pm 244, \pm 254, \\ & \pm 256, \pm 262, \pm 268, \pm 274, \pm 276, \pm 278, \pm 284, \pm 290, \pm 294, \pm 302, \pm 304, \\ & \pm 312, \pm 316, \pm 326, \pm 328, \pm 330, \pm 332, \pm 336, \pm 340, \pm 346, \pm 348, \pm 350, \\ & \pm 352, \pm 354, \pm 358, \pm 362, \pm 366, \pm 380, \pm 384, \text{ or } \pm 388 \pmod{792} \end{aligned} \tag{21}$$

and

$$n = 2kg + 6, \text{ where } g \text{ is odd and } k \text{ is even.} \tag{22}$$

Now, by Lemma 1 and (22), we get

$$24P_n + 1 = 24P_{2kg+6} + 1 \equiv 24(-1)^{g(k+1)} P_6 + 1 \pmod{Q_k} \equiv -1679 \pmod{Q_k}.$$

Hence, the Jacobi symbol

$$\left(\frac{24P_n + 1}{Q_k}\right) = \left(\frac{-1679}{Q_k}\right) = \left(\frac{Q_k}{1679}\right). \tag{23}$$

But, modulo 1679, the sequence $\{Q_n\}$ has period 792. Therefore, by (21) and (4), we get

$$\begin{aligned} Q_k \equiv\; & 577, 1132, 973, 485, 143, 1019, 923, 737, 141, 109, 513, 97, 329, 1015, \\ & 829, 601, 1098, 577, 1351, 1144, 513, 485, 362, 348, 1382, 1569, 1316, \\ & 316, 808, 163, 879, 1015, 1611, 1604, 973, 925, 1316, 923, 1151, 1019, \\ & 1589, 1382, 766, 1535, 1604, 329, 370, 163, 76, 1404, 26, 1385, 97, 122, \\ & 1535, 944, 1613, 143, 1589, 141, 1144, 1385, 1132, 370, 601, 1098, 1267, \\ & 582, 316, 109, 1175, 362, 348, 47, 1613, 766, 925, 582, 1351, 808, 139, 26, \\ & 76, 879, 1267, 122, 1569, \text{ or } 1175 \pmod{1679}, \text{ respectively.} \end{aligned}$$

And for all these values of k, the Jacobi symbol

$$\left(\frac{Q_k}{1679}\right) = -1. \tag{24}$$

From (23) and (24), it follows that

$$\left(\frac{24P_n + 1}{Q_k}\right) = -1 \text{ for } n \neq 6,$$

showing that $24P_n + 1$ is not a perfect square. Hence the lemma.

Lemma 7: Suppose $n \equiv 0 \pmod{2 \cdot 3 \cdot 7^2 \cdot 13}$. Then $24P_n + 1$ is a perfect square if and only if $n = 0$.

Proof: If $n = 0$, then we have $24P_n + 1 = 24P_0 + 1 = 1^2$. Conversely, suppose $n \equiv 0$ (mod $2 \cdot 3 \cdot 7^2 \cdot 13$) and for $n \neq 0$ put $n = 2 \cdot 7^2 \cdot 13 \cdot 3^r \cdot z$, where $r \geq 1$ and $3 \nmid z$. We choose m as follows:

$$m=\begin{cases}13\cdot 3^r & \text{if } r\equiv \pm 1 \pmod 4 \text{ according as } z\equiv \pm 1 \pmod 3,\\ 7\cdot 3^r & \text{if } r\equiv \pm 3 \pmod 4 \text{ according as } z\equiv \pm 1 \pmod 3,\\ 7^2\cdot 3^r & \text{if } r\equiv 0 \pmod 4, z\equiv 1 \pmod 3 \text{ or } r\equiv 2 \pmod 4, z\equiv 2 \pmod 3,\\ 3^r & \text{if } r\equiv 2 \pmod 4, z\equiv 1 \pmod 3 \text{ or } r\equiv 0 \pmod 4, z\equiv 2 \pmod 3.\end{cases}$$

Then $n=2m(3k\pm 1)$ for some integer k and odd m. Since, for $r\geq 1$, we have $3^r\equiv 3, 9, 27$, or 21 (mod 30) according as $r=1, 2, 3$, or 0 (mod 4), it follows that

$$m\equiv \pm 9 \pmod{30} \text{ according as } z\equiv \pm 1 \pmod 3. \tag{25}$$

Therefore, by Lemma 1, (4), (6), and the fact that m is odd, we have

$$\begin{aligned}24P_n+1=24P_{2(3m)k\pm 2m}&\equiv 24(-1)^{k(3m+1)}P_{\pm 2m}+1 \pmod{Q_{3m}}\\ &\equiv \pm 24P_{2m}+1 \pmod{Q_m^2+6P_m^2} \text{ according as } z\equiv \pm 1 \pmod 3.\end{aligned}$$

Letting $w_m=Q_m^2+6P_m^2$ and using (5), (7), and (8), we obtain the Jacobi symbol:

$$\begin{aligned}\left(\frac{24P_n+1}{w_m}\right)&=\left(\frac{\pm 24P_{2m}+1}{w_m}\right)=\left(\frac{\pm 48Q_mP_m-Q_m^2+2P_m^2}{w_m}\right)=\left(\frac{\pm 48Q_mP_m+8P_m^2}{w_m}\right)\\ &=\left(\frac{2}{w_m}\right)\left(\frac{P_m}{w_m}\right)\left(\frac{\pm 6Q_m+P_m}{w_m}\right)=\left(\frac{\pm 6Q_m+P_m}{w_m}\right)=-\left(\frac{w_m}{\pm 6Q_m+P_m}\right)\\ &=-\left(\frac{(\pm 6Q_m+P_m)(\pm 6Q_m-P_m)+217P_m^2}{\pm 6Q_m+P_m}\right)=-\left(\frac{217}{\pm 6Q_m+P_m}\right)\\ &=-\left(\frac{6Q_m\pm P_m}{217}\right)=-\left(\frac{H_m}{217}\right), \text{ where } H_m=6Q_m\pm P_m.\end{aligned}$$

But since

$$\text{modulo 217, the sequence } \{H_m\} \text{ is periodic with period 30.} \tag{26}$$

That is, $H_{n+30u}\equiv H_n$ (mod 217) for all integers $u\geq 0$. And $H_{\pm 9}=6Q_{\pm 9}\pm P_{\pm 9}\equiv \pm 12$ (mod 217). Therefore, by (25) and (26), we get

$$\left(\frac{24P_n+1}{w_n}\right)=-\left(\frac{\pm 12}{217}\right)=-1.$$

As a consequence of Lemmas 2-7, we have the following lemmas.

Lemma 8: Suppose $n\equiv 0, \pm 1, 2, \pm 3, 4$, or 6 (mod 152880). Then $24P_n+1$ is a perfect square if and only if $n\equiv 0, \pm 1, 2, \pm 3, 4$, or 6.

Lemma 9: $24P_n+1$ is not a perfect square if $n\not\equiv 0, \pm 1, 2, \pm 3, 4$, or 6 (mod 152880).

Proof: We prove the lemma in different steps, eliminating at each stage certain integers n congruent modulo 152880 for which $24P_n+1$ is not a square. In each step, we choose an integer m such that the period k (of the sequence $\{P_n\}$ mod m) is a divisor of 152880 and thereby eliminate certain residue classes modulo k. For example:

(a) Mod 41. The sequence $\{P_n\}$ mod 41 has period 10. We can eliminate $n\equiv 8$ (mod 10), since $24P_8+1\equiv 35$ (mod 41) and 35 is a quadratic nonresidue modulo 41. There remain $n\equiv 0$, 1, 2, 3, 4, 5, 6, 7, and 9 (mod 10) or, equivalently $n \equiv 0$, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, and 19 (mod 20).

(b) Mod 29. The sequence $\{P_n\}$ mod 29 has period 20. We can eliminate $n \equiv 7$, 12, 13, 14, 16, and 18 (mod 20), since they imply, respectively, $24P_n + 1 \equiv 26$, 11, 26, 3, 3, and 11 (mod 29). There remain $n \equiv 0$, 1, 2, 3, 4, 5, 6, 9, 10, 11, 15, 17, or 19 (mod 20) or, equivalently, $n \equiv 0$, 1, 2, 3, 4, 5, 6, 9, 10, 11, 15, 17, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31, 35, 37, or 39 (mod 40).

Similarly, we can eliminate the remaining values of n. After reaching modulo 152880, if there remain any values of n, we eliminate them in the higher modulos (i.e., in the multiples of 152880). We tabulate these in Tables A and B.

4. MAIN THEOREM

Theorem 1:

(a) P_n is a generalized pentagonal number only for $n = 0, \pm 1, 2, \pm 3, 4$, or 6.

(b) P_n is a pentagonal number only for $n = \pm 1, \pm 3, 4$, or 6.

Proof:

(a) From Lemmas 8 and 9, the first part of the theorem follows.

(b) Since an integer N is pentagonal if and only if $24N + 1 = (6m-1)^2$, where m is a positive integer, and since $P_0 = 0$, $P_2 = 2$, we have $24P_0 + 1 \neq (6m-1)^2$ and $24P_2 + 1 \neq (6m-1)^2$ for positive integer m, from which it follows that P_0 and P_2 are not pentagonal.

5. SOLUTIONS OF CERTAIN DIOPHANTINE EQUATIONS

If D is a positive integer that is not a perfect square, it is well known that $x^2 - Dy^2 = \pm 1$ is called the Pell equation and that if $x_1 + y_1\sqrt{D}$ is the fundamental solution of it (i.e., x_1 and y_1 are least positive integers), then $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$ is also a solution of the same equation; conversely, every solution of it is of this form.

Now, by (5), we have $Q_n^2 = 2P_n^2 + (-1)^n$ for every n. Therefore, it follows that

$$Q_{2n} + \sqrt{2}P_{2n} \text{ is a solution of } x^2 - 2y^2 = 1, \tag{27}$$

while

$$Q_{2n+1} + \sqrt{2}P_{2n+1} \text{ is a solution of } x^2 - 2y^2 = -1. \tag{28}$$

Thus, the complete set of solutions of the equations $x^2 - 2y^2 = \pm 1$ is given by

$$x = \pm Q_n,\ y = \pm P_n. \tag{29}$$

Theorem 2: The solution set of the Diophantine equation

$$2x^2 = y^2(3y-1)^2 - 2 \tag{30}$$

is $\{(\pm 1, 1), (\pm 7, 2)\}$.

Proof: Writing $Y = y(3y-1)/2$, equation (30) reduces to the form

$$x^2 - 2Y^2 = -1, \tag{31}$$

whose solutions are, by (28), $Q_{2n+1} + \sqrt{2}P_{2n+1}$ for any integer n.

TABLE A

Modulus m	Period k	Required values of n where $\left(\frac{24P_n+1}{m}\right)=-1$	Left out values of n (mod t) where t is a positive integer
41	10	8.	0, ±1, 2, ±3, 4, 5 or 6 (mod 10)
29	20	7, 12, 13, 14 and 16.	0, ±1, 2, ±3, 4, ±5, 6, ± 9 or 10(mod 20)
19	40	5, 15, 17, 19, 21, 22, 23, 25, 26 and 35.	0, ±1, 2, ±3, 4, 6, ± 9, ±10, ±11 or 20 (mod 40)
59	40	24.	
241	80	±9, ±10, ±29, 30, ±31, ±39, 44 and 50.	0, ±1, 2, ±3, 4, 6, ±11, ±20, ±37, 40, 42 or 46 (mod 80)
31	30	±7, ±11, 12, 14, 24 and 26.	0, ±1, 2, ±3, 4, 6, ±60, 100, ±117, 120 or 122 (mod 240)
269	60	±9, ±17, ±21 and 22.	
601	120	46.	
2281	120	20 and 40.	
1153	48	±5, 8, 28, 30 and 32.	
239	14	±5, 7, 8 and 10.	0, ±1, 2, ±3, 4, 6, 420, 840 or 1260 (mod 1680)
13	28	±11, 16, 20 and 26.	
113	56	±25, ±27, 30, 40 and 46.	
337	56	12 and 18.	
71	70	60 and 62.	
83	168	28, ±69 and ±71.	
139	280	42.	
281	280	126.	
37633	336	±165 and 170.	
79	26	±7, 10, 13, 14, 20 and 22.	0, ±1, 2, ±3, 4, 6, 5460, 10920 or 16380 (mod 21840).
599	26	8, ±9, 16 and 24.	
313	78	±11, 18, ±25, ±27, 28, ±29, ±31, 32, ±37, 38, 58 and 64.	
521	260	±21, ±23, 44, 80, ±83, 160, 186, 240 and 246.	
1949	260	±37, ±57, ±63, ±81, 82 and 122.	
1091	312	52, 54 and 168.	
181	364	168, 286 and 338.	
1471	98	±11, 14, ±15, 16, ±17, 18, ±27, 28, ±29, 30, ±39, 46, 48, 56, 58, 60 and 76.	0, ±1, 2, ±3, 4, 6, 38220, 76440 or 114660 (mod 152880).
293	196	±25, ±31, ±53, ±55, 84, ±85, 86, 88, 140 and 172.	
587	1176	±335, 338, 510, 678, 756, 846, 1012 and 1014.	
2939	5880	2520 and 2522.	

We now eliminate: $n \equiv 38220, 76440,$ or $114660 \pmod{152880}$.
Or equivalently: $n \equiv 38220, 76440, 114660, 191100, 229320,$ or $267540 \pmod{305760}$.

TABLE B

Modulus m	Period k	Required values of n where $\left(\frac{24P_n+1}{m}\right)=-1$	Left out values of n (mod t) where t is a positive integer
97	96	±12, 36 and 60.	±76440 (mod 305760) or equivalently ±76440, ±229320 (mod 611520)
449	448	56, 168.	Completely eliminated under modulo 611520.
2689	1344	840, 1176.	

Now $x=a$, $y=b$ is a solution of (30) $\Leftrightarrow a+\sqrt{2}b(3b-1)/2$ is a solution of (31) $\Leftrightarrow a=Q_{2n+1}$ and $b(3b-1)/2=P_{2n+1}$ for some integer n. But we know by Theorem 1(a) that P_k is generalized pentagonal if and only if $k=0, \pm 1, 2, \pm 3, 4$, or 6. Therefore, we have either

(i) $a=Q_{-1}=-1$, $b(3b-1)/2=P_{-1}=1$; (ii) $a=Q_1=1$, $b(3b-1)/2=P_1=1$;
(iii) $a=Q_{-3}=-7$, $b(3b-1)/2=P_{-3}=5$; (iv) $a=Q_3=7$, $b(3b-1)/2=P_3=5$.

Solving the above equations, we get the required solution set of equation (30).

We can prove the following theorem in a similar manner.

Theorem 3: The solution set of the Diophantine equation $2x^2=y^2(3y-1)^2+2$ is

$$\{(\pm 1, 0), (\pm 3, -1), (\pm 17, 3), (\pm 99, -280)\}.$$

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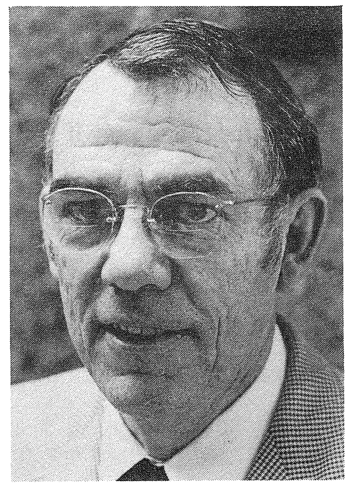
❖❖❖

A Special Tribute to Calvin T. Long

Teacher • Researcher • Mentor • Leader • Friend

Calvin T. Long was born in Rupert, Idaho, where he received his elementary and secondary education. Inspired by his teachers, he continued his education and was granted a B. S. degree from the University of Idaho in 1950, an M. S. degree from the University of Oregon in 1952 and a Ph.D. under the direction of Professor Ivan Niven, from the University of Oregon in 1955. After graduation, he spent one year as an analyst in Washington, D. C. working for the National Security Agency. In 1956, Cal accepted a position in the Mathematics Department at Washington State University (WSU) in Pullman, Washington, where he remained until his retirement in 1992.



It would be impossible to list all of Professor Long's accomplishments during his tenure at WSU so we will try to list only what we consider to be the most important ones. During the 1960s and early 1970s Cal served as director or associate director for several NSF-funded institutes for elementary, junior and high school mathematics teachers. This led to his deep interest in mathematics education. From 1970-78, he served as department chairman.

As a teacher, his students both at the graduate and undergraduate level respected him. He was a taskmaster but had a good sense of humor, sound scholarship and the ability to lead his students to their best efforts in an uncompromising way by insisting on excellence. For his efforts, Professor Long received the President's Faculty Excellence Award for Teaching in 1987 and was one of WSU's Case Award Nominees and Centennial Lecturers. He was also a visiting professor at three foreign and two American universities. During his career he directed 27 masters students and was the thesis advisor for five doctoral students.

As a researcher, he was the author or co-author of at least twenty-four grant proposals that funded programs or institutes related to mathematics education. He was the author of several books on number theory and mathematics education. He did extensive reviewing and refereeing of research papers and was an associate editor for two mathematics journals. He had more than 80-refereed publications and gave more than 150-invited lectures throughout the United States, Canada, Australia, New Zealand and Germany. He has also given at least 50 invited colloquium talks. Professor Long is a member of many honor societies, including Phi Beta Kappa. He is also an active member of many mathematical societies, including the Mathematical Association of America (MAA), the American Mathematical Society (AMS), the Fibonacci Association and the National Council of Teachers of Mathematics. He was elected Vice-Chairman, Chairman and Governor of the Northwest Section of the MAA. He served on numerous local, regional and national committees. For his dedication to his profession, he received the Certificate of Meritorious Service from the MAA in January of 1991.

As a mentor, he was always there for his fellow teachers as well as for his current and former students. As a leader, he was a state coordinator for the American High School Mathematics Examination, he was one of the organizers of the WSU Mathematics Honors Scholarship Competition

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ON THE REPRESENTATION OF THE INTEGERS AS A DIFFERENCE OF SQUARES

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1. INTRODUCTION

In recent times a number of authors (see [1]-[4]) have revisited the well-known results of Fermat and Jacobi in connection with the polygonal representation of the integers. In the papers cited an alternate derivation for such formulas giving the total number of representations of an integer as the sum of either two triangular or square numbers was provided. These enumerations, which are given in terms of elementary divisor functions, were deduced as a consequence of the Gauss-Jacobi triple product identity. In contrast to sums of polygonal numbers, the author has investigated within [5] the representation of the integers as a difference of two triangular numbers. By use of a purely combinatorial argument, it was shown that the number of such representations of an integer n was exactly equal to the number of odd divisors of n. In this note we propose to extend the methods employed in [5] to the case of squares to prove the following result.

Theorem 1.1: The number $s(n)$ of representations of a positive integer as a difference of the squares of two nonnegative integers is given by

$$s(n)=\frac{1}{2}\left(d_0(n)+(-1)^{n+1}d_1(n)+\frac{1+(-1)^{d(n)+1}}{2}\right), \tag{1}$$

where $d(n)$ is the total number of divisors of n and, for each $i\in\{0,1\}$,

$$d_i(n)=\sum_{d\mid n,\, d\equiv i \bmod 2} 1.$$

To facilitate the result, we shall need a preliminary definition and technical lemma.

Definition 1.1: For a given $n\in\mathbb{N}\setminus\{0\}$, a factorization $n=ab$, with $a,b\in\mathbb{N}\setminus\{0\}$ is said to be nontrivial if $a\neq 1,n$. Two such factorizations, $a_1b_1=a_2b_2=n$, are distinct if $a_1\neq a_2,b_2$.

The following result, which concerns counting the total number of distinct nontrivial factorizations, $ab=n$, may be known; however, interested readers can consult [5] for a proof.

Lemma 1.1: Let n be an integer greater than unity and $d(n)$ the number of divisors of n. Then the total number $N(n)$ of nontrivial distinct factorizations of n is given by

$$N(n)=\begin{cases}\dfrac{d(n)-2}{2} & \text{for nonsquare } n,\\[2mm] \dfrac{d(n)-1}{2} & \text{for square } n.\end{cases}$$

2. PROOF OF THEOREM 1.1

Our first goal will be to determine whether, for a given $n \in \mathbb{N}\setminus\{0\}$, there exists $x, y \in \mathbb{N}$ such that $n = x^2 - y^2$. To analyze the solvability of this diophantine equation, suppose $n = ab$, where $a, b \in \mathbb{N}\setminus\{0\}$, and consider the following system of simultaneous linear equations

$$\begin{aligned} x - y &= a \\ x + y &= b \end{aligned} \tag{2}$$

whose general solution is given by

$$(x, y) = \left(\frac{a+b}{2}, \frac{b-a}{2}\right).$$

Now, for there to exist a representation of n as a difference of two squares, one must be able to find a factorization $ab = n$ for which (2) will yield a solution (x, y) in integers.

Remark 2.1: We note that it is sufficient to consider only (2) since, if for a chosen factorization $ab = n$ an integer solution pair (x, y) is found, then the corresponding representation $n = x^2 - y^2$ is also obtained if the right-hand side of (2) is interchanged. Indeed, one finds upon solving

$$\begin{aligned} x' - y' &= b \\ x' + y' &= a \end{aligned}$$

that $x' = \frac{a+b}{2}$ and $y' = \frac{a-b}{2}$. Thus, $x' = x$ while $y' = -y$, which yields an identical difference of squares representation.

We deal with the existence or otherwise of those factorizations $ab = n$, which gives rise to the integer solution pair (x, y) of (2). It is clear from the general solution of (2) that, for x to be a positive integer a, b must at least be chosen so that $a + b$ is an even integer. Clearly, this can only be achieved if a and b are of the same parity. Furthermore, such a chose of a and b will also ensure that $y = x - a$ is also an integer. With this reasoning in mind, it will be convenient to consider the following cases separately.

Case 1: $n = 4k + 2$, $k \in \mathbb{N}$. In this instance, if $ab = 2(2k+1)$, then one cannot possibly find an a and b of the same parity, so no integer solution (x, y) of (2) can be found. Consequently, $s(4k+2) = 0$.

Case 2: $n \neq 4k + 2$. Clearly, $n = 2^m(2k+1)$ for some $n \in \mathbb{N}\setminus\{1\}$ and $s \in \mathbb{N}$. Considering first $m = 0$, it is immediate that all factorizations $ab = 2k+1$ will produce integer solutions to (2) since a and b are odd. Alternatively, when $m > 1$ one can always construct, for every factorization $cd = 2k+1$, an a and b of the form $(a, b) = (2^i c, 2^{m-i} d)$ with $i \in \{1, 2, \ldots., m-1\}$ that will produce an integer solution of (2). Hence, for the m and k prescribed above, one can conclude that $s(2^m(2k+1)) > 0$.

Having determined the set of integers n which are of the form $n = x^2 - y^2$, we can now address the problem of finding the exact number $s(n)$ of such representations. Primarily, this will entail determining whether any duplication occurs between the representations generated from the various distinct factorizations discussed in Case 2. To this end, we need to demonstrate that if in $\mathbb{Z}\setminus\{0\}$ $a_i b_i = a_j b_j$, with $a_i \neq a_j, b_j$ for $i \neq j$, then one has $a_i + b_i \neq a_j + b_j$. Suppose to the contary that $a_i + b_i = a_j + b_j$, then there must exist an $r \in \mathbb{Z}\setminus\{0\}$ such that $a_j = a_i + r$ and $b_i = b_j + r$.

Substituting these equations into the equality $a_ib_i = a_jb_j$, one finds $a_i(b_j + r) = (a_i + r)b_j$. Hence, r must be a nonzero integer solution of

$$r(a_i - b_j) = 0. \tag{3}$$

However, this is impossible as $r = 0$ is the only possible solution of (3) since $a_i - b_j \neq 0$; a contradiction. Consequently, if for two distinct factorizations $a_ib_i = a_jb_j = n$, one solves (2) to produce corresponding integer solutions (x_i, y_i) and (x_j, y_j), then we must have $x_i = (a_i + b_i)/2 \neq (a_j + b_j)/2 = x_j$ and so $y_i = x_i - a \neq x_j - a = y_j$. Thus, in order to calculate $s(n)$ for an $n \neq 4k+2$, one must determine the total number of distinct factorizations discussed in Case 2. Considering when n is odd we have, from Lemma 1.1, $N(n)$ nontrivial distinct factorizations. However, as $(a, b) = (1, n)$ contributes a representation, one has $s(n) = N(n) + 1$, which we write here using $d(n) = d_0(n) + d_1(n)$ as

$$s(n) = \begin{cases} \frac{1}{2}(d_0(n) + d_1(n)) & \text{for nonsquare } n, \\ \frac{1}{2}(d_0(n) + d_1(n) + 1) & \text{for square } n. \end{cases} \tag{4}$$

Suppose n is even, then, as was observed previously, $s(n)$ is equal to the total number of distinct factorizations $a_ib_i = n$ were both a_i and b_i are even. Denoting the number of distinct factorizations of $n = a_ib_i$ with a_i and b_i of opposite parity by $N'(n)$, observe that $s(n)$ must be equal to the difference between the number of distinct factorizations of n and $N'(n)$, that is, $s(n) = N(n) + 1 - N'(n)$. To determine $N'(n)$, consider an arbitrary factorization (c_i, d_i) **(possibly trivial)** of the odd number $2^{-m}n$. Now, if $2^{-m}n$ is not a perfect square, then $(2^m c_i, d_i)$ and $(c_i, 2^m d_i)$ must be distinct factorizations of n, which cannot be duplicated by the use of an alternate factorization (c_j, d_j) of $2^{-m}n$. Thus, from Lemma 1.1, there are $2\left(\frac{d(2^{-m}n)}{2}\right)$ distinct factorizations $a_ib_i = n$ having a_i and b_i of opposite parity. Similarly, if $2^{-m}n$ is a square, then $(2^m c_i, d_i)$ and $(c_i, 2^m d_i)$ will be distinct factorizations provided $c_i \neq d_i$, and so again by Lemma 1.1 we have, counting the single contribution from $(2^m c_i, c_i)$, precisely $2\left(\frac{d(2^{-m}n)+1}{2} - 1\right) + 1$ distinct factorizations $a_ib_i = n$ with a_i and b_i of opposite parity. Consequently, in any case, $N'(n) = d(2^{-m}n)$. Now, observing that $d(2^{-m}n) = d_1(n)$ and $d(n) = d_0(n) + d_1(n)$, we obtain, for an even $n \neq 4k+2$, the following expression:

$$s(n) = \begin{cases} \frac{1}{2}(d_0(n) - d_1(n)) & \text{for nonsquare } n, \\ \frac{1}{2}(d_0(n) - d_1(n) + 1) & \text{for square } n. \end{cases} \tag{5}$$

Recalling that $d(n)$ is odd if and only if n is a square, we find

$$\frac{1 + (-1)^{d(n)+1}}{2} = \begin{cases} 0 & \text{for nonsquare } n, \\ 1 & \text{for square } n. \end{cases}$$

Thus, one can combine equations (4) and (5) into a single expression independent of the parity of n as indicated in (1). Finally, we show that (1) holds for $n = 4k+2$. In this instance, as n cannot be a square, $\frac{1}{2}(1 + (-1)^{d(n)+1}) = 0$; moreover, $d_0(2(2k+1)) = d_1(2(2k+1))$, since every odd divisor d of n is in one-to-one correspondence with an even divisor of n, namely, $2d$. Thus, from (1), we find that $s(2(2k+1)) = \frac{1}{2}(d_0(2(2k+1)) - d_1(2(2k+1))) = 0$ as required.

Example 2.1: For a given integer n, whose prime factorization is known, one can determine all of the $s(n)$ representations of n as a difference of squares from the factorizations $ab = n$ with $a \geq b > 0$ and $a \equiv \pm b \pmod 2$, using $(x, y) = (\frac{a+b}{2}, \frac{a-b}{2}$. To illustrate this, we shall calculate the representations in the case of a square and nonsquare number. Beginning with, say $n = 2^2 \cdot 5 \cdot 7$, we have that $d_0(140) = 8$, $d_1(140) = 4$, and $d(140)$ even, so $s(140) = \frac{1}{2}(8-4) = 2$. Thus, from the two factorizations $(a, b) \in \{(2 \cdot 7, 2 \cdot 5), (2 \cdot 5 \cdot 7, 2)\}$, we find that $140 = 12^2 - 2^2$, $36^2 - 34^2$. In the case of $n = (2 \cdot 5 \cdot 7)^2$, we have $d_0(4900) = 18$, $d_1(4900) = 9$, $d(4900)$ odd; therefore, $s(4900) = \frac{1}{2}(18 - 9 + 1) =$. Thus, again from the five factorizations $(a, b) \in \{(2 \cdot 5^2 \cdot 7^2, 2), (2 \cdot 5^2 \cdot 7, 2 \cdot 7), (2 \cdot 5 \cdot 7, 2 \cdot 5 \cdot 7), (2 \cdot 7^2, 2 \cdot 5^2), (2 \cdot 5 \cdot 7^2, 2 \cdot 5)\}$, we now obtain that $4900 = 1226^2 - 1224^2$, $250^2 - 240^2, 70^2 - 0^2, 74^2 - 24^2, 182^2 - 168^2$.

To conclude, we present a simple application of Theorem 1.1 for counting the number of those partitions of an integer whose summands form a sequence of consecutive odd integers. Note that for an odd integer n we do not count $n = 0 + n$ as a partition of the required type as zero is not an odd integer.

Corollary 2.1: If $p_o(n)$ denotes the number of partitions of a positive integer n having summands consisting of consecutive odd integers, then

$$p_o(n) = s(n) + \frac{(-1)^n - 1}{2}.$$

Proof: Recalling that the m^{th} perfect square is equal to the sum of the first m odd integers, one sees that the representation $n = x^2 - y^2$ gives a partition of the required form, provided that $x - y > 1$. Moreover, as a consecutive square difference representation can only occur for an odd integer, we clearly must have $p_o(n) = s(n) - 1$ for odd n and $p_o(n) = s(n)$ for even n. □

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SEQUENCES RELATED TO RIORDAN ARRAYS

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1. INTRODUCTION

The concept of a Riordan array was defined in [4] as follows: Let $\mathscr{F} = \mathbf{R}[x]$ be a ring of formal power series with real coefficients in some indeterminate x. Let $g(x) \in \mathscr{F}$ and let $f(x) = \sum_{k=0}^{\infty} f_k x^k \in \mathscr{F}$ with $f_0 = 0$ (in this paper we assume $f_1 \neq 0$). Let $d_0(x) = g(x)$, $d_k = g(x)(f(x))^k$, and $d_{n,k} = [x^n]d_k(x)$, where $[x^n]d_k(x)$ means the coefficients of x^n in the expansion of $d_k(x)$ in x. Then an infinite lower triangular array, $D = \{d_{n,k} \,|\, k, n \in \mathrm{N}, k \leq n\}$, is obtained. We also write $D = (g(x), f(x))$ and call D a Riordan array. In this paper we obtain some new relations between two sequences and some new inverse relations by using Riordan arrays. Some results are a generalization of [2] and [3].

2. SEQUENCES RELATED TO RIORDAN ARRAYS

Let $a(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{F}$ and $D = (g(x), f(x))$. Let

$$h(x) = \frac{1}{g(x)} = \sum_{k=0}^{\infty} h_k x^k \in \mathscr{F},$$

$$A(x) = a(f(x)) = \sum_{k=0}^{\infty} A_k x^k \in \mathscr{F},$$

and

$$s(x) = g(x)A(x) = \sum_{k=0}^{\infty} s_k x^k \in \mathscr{F}.$$

Theorem 1: We have

$$A_n = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} d_{i,k} h_{n-i} \right) a_k. \tag{1}$$

Proof: By Theorem 1.1 in [5], we have

$$\sum_{k=0}^{\infty} d_{n,k} a_k = [x^n] g(x) a(f(x)) = s_n.$$

From $s(x) = g(x)A(x)$, $A(x) = s(x)h(x)$, we have

$$A_n = \sum_{i=0}^{n} s_i h_{n-i} = \sum_{i=0}^{n} \left(\sum_{k=0}^{\infty} d_{i,k} a_k \right) h_{n-i} = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} d_{i,k} h_{n-i}) \right) a_k..$$

This completes the proof. □

Theorem 2: We have

$$a_n = \sum_{k=0}^{\infty} \bar{d}_{n,k} A_k, \tag{2}$$

where $\bar{d}_{n,k}$ can be obtained by using one of the following Lagrange inversion formulas (see [1], pp. 148-52):

$$\bar{d}_{n,k} = \frac{k}{n}[x^{n-k}]\left(\frac{f(x)}{x}\right)^{-n}; \tag{3}$$

$$\bar{d}_{n,k} = k\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j}{n+j}\binom{n-k}{j} f_1^{-n-j}[x^{n-k+j}](f(x))^j. \tag{4}$$

Proof: By $A(x) = a(f(x))$, we have $a(x) = A(\bar{f}(x))$, where $\bar{f}(f(x)) = f(\bar{f}(x)) = x$ and $\bar{f}(0) = 0$. By [1] and Theorem 1.1 in [5], we obtain $a_n = \sum_{k=0}^{\infty} \bar{d}_{n,k} A_k$, in which

$$\bar{d}_{n,k} = [x^n](\bar{f}(x))^k = \frac{k}{n}[x^{n-k}]\left(\frac{f(x)}{x}\right)^{-n}$$

or

$$\bar{d}_{n,k} = [x^n](\bar{f}(x))^k = k\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j}{n+j}\binom{n-k}{j} f_1^{-n-j}[x^{n-k+j}](f(x))^j.$$

This completes the proof. □

We can combine Theorems 1 and 2 to obtain a generator of an inverse relation.

Theorem 3: We have the following inverse relation,

$$\begin{cases} A_n = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{n} d_{i,k}h_{n-i}\right)a_k, \\ a_n = \sum_{k=0}^{\infty} \bar{d}_{n,k}A_k, \end{cases} \tag{5}$$

where $\bar{d}_{n,k}$ can be obtained by using (3) or (4).

In addition, we obtain many new identities by using (1) or (2). The interested reader can consult [2] and [3].

Example 1: Let $g(x) = \frac{1}{1-ax}$ and $f(x) = \frac{b^l x^l}{(1-ax)^s}$. Then $h(x) = 1 - ax$ and

$$d_{n,k} = [x^n]\frac{1}{1-ax}\left(\frac{b^l x^l}{(1-ax)^s}\right)^k = b^{kl}a^{n-lk}\binom{n+(s-l)k}{sk}.$$

By (1), we have

$$\begin{aligned} A_n &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{n} b^{kl}a^{i-lk}\binom{i+(s-l)k}{sk}h_{n-i}\right)a_k \\ &= \sum_{k=0}^{\infty}\left(b^{kl}a^{n-lk}\binom{n+(s-l)k}{sk} - b^{kl}a^{n-lk}\binom{n+(s-l)k-1}{sk}\right)a_k = \sum_{k=0}^{\infty} b^{kl}a^{n-kl}\binom{n+(s-l)k-1}{sk-1}a_k. \end{aligned}$$

By (2) and (3), we have

$$\bar{d}_{n,k} = \frac{k}{n}[x^{n-k}]\left(\frac{b^l x^l}{x(1-ax)^s}\right)^{-n} = (-1)^{nl-k} a^{nl-k} b^{-nl} \frac{k}{n}\binom{sn}{nl-k},$$

$$a_n = \sum_{k=0}^{\infty} (-a)^{nl-k} b^{-nl} \frac{k}{n}\binom{sn}{nl-k} A_k.$$

So we obtain the following inverse relation:

$$\begin{cases} A_n = \sum_{k=0}^{\infty} b^{kl} a^{n-kl} \binom{n+(s-l)k-1}{sk-1} a_k, \\ a_n = \sum_{k=0}^{\infty} (-a)^{nl-k} b^{-nl} \frac{k}{n}\binom{sn}{nl-k} A_k. \end{cases}$$

Letting $s = l = 1$, $a = t$, and $b = s$, we can obtain Theorems 3 and 4 in [3]. □

Example 2: Let $D_1 = (1, \log(1-x)) = (d^1_{n,k})$ and $D_2 = (1, (1-\mathrm{e}^x)) = (d^2_{n,k})$. Then

$$d^1_{n,k} = [x^n](\log(1-x))^k = (-1)^k [x^n]\left(\log\left(\frac{1}{1-x}\right)\right)^k = (-1)^k \frac{k!}{n!} s_1(n, k)$$

and

$$d^2_{n,k} = [x^n](1-\mathrm{e}^x)^k = (-1)^k [x^n](\mathrm{e}^x - 1)^k = (-1)^k \frac{k!}{n!} s_2(n, k).$$

From $A(x) = a(\log(1-x))$, we find $a(x) = A(1-\mathrm{e}^x)$. So by (1) we have

$$\begin{cases} A_n = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{n!} s_1(n, k) a_k, \\ a_n = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{n!} s_2(n, k) A_k, \end{cases}$$

where $s_1(n, k)$ and $s_2(n, k)$ are the Stirling numbers of both kinds and have the following generating functions (see [5]), respectively:

$$\left(\log\frac{1}{1-x}\right)^m = \sum_{n=0}^{\infty} \frac{m!}{n!} s_1(n, m) x^n; \quad (\mathrm{e}^x - 1)^m = \sum_{n=0}^{\infty} \frac{m!}{n!} s_2(n, m) x^n. \quad \square$$

3. SEQUENCES RELATED TO EXPONENTIAL RIORDAN ARRAYS

Let

$$f(x) = \sum_{k=0}^{\infty} f_k \frac{x^k}{k!}.$$

We introduce a new notation, $\langle x^k \rangle f(x) = f_k$, and assume $f_0 = 0$, $f_1 \neq 0$. Let

$$g(x) = \sum_{k=0}^{\infty} g_k \frac{x^k}{k!}, \; g_0 \neq 0.$$

For an infinite lower triangular array $E = \{\mathrm{e}_{n,k} \mid n, k \in \mathrm{N}, \; k \leq n\}$, if

$$\mathrm{e}_{n,k} = \langle x^n \rangle g(x) \frac{(f(x))^k}{k!} \quad (k \geq 0),$$

for fixed k, then we write $E = \langle g(x), f(x) \rangle$ and say that $\langle g(x), f(x) \rangle$ is an exponential Riordan array.

Let

$$b(x) = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}$$

and let $E = \langle g(x), f(x) \rangle$ be an exponential Riordan array. Let

$$p(x) = \frac{1}{g(x)} = \sum_{k=0}^{\infty} p_k \frac{x^k}{k!}, \quad B(x) = b(f(x)) = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!},$$

and

$$q(x) = g(x)B(x) = \sum_{k=0}^{\infty} q_k \frac{x^k}{k!}.$$

For the exponential Riordan arrays, we have the following theorem as Theorem 1.1 in [5].

Theorem 4: We have

$$\sum_{k=0}^{\infty} \mathrm{e}_{n,k} b_k = \langle x^n \rangle g(x) b(f(x)). \tag{6}$$

Proof:

$$\sum_{k=0}^{\infty} \mathrm{e}_{n,k} b_k = \sum_{k=0}^{\infty} \langle x^n \rangle g(x) \frac{(f(x))^k}{k!} b_k = \langle x^n \rangle g(x) b(f(x)). \quad \square$$

Example 3: Let $E = \langle \mathrm{e}^x, -x \rangle$ be an exponential Riordan array. Then

$$\mathrm{e}_{n,k} = \langle x^n \rangle \mathrm{e}^x \frac{(-x)^k}{k!} = (-1)^k \binom{n}{k}.$$

For

$$b(x) = \frac{\mathrm{e}^{ax} - \mathrm{e}^{bx}}{a-b} = \sum_{n=0}^{\infty} F_n \frac{x^n}{n!},$$

where $a, b = (1 \pm \sqrt{5})/2$ and F_n is the n^{th} Fibonacci number defined by $F_{n+1} = F_n + F_{n-1}$, $F_0 = 0$, $F_1 = 1$ (see [2]), by (6) we have

$$\sum_{k=0}^{\infty} (-1)^k \binom{n}{k} F_k = \langle x^n \rangle \mathrm{e}^x b(-x) = \langle x^n \rangle \mathrm{e}^x \frac{\mathrm{e}^{-ax} - \mathrm{e}^{-bx}}{a-b} = \langle x^n \rangle - b(x) = -F_n,$$

that is,

$$\sum_{k=0}^{\infty} (-1)^{k+1} \binom{n}{k} F_k = F_n.$$

This is (8) in [2]. $\square$

By (6), we can obtain many new identities. The interested reader can refer to the related documents.

Theorem 5: We have

$$B_n = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} \mathrm{e}_{i,k} p_{n-i} \right) b_k. \tag{7}$$

Proof: The proof is similar to that of Theorem 1. □

Theorem 6: We have

$$b_n = \sum_{k=0}^{\infty} \bar{e}_{n,k} B_k, \tag{8}$$

where $\bar{e}_{n,k}$ can be obtained by using one of the following Lagrange inversion formulas (see [1], 148-52):

$$\bar{e}_{n,k} = \binom{n-1}{k-1} \langle x^{n-k} \rangle \left(\frac{f(x)}{x} \right)^{-n}. \tag{9}$$

$$\bar{e}_{n,k} = \frac{n!}{(k-1)!} \binom{2n-k}{n} \sum_{j=1}^{n-k} \frac{(-1)^j}{(n+j)(n-k+j)!} \binom{n-k}{j} f_1^{-n-j} \langle x^{n-k+j} \rangle (f(x))^j. \tag{10}$$

Proof: From $B(x) = b(f(x))$, we have $b(x) = B(\bar{f}(x))$, where $\bar{f}(f(x)) = f(\bar{f}(x)) = x$. So

$$b_n = \langle x^n \rangle B(\bar{f}(x)) = \sum_{k=0}^{\infty} \bar{e}_{n,k} B_k,$$

where

$$\begin{aligned} \bar{e}_{n,k} &= \langle x^n \rangle \frac{(\bar{f}(x))^k}{k!} = [x^n] \frac{n!}{k!} (\bar{f}(x))^k \\ &= \frac{(n-1)!}{(k-1)!} [x^{n-k}] \left(\frac{f(x)}{x} \right)^{-n} = \binom{n-1}{k-1} \langle x^{n-k} \rangle \left(\frac{f(x)}{x} \right)^{-n} \end{aligned}$$

or

$$\begin{aligned} \bar{e}_{n,k} &= [x^n] \frac{n!}{k!} (\bar{f}(x))^k = \frac{n!}{k!} k \binom{2n-k}{n} \sum_{j=1}^{n-k} \frac{(-1)^j}{n+j} \binom{n-k}{j} f_1^{-n-j} [x^{n-k+j}] (f(x))^j \\ &= \frac{n!}{(k-1)!} \binom{2n-k}{n} \sum_{j=1}^{n-k} \frac{(-1)^j}{(n+j)(n-k+j)!} \binom{n-k}{j} f_1^{-n-j} \langle x^{n-k+j} \rangle (f(x))^j. \quad \square \end{aligned}$$

Theorem 7: As in Theorem 3, we have the following inverse relation,

$$\begin{cases} B_n = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} e_{i,k} p_{n-i} \right) b_k, \\ b_n = \sum_{k=0}^{\infty} \bar{e}_{n,k} B_k, \end{cases}$$

where $\bar{e}_{n,k}$ can be obtained by using (9) or (10).

Example 4: Let $\langle g(x), f(x) \rangle = \left\langle 1, \log \frac{1}{1-x} \right\rangle$. Then

$$e_{n,k} = \langle x^n \rangle \frac{(\log \frac{1}{1-x})^k}{k!} = \frac{1}{k!} \langle x^n \rangle \left(\log \frac{1}{1-x} \right)^k = s_1(n,k).$$

By (7), we have

$$B_n = \sum_{k=0}^{\infty} s_1(n,k) b_k.$$

By (10), we have

$$\bar{e}_{n,k} = \frac{n!}{(k-1)!}\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j}{(n+j)(n-k+j)!}\binom{n-k}{j}\langle x^{n-k+j}\rangle\left(\log\frac{1}{1-x}\right)^j$$

$$= \frac{n!}{(k-1)!}\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j}{(n+j)(n-k+j)!}\binom{n-k}{j}j!s_1(n-k+j,j).$$

By (8), we have

$$b_n = \sum_{k=0}^{n}\frac{n!}{(k-1)!}\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j j!}{(n+j)(n-k+j)!}\binom{n-k}{j}s_1(n-k+j,j)B_k.$$

Therefore, we obtain the following inverse relation:

$$\begin{cases} B_n = \sum_{k=0}^{\infty} s_1(n,k)b_k, \\ b_n = \sum_{k=0}^{n}\frac{n!}{(k-1)!}\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j j!}{(n+j)(n-k+j)!}\binom{n-k}{j}s_1(n-k+j,j)B_k. \end{cases} \qquad \square$$

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ON THE INFINITUDE OF COMPOSITE NSW NUMBERS

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1. MOTIVATION

The NSW numbers (named in honor of Newman, Shanks, and Williams [3]) were studied approximately 20 years ago in connection with the order of certain simple groups. These are the numbers f_n which satisfy the recurrence

$$f_{n+1} = 6f_n - f_{n-1} \tag{1}$$

with initial conditions $f_1 = 1$ and $f_2 = 7$.

These numbers have also been studied in other contexts. For example, Bonin, Shapiro, and Simion [2] discuss them in relation to Schröder numbers and combinatorial statistics on lattice paths.

Recently, Barcucci et al. [1] provided a combinatorial interpretation for the NSW numbers by defining a certain regular language $\mathscr{L}$ and studying particular properties of $\mathscr{L}$. They close their note by asking two questions:

1. Do there exist infinitely many f_n prime?
2. Do there exist infinitely many f_n composite?

The goal of this paper is to answer the second question affirmatively, but in a much broader context. Fix an integer $k \geq 2$ and consider the sequence of values satisfying $f_{n+1} = kf_n - f_{n-1}$, $f_1 = 1$, and $f_2 = k+1$. Then we have the following theorem.

Theorem 1.1: For all $m \geq 1$ and all $n \geq 0$, $f_m | f_{(2m-1)n+m}$.

2. THE NECESSARY TOOLS

To prove Theorem 1.1, we need to develop a few key tools. First, let α be a zero of $x^2 - kx + 1$, the characteristic polynomial of the recurrence. If $\alpha \in \mathbb{Q}$ (the rational numbers), then we may assume that $\alpha = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ and $(m, n) = 1$. Hence, we have $m^2 - kmn + n^2 = 0$ or $m^2 = kmn - n^2$. It is clear then that $m | n^2$ and $n | m^2$, so that $\frac{m}{n} = \pm 1$ because $(m, n) = 1$. Therefore, $\mathbb{Z}[\alpha] \cap \mathbb{Q} = \mathbb{Z}$.

Now define congruence in $\mathbb{Z}[\alpha]$ by writing $\lambda \equiv \mu \pmod{\nu}$ for $\lambda, \mu, \nu \in \mathbb{Z}[\alpha]$ to mean that $\frac{(\lambda-\mu)}{\nu} \in \mathbb{Z}[\alpha]$, where $\nu \neq 0$. Note that if $\lambda, \mu, \nu \in \mathbb{Z}$ and $\lambda \equiv \mu \pmod{\nu}$ by this definition, then $\frac{(\lambda-\mu)}{\nu} \in \mathbb{Z}[\alpha] \cap \mathbb{Q}$, which implies $\frac{(\lambda-\mu)}{\nu} \in \mathbb{Z}$, so that $\lambda \equiv \mu \pmod{\nu}$ by the conventional definition of congruence.

Also, note that if $\gamma \in \mathbb{Q}(\alpha)$ and $\lambda, \mu, \nu, \gamma\lambda, \gamma\mu, \gamma\nu \in \mathbb{Z}[\alpha]$, then $\lambda \equiv \mu \pmod{\nu}$ implies $\gamma\lambda \equiv \gamma\mu \pmod{\gamma\nu}$.

Now we are ready to complete the proof of Theorem 1.1.

Proof: We first handle the case $k=2$ separately. In this case, it is easy to show that $f_n = 2n-1$ for $n \geq 1$. Then $f_{(2m-1)n+m} = 2((2m-1)n+m)-1 = (2m-1)(2n+1)$ and $f_m \mid f_{(2m-1)n+m}$ clearly.

Next, we assume that $k > 2$. Since α is a zero of $x^2 - kx + 1$, α is neither 0 nor 1. Also, $\alpha^2 + 1 = k\alpha$. Note that $\beta = 1/\alpha$ is the other zero of $x^2 - kx + 1$, and $\alpha + \beta = k$. Since α and β are distinct, we know that $f_n = A\alpha^n + B\beta^n$ for some constants A and B, and since $f_0 = -1$ and $f_1 = 1$, we have $A + B = -1$ and $A\alpha + B\beta = 1$. Solving these two equations yields

$$A = \frac{1+\beta}{\alpha-\beta} \quad \text{and} \quad B = -\frac{1+\alpha}{\alpha-\beta}.$$

Therefore,

$$\begin{aligned} f_m &= \frac{1}{\alpha-\beta}((1+\beta)\alpha^m - (1+\alpha)\beta^m) = \frac{1}{\alpha-\beta}\left(\left(1+\frac{1}{\alpha}\right)\alpha^m - (1+\alpha)\beta^m\right) \\ &= \frac{1}{\alpha-\beta}((1+\alpha)\alpha^{m-1} - (1+\alpha)\beta^m) = \frac{1+\alpha}{\alpha-\beta}(\alpha^{m-1} - \beta^m). \end{aligned}$$

Now let $U_m = \alpha^{m-1} - \beta^m \in \mathbb{Z}[\alpha]$ $(\beta = k - \alpha)$, where $m \geq 1$. Then $\alpha^{m-1} \equiv \beta^m \pmod{U_m}$ implies $\alpha^{2m-1} = \alpha^m \alpha^{m-1} \equiv \alpha^m \beta^m \equiv 1 \pmod{U_m}$ and $\beta^{2m-1} = \beta^m \beta^{m-1} \equiv \alpha^{m-1}\beta^{m-1} \equiv 1 \pmod{U_m}$. Hence,

$$\begin{aligned} U_{(2m-1)n+m} &= \alpha^{(2m-1)n+m-1} - \beta^{(2m-1)n+m} \\ &\equiv \beta^m(\alpha^{(2m-1)n} - \beta^{(2m-1)n}) \pmod{U_m} \\ &\equiv 0 \pmod{U_m}. \end{aligned}$$

Therefore,

$$\left(\frac{1+\alpha}{\alpha-\beta}\right)U_{(2m-1)n+m} \equiv 0 \left(\mathrm{mod}\left(\frac{1+\alpha}{\alpha-\beta}\right)U_m\right)$$

or $f_m \mid f_{(2m-1)n+m}$. □

3. CLOSING THOUGHTS

We close by noting that this theorem proves $f_m \mid f_{(2m-1)n+m}$ for a variety of well-known sequences $\{f_m\}_{m=1}^{\infty}$ other than the NSW numbers, including the odd numbers $(k=2)$, the Lucas numbers L_{2n} $(k=3)$, and the Fibonacci numbers F_{4n+2} $(k=7)$.

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A NOTE ON A DIOPHANTINE EQUATION CONSIDERED BY POWELL

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(*Submitted April 2000*)

1. INTRODUCTION

B. J. Powell [2] conjectured that, for any positive even integer m and for any positive integer n sufficiently large for which $mn+1=q$, where q is a prime, the Diophantine equation

$$Ax^n + By^n = Cz^n \tag{1}$$

has no solutions in positive integers x, y, and z, where A, B, and C are natural numbers. Q. Sun [4] provided evidence in favor of this conjecture by showing that, if

$$(A \pm B \pm C)(A \pm B)(A \pm C)(B \pm C) \neq 0,$$

where A, B, and C are even nonzero integers, then, if

$$n > [(|A| + |B| + |C|)^{\phi(m)} - 1]/m$$

and $q = nm+1$ is prime, $\phi(m)$ is Euler's totient function, then equation (1) has only the trivial solution $xyz = 0$.

A corollary of a theorem in the paper of G. Faltings [1] is that equation (1) has only finitely many solutions, coprime in pairs, for $n > 3$.

Thus, there has been considerable interest in the problem of the solvability of (1) in integers. In this paper we find further conditions under which (1) has no nontrivial natural number solutions. Various relationships between variables in the equation are found excluding the possibility of nontrivial solutions.

In the following we denote by $\mathbb{N}$ the set of nonzero positive integers and we denote by $\mathbf{Z}$ the set of integers.

2. RESULTS

We consider the Diophantine equation

$$Ax^n + Y = Cz^n, \tag{2}$$

where A, C, Y, x, n, and z are all integers. Clearly, (1) is a special case of (2). Therefore, if (2) has no solutions for $Y = By^n$, then (1) has no solutions.

Two lemmas are given before our main theorem. This Theorem 1 specifies conditions on a prime natural number, such that if these conditions hold then (2) has no solutions and therefore the corresponding version of (1) has no solution for a specific choice of variables.

Lemma 1: If A, C, Y, x, z, $n \in \mathbb{N}$ in equation (2), then $A > C$ implies $x < z$.

Proof: If $A > C$, then $x^n < \frac{A}{C}x^n < z^n$ because $Ax^n < Cz^n$. So $x < z$. □

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* We must sadly report that the author of this article recently passed away so any questions or concerns should be sent to the editor.

The following lemma is an application of the binomial theorem and is crucial to our arguments.

Lemma 2: Suppose q is a prime, $N, M, x, z \in \mathbb{Z}$ and $A, C, q, u, t \in \mathbb{N}$. $Ax^m + q^t M = Cz^m$ and $Ax^r + q^u N = Cz^r$, where $q^t | C^m A^r - C^r A^m$ (which is satisfied, in particular, by $q^t | C - A$), and $(q, m) = (q, C) = (q, A) = (q, x) = (q, N) = 1$, then $u \geq t$.

Proof: Now

$$z^{mr} = \left(\frac{A}{C}x^m + \frac{q^t M}{C}\right)^r = \left(\frac{A}{C}x^r + \frac{q^u N}{C}\right)^m.$$

So, from the binomial theorem, we find that

$$N_1 q^{2t} + N_2 q^{2u} = (C^m A^r - C^r A^m)x^{rm} + rC^m A^{r-1} x^{m(r-1)} q^t M - mC^r A^{m-1} x^{r(m-1)} q^u N$$

for particular $N_1, N_2 \in \mathbb{Z}$.

So $t > u$ implies $q | mCAxN$, which implies $q|m$ or $q|C$ or $q|A$ or $q|x$ or $q|N$; which are all contradictions. □

Theorem 1: Let q be a prime, when there exists $t, n \in \mathbb{N}$, $n > q-1$ such that $q^t | A - C, q^t \| Y$ then there are no solutions to the Diophantine equation (2) for $(q, n) = (q, Ax) = (q, Cz) = 1$, where q^t does not divide $z^{q-1} - x^{q-1}$. In particular, this last condition holds if either

(a) $t > (q-1)\log_q z$ and $A > C, Y, x, z \in \mathbb{N}$, or,

(b) if q^2 does not divide $z - x$ but $q | z - x, Y \in \mathbb{Z}$.

Proof: Now we may assume

$$Ax^n + q^t M = Cz^n \tag{3}$$

and

$$Ax^{n-q+1} + q^u N = Cz^{n-q+1}, \tag{4}$$

where $(q, N) = 1$ by the lesser Fermat theorem. Therefore, from Lemma 2,

$$u = t + k, \text{ where } k \in \mathbb{N} \cup \{0\}.$$

Hence, after multiplying (4) by x^{q-1}, we obtain that

$$Ax^n + q^{t+k+q-1} N_q = Cz^{n-q+1} x^{q-1} \tag{5}$$

for $N_q = x^{q-1} N \in \mathbb{Z}$. Then, after subtracting (3) from (5), we find that

$$q^{t+k+q-1} N_q = Cz^{n-q+1}(x^{q-1} - z^{q-1}) + q^t M.$$

Therefore, $q^t | z^{q-1} - x^{q-1}$, since $(C, q) = (x, q) = (z, q) = 1$. Thus, (2) has no solutions for $q^t > z^{q-1}$ when $A > C$, because $z > x$ from Lemma 1. After taking logarithms, this is equivalent to $t > (q-1)\log_q z$.

Now assume that the conditions of case (b) hold, then

$$\begin{aligned}\frac{z^{q-1} - x^{q-1}}{z - x} &= z^{q-2} + xz^{q-3} + \cdots + x^{q-2} \\ &\equiv (q-1)z^{q-2} \pmod{q}.\end{aligned}$$

Hence, the left-hand side is not divisible by q, because $(q-1,q)=(q,z)=1$. So q^2 does not divide $z^{q-1}-x^{q-1}$. $\square$

3. EXAMPLES

As an example of our Theorem 1, we show that the following Diophantine problem is not solvable, where the details are given in our Theorem 2.

Theorem 2: If $x, y, t, k \in \mathbb{N}$, then the Diophantine equation $(1+3^t)x^5+3^t y^5 = 5^{5k}$ has no solution for $(x,3)=(y,3)=(k,3)=1$ and $t>4k$.

Proof: The conditions of Theorem 1(a) are satisfied for $q=3$, $A=(1+3^t)$, $z=5^k$, and $n=5$, since $t>4k>2k\log_3(5)$. $\square$

It would be very interesting to find an elementary proof of Fermat's last theorem; to this end, we provide a new elementary sufficient condition for Fermat's last theorem to hold. This condition is given as Theorem 3. A more general result can be found in the paper by K. A. Ribet [3]. However, Theorem 2 provides a purely elementary route to a solution of the Diophantine equation under consideration.

Theorem 3: Fermat's last theorem holds if there are no $t \in \mathbb{N}$ and $x,y,z \in \mathbb{Z}$, such that, for $p \geq 3$ a prime, $x^p+2^t y^p = z^p$ and $z-x=2^t L$ for some $L \in \mathbb{Z}$ with $(x,2y)=(2y,z)=(x,z)=(y,2)=1$.

Proof: It is well known that to prove Fermat's last theorem we need only show that there are no integer solutions to the Diophantine equation $x^p+y^p+z^p=0$, with $(x,y)=(y,z)=(x,z)=1$. Hence, we may assume that

$$x^p+2^t y_1^p = z^p, \tag{6}$$

where x, z, $y_1 \in \mathbb{Z}$, with $(2y_1,z)=(x,z)=(2y_1,x)=(y_1,2)=1$, by rearranging the variables if necessary, since two of the x, y, z must be odd. All the conditions of Theorem 1 are satisfied for equation (6) for the case $q=2$. Because x, z are odd, we may assume that $z-x=2(2J+1)$ or $z-x=4K$, where $J, K \in \mathbb{Z}$. In the first case, $2|z-x$ but 4 does not divide $z-x$. Consequently, from Theorem 1, Fermat's theorem holds in this case. Hence, $z-x=4L^1$ for $L^1 \in \mathbb{Z}$. Hence,

$$\begin{aligned}
2^t y^p &= z^p - x^p \\
&= (z-x)(z^{p-1}+xz^{p-2}+\cdots+x^{p-1}) \\
&= 4L^1((x+4L^1)^{p-1}+x(x+4L^1)^{p-2}+\cdots+x^{p-1}) \\
&= 4L^1F,
\end{aligned}$$

thus defining F.

So $2^{t-2}y^p = L^1F$. Suppose $2|F$, then $2|px^{p-1}$, which implies $2|p$ or $2|x$; a contradiction. Thus, $2^{t-2}|L^1$ and $(2,F)=1$. Therefore, there exists $L \in \mathbb{Z}$ such that $z-x=2^tL$. $\square$

The paper by K. A. Ribet [3] states that the Diophantine equation $x^n+q^ty^n+z^n=0$, where q and n are distinct prime numbers and $q \in \{3,5,7,11,13,17,19,23,29,53,59\}$ does not have any solution for $n>11$. Theorem 4 uses Theorem 1 to obtain, in particular, an asymptotic result concerning the related Diophantine equation

$$x^n+q^ty^n=z^n \tag{7}$$

for infinitely many q and n. Note that Theorem 4 is not a special case of Q. Sun's aforementioned theorem, because in Theorem 4 n is not required to satisfy the constraint that $nm+1$ is a prime for suitable m, but n is required, in particular, to be coprime with respect to q; which is a weaker constraint.

Theorem 4: The Diophantine equation (7), with q, x, n, t, y, and z natural numbers, q an odd prime, and $(q,n)=(q,x)=(q,y)=(q,z)=1$, has no solutions if

$$\frac{z^n-x^n}{z^{q-1}-x^{q-1}}>y^n.$$

In particular, no solution exists if

$$x>y^{\frac{n}{n-1}}(q-1)^{\frac{1}{n-1}}z^{\frac{q-1}{n-1}}.$$

Hence, there exists a positive integer $N(x,q,y,z)$ such that there are no solutions to (7) for $n>N$ and $x>y$.

Proof: Suppose $q^t>z^{q-1}-x^{q-1}$, then q^t does not divide (fully) $z^{q-1}-x^{q-1}$. Therefore, from Theorem 1, there is no solution to the Diophantine equation (7) when

$$\frac{z^n-x^n}{y^n}=q^t>z^{q-1}-x^{q-1},$$

which is equivalent to

$$\frac{z^n-x^n}{z^{q-1}-x^{q-1}}>y^n, \tag{8}$$

given that (7) holds.

So there is no solution to (7), given that (8) holds, which, after canceling the factor $z-x$, results in

$$\frac{z^{n-1}+xz^{n-2}+\cdots+x^{n-1}}{z^{q-2}+xz^{q-3}+\cdots+x^{q-2}}>y^n.$$

Noting that $z>x$, this is satisfied if

$$\frac{nx^{n-1}}{(q-1)z^{q-1}}>y^n,$$

which occurs if and only if

$$\frac{\log(n)}{n}+\frac{n-1}{n}\log(x)>\log(y)+\frac{1}{n}\log((q-1)z^{q-1}).$$

This, in turn, is satisfied if

$$\log(x)>\frac{n}{n-1}\log(y)+\frac{1}{n-1}\log((q-1)z^{q-1}).$$

Consequently (7) has no solutions when

$$x>y^{\frac{n}{n-1}}(q-1)^{\frac{1}{n-1}}z^{\frac{q-1}{n-1}}.$$

In particular, let y, q, and z be all fixed, then, given $\varepsilon>0$, there is $M>0$ such that there are no solutions to (7) for $n>M$, $(n,q)=(x,q)=(y,q)=1$,

$$x>y(1+\varepsilon).$$

Thus, there exists $N > 0$ such that, if $n > N$, there are no solutions to (7) when $x > y$, because x and y are integers and ε may be chosen arbitrarily small. □

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Cal Long Active in Fibonacci Research Conferences

(Continued from page 242)

program, and he was a consultant to the Washington State Superintendent of Public Instruction, to the State Department of Education and to the National Science Foundation.

Concerning The Fibonacci Association, Cal is a Charter Member. He served on the Board of Directors from July 6, 1983 to June 19, 1999 and he was the President for the last fifteen years. He was a strong supporter of the Fibonacci Research Conferences, attending most of them and presenting papers. Under his leadership, the organization became stronger and more unified.

On the unprofessional side, Cal is an avid fisherman and lover of the outdoors. It was not unusual to see him fly casting in the lakes and streams or walking the trails of the idyllic Idaho wilderness and sometimes you could even see him boating down the rapids of the Snake River. Cal also has a beautiful tenor voice, which he put to good use as a member of his church choir, a member of the Vandeleers, a well known University of Idaho choral group, a member of the Eugene Gleesmen, during his graduate years, a member of the Pullman/Moscow Chorale and a member of the Idaho-Washington Symphony Chorale. Cal was also a very dedicated husband whose strongest supporter was his wife Jean on whom he always knew he could count on because her support was always there. Finally, Cal was a devoted father to his two children, Tracy and Greg.

Cal, for all that you have done in so many ways for so many people, we say thank you. Enjoy retirement and know that you have made a difference to so many people who have crossed your path.

THE LEAST INTEGER HAVING p FIBONACCI REPRESENTATIONS, p PRIME

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1. INTRODUCTION

Given a positive integer N, a representation of N as a sum of distinct Fibonacci numbers in descending order is a Fibonacci representation of N. Let $R(N)$ be the number of Fibonacci representations of N. For example, $R(58) = 7$, since 58 can be written as:

$$\begin{array}{lll} 55+3 & 34+21+3 & 34+13+8+3 \\ 55+2+1 & 34+21+2+1 & 34+13+8+2+1 \\ & & 34+13+5+3+2+1 \end{array}$$

Any positive integer N can be represented uniquely as the sum of distinct, nonconsecutive Fibonacci numbers; this representation is the Zeckendorf representation of N, denoted Zeck N. In particular, Zeck $58 = 55+3 = F_{10}+F_4$, in subscript notation.

The subscripts of the Fibonacci numbers appearing in Zeck N allow calculation of $R(N)$ by using reduction formulas [3], [4]. If Zeck $N = F_{n+k}+K$, where $K = F_n + \cdots + F_t < F_{n+1}$, then

$$R(N) = R(F_{n+2q}+K) = qR(K)+R(F_{n+1}-K-2), \quad k = 2q, \tag{1.1}$$

$$R(N) = R(F_{n+2q+1}+K) = (q+1)R(K), \quad k = 2q+1. \tag{1.2}$$

Further, subscripts in Zeck N can be shifted downward c to calculate $R(N-1)$,

$$R(N-1) = R(F_{n+k-c}+F_{n-c}+\cdots+F_{t-c}-1), \quad t \ge c+2. \tag{1.3}$$

Lastly, tables for $R(N)$ contain palindromic lists. For N within successive intervals $F_n \le N \le F_{n+1}-2$, the values for $R(N)$ satisfy the symmetric property

$$R(F_{n+1}-2-M) = R(F_n+M), \quad 0 \le M \le F_{n-1}, \quad n \ge 3. \tag{1.4}$$

The table for $R(N)$ repeats patterns within intervals and subintervals although with increasingly larger values; indeed, $R(N)$ appears fractal in nature. What interests us, however, is the inverse problem: Given a value n, write an integer N such that $R(N) = n$ or, most interesting of all, find the least N having exactly n representations as sums of distinct Fibonacci numbers.

Let A_n be the least positive integer having exactly n Fibonacci representations. Then $\{A_n\} = \{1, 3, 8, 16, 24, 37, 58, 63, \ldots\}$, but while the first 330 values for A_n are listed in [6], A_n is given by formula only for special values of n. However, when p is prime, all Fibonacci numbers used in Zeck A_p have even subscripts. The sequence $\{B_n\}$ of the next section arises from an attempt to make sense of $\{A_n\}$ when $n = p$ is prime.

2. EVEN-ZECK INTEGERS AND THE BOUNDING SEQUENCE $\{B_n\}$

If an integer N has a prime number of Fibonacci representations, then the subscripts of the Fibonacci numbers appearing in Zeck N have the same parity. Since $R(F_{2k+1}) = R(F_{2k})$, we concentrate upon even subscripts. We will call a positive integer whose Zeckendorf representation contains only even-subscripted Fibonacci numbers an *even-Zeck integer*.

Here we study a bounding sequence $\{B_n\}$, where $B_n \geq A_n$, $n \geq 1$. We let B_n be the least even-Zeck integer having exactly n Fibonacci representations. Note that $A_n = B_n$ whenever A_n is an even-Zeck integer.

We begin by listing even-Zeck N and computing $R(N)$ for N in our restricted domain. In Table 2.1, we underline the first occurrence of each value for $R(N)$ and list subscripts only for Zeck N. Notice that 2^k integers N have Zeck N beginning with $F_{2(k+1)}$. For N in the interval $F_{2k} \leq N \leq F_{2k+1} - 2$, $R(N)$ takes on values in a palindromic list which begins with $k = R(F_{2k})$ and ends with $k = R(F_{2k+1} - 2)$, with central value 2. Interestingly, every third entry for $R(N)$ is even.

TABLE 2.1. $R(N)$ for Even-Zeck N, $1 \leq N \leq 88$

$R(N)$	N	Zeck N	$R(N)$	N	Zeck N
$\underline{1}$	$\underline{1}$	2	5	55	10
$\underline{2}$	$\underline{3}$	4	4	56	10,2
1	4	4,2	$\underline{7}$	$\underline{58}$	10,4
$\underline{3}$	$\underline{8}$	6	3	59	10,4,2
2	9	6,2	$\underline{8}$	$\underline{63}$	10,6
3	11	6,4	5	64	10,6,2
1	12	6,4,2	7	66	10,6,4
$\underline{4}$	$\underline{21}$	8	2	67	10,6,4,2
3	22	8,2	7	76	10,8
$\underline{5}$	$\underline{24}$	8,4	5	77	10,8,2
2	25	8,4,2	8	79	10,8,4
5	29	8,6	3	80	10,8,4,2
3	30	8,6,2	7	84	10,8,6
4	32	8,6,4	4	85	10,8,6,2
1	33	8,6,4,2	5	87	10,8,6,4
			1	88	10,8,6,4,2

In Table 2.1, the listed values for $R(N)$ for $N = F_{10} + K$ can be obtained by writing the values (1), 4, 3, 5, 2, ..., from $R(N)$ for $N = F_8 + K$, interspersed with their sums: (1), $\underline{5}$, 4, $\underline{7}$, 3, $\underline{8}$, 5, $\underline{7}$, 2, ..., the first half of the palindromic sequence of $R(N)$ values for $N = F_{10} + K$, where, of course, the second half repeats. The first (1) arises from $R(F_t - 1) = 1$, $t \geq 1$; the algorithm computes $R(N)$ for even-Zeck N in the interval $F_{2k} \leq N \leq F_{2k+1} - 1$, using values obtained from the preceding interval for N.

Theorem 2.1: If N is an even-Zeck integer such that Zeck N ends in F_{2c}, $c \geq 2$, $F_{2k} \leq N \leq F_{2k+1} - 1$, and N^* is the even-Zeck integer preceding N, then

$$R(N) = R(N+1) + R(N^*). \tag{2.1}$$

Further, $R(N+1) = R(M)$ and $R(N^*) = R(M^*)$, where M^* is the even-Zeck integer preceding M in the interval $F_{2k-2} \leq M \leq F_{2k-1} - 1$.

Proof: We will use (1.3) to shift subscripts in computing $R(N+1)$ and $R(N^*)$. If $N = F_{2k} + \cdots + F_{2c+2p} + F_{2c}$, $c \geq 2$, then the even-Zeck integer preceding N is

$$\begin{aligned} N^* &= F_{2k} + \cdots + F_{2c+2p} + (F_{2c-2} + \cdots + F_4 + F_2) \\ &= F_{2k} + \cdots + F_{2n+2p} + (F_{2c-1} - 1) \\ &= N - F_{2c} + F_{2c-1} - 1 = N - F_{2c-2} - 1. \end{aligned} \tag{2.2}$$

While $(N-1)$ is not an even-Zeck integer, we can apply (1.3) to shift each subscript down $(2c-2)$ to obtain an even-Zeck integer,

$$\begin{aligned} R(N-1) &= R(F_{2k}+\cdots+F_{2c+2p}+F_{2c}-1) = R(F_{2k-2c+2}+\cdots+F_{2c+2p-2c+2}+F_{2c-2c+2}-1) \\ &= R(F_{2k-2c+2}+\cdots+F_{2p+2}+F_2-1) = R(F_{2k-2c+2}+\cdots+F_{2p+2}) = R(K), \end{aligned} \tag{2.3}$$

where K is an even-Zeck integer. Similarly, shifting subscripts down $2c-2$ in (2.2), we obtain $R(N^*) = R(N-1)$. From [3], $R(N) = R(N+1)+R(N-1)$ for any integer N such that Zeck N ends in F_{2c}, $c \geq 2$. The rest of Theorem 2.1 follows from similar subscript reductions, so that

$$R(N+1) = R(F_{2k-2}+\cdots+F_{2c+2p-2}+F_{2c-2}) = R(M), \tag{2.4}$$

and $R(N^*) = R(F_{2k-2}+\cdots+F_{2c+2p-2}+F_{2c-2}-F_{2c-4}-1) = R(M^*)$. $\square$

When we list the 2^k values for $R(N)$ for even-Zeck N in the interval $F_{2k} \leq N \leq F_{2k+1}-1$, the corresponding values for N can be found by numbering the entries for $R(N)$. For example, in Table 2.1, 66 is the 7^{th} entry in the interval $F_{10} \leq N \leq F_{11}-1$ (the 6^{th} entry after 55), and $6 = 2^2+2^1$ corresponds to $F_{2(2+1)}+F_{2(1+1)}$; Zeck $66 = F_{10}+F_6+F_4$. If $R(N)$ is the m^{th} entry in the interval $F_{2k} \leq N \leq F_{2k+1}-1$, and if $(m-1) = 2^p+\cdots+2^w$, then the associated even-Zeck integer N has Zeck $N = F_{2k}+F_{2(p+1)}+\cdots+F_{2(w+1)}$. Further, the list is palindromic; the m^{th} entry for $R(N)$ equals the $(2^{k-1}-m)^{\text{th}}$ entry.

Since A_p is an even-Zeck integer when p is prime, $B_p = A_p$ for prime p, and $B_n \geq A_n$ for all $n \geq 1$. The first occurrences of $R(N)$ in Table 2.1 give us $\{B_n\} = \{1, 3, 8, 21, 24, \_, 58, 63, \ldots\}$, where B_6 is as yet unknown. Table 2.2 lists the first 89 values for $\{B_n\}$, from computation of $R(N)$ for even-Zeck N, $1 \leq N < F_{23}$.

TABLE 2.2. B_n for $1 \leq n \leq 89$

n	B_n	n	B_n	n	B_n	n	B_n
1*	1	23*	1011	45	3134	67*	7166
2*	3	24	1063	46*	2990	68	7221
3*	8	25	1053	47*	2752	69*	7200
4	21	26	1045	48	6975	70	8158
5*	24	27*	1066	49*	2985	71*	7310
6	144	28	2608	50*	3019	72	18719
7*	58	29*	1050	51	6930	73*	7831
8*	63	30	1139	52	6917	74*	8187
9	147	31*	1160	53*	6967	75	7954
10	155	32	2650	54	19298	76*	7205
11*	152	33	2642	55*	3024	77	18295
12	173	34*	1155	56	7163	78	18164
13*	168	35	2663	57	6972	79*	7815
14	385	36	2807	58	7297	80	7959
15	398	37*	2647	59*	7349	81*	7925
16	461	38	6841	60	6933	82	18918
17*	406	39	2969	61*	7218	83*	18154
18*	401	40	2749	62	7836	84	18240
19*	435	41*	2736	63	7171	85	18112
20	1215	42	7145	64	7315	86	19083
21*	440	43*	2757	65	7208	87	18167
22	1016	44	2791	66	7899	88	18146
						89*	7920

In Table 2.2, * denotes $B_n = A_n$, which is always true for $n = F_k$ or L_k, and when n is prime. However, while we can have $B_n = A_n$ when n is composite, the most irregularly occurring values for B_n are when n is even.

Theorem 2.2: The following special values for n have $A_n = B_n$:

$$n = F_{k+1} \quad B_n = F_{2k} + F_{2k-4} + F_{2k-8} + F_{2k-12} + \cdots, \quad k \geq 2; \tag{2.5}$$

$$n = L_{k-1} \quad B_n = F_{2k} + F_{2k-6} + F_{2k-10} + F_{2k-14} + \cdots, \quad k \geq 3. \tag{2.6}$$

Proof: A_n has the above values for the given values of n from [1]. Since in these two cases A_n is an even-Zeck integer, $A_n = B_n$. □

From computation of the first 610 values for B_n, it appears that if Zeck n begins with F_k, that is, $F_k < n < F_{k+1}$, then Zeck B_n begins with F_{2k}, F_{2k+2}, or F_{2k+4}; this has not been proved. However, F_{m+1} is the largest value for $R(M)$ in the interval $F_{2m} \leq M \leq F_{2m+1}$, and all other values for $R(M)$ which appear in that interval have Zeck n beginning with F_m or a smaller Fibonacci number. Note that we are relating n and B_n in an interesting way, since the subscripts in Zeck N are used to compute $R(N)$.

3. PROPERTIES OF $\{B_n\}$

Theorem 3.1: If N is an even-Zeck integer such that $F_{2k} \leq N < F_{2k+1}$, and if $M = F_{k+1}^2 - 1$, then the three largest values occurring for $R(N)$ are:

$$\begin{array}{lll} R(N) = n & N = B_n & \\ F_{k+1} & M = F_{k+1}^2 - 1, & k \geq 2; \quad (3.1) \\ F_{k+1} - F_{k-4} & M + 5(-1)^k, & k \geq 6; \quad (3.2) \\ F_{k+1} - F_{k-4} - F_{k-8} & M + 39(-1)^k, & k \geq 9. \quad (3.3) \end{array}$$

For even-Zeck N in this interval, the following values for $R(N)$ do not occur:

$$R(N) = F_{k+1} - p, \ 1 \leq p \leq F_{k-4} + F_{k-8} - 1, \ k \geq 9, \tag{3.4}$$

except for $p = F_{k-4}$. In particular,

$$R(N) = F_{k+1} - 1, \ k \geq 7,$$

is a missing value.

Proof: From [1], M is the smallest integer having F_{k+1} Fibonacci representations; Zeck M appears in (2.5). Tables for $R(N)$ show palindromic behavior within each interval for N as well as "peaks" containing clusters of values where $N = B_n$. The "peak value" is the sum of two adjacent values for $R(M)$ at the "peak" of the preceding interval $F_{2k-2} \leq M < F_{2k-1}$ from the formation of the table for $R(N)$.

Table 3.1 exhibits behavior near the primary peak value $R(N) = F_{k+1}$ for the interval

$$F_{2k} + F_{2k-4} \leq N < F_{2k} + F_{2k-3}.$$

Recalling (2.1), when Zeck N ends in $F_{2c} \geq F_4$, $R(N) = R(N+1) + R(N^*)$, where N^* is the even-Zeck integer preceding N. Since we are looking at consecutive even-Zeck N in Table 3.1, the formula for each value of $R(N)$ can be proved by induction, $k \geq 6$.

TABLE 3.1. $R(N)$ for Even-Zeck N, $F_{2k}+F_{2k-4}\le N<F_{2k}+F_{2k-3}$

k odd: $M=F_{k+1}^2-1=F_{2k}+F_{2k-4}+\cdots F_{14}+F_{10}+F_6$

	$R(N)$	N	Zeck N ends with:
	$\dots$	$\dots$	$\dots$
	F_k+F_{k-5}	$M-8$	$\dots+F_{14}+F_{10}$
	L_{k-2}	$M-7$	$\dots+F_{14}+F_{10}+F_2$
$N=B_n$	$F_{k+1}-F_{k-4}$	$M-5$	$\dots+F_{14}+F_{10}+F_4$
	F_{k-1}	$M-4$	$\dots+F_{14}+F_{10}+F_4+F_2$
$N=B_n$	F_{k+1}	M	$\dots+F_{14}+F_{10}+F_6$
	F_k	$M+1$	$\dots+F_{14}+F_{10}+F_6+F_2$
	L_{k-1}	$M+3$	$\dots+F_{14}+F_{10}+F_6+F_4$
	F_{k-2}	$M+4$	$\dots+F_{14}+F_{10}+F_6+F_4+F_2$
	$F_{k+1}-L_{k-4}$	$M+13$	$\dots+F_{14}+F_{10}+F_8$
	$\dots$	$\dots$	$\dots$

k even: $M=F_{k+1}^2-1=F_{2k}+F_{2k-4}+\cdots+F_{12}+F_8+F_4$

	$R(N)$	N	Zeck N ends with:
	$\dots$	$\dots$	$\dots$
	$F_{k+1}-L_{k-4}$	$M-13$	$\dots+F_{12}+F_6+F_4$
	F_{k-2}	$M-12$	$\dots+F_{12}+F_6+F_4+F_2$
	L_{k-1}	$M-3$	$\dots+F_{12}+F_8$
	F_k	$M-2$	$\dots+F_{12}+F_8+F_2$
$N=B_n$	F_{k+1}	M	$\dots+F_{12}+F_8+F_4$
	F_{k-1}	$M+1$	$\dots+F_{12}+F_8+F_4+F_2$
$N=B_n$	$F_{k+1}-F_{k-4}$	$M+5$	$\dots+F_{12}+F_8+F_6$
	L_{k-2}	$M+6$	$\dots+F_{12}+F_8+F_6+F_2$
	F_k+F_{k-5}	$M+8$	$\dots+F_{12}+F_8+F_6+F_4$
	F_{k-3}	$M+9$	$\dots+F_{12}+F_8+F_6+F_4+F_2$
	$\dots$	$\dots$	$\dots$

We show that $R(N)=B_n$ for $n=F_{k+1}-F_{k-4}$ because we cannot get the same result for a smaller N. In Table 3.1, N is in the interval $F_{2k}+F_{2k-4}<N<F_{2k}+F_{2k-3}$. To have $R(N)=F_{k+1}-F_{k-4}$ for a smaller N, we must have $F_{2k}<N<F_{2k}+F_{2k-4}$. From (2.6), L_{k-1} is the largest value for $R(N)$ for even-Zeck N in the interval $F_{2k}+F_{2k-6}<N<F_{2k}+F_{2k-4}$, where $L_{k-1}=F_k+F_{k-2}<F_{k+1}-F_{k-4}=F_k+F_{k-2}+F_{k-5}$, so $R(N)=F_{k+1}-F_{k-4}$ cannot occur for $N<F_{2k}+F_{2k-4}$, establishing (3.2). Equation (3.3) follows in a similar manner. □

Corollary 3.1.1: For $n=F_{k+1}-F_{k-4}$ as in Theorem 3.1, $A_n=B_n$ for $k\ge 7$.

When N is any positive integer, $R(N)$ displays "peak" values near $R(N)=F_{k+1}$ similar to those listed in Table 3.1 for even-Zeck integers N. The three largest values for $R(N)$, when N is any positive integer, $F_{2k}\le N<F_{2k+1}$, are F_{k+1}, $F_{k+1}-F_{k-5}=4F_{k-2}$, and $F_{k+1}-F_{k-4}$. When $n=4F_{k-2}$, $A_n=M+8(-1)^{k+1}$ for $M=F_{k+1}^2-1$. The values for $R(N)=F_{k+1}-p$, $1\le p\le F_{k-5}-1$, $k\ge 6$, are missing for N in that interval.

A similar "secondary peak" in the lists for $R(N)$ clusters around L_{k-1}, both for N any positive integer and for N an even-Zeck integer; hence, Theorem 3.2.

Theorem 3.2: If $M = F_{2k} + F_{2k-6} + F_{2k-10} + \cdots = F_{2k} + F_{k-2}^2 - 1$, then when

$$n = L_{k-1} \qquad B_n = M, \qquad k \geq 5; \tag{3.5}$$

$$n = L_{k-1} - L_{k-6} \qquad B_n = M + 5(-1)^{k-1}, \quad k \geq 7; \tag{3.6}$$

$$n = L_{k-1} - L_{k-6} - L_{k-10} \qquad B_n = M + 39(-1)^{k-1}, \quad k \geq 11. \tag{3.7}$$

Corollary 3.2.1: For $n = L_{k-1} - L_{k-6}$ as in Theorem 3.2, $A_n = B_n$ for $k \geq 9$.

4. UNANSWERED QUESTIONS

Theorem 3.1 shows some values for $R(N)$ that are missing within each interval for even-Zeck N, $F_{2k} \leq N < F_{2k+1}$, $k \geq 9$. In what interval will those "missing values" first appear? The value $n = R(N)$ always occurs for some even-Zeck N, since, in the worst case scenario, $n = R(F_{2n})$. But when is $\{B_n\}$ complete?

Conjecture 3.1.3: If $R(N)$ is calculated for all even-Zeck N, $N < F_{2k+5}$, then $\{B_n\}$ is complete for $1 \leq n \leq F_k$. If $F_k < n < F_{k+1}$, then $F_{2k} < B_n < F_{2k+5}$.

Finding the least integer having p Fibonacci representations, p prime, is an unsolved problem.

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❖❖❖

SECOND-ORDER LINEAR RECURRENCES OF COMPOSITE NUMBERS

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In [3], W. Sierpinski proved that there are infinitely many odd integers k (Sierpinski numbers) such that $k2^n+1$ is a composite number for all $n \geq 0$, i.e., he found that the recurrence $u_{n+2} = 3u_{n+1}-2u_n$, $n \geq 0$, has infinitely many initial values $u_0 = k+1$ and $u_1 = 2k+1$ that give composite u_n for all $n \geq 0$. Analogously, R. L. Graham [1] and D. Knuth [2] found composite integers F_0, F_1, $(F_0, F_1) = 1$ for the Fibonacci-like sequence $\{F_n\}$, $n \geq 0$, $F_{n+2} = F_{n+1}+F_n$ such that F_n are all composite numbers.

In the construction of composite sequences, the authors [1]-[3] used the idea of a covering set, i.e., a set $P = \{p_1, p_2, \ldots, p_h\}$, $h \geq 1$, of prime numbers such that, for each $n \geq 0$, there exists at least one $p \in P$ such that $u_n \equiv 0 \bmod p$.

In this note we give a class of integers $a > 0$, b, $(a, b) = 1$ and find integers u_0, u_1, $(u_0, u_1) = 1$ such that the sequence $\{u_n\}$, $n \geq 0$, $u_{n+2} = au_{n+1}-bu_n$ with initial values u_0, u_1 contain only composite members. For even n, u_n has an algebraic decomposition while, for odd n, u_n has a covering set $P = \{p\}$.

To prove the main theorem, we need the following three lemmas.

Lemma 1: Let integers a, b be such that $\Delta = a^2-4b \neq 0$. Let integers v_0, v_1 be initial values for the recurrence $v_{n+2} = av_{n+1}-bv_n$, $n \geq 0$. Then for the sequence $\{u_n\}$, $n \geq 0$, and $u_0 = v_0w_0$, $u_2 = v_1w_1$, $u_{n+2} = au_{n+1}-bu_n$, $n \geq 0$, we have

$$u_{2n} = v_n w_n, \tag{1}$$

where

$$w_0 = k(2v_1 - av_0)/d, \quad w_1 = k(av_1 - 2bv_0)/d, \tag{2}$$

$d = (2v_1-av_0,\ av_1-2bv_0)$, k is an arbitrary integer and $w_{n+2} = aw_{n+1}-bw_n$.

Proof: Let w_0, w_1 be arbitrary integers. We prove that, if $u_{2n} = v_nw_n$, then w_0, w_1 satisfy (2). It is known that the sequence $\{x_n\}$, $n \geq 0$, satisfies the recurrence $x_{n+2} = ax_{n+1}-bx_n$ if and only if $x_n = A\alpha^n + B\beta^n$ for $n \geq 0$, where A, B are constants and α, β are the distinct roots of the characteristic polynomial z^2-az+b, since $\Delta = a^2-4b \neq 0$. So we have

$$v_n = A_1\alpha^n + B_1\beta^n, \ w_n = A_2\alpha^n + B_2\beta^n,$$

where $\alpha = (a+\Delta)/2$, $\beta = (a-\Delta)/2$, and

$$A_1 = (v_1-\beta v_0)/(\alpha-\beta), \quad B_1 = (\alpha v_0 - v_1)/(\alpha-\beta),$$
$$A_2 = (w_1-\beta w_0)/(\alpha-\beta), \quad B_2 = (\alpha w_0 - w_1)/(\alpha-\beta).$$

Furthermore,

$$v_nw_n = (A_1\alpha^n + B_1\beta^n)(A_2\alpha^n + B_2\beta^n) = A_1A_2\alpha^{2n} + (A_1B_2 + A_2B_1)\alpha^n\beta^n + B_1B_2\beta^{2n}.$$

So, if $A_1B_2 + A_2B_1 = 0$, the sequence $\{u_k\}$, $k \geq 0$, $u_k = A_1A_2\alpha^k + B_1B_2\beta^k$ satisfies $u_{k+2} = au_{k+1} - bu_k$ and $u_{2n} = v_nw_n$. Consider

$$0 = A_1B_2 + A_2B_1 = (v_1 - \beta v_0)(\alpha w_0 - w_1)/(\alpha-\beta)^2 + (\alpha v_0 - v_1)(w_1 - \beta w_0)/(\alpha-\beta)^2$$
$$= [(\alpha+\beta)(v_1w_0 + v_0w_1) - 2\alpha\beta v_0w_0 - 2v_1w_1]/(\alpha-\beta)^2.$$

Since $\alpha+\beta = a$, $\alpha\beta = b$, we have $a(v_1w_0 + v_0w_1) - 2bv_0w_0 - 2v_1w_1 = 0$, or

$$(av_1 - 2bv_0)w_0 = (2v_1 - av_0)w_1.$$

If $d = (2v_1 - av_0, av_1 - 2bv_0)$ and k is an arbitrary integer, then we have (2).

Lemma 2: Let $a > 1$, $m > 1$, and b be integers such that $a \equiv 0 \bmod m$ and u_0, u_1 are initial values for the recurrence $u_{n+2} = au_{n+1} - bu_n$, $n \geq 0$. If $u_1 \equiv 0 \bmod m$, then $u_{2n+1} \equiv 0 \bmod m$ for $n \geq 0$.

Proof: Consider the sequence $\{U_n\}$, where $U_0 = 0$, $U_1 = 1$, $U_{n+2} = aU_{n+1} - bU_n$, $n \geq 0$. It is known that $U_{2n} \equiv 0 \bmod a$ for $n \geq 1$. Since $u_{2n+1} = u_1U_{2n+1} - bu_0U_{2n}$ for $n \geq 0$, we have $u_{2n+1} \equiv 0 \bmod m$.

Lemma 3: Let integers $a > 0$ and b be such that $(a, b) = 1$, $\Delta = a^2 - 4b > 0$, and u_0, u_1 be initial values for $u_{n+2} = au_{n+1} - bu_n$, $n \geq 0$, such that $u_0 > 0$, $(b, u_1) = 1$, $(u_0, u_1) = 1$, and $u_1 > au_0/2$. Then $(u_n, u_{n+1}) = 1$ and $u_{n+1} > au_n/2$ for $n > 0$.

Proof: We prove this lemma by induction. We first prove that $(b, u_n) = 1$ for $n > 1$. By the condition of the lemma, $(b, u_1) = 1$. Let $(b, u_i) = 1$ for $1 < i \leq n$. For $i = n+1$, we have $(b, u_{n+1}) = (b, au_n - bu_{n-1}) = (b, au_n) = (b, u_n) = 1$. Since $(u_0, u_1) = 1$, let $(u_i, u_{i+1}) = 1$ for $1 \leq i \leq n$. For $i = n+1$, we have $(u_{n+1}, u_{n+2}) = (u_{n+1}, au_{n+1} + bu_n) = (u_{n+1}, u_n) = 1$. By the statement of Lemma 3, $u_1 > au_0/2$. Assume that $u_i > au_{i-1}/2$ is true for $1 < i \leq n$. Then, for $i = n+1$,

$$u_{n+1} = au_n - bu_{n-1} = au_n/2 + au_n/2 - bu_{n-1}$$
$$> au_n/2 + a(au_{n-1}/2)/2 - bu_{n-1} > au_n/2 + \Delta u_{n-1}/4 > au_{n-1}/2.$$

Thus, the lemma is proved.

We now proceed to prove the main theorem.

Theorem: Let odd $a > 2$ and b be integers such that $(a, b) = 1$ and let $\Delta = a^2 - 4b > 0$. Let p be an odd prime divisor of a such that the Legendre symbol $(b/p) = 1$ and let $t > 0$ be any solution of the congruence $x^2 \equiv b \bmod p$. Let $v_0 > 1$, $(a, v_0) = 1$, and $v_1 = tv_0 + kp$ for some positive k such that $(a, v_1) = (v_0, v_1) = (b, v_1) = 1$, $v_1 > av_0/2$. Let $d = (2v_1 - av_0, av_1 - 2bv_0)$.

Then the sequence $\{u_n\}$ with initial values $u_0 = (2v_0v_1 - av_0^2)/d$, $u_1 = (v_1^2 - bv_0^2)/d$, and $u_{n+2} = au_{n+1} - bu_n$ for $n \geq 0$ is a sequence of composite numbers.

Proof: By Lemma 1, $u_{2n} = v_nw_n$, $n \geq 0$. Here $v_{n+2} = av_{n+1} - bv_n$, $n \geq 0$, for given initial values v_0, v_1, and $w_{n+2} = aw_{n+1} - bw_n$, $n \geq 0$, for initial values $w_0 = (2v_1 - av_0)/d$, $w_1 = (av_1 - 2bv_0)/d$.

We have $u_0 = v_0w_0 = (2v_0v_1 - av_0^2)/d$, $u_2 = v_1w_1 = (av_1^2 - 2bv_0v_1)/d$. Hence,

$$u_1 = (u_2 + bu_0)/a = (av_1^2 - abv_0^2)/ad = (v_1^2 - bv_0^2)/d.$$

Since $t^2 \equiv b \bmod p$, $v_1 = tv_0 + kp$, and $(b, d) = 1$, we have $u_1 \equiv 0 \bmod p$. By Lemma 2, $u_{2m+1} \equiv 0 \bmod p$ for $n > 0$.

Further, $(u_0, u_1) \leq (u_0, au_1) = (u_0, u_2 + bu_0) = (u_0, u_2) = (v_0w_0, v_1w_1)$. Consider

$$(v_0, w_1) < (v_0, dw_1) = (v_0, av_1 - 2bv_0) = (v_0, av_1) = 1.$$

Analogously, $(w_0, v_1) \le (dw_0, v_1) = (2v_1 - v_0, v_1) = 1$. Since $(v_0, v_1) = 1$ and $(w_0, w_1) = 1$, we obtain $(u_0, u_1) = 1$, and by Lemma 3, $(u_n, u_{n+1}) = 1$ for $n \ge 0$.

Finally, consider

$$u_1 - au_0 / 2 = (v_1^2 - bv_0^2)/d - (2av_0v_1 - a^2v_0^2)/2d = (v_1 - av_0/2)^2/d + \Delta v_0^2/4d > 0.$$

By Lemma 3, $u_{n+1} > au_n/2$ for $n > 0$. Thus, the theorem is proved.

On the other hand, it is easy to prove that there are no primes p_0, p_1 such that $p_n = ap_{n-1} - bp_{n-2}$, $a > 0$, $(a, b) = 1$, and $a^2 - 4b > 0$ are primes for all $n > 1$.

Indeed, if $b \equiv 0 \bmod p_1$, then $p_2 = ap_1 - bp_0 \equiv 0 \bmod p_1$. Let $b \not\equiv 0 \bmod p_1$, then there is an $m \le p_1 + 1$ such that $U_m \equiv 0 \bmod p_1$, where $U_0 = 0$, $U_1 = 1$, $U_{n+2} = aU_{n+1} - bU_n$, $n \ge 0$. Since $p_{m+1} = p_1U_{m+1} - bp_0U_m$, we have $p_{m+1} \equiv 0 \bmod p_1$.

It is interesting to find a sequence of primes of maximal length for the Mersenne recurrence $p_{n+2} = 3p_{n+1} - 2p_n$ for $n \ge 0$, where p_0, $p_1 > p_0$ are given primes. The numerical search for small p_0, p_1 gives the sequence of nine primes $\{41, 71, 131, 251, 491, 971, 1931, 3851, 7691\}$. The more exact estimate for length N primes in the Mersenne recurrence uses

$$p_n = p_0M_{n+1} - 2p_{-1}M_n = p_0M_{n+1} - (3p_0 - p_1)M_n, \tag{3}$$

where $M_0 = 0$, $M_1 = 1$, $M_{n+2} = 3M_{n+1} - 2M_n$, $n \ge 0$. p_0, p_1 are given primes and $3p_0 - p_1 \ne 2^t$, $t > 0$. Let $m = \min_{q>2}\{\upsilon(q) : q | (3p_0 - p_1)\}$, q is prime, and let $\upsilon(q)$ be the minimal s such that $m_s \equiv 0 \bmod q$. Then by (3), $p_m \equiv 0 \bmod q$ and $N \le m - 1$. N is equal to the upper bound, e.g., for the sequence $\{3467, 6947, 13907, 27827, 55667, 111347, 222707, 445427, 890967\}$. Now, since $p_0 = 3467$, $p_1 = 6947$, and $11 | 3454 = 3p_0 - p_1$, we have $m = \upsilon(11) = 10$ and $N = 9$.

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A DIVISIBILITY PROPERTY OF BINARY LINEAR RECURRENCES

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INTRODUCTION

If A is a positive integer, let the polynomial $\lambda^2 - A\lambda - 1$ with discriminant $D = A^2 + 4$ have the roots:

$$\alpha = (A + \sqrt{D})/2, \quad \beta = (A - \sqrt{D})/2. \tag{1}$$

Define a primary binary linear recurrence $\{u_n\}$ and a secondary binary linear recurrence $\{v_n\}$ by

$$u_n = (\alpha^n - \beta^n)/(\alpha - \beta), \quad v_n = \alpha^n + \beta^n, \tag{2}$$

where $n \geq 0$. Equivalently, let

$$u_0 = 0, \ u_1 = 1, \ u_n = Au_{n-1} + u_{n-2} \tag{3}$$

and

$$v_0 = 2, \ v_1 = A, \ v_n = Av_{n-1} + v_{n-2} \tag{4}$$

for $n \geq 2$. Let

$$t = \begin{cases} D & \text{if } A \text{ is odd,} \\ D/4 & \text{if } A \equiv 0 \ (\text{mod } 4), \\ D/8 & \text{if } A \equiv 2 \ (\text{mod } 4). \end{cases} \tag{5}$$

Note that, in each case, t is an integer such that $t \equiv 1$ (mod 4).

Let $(\frac{a}{b})$ denote the Jacobi symbol.

In this note, we prove a divisibility property of the $\{u_n\}$ and of the $\{v_n\}$. In so doing, we generalize a recent result by V. Drobot [2] about Fibonacci numbers (the sequence $\{u_n\}$ with $A = 1$). It has been called to our attention that an alternate proof of Drobot's result follows from [1]. Note that, if $A = 2$, then the corresponding u_n sequence is called the *Pell* sequence, and is denoted P_n. Thus, we have $P_1 = 1$, $P_2 = 2$, $P_3 = 5$, $P_4 = 12$, $P_5 = 29$, and so forth.

THE MAIN RESULTS

Theorem 1: Let $\{u_n\}$ and t be defined as above. Let p be an odd integer such that $q = 2p - 1$ is prime, and $q \nmid t$. If $A \not\equiv 2$ (mod 4), let $(\frac{q}{t}) = -1$. If $A \equiv 2$ (mod 4) but $A > 2$, let $q \equiv \pm 1$ (mod 8) and $(\frac{q}{t}) = -1$ or $q \equiv \pm 3$ (mod 8) and $(\frac{q}{t}) = 1$. If $A = 2$, let $q \equiv \pm 3$ (mod 8). Then $q \mid u_p$. Furthermore, u_p is composite unless $u_p = q$, which can occur only in the cases $(A, p, q) = (1, 7, 13)$ (Fibonacci) or $(A, p, q) = (2, 3, 5)$ (Pell).

Proof: Equation (1) implies

$$\alpha - \beta = \sqrt{D} \quad \alpha\beta = -1. \tag{6}$$

Applying (2) and (6) with $n = p$, we obtain $\sqrt{D}u_p = \alpha^p - \beta^p$. Squaring and applying (6), we get

$$Du_p^2 = \alpha^{2p} + \beta^{2p} + 2. \tag{7}$$

Multiplying by 2^{2p-1} and applying (1), we have

$$2^{2p-1}Du_p^2 = \frac{1}{2}\{(A+\sqrt{D})^{2p}+(A-\sqrt{D})^{2p}\}+4^p \tag{8}$$

If we expand the right member of (8) via the binomial theorem and then simplify, we obtain

$$2^{2p-1}Du_p^2 = A^{2p}+\sum_{k=1}^{p-1}\binom{2p}{2k}A^{2p-2k}D^k+D^p+4^p. \tag{9}$$

Since $q=2p-1$ is prime by hypothesis, we have

$$q\,\left|\,\binom{2p}{2k}\right. \text{ for } 1\le k\le p-1.$$

Furthermore, by Fermat's Little Theorem, we have $A^{2p}\equiv A^2 \pmod q$, $4^p\equiv 4 \pmod q$, $2^{2p-1}\equiv 2 \pmod q$. Thus, we have

$$2Du_p^2\equiv A^2+4+D^p\equiv D+D^p\equiv D(1+D^{p-1}) \pmod q,$$

which yields $2u_p^2\equiv 1+D^{p-1} \pmod q$.

Since $p-1=(q-1)/2$, Euler's criterion yields

$$2u_p^2\equiv 1+\left(\frac{D}{q}\right) \pmod q.$$

Therefore, to prove that $q|u_p$, it suffices to show that $(\frac{D}{q})=-1$. If $A\not\equiv 2 \pmod 4$, then $(\frac{D}{q})=(\frac{t}{q})$. Since $t\equiv 1 \pmod 4$ and $t>1$, we have

$$\left(\frac{t}{q}\right)=\left(\frac{q}{t}\right)=-1$$

by hypothesis, so we are done.

If $A=2$, so that $D=8$, then

$$\left(\frac{D}{q}\right)=\left(\frac{8}{q}\right)=\left(\frac{2}{q}\right)=-1$$

since $q\equiv -3 \pmod 8$ by hypothesis. More generally, if $A\equiv 2 \pmod 4$ but $A>2$, then

$$\left(\frac{D}{q}\right)=\left(\frac{8t}{q}\right)=\left(\frac{2t}{q}\right)=\left(\frac{2}{q}\right)\left(\frac{t}{q}\right)=\left(\frac{2}{q}\right)\left(\frac{q}{t}\right),$$

since $t\equiv\pm1 \pmod 4$. By hypothesis, we have $(\frac{2}{q})=-(\frac{q}{t})$ so we are done.

The last sentence of the conclusion of Theorem 1 is now an easy corollary.

Remarks: If $A=1$, then $u_n=F_n$. (This was the case considered in [2].) If t is composite, then the determination of congruence conditions on $q \pmod t$ such that $(\frac{q}{t})=(\frac{t}{q})=\pm1$ may be achieved by factoring t as a product of primes and then applying the Chinese Remainder Theorem.

Corollary 1: If P_n denotes the n^{th} Pell number, the integer $p>3$, $p\equiv 3 \pmod 4$, and $q=2p-1$ is prime, then $q|u_p$ and $q<u_p$.

Proof: This follows from Theorem 1, with $A=2$.

We now present an analogous theorem regarding $\{v_n\}$, namely,

Theorem 2: Let $\{v_n\}$ and t be defined as in the Introduction. Let p be an odd integer such that $q=2p+1$ is prime and $q \nmid t$. If $A \not\equiv 2 \pmod 4$, let $\left(\frac{q}{t}\right)=-1$. If $A \equiv 2 \pmod 4$ but $A>2$, let $q \equiv \pm 1 \pmod 8$ and $\left(\frac{q}{t}\right)=-1$ or $q \equiv \pm 3 \pmod 8$ and $\left(\frac{q}{t}\right)=1$. If $A=2$, let $q \equiv \pm 3 \pmod 8$. Then $q \mid v_{p+1}$.

Proof: The proof is similar to that of Theorem 1 and is therefore omitted here.

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THE PRIME NUMBER MAZE

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1. INTRODUCTION TO THE MAZE

This paper introduces a fascinating maze based solely on the distribution of the prime numbers. Although it was originally designed as a simple puzzle, the maze revealed some rather startling properties of the primes. The rules are so simple and natural that traversing the maze seems more like exploring a natural cave formation than a maze of human design.

We will describe this maze using the language of graph theory. In particular, we first define an undirected graph G_0 with the set of all prime numbers as the vertex set. There will be an edge connecting two prime numbers iff their binary representations have a Hamming distance of 1. That is, two primes are connected iff their binary representations differ by exactly one digit.

The natural starting point is the smallest prime, $2 = 10_2$. Following the graph G_0 amounts to changing one binary digit at a time to form new prime numbers. The following sequence demonstrates how we can get to larger and larger prime numbers by following the edges of G_0.

$$\begin{aligned} 10_2 &= 2 \\ 11_2 &= 3 \\ 111_2 &= 7 \\ 101_2 &= 5 \\ 1101_2 &= 13 \\ 11101_2 &= 29 \\ 111101_2 &= 61 \\ 110101_2 &= 53 \\ 100101_2 &= 37 \\ 1100101_2 &= 101 \end{aligned}$$

Actually, we can get to large primes much faster, since the Hamming distance between 3 and $4099 = 1000000000011_2$ is just 1. However, the above example illustrates that we can get to 101 even if we add the restriction that the numbers increase at most one binary digit at a time. Even with this restriction, it is possible to reach 4099, but it requires a total of 46 steps.

We can include this restriction by considering a directed graph, G_1, whose vertices are again the prime numbers. There is an edge from p to q iff the Hamming distance is 1, and $3p \geq q$. Note that this always permits changing a 1 bit to a 0 bit, since $q < p$ implies $3p \geq q$. However, if a 0 bit is changed to a 1 bit, then the condition $3p \geq q$ insures that $q - p$ (which will be a power of 2) will be no more than twice the original number p.

The directed graph G_1 is easier to analyze than the graph G_0, since at any given vertex only a finite number of edges is possible. We define the *valence* of a prime number p to be the number of edges leaving the vertex p on G_1. It is not hard to have a computer map the first 70 steps (from 2) to determine which primes are attainable. A very small portion of the map is shown in Figure 1.

By a lucky coincidence, the distribution of the prime numbers is exactly what is needed to keep this graph interesting. As N increases, its number of bits grows as $\log_2 N$, so to compute the valence of N, we will need to test $\lfloor \log_2 N \rfloor + 1$ numbers for primality. However, by the prime number theorem [4], only about $1/\ln N$ of these numbers will be prime. So, heuristically, the expected value of the valence will remain roughly constant throughout the entire graph.

Figure 1 shows all of the primes that can be reached from the prime 2 without having to go to primes larger than 1024. However, this does not show all of the primes less than 1024 that can be reached from 2. The number 353 can be reached, but not without first attaining the prime $353 + 2^{27} + 2^{392} + 2^{441}$.

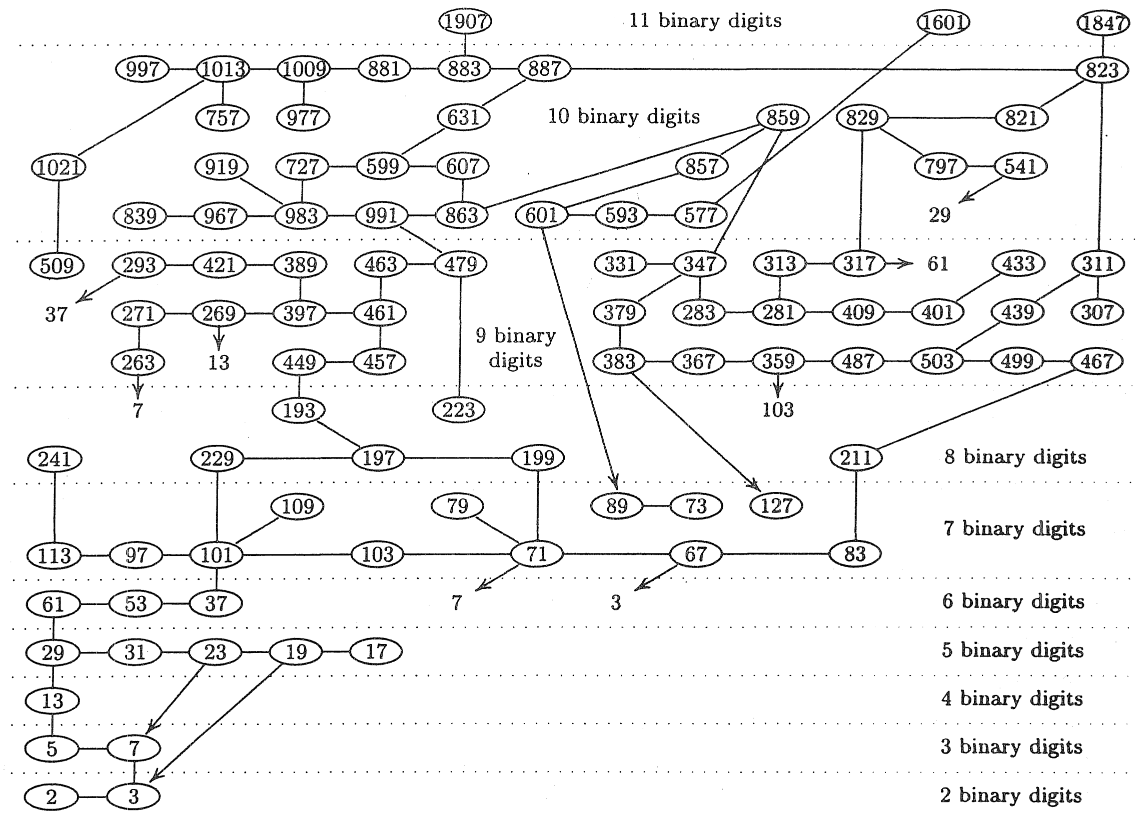

FIGURE 1. The First 9 Levels of the Directed Graph G_2

The example 353 shows how the directed graph G_1 can make back-tracking very difficult. Although only a finite number of primes can be reached from a given prime, there may in fact be an infinite number of primes from which one could get to a given prime. Some of the numbers involved will be very large, so one must be content with knowing that they are "probably prime" via the Miller-Rabin strong pseudoprime test. Since the probability of a composite number passing this test is about 4^{-100} [7], we can be fairly confident that the pseudoprimes needed to get to 353 are indeed prime.

2. THE PARTITIONING OF THE PRIMES

The prime number 11 is ominously missing in Figure 1. This begs the question as to whether one can reach the prime 11 via a much larger prime, as in the case of 353. Obviously, 11 is in the same connected component as 2 in G_0, since there is an edge between 11 and 3. But can we get from 2 to 11 in G_1?

For each prime p, let us define G_p to be the subgraph of G_1 consisting of all vertices and edges that can be reached starting from the prime p. Note that there are many instances when G_p is a finite graph. For example, G_{73} consists of just two vertices, 73 and 89, and the bidirectional edge connecting them. The question is whether 11 is a vertex of G_2. A simple parity argument shows that it is not.

Definition: Let $p > 3$ be a prime number. We say that p is of *correct parity* if either $p \equiv 2 \bmod 3$ and p has an even number of 1 bits in its binary representation, or $p \equiv 1 \bmod 3$ and p has an odd number of 1 bits. We say that $p > 3$ is of *incorrect parity* if p is not of correct parity. We do not define parity for the primes 2 and 3. Note that 5 and 7 are of correct parity, but 11 is of incorrect parity.

Proposition 1: If an edge in G_0 connects two primes $p > 3$ and $q > 3$, then p and q have the same parity. In particular, all of the vertices of G_2, besides 2 and 3, are of the correct parity.

Proof: If an edge connects p and q, their binary representation differs by exactly one digit. Thus, one of the primes will have an even number of 1 bits, while the other will have an odd number.

Also, since p and q differ by a power of 2, they cannot be congruent mod 3. Neither can be congruent to 0 mod 3, for both p and q are primes > 3. Thus, one of the primes is congruent to 1 mod 3, while the other is congruent to 2 mod 3. By the way that we defined the parity, if either p or q is of correct parity, then the other must also be of correct parity.

Finally, we notice that in the graph of G_2, 2 only can go to 3, which can only go to 7. Thus, any other vertex in G_2 must be reached from 7 without going through 2 or 3. Since 7 has the correct parity, any prime > 3 in G_2 must also be of the correct parity. □

With this proposition and the fact that 11 has incorrect parity, one sees that 11 is not a vertex of G_2. In fact, if we delete the vertex 3 from the graph of G_0, together with all edges connecting to 3, then the resulting graph consists of 2 and at least two large disconnected subgraphs. It is highly probable that these subgraphs are both infinite. The connected components of G_0—$\{3\}$ form a partition of the prime numbers. By convention, we will include 2 and 3 in the partition that contains the vertex 7.

The parity argument shows that there must be at least two partitions. We will call the partition containing the first 4 primes the *α-partition*, which would of course contain all vertices of G_2. A second partition, the *β-partition*, contains the prime 11. All primes in the β-partition would have incorrect parity.

3. ISOLATED PRIMES

In asking how many partitions there are, one must ask whether there is any prime p totally isolated from any other primes in G_0. In order for this to happen, $p + 2^n$ must always be composite whenever $2^n > p$. This is closely related to two other problems: the Polignac-Erdös problem and the Sierpiński problem.

In 1849, Polignac conjectured that every odd integer > 1 could be expressed in the form $2^n + p$ (see [10]). In 1950, Paul Erdös [3] disproved this conjecture, and in fact proved that there is an arithmetic progression of odd numbers, no term of which is of the form $2^n + p$. In fact, no term in this sequence is of the form $2^n \pm p$, where p is a prime. If we considered negative terms in this arithmetic progression, and found a term $-k$ such that k is prime, then k would be a candidate for an isolated prime.

In 1960, Sierpiński [9] asked: for what numbers k is $2^m \cdot k + 1$ composite for all $m \geq 1$. Such numbers are called *Sierpiński numbers*. The smallest Sierpiński number is believed to be 78557, but there are several smaller candidates for which no prime of the form $2^m \cdot k + 1$ is known [1].

Sierpiński showed that, if k belongs to one of several arithmetic progressions, then any term of the sequence $k+1,\ 2k+1,\ 4k+1,\ \ldots,\ 2^m \cdot k + 1$ is divisible by one of a set of 6 or 7 fixed primes. The set of primes is called the *covering set* for the Sierpiński number. The number 78557 has the covering set {3, 5, 7, 13, 19, 37, 73}, while the next known Sierpiński number, 271129, uses {3, 5, 7, 13, 17, 241} as its covering set [5].

The relationship between the Sierpiński numbers and the Polignac-Erdös numbers is given in [10]. Since the Polignac-Erdös numbers are in turn related to the isolated primes, there is a direct connection between the Sierpiński numbers and the isolated primes. The following proposition is taken from [10].

Proposition 2: Let k be a Sierpiński number with a covering set S. Then, for all n, $k + 2^n$ will be divisible by some prime in S.

Proof: Let N be the product of the odd primes in the set S. If we let $L = \phi(N)$, then N will divide the Mersenne number $2^L - 1$ by Euler's theorem. We then have that, for all m,

$$\gcd(2^m \cdot k + 1, N) > 1.$$

Multiplying the first part by 2^{L-m} gives

$$\gcd(2^L \cdot k + 2^{L-m}, N) > 1.$$

Since $2^L \equiv 1 \bmod N$, we can replace $2^L \cdot k$ with k and write n for $L - m$ to give us

$$\gcd(k + 2^n, N) > 1.$$

Hence, for all n, $k + 2^n$ is divisible by some prime in S. Note that this process is reversible, so any covering set which shows that $k + 2^n$ is always composite will show that k is a Sierpiński number. □

This proposition makes it clear how to search for isolated prime numbers. We need to find a Sierpiński number that is prime, and for which changing any 1 to a zero in its binary representation results in a composite number. A quick search through the known Sierpiński numbers [11] reveals tat 2131099 satisfies both the extra conditions, and so 2131099 is an isolated prime.

However, 2131099 may not be the smallest isolated prime. The prime 19249 is still a candidate for being Sierpiński. If a covering set is discovered for this number, it will be the smallest isolated prime.

A natural question that arises is whether there is an infinite number of isolated primes. To answer this question, we introduce two more sets of numbers related to the Sierpiński numbers, the Riesel numbers, and the Brier numbers.

Definition: A *Riesel number* is a number k for which $2^n \cdot k - 1$ is composite for all $n > 0$. A *Brier number* is a number that is both Sierpiński and Riesel.

We can use an argument similar to that in Proposition 2 to show that, if k is a Riesel number with a covering set S, then $k - 2^n$ will always have a divisor in the set S.

In 1998, Eric Brier [2] discovered the 41-digit number,

$$29364695660123543278115025405114452910889,$$

and suggested that it might be the smallest such number. However, this record for the smallest known Brier number has been beaten numerous times by Keller and Nash [6] and by Gallot in [8]. The current record is the 27-digit Brier number,

$$B = 878503122374924101526292469,$$

using the covering set

$$S = \{3, 5, 7, 11, 13, 17, 19, 31, 37, 41, 61, 73, 97, 109, 151, 241, 257, 331, 61681\}.$$

Just one Brier number is sufficient to prove the following proposition.

Proposition 3: There is an infinite number of isolated primes.

Proof: Let B be the above Brier number, and let $N = 2^{17}$ times the product of the primes in S. Since B and N are coprime, by Dirichlet's theorem [7] there is an infinite number of primes of the form $aN + B$ with a a positive integer. All that needs to be shown is that these primes are all isolated. In fact, we can prove that $aN + B \pm 2^n$ is composite for all $a \geq 0$ and $n \geq 0$. Note that

$$\begin{array}{llll}
\text{if } n \equiv 0 \pmod{2}, & 3|aN+B-2^n; & \text{if } n \equiv 1 \pmod{2}, & 3|aN+B+2^n; \\
\text{if } n \equiv 2 \pmod{3}, & 7|aN+B-2^n; & \text{if } n \equiv 0 \pmod{4}, & 5|aN+B+2^n; \\
\text{if } n \equiv 7 \pmod{12}, & 13|aN+B-2^n; & \text{if } n \equiv 6 \pmod{8}, & 17|aN+B+2^n; \\
\text{if } n \equiv 13 \pmod{24}, & 241|aN+B-2^n; & \text{if } n \equiv 1 \pmod{5}, & 31|aN+B+2^n; \\
\text{if } n \equiv 1 \pmod{48}, & 97|aN+B-2^n; & \text{if } n \equiv 0 \pmod{10}, & 11|aN+B+2^n; \\
\text{if } n \equiv 9 \pmod{16}, & 257|aN+B-2^n; & \text{if } n \equiv 18 \pmod{20}, & 41|aN+B+2^n; \\
\text{if } n \equiv 0 \pmod{9}, & 73|aN+B-2^n; & \text{if } n \equiv 34 \pmod{40}, & 61681|aN+B+2^n; \\
\text{if } n \equiv 15 \pmod{18}, & 19|aN+B-2^n; & \text{if } n \equiv 12 \pmod{15}, & 151|aN+B+2^n; \\
\text{if } n \equiv 3 \pmod{36}, & 37|aN+B-2^n; & \text{if } n \equiv 22 \pmod{30}, & 331|aN+B+2^n; \\
\text{if } n \equiv 21 \pmod{36}, & 109|aN+B-2^n; & \text{if } n \equiv 2 \pmod{60}, & 61|aN+B+2^n;
\end{array}$$

so the only case left to consider is if $aN + B - 2^n$ happens to be one of the primes in the set S. If $n < 17$, we have $aN + B - 2^n > B - 2^{17}$, which is of course greater than all the primes in S. If, on the other hand, $n \geq 17$, then

$$aN + B - 2^n \equiv B \equiv 67573 \pmod{2^{17}},$$

which is again greater than all of the primes in S. Thus, $aN + B \pm 2^n$ is always composite, and so there is an infinite number of isolated primes.

In the search for isolated primes, a few primes were discovered that were *almost* isolated, meaning that there was only one edge in G_1 directed away from the prime p rather than toward it. The prime 36652489 is a Sierpiński number, so we can tell that the only edge in G_0 is one that connects to the prime 3098057. Yet this is a directed edge in G_1, so there are no edges that connect a prime number to the prime 36652489. Hence, for $p \neq 36652489$, the vertex 36652489 is not in G_p. Ironically, $G_{36652489}$ not only contains 36652489, it also contains the vertex 2; hence, G_2 is a strict subgraph of $G_{36652489}$.

One could also ask whether there are any finite partitions of G_0 other than the isolated primes. We may never be able to answer this question, since such a partition would have to contain a prime p that is "almost Sierpiński," that is, $p+2^n$ would be composite for all n with one exception, that being another member of the partition. The one exception would preclude the possibility for a covering set for p. Without a covering set, proving $p+2^n$ is composite for all other n would be at least as difficult as proving that there are exactly 4 Fermat primes. A computer search will likely produce some "candidates" for finite partitions, but no amount of computation would be able to prove that the partition is really finite.

4. SOME CONJECTURES ABOUT THE MAZE

Conjecture 1: All Fermat primes are vertices in G_2.

This is a very safe conjecture, for it is almost certain that the only Fermat primes are 3, 5, 17, 257, and 65537, which can be verified to be in G_2. Furthermore, any Fermat prime will have the correct parity. The first three primes show up quickly in Figure 1, but getting to 257 requires as many as 627 steps in the maze, since one first must reach the number $2^{91}+2^{26}+769$. The prime 65537 requires first getting to $2^{268}+2^{100}+2^{98}+2^{83}+3$. Finding the shortest path to these primes remains an unsolved problem.

Conjecture 2: All Mersenne primes are vertices in G_2.

The binary representation of the Mersennes makes them the natural goal for this maze of primes, and by a fortunate coincidence all Mersenne primes have the correct parity. Besides the easy ones found in Figure 1, 8191 requires exactly 38 steps, 131071 requires 48 steps, and $2^{19}-1$ requires 62 steps. The shortest path to $2^{31}-1$ is unknown, since one must first reach $2^{74}+2^{31}-1$. Getting to $2^{61}-1$ and $2^{89}-1$ are straightforward; however, getting to $2^{107}-1$ requires first going to $2^{135}+2^{107}-2^{47}-2^{33}-4097$. Reaching $2^{127}-1$ requires first getting to $2^{182}+2^{127}-1$. By backtracking, a computer has verified that $2^{521}-1$ is in G_2, but the smallest neighbor to $2^{607}-1$ is $2^{1160}+2^{607}-1$, which is currently too large for the computer to handle.

Conjecture 3: There are four infinite partitions of $G_0-\{3\}$ that contain primes less than 1000.

Proving this conjecture if the fundamental unsolved problem of this maze. We have already seen using parity that there are at least two main partitions, the α-partition and the β-partition. But as we explore G_0, two more partitions seem to crop up. Although there is no proof that these extra partitions do not connect in some way to the α-partition or the β-partition, there is very strong evidence that no such connection is possible, hence the conjecture. A table of the four partitions that seem to exist is shown below.

The conjectured partitions			
	Lowest prime	Starting point	Comments
α-partition	2	2	Main maze
β-partition	11	547	Can go from $\beta \to \alpha$ via 3
γ-partition	277	4957	Incorrect parity
δ-partition	683	35759	Correct parity

The primes less than 16000 in the δ-partition are {683, 2699, 2729, 2731, 6827, 8363, 8747, 8867, 10427, 1067, 10799, 10859, 10883, 10889, 10891, 10937, 10939, 10979, 10987, 11003, 11171, 11177, 11243, 11939, 12011, 12203, 14891, 15017, 15083, ...}. All other primes <16000 of correct parity are in the α-partition.

Likewise, {277, 337, 349, 373, 853, 1093, 1109, 1117, 1237, 1297, 1301, 1303, 1362, 1367, 1373, 1381, 1399, 1429, 1489, 1493, 1621, 1861, 1873, 1877, 1879, 2389, 3413, 3541, 4177, 4357, 4373, 4421, 4423, 4441, 4447, 4549, 4561, 4567, 4597, 4933, 4951, 4957, 5077, 5189, 5197, 5209, 5233, 5237, 5333, 5381, 5393, 5399, 5407, 5413, 5431, 5437, 5441, 5443, 5449, 5471, 5477, 5479, 5501, 5503, 5521, 5527, 5557, 5569, 5573, 5581, 5591, 5623, 5653, 5701, 5717, 5749, 5953, 5981, 6007, 6037, 6101, 6133, 6229, 6421, 6469, 6481, 6997, 7237, 7253, 7477, 7489, 7507, 7517, 7537, 7541, 7549, 7573, 7621, 7639, 7669, 8017, 8053, 10069, 12373, 12613, 12637, 12757, 13381, 13397, 13399, 13591, 13597, 13633, 13649, 13669, 13681, 13687, 13693, 13781, 13789, 14149, 14173, 14197, 14293, 15733, ...} are in the γ-partition. All other primes <16000 of incorrect parity, with the possible exception of 6379, are in the β-partition. (Analyzing 6379 requires working with numbers larger than 2^{1396}, which takes too long to determine which of these two sectors it is in.)

This table includes the *starting point* for each partition. The starting point is the smallest prime s in the partition for which G_s apparently contains an infinite number of the vertices of the partition. In other words, for all smaller values of p in the partition, G_p produces a finite graph. (For the primes in the β-partition, we would delete the vertex 3 before computing G_p.) It would be tempting to think that G_s would contain all of the vertices of the partition, but the almost isolated" primes in the partition, such as 36652489, would be excluded. Hence, the most we could say is that G_s contains almost all of the vertices of the partition. In fact, all primes less than 16000, with the possible exception of 6379, are in either G_2, G_{547}, G_{4957}, or G_{35759}. Furthermore, for all primes less than 50000, G_p is either finite or contains one of the four graphs. Thus, if there were a fifth infinite partition, the starting point would have to be larger than 50000. So the four partitions in the above table are the first four partitions in every sense.

5. CONCLUSION

It is amazing that the simple rules of the prime maze can raise so many theoretical questions. What started out as a simple puzzle turned into a fountain of problems, some of them solvable, while others may never be solved. It is ironic that the solution to some of the problems, such as finding an infinite number of isolated primes, turns out not involving the binary number system but rather just the powers of two. Therefore, the results of the prime number maze is likely to have significance in other areas of number theory.

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FIBONACCI-LUCAS QUASI-CYCLIC MATRICES

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1. INTRODUCTION

Matrices such as

$$R = R(D; x_1, x_2, \dots, x_n) = \begin{pmatrix} x_1 & Dx_n & \cdots & Dx_2 \\ x_2 & x_1 & \cdots & Dx_3 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{pmatrix} \tag{1}$$

are called quasi-cyclic matrices. These matrices were introduced and studied in [2] and [5]. We can obtain these matrices by multiplying every element of the upper triangular part (not including the diagonal) of the cyclic matrices (see [4])

$$C = \begin{pmatrix} x_1 & x_n & \cdots & x_2 \\ x_2 & x_1 & \cdots & x_3 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{pmatrix}$$

by D.

In this paper we will prove that, for $n \geq 2$,

$$\det(R(L_n; F_{2n-1}, F_{2n-2}, \dots, F_n)) = 1,$$

where L_n and F_n denote, as usual, the n^{th} Lucas and Fibonacci numbers, respectively, and $\det(R)$ denotes the determinant of R. In addition, if we let

$$R_{n,k} = R(L_n; F_{2n-1+k}, F_{2n-2+k}, \dots, F_{n+k})$$

for integral k, then

$$\det(R_{n,k}) = (-1)^{n-1} L_n F_k^n + F_{k-1}^n.$$

The motivation for studying these determinants comes from Pell's equation. It is well known that the solution of Pell's equation $x^2 - dy^2 = \pm 1$ is closely related to the unit of the quadratic field $Q(\sqrt{d})$. We may extend the conclusion to fields of higher degree. If we rewrite $x^2 - dy^2 = \pm 1$ as

$$\det\begin{pmatrix} x & dy \\ y & x \end{pmatrix} = \pm 1,$$

we can easily do this. The equation

$$\det\begin{pmatrix} x_1 & Dx_n & \cdots & Dx_2 \\ x_2 & x_1 & \cdots & Dx_3 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{pmatrix} = \pm 1 \tag{2}$$

is called Pell's equation of degree n. Using our results, we can obtain solutions to an infinite family of Pell equations of higher degree based on Fibonacci and Lucas numbers.

To prove our results, we will need two propositions. These two propositions came from [2] and [5].

Proposition 1:

$$\det(R) = \prod_{k=0}^{n-1} \left(\sum_{i=1}^{n} x_i d^{i-1} \varepsilon^{k(i-1)} \right), \tag{3}$$

where $d = \sqrt[n]{D}$ and $\varepsilon = e^{2\pi i/n}$. Also, each factor $\sum_{i=1}^{n} x_i d^{i-1} \varepsilon^{k(i-1)}$ of the right-hand side of (3) is an eigenvalue of the matrix R.

Proposition 2: Let n and D be fixed. Then the sum, difference, and product of two quasi-cyclic matrices is also quasi-cyclic. The inverse of a quasi-cyclic matrix is quasi-cyclic.

2. THE MAIN RESULTS AND THEIR PROOFS

We are now ready to state and prove the first theorem.

Theorem 1: Let $n \geq 2$. Then

$$\det(R(L_n; F_{2n-1}, F_{2n-2}, \dots, F_n)) = 1, \tag{4}$$

where L_n and F_n denote, as usual, the n^{th} Lucas and Fibonacci numbers, respectively.

Proof: For $n = 2$, we have that

$$\det(R(L_2; F_3, F_2)) = \det\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = 1,$$

so the result of the theorem holds. If $n > 2$, let

$$T = \begin{pmatrix} 1 & -1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{5}$$

By multiplication of matrices and properties of Fibonacci and Lucas numbers, we have

$$RT = \begin{pmatrix} F_{2n-1} & F_{2n-2} & (-1)^n & & \\ \vdots & \vdots & & \ddots & \\ \vdots & \vdots & & & (-1)^n \\ F_{n+1} & F_n & 0 & \cdots & 0 \\ F_n & F_{n-1} & 0 & \cdots & 0 \end{pmatrix}. \tag{6}$$

Taking the determinant of both sides of (6) and noting that $\det(T) = 1$, we have

$$\begin{aligned} \det(R) &= \det(R)\det(T) = \det(RT) \\ &= (-1)^{2n-4} \det \begin{pmatrix} F_{n+1} & F_n & 0 & \cdots & 0 \\ F_n & F_{n-1} & 0 & \cdots & 0 \\ F_{2n-1} & F_{2n-2} & & & \\ \vdots & \vdots & & (-1)^n I_{n-2} & \\ F_{n+2} & F_{n+1} & & & \end{pmatrix} \end{aligned}$$

$$= \det\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \det((-1)^n I_{n-2})$$
$$= (F_{n+1}F_{n-1} - F_n^2)(-1)^{n(n-2)} = (-1)^n(-1)^n = 1,$$

where I_n denotes the identity matrix of order n. Thus, Theorem 1 is true.

Corollary 1: If $D = L_n$, then $(F_{2n-1}, F_{2n-2}, \ldots, F_n)$ is a solution of Pell's equation (3).

Corollary 2: Let $d = \sqrt[n]{L_n}$, $\varepsilon = e^{2\pi i/n}$. Then

$$\prod_{k=0}^{n-1} \left(\sum_{i=1}^{n} F_{2n-i} d^{i-1} \varepsilon^{k(i-1)} \right) = 1.$$

Proof: This is obvious by Theorem 1 and Proposition 1.

We now make the following conclusion.

Theorem 2: The matrix $R = R(L_n; F_{2n-1}, \ldots, F_n)$ is invertible. In addition,

$$R^{-1} = (-1)^{n-1}(I + E - E^2), \tag{7}$$

where $I = I_n$ and

$$E = E_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & L_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Proof: Since $\det(R) = 1 \neq 0$, the inverse R^{-1} exists. Obviously,

$$R = R(L_n; F_{2n-1}, \ldots, F_n) = F_{2n-1}I + F_{2n-2}E + F_{2n-3}E^2 + \cdots + F_nE^{n-1}.$$

Hence,

$$\begin{aligned}
&R(-1)^{n-1}(I + E - E^2) \\
&= (-1)^{n-1}(F_{2n-1}I + F_{2n-2}E + F_{2n-3}E^2 + \cdots + F_nE^{n-1} + F_{2n-1}E + F_{2n-2}E^2 + \cdots \\
&\qquad + F_{n+1}E^{n-1} + F_nE^n - F_{2n-1}E^2 - \cdots - F_{n+2}E^{n-1} - F_{n+1}E^n - F_nE^{n+1}) \\
&= (-1)^{n-1}(F_{2n-1}I + F_{2n-2}E + F_{2n-1}E + F_nE^n - F_{n+1}E^n - F_nE^{n+1}) \\
&= (-1)^{n-1}(F_{2n-1}I + F_{2n}E + F_nL_nI - F_{n+1}L_nI - F_nL_nE) \\
&= (-1)^{n-1}(F_{2n-1}I + F_{2n}I + F_{2n-1}E - F_{2n+1}I - (-1)^nI) \\
&= (-1)^{n-1}(-1)^{n+1}I = I.
\end{aligned}$$

In the above, the three following facts have been used:

1. $F_{2n-3} + F_{2n-2} - F_{2n-1} = 0, \ldots, F_n + F_{n+1} - F_{n+2} = 0$. This is obvious from the definition of Fibonacci numbers.
2. $E^n = L_nI$, $E^{n+1} = L_nE$. This can be verified easily by multiplication of matrices.
3. $L_nF_n = F_{2n}$ and $L_nF_{n+1} = F_{2n+1} + (-1)^n$. These are well-known properties of Fibonacci and Lucas numbers.

Corollary 3: Let $n \geq 3$ be an odd number and $D = L_n$. Then

$$(x_1, x_2, x_3, x_4, \ldots, x_n) = (1, 1, -1, 0, \ldots, 0)$$

is a solution of Pell's equation (3) of degree n.

Let $n \geq 4$ be an even number and $D = L_n$. Then

$$(x_1, x_2, x_3, x_4, \ldots, x_n) = (-1, -1, 1, 0, \ldots, 0)$$

is a solution of Pell's equation (3) of degree n.

Proof: Based on Theorem 2, when n is odd, we have

$$\det(R(L_n; 1, 1, -1, 0, \ldots, 0))\det(R(L_n; F_{2n-1}, \ldots, F_n)) = \det(I) = 1$$

and

$$\det(R(L_n; F_{2n-1}, \ldots, F_n)) = 1$$

from Theorem 1, so

$$\det(R(L_n; 1, 1, -1, 0, \ldots, 0)) = 1$$

and, by definition of solution, the conclusion is true. For even n, the proof is similar.

3. MORE RESULTS ABOUT THE DETERMINANTS

Let $R_{n,k} = R(L_n; F_{2n-1+k}, F_{2n-2+k}, \ldots, F_{n+k})$, $k = 0, \pm 1, \pm 2, \ldots$, be square matrices of degree n. Then Theorem 1 has the form $\det(R_{n,0}) = 1$. For $\det(R_{n,1})$, $\det(R_{n,2}), \ldots, \det(R_{n,-1})$, $\det(R_{n,-2})$, ..., we can also obtain corresponding results, but the values of these determinants are not 1, so that the inverses $R_{n,k}^{-1}$ of $R_{n,k}$, $k = \pm 1, \pm 2$, ..., are not matrices with integer elements.

Theorem 3:

$$\begin{aligned}
\det(R_{n,-2}) &= 2^n - L_n, \\
\det(R_{n,-1}) &= (-1)^{n-1}(L_n - 1), \\
\det(R_{n,0}) &= 1, \\
\det(R_{n,1}) &= (-1)^{n-1}L_n, \\
\det(R_{n,2}) &= (-1)^{n-1}L_n + 1.
\end{aligned}$$

The result in the middle of Theorem 3, i.e., $\det(R_{n,0}) = 1$ is just Theorem 1. The other results are closely related to L_n, so we list them here. In fact, they can be deduced from the more extensive following results.

Theorem 4: Let $n \geq 2$ be an integer and let k be an integer. Then

$$\det(R_{n,k}) = (-1)^{n-1}L_n F_k^n + F_{k-1}^n.$$

To prove Theorem 4, set

$$g_{n,k} = \begin{vmatrix} F_{2n+k-1} & F_{2n+k-2} & (-1)^n F_{k-1} & & \\ F_{2n+k-2} & F_{2n+k-3} & (-1)^n F_k & \ddots & \\ \vdots & \vdots & & \ddots & (-1)^n F_{k-1} \\ \vdots & \vdots & & & (-1)^n F_k \\ F_{n+k} & F_{n+k-1} & 0 & \cdots & 0 \end{vmatrix}, \tag{8}$$

$$h_{n,k}=\begin{vmatrix} F_{2n+k-1} & (-1)^n F_k & (-1)^n F_{k-1} & & \\ F_{2n+k-2} & 0 & (-1)^n F_k & \ddots & \\ \vdots & \vdots & & \ddots & (-1)^n F_{k-1} \\ \vdots & \vdots & & & (-1)^n F_k \\ F_{n+k} & 0 & 0 & \cdots & 0 \end{vmatrix}, \tag{9}$$

where the elements in the middle are zero in every determinant. Now the proof of Theorem 4 consists of the following four points:

1. $\det(R_{n,k}) = g_{n,k} + h_{n,k}$;
2. $g_{n,k} = F_{k-1}^n + (-1)^{n-1} F_{n-1} F_k^n + (-1)^n F_k^{n-1} F_n F_{k-1}$;
3. $h_{n,k} = (-1)^{n-1} F_{n+k} F_k^{n-1}$;
4. $g_{n,k} + h_{n,k} = (-1)^{n-1} L_n F_k^n + F_{k-1}^n$.

We can obtain the above four points from five lemmas.

Lemma 1: Suppose $g_{n,k}$ and $h_{n,k}$ are defined as in (8) and (9). Then $\det(R_{n,k}) = g_{n,k} + h_{n,k}$.

Proof: Let T be as in (5). Then, by properties of determinants, we have

$$\begin{aligned} \det(R_{n,k}) &= \det(R_{n,k})\det(T) \\ &= \det(R_{n,k}\cdot T) \\ &= \begin{vmatrix} F_{2n+k-1} & F_{2n+k-2}+(-1)^n F_k & (-1)^n F_{k-1} & & \\ F_{2n+k-2} & F_{2n+k-3} & (-1)^n F_k & \ddots & \\ \vdots & \vdots & & \ddots & (-1)^n F_{k-1} \\ \vdots & \vdots & & & (-1)^n F_k \\ F_{n+k} & F_{n+k-1} & 0 & \cdots & 0 \end{vmatrix} \\ &= g_{n,k} + h_{n,k}. \end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2 (the recurrence of $g_{n,k}$):

$$g_{n,k} = (-1)^{n+k} F_{n-1} F_k^{n-2} + F_{k-1} g_{n-1,k}. \tag{10}$$

Proof: By subtracting the second column from the first column of $g_{n,k}$, the first column becomes $(F_{2n+k-3}, F_{2n+k-4}, \ldots, F_{n+k-2})^T$ by the properties of Fibonacci numbers, where T in the superscript denotes the transpose of a matrix or vector. By subtracting the first column from the second column, and so on, after $n+k-1$ subtractions between the two columns, the first two columns become

$$\begin{pmatrix} F_{n-1} & F_{n-2} & \cdots & F_0 \\ F_n & F_{n-1} & \cdots & F_1 \end{pmatrix}^T.$$

Next we exchange the first two columns if $n+k$ is even and we keep the matrix if $n+k$ is odd. Hence, the first two columns become

$$\begin{pmatrix} F_n & F_{n-1} & \cdots & F_1 \\ F_{n-1} & F_{n-2} & \cdots & F_0 \end{pmatrix}^T.$$

Thus,

$$g_{n,k} = (-1)^{n+k-1}\begin{vmatrix} F_n & F_{n-1} & (-1)^n F_{k-1} & \ddots & \\ F_{n-1} & F_{n-2} & (-1)^n F_k & \ddots & \\ \vdots & \vdots & & \ddots & (-1)^n F_{k-1} \\ \vdots & \vdots & & & (-1)^n F_k \\ F_1 & F_0 & 0 & \cdots & 0 \end{vmatrix}$$

$$= (-1)^{k-1}\begin{vmatrix} F_n & F_{n-1} & F_{k-1} & \ddots & \\ F_{n-1} & F_{n-2} & F_k & \ddots & \\ \vdots & \vdots & & \ddots & F_{k-1} \\ \vdots & \vdots & & & F_k \\ F_1 & F_0 & 0 & \cdots & 0 \end{vmatrix}.$$

Expanding the last determinant by the first row and noting that $F_0 = 0$, we have

$$g_{n,k} = (-1)^{k-1}\left(-F_{n-1}\begin{vmatrix} F_{n-1} & F_k & F_{k-1} & \ddots & \\ & & F_k & \ddots & \\ \vdots & \vdots & & \ddots & F_{k-1} \\ \vdots & \vdots & & & F_k \\ F_1 & F_0 & 0 & \cdots & 0 \end{vmatrix} + F_{k-1}\begin{vmatrix} F_{n-1} & F_{n-2} & F_{k-1} & \ddots & \\ & & F_k & \ddots & \\ \vdots & \vdots & & \ddots & F_{k-1} \\ \vdots & \vdots & & & F_k \\ F_1 & F_0 & 0 & \cdots & 0 \end{vmatrix}\right)$$

$$= (-1)^{k-1}(-F_{n-1}(-1)^{1+n-1}F_1F_k^{n-2} + F_{k-1}(-1)^{k-1}g_{n-1,k}) = (-1)^{n+k}F_{n-1}F_k^{n-2} + F_k g_{n-1,k}.$$

Thus, Lemma 2 is proved.

Lemma 3:

$$g_{n,k} = F_{k-1}^n + (-1)^{n-1}F_{n-1}F_k^n + (-1)^n F_n F_k^{n-1}F_{k-1}. \tag{11}$$

Proof: By induction on n.

(A) On the one hand, by the definition of $g_{n,k}$, we have

$$g_{2,k} = \begin{vmatrix} F_{3+k} & F_{2+k} \\ F_{2+k} & F_{1+k} \end{vmatrix} = F_{3+k}F_{1+k} - F_{2+k}^2 = (-1)^{k+2-1} = (-1)^{k-1}.$$

On the other hand, the right side of (11) becomes

$$F_{k-1}^2 + (-1)F_{2-1}F_k^2 + (-1)^2F_2F_k^{2-1}F_{k-1} = F_{k-1}^2 - F_k^2 + F_{k-1}F_k$$
$$= F_{k-1}^2 - F_{k-1}F_k - F_k^2 = F_{k-1}F_{k+1} - F_k^2 = (-1)^{k-1}.$$

Hence, Lemma 3 holds when $n = 2$.

(B) Assume (11) holds for $n-1$, i.e.,

$$g_{n-1,k} = F_{k-1}^{n-1} + (-1)^{n-2}F_{n-2}F_k^{n-1} + (-1)^{n-1}F_{n-1}F_k^{n-1}F_{k-1}. \tag{12}$$

We will prove that (11) holds for n. By (12) and recurrence (10), we have

$$g_{n-1,k} = (-1)^{n+k}F_{n-1}F_k^{n-2} + F_{k-1}(F_{k-1}^{n-1} + (-1)^{n-2}F_{n-2}F_k^{n-1} + (-1)^{n-1}F_{n-1}F_k^{n-2}F_{k-1})$$
$$= F_{k-1}^n + (-1)^{n-1}(-1)^{k-1}F_{n-1}F_k^{n-2} + (-1)^{n-2}F_{n-2}F_k^{n-1}F_{k-1} + (-1)^{n-1}F_{n-1}F_k^{n-2}F_{k-1}^2$$
$$= F_{k-1}^n + (-1)^{n-1}F_k^{n-2}((-1)^{k-1}F_{n-1} - (F_n - F_{n-1})F_{k-1}F_k + F_{n-1}F_{k-1}^2)$$
$$= F_{k-1}^n + (-1)^n F_k^{n-2}((-1)^{k-1}F_{n-1} - (F_n - F_{n-1})F_{k-1}F_k + F_{n-1}F_{k-1}^2)$$

$$= F_{k-1}^n + (-1)^n F_n F_k^{n-1} F_{k-1} + (-1)^{n-1} F_{n-1} F_k^{n-2}(F_k^2 - F_{k-1}F_{k+1} + F_k F_{k-1} + F_{k-1}^2)$$
$$= F_{k-1}^n + (-1)^n F_n F_k^{n-1} F_{k-1} + (-1)^{n-1} F_{n-1} F_k^n.$$

Hence, (11) holds for n. According to the induction principle, (11) holds for any number $n \geq 2$. Thus, Lemma 3 is true.

Corollary 4: $g_{n,n} = F_{n-1}^n$.

Proof: Let $k = n$ in (11).

Lemma 4:

$$h_{n,k} = (-1)^{n-1} F_{n+k} F_k^{n-1}. \tag{13}$$

Proof: We obtain this by expanding the n^{th} row of the right side of (9).

Lemma 5:

$$g_{n,k} + h_{n,k} = (-1)^{n-1} L_n F_k^n + F_{k-1}^n. \tag{14}$$

Proof: By (11) and (13), and noticing that $F_{n+k} = F_{n+1}F_k + F_n F_{k-1}$, we have

$$g_{n,k} + h_{n,k} = F_{k-1}^n + (-1)^{n-1} F_{n-1} F_k^n + (-1)^n F_n F_k^{n-1} F_{k-1} + (-1)^{n-1}(F_{n+1}F_k + F_n F_{k-1}) F_k^{n-1}$$
$$= F_{k-1}^n + (-1)^{n-1} F_{n-1} F_k^n + (-1)^{n-1} F_{n+1} F_k^n$$
$$= F_{k-1}^n + (-1)^{n-1}(F_{n-1} + F_{n+1}) F_k^n = F_{k-1}^n + (-1)^{n-1} L_n F_k^n.$$

Hence, Lemma 5 holds.

Corollary 5:

$$\det(R_{n,n}) = \begin{vmatrix} F_{3n-1} & L_n F_{2n} & \cdots & L_n F_{3n-2} \\ F_{3n-2} & F_{3n-1} & \cdots & L_n F_{3n-3} \\ \cdots & \cdots & \cdots & \cdots \\ F_{2n} & F_{2n+1} & \cdots & F_{3n-1} \end{vmatrix} = F_{n-1}^n + (-1)^{n-1} F_{2n} F_n^{n-1}.$$

Proof: Let $k = n$ in Theorem 4 and note that $F_{2n} = L_n F_n$.

Remark: We can verify that our lemmas and Theorem 4 are also true for negative k.

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PARTITION FORMS OF FIBONACCI NUMBERS

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In the notation of Comtet [1], define the partitions of integer n as $n = \sum ik_i$, where $i \geq 1$ is a summand and $k_i \geq 0$ is the frequency of summand i. It is known that the number of subsets of an n-element set is 2^n and

$$2^n = \sum_{\sum ik_i = n+1} \frac{(\sum k_i)!}{\prod k_i!}, \tag{1}$$

because of

$$\sum_{\substack{\sum ik_i = n+1 \\ \sum k_i = k}} \frac{1}{\prod k_i!} = \frac{1}{k!}\binom{n}{k-1}.$$

Equation (1) shows that the number of subsets of an n-element set is related to the number of summands in partitions of n. It is surprising that the sums on the right of identity (1) become Fibonacci numbers when some summands of the partitions of n no longer appear.

By means of generating functions, this article obtains the following result.

Theorem: For any $n \geq 1$, Fibonacci numbers satisfy

(a)
$$F_n = \sum_{\substack{\sum ik_i = n+1 \\ k_1 = 0}} \frac{(\sum k_i)!}{\prod k_i!}, \tag{2}$$

(b)
$$F_n = \sum_{\substack{\sum ik_i = n \\ \text{all } k_{2i} = 0}} \frac{(\sum k_i)!}{\prod k_i!}. \tag{3}$$

For example, the partitions of the integer 7 are

$7,\ 1+6,\ 2+5,\ 3+4,\ 1+1+5,\ 1+2+4,\ 1+3+3,\ 2+2+3,\ 1+1+1+4,\ 1+1+2+3,$
$1+2+2+2,\ 1+1+1+1+3,\ 1+1+1+2+2,\ 1+1+1+1+1+2,\ 1+1+1+1+1+1+1.$

From (2), we have

$$F_6 = \frac{1!}{1!} + \frac{2!}{1!\cdot 1!} + \frac{2!}{1!\cdot 1!} + \frac{3!}{2!\cdot 1!} = 1+2+2+3 = 8,$$

and from (3),

$$F_7 = \frac{1!}{1!} + \frac{3!}{2!\cdot 1!} + \frac{3!}{1!\cdot 2!} + \frac{5!}{4!\cdot 1!} + \frac{7!}{7!} = 1+3+3+5+7 = 13.$$

The Theorem can be proved easily by using the recurrence relations of Fibonacci numbers and the results of Bell polynomials $B_{n,k}$ [1]:

$$\frac{1}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} B_{n,k}\frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots,$$

and

$$\frac{1}{n!}B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\substack{\sum ik_i=n \\ \sum k_i=k,\, k_i\geq 0}} \frac{\prod x_i^{k_i}}{\prod k_i!(i!)^{k_i}}.$$

In this article, $[t^n]f(t)$ means the coefficient of t^n is in the formal series $f(t)$, so that

$$\sum_{n\geq 1} F_n t^n = \frac{t}{1-t-t^2} \quad \text{can be written as} \quad F_n = [t^n]\frac{t}{1-t-t^2}.$$

Proof of Theorem: (a) It is well known that $F_{n+2} = F_n + F_{n+1}$, $n \geq 1$, then

$$\begin{aligned} F_n &= [t^{n+2}]\frac{t}{1-t-t^2} - [t^{n+1}]\frac{t}{1-t-t^2} \\ &= [t^{n+1}]\frac{1-t}{1-t-t^2} = [t^{n+1}]\frac{1}{1-\left(\frac{t^2}{1-t}\right)} = [t^{n+1}]\frac{1}{1-(t^2+t^3+t^4+\cdots)} \\ &= \sum_{k\geq 1}[t^{n+1}](t^2+t^3+t^4+\cdots)^k = \sum_{k\geq 1}\sum_{\substack{\sum ik_i=n+1 \\ k_1=0,\, \sum k_i=k}}\left[\frac{k!}{\prod_{i\geq 1} k_i!}\right] = \sum_{\substack{\sum ik_i=n+1 \\ k_1=0}} \frac{(\sum k_i)!}{\prod k_i!}. \end{aligned}$$

(b) The proof is similar; notice that $F_n = F_{n+1} - F_{n-1}$, $n \geq 2$. Thus, for any $n \geq 2$,

$$\begin{aligned} F_n &= [t^{n+1}]\frac{t}{1-t-t^2} - [t^{n-1}]\frac{t}{1-t-t^2} \\ &= [t^n]\frac{1-t^2}{1-t-t^2} = [t^n]\frac{1}{1-\left(\frac{t}{1-t^2}\right)} = [t^n]\frac{1}{1-(t+t^3+t^5+t^7\cdots)} \\ &= \sum_{k\geq 1}[t^n](t+t^3+t^5+t^7\cdots)^k = \sum_{k\geq 1}\sum_{\substack{\sum ik_i=n \\ \text{all } k_{2i}=0,\, \sum k_i=k}}\left[\frac{k!}{\prod_{i\geq 1} k_i!}\right] = \sum_{\substack{\sum ik_i=n \\ \text{all } k_{2i}=0}} \frac{(\sum k_i)!}{\prod k_i!}. \end{aligned}$$

Remark 1: The number of summands on the right of (2) is $p(n+1)-p(n)$, and that of (3) is $q(n)$. Here, $p(n)$ is the number of partitions of n and $q(n)$ is the number of partitions of n into distinct summands, see [1].

Remark 2: It is well known that Fibonacci numbers have a simple combinatorial meaning, F_n is the number of subsets of $\{1, 2, 3, \ldots, n\}$ such that no two elements are adjacent. Comparing with (1), the Theorem shows that Fibonacci numbers have a kind of new combinatorial structure as a weighted sum over partitions.

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