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The Fibonacci Quarterly

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DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

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THE INTERVAL ASSOCIATED WITH A FIBONACCI NUMBER

Takao Komatsu
Faculty of Education, Mie University, Tsu, Mie 514-8507, Japan
e-mail: komatsu@edu.mie-u.ac.jp
(*Submitted October 2000-Final Revision February 2001*)

1. INTRODUCTION

It is well known that the n^{th} Fibonacci number F_n is given by the Binet-Moivre form $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, where $\alpha, \beta = (1 \pm \sqrt{5})/2$. Möbius [2], however, gave a different way to characterize a Fibonacci number. Let z be an integer with $z \ge 2$. Then z is a Fibonacci number if and only if the interval $[gz - 1/z, gz + 1/z]$ contains exactly one integer, where $g = \alpha = (1+\sqrt{5})/2$ is the golden number.

In this paper we shall give some criteria about a more general case.

2. CRITERION 1

As usual, let $\alpha = [a_0; a_1, a_2, \dots]$ denote the continued fraction expansion of α, where

$$\alpha = a_0 + 1/\alpha_1, \qquad a_0 = \lfloor \alpha \rfloor,$$
$$\alpha_n = a_n + 1/\alpha_{n+1}, \quad a_n = \lfloor \alpha_n \rfloor \quad (n = 1, 2, \dots).$$

The n^{th} convergent $p_n/q_n = [a_0; a_1, \dots, a_n]$ of α is given by the recurrence relations

$$p_n = a_n p_{n-1} + p_{n-2} \quad (n = 0, 1, \dots), \qquad p_{-2} = 0, \quad p_{-1} = 1,$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad (n = 0, 1, \dots), \qquad q_{-2} = 1, \quad q_{-1} = 0.$$

Let the sequence G_n be defined by $G_0 = 0$, $G_1 = 1$, $G_n = aG_{n-1} + G_{n-2}$ $(n = 2, 3, \dots)$. G_n is called the n^{th} *generalized Fibonacci number*. The Binet-Moivre form of G_n $(n = 0, 1, 2, \dots)$ is given by

$$G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where α and β are the solutions of the equation $x^2 - ax - 1 = 0$. Assume that $\alpha > \beta$. Then the continued fraction expansion of α is given by

$$\alpha = \frac{a + \sqrt{a^2+4}}{2} = [a; a, a, \dots]$$

and $G_n = q_{n-1} = p_{n-2}$ $(n \ge 0)$.

Theorem 1: Let $\alpha = (a + \sqrt{a^2+4})/2 = [a; a, a, \dots]$ with $a \ge 2$ (or $a = 1$ and $n \ge 2$). Then q is a generalized Fibonacci number is and only if the interval

$$\left[q\alpha - \frac{1}{aq}, \; q\alpha + \frac{1}{aq}\right]$$

contains exactly one integer p; explicitly $q = q_n = G_{n+1}$ and $p = p_n$.

Remark: In fact,

$$p_n - 1 < q_n\alpha - \frac{1}{aq_n} < p_n < q_n\alpha + \frac{1}{aq_n} < p_n + 1.$$

Proof: In general, by [3],

$$\left|\alpha-\frac{p_n}{q_n}\right|=\frac{1}{q_n(\alpha_{n+1}q_n+q_{n-1})}<\frac{1}{a_{n+1}q_n^2}$$

because $\alpha_{n+1}=[a_{n+1};a_{n+2},\dots]$. When $\alpha=(\sqrt{5}+1)/2=[1;1,1,\dots]$ and $n\ge 2$, a more precise upper bound is possible. Namely, from

$$\alpha_{n+1}+\frac{q_{n-1}}{q_n}\ge\frac{\sqrt{5}+1}{2}+\frac{1}{2}>2,$$

we have $|\alpha-p_n/q_n|<1/(2q_n^2)$.

Returning to $\alpha=(a+\sqrt{a^2+4})/2=[a;a,a,\dots]$, if $q=q_n$ $(=G_{n+1})$ and $p=p_n$, then $|\alpha-p/q|$ $<1/(aq^2)$, which is equivalent to

$$q\alpha-\frac{1}{aq}<p<q\alpha+\frac{1}{aq}.$$

Furthermore,

$$p_n+1>q_n\alpha+1-\frac{1}{aq_n}\ge q_n\alpha+\frac{1}{aq_n}\quad\text{and}\quad p_n-1<q_n\alpha-1+\frac{1}{aq_n}\le q_n\alpha-\frac{1}{aq_n}.$$

Notice that when $a=1$ and $n=0,1$, the interval contains two integers. In fact,

$$q_n\alpha-\frac{1}{q_n}=\alpha-1<1<2<\alpha+1=q_n\alpha+\frac{1}{q_n}.$$

On the other hand, suppose that p/q satisfies $|\alpha-p/q|<1/(aq^2)$. We shall follow a similar step to the proof of Theorem 184 in [1]. Assume that $p/q=[b_0;b_1,\dots,b_n]$. Then

$$\alpha-\frac{p}{q}=\frac{\varepsilon(-1)^n}{q^2}\quad\left(0<\varepsilon<\frac{1}{a}\right).$$

Set

$$\omega=\frac{P_{n-1}-\alpha Q_{n-1}}{\alpha Q_n-P_n},\ \text{ i.e., }\ \alpha=\frac{\omega P_n+P_{n-1}}{\omega Q_n+Q_{n-1}},$$

where $P_n/Q_n=p/q=[b_0;b_1,\dots,b_n]$. Then

$$\frac{\varepsilon(-1)^n}{q^2}=\alpha-\frac{P_n}{Q_n}=\frac{(-1)^n}{Q_n(\omega Q_n+Q_{n-1})}.$$

Letting $\varepsilon=Q_n/(\omega Q_n+Q_{n-1})$, we have

$$\omega=\frac{1}{\varepsilon}-\frac{Q_{n-1}}{Q_n}>a-1\ge 1\ (a\ge 2).$$

Notice again that we can set $a=2$ instead of $a=1$ when $\alpha=(\sqrt{5}+1)/2=[1;1,1,\dots]$ and $n\ge 2$. Therefore, by Theorem 172 in [1], P_{n-1}/Q_{n-1} and P_n/Q_n are two consecutive convergents to α.

3. CRITERION 2

As Möbius proved, unless α is the golden number, the number of integers included in the interval $\left[q\alpha-\frac{1}{q},q\alpha+\frac{1}{q}\right]$ may be more than one. For the generalized α, the following criterion holds.

Theorem 2: Let $\alpha=(a+\sqrt{a^2+4})/2=[a;a,a,\dots]$. Then the solutions (p,q) of the inequality $q\alpha-1/q<p<q\alpha+1/q$ using positive integers p and q are as follows:

$$(p_n,q_n),\dots,(tp_n,tq_n),\ (p_n+p_{n-1},q_n+q_{n-1}),\ (p_{n+1}-p_n,q_{n+1}-q_n)\ \ (n\ge 0),$$

where $t=\lfloor\sqrt{a}\rfloor$.

Proof: Let q be an integer with $q_n\le q<q_{n+1}$. First, we will show that if $|q\alpha-p|\le 1/q$ then the form of q must be iq_n or $q_{n+1}-iq_n$ $(i=1,2,\dots,a_{n+1}-1)$. By Lemma 2.1 and Theorem 3.3 in [4], we have

$$\{u_1\alpha\}<\{u_2\alpha\}<\cdots<\{u_{q_{n+1}-1}\alpha\}\quad\text{if } n \text{ is even,}$$
$$\{u_1\alpha\}>\{u_2\alpha\}>\cdots>\{u_{q_{n+1}-1}\alpha\}\quad\text{if } n \text{ is odd,}$$

where $\{u_1,u_2,\dots,u_{q_{n+1}-1}\}=\{1,2,\dots,q_{n+1}-1\}$ is a set with $u_j\equiv jq_n \pmod{q_{n+1}}$ $(j=1,2,\dots,q_{n+1}-1)$. Explicitly,

$$\begin{aligned}\{q_n\alpha\}&<\{2q_n\alpha\}<\cdots<\{a_{n+1}q_n\alpha\}<\{(q_n-q_{n-1})\alpha\}<\{(2q_n-q_{n-1})\alpha\}\\&<\cdots<\{(q_{n+1}-2q_n+q_{n-1})\alpha\}<\{(q_{n+1}-q_n+q_{n-1})\alpha\}<\{(q_{n+1}-a_{n+1}q_n)\alpha\}\\&<\cdots<\{(q_{n+1}-2q_n)\alpha\}<\{(q_{n+1}-q_n)\alpha\},\end{aligned}$$

if n is even; similar if n is odd. Since

$$\|(q_n-q_{n-1})\alpha\|=|(q_n-q_{n-1})\alpha-(p_n-p_{n-1})|=\frac{\alpha_{n+1}+1}{\alpha_{n+1}q_n+q_{n-1}}\ge\frac{1}{q_n}$$

and

$$\|(q_{n+1}-q_n+q_{n-1})\alpha\|=|(q_{n+1}-q_n+q_{n-1})\alpha-(p_{n+1}-p_n+p_{n-1})|=\frac{\frac{1}{\alpha_{n+2}}+\alpha_{n+1}+1}{\alpha_{n+1}q_n+q_{n-1}}\ge\frac{1}{q_n},$$

there does not exist a q satisfying $q\alpha-1/q<p<q\alpha+1/q$ unless the form of q is $q=iq_n$ or $q=q_{n+1}-iq_n$ $(i=1,2,\dots,a_{n+1}-1)$.

$$\left|\alpha-\frac{kp_n}{kq_n}\right|=\left|\alpha-\frac{p_n}{q_n}\right|=\frac{1}{q_n(\alpha_{n+1}q_n+q_{n-1})}\le\frac{1}{k^2q_n^2}$$

holds if and only if

$$k\le\sqrt{\alpha_{n+1}+\frac{q_{n-1}}{q_n}}.$$

When $\alpha=[a;a,a,\dots]$, we have

$$\sqrt{a}<\sqrt{\alpha}=\sqrt{\alpha_1+\frac{q_{-1}}{q_0}}<\sqrt{\alpha_3+\frac{q_1}{q_2}}<\cdots<\sqrt{\alpha_4+\frac{q_2}{q_3}}<\sqrt{\alpha_2+\frac{q_0}{q_1}}=\frac{\alpha}{\sqrt{a}}.$$

Since $\lfloor\sqrt{a}\rfloor=\lfloor\alpha/\sqrt{a}\rfloor=1$ $(a=1)$ and $\alpha/\sqrt{a}<\sqrt{a+1}$ $(a\ge 2)$, we obtain

$$\lfloor\sqrt{a}\rfloor=\left\lfloor\sqrt{\alpha_{n+1}+\frac{q_{n-1}}{q_n}}\right\rfloor=\left\lfloor\frac{\alpha}{\sqrt{a}}\right\rfloor.$$

Let $a_{n+1}\ge 4$. Then, since $a_{n+1}\ge i+2\ge i^2/(i-1)$ for $i=2,3,\dots,a_{n+1}-2$, we have $(i-1)\alpha_{n+2}a_{n+1}\ge i^2\alpha_{n+2}$, yielding $((i-1)\alpha_{n+2}+1)q_{n+1}>(i^2\alpha_{n+2}+i+1)q_n$. Thus,

$$\|(q_{n+1}-iq_n)\alpha\| = \frac{1}{\alpha_{n+2}q_{n+1}+q_n} + \frac{i}{\alpha_{n+1}q_n+q_{n-1}} = \frac{i\alpha_{n+2}+1}{\alpha_{n+2}q_{n+1}+q_n} > \frac{1}{q_{n+1}-iq_n}.$$

Since

$$\alpha > a+\frac{1}{a}-1 = \frac{p_1}{q_1}-1 \geq \frac{p_n}{q_n}-1 = \frac{q_{n+1}}{q_n}-1$$

for $n \geq 0$, we have

$$|(q_{n+1}-q_n)\alpha-(p_{n+1}-p_n)| = \frac{\alpha+1}{\alpha q_{n+1}+q_n} < \frac{1}{q_{n+1}-q_n},$$

yielding

$$\left|\alpha-\frac{p_{n+1}-p_n}{q_{n+1}-q_n}\right| < \frac{1}{(q_{n+1}-q_n)^2}.$$

Since

$$\alpha < a+1 = \frac{p_0}{q_0}+1 \leq \frac{p_n}{q_n}+1 = \frac{q_{n+1}}{q_n}+1$$

for $n \geq 1$, we have

$$\|(q_{n+1}-(a_{n+1}-1)q_n)\alpha\| = |(q_n+q_{n-1})\alpha-(p_n+p_{n-1})| = \frac{\alpha-1}{\alpha q_n+q_{n-1}} < \frac{1}{q_n+q_{n-1}},$$

yielding

$$\left|\alpha-\frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right| < \frac{1}{(q_n+q_{n-1})^2}.$$

For $n=0$,

$$\left|\alpha-\frac{p_0+1}{q_0}\right| = |\alpha-(a+1)| < 1 = \frac{1}{q_0^2}.$$

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SOME IDENTITIES INVOLVING THE POWERS OF THE GENERALIZED FIBONACCI NUMBERS

Feng-Zhen Zhao and Tianming Wang
Dalian University of Technology, 116024 Dalian, P.R. China
(*Submitted October 2000-Final Revision May 2001*)

1. INTRODUCTION

In this paper, we are interested in the generalized Fibonacci and Lucas numbers

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n, \tag{1}$$

where $\alpha = (p+\sqrt{p^2-4q})/2$, $\beta = (p-\sqrt{p^2-4q})/2$, and p and q are real numbers with $pq \neq 0$ and $p^2 - 4q > 0$. For $p = -q = 1$, $\{U_n\}$ and $\{V_n\}$ are the classical Fibonacci sequence $\{F_n\}$and the classical Lucas sequence $\{L_n\}$, respectively. It is obvious that the sequences $\{U_n\}$ and $\{V_n\}$ satisfy the linear recurrence relation $W_n = pW_{n-1} - qW_{n-2}$, $n \geq 2$. In [1], Zhang discussed the calculation of the summation

$$\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}U_{a_2}\ldots U_{a_k}. \tag{2}$$

This problem is very interesting and can help us to find some convolution properties. Zhang [1] gave a method for calculating (2) and obtained a series of identities involving the generalized Fibonacci numbers. For instance, he proved that

$$\sum_{a+b=n} U_aU_b = \frac{U_1}{p^2-4q}[(n-1)pU_n - 2nqU_{n-1}], \; n \geq 1, \tag{3}$$

$$\begin{aligned}\sum_{a+b+c=n} U_aU_bU_c = \frac{U_1^2}{2(p^2-4q)^2}[&((p^3-4pq)n^2 - (3p^3-6pq)n + (2p^3+4pq))U_{n-1} \\ &+ ((4q^2-p^2q)n^2 + 3p^2qn - (2p^2q+4q^2))U_{n-2}], \; n \geq 2.\end{aligned} \tag{4}$$

For the powers of the generalized Fibonacci numbers, are there results similar to (3) and (4)? It seems that this has not been studied. The purpose of this paper is to investigate the calculation of the summation of the forms

$$\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}^2U_{a_2}^2\ldots U_{a_k}^2 \quad \text{and} \quad \sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}^3U_{a_2}^3\ldots U_{a_k}^3.$$

Using Zhang's method, we will establish some identities involving the squares and cubics of the generalized Fibonacci numbers.

2. MAIN RESULTS

Consider the generating functions of $\{U_n^2\}$ and $\{U_n^3\}$:

$$G(x) = \sum_{n=0}^{\infty} U_n^2x^n \quad \text{and} \quad H(x) = \sum_{n=0}^{\infty} U_n^3x^n.$$

By using (1) and the geometric series formula, we have

$$G(x)=\frac{1}{\alpha-\beta}\left[\frac{\alpha x}{(1-\alpha^2x)(1-\alpha\beta x)}-\frac{\beta x}{(1-\beta^2x)(1-\alpha\beta x)}\right],\ |x|<\min\left(\frac{1}{|\alpha^2|},\frac{1}{|\beta^2|},\frac{1}{|q|}\right),$$

and

$$H(x)=\frac{1}{(\alpha-\beta)^2}\left[\frac{U_3x}{(1-\alpha^3x)(1-\beta^3x)}-\frac{3qx}{(1-\alpha qx)(1-\beta qx)}\right],\ |x|<\min\left(\frac{1}{|\alpha^3|},\frac{1}{|\beta^3|},\frac{1}{|\alpha q|},\frac{1}{|\beta q|}\right).$$

Define

$$F(x)=\frac{G(x)}{x}=\sum_{n=1}^{\infty}U_n^2x^{n-1},\quad F_1(x)=\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\alpha^nU_nx^{n-1},\quad F_2(x)=\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\beta^nU_nx^{n-1}. \tag{5}$$

$$E(x)=\frac{H(x)}{x}=\sum_{n=1}^{\infty}U_n^3x^{n-1},\quad E_1(x)=\frac{1}{(\alpha-\beta)^2}\sum_{n=1}^{\infty}U_{3n}x^{n-1},\quad E_2(x)=\frac{-3}{(\alpha-\beta)^2}\sum_{n=1}^{\infty}U_nq^nx^{n-1}. \tag{6}$$

For $F(x), F_1(x), F_2(x), E(x), E_1(x)$, and $E_2(x)$, we have the following lemmas.

Lemma 1: If $F(x)$, $F_1(x)$, and $F_2(x)$ are defined by (5), then they satisfy:

$$\begin{aligned}F^2(x)=&\frac{1}{(\alpha-\beta)^3}[(p-2\alpha^2\beta x)F_1'(x)-4\alpha^2\beta F_1(x)+(p-2\alpha\beta^2x)F_2'(x)-4\alpha\beta^2F_2(x)]\\&-\frac{2\alpha\beta}{(\alpha-\beta)^4}\left[\frac{\alpha^4}{(\alpha^2-\beta^2)(1-\alpha^2x)}-\frac{\beta^4}{(\alpha^2-\beta^2)(1-\beta^2x)}-\frac{\alpha\beta}{1-\alpha\beta x}-\frac{\alpha\beta}{(1-\alpha\beta x)^2}\right];\end{aligned}\tag{7}$$

$$\begin{aligned}F^3(x)=&\frac{(p-2\alpha^2\beta x)^2F_1''(x)-14\alpha^2\beta(p-2\alpha^2\beta x)F_1'(x)+32\alpha^4\beta^2F_1(x)}{2(\alpha-\beta)^6}\\&-\frac{(p-2\alpha\beta^2x)^2F_2''(x)-14\alpha\beta^2(p-2\alpha\beta^2x)F_2'(x)+32\alpha^2\beta^4F_2(x)}{2(\alpha-\beta)^6}\\&-\frac{3\alpha\beta}{(\alpha-\beta)^6}\left[\frac{\alpha^6}{(\alpha^2-\beta^2)(1-\alpha^2x)^2}-\frac{4\alpha^6\beta}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\alpha^2x)}\right.\\&-\frac{4\alpha\beta^6}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\beta^2x)}+\frac{4\alpha^5\beta^2+4\alpha^2\beta^5}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\alpha\beta x)}\\&\left.-\frac{\beta^6}{(\alpha^2-\beta^2)(1-\beta^2x)^2}+\frac{3\alpha^2\beta^2}{(1-\alpha\beta x)^2}+\frac{2\alpha^2\beta^2}{(1-\alpha\beta x)^3}\right].\end{aligned}\tag{8}$$

Proof: It is clear that

$$F^2(x)=F_1^2(x)+F_2^2(x)-2F_1(x)F_2(x)\quad\text{and}\quad F^3(x)=F_1^3(x)-F_2^3(x)-3F_1(x)F_2(x)F(x).$$

Using the definition of $F_1(x)$ and the derivative of $F_1(x)$, and noticing that $\alpha+\beta=p$, we get

$$F_1^2(x)=\frac{1}{(\alpha-\beta)^3}[(p-2\alpha^2\beta x)F_1'(x)-4\alpha^2\beta F_1(x)]. \tag{9}$$

Following the same pattern, we get

$$F_2^3(x)=\frac{1}{(\alpha-\beta)^3}[(p-2\alpha\beta^2x)F_2'(x)-4\alpha\beta^2F_2(x)].$$

In the meantime, it is very easy to show that

$$F_1(x)F_2(x)=\frac{\alpha\beta}{(\alpha-\beta)^4}\left[\frac{\alpha^4}{(\alpha-\beta)(1-\alpha^2x)}-\frac{\beta^4}{(\alpha^2-\beta^2)(1-\beta^2x)}-\frac{\alpha\beta}{1-\alpha\beta x}-\frac{\alpha\beta}{(1-\alpha\beta x)^2}\right],$$

where

$$|x|<\min\left(\frac{1}{|\alpha|^2},\frac{1}{|\beta|^2},\frac{1}{|q|}\right).$$

Thus, (7) holds.

Differentiating in (9), we have

$$2F_1(x)F_1'(x)=\frac{1}{(\alpha-\beta)^3}[(p-2\alpha^2\beta x)F_1''(x)-6\alpha^2\beta F_1'(x)].$$

Applying (9) again, we have

$$F_1^3(x)=\frac{(p-2\alpha^2\beta x)F_1''(x)-14\alpha^2\beta(p-2\alpha^2\beta x)F_1'(x)+32\alpha^4\beta^2F_1(x)}{2(\alpha-\beta)^6}.$$

Following the same way, we have

$$F_2^3(x)=\frac{(p-2\alpha\beta^2x)F_2''(x)-14\alpha\beta^2(p-2\alpha\beta^2x)F_2'(x)+32\alpha^2\beta^4F_2(x)}{2(\alpha-\beta)^6}.$$

On the other hand, after careful calculus, one can verify that

$$\begin{aligned}F_1(x)F_2(x)F(x)=\frac{\alpha\beta}{(\alpha-\beta)^6}\Bigg[&\frac{\alpha^6}{(\alpha^2-\beta^2)(1-\alpha^2x)^2}-\frac{4\alpha^6\beta}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\alpha^2x)}\\&-\frac{4\alpha\beta^6}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\beta^2x)}+\frac{4\alpha^5\beta^2+4\alpha^2\beta^5}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\alpha\beta x)}\\&-\frac{\beta^6}{(\alpha^2-\beta^2)(1-\beta^2x)^2}+\frac{3\alpha^2\beta^2}{(1-\alpha\beta x)^2}+\frac{2\alpha^2\beta^2}{(1-\alpha\beta x)^3}\Bigg].\end{aligned}$$

Therefore, (8) holds. □

Lemma 2: If $E(x)$, $E_1(x)$, and $E_2(x)$ are defined by (6), then they satisfy:

$$\begin{aligned}E^2(x)=&\frac{(-2q^3x+V_3)E_1'(x)}{(p^2-4q)^2U_3}-\frac{4q^3E_1(x)}{(p^2-4q)^2U_3}+\frac{[6q^2x+3p(q+2)]E_2'(x)}{(p^2q+2p^2+4)(p^2-4q)}\\&-\frac{12qE_2(x)}{(p^2q+2p^2+4)(p^2-4q)}-\frac{6q}{p(p^2-4q)^{7/2}}\left[\frac{\alpha^6}{1-\alpha^3x}-\frac{\alpha^2q(V_2+q)}{1-\alpha qx}\right.\\&\left.+\frac{\beta^2q(V_2+q)}{1-\beta qx}-\frac{\beta^6}{1-\beta^3x}\right].\end{aligned}\tag{10}$$

$$\begin{aligned}E^3(x)=&\frac{1}{2(p^2-4q)^4U_2^3}[(V_3-2q^3x)^2E_1''(x)-14q^3(V_3-2q^3x)E_1'(x)+32q^6E_1(x)]\\&+\frac{1}{2(p^2q+2p^2+4)^2(p^2-4q)^2}\{[3p(q+2)+6q^2x]^2E_2''(x)\\&+[3p(q+2)+6q^2x](6q^2+12q)E_2'(x)+288q^2E_2(x)\}\end{aligned}$$

$$-\frac{9q}{p(p^2-4q)^5}\left\{\frac{\alpha^9}{(1-\alpha^3x)^2}+\left[\frac{\alpha^3q^3(V_2+q)}{\alpha^2-\beta^2}-\frac{\alpha^6q(V_2+q)}{\alpha-\beta}-\frac{\alpha^3q^3V_3}{\alpha^3-\beta^3}\right]\frac{1}{1-\alpha^3x}\right.$$
$$+\left(\frac{\alpha^4q^2}{\alpha-\beta}+\frac{\alpha q^4}{\alpha^2-\beta^2}\right)\frac{V_2+q}{1-\alpha qx}-\left(\frac{\beta^4q^2}{\alpha-\beta}+\frac{\beta q^4}{\alpha^2-\beta^2}\right)\frac{V_2+q}{1-\beta qx}$$
$$\left.+\left[\frac{\beta^6q(V_2+q)}{\alpha-\beta}-\frac{\beta^3q^3(V_2+q)}{\alpha^2-\beta^2}+\frac{\beta^3q^3V_3}{\alpha^3-\beta^3}\right]\frac{1}{1-\beta^3x}+\frac{\beta^9}{(1-\beta^3x)^2}\right\}+\frac{27q^2}{p(p^2-4q)^5} \tag{11}$$
$$\times\left\{\frac{\alpha^9}{(\alpha^2-\beta^2)(1-\alpha^3x)}+\left[\frac{\alpha V_1q^2(V_2+q)-\alpha^6q}{\alpha-\beta}-\frac{\beta^3q^3}{\alpha^2-\beta^2}\right]\frac{1}{1-\alpha qx}-\frac{\alpha^3q(V_2+q)}{(1-\alpha qx)^2}\right.$$
$$\left.+\left[\frac{\beta^6q-\beta V_1q^2(V_2+q)}{\alpha-\beta}+\frac{\alpha^3\beta^3}{\alpha^2-\beta^2}\right]\frac{1}{1-\beta qx}-\frac{\beta^3q(V_2+q)}{(1-\beta qx)^2}-\frac{\beta^9}{(\alpha^2-\beta^2)(1-\beta^3x)}\right\}.$$

The proof of Lemma 2 is similar to that of Lemma 1 and therefore is omitted here.

From the lemmas, we can obtain the main results of this paper. We now state and prove the following new results.

Theorem 1: Let $\{U_n\}$ be the generalized Fibonacci sequence. Then we have

$$\sum_{a+b=n} U_a^2U_b^2=\frac{[-2nqU_{n-1}+p(n-1)U_n]V_n}{p^2-4q}-\frac{2q}{p^2-4q}\left(\frac{U_{2n+2}}{U_2}-nq^{n-1}\right),\ n\geq 1, \tag{12}$$

$$\sum_{a+b+c=n} U_a^2U_b^2U_c^2=\frac{1}{2(p^2-4q)^3}[p^2n(n-1)U_{n+1}^2-2pq(2n+3)U_nU_{n+1}$$
$$+4q^2(n^2+10n-9)U_{n-1}U_{n+1}]-\frac{3q}{(p^2-4q)^3}\left[\frac{(n-1)U_{2n+2}}{U_2}\right. \tag{13}$$
$$\left.+\frac{4q(q^{n-1}V_3-V_{2n+1})}{U_2(p^2-4q)}+(n+3)(n-1)q^n\right],\ n\geq 2.$$

Proof: To show that this theorem is valid, comparing the coefficients on both sides of (7) and (8), and noticing that (1), $\alpha+\beta=p$, $\alpha\beta=q$, and $(\alpha-\beta)^2=p^2-4q$, we get identities (12) and (13). □

Corollary 1: Let $\{U_n\}$ be the generalized Fibonacci sequence and k be a positive integer. Then

$$\sum_{a+b=n} U_{ak}^2U_{bk}^2=\frac{[-2nq^kU_{nk-k}+(n-1)V_kU_{nk}]V_{nk}U_k}{p^2-4q}-\frac{2q^kU_k^2}{p^2-4q}\left(\frac{U_{2nk+2k}}{U_{2k}}-nq^{nk-k}\right),\ n\geq 1, \tag{14}$$

$$\sum_{a+b+c=n} U_{ak}^2U_{bk}^2U_{ck}^2=\frac{1}{2(p^2-4q)^3U_k^2}[n(n-1)V_k^2U_{nk+k}^2$$
$$-2V_kq^k(2n+3)U_{nk}U_{nk+k}+4q^{2k}(n^2+10n-9)U_{nk-k}U_{nk+k}] \tag{15}$$
$$-\frac{3q^k}{U_{2k}(p^2-4q)^3}\left[(n-1)U_{2nk+2k}+\frac{4(q^{nk}V_{3k}-q^kV_{2nk+k})}{U_k(p^2-4q)}+(n+3)(n-1)q^{nk}U_{2k}\right],$$

for $n\geq 2$.

Proof: Let

$$U'_n=\frac{(\alpha^k)^n-(\beta^k)^n}{\alpha^k-\beta^k}=\frac{U_{nk}}{U_k},\quad V'_n=\alpha^{nk}+\beta^{nk}=V_{nk}. \tag{16}$$

It is clear that the sequences $\{U'_n\}$ and $\{V'_n\}$ satisfy the linear recurrence relation

$$W_n=V_kW_{n-1}-q^kW_{n-2},\ n\geq 2.$$

Now, we apply Theorem 1 to the sequences $\{U'_n\}$ and $\{V'_n\}$. In (12), if U_n, V_n, p, and q are replaced by U'_n, V'_n, V_k, and q^k, respectively, one has that

$$\sum_{a+b=n}U'^2_aU'^2_b=\frac{[-2nq^kU'_{n-1}+V_k(n-1)U'_n]V'_n}{V_k^2-4q^k}-\frac{2q^k}{V_k^2-4q^k}\left(\frac{U'_{2n+2}}{U'_2}-nq^{nk-k}\right).$$

Due to (16) and $V_k^2-4q^k=U_k^2(p^2-4q)$, we get (14). Using a similar method, we get (15). □

Theorem 2: Let $\{U_n\}$ be the generalized Fibonacci sequence. Then

$$\begin{aligned}\sum_{a+b=n}U_a^3U_b^3=&\frac{-2nq^3U_{3(n-1)}+(n-1)V_3U_{3n}}{(p^2-4q)^3U_3}-\frac{9q^n(2nqU_{n-1}-pU_n)}{(p^2-4q)^3}\\&-\frac{6q[U_{3n}-(V_2+q)q^{n-1}U_n]}{p(p^2-4q)^3},\end{aligned} \tag{17}$$

$$\begin{aligned}\sum_{a+b+c=n}U_a^3U_b^3U_c^3=&\frac{(n-2)[(n-1)V_3^2U_{3n}-2(2n+1)q^3V_3U_{3n-3}]+4(n^2-1)q^6U_{3n-6}}{2(p^2-4q)^5U_3^2}\\&-\frac{27q^n}{(p^2-4q)^5}[4q^2(n^2-1)U_{n-2}-2pq(7n-12)U_{n-1}+p^2(n-1)(n-2)U_n]\\&-\frac{9q}{p(p^2-4q)^5}\left[(n-2)V_{3n}+\frac{(V_2+q)q^3U_{3n-6}}{p}-q(V_2+q)U_{3n-3}-\frac{q^3V_3U_{3n-6}}{U_3}\right.\\&\left.+(V_2+q)q^{n-1}U_{n+1}+\frac{(V_2+q)q^{n+1}U_{n-2}}{p}\right]+\frac{27q^2}{p(p^2-4q)^5}\left[\frac{U_{3n}}{p}-q^{n-2}U_{n+3}\right.\\&\left.-V_1(V_2+q)q^{n-1}U_{n-2}-\frac{q^{n+3}U_{n-6}}{p}-(n-2)(V_2+q)q^{n+1}V_{n+1}\right].\end{aligned}$$

Proof: Comparing the coefficients of x^{n-2} and x^{n-3} on both sides of (10) and (11), respectively, we get Theorem 2. □

Corollary 2: Let $\{U_n\}$ be the generalized Fibonacci sequence and k be a positive integer. Then

$$\begin{aligned}\sum_{a+b=n}U_{ak}^3U_{bk}^3=&\frac{-2nq^{3k}U_{3k(n-1)}+(n-1)V_{3k}U_{3kn}}{U_{3k}(p^2-4q)^3}-\frac{9q^{kn}(2nq^kU_{nk-k}-V_kU_{kn}}{U_k(p^2-4q)^3}\\&-\frac{6q^k[U_{3kn}-(V_{2k}q^{kn-k}+q^{kn})U_{nk}]}{U_kV_k(p^2-4q)^3},\end{aligned}$$

$$\sum_{a+b+c=n} U_{ak}^3 U_{bk}^3 U_{ck}^3 = \frac{1}{2U_{3k}^2(p^2-4q)^5}[(n-1)(n-2)V_{3k}^2 U_{3kn} - 2(n-2)(2n+1)q^{3k}U_{3kn-3k}V_{3k}$$
$$+4(n^2-1)q^{6k}U_{3kn-6k}] - \frac{27q^{nk}}{U_k^2(p^2-4q)^5}[4q^{2k}(n^2-1)U_{kn-2k}$$
$$-2V_k q^k(7n-12)U_{kn-k} + V_k^2(n-1)(n-2)U_{nk}] - \frac{9q^k}{U_kV_k(p^2-4q)^5}$$
$$\times\left[(n-2)V_{3kn} + \frac{q^{3k}(V_{2k}+q^k)U_{3kn-6k}}{U_kV_k} - \frac{q^k(V_{2k}+q^k)U_{3kn-3k}}{U_k}\right.$$
$$\left.-\frac{q^{3k}V_{3k}U_{3kn-6k}}{U_{3k}} + \frac{q^{kn-k}(V_{2k}+q^k)U_{kn+k}}{U_k} + \frac{q^{kn+k}(V_{2k}+q^k)U_{kn-2k}}{U_kV_k}\right]$$
$$+\frac{27q^{2k}}{U_k^2V_k(p^2-4q)^5}\left[\frac{U_{3kn}}{V_k} - q^{kn-2k}U_{kn+3k} - V_k(V_{2k}+q^k)q^{kn-k}U_{kn-2k}\right.$$
$$\left.-\frac{q^{kn+3k}U_{kn-6k}}{V_k} - (n-2)(V_{2k}+q^k)q^{kn+k}V_{kn+k}U_k\right].$$

We note that Corollaries 1 and 2 are generalizations of Theorems 1 and 2, respectively.

Finally, we can find some congruences from Theorems 1 and 2 according to the particular choices of p and q. For example, setting $p=-q=1$ in (12) and $F_2=1$, we obtain

$$\sum_{a+b=n} F_a^2F_b^2 = \frac{[2nF_{n-1}+(n-1)F_n]L_n}{5} + \frac{2}{5}(F_{2n+2}+n(-1)^n).$$

Setting $p=-q=1$ in (17), we have

$$\sum_{a+b=n} F_a^3F_b^3 = \frac{nF_{3n-3}+2(n+2)F_{3n}+21(-1)^nF_n+18(-1)^nF_{n-1}}{125}.$$

Hence,

$$[2nF_{n-1}+(n-1)F_n]L_n + 2(F_{2n+2}+n(-1)^n) \equiv 0 \pmod 5,\ \ n\ge 1,$$

and

$$nF_{3n-3}+2(n+2)F_{3n}+(-1)^n(21nF_n+18nF_{n-1}) \equiv 0 \pmod{125},\ \ n\ge 1.$$

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COMPLEX FACTORIZATIONS OF THE FIBONACCI AND LUCAS NUMBERS

Nathan D. Cahill, John R. D'Errico, and John P. Spence
Eastman Kodak Company, 343 State Street, Rochester, NY 14650
{nathan.cahill, john.derrico, john.spence]@kodak.com
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1. INTRODUCTION

It is always fascinating to see what results when seemingly different areas of mathematics overlap. This article reveals one such result; number theory and linear algebra (with the help of orthogonal polynomials) are intertwined to yield complex factorizations of the Fibonacci and Lucas numbers. In Sections 2 and 3, respectively, we derive these complex factorizations:

$$F_n = \prod_{k=1}^{n-1}\left(1-2i\cos\frac{\pi k}{n}\right),\ n \geq 2, \tag{1.1}$$

and

$$L_n = \prod_{k=1}^{n}\left(1-2i\cos\frac{\pi(k-\frac{1}{2})}{n}\right),\ n \geq 1. \tag{1.2}$$

Along the way, we also establish the general forms:

$$F_n = i^{n-1}\frac{\sin(n\cos^{-1}(-\frac{i}{2}))}{\sin(\cos^{-1}(-\frac{i}{2}))},\ n \geq 1, \tag{1.3}$$

and

$$L_n = 2i^n\cos(n\cos^{-1}(-\tfrac{i}{2})),\ n \geq 1. \tag{1.4}$$

A simple proof of (1.1) can be obtained by considering the roots of Fibonacci polynomials (see Webb and Parberry [7] and Hoggatt and Long [3]). This paper proves (1.1) by considering how the Fibonacci numbers can be connected to Chebyshev polynomials by determinants of a sequence of matrices, and then illustrates a connection between the Lucas numbers and Chebyshev polynomials (and hence proves (1.2)) by using a slightly different sequence of matrices. (1.3) and (1.4) are not new developments (see Morgado [4] and Rivlin [5]); however, they are of interest here because they fall out of the derivations of (1.1) and (1.2) quite naturally.

In order to simplify the derivations of (1.1) and (1.2), we present the following lemma (the proof is included for completeness).

Lemma 1: Let $\{\mathbf{H}(n), n = 1, 2, \dots\}$ be a sequence of tridiagonal matrices of the form:

$$\mathbf{H}(n) = \begin{pmatrix} h_{1,1} & h_{1,2} & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & & \\ & h_{3,2} & h_{3,3} & \ddots & \\ & & \ddots & \ddots & h_{n-1,n} \\ & & & h_{n,n-1} & h_{n,n} \end{pmatrix}. \tag{1.5}$$

Then the successive determinants of $\mathbf{H}(n)$ are given by the recursive formula:

$$\begin{aligned} &|\mathbf{H}(1)| = h_{1,1}, \\ &|\mathbf{H}(2)| = h_{1,1}h_{2,2} - h_{1,2}h_{2,1}, \\ &|\mathbf{H}(n)| = h_{n,n}|\mathbf{H}(n-1)| - h_{n-1,n}h_{n,n-1}|\mathbf{H}(n-2)|. \end{aligned} \tag{1.6}$$

Proof: We prove Lemma 1 by the second principle of finite induction, computing all determinants by cofactor expansion. For the basis step, we have:

$$|\mathbf{H}(1)| = |(h_{1,1})| = h_{1,1},$$

$$|\mathbf{H}(2)| = \left|\begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix}\right| = h_{1,1}h_{2,2} - h_{1,2}h_{2,1},$$

$$|\mathbf{H}(3)| = \left|\begin{pmatrix} h_{1,1} & h_{1,2} & 0 \\ h_{2,1} & h_{2,2} & h_{2,3} \\ 0 & h_{3,2} & h_{3,3} \end{pmatrix}\right| = h_{3,3}|\mathbf{H}(2)| - h_{2,3}\left|\begin{pmatrix} h_{1,1} & h_{1,2} \\ 0 & h_{3,2} \end{pmatrix}\right| = h_{3,3}|\mathbf{H}(2)| - h_{2,3}h_{3,2}|\mathbf{H}(1)|.$$

For the inductive step, we assume $|\mathbf{H}(k)| = h_{k,k}|\mathbf{H}(k-1)| - h_{k-1,k}h_{k,k-1}|\mathbf{H}(k-2)|$ for $3 \le k \le n$. Then

$$|\mathbf{H}(k+1)| = \left|\begin{pmatrix} h_{1,1} & h_{1,2} & & & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & & & & \\ & h_{3,2} & h_{3,3} & \ddots & & & \\ & & \ddots & \ddots & h_{k-2,k-1} & & \\ & & & h_{k-1,k-2} & h_{k-1,k-1} & h_{k-1,k} & \\ & & & & h_{k,k-1} & h_{k,k} & h_{k,k+1} \\ & & & & & h_{k+1,k} & h_{k+1,k+1} \end{pmatrix}\right|$$

$$= h_{k+1,k+1}|\mathbf{H}(k)| - h_{k,k+1}\left|\begin{pmatrix} h_{1,1} & h_{1,2} & & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & & & \\ & h_{3,2} & h_{3,3} & \ddots & & \\ & & \ddots & \ddots & h_{k-2,k-1} & \\ & & & h_{k-1,k-2} & h_{k-1,k-1} & h_{k-1,k} \\ & & & & 0 & h_{k+1,k} \end{pmatrix}\right|$$

$$= h_{k+1,k+1}|\mathbf{H}(k)| - h_{k,k+1}h_{k+1,k}|\mathbf{H}(k-1)|. \ \square$$

2. COMPLEX FACTORIZATION OF THE FIBONACCI NUMBERS

In order to derive (1.1), we introduce the sequence of matrices $\{\mathbf{M}(n), n = 1, 2, \ldots\}$, where $\mathbf{M}(n)$ is the $n \times n$ tridiagonal matrix with entries $m_{k,k} = 1$, $1 \le k \le n$, and $m_{k-1,k} = m_{k,k-1} = i$, $2 \le k \le n$. That is,

$$\mathbf{M}(n) = \begin{pmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & 1 \end{pmatrix}. \tag{2.1}$$

According to Lemma 1, successive determinants of $\mathbf{M}(n)$ are given by the recursive formula:

$$\begin{aligned} &|\mathbf{M}(1)| = 1, \\ &|\mathbf{M}(2)| = 1^2 - i^2 = 2, \\ &|\mathbf{M}(n)| = 1|\mathbf{M}(n-1)| - i^2|\mathbf{M}(n-2)| = |\mathbf{M}(n-1)| + |\mathbf{M}(n-2)|. \end{aligned} \tag{2.2}$$

Clearly, this is also the Fibonacci sequence, starting with F_2. Hence,

$$F_n = |\mathbf{M}(n-1)|, \quad n \geq 2. \tag{2.3}$$

There are a variety of ways to compute the matrix determinant (see Golub and Van Loan [2] for more details). In addition to the method of cofactor expansion, the determinant of a matrix can be found by taking the product of its eigenvalues. Therefore, we will compute the spectrum of $\mathbf{M}(n)$ in order to find an alternate formulation for $|\mathbf{M}(n)|$.

We now introduce another sequence of matrices $\{\mathbf{G}(n), n = 1, 2, \ldots\}$, where $\mathbf{G}(n)$ is the $n \times n$ tridiagonal matrix with entries $g_{k,k} = 0$, $1 \leq k \leq n$, and $g_{k-1,k} = g_{k,k-1} = 1$, $2 \leq k \leq n$. That is,

$$\mathbf{G}(n) = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}. \tag{2.4}$$

Note that $\mathrm{M}(n) = \mathbf{I} + i\mathbf{G}(n)$. Let λ_k, $k = 1, 2, \ldots, n$, be the eigenvalues of $\mathbf{G}(n)$ (with associated eigenvectors $\mathbf{x}_k$). Then, for each j,

$$\mathbf{M}(n)\mathbf{x}_j = [\mathbf{I} + i\mathbf{G}(n)]\mathbf{x}_j = \mathbf{I}\mathbf{x}_j + i\mathbf{G}(n)\mathbf{x}_j = \mathbf{x}_j + i\lambda_j\mathbf{x}_j = (1 + i\lambda_j)\mathbf{x}_j.$$

Therefore, $\mu_k = 1 + i\lambda_k$, $k = 1, 2, \ldots, n$, are the eigenvalues of $\mathbf{M}(n)$. Hence,

$$|\mathbf{M}(n)| = \prod_{k=1}^{n}(1 + i\lambda_k), \quad n \geq 1. \tag{2.5}$$

In order to determine the λ_k's, we recall that each λ_k is a zero of the characteristic polynomial $p_n(\lambda) = |\mathbf{G}(n) - \lambda\mathbf{I}|$. Since $\mathbf{G}(n) - \lambda\mathbf{I}$ is a tridiagonal matrix, i.e.,

$$\mathbf{G}(n) - \lambda\mathbf{I} = \begin{pmatrix} -\lambda & 1 & & & \\ 1 & -\lambda & 1 & & \\ & 1 & -\lambda & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -\lambda \end{pmatrix}, \tag{2.6}$$

we use Lemma 1 to establish a recursive formula for the characteristic polynomials of $\{\mathbf{G}(n), n = 1, 2, \ldots\}$:

$$\begin{aligned} &p_1(\lambda) = -\lambda, \\ &p_2(\lambda) = \lambda^2 - 1, \\ &p_n(\lambda) = -\lambda p_{n-1}(\lambda) - p_{n-2}(\lambda). \end{aligned} \tag{2.7}$$

This family of characteristic polynomials can be transformed into another family $\{U_n(x), n \geq 1\}$ by the transformation $\lambda \equiv -2x$:

$$\begin{aligned} U_1(x) &= 2x, \\ U_2(x) &= 4x^2-1, \\ U_n(x) &= 2xU_{n-1}(x) - U_{n-2}(x). \end{aligned} \tag{2.8}$$

The family $\{U_n(x), n \geq 1\}$ is the set of Chebyshev polynomials of the second kind. It is a well-known fact (see Rivlin [5]) that defining $x \equiv \cos\theta$ allows the Chebyshev polynomials of the second kind to be written as:

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{2.9}$$

From (2.9), we can see that the roots of $U_n(x) = 0$ are given by $\theta_k = \frac{\pi k}{n+1}$, $k = 1, 2, \ldots, n$, or $x_k = \cos\theta_k = \cos\frac{\pi k}{n+1}$, $k = 1, 2, \ldots, n$. Applying the transformation $\lambda \equiv -2x$, we now have the eigenvalues of $\mathbf{G}(n)$:

$$\lambda_k = -2\cos\frac{\pi k}{n+1}, \; k = 1, 2, \ldots, n. \tag{2.10}$$

Combining (2.3), (2.5), and (2.10), we have

$$F_{n+1} = |\mathbf{M}(n)| = \prod_{k=1}^{n}\left(1 - 2i\cos\frac{\pi k}{n+1}\right), \; n \geq 1, \tag{2.11}$$

which is identical to the complex factorization (1.1).

From (2.6), we can think of Chebyshev polynomials of the second kind as being generated by determinants of successive matrices of the form

$$\mathbf{A}(n,x) = \begin{pmatrix} 2x & 1 & & & \\ 1 & 2x & 1 & & \\ & 1 & 2x & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2x \end{pmatrix}, \tag{2.12}$$

where $\mathbf{A}(n, x)$ is $n \times n$. If we note that $\mathbf{M}(n) = i\mathbf{A}(n, -\frac{i}{2})$, then we have:

$$|\mathbf{M}(n)| = i^n \left|\mathbf{A}\left(n, -\frac{i}{2}\right)\right| = i^n U_n\left(-\frac{i}{2}\right). \tag{2.13}$$

Combining (2.3), (2.9), and (2.13) yields

$$F_{n+1} = i^n \frac{\sin\left((n+1)\cos^{-1}(-\frac{i}{2})\right)}{\sin\left(\cos^{-1}(-\frac{i}{2})\right)}, \; n \geq 1 \tag{2.14}$$

Since it is also true that

$$F_1 = 1 = i^0 \frac{\sin\left(\cos^{-1}(-\frac{i}{2})\right)}{\sin\left(\cos^{-1}(-\frac{i}{2})\right)},$$

(1.3) holds.

3. COMPLEX FACTORIZATION OF THE LUCAS NUMBERS

The process by which we derive (1.2) is similar to that of the derivation of (1.1), but it has its own intricacies. Consider the sequence of matrices $\{\mathbf{S}(n), n = 1, 2, \ldots\}$, where $\mathbf{S}(n)$ is the $n \times n$ tri-diagonal matrix with entries $s_{1,1} = \frac{1}{2}$, $s_{k,k} = 1$, $2 \leq k \leq n$, and $s_{k-1,k} = s_{k,k-1} = i$, $2 \leq k \leq n$. That is,

$$\mathbf{S}(n) = \begin{pmatrix} \frac{1}{2} & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & 1 \end{pmatrix}. \tag{3.1}$$

According to Lemma 1, successive determinants of $\mathbf{S}(n)$ are given by the recursive formula:

$$\begin{aligned} &|\mathbf{S}(1)| = \tfrac{1}{2}, \\ &|\mathbf{S}(2)| = \tfrac{1}{2} - i^2 = \tfrac{3}{2}, \\ &|\mathbf{S}(n)| = 1|\mathbf{S}(n-1)| - i^2|\mathbf{S}(n-2)| = |\mathbf{S}(n-1)| + |\mathbf{S}(n-2)|. \end{aligned} \tag{3.2}$$

Clearly, each number in this sequence is half of the corresponding Lucas number. We have

$$L_n = 2|\mathbf{S}(n)|, \; n \geq 1. \tag{3.3}$$

Unlike the derivation in the previous section, we will not compute the spectrum of $\mathbf{S}(n)$ directly. Instead, we will first note the following:

$$|\mathbf{S}(n)| = \frac{1}{2}\left|(\mathbf{I} + \mathbf{e}_1\mathbf{e}_1^{\mathrm{T}})\mathbf{S}(n)\right|, \tag{3.4}$$

where $\mathbf{e}_j$ is the j^{th} column of the identity matrix. (This is true because $|\mathbf{I} + \mathbf{e}_1\mathbf{e}_1^{\mathrm{T}}| = 2$.) Furthermore, we can express the right-hand side of (3.4) in the following way:

$$\frac{1}{2}\left|(\mathbf{I} + \mathbf{e}_1\mathbf{e}_1^{\mathrm{T}})\mathbf{S}(n)\right| = \frac{1}{2}\left|\mathbf{I} + i\left(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}}\right)\right|, \tag{3.5}$$

where $\mathbf{G}(n)$ is the matrix given in (2.4). Let γ_k, $k = 1, 2, \ldots, n$ be the eigenvalues of $\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}}$ (with associated eigenvectors $\mathbf{y}_k$). Then, for each j,

$$(\mathbf{I} + i(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}}))\mathbf{y}_j = \mathbf{I}\mathbf{y}_j + i(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}})\mathbf{y}_j = \mathbf{y}_j + i\gamma_j\mathbf{y}_j = (1 + i\gamma_j)\mathbf{y}_j.$$

Therefore,

$$\frac{1}{2}\left|\mathbf{I} + i(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}})\right| = \frac{1}{2}\prod_{k=1}^{n}(1 + i\gamma_k). \tag{3.6}$$

In order to determine the γ_k's, we recall that each γ_k is a zero of the characteristic polynomial $q_n(\gamma) = |\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}} - \gamma\mathbf{I}|$. Since $\left|(\mathbf{I} - \frac{1}{2}\mathbf{e}_1\mathbf{e}_1^{\mathrm{T}})\right| = \frac{1}{2}$, we can alternately represent the characteristic polynomial as

$$q_n(\gamma) = 2\left|\left(\mathbf{I} - \tfrac{1}{2}\mathbf{e}_1\mathbf{e}_1^{\mathrm{T}}\right)\left(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}} - \gamma\mathbf{I}\right)\right|. \tag{3.7}$$

Since $q_n(\gamma)$ is twice the determinant of a tridiagonal matrix, i.e.,

$$q_n(\gamma) = 2\left|\left(\mathbf{I} - \tfrac{1}{2}\mathbf{e}_1\mathbf{e}_1^{\mathrm{T}}\right)\left(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}} - \gamma\mathbf{I}\right)\right| = 2\left|\begin{pmatrix} -\frac{\gamma}{2} & 1 & & & \\ 1 & -\gamma & 1 & & \\ & 1 & -\gamma & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -\gamma \end{pmatrix}\right|, \tag{3.8}$$

we can use Lemma 1 to establish a recursive formula for $q_n(\gamma)$:

$$\begin{aligned} q_1(\gamma) &= -\tfrac{\gamma}{2}, \\ q_2(\gamma) &= \tfrac{\gamma^2}{2} - 1, \\ q_n(\gamma) &= -\gamma q_{n-1}(\gamma) - q_{n-2}(\gamma). \end{aligned} \tag{3.9}$$

This family of polynomials can be transformed into another family $\{T_n(x),\ n \geq 1\}$ by the transformation $\gamma \equiv -2x$:

$$\begin{aligned} T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x). \end{aligned} \tag{3.10}$$

The family $\{T_n(x),\ n \geq 1\}$ is the set of Chebyshev polynomials of the first kind. Rivlin [5] shows that defining $x \equiv \cos\theta$ allows the Chebyshev polynomials of the first kind to be written as

$$T_n(x) = \cos n\theta. \tag{3.11}$$

From (3.11), we can see that the roots of $T_n(x) = 0$ are given by

$$\theta_k = \frac{\pi\left(k - \frac{1}{2}\right)}{n},\ k = 1, 2, \ldots, n, \quad \text{or} \quad x_k = \cos\theta_k = \cos\frac{\pi\left(k - \frac{1}{2}\right)}{n},\ k = 1, 2, \ldots, n.$$

Applying the transformation $\gamma \equiv -2x$ and considering that the roots of (3.7) are also roots of $|\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}} - \gamma\mathbf{I}| = 0$, we now have the eigenvalues of $\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}}$:

$$\gamma_k = -2\cos\frac{\pi\left(k - \frac{1}{2}\right)}{n},\ k = 1, 2, \ldots, n. \tag{3.12}$$

Combining (3.3)-(3.6) and (3.12), we have

$$L_n = \prod_{k=1}^{n}\left(1 - 2^i \cos\frac{\pi\left(k - \frac{1}{2}\right)}{n}\right),\ n \geq 1, \tag{3.13}$$

which is identical to the complex factorization (1.2).

From (3.8), we can think of Chebyshev polynomials of the first kind as being generated by determinants of successive matrices of the form

$$\mathbf{B}(n, x) = \begin{pmatrix} x & 1 & & & \\ 1 & 2x & 1 & & \\ & 1 & 2x & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2x \end{pmatrix}, \tag{3.14}$$

where $\mathbf{B}(n, x)$ is $n \times n$. If we note that $\mathbf{S}(n) = i\mathbf{B}(n, -\frac{i}{2})$, then we have

$$|\mathbf{S}(n)| = i^n \left|\mathbf{B}\left(n, -\frac{1}{2}\right)\right| = i^n T_n\left(-\frac{i}{2}\right). \tag{3.15}$$

Combining (3.3), (3.11), and (3.15) yields

$$L_n = 2i^n \cos\left(n \cos^{-1}\left(-\frac{i}{2}\right)\right),\ n \geq 1, \tag{3.16}$$

which is exactly (1.4).

4. CONCLUSION

This method of exploiting special properties of the Chebyshev polynomials allows us to find other interesting factorizations as well. For instance, the factorizations

$$F_{2n+2} = \prod_{k=1}^{n}\left(3 - 2\cos\frac{\pi k}{n+1}\right),\ n \geq 1, \tag{4.1}$$

$$n = \prod_{k=1}^{n-1}\left(2 - 2\cos\frac{\pi k}{n}\right),\ n \geq 2, \tag{4.2}$$

and

$$2^{1-n} = \prod_{k=1}^{n}\left(1 - \cos\frac{\pi\left(k - \frac{1}{2}\right)}{n}\right),\ n \geq 1, \tag{4.3}$$

can be derived with judicious choices of entries in tridiagonal matrices (Strang [6] presents a family of tridiagonal matrices that can be used to derive (4.1)). It is also possible to compare these factorizations with the Binet-like general formulas (see Burton [1]) for second-order linear recurrence relations in order to determine which products converge to zero, converge to a non-zero number, or diverge as n approaches infinity.

One final note: an interesting (but fairly straightforward) problem for students of complex variables is to prove that (1.3) is equivalent to Binet's formula for the Fibonacci Numbers.

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INEQUALITIES AMONG RELATED PAIRS OF FIBONACCI NUMBERS

K. T. Atanassov
CLBME-Bulgarian Academy of Sciences, PO Box 12, Sofia-1113, Bulgaria

Ron Knott
RDK Internet Consultancy, 1 Upperton Road, Guildford, GU2 7NB, UK

Kiyota Ozeki
Utsunomiya University, 350 Mine-machi, Utsunomiya City, Japan

A. G. Shannon
KvB Institute of Technology, North Sydney, 2060, and
Warrane College, The University of New South Wales, 2033, Australia

László Szalay
University of West Hungary, H-9400, Sopron, Bajcsy Zs, út 4, Hungary
(Submitted November 2000-Final Revision January 2001)

1. INTRODUCTION

What may be called "Fibonacci inequalities" have been studied in a variety of contexts. These include metric spaces [6], diophantine approximation [7], fractional bounds [5], Fibonacci numbers of graphs [1], and Farey-Fibonacci fractions [2]. Here we consider Fibonacci inequalities. The results are the "best possible" and we relate them through the sequence $\{m_r\}_{r=0}^{n}$ defined by

$$m_r = F_r F_{n+1-r},$$

where n is a fixed natural number and $F_1, F_2, F_3, \ldots,$ are the ordinary Fibonacci numbers.

2. MAIN RESULT

Theorem: For every natural number k, the following inequalities for the elements of the sequence $\{m_k\}_{k=0}^{n}$ are valid:

(a) if $n = 4k$, then

$$F_1F_{4k} > F_3F_{4k-2} > \cdots > F_{2k-1}F_{2k+2} > F_{2k}F_{2k+1} > F_{2k-2}F_{2k+3} > \cdots > F_2F_{4k-1};$$

(b) if $n = 4k+1$, then

$$F_1F_{4k+1} > F_3F_{4k-1} > \cdots > F_{2k-1}F_{2k+3} > F_{2k+1}F_{2k+1} > F_{2k}F_{2k+2} > \cdots > F_2F_{4k};$$

(c) if $n = 4k+2$, then

$$F_1F_{4k+2} > F_3F_{4k} > \cdots > F_{2k+1}F_{2k+2} > F_{2k}F_{2k+3} > F_{2k-2}F_{2k+5} > \cdots > F_2F_{4k+1};$$

(d) if $n = 4k+3$, then

$$F_1F_{4k+3} > F_3F_{4k+1} > \cdots > F_{2k+1}F_{2k+3} > F_{2k+2}F_{2k+2} > F_{2k}F_{2k+4} > \cdots > F_2F_{4k+2}.$$

Examples when $k = 3$:

(a) $F_1F_{12} = 144 > F_3F_{10} = 110 > F_5F_8 = 105 > F_6F_7 = 104 > F_4F_9 = 102 > F_2F_{11} = 89;$

(b) $F_1F_{13} = 233 > F_3F_{11} = 178 > F_5F_9 = 170 > F_7F_7 = 169 > F_6F_8 = 168$
$> F_4F_{10} = 165 > F_2F_{12} = 144;$

(c) $F_1F_{14} = 377 > F_3F_{12} = 288 > F_7F_8 = 273 > F_6F_9 = 272 > F_4F_{11} = 267 > F_2F_{13} = 233;$

(d) $F_1F_{15} = 610 > F_3F_{13} = 466 > F_7F_9 = 442 > F_8F_8 = 441 > F_6F_{10} = 440$
$> F_4F_{12} = 432 > F_2F_{14} = 377.$

Proof of Theorem: We shall use induction simultaneously on n, that is, on k and i. For $k = 1$, we have

$$\begin{aligned}
&F_1F_4 = 1\times 3 > 2\times 1 = F_3F_2,\\
&F_1F_5 = 1\times 5 > 2\times 2 = F_3F_3 > 1\times 3 = F_2F_4,\\
&F_1F_6 = 1\times 8 > 2\times 3 = F_3F_4 > 1\times 5 = F_2F_5,\\
&F_1F_7 = 1\times 13 > 2\times 5 = F_3F_5 > 3\times 3 = F_4F_4 > 1\times 8 = F_2F_5.
\end{aligned}$$

Assume that inequalities (a), (b), (c), and (d) are true for some $k \geq 1$. Then we must prove that the inequalities are true for $k+1$.

In particular, from the truth of (d), it follows that, for every i, $1 \leq i \leq k$:

$$F_{2i+1}F_{4k-2i+3} > F_{2i+2}F_{4k-2i+2}. \tag{2.1}$$

We shall discuss case (a), but the other cases are proved similarly.

First, we see that

$$F_1F_{4k+4} - F_3F_{4k+2} = F_{4k+3} + F_{4k+2} - 2F_{4k+2} = F_{4k+3} - F_{4k+2} > 0,$$

that is, the inequality

$$F_{2i-1}F_{4k-2i+6} > F_{2i+1}F_{4k-2i+4} \tag{2.2}$$

is valid for $i = 1$.

Let us assume that, for some i, $1 \leq i \leq k$, inequality (2.2) is true. Then we must prove that the inequality

$$F_{2i+1}F_{4k-2i+4} > F_{2i+3}F_{4k-2i+2} \tag{2.3}$$

is also true.

But

$$\begin{aligned}
F_{2i+1}F_{4k-2i+4} - F_{2i+3}F_{4k-2i+2} &= F_{2i+1}F_{4k-2i+3} + F_{2i+1}F_{4k-2i+2} - F_{2i+1}F_{4k-2i+2} - F_{2i+2}F_{4k-2i+2}\\
&= F_{2i+1}F_{4k-2i+3} - F_{2i+2}F_{4k-2i+2} > 0
\end{aligned}$$

because of the inductive assumption for (d) in case (2.1). Therefore, inequality (3) holds, from which the truth of (a) follows.

3. DISCUSSION

By analogy with the extremal problems discussed in [3], we can formulate the following.

Corollary: For every natural number n the maximal element of the sequence $\{m_k\}_{k=0}^{n}$ is F_1F_n.

Proof of Corollary: Equation (I_{26}) of [8] can be rewritten as

$$F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1}$$

to show that $F_n > F_{n-k}F_{k+1}$, $1 \leq k \leq n$, which gives the required result.

More generally, F_1F_n is the maximal element of the set

$$M = \{F_{i_1}F_{i_2}, \dots, F_{i_{k-1}}F_{i_k}\},$$

where $n = i_1 + i_2 + \cdots + i_k$, $1 < k \le n$, is a positive partition into k parts. The result can be proved by induction.

4. CONCLUDING COMMENTS

A somewhat analogous result was proposed by Bakinova [4] and proved by Mascioni [9]:

$$\frac{F_{k+1}}{F_1} < \frac{F_{k+3}}{F_3} < \frac{F_{k+5}}{F_5} < \cdots < \alpha^k < \cdots < \frac{F_{k+6}}{F_6} < \frac{F_{k+4}}{F_4} < \frac{F_{k+2}}{F_2}.$$

The results can be generalized to the sequence $\{u_n(0, 1; p, q)\}$ defined by the recurrence

$$u_n = pu_{n-1} - qu_{n-2},\ n \ge 2,$$

with $u_0 = 0$, $u_1 = 1$, $p \ne 0$, $p \in \mathbb{Z}$, $\Delta = p^2 + 4q > 0$, in which case a neater proof comes from the use of the Binet forms for the general terms and hyperbolic cosines and sines. The result can also be extended to related triples of Fibonacci numbers.

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A SUMMATION FORMULA FOR POWER SERIES USING EULERIAN FRACTIONS*

Xinghua Wang
Math. Department, Zhejiang University, Hangzhou 310028, China

Leetsch C. Hsu
Math. Institute, Dalian University of Technology, Dalian 116024, China
(Submitted November 2000)

1. INTRODUCTION

This paper is concerned with the summation problem of power series of the form

$$S_a^b(x,f) := \sum_{a \le k < b} f(k)x^k, \tag{1.1}$$

where $f(t)$ is a differentiable function defined on the real number interval $[a,b)$, and x may be a real or complex number with $x \ne 0$ and $x \ne 1$. Obviously, the case for $x=1$ of (1.1) could be generally treated by means of the well-known Euler-Maclaurin summation formula. The object of this paper is to find a general summation formula for (1.1) that could be applied readily to some interesting special cases, e.g., those with $f(t)=t^\lambda$ $(\lambda \in R)$, $f(t)=\log t$ $(t \ge 1)$, and $f(t)=q^{t^2}$ $(0<q<1)$, respectively. Related results will be presented in Sections 3-5.

Recall that for the particular case $f(t)=t^p$ and $[a,b)=[0,\infty)$ with p being a positive integer, we have the classical result (cf. [1], [2], [4])

$$S_0^\infty(x) = \sum_{k=0}^\infty k^p x^k = \frac{A_p(x)}{(1-x)^{p+1}}, \quad (|x|<1), \tag{1.2}$$

where $A_p(x)$ is the Eulerian polynomial of degree p, and may be written explicitly in the form (cf. Comtet [2], §6.5)

$$A_p(x) = \sum_{k=1}^p A(p,k)x^k, \quad A_0(x)=1,$$

with

$$A(p,k) = \sum_{j=0}^k (-1)^j \binom{p+1}{j}(k-j)^p \quad (1 \le k \le p),$$

$A(p,k)$ being known as Eulerian numbers.

As is known, various methods have been proposed for computing the sum of the so-called arithmetic-geometric progression (cf. [2], [3], [4])

$$S_0^n(x) = \sum_{k=0}^n k^p x^k. \tag{1.3}$$

This is a partial cum of (1.2) with ∞ being replaced by n. For $k = 0, 1, 2, \ldots,$ denote

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$$a_k(x) = \frac{A_k(x)}{(1-x)^{k+1}}, \tag{1.4}$$

and call $a_k(x)$ a Eulerian fraction with $x \neq 1$. Then the right-hand side of (1.2) is precisely $a_p(x)$, and one can also have a closed formula for (1.3) using $a_k(x)$'s, namely,

$$S_0^n(x) = a_p(x) - x^{n+1}\sum_{k=0}^{p}\binom{p}{k}a_k(x)(n+1)^{p-k}. \tag{1.5}$$

This is known as a refinement of DeBruyn's formula for (1.3) (cf. Hsu & Tan [5]).

Both (1.2) and (1.5) may suggest that Eulerian fractions $a_k(x)$ $(k = 0, 1, 2, \dots)$ would play an important role in solving the summation problem of (1.1). That this prediction is true will be justified in Section 3.

2. AN EXTENSION OF EULERIAN FRACTIONS

We shall introduce a certain linear combination of Eulerian fractions that will be used for the construction of a summation formula for (1.1). As before, we always assume $x \neq 0, 1$.

First, Eulerian polynomials $A_k(x)$ may be defined via the exponential generating function (cf. [2], (6.5.10))

$$\sum_{k=0}^{\infty} A_k(x)\frac{t^k}{k!} = \frac{1-x}{1-xe^{t(1-x)}}. \tag{2.1}$$

Substituting $t/(1-x)$ for t, we obtain the generating function for Eulerian fractions:

$$\sum_{k=0}^{\infty} a_k(x)\frac{t^k}{k!} = \frac{1}{1-xe^t}. \tag{2.2}$$

Also, we may write (cf. Hsu & Shiue [4])

$$a_k(x) = \sum_{j=0}^{k} j!\,S(k,j)\frac{x^j}{(1-x)^{j+1}}, \tag{2.3}$$

where $S(k, j)$ are Stirling numbers of the second kind.

Multiplying both sides of (2.2) by $(1-xe^t)$, one can verify that (2.2) implies the recurrence relations $a_0(x) = 1/(1-x)$ and

$$a_k(x) = \frac{x}{1-x}\sum_{j=0}^{k-1}\binom{k}{j}a_j(x) \quad (k \geq 1). \tag{2.4}$$

Now, let us define a polynomial in z of degree k via a certain linear combination of $a_j(x)$'s, i.e.,

$$a_k(z, x) := \sum_{j=0}^{k}\binom{k}{j}a_j(x)z^{k-j}, \tag{2.5}$$

where $a_0(z, x) = a_0(x) = 1/(1-x)$. Using (2.2), we may easily obtain a generating function for $a_k(z, x)$:

$$\sum_{k=0}^{\infty} a_k(z, x)\frac{t^k}{k!} = \frac{e^{zt}}{1-xe^t}. \tag{2.6}$$

Moreover, some simple properties of $a_k(z, x)$ may be derived easily from (2.5), (2.4), and (2.6), namely,

(a) $a_k(0,x) = a_k(x) \quad (k \geq 0)$. (2.7)

(b) $a_k(1,x) = \dfrac{1}{x} a_k(x) \quad (k \geq 1)$. (2.8)

(c) $\dfrac{\partial a_k(z,x)}{\partial z} = k a_{k-1}(z,x) \quad (k \geq 1)$. (2.9)

(d) $a_k\left(z, \dfrac{1}{x}\right) = (-1)^{k+1} a_k(1-z,x)x$. (2.10)

Obviously, (a) and (b) imply that $a_k(z,x)$ may be regarded as an extension of $a_k(x)$. Also (d) is inferred easily from (2.6), and the relation

$$\frac{e^{zt}}{1-(1/x)e^t} = \frac{(-x)e^{-(1-z)t}}{1-xe^{-t}}.$$

(e) For $0 < x < 1$, the function $a_k(z,x)$ is positive and monotonically increasing with $z \geq 0$. For $x > 1$, so is the function $(-1)^{k+1} a_k(1-z,x)$ with $z \geq 0$.

In fact, the first statement of (e) follows from (2.3), (2.5), and (2.9). The second statement is inferred from (2.10) and the first statement since

$$(-1)^{k+1} a_k(1-z,x) = a_k\left(z, \frac{1}{x}\right)\frac{1}{x} > 0 \quad (x > 1).$$

Finally, to use the latter in the next section, we need to make the function $a_k(z,t)$ $(0 \leq z < 1)$ periodic of period unity for $z \in R$ (the set of real numbers). In other words, we have to define

$$a_k^*(z,x) := \begin{cases} a_k(z,x) & \text{when } 0 \leq z < 1, \\ a_k^*(z-1,x) & \text{for all } z \in R. \end{cases} \tag{2.11}$$

Also, we shall need

Lemma 2.1 (cf. Wang [8]): For $k \geq 1$, $a_k^*(z,x)x^{-[z]}$ is an absolutely continuous function of z in R, where $[z]$ denotes the integer part of z so that $z-1 < [z] \leq z$.

Proof: It suffices to verify the continuity property at integer points $z = j$. Clearly, using (2.11) and (2.8), we have

$$a_k^*(j+,x)x^{-[j+]} = a_k^*(j,x)x^{-[j]} = a_k(0,x)x^{-j} = a_k(x)x^{-j},$$
$$a_k^*(j-,x)x^{-[j-]} = a_k(1,x)x^{-j+1} = a_k(x)x^{-j}.$$

Since $a^*(z,x)x^{-[z]}$ is a piece-wise polynomial in z, it is clear that $a_k^*(z,x)e^{-[z]}$ is an absolutely continuous function of z.

3. A SUMMATION FORMULA FOR (1.1)

A basic result is contained in the following theorem.

Theorem 3.1: Let $f(z)$ be a real function continuous together with its m^{th} derivative on $[a,b]$ $(m \geq 1)$. Then, for $x \neq 0, 1$ we have

$$\sum_{a\le k<b} f(k)x^k = \sum_{k=0}^{m-1} \frac{-1}{k!}\left[a_k^*(-z, x)x^{-[-z]}f^{(k)}(z)\right]_{z=a}^{z=b} + \frac{1}{(m-1)!}\int_a^b a_{m-1}^*(-z, x)x^{-[-z]}f^{(m)}(z)\,dz, \tag{3.1}$$

where the notation $[F(z)]_{z=a}^{z=b} := F(b)-F(a)$ is adopted.

Proof: We shall prove (3.1) by using integration by parts for a certain Riemann-Stieltjes integral. The basic idea is very similar to that of proving the general Euler-Maclaurin sum formula with an integral remainder (cf. Wang [7]).

In what follows, all the integrals are taken with respect to the independent variable z. Denote the remainder term of (3.1) by

$$R_m = \frac{1}{(m-1)!}\int_a^b a_{m-1}^*(-z, x)x^{-[-z]}f^{(m)}(z)\,dz. \tag{3.2}$$

By Lemma 2.1 and (2.9), we see that (3.2) may be rewritten as Riemann-Stieltjes integrals in the following forms:

$$R_m = \frac{-1}{m!}\int_a^b f^{(m)}(z)\,d\left(a_m^*(-z, x)x^{-[-z]}\right) \quad (m\ge 1); \tag{3.3}$$

$$R_m = \frac{1}{(m-1)!}\int_a^b a_{m-1}^*(-z, x)x^{-[-z]}\,df^{(m-1)}(z) \quad (m\ge 1). \tag{3.4}$$

The form of (3.3) suggests that one may even supply the definition of R_0 via (3.3) by setting $m=0$ in the right-hand side of (3.3). Thus, one may find that the case $m=0$ of (3.3) just gives the power series $S_a^b(x, f)$ as defined by (1.1):

$$R_0 = \frac{1}{x-1}\int_a^b f(z)\,d\left(x^{-[-z]}\right) = \frac{1}{x-1}\sum_{a\le k<b} f(k)(x^{k+1}-x^k) = \sum_{a\le k<b} f(k)x^k.$$

Now, starting with (3.4) and using integration by parts, we obtain

$$R_m = \frac{1}{(m-1)!}\left[a_{m-1}^*(-z, x)x^{-[-z]}f^{(m-1)}(z)\right]_{z=a}^{z=b} - \frac{1}{(m-1)!}\int_a^b f^{(m-1)}(z)\,d\left(a_{m-1}^*(-z, x)x^{-[-z]}\right),$$

where the last term may be denoted by R_{m-1} in accordance with (3.3). Consequently, by recursion we find

$$R_m = \sum_{k=0}^{m-1}\frac{1}{k!}\left[a_k^*(-z, x)x^{-[-z]}f^{(k)}(z)\right]_{z=a}^{z=b} + R_0.$$

This is precisely equivalent to (3.1), and the theorem is proved. □

Remark 3.2: As regards formula (3.1) and its applications, some earlier and rudimentary results containing a different form of it in terms of Stirling numbers instead of Eulerian fractions appeared in Wang [8] and in Wang and Shen [9]. Also, it may be worth mentioning that (3.1) can be used to treat trigonometric sums with summands like $f(k)r^k\cos k\theta$ and $f(k)r^k\sin k\theta$ when taking $x = re^{i\theta}$ $(i^2=-1, r>0, 0<\theta<2\pi)$.

4. FORMULAS WITH ESTIMABLE REMAINDERS

Throughout this section, we assume $x>0$, $x\neq 1$, and $[a,b]=[M,N]$, where M and N are integers with $0\le M<N$. Recalling (2.11) and (2.7), we find

$$\left[a_k^*(-z,x)x^{-[-z]}f^{(k)}(z)\right]_{z=M}^{z=N}=a_k(x)\left[x^N f^{(k)}(N)-x^M f^{(k)}(M)\right].$$

Consider the remainder given (3.1):

$$R_m=\frac{1}{(m-1)!}\int_M^N a_{m-1}^*(-z,x)x^{-[-z]}f^{(m)}(z)dz. \tag{4.1}$$

Setting $f^{(m)}(z)\equiv 1$, we can see that the integrand function of the above integral keeps definite (either positive or negative) sign, in accordance with (2.11) and property (e) of Section 2. In fact, for the case $f^{(m)}(z)\equiv 1$,

$$\begin{aligned}R_m&=\frac{1}{(m-1)!}\sum_{n=M}^{N-1}\int_n^{n+1}a_{m-1}^*(-z,x)x^{-[-z]}dz\\&=\frac{1}{(m-1)!}\sum_{n=M}^{N-1}\int_n^1 a_{m-1}(1-z,x)x^{n+1}dz\\&=\frac{1}{m!}\sum_{n=M}^{N-1}[a_m(1,x)-a_m(0,x)]x^{n+1}\\&=\frac{1}{m!}a_m(x)\sum_{n=M}^{N-1}(x^n-x^{n+1})=\frac{a_m(x)}{m!}(x^M-x^N).\end{aligned}$$

Clearly, the integrand $a_{m-1}(1-z,x)x^{n+1}$ $(0\le z\le 1)$ shown above has a definite sign whenever $x>1$ or $0<x<1$.

Therefore, applying the mean value theorem to the integral (4.1), we are led to the following theorem.

Theorem 4.1: Let $f(z)$ have the m^{th} continuous derivative $f^{(m)}(z)$ $(m\ge 1)$ on $[M,N]$. Then, for $x>0$ with $x\neq 1$, there exists a number $\xi\in(M,N)$ such that

$$\sum_{k=M}^{N-1}f(k)x^k=\sum_{k=0}^{m-1}\frac{a_k(x)}{k!}[x^M f^{(k)}(M)-x^N f^{(k)}(N)]+\frac{a_m(x)}{m!}(x^M-x^N)f^{(m)}(\xi). \tag{4.2}$$

As a simple example, for the case $M=0$, $N\to\infty$, $0<x<1$, and $f(t)=t^p$ with p being a positive integer, we may choose $m=p+1$ and find that

$$\lim_{N\to\infty}x^N f^{(k)}(N)=0,\ 0\le k\le p,$$

so that (4.2) yields

$$\sum_{k=0}^{\infty}k^p x^k=a_p(x).$$

Also, for the case $N<\infty$, we have

$$\sum_{k=0}^{p}\frac{a_k(x)}{k!}x^N f^{(k)}(N)=x^N\sum_{k=0}^{p}\binom{p}{k}N^{p-k}=x^N a_p(N,x),$$

so that (4.2) implies the result

$$\sum_{k=0}^{N-1} x^p x^k = a_p(x) - a_p(N, x)x^N$$

which is precisely the formula (1.5) with $n = N-1$.

The next theorem will provide a more available form for the remainder of the summation formula.

Theorem 4.2: Let $f(z)$ have the $(m+1)^{\text{th}}$ continuous derivative on $[M. N]$. Suppose that either of the following two conditions is satisfied with respect to the sum $S_M^N(x, f)$:

(I) For $x > 1$, $f^{(m)}(z)$ and $f^{(m+1)}(z)$ are of the same sign in (M, N).

(II) For $0 < x < 1$, $f^{(m)}(z)$ and $f^{(m+1)}(z)$ keep opposite signs in (M, N).

Then there is a number $\theta \in (0, 1)$ such that

$$\sum_{k=M}^{N-1} f(k)x^k = \sum_{k=0}^{m-1} \frac{a_k(x)}{k!}[x^M f^{(k)}(M) - x^N f^{(k)}(N)] + \theta \frac{a_m(x)}{m!}[x^M f^{(m)}(M) - x^N f^{(m)}(N)]. \quad (4.3)$$

Proof: Replacing m by $m+1$ in expression (4.1) and using integration by parts, one may find (cf. the proof of Theorem 3.1)

$$R_m = \frac{1}{m!} a_m(x)[x^M f^{(m)}(M) - x^N f^{(m)}(N)] + R_{m+1}.$$

Using property (e) in Section 2 and formula (4.1) for R_m, and also recalling the derivation of (4.2), one may observe that each of the conditions (I) and (II) implies that R_m and R_{m+1} have opposite signs. Consequently, there is a number θ $(0 < \theta < 1)$ such that

$$R_m = \theta \frac{a_m(x)}{m!}[x^M f^{(m)}(M) - x^N f^{(m)}(N)]. \quad \square$$

5. EXAMPLES AND REMARKS

Here we provide three illustrative examples that indicate the application of the results proved in Section 4.

Example 5.1: Let $f(t) = t^\lambda$ $(\lambda > 0, \lambda \in R)$ and choose $m > \lambda$. It is clear that $f(t)$ satisfies condition (II) of Theorem 4.2 on the interval $(0, \infty)$. Consequently, we can apply the theorem to the sum $S_M^N(x, f)$ with $[M, N] \subset (0, \infty)$ and $0 < x < 1$, getting

$$\sum_{k=M}^{N-1} k^\lambda x^k = \sum_{k=0}^{m-1} \binom{\lambda}{k} a_k(x)(x^M M^{\lambda-k} - x^N N^{\lambda-k}) + \theta \binom{\lambda}{m} a_m(x)(x^M M^{\lambda-m} - x^N N^{\lambda-m}), \quad (5.1)$$

where θ is a certain number with $0 < \theta < 1$.

Let us consider the generalized Riemann ζ-function

$$\zeta(s, x) := \sum_{k=1}^{\infty} k^{-s} x^k \quad (0 < x < 1, s \in R) \quad (5.2)$$

and choose $m > -s$, then the function $\zeta(s, x)$ can be approximated by its partial sum with an estimable remainder, viz.,

$$\zeta(s,x)=\sum_{k=1}^{N-1}k^{-s}x^k+x^N\left\{\sum_{k=0}^{m-1}\binom{-s}{k}a_k(x)N^{-s-k}+\theta\binom{-s}{m}a_m(x)N^{-s-m}\right\}. \tag{5.3}$$

Actually, this follows from (5.1) and the fact that

$$\lim_{n\to\infty}\binom{-s}{k}a_k(x)x^n n^{-s-k}=0 \quad (0\le k\le m).$$

Remark 5.2: For the general case in which x is a complex number $(x\neq 1)$, the remainder of formula (5.1) has to be replaced by its integral form, viz.,

$$R_m=m\binom{\lambda}{m}\int_M^N a_{m-1}^*(-z,x)x^{-[-z]}z^{\lambda-m}\,dz. \tag{5.4}$$

In particular, for the case $x=e^{i\alpha}$ $(i^2=-1, 0<\alpha<2\pi)$, $m=1$, and $M=1$, we have $a_0^*(-z,x)=a_0(x)=1/(1-x)$, and the remainder given by (5.4) has a simple estimate

$$|R_1|=O(N^{\lambda-1}) \quad (N\to\infty).$$

Remark 5.3: It is known that, as a nontrivial example treated by Olver [6], the estimation of the sum

$$S(\alpha,\beta,N)=\sum_{k=1}^{N-1}k^{\alpha}(e^{i\beta})^k \quad (i^2=-1),$$

where α and β are fixed real numbers with $\alpha>\beta$, $\beta\neq 0$, and $e^{i\beta}\neq 1$, has the expression

$$S(\alpha,\beta,N)=\frac{e^{iN\beta}}{e^{i\beta}-1}N^{\alpha}+O(N^{\alpha-1})+O(1). \tag{5.5}$$

Evidently, this is readily implied by (5.1) with $x=e^{i\beta}$, $\lambda=\alpha$, $m=1$, $M=1$, and the remainder being replaced by (5.4). Also, a more precise estimate may be obtained by taking $m=2$.

Example 5.4: Define the function $\Lambda(x)$ by the following:

$$\Lambda(x):=\sum_{k=2}^{\infty}(\log k)x^k \quad (0<x<1). \tag{5.6}$$

Then, for any given $m>1$, $\Lambda(x)$ can be approximated by its partial sum with an estimable remainder, viz.,

$$\Lambda(x)=\sum_{k=2}^{N-1}(\log k)x^k-x^N\left\{\sum_{k=1}^{m-1}\frac{(-1)^k a_k(x)}{kN^k}+\theta\frac{(-1)^m a_m(x)}{mN^m}\right\}, \tag{5.7}$$

where $0<\theta<1$.

Evidently the remainder term of (5.7) is obtained from an application of Theorem 4.2 to the function $f(t)=\log t$ with M and N being replaced by N and ∞, respectively.

Example 5.5: Let us take $f(t)=q^{t^2}$ with $0<q<1$, so that we are now concerned with the computation or estimation of Jacobi-type power series

$$J(x,q)=\sum_{k=0}^{\infty}q^{k^2}x^k \quad (x>0, x\neq 1) \tag{5.8}$$

which occurs as an important part in the well-known Jacobi triple-product formula

$$\prod_{k=1}^{\infty}(1+q^{2k-1}x)(1+q^{2k-1}x^{-1})(1-q^{2k})=\sum_{-\infty}^{\infty}q^{k^2}x^k. \tag{5.9}$$

It is known that the k^{th} derivative of $f(t)=q^{t^2}$ with respect to t may be expressed in the form

$$f^{(k)}(t)=q^{t^2}\sum_{j=0}^{[k/2]}\frac{k!}{j!(k-2j)!}(\log q)^{k-j}(2t)^{k-2j} \tag{5.10}$$

so that $f^{(k)}(t)\to 0$ as $t\to\infty$.

Now, applying Theorem 4.1 to $S_N^{\infty}(x,f)$, we easily obtain

$$J(x,q)=\sum_{k=0}^{N-1}q^{k^2}x^k+x^N\left\{\sum_{k=0}^{m-1}\frac{a_k(x)}{k!}f^{(k)}(N)+\frac{a_m(x)}{m!}f^{(m)}(\xi)\right\}, \tag{5.11}$$

where $\xi\in(N,\infty)$, and $f^{(k)}(N)$ and $f^{(m)}(\xi)$ are given by (5.10) with $t=N$ and $t=\xi$, respectively. Certainly the right-hand side of (5.11) without the last term $x^N a_m(x)f^{(m)}(\xi)/m!$ may be used as an approximation to $J(x,q)$ by taking large N.

Remark 5.6: As is known, Binet's formulas express both Fibonacci numbers F_k and Lucas numbers L_k in powers of the quantities $(1\pm\sqrt{5})/2$ with exponent $k+1$. Therefore, one may see that, under certain conditions for $f(t)$, various finite series of the forms $\Sigma_k f(k)F_k$ and $\Sigma_k f(k)L_k$ can also be computed by means of Theorems 4.1 and 4.2.

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q-FIBONACCI POLYNOMIALS

Johann Cigler
Institut für Mathematematik, Universität Wien, A-1090 Wien, Strudlhofgasse 4, Austria
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0. INTRODUCTION

Let MC be the monoid of all Morse code sequences of dots $a(:=\bullet)$ and dashes $b(:=-)$ with respect to concatenation. MC consists of all words in a and b. Let P be the algebra of all polynomials $\sum_{v\in MC}\lambda_v v$ with real coefficients.

We are interested in:

a) polynomials in P which we call *abstract Fibonacci polynomials*. They are defined by the recursion

$$F_n(a,b)=aF_{n-1}(a,b)+bF_{n-2}(a,b)$$

with initial values $F_0(a,b)=0,\ F_1(a,b)=\varepsilon$.

b) polynomials $F_n(x,s,q)$ in real variables x and s which we call *q-Fibonacci polynomials.* They are defined by the recursion

$$F_n(x,s,q)=xF_{n-1}(x,s,q)+t(q^{n-2}s)F_{n-2}(x,s,q)$$

with initial values $F_0(x,s,q)=0,\ F_1(x,s,q)=1$, where $t(s)\neq 0$ is a function of a real variable s and $q\neq 0$ is a real number.

We show how these classes of polynomials are connected, generalize some well-known theorems about the classical Fibonacci polynomials, and study some examples. Related results have been obtained previously by Al-Salam and Ismail [1], Andrews, Knopfmacher, and Paule [4], Carlitz [6], Ismail, Prodinger, and Stanton [11], and Schur [12]. I want to thank Peter Paule for his suggestion to formulate all results in terms of Morse code sequences. My thanks are also due to the referee for drawing my attention to the paper of Ismail, Prodinger, and Stanton [11] and to the polynomials of Al-Salam and Ismail [1]. Most of the cited papers are inspired by the Rogers-Ramanujan identities (e.g., [1], [4], [11], [12]) or by the connections with the general theory of orthogonal polynomials (e.g., [1], [3], [11], [13]), but the aim of this paper is to emphasize the analogy with elementary results on Fibonacci numbers and Fibonacci polynomials and to give as simple and transparent proofs as possible.

1. ABSTRACT FIBONACCI POLYNOMIALS

Let MC be the set of all Morse code sequences of dots $(\bullet)$ and dashes $(-)$. We interpret MC as a monoid with respect to concatenation whose unit element is the empty sequence ε. If we write a for a dot and b for a dash, then MC consists of all words in a and b. Let P be the corresponding monoid algebra over $\mathbb{R}$, i.e., the algebra of all finite sums (polynomials) $\sum_{v\in MC}\lambda_v v$ with real coefficients.

An important element of P is the binomial

$$(a+b)^n=\sum_{k=0}^{n}C_k^n(a,b). \tag{1.1}$$

Here $C_k^n(a, b)$ is the sum of all words with k dashes and $n-k$ dots. It is characterized by the boundary values $C_k^0(a, b) = \delta_{k,0}$ and $C_0^n(a, b) = a^n$ and each of the two recursions

$$C_k^{n+1}(a, b) = bC_{k-1}^n(a, b) + aC_k^n(a, b) \tag{1.2}$$

or

$$C_k^{n+1}(a, b) = C_{k-1}^n(a, b)b + C_k^n(a, b)a. \tag{1.3}$$

We are mainly interested in a class of polynomials which we call *abstract Fibonacci polynomials*. They are defined by the recursion

$$F_n(a, b) = aF_{n-1}(a, b) + bF_{n-2}(a, b) \tag{1.4}$$

and the initial values $F_0(a, b) = 0$ and $F_1(a, b) = \varepsilon$. This sequence begins with 0, ε, a, a^2+b, $a^3+ab+ba$, ...

If we define the length of an element $v \in MC$ as $2k+l$, where k is the number of dashes (elements b) and l is the number of dots (elements a) occurring in v, then it is easily shown by induction that $F_n(a, b)$ is the sum of all words in MC of length $n-1$.

It is now easy to see that they satisfy also another recursion:

$$F_n(a, b) = F_{n-1}(a, b)a + F_{n-2}(a, b)b. \tag{1.5}$$

To this end, consider all words of length $n-1$ which end with a and those which end with b.

Both recurrences are special cases of the formula

$$F_{m+n}(a, b) = F_{m-1}(a, b)bF_n(a, b) + F_m(a, b)F_{n+1}(a, b), \tag{1.6}$$

which follows from the fact that each word w of length $m+n-1$ can be factored uniquely either as $w = ubv$, where u has length $m-2$ and v has length $n-1$, or as $w = xy$, where x has length $m-1$ and y has length n.

There is a simple formula connecting these Fibonacci polynomials with the $C_k^n(a, b)$:

Theorem 1.1: The abstract Fibonacci polynomials are given by

$$F_n(a, b) = \sum_{k=0}^{n-1} C_k^{n-k-1}(a, b). \tag{1.7}$$

For $F_n(a, b)$ is the sum of all monomials $v \in MC$ with length $n-1$. If such a monomial has exactly k dashes, then it has $n-1-2k$ dots; therefore, $n-k-1$ letters and the sum over all such words is $C_k^{n-k-1}(a, b)$.

Consider now the homomorphism $\phi: P \to \mathbb{R}[x, s]$ defined by $\phi(a) = x$, $\phi(b) = s$, where x and s are commuting variables. Let $F_n(x, s) := \phi(F_n(a, b))$. Then we get the classical Fibonacci polynomials defined by

$$F_n(x, s) = xF_{n-1}(x, s) + sF_{n-2}(x, s)$$

with $F_0(x, s) = 0$, $F_1(x, s) = 1$. Since

$$\phi(C_k^n(a, b)) = \binom{n}{k} s^k x^{n-k},$$

we get from (1.7) the well-known formula

$$F_n(x, s) = \sum_{k=0}^{n-1} \binom{n-k-1}{k} s^k x^{n-2k-1}.$$

2. A CLASS OF q-FIBONACCI POLYNOMIALS

Now we consider Morse code sequences which are defined on some interval $\{m, m+1, \ldots, m+k-1\} \subseteq \mathbb{Z}$. In this case, we say that the sequence starts at place m. We want to associate a weight to such a sequence in the following way: Let $t(s) \neq 0$ be a function of a real variable s and let $q \neq 0$ be a real number. Let v be a Morse code sequence on some interval $\{m, m+1, \ldots, m+k-1\}$. If the place $i \in \{m, m+1, \ldots, m+k-1\}$ is occupied by a dot, we set $w(i) = x$; if it is the endpoint of a dash, we set $w(i) = t(q^i s)$. In the other cases, let $w(i) = 1$. Now the weight of v is defined as the product of the weights of all places of the interval, i.e.,

$$w(v) = \prod_{i=m}^{m+k-1} w(i).$$

If, e.g., the sequence $-\bullet-\bullet\bullet--\bullet$ starts at $m=4$, its weight is $x^4 t(q^5 s)t(q^8 s)t(q^{12}s)t(q^{14}s)$. The weight of all Morse code sequences with length $n-1$ starting at $m=0$ is denoted by $F_n(x, s, q)$ and is our q-analog of the Fibonacci polynomials.

We can immediately deduce a recursion for $F_n(x, s, q)$.

We show that the recurrence

$$F_n(x, s, q) = xF_{n-1}(x, s, q) + t(q^{n-2}s)F_{n-2}(x, s, q) \tag{2.1}$$

holds with initial values $F_0(x, s, q) = 0$, $F_1(x, s, q) = 1$. This recursion means that we can split a Morse code sequence of length $n-1$ into two parts, those with a dot in the last position and those with a dash there. In the first case, the dot has weight x and the sequence in front has the weight $F_{n-1}(x, s, q)$. Since the weights are multiplicative, the first term is explained. A dash at the end gives the weight $t(q^{n-2}s)$. Since the dash occupies two positions, the sequence in front of the dash now has length $n-2$ and we get the second term in the formula.

If we split the Morse code sequences into those with a dot in the first position and those with a dash in the beginning, we get in the same way the recursion

$$F_n(x, s, q) = xF_{n-1}(x, qs, q) + t(qs)F_{n-2}(x, q^2 s, q) \tag{2.2}$$

with the same initial conditions as before.

Let

$$A(x, s) = \begin{pmatrix} 0 & 1 \\ t(s) & x \end{pmatrix} \tag{2.3}$$

and

$$M_n(x, s) = A(x, q^{n-1}s)A(x, q^{n-2}s) \cdots A(x, s). \tag{2.4}$$

Then we get

$$M_n(x, s) = \begin{pmatrix} t(s)F_{n-1}(x, qs, q) & F_n(x, s, q) \\ t(s)F_n(x, qs, q) & F_{n+1}(x, s, q) \end{pmatrix}. \tag{2.5}$$

From (2.4), it follows that the matrices $M_n(x, s)$ satisfy the relation

$$M_{k+n}(x, s) = M_k(x, q^n s)M_n(x, s). \tag{2.6}$$

If we extend this to negative indices—which is uniquely possible—we get

$$M_{-k}(x, s) = (M_k(x, q^{-k}s))^{-1};$$

therefore,

$$M_{-n}(x,s)=\frac{1}{d_n(q^{-n}s)}\begin{pmatrix} F_{n+1}(x,q^{-n}s,q) & -F_n(x,q^{-n}s,q) \\ -t(q^{-n}s)F_n(x,q^{-n+1}s,q) & t(q^{-n}s)F_{n-1}(x,q^{-n+1}s,q)\end{pmatrix},$$

which implies

$$F_{-n}(x,s,q)=(-1)^{n-1}\frac{F_n(x,q^{-n}s,q)}{t\left(\frac{s}{q}\right)t\left(\frac{s}{q^2}\right)\cdots t\left(\frac{s}{q^n}\right)}. \tag{2.7}$$

Taking determinants in (2.5), we obtain the *q-Cassini formula*

$$F_{n-1}(x,qs,q)F_{n+1}(x,s,q)-F_n(x,s,q)F_n(x,qs,q)=(-1)^n t(qs)\cdots t(q^{n-1}s). \tag{2.8}$$

This is a special case of the following theorem.

Theorem 2.1 (q-Euler-Cassini formula): The q-Fibonacci polynomials satisfy the polynomial identity

$$\begin{aligned}F_{n-1}(x,qs,q)F_{n+k}(x,s,q)-F_n(x,s,q)F_{n+k-1}(x,qs,q)\\ =(-1)^n t(qs)\cdots t(q^{n-1}s)F_k(x,q^n s,q).\end{aligned} \tag{2.9}$$

This formula is an immediate consequence of (2.6) if we write it in the form

$$M_{k+n}(x,s)M_n(x,s)^{-1}=M_k(x,q^n s)$$

and compare the upper right entries of the matrices.

A more illuminating proof results from an imitation of the construction given in [14]: Consider all pairs of Morse code sequences of the form (u,v), where u starts at 0 and has length $n+k-1$ for some $k\geq 1$ and v starts at 1 and has length $n-2$. If there is a place i, $0\leq i\leq n-2$, where a dot occurs in one of the sequences, there is also a minimal $i_{\min}$ with this property. Then we exchange the sequences starting at $i_{\min}+1$.

Thus, to each pair (u,v) there is associated a pair $(\hat{u},\hat{v})$, where $\hat{u}$ starts at 0 and has length $n-1$ and $\hat{v}$ starts at 1 and has length $n+k-2$. It is clear that the weights of the pairs are the same, $w(u)w(v)=w(\hat{u})w(\hat{v})$. The only pairs where this bijection fails are, for even n, those where v has only dashes and in u all places up to $n-1$ are occupied by dashes. The weight of these pairs is $t(qs)\cdots t(q^{n-1}s)F_k(x,q^n s,q)$. If n is odd, then this bijection fails at those pairs $(\hat{u},\hat{v})$ where $\hat{u}$ has only dashes and in $\hat{v}$ all places up to $n-1$ are occupied by dashes. Thus, the q-Euler-Cassini formula is proved.

Corollary 2.2: In the special case $t(s)=s$, the Euler-Cassini formula reduces to

$$F_{n-1}(x,qs,q)F_{n+k}(x,s,q)-F_n(x,s,q)F_{n+k-1}(x,qs,q)=(-1)^n q^{\binom{n}{2}}s^{n-1}F_k(x,q^n s,q).$$

This corollary was first proved by Andrews, Knopfmacher, and Paule [4] with other methods. Another proof for the more general polynomials of Al-Salam and Ismail was given in [11], and yet another combinatorial proof was recently obtained by Berkovich and Paule [5].

Remark: If we choose a function $x(s)$ instead of the constant x and define the weight of the place i as $x(q^i s)$ if the place is occupied by a dot, then we get a polynomial $K_n(s)$ as the weight of the set of all Morse code sequences of length n starting at position 0. We call it the *q-continuant* corresponding to the set of all Morse code sequences of length n, since for $t(s) \equiv 1$ and $x(q^k s) = x_{k+1}$ we obtain the continuants considered in [10].

The continuant is intimately connected with continued fractions. If we set $x(q^i s) = x_i$ and $t(q^i s) = y_i$ and write

$$x_0 + \cfrac{y_1}{x_1 + \cfrac{y_2}{x_2 + \cdots}} = x_0 + \frac{y_1}{x_1 +}\frac{y_2}{x_2 +}\frac{y_3}{x_3 +}\cdots,$$

then it is easy to see that

$$x_0 + \frac{y_1}{x_1 +}\frac{y_2}{x_2 +}\cdots\frac{y_n}{x_n} = \frac{K_{n+1}(s)}{K_n(qs)}.$$

As a special case, we obtain

$$\frac{F_{n+2}(1, s, q)}{F_{n+1}(1, qs, q)} = 1 + \frac{t(qs)}{1+}\frac{t(q^2 s)}{1+}\cdots\frac{t(q^n s)}{1}.$$

If we let $n \to \infty$, it is easy to see that, at least in the case where $t(s)$ is a formal power series with $t(0) = 0$, we have $\lim_{n\to\infty} F_n(x, s, q) = F_\infty(x, s, q)$ in the sense that the coefficients of each power q^k remain constant beginning with some index $n(k)$. Therefore, we obtain the infinite continued fraction

$$\frac{F_\infty(1, s, q)}{F_\infty(1, qs, q)} = 1 + \frac{t(qs)}{1+}\frac{t(q^2 s)}{1+}\cdots$$

and the functional equation $F_\infty(x, s, q) = F_\infty(x, qs, q) + t(qs)F_\infty(x, q^2 s, q)$.

3. q-FIBONACCI OPERATORS

Now we want to establish a connection between the abstract Fibonacci polynomials and the q-Fibonacci polynomials. To this end, we consider the ring R of linear operators on the vector space of polynomials $\mathbb{R}[x, s]$. We are interested only in multiplication operators with polynomials and the operator η in R defined by $\eta f(x, s) = f(x, qs)$.

We define a homomorphism $\Phi : P \to R$ by

$$\Phi(a) = x\eta, \quad \Phi(b) = t(qs)\eta^2.$$

Then we have

$$\Phi(F_n(a, b)) = F_n(x, s, q)\eta^{n-1}. \tag{3.1}$$

This is easily verified by induction from

$$\begin{aligned}\Phi(F_n(a, b)) &= \Phi(a)\Phi(F_{n-1}(a, b)) + \Phi(b)\Phi(F_{n-2}(a, b)) \\ &= x\eta F_{n-1}(x, s, q)\eta^{n-2} + t(qs)\eta^2 F_{n-2}(x, s, q)\eta^{n-3} \\ &= xF_{n-1}(x, qs, q)\eta^{n-1} + t(qs)F_{n-2}(x, q^2 s, q)\eta^{n-1} \\ &= (xF_{n-1}(x, qs, q) + t(qs)F_{n-2}(x, q^2 s, q))\eta^{n-1} = F_n(x, s, q)\eta^{n-1}.\end{aligned}$$

As a special case, we see that applying the Fibonacci operators to the polynomial 1 we get

$$F_n(x, s, q) = F_n(x, s, q)\eta^{n-1}1 = \Phi(F_n(a, b))1. \tag{3.2}$$

4. THE q-FIBONACCI POLYNOMIALS OF L. CARLITZ

If we choose $t(s)=\frac{s}{q}$, we get the q-Fibonacci polynomials of Carlitz [6]. They are a special case of the Al-Salam and Ismail polynomials $U_n(x;a,b)$ introduced in [1], which are defined by $U_{n+1}(x;a,b)=x(1+aq^n)U_n(x;a,b)-bq^{n-1}U_{n-1}(x;a,b)$ for $n\geq 1$ with initial values $U_0(x;a,b)=1$, $U_1(x;a,b)=x(1+a)$. It is clear that $F_n(x,s,q)=U_{n-1}(x;0,-sq^{-1})$.

In this case, the recurrences are

$$F_n(x,s,q)=xF_{n-1}(x,s,q)+q^{n-3}sF_{n-2}(x,s,q),$$
$$F_0(x,s,q)=0,\ F_1(x,s,q)=1,$$

and

$$F_n(x,s,q)=xF_{n-1}(x,qs,q)+sF_{n-2}(x,q^2s,q),$$
$$F_0(x,s,q)=0,\ F_1(x,s,q)=1.$$

The matrix form reduces to

$$M_n(x,s)=\begin{pmatrix} sF_{n-1}(x,qs,q) & F_n(x,s,q) \\ sF_n(x,qs,q) & F_{n+1}(x,s,q)\end{pmatrix}$$

and the Cassini formula is

$$F_{n+1}(x,s,q)F_{n-1}(x,qs,q)-F_n(x,s,q)F_n(x,qs,q)=(-1)^n q^{\binom{n-1}{2}}s^{n-1}.$$

Remark: A special case of these polynomials has already been obtained by Schur [12]. In [4], these Schur polynomials are called e_n and d_n. In our terminology, these are $e_n=F_n(1,q,q)$ and $d_n=F_{n-1}(1,q^2,q)$.

For the following, we need the Gaussian q-binomial coefficients $\left[{n \atop k}\right]$ (cf., e.g., [3] or [7]). We define them by

$$\begin{bmatrix} n \\ k \end{bmatrix}=\prod_{i=1}^{k}\frac{q^{n-i+1}-1}{q^i-1} \text{ for } n\in\mathbb{Z} \text{ and } k\in\mathbb{N}.$$

They satisfy the following recursions:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}=q^k\begin{bmatrix} n \\ k \end{bmatrix}+\begin{bmatrix} n \\ k-1 \end{bmatrix}$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}=\begin{bmatrix} n \\ k \end{bmatrix}+q^{n-k+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}.$$

The q-binomial theorem (see, e.g., [3] or [7]) states that, for $n\in\mathbb{N}$,

$$(A+B)^n=\sum_{k\geq 0}\begin{bmatrix} n \\ k \end{bmatrix}B^kA^{n-k} \text{ if } AB=qBA.$$

Now we have $x\eta\cdot s\eta^2=qs\eta^2\cdot x\eta$ or, in other words, $\Phi(a)\Phi(b)=q\Phi(b)\Phi(a)$. This may be stated in the following way

$$\Phi(C_k^n(a,b))=\begin{bmatrix} n \\ k \end{bmatrix}(s\eta^2)^k(x\eta)^{n-k}. \tag{4.1}$$

Therefore, from (1.7), we get

$$F_n(x,s,q)=\Phi(F_n(a,b))1=\sum_{k=0}^{n-1}\begin{bmatrix}n-k-1\\k\end{bmatrix}(s\eta^2)^k(x\eta)^{n-2k-1}1$$

or, equivalently,

$$F_n(x,s,q)=\sum_{k=0}^{n-1}\begin{bmatrix}n-k-1\\k\end{bmatrix}q^{2\binom{k}{2}}s^kx^{n-2k-1}. \tag{4.2}$$

5. ANOTHER INTERESTING CASE

Another interesting special case is given by

$$t(s)=\frac{4qs}{(1+s)(1+qs)}.$$

We denote the corresponding Fibonacci polynomials by $f_n(x,s,q)$. Here we get

$$(x\eta+t(qs)\eta^2)^n1=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}\frac{4^kq^{2\binom{k+1}{2}}s^k}{\prod_{j=1}^{k}(1+q^js)(1+q^{n+j}s)}. \tag{5.1}$$

If we set

$$d_{n,k}(s)=\frac{4^kq^{2\binom{k+1}{2}}s^k}{\prod_{j=1}^{k}(1+q^js)(1+q^{n+j}s)},$$

we can write this in the form

$$(x\eta+t(qs)\eta^2)^n1=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}d_{n,k}(s).$$

It is easy to see that

$$d_{n,k}(qs)=q^k\frac{1+qs}{1+q^{k+1}s}d_{n+1,k}(s)$$

and

$$t(qs)d_{n,k-1}(q^2s)=\frac{(1+q^{n+2}s)}{1+q^{k+1}s}d_{n+1,k}(s).$$

We have to show that

$$(x\eta+t(qs)\eta^2)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}d_{n,k}(s)=\sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}x^{n+1-k}d_{n+1,k}(s).$$

The left-hand side is

$$\begin{aligned}&\sum_{k=0}^{n}x\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}d_{n,k}(qs)+\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}t(qs)d_{n,k}(q^2s)\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n+1-k}d_{n,k}(qs)+\sum_{k=0}^{n}\begin{bmatrix}n\\k-1\end{bmatrix}x^{n+1-k}t(qs)d_{n,k-1}(q^2s)\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n+1-k}q^k\frac{1+qs}{1+q^{k+1}s}d_{n+1,k}(s)+\sum_{k=0}^{n}\begin{bmatrix}n\\k-1\end{bmatrix}x^{n+1-k}\frac{1+q^{n+2}s}{1+q^{k+1}s}d_{n+1,k}(s).\end{aligned}$$

The recurrences of the q-binomial coefficients imply

$$\begin{bmatrix} n \\ k \end{bmatrix} q^k \frac{1+qs}{1+q^{k+1}s} + \begin{bmatrix} n \\ k-1 \end{bmatrix} \frac{1+q^{n+2}s}{1+q^{k+1}s} = \begin{bmatrix} n+1 \\ k \end{bmatrix},$$

from which the right-hand side follows.

From (5.1), it follows that

$$\Phi(C_k^n(a,b)) = \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} \frac{4^k q^{2\binom{k+1}{2}} s^k}{\prod_{j=1}^{k} (1+q^j s)(1+q^{n+j}s)}.$$

In this case, (1.7) implies the following theorem.

Theorem 5.1: The q-Fibonacci polynomials $f_n(x,s,q)$ are given by

$$\begin{aligned} f_n(x,s,q) &:= \Phi(F_n(a,b))1 \\ &= \sum_{k=0}^{n-1} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} x^{n-2k-1} \frac{4^k q^{2\binom{k+1}{2}} s^k}{\prod_{j=1}^{k} (1+q^j s)(1+q^{n-k+j-1}s)}. \end{aligned} \tag{5.2}$$

6. A CONNECTION WITH THE CATALAN NUMBERS

The classical Fibonacci polynomials are intimately related to the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Fibonacci polynomials $F_n(x,1)$, $n > 0$, are a basis of the vector space of polynomials. If we define the linear functional L by $L(F_{n+1}) = \delta_{n,0}$, then we get $L(x^{2n+1}) = 0$ and $L(x^{2n}) = (-1)^n C_n$.

We will now sketch how this fact can be generalized. The polynomials $F_n(x,s,q)$, $n > 0$, are a basis of the vector space $\mathbb{P}$ of all polynomials in x whose coefficients are rational functions in s and q. We can therefore define a linear functional L on $\mathbb{P}$ by

$$L(F_n) = \delta_{n,1}. \tag{6.1}$$

Let

$$\hat{F}_n(x,s,q) = \frac{F_n(x,s,q)}{t(s)t(qs)\cdots t(q^{n-1}s)}.$$

Then we have $x\hat{F}_n = t(q^n s)\hat{F}_{n+1} - \hat{F}_{n-1}$.

Now, define the numbers

$$a_{n,k} = (-1)^{\lceil \frac{n+k}{2} \rceil} L(x^n \hat{F}_{k+1}),$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x. They satisfy

$$\begin{aligned} a_{0,k} &= \delta_{0,k} \\ a_{n,k} &= a_{n-1,k-1} + t(q^{k+1}s)a_{n-1,k+1}, \end{aligned} \tag{6.2}$$

where $a_{n,k} = 0$ if $k < 0$.

These numbers have an obvious combinatorial interpretation (consider, e.g., [8], [9]). Consider all nonnegative lattice paths in $\mathbb{R}^2$ that start in $(0, 0)$ with upward steps $(1, 1)$ and downward steps $(1, -1)$. We associate to each upward step ending on the height k the weight 1 and to each downward step ending on the height k the weight $t(q^{k+1}s)$. The weight of the path is the product of the weights of all steps of the path.

Then $a_{n,k}$ is the weight of all lattice paths from $(0, 0)$ to (n, k). It is clear that $a_{2n+1,0} = 0$. If we set $a_{2n,0} = C_n(s, q)$, then $C_n(s, q)$ is a q-analog of the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

because it is well known that the number of such paths equals C_n (cf., e.g., [10]). It is easy to give a recurrence for these q-Catalan numbers. To this end, decompose each lattice path from $(0, 0)$ to $(2n, 0)$ into the first path which returns to the x-axis and the rest path. The first path goes from $(0, 0)$ to $(2k+2, 0)$, $0 \le k \le n-1$, and consists of a rising segment followed by a path from $(0, 0)$ to $(2k, 0)$ (but one level higher) and a falling segment. Thus,

$$C_n(s, q) = \sum_{k=0}^{n-1} C_k(qs, q)t(qs)C_{n-k-1}(s, q).$$

This is a q-analog of the recursion

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}, \quad C_0 = 1,$$

for the classical Catalan numbers.

For

$$t(s) = \frac{4qs}{(1+s)(1+qs)},$$

the corresponding q-Catalan numbers $C_n(1, q)$ have been found by Andrews [2] and are given by the explicit formula

$$C_n(1, q) = \frac{1}{[n+1]}\begin{bmatrix} 2n \\ n \end{bmatrix}\frac{(2q)^{2n}}{(1+q)(1+q^{n+1})\prod_{j=2}^{n}(1+q^j)^2}.$$

The q-Catalan numbers appear also as coefficients of the following power series associated with the Fibonacci polynomials. Consider the q-Fibonacci polynomials corresponding to $-zt(s)$ in place of $t(s)$. Then we have $F_n(1, s, q, z) = F_{n-1}(1, qs, q, z) - zt(qs)F_{n-2}(1, q^2s, q, z)$ If we define

$$g_n(s, z) = \frac{F_{n-1}(1, qs, q, z)}{F_n(1, s, q, z)},$$

then we have $g_n(s, z) = 1 + zt(qs)g_{n-1}(qs, z)g_n(s, z)$.

For $n \to \infty$, these formal power series in z converge coordinatewise toward a formal power series $g(s, z)$ which satisfies $g(s, z) = 1 + zt(qs)g(s, z)g(qs, z)$. Comparing coefficients we see that $g(s, z) = \Sigma C_n(s, q)z^n$, where the $C_n(s, q)$ are the q-Catalan numbers defined by

$$C_n(s, q) = \sum_{k=0}^{n-1} C_k(qs, q)t(qs)C_{n-k-1}(s, q).$$

This result is also an easy consequence of Theorem 5.8.2 in [3].

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ON LACUNARY RECURRENCES

Paul Thomas Young
Dept. of Mathematics, University of Charleston, Charleston, SC 29424
e-mail: paul@math.cofc.edu
(Submitted November 2000)

1. INTRODUCTION

Let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence which satisfies a linear recurrence of order $k+1$. We are herein concerned with the lacunary subsequences $\{a_{mn+b}\}_{n=-\infty}^{\infty}$, where m and b are fixed integers, so called because they consist of the terms from $\{a_n\}$ with *lacunae*, or gaps, of length m between them. In [5], [2], and [3] it has been shown that, for any m and b, the subsequences $\{a_{mn+b}\}$ also satisfy a linear recurrence of order $k+1$. In this note we shall express the coefficients of this recurrence in terms of generalized Dickson polynomials, by means of their functional equations, and present some applications of this description. As corollaries to our main theorem we give generalizations, to prime power moduli, of the known result ([5], Theorem 4) that whenever p is prime, the subsequences $\{a_{p^r n+b}\}_{n=0}^{\infty}$ satisfy the same linear recurrence modulo p as is satisfied by $\{a_n\}$. We conclude with an analog of Howard's tribonacci identity ([3], Theorem 3.1) for tetranacci sequences.

2. THE MAIN RESULT

Let our sequence $\{a_n\}$ satisfy a linear recurrence of order $k+1$, say

$$a_{n+k} = x_1 a_{n+k-1} - x_2 a_{n+k-2} + \cdots - (-1)^k x_k a_n + (-1)^k a\, a_{n-1}, \tag{2.1}$$

where a is a unit in some integral domain R and $x_1, x_2, \ldots, x_k$ are indeterminates over R. (By use of evaluation homomorphisms $R[x_1, \ldots, x_k] \to R$, one may also regard $x_1, x_2, \ldots, x_k$ as elements of R). If we are given some initial conditions, say $a_0, a_1, \ldots, a_k \in R[x_1, \ldots, x_k]$, then the recurrence (2.1) may be used to define a_n for all integers n, and for any integer b we have a formal power series identity

$$\sum_{n=0}^{\infty} a_{n+b} T^n = \frac{Q(T)}{P(T)} \tag{2.2}$$

in the formal power series ring $R[x_1, x_2, \ldots, x_k][[T]]$, where

$$P(T) = 1 - x_1 T + x_2 T^2 + \cdots + (-1)^k x_k T^k - (-1)^k a T^{k+1} \tag{2.3}$$

is the characteristic polynomial of the recurrence (2.1) and $Q(T)$ is some polynomial of degree at most k.

Now let K be the quotient field of the polynomial ring $R[x_1, x_2, \ldots, x_k]$. Then over some finite extension field L of K the polynomial $P(T)$ splits into the product

$$P(T) = \prod_{j=0}^{k} (1 - \alpha_j T). \tag{2.4}$$

It follows that $x_j = \sigma_j(\alpha_0, \ldots, \alpha_k)$ for $1 \le j \le k$ and $a = \sigma_{k+1}(\alpha_0, \ldots, \alpha_k)$, where σ_j denotes the j^{th} elementary symmetric function in $k+1$ indeterminates.

For $1 \le i \le k$, let $P^{(i)}(T)$ be the polynomial in $R[x_1, \ldots, x_k][T]$ of degree $l = \binom{k+1}{i}$ with constant term 1 whose reciprocal roots are all products of the form $\alpha_{j_1} \ldots \alpha_{j_i}$, where $0 \le j_1 < \cdots < j_i \le k$. The coefficients of $P^{(i)}$ are symmetric functions of $\alpha_0, \ldots, \alpha_k$, and therefore there are polynomials $y_{j,i}$ in $R[x_1, \ldots, x_k]$ such that

$$P^{(i)}(T) = 1 - y_{1,i}T + y_{2,i}T^2 + \cdots + (-1)^l y_{l,i}T^l, \tag{2.5}$$

with $y_{1,i} = x_i$ and $y_{l,i} = a^i$. The generalized Dickson polynomials $D_m^{(i)}$ (over R) are then defined for $m > 0$ by the expansion

$$\frac{dP^{(i)}}{P^{(i)}} = -\sum_{m=1}^{\infty} D_m^{(i)}(x_1, \ldots, x_k, a)T^m \frac{dT}{T} \tag{2.6}$$

in $R[x_1, \ldots, x_k][[T]]$ (cf. [6], eq. (1.6)). The usual Dickson polynomials $D_m(x, a)$ are obtained in this way from $P(T) = 1 - xT + aT^2$ with $i = k = 1$, and if R is a finite field then this definition of generalized Dickson polynomials agrees with that given in [4]. From the generating form (2.6), we may derive for $m > 0$ the functional equations (cf. [6], eq. 2.5))

$$D_m^{(i)}(x_1, \ldots, x_k, a) = \sigma_i(\alpha_0^m, \ldots, \alpha_k^m) \quad (1 \le i \le k) \tag{2.7}$$

and the identity $a^m = \alpha_0^m \alpha_1^m \ldots \alpha_k^m$. These relations may be used to define the polynomials $D_m^{(i)}(x_1, \ldots, x_k, a) \in R[x_1, \ldots, x_k]$ for all integers m; specifically, we have

$$D_0^{(i)}(x_1, \ldots, x_k, a) = \binom{k+1}{i} \tag{2.8}$$

for $m = 0$, and for any integer m we have

$$D_m^{(i)}(x_1, \ldots, x_k, a) = a^m D_{-m}^{(j)}(x_1, \ldots, x_k, a), \tag{2.9}$$

where $i + j = k + 1$. With this definition, the polynomial $D_m^{(i)}$ is a polynomial of total degree $|m|$ in $R[x_1, \ldots, x_k]$ for every integer m. Now we are ready to state the main theorem.

Theorem 1: Let $\{a_n\}$ satisfy the linear recurrence (2.1) in $R[x_1, \ldots, x_k]$. Then, for any integers m and b, the lacunary subsequence $\{a_{mn+b}\}$ in $R[x_1, \ldots, x_k]$ satisfies the recurrence

$$a_{km+b} = \left(\sum_{i=1}^{k} (-1)^{i-1} D_m^{(i)}(x_1, \ldots, x_k, a) a_{(k-i)m+b}\right) + (-1)^k a^m a_{b-m}.$$

Proof: Let m and b be given. If $m = 0$, the statement of the theorem reduces to the very well-known identity

$$\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} = 0 \tag{2.10}$$

by (2.8). Assuming the theorem is true for m, it follows also for $-m$ by (2.9); therefore, it suffices to assume m is positive. Consider the generating function (2.2) for the sequence $\{a_{n+b}\}$. Define the linear operator φ on $R[x_1, \ldots, x_k][[T]]$ by

$$\varphi f(T) = \frac{1}{m}(f(T) + f(\theta T) + f(\theta^2 T) + \cdots + f(\theta^{m-1}T)) \tag{2.11}$$

where θ is a primitive m^{th} root of unity in some finite extension of K. Since

$$\varphi(T^j) = \begin{cases} T^j, & \text{if } m \text{ divides } j, \\ 0, & \text{otherwise,} \end{cases} \tag{2.12}$$

we have

$$\varphi\left(\sum_{n=0}^{\infty} a_{n+b} T^n\right) = \sum_{n=0}^{\infty} a_{mn+b} T^{mn}, \tag{2.13}$$

which is the generating function for our lacunary subsequence.

By virtue of the factorization (2.4), we have a partial fraction decomposition

$$\frac{Q(T)}{P(T)} = \sum_{i=0}^{k} \frac{c_i}{(1-\alpha_i T)^{e_i}}, \tag{2.14}$$

valid as a power series identity in the subring $R[x_1, \ldots, x_k][\![T]\!]$ of $L[\![T]\!]$, where the exponents are defined by setting e_i equal to 1 plus the number of α_j with $\alpha_j = \alpha_i$ and $j < i$ (so, e.g., all e_i are 1 if and only if all α_i are distinct). Then we compute in $L(\theta)[\![T]\!]$

$$\begin{aligned} \varphi\left(\frac{Q(T)}{P(T)}\right) &= \frac{1}{m}\sum_{j=0}^{m-1}\sum_{i=0}^{k} \frac{c_i}{(1-\alpha_i\theta^j T)^{e_i}} = \frac{1}{m}\sum_{i=0}^{k}\sum_{j=0}^{m-1} \frac{c_i}{(1-\alpha_i\theta^j T)^{e_i}} \\ &= \frac{1}{m}\sum_{i=0}^{k} \frac{Q_i(T^m)}{(1-\alpha_i^m T^m)^{e_i}} = \frac{\widetilde{Q}(T^m)}{\widetilde{P}(T^m)}, \end{aligned} \tag{2.15}$$

with each Q_i a polynomial of degree less than e_i, and $\widetilde{Q}$ therefore a polynomial of degree at most k. It follows by comparison with (2.13) that $\widetilde{P}$ is the characteristic polynomial for the recurrent sequence $\{a_{mn+b}\}$, where $\widetilde{P}(T^m) = \prod_{i=0}^{k}(1-\alpha_i^m T^m)$. If we write

$$\widetilde{P}(T) = 1 - y_1 T + y_2 T^2 + \cdots + (-1)^k y_k T^k - (-1)^k y_{k+1} T^{k+1}, \tag{2.16}$$

then we have $y_i = \sigma_i(\alpha_0^m, \ldots, \alpha_k^m)$ for $1 \le i \le k$ and $y_{k+1} = \alpha_0^m \alpha_1^m \ldots \alpha_k^m$. Hence, by the functional equations (2.7), we have $y_i = D_m^{(i)}(x_1, \ldots, x_k, a)$ for $1 \le i \le k$ and $y_{k+1} = a^m$, giving the result.

Remarks: In Theorem 1 we have assumed a is a unit in R; however, this assumption is needed only to ensure that a_n and $D_n^{(i)}$ are elements of R when n is negative. The recurrence given in the theorem remains valid in $R[x_1, \ldots, x_k]$ if a is an arbitrary element of R (even if $a = 0$), or in $R[x_1, \ldots, x_k, a]$ if a is regarded as an indeterminate over R, provided $b \ge m \ge 0$. It is equally valid for arbitrary integers m and b if interpreted as a recurrence in the Laurent ring $R[x_1, \ldots, x_k, a, a^{-1}]$.

3. CONGRUENCES FOR LACUNARY RECURRENCES

It is known ([5], Theorem 4) that, if $\{a_n\}_{n=0}^{\infty}$ is a linearly recurrent sequence in $\mathbb{Z}$ and p is prime, then the subsequence $\{a_{p^r n+b}\}_{n=0}^{\infty}$ satisfies the same linear recurrence modulo p as is satisfied by $\{a_n\}$. Theorem 1 and results of [6] give rise to some generalizations of this result.

Corollary 2: Let $\{a_n\}_{n=0}^{\infty}$ satisfy a linear recurrence

$$a_{n+k} = x_1 a_{n+k-1} - x_2 a_{n+k-2} + \cdots - (-1)^k x_k a_n + (-1)^k a\, a_{n-1}$$

in $R[x_1, \ldots, x_k, a]$, and let $\{a_n'\}_{n=0}^{\infty}$ satisfy the linear recurrence

$$a_{n+k}' = x_1^p a_{n+k-1}' - x_2^p a_{n+k-2}' + \cdots - (-1)^k x_p^k a_n' + (-1)^k a^p a_{n-1}'$$

in $R[x_1, \dots, x_k, a]$. Then for any prime p and any positive integers b, d, m, and r, the two lacunary subsequences $\{a_{mp^r n+b}\}_{n=0}^{\infty}$ and $\{a'_{mp^{r-1}n+d}\}_{n=0}^{\infty}$ in $R[x_1, \dots, x_k, a]$ satisfy the same recurrence modulo $p^r R[x_1, \dots, x_k, a]$.

Proof: In Theorem 2 of [6], we showed that the differential form (2.6) is an invariant differential on the multiplicative formal group law over the polynomial ring $R[x_1, \dots, x_k, a]$, from which one may deduce the congruences

$$D_{mp^r}^{(i)}(x_1, \dots, x_k, a) \equiv D_{mp^{r-1}}^{(i)}(x_1^p, \dots, x_k^p, a^p) \pmod{p^r R[x_1, \dots, x_k, a]} \tag{3.1}$$

in $R[x_1, \dots, x_k, a]$. Since $a^{mp^r} = (a^p)^{mp^{r-1}}$, the corollary then follows from Theorem 1 and the observation that the left members of the congruences (3.1) are the coefficients of the recurrence for $\{a_{mp^r n+b}\}$ and the right members of the congruences (3.1) are the coefficients of the recurrence for $\{a'_{mp^{r-1}n+d}\}$.

Taking $m = r = 1$ in the above Corollary 2 yields a polynomial congruence which implies Theorem 3 of [5] and the main result of [1]. We now consider another generalization.

Corollary 3: Let $\{a_n\}_{n=0}^{\infty}$ satisfy the linear recurrence

$$a_{n+k} = x_1 a_{n+k-1} - x_2 a_{n+k-2} + \cdots - (-1)^k x_k a_n + (-1)^k a\, a_{n-1}$$

in $\mathbb{Z}$. Then, for any prime p and any positive integers b, d, m, and r, the two lacunary subsequences $\{a_{mp^r n+b}\}_{n=0}^{\infty}$ and $\{a_{mp^{r-1}n+d}\}_{n=0}^{\infty}$ in $\mathbb{Z}$ satisfy the same recurrence modulo p^r.

Proof: In Theorem 3 of [6], we showed that, for any integers $x_1, \dots, x_k, a$, the differential form (2.6) is an invariant differential on the multiplicative formal group law over $\mathbb{Z}$, from which one may deduce the congruences

$$D_{mp^r}^{(i)}(x_1, \dots, x_k, a) \equiv D_{mp^{r-1}}^{(i)}(x_1, \dots, x_k, a) \pmod{p^r \mathbb{Z}} \tag{3.2}$$

for any integers $x_1, \dots, x_k, a$. Since $a^{mp^r} \equiv a^{mp^{r-1}} \pmod{p^r}$, the corollary then follows from Theorem 1 and the observation that the left members of the congruences (3.2) are the coefficients of the recurrence for $\{a_{mp^r n+b}\}$ and the right members of the congruences (3.2) are the coefficients of the recurrence for $\{a_{mp^{r-1}n+d}\}$.

The $r = 1$ case of this theorem contains the result of [1] and Theorems 3 and 4 of [5]. To illustrate the general case, consider the example of the tribonacci sequence $\{P_n\}$ defined by the recurrence

$$P_{n+2} = P_{n+1} + P_n + P_{n-1} \tag{3.3}$$

with P_0, P_1, P_2 arbitrary integers. As a special case of Theorem 1, we have Howard's general formula (see [3], eq. (3.6)) for the lacunary subsequences $\{P_{mn+b}\}$ which implies, for example,

$$P_{n+4} = 3P_{n+2} + P_n + P_{n-2}, \tag{3.4}$$

$$P_{n+8} = 11P_{n+4} + 5P_n + P_{n-4}, \tag{3.5}$$

$$P_{n+16} = 131P_{n+8} - 3P_n + P_{n-8}, \tag{3.6}$$

$$P_{n+32} = 17155P_{n+16} + 253P_n + P_{n-16}. \tag{3.7}$$

We observe that the recurrence coefficients in (3.3) and (3.4) agree modulo 2, while those in (3.4) and (3.5) agree modulo 2^2, those in (3.5) and (3.6) agree modulo 2^3, and those in (3.6) and (3.7) agree modulo 2^4, as predicted by Corollary 3 for $p=2$. For $p=3$ one has

$$P_{n+6} = 7P_{n+3} - 5P_n + P_{n-3}, \tag{3.8}$$

$$P_{n+18} = 241P_{n+9} - 23P_n + P_{n-9}, \tag{3.9}$$

$$P_{n+54} = 13980895P_{n+27} + 4459P_n + P_{n-27}, \tag{3.10}$$

with the recurrence coefficients in (3.3) and (3.8) agreeing modulo 3, those in (3.8) and (3.9) agreeing modulo 3^2, and those in (3.9) and (3.10) agreeing modulo 3^3. Once more,

$$P_{n+10} = 21P_{n+5} + P_n + P_{n-5}, \tag{3.11}$$

$$P_{n+50} = 4132721P_{n+25} + 2201P_n + P_{n-25}, \tag{3.12}$$

with the recurrence coefficients in (3.3) and (3.11) agreeing modulo 5, and those in (3.11) and (3.12) agreeing modulo 5^2.

The system of congruences (3.2) implies that $\{D^{(i)}_{mp^r}(x_1, \dots, x_k, a)\}_{r=0}^{\infty}$ is a Cauchy sequence in the ring $\mathbb{Z}_p$ of p-adic integers for fixed $x_1, \dots, x_2$, a, m, i, and any prime p, and therefore converges p-adically to some limit $H^{(i)}_m$. Combining Theorem 1 with the complete statement of Theorem 3 in [6] therefore allows a p-adic restatement of Corollary 3.

Corollary 3 (alternate version): Let $\{a_n\}_{n=0}^{\infty}$ satisfy the linear recurrence

$$a_{n+k} = x_1 a_{n+k-1} - x_2 a_{n+k-2} + \cdots - (-1)^k x_k a_n + (-1)^k a\, a_{n-1}$$

in $\mathbb{Z}$ and let p be any prime. Then, for any positive integer m, there exist algebraic integers $H^{(1)}_m$, $\dots$, $H^{(k)}_m$, A_m in $\mathbb{Z}_p$, which depend only on $x_1, \dots, x_k, a$ (mod p), such that the lacunary subsequences $\{b_n\} = \{a_{mp^r n+d}\}$ satisfy

$$b_{n+k} \equiv H^{(1)}_m b_{n+k-1} - H^{(2)}_m b_{n+k-2} + \cdots - (-1)^k H^{(k)}_m b_n + (-1)^k A_m b_{n-1} \pmod{p^{r+1}\mathbb{Z}_p}$$

for all nonnegative integers r and d.

This version of the corollary says that associated to any integral linear recurrent sequence $\{a_n\}_{n=0}^{\infty}$ there is, for each positive integer m and each prime p, a *single* recurrence (with p-adic coefficients) that is satisfied modulo $p^{r+1}\mathbb{Z}_p$ by *every* lacunary subsequence $\{a_{mp^r n+d}\}_{n=0}^{\infty}$. As an illustration of the idea, from (3.7), we note that the recurrence

$$b_{n+2} = 17155b_{n+1} + 253b_n + b_{n-1} \tag{3.13}$$

is satisfied modulo 2^{r+1} by $\{b_n\} = \{P_{2^r n+d}\}$ for $r = 0, 1, 2, 3, 4$; analogous examples of this type for lacunary subsequences of $\{P_n\}$ are given by (3.10) for $p=3$ and $r = 0, 1, 2, 3$, and by (3.12) for $p=5$ and $r = 0, 1, 2$. A natural question to ask is: When will the "universal" p-adic recurrences of the corollary, which hold for all r, actually have integer coefficients?

This question may be answered to some extent in the case of second-order recurrences ($k=1$) using the results of [7], where systems of congruences

$$D^{(1)}_{mp^r}(x, a) \equiv B \pmod{p^{r+1}} \tag{3.14}$$

for integer values of B were classified. In particular, combining Theorem 1 of the present paper with Theorem 1 of [7] yields the following corollary.

Corollary 4: Let $\{a_n\}_{n=0}^{\infty}$ satisfy the second-order linear recurrence

$$a_{n+1} = xa_n - a\,a_{n-1}$$

for integers x and a. Then, for every prime p, there exists an integer m and integers H_m and A_m such that the recurrence

$$b_{n+1} = H_m b_n - A_m b_{n-1}$$

is satisfied modulo p^{r+1} by the lacunary subsequence $\{b_n\} = \{a_{mp^r n+d}\}$ for all nonnegative integers r and d. Furthermore, $H_m \in \{-2, -1, 0, 1, 2\}$ and $A_m \in \{-1, 0, 1\}$.

The means for determining the integers m, H_m, and A_m are outlined in the corollary to Theorem 2 of [7]. A few examples involving the Fibonacci sequence $\{F_n\}$ are:

$$F_{n+m\cdot 2^r} \equiv -F_n - F_{n-m\cdot 2^r} \pmod{2^{r+1}} \text{ if } 3\nmid m; \tag{3.15}$$

$$F_{n+m\cdot 2^r} \equiv 2F_n - F_{n-m\cdot 2^r} \pmod{2^{r+1}} \text{ if } 3\mid m; \tag{3.16}$$

$$F_{n+m\cdot 3^r} \equiv -F_{n-m\cdot 3^r} \pmod{3^{r+1}} \text{ if } m \equiv \pm 2 \pmod 8; \tag{3.17}$$

$$F_{n+m\cdot 5^r} \equiv -2F_n - F_{n-m\cdot 5^r} \pmod{5^{r+1}} \text{ if } m \equiv 2 \pmod 4; \tag{3.18}$$

$$F_{n+m\cdot 7^r} \equiv -F_{n-m\cdot 7^r} \pmod{7^{r+1}} \text{ if } m \equiv \pm 4 \pmod{16}. \tag{3.19}$$

4. TETRANACCI SEQUENCES

In Theorem 2.1 of [3], Howard showed that if $\{a_n\}$ satisfies the recurrence (2.1) over $\mathbb{C}$ then, for any integers m and b, the lacunary subsequence $\{a_{mn+b}\}$ satisfies the recurrence

$$a_{km+b} = \sum_{j=1}^{k+1} (-1)^{j-1} c_{m,\,jm} a_{(k-j)m+b}, \tag{4.1}$$

where the numbers $c_{m,\,jm}$ are independent of the initial conditions $a_0, a_1, \ldots, a_k$, and are defined by a certain generating function. The identity $c_{m,(k+1)m} = a^m$ was shown in Lemma 2.2 of [3]; the result of Theorem 1 above shows that $c_{m,\,jm} = D_m^{(j)}(x_1, \ldots, x_k, a)$ for $1 \le j \le k$. In the tribonacci case ($k = 2$), Howard showed (see Lemma 3.2 of [3]) that $c_{m,m} = D_m$ and $c_{m,2m} = a^m D_{-m}$, where $D_m = D_m^{(1)}(x_1, x_2, a)$. This produces the beautiful identity (cf. [3], eq. (1.5))

$$a_{n+2m} = D_m a_{n+m} - a^m D_{-m} a_n + a^m a_{n-m}, \tag{4.2}$$

which is valid for all integers m and n; observe that $\{a_m\}$ and $\{D_m\}$ satisfy the *same* third-order recurrence. We remark that the two identities of Lemma 3.2 in [3] are generalized to arbitrary k by (2.9) and Theorem 1; specifically, we have

$$c_{m,m} = D_m \quad \text{and} \quad c_{m,km} = a^m D_{-m}, \tag{4.3}$$

where $D_m = D_m^{(1)}(x_1, \ldots, x_k, a)$. In the tetranacci case ($k = 3$), equation (4.3) expresses all but the central coefficient $c_{m,2m}$ in terms of a and D_m. Whereas $\{a_m\}$ and $\{D_m\}$ both satisfy the same fourth-order recurrence, this central coefficient $\{c_{m,2m}\}$ unfortunately satisfies a recurrence of order $\binom{4}{2}=6$. This suggests that perhaps there is no general simple analog of (4.2) for recurrences whose order exceeds three. However, by means of the functional equations (2.7), one may easily verify that $c_{m,2m}=D_m^{(2)}(x_1,\ldots,x_k,a)=(D_m^2-D_{2m})/2$ over any integral domain R of characteristic not equal to 2. Therefore, we may state the following analog of Theorem 3.1 in [3] for tetranacci sequences.

Theorem 5: Let $\{a_n\}$ satisfy the linear recurrence

$$a_{n+3}=x_1a_{n+2}-x_2a_{n+1}+x_3a_n-a\,a_{n-1}$$

in $R[x_1,x_2,x_3]$, where the characteristic of the integral domain R is different from 2 and a is a unit in R. Then, for any integers m and n, we have the identity

$$a_{n+3m}=D_ma_{n+2m}-\frac{1}{2}(D_m^2-D_{2m})a_{n+m}+a^mD_{-m}a_n-a^ma_{n-m}$$

in $R[x_1,x_2,x_3]$, where $D_m=D_m^{(1)}(x_1,x_2,x_3,a)$.

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APPLICATIONS OF MATRIX THEORY TO CONGRUENCE PROPERTIES OF k^{th}-ORDER F-L SEQUENCES

Chizhong Zhou
Department of Computer Science, Yueyang Normal University
Yueyang, Hunan 414000, P.R. China
e-mail: chizhongz@163.com
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1. INTRODUCTION

For convenience, we quote some notations and symbols in [7]: Let the sequence $\{w_n\}$ be defined by the recurrence relation

$$w_{n+k} = a_1 w_{n+k-1} + \cdots + a_{k-1} w_{n+1} + a_k w_n \tag{1.1}$$

and the initial conditions

$$w_0 = c_0,\ w_1 = c_1, \ldots, w_{k-1} = c_{k-1}, \tag{1.2}$$

where $a_1, \ldots, a_k$ and $c_0, \ldots, c_{k-1}$ are complex constants. Then we call $\{w_n\}$ a k^{th}-order **Fibonacci-Lucas sequence** or, simply, an **F-L sequence**, call every w_n an **F-L number**, and call

$$f(x) = x^k - a_1 x^{k-1} - \cdots - a_{k-1} x - a_k \tag{1.3}$$

the characteristic polynomial of $\{w_n\}$. A number α satisfying $f(\alpha) = 0$ is called a characteristic root of $\{w_n\}$. If $a_k \neq 0$, we may consider $\{w_n\}$ as $\{w_n\}_{-\infty}^{+\infty}$. We denote $\mathbb{Z}(a_k) = \mathbb{Z}$ for $a_k \neq 0$ or $\mathbb{Z}^+ \cup \{0\}$ for $a_k = 0$. The set of F-L sequences satisfying (1.1) is denoted by $\Omega(a_1, \ldots, a_k)$ and also by $\Omega(f(x))$. Let $\{u_n^{(i)}\}$ $(0 \le i \le k-1)$ be a sequence in $\Omega(f(x))$ with the initial conditions $u_n^{(i)} = \delta_{ni}$ for $0 \le n \le k-1$, where δ is the Kronecker function. Then we call $\{u_n^{(i)}\}$ the i^{th} **basic sequence** in $\Omega(f(x))$, and also call $\{u_n^{(k-1)}\}$ the **principal sequence** in $\Omega(f(x))$ for its importance. In [3], M. E. Waddill considered the congruence properties modulo m of the k^{th}-order F-L sequence $\{M_n\} \in \Omega(1, \ldots, 1)$ with initial conditions $M_0 = M_1 = \cdots = M_{k-3} = 0$ and $M_{k-2} = M_{k-1} = 1$. In this paper we apply matrix techniques to research the congruence properties modulo m of the general k^{th}-order F-L sequence $\{w_n\} \in \Omega(a_1, \ldots, a_k) = \Omega(f(x))$, where $a_1, \ldots, a_k \in \mathbb{Z}$. In Section 2 we give required preliminaries. By using matrix techniques, in Section 3 we discuss the congruence properties of F-L sequences and get a series of general results. In Section 4 we apply our general results to the special case of second-order F-L sequences. As examples, two more interesting theorems are given.

2. PRELIMINARIES

Let $\{w_n\} \in \Omega(a_1, \ldots, a_k) = \Omega(f(x))$. Denote $\operatorname{col} w_n = (w_{n+k-1}, w_{n+k-2}, \ldots, w_n)^T$. Then, from (1.1), we have

$$\operatorname{col} w_{n+1} = A \operatorname{col} w_n, \tag{2.1}$$

where

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_{k-1} & a_k \\ 1 & & & & \\ & 1 & & & \\ & & \cdots & & \\ & & & 1 & \end{pmatrix} \tag{2.2}$$

is called the **associated matrix** of $\{w_n\}$, also of $f(x)$. And we also denote $\Omega(a_1, \ldots, a_k)$.by $\Omega(A)$. Note that in A the entry in the i^{th} row and j^{th} column is 0 if $i > 1$ and $i \neq j+1$.

Theorem 2.1: Let $\{w_n\} \in \Omega(A)$. Then, for $n \in \mathbb{Z}(a_k)$,

$$\text{col } w_n = A^n \text{ col } w_0. \tag{2.3}$$

For simplicity, in this paper we prove all theorems only for $\mathbb{Z}(a_k) = \mathbb{Z}$.

Proof: If $n \geq 0$, then (2.3) can be proved by induction and by using (2.1). If $n \geq 0$, again by induction and by using (2.1), we can easily verify col $w_{m+n} = A^m$ col w_n for $m \geq 0$. Taking $m = -n$ we get col $w_0 = A^{-n}$ col w_n, whence (2.3) also holds for $n < 0$. □

Theorem 2.2: Let $\{u_n^{(i)}\}$ $(i = 0, 1, \ldots, k-1)$ be the i^{th} basic sequence in $\Omega(a_1, \ldots, a_k) = \Omega(A)$. Then, for $n \in \mathbb{Z}(a_k)$,

$$A^n = (\text{col } u_n^{(k-1)}, \text{col } u_n^{(k-2)}, \ldots, \text{col } u_n^{(0)}). \tag{2.4}$$

Proof: From (2.3), the right-hand side of (2.4) is equal to (let I be the identity matrix)

$$\begin{aligned}&(A^n \text{ col } u_0^{(k-1)}, A^n \text{ col } u_0^{(k-2)}, \ldots, A^n \text{ col } u_0^{(0)})\\ &= A^n(\text{col } u_0^{(k-1)}, \text{col } u_0^{(k-2)}, \ldots, \text{col } u_0^{(0)}) = A^n I = A^n. \ \square\end{aligned}$$

Remark 2.3: Equation (2.4) was shown in [9] and [1]. Its equivalent form was shown as (4) in [4], where U_n is equal to $u_{n+1}^{(k-1)}$ in (2.4). It may be seen that, owing to the introduction of the basic sequences, it is more convenient to use (2.4) than to use (4) in [4].

Substituting (2.4) into (2.3) and comparing the k^{th} row on both sides, we get the following corollary which was stated in [7].

Corollary 2.4: Let $\{u_n^{(i)}\}$ $(i = 0, 1, \ldots, k-1)$ be the i^{th} basic sequence in $\Omega(a_1, \ldots, a_k) = \Omega(A)$ and let $\{w_n\} \in \Omega(A)$. Then $\{w_n\}$ can be represented uniquely as

$$w_n = \sum_{i=0}^{k-1} w_i u_n^{(i)}. \tag{2.5}$$

The following theorem gives a technique for generating F-L sequences by using the matrix other than the associated matrix. The method of proof is quoted from [9].

Theorem 2.5: Let $X_n = (x_{n1}, x_{n2}, \ldots, x_{nk})^T$ be a vector over $\mathbb{C}$ and let B be a square matrix of order k over $\mathbb{C}$. If

$$|xI - B| = f(x) = x^k - a_1 x^{k-1} - \cdots - a_{k-1}x - a_k$$

and

$$X_n = B^n X_0,$$

then, for $n \in \mathbb{Z}(a_k)$,

(1) $\{x_{nj}\}_n \in \Omega(a_1, \ldots, a_k) = \Omega(f(x))$ $(j = 1, \ldots, k)$ or, simply,

$$\{X_n\}_n \in \Omega(a_1, \ldots, a_k) = \Omega(f(x)).$$

(Naturally, we can generalize the concept of an F-L sequence to that of an F-L vector sequence.)

(2)

$$B^n = u_n^{(k-1)}B^{k-1} + u_n^{(k-2)}B^{k-2} + \cdots + u_n^{(1)}B + u_n^{(0)}I. \tag{2.6}$$

Specifically,

$$A^n = u_n^{(k-1)}A^{k-1} + u_n^{(k-2)}A^{k-2} + \cdots + u_n^{(1)}A + u_n^{(0)}I, \tag{2.7}$$

where $\{u_n^{(i)}\}$ is the i^{th} $(i = 0, \ldots, k-1)$ basic sequence in $\Omega(a_1, \ldots, a_k)$ and A is the associated matrix of $f(x)$.

Proof: By the Cayley-Hamilton Theorem, we have $B^k = a_1B^{k-1} + \cdots + a_{k-1}B + a_kI$, whence

$$B^{n+k} = a_1B^{n+k-1} + \cdots + a_{k-1}B^{n+1} + a_kB^n. \tag{2.8}$$

Multiplying by X_0, we obtain $X_{n+k} = a_1X_{n+k-1} + \cdots + a_{k-1}X_{n+1} + a_kX_n$. This means that (1) holds. Denote $B^n = (b_{ij})_{1\le i,\, j\le k}$. Then (2.8) implies $b_{ij}^{(n+k)} = a_1b_{ij}^{(n+k-1)} + \cdots + a_{k-1}b_{ij}^{(n+1)} + a_kb_{ij}^{(n)}$. Therefore, $\{b_{ij}^{(n)}\}_n \in \Omega(f(x))$. By (2.5), it follows that

$$b_{ij}^{(n)} = \sum_{r=0}^{k-1} b_{ij}^{(r)}u_n^{(r)},$$

which is equivalent to (2.6). □

The following theorem is called the **Theorem of Constructing Identities (TCI)** in matrix form. TCI in polynomial form was proved in [6].

Theorem 2.6 (TCI of matrix form): Let $\Omega(a_1, \ldots, a_k) = \Omega(A)$. If

$$\sum_{i=0}^{s} d_iA^{n_i} = \sum_{j=0}^{t} e_jA^{p_j} \tag{2.9}$$

holds, where $n_i, p_j \in \mathbb{Z}(a_k)$ and $d_i, e_j \in \mathbb{C}$, $i = 0, \ldots, s$ and $j = 0, \ldots, t$, then

$$\sum_{i=0}^{s} d_i \operatorname{col} w_{n_i} = \sum_{j=0}^{t} e_j \operatorname{col} w_{p_j} \tag{2.10}$$

holds for any $\{w_n\} \in \Omega(A)$. Specifically,

$$\sum_{i=0}^{s} d_iw_{n_i} = \sum_{j=0}^{t} e_jw_{p_j} \tag{2.11}$$

holds for any $\{w_n\} \in \Omega(A)$. Conversely, if (2.11) holds for any $\{w_n\} \in \Omega(A)$, then (2.9) holds.

Proof: Multiplying (2.9) by $\operatorname{col} w_0$ and using (2.3), we get (2.10), then (2.11). Conversely, if (2.11) holds for any $\{w_n\} \in \Omega(A)$, then it holds for every basic sequence $\{u_n^{(i)}\} \in \Omega(A)$ $(i = 0, \ldots, k-1)$. By using (2.5) and (2.7), we can prove that (2.9) holds. □

The following lemma was proved in [6]. It can also be proved by using the TCI of matrix form.

Lemma 2.7: Let $\{u_n^{(i)}\}$ $(i=0,\dots,k-1)$ be the i^{th} basic sequence in $\Omega=\Omega(a_1,\dots,a_k)=\Omega(A)$ and let $\{w_n\}$ be any sequence in Ω. Then, for $m,n\in\mathbb{Z}(a_k)$,

$$w_{m+n}=\sum_{i=0}^{k-1}u_m^{(i)}w_{n+i}. \tag{2.12}$$

Remark 2.8: For convenience, we rewrite (2.12) as

$$w_{m+n}=A_m\operatorname{col}w_n, \tag{2.13}$$

where $A_m=(u_m^{(k-1)},u_m^{(k-2)},\dots,u_m^{(0)})$.

3. CONGRUENCE PROPERTIES OF F-L SEQUENCES

In the subsequent discussions we deal with the integer sequences in $\Omega(a_1,\dots,a_k)=\Omega(A)=\Omega(f(x))$, where $a_1,a_2,\dots,a_k\in\mathbb{Z}$. The Cayley-Hamilton Theorem gives

$$A^k=a_1A^{k-1}+a_2A^{k-2}+\cdots+a_{k-1}A+a_kI. \tag{3.1}$$

Let $\mathbb{M}$ be the ring of integer matrices of order k. Let $m\in\mathbb{Z}^+$, $m>1$, and let (m) be the principal ideal generated by m over $\mathbb{M}$. For $M,N\in\mathbb{M}$, define $M\equiv N \pmod m$ if $M-N\in(m)$. Let $\{w_n\}\in\Omega(A)$. If there exists $t\in\mathbb{Z}^+$ such that

$$A^t\equiv I \pmod m, \tag{3.2}$$

then we call the least positive integer t satisfying (3.2) the **order of A modulo *m*** and denote $t=\operatorname{ord}_m(A)$. If there exist integers $t>0$ and $n_0\ge 0$ such that

$$w_{n+t}\equiv w_n \pmod m \ \text{ iff } n\ge n_0, \tag{3.3}$$

then we call $\{w_n\}$ **periodic modulo *m*** and call the least positive integer t satisfying (3.3) the **period of $\{w_n\}$ modulo *m***, and denote $t=P(m,w_n)$. If $n_0=0$, we call $\{w_n\}$ **purely periodic**. The following lemma is obvious.

Lemma 3.1:

(1) If an integer $t>0$ satisfies (3.2), then $\operatorname{ord}_m(A)\,|\,t$.

(2) If an integer $t>0$ satisfies (3.3), then $P(m,w_n)\,|\,t$.

Lemma 3.2: Let $\Omega(a_1,\dots,a_k)=\Omega(A)$. Then $\operatorname{ord}_m(A)$ exists iff $(m,a_k)=1$.

Proof: Assume that $\operatorname{ord}_m(A)$ exists. Then (3.2) holds. Taking determinants on both of its sides and noting (2.2), we get $(-1)^{(k-1)t}a_k^t\equiv 1 \pmod m$. This implies $(m,a_k)=1$. Conversely, assume $(m,a_k)=1$. Then there exists an integer b being the inverse of a_k (mod m). Whence, from (3.1), we have $Ab(A^{k-1}-a_1A^{k-2}-\cdots-a_{k-1}I)\equiv I \pmod m$. This means that there exists a matrix B which is the inverse of A (mod m). Since among $I,A,\dots,A^s,\dots$ (mod m) there are at most m^{k^2} different residues, there exist $r>s\ge 0$ such that $A^r\equiv A^s \pmod m$. Multiplying by B^s, we obtain $A^{r-s}\equiv I \pmod m$, so $\operatorname{ord}_m(A)$ exists. □

Theorem 3.3: Let $\Omega=\Omega(a_1,\dots,a_k)=\Omega(A)$ and let $\{u_n\}$ be the principal sequence in Ω. If $(m,a_k)=1$, then $\{u_n\}$ is purely periodic and $P(m,u_n)=\operatorname{ord}_m(A)$.

Proof: From Lemma 3.2, $t'=\operatorname{ord}_m(A)$ exists since $(m,a_k)=1$. Then (3.2) implies that, for any $n\ge 0$, $A^{n+t'}\equiv A^n \pmod m$ holds. From TCI, for any $n\ge 0$, $u_{n+t'}\equiv u_n \pmod m$ holds. Thus,

$\{u_n\}$ is purely periodic and, by Lemma 3.1, $t = P(m, u_n) | t'$. Conversely, since any $\{w_n\} \in \Omega$ can be represented linearly by $\{u_n\}$ over the ring of integers (see [7], Lemma 2.5), the congruence $w_{n+t} \equiv w_n \pmod m$ holds for any $\{w_n\} \in \Omega$. Whence the converse of TCI implies that $A^{n+t} \equiv A^n \pmod m$ holds. Multiplying by A^{-n} (from the proof of Lemma 3.2, A^{-1} exists), we get (3.2). Thus, Lemma 3.1 implies $t' | t$. Summarizing the above, we obtain $t = t'$. □

Corollary 3.4: Let $\Omega = \Omega(a_1, \ldots, a_k) = \Omega(A)$ and let $\{u_n\}$ be the principal sequence in Ω. If $(m, a_k) = 1$, then any $\{w_n\} \in \Omega$ is purely periodic and $P(m, w_n) | P(m, u_n) = \operatorname{ord}_m(A)$.

For what sequences $\{w_n\}$ in $\Omega(A)$ besides the principal sequence will the equality $P(m, w_n) = \operatorname{ord}_m(A)$ hold? To give an answer on the sufficient condition for the question, we introduce the **Hankel matrix** and **Hankel determinant** of $\{w_n\}$, which are defined by, respectively, $H(w_n) = (\operatorname{col} w_{n+k-1}, \operatorname{col} w_{n+k-2}, \ldots, \operatorname{col} w_n)$ and $\det H(w_n)$.

Theorem 3.5: Let $\Omega = \Omega(a_1, \ldots, a_k) = \Omega(A)$. Let $\{u_n\}$ be the principal sequence in Ω and let $\{w_n\}$ be any sequence in Ω. Assume $(m, a_k) = (m, \det H(w_0)) = 1$. Then $P(m, w_n) = P(m, u_n) = \operatorname{ord}_m(A)$.

Proof: From $(m, a_k) = 1$, Theorem 3.3, and Corollary 3.4, we conclude that $\{w_n\}$ is purely periodic and $P(m, w_n) | P(m, u_n) = \operatorname{ord}_m(A)$. Thus, we need only prove that $P(m, u_n) | P(m, w_n)$. Equation (2.13) gives $w_{n+i} = A_n \operatorname{col} w_i$. Whence

$$(w_{n+k-1}, \ldots, w_{n+1}, w_n) = A_n(\operatorname{col} w_{k-1}, \ldots, \operatorname{col} w_1, \operatorname{col} w_0). \tag{3.4}$$

The equality (3.4) can be considered a system of linear equations in unknowns $u_n^{(i)}$ ($i = 0, \ldots, k-1$). The coefficient determinant of the system is $\det(\operatorname{col} w_{k-1}, \ldots, \operatorname{col} w_1, \operatorname{col} w_0) = \det H(w_0)$. Since $(m, \det H(w_0)) = 1$, we can solve $u_n = u_n^{(k-1)} \equiv b_1 w_{n+k-1} + \cdots + b_{k-1} w_1 + b_k w_0 \pmod m$. Hence, $P(m, u_n) | P(m, w_n)$. □

For more detailed consideration on the periodicity, we introduce the following concepts: Let $\{w_n\} \in \Omega(A)$. If there exists $s \in \mathbb{Z}^+$ such that

$$A^s \equiv cI \pmod m, \tag{3.5}$$

where $c \in \mathbb{Z}$ and $(m, c) = 1$, then we call the least positive integer s satisfying (3.5) the **constrained order of A modulo m**, call c a **multiplier of A modulo m**, and denote $s = \operatorname{ord}'_m(A)$. Correspondingly, if there exist integers $s > 0$ and $n_0 \geq 0$ such that

$$w_{n+s} \equiv c w_n \pmod m \text{ iff } n \geq n_0, \tag{3.6}$$

where c is an integer independent of n and $(m, c) = 1$, then we call the least positive integer s satisfying (3.6) the **constrained period of $\{w_n\}$ modulo m**, call c a **multiplier of $\{w_n\}$ modulo m**, and denote $s = P'(m, w_n)$. If $n_0 = 0$, we call $\{w_n\}$ **purely constrained periodic**. We point out that the definition of "constrained period" has generalized and improved the definition in [2]. Similarly to Lemma 3.1, the following lemma is obvious.

Lemma 3.6:

(1) If an integer $s > 0$ satisfies (3.5), then $\operatorname{ord}'_m(A) | s$.

(2) If an integer $s > 0$ satisfies (3.6), then $P'(m, w_n) | s$.

Clearly, if $\text{ord}_m(A)$ exists, then $\text{ord}'_m(A)$ must exist [especially in the case $c \equiv 1 \pmod{m}$]. Hence, from 3.2, we obtain

Lemma 3.7: Let $\Omega(a_1, \ldots, a_k) = \Omega(A)$. Then $\text{ord}'_m(A)$ exists iff $(m, a_k) = 1$.

By induction on j, we can easily prove

Lemma 3.8: Let s and c be the constrained period and a multiplier of $\{w_n\}$ modulo m, respectively; that is to say that (3.6) holds. Then, for $j \geq 0$ and $n \geq n_0$, we have

$$w_{n+js} \equiv c^j w_n \pmod{m}. \tag{3.7}$$

Theorem 3.9: Let $\Omega = \Omega(a_1, \ldots, a_k) = \Omega(A)$, let $\{u_n\}$ be the principal sequence in Ω, and let $\{w_n\}$ be any sequence in Ω. If $(m, a_k) = 1$, then

(1) $\{u_n\}$ and $\{w_n\}$ are purely constrained periodic and $P'(m, w_n) \mid P'(m, u_n) = \text{ord}'_m(A)$.

(2) u_{s+k-1}, where $s = P'(m, u_n) = \text{ord}'_m(A)$, is a multiplier of $\{u_n\}$ (mod m).

Proof:

(1) The proof is similar to the proofs of Theorem 3.3 and Corollary 3.4.

(2) Take $n = k-1$ in the congruence $u_{n+s} \equiv cu_n \pmod{m}$ and note that $u_{k-1} = 1$. □

Theorem 3.10: Let $\{u_n\}$ be the principal sequence in $\Omega(a_1, \ldots, a_k)$ and let $(m, a_k) = 1$. Denote $P'(m, u_n) = s$, $u_{s+k-1} = c$, and $\text{ord}_m(c) = r$. Then

(1) $P(m, u_n) = rs$.

(2) The structure of $\{u_n \pmod{m}\}$ in a period is as follows:

$$\begin{array}{llllll} 0, \ldots, 0, 1, & u_k, & u_{k+1}, & \ldots, u_{s-1}, & \\ 0, \ldots, 0, c, & cu_k, & cu_{k+1}, & \ldots, cu_{s-1}, & \pmod{m} \\ \quad \cdots \quad \cdots & \cdots & & \cdots & \\ 0, \ldots, 0, c^{r-1}, & c^{r-1}u_k, & c^{r-1}u_{k+1}, & \ldots, c^{r-1}u_{s-1}. & \end{array}$$

Proof:

(1) Let $P(m, u_n) = t$. From $u_{n+t} \equiv u_n \pmod{m}$ and Lemma 3.6, we have $s \mid t$. Then $t = r_1 s$. On the other hand, Theorem 3.9 implies that c is a multiplier of $\{u_n\}$ (mod m). Equation (3.7) implies that

$$u_{n+js} \equiv c^j u_n \pmod{m}. \tag{3.8}$$

Taking $j = \text{ord}_m(c) = r$, we have $u_{n+rs} \equiv u_n \pmod{m}$. Whence Lemma 3.1 gives $t \mid rs$, that is, $r_1 s \mid rs$. Now we need only prove that $r_1 = r$. If this were not the case, then $r_1 < r$. Let A be the associated matrix of $\{u_n\}$. Theorem 3.9 implies that $A^s \equiv c \pmod{m}$. Theorem 3.3 implies that $A^t \equiv I \pmod{m}$, that is, $A^{r_1 s} = (A^s)^{r_1} \equiv c^{r_1} I \equiv I \pmod{m}$. This contradicts $\text{ord}_m(c) = r$.

(2) In (3.8), let $j = 0, 1, \ldots, r-1$ and let $n = 0, 1, \ldots, s-1$; then we have the required result. □

Corollary 3.11: Let $\{u_n\}$ be the principal sequence in $\Omega(a_1, \ldots, a_k)$ and let $(m, a_k) = 1$. Then $P'(m, u_n)$ is the least integer s such that $s > k-1$ and

$$u_s \equiv u_{s+1} \equiv \cdots \equiv u_{s+k-2} \equiv 0 \pmod{m}. \tag{3.9}$$

As an example, we let $\{u_n\}$ be the principal sequence in $\Omega(1, 1, 1)$. By calculating, we obtain

$$\{u_n \pmod 7\} = \{0, 0, 1, 1, 2, 4, 0, 6, 3, 2, 4, 2, 1, 0, 3, 4, \underline{0, 0, 4}, \ldots\}.$$

Therefore, $s = P'(7, u_n) = 16$, $c = u_{s+2} \equiv 4 \pmod 7$. Since $4^2 \equiv 2$ and $4^3 \equiv 1 \pmod 7$, we obtain $r = \text{ord}_7(c) = 3$, and so $t = P(7, u_n) = rs = 48$. Furthermore, from Theorem 3.10, we can get

$$\begin{aligned} &u_n \equiv 0 \pmod 7 \text{ iff } n \equiv 0, 1, 6, 13 \pmod{16}, \\ &u_n \equiv 1 \pmod 7 \text{ iff } n \equiv 2, 3, 12, 20, 25, 27, 37, 42, 47 \pmod{48}, \\ &\ldots. \end{aligned}$$

Another application example can be found in [8]. The above numerical results can be used to verify the following theorem.

Theorem 3.12: Let $\Omega = \Omega(a_1, \ldots, a_k) = \Omega(A) = \Omega(f(x))$, let $\{u_n\}$ be the principal sequence in Ω, and let $\{w_n\}$ be any sequence in Ω. Assume that $(m, a_k) = 1$. Denote $P(m, u_n) = s$, $u_{s+k-1} = c$, and $\text{ord}_m(c) = r$.

(1) If $(m, c-1) = 1$, then, for all integers $n \ge 0$,

$$\sum_{j=0}^{r-1} w_{n+js} \equiv 0 \pmod m. \tag{3.10}$$

(2) If $(m, f(1)) = 1$, then, for $a_0 = -1$ and, for all integers $n \ge 0$,

$$\sum_{j=0}^{s-1} w_{n+j} \equiv f(1)^{-1}(c-1)\sum_{j=0}^{k-1}(a_0 + a_1 + \cdots + a_j)w_{n+k-1-j} \pmod m. \tag{3.11}$$

Specifically,

$$\sum_{j=0}^{s-1} u_{is+j} \equiv f(1)^{-1}(1-c)c^i \pmod m \quad (i \ge 0). \tag{3.12}$$

Proof:

(1) From (1) of Theorem 3.9 and (3.7), we have

$$(c-1)\sum_{j=0}^{r-1} w_{n+js} \equiv (c-1)\sum_{j=0}^{r-1} c^j w_n = (c^r - 1)w_n \equiv 0 \pmod m.$$

Then (3.10) follows from the above congruence and $(m, c-1) = 1$.

(2) From $f(A) = 0$, we have

$$\begin{aligned} -f(1)I &= f(A) - f(1)I \\ &= (A-1)((A^{k-1} + \cdots + A + I) - a_1(A^{k-2} + \cdots + A + I) - \cdots - a_{k-1}I). \end{aligned}$$

Whence, from $(m, f(1)) = 1$, we get

$$(A-I)^{-1} \equiv f(1)^{-1}\sum_{j=0}^{k-1}(a_0 + a_1 + \cdots + a_j)A^{k-1-j}.$$

On the other hand, from Theorem 3.9 and (3.5), we have

$$(A-I)(A^{s-1} + A^{s-2} + \cdots + A + I) = A^s - I \equiv (c-1)I \pmod m.$$

Whence

$$A^{s-1}+A^{s-2}+\cdots+A+I\equiv(c-1)(A-I)^{-1}$$
$$\equiv(c-1)f(1)^{-1}\sum_{j=0}^{k-1}(a_0+a_1+\cdots+a_j)A^{k-1-j}\pmod m,$$

multiplying it by A^n, by TCI we get (3.11). □

Note: Since $(m,a_k)=1$, the inverse of A (mod m) exists, which is

$$A^{-1}\equiv a_k^{-1}(A^{k-1}-a_1A^{k-2}-\cdots-a_{k-1}I)\pmod m.$$

Similarly, the sequence $\{w_n \pmod m\}\in\Omega(A)$ can be extended to $n<0$ by using the recurrence. Under this definition, the last theorem and the subsequent theorems, which hold for $n\ge 0$, will hold for $n\in\mathbb{Z}$.

Corollary 3.13: Under the conditions of Theorem 3.12, let $t=rs$. If ***(a)*** $(m,c-1)=1$, or if ***(b)*** $m|(c-1)$ and $(m,f(1))=1$, then

$$\sum_{j=0}^{t-1}w_{n+j}\equiv\sum_{j=0}^{s-1}w_{n+j}\equiv 0\pmod m. \tag{3.13}$$

Proof: We have $\sum_{j=0}^{t-1}w_{n+j}=\sum_{i=0}^{s-1}\sum_{j=0}^{r-1}w_{n+i+js}$. So (3.13) is proved by using (3.10) for (a) or by using (3.11) for (b). □

Remark 3.14:

(1) If we change $P'(m,u_n)=s$ and $u_{s+k-1}=c$ so that $P'(m,w_n)=s$ and c is a multiplier of $\{w_n\}$ modulo m, respectively, then (3.10) and (3.13) still hold because (3.7) still holds. But at this time we cannot conclude that (3.11) holds.

(2) If neither conditions (a) nor (b) are fulfilled, (3.13) may not hold. For example: It is clear that $\{n\}$ is the principal sequence in $\Omega(2,-1)=\Omega(f(x))$. Thus, $f(1)=0$.

$$\{n \pmod{10}\}=\{0,1,2,3,4,5,6,7,8,9,\underline{0,1},\dots\}$$

implies $s=10$ and $c\equiv 1$ (mod 10). Hence, neither condition (a) nor condition (b) is fulfilled. We have $0+1+2+\cdots+9\equiv 5\not\equiv 0$ (mod 10), i.e., (3.13) does not hold.

Theorem 3.15: Let $\{u_n\}$ be the principal sequence in $\Omega(a_1,\dots,a_k)=\Omega(A)$ and $(m,a_k)=1$. Set $P'(m,u_n)=s$. Then, for $j>0$, we have

(1)

$$u_{js-1}\equiv a_k^{j-1}u_{s-1}^{j}\pmod{m^2}. \tag{3.14}$$

(2)

$$u_{js+d}\equiv j(a_ku_{s-1})^{j-1}u_{s+d}\pmod{m^2}\quad(0\le d\le k-2). \tag{3.15}$$

Proof: Let $\{u_n^{(i)}\}$ $(i=0,\dots,k-1)$ be the i^{th} basic sequence in $\Omega(A)$. Clearly, $u_{n+1}^{(0)}=a_ku_n^{(k-1)}=a_ku_n$. Denote $u_{s+k-1}=c$. We shall prove the theorem by induction. For $j=1$, (3.14) and (3.15) are trivial. Assume that both (3.14) and (3.15) hold for j. We want to prove that they also hold for $j+1$.

(1) From (2.12), we have $u_{(j+1)s-1}=\sum_{i=0}^{k-1}u_{js}^{(i)}u_{s-1+i}$. Theorem 3.9 and (3.7) imply that $u_{js}^{(i)}\equiv c^ju_0=0$ and $u_{s-1+i}\equiv cu_{i-1}=0$ (mod m) for $1\le i\le k-1$. Then, by the induction hypothesis,

$$u_{(j+1)s-1} \equiv u_{js}^{(0)} u_{s-1} = a_k u_{js-1} u_{s-1} \equiv a_k (a_k^{j-1} u_{s-1}^j) u_{s-1} = a_k^j u_{s-1}^{j+1} \pmod{m^2}.$$

(2) Again from (2.12), we have $u_{(j+1)s+d} = \sum_{i=0}^{k-1} u_s^{(i)} u_{js+d+i}$. From (3.9) and the recurrence (1.1), we obtain $c = u_{s+k-1} \equiv a_k u_{s-1} \pmod{m}$. Whence, from (3.7), we obtain $u_{js+d+i} \equiv c^j u_{d+i} \equiv (a_k u_{s-1})^j u_{d+i} \pmod{m}$ and $u_s^{(i)} \equiv c u_0^{(i)} = 0 \pmod{m}$ for $1 \le i \le k-1$. It follows that

$$u_{(j+1)s+d} \equiv u_s^{(0)} u_{js+d} + \sum_{i=1}^{k-1} u_s^{(i)} (a_k u_{s-1})^j u_{d+i} \pmod{m^2}.$$

Since $u_d = 0$ for $0 \le d \le k-2$, the last expression can be rewritten as

$$u_{(j+1)s+d} \equiv u_s^{(0)} u_{js+d} + (a_k u_{s-1})^j \sum_{i=0}^{k-1} u_s^{(i)} u_{d+i} \pmod{m^2}.$$

Thus, by (2.12), we get

$$u_{(j+1)s+d} \equiv u_s^{(0)} u_{js+d} + (a_k u_{s-1})^j u_{s+d} \pmod{m^2}.$$

Since $u_s^{(0)} = a_k u_{s-1}$, the conclusion follows by the induction hypothesis. □

We point out that Theorem 3.12 and Corollary 3.13 have generalized Theorem 12 in [3], while Theorem 3.15 has generalized Theorem 7 in [3].

4. THE CASE OF $k = 2$

For $k = 2$, the principal sequence $u_n = u_n^{(1)}$ in $\Omega(a, b)$ satisfies $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = au_{n+1} + bu_n$ for $n \ge 0$. The 0th basic sequence $u_n^{(0)}$ satisfies $u_0^{(0)} = 1$, $u_1^{(0)} = 0$. and the same recurrence. We assume $b \ne 0$, since $b = 0$ is less interesting. Clearly, $u_n^{(0)} = bu_{n-1}$. The associated matrix is

$$A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

Our conclusions for general k can be easily transferred to the case of $k = 2$, for example:

Theorem 2.2 gives that, for $n \in \mathbb{Z}$,

$$A^n = \begin{pmatrix} u_{n+1} & bu_n \\ u_n & bu_{n-1} \end{pmatrix}. \tag{4.1}$$

Theorem 2.5 give that, for $n \in \mathbb{Z}$,

$$A^n = u_n A + bu_{n-1} I. \tag{4.2}$$

Corollary 3.11 given that, if $(m, b) = 1$, then $P'(m, u_n)$ is the least integer s such that $s > 1$ and $u_s \equiv 0 \pmod{m}$.

We do not enumerate all of them. Instead, we focus our mind on obtaining more interesting conclusions. Because of limited space, as examples we give only those for Theorems 3.12 and 3.15.

Theorem 4.1: Let $\{F_n\}$ be the Fibonacci sequence, i.e., the principal sequence in $\Omega = \Omega(1, 1)$, and let $\{w_n\}$ be any sequence in Ω. Let $p > 3$ be a prime. Then, for all integer $n \in \mathbb{Z}$:

(1)

$$w_n + w_{n+p} + w_{n+2p} + w_{n+3p} \equiv 0 \pmod{F_p}. \tag{4.3}$$

(2)

$$\sum_{j=0}^{p-1} w_{n+j} \equiv (F_{p-1}-1)w_{n+1} \pmod{F_p}. \tag{4.4}$$

Proof: In Theorem 3.12, take $m=F_p$. Then, from (4.2), $A^p \equiv F_{p-1}$ (mod m), where, as is well known, $(m, F_{p-1}) = (F_p, F_{p-1}) = 1$. Lemma 3.6 implies $P'(m, F_n) = s|p$. Since $s>1$ and p is prime, we have $s=p$. And the multiplier $c \equiv F_{p-1} \equiv F_{p+1}$ (mod m) (or, it can be obtained by Theorem 3.9 directly). It is well known that

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n. \tag{4.5}$$

Whence $c^2 \equiv F_{p+1}^2 \equiv (-1)^p = -1$ (mod m). Thus, $r = \operatorname{ord}_m(c) = 4$. From Theorem 3.12:

(1) To prove (4.3), it is sufficient to prove $d=(m,c-1)=(F_p, F_{p+1}-1)=1$. Let $p=2q+1$ and let L_n be the n^{th} Lucas number. Then $F_p = F_{q+1}^2 + F_q^2$ and

$$\begin{aligned} F_{p+1}-1 &= F_{q+1}L_{q+1} - 1 = F_{q+1}(F_{q+1}+2F_q) - (-1)^q(F_{q+1}^2 - F_{q+1}F_q - F_q^2) \\ &= \begin{cases} 3F_{q+1}F_q + F_q^2 & \text{for } 2|q, \\ 2F_{q+1}^2 + F_{q+1}F_q - F_q^2 & \text{otherwise.} \end{cases} \end{aligned}$$

For even q,

$$d = (F_{q+1}^2 + F_q^2, 3F_{q+1}F_q + F_q^2) = (F_{q+1}^2 + F_q^2, F_q(3F_{q+1} + F_q)).$$

Since $(F_{q+1}^2 + F_q^2, F_{q+1}) = (F_q^2, F_{q+1}) = 1$ and, by the same reasoning, $(F_{q+1}^2 + F_q^2, F_q) = 1$, we have $d = (F_{q+1}^2 + F_q^2, 3F_{q+1} + F_q)$.

For odd q, we also have

$$\begin{aligned} d &= (F_{q+1}^2 + F_q^2, 2F_{q+1}^2 + F_{q+1}F_q - F_q^2) = (F_{q+1}^2 + F_q^2, F_{q+1}(3F_{q+1} + F_1)) \\ &= (F_{q+1}^2 + F_q^2, 3F_{q+1} + F_q). \end{aligned}$$

Thus,

$$\begin{aligned} d &= (F_{q+1}(F_{q+1} - 3F_q), 3F_{q+1} + F_q) = (F_{q+1} - 3F_q, 3F_{q+1} + F_q) \\ &= (F_{q+1} - 3F_q, 10F_q) = (F_{q+1} - 3F_q, 10) = (-L_{q-1}, 10). \end{aligned}$$

The fact that $\{L_n \pmod 5\} = \{2, 1, 3, 4, 2, 1, \dots\}$ implies that $(L_{q-1}, 5) = 1$. And the fact that $\{L_n \pmod 2\} = \{0, 1, 1, 0, 1, 1, \dots\}$ implies that $2|L_{q-1}$ iff $3|(q-1)$, i.e., $3|(p-3)/2$. Whence, $3|p$. This is also impossible. Hence, $d=1$.

(2) Here $f(x) = x^2 - x - 1$ and $f(1) = -1$. Whence $(m, f(1)) = 1$ holds. Hence, (4.4) holds by (3.11). □

The following theorem implies a possible generalization and an alternative proof of Theorem 3.15.

Theorem 4.2: Let $\{u_n\}$ be the principal sequence in $\Omega = \Omega(a,b) = \Omega(A)$ and $\{w_n\}$ be any sequence in Ω. Assume $(m,b)=1$. Denote $P'(m, u_n) = s$. Then, for $j>0$ and $d \geq 0$, we have

(1)

$$w_{js-1} \equiv b^{j-1}u_{s-1}^j w_1 + (bu_{s-1})^{j-1}(ju_s - au_{s-1})w_0 \pmod{m^2}. \tag{4.6}$$

(2)

$$w_{js+d} \equiv (bu_{s-1})^j w_d + j(bu_{s-1})^{j-1} u_s w_{d+1} \pmod{m^2}. \tag{4.7}$$

Proof: From (4.2), $A^s = u_s A + bu_{s-1} I$. Since $m|u_s$, we have $A^{js} = (bu_{s-1})^j I + j(bu_{s-1})^{j-1} u_s A$ $(\bmod\, m^2)$. Whence

$$A^{js+d} = (bu_{s-1})^j A^d + j(bu_{s-1})^{j-1} u_s A^{d+1} \pmod{m^2}. \tag{4.8}$$

If $d \ge 0$, then (4.7) follows from TCI. For $d = -1$, from $A^2 - aA - bI = 0$, we get $A(A - aI) \equiv b$ $(\bmod\, m^2)$. Whence $(m, b) = 1$ gives $A^{-1} \equiv b^{-1}(A - aI)$ $(\bmod\, m^2)$. And (4.8) becomes $A^{js-1} = b^{j-1} u_{s-1}^j A + (bu_{s-1})^{j-1}(ju_s - au_{s-1}) I$ $(\bmod\, m^2)$. Thus, (4.6) follows from TCI. □

It is easy to see that when $\{w_n\} = \{u_n\}$ and $d = 0$ the conclusions of the last theorem agree with those of Theorem 3.15.

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ON SOME RECIPROCAL SUMS OF BROUSSEAU: AN ALTERNATIVE APPROACH TO THAT OF CARLITZ

R. S. Melham
Department of Math. Sciences, University of Technology, Sydney
PO Box 123, Broadway, NSW 2007, Australia
(*Submitted November 2000*)

1. INTRODUCTION

In [2], it was shown that

$$\sum_{n=1}^{\infty}\frac{1}{F_n}=3+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_nF_{n+1}F_{n+2}},\tag{1.1}$$

and, using the same approach, it can be shown that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_n}=1-\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}F_{n+2}}.\tag{1.2}$$

Let m be a positive integer, and define the sums

$$S(1,\dots,m)=\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}\cdots F_{n+m}},\quad m\ge 1,\tag{1.3}$$

and

$$T(1,\dots,m)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_nF_{n+1}\cdots F_{n+m}},\quad m\ge 1.\tag{1.4}$$

In [1] (see equations 20, 21, and 22), Brousseau proved that

$$T(1,2)=\frac{5}{12}-\frac{3}{2}S(1,2,3,4),\tag{1.5}$$

$$S(1,2,3,4)=\frac{97}{2640}+\frac{40}{11}T(1,2,3,4,5,6),\tag{1.6}$$

and

$$T(1,2,3,4,5,6)=\frac{589}{1900080}-\frac{273}{29}S(1,2,3,4,5,6,7,8).\tag{1.7}$$

As an application, he computed the value of the sum $\sum_{n=1}^{\infty}\frac{1}{F_n}$ to twenty-five decimal places.

Our aim in this paper is to establish explicit formulas that extend (1.5)-(1.7). Specifically, we obtain formulas

$$T(1,\dots,m)=r_1+r_2S(1,\dots,m,\ m+1,\ m+2),\quad m\ge 1,\tag{1.8}$$

and

$$S(1,\dots,m)=r_3+r_4T(1,\dots,m,\ m+1,\ m+2),\quad m\ge 1,\tag{1.9}$$

where the r_i are rational numbers that depend on m. Among other things, Carlitz [3] attacked the same problem with the use of generating functions and Fibonomial coefficients. Here we provide an alternative and more transparent approach, with the use of only simple identities that involve the Fibonacci numbers.

2. PRELIMINARY RESULTS

We require the following results:

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+2}}=1; \tag{2.1}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_nF_{n+2}}=-1+2T(1); \tag{2.2}$$

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}F_{n+3}}=-\frac{1}{4}+S(1,2); \tag{2.3}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_nF_{n+1}F_{n+3}}=\frac{1}{4}; \tag{2.4}$$

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}\cdots F_{n+m-1}F_{n+m+1}}=\frac{(-1)^{m-1}}{F_1\cdots F_mF_{m+1}^2}+\frac{(-1)^m+F_m}{F_{m+1}}S(1,2,\dots,m),\quad m\geq 1; \tag{2.5}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_nF_{n+1}\cdots F_{n+m-1}F_{n+m+1}}=\frac{(-1)^{m}}{F_1\cdots F_mF_{m+1}^2}+\frac{(-1)^{m-1}+F_m}{F_{m+1}}T(1,2,\dots,m),\quad m\geq 1; \tag{2.6}$$

$$\begin{aligned}&F_{n+m}F_{n+m+2}-F_nF_{n+m+1}\\&=(-1)^{m-1}F_{m+1}F_{n+m+1}F_{n+m+2}+[1+(-1)^mF_{m+2}]F_{n+m}F_{n+m+2}-(-1)^{n-1}F_{m+2};\end{aligned} \tag{2.7}$$

$$\begin{aligned}&F_{n+m}F_{n+m+2}+F_nF_{n+m+1}\\&=(-1)^{m}F_{m+1}F_{n+m+1}F_{n+m+2}+[1+(-1)^{m-1}F_{m+2}]F_{n+m}F_{n+m+2}+(-1)^{n-1}F_{m+2}.\end{aligned} \tag{2.8}$$

Formulas (2.1)-(2.3) can be obtained from [1]. More precisely: (2.1) occurs as (4); (2.2) follows if we use (3) to evaluate

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+k}}$$

for $k=1$ and $k=2$; and (2.3) is the first entry in Table III. Formula (2.4) follows from (2.4) in [4]. Again, turning to [4], we see that identity (3.3) therein and its counterpart for T yield (2.5) and (2.6) for $m\geq 3$. We can verify the validity of (2.5) and (2.6) for $m=1$ and 2 by simply substituting these values and comparing the outcomes with (2.1)-(2.4). Finally, (2.7) and (2.8) can be established with the use of the Binet form for F_n.

3. THE RESULTS

Our results are contained in the theorem that follows.

Theorem: Let $m\geq 1$ be an integer. Then

$$S(1,\dots,m)=\frac{1}{(1+(-1)^{m-1}-L_{m+1})}\left[\frac{(-1)^{m-1}F_{m+2}-F_{2m+3}}{F_1\cdots F_{m+2}}-F_{m+1}F_{m+2}T(1,\dots,m+2)\right], \tag{3.1}$$

$$T(1,\dots,m)=\frac{1}{(1+(-1)^{m-1}+L_{m+1})}\left[\frac{(-1)^{m-1}F_{m+2}+F_{2m+3}}{F_1\cdots F_{m+2}}-F_{m+1}F_{m+2}S(1,\dots,m+2)\right]. \tag{3.2}$$

Proof: Let $m \geq 1$ be an integer. Then, due to telescoping, we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{F_n \cdots F_{n+m-1}F_{n+m+1}} - \frac{1}{F_{n+1} \cdots F_{n+m}F_{n+m+2}}\right) = \frac{1}{F_1 \cdots F_m F_{m+2}}. \tag{3.3}$$

Alternatively, with the use of (2.7), this sum can be written as

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{F_{n+m}F_{n+m+2} - F_nF_{n+m+1}}{F_n \cdots F_{n+m+2}} \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{m-1}F_{m+1}F_{n+m+1}F_{n+m+2} + [1+(-1)^m F_{m+2}]F_{n+m}F_{n+m+2} - (-1)^{n-1}F_{m+2}}{F_n \cdots F_{n+m+2}} \qquad (3.4) \\ &= (-1)^{m-1}F_{m+1}S(1, \ldots, m) + [1+(-1)^m F_{m+2}]\sum_{n=1}^{\infty}\frac{1}{F_n \cdots F_{n+m-1}F_{n+m+1}} - F_{m+2}\, T(1, \ldots, m+2). \end{aligned}$$

Finally, after using (2.5) to substitute for

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1} \cdots F_{n+m-1}F_{n+m+1}},$$

we equate the right sides of (3.3) and (3.4), and then solve for $S(1, \ldots, m)$ to obtain (3.1). In the course of the algebraic manipulations, we make use of the well-known identities $L_n = F_{n-1} + F_{n+1}$, $F_n^2 + F_{n+1}^2 = F_{2n+1}$, and $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

Since the proof of (3.2) is similar, we merely give an outline. To begin, we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{F_n \cdots F_{n+m-1}F_{n+m+1}} + \frac{1}{F_{n+1} \cdots F_{n+m}F_{n+m+2}}\right) = \frac{1}{F_1 \cdots F_m F_{m+2}}. \tag{3.5}$$

Next, we write the left side of (3.5) as

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{F_{n+m}F_{n+m+2} + F_nF_{n+m+1}}{F_n \cdots F_{n+m+2}}\right). \tag{3.6}$$

Finally, we make use of (2.8) and (2.6), and then solve for $T(1, \ldots, m)$. This completes the proof of the theorem. □

If we substitute $m = 4$ into (3.1), we obtain (1.6). Likewise, if we substitute $m = 2$ and $m = 6$ into (3.2), we obtain (1.5) and (1.7), respectively.

4. CONCLUDING COMMENTS

Our results (3.1) and (3.2) do not produce (1.1) and (1.2), which, as stated in the Introduction, can be arrived at independently. Interestingly, due to our alternative approach, our main results are more simply stated than the corresponding results in [3]. See, for example, (5.8) in [3]. Incidentally, there is a typographical error in the last formula on page 464, where $-\frac{2}{3}$ should be replaced by $\frac{2}{3}$.

Finally, we refer the interested reader to the recent paper [5], where Rabinowitz discusses algorithmic aspects of certain finite reciprocal sums.

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SOME FRACTALS IN GOLDPOINT GEOMETRY

J. C. Turner
University of Waikato, Hamilton, New Zealand
(*Submitted December 2000-Final Revision February 2002*)

Dedicated to the memory of Herta T. Freitag

1. INTRODUCTION

Problems of interest in goldpoint[1] geometry [1] arise from study of tile-figures that are obtained when goldpoints are marked on sides of triangles, squares, pentagons, etc. and joined by lines in various ways. Many combinatoric problems arise naturally in the course of such studies. Another type of problem is to determine how to combine collections of golden tiles in jig-saw fashion, so that they tile a given geometric figure (or the whole plane) with goldpoint marks on touching sides corresponding everywhere.

Examples of these types of problems are the following:

(i) Find how many different golden tiles can be formed from regular polygons; that is, find how many inequivalent golden triangles, squares, pentagons, etc. there are.

(ii) Given a regular hexagon, find how many different ways it can be tiled by equilateral golden triangles, jig-saw fashion.

In this paper I introduce a new type of problem into goldpoint geometry. I study a variety of fractals which are achieved by using as base the segment [0, 1], and a motif which involves the goldpoints of that segment.[2]

In Sections 2 and 3, the goldpoint dust set and snowflake are defined, and some of their properties are derived.

In the following section, I describe goldpoint fractals which I dedicate to the memory of the inspirational American mathematician Herta T. Freitag, who passed away early in 2000 in her 91st year.

In the final section, I present studies of fractals which are based on the regular pentagon. It is well-known (indeed the knowledge goes back to extreme antiquity, since it is mentioned in cabalistic literature) that the golden mean occurs frequently in the geometry of the pentagon [3] and its accompanying pentagram star. It is hoped that the results given below on pentagon fractals will add to existing literature on the pentagram.

2. THE GOLDPOINT DUST SET

We define the goldpoint dust set (the *gp-dust set*) by prescribing an infinite process similar to that used to produce Cantor's fractal set.

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[1] A point P in a segment AB is a *goldpoint* of AB if AP/PB is either α or $1/\alpha$ (α is the golden ratio).

[2] The terms 'base' and 'motif' are now well known. Excellent references for these terms, and for several of the analytic techniques used in this paper are [2] and [4].

We take the unit line-segment [0, 1] on the x-axis, and compute its goldpoints, which are at points $(\alpha^{-1}, 0)$ and $(\alpha^{-2}, 0)$; call these points G_1 and G_2, respectively. Then we discard all points in the open set of the segment (G_1G_2).

Next we compute the positions of the goldpoints H_1, H_2 and H_3, H_4 of the two remaining segments $[G_1, 1]$ and $[0, G_2]$, respectively. Then we discard the two open sets between these two pairs of goldpoints.

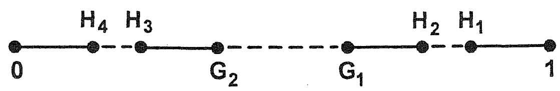


FIGURE 1. Stage 2 of the formation of the goldpoint dust set

We continue this process *ad infinitum*, at each stage discarding all the central open sets between pairs of goldpoints.

The limiting set of points is called the goldpoint dust set of [0, 1]. All points in it, except the two endpoints, are goldpoints of some segment in [0, 1].

Some properties of points in the gp-dust set are described next.

G_1, G_2 are the goldpoints of line segment [0, 1], and G_3, G_4 are the goldpoints of $[0G_2]$.

Measuring lengths from 0, and writing G_i for $\|[0G_i]\|$, we find:

$$\begin{aligned} G_1 &= 1/\alpha^2 + 1/\alpha^3 = 1/\alpha &&= F_{-1}\alpha + F_{-2} \\ G_2 &= 1/\alpha^2 = -\alpha + 2 &&= F_{-2}\alpha + F_{-3} \\ G_3 &= 1/\alpha^4 + 1/\alpha^5 = 1/\alpha^3 &&= F_{-3}\alpha + F_{-4} \\ G_4 &= 1/\alpha^4 &&= F_{-4}\alpha + F_{-5} \end{aligned}$$

and so on.

Similarly, H_1, H_2 are the goldpoints of line segment $[G_1, 1]$, and for them we find:

$$\begin{aligned} H_1 &= 1/\alpha + 1/\alpha^4 + 1/\alpha^5 = 1/\alpha + 1/\alpha^3 \\ H_2 &= 1/\alpha + 1/\alpha^4 \end{aligned}$$

It may be noted that:

G_1 is a goldpoint of [0, 1] (given),
G_1 is a goldpoint of $[G_2H_2]$ (since $G_2G_1 = \alpha^{-3}$ and $G_1H_2 = \alpha^{-4}$),
G_1 is a goldpoint of $[G_3H_1]$ (since $G_3G_1 = \alpha^{-2}$ and $G_1H_1 = \alpha^{-3}$).

It follows that, as the process of discarding central open segments continues, all of the points left in the dust set are goldpoints (0 and 1 are excluded); in the limit, each point is a goldpoint an infinite number of times, with respect to pairs of other points in the dust set. It might be appropriate to call this the gold-dust set.

It is evident from the above analysis that each goldpoint in the dust set can be expressed uniquely in α-nary form thus:

$$\text{goldpoint} = 0.c_1c_2c_3\cdots \equiv c_1\alpha^{-1} + c_2\alpha^{-2} + c_3\alpha^{-3} + \cdots,$$

where all the c_i coefficients are zero or unity, and with no pair of adjacent coefficients being (1, 1).*

* If in the calculation of a goldpoint we obtain both $c_i = 1$ and $c_{i+1} = 1$, we are required to combine the adjacent terms, using $\alpha^{-i} + \alpha^{-(i+1)} = \alpha^{-(i-1)}$.

Examples:

$$G_1 = 0.1,\ G_2 = 0.01,\ G_3 = 0.001,\ \text{etc.}$$
$$H_1 = G_1 + \alpha^{-4} + \alpha^{-5} = G_1 + \alpha^{-3} = 0.101,$$
$$H_2 = G_1 + \alpha^{-4} = 0.1001.$$

The goldpoint dust set is the set of all points in $(0, 1)$ which have this type of α-ternary form (reminiscent of maximal Zeckendorf representations of n in terms of the Fibonacci numbers).

3. THE GOLDPOINT SNOWFLAKE

The following diagrams show how a snowflake fractal (*à la* von Koch, 1904) can be constructed from a line segment base, and the motif given as phase 1 in Figure 2. G and H are the goldpoints of the line segment $[0, 1]$. Phases 2 and 5 indicate how the fractal develops. Since $OG = H1 = 1/\alpha^2$, and the reduction factor is $r = \alpha^2$ at each step, the length of the perimeter of the snowflake at phase p is $P_p = (4/\alpha^2)^p$ for $p = 0, 1, 2, \ldots$.

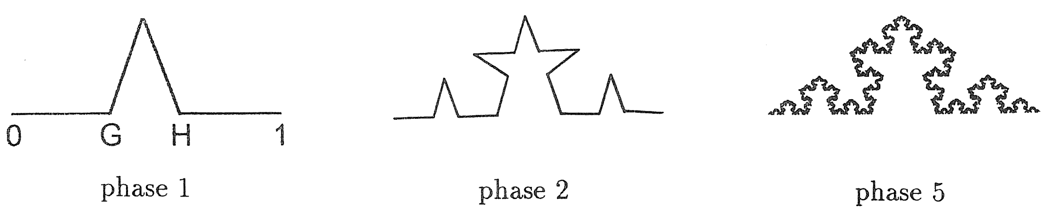


FIGURE 2. Development phases of the goldpoint snowflake

Fractal (or self-similarity) dimension

At each step, from each segment $m = 4$ new segments are formed, with length reduction factor $r = \alpha^2$ in every case. Hence, the fractal dimension of the goldpoint snowflake is

$$d = \frac{\log m}{\log r} = \frac{\log 4}{2\log \alpha} = 1.44042\ldots.$$

4. HERTA'S SHIELD, STAR JEWEL AND COMB

In the last few months of Herta Freitag's life, I sent her three goldpoint fractal diagrams, which I hoped would amuse her. The shield (I said) was for her protection, and was drawn on her 90th birthday card. The jewel for her dress and the comb for her hair were sent later with get-well messages. Sadly, my shield did not avail her for long; however, I was sure that she would appreciate the diagrams and look for the relationships to the golden mean that are evident within them.

Both the shield and the star jewel are developed with goldpoint snowflakes on the sides of an equilateral triangle. The shield is exterior to the triangle; the jewel is interior to it (see Figs. 3 and 4).

Figure 5 shows Herta's goldpoint comb; I imagined it to be made of ivory. In the limit, it has an infinite number of teeth, the prong points forming a set of Hausdorff measure zero and equivalent to the gp-dust set. I don't know what it would have done to her hair. It is easy to see how the comb is built up of rectangles erected upon line segments parallel to those 'left in' during the process of obtaining the goldpoint set (see Fig. 2). Upon each segment, a golden rectangle is constructed, with the horizontal segment being the larger side.

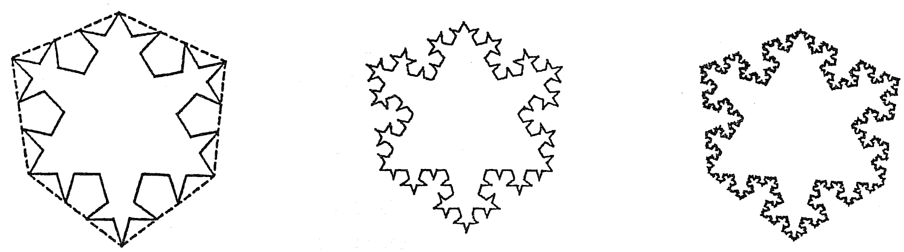


FIGURE 3. Herta's goldpoint shield (phases 2, 3, and 5)
[The dotted bounding-polygon is added in 2 to demonstrate the shield's outer shape.]

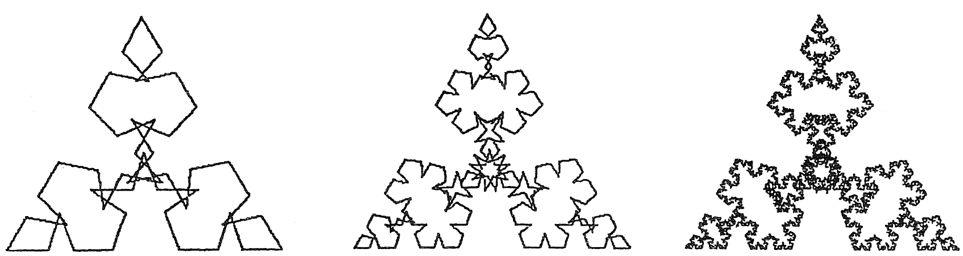


FIGURE 4. Herta's star jewel (phases 2, 3, and 5)

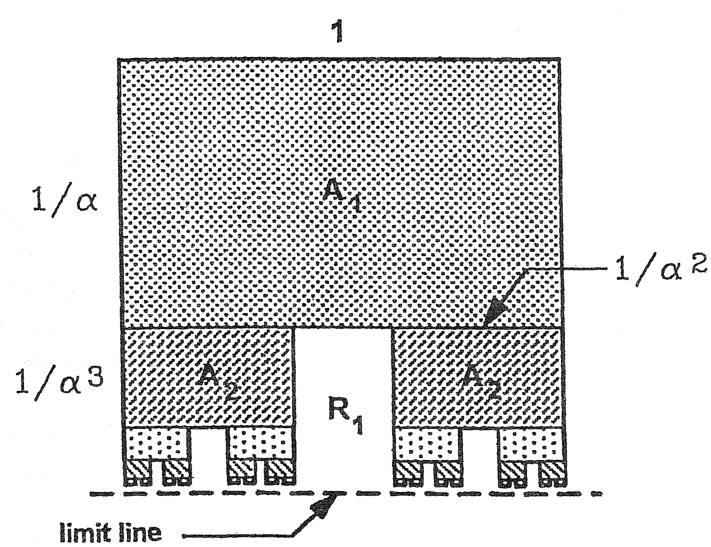


FIGURE 5. Herta's goldpoint comb

Figure 5 shows how the short sides of the rectangles have lengths in the sequence:

$$\frac{1}{\alpha}, \frac{1}{\alpha^3}, \frac{1}{\alpha^5}, \frac{1}{\alpha^7}, \ldots.$$

This is a geometric progression of common ratio $1/\alpha^2$, and its infinite sum is 1. Therefore, the goldpoint comb has height 1 and it covers (in the limit, and except for the limit line) a square of side 1.

Thus, the unit square of the comb is tiled by golden rectangles in an interesting way.

If we check the 'hole' or 'spaces' in the comb, we see that they are also rectangles, all standing on the horizontal limit line where the teeth 'end'. Again checking the dimensions, we see that each of these rectangles is also a golden rectangle. Moreover, the largest 'hole' rectangle is equal to the second largest ivory rectangle; the second largest 'hole' rectangle is equal to the third largest ivory rectangle; and so on.

The area (A) of the ivory, and the area (H) of the 'holes'

Working directly from Figure 5 we get, for the total ivory in the comb:

$$A = 1\times\frac{1}{\alpha} + 2\times\frac{1}{\alpha^5} + 4\times\frac{1}{\alpha^9} + 8\times\frac{1}{\alpha^{13}} + \cdots = \frac{1}{\alpha}\sum_{i=1}^{\infty}\left(\frac{2}{\alpha^4}\right)^i = \frac{\alpha^2}{3}.$$

Then, for the area of the 'holes' in the comb:

$$H = 1 - A = 1 - \frac{1}{3}\alpha^2 = \frac{1}{3\alpha^2}. \text{ [Check: } (\alpha^2 + \alpha^{-2}) = 3.]$$

5. THE GOLDPOINT MOTIF TRIANGLE, AND PENTAGON FRACTALS

In this final section we first analyze the goldpoint motif triangle, showing various ways by which it can be partitioned.

Then we take a regular pentagon and study some of its goldpoint properties. We show how a fractal of pentagon fractals can be constructed within it, and point out one or two of the properties of this object.

Properties of the goldpoint motif bounding triangle

In Figure 6(a) below, the goldpoint motif *AGCHB* is shown, together with its bounding triangle *ABC*. (It was also shown in Fig. 2 above.) This triangle partitions into two (108°, 36°, 36°) triangles, viz. *AGC* and *BHC*, which we call *S*-triangles, and a (36°, 72°, 72°) triangle, *GHC*, which we call a *T*-triangle. We shall use the convention S_i to describe an *S*-triangle drawn on a base line segment of length $1/\alpha^i$, $i = 0, 1, 2, \ldots$; similarly, we shall use T_i for the *T*-triangles drawn on such base line segments.

When making the analyses and calculations, we shall have recourse to the formulas given at the beginning of Section 2, and also to the following trigonometric relations:

θ	36°	72°
$\sin\theta$	$\sqrt{\alpha+2}/(2\alpha)$	$(1/2)\sqrt{\alpha+2}$
$\cos\theta$	$\alpha/2$	$1/(2\alpha)$
$\tan\theta$	$\sqrt{\alpha+2}/\alpha^2$	$\alpha\sqrt{\alpha+2}$

The goldpoint motif triangle, and some partitions of it

Figure 6(a) is used to demonstrate several partition properties of the goldpoint motif triangle. Figure 6(b) shows how the triangle can be partitioned by pentagrams and S-triangles of diminishing sizes and with sides $1/\alpha^i$. Various calculations and comments on these figures are given below the diagrams.

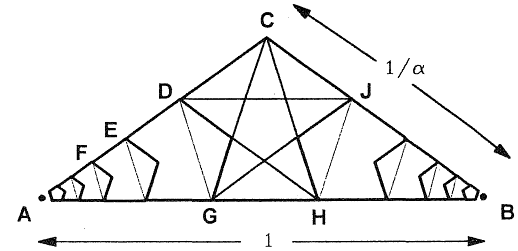


FIGURE 6(a). The motif triangle and some dividing lines

If $AB = 1$, then it is immediately seen that ABC is an S_0 triangle, which is partitioned by GC and HC into two S_1 and one T_3 triangles (since $AC = BC = 1/\alpha$ and $GH = 1/\alpha^3$). Thus, $S_0 = 2S_1 \cup T_3$.

The area of triangle ABC is $(1/2)AC\sin 36° = \sqrt{\alpha+2}/(4\alpha^2)$.

Other partitions of ABC can be seen in the constructions. For example, the two T_3 triangles ADG and BJH together with the central pentagon P_3 on GH. Another is the set of decreasing and overlapping pentagons, on sides CD, DE, EF, ... and, similarly, on the right side of center, whose union limitingly fills triangle ABC.

Finally, we observe that since an S-triangle can be partitioned into a T-triangle and an S-triangle (e.g., $ABC = AGC \cup GCB$), by repeated divisions ABC can be partitioned into a sequence of diminishing S-triangles; or else, similarly, into a sequence of diminishing T-triangles. We won't spell out their relative sizes, but point out that they are all in ratios of powers of α.

Figure 6(b) demonstrates how the golden motif triangle can be partitioned into an attractive double sequence of diminishing pentagrams, with sides in diminishing powers of α, together with sequences of diminishing S-triangles.

Proposition: Every pentagram vertex (except C) is a double goldpoint with respect to two pairs of pentagram vertices.

Proof: By inspection of the largest pair of pentagrams, and induction.

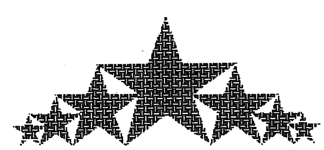


FIGURE 6(b). Pentagrams and S-triangles constructed in the motif triangle

The complement in ΔABC of the infinite set of (interiors of) pentagrams is an infinite set S of S-triangles, being $3S_3 \cup 8S_4 \cup 8S_5 \cup \cdots$. This can be regarded as phase 1 of a fractal. In the next phase, every S-triangle in phase 1 provides a similar figure, all S-triangles in it being reduced by α^{-3}.

The dust set of this fractal is the set of all vertices of S-triangles produced in this multiply-infinite recurrence process.

Some properties of the regular pentagon, with goldpoints and partitions

The next two figures, 6(c) and 6(d), show regular pentagons, of side 1, with various construction lines upon them.

In Figure 6(c), ΔABC is a T_1-triangle, so $AC = 1/(2\cos 72) = \alpha$. From ΔAGD, we get $AG = 1/(2\cos 54) = \alpha/\sqrt{\alpha+2}$ and $GD = (1/2)\tan 54 = \alpha^2/2\sqrt{\alpha+2}$. Also, $CD = (1/2)\tan 72 = (1/2)\alpha\sqrt{\alpha+2}$.

By similar pentagons, $G'D = \alpha^{-3}GD$ and $G''A = \alpha G'/\sin 54 = 2/\sqrt{\alpha+2}$.

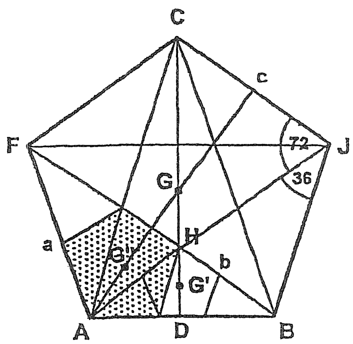


FIGURE 6(c). A regular pentagon

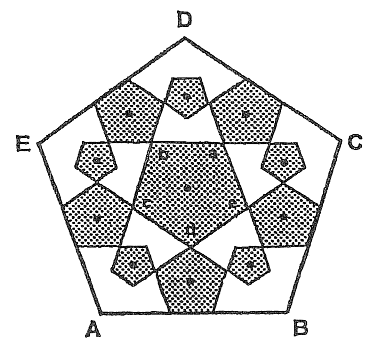


FIGURE 6(d). A fractal of pentagons

Proposition:

(i) $GG' = GG''$.

(ii) G is a goldpoint of CG'.

Proof:

(i)

$$\begin{aligned} GG' &= GD - G'D \\ &= \alpha^2/2\sqrt{\alpha+2} - 1/(2\alpha\sqrt{\alpha+2}) \\ &= 1/\sqrt{\alpha+2} \text{ (since } \alpha^2 - 1/\alpha = 2) \end{aligned}$$

and

$$\begin{aligned} GG'' &= GA - G''A \\ &= \alpha/\sqrt{\alpha+2} - 1/(\alpha\sqrt{\alpha+2}) \\ &= 1/\sqrt{\alpha+2} \text{ (since } \alpha - 1/\alpha = 1). \end{aligned}$$

Therefore

$$GG' = GG''.$$

(ii)

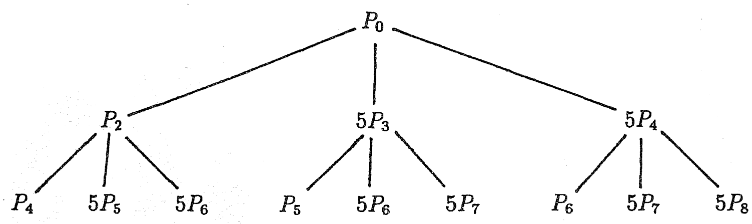
$$\begin{aligned} GG'/GC &= GG'/GA \\ &= (1/\sqrt{\alpha+2})\cdot(\sqrt{\alpha+2})/\alpha = 1/\alpha. \end{aligned}$$

Other results about goldpoints in a pentagon construction may be found on page 28 in [2]. Let us turn to Figure 6(d) and examine the fractal of pentagons.

It is evident how Figure 6(d) can be obtained from Figure 6(c). The shaded pentagon P_2 is replaced by its inner pentagon (a P_4), and then the two small pentagons are replicated around pentagon *abcde* (a P_2).

Looking at Figure 6(d), we see $5P_3$'s and $5P_4$'s arranged alternately with their centers on a circle by Proposition (i) above, and with a pentagon P_2 in the middle. We can regard this as a motif for constructing a fractal of pentagons in the interior of pentagon *ABCDE*.

Thus, to arrive at phase 1, we must remove all points in the unshaded regions, together with the perimeter of *ABCDE*. Then, to arrive at phase 2, we repeat the above constructions and removals in each of the eleven shaded pentagons. What remains will be 121 shaded pentagons, each scaled by a factor of α^i, $i = 2, 3$, or 4 according to its construction. From the tree diagram below, we see that the distribution of pentagons will then be $1P_4$, $10P_5$, $35P_6$, $50P_7$, $25P_8$.



Evidently, this process can be continued indefinitely. And formulas can be computed for the coefficients on the tree and for reduction factors in areas when passing from phase i to phase $i+1$.

The dust set of the fractal is the set of points in *ABCDE* which are not removed by this infinite process. A moment's thought shows that this set consists of the centers of all the pentagons constructed in the 'whole' process. And the set consists of a *cosmos* of points arranged in circles, with similar, reduced, circles arranged around each of them, and so on *ad infinitum*. Because of the similarity of this system with Ptolomy's model of the Universe, we name this dust set the **Ptolomaic dust set**.

The next two figures show phases of the interior and exterior fractals which are constructed on a regular pentagon using the goldpoint motif on its sides.

Phase 2 of Figure 6(e) shows an attractive clover-leaf arrangement of five leaves, each of three P_3 pentagons, formed in P_2's and arranged around a central P_2.

Phase 4 shows clearly how the interior goldpoint fractal of a regular pentagon is equal to the exterior goldpoint fractal of its pentagram.

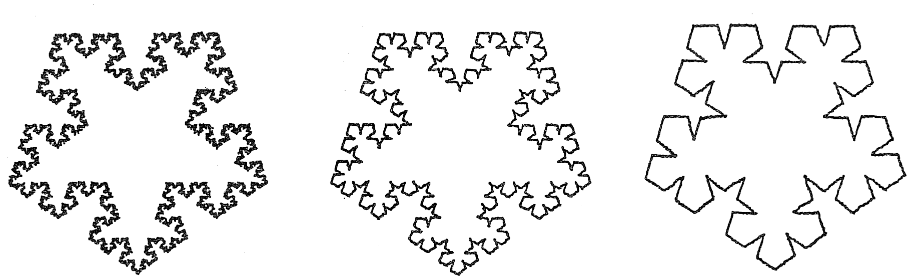


FIGURE 6(e). Interior goldpoint fractal of a pentagon (phases 2, 3, and 4)

It is clear from the phase 1 diagram of Figure 6(f) that the exterior goldpoint fractal of a regular pentagon is bounded by a regular pentagon. It is easy to prove this using angle values of the S- and T-triangles which touch the boundary. We believe this property of a von Koch-type fractal having a bounding polygon which is similar to the generating polygon to be unique.

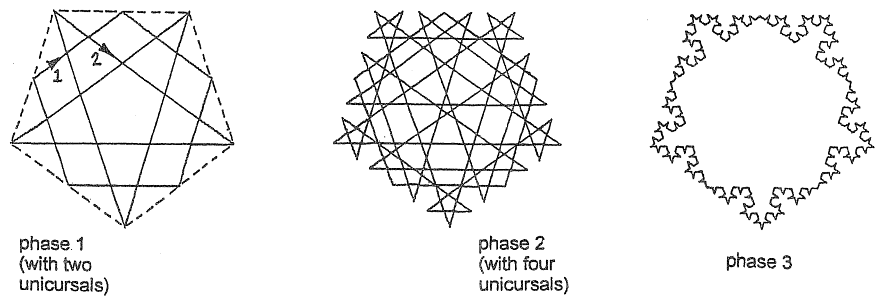


FIGURE 6(f). Exterior goldpoint fractal of a pentagon (phases 1, 2, and 3)

A final interesting comment is the following: the sharp boundary points in the phase 1 diagram can be connected by a unicursal polygon of chords of the diagram (each chord begins and ends along an arm of a point-angle), whereas the sharp boundary points of the phase 2 diagram require two such unicursal polygons to join them all up. In phase n, there will be 2^n unicursal polygons required. The unicursal perimeters can be calculated in terms of α, given that P_0 has side length 1. For example, in P_1, the unicursals have perimeters 5 and $5(7-3\alpha)$, respectively.

ACKNOWLEDGMENT

I wish to thank Dr. Hans Walser for his valuable comments on this paper. Dr. Walser is the author of a book [5] entitled *Der Goldene Schnitt* [The Golden Section], which contains a chapter that presents other fractals involving the golden ratio in their construction.

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AMS Classification Numbers: 11B37, 11B39, 10A35

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DIVISIBILITY PROPERTIES BY MULTISECTION

Tamás Lengyel
Occidental College, 1600 Campus Road, Los Angeles, CA 90041
(Submitted December 2000-Final Revision April 2001)

1. INTRODUCTION

The p-adic order, $\nu_p(r)$, of r is the exponent of the highest power of a prime p which divides r. We characterize the p-adic order $\nu_p(F_n)$ of the F_n sequence using multisection identities. The method of multisection is a helpful tool in discovering and proving divisibility properties. Here it leads to invariants of the modulo p^2 Fibonacci generating function for $p \neq 5$. The proof relies on some simple results on the periodic structure of the series F_n.

The periodic properties of the Fibonacci and Lucas numbers have been extensively studied (e.g., [13]). (For a general discussion of the modulo m periodicity of integer sequences, see [8].) The smallest positive index n such that $F_n \equiv 0 \pmod p$ is called the rank of apparition (or rank of appearance, or Fibonacci entry-point) of prime p and is denoted by $n(p)$. The notion of rank of apparition $n(m)$ can be extended to arbitrary modulus $m \geq 2$. The order of p in $F_{n(p)}$ will be denoted by $e = e(p) = \nu_p(F_{n(p)}) \geq 1$. Interested readers might consult [6] and [9] for a list of relevant references on the properties of $\nu_p(F_n)$.

The main focus of this paper is the multisection based derivation of some important divisibility properties of F_n (Theorem A) and L_n (Theorem D). A result similar to Theorem A was obtained by Halton [4]. This latter approach expresses the p-adic order of generalized binomial coefficients in terms of the number of "carries." Theorem A can be generalized to include other linear recurrent sequences and a proof without using generating functions was given in Exercise 3.2.2.11 of [6]. The latter approach is implicitly based on multisections.

The generating functions of the Fibonacci and Lucas numbers are

$$f(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} \quad \text{and} \quad h(x) = \sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2},$$

respectively. In this paper the general coefficients of these generating functions will be determined by multisection identities, as we prove

Theorem A [9]: For all $n \geq 0$, we have

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3, \\ 1, & \text{if } n \equiv 3 \pmod 6, \\ 3, & \text{if } n \equiv 6 \pmod{12}, \\ \nu_2(n)+2 & \text{if } n \equiv 0 \pmod{12}, \end{cases}$$

$$\nu_5(F_n) = \nu_5(n),$$

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + e(p), & \text{if } n \equiv 0 \pmod{n(p)}, \\ 0, & \text{if } n \not\equiv 0 \pmod{n(p)}, \end{cases} \quad \text{if } p \neq 2 \text{ and } 5.$$

The cases $p = 2$ and $p = 5$ are discussed in Sections 2 and 3, respectively. The general case is completed in Section 4. The case of $p = 2$ has been discussed in [5] using a different approach.

The multisection based technique offers a simplified treatment of this case. We extend the method to the Lucas numbers in Section 5.

By the m-section of a power series $g(x)=\sum_{n=0}^{\infty}a_nx^n$ we mean the extraction of the sum of terms a_lx^l in which l is divisible by m. We use the resulting power series $g_m(x)=\sum_{n=0}^{\infty}a_{mn}x^{mn}$ in its modified form $g_m(x^{1/m})=\sum_{n=0}^{\infty}a_{mn}x^n$ and call it the m-section as well. The corresponding sequence $\{a_{mn}\}_{n=0}^{\infty}$ of coefficients is referred to as the m-section of the sequence $\{a_n\}_{n=0}^{\infty}$. The notion of m-section can be generalized to form a sum of terms with index l ranging over a fixed congruence class of integers modulo m. It will be used in Sections 2 and 5. There are various general multisection identities (cf. [10, p.131] or [1, p. 84]), and they can be helpful in proving divisibility patterns (e.g., [2]). The m-section of the Fibonacci sequence leads to the form

$$\sum_{n=0}^{\infty}F_{mn}x^n=\frac{F_mx}{1-L_mx+(-1)^mx^2}. \tag{1}$$

The denominators are referred to as Lucas factors. For other applications of Lucas factors, see [11].

The present proof of Theorem A is based on a multisection invariant. In fact, we will see in (5), (13), and (14) that $x/(1-x)^2$ or $x/(1+x)^2$ is an invariant of the properly sected Fibonacci generating function taken mod p^2 for every prime $p\neq 5$. The power of p can be improved easily.

We shall need some facts on the location of zeros in the series $\{F_n \bmod m\}_{n\geq 0}$.

Theorem B (Theorem 3 in [13]): The terms for which $F_n\equiv 0 \pmod m$ have subscripts that form a simple arithmetic progression. That is, for some positive integer $d=d(m)$ and for $x=0,1,2,\ldots$, $n=x\cdot d$ gives all n with $F_n\equiv 0 \pmod p$ unless l is a multiple of $n(p)$.

Note that $d(m)$ is exactly $n(m)$, and $d(p^i)=d(p)=n(p)$ for all $1\leq i\leq e(p)$. It also follows that $F_l\not\equiv 0 \pmod p$ unless l is a multiple of $n(p)$.

We denote the *modulo m* period of the Fibonacci series by $\pi(m)$. Gauss proved that the ratio $\frac{\pi(p)}{n(p)}$ is 1, 2, or 4. In fact, we get

Lemma C [9]: The ratio $\frac{\pi(p)}{n(p)}$ can be characterized fully in terms of $x\equiv F_{n(p)-1}\equiv F_{n(p)+1} \pmod p$ by

$$\pi(p)=\begin{cases} n(p), & \text{iff } x\equiv 1 \pmod p,\\ 2n(p), & \text{iff } x\equiv -1 \pmod p,\\ 4n(p), & \text{iff } x^2\equiv -1 \pmod p.\end{cases}$$

In the first case, p must have the form $10l\pm 1$ while the third case requires that $p=4l+1$.

We also will repeatedly use two identities (cf. (23) and (24) in [12]) for the Lucas numbers with arbitrary integers $h\geq 0$:

$$L_{2h}=2(-1)^h+5F_h^2, \tag{2}$$

$$L_h^2=4(-1)^h+5F_h^2. \tag{3}$$

It is worth noting that our proofs of Theorems A and D rely on three congruences for the Lucas numbers (cf. Lemmas 1, 2, and 3) which, in turn, can be improved significantly (cf. Lemmas 1′, 2′, and 3′) using the theorems.

2. THE CASE OF $p = 2$

By adding together the six 6-sections $\sum_{n=0}^{\infty} F_{6n+l}x^{6n+l}$, $l = 0, 1, \ldots, 5$, of the generating function $f(x)$, we obtain

$$f(x) = \frac{x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 - 5x^7 + 3x^8 - 2x^9 + x^{10} - x^{11}}{1 - 18x^6 + x^{12}}$$

which is equivalent to the recurrence relation $F_{n+12} = 18F_{n+6} - F_n$, $F_0 = 0$, $F_1 = 1, \ldots, F_{11} = 89$. This immediately implies that

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3, \\ 1, & \text{if } n \equiv 3 \pmod 6, \\ 3, & \text{if } n \equiv 6 \pmod{12}. \end{cases}$$

It remains to be proven that

$$\nu_2(F_{12 \cdot n}) = \nu_2(n) + 4. \tag{4}$$

To this end, first we note that

Lemma 1: $L_{12 \cdot 2^k} \equiv 2 \pmod{2^2}$ for all $k \geq 0$.

In fact, the modulo 4 period of F_n is 6, and this implies $L_{6j} \equiv 2F_{6j+1} \equiv 2 \pmod 4$ for every integer $j \geq 0$.

By identity (1), we obtain that, for all $k \geq 0$,

$$\sum_{n=0}^{\infty} \frac{F_{12 \cdot 2^k n}}{F_{12 \cdot 2^k}} x^n = \frac{x}{1 - L_{12 \cdot 2^k} x + x^2} \equiv \frac{x}{(1-x)^2} \equiv \sum_{n=1}^{\infty} nx^n \pmod{2^2}. \tag{5}$$

We have $F_{12} = 144 = 2^4 \cdot 9$. By setting $k = 0$ and $n = 2$ in (5) it follows that $F_{12 \cdot 2} / F_{12} \equiv 2$ (mod 2^2), thus $\nu_2(F_{24}) = \nu_2(F_{12}) + 1 = 5$. In general, we use $n = 2$ and observe that

$$\nu_2(F_{12 \cdot 2^{k+1}}) = \nu_2(F_{12 \cdot 2^k}) + 1 = \cdots = \nu_2(F_{12}) + k + 1 = 4 + \nu_2(2^{k+1})$$

follows by a simple inductive argument. We complete the proof of (4) by noting that, for n odd, $\nu_2(F_{12 \cdot 2^k n}) = \nu_2(F_{12 \cdot 2^k})$ holds by (5). $\square$

A sharper version of Lemma 1 can be derived from Theorem A (once it has been proven).

Lemma 1': $L_{12 \cdot 2^k} \equiv 2 \pmod{2^{2k+6}}$ for all $k \geq 0$.

Proof of Lemma 1': We note that $L_{12 \cdot 2^k} \equiv 2 \pmod{2^{k+3}}$ can be derived easily from the periodicity of F_n, for $L_{12 \cdot 2^k} \equiv 2F_{12 \cdot 2^k + 1} \equiv 2 \pmod{2^{k+3}}$ as $\pi(2^l) = 12 \cdot 2^{l-3}$, $l \geq 1$. We notice, however, that the sharper $L_{12} = 322 \equiv 2 \pmod{2^6}$ also holds. Moreover, identity (2) yields $L_{12 \cdot 2^{k+1}} \equiv 2$ (mod $F_{12 \cdot 2^k}^2$), and we derive that $L_{12 \cdot 2^{k+1}} \equiv 2 \pmod{(2^{4+k})^2}$ using Theorem A. Accordingly, we can replace the exponent of p in identity (5). $\square$

3. THE CASE OF $p = 5$

This case is a little more involved. We will find $\nu_5(F_{5^k n})$, $k \geq 1$, in terms of $\nu_5(F_{5^k})$ in three steps. In the first two, we assume that $(n, 5) = 1$, then we deal with the case of $n = 5$.

First, we take the 5-section of $f(x)$ and obtain

$$\sum_{n=0}^{\infty}\frac{F_{5n}}{F_5}x^n=\frac{x}{1-11x-x^2}\equiv\frac{x}{1-x-x^2}\equiv\sum_{n=1}^{\infty}F_nx^n \pmod 5,$$

which guarantees that $\nu_5(F_{5n})=\nu_5(F_5)$ if $(n,5)=1$. In the second step, we try to generalize this relation for indices of the form 5^kn, $(n,5)=1$, $k\geq 2$. We shall need the following lemma.

Lemma 2: $L_{5^{k+1}}-L_{5^k}\equiv 0 \pmod{25}$ for $k\geq 1$.

Proof of Lemma 2: By identity (3) we have, for $k\geq 1$, that $L_{5^{k+1}}^2-L_{5^k}^2\equiv 0 \pmod{F_{5^k}^2}$. It follows that

$$(L_{5^{k+1}}-L_{5^k})(L_{5^{k+1}}+L_{5^k})\equiv 0 \pmod{25} \tag{6}$$

by Theorem B. Clearly,

$$L_{5^{k+1}}\equiv L_{5^k}\equiv L_5\equiv 1 \pmod 5, \tag{7}$$

thus the factor $L_{5^{k+1}}+L_{5^k}$ cannot be a multiple of 5. Therefore, $L_{5^{k+1}}-L_{5^k}\equiv 0 \pmod{25}$ by identity (6). □

We note that $\nu_5(F_{25})=2$. It is true that, for $k\geq 1$,

$$\sum_{n=0}^{\infty}\left(\frac{F_{5^{k+1}n}}{F_{5^{k+1}}}-\frac{F_{5^kn}}{F_{5^k}}\right)x^n=\frac{x}{1-L_{5^{k+1}}x-x^2}-\frac{x}{1-L_{5^k}x-x^2}$$
$$=(L_{5^{k+1}}-L_{5^k})\frac{x}{1-L_{5^{k+1}}x-x^2}\frac{x}{1-L_{5^k}x-x^2}.$$

The first factor is divisible by 25 according to Lemma 2. For $(n,5)=1$, we get

$$\nu_5(F_{5^kn}/F_{5^k})=\nu_5(F_{5^{k-1}n}/F_{5^{k-1}})=\cdots=\nu_5(F_{5n}/F_5)=0, \tag{8}$$

i.e., $\nu_5(F_{5^kn})=\nu_5(F_{5^k})$ by induction on $k\geq 1$.

Now we turn to the case of $n=5$. For $k\geq 1$ and $n=5$, we get that $F_{5^{k+2}}/F_{5^{k+1}}\equiv F_{5^{k+1}}/F_{5^k}$ (mod 25); therefore,

$$\nu_5(F_{5^{k+2}})=\nu_5(F_{5^{k+1}})+1=\cdots=\nu_5(F_5)+k+1.$$

by induction using $\nu_5(F_{25}/F_5)=1$. The proof of the case $p=5$ is now complete. □

Note that, once it is proven, Theorem A guarantees the much stronger lemma.

Lemma 2': $L_{5^{k+1}}\equiv L_{5^k} \pmod{5^{2k}}$ for $k\geq 1$.

We note that an alternative derivation of (8) is possible by (7) but without using Lemma 2:

$$\frac{x}{1-L_{5^{k+1}}x-x^2}\frac{x}{1-L_{5^k}x-x^2}\equiv\sum_{n=0}^{\infty}F_n^{(2)}x^n \pmod 5$$

with $F_n^{(2)}$ being the 2-fold convolution of the sequence F_n. The m-fold convolution of the sequence F_n is defined by

$$F_n^{(m)}=\sum_{i_1+i_2+\cdots+i_m=n}F_{i_1}F_{i_2}\ldots F_{i_m},$$

which has the generating function $[f(x)]^m$. Note that, by identity (7.61) on page 354 in [3], $F_n^{(2)} = \frac{1}{5}(2nF_{n+1} - (n+1)F_n) = \frac{n}{5}(2F_{n+1} - F_n) - \frac{1}{5}F_n = \frac{n}{5}L_n - \frac{1}{5}F_n$. We can easily find the period of $F_n^{(m)}$ by the general theory (cf. [8]) or by simple inspection. The latter approach also provides us with the actual elements of the period. It is clear that 100 is the modulo 25 period of $nL_n - F_n$, and $nL_n - F_n$ is divisible by 25 if n is divisible by 5. It follows that $5|F_n^{(2)}$ if $5|n$.

4. THE GENERAL CASE

In this section p is a prime different from 2 and 5, and n denotes an integer for which $\nu_p(n)$ is either 0 or 1. We will use either an $n(p)p^k$- or a $2n(p)p^k$-section in obtaining the required divisibility properties. First, we prove

Lemma 3: For any prime $p \equiv 3 \pmod 4$,

$$L_{n(p)p^k} \equiv \begin{cases} 2 \pmod{p^2}, & \text{if } \pi(p)/n(p) = 1, \\ -2 \pmod{p^2}, & \text{if } \pi(p)/n(p) = 2. \end{cases}$$

Proof: Formula (3) yields that, if $h \ge 0$ is even, then $L_{2h}^2 - L_h^2 \equiv 0 \pmod{F_h^2}$. Note that $n(p)$ is even for $p \equiv 3 \pmod 4$ (see [13]). By setting $h = n(p)p^k$ we obtain

$$(L_{2n(p)p^k} - L_{n(p)p^k})(L_{2n(p)p^k} + L_{n(p)p^k}) \equiv 0 \pmod{p^2}. \tag{9}$$

Therefore, either

$$L_{2n(p)p^k} \equiv L_{n(p)p^k} \pmod{p^2} \tag{10}$$

or

$$L_{2n(p)p^k} \equiv -L_{n(p)p^k} \pmod{p^2}, \tag{11}$$

for otherwise both $L_{2n(p)p^k} - L_{n(p)p^k}$ and $L_{2n(p)p^k} + L_{n(p)p^k}$ will be divisible by p. This would lead to $L_{n(p)p^k} \equiv 0 \pmod p$, which is impossible as $L_{n(p)p^k} \equiv 2F_{n(p)p^k+1} \pmod p$. According to identity (2), $L_{2n(p)} = 2 + 5F_{n(p)}^2$, which yields $L_{2n(p)} \equiv 2 \pmod{p^2}$ and also

$$L_{2n(p)p^k} \equiv 2 \pmod{p^2} \tag{12}$$

by Theorem B [13].

If $\pi(p)/n(p) = 1$, then $F_{n(p)+1} \equiv 1 \pmod p$ by Lemma C, and we get $L_{2n(p)} \equiv L_{n(p)} \equiv 2 \pmod p$ and, similarly, $L_{2n(p)p^k} \equiv L_{n(p)p^k} \equiv 2F_{2n(p)p^k+1} \equiv 2 \pmod p$, leading to (10). If $\pi(p)/n(p) = 2$, then $F_{n(p)+1} \equiv -1 \pmod p$ and $L_{2n(p)} \equiv -L_{n(p)} \equiv 2 \pmod p$ and $L_{2n(p)p^k} \equiv -L_{n(p)p^k} \equiv 2 \pmod p$ corresponding to (11). □

We are now able to finish the proof of Theorem A. In the case of $\pi(p)/n(p) = 1$ and 2, we can use

$$\sum_{n=0}^{\infty} \frac{F_{n(p)p^k n}}{F_{n(p)p^k}} x^n = \frac{x}{1 - L_{n(p)p^k}x + x^2} \equiv \frac{x}{(1 \pm x)^2} \equiv \sum_{n=1}^{\infty} (\mp 1)^{n-1} n x^n \pmod{p^2}, \tag{13}$$

which proves $\nu_p(F_{n(p)p^k n}) = \nu_p(F_{n(p)p^k}) + \nu_p(n)$ for $\nu_p(n) \le 1$. In particular, by setting $n = p$, we obtain $\nu_p(F_{n(p)p^{k+1}}) = \nu_p(F_{n(p)p^k}) + 1$, and $\nu_p(F_{n(p)p^{k+1}}) = e(p) + k + 1$ follows by induction on $k \ge 0$. In summary, we derived that $\nu_p(F_{n(p)p^k n}) = e(p) + k + \nu_p(n)$ and the proof is now complete.

On the other hand, if $\pi(p)/n(p)=4$, then we switch from using an $n(p)p^k$-section to a $2n(p)p^k$-section. By the duplication formula (cf. [3] or [12]), we get $F_{2n(p)p^k n}=F_{n(p)p^k n}L_{n(p)p^k n}$ for any integer $n>0$. This yields $\nu_p(F_{2n(p)p^k n})=\nu_p(F_{n(p)p^k n})$. We consider

$$\sum_{n=0}^{\infty}\frac{F_{2n(p)p^k n}}{F_{2n(p)p^k}}x^n=\frac{x}{1-L_{2n(p)p^k}x+x^2}.$$

Identity (12) implies that

$$\sum_{n=0}^{\infty}\frac{F_{2n(p)p^k n}}{F_{2n(p)p^k}}x^n\equiv\frac{x}{(1-x)^2}\equiv\sum_{n=1}^{\infty}nx^n \pmod{p^2}. \tag{14}$$

The proof can be concluded as above for

$$\begin{aligned}\nu_p(F_{n(p)p^k n})&=\nu_p(F_{2n(p)p^k n})=\nu_p(F_{2n(p)})+k+\nu_p(n)\\&=\nu_p(F_{n(p)})+k+\nu_p(n)=e(p)+k+\nu_p(n).\ \square\end{aligned}$$

By means similar to Lemma 1′, we can prove a stronger version of Lemma 3.

Lemma 3′: For any prime $p\equiv 3 \pmod 4$,

$$L_{n(p)p^k}\equiv\begin{cases}2 \pmod{p^{2(k+e(p))}}, & \text{if } \pi(p)/n(p)=1,\\ -2 \pmod{p^{2(k+e(p))}}, & \text{if } \pi(p)/n(p)=2.\end{cases}$$

Proof: We know that $\nu_p(F^2_{n(p)p^k})=2(k+2(p))$ by Theorem A. Thus, we can replace p^2 by $p^{2(k+e(p))}$ in identities (9)-(14). $\square$

We note that, according to Lemmas 1′ and 3′, the denominators of the multisection identities (5), (13), and (14) have either 1 or -1 as a double root modulo some p-power with exponent $2k+6$ or $2(k+2(p))$. This observation, combined with the remarks made in the proofs of the lemmas, helps in obtaining the full description of the structure of the periods of the corresponding multisected sequences [cf. (5), (13), and (14)] with respect to the above-mentioned p-power moduli ($p\neq 5$).

5. LUCAS NUMBERS

By using methods we applied to the Fibonacci sequence, we obtain

$$\sum_{n=0}^{\infty}L_nx^n=\frac{2+x+3x^2+4x^3+7x^4+11x^5-18x^6+11x^7-7x^8+4x^9-3x^{10}+x^{11}}{1-18x^6+x^{12}},$$

which proves that

$$\nu_2(L_n)=\begin{cases}0, & \text{if } n\equiv 1,2 \pmod 3,\\ 2, & \text{if } n\equiv 3 \pmod 6,\\ 1, & \text{if } n\equiv 0 \pmod 6.\end{cases}$$

If $p=5$, then the modulo 5 periodic pattern of L_n is 2, 1, 3, 4, and thus $5\nmid L_n$.

If $p\neq 2$ or 5, then the order $\nu_p(L_n)$ can be derived easily by the duplication formula and Theorem A (see [9]). Here, for the sake of uniformity, we use multisection identities. We need the companion multisection identity to (1) for the Lucas sequence:

$$h_m(x) = \sum_{n=0}^{\infty} L_{mn} x^n = \frac{2 - L_m x}{1 - L_m x + (-1)^m x^2}. \tag{15}$$

As $L_n = F_{2n} / F_n$, we see that L_n is divisible by p only if $2n$ is a multiple of $n(p)$ while n is not; in other words, if n is an odd multiple of $n(p)/2$. This implies that we have to deal only with the case in which $n(p)$ is even. The generalized $\frac{n(p)}{2}$-sected Lucas sequence will suffice to prove

Theorem D: If $p \neq 2$ and $\pi(p)/n(p) \neq 4$, then, for every $k \geq 0$ and $m = (n(p)/2)p^k$,

$$l(x) = \sum_{2 \nmid n} \frac{L_{mn}}{L_m} x^n \equiv \begin{cases} \frac{x(1+x^2)}{(1-x^2)^2} \equiv \sum_{2 \nmid n} n x^n & (\bmod p^2), \quad \text{if } \pi(p)/n(p) = 1, \\ \frac{x(1-x^2)}{(1+x^2)^2} \equiv \sum_{2 \nmid n} (-1)^{\frac{n-1}{2}} n x^n & (\bmod p^2) \quad \text{if } \pi(p)/n(p) = 2, \end{cases}$$

yielding $\nu_p(L_n) = \nu_p(n) + e(p)$ if $n \equiv n(p)/2 \pmod{n(p)}$.

Proof: Note that the conditions guarantee that $n(p)$ is even. We discuss the case in which $\pi(p)/n(p) = 1$ with $k = 0$ only, while the other cases can be carried out similarly. We note that

$$L_{n(p)/2} l(x) = h_{n(p)/2}(x) - h_{n(p)}(x^2).$$

It is known that $n(p)/2$ is odd if $\pi(p)/n(p) = 1$ (cf. [9]). The common denominator of the above difference can be simplified. In fact, according to identity (15), the denominator of $h_{n(p)}(x^2)$ is

$$1 - L_{n(p)} x^2 + x^4 = 1 - (L_{n(p)/2}^2 + 2)x^2 + x^4$$

by $L_{n(p)} = L_{n(p)/2}^2 - 2(-1)^{n(p)/2}$, which follows from (2) and (3). We get

$$1 - L_{n(p)} x^2 + x^4 = (1 - x^2)^2 - L_{n(p)/2}^2 x^2 \equiv (1 - x^2)^2 \pmod{p^2}.$$

Finally, it is easy to see that $l(x)$ simplifies to

$$\frac{x(1+x^2)}{(1-x^2)^2} \pmod{p^2}. \quad \square$$

The exponent of p can be increased to $2(k + e(p))$ in the above proof and therefore in the theorem also.

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THE SUM OF THE SQUARES OF TWO GENERALIZED FIBONACCI NUMBERS

F. T. Howard
Wake Forest University, Box 7388 Reynolda Station, Winston-Salem, NC 27109
(*Submitted January 2001-Final Revision March 2001*)

1. INTRODUCTION

The following identity is well known:

$$F_{n+1}^2 + F_n^2 = F_{2n+1}. \tag{1.1}$$

Recently, Melham [6] proved the generalization

$$F_{n+k+1}^2 + F_{n-k}^2 = F_{2k+1}F_{2n+1} \tag{1.2}$$

for all integers n and k, and he also proved

$$L_{n+k+1}^2 + L_{n-k}^2 = 5F_{2k+1}F_{2n+1}. \tag{1.3}$$

Formula (1.2) appears to be a special case of the more general formula

$$F_n^2 + (-1)^{n+j-1}F_j^2 = F_{n-j}F_{n+j} \tag{1.4}$$

which appears without proof in [3, p. 59]. Obviously, (1.4) implies (1.2); we will show later in the paper that (1.2) also implies (1.4). Our main purpose, however, is to extend (1.4) to the generalized Fibonacci sequence $\{w_n\} = \{w_n(a, b; p, q)\}$ defined by

$$w_0 = a,\ w_1 = b;\ w_n = pw_{n-1} - qw_{n-2}\ (n \geq 2), \tag{1.5}$$

where a, b, p, and q are arbitrary complex numbers, with $q \neq 0$. The numbers w_n have been studied by Horadam (see, e.g., [4]), and some special cases were investigated by Lucas [5]. Obviously the definition can be extended to include negative subscripts; that is, for $n = 1, 2, 3, \ldots$, define $w_{-n} = (pw_{-n+1} - w_{-n+2})/q$. A useful and interesting special case is $\{u_n\} = \{w_n(0, 1; p, q)\}$; that is,

$$u_0 = 0,\ u_1 = 1;\ u_n = pu_{n-1} - qu_{n-2}. \tag{1.6}$$

One of the results in the present paper is

$$w_n^2 - q^{n-j}w_j^2 = u_{n-j}(bw_{n+j} - qaw_{n+j-1}), \tag{1.7}$$

which is valid for arbitrary a, b, p, q, and for all integers n and j. Formula (1.7) contains (1.1)-(1.4) as special cases. In fact, we will prove a more general result (Theorem 3.1 below) that contains (1.7) as a special case.

2. A BASIC IDENTITY

The following formula is essential for the proof of (1.7).

Theorem 2.1: For arbitrary a, b, p, q, and for all integers m and n, $w_{m+n+1} = w_{m+1}u_{n+1} - qw_mu_n$, where w_k and u_k are defined by (1.5) and (1.6), respectively.

Proof: We will first motivate Theorem 2.1 by showing how it can be derived, without prior knowledge, by a combinatorial argument, if we put some restrictions on a, b, p, q, and the subscripts. We will then verify the theorem by means of Binet formulas, and all the restrictions will be removed. We note that there has been some recent interest in proving Fibonacci identities by means of combinatorial arguments [1].

Assume $p>0$, $-q>0$, $a>0$, $b\geq ap$, and suppose we have a sequence of towns labeled X, 0, 1, 2, 3, ... Starting at town X, a driver wants to reach town n under the following conditions: (1) There are exactly a different routes from town X to town 0; (2) There are exactly b different routes from town X to town 1 (including through town 0); (3) If $k>1$, the driver cannot go directly from town X to town k; (4) Once town k has been reached, for any $k\geq 0$, there are only two ways to continue—the driver can go to town $k+1$ in p different ways, or he can bypass town $k+1$ and go directly to town $k+2$ in $-q$ different ways. Let r_n be the number of different routes from town X to town n. Then $r_0=a$, $r_1=b$, and for $n>1$, $r_n=pr_{n-1}-qr_{n-2}$. Thus, $r_n=w_n$, and it is clear that the number of ways to go from town k to town $k+n$, for $k\geq 0$, is $w_{n+1}(0,1;p,q)=u_{n+1}$.

If the driver reaches town $m+n+1$, there are two cases:

Case 1. The driver goes through town $m+1$. She can reach town $m+1$ in w_{m+1} ways, and then she can continue to town $m+n+1$ in u_{n+1} ways.

Case 2. The driver bypasses town $m+1$. She can reach town m in w_m ways, and then there are $-q$ ways to reach town $m+2$. From town $m+2$, the driver can continue to town $m+n+1$ in u_n ways.

Therefore, the number of different routes from town X to town $m+n+1$ is

$$w_{m+n+1}=w_{m+1}u_{n+1}-qw_mu_n,$$

and Theorem 2.1 is true with the given restrictions on a, b, p, q, and the subscripts. By a remarkable theorem of Bruckman and Rabinowitz [2], if an identity involving generalized Fibonacci numbers is true for all positive subscripts, it is true for all nonpositive subscripts as well. Thus, the identity is true for all n and m.

Now we can remove all restrictions on a, b, p, and q by looking at the Binet forms of w_n and u_n. Let α and β be the roots of $x^2-px+q=0$. Then $\alpha\beta=q$, and the Binet forms are (for some constants A_1, A_2, B_1, B_2):

$$w_n=A_1\alpha^n+A_2\beta^n,\ \ u_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},\ \text{if } \alpha\neq\beta, \tag{2.1}$$

$$w_n=B_1\alpha^n+B_2n\alpha^n,\ \ u_n=n\alpha^{n-1},\ \text{if } \alpha=\beta. \tag{2.2}$$

If each of the numbers in Theorem 2.1 is replaced by its Binet form (2.1) or (2.2), we can verify that Theorem 2.1 is valid with no restrictions on a, b, p, or q. This completes the proof. □

We note that the actual values of A_1, A_2, B_1, B_2 are not needed in the above proof. However, for completeness we give the values here:

$$\text{If } \alpha\neq\beta, \text{ then } A_1=\frac{b-a\beta}{\alpha-\beta} \text{ and } A_2=\frac{a\alpha-b}{\alpha-\beta}. \ \text{ If } \alpha=\beta, \text{ then } B_1=a \text{ and } B_2=\frac{b-a\alpha}{\alpha}.$$

We also note that Theorem 2.1 can be proved by induction on n.

Corollary 2.1: For arbitrary a, b, p, q, and all integers n, $w_n = bu_n - qau_{n-1}$.

Proof: In Theorem 2.1, replace m by 0, and replace n by $(n-1)$. □

3. THE MAIN RESULT

In this section we assume that $w_n = w_n(a, b; p, q)$ is defined by (1.5), and we assume that $v_n = w_n(c, d; p, q)$ for arbitrary c and d. That is,

$$v_0 = c, \;\; v_1 = d, \;\; \text{and} \;\; v_n = pv_{n-1} - qv_{n-2}. \tag{3.1}$$

Theorem 3.1: For arbitrary a, b, c, d, p, q, and for all integers m, n, k,

$$v_{m+k}w_{n+k} - q^k v_m w_n = u_k(bv_{m+n+k} - qav_{m+n+k-1})$$

where v_j, w_j, and u_j are defined by (3.1), (1.5), and (1.6), respectively.

Proof: We first show the theorem is true for all integers $k \geq 0$ by using induction on k. The case $k = 0$ is trivial; if $k = 1$, then by the corollary to Theorem 2.1,

$$\begin{aligned} v_{m+1}w_{n+1} - qv_m w_n &= v_{m+1}(bu_{n+1} - qau_n) - qv_m(bu_n - qau_{n-1}) \\ &= b(v_{m+1}u_{n+1} - qv_m u_n) - qa(v_{m+1}u_n - qv_m u_{n-1}) \\ &= bv_{m+n+1} - qav_{m+n}, \end{aligned}$$

with the last equality following from Theorem 2.1. Since $u_1 = 1$, we see that Theorem 3.1 is true for $k = 1$. Assume Theorem 3.1 is true for $k = 0, 1, \ldots, j$. Then

$$\begin{aligned} v_{m+j+1}w_{n+j+1} - q^{j+1}v_m w_n &= (v_{m+j+1}w_{n+j+1} - qv_{m+j}w_{n+j}) + (qv_{m+j}w_{n+j} - q^{j+1}v_m w_n) \\ &= (bv_{m+n+2j+1} - qav_{m+n+2j}) + qu_j(bv_{m+n+j} - qav_{m+n+j-1}) \\ &= b(v_{m+n+2j+1} + qu_j v_{m+n+j}) - qa(v_{m+n+2j} + qu_j v_{m+n+j-1}). \end{aligned} \tag{3.2}$$

Now in Theorem 2.1, if we first replace n by j and then replace m by $m+n+j$, we have

$$v_{m+n+2j+1} + qv_{m+n+j}u_j = v_{m+n+j+1}u_{j+1}, \tag{3.3}$$

and if we first replace n by j and then replace m by $m+n+j-1$, we have

$$v_{m+n+2j+1} + qv_{m+n+j-1}u_j = v_{m+n+j}u_{j+1}. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we have

$$v_{m+j+1}w_{n+j+1} - q^{j+1}v_m w_n = u_{j+1}(bv_{m+n+j+1} - qav_{m+n+j}),$$

and Theorem 3.1 is valid for $k = j+1$. By induction, Theorem 3.1 is valid for all $k \geq 0$ and all integers m and n.

We now want to show Theorem 3.1 is valid for all integers k. Clearly $u_{-1} = -q^{-1}$, and it is easy to prove by induction that $u_{-k} = -q^{-k}u_k$ for all integers k. In Theorem 3.1, replace m by $m-k$ and replace n by $n-k$ to get

$$v_m w_n - q^k v_{m-k}w_{n-k} = u_k(bv_{m+n-k} - qav_{m+n-k-1}),$$

so that

$$v_{m-k}w_{n-k} - q^{-k}v_m w_n = u_{-k}(bv_{m+n-k} - qav_{m+n-k-1})$$

and we see that Theorem 3.1 is valid for all integers k. This completes the proof. □

Corollary 3.1: For arbitrary a, b, c, d, p, q, and for all integers n and j,

$$v_n w_n - q^{n-j} v_j w_j = u_{n-j}(b v_{n+j} - qa v_{n+j-1}),$$

where v_k, w_k, and u_k are defined by (3.1), (1.5), and (1.6), respectively.

Proof: First rewrite Theorem 3.1 by replacing both m and n by j. In the resulting equation, replace k by $(n-j)$ to obtain Corollary 3.1. □

Corollary 3.2: For all integers n and j,

$$L_n F_n + (-1)^{n+j+1} L_j F_j = L_{n+j} F_{n-j}, \tag{3.5}$$

$$L_n F_n + (-1)^{n+j} L_j F_j = L_{n-j} F_{n+j}. \tag{3.6}$$

Proof: Equation (3.5) follows from Corollary 3.1, when $v_n = L_n$ and $w_n = F_n$. Formula (3.6) follows from (3.5): replace j by $-j$, and use $L_{-j} = (-1)^j L_j$, $F_{-j} = (-1)^{j+1} F_j$. □

Corollary 3.3: For arbitrary a, b, p, q, and for all integers n and j,

$$w_n^2 - q^{n-j} w_j^2 = u_{n-j}(b w_{n+j} - qa w_{n+j-1}),$$

where w_k and u_k are defined by (1.5) and (1.6), respectively.

Proof: In Corollary 3.1, let $v_k = w_k$ for all integers k. □

Corollary 3.4: For all integers n and j,

$$F_n^2 + (-1)^{n+j-1} F_j^2 = F_{n-j} F_{n+j},$$

$$L_n^2 + (-1)^{n+j-1} L_j^2 = 5 F_{n-j} F_{n+j}.$$

In the final corollary, which follows directly from Theorem 3.1, we let $G_n = w_n(c, d; 1, -1)$, with c and d arbitrary. That is

$$G_0 = c,\ G_1 = d, \text{ and } G_n = G_{n-1} + G_{n-2} \tag{3.7}$$

for all n. For example, $G_n = F_n$ if $c = 0$, $d = 1$, and $G_n = L_n$ if $c = 2$, $d = 1$.

Corollary 3.5: For all integers m, n, and k,

$$G_{m+k} F_{n+k} + (-1)^{k+1} G_m F_n = F_k G_{m+n+k},$$

where G_n is defined by (3.7) for all n.

4. EQUIVALENCE OF (1.2) AND (1.4)

The following theorem generalizes Melham's results (1.2) and (1.3), and it proves that (1.2) and (1.4) are equivalent.

Theorem 4.1: For arbitrary a, b, p, q, and for all integers n and k,

$$w_{n+k+1}^2 - q^{2k+1} w_{n-k}^2 = u_{2k+1}(b w_{2n+1} - qa w_{2n}), \tag{4.1}$$

$$w_{n+k}^2 - q^{2k} w_{n-k}^2 = u_{2k}(b w_{2n} - qa w_{2n-1}), \tag{4.2}$$

where w_j and u_j are defined by (1.5) and (1.6), respectively; also, (4.1) and (4.2) are equivalent.

Proof: It is clear that (4.1) and (4.2) together are equivalent to Corollary 3.3, so (4.1) and (4.2) are valid formulas. To see that (4.1) and (4.2) are equivalent to each other, we first assume that (4.1) holds for all integers n and k. From Corollary 3.3, we have

$$qw_{n+k}^2 = w_{n+k+1}^2 - bw_{2n+2k+1} + qaw_{2n+2k}. \tag{4.3}$$

Subtracting $q^{2k+1}w_{n-k}^2$ from both sides of (4.3) yields

$$\begin{aligned} q(w_{n+k}^2 - q^{2k}w_{n-k}^2) &= (w_{n+k+1}^2 - q^{2k+1}w_{n-k}^2) - bw_{2n+2k+1} + qaw_{2n+2k} \\ &= u_{2k+1}(bw_{2n+1} - qaw_{2n}) - bw_{2n+2k+1} + qaw_{2n+2k} \\ &= -b(w_{2n+2k+1} - u_{2k+1}w_{2n+1}) + qa(w_{2n+2k} - u_{2k+1}w_{2n}) \\ &= qbu_{2k}w_{2n} - q^2au_{2k}w_{2n-1}, \end{aligned}$$

with the last equality following from Theorem 2.1. Thus, since $q \neq 0$,

$$w_{n+k}^2 - q^{2k}w_{n-k}^2 = u_{2k}(bw_{2n} - qaw_{2n-1}),$$

and (4.1) implies (4.2). The proof that (4.2) implies (4.1) is entirely similar. □

ACKNOWLEDGMENT

The author thanks Calvin Long, Ray Melham, and an anonymous referee for their interest in this paper, and for their suggestions.

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AMS Classification Number: 11B39

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2003. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-951 ***Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA***

The sequence $\langle u_n \rangle$ is defined by the recurrence

$$u_{n+1} = \frac{3u_n + 1}{5u_n + 3}$$

with the initial condition $u_1 = 1$. Express u_n in terms of Fibonacci and/or Lucas numbers.

B-952 ***Proposed by Scott H. Brown, Auburn University, Montgomery, AL***

Show that

$$10F_{10n-5} + 12F_{10n-10} + F_{10n-15} = 25F_{2n}^5 + 25F_{2n}^3 + 5F_{2n}$$

for all integers $n \geq 2$.

B-953 ***Proposed by Harvey J. Hindin, Huntington Station, NY***

Show that

$$(F_n)^4 + (F_{n+1})^4 + (F_{n+2})^4$$

is never a perfect square. Similarly, show that

$$(qW_n)^4 + (pW_{n+1})^4 + (W_{n+2})^4$$

is never a perfect square, where W_n is defined for all integers n by $W_n = pW_{n-1} - qW_{n-2}$ and where $W_0 = a$ and $W_1 = b$.

B-954 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Let n be a nonnegative integer. Show that

$$\sqrt{(\sqrt{5}+2)(\sqrt{5}F_{2n+1}-2)} = L_{2\lfloor n/2\rfloor+1} + \sqrt{5}F_{2\lceil n/2\rceil},$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor- and ceiling-function, respectively.

B-955 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI***

Prove that

$$1 < \frac{F_{2n}}{\sqrt{1+F_{2n}^2}} + \frac{1}{\sqrt{1+F_{2n+1}^2}} + \frac{1}{\sqrt{1+F_{2n+2}^2}} < \frac{3}{2}$$

for all integers $n \geq 0$.

SOLUTIONS

A Fibonacci Sine

B-935 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (Vol. 40, no. 2, May 2002)***

Prove that

$$8\sin\left(\frac{F_3}{2}\right)\sin\left(\frac{F_9}{2}\right)\sin\left(\frac{F_{12}}{2}\right) < 1,$$

where the arguments are measured in degrees.

Solution by Walther Janous, Innsbruck, Austria

In what follows, we shall prove a stronger inequality. We start from the familiar inequality

$$\frac{\sin(x)}{x} < 1$$

valid for all $x \neq 0$.

We observe that, for α measured in degrees, there holds

$$\sin(\alpha) = \sin\left(\frac{\alpha \cdot \pi}{180}\right).$$

Therefore,

$$\sin\left(\frac{F_3}{2}\right)\cdot\sin\left(\frac{F_9}{2}\right)\cdot\sin\left(\frac{F_{12}}{2}\right) < \frac{F_3}{2}\cdot\frac{F_9}{2}\cdot\frac{F_{12}}{2}\cdot\left(\frac{\pi}{180}\right)^3,$$

that is

$$\sin\left(\frac{F_3}{2}\right)\cdot\sin\left(\frac{F_9}{2}\right)\cdot\sin\left(\frac{F_{12}}{2}\right) < \frac{17\pi^3}{81000},$$

whence, finally,

$$8\cdot\sin\left(\frac{F_3}{2}\right)\cdot\sin\left(\frac{F_9}{2}\right)\cdot\sin\left(\frac{F_{12}}{2}\right)<\frac{17\pi^3}{10125}=0.05205992133$$

and the proof is complete.

Several solvers proved that $8\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\le 1$, *where* $x+y+z=\pi$ *with equality occurring when* $x=y=z=\frac{\pi}{3}$.

Also solved by Paul Bruckman, Scott Brown, Haci Civciv, José Luis Diáz-Barrero & Juan José Egozcue (jointly), M. Deshpande, L. A. G. Dresel, Douglas Iannucci, John Jaroma, H.-J. Seiffert, and the proposer.

Exclusive Roots

B-936 ***Proposed by José Luis Diáz & Juan José Egozcue, Terrassa, Spain (Vol. 40, no. 2, May 2002)***

Let n be a nonnegative integer. Show that the equation

$$x^5+F_{2n}x^4+2(F_{2n}-2F_{n+1}^2)x^3+2F_{2n}(F_{2n}-2F_{n+1}^2)x^2+F_{2n}^2x+F_{2n}^3=0$$

has only integer roots.

Solution by Maitland A. Rose, University of South Carolina, Sumter, SC

Use is made of the identities $F_{2n}=F_nL_n$ and $F_n+L_n=2F_{n+1}$, which leads to

$$F_n^2+L_n^2=-2(F_{2n}-2F_{n+1}^2).$$

We note that

$$\begin{aligned}&(x-F_n)(x+F_n)(x-L_n)(x+L_n)(x+F_{2n})\\&=(x^2-F_n^2)(x^2-L_n^2)(x+F_{2n})\\&=(x^4-x^2(F_n^2+L_n^2)+F_n^2L_n^2)(x+F_{2n})\\&=x^5+F_{2n}x^4-(F_n^2+L_n^2)x^3-F_{2n}(F_n^2+L_n^2)x^2+F_n^2L_n^2x+F_{2n}F_n^2L_n^2\\&=x^5+F_{2n}x^4+2(F_{2n}-2F_{n+1}^2)x^3+2F_{2n}(F_{2n}-2F_{n+1}^2)x^2+F_{2n}^2x+F_{2n}^3.\end{aligned}$$

The roots of the given equation are the integers $\pm F_n$, $\pm L_n$, and $-F_{2n}$.

Pentti Haukkanen and Walther Janous used Mathematica *and* Derive, *respectively, to do the calculations and found the same roots.*

Also solved by Paul Bruckman, Charles Cook, Haci Civciv & Naim Tuglu (jointly), M. N. Deshpande, L. A. G. Dresel, Steve Edwards, Ovidiu Furdui, Pentti Haukkanen, Walther Janous, Harris Kwong, Don Redmond, Jaroslav Seibert, H.-J. Seiffert, James Sellers, and the proposers.

Some Identities

B-937 ***Proposed by Paul Bruckman, Sacramento, CA (Vol. 40, no. 2, May 2002)***

Prove the following identities:

(a) $(F_n)^2+(F_{n+1})^2+4(F_{n+2})^2=(F_{n+3})^2+(L_{n+1})^2$;

(b) $(L_n)^2+(L_{n+1})^2+4(L_{n+2})^2=(L_{n+3})^2+(5F_{n+1})^2$.

Solution by Jaroslav Seibert, University of Hradec Králové, The Czech Republic

We will prove the more general identity,

$$G_n^2 + G_{n+1}^2 + 4G_{n+2}^2 = G_{n+3}^2 + (G_n + G_{n+2})^2,$$

where $\{G_n\}_{n=1}^{\infty}$ is an arbitrary sequence satisfying the recurrence $G_{n+2} = G_{n+1} + G_n$.

The more general identity may be written as

$$\begin{aligned}
&G_n^2 + G_{n+1}^2 + 4G_{n+2}^2 - G_{n+3}^2 - (G_n + G_{n+2})^2 \\
&= G_n^2 + G_{n+1}^2 + 4(G_n + G_{n+1})^2 - (G_n + 2G_{n+1})^2 - (2G_n + G_{n+1})^2 \\
&= G_n^2 + G_{n+1}^2 + 4G_n^2 + 8G_nG_{n+1} + 4G_{n+1}^2 - G_n^2 - 4G_nG_{n+1} - 4G_{n+1}^2 - 4G_n^2 - 4G_nG_{n+1} - G_{n+1}^2 = 0.
\end{aligned}$$

If we put $G_n = F_n$, then $F_n + F_{n+2} = L_{n+1}$ and we obtain (a).

If we put $G_n = L_n$, then $L_n + L_{n+2} = 5F_{n+1}$ and we obtain (b).

Also solved by Scott Brown, Mario Catalani, Haci Civciv, Charles Cook, Kenneth Davenport, M. N. Deshpande, José Luis Diáz-Barrero & Juan José Egozcue (jointly), L. A. G. Dresel, Steve Edwards, Ovidiu Furdui (two solutions), Pentti Haukkanen, Walther Janous, Muneer Jebreel, Harris Kwong, William Moser, Maitland Rose, H.-J. Seiffert, James Sellers, and the proposer.

Series Problem

B-938 *Proposed by Charles K. Cook, University of South Carolina at Sumpter, Sumpter, SC (Vol. 40, no. 2, May 2002)*

Find the smallest positive integer k for which the given series converges and find its sum

(a) $\sum_{n=1}^{\infty} \frac{nF_n}{k^n}$;

(b) $\sum_{n=1}^{\infty} \frac{nL_n}{k^n}$.

Solution by Don Redmond, Southern Illinois University at Carbondale, Carbondale, IL

Let G_n denote either F_n or L_n. If α, β represent the solutions to the quadratic $x^2 - x - 1 = 0$, as usual, then for appropriate values of c and d we have $G_n = c\alpha^n + d\beta^n$. Then, if the series converges, we have

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{nG_n}{k^n} &= \sum_{n=1}^{+\infty} \frac{n(c\alpha^n + d\beta^n)}{k^n} = c\sum_{n=1}^{+\infty} \frac{n\alpha^n}{k^n} + d\sum_{n=1}^{+\infty} \frac{n\beta^n}{k^n} \\
&= c\frac{(\alpha/k)}{(1-\alpha/k)^2} + d\frac{(\beta/k)}{(1-\beta/k)^2} = c\frac{k\alpha}{(k-\alpha)^2} + d\frac{k\beta}{(k-\beta)^2}.
\end{aligned}$$

Since these are geometric series, we see that they converge if $\max(|\alpha/k|, |\beta/k|) < 1$, which gives that the least integer k that yields convergence is $k = 2$.

From the definition of α and β, we see that $2 - \alpha = \beta^2$ and $2 - \beta = \alpha^2$. Thus,

$$\sum_{n=1}^{+\infty} \frac{nG_n}{2^n} = c\frac{2\alpha}{\beta^4} + d\frac{2\beta}{\alpha^4} = 2\frac{c\alpha^5 + d\beta^5}{(\alpha\beta)^4} = 2G_5.$$

Thus, the answer to (a) is $2F_5 = 10$ and the answer to (b) is $2L_5 = 22$. $\square$

Problem B-670 also considers these sums.

Also solved by Paul Bruckman , Mario Catalani, Haci Civciv & Naim Tuglu (jointly), Kenneth Davenport, M. N. Deshpande, José Luis Diáz-Barrero & Juan José Egozcue (jointly), L. A. G. Dresel, Steve Edwards, Ovidiu Furdui, Douglas Iannucci, Walther Janous, John Jaroma, Harris Kwong, Kathleen Lewis, Jaroslav Seibert, H.-J. Seiffert, James Sellers, and the proposer.

Identities Problem

B-939 *Proposed by N. Gauthier, Royal Military College of Canada (Vol. 40, no. 2, May 2002)*

For $n \geq 0$ and s arbitrary integers, with

$$f(l,m;n) \equiv f(l,m) = (-1)^{n-l}\binom{n}{l}\binom{n}{m},$$

prove the following identities:

(a) $2^n F_{n+s} = \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m,m) F_{l+s}$;

(b) $3 \cdot 2^{n-1} n F_{n+s+2} = \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m,m)[(l-2m)F_{l+s} + mF_{l+s-1}]$.

Solution by H.-J. Seiffert, Berlin, Germany

For $(x,y) \in R^2$, let

$$S_n(x,y) = \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m,m) x^l y^m$$
$$= \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} (-1)^{n-1+m} \binom{n}{l-3m}\binom{n}{m} x^l y^m.$$

Changing the summations and reindexing gives

$$S_n(x,y) = \sum_{m=0}^{n} \sum_{l=3m}^{n+3m} (-1)^{n-1+m} \binom{n}{l-3m}\binom{n}{m} x^l y^m$$
$$= \sum_{m=0}^{n} \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k}\binom{n}{m} x^{k+3m} y^m,$$

which turns out to be the product of two sums. By the binomial theorem,

$$S_n(x,y) = (1+x^3y)^n(x-1)^n. \tag{1}$$

Proof of (a): From the known Binet form of the Fibonacci numbers, we see that the right-hand side of the desired identity equals

$$A_n = \frac{1}{\sqrt{5}}(S_n(\alpha,1)\alpha^s - S_n(\beta,1)\beta^s).$$

Since $1+\alpha^3 = 2\alpha^2$, $1+\beta^3 = 2\beta^2$, $\alpha - 1 = -\beta$, $\beta - 1 = -\alpha$, and $-\alpha\beta = 1$, by (1), we have

$$A_n = 2^n F_{n+s}.$$

Proof of (b): Let

$$B_n := \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m, m)(l-3m)F_{l+s}$$

and

$$C_n := \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m, m) m F_{l+s+1}.$$

Consider the function

$$T_n(x, y) := S_n(x, x^{-3}y) = (1+y)^n(x-1)^n.$$

If

$$U_n(x, y) := \frac{\partial T_n(x, y)}{\partial x} = n(1+y)^n(x-1)^{n-1} \tag{2}$$

and

$$V_n(x, y) := \frac{\partial T_n(x, y)}{\partial y} = n(1+y)^{n-1}(x-1)^n \tag{3}$$

denote the partial derivatives of T_n, then, by the definition of S_n,

$$B_n = \frac{1}{\sqrt{5}}(U_n(\alpha, \alpha^3)\alpha^{s+1} - U_n(\beta, \beta^3)\beta^{s+1}),$$

or, by (2), $B_n = 2^n n F_{n+s+2}$. Similarly, from the definition of S_n and (3), one finds

$$C_n = \frac{1}{\sqrt{5}}(V_n(\alpha, \alpha^3)\alpha^{s+4} - V_n(\beta, \beta^3)\beta^{s+4}),$$

or $C_n = 2^{n-1} n F_{n+s+2}$. It follows that $B_n + C_n = 3 \cdot 2^{n-1} n F_{n+s+2}$, which proves the requested identity, because

$$(l-3m)F_{l+s} + mF_{l+s+1} = (l-2m)F_{l+s} + mF_{l+s-1}.$$

Also solved by Paul Bruckman and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

Notice: *Dr. Florian Luca is the new Advanced Problems Editor. He requests that all proposals and solutions be e-mailed to fluca@matmor.unam.mx. Please use one of the following file types: tex, dvi, ps, doc, html, pdf, etc.*

PROBLEMS PROPOSED IN THIS ISSUE

H-593 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Let $p > 5$ be a prime. Prove the congruence

$$2\sum_{k=0}^{[(p-5)/10]} \frac{(-1)^k}{2k+1} \equiv (-1)^{(p-1)/2}\frac{2^{p-1}-L_p}{p} \pmod{p}.$$

H-594 ***Proposed by Mario Catalani, University of Torino, Torino, Italy***

Consider the generalized Fibonacci and Lucas polynomials:

$$F_{n+1}(x,y) = xF_n(x,y) + yF_{n-1}(x,y),\ F_0(x,y)=0,\ F_1(x,y)=1;$$
$$L_{n+1}(x,y) = xL_n(x,y) + yL_{n-1}(x,y),\ L_0(x,y)=2,\ L_1(x,y)=x.$$

Assume $y \neq 0,\ 2x^2 - y \neq 0$. We will write F_n and L_n for $F_n(x,y)$ and $L_n(x,y)$, respectively. Show that:

1. $$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^k y^{-2k} F_{3k} = \frac{xF_{2n+1} - yF_{2n} + (-x)^{n+2}F_n + (-x)^{n+1}yF_{n-1}}{y^n(2x^2-y)};$$

2. $$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^k y^{-2k} L_{3k} = \frac{xL_{2n+1} - yL_{2n} + (-x)^{n+2}L_n + (-x)^{n+1}yL_{n-1}}{y^n(2x^2-y)}.$$

H-595 ***Proposed by José Díaz-Barrero & Juan Egozcue, Barcelona, Spain***

Let ℓ, n be positive integers. Prove that

$$\sum_{k=0}^{n} \binom{k+\ell+1}{k+1} \left\{ \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} P_n^{j-k-1} \right\} = P_n^{\ell+1} - 1,$$

where P_n is the n^{th} Pell number, i.e., $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 2$.

SOLUTIONS

A Convoluted Problem

H-583 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 40, no. 2, May 2002)***

A Theorem on Generalized Fibonacci Convolutions

This is a generalization of Problem B-858 by W. Lang (*The Fibonacci Quarterly* **36.3**, 1998).

Let $n \geq 0$, a, b be integers; also let A, B be arbitrary yet known real numbers and consider the generalized Fibonacci sequence $\{G_n \equiv A\alpha^n + B\beta^n\}_{n=-\infty}^{\infty}$, where

$$\alpha = \frac{1}{2}[1+\sqrt{5}],\ \beta = \frac{1}{2}[1-\sqrt{5}].$$

For m a nonnegative integer, prove the following generalized convolution theorem for the sequences $\{(a+n)^m\}_{n=-\infty}^{\infty}$ and $\{G_n\}_{n=-\infty}^{\infty}$,

$$\sum_{k=0}^{n}(a+k)^m G_{b-a-k} = \sum_{l=0}^{m} l![c_l^m(a)G_{b-a+l+1} - c_l^m(a+n+1)G_{b-a-n+1+l}],$$

where the set of coefficients $\{c_l^m(v);\ 0 \leq m;\ 0 \leq l \leq m;\ v = a \text{ or } a+n+1\}$ satisfies the following second-order linear recurrence relation

$$c_l^{m+1}(v) = (v+l)c_l^m(v) + c_{l-1}^m(v);\ c_{l=0}^{m=0}(v) = 1, c_{l=0}^{m=1}(v) = v, c_{l=1}^{m=1}(v) = 1$$

with the understanding that $c_{-1}^m(v) \equiv 0$ and that $c_{m+1}^m(v) \equiv 0$.

Prob. B-858 follows as a special case if one sets $a=0$, $m=1$, $b=n$, and $A=-B=(\alpha-\beta)^{-1}$ in the above theorem. Indeed, one then gets that

$$G_n = F_n,\ c_0^1(0) = 0,\ c_1^1(0) = 1,\ c_0^1(n+1), \text{ and } c_1^1(n+1) = 1$$

and the result follows directly.

Solution by Paul S. Bruckman, Berkeley, CA

For typographical clarity, we change the summation variable "l" to "j" and we also change the notation "$c_j^m(x)$" to "$c(x; j, m)$". We also note that there is a misprint in the statement of the problem. The correct expression in the right member of the statement of the problem (as modified by the indicated changes in notation) is as follows:

$$\sum_{j=0}^{m} j![c(\alpha; j, m)\,G_{b-a+j+2} - c(\alpha+n+1; j, m)\,G_{b-a-n+j+1}].$$

We employ the standard finite difference operators Δ and $E \equiv 1+\Delta$, where the operand is x. We first demonstrate the following result:

$$c(x; j, m) = \Delta^j / j! = (x^m). \tag{1}$$

Proof of (1): Let $d(x; j, m) = \Delta^j / j!(x^m)$, $0 \leq j \leq m$ for all real x. Clearly, $d(x; j, m)$ is a polynomial in x. Note that $d(x; 0, m) = x^m$. Also, $d(x; m, m) = 1$ for all m and x, and $d(x; 0, 1) = x$. Thus, the boundary conditions satisfied by the $c(x; j, m)$ are also satisfied by the $d(x; j, m)$.

Next, note that

$$\begin{aligned}
d(x;j,m+1) &= \Delta^j/j!(x^{m+1}) = (1/j!)\sum_{k=0}^{j} {}_jC_k(-1)^k(x+j-k)^{m+1} \\
&= \{(x+j)/j!\}\sum_{k=0}^{j} {}_jC_k(-1)^k(x+j-k)^m - \{j/j!\}\sum_{k=1}^{j} {}_{j-1}C_{k-1}(-1)^k(x+j-k)^m \\
&= \{(x+j)/j!\}\sum_{k=0}^{j} {}_jC_k(-1)^k(x+j-k)^m + \{1/(j-1)!\}\sum_{k=0}^{j-1} {}_{j-1}C_k(-1)^k(x+j-1-k)^m \\
&= (x+j)\{\Delta^j/j!\}(x^m) + \{\Delta^{j-1}/(j-1)!\}(x^m) = (x+j)d(x;j,m) + d(x;j-1,m).
\end{aligned}$$

This is the same recurrence as the one satisfied by the $c(x;j,m)$. Since the two-dimensional sequences $c(x;j,m)$ and $d(x;j,m)$ satisfy the same recurrence and have the same boundary conditions, they must be identical. This establishes (1). □

Therefore, the left member of the putative identity (denoted as $\mathfrak{L}$) is transformed as follows:

$$\begin{aligned}
\mathfrak{L} &= \sum_{k=0}^{n} E^k(x^m)G_{b-a-k}\Big|_{x=\alpha} = \sum_{k=0}^{n} E^k(x^m)\{A\alpha^{b-a-k} + B\beta^{b-a-k}\}\Big|_{x=\alpha} \\
&= [A\alpha^{b-a}\{(E/\alpha)^{n+1}-1\}/(E/\alpha-1)\} + B\beta^{b-a}\{(E/\beta)^{n+1}-1\}/(E/\beta-1)\}](x^m)\Big|_{x=\alpha}
\end{aligned}$$

or

$$\mathfrak{L} = [A\alpha^{b-a-n}\{E^{n+1}-\alpha^{n+1})/(E-\alpha)\} + B\beta^{b-a-n}\{(E^{n+1}-\beta^{n+1})/(E-\beta)\}](x^m)\Big|_{x=\alpha}. \tag{2}$$

On the other hand, if $\mathfrak{R}$ represents the right member of the (corrected) putative identity, then

$$\mathfrak{R} = \sum_{k=0}^{m}\Delta^k(x^m)\{A\alpha^{b-a+2+k} + B\beta^{b-a+2+k}\}\Big|_{x=\alpha} - \sum_{k=0}^{m}\Delta^k(x^m)\{A\alpha^{b-a+1-n+k} + B\beta^{b-a+1-n+k}\}\Big|_{x=\alpha+n+1}$$

$$\begin{aligned}
\mathfrak{R} &= [A\alpha^{b-a+2}\{(\Delta\alpha)^{m+1}-1\}/(\Delta\alpha-1) + B\beta^{b-a+2}\{(\Delta\beta)^{m+1}-1\}/(\Delta\beta-1)\}](x^m)\Big|_{x=\alpha} \\
&\quad - [A\alpha^{b-a+1-n}\{(\Delta\alpha)^{m+1}-1\}/(\Delta\alpha-1) + B\beta^{b-a+1-n}\{(\Delta\beta)^{m+1}-1\}/(\Delta\beta-1)](x^m)\Big|_{x=\alpha+n+1}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{R} &= [A\alpha^{b-a+1-n}\{(\Delta\alpha)^{m+1}-1\}/(\Delta\alpha-1)(\alpha^{n+1}-E^{n+1}) \\
&\quad + B\beta^{b-a+1-n}\{(\Delta\beta)^{m+1}-1\}/(\Delta\beta-1)(\beta^{n+1}-E^{n+1})](x^m)\Big|_{x=\alpha}.
\end{aligned}$$

Now note that $\Delta^{m+1}(x^m)=0$. Also, $\Delta\alpha-1=(E-1)\alpha-1=E\alpha-\alpha^2$, and $\Delta\beta-1=(E-1)\beta-1=E\beta-\beta^2$. Therefore, we see that

$$\mathfrak{R} = A\alpha^{b-a-n}\{(E^{n+1}-\alpha^{n+1})/(E-\alpha)\} + B\beta^{b-a-n}\{(E^{n+1}-\beta^{n+1})/(E-\beta)\}(x^m)]\Big|_{x=\alpha}. \tag{3}$$

Comparison of (2) and (3) shows that $\mathfrak{L}=\mathfrak{R}$. Q.E.D.

Also solved by the proposer.

Find Your Identity

H-584 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 40, no. 2, May 2002)

Prove the following identity:

$$\begin{aligned}
&(F_{n+4}+L_{n+3})^5 + (F_n+L_{n+1})^5 + (2F_{n+1}+L_{n+2})^5 \\
&= (2F_{n+3}+L_{n+2})^5 + (F_{n+2})^5 + (5F_{n+2})^5 + 1920F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}.
\end{aligned}$$

Solution by the proposer

We begin with the following identity:

$$\begin{aligned}&(a+b+c+d-e)^5+(a+b+c-d+e)^5+(a+b-c+d+e)^5+(a-b+c+d+e)^5\\&+(-a+b+c-d+e)^5+(a+b+c-d-e)^5+(a+b-c+d-e)^5+(a-b+c+d-e)^5\\&+(-a+b+c+d-e)^5+(a+b-c-d+e)^5+(a-b+c-d+e)^5+(-a+b+c-d+e)^5 \quad (*)\\&+(a-b-c+d+e)^5+(-a-b-c+d+e)^5+(-a+b+c+d+e)^5+(a+b+c+d+e)^5\\&=1920abcde.\end{aligned}$$

We replace a, b, c, d, and e by x_1, x_2, x_3, x_4, and x_5, respectively. We may prove $(*)$ (as thus modified) by expanding

$$(x_1+x_2+x_3+x_4+x_5)^5=\sigma_5+5\sigma_{14}+10\sigma_{23}+20\sigma_{113}+30\sigma_{122}+60\sigma_{1112}+120\sigma_{11111},$$

where $\sigma_{abc}=\sum u^a v^b w^c$, for example (with u, v, and w representing the x_i's over all possible permutations), with similar definitions for other quantities. Then we note that in the sum of the 16 terms indicated in $(*)$, the terms involving $(x_1)^5$ vanish, since their coefficient is $+1$ 8 times and -1 8 times. The terms involving $(x_1)^4$ also vanish, since their coefficients are x_i twice and $-x_i$ twice (for each $i=2,3,4,$ or 5). The terms involving $(x_1)^3$ have two kinds of coefficients: $(x_i)^2$ and $-(x_i)^2$; also, x_ix_j and $-x_ix_j$, where i or $j=2,3,4,$ or 5, $i\neq j$. In either case, each sign occurs an equal number of times, and so the term vanishes. The *remaining* terms involving $(x_1)^2$ have two kinds of coefficients: $x_i(x_j)^2$ and $-x_i(x_j)^2$; also $x_ix_jx_k$ and $-x_ix_jx_k$. Here, $i, j,$ or $k=2,3,4,$ or 5, with $i, j,$ and k distinct. In either case, the positive terms again cancel the negative ones, and so the terms involving $(x_1)^2$ all vanish. Finally, the *remaining* terms involving the first powers x_1 have coefficients $x_2x_3x_4x_5$ or $-x_2x_3x_4x_5$ for each of the 16 terms, but are such that the total term is always positive. Therefore, the total coefficient of the product $x_1x_2x_3x_4x_5$ is $16*120=1920$. By symmetry, the sum is therefore equal to $1920x_1x_2x_3x_4x_5$. Thus, $(*)$ is proved.

In particular, if we set $a=F_n$, $b=F_{n+1}$, $c=F_{n+2}$, $d=F_{n+3}$, and $e=F_{n+4}$, we obtain (after some simplification) the indicated result.

Also solved by K. Davenport, L. A. G. Dresel, O. Furdui, N. Tuglu, and H. Civciv.

A *D*-Sequence

H-585 ***Proposed by Herrmann Ernst, Siegburg, Germany (Vol. 40, no. 4, August 2002)***

Let (d_n) denote a sequence of positive integers d_n with $d_1\geq 3$ and $d_{n+1}-d_n\geq 1$, $n=1,2,\ldots$. We introduce the following sets of sequences (d_n):

$$A=\left\{(d_n)\colon\ \sum_{k=1}^{\infty}\frac{1}{F_{d_k}}\leq 1\right\};$$

$$B=\left\{(d_n)\colon\ \frac{1}{F_{d_n}}<\sum_{k=n}^{\infty}\frac{1}{F_{d_k}}<\frac{1}{F_{d_n-1}}\text{ for all }n\in N\right\};$$

$$C=\left\{(d_n)\colon\ 0\leq\frac{1}{F_{d_n-1}}-\frac{1}{F_{d_n}}-\frac{1}{F_{d_{n+1}-1}}\text{ for all }n\in N\right\}.$$

Show that:

(a) there is a bijection $f:]0, 1] \to B$, $f(x) = (d_n(x))_{n=1}^{\infty}$;

(b) B is a subset of A with $A \setminus B \neq \emptyset$;

(c) C is a subset of B with $B \setminus C \neq \emptyset$.

Solution by Paul S. Bruckman, Berkeley, CA

A sequence $(d_n)_{n=1}^{\infty}$ of positive integers is called a <u>D-sequence</u> iff $d_1 \geq 3$ and $d_{n+1} - d_n \geq 1$, $n \in N$. Let Δ denote the set of all D-sequences. Also, for typographical convenience, we write $F(k)$ for F_k. We also write $S_{n,M} = \sum_{k=n}^{M} 1/F(d_k)$, and $S_n = S_{n,\infty}$ for all $(d_n) \in \Delta$. For a given $\delta = (d_n) \in \Delta$, we may also write $S_n(\delta) = S_n$. We may characterize A, B, and C as follows:

$A = \{\delta = (d_n) \in \Delta : S_1(\delta) \geq 1\}$;

$B = \{\delta = (d_n) \in \Delta : 1/F(d_n) < S_n(\delta) \leq 1/F(d_n - 1) \text{ for all } n \in N\}$;

$C = \{\delta = (d_n) \in \Delta : 0 \leq 1/F(d_n - 1) - 1/F(d_n) - 1/F(d_{n+1} - 1) \text{ for all } n \in N\}$.

Note the slight modification in the definition of B ("$\leq$" instead of "$<$" in the second inequality defining B.

Proof of (a): Suppose $x_1 \in (0, 1)$ is given. Then there exists $d_1 \in N$, $d_1 \geq 3$, such that $1/F(d_1) < x_1 \leq 1/F(d_1 - 1)$. Let $x_2 = x_1 - 1/F(d_1)$. Note that $0 < x_2 < x_1 < 1$. We continue in this fashion; generally, we define the sequence (x_n) as follows: $x_{n+1} = x_n - 1/F(d_n)$, $1/F(d_n) < x_n \leq 1/F(d_n - 1)$, $d_{n+1} > d_n$, $n \in N$. Note that (x_n) is a decreasing sequence, bounded below by zero. Since $d_n \to \infty$ as $n \to \infty$, we see that x_n is arbitrarily small. Hence, $\lim_{n\to\infty} x_n = 0$. By iteration, $x_1 = 1/F(d_1) + x_2 = 1/F(d_1) + 1/F(d_2) + x_3 = \cdots = S_{1,M} + x_{M+1}$ for all $M \in N$. Allowing $M \to \infty$, we deduce that $x_1 = S_1$, where the D-sequence (d_n) is uniquely determined by the construction indicated above. Note, however, that the maximum value of S_1 over the domain Δ is $\sum_{n=3}^{\infty} 1/F_n = \sigma$, say, where $\sigma \approx 1.3599$. In other words, there is *not* a one-to-one correspondence between $(0, 1)$ and Δ, the set of all possible D-sequences. There are sequences $\delta \in \Delta$ such that $S_1(\delta) \geq 1$. We may use the same construction as before, if $1 \leq x_1 \leq \sigma$. For example,

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{34} + \frac{1}{89} + \frac{1}{987} + \frac{1}{196418} + \frac{1}{2178309} + \cdots$$
$$= \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_6} + \frac{1}{F_9} + \frac{1}{F_{11}} + \frac{1}{F_{16}} + \frac{1}{F_{27}} + \frac{1}{F_{32}} + \cdots,$$

corresponding to the D-sequence $\delta_1 = (3, 4, 6, 9, 11, 16, 27, 32, \ldots)$, such that $S_1(\delta_1) = 1$.

We are to establish that if $x_1 \in (0, 1)$ then there exists a unique $\delta \in B$ such that $S_n(\delta) \leq 1/F(d_n - 1)$ for all $n \in N$; the other condition for $\delta \in B$, namely that $1/F(d_n) < S_n(\delta)$ for all $n \in N$, is automatically satisfied. We already know how to effect the construction of the unique $\delta \in \Delta$ such that $x_1 = S_1(\delta)$. It only remains to show that, for such δ, $S_n(\delta) \leq 1/F(d_n - 1)$ for all $n \in N$. Note that $x_1 = S_{1,n-1}(\delta) + x_n = x_1 - S_n(\delta) + x_n$ for all $n \in N$. This implies that $x_n = S_n(\delta)$ for all $n \in N$. By our construction, $x_n = S_n(\delta) \leq 1/F(d_n - 1)$ for all $n \in N$. This completes the proof of part (a).

Note: Although, for a given $x_1 \in (0, 1)$, there exists a unique $\delta \in B$ corresponding to x_1 (as provided by our construction, and such that $S_1(\delta) = x_1$), there may be other $\delta \in \Delta \setminus B$, say δ', such that $S_1(\delta') = x_1$. An illustration of this is provided by $x_1 = \rho = 1/F_4 + 1/F_6 + 1/F_8 + \cdots \approx 0.5354$. Clearly, this is generated by the sequence $\delta' = \delta'(\rho) = (4, 6, 8, \ldots)$, which is an element of Δ. However, it is easily verified that δ' is not an element of B, since $1/F(d_1 - 1) = 1/F(3) = 1/2 < S_1(\delta') = \rho = 0.5354$. Our construction, however, yields the alternative sequence $\delta = \delta(\rho) = (3, 9, 13, 15, 24, 27, 31, 35, 37, 39, 42, 49, \ldots)$, which also has $S_1(\delta) = \rho = 0.5354$ and is, moreover, an element of B (this is true by the nature of the construction).

Proof of (b): Suppose $\delta \in B$. Then $S_1(\delta) \le 1/F(d_1 - 1) \le 1/F(2) = 1$; hence, $\delta \in A$. Thus, $B \subseteq A$.

As we have seen, $\delta' \in \Delta \setminus B$, where $\delta' = (4, 6, 8, \ldots) = (2n)_{n=2}^{\infty}$, but $S_1(\delta') = \rho \approx 0.5354 < 1$, so $\delta_1 \in A$. Hence, $\delta' \in A \setminus B$ and $A \setminus B \ne \varnothing$.

Proof of (c): Suppose $\delta \in C$. Then, for all $n \in N$, $1/F(d_n - 1) \ge 1/F(d_n) + 1/F(d_{n+1} - 1)$. By iteration, $1/F(d_n - 1) \ge 1/F(d_n) + 1/F(d_{n+1}) + \cdots + 1/F(d_M) + 1/F(d_{M+1} - 1)$ for all M, n with $M \ge n \ge 1$. Thus, $1/F(d_n - 1) > S_{n,M}(\delta)$ for all such M, n. Allowing $M \to \infty$, it follows that $1/F(d_n - 1) \ge S_n(\delta)$ for all $n \in N$. Therefore, $\delta \in B$, which shows that $C \subseteq B$.

We display an example of a sequence $\delta'' \in B \setminus C$.

We let $\delta'' = (6, 8, 10, 12, 15, 18, 20, 22, 24, 29, \ldots)$ represent the element of B determined by our construction, such that $S_1(\delta'') = 0.2$. By definition, $\delta'' \in B$. However,

$$\frac{1}{F(d_1 - 1)} - \frac{1}{F(d_1)} - \frac{1}{F(d_2 - 1)} = \frac{1}{F(5)} - \frac{1}{F(6)} - \frac{1}{F(7)} = \frac{1}{5} - \frac{1}{8} - \frac{1}{13} = \frac{-1}{520} < 0,$$

which shows that $\delta'' \notin C$; hence, $\delta'' \in B \setminus C$ and $B \setminus C \ne \varnothing$. This completes the proof of part (c).

Note: More generally, $1/F(2n-1) - 1/F(2n) - 1/F(2n+1) = -1/F(2n-1)F(2n)F(2n+1) < 0$, after simplification. Thus, given $\delta = (d_n) \in \Delta$ with $d_k = 2n$ and $d_{k+1} = 2n+2$, say, then $\delta \notin C$; i.e., if $d_k \in \delta \in C$ and d_k is even, then $d_{k+1} - d_k \ge 3$.

Also solved by the proposer.

Note: Problem **H-582** (proposed by Ernst Herrman and solved by Paul S. Bruckman) will appear in the May 2003 issue of this quarterly.

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