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The primary function of THE FIBONACCI QUARTERLY is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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# CONVERGENCE OF r-GENERALIZED FIBONACCI SEQUENCES AND AN EXTENSION OF OSTROWSKI'S CONDITION 

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## 1. INTRODUCTION

Let $a_{0}, \ldots, a_{r-1}\left(r \geq 2, a_{r-1} \neq 0\right)$ be fixed real numbers. An $r$-generalized Fibonacci sequence $\left\{V_{n}\right\}_{n=0}^{+\infty}$ is defined by the linear recurrence relation of order $r$,

$$
\begin{equation*}
V_{n+1}=a_{0} V_{n}+a_{1} V_{n-1}+\cdots+a_{r-1} V_{n-r+1}, \text { for } n \geq r-1 \tag{1}
\end{equation*}
$$

where $V_{0}, \ldots, V_{r-1}$ are specified by the initial conditions. In the sequel we refer to these sequences as sequences (1) or (1). When $a_{i}(0 \leq i \leq r-1)$ are nonnegative and $\operatorname{gcd}\left\{i+1 ; a_{1}>0\right\}=1$, where gcd means the greatest common divisor, it was established in [10] that the characteristic polynomial $P(X)=X^{r}-a_{0} X^{r-1}-\cdots-a_{r-2} X-a_{r-1}$ has a unique positive zero $q$ and $|\lambda|<q$ for any other zero $\lambda$ of $P(X)$. And in [2] and [8] it was shown, by two different methods, that the limit of the ratio $V_{n} / q^{n}$ exists if and only if the Ostrowski condition $\operatorname{gcd}\left(i+1 ; a_{i}>0\right\}=1$ is satisfied.

The purpose of this paper is to study the extended Ostrowski condition by considering (C): $\operatorname{gcd}\left\{i+1 ; a_{i} \neq 0\right\}=1$ for sequences (1) in the case of real coefficients (Section 2). We apply Hörner's diagram to the convergence of sequences (1) (Section 3). An extension of (C) to the case of real coefficients is studied in Section 4. Finally, some concluding remarks are given in Section 5.

## 2. CONDITION (C) FOR SEQUENCES (1)

The Hörner diagram for a given polynomial $P(X)=a_{0} X^{n}+\cdots+a_{n-1} X+a_{n}$, where $a_{0}, a_{1}, \ldots$, $a_{n}$ are real numbers, is a process for computing the value of $P(\xi)$ for every $x=\xi$. Its main idea consists of writing $P(\xi)=\left(\cdots\left(\left(a_{0} \xi+a_{1}\right) \xi+a_{2}\right) \xi+\cdots\right) \xi+a_{n}$. Therefore, we can consider the finite sequence $\left\{\beta_{j}\right\}_{0 \leq j \leq n}$ defined as follows:

$$
\beta_{0}=a_{0}, \beta_{1}=\beta_{0} \xi, \beta_{2}=a_{2}+\beta_{1} \xi, \ldots, \beta_{n}=a_{n}+\beta_{n-1} \xi .
$$

Hence, we derive that $\beta_{n}=P(\xi)$ and $P(X)=Q(X)(X-\xi)+P(\xi)$, where $Q(X)=\beta_{0} X^{n-1}+\cdots+$ $\beta_{n-2} X+\beta_{n-1}$.

Suppose that sequence (1) converges. For $\lim _{n \rightarrow+\infty} V_{n} \neq 0$, we have $a_{0}+a_{1}+\cdots+a_{r-1}=1$. Suppose also that

$$
\begin{equation*}
a_{0}+a_{1}+\cdots+a_{r-1}=1 . \tag{2}
\end{equation*}
$$

## CONVERGENCE OF $r$-GENERALIZED FIBONACCI SEQUENCES AND AN EXTENSION OF OSTROWSKI'S CONDITION

Set $b_{i}=\sum_{j=i}^{r-1} a_{j}=\beta_{i}$ and $d=\operatorname{gcd}\left\{j+1 ; a_{j} \neq 0\right\}$. Then $b_{i}=\beta_{i}$ for $\xi=1$ and condition (2) implies that $b_{0}=1$. Assume that the following condition is satisfied:

$$
\begin{equation*}
\sum_{j=0}^{r-1} b_{j} \neq 0 \tag{3}
\end{equation*}
$$

By direct computation, we can verify that we have

$$
\begin{equation*}
V_{n}+b_{1} V_{n-1}+\cdots+b_{r-1} V_{n-r+1}=V_{r-1}+b_{1} V_{r-2}+\cdots+b_{r-1} V_{0} \tag{4}
\end{equation*}
$$

Thus,

$$
\lim _{n \rightarrow+\infty} V_{n}=\frac{\sum_{j=0}^{r-1}\left(\sum_{j=k}^{r-1} a_{k}\right) V_{j}}{\sum_{j=0}^{r-1}(j+1) a_{j}}
$$

This expression was established in [2] and [8]. If (3) is not satisfied, the characteristic polynomial takes the form $P(X)=(X-1)\left(X^{r-1}+b_{1} X^{r-2}+\cdots+b_{r-1}\right)$. Hence, $\lambda=1$ is of multiplicity $\geq 2$. Then $\left\{V_{n}\right\}_{n=0}^{+\infty}$ does not converge for any choice of the initial conditions.

In the case of nonnegative coefficients satisfying (2), it was shown in [2] and [8] that $\lim _{n \rightarrow+\infty} V_{n}$ exists for any choice of the initial conditions if and only if $(C)$ is satisfied. Let us establish that $(C)$ is still necessary in the case of arbitrary real coefficients. In [9] it was established that the combinatorial form of a sequence (1) is given by

$$
\begin{equation*}
V_{n}=A_{0} \rho(n, r)+A_{1} \rho(n-1, r)+\cdots+A_{r-1} \rho(n-r+1, r) \tag{5}
\end{equation*}
$$

for any $n \geq r$, where $A_{m}=a_{r-1} V_{m}+\cdots+a_{m} V_{r-1}$ and

$$
\begin{equation*}
\rho(n, r)=\sum_{k_{0}+2 k_{1}+\cdots+r k_{r-1}=n-r} \frac{\left(k_{0}+\cdots+k_{r-1}\right)!}{k_{0}!k_{1}!\ldots k_{r-1}!} a_{0}^{k_{0}} a_{1}^{k_{1}} \ldots a_{r-1}^{k_{r-1}} \tag{6}
\end{equation*}
$$

with $\rho(r, r)=1$ and $\rho(n, r)=0$, if $n \geq r-1$. For $V_{0}=\cdots=V_{r-2}=0$ and $V_{r-1}=1$, we have $V_{n}=$ $\rho(n+1, r)$ for $n \geq 0$. In the case of nonnegative coefficients, the sequence

$$
\left\{\frac{\rho(n, r)}{q^{n-r}}\right\}_{n=0}^{+\infty}
$$

where $q$ is the unique positive characteristic root, converges with

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \frac{\rho(n, r)}{q^{n-r}}=\frac{1}{1+b_{1}^{\prime}+\cdots+b_{r-1}^{\prime}} \tag{7}
\end{equation*}
$$

where $b_{k}^{\prime}=\sum_{j=k}^{r-1} \frac{a_{j}}{q^{j+1}}$ (see [9]).
(The combinatorial form of sequence (1) has been studied by various methods and techniques; see, e.g., [6], [7], [9], and [11].)

Suppose that $a_{0}, \ldots, a_{r-1}$ are real numbers and let $a_{j_{0}}, a_{j_{1}}, \ldots, a_{j_{s}}$ be the nonvanishing coefficients $\left(a_{j_{s}}=a_{r-1}\right.$ or $\left.j_{s}=r-1\right)$. Then (6) takes the form

$$
\rho(n, r)=\sum_{\left(i_{0}+1\right) k_{i_{0}}+\left(i_{1}+1\right) k_{i_{1}}+\cdots+\left(i_{s}+1\right) k_{i_{s}}=n-r .} \frac{\left(k_{i_{0}}+\cdots+k_{i_{s}}\right)!}{k_{i_{0}}!k_{i_{1}}!\ldots k_{i_{s}}!} a_{i_{0}}^{k_{i_{0}}} a_{i_{1}}^{k_{i_{1}}} \ldots a_{i_{s}}^{k_{i_{s}}}
$$

Thus, we deduce that $\rho(n, r)=0$ for $n<r$ or $n \neq k d(k \in \mathbb{N})$, where $d=\operatorname{gcd}\left\{j+1 ; a_{j} \neq 0\right\}$. For $d=\operatorname{gcd}\left\{j+1 ; a_{j} \neq 0\right\} \geq 2$, it was shown in [8] that the sequence (1) has $d$ subsequences of type

CONVERGENCE OF $r$-GENERALIZED FIBONACCI SEQUENCES AND AN EXTENSION OF OSTROWSKI'S CONDITION
(1) in the case of nonnegative coefficients. For $a_{0}, \ldots, a_{r-1}$ real, we can derive from ( $C$ ) that the sequence (1) also owns $d$ subsequences $\left\{V_{n}^{(j)}\right\}_{n \geq 0}(0 \leq j \leq d-1)$ of type (1) defined as follows: $V_{n}^{(j)}=V_{n d+j}=A_{j} \rho(n d, r)+A_{d+j} \rho((n-1) d, r)+\cdots+A_{r-d+j} \rho((n+1) d-r, r)$ for $0 \leq j \leq d-1$. So, if the sequence (1) converges for any choice of initial conditions, we have $V_{n}=V_{n}^{(j)}$ for any $j$, which implies that $d=\operatorname{gcd}\left\{j+1 ; a_{j} \neq 0\right\}=1$.

Proposition 2.1: Let $\left\{V_{n}\right\}_{n \geq 0}$ be a sequence (1), where $a_{0}, \ldots, a_{r-1}$ are real numbers satisfying (2). If $\left\{V_{n}\right\}_{n \geq 0}$ converges for any choice of the initial conditions, then condition $(C)$ is satisfied.

The following example allows us to see that condition $(C)$ is not sufficient for the convergence of a sequence (1), in the case of arbitrary real coefficients, with (2).

Example 2.1: Let $\left\{V_{n}\right\}_{n \geq 0}$ be a sequence (1) whose characteristic polynomial is

$$
P(X)=X^{3}-a_{0} X^{2}-a_{1} X-a_{2}
$$

with $a_{0}=2+v, a_{1}=-(1+2 v)$, and $a_{2}=v(v \neq 0,-2)$. Thus, $\sum_{j=0}^{2} a_{j}=1$ and ( $\left.C\right)$ is satisfied. Because the multiplicity of the characteristic root $\lambda=1$ is 2 , the sequence $\left\{V_{n}\right\}_{n \geq 0}$ does not converge for any choice of initial conditions.

## 3. CONVERGENCE OF SEQUENCES (1)

Hörner's diagram is used for practical computations of values of polynomials (see, e.g., [1]). In this section we apply this method to the convergence of some sequences (1), where the role of the initial conditions is considered.

Let $\left\{V_{A}(n)\right\}_{n \geq 0}$ be a sequence (1) whose initial conditions are $A=\left(\alpha_{0}, \ldots, \alpha_{r-1}\right)$. Let $\lambda_{1}, \ldots$, $\lambda_{s}$ be its real characteristic roots with multiplicities $m_{1}, \ldots, m_{s}$, respectively. Because the coefficients and initial conditions are real numbers, we deduce that if $\lambda=\lim _{n \rightarrow+\infty} \frac{V_{A}(n+1)}{V_{A}(n)}$ exists, then $\lambda$ is a real characteristic root.

Proposition 3.1: Let $\left\{V_{A}(n)\right\}_{n \geq 0}$ be a sequence (1) whose coefficients and initial conditions are real numbers. Suppose that

$$
\begin{equation*}
\sum_{j=0}^{k} a_{j} \leq 1 \text { for } 0 \leq k \leq r-1 \tag{8}
\end{equation*}
$$

If $\lim _{n \rightarrow+\infty} \frac{V_{A}(n+1)}{V_{A}(n)}$ exists and is positive, then $\left\{V_{A}(n)\right\}_{n \geq 0}$ converges.
Proof: Condition (8) implies that $b_{0}=1$ and $b_{k}=1-\sum_{j=0}^{k-1} a_{j} \geq 0$. Hence, from the Hörner diagram we deduce that, for any real zero $\lambda$ of the characteristic $P(X)$, we have $\lambda \leq 1$. Since

$$
V_{A}(n)=\sum_{l=1}^{s} \sum_{j=0}^{m_{s}-1} \beta_{l, j} n^{j} \lambda_{l}^{n}
$$

where $\left|\lambda_{1}\right| \geq\left|\lambda_{2}\right| \geq \cdots \geq\left|\lambda_{i}\right| \geq \cdots \geq\left|\lambda_{k}\right|$ and $\beta_{l, j}$ are obtained from initial condition A (see [2]), it follows that when $\lim _{n \rightarrow+\infty} \frac{V_{A}(n+1)}{V_{A}(n)}=\lambda_{i}$ exists and is positive, we have

$$
0<\lim _{n \rightarrow+\infty} \frac{V_{A}(n+1)}{V_{A}(n)}=\lambda_{i} \leq 1 \text { and } V_{A}(n)=\sum_{l=i}^{s} \sum_{j=0}^{m_{s}-1} \beta_{l, j} n^{j} \lambda_{l}^{n} \text { with } \sum_{j=0}^{m_{i}-1} \beta_{i, j} n^{j} \neq 0
$$

For $\lambda_{i}<1$, we deduce that $\lim _{n \rightarrow+\infty} V_{A}(n)=0$. For $\lambda_{i}=1$, condition (8) implies that $\lambda_{i}=1$ is a simple characteristic root. Also, $\left|\lambda_{j}\right|<\lambda_{i}=1$ for $j>i$. Therefore, Binet's formula implies that $\left\{V_{A}(n)\right\}_{n \geq 0}$ converges.

Remark 3.1: We can also use Descartes' rule of signs to derive the convergence of $\left\{V_{A}(n)\right\}_{n \geq 0}$. More precisely, we have $P(X)=\left(b_{0} X^{r-1}+\cdots+b_{r-1}\right)(X-1)+P(1)$, where $b_{0}=1, b_{k}=1-\sum_{j=0}^{r-2} a_{j}$ $\geq 0$, and $P(1)=1-\sum_{j=0}^{r-2} a_{j} \geq 0$, by (9). From Descartes' rule, we have $Q(x)>0$ for every $x \geq 0$. Thus, $P(x)>0$ for every $x>1$. Hence, $\lambda \leq 1$ for every positive zero $\lambda$ of $P(X)$.

Proposition 3.2: Let $\left\{V_{A}(n)\right\}_{n \geq 0}$ be a sequence (1) whose coefficients and initial conditions are real numbers. Suppose that

$$
\begin{equation*}
a_{0} \geq-1, \sum_{j=0}^{k}(-1)^{j+1} a_{j} \leq 1 \text { for } 1 \leq k \leq r-2, \sum_{j=0}^{r-2}(-1)^{j+1} a_{j}<1 \tag{9}
\end{equation*}
$$

If $\lim _{n \rightarrow+\infty} \frac{V_{A}(n+1)}{V_{A}(n)}$ exists and is negative, then $\left\{V_{A}(n)\right\}_{n \geq 0}$ converges. More precisely, we have $\lim _{n \rightarrow+\infty} V_{A}(n)=0$.

Proof: We have $Q(X)=(-1)^{r} P(-X)$; thus, $\lambda$ is a zero of $P(X)$ if and only if $-\lambda$ is a zero of $Q(X)$. Set $Q(X)=\left(b_{0} X^{r-1}+\cdots+b_{r-1}\right)(X-1)+Q(1)$; expression (9) implies that $b_{0}=1, b_{k}=$ $1+\sum_{j=0}^{k-1}(-1)^{j} a_{j} \geq 0(k=1, \ldots, r-2)$, and $b_{r-1}=1+\sum_{j=0}^{r-2}(-1)^{j} a_{j}>0$. We now have $Q(1) \neq 0$ and Hörner's diagram implies that $\lambda<1$ for any real zero $\lambda$ of $Q(X)$. Thus, for any real zero $\lambda$ of $P(X)$, we have also $\lambda>-1$. Since $\lim _{n \rightarrow+\infty} \frac{V_{A}(n+1)}{V_{A}(n)}$ exists, it follows from Binet's formula that $\left\{V_{A}(n)\right\}_{n \geq 0}$ converges with $\lim _{n \rightarrow+\infty} V_{A}(n)=0$.
Example 3.1: Let $\left\{V_{A}(n)\right\}_{n \geq 0}$ be a sequence (1) defined by

$$
V_{A}(n+1)=\frac{9}{20} V_{A}(n)+\frac{18}{20} V_{A}(n-1) \text { for } n \geq 1
$$

It is easy to see that $a_{0}=\frac{9}{20}$ and $a_{1}=\frac{18}{20}$ satisfy condition (9). For $A=\left(1,-\frac{3}{4}\right)$, we have

$$
\lim _{n \rightarrow+\infty} \frac{V_{A}(n+1)}{V_{A}(n)}=-\frac{3}{4}<0
$$

Thus, $\left\{V_{A}(n)\right\}_{n \geq 0}$ converges with $\lim _{n \rightarrow+\infty} V_{A}(n)=0$. For any $A \neq\left(1 \alpha,-\frac{3 \alpha}{4}\right)$, where $\alpha \neq 0$ is a real number, we have

$$
\lim _{n \rightarrow+\infty} \frac{V_{A}(n+1)}{V_{A}(n)}=\frac{6}{5}>0
$$

and $\left\{V_{A}(n)\right\}_{n \geq 0}$ diverges.

## 4. EXTENSION OF (C) AND CONVERGENCE OF (1)

Let $\left\{V_{n}\right\}_{n \geq 0}$ be a sequence (1), where $a_{0}, \ldots, a_{r-1}$ are real numbers satisfying (2). Then $P(X)=(X-1) Q(X)$, where $Q(X)=b_{0} X^{r-1}+b_{1} X^{r-2}+\cdots+b_{r-1}$ with $b_{k}=\sum_{j=k}^{r-1} a_{j}$, where $b_{0}=1$. Suppose that $b_{j} \neq 0(1 \leq j \leq r-1)$ and set

$$
H=H(Q)=\max \left\{\left|\frac{b_{1}}{b_{0}}=b_{1}\right|,\left|\frac{b_{2}}{b_{1}}\right|, \ldots,\left|\frac{b_{r-1}}{b_{r-2}}\right|\right\} .
$$

Let $R(X)=X^{r-1}-X^{r-2}-\cdots-X-1$ and let $\widetilde{q}>0$ be its unique positive zero. Then $\widetilde{q}>0$ is also a solution of the equation $X^{r}-2 X^{r-1}+1=0$. A straightforward computation allows us to derive that

$$
\frac{2(r-1)}{r} \leq \widetilde{q}<2 .
$$

Lemma 4.1: Let $\widetilde{q}>0$ be the unique positive zero of $R(X)=X^{r-1}-X^{r-2}-\cdots-X-1$ and $M>0$. Then the following two conditions are equivalent:

$$
\begin{gather*}
M \widetilde{q}<1 ;  \tag{10}\\
M<1 \text { and } M^{r}-2 M+1>0 . \tag{11}
\end{gather*}
$$

Proof: It is clear that $\widetilde{q} \geq 1$. Suppose that $M \widetilde{q}<1$. Then we have $0<M<1 / \widetilde{q} \leq 1$. Since $g(x)=x^{r}-2 x^{r-1}+1$ is a nondecreasing function on $[\widetilde{q},+\infty)$, we have $g(\widetilde{q})=0<g(1 / M)$. Thus, we have $M^{r}-2 M+1>0$. Conversely, suppose that $0<M<1$ and $M^{r}-2 M+1>0$. Then $0<\left(M^{r}-2 M+1\right) / M^{r}=g(1 / M)$ and $1 / M>1$. Since $g(x) \leq 0$ for $1 \leq x \leq \widetilde{q}$, we must have $1 / M>\widetilde{q}$, i.e., $M \widetilde{q}<1$.

Lemma 4.2: Let $Q(X)=b_{0} X^{r-1}+b_{1} X^{r-2}+\cdots+b_{r-1}$. Assume that $b_{0}=1$ and $b_{j} \neq 0$ for $1 \leq j \leq$ $r-1$. Then the zeros of $Q(X)$ have modulus bounded by $H \widetilde{q}$.

Proof: For every real number $X$, we have

$$
\begin{aligned}
|Q(X)| & \geq\left|X^{r-1}\right|-\left|b_{1} X^{r-2}\right|-\cdots-\left|b_{r-1}\right| \\
& =\left|X^{r-1}\right|-\left|\frac{b_{1}}{b_{0}} X^{r-2}\right|-\left|\frac{b_{2}}{b_{1}} \frac{b_{1}}{b_{0}} X^{r-3}\right|-\cdots-\left|\frac{b_{r-1}}{b_{r-2}} \cdots \frac{b_{1}}{b_{0}}\right| \\
& \geq|X|^{r-1}-H X^{r-2}-H^{2} X^{r-3}-\cdots-H^{r-1}
\end{aligned}
$$

If $X=z H \widetilde{q}$, where $|z|>1$, then

$$
|Q(X)| \geq|z|^{r-1} H^{r-1} \widetilde{q}^{r-1}-H|z|^{r-2} H^{r-2} \widetilde{q}^{r-2}-\cdots-H^{r-1}=H^{r-1} R(|z| \widetilde{q})>0 .
$$

Suppose that $Q(1) \neq 0$. Let $X=\alpha Y(\alpha>0)$ and let

$$
Q_{\alpha}(Y)=Y^{r-1}+\frac{b_{1}}{\alpha} Y^{r-2}+\frac{b_{2}}{\alpha^{2}} Y^{r-3}+\cdots+\frac{b_{r-1}}{\alpha^{r-1}} .
$$

If $y_{0}$ is a zero of $Q_{\alpha}(X)$, then $x_{0}=\alpha y_{0}$ is a zero of $Q(X)$ and $H_{\alpha}=H\left(Q_{\alpha}\right)=\frac{H}{\alpha}$. Let $\alpha>0$ be such that $H_{\alpha}<1$ and $H_{\alpha}^{r}-2 H_{\alpha}+1>0$. Then Lemma 4.2 implies that the zeros of $Q_{\alpha}(Y)$ are of modulus $<1$ and those of $Q(X)$ are of modulus $<\alpha$. Let

$$
\alpha_{0}=\inf \left\{\alpha>0 ; H_{\alpha}<1 \text { and } H_{\alpha}^{r}-2 H_{\alpha}+1>0\right\} .
$$

Elementary computation using the function $f(x)=x^{r}-2 x+1$ allows us to deduce that $\alpha_{0}=\frac{H}{x_{0}}$, where $x_{0} \neq 1$ is the other positive zero of the equation $x^{r}-2 x+1=0$. Thus, we can formulate the following result.

Proposition 4.1: Let $Q(X)=X^{r-1}+b_{1} X^{r-2}+\cdots+b_{r-1}$ satisfy $Q(1) \neq 0$. Assume that the $b_{j}$ 's are not zero. Then, for any $\lambda$ of $Q(X)$, we have $|\lambda|>\frac{H}{x_{0}}$, where $x_{0} \neq 1$ is the positive zero of $x^{r}-2 x+1=0$

The connection between ( $C$ ) and (10) may be expressed as follows.
Corollary 4.1: Let $\widetilde{q}$ be the unique positive zero of $R(X)=X^{r-1}-X^{r-2}-\cdots-X-1$. Assume that the $b_{j}$ 's are not zero and that

$$
H=\max \left\{\left|b_{1}\right|,\left|\frac{b_{2}}{b_{1}}\right|, \ldots,\left|\frac{b_{r-1}}{b_{r-2}}\right|\right\} .
$$

Then, for $M=H$, condition (10) implies condition (C).
Proof: Suppose that condition (10) is satisfied. Then Lemma 4.1 implies that $H<1$. If $a_{0}=0$, we can deduce that $b_{0}=b_{1}=1$ and thus $H \geq 1$, which gives a contradiction.

For the convergence of sequences (1) in the case of arbitrary real coefficients, condition (10) for $M=H$ may replace ( $C$ ) considered in the case of nonnegative coefficients. More precisely, we have the following result.
Proposition 4.2: Let $\left\{V_{n}\right\}_{n \geq 0}$ be a sequence (1), where $a_{0}, \ldots, a_{r-1}$ are real numbers satisfying (2). Assume that Hörner's $b_{j}$ 's are not zero and that

$$
H=\max \left\{\left|b_{1}\right|,\left|\frac{b_{2}}{b_{1}}\right|, \ldots,\left|\frac{b_{r-1}}{b_{r-2}}\right|\right\} .
$$

Then, if (10) is satisfied for $M=H$, the sequence $\left\{V_{n}\right\}_{n \geq 0}$ converges for any choice of initial conditions.

Proof: Set $C=V_{r-1}+b_{1} V_{r-2}+\cdots+b_{r-1} V_{0}$ and $L=\frac{C}{1+b_{1}+\cdots+b_{r-1}}$. Consider the sequence $\left\{W_{n}\right\}_{n \geq 0}$ defined by $W_{n}=V_{n}-L$. From (4), we deduce that $W_{n}=-b_{1} W_{n-1}+\cdots-b_{r-1} W_{n-r+1}$ for $n \geq r-1$. Thus, $\left\{W_{n}\right\}_{n \geq 0}$ is also a sequence (1) of order $r-1$ whose combinatorial expression defined by (5) and (6) is

$$
W_{n}=B_{1} \rho^{\prime}(n, r-1)+B_{1} \rho^{\prime}(n-1, r-1)+\cdots+B_{r-1} \rho^{\prime}(n-r+2, r-1) \text { for } n \geq r-1 \text {, }
$$

where $B_{m}=-b_{r-1} W_{m}-\cdots-b_{m} W_{r-1}(m=1, \ldots, r-1)$ and

$$
\rho^{\prime}(n, r-1)=\sum_{k_{1}+2 k_{1}+\cdots+(r-1) k_{r-1}=n-r+1} \frac{\left(k_{1}+\cdots+k_{r-1}\right)!}{k_{1}!\ldots k_{r-1}!} c_{1}^{k_{1}} \ldots c_{r-1}^{k_{r-1}},
$$

where $c_{j}=-b_{j}, \rho^{\prime}(k, k)=1$, and $\rho^{\prime}(n, k)=0$ if $n \geq k-1$. Therefore, $\left\{V_{n}\right\}_{n \geq 0}$ converges for any choice of initial conditions if and only if $\lim _{n \rightarrow+\infty} W_{n}=0$ if and only if $\lim _{n \rightarrow+\infty} \rho^{\prime}(n, r-1)=0$. Suppose $b_{j} \neq 0(1 \leq j \leq r-1)$. Then

$$
\left|b_{1}\right|^{k_{1}} \cdots\left|b_{r-1}\right|^{k_{r-1}}=\left|b_{1}\right|^{k_{1}+\cdots+k_{r-1}}\left|\frac{b_{2}}{b_{1}}\right|^{k_{2}+\cdots+k_{r-1}} \cdots\left|\frac{b_{r-1}}{b_{r-2}}\right|^{k_{r-1}}
$$

Thus, we have

$$
\begin{equation*}
\left|\rho^{\prime}(n, r-1)\right| \leq H^{n-r+1} \sum_{k_{1}+2 k_{1}+\cdots+(r-1) k_{r-1}=n-r+1} \frac{\left(k_{1}+\cdots+k_{r-1}\right)!}{k_{1}!\ldots k_{r-1}!} . \tag{12}
\end{equation*}
$$

From expression (7) we derive that the right-hand side of (12) is asymptotically equivalent to the expression

$$
\frac{(H \widetilde{q})^{n-r+1}}{\widetilde{q}^{-1}+2 \widetilde{q}^{-2}+\cdots+(r-1) \widetilde{q}^{-r+1}}
$$

(see Theorem 3.2 of [9]). The conclusion follows from (10).
Condition (10) is not necessary for the convergence of a sequence (1), as is shown in the following example.

Example 4.1: Let $\left\{V_{n}\right\}_{n \geq 0}$ be a sequence (1), where $r=3$ and $a_{0}=1-\mu, a_{1}=\mu-\alpha, a_{2}=\alpha$ with $\mu \neq 0$ and $\alpha \neq 0$. Then $a_{0}+a_{1}+a_{2}=1, b_{1}=\mu$, and $b_{2}=\alpha$. For example, if $\mu=\frac{9}{10}$ and $\alpha=\frac{2}{10}$, we deduce that $\lambda_{0}=1, \lambda_{1}=\frac{2}{5}$, and $\lambda_{2}=\frac{1}{2}$ are simple zeros of $P(X)$. Thus, the sequence $\left\{V_{n}\right\}_{n \geq 0}$ converges. Meanwhile, in this case we have $H=\frac{9}{2}$, and $\widetilde{q}=\frac{1+\sqrt{5}}{2}$ is the solution of $x^{2}=x+1$, so $H \widetilde{q}>1$. Other values of $\mu$ and $\alpha$ may give the same conclusion.

## 5. CONCLUDING REMARKS

Let us consider the following classical lemma (see, e.g., [5] and [10]).
Lemma 5.1: Let $R(X)=b_{0} X^{s}+b_{1} X^{s-1}+\cdots+b_{s}\left(b_{0} \neq 0\right)$ be a polynomial of real coefficients. Assume that the $b_{j}$ 's are not zero. Set

$$
\begin{gathered}
M_{1}(R)=\max \left\{1, \sum_{j=1}^{s}\left|\frac{b_{j}}{b_{0}}\right|\right\}, M_{2}(R)=\sum_{j=0}^{s-1}\left|\frac{b_{j+1}}{b_{j}}\right|, \\
M_{3}(R)=\max \left\{\left|\frac{b_{j}}{b_{j-1}}\right|^{1 / j} ; 1 \leq j \leq s\right\}, M_{4}(R)=\max \left\{\left|\frac{b_{s}}{b_{s-1}}\right|, 2\left|\frac{b_{j}}{b_{j-1}}\right| ; 1 \leq j \leq s-1\right\} .
\end{gathered}
$$

Thus, $|\lambda| \leq M_{j}(R)(j=1,2,3,4)$ for any zero $\lambda$ of $R(X)$.
Condition (2) implies that $P(X)=(X-1) Q(X)$, where $Q(X)=b_{0} X^{r-1}+b_{1} X^{r-2}+\cdots+b_{r-1}$ with $b_{k}=\sum_{j=k}^{r-1} a_{j}$ and $b_{0}=1$. Thus, if $a_{0}=0$, we have $b_{0}=b_{1}$, which implies that $M_{j}(Q) \geq 1$ for $j=2,3,4$. In particular, if $M_{j}(Q)<1(j=2,3,4)$, we deduce that $a_{0} \neq 0$, and (C) is satisfied.

Proposition 5.1: Let $\left\{V_{n}\right\}_{n \geq 0}$ be a sequence (1) whose coefficients are real numbers satisfying (2). Let $Q(X)=b_{0} X^{r-1}+b_{1} X^{r-2}+\cdots+b_{r-1}$, where $b_{k}=\sum_{j=k}^{r-1} a_{j}$. Assume that the $b_{j}$ 's are not zero. Then, if $M_{j}(Q)<1$ for some $j=2,3,4$, the sequence $\left\{V_{n}\right\}_{n \geq 0}$ converges for any choice of initial conditions.

The convergence of a sequence (1) has been studied in [3] and [4] for $r=2,3$. Proposition 5.1 extends Theorem 2 of [3] and Theorem 1 of [4] to $r \geq 2$.

Remark 5.1: Let $\left\{V_{n}\right\}_{n \geq 0}$ be a sequence (1) and set

$$
M=\max \left\{\left|b_{j}\right|^{1 / j} ; j=1, \ldots, r-1\right\}
$$

Assume that the $b_{j}$ 's are not zero. Then all results of Section 4 are still valid if we substitute $M$ for

$$
H=\max \left\{\left|b_{1}\right|,\left|\frac{b_{2}}{b_{1}}\right|, \ldots,\left|\frac{b_{r-1}}{b_{r-2}}\right|\right\} .
$$

Also note that $H \leq M_{4}(Q)$, where

$$
M_{4}(Q)=\max \left\{\left|\frac{b_{s}}{b_{s-1}}\right|, 2\left|\frac{b_{j}}{b_{j-1}}\right| ; 1 \leq j \leq s-1\right\}
$$

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# A NOTE ON A CLASS OF COMPUTATIONAL FORMULAS INVOLVING THE MULTIPLE SUM OF RECURRENCE SEQUENCES 

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## 1. INTRODUCTION

The second-order linear recurrence sequence $U=\left\{U_{n}\right\}, n=0,1,2, \ldots$, is defined by integers $a, b, U_{0}, U_{1}$ and by the recursion $U_{n+2}=b U_{n+1}+a U_{n}$ for $n \geq 0$. We suppose that $a b \neq 0$ and not both $U_{0}$ and $U_{1}$ are zero. If $\alpha$ and $\beta$ denote the roots of the characteristic polynomial $x^{2}-b x-a$ of the sequence $U$, then we have the Binet formula (see [1]):

$$
U_{n}=\frac{A \alpha^{n}-B \beta^{n}}{\alpha-\beta},
$$

where $A=U_{1}-U_{0} \beta$ and $B=U_{1}-U_{0} \alpha$. The generating function is

$$
\sum_{n=0}^{\infty} U_{n} x^{n}=\frac{U_{0}+\left(U_{1}-U_{0} b\right) x}{1-b x-a x^{2}} .
$$

If $U_{0}=0, U_{1}=1$, then the sequence $\mathscr{F}=\left\{U_{n}\right\}$ is called the generalized Fibonacci sequence, and $\mathscr{F}_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}$.

In order to express our results, we denote by $\sigma_{i, j}(n, k)(i, j$, and $k$ are nonnegative integers) the summation of all products of choosing $j$ elements from $n+2 k-1, n+2 k-2, \ldots, n+2 k-i+1$ but not containing any two consecutive elements. We note that $\sigma_{i, j}(n, k)=0$ if $j<0$ or $j>\left[\frac{i}{2}\right]$, $\sigma_{i, 0}(n, k)=1(i \geq 0), \sigma_{i, 1}(n, k)=\frac{1}{2}(i-1)(2 n+4 k-i)(i \geq 1)$. For example, when $i=6$, we have

$$
\sigma_{6,0}(n, k)=1,
$$

$$
\sigma_{6,1}(n, k)=(n+2 k-1)+(n+2 k-2)+(n+2 k-3)+(n+2 k-4)+(n+2 k-5),
$$

$$
\sigma_{6,2}(n, k)=(n+2 k-1)(n+2 k-3)+(n+2 k-1)(n+2 k-4)+(n+2 k-1)(n+2 k-5)
$$

$$
+(n+2 k-2)(n+2 k-4)+(n+2 k-2)(n+2 k-5)+(n+2 k-3)(n+2 k-5)
$$

$$
\sigma_{6,3}(n, k)=(n+2 k-1)(n+2 k-3)(n+2 k-5) .
$$

It is easy to prove that

$$
(n+2 k-1) \sigma_{2 k-2, k-1}(n, k-1)=\sigma_{2 k, k}(n, k) \quad(k \geq 1)
$$

and

$$
(n+2 k-1) \sigma_{k+i-2, i-1}(n, k-1)+\sigma_{k+i-1, i}(n+1, k-1)=\sigma_{k+i, i}(n, k) \quad(1 \leq i \leq k, k \geq 2) .
$$

Recently, W. Zhang [2] obtained the following result: Let $U=\left(U_{n}\right\}$ be defined as above. If $U_{0}=0$, then for any positive integer $k \geq 2$, we have

$$
\sum_{a_{1}+a_{2}+\cdots+a_{k}=n} U_{a_{1}} U_{a_{2}} \ldots U_{a_{k}}=\frac{U_{1}^{k-1}}{\left(b^{2}+4 a\right)^{k-1}(k-1)!}\left[g_{k-1}(n) U_{n-k+1}+h_{k-1}(n) U_{n-k}\right],
$$

where the summation is taken over all $n$-tuples with positive integer coordinates $\left(a_{1}, a_{2}, \ldots, a_{k}\right)$ such that $a_{1}+a_{2}+\cdots+a_{k}=n$, and he pointed out that $g_{k-1}(x)$ and $h_{k-1}(x)$ are two effectively computable polynomials of degree $k-1$, their coefficients depending only on $a, b$, and $k$.

In this paper, we obtain

$$
g_{k-1}(n)=\sum_{i=0}^{k-1}(2 a)^{i} b^{k-i-1}\langle n-k+1\rangle_{k-i-1} \sigma_{k+i-1, i}(n-k+1, k-1) \mathscr{F}_{k-i} \quad(k \geq 1)
$$

and

$$
h_{k-1}(n)=a \sum_{i=0}^{k-1}(2 a)^{i} b^{k-i-1}\langle n-k+1\rangle_{k-i-1} \sigma_{k+i-1, i}(n-k+1, k-1) \mathscr{F}_{k-i-1} \quad(k \geq 1),
$$

where $\langle n\rangle_{k}=n(n+1) \cdots(n+k-1)$ with $\langle n\rangle_{0}=1$. We also give the congruence relation

$$
g_{k-1}(n) U_{n-k+1}+h_{k-1}(n) U_{n-k} \equiv 0\left(\bmod (k-1)!\left(b^{2}+4 a\right)^{k-1}\right)(k \geq 1),
$$

which generalizes the results presented in [2].

## 2. THE RESULTS AND THEIR PROOFS

In this section, with $U_{0}=0$, let

$$
G_{k}(x)=\left(\frac{U_{1}}{1-b x-a x^{2}}\right)^{k}=\sum_{n=0}^{\infty} U_{n}^{(k)} x^{n-1}
$$

Then

$$
\sum_{a_{1}+a_{2}+\cdots+a_{m}=n} U_{a_{1}}^{\left(k_{1}\right)} U_{a_{2}}^{\left(k_{2}\right)} \cdots U_{a_{m}}^{\left(k_{m}\right)}=U_{n-m+1}^{\left(k_{1}+k_{2}+\cdots+k_{m}\right)} .
$$

Taking $k_{1}=k_{2}=\cdots=k_{m}=1$, we have
Lemma 1: $\sum_{a_{1}+a_{2}+\cdots+a_{m}=n} U_{a_{1}} U_{a_{2}} \cdots U_{a_{m}}=U_{n-m+1}^{(m)}$.
Theorem 1: $\quad U_{n}^{(k+1)}=\frac{U_{1}}{k\left(b^{2}+4 a\right)}\left\{n b U_{n+1}^{(k)}+2 a(n+2 k-1) U_{n}^{(k)}\right\} \quad(k \geq 1)$.
Proof:

$$
\frac{d}{d x}\left(G_{k}(x)(b+2 a x)^{k}\right)=G_{k}^{\prime}(x)(b+2 a x)^{k}+G_{k}(x) k(b+2 a x)^{k-1} 2 a
$$

and

$$
\begin{aligned}
& \frac{d}{d x}\left(G_{k}(x)(b+2 a x)^{k}\right)=\frac{d}{d x}\left(\frac{U_{1}(b+2 a x)}{1-b x-a x^{2}}\right)^{k} \\
& =k U_{1}\left(\frac{U_{1}(b+2 a x)}{1-b x-a x^{2}}\right)^{k-1} \frac{2 a\left(1-b x-a x^{2}\right)+(b+2 a x)^{2}}{\left(1-b x-a x^{2}\right)^{2}}
\end{aligned}
$$

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$$
\begin{aligned}
& =k(b+2 a x)^{k-1} U_{1}\left(\frac{U_{1}}{1-b x-a x^{2}}\right)^{k-1} \frac{2 a^{2} x^{2}+2 a b x+b^{2}+2 a}{\left(1-b x-a x^{2}\right)^{2}} \\
& =k(b+2 a x)^{k-1} U_{1}\left(\frac{U_{1}}{1-b x-a x^{2}}\right)^{k-1} \frac{-2 a\left(1-b x-a x^{2}\right)+b^{2}+4 a}{\left(1-b x-a x^{2}\right)^{2}} .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& G_{k}^{\prime}(x)(b+2 a x)^{k}+G_{k}(x) k(b+2 a x)^{k-1} 2 a \\
& =k(b+2 a x)^{k-1} U_{1}\left(\frac{U_{1}}{1-b x-a x^{2}}\right)^{k-1} \frac{-2 a\left(1-b x-a x^{2}\right)+b^{2}+4 a}{\left(1-b x-a x^{2}\right)^{2}} .
\end{aligned}
$$

Therefore,

$$
G_{k}^{\prime}(x) U_{1}(b+2 a x)+2 a k U_{1} G_{k}(x)=-2 a k U_{1} G_{k}(x)+\left(b^{2}+4 a\right) k G_{k+1}(x) .
$$

This concludes the proof of Theorem 1.
Theorem 2: $U_{n}^{(k+1)}=\frac{U_{1}^{k}}{k!\left(b^{2}+4 a\right)^{k}} \sum_{i=0}^{k}(2 a)^{i} b^{k-i}\langle n\rangle_{k-i} \sigma_{k+i, i}(k) U_{n+k-i} \quad(k \geq 0)$.
Proof: This theorem can be proved by induction. When $k=0$, the theorem is trivial. When $k=1$, the theorem is true by applying Theorem 1. Assume the theorem is true for a positive integer $k-1$, then

$$
\begin{aligned}
& U_{n}^{(k+1)}=\frac{U_{1}}{k\left(b^{2}+4 a\right)}\left\{n b U_{n+1}^{(k)}+2 a(n+2 k-1) U_{n}^{(k)}\right\} \\
&= \frac{U_{1}}{k\left(b^{2}+4 a\right)}\left\{n b \frac{U_{1}^{k-1}}{(k-1)!\left(b^{2}+4 a\right)^{k-1}} \sum_{i=0}^{k-1}(2 a)^{i} b^{k-i-1}\langle n+1\rangle_{k-i-1, i} \sigma_{k+i-1, i}(n+1, k-1) U_{n+k-i}\right. \\
&\left.+2 a(n+2 k-1) \frac{U_{1}^{k-1}}{(k-1)!\left(b^{2}+4 a\right)^{k-1}} \sum_{i=0}^{k-1}(2 a)^{i} b^{k-i-1}\langle n\rangle_{k-i-1} \sigma_{k+i-1, i}(n, k-1) U_{n+k-i-1}\right\} \\
&= \frac{U_{1}^{k}}{k!\left(b^{2}+4 a\right)^{k}}\left\{\sum_{i=0}^{k-1}(2 a)^{i} b^{k-n}\langle n+1\rangle_{k-i-1} \sigma_{k+i-1, i}(n+1, k-1) U_{n+k-i}\right. \\
&\left.+\sum_{i=0}^{k-1}(2 a)^{i+1} b^{k-i-1}\langle n\rangle_{k-i-1}(n+2 k-1) \sigma_{k+i-1, i}(n, k-1) U_{n+k-i-1}\right\} \\
&= \frac{U_{1}^{k}}{k!\left(b^{2}+4 a\right)^{k}}\left\{\sum_{i=0}^{k-1}(2 a)^{i} b^{k-i}\langle n\rangle_{k-i} \sigma_{k+i-1, i}(n+1, k-1) U_{n+k-i}\right. \\
&\left.+\sum_{i=0}^{k}(2 a)^{i} b^{k-i}\langle n\rangle_{k-i}(n+2 k-1) \sigma_{k+i-2, i-1}(n, k-1) U_{n+k-i}\right\} \\
&= \frac{U_{1}^{k}}{k!\left(b^{2}+4 a\right)^{k}}\left\{b^{k}\langle n\rangle_{k} \sigma_{k-1,0}(n+1, k-1) U_{n+k}+\sum_{i=1}^{k-1}(2 a)^{i} b^{k-i}\langle n\rangle_{k-i} U_{n+k-i}\left[\sigma_{k+i-1, i}(n, k-1)\right.\right. \\
&\left.\left.+(n+2 k-1) \sigma_{k+i-2, i-1}(n, k-1)\right]+(2 a)^{k}(n+2 k-1) \sigma_{2 k-2, k-1}(n, k-1) U_{n}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{U_{1}^{k}}{k!\left(b^{2}+4 a\right)^{k}}\left\{b^{k}\langle n\rangle_{k} \sigma_{k, 0}(n+1, k) U_{n+k}+\sum_{i=1}^{k-1}(2 a)^{i} b^{k-i}\langle n\rangle_{k-i} U_{n+k-i} \sigma_{k+i, i}(n, k)+(2 a)^{k} \sigma_{2 k, k}(n, k) U_{n}\right\} \\
& =\frac{U_{1}^{k}}{k!\left(b^{2}+4 a\right)^{k}} \sum_{i=0}^{k}(2 a)^{i} b^{k-i}\langle n\rangle_{k-i} \sigma_{k+i, i}(n, k) U_{n+k-i} .
\end{aligned}
$$

That is, the theorem is also true for $k$. This proves the Theorem 2.
Lemma 2: $U_{m+k}=\mathscr{F}_{k+1} U_{m}+a \mathscr{F}_{k} U_{m-1}(k \geq 0, m \geq 1)$.
Proof: Use Binet's formula.
Theorem 3: $\quad U_{n}^{(k+1)}=\frac{U_{1}^{k}}{k!\left(b^{2}+4 a\right)^{k}} \sum_{i=0}^{k}(2 a)^{i} b^{k-i}\langle n\rangle_{k-i} \sigma_{k+i, i}(n, k)\left(\mathscr{F}_{k-i+1} U_{n}+a \mathscr{F}_{k-i} U_{n-1}\right) \quad(k \geq 0)$.
Proof: Use Theorem 2 and Lemma 2.
Theorem 4: $\sum_{a_{1}+a_{2}+\cdots+a_{k}=n} U_{a_{1}} U_{a_{2}} \ldots U_{a_{k}}$

$$
\begin{aligned}
& =\frac{U_{1}^{k-1}}{\left(b^{2}+4 a\right)^{k-1}(k-1)!}\left\{\left[\sum_{i=0}^{k-1}(2 a)^{i} b^{k-i-1}\langle n-k+1\rangle_{k-i-1} \sigma_{k+i-1, i}(n-k+1, k-1) \mathscr{F}_{k-i}\right] U_{n-k+1}\right. \\
& \left.+a\left[\sum_{i=0}^{k-1}(2 a)^{i} b^{k-i-1}\langle n-k+1\rangle_{k-i-1} \sigma_{k+i-1, i}(n-k+1, k-1) \mathscr{F}_{k-i-1}\right] U_{n-k}\right\}(k \geq 1) .
\end{aligned}
$$

Proof: Noting Lemma 1 and Theorem 3, we have

$$
\begin{aligned}
& \sum_{a_{1}+a_{2}+\cdots+a_{k}=n} U_{a_{1}} U_{a_{2}} \ldots U_{a_{k}}=U_{n-k+1}^{(k)} \\
= & \frac{U_{1}^{k-1}}{(k-1)!\left(b^{2}+4 a\right)^{k-1}} \sum_{i=0}^{k-1}(2 a)^{i} b^{k-i-1}\langle n-k+1\rangle_{k-1-i} \sigma_{k-1+i, i}(n-k+1, k-1) \\
& \times\left(\mathscr{F}_{k-i} U_{n-k+1}+a \mathscr{F}_{k-i-1} U_{n-k}\right) \\
= & \frac{U_{1}^{k-1}}{\left(b^{2}+4 a\right)^{k-1}(k-1)!}\left\{\left[\sum_{i=0}^{k-1}(2 a)^{i} b^{k-i-1}\langle n-k+1\rangle_{k-i-1} \sigma_{k+i-1, i}(n-k+1, k-1) \mathscr{F}_{k-i}\right] U_{n-k+1}\right\} \\
& \left.+a\left[\sum_{i=0}^{k-1}(2 a)^{i} b^{k-i-1}\langle n-k+1\rangle_{k-i-1} \sigma_{k+i-1, i}(n-k+1, k-1) \mathscr{F}_{k-i-1}\right] U_{n-k}\right\} .
\end{aligned}
$$

From this theorem, we can get the expression of $g_{k-1}(n)$ and $h_{k-1}(n)$, namely,

$$
g_{k-1}(n)=\sum_{i=0}^{k-1}(2 a)^{i} b^{k-i-1}\langle n-k+1\rangle_{k-i-1} \sigma_{k+i-1, i}(n-k+1, k-1) \mathscr{F}_{k-i} \quad(k \geq 1)
$$

and

$$
h_{k-1}(n)=a \sum_{i=0}^{k-1}(2 a)^{i} b^{k-i-1}\langle n-k+1\rangle_{k-i-1} \sigma_{k+i-1, i}(n-k+1, k-1) \mathscr{F}_{k-i-1} \quad(k \geq 1) .
$$

Theorem 5: $g_{k-1}(n) U_{n-k+1}+h_{k-1}(n) U_{n-k} \equiv 0\left(\bmod (k-1)!\left(b^{2}+4 a\right)^{k-1}\right) \quad(k \geq 1)$.

This result is a generalization of Corollary 2 of [2]. When $U_{1}=a=b=1$ and $k=1,2,3$, respectively, this result becomes (i)-(iii) of Corollary 2 of [2].

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## THE PASSING OF THREE FIBONACCI ASSOCIATION FRIENDS

We were all deeply saddened to learn of the recent deaths of Joe Arkin, Daniel Fielder and Peter Kiss. These three long-time members of the Fibonacci Association will be greatly missed.

# EULERIAN POLYNOMIALS AND RELATED EXPLICIT FORMULAS 

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## 1. INTRODUCTION

For $\alpha \in \mathbb{C} \backslash\{1\}$ we write, as in [2],

$$
\frac{1-\alpha}{e^{t}-\alpha}=\sum_{n=0}^{\infty} H_{n}(\alpha) \frac{t^{n}}{n!},
$$

from which it follows that $H_{n}(\alpha)(n=0,1, \ldots)$ are uniquely determined by

$$
\begin{equation*}
H_{0}(\alpha) \equiv 1, \quad H_{n}(\alpha)=\frac{1}{\alpha-1} \sum_{k=0}^{n-1}\binom{n}{k} H_{k}(\alpha) \quad(n \geq 1) . \tag{1}
\end{equation*}
$$

The Eulerian polynomials $R_{n}(\alpha)(n=0,1, \ldots)$ are defined by $R_{n}(\alpha)=(\alpha-1)^{n} H_{n}(\alpha)$ as Euler first discussed them in [4]. For $n \geq 1$, as is easily seen from (1), $R_{n}(\alpha)$ is a polynomial in $\alpha$ of degree $n-1$ with integer coefficients and was expressed by Euler in [4] as

$$
\begin{equation*}
R_{n}(\alpha)=\sum_{k=1}^{n} A_{k}^{n} \alpha^{k-1}, \tag{2}
\end{equation*}
$$

where the integers $A_{k}^{n}(1 \leq k \leq n)$ are known as Eulerian numbers (see also [3, p. 51]). Later, Frobenius [5] gave another expression for $R_{n}(\alpha)$ as

$$
\begin{equation*}
R_{n}(\alpha)=\sum_{k=1}^{n} k!S_{k}^{n}(\alpha-1)^{n-k}, \tag{3}
\end{equation*}
$$

where $S_{k}^{n}(1 \leq k \leq n)$ denote the Stirling numbers of the second kind (see also [3, p. 244]).
The object of this paper is to obtain one more expression for $R_{n}(\alpha)$ in terms of an array of integers $C_{k}^{n}$ closely related to the central factorial numbers (see [6, §6.5]). By means of the new expression for $R_{n}(\alpha)$, we derive explicit formulas for Bernoulli and Euler numbers and others, and unify some known results, in terms of these $C_{k}^{n}$.

## 2. A NEW EXPRESSION FOR $\mathbb{R}_{n}(\alpha)$

We define an array of integers $C_{k}^{n}$ in the following way: for integers $r, k \geq 1$,

$$
C_{k}^{n}= \begin{cases}\frac{1}{k} \sum_{j=1}^{k}(-1)^{k-j}\binom{2 k}{k-j} j^{2 r} & \text { if } n=2 r-1,  \tag{4}\\ k C_{k}^{2 r-1} & \text { if } n=2 r .\end{cases}
$$

Clearly, $C_{1}^{2 r-1}=C_{1}^{2 r}=1$. We make the convention that $C_{0}^{2 r-1}=C_{0}^{2 r}=0$.

These integers $C_{k}^{n}$ are closely related to the central factorial numbers of the second kind $T(n, k)$ defined as in [6, p. 212]. Indeed, for $r, k \geq 1, C_{k}^{2 r-1}=(2 k-1)!T(2 r, 2 k)$. Thus, it follows from the properties of $T(2 r, 2 k)$ (see also [1, pp. 428-29]) that

$$
C_{k}^{2 r-1}= \begin{cases}(2 r-1)! & \text { if } k=r  \tag{5}\\ 0 & \text { if } k \geq r+1\end{cases}
$$

Moreover, the second formula in the definition (4) together with

$$
\begin{equation*}
C_{k}^{2 r+1}=2(2 k-1) C_{k-1}^{2 r}+k C_{k}^{2 r} \tag{6}
\end{equation*}
$$

gives the recurrence for $C_{k}^{n}$. We may also derive (5) and (6) directly from the definition.
The new expression of $R_{n}(\alpha)$ given below contains the powers of $\alpha$ as in (2) and also that of $\alpha-1$ as in (3). Moreover, the number of the terms in the summation is about half of that in (2) and (3).
Theorem 1: For an integer $r \geq 1$ :

$$
\begin{gather*}
R_{2 r-1}(\alpha)=\sum_{k=1}^{r} C_{k}^{2 r-1} \alpha^{k-1}(\alpha-1)^{2 r-2 k}  \tag{7}\\
R_{2 r}(\alpha)=(1+\alpha) \sum_{k=1}^{r} C_{k}^{2 r} \alpha^{k-1}(\alpha-1)^{2 r-2 k} \tag{8}
\end{gather*}
$$

Proof: Clearly, from (1), $R_{1}(\alpha)=1$ and $R_{2}(\alpha)=1+\alpha$. For the general case, the proof is by induction on $r \geq 1$ using the recurrence

$$
\begin{equation*}
R_{n+1}(\alpha)=(n+1) \alpha R_{n}(\alpha)+(1-\alpha) \frac{d}{d \alpha}\left(\alpha R_{n}(\alpha)\right) \tag{9}
\end{equation*}
$$

for $n \geq 1$ (see [2], [5]). If (7) is true, then by (9),

$$
\begin{aligned}
R_{2 r}(\alpha) & =(2 r) \alpha R_{2 r-1}(\alpha)+(1-\alpha) \frac{d}{d \alpha}\left(\alpha R_{2 r-1}(\alpha)\right) \\
& =\sum_{k=1}^{r} k C_{k}^{2 r-1}(1+\alpha) \alpha^{k-1}(\alpha-1)^{2 r-2 k},
\end{aligned}
$$

which by (4) equals the right-hand side of (8). If (8) is true, then by (9) again,

$$
\begin{aligned}
R_{2 r+1}(\alpha) & =(2 r+1) \alpha R_{2 r}(\alpha)+(1-\alpha) \frac{d}{d \alpha}\left(\alpha R_{2 r}(\alpha)\right) \\
& =\sum_{k=1}^{r} C_{k}^{2 r}\left\{2 \alpha(2 k+1)+k(\alpha-1)^{2}\right\} \alpha^{k-1}(\alpha-1)^{2 r-2 k} \\
& =C_{1}^{2 r}(\alpha-1)^{2 r}+C_{r}^{2 r} 2(2 r+1) \alpha^{r}+\sum_{k=2}^{r}\left\{2(2 k-1) C_{k-1}^{2 r}+k C_{k}^{2 r}\right\} \alpha^{k-1}(\alpha-1)^{2 r-2 k+2},
\end{aligned}
$$

which by (5) and (6) equals the right-hand side of (7) with $r$ replaced by $r+1$. This completes the proof of the theorem.

Some classical formulas involving the Eulerian numbers have their counterparts in the integers $C_{k}^{n}$. Analogous to an identity of Worpitzky (see [3, p. 243]), we have the following theorem.

Theorem 2: For an integer $r \geq 1$ :

$$
\begin{equation*}
x^{2 r-1}=\sum_{k=1}^{r} C_{k}^{2 r-1}\binom{x+k-1}{2 k-1} \tag{10}
\end{equation*}
$$

Proof: Let $\Delta$ be the difference operator defined by $\Delta f(x)=f(x+1)-f(x)$. Following an idea of Frobenius in [5], we have by a property of $S_{j}^{n}$ (see [3, p. 207]) and (3)

$$
x^{n}=\sum_{j=1}^{n} j!S_{j}^{n}\binom{x}{j}=\sum_{j=1}^{n} j!S_{j}^{n} \Delta^{n-j}\binom{x}{n}=R_{n}(I+\Delta)\binom{x}{n} .
$$

Thus, by (7),

$$
x^{2 r-1}=\sum_{k=1}^{r} C_{k}^{2 r-1}(I+\Delta)^{k-1} \Delta^{2 r-2 k}\binom{x}{2 r-1}=\sum_{k=1}^{r} C_{k}^{2 r-1}\binom{x+k-1}{2 k-1} .
$$

In connection with the Bernoulli polynomials $B_{n}(x)$ and the Bernoulli numbers $B_{n}=B_{n}(0)$ analogous to

$$
\frac{1}{n}\left\{B_{n}(x)-B_{n}\right\}=\sum_{j=1}^{n-1} A_{j}^{n-1}\binom{x+j-1}{n},
$$

we have the following theorem.
Theorem 3: For an integer $r \geq 1$ :

$$
\begin{gather*}
\frac{1}{2 r}\left\{B_{2 r}(x)-B_{2 r}\right\}=\sum_{k=1}^{r} C_{k}^{2 r-1}\binom{x+k-1}{2 k},  \tag{11}\\
\frac{1}{2 r+1} B_{2 r+1}(x)=(2 x-1) \sum_{k=1}^{r} \frac{1}{2 k+1} C_{k}^{2 r}\binom{x+k-1}{2 k} . \tag{12}
\end{gather*}
$$

Proof: Since both sides of (11) are polynomials in $x$, it suffices to assume that $x$ equals an integer $m \geq 1$. Then it follows from (10) using formulas in [3, pp. 10 and 155] that

$$
\frac{1}{2 r}\left\{B_{2 r}(m)-B_{2 r}\right\}=\sum_{j=1}^{m-1} j^{2 r-1}=\sum_{k=1}^{r} C_{k}^{2 r-1} \sum_{j=1}^{m-1}\binom{k+j-1}{2 k-1}=\sum_{k=1}^{r} C_{k}^{2 r-1}\binom{m+k-1}{2 k} .
$$

Similarly,

$$
\begin{aligned}
\frac{1}{2 r+1} B_{2 r+1}(m) & =\sum_{j=1}^{m-1} j^{2 r}=\sum_{k=1}^{r} C_{k}^{2 r} \sum_{j=1}^{m-1} \frac{j}{k}\binom{k+j-1}{2 k-1} \\
& =\sum_{k=1}^{r} C_{k}^{2 r} \sum_{j=1}^{m-1}\left[\binom{k+j-1}{2 k}+\binom{k+j}{2 k}\right] \\
& =\sum_{k=1}^{r} C_{k}^{2 r}\left[\binom{m+k-1}{2 k+1}+\binom{m+k}{2 k+1}\right] \\
& =\sum_{k=1}^{r} C_{k}^{2 r} \frac{2 m-1}{2 k+1}\binom{m+k-1}{2 k} .
\end{aligned}
$$

As a simple and interesting consequence of Theorem 3, we derive some explicit formulas for Bernoulli numbers which may be compared with those in Theorems 5 and 6 below.

Theorem 4: For an integer $r \geq 1$ :

$$
\begin{align*}
& B_{2 r}=\sum_{k=1}^{r}(-1)^{k-1} \frac{(k-1)!k!}{(2 k+1)!} C_{k}^{2 r}  \tag{13}\\
& B_{2 r+2}=2 \sum_{k=1}^{r}(-1)^{k} \frac{k!(k+1)!}{(2 k+3)!} C_{k}^{2 r} \tag{14}
\end{align*}
$$

Proof: We obtain (13) by differentiating both sides of (12) and then evaluating at $x=0$. Moreover, we have, by (6),

$$
\begin{aligned}
B_{2 r+2} & =\sum_{k=1}^{r+1}(-1)^{k-1} \frac{(k-1)!k!}{(2 k+1)!} k\left\{2(2 k-1) C_{k-1}^{2 r}+k C_{k}^{2 r}\right\} \\
& =\sum_{k=1}^{r}(-1)^{k} \frac{k!(k+1)!}{(2 k+3)!} 2\{(k+1)(2 k+1)-k(2 k+3)\} C_{k}^{2 r},
\end{aligned}
$$

from which (14) follows.
From the proof of Theorem 3 we have, in particular,

$$
\begin{gather*}
\sum_{j=1}^{m} j^{2 r-1}=\sum_{k=1}^{r} C_{k}^{2 r-1}\binom{m+k}{2 k}, \\
\sum_{j=1}^{m} j^{2 r}=(2 m+1) \sum_{k=1}^{r} \frac{1}{2 k+1} C_{k}^{2 r}\binom{m+k}{2 k} . \tag{15}
\end{gather*}
$$

We refer to [7] in which (13) and (15) have been given.

## 3. BERNOULLI AND EULER NUMBERS

We recall that

$$
\sec t=\sum_{r=0}^{\infty}(-1)^{r} E_{2 r} \frac{t^{2 r}}{(2 r)!}, \quad \tan t=\sum_{r=1}^{\infty} T_{2 r-1} \frac{t^{2 r-1}}{(2 r-1)!},
$$

where $E_{2 r}$ are known as the Euler numbers and $T_{2 r-1}$ as the tangent numbers. The Bernoulli numbers can be obtained by

$$
B_{2 r}=(-1)^{r-1} \frac{2 r}{4^{r}\left(4^{r}-1\right)} T_{2 r-1} .
$$

Since

$$
\sec t+\tan t=\frac{2 e^{i t}}{e^{2 i t}+1}-i \frac{e^{2 i t}-1}{e^{2 i t}+1}=1+(1+i) \sum_{n=1}^{\infty} H_{n}(i) \frac{(i t)^{n}}{n!},
$$

where $i=\sqrt{-1}$, it follows that, for $r \geq 1$,

$$
\begin{gather*}
E_{2 r}=(1+i) H_{2 r}(i),  \tag{16}\\
T_{2 r-1}=(-1)^{r}(1-i) H_{2 r-1}(i) . \tag{17}
\end{gather*}
$$

Moreover, it is easy to verify that

$$
\begin{equation*}
T_{2 r-1}=(-1)^{r} 2^{2 r-1} H_{2 r-1}(-1)=(-1)^{r-1} R_{2 r-1}(-1) \tag{18}
\end{equation*}
$$

See also [2, p. 257] and [3, p. 259].

Theorem 5: For an integer $r \geq 1$ :

$$
\begin{align*}
T_{2 r-1} & =\sum_{k=1}^{r}(-1)^{r-k} \frac{1}{2^{k-1}} C_{k}^{2 r-1},  \tag{19}\\
T_{2 r-1} & =\sum_{k=1}^{r}(-1)^{r-k} 2^{2 r-2 k} C_{k}^{2 r-1} \tag{20}
\end{align*}
$$

and

$$
\begin{gather*}
T_{2 r+1}=\sum_{k=1}^{r}(-1)^{r-k} \frac{k+1}{2^{k-1}} C_{k}^{2 r},  \tag{21}\\
T_{2 r+1}=\sum_{k=1}^{r}(-1)^{r-k} 2^{2 r-2 k+1} C_{k}^{2 r} . \tag{22}
\end{gather*}
$$

Proof: We have, by (7) and (17),

$$
T_{2 r-1}=(-1)^{r} 2 \sum_{k=1}^{r} C_{k}^{2 r-1} \frac{i^{k}}{(i-1)^{2 k}},
$$

from which (19) follows. Moreover, we have, by (6),

$$
\begin{aligned}
T_{2 r+1} & =\sum_{k=1}^{r+1}(-1)^{r-k+1} \frac{1}{2^{k-1}}\left\{2(2 k-1) C_{k-1}^{2 r}+k C_{k}^{2 r}\right\} \\
& =\sum_{k=1}^{r}(-1)^{r-k} \frac{1}{2^{k}}\{2(2 k+1)-2 k\} C_{k}^{2 r},
\end{aligned}
$$

from which (21) follows. We obtain (20) and (22) similarly using (18) instead.
Theorem 6: For an integer $r \geq 1$ :

$$
\begin{gather*}
E_{2 r}=\sum_{k=1}^{r}(-1)^{k} \frac{1}{2^{k-1}} C_{k}^{2 r},  \tag{23}\\
E_{2 r+2}=\sum_{k=1}^{r}(-1)^{k-1} \frac{k^{2}+3 k+1}{2^{k-1}} C_{k}^{2 r} . \tag{24}
\end{gather*}
$$

Proof: We have, by (8) and (16),

$$
E_{2 r}=2 \sum_{k=1}^{r} C_{k}^{2 r} \frac{i^{k}}{(i-1)^{2 k}},
$$

from which (23) follows. Moreover, we have, by (6),

$$
\begin{aligned}
E_{2 r+2} & =\sum_{k=1}^{r+1}(-1)^{k} \frac{k}{2^{k-1}}\left\{2(2 k-1) C_{k-1}^{2 r}+k C_{k}^{2 r}\right\} \\
& =\sum_{k=1}^{r}(-1)^{k-1} \frac{1}{2^{k-1}}\left\{(k+1)(2 k+1)-k^{2}\right\} C_{k}^{2 r},
\end{aligned}
$$

from which (24) follows.
The formulas (21) and (23) can be found in [3, p. 259] where no proofs are given. We refer to [1, pp. 479-80] for other explicit formulas for $T_{2 r-1}$ and $E_{2 r}$.

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# REFLECTIONS ON THE LAMBDA TRIANGLE 

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## 1. INTRODUCTION

Kappraff [2] described the panels in the pavement of the Reading Room of the Library on the second floor of the San Lorenzo church complex in Florence. Work on the library was begun in 1523 by Pope Clement VII, Giulio di Medici, as a monument to his uncle, Lorenzo di Medici. The library was one of the few successes of Clement's disastrous reign, characterized as it was by bad political decisions (see [1], [11]). In the Timaeus panel of the library, Michelangelo, the designer of the library, used the number relations (the scale) of the lambda figure which had previously been used as the musical system studied by Pythagoras [4].

Kappraff used the lambda triangle in Table 1 "found in Plato's Timaeus and referred to there as the World Soul." Strictly speaking, the lambda diagram displayed in Table 1 is that given in Taylor [10] but with the empty space between the two slanting lines $\Lambda$ (hence the designation lambda) filled in a methodical and obvious way. Plato himself does not appear to have used the lambda figure as such though he used the two generating scales $1,2,4,8$ and $1,3,9,27$ shown by the slanting lines to describe the creation by the Demiurge of the World Soul. These scales are represented linearly (essentially in one line) in the commentary on the Timaeus [5].

TABLE 1. The Lambda Triangle

|  |  |
| :---: | :---: |
|  |  |
|  | $4 /$ |
|  | / 12 |

The formation is obvious and one cannot resist the temptation to portray the associated leftand right-triangular arrays (Tables 2 and 3). Clearly, these arrays may be extended infinitely.

TABLE 2. Left-Triangular Lambda Array TABLE 3. Right-Triangular Lambeda Array

| 1 | 0 | 0 | 0 |
| ---: | ---: | ---: | ---: |
| 2 | 3 | 0 | 0 |
| 4 | 6 | 9 | 0 |
| 8 | 12 | 18 | 27 |

$\begin{array}{llll}0 & 0 & 0 & 1\end{array}$
$0 \quad 0 \quad 2 \quad 3$
$\begin{array}{llll}0 & 4 & 6 & 9\end{array}$
$\begin{array}{llll}8 & 12 & 18 & 27\end{array}$
It is the purpose of this paper to describe some of the properties of these arrays and triangles.

## 2. LAMBDA TRIANGLES

The elements, $u_{n, m}$, of the left-triangular array satisfy the partial difference equation

$$
\begin{equation*}
u_{n, m}=u_{n, m-1}+u_{n-1, m-1}, \quad n>0,0<m \leq n, \tag{2.1}
\end{equation*}
$$

with boundary conditions $u_{n, 0}=2^{n-1}, u_{n, m}=0$ when $n<0$ and $m>n$, and general term

$$
\begin{equation*}
u_{n, m}=2^{n-m} 3^{m-1}, \tag{2.2}
\end{equation*}
$$

where $n, m$ represent the rows and columns, respectively. We can see that the row sums, $1,5,19$, $65,211, \ldots$ (Sequence M3887 of [8]), are given by the second-order homogeneous linear recurrence relation

$$
\begin{align*}
v_{n} & =5 v_{n-1}-6 v_{n-2}, \quad n \geq 3, v_{1}=1, v_{2}=5, \\
& =3^{n}-2^{n}, \quad n \geq 1 . \tag{2.3}
\end{align*}
$$

The partial column sums are displayed in Table 4.
TABLE 4. Partial Column Sums of Left-Triangular Lambda Array

| 1 |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: |
| 3 | 3 |  |  |  |  |
| 7 | 9 | 9 |  |  |  |
| 15 | 21 | 27 | 27 |  |  |
| 31 | 45 | 63 | 81 | 81 |  |
| 63 | 93 | 135 | 189 | 243 | 243 |

The elements in the cells of Table 4 satisfy the partial recurrence relation

$$
\begin{equation*}
w_{n, m}=w_{n, m-1}+w_{n-1, m}-w_{n-2, m-1}, \quad n \geq m>1, \tag{2.4}
\end{equation*}
$$

with general term

$$
\begin{equation*}
w_{n, m}=3^{m-1}\left(2^{n-m+1}-1\right) \tag{2.5}
\end{equation*}
$$

We now develop more general properties by means of the polynomials associated with the numbers in lambda triangles.

## 3. ABSTRACT LAMBDA TRIANGLES

Kappraff's array (Table 1) may be readily abstracted and extended as in Table 5 ( $a, b$ integers $>0$ ):

TABLE 5. Abstract Lambda Triangle


The abstract lambda polynomials $\mathscr{L}_{m}(x)$ (where $\mathscr{L}_{1}(x)=1$ ) may be easily read off from the rows of Table 5. To illustrate the situation we have

$$
\mathscr{L}_{5}(x)=a^{4}+a^{3} b x+a^{2} b^{2} x^{2}+a b^{3} x^{3}+b^{4} x^{4}=\frac{a^{5}-b^{5} x^{5}}{a-b x}
$$

Interchanging $a$ and $b$, we get the abstract reciprocal lambda polynomials $l_{m}(x)$ (with $l_{1}(x)=1$ ).
Recurrence relations are, respectively,

$$
\begin{equation*}
\mathscr{L}_{m+2}(x)=(a+b x) \mathscr{L}_{m+1}(x)-a b x \mathscr{L}_{m}(x), \tag{3.1}
\end{equation*}
$$

$$
\begin{equation*}
l_{m+2}(x)=(b+a x) l_{m+1}(x)-a b x l_{m}(x) . \tag{3.2}
\end{equation*}
$$

Generating functions are, respectively,

$$
\begin{gather*}
\sum_{m=1}^{\infty} \mathscr{L}_{m}(x) y^{m-1}=\left\{1-\left[(a+b x) y-a b x y^{2}\right]\right\}^{-1}  \tag{3.3}\\
\left.\sum_{m=1}^{\infty} l_{m}(x) y^{m-1}=\left\{1-[b+a x) y-a b x y^{2}\right]\right\}^{-1} \tag{3.4}
\end{gather*}
$$

Properties of these polynomials may be developed to include, for example:
(i) Other fundamental features such as Binet forms, Simson's formulas, closed forms;
(ii) Convolutions $\mathscr{L}_{m}^{(k)}(x), l_{m}^{(k)}(x)$;
(iii) Rising and descending polynomials.

We do this in Section 4 by considering a case closer to the original lambda triangle, namely, when $a=n, b=n+1$.

## 4. GENERALIZED LAMBDA POLYNOMIALS

We consider generalized lambda polynomials, $\Lambda_{m}(x)$, and reciprocal lambda polynomials, $\lambda_{m}(x)$, associated with the generalized lambda triangle of Table 6 , which should be compared with Table 1.

## TABLE 6. Generalized Lambda Triangle



The two classes of polynomials are related by

$$
\begin{align*}
& \lambda_{m}(x)=x^{m-1} \Lambda_{m}\left(\frac{1}{x}\right), \\
& \Lambda_{m}(x)=x^{m-1} \lambda_{m}\left(\frac{1}{x}\right) . \tag{4.1}
\end{align*}
$$

## 4.1 $\Lambda_{m}(x)$ Polynomials

Basic properties of $\Lambda_{m}(x)$ are listed succinctly hereunder:

$$
\begin{align*}
& \Lambda_{0}(x)=0 \\
& \Lambda_{1}(x)=1 \\
& \Lambda_{2}(x)=n+(n+1) x \\
& \Lambda_{3}(x)=n^{2}+n(n+1) x+(n+1)^{2} x^{2}  \tag{4.2}\\
& \Lambda_{4}(x)=n^{3}+n^{2}(n+1) x+n(n+1)^{2} x^{2}+(n+1)^{3} x^{3} \\
& \Lambda_{5}(x)=n^{4}+n^{3}(n+1) x+n^{2}(n+1)^{2} x^{2}+n(n+1)^{3} x^{3}+(n+1)^{4} x^{4}
\end{align*}
$$

Setting $x=1, m>0$, we obtain the sequence of coefficient sums, thus (observe the binomial coefficients):

$$
\begin{equation*}
\left\{\Lambda_{m}(1)\right\}=\left\{1,2 n+1,3 n^{2}+3 n+1,4 n^{3}+6 n^{2}+4 n+1, \ldots\right\} . \tag{4.3}
\end{equation*}
$$

## Recurrence relations:

homogeneous:

$$
\begin{equation*}
\Lambda_{m+2}(x)=[n+(n+1) x] \Lambda_{m+1}(x)-n(n+1) x \Lambda_{m}(x) . \tag{4.4}
\end{equation*}
$$

inhomoegneous:

$$
\begin{equation*}
\Lambda_{m+1}(x)=n \Lambda_{m}(x)+[(n+1) x]^{m} \quad(m \geq 0) \tag{4.5}
\end{equation*}
$$

## Roots of characteristic equation:

$$
\begin{equation*}
n,(n+1) x . \tag{4.6}
\end{equation*}
$$

Closed form:

$$
\begin{equation*}
\Lambda_{m}(x)=\sum_{j=0}^{m-1} n^{j}[(n+1) x]^{m-1-j} . \tag{4.7}
\end{equation*}
$$

Binet form:

$$
\begin{equation*}
\Lambda_{m}(x)=\frac{[(n+1) x]^{m}-n^{m}}{(n+1) x-n} \tag{4.8}
\end{equation*}
$$

Simson's formula:

$$
\begin{equation*}
\Lambda_{m+1}(x) \Lambda_{m-1}(x)-\Lambda_{m}^{2}(x)=-[n(n+1) x]^{m-1} \quad(m \geq 1) . \tag{4.9}
\end{equation*}
$$

## Generating function:

$$
\begin{equation*}
\sum_{m=1}^{\infty} \Lambda_{m}(x) y^{m-1}=\left\{1-\left[(n+(n+1) x) y-n(n+1) x y^{2}\right]\right\}^{-1} \tag{4.10}
\end{equation*}
$$

### 4.2 Reciprocal $\lambda_{m}(x)$ Polynomials

$$
\begin{aligned}
& \lambda_{0}(x)=0 \\
& \lambda_{1}(x)=1 \\
& \lambda_{2}(x)=(n+1)+n x \\
& \lambda_{3}(x)=(n+1)^{2}+n(n+1) x+n^{2} x^{2} \\
& \lambda_{4}(x)=(n+1)^{3}+n(n+1)^{2} x+n^{2}(n+1) x^{2}+n^{3} x^{3} \\
& \lambda_{5}(x)=(n+1)^{4}+n(n+1)^{3} x+n^{2}(n+1)^{2} x^{2}+n^{3}(n+1) x+n^{4} x^{4} \\
& \ldots
\end{aligned}
$$

Setting $x=1, m>0$, we obtain the sequence of coefficient sums, thus (observe the binomial coefficients):

$$
\begin{equation*}
\left\{\lambda_{m}(1)\right\}=\left\{1,2 n+1,3 n^{2}+3 n+1,4 n^{3}+6 n^{2}+4 n+1, \ldots\right\}=\left\{\Lambda_{m}(1)\right\} . \tag{4.12}
\end{equation*}
$$

## Recurrence relations:

homogeneous:

$$
\begin{equation*}
\lambda_{m+2}(x)=[(n+1)+n x] \lambda_{m+1}(x)-n(n+1) x \lambda_{m}(x) . \tag{4.13}
\end{equation*}
$$

inhomogeneous:

$$
\begin{equation*}
\lambda_{m+1}(x)=(n+1) \lambda_{m}(x)+[(n+1) x]^{m} \quad(m \geq 0) . \tag{4.14}
\end{equation*}
$$

## Roots of characteristic equation:

$$
\begin{equation*}
n+1, n x \tag{4.15}
\end{equation*}
$$

## Closed form:

$$
\begin{equation*}
\lambda_{m}(x)=\sum_{j=0}^{m-1}(n+1)^{j}[n x]^{m-1-j} . \tag{4.16}
\end{equation*}
$$

## Binet form:

$$
\begin{equation*}
\lambda_{m}(x)=\frac{(n x)^{m}-(n+1)^{m}}{n x-(n+1)} . \tag{4.17}
\end{equation*}
$$

Simson's formula:

$$
\begin{equation*}
\lambda_{m+1}(x) \lambda_{m-1}(x)-\lambda_{m}^{2}(x)=-[n(n+1) x]^{m-1} \quad(m \geq 1) . \tag{4.18}
\end{equation*}
$$

## Generating function:

$$
\begin{equation*}
\sum_{m=1}^{\infty} \lambda_{m}(x) y^{m-1}=\left\{1-\left[(n+1+n x) y-n(n+1) x y^{2}\right]\right\}^{-1} \tag{4.19}
\end{equation*}
$$

## 5. RELATED POLYNOMIALS

In this section, polynomial properties of related convolutions and of rising and falling diagonals are sketched.

### 5.1 Convolutions

There are two types of lambda convolution polynomials which are related by

$$
\begin{align*}
& \lambda_{m}^{(k)}(x)=x^{m-1} \Lambda_{m}^{(k)}\left(\frac{1}{x}\right),  \tag{5.1}\\
& \Lambda_{m}^{(k)}(x)=x^{m-1} \lambda_{m}^{(k)}\left(\frac{1}{x}\right), \tag{5.2}
\end{align*}
$$

in which $\Lambda_{m}^{(k)}(x)$ and $\lambda_{m}^{(k)}(x)$ are the $k^{\text {th }}$ convolutions of $\Lambda_{m}(x)$ and $\lambda_{m}(x)$, respectively, and $\Lambda_{m}^{(k)}(x)$ is defined in terms of a generating function

$$
\begin{equation*}
\sum_{m=1}^{\infty} \Lambda_{m}^{(k)}(x) y^{m-1}=\left\{1-\left[(n+(n+1) x) y-n(n+1) x y^{2}\right]\right\}^{-(k+1)} \tag{5.3}
\end{equation*}
$$

whence we get the recurrence relation

$$
\begin{equation*}
\Lambda_{m}^{(k)}(x)=\Lambda_{m}^{(k+1)}(x)-(n+(n+1) x) \Lambda_{m-1}^{(k+1)}(x)+n(n+1) x \Lambda_{m-2}^{(k+1)}(x) \tag{5.4}
\end{equation*}
$$

For instance, when $k=1$ :

$$
\begin{align*}
& \Lambda_{0}^{(1)}(x)=0 \text { (definition) } \\
& \Lambda_{1}^{(1)}(x)=1 \\
& \Lambda_{2}^{(1)}(x)=2 n+2(n+1) x  \tag{5.5}\\
& \Lambda_{3}^{(1)}(x)=3 n^{2}+4 n(n+1) x+3(n+1)^{2} x^{2} \\
& \Lambda_{4}^{(1)}(x)=4 n^{3}+6 n^{2}(n+1) x+6 n(n+1)^{2} x^{2}+4(n+1)^{3} x^{3}
\end{align*}
$$

Analogously to (5.3) there is a generating function for $\lambda_{m}^{(k)}(x)$ with $n \leftrightarrow n+1$.
If we consider $\partial\left(\sum_{m=1}^{\infty} \Lambda_{m}^{(k)}(x) y^{m-1}\right) / \partial y$, then we get

$$
\begin{align*}
(m-1) \Lambda_{m}^{(k-1)}(x) & =k\{(n+(n+1) x)-2 n(n+1) x y\}  \tag{5.6}\\
& =k\left\{(n+n(n+1) x) \Lambda_{m-1}^{(k)}(x)-2 n(n+1) x \Lambda_{m-2}^{(k)}(x)\right\} . \tag{5.7}
\end{align*}
$$

Replace $k$ by $k-1$ in Equation (5.4):

$$
\begin{equation*}
\Lambda_{m}^{(k-1)}(x)=\Lambda_{m}^{(k)}(x)-(n+(n+1) x) \Lambda_{m-1}^{(k)}(x)+n(n+1) \Lambda_{m-2}^{(k)}(x) . \tag{5.8}
\end{equation*}
$$

Now eliminate $\Lambda_{m}^{(k)}(x)$ from (5.7) and (5.8) to get the recurrence

$$
(m-1) \Lambda_{m}^{(k)}(x)=[k+m-1](n+(n+1) x) \Lambda_{m-1}^{(k)}(x)-n(n+1)[2 k+m-1] \Lambda_{m-2}^{(k)}(x) .
$$

From this, with $k=1, m \rightarrow m+1$, we can get

$$
\begin{equation*}
m \Lambda_{m+1}^{(1)}(1)=(m+1)(2 n+1) \Lambda_{m}^{(1)}(1)-(m+2) n(n+1) \Lambda_{m-1}^{(1)}(1) . \tag{5.9}
\end{equation*}
$$

Let $n=2$ in Equation (5.9). Then

$$
\begin{equation*}
m \Lambda_{m+1}^{(1)}(1)=5(m+1) \Lambda_{m}^{(1)}(1)-6(m+2) \Lambda_{m-1}^{(1)}(1) . \tag{5.10}
\end{equation*}
$$

Notice that in $\left\{\Lambda_{m}^{(1)}(x)\right\}$ (reference (5.5) above) the numerical coefficients form a neat triangle as displayed in Table 7, in which the row sums are the tetrahedral numbers $\binom{n+3}{3}$ (that is, $1,4,10,20$, $35, \ldots$ ) and the rising diagonal sums belong to Sequence 1349 of [8] with general terms $\frac{1}{4}\binom{n+3}{3}, n$ odd, and $n(n+2)(n+4) / 24, n$ even.

## TABLE 7. Lambda Convolution Coefficients



### 5.2 Rising and Descending Polynomials

Denote the rising and descending polynomials of $\Lambda_{m}(x)$ and $\lambda_{m}(x)$ by $R_{m}(x)$ and $r_{m}(x)$ and $D_{m}(x)$ and $d_{m}(x)$, respectively. They are related, in each case, by the interchange of $n$ and $n+1$.

## $\Lambda_{m}(x)$ Rising

$$
\begin{align*}
& R_{1}(x)=1 \\
& R_{2}(x)=n \\
& R_{3}(x)=n^{2}+(n+1) x \\
& R_{4}(x)=n^{3}+n(n+1) x  \tag{5.11}\\
& R_{5}(x)=n^{4}+n^{2}(n+1) x+(n+1)^{2} x^{2} \\
& R_{6}(x)=n^{5}+n^{3}(n+1) x+n(n+1)^{2} x^{2}
\end{align*}
$$

Setting $n=2$ and $x=1$, we obtain the sequence

$$
\begin{equation*}
\left\{R_{m}(1)\right\}=\{1,2,7,14,37,74,175,350, \ldots\} \tag{5.12}
\end{equation*}
$$

## Recurrence relations:

homogeneous:

$$
\begin{align*}
R_{2 m+1}(x) & =\left[n^{2}+(n+1) x\right] R_{2 m-1}(x)-n^{2}(n+1) x R_{2 n-3}(x) \quad(m \geq 2),  \tag{5.13}\\
R_{2 m}(x) & =n R_{2 n-1}(x) \quad(m \geq 1) . \tag{5.14}
\end{align*}
$$

inhomogeneous:

$$
\begin{equation*}
R_{2 m+1}(x)=n R_{2 m}(x)+((n+1) x)^{m} \quad(m \geq 0) . \tag{5.15}
\end{equation*}
$$

$\lambda_{m}(x)$ Rising

$$
\begin{align*}
& r_{1}(x)=1 \\
& r_{2}(x)=n+1 \\
& r_{3}(x)=(n+1)^{2}+n x \\
& r_{4}(x)=(n+1)^{3}+n(n+1) x  \tag{5.16}\\
& r_{5}(x)=(n+1)^{4}+n(n+1)^{2} x+n^{2} x^{2} \\
& r_{6}(x)=(n+1)^{5}+n(n+1)^{3} x+n^{2}(n+1) x^{2}
\end{align*}
$$

Setting $n=2$ and $x=1$, we obtain the sequence

$$
\begin{equation*}
\left\{r_{m}(1)\right\}=\{1,3,11,33,103,309,935, \ldots\} . \tag{5.17}
\end{equation*}
$$

## Recurrence relations:

homogeneous:

$$
\begin{align*}
r_{2 m+1}(x) & =\left[(n+1)^{2}+n x\right] r_{2 m-1}(x)-n(n+1)^{2} x r_{2 m-3}(x) \quad(m \geq 2),  \tag{5.18}\\
r_{2 m}(x) & =(n+1) r_{2 m-1}(x) \quad(m \geq 1) \tag{5.19}
\end{align*}
$$

inhomogeneous:

$$
\begin{equation*}
r_{2 m+1}(x)=(n+1) r_{2 m}(x)+(n x)^{m} \quad(m \geq 0) \tag{5.20}
\end{equation*}
$$

Observe from (5.14) and (5.19) the link

$$
\begin{equation*}
(n+1) R_{2 m}(x) r_{2 m-1}(x)=n R_{2 m-1}(x) r_{2 m}(x) \quad(m \geq 1) \tag{5.21}
\end{equation*}
$$

A quasi-reciprocal relationship between $R_{m}(x)$ and $r_{m}(x)$ can be evolved subject to certain provisos regarding $n$ and $n+1$. For example,

$$
R_{5}(x)=x^{2} r_{5}\left(\frac{1}{x}\right) \text { if } n^{2} \rightarrow n, n+1 \rightarrow(n+1)^{2}
$$

Check for $r_{5}(x)$ and $R_{5}\left(\frac{1}{x}\right)$. Likewise, look at $R_{6}(x)$ and $r_{6}\left(\frac{1}{x}\right)$, and $r_{6}(x)$ and $R_{6}\left(\frac{1}{x}\right)$.
Patterns for $m$ odd and $m$ even emerge.

## $\Lambda_{m}(x)$ Descending

Clearly, $D_{m}(x)=n^{m-1}(1-(n+1) x)^{-1}$, so

$$
\begin{equation*}
D_{m}(x)=n D_{m-1}(x) \tag{5.22}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d D_{m}(x)}{d x}=(n+1) n^{m-1}(1-(n+1) x)^{-2} \tag{5.23}
\end{equation*}
$$

If

$$
D \equiv D(x, y) \equiv \sum_{m=1}^{\infty} D_{m}(x) y^{m-1}=(1-(n+1) x y)^{-1}
$$

then

$$
\begin{equation*}
\frac{\partial D / \partial y}{\partial D / \partial x}=\frac{x}{y} . \tag{5.24}
\end{equation*}
$$

## $\lambda_{m}(x)$ Descending

Obviously, $d_{m}(x)=(n+1)^{m-1}(1-n x)^{-1}$, so

$$
d_{m}(x)=(n+1) d_{m-1}(x)
$$

and

$$
\frac{d d_{m}(x)}{d x}=n(n+1)^{m-1}(1-n x)^{-2} .
$$

If

$$
d \equiv d(x, y) \equiv \sum_{n=1}^{\infty} d_{m}(x) y^{m-1}=(1-n x y)^{-1}
$$

then

$$
\frac{\partial d / \partial y}{\partial d / \partial x}=\frac{x}{y} .
$$

Hence,

$$
\frac{\partial D}{\partial x} \frac{\partial d}{\partial y}=\frac{\partial D}{\partial y} \frac{\partial d}{\partial x}
$$

and

$$
\begin{equation*}
\frac{\frac{\partial D_{m}(x)}{\partial x}}{\frac{\partial d_{m}(x)}{\partial x}}=\left(\frac{n}{n+1}\right)^{m-1}\left[\frac{1-n x}{1-(n+1) x}\right]^{2} . \tag{5.25}
\end{equation*}
$$

## Special Case

Putting $n=2$ in the results of Sections 4 and 5, we obtain the particular cases for the original configuration in Table 1.

Further investigation of rising and descending polynomials could be undertaken; for example, the establishment of closed summation forms for $R_{m}(x)$ and $r_{m}(x)$.

## 6. FIBONACCI-LAMBDA TRIANGLES

### 6.1 Fibonacci-Lambda Polynomials

Suppose now that we replace $\alpha$ and $b$ in Section 3 by $\alpha$ and $\beta$, respectively, where $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$. We then have a triangle whose row sums are, successively,

$$
\begin{array}{r}
1=\frac{\alpha^{1}-\beta^{1}}{\alpha-\beta}=F_{1} \\
\alpha+\beta=\frac{\alpha^{2}-\beta^{2}}{\alpha-\beta}=F_{2} \\
\alpha^{2}+\alpha \beta+\beta^{2}=\frac{\alpha^{3}-\beta^{3}}{\alpha-\beta}=F_{3}  \tag{6.1}\\
\alpha^{3}+\alpha^{2} \beta+\alpha \beta^{2}+\beta^{3}=\frac{\alpha^{4}-\beta^{4}}{\alpha-\beta}=F_{4}
\end{array}
$$

so that the $n^{\text {th }}$ row sums to

$$
F_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta} .
$$

The Fibonacci-lambda polynomials $\Phi_{m}(x)$ will then have the recurrence relations

$$
\begin{align*}
& \Phi_{m+2}(x)=(\alpha+\beta x) \Phi_{m+1}(x)-\alpha \beta x \Phi_{m}(x), \quad m \geq 0,  \tag{6.2}\\
& \Phi_{m+1}(x)=\alpha \Phi_{m}(x)+(\beta x)^{m}, \quad m \geq 0 . \tag{6.3}
\end{align*}
$$

The first few examples are

$$
\begin{align*}
& \Phi_{0}(x)=0 \\
& \Phi_{1}(x)=1 \\
& \Phi_{2}(x)=\alpha-\beta x \\
& \Phi_{3}(x)=\alpha^{2}+\alpha \beta x+\beta^{2} x^{2}  \tag{6.4}\\
& \Phi_{4}(x)=\alpha^{3}+\alpha^{2} \beta x+\alpha \beta^{2} x^{2}+\beta^{3} x^{3} \\
& \Phi_{5}(x)=\alpha^{4}+\alpha^{3} \beta x+\alpha^{2} \beta^{2} x^{2}+\alpha \beta^{3} x^{3}+\beta^{4} x^{4}
\end{align*}
$$

Clearly,

$$
\Phi_{m}(1)=F_{m} .
$$

## 6.2 "Fibonacci-Lucas Triangle"

To continue the Fibonacci theme in this section, we next form the triangle with elements $b_{i, j}$ (where $i$ refers to rows and $j$ to columns) defined by

$$
\begin{equation*}
b_{i, j}=b_{i-1, j}+b_{i-1, j-1}, \quad i \geq 2,0<j<i \tag{6.5}
\end{equation*}
$$

with boundary conditions

$$
\begin{equation*}
b_{i, 0}=F_{i+2}, i \geq 0 ; \quad b_{i, i}=L_{i+1}, i \geq 1 ; \quad b_{i, j}=0, j>i, \tag{6.6}
\end{equation*}
$$

in which $L_{n}=\alpha^{n}+\beta^{n}$ represents the Lucas numbers. This yields the formation in Table 8. Note that (6.5) and (6.6) lead to $b_{i, 1}=F_{i+3}=b_{i+1,0}, i \geq 1$.

TABLE 8. "Fibonacci-Lucas Triangle"

| 1 |  |  |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 2 | 3 |  |  |  |  |  |  |
| 3 | 5 | 4 |  |  |  |  |  |
| 5 | 8 | 9 | 7 |  |  |  |  |
| 8 | 13 | 17 | 16 | 11 |  |  |  |
| 13 | 21 | 30 | 33 | 27 | 18 |  |  |
| 21 | 34 | 51 | 63 | 60 | 45 | 29 |  |
| 34 | 55 | 85 | 114 | 123 | 105 | 74 | 47 |

This is termed a "Fibonacci-Lucas triangle" to distinguish it from the Fibonacci and Lucas triangles already $n$ the literature [7]. The vertical and sloping sides of this triangle clearly have Fibonacci and Lucas numbers as their elements, but there are other connections, too.

### 6.3 Difference Operators

Instead of considering sums along rows, diagonals, and columns, we here look at differences between rows and columns by means of the row and column difference operators defined by

$$
\begin{align*}
& \Delta_{r} b_{i, j}=b_{i+1, j}-b_{i, j},  \tag{6.7}\\
& \Delta_{c} b_{i, j}=b_{i, j+1}-b_{i, j} . \tag{6.8}
\end{align*}
$$

For example,

$$
\begin{aligned}
\Delta_{r} b_{i, 0} & =b_{i+1,0}-b_{i, 0} & & \text { by (6.7) } \\
& =F_{i+3}-F_{i+2} & & \text { by (6.6) } \\
& =F_{i+1} & & \\
& =b_{i-1,0} & & \text { by (6.6) } \\
& =b_{i, 1}-b_{i, 0} & & \text { by (6.5) and } b_{i-1,1}=b_{i, 0} \\
& =\Delta_{c} b_{i, 0} & & \text { by (6.8). }
\end{aligned}
$$

More generally, $\Delta_{r}, \Delta_{c}$ are commutative operations:

$$
\begin{align*}
\Delta_{r} \Delta_{c} b_{i, j} & =\Delta_{r}\left(b_{i, j+1}-b_{i, j}\right) & & \text { by (6.8) } \\
& =\left(b_{i+1, j+1}-b_{i, j+1}\right)=\left(b_{i+1, j}-b_{i, j}\right) & & \text { by (6.7) } \\
& =\left(b_{i+1, j+1}-b_{i+1, j}\right)-\left(b_{i, j+1}-b_{i, j}\right) & & \\
& =\Delta_{c} b_{i+1, j}-\Delta_{c} b_{i, j} & & \text { by (6.8) }  \tag{6.8}\\
& =\Delta_{c} \Delta_{r} b_{i, j} & & \text { by (6.7). }
\end{align*}
$$

Other results can be investigated. For instance,

$$
\begin{equation*}
\Delta_{r}^{j} b_{i, j}=F_{i+2} . \tag{6.9}
\end{equation*}
$$

We can prove (6.9) by means of mathematical induction on $i$ and $j$.
By reversing the columns in Table 8 (that is, by making the Lucas numbers the left-hand exterior sloping side), one can also study these and other properties for a "Lucas-Fibonacci triangle"; this is a topic for further research. Are there, one might ask, any interesting relationships between the "Fibonacci-Lucas" and the "Lucas-Fibonacci" triangles?

## 7. CONCLUSION

### 7.1 Binary Extensions

These lambda-type triangles can be extended indefinitely. For instance, we can construct a triangle of binary numbers as in Table 9.

## TABLE 9. Binary Triangle

| 1 |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- |
| 10 | 11 |  |  |  |
| 100 | 101 | 111 |  |  |
| 1000 | 1001 | 1011 | 1111 |  |
| 10000 | 10001 | 10011 | 10111 | 11111 |

### 7.2 Determinants

Two other properties which are commonly examined are the values of corresponding determinants and modular arrays. The original left- and right-triangular lambda matrices (in Tables 2 and 3) have determinants with values which are powers of 3 and 2 , respectively.

### 7.3 Modular Triangles

The displays in Tables 10 and 11 represent the original extended lambda triangle (Table 1) modulo 5 and modulo 7, respectively. Table 10 has symmetry in its odd rows and Table 11 has neat patterns of cycles. Further research could involve seeking a modulus which could produce remainders to develop specific patterns such as Sirpinski triangles [9], arrowhead curves [7], or the partitioning of the triangles into square arrays [3].

TABLE 10. Lambda Triangle Modulo 5


TABLE 11. Lambda Triangle Modulo 7


### 7.4 Ongoing Research

The purpose of this paper has been to explore some of the properties associated with the lambda triangle. In doing so, several ideas for further research have been suggested for the interested reader. Finally, in this spirit, one might extend the previous knowledge through negative numbers, that is, start with $-2,-4,-8, \ldots$ and $-3,-9,-27, \ldots$ (as in Table 1 with common vertex 1). All this has no physical or artistic relation to our original Timaeus panel. Indeed it is a world away from Plato and Michelangelo.

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## REPORT ON THE TENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

The Tenth International Conference on Fibonacci Numbers and Their Applications held at Northern Arizona University in Flagstaff, Arizona, from June 24-28, 2002, found over 70 enthusiastic Fibonacci number lovers from Australia, Canada, England, Germany, Italy, Japan, Mexico, New Zealand, Poland, Romania, Scotland, and the USA gathered together to hear over 50 excellent presentations. The gathering was attended by both old and new Fibonacci friends, but it was sadly noted that several regulars were unable to be with us this year. They were both warmly remembered and greatly missed. A special thanks to organizer Cal Long and all the folks at Northern Arizona University for their hospitality and generosity in hosting this outstanding conference.

Monday through Wednesday morning found us savoring a variety of talks on things theoretical, operational, and applicable of a Fibonacci and related nature, with members sharing ideas while renewing old friendships and forming new ones.

Later on Wednesday the group was doubly treated. After the morning talks, we were entertained by mathemagician Art Benjamin's most impressive presentation; displaying his skills and cleverness by mentally performing challenging mathematical manipulations and zapping out magic squares as if (yes!) by magic. After graciously sharing some of the secrets of his wizardry with us, he dazzled one and all by mentally and accurately multiplying two five-place numbers to terminate his mesmerizing performance.

That afternoon we were bussed to our second wonder of the day: The Grand Canyon. Here we were able to spend several hours gazing at nature's wondrous spectacle. Oh to be a condor for an hour! In the evening a steak dinner was catered for us as we exchanged social and mathematical dialog to the background of exquisite scenic wonder at the edge of the Canyon. On the way back to the campus, we were able to witness a magnificent display of stars but an arm length away in the clear Arizona night sky.

On Thursday and Friday it was back to many more interesting, informative presentations and during the breaks we were treated to Peter Anderson's marvelous computer display of the many photographs he took of association members and their families enjoying the Canyon.

The closing banquet on Friday night terminated with a special tribute to Calvin T. Long for his very distinguished career of 50 years as teacher, mentor, and researcher, as well as valued friend, contributor to, and supporter of The Fibonacci Association. He was both praised and roasted by President Fred T. Howard and former editor Gerald E. Bergum. After much laughter and tears, Cal received a standing ovation from this proud and grateful group of his friends and colleagues.

After over an hour of cordial good-byes, everyone eventually drifted away vowing that, Lord willing, we'll all meet again in Braunschweig, Germany, in 2004.

Charles K. Cook

# COMBINATORIAL MATRICES AND LINEAR RECURSIVE SEQUENCES 

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## 1. INTRODUCTION

Various authors (see, e.g., [5], [7], [16], [17]) have studied number theoretic properties associated with the matrix $S(n)$, defined in effect by

$$
\begin{equation*}
S(n)=\left[s_{i, j}\right]_{n \times n} \tag{1.1}
\end{equation*}
$$

where

$$
\begin{equation*}
s_{i, j}(n) \equiv s_{i, j}=(-1)^{n-i}\binom{j-1}{n-i} p^{i+j-n-1} q^{n-i} \tag{1.2}
\end{equation*}
$$

where $p, q$ are arbitrary integers. These properties have generally been in the context of secondorder linear recursive sequences, particularly the Fibonacci numbers. We note that, for Horadam's generalized sequence $\left\{w_{n}\right\}=\left\{w_{n}(a, b ; p, q)\right\}$ [13], we have the recurrence relation

$$
\begin{equation*}
w_{n}=p w_{n-1}-q w_{n-2}, \quad n \geq 2 \tag{1.3}
\end{equation*}
$$

with initial conditions $w_{0}=a, w_{1}=b$. For the matrix $S$, we have the comparable partial recurrence relation

$$
\begin{equation*}
s_{i, j}=p s_{i, j-1}-q s_{i+1, j-1} \tag{1.4}
\end{equation*}
$$

We define the combinatorial matrix [2]: $S_{p, q}(n ; 2)=\left[\left|s_{i, j}(n)\right|\right]_{n \times n}$.
The purpose of this paper is to show how higher-order sequences arise quite naturally from $S(n)$ and to suggest problems for analogous further research arising out of further generalizations of the binomial coefficients. For notational purposes, we consider $S_{p, q}(n ; r)$, where $S_{p, q}(n ; 2)=$ $S(n)$ above, and for simplicity we take the absolute values of the numbers in the cells of each matrix.

## 2. PRELIMINARY OBSERVATIONS

We now have

$$
S_{1,-1}(7 ; 2)=\left(\begin{array}{rrrrrrr}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 6 \\
0 & 0 & 0 & 0 & 1 & 5 & 15 \\
0 & 0 & 0 & 1 & 4 & 10 & 20 \\
0 & 0 & 1 & 3 & 6 & 10 & 15 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)
$$

We observe that the falling diagonal sums are the Fibonacci numbers $\{1,1,2,3,5,8,13\}$ and the rising diagonal sums are the binomial coefficients $\{7,21,35,21,7,1\}$. Similarly,

$$
S_{2,-1}(7 ; 2)=\left(\begin{array}{rrrrrrr}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 12 \\
0 & 0 & 0 & 0 & 1 & 10 & 60 \\
0 & 0 & 0 & 1 & 8 & 40 & 160 \\
0 & 0 & 1 & 6 & 24 & 80 & 240 \\
0 & 1 & 4 & 12 & 32 & 80 & 192 \\
1 & 2 & 4 & 8 & 16 & 32 & 64
\end{array}\right)
$$

Other generalizations can be pursued. For instance,

$$
\begin{equation*}
S_{2^{k},-1}^{\prime} S_{2^{k},-1}=S_{2^{k+1},-1}, \quad k \geq 0 \tag{2.1}
\end{equation*}
$$

where

$$
S_{2^{k},-1}^{\prime}=S_{2^{k},-1} E
$$

in which $E$ is the elementary (self-inverse) matrix

$$
\begin{gathered}
E=\left[e_{i, j}\right]_{n \times n} \\
e_{i, j}= \begin{cases}1 & \text { if } j=n+1 \\
0 & \text { otherwise }\end{cases}
\end{gathered}
$$

$E$ is the unit matrix with rows reversed. It is used again in Section 5. An example of (2.1) when $k=1$ is

$$
\left(\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
6 & 1 & 0 & 0 \\
12 & 4 & 1 & 0 \\
8 & 4 & 2 & 1
\end{array}\right)\left(\begin{array}{rrrr}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 6 \\
0 & 1 & 4 & 12 \\
1 & 2 & 4 & 8
\end{array}\right)=\left(\begin{array}{rrrr}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 12 \\
0 & 1 & 8 & 48 \\
1 & 4 & 16 & 64
\end{array}\right)
$$

The falling (from left to right) diagonal sums in these matrices are generalized Pell numbers, $\left\{P_{n}\right\}$, defined in turn by the second-order linear recurrence relations

$$
\begin{equation*}
P_{n}=2^{k} P_{n-1}+P_{n-2}, \quad n \geq 2, k \geq 0 \tag{2.2}
\end{equation*}
$$

with initial conditions $P_{0}=0, P_{1}=1$. When $k=0,1$, we have the ordinary Fibonacci and Pell numbers, respectively.

In what follows, we use Bondarenko's notation $\binom{n}{m}_{r}$ for the number of different ways of distributing $m$ objects among $n$ cells where each cell may contain at most $r-1$ objects [3]:

$$
\begin{aligned}
& \binom{n}{0}_{r}=\binom{n}{r-1}_{r}=1, \\
& \binom{n}{m}_{r}=\binom{n}{(r-1) n-m}_{r}, \\
& \binom{n}{m}_{r}= \begin{cases}0, & n<0, m<0, \text { or } m>(r-1) n \\
1, & n=m=0\end{cases} \\
& \binom{n}{m}_{r}=\sum_{i=1}^{r}\binom{n-1}{m-i+1}_{r}
\end{aligned}
$$

## 3. THE $A$ AND $S$ MATRICES

We define the $A$ and $S$ matrices by

$$
\begin{equation*}
A(n ; r)=\left[\binom{n-j}{j-i}_{r}\right] \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
S(n ; r)=\left[\binom{j-1}{n-i}_{r}\right] \tag{3.2}
\end{equation*}
$$

For related developments, see [4], [8], [18]. As examples, we now look at $S(7 ; 2) \equiv S_{1,-1}(7 ; 2)$ and the associated matrix $A(7 ; 2)$,

$$
A(7 ; 2)=\left(\begin{array}{lllllll}
1 & 5 & 6 & 1 & 0 & 0 & 0 \\
0 & 1 & 4 & 3 & 0 & 0 & 0 \\
0 & 0 & 1 & 3 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Then it is readily verified that

$$
\begin{equation*}
A(7 ; 2) S(7 ; 2)=S(7 ; 3) \tag{3.3}
\end{equation*}
$$

where

$$
S(7 ; 3)=\left(\begin{array}{rrrrrrr}
0 & 0 & 0 & 1 & 10 & 45 & 141 \\
0 & 0 & 0 & 3 & 16 & 51 & 126 \\
0 & 0 & 1 & 6 & 19 & 45 & 90 \\
0 & 0 & 2 & 7 & 16 & 30 & 50 \\
0 & 1 & 3 & 6 & 10 & 15 & 21 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)
$$

with falling diagonal sums $\{1,1,2,4,7,13,24\}$ which is a subset of the set of $n$-step self-avoiding walks on a Manhattan lattice, and the elements satisfy the linear third-order recurrence relation $u_{n}=u_{n-1}+u_{n-2}+u_{n-3}, n \geq 3$, with $u_{0}=0, u_{1}=1, u_{2}=1$ (see [21]). Next, let

$$
A(7 ; 3)=\left(\begin{array}{rrrrrrr}
1 & 5 & 10 & 7 & 1 & 0 & 0 \\
0 & 1 & 4 & 6 & 2 & 0 & 0 \\
0 & 0 & 1 & 3 & 3 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

and

$$
S(7 ; 4)=\left(\begin{array}{rrrrrrr}
0 & 0 & 1 & 10 & 44 & 135 & 336 \\
0 & 0 & 2 & 12 & 40 & 101 & 216 \\
0 & 0 & 3 & 12 & 31 & 65 & 120 \\
0 & 1 & 4 & 10 & 20 & 35 & 56 \\
0 & 1 & 3 & 6 & 10 & 15 & 21 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right) .
$$

Then

$$
\begin{equation*}
A(7 ; 3) S(7 ; 2)=S(7 ; 4) \tag{3.4}
\end{equation*}
$$

More generally,
Theorem 1: $A(n ; r) S(n ; 2)=S(n ; r+1)$.
Proof:

$$
\begin{aligned}
A(n ; r) S(n ; 2) & =\left[\sum_{k=1}^{n}\binom{n-k}{k-i}_{r}\binom{j-1}{n-k}\right] \\
& =\left[\sum_{k=0}^{n}\binom{n-k}{k-i}_{r}\binom{j-1}{n-k}\right] \quad \text { from the definition of }\binom{n}{m}_{r}, \\
& \left.=\left[\sum_{k=0}^{n}\binom{k}{n-k-i}\right)_{r}\binom{j-1}{k}\right] \text { reversing the order of summation, } \\
& =\left[\binom{j-1}{n-i}_{r+1}\right] \quad \text { from Equation (1.15) of }[3], \\
& =S(n ; r+1) .
\end{aligned}
$$

The elements of $S$ and $A$ can be rearranged to form generalized Pascal triangles (see [19], [22], [25]). They can also be made into tetrahedrons with Pascal's triangle as one section (see [11], [12], [21]). Ericksen [9] has elaborated the principal properties of Bondarenko's coefficients in a pyramid.

## 4. RECURSIVE SEQUENCES

The rising diagonal sums associated with each of the $r^{\text {th }}$ rows in the triangles of Section 3 yield the Fibonacci sequences and their generalizations; that is, the rising diagonals associated with the combined second rows yield the Fibonacci numbers. We can express this by the following theorem.

## Theorem 2:

$$
\sum_{k=0}^{\lfloor(r-1) n / r\rfloor}\binom{n-k}{k}_{r}=U_{n+1}
$$

in which $\left\{U_{n}\right\}$ is the generalized Fibonacci sequence of arbitrary order $r$ defined by the recurrence relation

$$
U_{n}=\sum_{j=1}^{r} U_{n-j}, n>1,
$$

with initial conditions $U_{-n}=0, n=0,1,2, \ldots, r-2, U_{1}=1$.
Proof: Consider

$$
d_{n}=\sum_{k=0}^{n}\binom{n-k}{k}_{r}=\sum_{k=0}^{\lfloor(r-1) n / r\rfloor}\binom{n-k}{k}_{r}
$$

from a consideration of the zero terms in the upper portion of the $\binom{n}{m}_{r}$ array. Then $d_{0}=1, d_{n}=0$ for $n<0$ and, for $n>0$,

$$
\begin{aligned}
d_{n} & =\sum_{k=0}^{n} \sum_{i=1}^{r}\binom{n-k-1}{k-i+1}_{r}=\sum_{i=1}^{r} \sum_{k=0}^{n}\binom{n-k-1}{k-i+1}_{r} \\
& =\sum_{i=1}^{r} \sum_{k=-i+1}^{n-i+1}\binom{n-i-k}{k}_{r} \text { changing the summation index to } j=k-i+1 \text { then reverting to } k, \\
& =\sum_{i=1}^{r} \sum_{k=0}^{n-i}\binom{n-i-k}{k}_{r} \quad \text { using the boundary conditions, } \\
& =\sum_{i=1}^{r} d_{n-i} .
\end{aligned}
$$

Thus, $d_{n}$ satisfies the generalized Fibonacci recurrence relation of arbitrary order $r$ with the given initial conditions.

Basically, this theorem says that each element in the $\binom{n}{m}_{r}$ array is the sum of $r$ elements above and to the left of it, and that $r$ consecutive diagonals are needed to obtain all the terms required to form the elements of the next diagonal.

When $r=2$, the theorem reduces to a familiar expression for the Fibonacci numbers, namely,

$$
\begin{equation*}
F_{n+1}=\sum_{m=0}^{\lfloor n / 2\rfloor}\binom{n-m}{m} \tag{4.1}
\end{equation*}
$$

and when $r=3$, we get equation (4.1) of [21]:

$$
\begin{equation*}
U_{n+1}=\sum_{m=0}^{\lfloor n / 2\rfloor} \sum_{j=0}^{\lfloor n / 3\rfloor}\binom{n-m-j}{m+j}\binom{m+j}{j}=\sum_{k=0}^{\lfloor n / 2\rfloor}\binom{n-k}{k}_{3} . \tag{4.2}
\end{equation*}
$$

## 5. INVERSE MATRICES

The inverse matrices have some neat properties. For instance, for absolute values of the entries, we have

$$
\begin{equation*}
S^{-1}=E S E \tag{5.1}
\end{equation*}
$$

where $E$ is the elementary matrix defined in Section 2. Of more interest is

$$
A_{1,-1}^{-1}(7 ; 2)=\left(\begin{array}{rrrrrrr}
1 & -5 & 14 & -28 & 42 & -42 & 0  \tag{5.2}\\
0 & 1 & -4 & 9 & -14 & 14 & 0 \\
0 & 0 & 1 & -3 & 5 & -5 & 0 \\
0 & 0 & 0 & 1 & -2 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

The absolute values of the elements of the columns of $A^{-1}$ can be rearranged to form the rows of Table 1 . The row and column headed $M$ refer to the corresponding sequence in Sloane and Plouffe [23].

The elements $a_{i, j}$ in Table 1 satisfy the partial recurrence relation

$$
a_{i, j}=a_{i-1, j}+a_{i+1, j-1}, i, j \geq 1
$$

with boundary conditions

$$
a_{i, 0}=1, \quad a_{0, j}=\frac{\binom{2 j}{j}}{j+1}=c_{j},
$$

a Catalan number [14]. A general solution of this is given by

$$
a_{i, j}=\frac{i+1}{1+j+1}\binom{2 j+1}{j} .
$$

TABLE 1. Elements of the Inverse Associated Matrix

| $i \backslash j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |  | $M$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | 1 | 2 | 5 | 14 | 42 | 132 |  | 1459 |
| 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1459 |  |
| 2 | 1 | 3 | 9 | 28 | 90 | 297 | 1001 | 2809 |  |
| 3 | 1 | 4 | 14 | 48 | 165 | 572 | 2002 |  | 3483 |
| 4 | 1 | 5 | 20 | 75 | 275 | 1001 | 3640 |  | 3904 |
| 5 | 1 | 6 | 27 | 110 | 429 | 1638 | 6188 | 4177 |  |
| 6 | 1 | 7 | 35 | 154 | 637 | 2548 | 9996 | 4413 |  |
|  |  |  |  |  |  |  |  |  |  |
| $M$ |  | 1356 | 3841 | 4929 | 5277 | - |  |  |  |

Note that the rising diagonals in Table 1 generate the Catalan numbers. The elements in Table 1 correspond to the number of two element lattice permutations, where the permutation represents a path through a lattice where the path does not cross a diagonal [6]. Since there are some intersections among the sequences in Table 1, a topic for further research could be to consider if these are the only intersections (cf. [24]).

Bondarenko's generalization of the binomial coefficient takes no account of the order across or within cells. Further research could accommodate this order and then apply these extensions to other combinatorial applications along the lines of the work of Letac and Takács [15] who, in effect, related the permutations associated with Bondarenko's $\binom{3}{m}_{3}$ to random walks along the edges of a dodecahedron or the connections of combinatorial matrices to planar networks [10]. Such research should lead to generalizations of the Fibonacci sequence which would be different from the $\left\{U_{n}\right\}$ discussed here and the standard generalizations of Philippou and his colleagues [20].

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# ON CHEBYSHEV POLYNOMIALS AND FIBONACCI NUMBERS* 

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## 1. INTRODUCTION AND RESULTS

As usual, Chebyshev polynomials of the first and second kind, $T(x)=\left\{T_{n}(x)\right\}$ and $U(x)=$ $\left\{U_{n}(x)\right\}(n=0,1,2, \ldots)$, are defined by the second-order linear recurrence sequences

$$
\begin{equation*}
T_{n+2}(x)=2 x T_{n+1}(x)-T_{n}(x) \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
U_{n+2}(x)=2 x U_{n+1}(x)-U_{n}(x) \tag{2}
\end{equation*}
$$

for $n \geq 0, T_{0}(x)=1, T_{1}(x)=x, U_{0}(x)=1$, and $U_{1}(x)=2 x$. These polynomials play a very important role in the study of the orthogonality of functions (see [1]), but regarding their arithmetical properties, we know very little at present. We do not even know whether there exists any relation between Chebyshev polynomials and some famous sequences. In this paper, we want to prove some identities involving Chebyshev polynomials, Lucas numbers, and Fibonacci numbers. For convenience, we let $T_{n}^{(k)}(x)$ and $U_{n}^{(k)}(x)$ denote the $k^{\text {th }}$ derivatives of $T_{n}(x)$ and $U_{n}(x)$ with respect to $x$. Then we can use the generating functions of the sequences $T_{n}(x)$ and $U_{n}(x)$, and their partial derivatives, to prove the following three theorems.

Theorem 1: Let $U_{n}(x)$ be defined by (2). Then, for any positive integer $k$ and nonnegative integer $n$, we have the identity

$$
\sum_{a_{1}+a_{2}+\cdots+a_{k+1}=n} \prod_{i=1}^{k+1} U_{a_{i}}(x)=\frac{1}{2^{k} \cdot k!} U_{n+k}^{(k)}(x)
$$

where the summation is over all $k+1$-dimension nonnegative integer coordinates $\left(a_{1}, a_{2}, \ldots, a_{k+1}\right)$ such that $a_{1}+a_{2}+\cdots+a_{k+1}=n$.

Theorem 2: Under the conditions of Theorem 1, we have

$$
\sum_{a_{1}+\cdots+a_{k+1}=n+2 k+2} \prod_{i=1}^{k+1}\left(a_{i}+1\right) U_{a_{i}}(x)=\frac{1}{2^{2 k+1} \cdot(2 k+1)!} \sum_{h=0}^{k+1}(-1)^{h}\binom{k+1}{h} U_{n+4 k+3-2 h}^{(2 k+1)}(x)
$$

where $\binom{k}{h}=\frac{k!}{h!(k-h)!}$.
Theorem 3: Under the conditions of Theorem 1, we also have

$$
\sum_{a_{1}+\cdots+a_{k+1}=n+k+1} \prod_{i=1}^{k+1} T_{a_{i}}(x)=\frac{1}{2^{k} \cdot k!} \sum_{h=0}^{k+1}(-x)^{h}\binom{k+1}{h} U_{n+2 k+1-h}^{(k)}(x)
$$

From these theorems, we may immediately deduce the following corollaries.
Corollary 1: Let $F_{n}$ be the $n^{\text {th }}$ Fibonacci number. Then, for any positive integer $k$ and nonnegative integer $n$, we have the identities:

[^1]\[

$$
\begin{aligned}
& \sum_{a_{1}+a_{2}+\cdots+a_{k+1}=n} F_{a_{1}+1} \cdot F_{a_{2}+1} \cdots \cdots \cdot F_{a_{k+1}+1}=\frac{(-i)^{n}}{2^{k} \cdot k!} U_{n+k}^{(k)}\left(\frac{i}{2}\right), \\
& \sum_{a_{1}+a_{2}+\cdots+a_{k+1}=n} F_{2\left(a_{1}+1\right)} \cdot F_{2\left(a_{2}+1\right)} \cdots \cdot F_{2\left(a_{k+1}+1\right)}=\frac{(-1)^{n}}{2^{k} \cdot k!} U_{n+k}^{(k)}\left(\frac{-3}{2}\right), \\
& \sum_{a_{1}+a_{2}+\cdots+a_{k+1}=n} F_{3\left(a_{1}+1\right)} \cdot F_{3\left(a_{2}+1\right)} \cdots \cdots F_{3\left(a_{k+1}+1\right)}=\frac{2 i^{n}}{k!} U_{n+k}^{(k)}(-2 i),
\end{aligned}
$$
\]

where $i^{2}=-1$. In particular, for $k=2$, we have the identities:

$$
\begin{aligned}
& \sum_{a+b+c=n} F_{a+1} \cdot F_{b+1} \cdot F_{c+1}=\frac{1}{50}\left[(n+2)(5 n+17) F_{n+3}-6(n+3) F_{n+2}\right], \\
& \sum_{a+b+c=n} F_{2(a+1)} \cdot F_{2(b+1)} \cdot F_{2(c+1)}=\frac{1}{50}\left[18(n+3) F_{2 n+4}+(n+2)(5 n-7) F_{2 n+6}\right], \\
& \sum_{a+b+c=n} F_{3(a+1)} \cdot F_{3(b+1)} \cdot F_{3(c+1)}=\frac{1}{50}\left[(n+2)(5 n+8) F_{3 n+9}-6(n+3) F_{3 n+6}\right] .
\end{aligned}
$$

Corollary 2: Under the conditions of Corollary 1, we have:

$$
\begin{aligned}
& \quad \sum_{a_{1}+\cdots+a_{k+1}=n+2 k+2}\left(a_{1}+1\right) \cdots\left(a_{k+1}+1\right) \cdot F_{a_{1}+1} \cdots F_{a_{k+1}+1} \\
& =\frac{(-i)^{n+2 k+2}}{2^{2 k+1} \cdot(2 k+1)!} \sum_{h=0}^{k+1}(-1)^{h}\binom{k+1)}{h} U_{n+4 k+3-2 h}^{(2 k+1)}\left(\frac{i}{2}\right), \\
& \\
& \sum_{a_{1}+\cdots+a_{k+1}=n+2 k+2}\left(a_{1}+1\right) \cdots\left(a_{k+1}+1\right) \cdot F_{2\left(a_{1}+1\right)} \cdots F_{2\left(a_{k+1}+1\right)} \\
& =\frac{(-1)^{n}}{2^{2 k+1} \cdot(2 k+1)!} \sum_{h=0}^{k+1}(-1)^{h}\binom{k+1}{h} U_{n+4 k+3-2 h}^{(2 k+1)}\left(\frac{-3}{2}\right), \\
& \\
& \sum_{a_{1}+\cdots+a_{k+1}=n+2 k+2}\left(a_{1}+1\right) \cdots\left(a_{k+1}+1\right) \cdot F_{3\left(a_{1}+1\right)} \cdots F_{3\left(a_{k+1}+1\right)} \\
& =\frac{i^{n+2 k+2}}{2^{k} \cdot(2 k+1)!} \sum_{h=0}^{k+1}(-1)^{h}\binom{k+1}{h} U_{n+4 k+3-2 h}^{(2 k+1)}(-2 i) .
\end{aligned}
$$

Corollary 3: Let $L_{n}$ be the $n^{\text {th }}$ Lucas numbers. Then, for any positive integer $k$ and nonnegative integer $n$, we have the identities:

$$
\begin{aligned}
& \sum_{a_{1}+\cdots+a_{k+1}=n+k+1} L_{a_{1}} \cdot L_{a_{2}} \cdots \cdot L_{a_{k+1}}=\frac{(-i)^{n+k+1}}{2^{-1} \cdot k!} \sum_{h=0}^{k+1}\left(\frac{-i}{2}\right)^{h}\binom{k+1}{h} U_{n+2 k+1-h}^{(k)}\left(\frac{i}{2}\right), \\
& \sum_{a_{1}+\cdots+a_{k+1}=n+k+1} L_{2 a_{1}} \cdot L_{2 a_{2}} \cdots \cdot L_{2 a_{k+1}}=\frac{(-i)^{n+k+1}}{2^{-1} \cdot k!} \sum_{h=0}^{k+1}\left(\frac{3}{2}\right)^{h}\binom{k+1}{h} U_{n+2 k+1-h}^{(k)}\left(\frac{-3}{2}\right), \\
& \sum_{a_{1}+\cdots+a_{k+1}=n+k+1} L_{3 a_{1}} \cdot L_{3 a_{2}} \cdots \cdot L_{3 a_{k+1}}=\frac{i^{n+k+1}}{2^{-1} \cdot k!} \sum_{h=0}^{k+1}(2 i)^{h}\binom{k+1}{h} U_{n+2 k+1-h}^{(k)}(-2 i),
\end{aligned}
$$

where $i^{2}=-1$. In particular, for $k=2$, we have the identities:

$$
\begin{aligned}
& \sum_{a+b+c=n+3} L_{a} \cdot L_{b} \cdot L_{c}=\frac{n+5}{2}\left[(n+10) F_{n+3}+2(n+7) F_{n+2}\right], \\
& \sum_{a+b+c=n+3} L_{2 a} \cdot L_{2 b} \cdot L_{2 c}=\frac{n+5}{2}\left[3(n+10) F_{2 n+5}+(n+16) F_{2 n+4}\right], \\
& \sum_{a+b+c=n+3} L_{3 a} \cdot L_{3 b} \cdot L_{3 c}=\frac{n+5}{2}\left[4(n+10) F_{3 n+7}+3(n+9) F_{3 n+6}\right] .
\end{aligned}
$$

Corollary 4: For any nonnegative integer $n$, we have the congruence

$$
(n+2)(5 n+8) F_{3 n+9} \equiv 6(n+3) F_{3 n+6} \bmod 400 .
$$

These corollaries are generalizations of [2].

## 2. PROOF OF THE THEOREMS

In this section we shall complete the proofs of the theorems. First, note that (see [1], (2.1.1))

$$
T_{n}(x)=\frac{1}{2}\left[\left(x+\sqrt{x^{2}-1}\right)^{n}+\left(x-\sqrt{x^{2}-1}\right)^{n}\right]
$$

and

$$
U_{n}(x)=\frac{1}{2 \sqrt{x^{2}-1}}\left[\left(x+\sqrt{x^{2}-1}\right)^{n+1}-\left(x-\sqrt{x^{2}-1}\right)^{n+1}\right],
$$

so we can easily deduce that the generating function of $T(x)$ and $U(x)$ are

$$
\begin{equation*}
G(t, x)=\frac{1-x t}{1-2 x t+t^{2}}=\sum_{n=0}^{+\infty} T_{n}(x) \cdot t^{n} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
F(x, t)=\frac{1}{1-2 x t+t^{2}}=\sum_{n=0}^{+\infty} U_{n}(x) \cdot t^{n}, \tag{4}
\end{equation*}
$$

respectively. Then from (4) we have

$$
\begin{gather*}
\frac{\partial F(t, x)}{\partial x}=\frac{2 t}{\left(1-2 x t+t^{2}\right)^{2}}=\sum_{n=0}^{\infty} U_{n+1}^{(1)}(x) \cdot t^{n+1}, \\
\frac{\partial^{2} F(t, x)}{\partial x^{2}}=\frac{2!\cdot(2 t)^{2}}{\left(1-2 x t+t^{2}\right)^{3}}=\sum_{n=0}^{\infty} U_{n+2}^{(2)}(x) \cdot t^{n+2},  \tag{5}\\
\frac{\partial^{k} F(t, x)}{\partial x^{k}}=\frac{k!\cdot(2 t)^{k}}{\left(1-2 x t+t^{2}\right)^{k+1}}=\sum_{n=0}^{\infty} U_{n+k}^{(k)}(x) \cdot t^{n+k},
\end{gather*}
$$

where we have used the fact that $U_{n}(x)$ is a polynomial of degree $n$.
Therefore, from (5) we get

$$
\begin{align*}
& \sum_{n=0}^{\infty}\left(\sum_{a_{1}+\cdots+a_{k+1}=n} U_{a_{1}}(x) \cdot U_{a_{2}}(x) \cdots \cdot U_{a_{k+1}}(x)\right) \cdot t^{n}=\left(\sum_{n=0}^{\infty} U_{n}(x) \cdot t^{n}\right)^{k+1}  \tag{6}\\
& =\frac{1}{\left(1-2 x t+t^{2}\right)^{k+1}}=\frac{1}{k!(2 t)^{k}} \frac{\partial^{k} F(t, x)}{\partial x^{k}}=\frac{1}{2^{k} \cdot k!} \sum_{n=0}^{\infty} U_{n+k}^{(k)}(x) \cdot t^{n} .
\end{align*}
$$

Equating the coefficients of $t^{n}$ on both sides of equation (6), we obtain the identity

$$
\sum_{a_{1}+a_{2}+\cdots a_{k+1}=n} U_{a_{1}}(x) \cdot U_{a_{2}}(x) \cdots \cdot U_{a_{k+1}}(x)=\frac{1}{2^{k} \cdot k!} \cdot U_{n+k}^{(k)}(x) .
$$

This proves Theorem 1.
Now we prove Theorem 3. Multiplying both sides of (5) by $(1-x t)^{k+1}$ gives

$$
\begin{equation*}
\frac{(1-x t)^{k+1}}{\left(1-2 x t+t^{2}\right)^{k+1}}=\frac{1}{2^{k} \cdot k!} \sum_{n=0}^{\infty} U_{n+k}^{(k)}(x) \cdot t^{n}(1-x t)^{k+1} . \tag{7}
\end{equation*}
$$

Note that

$$
(1-x t)^{k+1}=\sum_{h=0}^{k+1}(-x)^{h} t^{h}\binom{k+1}{h} .
$$

Comparing the coefficients of $t^{n+k+1}$ on both sides of equation (7), we obtain Theorem 3.
To prove Theorem 2, we note that $\frac{d\left(T_{n}(x)\right)}{d x}=n U_{n-1}(x)$ and

$$
\frac{\partial G(t, x)}{\partial x}=\frac{t-t^{3}}{\left(1-2 x t+t^{2}\right)^{2}}=\sum_{n=0}^{\infty} T_{n+1}^{(1)}(x) \cdot t^{n+1}
$$

or

$$
\begin{equation*}
\frac{1-t^{2}}{\left(1-2 x t+t^{2}\right)^{2}}=\sum_{n=0}^{\infty}(n+1) U_{n}(x) \cdot t^{n} . \tag{8}
\end{equation*}
$$

Taking $k=2 m+1$ in (5), then multiplying by $\left(1-t^{2}\right)^{m+1}$ on both sides of (5), we can also get

$$
\begin{equation*}
\frac{\left(1-t^{2}\right)^{m+1}}{\left(1-2 x t+t^{2}\right)^{2 m+2}}=\frac{1}{2^{2 m+1} \cdot(2 m+1)!} \sum_{n=0}^{\infty} U_{n+2 m+1}^{(2 m+1)}(x) \cdot t^{n}\left(1-t^{2}\right)^{m+1} . \tag{9}
\end{equation*}
$$

Combining (8) and (9), we may immediately obtain the identity

$$
\begin{align*}
& \sum_{a_{1}+\cdots+a_{m+1}=n+2 m+2}\left(a_{1}+1\right) \cdots\left(a_{m+1}+1\right) \cdot U_{a_{1}}(x) \ldots U_{a_{m+1}}(x)  \tag{x}\\
& =\frac{1}{2^{2 m+1} \cdot(2 m+1)!} \sum_{h=0}^{m+1}(-1)^{h}\binom{m+1}{h} U_{n+4 m+3-2 h}^{(2 m+1)}(x) .
\end{align*}
$$

This completes the proof of Theorem 2.
Proof of the Corollaries: Taking $x=\frac{i}{2}, \frac{-3}{2}$, and $-2 i$ in Theorems 1-3, respectively, and noting that

$$
\begin{aligned}
& U_{n}\left(\frac{i}{2}\right)=i^{n} F_{n+1}, U_{n}\left(\frac{-3}{2}\right)=(-1)^{n} F_{2(n+1)}, U_{n}(-2 i)=\frac{(-i)^{n}}{2} F_{3(n+1)}, \\
& T_{n}\left(\frac{i}{2}\right)=\frac{i^{n}}{2} L_{n}, T_{n}\left(\frac{-3}{2}\right)=\frac{(-1)^{n}}{2} L_{2 n}, T_{n}(-2 i)=\frac{(-i)^{n}}{2} L_{3 n}, \\
& F_{n+2}=F_{n+1}+F_{n}, \\
& \left(1-x^{2}\right) U_{n}^{\prime}(x)=(n+1) U_{n-1}(x)-n x U_{n}(x),
\end{aligned}
$$

and

$$
\left(1-x^{2}\right) U_{n}^{\prime \prime}(x)=3 x U_{n}^{\prime}(x)-n(n+2) U_{n}(x),
$$

we may immediately deduce Corollaries $1-3$. Corollary 4 follows from Corollary 1 and the fact that $2 \mid F_{3(a+1)}$ for all integers $a \geq 0$.

Remark: For any positive integer $m \geq 4$, using our theorems, we can also give an exact calculating formula for the general sums

$$
\sum_{a_{1}+\cdots+a_{k}=n} \prod_{i=1}^{k} F_{m\left(a_{i}+1\right)} \text { and } \sum_{a_{1}+\cdots+a_{k}=n+k} \prod_{i=1}^{k} L_{m a_{i}},
$$

but in these cases the computations are more complex.

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# ON THE NUMBER OF PERMUTATIONS WITHIN A GIVEN DISTANCE 

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## 1. $\mathbb{N}^{\prime} /$ RODUCTION AND RESULTS

The set $\Pi^{n}$ of all permutations of $(1,2, \ldots, n)$, i.e., of all one-to-one mappings $\pi$ from $N=$ $\{1,2, \ldots, n\}$ onto $N$, can be made to a metric space by defining

$$
\left\|\pi=\pi^{\prime}\right\|=\max \left\{\left|\pi(i)-\pi^{\prime}(i)\right|: 1 \leq i \leq n\right\} .
$$

This space has been studied by Lagrange [1] with emphasis on the number of points contained in a sphere with radius $k$ around the identity, i.e.,

$$
\varphi(k ; n)=\left|\left\{\pi \in \mathbb{\Pi}^{n}:|\pi(i)-i| \leq k, 1 \leq i \leq n\right\}\right|
$$

where $|A|$ denotes the cardinality of the set $A$.
These numbers have been calculated in [1] for $k \in\{1,2,3\}$ and all $n \in \mathbb{N}$, the set of positive integers. For $k=1$, it is fairly easy to show that $\varphi(1 ; n-1), n \in \mathbb{N}, \varphi(1 ; 0)=1$, is the sequence of Fibonacci numbers. For $k=2$ and $k=3$, the enumeration is based on quite involved recurrences. The corresponding sequences are listed in Sloane and Puffle [4] as series M1600 and M1671, respectively.

The main purpose of this note is to supplement these findings by providing a closed formula for $\varphi(k ; n)$ when $k+2 \leq n \leq 2 k+2$. Note that, for $n \leq k+1$, one obviously has $\varphi(k ; n)=n!$; thus, the cases $n \geq 2 k+3, k \geq 4$, remain unresolved.

As a by-product, we obtain a formula for the permanent of specially patterned $(0,1)$-matrices. The connection to the problem above is as follows: Let $n, k \in \mathbb{N}, k \leq n-1$, be fixed, and for $i \in N, B_{i}=\{j \in \mathbb{Z}: i-k \leq j \leq i+k\} \cap N$, where $\mathbb{Z}$ is the set of all integers.

Then $\varphi(k ; n)$ is the same as the number of systems of distinct representatives for the set $\left\{B_{1}, B_{2}, \ldots, B_{n}\right\}$. Defining now for $i, j \in N$

$$
a_{i j}= \begin{cases}1, & j \in B_{i} \\ 0, & j \notin B_{i}\end{cases}
$$

one has, for the permanent of the matrix $A=\left(a_{i j}\right)$ (cf. Minc [2], p. 31),

$$
\begin{equation*}
\operatorname{Per}(A)=\varphi(k ; n) \tag{1.1}
\end{equation*}
$$

Remarll: The recurrence formula for $\varphi(2 ; n)$ has also been derived by Minc using properties of permanents (see [2], p. 49, Exercise 16).

The matrix $A$ defined in this way is symmetric and has, when $k+2 \leq n \leq 2 k+2$, the block structure

$$
A=\left(\begin{array}{ccc}
\mathbb{1}_{m \times m} & \mathbb{1}_{m \times s} & \Delta_{m \times m}  \tag{1.2}\\
\mathbb{1}_{s \times m} & \mathbb{1}_{s \times s} & 1_{s \times m} \\
\Delta_{m \times m}^{T} & \mathbb{1}_{m \times s} & \mathbb{1}_{m \times m}
\end{array}\right)
$$

where $m=n-1-k, s=2 k+2-n, 1_{a \times b}$ is the $a \times b$-matrix with all elements equal to one and $\Delta_{m \times m}$ is the $m \times m$-matrix with zeros on and above the diagonal and ones under the diagonal. For $n=2 k+2$, the second row and column blocks cancel. The matrix $\Delta_{m \times m}$ has been studied by Riordan ([3], p. 211 ff .) in connection with the rook problem. Riordan proved that the numbers of ways to put $r$ non-attacking rooks on a triangular chessboard are given by the Stirling number of the second kind. This will be crucial for the calculation of $\varphi(k ; n)$ and of $\operatorname{Per}(A)$ for matrices $A$ of a slightly more general structure than that given in (1.2). The results we will prove in Section 2 are as follows: Let $S_{r}^{n}$ denote the Stirling numbers of the second kind, i.e., the number of ways to partition an $n$-set into $r$ nonempty subsets.
Theorem 1: Let $k, n \in \mathbb{N}, k+2 \leq n \leq 2 k+2, m=n-k-1$. Then

$$
\varphi(k ; n)=\sum_{r=0}^{m}(-1)^{m-r}(n-2 m+r)!(n-2 m+r)^{m} S_{r+1}^{m+1} .
$$

Furthermore, let the matrix $A_{\Delta}$ be defined as

$$
A_{\Delta}=\left(\begin{array}{lll}
1_{m_{2} \times m_{1}} & 1_{m_{2} \times m_{3}} & \Delta_{m_{2} \times m_{2}}  \tag{1.3}\\
1_{m_{3}} \times m_{1} & 1_{m_{3} \times m_{3}} & 1_{m_{3} \times m_{2}} \\
\Delta_{m_{1} \times m_{1}}^{T} & 1_{m_{1} \times m_{3}} & 1_{m_{1} \times m_{2}}
\end{array}\right),
$$

where $n \in \mathbb{N}, n=m_{1}+m_{2}+m_{3}, m_{i} \in \mathbb{N}\{0\}, 1 \leq i \leq 3, \Delta_{a \times a}$ as above; for $m_{i}=0$, the corresponding row and column blocks cancel.
Theorem 2: Let $A_{\Delta}$ be defined by (1.3). Then

$$
\operatorname{Per}\left(A_{\Delta}\right)=\sum_{r=0}^{m_{1}}(-1)^{m_{1}-r}\left(m_{3}+r\right)!\left(m_{3}+r\right)^{m_{2}} S_{r+1}^{m_{1}+1} .
$$

## Remarks:

(a) Since the permanent is invariant with respect to transposing a matrix and to multiplication by permutation matrices, $A_{\Delta}$ as given in (1.3) is only a representative of a set of matrices for which Theorem 2 holds. In particular, it follows that, for all $m_{1}, m_{2}, m_{3} \in \mathbb{N}\{0\}$,

$$
\sum_{r=0}^{m_{1}}(-1)^{m_{1}-r}\left(m_{3}+r\right)!\left(m_{3}+r\right)^{m_{2}} S_{r+1}^{m_{1}+1}=\sum_{r=0}^{m_{2}}(-1)^{m_{2}-r}\left(m_{3}+r\right)!\left(m_{3}+r\right)^{m_{1}} S_{r+1}^{m_{2}+1} .
$$

Specializing further one gets, for $m_{1}=0, m_{3}=1, m_{2}+1=m$, the well-known relation

$$
1=\sum_{r=1}^{m}(-1)^{m-r} r!S_{r}^{m}
$$

(b) Since the matrix $A$ given in (1.2) is a special case of the matrix $A_{\Delta}$, in view of (1.1), Theorem 1 is a special case of Theorem 2. Therefore, we have to prove only Theorem 2.

## 2. PROOFS

By a suitable identification of the rook problem discussed in Riordan [3], chapters 7 and 8, with the problem considered here, part of the proof of Theorem 2 could be derived from results in
[3]. In view of a certain consistence of the complete proof, we prefer however to develop the necessary details from the beginning.

The problem of determining $\varphi(k ; n)$ can be seen as a problem of finding the cardinality of an intersection of unions of sets. We will do this by applying the principle of inclusion and exclusion to its complement. Therefore, the sets $\Pi_{i j}^{n}=\left\{\pi \in \Pi^{n}: \pi(i)=j\right\}, i, j \in N=\{1,2, \ldots, n\}$ are relevant. Let $\mathscr{P}_{k}(J)$ for $J \subset \mathbb{N}$ denote the set of all $I \subset J$ with $|I|=k$ and $\underset{\neq}{k}$ the set of all $k$-tuples in $\mathbb{N}^{k}$ with pairwise different components. For $k, n \in \mathbb{N}, k \leq n,\left(i_{1}, i_{2}, \ldots, i_{k}\right) \in \mathbb{N}_{\neq}^{k} \cap N^{k}$, and $j_{v} \in N, 1 \leq v \leq k$, one obviously has

$$
\left|\bigcap_{v=1}^{k} \Pi_{i_{i, j}}^{n}\right|= \begin{cases}(n-k)! & \text { if }\left(j_{1}, j_{2}, \ldots, j_{k}\right) \in \mathbb{N}_{\neq}^{k} \\ 0 & \text { otherwise }\end{cases}
$$

Therefore, one gets from the principle of inclusion and exclusion that, for $k, n \in \mathbb{N}, k \leq n, J \subset N$ with $|J|=k$, and $B_{i} \subset N, i \in J$,

$$
\begin{equation*}
\left|\bigcup_{i \in J} \bigcup_{j \in B_{i}} \Pi_{i j}^{n}\right|=\sum_{r=1}^{k}(-1)^{r-1}(n-r)!\sum_{I \in \mathscr{P}_{r}(J)}\left|\left\{\left(j_{1}, \ldots, j_{r}\right) \in \mathbb{N}_{\neq}^{r}: j_{i} \in B_{i} \forall i \in I\right\}\right| . \tag{2.1}
\end{equation*}
$$

For the sets on the right-hand side of (2.1), it holds that

$$
\begin{equation*}
\left|\left\{\left(j_{1}, \ldots, j_{r}\right) \in \mathbb{N}_{\neq}^{r}: j_{i} \in B_{i} \forall_{i} \in I\right\}\right|=\frac{1}{(n-k)!}\left|\bigcap_{i \in J}\left\{\pi \in \Pi^{n}: \pi(i) \in B_{i}\right\}\right| . \tag{2.2}
\end{equation*}
$$

For $n \in \mathbb{N}, k \in \mathbb{N}_{\cup}\{0\}, k \leq n, B_{1}, B_{2}, \ldots, B_{n} \subset N$, let

$$
R_{k}^{n}\left(B_{1}, \ldots, B_{n}\right)=\left\{\begin{array}{cc}
\sum_{j \in \mathscr{F}_{k}(N)}\left|\left\{\left(j_{1}, \ldots, j_{k}\right) \in \mathbb{N}_{*}^{k}: j_{i} \in B_{i} \forall i \in J\right\}\right|, & \text { for } k \geq 1,  \tag{2.3}\\
1, & \text { for } k=0 .
\end{array}\right.
$$

[If one considers a chessboard on which pieces may be placed only on positions ( $i, j$ ) for which $j \in B_{i}$, then $R_{k}^{n}\left(B_{1}, \ldots, B_{n}\right)$ is the number of ways of putting $k$ non-attacking rooks on this board.]

Lemma 1: Let $k, n \in \mathbb{N}, j \leq n, B_{i} \subset N$ for $i \in N$. Then it holds that

$$
\sum_{J \in \mathscr{P}_{k}(N)}\left|\bigcup_{i \in J} \bigcup_{j \in B_{i}} \Pi_{i j}^{n}\right|=\sum_{r=1}^{k}(-1)^{r-1}(n-r)!\binom{n-r}{k-r} R_{r}^{n}\left(B_{1}, \ldots, B_{n}\right) .
$$

Proof: With the help of (2.1), one gets

$$
\begin{aligned}
& \sum_{J \in \mathcal{P}_{k}(N)}\left|\bigcup_{i \in J} \bigcup_{j \in B_{i}} \Pi_{i j}^{n}\right| \\
= & \sum_{r=1}^{k}(-1)^{r-1}(n-r)!\sum_{J \in \mathscr{P}_{k}(N)} \sum_{I \in \mathscr{F}_{r}(\mathcal{J}}\left|\left\{\left(j_{1}, \ldots, j_{r}\right) \in \mathbb{N}_{\neq}^{r}: j_{i} \in B_{i} \forall i \in I\right\}\right| \\
= & \sum_{r=1}^{k}(-1)^{r-1}(n-r)!\sum_{I \in \mathscr{P}_{r}(N)}\left|\left\{\left(j_{1}, \ldots, j_{r}\right) \in \mathbb{N}_{\neq}^{r}: j_{i} \in B_{i} \forall i \in I\right\} \|\left\{J \in \mathscr{P}_{k}(N): I \subset J\right\}\right| \\
= & \sum_{r=1}^{k}(-1)^{r-1}(n-r)!\binom{n-r}{k-r} R_{r}^{n}\left(B_{1}, \ldots, B_{n}\right) .
\end{aligned}
$$

In the next lemma it is shown how the numbers $R_{k}^{n}\left(B_{1}, \ldots, B_{n}\right)$ are related to $R_{k}^{n}\left(B_{1}^{c}, \ldots, B_{n}^{c}\right)$, where $B_{i}^{c}$ denotes the complement of $B_{i}$ w.r.t. N. (In terms of the rook problem, one thus considers the complement of the chessboard.) The lemma is equivalent to Theorem 2 in Riordan ([3], p. 180).

Lemma 2: Let $k, n \in \mathbb{N}, k \leq n, B_{i} \subset N, i \in N$. Then it holds that

$$
R_{k}^{n}\left(B_{1}, \ldots, B_{n}\right)=\sum_{r=0}^{k}(-1)^{r}(k-r)!\binom{n-r}{n-k}\binom{n-r}{k-r} R_{r}^{n}\left(B_{1}^{c}, \ldots, B_{n}^{c}\right) .
$$

Proof: By (2.2) and (2.3), one has

$$
\begin{aligned}
(n-k)!R_{k}^{n}\left(B_{1}, \ldots, B_{n}\right) & =\sum_{J \in \mathscr{P}_{k}(N)}\left|\bigcap_{i \in J}\left\{\pi \in \Pi^{n}: \pi(i) \in B_{i}\right\}\right| \\
& =\sum_{J \in \mathcal{P}_{k}(N)}\left(n!-\left|\bigcup_{i \in J} \bigcup_{j \in \tilde{B}_{i}} \Pi_{i j}^{n}\right|\right) .
\end{aligned}
$$

The assertion then follows with the help of Lemma 1.
Lemma 2 will become useful for calculating $\operatorname{Per}\left(A_{\Delta}\right)$ in the following manner: Let $A_{\Delta}=\left(a_{i j}\right)$ and put $B_{i}=\left\{j \in N: a_{i j}=1\right\}$. Since by (2.2) and (2.3)

$$
\operatorname{Per}\left(A_{\Delta}\right)=\sum_{\pi \in \Pi^{n}} \prod_{i=1}^{n} a_{i, \pi(i)}=\left|\left\{\pi \in \Pi^{n}: \prod_{i=1}^{n} a_{i, \pi(i)}=1\right\}\right|=R_{n}^{n}\left(B_{1}, \ldots, B_{n}\right),
$$

one obtains from Lemma 2 that

$$
\begin{equation*}
\operatorname{Per}\left(A_{\Delta}\right)=\sum_{r=0}^{n}(-1)^{r}(n-r)!R_{r}^{n}\left(B_{1}^{c}, \ldots, B_{n}^{c}\right) . \tag{2.4}
\end{equation*}
$$

The matrix corresponding to $B_{1}^{c}, \ldots, B_{n}^{c}$ is $\bar{A}_{\Delta}=1_{n \times n}-A_{\Delta}$, which is easier to handle because it has mainly blocks of zero-matrices. A further simplification is obtained by considering instead of $\bar{A}_{\Delta}$ the matrix

$$
\hat{A}_{\Delta}=\left(\begin{array}{ccc}
\hat{\Delta}_{m_{1} \times m_{1}} & 0_{m_{1} \times m_{2}} & 0_{m_{1} \times m_{3}}  \tag{2.5}\\
0_{m_{2} \times m_{1}} & \hat{\Delta}_{m_{2} \times m_{2}} & 0_{m_{2} \times m_{3}} \\
0_{m_{3} \times m_{1}} & 0_{m_{3} \times m_{2}} & 0_{m_{3} \times m_{3}}
\end{array}\right),
$$

where $\hat{\Delta}_{a \times a}=1_{a \times a}-\Delta_{a, a}^{T} \cdot \hat{A}_{\Delta}$ is obtained from $\overline{A_{\Delta}}$ by suitable permutations of rows and columns. By Remark (a) one has $\operatorname{Per}\left(\bar{A}_{\Delta}\right)=\operatorname{Per}\left(\hat{A}_{\Delta}\right)$.

Now we turn to the special structure related to the matrices of the form $\hat{\Delta}_{m \times m}$, that is, we consider $B_{i}=\{1,2, \ldots, i\}, i \in N_{m}=\{1,2, \ldots, m\}$. One can easily show by induction on $k$ that

$$
\left.\mid\left\{j_{1}, \ldots, j_{k}\right) \in \mathbb{N}_{\neq}^{k}: j_{v} \in D_{v}, 1 \leq v \leq k\right\} \mid=\prod_{v=1}^{k}\left(\left|D_{v}\right|+1-v\right)
$$

if $k, m \in \mathbb{N}, k \leq m$, and $D_{1}, \ldots, D_{k} \subset N_{m}$ such that $D_{v} \subset D_{v+1}, 1 \leq v<k$, so that

$$
\begin{equation*}
R_{k}^{m}\left(B_{1}, \ldots, B_{m}\right)=\sum_{1 s_{i_{1}}<\cdots<i_{k} \leq m} \prod_{v=1}^{k}\left(i_{v}+1-v\right) \tag{2.6}
\end{equation*}
$$

We denote the right-hand side of (2.6) by $\alpha_{k}^{m}, 1 \leq k \leq m, \alpha_{0}^{m}=1, \alpha_{k}^{m}=0$, for $k<0$ or $k>m$.
Lemma 3: For $\alpha_{k}^{m}$ defined as above, it holds that
(a) $\alpha_{k}^{m}=\alpha_{k}^{m-1}+(m+1-k) \alpha_{k-1}^{m-1}$ for all $k \in \mathbb{Z}, m \in \mathbb{N}, m \geq 2$.
(b) $\alpha_{k}^{m}=S_{m+1-k}^{m+1}$ for all $m \in \mathbb{N}, k \in \mathbb{N},\{0\}, k \leq m$.

Proof: Part (a) follows immediately from the definition of $\alpha_{k}^{m}$. Assertion (b) obviously holds true for $m=1$. Since the Stirling numbers of the second kind satisfy the recursion $S_{k}^{m}=S_{k-1}^{m-1}+$ $k S_{k}^{m-1}$, the assertion is a consequence of (a).

It now follows from Lemma 3 and (2.6) that, for $B_{i}=\{1,2, \ldots, i\}, 1 \leq i \leq m$,

$$
R_{k}^{m}\left(B_{1}, \ldots, B_{m}\right)= \begin{cases}S_{m+1-k}^{m+1}, & \text { for all } m \in \mathbb{N}, k \in \mathbb{N},\{0\}, k \leq m  \tag{2.7}\\ 0, & \text { otherwise }\end{cases}
$$

To deal with the two $\Delta$-blocks of the matrix $\hat{A}_{\Delta}$, the following lemma is helpful.
Lemma 4: Let $m_{1}, m_{2}, n \in \mathbb{N}, n \geq m_{1}+m_{2}$, and $C_{1}, C_{2}, \ldots, C_{n} \subset N$ such that:
(a) $C_{i} \subset\left\{1,2, \ldots, m_{1}\right\}, 1 \leq i \leq m_{1}$;
(b) $C_{i} \subset\left\{m_{1}+1, \ldots, m_{1}+m_{2}\right\}, m_{1}+1 \leq i \leq m_{1}+m_{2}$;
(c) $C_{i}=\emptyset, m_{1}+m_{2}+1 \leq i \leq n$.

Furthermore, let $D_{i}=\left\{j \in\left\{1, \ldots, m_{2}\right\}: j+m_{1} \in C_{i+m_{1}}\right\}, 1 \leq i \leq m_{2}$. Then it holds that

$$
R_{k}^{n}\left(C_{1}, \ldots, C_{n}\right)= \begin{cases}\sum_{v=0}^{k} R_{v}^{m_{1}}\left(C_{1}, \ldots, C_{m_{1}}\right) R_{k-v}^{m_{2}}\left(D_{1}, \ldots, D_{m_{2}}\right), & 0 \leq k \leq m_{1}+m_{2} \\ 0, & m_{1}+m_{2}+1 \leq k \leq n\end{cases}
$$

Proof: Let $N_{1}=\left\{1, \ldots, m_{1}\right\}, N_{2}=\left\{m_{1}+1, \ldots, m_{1}+m_{2}\right\}, N_{3}=\left\{m_{1}+m_{2}+1, \ldots, n\right\}$ and, for $J \in$ $\mathscr{P}_{k}(N), f_{k}(J)=\left|\left\{\left(j_{1}, \ldots, j_{k}\right) \in \mathbb{N}_{\neq}^{k}: j_{i} \in C_{i} \forall i \in J\right\}\right|$. Since $C_{i}=\emptyset$ for $i \in N_{3}$, one has $f_{k}(J)=0$ if $J \in \mathscr{P}_{k}(N)$ and $J \cap N_{3} \neq \emptyset$. This implies

$$
R_{k}^{n}\left(C_{1}, \ldots, C_{n}\right)=\sum_{r=0}^{k} \sum_{J_{1} \in \mathscr{F}_{r}\left(N_{1}\right)} \sum_{J_{2} \in \mathscr{F}_{k-r}\left(N_{2}\right)} f_{k}\left(J_{1} \cup J_{2}\right)
$$

Since $\left(\bigcup_{i \in N_{1}} C_{i}\right) \cap\left(\bigcup_{i \in N_{2}} C_{i}\right)=\emptyset$ one has, for $J_{1} \in \mathscr{P}_{r}\left(N_{1}\right), J_{2} \in \mathscr{P}_{k-r}\left(N_{2}\right)$, that

$$
f_{k}\left(J_{1} \cup J_{2}\right)=\left|\left\{\left(j_{1}, \ldots, j_{r}\right) \in \mathbb{N}_{\neq t}^{r}, j_{i} \in C_{i} \forall i \in J_{1}\right\}\right|\left|\left\{\left(j_{r+1}, \ldots, j_{k}\right) \in \mathbb{N}_{\neq}^{k-r}: j_{i} \in C_{i} \forall i \in J_{2}\right\}\right|
$$

The assertion then follows from

$$
R_{r}^{m_{1}}\left(C_{1}, \ldots, C_{m_{1}}\right)=\sum_{J_{1} \in \mathscr{P}_{r}\left(N_{1}\right)}\left|\left\{\left(j_{1}, \ldots, j_{r}\right) \in \mathbb{N}_{\neq}^{r}: j_{i} \in C_{i} \forall i \in J_{1}\right\}\right|
$$

and

$$
R_{k-r}^{m_{2}}\left(D_{1}, \ldots, D_{m_{2}}\right)=\sum_{J_{2} \in \mathscr{S}_{k-r}\left(N_{2}\right)}\left|\left\{\left(j_{r+1}, \ldots, j_{k}\right) \in \mathbb{N}_{\neq}^{k-r}: j_{i} \in C_{i} \forall i \in J_{2}\right\}\right| .
$$

Finally, the following identity will become useful:

$$
\begin{equation*}
\sum_{r=0}^{m}(-1)^{r}(n-r)!S_{m+1-r}^{m+1}=(n-m)!(n-m)^{m} \text { for } m, n \in \mathbb{N}\{0\}, n \geq m \tag{2.8}
\end{equation*}
$$

Identity (2.8) can easily be proved by induction on $m$ using the recurrence formula for the Stirling numbers. Now we are ready to prove Theorem 2. Consider the matrix $\hat{A}_{\Delta}=\left(\hat{a}_{i j}\right)$ defined in (2.5). Putting

$$
C_{i}= \begin{cases}\{1, \ldots, i\}, & 1 \leq i \leq m_{1}, \\ \left\{m_{1}+1, \ldots, m_{1}+i-m_{1}\right\}, & m_{1}+1 \leq i \leq m_{1}+m_{2}, \\ \emptyset, & m_{1}+m_{2}+1 \leq i \leq n,\end{cases}
$$

one has $\hat{a}_{i j}=1$ if and only if $j \in C_{i}$. Note that for $C_{1}, \ldots, C_{n}$ the assumptions of Lemma 4 are satisfied and that $D_{i}=\{1, \ldots, i\}$ for $1 \leq i \leq m_{2}$. Put $n-m_{1}-m_{2}=m_{3}$. Then, from (2.4), Lemma 4, (2.7), and (2.8), one gets that

$$
\begin{aligned}
\operatorname{Per}\left(A_{\Delta}\right) & =\sum_{r=0}^{n}(-1)^{r}(n-r)!R_{r}^{n}\left(C_{1}, \ldots, C_{n}\right) \\
& =\sum_{r=0}^{m_{1}+m_{2}}(-1)^{r}(n-r)!\sum_{v=0}^{r} R_{v}^{m_{1}}\left(C_{1}, \ldots, C_{m_{1}}\right) R_{r-v}^{m_{2}}\left(D_{1}, \ldots, D_{m_{2}}\right) \\
& =\sum_{r=0}^{m_{1}+m_{2}}(-1)^{r}(n-r)!\sum_{v=0}^{r} S_{m_{1}+1-v}^{m_{1}+1} S_{m_{2}+1-r+v}^{m_{2}+1}=\sum_{v=0}^{m_{1}+m_{2}} S_{m_{1}+1-v}^{m_{1}+1} \sum_{r=v}^{m_{1}+m_{2}}(-1)^{r}(n-r)!S_{m_{2}+1-r+v}^{m_{2}+1} \\
& =\sum_{v=0}^{m_{1}} S_{m_{1}+1-v}^{m_{1}+1} \sum_{r=0}^{m_{2}}(-1)^{r+v}(n-r-v)!S_{m_{2}+1-r}^{m_{2}+1}=\sum_{v=0}^{m_{1}}(-1)^{v} S_{m_{1}+1-v}^{m_{1}+1}\left(n-v-m_{2}\right)!\left(n-v-m_{2}\right)^{m_{2}} \\
& =\sum_{v=0}^{m_{1}}(-1)^{m_{1}-v}\left(n-m_{1}+v-m_{2}\right)!\left(n-m_{1}+v-m_{2}\right)^{m_{2}} S_{v+1}^{m_{1}+1} \\
& =\sum_{v=0}^{m_{1}}(-1)^{m_{1}-v}\left(m_{3}+v\right)!\left(m_{3}+v\right)^{m_{2}} S_{v+1}^{m_{1}+1} .
\end{aligned}
$$

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# GENERALIZATION OF A THEOREM OF DROBOT 

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It is well known that the Fibonacci number $F_{n}$ can be a prime only if $n=4$ or $n=p$, where $p$ is an odd prime. Throughout this paper, $p$ will denote a prime. In a very interesting paper, Drobot [2] proved that $F_{p}$ is composite for certain primes $p$. In particular, he proved that if $p>7, p \equiv 2$ or $4(\bmod 5)$, and $2 p-1$ is also a prime, then $2 p-1 \mid F_{p}$ and $F_{p}>2 p-$. For example, $37 \mid F_{19}=$ $4181=37 \cdot 113$.

A similar result was proved by Euler and, independently, by Lagrange about the Mersenne numbers. It is easy to see that the Mersenne number $M_{n}=2^{n}-1$ can be a prime only if $n$ is a prime. Euler and Lagrange proved that, if $p \equiv 3(\bmod 4)$ and $2 p+1$ is also a prime, then $2 p+1 \mid M_{p}=2 p+1$. A proof of this result is given in [5, pp. 90-91].

The primality of Mersenne numbers is of interest because of the following relationship to even perfect numbers. A positive integer is perfect if it is equal to the sum of its proper divisors. Euclid and Euler proved that the even integer $n$ is perfect if and only if $n$ is of the form $2^{p-1}\left(2^{p}-1\right)$, where $2^{p}-1$ is a Mersenne prime. Euclid proved that this condition is sufficient for $n$ to be a perfect number and Euler proved the necessity of this condition. At the present time only thirty-eight Mersenne primes are known, with the largest known Mersenne prime being $2^{6972593}-1$, which has over two million digits. A list of all known Mersenne primes is given in the web site

## http://www.utm.edu/research/primes/glossary/Mersennes.html.

We will prove a theorem which generalizes both of the results given above concerning the compositeness of $F_{p}$ and $M_{p}$. Before presenting this theorem, we will need the following definition and results involving Lucas sequences.
Definition 1: The Lucas sequence $u(a, b)$ is a second-order linear recurrence satisfying the relation $u_{n+2}=a u_{n+1}+b u_{n}$ and having initial terms $u_{0}=0, u_{1}=1$, where $a$ and $b$ are integers.

We let $D=a^{2}+4 b$ be the discriminant of $u(a, b)$. Associated with $u\left(a_{\mathrm{n}} b\right)$ is the characteristic polynomial $f(x)=x^{2}-a x-b$ with characteristic roots $\alpha$ and $\beta$. Then, by the Binet formula

$$
\begin{equation*}
u_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta} . \tag{1}
\end{equation*}
$$

We have the following theorem concerning the divisibility of $u_{n}$ by the prime $p$.
Theorem 1: Let $u(a, b)$ be a Lucas sequence. Let $p$ be an odd prime such that $p \nmid b D$. Then

$$
\begin{equation*}
p \mid u_{p-(D / p)} \tag{2}
\end{equation*}
$$

where $(D / p)$ is the Legendre symbol. Moreover,

$$
\begin{equation*}
p \mid u_{(p-(D / p)) / 2} \tag{3}
\end{equation*}
$$

if and only if $(-b / p)=1$.

Proof: Proofs of (2) are given in [4, pp. 290, 296-97] and [1, pp. 44-45]. A proof of (3) is given in [3, p. 441].

We are now ready for our main result, Theorem 2. The results by Drobot and by Euler and by Lagrange on the compositeness of $F_{p}$ and $M_{p}$ will then be given as corollaries of Theorem 2.

Theorem 2: Let $u(a, b)$ be a Lucas sequence. Let $p$ be an odd prime such that $p \nmid b$.
(a) If $2 p-1$ is a prime, $(D /(2 p-1))=-1$, and $(-b /(2 p-1))=1$, then $2 p-1 \mid u_{p}$.
(b) If $2 p+1$ is a prime, $(D /(2 p+1))=1$, and $(-b /(2 p+1))=1$, then $2 p+1 \mid u_{p}$.

Proof: (a) $\operatorname{By}$ (3), $2 p-1 \mid u_{(2 p-1+1) / 2}=u_{p}$. (b) $\operatorname{By}$ (3), $2 p+1 \mid u_{(2 p+1-1) / 2}=u_{p}$.
Corollary 1 (Drobot): Let $p$ be a prime such that $p>7, p \equiv 2$ or $4(\bmod 5)$, and $2 p-1$ is a prime. Then $2 p-1 \mid F_{p}$ and $F_{p}>2 p-1$.

Proof: Note that $\left\{F_{n}\right\}=u(1,1)$ and $D=5$. It is clear from (1) that if $p>7$, then $F_{p}>2 p-1$. If $p \equiv 2(\bmod 5)$, then $2 p-1 \equiv 3(\bmod 5)$, while if $p \equiv 4(\bmod 5)$, then $2 p-1 \equiv 2(\bmod 5)$. By the law of quadratic reciprocity, if $2 p-1 \equiv 2$ or $3(\bmod 5)$, then

$$
(D /(2 p-1))=(5 /(2 p-1))=-1 .
$$

Since $p \equiv 1$ or $3(\bmod 4)$, it follows that $2 p-1 \equiv 1(\bmod 4)$. Hence,

$$
(-b /(2 p-1))=(-1 /(2 p-1))=1 .
$$

It now follows from Theorem 2(a) that

$$
2 p-1 \mid F_{p} .
$$

Corollary 2 (Euler and Lagrange): Let $p$ be a prime such that $p>3, p \equiv 3(\bmod 4)$, and $2 p+1$ is a prime. Then $2 p+1 \mid M_{p}$ and $M_{p}>2 p+1$.

Proof: It is clear that if $p>3$, then $M_{p}=2^{p}-1>2 p+1$. Consider the Lucas sequence $u(3,-2)$. Then $D=1$ and, by the Binet formula (1),

$$
u_{n}=\frac{2^{n}-1}{2-1}=2^{n}-1=M_{n} .
$$

Moreover,

$$
(D /(2 p+1))=(1 /(2 p+1))=1 .
$$

It also follows from the fact that $p \equiv 3$ or $7(\bmod 8)$ that $2 p+1 \equiv 7(\bmod 8)$. Thus,

$$
(-b /(2 p+1))=(2 /(2 p+1))=1 .
$$

It now follows from Theorem 2(b) that

$$
2 p+1 \mid u_{p}=M_{p}
$$

Remark: Primes $p$ such that $2 p+1$ is also a prime are called Sophie Germain primes of the first kind, while primes $p$ such that $2 p-1$ is a prime are called Sophie Germain primes of the second kind. It is not known whether there exist infinitely many Sophie Germain primes of the first or second kind. At the present time, the largest known Sophie Germain prime of the first kind is $3714089895285 \cdot 2^{60000}-1$ with 18075 digits, and the largest known Sophie Germain prime of the
second kind is $16769025 \cdot 2^{34071}+1$ with 10264 digits. For a list of the largest known Sophie Germain primes, see the web sites
http://www.utm.edu/research/primes/lists/top20/SophieGermain.html
and
http://ksc9.th.com/warut/cunningham.html.

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# ON THE PRODUCT OF LINE-SEQUENCES 

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For consistency, we adopt the same notations and formats developed in our previous work on line-sequences, see [2].

A line-sequence is expressed as

$$
\begin{equation*}
\bigcup_{u_{0}, u_{1}}(c, b): \ldots u_{-3}, u_{-2}, u_{-1},\left[u_{0}, u_{1}\right], u_{2}, u_{3}, u_{4}, \ldots, \tag{1}
\end{equation*}
$$

where $u_{n}, n \in Z$, denotes the $n^{\text {th }}$ term, the generating pair is given by $\left[u_{0}, u_{1}\right]$, and the recurrence relation is

$$
\begin{equation*}
c u_{n}+b u_{n+1}=u_{n+2}, \tag{2}
\end{equation*}
$$

where $c, b \in R$ are not zero. Since (2) is valid for any value of $n$, we also have

$$
c u_{n+1}+b u_{n+2}=u_{n+3} .
$$

From these two relations, we find

$$
\begin{align*}
& b=\left(u_{n} u_{n+3}-u_{n+1} u_{n+2}\right) /\left(u_{n} u_{n+2}-\left(u_{n+1}\right)^{2}\right),  \tag{3}\\
& c=\left(\left(u_{n+2}\right)^{2}-u_{n+1} u_{n+3}\right) /\left(u_{n} u_{n+1}-\left(u_{n+1}\right)^{2}\right) . \tag{4}
\end{align*}
$$

The product (see, e.g., [1], [4], [5]), abbreviated as "product" here, of two line-sequences does not necessarily satisfy a recurrence relation. We will give some conditions under which it does.

A generalized Fibonacci line-sequence is given by

$$
\begin{equation*}
\bigcup_{0,1}(c, b): \ldots[0,1], b, c+b^{2}, \ldots \tag{5}
\end{equation*}
$$

and a generalized Lucas line-sequence is given by

$$
\begin{equation*}
\bigcup_{2, b}(c, b): \ldots[2, b], 2 c+b^{2}, 3 c b+b^{3}, \ldots \tag{6}
\end{equation*}
$$

see (4.3) and (4.12) in [2]. Let

$$
\begin{equation*}
\bigcup_{0, b}(y, x)=\bigcup_{0,1}(c, b) \bigcup_{2, b}(c, b) . \tag{7}
\end{equation*}
$$

Substituting (5) and (6) into (7) and multiplying corresponding terms produces

$$
\begin{equation*}
\bigcup_{0, b}(y, x): \ldots[0, b], 2 c b+b^{3}, 3 c^{2} b+4 c b^{3}+b^{5}, \ldots \tag{8}
\end{equation*}
$$

Putting $n=0$ in (3) and (4) and applying to (8), we obtain

$$
\begin{equation*}
x=2 c+b^{2}, y=-c^{2} . \tag{9}
\end{equation*}
$$

So (7) becomes

$$
\begin{equation*}
\bigcup_{0, b}\left(-c^{2}, 2 c+b^{2}\right)=\bigcup_{0,1}(c, b) \bigcup_{2, b}(c, b) . \tag{10}
\end{equation*}
$$

Let

$$
\begin{equation*}
\bigcup_{-b, 0}(y, x)=\bigcup_{1,0}(c, b) \bigcup_{-b, 2 c}(c, b) . \tag{11}
\end{equation*}
$$

Following the same procedure, we find

$$
\begin{equation*}
\bigcup_{-b, 0}\left(-c^{2}, 2 c+b^{2}\right)=\bigcup_{1,0}(c, b) \bigcup_{-b, 2 c}(c, b) \tag{12}
\end{equation*}
$$

From (10) and (12), we have the following pair:

$$
\begin{align*}
& \bigcup_{1,0}\left(-c^{2}, 2 c+b^{2}\right)=-(1 / b) \bigcup_{1,0}(c, b) \bigcup_{-b, 2 c}(c, b),  \tag{13}\\
& \bigcup_{0,1}\left(-c^{2}, 2 c+b^{2}\right)=(1 / b) \bigcup_{0,1}(c, b) \bigcup_{2, b}(c, b) . \tag{14}
\end{align*}
$$

So we obtain the formula:

$$
\begin{align*}
\bigcup_{i, j}\left(-c^{2}, 2 c+b^{2}\right) & =i \bigcup_{1,0}\left(-c^{2}, 2 c+b^{2}\right)+j \bigcup_{0,1}\left(-c^{2}, 2 c+b^{2}\right) \\
& =(1 / b)\left[-i \bigcup_{1,0}(c, b) \bigcup_{-b, 2 c}(c, b)+j \bigcup_{0,1}(c, b) \bigcup_{2, b}(c, b)\right] . \tag{15}
\end{align*}
$$

Example: Let $c=b=1$ in (15) and put $M_{i, j}=\bigcup_{i, j}(-1,3)$ and $F_{i, j}=\bigcup_{i, j}(1,1)$, then

$$
\begin{equation*}
M_{i, j}=-i F_{1,0} F_{-1,2}+j F_{0,1} F_{2,1} \tag{16}
\end{equation*}
$$

where $M$ denotes Morgan-Voyce numbers, see (1) in [3].
Let $m_{i, j ; n}$ and $f_{i, j ; n}$ be the $n^{\text {th }}$ term of $M_{i, j}$ and $F_{i, j}$, respectively. Then

$$
\begin{equation*}
m_{i, j ; n}=-i f_{1,0 ; n} f_{-1,2 ; n}+j f_{0,1 ; n} f_{2,1 ; n}=-i f_{n-1} l_{n-1}+j f_{n} l_{n} \tag{17}
\end{equation*}
$$

where $f_{n}$ and $l_{n}$ denote the $n^{\text {th }}$ Fibonacci and the $n^{\text {th }}$ Lucas numbers, respectively. In particular,

$$
\begin{gather*}
m_{1,0 ; n}=-f_{n-1} l_{n-1}=-f_{2 n-2}  \tag{18}\\
m_{0,1 ; n}=f_{n} l_{n}=f_{2 n} \tag{19}
\end{gather*}
$$

Since the generating function of $M_{i, j}$ is $(j-i t) /\left(1-3 t+t^{2}\right)$, we have

$$
\begin{equation*}
t /\left(1-3 t+t^{2}\right)=\sum_{n \geq 1} f_{2 n-2} t^{n-1} \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
1 /\left(1-3 t+t^{2}\right)=\sum_{n \geq 1} f_{2 n} t^{n-1} \tag{21}
\end{equation*}
$$

For $M_{1,1}$,

$$
\begin{equation*}
(1-t) /\left(1-3 t+t^{2}\right)=\sum_{n \geq 1} f_{2 n-1} t^{n-1} \tag{22}
\end{equation*}
$$

and for $M_{-1,1}$,

$$
\begin{equation*}
(1+t) /\left(1-3 t+t^{2}\right)=\sum_{n \geq 1} l_{2 n-1} t^{n-1} \tag{23}
\end{equation*}
$$

## ACKNOWLEDGMENT

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# FIBONACCI TREE IS CRITICALLY BALANCED-A NOTE* 

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## 1. $\operatorname{INTRODUCTION}$

To continue a previous note [2] (also [3]) on the morphology of self-similar trees, we reconsider, as simple model trees (see [2] for motivations), the sequence of binary trees $S_{k}=S_{k}(a, b)$, $k=1,2, \ldots$, defined recursively for relatively prime integers $a, b$ such that $1 \leq a \leq b: S_{1}, \ldots, S_{b}$ are just one-leaf trees, and, for $k \geq b+1$, the left subtree of $S_{k}$ is given by $S_{k-a}$ and the right by $S_{k-b}$. Put $c=\frac{b}{a}$. When $c=2$, we have $S_{k}(1,2)$, the Fibonacci tree (of order $k$ ).

Denote the number of leaves in $S_{k}$ by $n_{k}=n_{k}(c)$ and write

$$
\left\{\begin{array}{l}
\lambda_{k}=\lambda_{k}(c)=\frac{n_{k-a}}{n_{k}}(k \geq b+1), \\
\lambda=\lambda(c)=\lim _{k \rightarrow \infty} \lambda_{k},
\end{array}\right.
$$

then $\lambda_{k}:\left(1-\lambda_{k}\right)$ may be considered as a left-to-right weight-proportion in $S_{k}$.
The average path length $L_{k}=L_{k}(c)$ (i.e., the average number of branchings along the path from the root to a leaf) of $S_{k}$ is the sum of the lengths of all the paths from the root to leaves divided by $n_{k}$.

In Section 2 we show the following relation:

$$
G(c) H(c)=1,
$$

where

$$
\left\{\begin{array}{l}
G(c)=\lim _{k \rightarrow \infty} \frac{L_{k}}{\log n_{k}}, \\
H(c)=-\lambda \log \lambda-(1-\lambda) \log (1-\lambda) .
\end{array}\right.
$$

("log" is to the base 2 , while " $\ln$ " is to the base $e$.)
That is, we show that the normalized $L_{k}, L_{k} / \log n_{k}$, converges and the limit equals $(H(c))^{-1}$, the inverse of the entropy of the distribution $\lambda, 1-\lambda$. Roughly, $G(c)$ and $H(c)$ express the asymptotic growth and breadth indices, respectively, of the tree.

We will then observe in Section 3 some simple balance properties of $S_{k}$ and show that the $c$ maximizing $G(c)$ but maintaining $S_{k}$ balanced for every $k$ is equal to 2 .

## 2. A LIMITING RELATION

The following lemma was implicitly shown in [2] and will be used in the sequel.

## Lemma 1:

(a) $\lambda^{b}=(1-\lambda)^{a}$;
(b) $\lambda=\lambda(c)(1 \leq c)$ is less than 1 and strictly monotone increasing, and $\lambda(1)=\frac{1}{2}, \lambda(2)=\frac{\sqrt{5}-1}{2}$;

[^2](c) $\frac{1}{k} \log n_{k} \rightarrow \frac{1}{a}(-\log \lambda)$ as $k \rightarrow \infty$;
(d) $\left|\lambda_{k}-\lambda\right| \rightarrow 0$ exponentially fast as $k \rightarrow \infty$.

Theorem 1: $G(c) H(c)=1$.
Proof: It is easy to see that the recursive structure of $S_{k}$ implies

$$
\begin{equation*}
L_{k}=\lambda_{k} L_{k-a}+\left(1-\lambda_{k}\right) L_{k-b}+1(k \geq b+1) \tag{1}
\end{equation*}
$$

( $L_{1}=\cdots=L_{b}=0$ ), which we are going to compare with the following equation with constant coefficients:

$$
\begin{equation*}
x_{k}=\lambda x_{k-a}+(1-\lambda) x_{k-b}+1(k \geq b+1) \tag{2}
\end{equation*}
$$

( $x_{1}=\cdots=x_{b}=0$ ).
Remark: Kapoor and Reingold [4] treated, in a different way, a general recurrence, including (1), derived from the binary trees with costs $a$ and $b$ on the left and right branches.

The characteristic equation $\lambda t^{-a}+(1-\lambda) t^{-b}=1$ of the homogeneous

$$
\begin{equation*}
y_{k}=\lambda y_{k-a}+(1-\lambda) y_{k-b} \tag{3}
\end{equation*}
$$

clearly has root 1 , and it can be shown that $|\alpha|<1$ for every other root $\alpha$. Therefore, the general solution of (3) is given by $y_{k}=C_{1}+\varepsilon_{k}$, where $C_{1}$ is a constant and $\varepsilon_{k} \rightarrow 0(k \rightarrow \infty)$.

As a particular solution of (2), we have

$$
x_{k}=\frac{(-\log \lambda)}{a H(c)} k \quad(k \geq 1)
$$

In fact, the right-hand side of (2) then becomes

$$
\begin{aligned}
& \lambda \frac{(-\log \lambda)}{a H(c)}(k-a)+(1-\lambda) \frac{(-\log \lambda)}{a H(c)}(k-b)+1 \\
= & \frac{(-\log \lambda)}{a H(c)} k+\frac{1}{a H(c)}(a \lambda \log \lambda+b(1-\lambda) \log \lambda-a \lambda \log \lambda-a(1-\lambda) \log (1-\lambda)) \\
= & \frac{(-\log \lambda)}{a H(c)} k+\frac{1-\lambda}{a H(c)} \log \left\{\frac{\lambda^{b}}{(1-\lambda)^{a}}\right\}=\frac{(-\log \lambda)}{a H(c)} k \quad[\text { by Lemma 1(a) ] } \\
= & x_{k} .
\end{aligned}
$$

The solution of (2) is therefore given by

$$
\begin{equation*}
x_{k}=\frac{(-\log \lambda)}{a H(c)} k+C_{1}+\varepsilon_{k}, \tag{4}
\end{equation*}
$$

which we regard as the solution satisfying the initial condition $x_{1}=\cdots=x_{k}=0$.
Subtract (2) from (1) to get

$$
L_{k}-x_{k}=\lambda_{k}\left(L_{k-a}-x_{k-a}\right)+\left(1-\lambda_{k}\right)\left(L_{k-b}-x_{k-b}\right)+\left(\lambda_{k}-\lambda\right)\left(x_{k-a}-x_{k-b}\right),
$$

then

$$
\begin{equation*}
\left|L_{k}-x_{k}\right| \leq \lambda_{k}\left|L_{k-a}-x_{k-a}\right|+\left(1-\lambda_{k}\right)\left|L_{k-b}-x_{k-b}\right|+C_{2}\left|\lambda_{k}-\lambda\right|, \tag{5}
\end{equation*}
$$

since we can write $\left|x_{k-a}-x_{k-b}\right| \leq C_{2}$ from (4).

Now we prove by induction on $k$ that

$$
\begin{equation*}
\left|L_{k}-x_{k}\right| \leq C_{3} \ln k \quad(k \geq 1) \tag{6}
\end{equation*}
$$

for some constant $C_{3}$. Trivially true for $k=1, \ldots, b$, since $L_{k}=x_{k}=0$ for those $k$. Suppose $k \geq$ $b+1$, then $\frac{a}{k} \leq \frac{b}{k}<1$. By the induction hypothesis, (5), and the inequality $\ln (1-x) \leq x$, we have

$$
\begin{aligned}
& \left|L_{k}-x_{k}\right| \leq C_{3} \lambda_{k} \ln (k-a)+C_{3}\left(1-\lambda_{k}\right) \ln (k-b)+C_{2}\left|\lambda_{k}-\lambda\right| \\
& =C_{3}\left\{\lambda_{k}\left(\ln k+\ln \left(1-\frac{a}{k}\right)\right)+\left(1-\lambda_{k}\right)\left(\ln k+\ln \left(1-\frac{b}{k}\right)\right)\right\}+C_{2}\left|\lambda_{k}-\lambda\right| \\
& \leq C_{3}\left\{\ln k-\frac{1}{k}\left(a \lambda_{k}+b\left(1-\lambda_{k}\right)\right)\right\}+C_{2}\left|\lambda_{k}-\lambda\right| \\
& \leq C_{3} \ln k-\frac{a C_{3}}{k}+C_{2}\left|\lambda_{k}-\lambda\right| \leq C_{3} \ln k,
\end{aligned}
$$

where the last inequality holds because, by Lemma 1(d), we could have chosen $C_{3}$ large enough so that $-\frac{a C_{3}}{k}+C_{2}\left|\lambda_{k}-\lambda\right| \leq 0$ for $k \geq b+1$.

From (4) and (6), we obtain

$$
\left|L_{k}-\frac{(-\log \lambda)}{a H(c)} k-C_{1}-\varepsilon_{k}\right| \leq C_{3} \ln k ;
$$

hence,

$$
\left|\frac{L_{k}}{\log n_{k}}-\frac{1}{H(c)} \frac{(-\log \lambda)}{a} \frac{k}{\log n_{k}}-\frac{C_{1}+\varepsilon_{k}}{\log n_{k}}\right| \leq C_{3}\left(\frac{k}{\log n_{k}}\right)\left(\frac{\ln k}{k}\right) .
$$

Therefore, $\frac{L_{k}}{\log n_{k}} \rightarrow \frac{1}{H(c)}(k \rightarrow \infty)$ by Lemma $1(\mathrm{c})$.

## 3. CRITICAL BALANCE

A most pleasing, though rather vague, concept concerning the form of a tree might be the concept of being "balanced as a whole."

One natural definition of "balancedness" (let us call it "w-balanced") of the trees $S_{k}$ is:
$\left\{S_{k}\right\}$ is said to be $w$-balanced if $n_{k} \geq n_{k-a}+n_{k-2 a}$ for every $k \geq b+a+1$ (see [2]).
(Remark: $b+a+1$ is the minimum $k$ such that $n_{k} \geq 3$.)
Note that the definition takes this form to refer to the sequence $\left\{S_{k}\right\}$ not to individual $S_{k}$ for reason of compactness. Also note that the definition may be viewed as stemming from the fact that the condition $n_{k} \geq n_{k-a}+n_{k-2 a}$ can be written as

$$
n_{k-a}-\left(n_{k}-n_{k-a}\right) \leq\left(n_{k}-n_{k-2 a}\right)-n_{k-2 a},
$$

meaning that the division $n_{k-a}:\left(n_{k}-n_{k-a}\right)$ of $n_{k}$ is balanced better than or equally to the division $n_{k-2 a}:\left(n_{k}-n_{k-2 a}\right)$.

Another pretty concept of balancedness of a binary tree is due to Adelson-Velskii and Landis [1]. Denote the height of $S_{k}$ by $h_{k}=h_{k}(c)$, then their definition adapted to $S_{k}$ is:
$\left\{S_{k}\right\}$ is said to be $h$-balanced if $h_{k-a}-h_{k-b} \leq 1$ for every $k \geq b+a+1$.

We know from [2] that $h_{k}=\left\lceil\frac{k-b}{a}\right\rceil(k \geq b)$.
It should be mentioned here that, according to Nievergelt and Wong [5], $\left\{S_{k}\right\}$ may be called " $\alpha$-balanced" $\left(0<\alpha \leq \frac{1}{2}\right)$ if $\frac{n_{k-b}}{n_{k}} \geq \alpha$ holds for every $k \geq b+a+1$ and they showed that

$$
\left(\frac{L_{k}}{\log n_{k}}\right)(-\alpha \log \alpha-(1-\alpha) \log (1-\alpha)) \leq 1
$$

for $\alpha$-balanced $\left\{S_{k}\right\}$ [in place of $\left.G(c) H(c)=1\right]$.

## Lemma 2:

(a) $\left\{S_{k}\right\}$ is w-balanced if and only if $c \leq 2$.
(b) $\left\{S_{k}\right\}$ is h-balanced if and only if $c \leq 2$.
(c) $n_{k}=n_{k-b}+n_{k-2 a}$ for every $k \geq b+a+1$ if and only if $c=2$.
(d) $h_{k-a}-h_{k-b}=1$ for every $k \geq b+a+1$ if and only if $c=2$.

Proof: The proof is simple, comprising the following pieces $1 \sim 5$.

1. We first note that $n_{k}=n_{k-a}+n_{k-b}$, and hence the "if" part of (c) is obvious.
2. There are (infinitely) many $i$ such that $n_{i}<n_{i+1}$. So, if $c<2$ (i.e., $b<2 a$ ), we have $n_{k-2 a}<n_{k-b}$ for (infinitely) many $k$, and if $c>2$ (i.e., $b>2 a$ ), we have $n_{k-2 a}>n_{k-b}$ for (infinitely) many $k$. This proves the "only if" parts of (a) and (c). An alternative proof is: Divide both sides of $n_{k} \geq n_{k-a}+n_{k-2 a}$ by $n_{k}$ to obtain

$$
1 \geq\left(\frac{n_{k-a}}{n_{k}}\right)+\left(\frac{n_{k-a}}{n_{k}}\right)\left(\frac{n_{k-2 a}}{n_{k-a}}\right) .
$$

Let $k \rightarrow \infty$, then $1 \geq \lambda(c)+(\lambda(c))^{2}$. Therefore, we deduce $\lambda(c) \leq \frac{\sqrt{5}-1}{2}$, and using Lemma 1 (b) finishes the proof of those parts.
3. Proof of the "if" part of (a). Suppose $k \geq b+a+1$. Since $b \leq 2 a$ by $c \leq 2$, we have $n_{k-b} \geq n_{k-2 a}$. Hence, $\left\{S_{k}\right\}$ is w-balanced.
4. Suppose $c<2$. Then $b \leq 2 a-1$. Take $k=b+i a(i \geq 2)$ to see that

$$
\begin{aligned}
0 & \leq h_{k-a}-h_{k-b}=\left\lceil\frac{(k-a)-b}{a}\right\rceil-\left\lceil\frac{(k-b)-b}{a}\right\rceil=(i-1)-\left\lceil\frac{i a-b}{a}\right\rceil \\
& \leq(i-1)-\left\lceil\frac{i a-(2 a-1)}{a}\right\rceil=(i-1)-(i-2)-\left\lceil\frac{1}{a}\right\rceil=0 .
\end{aligned}
$$

That is, $h_{k-a}-h_{k-b}=0$ holds for (infinitely) many $k$.
Suppose $c>2$. Then $b \geq 2 a+1$. In this case, taking $k=b+i a+1(i \geq 2)$ leads us to $h_{k-a}-h_{k-b}=i-(i-2)=2$. That is, $h_{k-a}-h_{k-b}=2$ holds for (infinitely) many $k$.

The two remarks above prove the "only if" parts of (b) and (d).
5. Proof of the "if" parts of (b) and (d). Suppose $b+a+1 \leq k \leq b+2 a$. Then, since $b+1 \leq$ $k-a \leq b+a$, we have $h_{k-a}-h_{k-b}(=1-0$ or $1-1) \leq 1$. (Furthermore, if $c=2$, then $k-b \leq b$ and $h_{k-a}-h_{k-b}=1-0=1$.)

Suppose next that $k \geq b+2 a+1$. From $b \leq 2 a$, we have

$$
\frac{(k-a)-b}{a} \leq \frac{(k-b)-b}{a}+1,
$$

and hence, by noting that $k-b \geq 2 a+1 \geq b+1$, we have $h_{k-a} \leq h_{k-b}+1$. Therefore, $\left\{S_{k}\right\}$ is hbalanced. (Furthermore, if $c=2$, then $h_{k-a}=h_{k-b}+1$.)

The (asymptotic) average growth function $G(c)$ is strictly monotone increasing because the entropy $H(c)$ is strictly monotone decreasing. Therefore, the $c$ maximizing $G(c)$ while keeping the $S_{k}$ balanced for every $k$ equals 2 .

## SUMMARY

Summarizing, we may say that the Fibonacci tree is critically balanced, and in this sense the Golden-cut point $\lambda(2)$ might be interpreted as the critical balancing point.

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# GENERALIZED FIBONACCI SEQUENCES AND LINEAR CONGRUENCES 

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## 1. INTRODUCTION

There exists a very wide literature about the generalized Fibonacci sequences (see, e.g., [3], where interesting applications to number theory are also shown, and [2], where such sequences are treated as a particular case of a more general class of sequences of numbers). In this paper we start by defining some particular generalized Fibonacci sequences (denoted by $\left\{U_{n}(c-1,-c)\right\}_{n \in \mathbb{N}}$, $c \in \mathbb{N}$ ) and by studying their properties. In particular, we find interesting relations between a generic term $U_{n}(c-1,-c), n \in \mathbb{N}$, and $U_{n+1}(c-1,-c)$ and show a nice connection between the numbers $U_{n}(c-1,-c)$ and their expression in the $c$-ary enumeration system. After this, we give an estimate of the value of the logarithm of $U_{n}(c-1,-c)$ on the basis $c$.

Successively, we apply the properties of the sequences $\left\{U_{n}(c-1,-c)\right\}_{n \in \mathbb{N}}$ to the study of the number of solutions of linear equations in $\mathbb{Z}_{r}, r \in \mathbb{N}$.

Finally, we briefly show the principal characteristics of another class of generalized Fibonacci sequences, $\left\{U_{n}(c+1, c)\right\}_{n \in \mathbb{N}}, c \in \mathbb{N} \backslash\{1\}$.

## 2. GENERALIZED FIBONACCI SEQUENCES: THE SEQUENCES $\left\{\boldsymbol{U}_{\boldsymbol{n}}(\boldsymbol{c}-\mathbf{1},-\boldsymbol{c})\right\}_{n \in \mathbb{N}}$

For each pair $(h, k), h, k \in \mathbb{C}$ of complex numbers such that $k\left(h^{2}-4 k\right) \neq 0$, we denote by $\left\{U_{n}(h, k)\right\}_{n \in \mathbb{N}}$ the generalized Fibonacci sequence defined as follows:

$$
\forall n \in \mathbb{N}, n \geq 2, U_{n}(h, k)=h U_{n-1}(h, k)-k U_{n-2}(h, k), U_{0}(h, k)=0, U_{1}(h, k)=1 .
$$

An explicit expression of the $n^{\text {th }}$ term of $\left\{U_{n}(h, k)\right\}_{n \in \mathbb{N}}$ for generic $n \in \mathbb{N} \cup\{0\}$ is given by the Binet formula

$$
U_{n}(h, k)=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta},
$$

where

$$
\alpha=\frac{h+\sqrt{h^{2}-4 k}}{2} \text { and } \beta=\frac{h-\sqrt{h^{2}-4 k}}{2}
$$

are the distinct roots of the polynomial $x^{2}-h x+k \in \mathbb{C}[x]$, called the characteristic polynomial of the sequence. Moreover, for every integer $n \in \mathbb{N} \cup\{0\}$, we have

$$
\alpha \cdot \frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}+\beta^{n}=\frac{\alpha^{n+1}-\alpha \beta^{n}+\alpha \beta^{n}-\beta^{n+1}}{\alpha-\beta}=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}
$$

We then obtain

$$
\begin{equation*}
\forall n \in \mathbb{N} \cup\{0\}, \quad \alpha \cdot U_{n}(h, k)+\beta^{n}=U_{n+1}(h, k) . \tag{1}
\end{equation*}
$$

As the role played by $\alpha$ and $\beta$ in the Binet formulas is symmetric, the following equalities are also true:

$$
\begin{equation*}
\forall n \in \mathbb{N} \cup\{0\}, \quad \beta \cdot U_{n}(h, k)+\alpha^{n}=U_{n+1}(h, k) . \tag{2}
\end{equation*}
$$

As a particular case, let us consider now the generalized Fibonacci sequences of the form $\left\{U_{n}(c-1,-c)\right\}_{n \in \mathbb{N}}, c$ being a positive integer; from the equalities $h=c-1$ and $k=-c$, we easily obtain $\alpha=c$ and $\beta=-1$. Then, for all $n \in \mathbb{N} \cup\{0\}$, from the Binet formula we have

$$
U_{n}(c-1,-c)=\frac{c^{n}-(-1)^{n}}{c+1}
$$

while equalities (1) and (2) show, respectively, that

$$
\begin{equation*}
\forall n \in \mathbb{N} \cup\{0\}, U_{n+1}(c-1,-c)=c U_{n}(c-1,-c)+(-1)^{n} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\forall n \in \mathbb{N} \cup\{0\}, U_{n}(c-1,-c)+U_{n+1}(c-1,-c)=c^{n} \tag{4}
\end{equation*}
$$

The first terms of some of such generalized Fibonacci sequences, corresponding to fixed values of $c$, are:

$$
\begin{aligned}
& \left\{U_{n}(0,-1)\right\}_{n \in \mathbb{N}}: 0,1,0,1,0,1,0,1,0,1,0,1, \ldots ; \\
& \left\{U_{n}(1,-2)\right\}_{n \in \mathbb{N}}: 0,1,1,3,5,11,21,43,85,171,341,683, \ldots \\
& \left\{U_{n}(2,-3)\right\}_{n \in \mathbb{N}}: 0,1,2,7,20,61,182,547,1640,4921, \ldots ; \\
& \left\{U_{n}(3,-4)\right\}_{n \in \mathbb{N}}: 0,1,3,13,51,205,819,3277,13107,52429, \ldots ; \\
& \left\{U_{n}(5,-6)\right\}_{n \in \mathbb{N}}: 0,1,5,31,185,1111,6665,39991,239945, \ldots
\end{aligned}
$$

## 3. $\left\{U_{n}(c-1,-c)\right\}_{n \in N}(c \geq 2)$ IN THE $c$-ARY ENUMERATION SYSTEM

Theorem: Let $c \geq 2$ be a fixed integer; then, for each fixed integer $m \geq 2$, the two following assertions are equivalent:
(a) $\exists n \in \mathbb{N}: m=U_{n}(c-1,-c)$;
(b) in the $c$-ary enumeration system, the expression of $m$ is either of the form $(c-1) 0(c-1) \ldots 0(c-1)$ or of the form $(c-1) 0(c-1) \ldots 0(c-1) 1$.
Moreover, when for a given $m$ the two assertions are satisfied, we have $m=U_{d+1}(c-1,-c)$, where $t$ denotes the number of digits of $m$ which appear when it is written in the $c$-ary enumeration system.

The theorem can be proven by noticing that, for every $n \in \mathbb{N} \cup\{0\}$, we have the recursion $U_{n+1}(c-1,-c)=c U_{n}(c-1,-c)+(-1)^{n}$. Hence, if (a) is satisfied, assertion (b) straightforwardly follows by induction from the first few terms:

$$
\begin{aligned}
U_{2}(c-1,-c) & =c \cdot 1-1=\mathfrak{c}-\mathbb{1} ; \\
U_{3}(c-1,-c) & =c \cdot(c-1)+1=10 \cdot(c-1)+\mathbb{1}=(c-1) 0+\mathbb{1}=(c-1) \mathbb{1} ; \\
U_{4}(c-1,-c) & =c \cdot U_{3}(c-1,-c)-1=10 \cdot[(c-1) 1]-1=(c-1) 10-1 \\
& =(c-1) 0(c-1) ; \\
U_{5}(c-1,-c) & =c \cdot U_{4}(c-1,-c)+1=10 \cdot[(\mathbb{c}-1) 0(c-1)]+\mathbb{1} \\
& =(\mathfrak{c}-\mathbb{1}) 0(c-1) 0+1=(c-1) 0(c-1) \mathbb{1} ;
\end{aligned}
$$

$$
\begin{aligned}
U_{6}(c-1,-c) & =c \cdot U_{5}(c-1,-c)-1=10 \cdot[(c-1) 0(c-1) 1]-1 \\
& =(c-1) 0(c-1) 10-1=(c-1) 0(c-1) 0(c-1) .
\end{aligned}
$$

(For the sake of clarity, the convention was adopted of writing the $c$-ary expressions in boldface characters; the dot denotes multiplication.) Conversely, if (b) is satisfied, $m$ is clearly seen to be a term of the sequence $\left\{U_{n}(c-1,-c)\right\}_{n \in \mathbb{N}}$ by applying a finite number of times the recursion $U_{n+1}(c-1,-c)=c U_{n}(c-1,-c)+(-1)^{n}$, and assertion (a) follows.

Moreover, it is clear that, for every $n \geq 2$, the number of digits of $U_{n+1}(c-1,-c)$ when it is written in the $c$-ary system is one unit larger than the number of digits of $U_{n}(c-1,-c)$ when it is expressed in the same system. Since in the $c$-ary system the number $U_{2}(c-1,-c)$ is expressed by the only digit $c-1$, the second part of the theorem follows by induction.

## 4. AN ESTIMATE OF $\log _{c}\left(U_{n}(c-1,-c)\right)(c \geq 2, n \geq 1)$

For any $c \geq 2$ and $n \geq 1$, we know that

$$
U_{n}(c-1,-c)=\frac{c^{n}-(-1)^{n}}{c+1}
$$

hence, we have $\log _{c}\left(U_{n}(c-1,-c)\right)=\log _{c}\left(c^{n}-(-1)^{n}\right)-\log _{c}(c+1)$, which is equal to

$$
\log _{c}\left[c^{n}\left(1-\frac{(-1)^{n}}{c^{n}}\right)\right]-\log _{c}\left[c\left(1+\frac{1}{c}\right)\right]=n-1+\log _{c}\left(1-\frac{(-1)^{n}}{c^{n}}\right)-\log _{c}\left(1+\frac{1}{c}\right)
$$

Now we suppose $c$ fixed and consider $\log _{c}\left(U_{n}(c-1,-c)\right)$ as a function of $n$. Since

$$
\frac{\ln (1+y)}{y}=1+o(1) \text { as } y \rightarrow 0
$$

we have $\ln (1+y)=y+o(y)(y \rightarrow 0) ; \log _{c}(1+y)=\frac{y}{\ln c}+o(y)(y \rightarrow 0)$. Then, for $n \rightarrow+\infty$, we can write

$$
\log _{c}\left(1-\frac{(-1)^{n}}{c^{n}}\right)=\frac{(-1)^{n-1}}{c^{n} \ln c}+o\left(\frac{1}{c^{n}}\right)(n \rightarrow+\infty)
$$

On the other hand, for every positive real number $x$, the following inequalities hold: $0<\ln (1+x)$ $<x$; hence, we have $0<\log _{c}(1+x)<\frac{x}{\ln c}$. Taking $x=\frac{1}{c}$, we obtain

$$
0<\log _{c}\left(1+\frac{1}{c}\right)<\frac{1}{c \ln c} .
$$

Then, from the above equalities we have, when setting $\gamma(c)=\log _{c}\left(1+\frac{1}{c}\right)$, the approximation of $\log _{c}\left(U_{n}(c-1,-c)\right)$ holding for $n$ large,

$$
\begin{aligned}
\log _{c}\left(U_{n}(c-1,-c)\right) & =n-1+\log _{c}\left(1-\frac{(-1)^{n}}{c^{n}}\right)-\log _{c}\left(1+\frac{1}{c}\right) \\
& =n-1-\gamma(c)+\frac{(-1)^{n-1}}{c^{n} \ln c}+o\left(\frac{1}{c^{n}}\right) \quad(n \rightarrow+\infty)
\end{aligned}
$$

where $0<\gamma(c)<\frac{1}{c \ln c}$.

## 5. LINEAR EQUATIONS $\mathbb{N} \mathbb{Z}_{r}$ AND THEIR RELATION WITH THE SEQUENCES $\left\{U_{n}(c-1,-c)\right\}_{n \in N}$

We consider the problem of finding the elements $\left(x_{1} ; x_{2} ; \ldots ; x_{k}\right) \in\left(\mathbb{Z}_{r}\right)^{k}$ which satisfy the congruence equation

$$
\begin{equation*}
\sum_{j=1}^{k} x_{j} \equiv a(\bmod r), \tag{5}
\end{equation*}
$$

and the constraining equalities

$$
\begin{equation*}
\operatorname{gcd}\left(x_{j}, r\right)=d_{j} ; j=1,2, \ldots, k, \tag{6}
\end{equation*}
$$

where $r$ and $k$ are fixed positive integers, $r$ is odd, $a \in \mathbb{Z}_{r}$, and $d_{1}, d_{2}, \ldots, d_{k}$ are $k$ divisors (not necessarily distinct) of $r$. Let us pose, for each prime divisor $p$ of $r, b_{p}=\#\left(\left\{j, 1 \leq j \leq k: p \nmid d_{j}\right\}\right)$, and let us assume that, for each $p, b_{p} \geq 2$.

Starting from formulas which give the total number $N_{a}$ of solutions of the above problem (see [1], eq. (3.37), and [4], ex. 3.8, p. 138), replacing in such formulas Ramanujan sums by their expressions as given by Hölder's equalities, i.e.,

$$
\forall m, n \in \mathbb{N}, c(m ; n)=\sum_{\substack{j=1 \\ \operatorname{gcd}(j, n)=1}}^{n}\left(e^{2 \pi i / n}\right)^{j m}=\frac{\varphi(n)}{\varphi(n / \operatorname{gcd}(n, m))} \cdot \mu(n / \operatorname{gcd}(n, m)),
$$

$\varphi$ and $\mu$ being, respectively, Euler's and Möbius' functions (see [5]), and then using basic properties of $\varphi$ and $\mu$ and applying (in reverse order) the distributive property of the product with respect to the sum, gives rise to the following equality:

$$
\begin{equation*}
N_{a}=\frac{\varphi\left(r / d_{1}\right) \varphi\left(r / d_{2}\right) \ldots \varphi\left(r / d_{k}\right)}{r} \cdot P_{a}, \tag{7}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{a}=\prod_{p \mid r, p \nmid a}\left[1-\frac{(-1)^{b_{p}}}{(p-1)^{b_{p}}}\right] \cdot \prod_{p|r, p| a}\left[1-\frac{(-1)^{b_{p}-1}}{(p-1)^{b_{p}-1}}\right] . \tag{8}
\end{equation*}
$$

The latter formula can be found in [5] for the special case $d_{1}=d_{2}=\cdots=d_{k}=1$ only. Compare equalities (7) and (8) also with [6].

Now we want to rewrite equality (8) in terms of the generalized Fibonacci sequences that we treated in the previous sections. First, we observe that, for each prime divisor $p$ of $r$, by applying the Binet formula to the terms of $\left\{U_{n}(c-1,-c)\right\}_{n \in \mathbb{N}}$ in the case in which $c=p-1$, we have, for each nonnegative integer $n$,

$$
U_{n}(p-2,1-p)=\frac{(p-1)^{n}-(-1)^{n}}{p}
$$

i.e., $p U_{n}(p-2,1-p)=(p-1)^{n}-(-1)^{n}$. Hence, from (8), we obtain

$$
\begin{aligned}
P_{a} & =\prod_{p|r, p| a}\left[\frac{(p-1)^{b_{p}}-(-1)^{b_{p}}}{(p-1)^{b_{p}}}\right] \cdot \prod_{p|r, p| a}\left[\frac{(p-1)^{b_{p}-1}-(-1)^{b_{p}-1}}{(p-1)^{b_{p}-1}}\right] \\
& =\prod_{p|r, p| a}\left[\frac{p \cdot U_{b_{p}}(p-2,1-p)}{(p-1)^{b_{p}}}\right] \cdot \prod_{p|r, p| a}\left[\frac{p \cdot U_{b_{p}-1}(p-2,1-p)}{(p-1)^{b_{p}-1}}\right]
\end{aligned}
$$

$$
\begin{equation*}
=\prod_{p \mid r}\left[\frac{p}{(p-1)^{b_{p}-1}}\right] \cdot \prod_{p \mid r, p \nmid a}\left[\frac{U_{b_{p}}(p-2,1-p)}{p-1}\right] \cdot \prod_{p|r, p| a} U_{b_{p}-1}(p-2,1-p) . \tag{9}
\end{equation*}
$$

Now let us fix a prime divisor $q$ of $r$ and let $u$ be a residue class in $\mathbb{Z}_{r}$ such that $q \nmid u$. We want to calculate the ratio of $P_{q u}$ to $P_{u}$. From expression (9) of $P_{a}$ for generic $a$, comparing the case in which $a=q u$ with the case in which $a=u$, we immediately obtain

$$
\begin{equation*}
\frac{P_{q u}}{P_{u}}=\frac{U_{b_{q}-1}(q-2,1-q)}{U_{b_{q}}(q-2,1-q) /(q-1)}=\frac{(q-1) U_{b_{q}-1}(q-2,1-q)}{U_{b_{q}}(q-2,1-q)} . \tag{10}
\end{equation*}
$$

Moreover, from (3), taking $c=q-1$ and $n=b_{q}-1$, we obtain

$$
U_{b_{q}}(q-2,1-q)=(q-1) U_{b_{q}-1}(q-2,1-q)+(-1)^{b_{q}-1}
$$

i.e., $(q-1) U_{b_{q}-1}(q-2,1-q)=U_{b_{q}}(q-2,1-q)+(-1)^{b_{q}}$, and hence

$$
\begin{equation*}
\frac{P_{q u}}{P_{u}}=\frac{U_{b_{q}}(q-2,1-q)+(-1)^{b_{q}}}{U_{b_{q}}(q-2,1-q)}=1+\frac{(-1)^{b_{q}}}{U_{b_{q}}(q-2,1-q)} . \tag{11}
\end{equation*}
$$

Equations (11) show that the ratio $P_{q u} / P_{u}$ depends on $q$, but is independent of $u$. They also show that, when $b_{q}$ is even, then $P_{q u}>P_{u}$, while when $b_{q}$ is odd, then $P_{q u}<P_{u}$. This means that a sum having an even number of addenda which are not multiples of $q$ tends to favor as possible results the multiples of $q$, while a sum having an odd number of addenda which are not multiples of $q$ tends to favor the numbers which are not multiples of $q$. Moreover, since $r$ is odd (which implies $q \geq 3$ ) and for $c \geq 2$ the integer $U_{n}(c-1,-c)$ tends to infinity as $n \rightarrow+\infty$, equations (11) show that the greater $b_{q}$, the nearer one to another are the values of $P_{q u}$ and $P_{u}$. This means that if in a sum there are many addenda which are not multiples of $q$, then the sum tends to favor significantly neither the multiples of $q$ nor the integers which are not multiples of $q$. More generally, in view of (7) and (8), the distribution in $\mathbb{Z}_{r}$ of the values of the expression $\sum_{j=1}^{k} x_{j}$ as $x_{1}, x_{2}, \ldots, x_{k}$ vary in $\mathbb{Z}_{r}^{*}$, tends to be a uniform distribution as $k$ tends to infinity (because $P_{a}$ tends to 1 and $N_{a}$ becomes independent of $a$ ).

Furthermore, if $q^{2} \nmid r$, then for each residue class $a$ in $\mathbb{Z}_{r}$ which is a multiple of $q$, there exist exactly $q-1$ classes $u$ in $\mathbb{Z}_{r}$ not multiples of $q$ such that $a \equiv q u(\bmod r)$. In this case, from equations (10), dividing $P_{q u} / P_{u}$ by $q-1$, we obtain the number

$$
\begin{equation*}
\frac{U_{b_{q}-1}(q-2,1-q)}{U_{b_{q}}(q-2,1-q)}, \tag{12}
\end{equation*}
$$

which, being independent of $a$, can be considered as the ratio of the number of the strings ( $x_{1}$; $x_{2} ; \ldots ; x_{k}$ ) such that $q \mid \sum_{j=1}^{k} x_{j}$ to the number of the strings $\left(x_{1} ; x_{2} ; \ldots ; x_{k}\right)$ such that $q \nmid \sum_{j=1}^{k} x_{j}$.

We now give an example of what was discussed in this section. Let the following problem be assigned:

$$
\sum_{j=1}^{7} x_{j} \equiv a(\bmod 3), \operatorname{gcd}\left(x_{j}, 3\right)=1 \text { for } j=1,2, \ldots, 7
$$

We want to calculate the ratio $N_{0} / N_{1}$.

By taking $q=3$ and $u=1$, we have $b_{q}=7$ and then, by (11), we can write

$$
\frac{N_{0}}{N_{1}}=\frac{N_{3}}{N_{1}}=\frac{P_{3}}{P_{1}}=1+\frac{(-1)^{7}}{U_{7}(1,-2)}=1-\frac{1}{43}=\frac{42}{43} .
$$

To obtain the ratio of the number of strings $\left(x_{1} ; x_{2} ; \ldots ; x_{7}\right) \in\left(\mathbb{Z}_{3}^{*}\right)^{7}$ such that $3 \mid \Sigma_{j=1}^{7} x_{j}$ to the number of strings $\left(x_{1} ; x_{2} ; \ldots ; x_{7}\right) \in\left(\mathbb{Z}_{3}^{*}\right)^{7}$ such that $3 \backslash \sum_{j=1}^{7} x_{j}$, we use expression (12) and find that this ratio is equal to $\frac{U_{6}(1,-2)}{U_{7}(1,-2)}$, i.e., to $\frac{21}{43}$.

## 6. THE SEQUENCES $\left\{U_{n}(c+1, c)\right\}_{n \in N}$

Another interesting class of generalized Fibonacci sequences is the set $\left\{U_{n}(c+1, c)\right\}_{n \in \mathbb{N}}$, i.e., of the sequences whose characteristic polynomial has $c$ and 1 as roots, $c$ being a positive integer not equal to 1 .

For all $n \in \mathbb{N} \cup\{0\}$, we have the Binet formulas

$$
U_{n}(c+1, c)=\frac{c^{n}-1}{c-1} ; \text { then } \forall n \in \mathbb{N}, U_{n}(c+1, c)=c^{n-1}+c^{n-2}+\cdots+c+1 .
$$

Some examples of such sequences are:

$$
\begin{aligned}
& \left\{U_{n}(3,2)\right\}_{n \in \mathbb{N}}: 0,1,3,7,15,31,63,127, \ldots ; \\
& \left\{U_{n}(4,3)\right\}_{n \mathbb{N}}: 0,1,4,13,40,121,364,1093, \ldots ; \\
& \left\{U_{n}(5,4)\right\}_{n \in \mathbb{N}}: 0,1,5,21,85,341,1365,5461, \ldots \\
& \left\{U_{n}(6,5)\right\}_{n \in \mathbb{N}}: 0,1,6,31,156,781,3906,19531, \ldots
\end{aligned}
$$

From equalities (1) and (2) we have, respectively,

$$
\forall n \in \mathbb{N} \cup\{0\}, U_{n+1}(c+1, c)=c U_{n}(c+1, c)+1
$$

and

$$
\forall n \in \mathbb{N} \cup\{0\}, U_{n+1}(c+1, c)=U_{n}(c+1, c)+c^{n}
$$

For a fixed $c$, it is clear that the terms of $\left\{U_{n}(c+1, c)\right\}_{n \in \mathbb{N}}$, if we exclude the first term 0 , are exactly the integers which in the $c$-ary system are written in the form 11..1. Moreover, for each $n \in \mathbb{N}$, the number of digits " 1 " that appear in the expression of $U_{n}(c+1, c)$ in the $c$-ary system is $n$.

For any $c \geq 2$ and $n \geq 1$, we have $\log _{c}\left(U_{n}(c+1, c)\right)=\log _{c}\left(c^{n}-1\right)-\log _{c}(c-1)$, which is equal to

$$
n-1+\log _{c}\left(1-\frac{1}{c^{n}}\right)-\log _{c}\left(1-\frac{1}{c}\right) .
$$

Since $\log _{c}(1+y)=\frac{y}{\ln c}+o(y)(y \rightarrow 0)$,

$$
\log _{c}\left(1-\frac{1}{c^{n}}\right)=-\frac{1}{c^{n} \ln c}+o\left(\frac{1}{c^{n}}\right)(n \rightarrow+\infty) .
$$

Further,

$$
-\frac{1}{c-1}<\ln \left(1-\frac{1}{c}\right)<0 .
$$

Therefore, we deduce

$$
-\frac{1}{(c-1) \ln c}<\log _{c}\left(1-\frac{1}{c}\right)<0 .
$$

Now we can write, setting

$$
\delta(c)=\left|\log _{c}\left(1-\frac{1}{c}\right)\right|=\log _{c}\left(1+\frac{1}{c-1}\right),
$$

the approximation to $\log _{c}\left(U_{n}(c+1, c)\right)$ holding for large $n$,

$$
\begin{aligned}
\log _{c}\left(U_{n}(c+1, c)\right) & =n-1+\log _{c}\left(1-\frac{1}{c^{n}}\right)-\log _{c}\left(1-\frac{1}{c}\right) \\
& =n-1+\delta(c)-\frac{1}{c^{n} \ln c}+o\left(\frac{1}{c^{n}}\right)(n \rightarrow+\infty),
\end{aligned}
$$

where $0<\delta(c)<\frac{1}{(c-1) \ln c}$.

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# APPLICATION OF MARKOV CHAINS PROPERTIES TO o-GENERALIZED FIBONACCI SEQUENCES 

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## 1. INTRODUCTION

The idea of $\infty$-generalized Fibonacci sequences began with Euler, who discussed Daniel Bernoulli's method of using linear recurrences to approximate roots of (mainly polynomial) equations (see [4], article 355). Recently, such sequences have been introduced and studied in [10], [11], and [14]. They are defined as follows: Let $\left\{a_{j}\right\}_{j=0}^{+\infty}$ be a sequence of real numbers and consider the sequence $\left\{V_{n}\right\}_{n \in \mathbb{Z}}$ defined by the following linear recurrence relation of order $\infty$,

$$
\begin{equation*}
V_{n+1}=\sum_{m=0}^{+\infty} a_{m} V_{n-m} \text { if } n \geq 0 \tag{1}
\end{equation*}
$$

where $\left\{V_{-j}\right\}_{j=0}^{+\infty}$ are specified by the initial conditions. We shall refer to them in the sequel as sequences (1). They are an extension of r-generalized Fibonacci sequences (see, e.g., [3], [8], and [9]) and their general term $V_{n}(n \geq 1)$ does not always exist. Hence, they were studied under some conditions on the sequences of coefficients $\left\{a_{j}\right\}_{j=0}^{+\infty}$ and the initial conditions $\left\{V_{-j}\right\}_{j=0}^{+\infty}$ (see [10], [11], and [14]).

The aim of this paper is to study the combinatoric expression of sequences (1) and extend the results of [13]. When the coefficients are nonnegative with sum 1 , this expression is derived from properties of Markov chains. By induction we see also that this expression is still valid for arbitrary coefficients (Section 2). For the case of arbitrary nonnegative coefficients, we give the asymptotic behavior of $V_{n}$ (Section 3).

## 2. MARKOV CHAINS AND COMBINATORIC EXPRESSION OF $V_{n}$

### 2.1 Fundamental Hypotheses

It was shown in [10], [11], and [14] that the general term $V_{n}$ of a sequence (1) does not exist in general. Therefore, we need some necessary hypotheses on $\left\{a_{n}\right\}_{n \geq 0}$ and $\left\{V_{-n}\right\}_{n \geq 0}$ which insure the existence of $V_{n}$ for every $n \geq 1$. In this paper we are interested in the following hypotheses:

- (H.1) For every $m$, we have $a_{m} \geq 0$ and there exists $k \geq m$ such that $a_{k}>0$;
- (H.2) There exists $C>0$ such that $a_{m} \leq C$ for any $m$;
- (H.3) The series $\sum_{m=0}^{+\infty}\left|V_{-m}\right|$ is convergent.

These hypotheses are compatible with the Markov chains formulation of sequences (1).

### 2.2 Sequences (1) and Markov Chains

Let $\left\{a_{j}\right\}_{j \geq 0}$ be a sequence of real numbers which satisfies (H.1). Suppose that the following condition is satisfied:

$$
\begin{equation*}
\sum_{m=0}^{+\infty} a_{m}=1 \tag{2}
\end{equation*}
$$

Condition (2) shows that (H.2) is trivially verified. Consider the following matrix:

$$
\left.P=\begin{array}{rccccccc} 
& & \cdots & -1 & 0 & 1 & 2 & \cdots \\
 \tag{3}\\
\vdots \\
-1 \\
0 \\
1 \\
2 & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots \\
\cdots & 0 & 1 & 0 & 0 & & 0 \\
\cdots & 0 & 0 & 1 & 0 & & 0 & \\
\vdots & \cdots & a_{2} & a_{1} & a_{0} & 0 & \cdots & \\
\cdots & a_{3} & a_{1} & a_{0} & 0 & \cdots & \\
\cdots & a_{4} & a_{3} & a_{2} & a_{1} & a_{0} & 0 & \cdots \\
\cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}\right) .
$$

If we set $P=(P(n, m))_{n, m \in \mathbb{Z}}$, we have $P(n, m)=\delta_{n, m}$ for $n, m \in\{\cdots,-1,0\}, P(n, m)=a_{n-m-1}$ for $n>0$ and $n-m-1 \geq 0$, and $P(n, m)=0$ elsewhere. Condition (2) shows that $P$ is a stochastic matrix. Therefore, $P$ is a transition matrix of a Markov chain $(\mathfrak{I})$ whose state space is $\mathbb{Z}=\{\cdots$, $-1,0,1, \cdots\}$. The states $\cdots,-2,-1,0$ are absorbing states and $1,2, \cdots$ are transient states.

Consider the following infinite vector $X=\left(\cdots, V_{-m}, \cdots, V_{0}, \cdots, V_{n}, \cdots\right)^{t}$. Then a sequence (1) can be written in the following matrix form:

$$
\begin{equation*}
X=P X \tag{4}
\end{equation*}
$$

The preceding infinite matrix product (4) is simply $V_{n}=\Sigma_{m<n} P(n, m) V_{m}$. In the same way, matrix $P^{2}=\left(P^{(2)}(n, m)\right)_{n, m \in \mathbb{Z}}$ is given by $P^{(2)}(n, m)=\sum_{m+1 \leq j \leq n-1} P(n, j) P(j, m)$ for every $m>0, n>0$. By induction, we also define the matrix $P^{k}=\left(P^{(k)}(n, m)\right)_{n, m \in \mathbb{Z}}$. Equation (4) shows that $X=P^{k} X$ for every $k \geq 1$. Thus,

$$
\begin{equation*}
X=Q_{k} X, \text { where } Q_{k}=\frac{P+P^{2}+\cdots+P^{k}}{k} . \tag{5}
\end{equation*}
$$

Properties of Césaro mean convergence, applied to the matrix sequence $\left\{P^{k}\right\}_{k \geq 1}$ (see, e.g., [6] and [7]), allows us to state the following proposition.
Proposition 2.2: Let $P$ be a stochastic matrix defined by (3). Then, the sequence $\left\{Q_{k}\right\}_{k \geq 1}$ given by (5) converges (when $k \rightarrow+\infty$ ) to the following matrix,

$$
\left.Q=\begin{array}{cccccccc} 
& & \cdots & -1 & 0 & 1 & \cdots &  \tag{6}\\
\vdots \\
-1 \\
0 \\
1 \\
2 & \left(\begin{array}{ccccc}
\ddots & \ddots & \ddots & \ddots & \vdots \\
& & \vdots \\
3 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 \\
\vdots & \cdots & \rho(1,-m) & \cdots & 1 \\
\cdots(1,0) & 0 & \cdots & 0 \\
\cdots & \rho(2,-m) & \cdots & \rho(2,0) & 0 \\
\cdots \\
\cdots & \rho(3,-m) & \cdots & \rho(3,0) & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots
\end{array}\right),
\end{array}\right)
$$

where $\rho(k,-m)$ for $k \geq 1$ and $m \geq 0$ is the probability of absorption of the system by the state $-m$ when it starts from $k$.

Relation (5) and Proposition 2.2 show that $X=Q X$, where $Q$ is the matrix given by (6). Therefore, using the matrix product (4), we prove the following extension of Theorem 2.2 of [13].

Theorem 2.3: Let $\left\{V_{n}\right\}_{n \in \mathbb{Z}}$ be a sequence (1) such that (H.1), (H.3), and (2) are verified. Then, for every $n \geq 1$, we have

$$
\begin{equation*}
V_{n}=\sum_{m=0}^{+\infty} \rho(n,-m) V_{-m} . \tag{7}
\end{equation*}
$$

Expression (7) gives $V_{n}(n \geq 1)$ as a linear combination of the initial conditions and the absorption probabilities $\rho(k,-m)(k \geq 1, m \geq 0)$.

### 2.3 Computation of the $\rho(n, m)$

The computation of $\rho(n, m)$ and $\rho(n,-m)$ is the same as in [13].
Case of $n>m>0$. In this case, $\rho(n, m)$ is the probability of reaching the transient state $m$ starting from the initial one $n$. The system, starting from $n$, will go to $m$ after one transition with the probability $P(n, m)=a_{n-m-1}$. We say that the system had made a jump of $n-m$ units. To go from $n$ to $m(n>m)$, the system must make $k_{j}$ jumps of $j+1$ units with probability $a_{j}(j \geq 0)$. Since the total displacement is $n-m$, we have $k_{0}+2 k_{1}+\cdots+(n-m) k_{n-m-1}=n-m$, and the total number of units of this displacement is $k_{0}+k_{1}+\cdots+k_{n-m-1}$. The number of ways to choose $k_{0}, k_{1}, \ldots, k_{n-m-1}$ is

$$
\frac{\left(k_{0}+k_{1}+\cdots+k_{n-m-1}\right)!}{k_{0}!k_{1}!\ldots k_{n-m-1}!}
$$

and the probability of each choice is $a_{0}^{k_{0}} a_{1}^{k_{1}} \ldots a_{n-m-1}^{k_{n-m}}$. Therefore, we have

$$
\begin{equation*}
\rho(n, m)=\sum_{\sum_{j=0}^{n-m-1}(j+1) k_{j}=n-m} \frac{\left(\sum_{j=0}^{n-m-1} k_{j}\right)!}{k_{0}!k_{1}!\ldots k_{n-m-1}!} a_{0}^{k_{0}} a_{1}^{k_{1}} \ldots a_{n-m-1}^{k_{n}-m-1} . \tag{8}
\end{equation*}
$$

From (8), we prove easily that

$$
\begin{equation*}
\rho(n, m)=\rho(n-m, 0) \text { and } \rho(0,0)=1 \tag{9}
\end{equation*}
$$

We note that for $n>m \geq 0$ we have

$$
\rho(n, m)=H_{n-m+1}^{(n-m+2)}\left(a_{0}, \ldots, a_{n-m+1}\right),
$$

where $\left\{H_{n-m+1}^{(s)}\left(a_{0}, \ldots, a_{s}\right)\right\}_{n \geq 0}$ is the sequence of multivariate Fibonacci polynomials of Philippou of order $s$ (see [1]).

Case of $n>0$ and $-m \leq 0$. In this case, $n$ is a transient state and $-m$ is an absorbing one. To go from $n$ to $-m$, the last transient state visited by the system is $s$, where $0<s<n$. And to go from $s$ to $-m$, the system must make only one jump with probability $a_{s+m-1}$. Since $\rho(n, s)$ is the probability of going from $n$ to $s$, we show that the probability of absorption of the system by the state $-m$ when it starts from $n>0$ is $\rho(n,-m)=a_{n+m-1}+\sum_{s=1}^{n} \rho(n, s) a_{s+m-1}$. Therefore, using (9), we establish the following expression:

$$
\begin{equation*}
\rho(n,-m)=\sum_{s=1}^{n} \rho(n-s, 0) a_{s+m-1} . \tag{10}
\end{equation*}
$$

### 2.4 Combinatoric Expression of $V_{n}(n \geq 1)$

The substitution of (10) in (7) allows us to obtain

$$
\begin{equation*}
V_{n}=\sum_{m=0}^{+\infty}\left\{\sum_{s=1}^{n} \rho(n-s, 0) a_{s+m-1}\right\} V_{-m} \tag{11}
\end{equation*}
$$

for every $n \geq 1$. The two hypotheses (H.2)-(H.3) show that we can make the permutation of the two sums in (11). Therefore, we prove the following result.
Theorem 2.4: Let $\left\{V_{n}\right\}_{n \in \mathbb{Z}}$ be a sequence (1) such that (H.1), (H.3), and (2) are verified. Then we have

$$
\begin{equation*}
V_{n}=\sum_{s=1}^{n} A_{s} \rho(n-s, 0) \tag{12}
\end{equation*}
$$

for every $n \geq 1$, where the $\rho(n-s, 0)$ are defined by (8)-(9) and $A_{s}=\sum_{m=0}^{+\infty} a_{s+m-1} V_{-m}$.
In particular, we have the following corollary.
Corollary 2.5: Let $\left\{V_{n}\right\}_{n \in \mathbb{Z}}$ be a sequence (1) such that (H.1) and (2) are satisfied. Suppose that $V_{0}=1$ and $V_{-m}=0$ for $m \geq 1$. Then, for every $n \geq 1$, we have

$$
\begin{equation*}
V_{n}=\rho(n, 0)=a_{0} \rho(n-1,0)+a_{1} \rho(n-2,0)+\cdots+a_{n-1} \rho(0,0) \tag{13}
\end{equation*}
$$

where the $\rho(n-s, 0)$ are defined by (8)-(9).
Expression (13) can also be obtained using the Markov chains techniques on the displacement of the system from the state $n$ to the state 0 , as was done in Subsection 2.3.

## 3. COMBINATORIC EXPRESSION OF $\boldsymbol{V}_{\boldsymbol{n}}$ IN THE GENERAL CASE

Let $\left\{V_{n}\right\}_{n \in \mathbb{Z}}$ be a sequence (1) whose coefficients $\left\{a_{j}\right\}_{j \geq 0}$ are arbitrary real numbers. Suppose that $\left\{\left|a_{j}\right|\right\}_{j \geq 0}$ and $\left\{V_{-j}\right\}_{j \geq 0}$ satisfy (H.1), (H.2), and (H.3). For every $n \geq 1$, we set

$$
\begin{equation*}
\rho(n, 0)=\sum_{\sum_{j=0}^{n=1}(j+1) k_{j}=n} \frac{\left(\sum_{j=0}^{n-1} k_{j}\right)!}{k_{0}!k_{1}!\ldots k_{n-1}!} a_{0}^{k_{0}} a_{1}^{k_{1}} \ldots a_{n-1}^{k_{n-1}}, \tag{14}
\end{equation*}
$$

with $\rho(0,0)=1$ and $\rho(-k, 0)=0$ for every $k \geq 1$. Thus, by induction on $n$, we prove that (13) is also verified by expression (14) of $\rho(n, 0)$. Consider the sequence $\left\{W_{n}\right\}_{n \in \mathbb{Z}}$ defined as follows: $W_{n}=V_{n}$ for $n \leq-1$ and

$$
W_{n}=\sum_{m=0}^{+\infty}\left\{\sum_{s=1}^{n} \rho(n-s, 0) a_{s+m-1}\right\} V_{-m}
$$

for $n \geq 1$. For $n=1$, a direct computation shows that we have $W_{1}=\sum_{m=0}^{+\infty} a_{m} V_{-m}=V_{1}$. Since (14) satisfies (13), we derive by a simple induction that $W_{n}=V_{n}$ for every $n \geq 1$. Therefore, we have the following general result.
Theorem 3.1: Let $\left\{V_{n}\right\}_{n \in \mathbb{Z}}$ be a sequence (1) whose coefficients $\left\{a_{j}\right\}_{j \geq 0}$ are arbitrary real numbers such that $\left\{\left|a_{j}\right|\right\}_{j \geq 0}$ and $\left\{V_{-j}\right\}_{j \geq 0}$ satisfy (H.1), (H.2), and (H.3). Then, for every $n \geq 1$, we have

$$
V_{n}=\sum_{s=1}^{n} A_{s} \rho(n-s, 0),
$$

where the $\rho(n-s, 0)$ are given by (14) and

$$
A_{s}=\sum_{m=0}^{+\infty} a_{s+m-1} V_{-m} .
$$

The combinatoric expression of $r$-generalized Fibonacci sequences has been established by various techniques and methods (see, e.g., [1], [5], [8], [13], and [15]). Theorem 3.1 is a generalization of such a combinatoric expression to 0 -generalized Fibonacci sequences.

## 4. ASYMIPTOTIC BEHAVIOR OF $\rho(n, 0)$

In this section we study the asymptotic behavior of $\rho(n, 0)$ when the coefficients $a_{j}(j \geq 0)$ are nonnegative real numbers.

Let $\left\{V_{n}\right\}_{n \in \mathbb{Z}}$ be a sequence (1) whose coefficients $\left\{a_{j}\right\}_{j \geq 0}$ are arbitrary nonnegative real numbers. Suppose that (H.1), (H.2), and (H.3) are verified. If $V_{0}=1$ and $V_{-m}=0$ for every $m \geq 1$, we derive from (7) that $V_{n}=\rho(n, 0)$ for every $n \geq 1$, where $\rho(n, 0)$ is given by (14). For $\sum_{k=0}^{+\infty} a_{k}=1$, it was established in [14] that the following condition ( $C$ ) : $\operatorname{gcd}\left\{j+1 ; a_{j}>0\right\}=1$, implies that $\lim _{n \rightarrow+\infty} V_{n}=0$ if $\Sigma_{m \geq 0}(m+1) a_{m}=+\infty$ and $\lim _{n \rightarrow+\infty} V_{n}=\Sigma_{m \geq 0} \Pi(m) V_{-m}$ if $\Sigma_{m \geq 0}(m+1) a_{m}<+\infty$, where $\Pi(m)=\sum_{k=m}^{+\infty} a_{k} / \Sigma_{k \geq 0}(k+1) a_{k}$ (see [14], Theorem 2.2). Therefore, we have the following proposition.

Proposition 4.1: Let $\left\{a_{j}\right\}_{j \geq 0}$ be a sequence of nonnegative real numbers that satisfies (H.1) and (2). Then, if ( $C$ ) is verified, we have

$$
\lim _{n \rightarrow+\infty} \rho(n, 0)=0 \text { for } \sum_{m \geq 0}(m+1) a_{m}=+\infty
$$

and

$$
\lim _{n \rightarrow+\infty} \rho(n, 0)=\frac{1}{\sum_{m \geq 0}(m+1) a_{m}} \text { for } \sum_{m \geq 0}(m+1) a_{m}<+\infty .
$$

Suppose now that $\sum_{k=0}^{+\infty} a_{k} \neq 1$ arbitrary. Hence, we have the following two cases.
Case 1: $\Sigma_{m \geq 0} a_{m}>1$. Let $R$ be the radius of convergence of $f(x)=\sum_{k=0}^{+\infty} a_{k} x^{k+1}$. Hypothesis (H.2) implies that $R \geq 1$. The function $f$ is nondecreasing on $[0, R[$ and

$$
l=\lim _{x \rightarrow R^{-}} f(x) \geq f(1)=\sum_{m \geq 0} a_{m}>1 .
$$

Therefore, there exists a unique $q>1$ such that $f\left(q^{-1}\right)=1$. Set $b_{m}=q^{-m-1} a_{m}$ and $W_{n}=q^{-n} V_{n}$. It is easy to see that

$$
\begin{equation*}
W_{n+1}=\sum_{m=0}^{+\infty} b_{m} W_{n-m} . \tag{15}
\end{equation*}
$$

Hence, $\left\{W_{n}\right\}_{n \in \mathbb{Z}}$ is also a sequence (1) with $\sum_{m \geq 0} b_{m}=1$. Since $q>1$, we have $\left|W_{-n}\right| \leq\left|V_{-n}\right|$, which proves that the initial conditions $\left\{W_{-n}\right\}_{n \geq 0}$ satisfy (H.3). Suppose that $\left\{a_{j}\right\}_{j \geq 0}$ satisfies (C). Since $\operatorname{gcd}\left\{j+1 ; a_{j}>0\right\}=\operatorname{gcd}\left\{j+1 ; b_{j}>0\right\}$, we show that $\left\{a_{j}\right\}_{j \geq 0}$ also satisfies (C). If we apply Proposition 4.1, we prove the following proposition.

Proposition 4.2: Let $\left\{a_{j}\right\}_{j \geq 0}$ be a sequence of nonnegative real numbers that satisfies (H.1), (H.2), and (C). Suppose that $\sum_{m=0}^{+\infty} a_{m}>1$. Then there exists a unique $q>1$ such that

$$
\lim _{n \rightarrow+\infty} \frac{\rho(n, 0)}{q^{n}}=0 \text { for } \sum_{m \geq 0}(m+1) a_{m} q^{-m-1}=+\infty
$$

and

$$
\lim _{n \rightarrow+\infty} \frac{\rho(n, 0)}{q^{n}}=\left\{\sum_{m \geq 0}(m+1) \frac{a_{m}}{q^{m+1}}\right\}^{-1} \text { for } \sum_{m \geq 0}(m+1) a_{m} q^{-m-1}<+\infty,
$$

where $\rho(n, 0)$ is given by (14).
Case 2: $\Sigma_{m \geq 0} a_{m}<1$. In this case, it was established in [14] that the series $\Sigma_{n \geq 0} V_{n}$ converges absolutely. Thus, the series $\Sigma_{m \geq 0} \rho(n, 0)$ is convergent, which implies that $\lim _{n \rightarrow+\infty} \rho(n, 0)=0$.

For $\sum_{m \geq 0}(m+1) a_{m} q^{-m-1}<+\infty$, the real number $q>1$ can be approximated as follows:

$$
q=\lim _{n \rightarrow+\infty} \sqrt[n]{\rho(n, 0)} .
$$

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# PERFECT SQUARES IN THE LUCAS NUMBERS 

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## 1. INTRODUCTION

We consider two sequences defined by the recursion relations

$$
\begin{align*}
& u_{0}=0, u_{1}=1, u_{n+2}=a u_{n+1}-b u_{n},  \tag{1}\\
& v_{0}=2, v_{1}=a, v_{n+2}=a v_{n+1}-b v_{n}, \tag{2}
\end{align*}
$$

where $a$ and $b$ are integers which are nonzero, $D=a^{2}-4 b \neq 0$. Then

$$
\begin{equation*}
u_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}, \quad v_{n}=\alpha^{n}+\beta^{n}, \tag{3}
\end{equation*}
$$

where $\alpha$ and $\beta$ are distinct roots of the polynomial $f(z)=z^{2}-a z+b$. Each $u_{n}$ is called a Lucas number, which is an integer. A Lucas sequence $\left\{u_{n}\right\}$ is called degenerate if the quotient of the roots of $f$ is a root of unity and nondegenerate otherwise. Throughout this paper we assume that $a$ and $b$ are coprime.

The problem of determining all the perfect squares in a Lucas sequence has been studied by several authors: Cohn, Halton, Shorey, Tijdeman, Ribenboim, Mcdaniel, among others. In 1964, Cohn [1], [2] proved that when $a=1$ and $b=-1$, the only squares in the sequence $\left\{u_{n}\right\}$ are $u_{0}=0, u_{1}=u_{2}=1$, and $u_{12}=144$, and the only squares in the sequence $\left\{v_{n}\right\}$ are $v_{1}=1$ and $v_{3}=4$. In 1969, by using the theory of elliptic curves, London and Finkelstein [5] proved that the only cubes in the Fibonacci sequence are $F_{0}=0, F_{1}=F_{2}=1$, and $F_{6}=8$. Shorey and Tijdeman [9] proved for nondegenerate Lucas sequences that given $d \neq 0$ and $e \geq 2$, where $d$ and $e$ are integers, if $u_{m}=d U^{e}$ with $U \neq 0$ ( $U$ integral), then $m$ is bounded by an effectively computable constant. In 1996, Ribenboim and Mcdaniel [8] proved that, if $a$ and $b$ are odd and coprime and if $D=a^{2}-4 b$ is positive, then $u_{n}$ is a perfect square only if $n=0,1,2,3,6$, or $12, v_{n}$ is a perfect square only if $n=1,3$, or 5 .

The aim of this paper is to give an elementary proof of a special case of the above result obtained by Shorey and Tijdeman [9]. Developing the argument of London and Finkelstein [5], we obtain the following results.

Proposition 1: Let $n \geq 0$ be an integer of the form $n=4 m+r$ with $0 \leq r<4$. If $u_{n}$ is a perfect square, then the rational point ( $D s^{2} / b^{2 m}$, Dst $/ b^{3 m}$ ) lies on the elliptic curve $y^{2}=x^{3}+4 D b^{r} x$, where $D=a^{2}-4 b, s^{2}=\left|u_{n}\right|, t=v_{n}$, all of which are prime to $b$.

Proposition 2: Let $0 \leq r<4$ be a fixed integer. If $b$ is even and the group of rational points on the elliptic curve $y^{2}=x^{3}+4 D b^{r} x$ has rank zero or rank one, then $u_{4 m+r}$ is a perfect square only for finitely many $m \geq 0$.

## 2. PROOFS OF PROPOSITIONS 1 AND 2

## Proof of Proposition $\mathbb{1}$

Let $\alpha$ and $\beta$ be distinct roots of the polynomial $f(z)=z^{2}-a z+b$. Since $\alpha \beta=b$ and $D=$ $(\alpha-\beta)^{2}$ we obtain, from (3), $v_{n}^{2}-D u_{n}^{2}=4 b^{n}$. Suppose that the $n^{\text {th }}$ term $u_{n}$ is a perfect square. Putting $\left|u_{n}\right|=s^{2}$ and $v_{n}=t$, from the equality above we have $t^{2}=D s^{4}+4 b^{n}$. Multiplying through by $D^{2} s^{2}$, we see

$$
(D s t)^{2}=\left(D s^{2}\right)^{3}+4 D\left(D s^{2}\right) b^{n}
$$

Writing $n=4 m+r$ with $0 \leq r<4$, we obtain

$$
\left(\frac{D s t}{b^{3 m}}\right)^{2}=\left(\frac{D s^{2}}{b^{2 m}}\right)^{3}+4 D b^{r}\left(\frac{D s^{2}}{b^{2 m}}\right)
$$

Next we shall show that $D s^{2} / b^{2 m}$ and $D s t / b^{3 m}$ are in lowest terms. Let $p$ be an arbitrary prime divisor of $b$. Then, from (1) and (2), we have $u_{n} \equiv a^{n-1}(\bmod p)$ and $v_{n} \equiv a^{n}(\bmod p)$. Since $a$ and $b$ are coprime, $u_{n} \neq 0(\bmod p)$ and $v_{n} \neq 0(\bmod p)$; furthermore, $D=a^{2}-4 b \equiv a^{2} \equiv 0(\bmod$ $p$ ). We have thus completed the proof.

Before proceeding to the proof of Proposition 2, we will need the following information.
Let $c$ be a nonzero integer and let C be the elliptic curve given by the equation $y^{2}=x^{3}+c x$. We denote by $\Gamma$ the additive group of rational points on C and by $O$ the zero element of $\Gamma$.
Definition 1: For $P=(x, y) \in \Gamma$, we write $x=p / q$ in lowest terms and define the logarithmic height of $P$ by

$$
h(P)=\log \max (|P|,|q|) .
$$

Definition 2: For $P \in \Gamma$, the quantity

$$
\hat{h}(P)=\lim _{n \rightarrow \infty} \frac{h\left(2^{n} P\right)}{4^{n}}
$$

is called the canonical height of $P$.
The following two fundamental theorems on the height are well known, so the proofs are omitted (see [4] or [10]).
Theorem 1: There is a constant $\kappa_{0}$ that depends on the elliptic curve $C$, so that

$$
\begin{equation*}
|h(2 P)-4 h(P)| \leq \kappa_{0} \text { for all } P \in \Gamma . \tag{4}
\end{equation*}
$$

Theorem 2 (Néron): There is a constant $\kappa_{1}$ that depends only on the elliptic curve C , so that for all positive integers $n$ and for all $P \in \Gamma$ we have

$$
\begin{equation*}
\left|h(n P)-n^{2} \hat{h}(P)\right| \leq \kappa_{1} \tag{5}
\end{equation*}
$$

Definition 3: For $P=(x, y)$ in $\Gamma$, we write $x=p / q$ in lowest terms and denote by $\lambda(P)$ the exponent of the highest power of 2 that divides the denominator $q$. By convention, we define $\lambda(O)=0$.
Lemma 1: Let $P \in \Gamma$ with $P \neq(0,0)$. If $\lambda(P) \neq 0$, then $\lambda(2 P)=\lambda(P)+2$.

Proof: We can write $P=(x, y)=\left(m / e^{2}, n / e^{3}\right)$, where $m / e^{2}$ and $n / e^{3}$ are in lowest terms with $e>0$. Then the $x$ coordinate of $2 P$ is given by

$$
x(2 P)=-2 x+\left(\frac{3 x^{2}+c}{2 y}\right)^{2}=\frac{\left(m-c e^{4}\right)^{2}}{(2 e n)^{2}}
$$

Since $e$ is even and $m, n$ are odd, $\lambda(2 P)=\lambda(P)+2$.
Lemma 2: Let $P_{1}$ and $P_{2}$ be in $\Gamma$ with $P_{1} \neq(0,0)$ and $P_{2} \neq(0,0)$. If $0 \leq \lambda\left(P_{1}\right)<\lambda\left(P_{2}\right)$, then $\lambda\left(P_{1}+P_{2}\right) \leq \lambda\left(P_{2}\right)$.

Proof: If $P_{1}=O$, then $\lambda\left(P_{1}+P_{2}\right)=\lambda\left(P_{2}\right)$. So let us write $P_{1}=\left(x_{1}, y_{1}\right)=\left(m / e^{2}, n / e^{3}\right)$ and $P_{2}=\left(x_{2}, y_{2}\right)=\left(\bar{m} / f^{2}, \bar{n} / f^{3}\right)$, where $m / e^{2}, n / e^{3}, \bar{m} / f^{2}$, and $\bar{n} / f^{3}$ are in lowest terms with $e>0$ and $f>0$. Then the $x$ coordinate of $P_{1}+P_{2}$ is given by

$$
\begin{aligned}
x\left(P_{1}+P_{2}\right) & =-x_{1}-x_{2}+\left(\frac{y_{1}-y_{2}}{x_{1}-x_{2}}\right)^{2} \\
& =\frac{\left(n f^{3}-\bar{n} e^{3}\right)^{2}-\left(m f^{2}-\bar{m} e^{2}\right)^{2}\left(m f^{2}+\bar{m} e^{2}\right)}{e^{2} f^{2}\left(m f^{2}-\bar{m} e^{2}\right)^{2}}
\end{aligned}
$$

Since $0 \leq \lambda\left(P_{1}\right)<\lambda\left(P_{2}\right)$, we can write $e=2^{s} e^{\prime}$ and $f=2^{t} f^{\prime}$, where $e^{\prime}$ and $f^{\prime}$ are odd and $s$ and $t$ are integers with $0 \leq s<t$. Then $x\left(P_{1}+P_{2}\right)$ becomes

$$
\frac{\left(2^{3 t-3 s} n f^{\prime 3}-\bar{n} e^{\prime 3}\right)^{2}-\left(2^{2 t-2 s} m f^{\prime 2}-\bar{m} e^{\prime 2}\right)^{2}\left(2^{2 t-2 s} m f^{\prime 2}+\bar{m} e^{\prime 2}\right)}{2^{2 t} e^{\prime 2} f^{\prime 2}\left(2^{2 t-2 s} m f^{\prime 2}-\bar{m} e^{\prime 2}\right)^{2}}
$$

Since $e^{\prime}, f^{\prime}, \bar{m}$, and $\bar{n}$ are odd, we have $\lambda\left(P_{1}+P_{2}\right) \leq 2 t$. Combining this with $\lambda\left(P_{2}\right)=2 t$, we obtain $\lambda\left(P_{1}+P_{2}\right) \leq \lambda\left(P_{2}\right)$.

Lemma 3: Assume that $\Gamma$ has rank one, and let $P$ be a generator for the infinite cyclic subgroup of $\Gamma$. Let $t_{0}$ denote the least positive value of the integer $t$ such that $\lambda(t P) \neq 0$. Then, for any integer $l \geq 0$, if $2^{l} t_{0} \leq n<2^{l+1} t_{0}$, then $\lambda(n P) \leq \lambda\left(2^{l} t_{0} P\right)$.

Proof: We use strong induction on $l$. First we show that the result is true for $l=0$. Suppose $t_{0} \leq n<2 t_{0}$. Then we can write $n=t_{0}+r$ with $0 \leq r<t_{0}$. Since $\lambda(r P)=0$ and $\lambda\left(t_{0} P\right)>0$, by Lemma 1 we have $\lambda(n P)=\lambda\left(t_{0} P+r P\right) \leq \lambda\left(t_{0} P\right)$.

Next we suppose that the result is true for each $l=0,1,2, \ldots, k$. For any integer $n$ satisfying $2^{k+1} t_{0} \leq n<2^{k+2} t_{0}$, there exists an integer $r$ such that $n=2^{k+1} t_{0}+r$ and $0 \leq r<2^{k+1} t_{0}$. The induction hypothesis gives $\lambda(r P) \leq \lambda\left(2^{k} t_{0} P\right)$. By Lemma 1 we have $\lambda\left(2^{k} t_{0} P\right)<\lambda\left(2^{k+1} t_{0} P\right)$. Therefore, $\lambda(r P)<\lambda\left(2^{k+1} t_{0} P\right)$; thus, by Lemma 2 we have $\lambda(n P)=\lambda\left(2^{k+1} t_{0} P+r P\right) \leq \lambda\left(2^{k+1} t_{0} P\right)$, which shows that the result is true for $l=k+1$. Hence, the result is true for every integer $l \geq 0$ and the proof is complete.

## Proof of Proposition 2

We put $R_{m}=\left(D s^{2} / b^{2 m}, D s t / b^{3 m}\right)$, where $s^{2}=\left|u_{4 m+r}\right|$ and $t=v_{4 m+r}$. Assume that $\Gamma$ has rank zero. Then it is a finite cyclic group, and so the rational point $R_{m}$ lies on the elliptic curve $C$ only for finitely many $m \geq 0$; therefore, $u_{4 m+r}$ is a perfect square only for finitely many $m \geq 0$.

Next assume that $\Gamma$ has rank one. Then $\Gamma \cong Z \oplus F$, where $Z$ is an infinite cyclic group and $F$ is a torsion group of order two or four (see [4] or [10]). Let $P \in \Gamma$ be a generator for $Z$ and $Q \in \Gamma$ for $F$. Now suppose that the rational point $R_{m}$ lies on the elliptic curve $C$. Then there are integers $i$ and $j$ such that

$$
\begin{equation*}
R_{m}=i P+j Q \tag{6}
\end{equation*}
$$

Since $4 Q=O$, where $O$ is the zero element of $\Gamma$, we obtain

$$
\begin{equation*}
4 R_{m}=4 i P \tag{7}
\end{equation*}
$$

The essential tool for the proof is the logarithmic height. Since $h(4 i P)=h(-4 i P)$, we can assume $i>0$ without loss of generality. Let $k_{0}$ be the least positive value of the integer $k$ such that $\lambda(k P) \neq 0$. Then there is an integer $l \geq 0$ such that $2^{l} k_{0} \leq 4 i<2^{l+1} k_{0}$. From Lemmas 1 and 3 , we find $\lambda(4 i P) \leq \lambda\left(2^{l} k_{0} P\right)=\lambda\left(k_{0} P\right)+2 l$. Since $\lambda(4 i P)=\lambda\left(4 R_{m}\right)>2 m$, putting $\lambda_{0}=\lambda\left(k_{0} P\right)$, we obtain $2 l>\lambda(4 i P)-\lambda_{0}>2 m-\lambda_{0}$. Hence, $4 i \geq 2^{l} k_{0}>2^{m-\lambda_{0} / 2}$.

Now, Theorem 2 tells us that there is a constant $K_{1}$ depending only on the elliptic curve $C$, so that

$$
\begin{equation*}
h(4 i P) \geq(4 i)^{2} \hat{h}(P)-K_{1}>2^{2 m-\lambda_{0}} \hat{h}(P)-K_{1} \tag{8}
\end{equation*}
$$

Next we estimate for $h\left(4 R_{m}\right)$. Let $\alpha$ and $\beta$ be distinct roots of the polynomial $f(z)=z^{2}-$ $a z+b$. Putting $\gamma=\max (|\alpha|,|\beta|) \geq 1$, we find

$$
\begin{aligned}
\left|b^{2 m}\right| & =|\alpha \beta|^{2 m} \leq \gamma^{4 m}, \\
\left|D s^{2}\right| & =\left|D u_{4 m+r}\right|=\left|\alpha-\beta \| \alpha^{4 m+r}-\beta^{4 m+r}\right| \\
& \leq(|\alpha|+|\beta|)\left(|\alpha|^{4 m+r}+|\beta|^{4 m+r}\right) \leq 4 \gamma^{4 m+4}
\end{aligned}
$$

Therefore, $h\left(R_{m}\right) \leq \log 4 \gamma^{4(m+1)}=4(m+1) \log \gamma+2 \log 2$. Hence, by Theorem 1 ,

$$
\begin{equation*}
h\left(4 R_{m}\right) \leq 16 h\left(R_{m}\right)+5 K_{0} \leq 64(m+1) \log \gamma+32 \log 2+5 K_{0} \tag{9}
\end{equation*}
$$

where $K_{0}$ is a constant depending only on the elliptic curve $C$.
It follows that, if the rational point $R_{m}$ lies on the elliptic curve $C$, then $m$ satisfies the following inequality:

$$
\begin{equation*}
64(m+1) \log \gamma+32 \log 2+5 K_{0}>2^{2 m-\lambda_{0}} \hat{h}(P)-K_{1} \tag{10}
\end{equation*}
$$

However, there exists a constant $N>0$ such that inequality (10) is false for every $m \geq N$, so the rational point $R_{m}$ is not found on $C$ for every $m \geq N$. We conclude from Proposition 1 that $u_{4 m+r}$ is not a perfect square for every $m \geq N$. We have thus completed the proof.

## 3. APPLICATIONS

Following Silverman and Tate [10], we describe how to compute the rank $r$ of the group $\Gamma$ of rational points on the elliptic curve $C: y^{2}=x^{3}+c x$ with integral coefficients. Let $\mathbb{Q}^{*}$ denote the multiplicative group of nonzero rational numbers, and let $\mathbb{Q}^{* 2}=\left\{u^{2}: u \in \mathbb{Q}^{*}\right\}$. Now consider the map $\varphi: \Gamma \rightarrow \mathbb{Q}^{*} / \mathbb{Q}^{* 2}$ defined by the rule:

$$
\begin{array}{ll}
\varphi(O)=1 & \left(\bmod \mathbb{Q}^{* 2}\right) \\
\varphi(0,0)=c & \left(\bmod \mathbb{Q}^{* 2}\right) \\
\varphi(x, y)=x & \left(\bmod \mathbb{Q}^{* 2}\right) \text { if } x \neq 0
\end{array}
$$

On the other hand, let $\bar{\Gamma}$ denote the group of rational points on the elliptic curve $\bar{C}: y^{2}=$ $x^{3}-4 c x$. Using the analogous map $\bar{\varphi}: \bar{\Gamma} \rightarrow \mathbb{Q}^{*} / \mathbb{Q}^{* 2}$, we obtain the formula for the rank of $\Gamma$ :

$$
\begin{equation*}
2^{r}=\frac{\# \varphi(\Gamma) \cdot \# \bar{\varphi}(\bar{\Gamma})}{4}, \tag{11}
\end{equation*}
$$

where $\# \varphi(\Gamma)$ and $\# \bar{\varphi}(\bar{\Gamma})$ denote the order of $\varphi(\Gamma)$ and the order of $\bar{\varphi}(\bar{\Gamma})$, respectively.
Next we describe how to determine the order of $\varphi(\Gamma)$. It is obvious from the rule of the map $\varphi$ that $\left\{1, c\left(\bmod \mathbb{Q}^{* 2}\right)\right\} \subset \varphi(\Gamma)$.

Now, for $P=(x, y) \in \Gamma$ with $y \neq 0$, the coordinates $x$ and $y$ are written in the form

$$
x=\frac{c_{1} M^{2}}{e^{2}}, \quad y=\frac{c_{1} M N}{e^{3}}
$$

in lowest terms with $M \neq 0$ and $e>0$, where $c_{1}$ is an integral divisor of $c$, so that $c=c_{1} c_{2}$. Here $M, e$, and $N$ must satisfy the equation

$$
\begin{equation*}
N^{2}=c_{1} M^{4}+c_{2} e^{4}, \tag{12}
\end{equation*}
$$

and also the conditions

$$
\begin{aligned}
\operatorname{gcd}(M, e) & =\operatorname{gcd}(N, e)=\operatorname{gcd}\left(c_{1}, e\right)=1, \\
\operatorname{gcd}\left(c_{2}, M\right) & =\operatorname{gcd}(M, N)=1
\end{aligned}
$$

Hence, for a factorization $c=c_{1} c_{2}$, if the equation $N^{2}=c_{1} M^{4}+c_{2} e^{4}$ has a solution ( $M, e, N$ ) with $M \neq 0$ that satisfies the side conditions above, then $c_{1}\left(\bmod \mathbb{Q}^{* 2}\right)$ is in $\varphi(\Gamma)$, otherwise it is not.

Proposition 3: Let $p$ be a prime and let $C$ be the elliptic curve $y^{2}=x^{3}-4 p x$. If $p \equiv 3(\bmod 4)$, then the group $\Gamma$ of rational points on $C$ has rank zero or rank one.

Proof: Since $c=-4 p$, the possibilities for $c_{1}$ are $c_{1}= \pm 1, \pm 2, \pm 4, \pm p \pm 2 p, \pm 4 p$. So we see that $\varphi(\Gamma) \subset\left\{ \pm 1, \pm 2, \pm p, \pm 2 p\left(\bmod \mathbb{Q}^{* 2}\right)\right\}$. We shall show first that $-1 \notin \Gamma$. Let us consider the equation

$$
\begin{equation*}
N^{2}=-M^{4}+4 p e^{4} . \tag{13}
\end{equation*}
$$

This implies the congruence $N^{2} \equiv-M^{4}(\bmod p)$. Since $p \equiv 3(\bmod 4)$, we have $(-1 / p)=-1$, where $(-1 / p)$ is the Legendre symbol of -1 for $p$; hence, the congruence above has no solutions with $M \not \equiv 0(\bmod p)$. So equation (13) has no solutions in integers with $\operatorname{gcd}(M, N)=1$. Similarly, the equation $N^{2}=-4 M^{4}+p e^{4}$ has no solutions in integers with $\operatorname{gcd}(M, N)=1$. Therefore, $-1 \notin \varphi(\Gamma)$, and hence $\# \varphi(\Gamma)=2$ or $\# \varphi(\Gamma)=4$.

On the other hand, let $\bar{C}$ be the elliptic curve $y^{2}=x^{3}+16 p x$, and let $\bar{\Gamma}$ denote the group of rational points on $\bar{C}$. Since $\bar{c}=16 p$, we have $\bar{\varphi}(\bar{\Gamma}) \subset\left\{1,2, p, 2 p\left(\bmod \mathbb{Q}^{* 2}\right)\right\}$. We shall show by contradiction that $2 \notin \bar{\varphi}(\bar{\Gamma})$. Let us consider the equation

$$
\begin{equation*}
N^{2}=2 M^{4}+8 p e^{4} \tag{14}
\end{equation*}
$$

Suppose equation (14) has a solution in integers with $M \neq 0$ and $\operatorname{gcd}(M, N)=1$. Then $N$ is even. Putting $N=2 N_{1}$, we have $2 N_{1}^{2}=M^{4}+4 p e^{4}$, showing that $M$ is even, contrary to the hypothesis that $M$ and $N$ are coprime. Hence, equation (14) has no solutions in integers with $\operatorname{gcd}(M, N)=1$. Similarly, the equation $N^{2}=8 M^{4}+2 p e^{4}$ has no solutions in integers with $\operatorname{gcd}(N, e)=1$. Thus, $2 \notin \bar{\varphi}(\bar{\Gamma})$, and so $\# \bar{\varphi}(\bar{\Gamma})=2$. By formula (11), we find

$$
2^{r}=\frac{\# \varphi(\Gamma) \cdot \# \bar{\varphi}(\bar{\Gamma})}{4}=1 \text { or } 2 .
$$

Therefore, $\Gamma$ has rank zero or rank one.
Proposition 4: Let $p, q$ be primes and let $C$ be the elliptic curve $y^{2}=x^{3}-4 p q x$. If $p \equiv 5(\bmod$ $8), q \equiv 3(\bmod 8)$, and $(p / q)=-1$, then the group $\Gamma$ of rational points on $C$ has rank zero or rank one.

Proof: Since $c=-4 p q$, we have $\varphi(\Gamma) \subset\left\{ \pm 1, \pm 2, \pm p, \pm q, \pm 2 p, \pm 2 q, \pm p q, \pm 2 p q\left(\bmod \mathbb{Q}^{*}\right)\right\}$. We shall show, for instance, that $-2 p \notin \Gamma$. The hypotheses give

$$
\left(\frac{-1}{p}\right)=1,\left(\frac{2}{p}\right)=\left(\frac{-1}{q}\right)=\left(\frac{2}{q}\right)=-1,\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right)=-1 .
$$

Hence, the congruence $N^{2} \equiv-2 p M^{4}(\bmod q)$ has no solutions with $M \not \equiv 0(\bmod q)$ because $(-2 p / q)=(-1 / q)(2 / q)(p / q)=-1$, so $N^{2}=-2 p M^{4}+2 q e^{4}$ has no solutions in integers with $\operatorname{gcd}(M, N)=1$. Therefore, $-2 p \notin \Gamma$. By using the same argument, we can show that $\varphi(\Gamma)$ does not have any elements of $\{-1, \pm 2, \pm p, \pm q,-2 p, p q, 2 p q\}$. Thus, we obtain $\# \varphi(\Gamma) \leq 4$.

On the other hand, let $\bar{C}$ be the elliptic curve $y^{2}=x^{3}+16 p q x$, and let $\bar{\Gamma}$ denote the group of rational points on $\bar{C}$. Since $\bar{c}=16 p q$, we have $\bar{\alpha}(\bar{\Gamma}) \subset\left\{1,2, p, q, 2 p, 2 q, p q, 2 p q\left(\bmod \mathbb{Q}^{* 2}\right)\right\}$. By using an argument similar to the one above, we can show that $p \notin \bar{\varphi}(\bar{\Gamma})$ and $q \notin \bar{\varphi}(\bar{\Gamma})$. Furthermore, by using an argument similar to the one we gave in the proof of Proposition 3, we can show that $\bar{\varphi}(\bar{\Gamma})$ does not have any elements of $\{2,2 p, 2 q, 2 p q\}$. Thus, we obtain $\# \varphi(\Gamma)=2$. Therefore, by formula (11), we find $2^{r} \leq 2$. In conclusion, $\Gamma$ has rank zero or rank one.

In addition, the following proposition holds. The proof is completely analogous to that of Proposition 4.
Proposition 5: Let $p, q$ be primes and let $C$ be the elliptic curve $y^{2}=x^{3}-4 p q x$. If $p \equiv 1(\bmod$ $8), q \equiv 7(\bmod 8)$, and $(p / q)=-1$, then the group $\Gamma$ of rational points on $C$ has rank zero or rank one.

Now let us consider the Lucas sequence determined by $u_{0}=0, u_{1}=1, u_{n+2}=a u_{n+1}-b u_{n}$, where $a$ and $b$ are coprime integers that are nonzero, $D=a^{2}-4 b \neq 0$. Assume that $b$ is even. If $D=-p<0$, where $p$ is a prime, then $p \equiv 3(\bmod 4)$. If $D=-p q<0$, where $p$ and $q$ are primes, then $(p, q) \equiv(3,5)(\bmod 8)$ or $(p, q) \equiv(1,7)(\bmod 8)$. Hence, the following three corollaries hold.

Corollary 1: Assume $b$ is even and $D=-p<0$, where $p$ is a prime. Then there are only finitely many perfect squares in the subsequence $\left\{u_{4 m}\right\}$.

Corollary 2: Assume $b$ is even and $D=-p q<0$, where $p$ and $q$ are primes with $(p / q)=-1$. Then there are only finitely many perfect squares in the subsequence $\left\{u_{4 m}\right\}$.
Corollary 3: Assume $b$ is of the form $b=(2 d)^{4}$ for some integer $d$. If $D=-p$, where $p$ is a prime, or if $D=-q r<0$, where $q$ and $r$ are primes with $(q / r)=-1$, then there are only finitely many perfect squares in the sequence $\left\{u_{n}\right\}$.

Proof: Suppose that the $n^{\text {th }}$ term, $u_{n}$, is a perfect square. As mentioned above, we have $t^{2}=D s^{4}+4(2 d)^{4 n}$, where $s^{2}=\left|u_{n}\right|$ and $t=v_{n}$. This implies

$$
\left\{\frac{D s t}{(2 d)^{3 n}}\right\}^{2}=\left\{\frac{D s^{2}}{(2 d)^{2 n}}\right\}^{3}+4 D\left\{\frac{D s^{2}}{(2 d)^{2 n}}\right\} .
$$

From Propositions 3, 4, and 5, we obtain that the elliptic curve $y^{2}=x^{3}+4 D x$ has rank zero or rank one. It follows that $u_{n}$ is a perfect square only for finitely many $n \geq 0$.

## ACKNOWLEDGMENT

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# ELEMENTARY PROBLEMS AND SOLUTIONS 

## Edited by

## Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by April 15, 2003. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

## BASIC FORMULAS

The Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ satisfy

$$
\begin{aligned}
& F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1 ; \\
& L_{n+2}=L_{n+1}+L_{n}, L_{0}=2, L_{1}=1 .
\end{aligned}
$$

Also, $\alpha=(1+\sqrt{5}) / 2, \beta=(1-\sqrt{5}) / 2, F_{n}=\left(\alpha^{n}-\beta^{n}\right) / \sqrt{5}$, and $L_{n}=\alpha^{n}+\beta^{n}$.

## PROBLEMS PROPOSED IN THIS ISSUE

## B-946 Proposed by Mario Catalani, University of Torino, Torino, Italy

Find the smallest positive integer $k$ such that the following series converge and find the value of the sums:

1. $\sum_{i=1}^{\infty} \frac{i^{2} F_{i} L_{i}}{k^{i}}$
2. $\sum_{i=1}^{\infty} \frac{i F_{i}^{2}}{k^{i}}$

## B-947 Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA

(a) Find a nonsquare polynomial $f(x, y, z)$ with integer coefficients such that $f\left(F_{n}, F_{n+1}, F_{n+2}\right)$ is a perfect square for all $n$.
(b) Find a nonsquare polynomial $g(x, y)$ with integer coefficients such that $g\left(F_{n}, F_{n+1}\right)$ is a perfect square for all $n$.

## B-948 Proposed by José Luis Diaz-Barrero \&\& Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain

Let $\ell$ be a positive integer greater than or equal to 2 . Show that, for $x>0$,

$$
\log _{F_{\ell+1} F_{\ell+2}+. F_{\ell+n}} x^{n^{2}} \leq \sum_{k=1}^{n} \log _{F_{\ell+k}} x .
$$

## B-949 Proposed by N. Gauthier, Royal Military College of Canada

For $l$ and $n$ positive integers, find closed form expressions for the following sums,

$$
S_{1} \equiv \sum_{k=1}^{n} 3^{n-k} F_{3^{k} \cdot 2 l}^{3} \quad \text { and } \quad S_{2} \equiv \sum_{k=1}^{n} 3^{n-k} L_{3^{k} \cdot(2 l+1)}^{3}
$$

## B-950 Proposed by Paul S. Bruckman, Berkeley, CA

For all primes $p>2$, prove that

$$
\sum_{k=1}^{p-1} \frac{F_{k}}{k} \equiv 0(\bmod p)
$$

where $\frac{1}{k}$ represents the residue $k^{-1}(\bmod p)$.

## SOLUTIONS

## An Inequality and an Equality Case

## B-930 Proposed by José Luis Díaz \& Juan José Egozcue, Terrassa, Spain (Vol. 40, no. 1, February 2002)

Let $n \geq 0$ be a nonnegative integer. Prove that $F_{n}^{L_{n}} L_{n}^{F_{n}} \leq\left(F_{n+1}^{F_{n+1}}\right)^{2}$. When does equality occur?

## Solution by H.-J. Seiffert, Berlin, Germany

We shall prove that, for all nonnegative integers $n$,

$$
\begin{equation*}
F_{n}^{L_{n}} L_{n}^{F_{n}} \leq F_{2 n}^{F_{n+1}} \leq F_{n+1}^{2 F_{n+1}} \tag{1}
\end{equation*}
$$

with equality on the left-hand side only when $n=0$ or $n=1$, and on the right-hand side only when $n=1$.

If $x$ and $y$ are distinct positive real numbers, then, by the weighted Arithmetic-Geometric Mean Inequality,

$$
x^{y} y^{x}<\left(\frac{2 x y}{x+y}\right)^{x+y}
$$

Since $\frac{2 x y}{x+y}<\sqrt{x y}<\frac{x+y}{2}$, we have

$$
\begin{equation*}
x^{y} y^{x}<(x y)^{(x+y) / 2}<\left(\frac{x+y}{2}\right)^{x+y} \tag{2}
\end{equation*}
$$

The cases $n=0$ and $n=1$ can be treated directly. If $n \geq 2$, then $0<F_{n}<L_{n}$ and (2) gives

$$
F_{n}^{L_{n}} L_{n}^{F_{n}}<\left(F_{n} L_{n}\right)^{\left(F_{n}+L_{n}\right) / 2}<\left(\frac{F_{n}+L_{n}}{2}\right)^{F_{n}+L_{n}}
$$

From ( $\mathrm{I}_{7}$ ) and ( $\mathrm{I}_{8}$ ) of [1], we know that $F_{n} L_{n}=F_{2 n}$ and $L_{n}=F_{n-1}+F_{n+1}$. The latter identity gives $F_{n}+L_{n}=2 F_{n+1}$ so that (1), including the conditions for equality, is proved.

## Reference

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Also solved By Paull Bruckman, Charles Cook, L.A. G. Dresel, Ovidiu Furdui, Walther Janous, Harris Kwong, Toufik Mansonir (partial solution), and the proposer.

## A Relatively Prime Couple

B-931 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (Vol. 40, no. 1, February 2002)
Prove that $\operatorname{gcd}\left(L_{n}, F_{n+1}\right)=1$ for all $n \geq 0$.
Solution by Charles $\mathbb{K}$. Cook, University of South Carolina at Sumier, Sumter, $S C$
First, note that $L_{n}=F_{n+1}+F_{n-1}$ and $F_{n}=F_{n+1}-F_{n-1}$. Let $d=\left(F_{n+1}, L_{n}\right)$. Then $d \mid F_{n+1}$ and $d \mid L_{n}=F_{n+1}+F_{n-1}$. It follows that $d \mid F_{n-1}$. Thus, $d \mid F_{n+1}-F_{n-1}$ and $F \mid F_{n}$. But $\left(F_{n}, F_{n+1}\right)=1$. Hence, $d=1$.

Also solved by Paul Bruckman, L.A.G. Dresel, Pentit Haukkanen, John Jaroma, Walther Janous, Harris Kwong, Toufik Mansour, Maitlland Rose, H.-J. Seiffert, and the proposer.

## A Strict Inequality and a Serious Series

## B-932 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

 (Vol. 40, no. 1, February 2002)Prove that
A) $\frac{F_{2} F_{4} \ldots F_{2 n}}{F_{1} F_{3} \ldots F_{2 n+1}}<\frac{1}{\sqrt{F_{2 n+1}}}$ for all $n \geq 1$ and $\left.\mathbb{B}\right) \sum_{k=1}^{\infty} \frac{F_{2} F_{4} \ldots F_{2 k}}{F_{1} F_{3} \ldots F_{2 k+1}}$ converges.

## Solution by Hapris Kwong, SUNY College at Fredonia, Fredonia, NY

The inequality in A) can be established by means of induction. To complete the inductive step, it suffices to prove that

$$
\frac{1}{\sqrt{F_{2 n+1}}} \cdot \frac{F_{2 n+2}}{F_{2 n+3}}<\frac{1}{\sqrt{F_{2 n+3}}}
$$

or, equivalently,

$$
F_{2 n+2}^{2}<F_{2 n+1} F_{2 n+3}
$$

Binet's formulas yield

$$
5 F_{2 n+2}^{2}=\left(\alpha^{2 n+2}-\beta^{2 n+2}\right)^{2}=\alpha^{4 n+4}-2(\alpha \beta)^{2 n+2}+\beta^{4 n+4}=L_{4 n+4}-2
$$

and, in a similar manner,

$$
5 F_{2 n+1} F_{2 n+3}=\alpha^{4 n+4}-2(\alpha \beta)^{2 n+1}\left(\alpha^{2}+\beta^{2}\right)+\beta^{4 n+4}=L_{4 n+4}+2 L_{3}
$$

thereby completing the induction.
To prove $\mathbb{B}$ ), we use a comparison test. Because of $\mathbf{A}$ ), it remains to show that

$$
\sum_{k=1}^{\infty} \frac{1}{\sqrt{F_{2 k+1}}}
$$

converges. We now apply a ratio test to complete the proof:

$$
\lim _{k \rightarrow \infty} \frac{1}{\sqrt{F_{2 k+3}}} / \frac{1}{\sqrt{F_{2 k+1}}}=\lim _{k \rightarrow \infty} \sqrt{\frac{F_{2 k+1}}{F_{2 k+3}}}=\frac{1}{\alpha}<1 .
$$

H.-J. Seiffert proved the sharper inequality

$$
\frac{F_{2} F_{4} \ldots F_{2 n}}{F_{1} F_{3} \ldots F_{2 n+1}} \leq \frac{1}{\sqrt{2 F_{2 n+1}}}
$$

with equality occurring only when $n=1$.
Also solved by Paul Bruckman, Charles Cook, L.A.G. Dresel, José Luis Díaz \& Juan José Egozcue (jointly), Pentti Haukkanen, Walther Janous, Toufik Mansour, H.-J. Seiffert, and the proposer.

## A Special Case of a More General Inequality

## B-933 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

 (Vol. 40, no. 1, February 2002)Prove that $F_{n}^{F_{n+1}}>F_{n+1}^{F_{n}}$ for all $n \geq 4$.

## Solution I by Pantelimon Stănică, Montgomery, AL

We prove the inequality by induction. For $n=4$, we need $3^{5}>5^{3}$, which is obviously true. Assuming the inequality true for $n$, we prove it for $n+1$. Thus,

$$
F_{n+1}^{F_{n+2}}=F_{n+1}^{F_{n+1}} F_{n+1}^{F_{n}}>F_{n+1}^{F_{n+1}} F_{n}^{F_{n+1}}=\left(F_{n+1} F_{n}\right)^{F_{n+1}}>F_{n+2}^{F_{n+1}},
$$

using induction and the fact that $F_{n} F_{n+1}>F_{n}+F_{n+1}=F_{n+2}$.
Further Comment: In fact, a much more general inequality is true (see, e.g., F. Qi \& L. Debnath, "Inequalities of Power-Exponential Functions," J. Ineq. Pure Appl. Math. 1.2 [2000]:art. 15):

If $e<x<y$, then $x^{y}>y^{x}$;
If $x<y<e$, then $x^{y}<y^{x}$.
The first inequality, taking $x=F_{n}, y=F_{n+1}, n \geq 4$, implies B-933.
Solution II by Paul S. Bruckman, Sacramento, CA
For brevity, write $a=F_{n}, b=F_{n+1}$.
Consider the analytic function $F(x)=x / \ln x, x>1$. Note that $F^{\prime}(x)=1 / \ln x-1 / \ln ^{2} x$; thus $F^{\prime}(x)=0$ iff $x=e$. Also, $F^{\prime \prime}(x)=-1 /\left(x \ln ^{2} x\right)+2 /\left(x \ln ^{3} x\right)$. Since $F^{\prime \prime}(e)=1 / e>0$, then $F$ attains a relative minimum at $x=e$; in fact, $F(e)=e$. Moreover, $F^{\prime}(x)>0$ if $x>e$, i.e., $F$ is increasing for all $x>e$.

In particular, if $n \geq 4$, we have $b \geq 5, b>a \geq 3>e$, so $b / \ln b>a / \ln a$. Equivalently, $b \ln a>$ $a \ln b$, which implies $a^{b}>b^{a}$. Q.E.D.

Incidentally, note that if $n<4, a<e$, then the indicated inequality is invalid. For $n=1,2,3$, respectively, we find that $1=1^{1}=1^{1}, 1=1^{2}<2^{1}=2$, and $8=2^{3}<3^{2}=9$.
Most solvers used variations of Solution II. H.-J. Seiffert improved the inequality by showing

$$
F_{n}^{F_{n+1}}>\left(\frac{15}{4 e}\right)^{2} F_{n+1}^{F_{n}} .
$$

Also solved by Charles Cook, José Luis Díaz-Barrero \& Juan José Egozcue (jointly), L.A.G. Dresel, Walther Janous, Toufik Mansour, H.-J. Seiffert, and the proposer.

## A Trigonometric Fibonacci Equality

## B-934 Proposed by N. Gauthier, Royal Military College of Canada

(Vol. 40, no. 1, February 2002)
Prove that

$$
2 \sum_{n=1}^{m} \sin ^{2}\left(\frac{\pi}{2} \frac{F_{m+1}}{F_{m} F_{n} F_{n+1}}\right) \sin \left(\pi \frac{F_{n} F_{m+1}}{F_{m} F_{n+1}}\right)=\sum_{n=1}^{m}(-1)^{n} \sin \left(\pi \frac{F_{m+1}}{F_{m} F_{n} F_{n+1}}\right) \cos \left(\pi \frac{F_{n} F_{m+1}}{F_{n+1} F_{m}}\right),
$$

where $m$ is a positive integer.
Solution by L. A. G. Dresel, Reading, England
For any given $m$, let $A_{n}=\pi F_{m+1} /\left(F_{m} F_{n} F_{n+1}\right)$ and $B_{n}=\pi F_{n} F_{m+1} /\left(F_{m} F_{n+1}\right)$. Using the trigonometric identities $2 \sin ^{2}(A / 2)=1-\cos A, \sin (B+A)=\cos A \sin B+\cos B \sin A$, and $\sin (B-A)=$ $\cos A \sin B-\cos B \sin A$, the proposition to be proved transforms to

$$
\sum \sin B_{n}=\sum \sin \left\{B_{n}+(-1)^{n} A_{n}\right\},
$$

the summations being from $n=1$ to $m$. Now

$$
\left\{B_{n}+(-1)^{n} A_{n}\right\}=\left\{\pi F_{m+1} /\left(F_{m} F_{n} F_{n+1}\right)\right\}\left\{\left(F_{n}\right)^{2}+(-1)^{n}\right\}
$$

and using identity (29) of [1], $\left(F_{n}\right)^{2}+(-1)^{n}=F_{n+1} F_{n-1}$, we obtain

$$
\left\{B_{n}+(-1)^{n} A_{n}\right\}=\pi F_{n-1} F_{m+1} /\left(F_{m} F_{n}\right)=B_{n-1} .
$$

Thus, it remains to prove $\Sigma\left(\sin B_{n}-\sin B_{n-1}\right)=0$, which reduces to $\left(\sin B_{m}-\sin B_{0}\right)=0$. But $B_{0}=0$ and $B_{m}=\pi$, so that $\sin B_{m}=\sin B_{0}=0$.

## Reference

1. S. Vajda. Fibonacci \& Lucas Numbers, and the Golden Section. Chichester: Ellis Horwood Ltd., 1989.
Also solved by Paul Bruckman, Ovidiu Furdui, Walther Janous, H.-J. Seiffert, and the proposer.

# ADVANCED PROBLEMS AND SOLUTIONS 

Ealited by<br>Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

## PROBLEMS PROPOSED IN THIS ISSUE

## H-589 Proposed by Robert DiSario, Bryant College, Smithfield, RI

Let $f(n)=F(F(n))$, where $F(n)$ is the $n^{\text {th }}$ Fibonacci number. Show that

$$
f(n)=\frac{(f(n-1))^{2}-(-1)^{F(n)}(f(n-2))^{2}}{f(n-3)}
$$

for $n>3$.

## H-590 Proposed by Florian Luca, Campus Morelia, Michoacan, Mexico

For any positive integer $k$, let $\phi(k), \sigma(k), \tau(k), \Omega(k), \omega(k)$ be the Euler function of $k$, the sum of divisors function of $k$, the number of divisors function of $k$, and the number of prime divisors function of $k$ (where the primes are counted with or without multiplicity), respectively.

1. Show that $n \mid \phi\left(F_{n}\right)$ holds for infinitely many $n$.
2. Show that $n \mid \sigma\left(F_{n}\right)$ holds for infinitely many $n$.
3. Show that $n \mid \tau\left(F_{n}\right)$ holds for infinitely many $n$.
4. Show that for no $n>1$ can $n$ divide either $\Omega\left(F_{n}\right)$ or $\omega\left(F_{n}\right)$.

## H-591 Proposed by H.-J. Seiffert, Berlin, Germany

Prove that, for all positive integers $n$,
(c)
(d)

$$
\begin{align*}
& 5^{n} F_{2 n-1}=\sum_{\substack{k=0 \\
5 \nmid 2 n-k+3}}^{2 n}(-1)^{[(4 n+3 k) / 5]}\binom{4 n+1}{k},  \tag{a}\\
& 5^{n} L_{2 n}= \sum_{\substack{k=0 \\
5 / 2 n-k+4}}^{2 n+1}(-1)^{[(4 n+3 k-3) / 5]}\binom{4 n+3}{k},  \tag{b}\\
& 5^{n-1} F_{2 n}= \sum_{\substack{k=0 \\
5 / 2 n-k+1}}(-1)^{[(8 n+k+3) / 5]}\binom{4 n-3}{k}, \\
& 5^{n-1} \mathbb{L}_{2 n+1}=\sum_{\substack{2 n-1}}^{2 n-1)^{[(8 n+k+2) / 5]}\binom{4 n-1}{k},}
\end{align*}
$$

where [ ] denotes the greatest integer function.

## H-592 Proposed by N. Gautheir \& J. R. Gosselin, Royal Military College of Canada

For integers $m \geq 1, n \geq 2$, let $X$ be a nontrivial $n \times n$ matrix such that

$$
\begin{equation*}
X^{2}=x X+y I, \tag{1}
\end{equation*}
$$

where $x, y$ are indeterminates and $I$ is a unit matrix. (By definition, a trivial matrix is diagonal.) Then consider the Fibonacci and Lucas sequences of polynomials, $\left\{F_{l}(x, y)\right\}_{l=0}^{\infty}$ and $\left\{L_{l}(x, y)\right\}_{l=0}^{\infty}$, defined by the recurrences

$$
\begin{array}{ll}
F_{0}(x, y)=0, & F_{1}(x, y)=1, \quad F_{l+2}(x, y)=x F_{l+1}(\dot{x}, y)+y F_{l}(x, y),  \tag{2}\\
L_{0}(x, y)=2, & L_{1}(x, y)=x, \\
L_{l+2}(x, y)=x L_{l+1}(x, y)+y L_{l}(x, y),
\end{array}
$$

respectively.
a. Show that

$$
X^{m}=a_{m} X+b_{m} y I \text { and that } X^{m}+(-y)^{m} X^{-m}=c_{m} I,
$$

where $a_{m}, b_{m}$, and $c_{m}$ are to be expressed in closed form as functions of the polynomials (2).
b. Now let

$$
f(\lambda ; x, y) \equiv|\lambda I-X| \equiv \sum_{m=0}^{n}(-1)^{n-m} \lambda_{n-m} \lambda^{m}
$$

be the characteristic (monic) polynomial associated to $X$, where the set of coefficients,

$$
\left\{\lambda_{l} \equiv \lambda_{l}(x, y) ; 0 \leq l \leq n\right\}
$$

is entirely determined from the defining relation for $f(\lambda ; x, y)$. For example, $\lambda_{0}=1, \lambda_{1}=\operatorname{tr}(X)$, $\lambda_{n}=\operatorname{det}(X)$, etc. Show that

$$
\sum_{m=1}^{n}(-1)^{m} \lambda_{n-m} F_{m}(x, y)=0 \text { and that } y \sum_{m=1}^{n}(-1)^{m} \lambda_{n-m} F_{m-1}(x, y)+\lambda_{n}=0 .
$$

## SOLUTIONS

## A Fine Product

## H-577 Proposed by Paul S. Bruckman, Sacramento, CA (Vol. 39, no. 5, November 2001)

Define the following constant: $C \equiv \prod_{p}\{1-1 / p(p-1)\}$ as an infinite product over all primes $p$.
(A) Show that

$$
\sum_{n=1}^{\infty} \mu(n) /\{n \phi(n)\}
$$

where $\mu(n)$ and $\phi(n)$ are the Möbius and Euler functions, respectively.

## Solution by Naim Tuglu, Turkey

$$
\sum_{n=1}^{\infty} \mu(n) /\{n \phi(n)\}=\lim _{m \rightarrow \infty} \sum_{d \mid m} \mu(d) /\{d \phi(d)\} .
$$

If $f$ is a multiplicative arithmetic function, then

$$
\sum_{d \mid r} \mu(d) f(d)=\prod_{p}(1-f(p)),
$$

where $p$ is prime less than $r$.
If the Euler function $\phi(r)$ is multiplicative, then $f(r)=\frac{1}{r \phi(r)}$ is a multiplicative function; so

$$
\sum_{d \mid m} \mu(d) /\{d \phi(d)\}=\prod_{p}\left\{1-\frac{1}{p \phi(p)}\right\} .
$$

If $p$ is prime, then $\phi(p)=p-1$, and we have

$$
\sum_{d \mid m} \mu(d) /\{d \phi(d)\}=\prod_{p}\left\{1-\frac{1}{p(p-1)}\right\},
$$

where $p$ is prime less than $m$; therefore,

$$
\sum_{n=1}^{\infty} \mu(n) /\{n \phi(n)\}=\lim _{m \rightarrow \infty} \prod_{p}\left\{1-\frac{1}{p(p-1)}\right\}
$$

is an infinite product over all primes $p$.

## Firm Matrices

## H-578 Proposed by N. Gauthier \& J. R. Gosselin, Royal Military College of Canada (Vol. 39, no. 5, November 2001)

In Problem B-863, S. Rabinowitz gave a set of four $2 \times 2$ matrices which are particular solutions of the matrix equation

$$
\begin{equation*}
X^{2}=X+I, \tag{1}
\end{equation*}
$$

where $I$ is the unit matrix [The Fibonacci Quarterly 36.5 (1998); solved by H. Kappus, 37.3 (1999)]. The matrices presented by Rabinowitz are not diagonal (i.e., they are nontrivial), have determinant -1 and trace +1 .
a. Find the complete set $\{X\}$ of the nontrivial solutions of (1) and establish whether the properties $\operatorname{det}(X)=-1$ and $\operatorname{tr}(X)=+1$ hold generally.
b. Determine the complete set $\{X\}$ of the nontrivial solutions of the generalized characteristic equation

$$
\begin{equation*}
X^{2}=x X+y I \tag{2}
\end{equation*}
$$

for the $2 \times 2$ Fibonacci matrix sequence $X^{n+2}=x X^{n+1}+y X^{n}, n=0,1,2, \ldots$, where $x$ and $y$ are arbitrary parameters such that $x^{2} / 4+y \neq 0$; obtain expressions for the determinant and for the trace.

## Solution by Walther Janous, Innsbruck, Austria

It is enough to deal with part $b$ (clearly containing the first question $a$ ).
Let the matrix $X$ under consideration be given as $\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$. Then for $X$ it has to hold that

$$
\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]^{2}-x \cdot\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]-y \cdot\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right],
$$

i.e. (upon factorization of the left-hand side),

$$
\left[\begin{array}{cc}
-a \cdot x-y+a^{2}+b \cdot c & -b \cdot(x-a-d) \\
-c \cdot(x-a-d) & -d \cdot x-y+b \cdot c+d^{2}
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right] .
$$

Therefore, we have to distinguish several cases.
CASE 1. $b=0$. Then

$$
\begin{array}{r}
{\left[\begin{array}{cc}
-a \cdot x-y+a^{2}+0 \cdot c & -0 \cdot(x-a-d) \\
-c \cdot(x-a-d) & -d \cdot x-y+0 \cdot c+d^{2}
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right],} \\
{\left[\begin{array}{cc}
-a \cdot x-y+a^{2} & 0 \\
-c \cdot(x-a-d) & -d \cdot x-y+d^{2}
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right] .}
\end{array}
$$

Case 1.1. $c=0$. Then

$$
\begin{aligned}
& {\left[\begin{array}{cc}
-a \cdot x-y+a^{2} & 0 \\
-0 \cdot(x-a-d) & -d \cdot x-y+d^{2}
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]} \\
& {\left[\begin{array}{cc}
-a \cdot x-y+a^{2} & 0 \\
0 & -d \cdot x-y+d^{2}
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]}
\end{aligned}
$$

yielding, for the entries $a$ and $d$, the possibilities

$$
\begin{aligned}
& {\left[a=\frac{\sqrt{\left(x^{2}+4 \cdot y\right)+x}}{2} \wedge d=\frac{\sqrt{\left(x^{2}+4 \cdot y\right)+x}}{2}, a=\frac{\sqrt{\left(x^{2}+4 \cdot y\right)+x}}{2} \wedge d=\frac{x-\sqrt{\left(x^{2}+4 \cdot y\right)}}{2}\right.} \\
& \left.a=\frac{x-\sqrt{\left(x^{2}+4 \cdot y\right)}}{2} \wedge d=\frac{\sqrt{\left(x^{2}+4 \cdot y\right)+x}}{2}, a=\frac{x-\sqrt{\left(x^{2}+4 \cdot y\right)}}{2} \wedge d=\frac{x-\sqrt{\left(x^{2}+4 \cdot y\right)}}{2}\right]
\end{aligned}
$$

All of these solutions yield desired matrices $X$ of type $\left[\begin{array}{cc}a & 0 \\ 0 & d\end{array}\right]$ having $\operatorname{det}(X)=a * d$ and $\operatorname{tr}(X)=$ $a+d$.

Remark: From these possibilities, it is easily derived that for part a the two stated properties $\operatorname{det}(X)=-1$ and $\operatorname{tr}(X)=+1$ do not hold in general!

Case 1.2. $c \neq 0$. Then $x-a-d=0$, i.e., $d=x-a$, whence

$$
\begin{array}{r}
{\left[\begin{array}{cc}
-a \cdot x-y+a^{2} & 0 \\
-c \cdot(x-a-(x-a)) & -(x-a) \cdot x-y+(x-a)^{2}
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right],} \\
{\left[\begin{array}{cc}
-a \cdot x-y+a^{2} & 0 \\
0 & -a \cdot x-y+a^{2}
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]}
\end{array}
$$

Therefore the entry $a$ has to be

$$
\left[a=\frac{\sqrt{\left(x^{2}+4 \cdot y\right)+x}}{2}, a=\frac{\sqrt{\left(x^{2}+4 \cdot y\right)}}{2}\right]
$$

with the corresponding possibilities of $d(=x-a)$ :

$$
\left[d=\frac{x-\sqrt{\left(x^{2}+4 \cdot y\right)}}{2}, d=\frac{x-\sqrt{\left(x^{2}+4 \cdot y\right)+x}}{2}\right]
$$

Hence, all matrices $X$ of type $\left[\begin{array}{ll}a & 0 \\ c & d\end{array}\right]$ are of the desired kind. Their determinants and traces are $a * d$ and $a+d$, respectively.
CASE 2. " $c=0$ ", and its only 'new' subcase $b \neq 0$, are dealt with in a manner similar to that shown above.

Thus, we are left with
CASE 3. $b \neq 0$ and $c \neq 0$. Then $x-a-d=0$, whence $d=x-a$ and, further,

$$
\begin{array}{r}
{\left[\begin{array}{cc}
-a \cdot x-y+a^{2}+b \cdot c & -b \cdot x+a \cdot b+b \cdot(x-a) \\
-c \cdot x+a \cdot c+c \cdot(x-a) & -(x-a) \cdot x-y+b \cdot c+(x-a)^{2}
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right],} \\
{\left[\begin{array}{cc}
-a \cdot x-y+a^{2}+b \cdot c & 0 \\
0 & -a \cdot x-y+a^{2}+b \cdot c
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right] .}
\end{array}
$$

Thus, we get (with $a$ and $b$ arbitrary)

$$
\left[c=\frac{a \cdot x+y-a^{2}}{b}\right] .
$$

Therefore, finally, the matrices $X$ for this case are

$$
\left[\begin{array}{cc}
a & b \\
\frac{a \cdot x+y-a^{2}}{b} & x-a
\end{array}\right] \text { having } \operatorname{tr}(X)=x \text {, and } \operatorname{det}\left[\begin{array}{cc}
a & b \\
\frac{a \cdot x+y-a^{2}}{b} & x-a
\end{array}\right]=-y \text {. }
$$

Also solved by P. Bruckman, M. Catalani, O. Furdui, J. Morrison, and H.-J. Seiffert.

## A Lesser Problem

## H-581 Proposed by José Luis Díaz, Polytechnic University of Catalunya, Spain (Vol. 40, no. 1, February 2002)

Let $n$ be a positive integer. Prove that
(a) $F_{n}^{F_{n+1}}+F_{n+1}^{F_{n+2}}+F_{n+2}^{F_{n}}<F_{n}^{F_{n}}+F_{n+1}^{F_{n+1}}+F_{n+2}^{F_{n+2}}$.
(b) $F_{n}^{F_{n+1}} F_{n+1}^{F_{n+2}} F_{n+2}^{F_{n}}<F_{n}^{F_{n}} F_{n+1}^{F_{n+1}} F_{n+2}^{F_{n+2}}$.

## Solution by the proposer

Part (a) trivially holds if $n=1,2$. In order to prove the general statement, we observe that

$$
\begin{aligned}
& \left(F_{n}^{F_{n}}+F_{n+1}^{F_{n+1}}+F_{n+2}^{F_{n+2}}\right)-\left(F_{n}^{F_{n+1}}+F_{n+1}^{F_{n+2}}+F_{n+2}^{F_{n}}\right) \\
& =\left[\left(F_{n}^{F_{n}}+F_{n+1}^{F_{n+1}}\right)-\left(F_{n}^{F_{n+1}}+F_{n+1}^{F_{n}}\right)\right]+\left[\left(F_{n+2}^{F_{n+2}}-F_{n+2}^{F_{n}}\right)-\left(F_{n+1}^{F_{n+2}}-F_{n+1}^{F_{n}}\right)\right] .
\end{aligned}
$$

Therefore, our statement will be established if we prove that, for $n \geq 3$,

$$
\begin{equation*}
F_{n}^{F_{n+1}}+F_{n+1}^{F_{n}}<F_{n}^{F_{n}}+F_{n+1}^{F_{n+1}} \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{n+1}^{F_{n+2}}-F_{n+1}^{F_{n}}<F_{n+2}^{F_{n+2}}-F_{n+2}^{F_{n}} \tag{2}
\end{equation*}
$$

hold.
In fact, we consider the integral

$$
I_{1}=\int_{F_{n}}^{F_{n+1}}\left(F_{n+1}^{x} \log F_{n+1}-F_{n}^{x} \log F_{n}\right) d x
$$

Since $F_{n}<F_{n+1}$ if $n \geq 3$, then, for $F_{n} \leq n \leq F_{n+1}$, we have $F_{n}^{x} \log F_{n}<F_{n+1}^{x} \log F_{n}<F_{n+1}^{x} \log F_{n+1}$ and $I_{1}>0$.

On the other hand, evaluating the integral, we obtain

$$
\begin{aligned}
I_{1} & =\int_{F_{n}}^{F_{n+1}}\left(F_{n+1}^{x} \log F_{n+1}-F_{n}^{x} \log F_{n}\right) d x=\left[F_{n+1}^{x}-F_{n}^{x}\right]_{F_{n}}^{F_{n+1}} \\
& =\left(F_{n}^{F_{n}}+F_{n+1}^{F_{n+1}}\right)-\left(F_{n}^{F_{n+1}}+F_{n+1}^{F_{n}}\right)
\end{aligned}
$$

and (1) is proved.
To prove (2), we consider the integral

$$
I_{2}=\int_{F_{n}}^{F_{n+2}}\left(F_{n+2}^{x} \log F_{n+2}-F_{n+1}^{x} \log F_{n+1}\right) d x
$$

Since $F_{n+1}<F_{n+2}$, then, for $F_{n} \leq x \leq F_{n+2}$, we have $F_{n+1}^{x} \log F_{n+1}<F_{n+2}^{x} \log F_{n+2}$ and $I_{2}>0$.
On the other hand, evaluating $I_{2}$, we obtain

$$
\begin{aligned}
I_{2} & =\int_{F_{n}}^{F_{n+2}}\left(F_{n+2}^{x} \log F_{n+2}-F_{n+1}^{x} \log F_{n+1}\right) d x=\left[F_{n+2}^{x}-F_{n+1}^{x}\right]_{F_{n}}^{F_{n+2}} \\
& =\left(F_{n+2}^{F_{n+2}}-F_{n+2}^{F_{n}}\right)-\left(F_{n+1}^{F_{n+2}}-F_{n+1}^{F_{n}}\right)
\end{aligned}
$$

This completes the proof of part (a).
We will prove part (b) of our statement using the weighted AM-GM-HM inequality [1]: "Let $x_{1}, x_{2}, \ldots, x_{n}$ be positive real numbers and let $w_{1}, w_{2}, \ldots, w_{n}$ be nonnegative real numbers that sum to 1. Then

$$
\begin{equation*}
\sum_{k=1}^{n} w_{k} x_{k} \geq \prod_{k=1}^{n} x_{k}^{w_{k}} \geq\left(\sum_{k=1}^{n} \frac{w_{k}}{x_{k}}\right)^{-1} \tag{3}
\end{equation*}
$$

Equality holds when $x_{1}=x_{2}=\cdots=x_{n}$."
The proof will be done in two steps. First, we will prove that

$$
\begin{equation*}
F_{n}^{F_{n+1}} F_{n+1}^{F_{n+2}} F_{n+2}^{F_{n}}<\left(\frac{F_{n}+F_{n+1}+F_{n+2}}{3}\right)^{F_{n}+F_{n+1}+F_{n+2}} \tag{4}
\end{equation*}
$$

In fact, setting

$$
x_{1}=F_{n}, \quad x_{2}=F_{n+1}, \quad x_{3}=F_{n+2}
$$

and

$$
w_{1}=\frac{F_{n+1}}{F_{n}+F_{n+1}+F_{n+2}}, w_{2}=\frac{F_{n+2}}{F_{n}+F_{n+1}+F_{n+2}}, w_{3}=\frac{F_{n}}{F_{n}+F_{n+1}+F_{n+2}}
$$

we have, from (3),

$$
\begin{aligned}
& F_{n}^{F_{n+1} /\left(F_{n}+F_{n+1}+F_{n+2}\right)} F_{n+1}^{F_{n+2} /\left(F_{n}+F_{n+1}+F_{n+2}\right)} F_{n+2}^{F_{n} /\left(F_{n}+F_{n+1}+F_{n+2}\right)} \\
& <\frac{F_{n} F_{n+1}}{F_{n}+F_{n+1}+F_{n+2}}+\frac{F_{n+1} F_{n+2}}{F_{n}+F_{n+1}+F_{n+2}}+\frac{F_{n+2} F_{n}}{F_{n}+F_{n+1}+F_{n+2}}
\end{aligned}
$$

or

$$
F_{n}^{F_{n+1}} F_{n+1}^{F_{n+2}} F_{n+2}^{F_{n}}<\left(\frac{F_{n} F_{n+1}+F_{n+1} F_{n+2}+F_{n+2} F_{n}}{F_{n}+F_{n+1}+F_{n+2}}\right)^{F_{n}+F_{n+1}+F_{n+2}}
$$

Inequality (4) will be established if we prove that

$$
\left(\frac{F_{n} F_{n+1}+F_{n+1} F_{n+2}+F_{n+2} F_{n}}{F_{n}+F_{n+1}+F_{n+2}}\right)^{F_{n}+F_{n+2}+F_{n+2}}<\left(\frac{F_{n}+F_{n+1}+F_{n+2}}{3}\right)^{F_{n}+F_{n+1}+F_{n+2}}
$$

or, equivalently,

$$
\frac{F_{n} F_{n+1}+F_{n+1} F_{n+2}+F_{n+2} F_{n}}{F_{n}+F_{n+1}+F_{n+2}}<\frac{F_{n}+F_{n+1}+F_{n+2}}{3}
$$

or

$$
\left(F_{n}+F_{n+1}+F_{n+2}\right)^{2}>3\left(F_{n} F_{n+1}+F_{n+1} F_{n+2}+F_{n+2} F_{n}\right),
$$

i.e.,

$$
F_{n}^{2}+F_{n+1}^{2}+F_{n+2}^{2}>F_{n} F_{n+1}+F_{n+1} F_{n+2}+F_{n+2} F_{n} .
$$

The last inequality will be proved in a straightforward manner. In fact, adding the inequalities $F_{n}^{2}+F_{n+1}^{2} \geq 2 F_{n} F_{n+1}, F_{n+1}^{2}+F_{n+2}^{2}>2 F_{n+1} F_{n+2}$, and $F_{n+2}^{2}+F_{n}^{2}>2 F_{n+2} F_{n}$ we have

$$
F_{n}^{2}+F_{n+1}^{2}+F_{n+2}^{2}>F_{n} F_{n+1}+F_{n+1} F_{n+2}+F_{n+2} F_{n}
$$

and the result is proved.
Finally, we will prove that

$$
\begin{equation*}
\left(\frac{F_{n}+F_{n+1}+F_{n+2}}{3}\right)^{F_{n}+F_{n+1}+F_{n+2}}<F_{n}^{F_{n}} F_{n+1}^{F_{n+1}} F_{n+2}^{F_{n+2}} . \tag{5}
\end{equation*}
$$

Setting $x_{1}=F_{n}, x_{2}=F_{n+1}, x_{3}=F_{n+2}, w_{1}=F_{n} /\left(F_{n}+F_{n+1}+F_{n+2}\right), w_{2}=F_{n+1} /\left(F_{n}+F_{n+1}+F_{n+2}\right)$, and $w_{3}=F_{n+2} /\left(F_{n}+F_{n+1}+F_{n+2}\right)$, and using GM-HM inequality, we have

$$
\begin{aligned}
\frac{F_{n}+F_{n+1}+F_{n+2}}{3} & =\frac{1}{\frac{1}{F_{n}+F_{n+1}+F_{n+2}}+\frac{1}{F_{n}+F_{n+1}+F_{n+2}}+\frac{1}{F_{n}+F_{n+1}+F_{n+2}}} \\
& <F_{n}^{F_{n} /\left(F_{n}+F_{n+1}+F_{n+2}\right) F_{n+1}^{F_{n+1}}\left(F_{n}+F_{n+1}+F_{n+2}\right) F_{n+2}^{F_{n+2} /\left(F_{n}+F_{n+1}+F_{n+2}\right)} .}
\end{aligned}
$$

Hence,

$$
\left(\frac{F_{n}+F_{n+1}+F_{n+2}}{3}\right)^{F_{n}+F_{n+1}+F_{n+2}}<F_{n}^{F_{n}} F_{n+1}^{F_{n+1}} F_{n+2}^{F_{n+2}}
$$

and (5) is proved.
This completes the proof of part (b) and we are done.

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