

## TABLE OF CONTENTS

On Rational Approximations by Pythagorean Numbers Carsten Elsner ..... 98
An Involutory Matrix of Eigenvectors

$\qquad$
David Callan and Helmut Prodinger ..... 105
Generalized Fibonacci Functions and Sequences of Generalized Fibonacci Functions Gwang-Yeon Lee, Jin-Soo Kim and Tae Ho Cho ..... 108
The Multiplicative Group Generated by the
Lehmer Numbers Florian Luca and Štefan Porubský ..... 122
Rounding the Solutions of Fibonacci-Like
Difference Equations

$\qquad$
Renato M. Capocelli and Paul Cull ..... 133
A Fibonacci Identity in the Spirit of Simson and Gelin-Cesàro

$\qquad$
R. S. Melham ..... 142
Computational Formulas for Convoluted Generalized
Fibonacci and Lucas Numbers

$\qquad$
Hong Feng and Zhizheng Zhang ..... 144
A Non-Integer Property of Elementary Symmetric Functions in Reciprocals of Generalised Fibonacci Numbers

$\qquad$
M. A. Nyblom ..... 152
The Existence of Special Multipliers of Second-Order
Recurrence Sequences

$\qquad$
Walter Carlip and Lawrence Somer 156
Stern's Diatomic Array Applied toFibonacci Representations
$\qquad$Marjorie Bicknell-Johnson169
Tribute to JoAnn Vine ..... 180
Elementary Problems and Solutions

$\qquad$ Edited by Russ Euler and Jawad Sadek ..... 181
Advanced Problems and Solutions Edited by Florian Luca ..... 187


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#### Abstract

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# ON RATIONAL APPROXIMATIONS BY PYTHAGOREAN NUMBER: 

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## 1. STATEMENT OF THE RESULTS

A famous result of Heilbronn states that for every real irrational $\xi$ and any $\varepsilon>0$ there are infinitely many integers $n$ satisfying

$$
\left\|\xi n^{2}\right\|<\frac{1}{n^{1 / 2-\varepsilon}}
$$

Here $\|\cdot\|$ denotes the distance to the nearest integer [3]. In view of our results below we reformulate Heilbronn's theorem as follows: There are infinitely many pairs of integers $m, k$ where $m$ is a perfect square such that the inequality

$$
|\xi m-k|<\frac{1}{m^{1 / 4-\varepsilon}}
$$

holds.
The Pythagorean numbers $x, y, z$ with $x^{2}+y^{2}=z^{2}$, where additionally $x$ and $y$ are coprime, play an important role in number theory since they were first investigated by the ancients. It is well-known that to every Pythagorean triplet $x, y, z$ of positive integers satisfying

$$
\begin{equation*}
x^{2}+y^{2}=z^{2}, \quad(x, y)=1, \quad x \equiv 0 \bmod 2 \tag{1.1}
\end{equation*}
$$

a pair of positive integers $a, b$ with $a>b>0$ corresponds such that

$$
\begin{equation*}
x=2 a b, y=a^{2}-b^{2}, z=a^{2}+b^{2},(a, b)=1, a+b \equiv 1 \bmod 2 \tag{1.2}
\end{equation*}
$$

hold ([2], Theorem 225). Moreover, there is a (1,1) correspondence between different values of $a, b$ and different values of $x, y, z$. The object of this paper is to investigate diophantine inequalities $|\xi y-x|$ for integers $y$ and $x$ from triplets of Pythagorean numbers. Since $x^{2}+y^{2}$ is required to be a perfect square - in what follows we write $x^{2}+y^{2} \in \square$ - we have a essential restriction on the rationals $x / y$ approximating a real irrational $\xi$. So one may not expect to get a result as strong as Heilbronn's theorem. Indeed, there are irrationals $\xi$ such that $|\xi y-x| \gg 1$ holds for all integers $x, y$ satisfying $x^{2}+y^{2} \in \square$. But almost all real irrationals $\xi$ (in the sense of the Lebesgue-measure) can be approximated in such a way that $|\xi y-x|$ tends to zero for a infinite sequence or pairs $x, y$ corresponding to Pythagorean numbers. In order to prove our results we shall make use of the properties of continued fraction expansions. By our first theorem we describe those real irrationals having good approximations by Pythagorean numbers.

Theorem 1.1: Let $\xi>0$ denote a real irrational number such that the quotients of the continued fraction expansion of at least one of the numbers $\eta_{1}:=\xi+\sqrt{1+\xi^{2}}$ and $\eta_{2}:=$ $\left(1+\sqrt{1+\xi^{2}}\right) / \xi$ are not bounded. Then there are infinitely many pairs of positive integers $x, y$ satisfying

$$
|\xi y-x|=o(1) \quad \text { and } \quad x^{2}+y^{2} \in \square
$$

Conversely, if the quotients of both of the numbers $\eta_{1}$ and $\eta_{2}$ are bounded, then there exists some $\delta>0$ such that

$$
|\xi y-x| \geq \delta
$$

holds for all positive integers $x, y$ where $x^{2}+y^{2} \in \square$.
It can easily be seen that the irrationality of $\xi$ does not allow the numbers $\eta_{1}$ and $\eta_{2}$ to be rationals. The following result can be derived from the preceding theorem and from the metric theory of continued fractions:
Corollary 1.1: To almost all real numbers $\xi$ (in the sense of the Lebesgue measure) there are infinitely many pairs of integers $x \neq 0, y>0$ satisfying

$$
|\xi y-x|=o(1) \quad \text { and } x^{2}+y^{2} \in \square
$$

Many exceptional numbers $\xi$ not belonging to that set of full measure are given by certain quadratic surds:
Corollary 1.2: Let $r>1$ denote some rational such that $\xi:=\sqrt{r^{2}-1}$ is an irrational number. Then the inequality

$$
\begin{equation*}
|\xi y-x|>\delta \tag{1.3}
\end{equation*}
$$

holds for some $\delta>0$ (depending only on $r$ ) and for all positive integers $x, y$ with $x^{2}+y^{2} \in \square$.
The lower bound $\delta$ can be computed explicitly. The corollary follows from Theorem 1.1 by setting $\xi:=\sqrt{r^{2}-1}$.

Taking $r=3 / 2$, we conclude that $\xi=\sqrt{5} / 2$ satisfies the condition of Corollary 1.2. Involving some refinements of the estimates from the proof of the general theorem, we find that (1.3) holds with $\delta=1 / 4$ for $\xi=\sqrt{5} / 2$.

Finally, we give an application to inhomogeneous diophantine approximations by Fibonacci numbers. Although $|y \sqrt{5} / 2-x|>\delta$ holds for all Pythagorean numbers $x, y$, this is no longer true in the case of inhomogeneous approximation. By the following result we estimate $\mid \xi y$ -$x-\eta \mid$ for infinitely many Pythagorean numbers $x$ and $y$, where $\xi$ and $\eta$ are given by $F_{k} \sqrt{5} / 2$ and $\pm F_{2 k} / \sqrt{5}$, respectively, for some fixed even integer $k$.
Theorem 1.2: Let $k \geq 2$ denote an even integer. Then,

$$
0<\frac{F_{k} \sqrt{5}}{2} \cdot\left(2 F_{n} F_{n+k}\right)-F_{k} F_{2 n+k}+(-1)^{n} \frac{F_{2 k}}{\sqrt{5}}<\frac{2^{2 n+1}}{5(1+\sqrt{5})^{2 n}}
$$

holds for all integers $n \geq 1$, and we have

$$
\left(2 F_{n} F_{n+k}\right)^{2}+\left(F_{k} F_{2 n+k}\right)^{2} \in \square \quad(n \geq 1)
$$

## 2. PROOF OF THEOREM 1.1

It can easily be verified that $\eta_{1}>1$ and $\eta_{2}>1$. One gets $\eta_{2}$ by substituting $1 / \xi$ for $\xi$ in $\eta_{1}$. First we assume that the sequence $a_{0}, a_{1}, a_{2}, \ldots$ of quotients from the continued fraction expansion $\eta_{1}=\left\langle a_{0} ; a_{1}, a_{2}, \ldots\right\rangle$ is not bounded. If $p_{n} / q_{n}$ denotes the $n^{t h}$ convergent of $\eta_{1}$, the inequality

$$
\begin{equation*}
\left|\eta_{1}-\frac{p_{n}}{q_{n}}\right|<\frac{1}{a_{n+1} q_{n}^{2}} \quad\left(n \geq n_{0}\right) \tag{2.1}
\end{equation*}
$$

holds, where $n_{0}$ is chosen sufficiently large. There exists some positive real number $\beta$ such that $\eta_{1}=1+2 \beta$; particularly we have $\eta_{1}>(1+\beta)\left(1+1 / p_{n}\right)$ for $n \geq n_{0}$. By $\eta_{1} q_{n}-p_{n}<1$, one gets

$$
\begin{equation*}
q_{n}<\frac{p_{n}+1}{\eta_{1}}<\frac{p_{n}}{1+\beta} \quad\left(n \geq n_{0}\right) . \tag{2.2}
\end{equation*}
$$

Let

$$
f(t):=\xi-\frac{1}{2}\left(t-\frac{1}{t}\right) \quad(t \geq 1)
$$

By straightforward computations it can easily be verified that

$$
\begin{equation*}
f\left(\eta_{1}\right)=0 \tag{2.3}
\end{equation*}
$$

For any two real numbers $t_{1}, t_{2}$ satisfying $1 \leq t_{1}<t_{2}$ there is some real number $\alpha$ with $t_{1} \leq \alpha \leq t_{2}$ such that

$$
\left|f\left(t_{2}\right)-f\left(t_{1}\right)\right|=\left|f^{\prime}(\alpha)\right| \cdot\left|t_{2}-t_{1}\right|
$$

holds. In the case when $n$ is even let $t_{1}=p_{n} / q_{n}$ and $t_{2}=\eta_{1}$, otherwise put $t_{1}=\eta_{1}, t_{2}=p_{n} / q_{n}$. Thus, for any even index $n \geq n_{0}$ we have $\eta_{1} \geq \alpha \geq p_{n} / q_{n}>1$, where the lower bound 1 follows immediately from $\eta_{1}>1$ and from (2.1). For any odd index $n$ it is clear that $\alpha \geq \eta_{1}$ holds. Therefore one gets

$$
\left|f\left(\eta_{1}\right)-f\left(\frac{p_{n}}{q_{n}}\right)\right|=\frac{1}{2}\left(1+\frac{1}{\alpha^{2}}\right) \cdot\left|\eta_{1}-\frac{p_{n}}{q_{n}}\right|
$$

where

$$
\begin{equation*}
\alpha>1 \quad\left(n \geq n_{0}\right) \tag{2.4}
\end{equation*}
$$

Applying (2.1), (2.3), and the definition of $f$, the inequality takes the form

$$
\left|\xi-\frac{1}{2}\left(\frac{p_{n}}{q_{n}}-\frac{q_{n}}{p_{n}}\right)\right|<\left(1+\frac{1}{\alpha^{2}}\right) \frac{1}{2 a_{n+1} q_{n}^{2}}
$$

Put $x:=p_{n}^{2}-q_{n}^{2}, y:=2 p_{n} q_{n}$. For $n$ tending to infinity the positive integers $p_{n}$ are not bounded, therefore we get infinitely many pairs $x, y$ of positive integers. By (2.2), $x>0$ holds for all sufficiently large indices $n$. Putting $x$ and $y$ into the above inequality and applying (2.4), we get

$$
\begin{equation*}
\left|\xi-\frac{x}{y}\right|<\frac{1}{a_{n+1} q_{n}^{2}}=\frac{4 p_{n}^{2}}{a_{n+1} y^{2}} \tag{2.5}
\end{equation*}
$$

where $x^{2}+y^{2}=\left(p_{n}^{2}+q_{n}^{2}\right)^{2} \in \square$. Using (2.2), we compute an upper bound for $p_{n}^{2}$ on the right side of (2.5): $p_{n}^{2}=x+q_{n}^{2}<x+p_{n}^{2} /(1+\beta)^{2}$, or, $p_{n}^{2}<(1+\beta)^{2} x / \beta(2+\beta)<(1+\beta)^{2} x / 2 \beta$ for $n \geq n_{0}$. Moreover, (2.5) gives $|\xi y-x|<y$, from which the estimate $x \leq(1+\xi) y$ follows immediately. Altogether we have proved that infinitely many pairs of positive integers $x, y$ with $x^{2}+y^{2} \in \square$ exists such that

$$
\left|\xi-\frac{x}{y}\right|<\frac{2(1+\xi)(1+\beta)^{2}}{\beta a_{n+1} y}
$$

holds, where any pair $x, y$ corresponds to some $n$. Finally, we restrict $n$ on integers from a subsequence corresponding to monotonously increasing partial quotients $a_{n+1}$. For $n$ tending to infinity, the assertion of the first part of the theorem concerning $\eta_{1}$ follows from

$$
|\xi y-x|<\frac{2(1+\xi)(1+\beta)^{2}}{\beta a_{n+1}}
$$

Next, if the sequence of quotients from the continued fraction expansion of $\eta_{2}$ is not bounded, we get by the same method infinitely many pairs $x, y$ of integers (where $y$ is even) satisfying $x^{2}+y^{2} \in \square$ and

$$
\left|\frac{y}{\xi}-x\right|<\frac{2(1+1 / \xi)(1+\beta)^{2}}{\beta a_{n+1}} \quad\left(2 \beta:=\eta_{2}-1, \eta_{2}=\left\langle a_{0} ; a_{1}, a_{2}, \ldots\right\rangle\right)
$$

This inequality can be simplified by

$$
|\xi x-y|<\frac{2(1+\xi)(1+\beta)^{2}}{\beta a_{n+1}}
$$

which completes the proof of the first part of the theorem.
In order to show the second part we now assume that both numbers, $\eta_{1}$ and $\eta_{2}$, have bounded partial quotients. It suffices to prove

$$
\begin{equation*}
|\xi y-x|>\delta \tag{2.6}
\end{equation*}
$$

for coprime Pythagorean integers $x$ and $y$ : if $|\xi y-x| \leq \delta$ for $(x, y)>1$, one may divide the inequality by $(x, y)$. Then we get a new pair of coprime integers with

$$
\left(\frac{x}{(x, y)}\right)^{2}+\left(\frac{y}{(x, y)}\right)^{2} \in \square
$$

which contradicts (2.6). From the hypothesis on $\eta_{1}$ we conclude that there is some positive real number $\delta_{1}$ satisfying

$$
\begin{equation*}
\left|\eta_{1}-\frac{a}{b}\right|>\frac{\delta_{1}}{b^{2}} \tag{2.7}
\end{equation*}
$$

for all positive coprime integers $a$ and $b$.
The first assertion we shall disprove states that there are infinitely many pairs of positive coprime integers $x, y$ such that $2 \mid y, x^{2}+y^{2} \in \square$, and

$$
\begin{equation*}
|\xi y-x|<\delta_{1}\left(\eta_{1}-1\right) \tag{2.8}
\end{equation*}
$$

By (1.1) and (1.2) we know that to every pair $x, y$ two integers $a, b$ correspond such that $y=2 a b, x=a^{2}-b^{2}, a>b,(a, b)=1$, and $a+b \equiv 1 \bmod 2$. Again we denote by $f(t)(t \geq 1)$ the function defined above. Using $\eta_{1}>1$ and $a / b>1$ it is clear that $f^{\prime}(t)$ is defined for all real numbers which are situated between $\eta_{1}$ and $a / b$. Therefore, corresponding to $a$ and $b$, a real number $\alpha$ exists satisfying

$$
\left(\alpha-\eta_{1}\right) \cdot\left(\alpha-\frac{a}{b}\right)<0 \text { and }\left|f\left(\eta_{1}\right)-f\left(\frac{a}{b}\right)\right|=\left|f^{\prime}(\alpha)\right| \cdot\left|\eta_{1}-\frac{a}{b}\right| .
$$

By (2.3) we find that

$$
\begin{equation*}
\left|\xi-\frac{x}{y}\right|=\left|\xi-\frac{a^{2}-b^{2}}{2 a b}\right|=\frac{1}{2}\left(1+\frac{1}{\alpha^{2}}\right) \cdot\left|\eta_{1}-\frac{a}{b}\right| . \tag{2.9}
\end{equation*}
$$

In what follows it is necessary to distinguish two cases.
Case 1: $\left|\eta_{1}-a / b\right| \geq 1$.
Using $1 / \alpha>0$, we conclude from (2.9) that $|\xi y-x|>y / 2$. For all sufficiently large integers $y$ this contradicts to our assumption from (2.8).
Case 2: $\left|\eta_{1}-a / b\right|<1$.
First, it follows from this hypothesis that $b<a /\left(\eta_{1}-1\right)$. Next, we estimate the right side of (2.9) by the inequality from (2.7):

$$
\left|\xi-\frac{x}{y}\right|>\frac{\delta_{1}}{2 b^{2}}>\frac{\delta_{1}\left(\eta_{1}-1\right)}{2 a b}
$$

Consequently we have, using $y=2 a b$,

$$
|\xi y-x|>\delta_{1}\left(\eta_{1}-1\right)
$$

which again is impossible by our assumption. So we have proved that there are at most finitely many pairs $x, y$ of positive coprime integers satisfying $2 \mid y, x^{2}+y^{2} \in \square$, and

$$
\begin{equation*}
|\xi y-x|<\delta_{1}\left(\eta_{1}-1\right) \tag{2.10}
\end{equation*}
$$

Since we may assume that the partial quotients of the number $\eta_{2}$ are also bounded, we get a similar result concerning the approximation of $1 / \xi$ : There are at most finitely many pairs $x, y$ of positive coprime integers with $2 \mid y, x^{2}+y^{2} \in \square$, and

$$
\begin{equation*}
\left|\frac{y}{\xi}-x\right|<\delta_{2}\left(\eta_{2}-1\right) \tag{2.11}
\end{equation*}
$$

where $\delta_{2}$ denotes some positive real number satisfying

$$
\left|\eta_{2}-\frac{a}{b}\right|>\frac{\delta_{2}}{b^{2}}
$$

for all coprime positive integers $a$ and $b$. Since $\xi$ is positive, the inequality from (2.11) can be transformed into

$$
|\xi x-y|<\delta_{2} \xi\left(\eta_{2}-1\right),
$$

which is satisfied at most by finitely many coprime Pythagorean numbers $x, y$ with $2 \mid y$. By (2.10) we complete the proof of the theorem.

## 3. PROOF OF THEOREM 1.2

Lemma 3.1: Let $k \geq 2$ and $n \geq 1$ denote integers, where $k$ is even. Then one has

$$
\begin{equation*}
F_{n+k}^{2}-F_{n}^{2}=F_{k} F_{2 n+k} \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(2 F_{n} F_{n+k}\right)^{2}+\left(F_{k} F_{2 n+k}\right)^{2} \in \square \tag{3.2}
\end{equation*}
$$

Proof: Throughout this final section we denote the number $(1+\sqrt{5}) / 2$ by $\rho$. We shall need Binet's formula

$$
\begin{equation*}
F_{m}=\frac{1}{\sqrt{5}}\left(\rho^{m}-\frac{(-1)^{m}}{\rho^{m}}\right) \quad(m \geq 1) \tag{3.3}
\end{equation*}
$$

Since $k$ is assumed to be even, one gets from (3.3):

$$
5\left(F_{n+k}^{2}-F_{n}^{2}\right)=\left(\rho^{n+k}-\frac{(-1)^{n}}{\rho^{n+k}}\right)^{2}-\left(\rho^{n}-\frac{(-1)^{n}}{\rho^{n}}\right)^{2}
$$

## ON RATIONAL APPROXIMATIONS BY PYTHAGOREAN NUMBERS

$$
=\left(\rho^{k}-\frac{1}{\rho^{k}}\right) \cdot\left(\rho^{2 n+k}-\frac{1}{\rho^{2 n+k}}\right)=5 F_{k} F_{2 n+k}
$$

This proves (3.1). Then the second assertion of the lemma follows easily, since one has

$$
\left(2 F_{n} F_{n+k}\right)^{2}+\left(F_{k} F_{2 n+k}\right)^{2}=\left(2 F_{n} F_{n+k}\right)^{2}+\left(F_{n+k}^{2}-F_{n}^{2}\right)^{2}=\left(F_{n+k}^{2}+F_{n}^{2}\right)^{2} \in \square
$$

Binet's formula (3.3) is a basic identity which also is used a several times to prove the inequalities in Theorem 1.2. Since $k \geq 2$ is assumed to be an even integer, one gets

$$
\begin{gathered}
\sqrt{5} F_{k} F_{n} F_{n+k}-F_{k} F_{2 n+k}+(-1)^{n} \frac{F_{2 k}}{\sqrt{5}} \\
=\frac{1}{5}\left(\rho^{k}-\frac{1}{\rho^{k}}\right) \cdot\left(\rho^{n}-\frac{(-1)^{n}}{\rho^{n}}\right) \cdot\left(\rho^{n+k}-\frac{(-1)^{n}}{\rho^{n+k}}\right)- \\
-\frac{1}{5}\left(\rho^{k}-\frac{1}{\rho^{k}}\right) \cdot\left(\rho^{2 n+k}-\frac{1}{\rho^{2 n+k}}\right)+\frac{(-1)^{n}}{5}\left(\rho^{2 k}-\frac{1}{\rho^{2 k}}\right) \\
=\frac{2}{5}\left(1-\frac{1}{\rho^{2 k}}\right) \frac{1}{\rho^{2 n}}
\end{gathered}
$$

It follows that the term on the left side represents a positive real number, which is bounded by $2 / 5 \rho^{2 n}$. By (3.2) from Lemma 3.1, this finishes the proof of the theorem.

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# AN INVOLUTORY MATRIX OF EIGENVECTORS 

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## 1. INTRODUCTION

A matrix with a full set of linearly independent eigenvectors is diagonalizable: if the $n$ by $n$ matrix $A$ has eigenvalues $\lambda_{j}$ with corresponding eigenvectors $u_{j}(1 \leq j \leq n)$, if $U=$ $\left(u_{1}\left|u_{2}\right| \ldots \mid u_{n}\right)$ and $D=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$, then $U$ is a diagonalizing matrix for $A: U^{-1} A U=$ $D$. Taking transposes shows that $\left(U^{-1}\right)^{t}$ is a diagonalizing matrix for $A^{t}$. Hence $U^{t}$ itself is a diagonalizing matrix for $A^{t}$ if $U^{2}$ is the identity matirx, or more generally, due to the scalability of eigenvectors, if $U^{2}$ is a scalar matrix.

The purpose of this note is to point out that the right-justified Pascal-triangle matrix $R=$ $\left.\binom{i-1}{n-j}\right)_{1 \leq i, j \leq n}$ is an example of this phenomenon. Let $a$ denote the golden ratio $(1+\sqrt{5}) / 2$.

The eigenvalues of $R^{t}$ (which of course are the same as the eigenvalues of $R$ ) were found in [1]: $\lambda_{i}=(-1)^{n-i} a^{2 i-n-1}, 1 \leq i \leq n$. The corresponding eigenvectors $u_{i}$ of $R^{t}$ were also found in [1] (here suitably scaled for our purposes): $u_{i}=\left(u_{i j}\right)_{1 \leq j \leq n}$ where

$$
u_{i j}=(-a)^{n-j} \sum_{k=1}^{j}(-1)^{i-k}\binom{i-1}{k-1}\binom{n-i}{j-k} a^{2 k-i-1} .
$$

Let $U=\left(u_{i j}\right)_{1 \leq i, j \leq n}$.
For example, when $n=5$,

$$
R=\left(\begin{array}{lllll}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 2 & 1 \\
0 & 1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1
\end{array}\right) \text { and } U=\left(\begin{array}{ccccc}
a^{4} & -4 a^{3} & 6 a^{2} & -4 a & 1 \\
-a^{3} & 3 a^{2}-a^{4} & -3 a+3 a^{3} & 1-3 a^{2} & a \\
a^{2} & -2 a+2 a^{3} & 1-4 a^{2}+a^{4} & 2 a-2 a^{3} & a^{2} \\
-a & 1-3 a^{2} & 3 a-3 a^{3} & 3 a^{2}-a^{4} & a^{3} \\
1 & 4 a & 6 a^{2} & 4 a^{3} & a^{4}
\end{array}\right) .
$$

Since the rows of $U$ are eigenvectors of $R^{t}, U^{t}$ is a diagonalizing matrix for $R^{t}$. By the first paragraph applied to $A=R^{t}, U$ will be a diagonalizing matrix for $R$ if we can show that

## AN INVOLUTORY MATRIX OF EIGENVECTORS

$\left(U^{t}\right)^{2}$ (equivalently $U^{2}$ ) is a scalar matrix. We now proceed to show that $U^{2}=\left(1+a^{2}\right)^{n-1} I_{n}$ and in fact this holds for arbitrary $a$. We use the notation $\left[x^{k}\right] p(x)$ to denote the coefficient of $x^{k}$ in the polynomial $p(x)$. Consider the generating function $U_{i}(z)=z(z-a)^{n-i}(a z+1)^{i-1}$. Using the binomial theorem to expand $U_{i}(z)$, it is immediate that

$$
U_{i}(z)=\sum_{j=1}^{n} u_{i j} z^{j}
$$

Now the $(i, k)$ entry of $U^{2}$ is

$$
\begin{aligned}
\left(U^{2}\right)_{i k} & =\sum_{j=1}^{n}\left[x^{k-1}\right](x-a)^{n-j}(a x+1)^{j-1} \cdot\left[z^{j-1}\right](z-a)^{n-i}(a z+1)^{i-1} \\
& =\left[x^{k-1}\right] \sum_{j=1}^{n}\left[z^{j-1}\right](x-a)^{n-j}(a x+1)^{j-1}(z-a)^{n-i}(a z+1)^{i-1} \\
& =\left[x^{k-1}\right](x-a)^{n-1} \sum_{j=1}^{n}\left[z^{j-1}\right]\left(\frac{a x+1}{x-a}\right)^{j-1}(z-a)^{n-i}(a z+1)^{i-1} \\
& =\left[x^{k-1}\right](x-a)^{n-1} \sum_{j=1}^{n}\left[z^{j-1}\right]\left(z \frac{a x+1}{x-a}-a\right)^{n-i}\left(a z \frac{a x+1}{x-a}+1\right)^{i-1} \\
& =\left[x^{k-1}\right](x-a)^{n-1}\left(\frac{a x+1}{x-a}-a\right)^{n-i}\left(a \frac{a x+1}{x-a}+1\right)^{i-1} \\
& =\left[x^{k-1}\right]\left(a x+1-a x+a^{2}\right)^{n-i}\left(a^{2} x+a+x-a\right)^{i-1} \\
& =\left[x^{k-1}\right]\left(1+a^{2}\right)^{n-i} x^{i-1}\left(1+a^{2}\right)^{i-1} \\
& =\left[x^{k-i}\right]\left(1+a^{2}\right)^{n-1} \\
& =\left(1+a^{2}\right)^{n-1} \delta_{k i},
\end{aligned}
$$

as desired.

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## AN INVOLUTORY MATRIX OF EIGENVECTORS

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# GENERALIZED FIBONACCI FUNCTIONS AND SEQUENCES OF GENERALIZED FIBONACCI FUNCTIONS 

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## 1. INTRODUCTION

We consider a generalization of the Fibonacci sequence which is called the $k$-Fibonacci sequence for a positive integer $k \geq 2$. The $k$-Fibonacci sequence $\left\{g_{n}^{(k)}\right\}$ is defined as

$$
g_{0}^{(k)}=g_{1}^{(k)}=\cdots=g_{k-2}^{(k)}=0, \quad g_{k-1}^{(k)}=1
$$

and for $n \geq k \geq 2$,

$$
g_{n}^{(k)}=g_{n-1}^{(k)}+g_{n-2}^{(k)}+\cdots+g_{n-k}^{(k)}
$$

We call $g_{n}^{(k)}$ the $n^{t h} k$-Fibonacci number. For example, if $k=2$, then $\left\{g_{n}^{(2)}\right\}$ is the Fibonacci sequence $\left\{F_{n}\right\}$. If $k=5$, then $g_{0}^{(5)}=g_{1}^{(5)}=g_{2}^{(5)}=g_{3}^{(5)}=0, g_{4}^{(5)}=1$, and the 5 -Fibonacci sequence is

$$
\left(g_{0}^{(5)}=0\right), 0,0,0,1,1,2,4,8,16,31,61,120,236,464,912, \ldots
$$

Let $E$ be a 1 by $(k-1)$ matrix whose entries are ones and let $I_{n}$ be the identity matrix of order $n$. Let $\mathrm{g}_{n}^{(k)}=\left(g_{n}^{(k)}, \ldots, g_{n+k-1}^{(k)}\right)^{T}$ for $n \geq 0$. For any $k \geq 2$, the fundamental recurrence relation, $n \geq k$,

$$
g_{n}^{(k)}=g_{n-1}^{(k)}+g_{n-2}^{(k)}+\cdots+g_{n-k}^{(k)}
$$

can be defined by the vector recurrence relation $g_{n+1}^{(k)}=Q_{k} g_{n}^{(k)}$, where

$$
Q_{k}=\left[\begin{array}{cc}
0 & I_{k-1}  \tag{1}\\
1 & E
\end{array}\right]
$$

We call $Q_{k}$ the $k$-Fibonacci matrix. By applying (1), we have $\mathrm{g}_{n+1}^{(k)}=Q_{k}^{n} \mathrm{~g}_{1}^{(k)}$. In [4], [6] and [7], we can find relationships between the $k$-Fibonacci numbers and their associated matrices. In [2], M. Elmore introduced the Fibonacci function following as:

$$
f_{0}(x)=\frac{e^{\lambda_{1} x}-e^{\lambda_{2} x}}{\sqrt{5}}, f_{n}(x)=f_{0}^{(n)}(x)=\frac{\lambda_{1}^{n} e^{\lambda_{1} x}-\lambda_{2}^{n} e^{\lambda_{2} x}}{\sqrt{5}}
$$

and hence $f_{n+1}(x)=f_{n}(x)+f_{n-1}(x)$, where

$$
\lambda_{1}=\frac{1+\sqrt{5}}{2} \text { and } \lambda_{2}=\frac{1-\sqrt{5}}{2}
$$

Here, $\lambda_{1}, \lambda_{2}$ are the roots of $x^{2}-x-1=0$.
In this paper, we consider a function which is a generalization of the Fibonacci function and consider sequences of generalized Fibonacci functions.

## 2. GENERALIZED FIBONACCI FUNCTIONS

For positive integers $l$ and $n$ with $l \leq n$, let $Q_{l, n}$ denote the set of all strictly increasing $l$-sequences from $\{1,2, \ldots, n\}$. For an $n \times n$ matrix $A$ and for $\alpha, \beta \in Q_{l, n}$, let $A[\alpha \mid \beta]$ denote the matrix lying in rows $\alpha$ and columns $\beta$ and let $A(\alpha \mid \beta)$ denote the matrix complementary to $A[\alpha \mid \beta]$ in $A$. In particular, we denote $A(\{i\} \mid\{j\})=A(i \mid j)$.

We define a function $G(k, x)$ by

$$
G(k, x)=\sum_{i=0}^{\infty} \frac{g_{i}^{(k)}}{i!} x^{i}
$$

Since

$$
\lim _{n \rightarrow \infty} \frac{g_{n}^{(k)}(n+1)}{g_{n+1}^{(k)}} \rightarrow \infty
$$

the function $G(k, x)$ is convergent for all real number $x$.
For fixed $k \geq 2$, the power series $G(k, x)$ satisfies the differential equation

$$
\begin{equation*}
G^{(k)}(k, x)-G^{(k-1)}(k, x)-\cdots-G^{\prime \prime}(k, x)-G^{\prime}(k, x)-G(k, x)=0 \tag{2}
\end{equation*}
$$

In [5], we can find that the characteristic equation $x^{k}-x^{k-1}-\cdots-x-1=0$ of $Q_{k}$ does not have multiple roots. So, if $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}$ are the roots of $x^{k}-x^{k-1}-\cdots-x-1=0$, then
$\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}$ are distinct. That is, the eigenvalues of $Q_{k}$ are distinct. Define $V$ to be the $k$ by $k$ Vandermonde matrix by

$$
V=\left[\begin{array}{cccc}
1 & 1 & \cdots & 1  \tag{2}\\
\lambda_{1} & \lambda_{2} & \cdots & \lambda_{k} \\
\vdots & \vdots & \vdots & \vdots \\
\lambda_{1}^{k-2} & \lambda_{2}^{k-2} & \cdots & \lambda_{k}^{k-2} \\
\lambda_{1}^{k-1} & \lambda_{2}^{k-1} & \ldots & \lambda_{k}^{k-1}
\end{array}\right]
$$

Then we have the following theorem.
Theorem 2.1: Let $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}$ be the eigenvalues of the $k$-Fibonacci matrix $Q_{k}$. Then, the initial-value problem $\sum_{i=0}^{k-1} G^{(i)}(k, x)=G^{(k)}(k, x)$, where $G^{(i)}(k, 0)=0$ for $i=0,1, \ldots, k-2$, and $G^{(k-1)}(k, 0)=1$ has the unique solution $G(k, x)=\sum_{i=1}^{k} c_{i} e^{\lambda_{i} x}$, where

$$
\begin{equation*}
c_{i}=(-1)^{k+i} \frac{\operatorname{det} V(k \mid i)}{\operatorname{det} V}, i=1,2, \ldots, k \tag{3}
\end{equation*}
$$

Proof: Since the characteristic equation of $Q_{k}$ is $x^{k}-x^{k-1}-\cdots-x-1=0$, it is clear that $c_{1} e^{\lambda_{1} x}+c_{2} e^{\lambda_{2} x}+\cdots+c_{k} e^{\lambda_{k} x}$ is a solution of (2.).

Now, we will prove that $c_{i}=\frac{1}{\operatorname{det} V}(-1)^{k+i} \operatorname{det} V(k \mid i), i=1,2, \ldots, k$. Since $G(k, x)=$ $c_{1} e^{\lambda_{i} x}+c_{2} e^{\lambda_{2} x}+\ldots c_{k} e^{\lambda_{k} x}$ and for $x=0, G^{(i)}(k, 0)=0$ for $i=0,1, \ldots, k-2, G^{(k-1)}(k, 0)=1$, we have

$$
\begin{aligned}
G(k, 0) & =c_{1}+c_{2}+\cdots+c_{k}=0 \\
G^{\prime}(k, 0) & =c_{1} \lambda_{1}+c_{2} \lambda_{2}+\cdots+c_{k} \lambda_{k}=0 \\
& \vdots \\
G^{(k-2)}(k, 0) & =c_{1} \lambda_{1}^{k-2}+c_{2} \lambda_{2}^{k-2}+\cdots+c_{k} \lambda_{k}^{k-2}=0 \\
G^{(k-1)}(k, 0) & =c_{1} \lambda_{1}^{k-1}+c_{2} \lambda_{2}^{k-1}+\cdots+c_{k} \lambda_{k}^{k-1}=1
\end{aligned}
$$

Let $\mathbf{c}=\left(c_{1}, c_{2}, \ldots, c_{k-1}, c_{k}\right)^{T}$ and $\mathbf{b}=(0,0, \ldots, 0,1)^{T}$. Then we have $V \mathbf{c}=\mathbf{b}$. Since the matrix $V$ is a. Vandermonde matrix and $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}$ are distinct, the matrix $V$ is nonsingular. For $i=1,2, \ldots, k$, the matrix $V(k \mid i)$ is also a Vandermonde matrix and nonsingular. Therefore,
by Cramer's rule, we have $c_{i}=(-1)^{k+i} \frac{\operatorname{det} V(k \mid i)}{\operatorname{det} V}, i=1,2, \ldots, k$ and the proof is complete.
We can replace the writing of (2) by the form

$$
G^{(k)}(k, x)=G^{(k-1)}(k, x)+\cdots+G^{\prime \prime}(k, x)+G^{\prime}(k, x)+G(k, x)
$$

This suggests that we use the notation $G_{0}(k, x)=G(k, x)$ and, for $i \geq 1, G_{i}(k, x)=G^{(i)}(k, x)$. Thus

$$
G_{n}(k, x)=G^{(n)}(k, x)=c_{1} \lambda_{1}^{n} e^{\lambda_{1} x}+c_{2} \lambda_{2}^{n} e^{\lambda_{2} x}+\cdots+c_{k} \lambda_{k}^{n} e^{\lambda_{k} x}
$$

gives us the sequence of functions $\left\{G_{n}(k, x)\right\}$ with the property that

$$
\begin{equation*}
G_{n}(k, x)=G_{n-1}(k, x)+G_{n-2}(k, x)+\cdots+G_{n-k}(k, x), \quad n \geq k \tag{4}
\end{equation*}
$$

where each $c_{i}$ is in (3). We shall refer to these functions as $k$-Fibonacci functions. If $k=2$, then $G(2, x)=f_{0}(x)$ is the Fibonacci function as in [2]. From (4), we have the following theorem.
Theorem 2.2: For the $k$-Fibonacci function $G_{n}(k, x)$,

$$
\begin{aligned}
G_{0}(k, 0) & =0=g_{0}^{(k)}, G_{1}(k, 0)=0=g_{1}^{(k)}, \ldots, G_{k-2}(k, 0)=0=g_{k-2}^{(k)} \\
G_{k-1}(k, 0) & =1=g_{k-1}^{(k)}, G_{k}(k, 0)=G_{0}(k, 0)+\cdots+G_{k-1}(k, 0)=1=g_{k}^{(k)} \\
g_{n}^{(k)} & =G_{n}(k, 0)=c_{1} \lambda_{1}^{n}+c_{2} \lambda_{2}^{n}+\cdots+c_{k} \lambda_{k}^{n} \\
& =g_{n-1}^{(k)}+g_{n-2}^{(k)}+\cdots+g_{n-k}^{(k)}, n \geq k
\end{aligned}
$$

where each $c_{i}$ is given by (3).
Let $\mathrm{G}_{n}(k, x)=\left(G_{n}(k, x), \ldots, G_{n+k-1}(k, x)\right)^{T}$. For $k \geq 2$, the fundamental recurrence realtion (4) can be defined by the vector recurrence relation $\mathbf{G}_{n+1}(k, x)=Q_{k} \mathbf{G}_{n}(k, x)$ and hence $\mathbb{G}_{n+1}(k, x)=Q_{k}^{n} \mathbf{G}_{1}(k, x)$.

Since $g_{k-1}^{(k)}=g_{k}^{(k)}=1$, we can replace the matrix $Q_{k}$ in (1) with

$$
Q_{k}=\left[\begin{array}{ccccc}
0 & g_{k-1}^{(k)} & 0 & \cdots & 0 \\
0 & 0 & g_{k-1}^{(k)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & g_{k-1}^{(k)} \\
g_{k-1}^{(k)} & g_{k-1}^{(k)} & \cdots & g_{k-1}^{(k)} & g_{k}^{(k)}
\end{array}\right]
$$

Then we can find the matrix $Q_{k}^{n}=\left[g_{i, j}^{\dagger}(n)\right]$ in [5] where, for $i=1,2, \ldots, k$ and $j=1,2, \ldots, k$,

$$
\begin{equation*}
g_{i, j}^{\dagger}(n)=g_{n+(i-2)}^{(k)}+\cdots+g_{n+(i-2)-(j-1)}^{(k)} \tag{5}
\end{equation*}
$$

We know that $g_{i, 1}^{\dagger}(n)=g_{n+i-2}^{(k)}$ and $g_{i, k}^{\dagger}(n)=g_{n+i-1}^{(k)}$. So, we have the following theorem.

Theorem 2.3: For nonnegative integers $n$ and $m, n+m \geq k$, we have

$$
G_{n+m+1}(k, x)=\sum_{j=1}^{k} g_{1, j}^{\dagger}(n) G_{m+j}(k, x)
$$

In particular,

$$
G_{k}(k, x)=\sum_{i=0}^{\infty} \frac{g_{i+k}^{(k)}}{i!} x^{i}
$$

Proof: Since $\mathbf{G}_{n+1}(k, x)=Q_{k}^{n} \mathbf{G}_{\mathbf{1}}(k, x)$,

$$
\begin{aligned}
G_{n+m+1}(k, x) & =Q_{k}^{n+m} \mathbf{G}_{1}(k, x)=Q_{k}^{n} \cdot Q_{k}^{m} \mathbf{G}_{1}(k, x) \\
& =Q_{k}^{n} \mathbf{G}_{m+1}(k, x)
\end{aligned}
$$

By applying (5), we have

$$
G_{n+m+1}(k, x)=g_{1,1}^{\dagger}(n) G_{m+1}(k, x)+\cdots+g_{1, k}^{\dagger}(n) G_{m+k}(k, x)
$$

Since $\sum_{i=0}^{k-1} G_{i}(k, x)=G_{k}(k, x)$ and

$$
\sum_{i=0}^{k-1} G_{i}(k, x)=g_{k}^{(k)}+g_{k+1}^{(k)} x+\frac{g_{k+2}^{(k)}}{2!} x^{2}+\cdots+\frac{g_{n+k}^{(k)}}{n!} x^{n}+\ldots
$$

we have

$$
G_{k}(k, x)=\sum_{i=0}^{\infty} \frac{g_{i+k}^{(k)}}{i!} x^{i}
$$

Note that $Q_{k}^{n+m}=Q_{k}^{m+n}$. Then we have the following corollary.
Corollary 2.4: For nonnegative integers $n$ and $m, n+m \geq k$, we have

$$
G_{n+m+1}(k, x)=\sum_{j=1}^{k} g_{1, j}^{\dagger}(m) G_{n+j}(k, x)
$$

We know that the characteristic polynomial of $Q_{k}$ is $\lambda^{k}-\lambda^{k-1}-\cdots-\lambda-1$. So, we have the following lemma.

Lemma 2.5: Let $\lambda^{k}-\lambda^{k-1}-\cdots-\lambda-1=0$ be the characteristic equation of $Q_{k}$. Then, for any root $\lambda$ of the characteristic equation, $n \geq k>0$, we have,

$$
\lambda^{n}=\sum_{j=1}^{k} g_{1, j}^{\dagger}(n) \lambda^{j-1}
$$

Proof: From (5) we have, for $j=1,2, \ldots, k$,

$$
g_{1, j}^{\dagger}(n)=g_{n-1}^{k}+g_{n-2}^{k}+\cdots+g_{n-j}^{k}
$$

It can be shown directly for $n=k$ that

$$
\begin{aligned}
\lambda^{k} & =g_{k}^{(k)} \lambda^{k-1}+\left(g_{k-1}^{(k)}+g_{k-2}^{(k)}+\cdots+g_{1}^{(k)}\right) \lambda^{k-2}+\cdots+\left(g_{k-1}^{(k)}+g_{k-2}^{(k)}\right) \lambda+g_{k-1}^{k} \\
& =\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1
\end{aligned}
$$

We show this by induction on $n$. Then

$$
\begin{aligned}
\lambda^{n+1}= & \lambda^{n} \cdot \lambda \\
= & \left(g_{1, k}^{\dagger}(n) \lambda^{k-1}+g_{1, k-1}^{\dagger}(n) \lambda^{k-2}+\cdots+g_{1,2}^{\dagger}(n) \lambda+g_{1,1}^{+}(n)\right) \lambda \\
= & g_{n}^{k} \lambda^{k}+\left(g_{n-1}^{(k)}+g_{n-2}^{(k)}+\cdots+g_{n-k+1}^{(k)}\right) \lambda^{k-1} \\
& +\left(g_{n-1}^{(k)}+\cdots+g_{n-k+2}^{(k)}\right) \lambda^{k-2}+\cdots+\left(g_{n-1}^{(k)}+\left(g_{n-2}^{(k)}\right) \lambda^{2}+g_{n-1}^{(k)} \lambda\right.
\end{aligned}
$$

Since $\lambda^{k}=\lambda^{k-1}+\cdots+\lambda+1$, we have

$$
\begin{aligned}
\lambda^{n+1}= & g_{n}^{(k)}\left(\lambda^{k-1}+\cdots+\lambda+1\right)+\left(g_{n-1}^{(k)}+g_{n-2}^{(k)}+\cdots+g_{n-k+1}^{(k)}\right) \lambda^{k-1}+ \\
& \left(g_{n-1}^{(k)}+g_{n-2}^{(k)}+\cdots+g_{n-k+2}^{(k)}\right) \lambda^{k-2}+\cdots+\left(g_{n-1}^{(k)}+g_{n-2}^{(k)}\right) \lambda^{2}+g_{n-1}^{(k)} \lambda \\
= & \left(g_{n}^{(k)}+g_{n-1}^{(k)}+\cdots+g_{n-k+1}^{(k)}\right) \lambda^{k-1}+\left(g_{n}^{(k)}+\cdots+g_{n-k+2}^{(k)}\right) \lambda^{k-2} \\
& +\cdots+\left(g_{n}^{(k)}+g_{n-1}^{(k)}\right) \lambda+g_{n}^{(k)} \\
= & g_{n+1}^{(k)} \lambda^{k-1}+\left(g_{n}^{(k)}+g_{n-1}^{(k)}+\cdots+g_{n-k+2}^{(k)}\right) \lambda^{k-2} \\
& +\cdots+\left(g_{n}^{(k)}+g_{n-1}^{(k)}\right) \lambda+g_{n}^{(k)} \\
= & g_{1, k}^{\dagger}(n+1) \lambda^{k-1}+g_{1, k-1}^{\dagger}(n+1) \lambda^{k-2}+g_{1, k-2}^{\dagger}(n+1) \lambda^{k-3} \\
& +\cdots+g_{1,2}^{\dagger}(n+1) \lambda+g_{1,1}^{\dagger}(n+1) \\
= & \sum_{j=1}^{k} g_{1, j}^{\dagger}(n+1) \lambda^{j-1} .
\end{aligned}
$$

Therefore, by induction of $n$, the proof is completed.
Theorem 2.6: Let $\lambda$ be a root of characteristic equation of $Q_{k}$. For positive integer $n$, we have

$$
G_{n}(k, \lambda)=\sum_{j=n}^{k} \alpha_{n j} \lambda^{j-1}
$$

where

$$
\alpha_{j, n}=\frac{g_{n+k}^{(k)}}{k!}+\frac{g_{n+j-1}^{(k)}}{(j-1)!}+\sum_{i=k+1}^{\infty} g_{1, j}^{\dagger}(i) \frac{g_{n+i}^{(k)}}{i!} .
$$

Proof: Since $\lambda^{k}=\lambda^{k-1}+\cdots+\lambda+1$ and by lemma 2.5, we have

$$
\begin{aligned}
G_{n}(k, \lambda)= & g_{n}^{(k)}+g_{n+1}^{(k)} \lambda+\frac{g_{n+2}^{(k)}}{2!} \lambda^{2}+\cdots+\frac{g_{2 n}^{(k)}}{n!} \lambda^{n}+\ldots \\
= & \left(g_{n}^{(k)}+\frac{g_{n+k}^{(k)}}{k!}+g_{11}^{\dagger}(k+1) \frac{g_{n+k+1}^{(k)}}{(k+1)!}+\cdots+g_{11}^{\dagger}(n) \frac{g_{2 n}^{(k)}}{n!}+\cdots\right)+ \\
& \left(g_{n+1}^{(k)}+\frac{g_{n+k}^{(k)}}{k!}+g_{12}^{\dagger}(k+1) \frac{g_{n+k+1}^{(k)}}{(k+1)!}+\cdots+g_{12}^{\dagger}(n) \frac{g_{2 n}^{(k)}}{n!}+\ldots\right) \lambda \\
& +\cdots+ \\
& \left(\frac{g_{n+k-1}^{(k)}}{(k-1)!}+\frac{g_{n+k}^{(k)}}{k!}+g_{1 k}^{\dagger}(k+1) \frac{g_{n+k+1}^{(k)}}{(k+1)!}+\cdots+g_{1 k}^{\dagger}(n) \frac{g_{2 n}^{(k)}}{n!}+\cdots\right) \lambda^{k-1} \\
= & \alpha_{1_{n}}+\alpha_{2_{n}} \lambda+\cdots+\alpha_{k_{n}} \lambda^{k-1} \\
= & \sum_{j=1}^{k} \alpha_{j_{n}} \lambda^{j-1},
\end{aligned}
$$

where

$$
\alpha_{j_{n}}=\frac{g_{n+k}^{(k)}}{k!}+\frac{g_{n+j-1}^{(k)}}{(j-1)!}+\sum_{i-k+1}^{\infty} g_{1, j}^{\dagger}(i) \frac{g_{n+i}^{(k)}}{i!}
$$

for $j=1,2, \ldots, k$, the proof is completed.
From theorem 2.3 and theorem 2.6, we have

$$
\begin{aligned}
G_{n}(k, x) & =\sum_{i=0}^{\infty} \frac{g_{n+i}^{(k)}}{i!} x^{i} \\
& =g_{1,1}^{\dagger}(n-1) G_{1}(k, x)+\cdots+g_{1, k}^{\dagger}(n-1) G_{k}(k, x) \\
& =\sum_{j=1}^{k} \alpha_{j_{n}} x^{j-1}
\end{aligned}
$$

where

$$
\alpha_{j_{n}}=\frac{g_{n+k}^{(k)}}{k!}+\frac{g_{n+j-1}^{(k)}}{(j-1)!}+\sum_{i=k+1}^{\infty} g_{1, j}^{\dagger}(i) \frac{g_{n+i}^{(k)}}{i!}
$$

for $j=1,2, \ldots, k$.

## 3. SEQUENCES OF GENERALIZED FIBONACCI FUNCTIONS

Matrix methods are a major tool in solving certain problems stemming from linear recurrence relations. In this section, the procedure will be illustrated by means of a sequence, and an interesting example will be given.

To begin with, we introduce the concept of the resultant of given polynomials [3]. Let $f(x)=\sum_{i=0}^{n} a_{i} x^{n-i}$ and $g(x)=\sum_{i=0}^{m} b_{i} x^{m-i}$ be polynomials, where $a_{0} \neq 0$ and $b_{0} \neq 0$. The presence of a common divisor for $f(x)$ and $g(x)$ is equivalent to the fact that there exists polynomials $p(x)$ and $q(x)$ such that $f(x) q(x)=g(x) p(x)$ where $\operatorname{deg} p(x) \leq n-1$ and deg $q(x) \leq m-1$. Let $q(x)=u_{0} x^{m-1}+\cdots+u_{m-1}$ and $p(x)=v_{0} x^{n-1}+\cdots+v_{n-1}$. The equality $f(x) q(x)=g(x) p(x)$ can be expressed in the form of a system of equations

$$
\begin{aligned}
a_{0} u_{0} & =b_{0} v_{0} \\
a_{1} u_{0}+a_{0} u_{1} & =b_{1} v_{0}+b_{0} v_{1} \\
a_{2} u_{0}+a_{1} u_{1}+a_{0} u_{2} & =b_{2} v_{0}+b_{1} v_{1}+b_{0} v_{2}
\end{aligned}
$$

The polynomials $f(x)$ and $g(x)$ have a common root if and only if this system of equations has a nonzero solution ( $u_{0}, u_{1}, \ldots, v_{0}, v_{1}, \ldots$ ). If, for example, $m=3$ and $n=2$, then the determinant of this system is of the form

$$
\left|\begin{array}{ccccc}
a_{0} & 0 & 0 & -b_{0} & 0 \\
a_{1} & a_{0} & 0 & -b_{1} & -b_{0} \\
a_{2} & a_{1} & a_{0} & -b_{2} & -b_{1} \\
0 & a_{2} & a_{1} & -b_{3} & -b_{2} \\
0 & 0 & a_{2} & 0 & -b_{3}
\end{array}\right|=\left|\begin{array}{ccccc}
a_{0} & a_{1} & a_{2} & 0 & 0 \\
0 & a_{0} & a_{1} & a_{2} & 0 \\
0 & 0 & a_{0} & a_{1} & a_{2} \\
b_{0} & b_{1} & b_{2} & b_{3} & 0 \\
0 & b_{0} & b_{1} & b_{2} & b_{3}
\end{array}\right|=|S(f(x), g(x))| .
$$

The matrix $S(f(x), g(x))$ is called the Sylvester matrix of polynomials $f(x)$ and $g(x)$. The determinant of $S(f(x), g(x))$ is called the resultant of $f(x)$ and $g(x)$ and is denoted by $R(f(x), g(x))$. It is clear that $R(f(x), g(x))=0$ if and only if the polynomials $f(x)$ and $g(x)$ have a common divisor, and hence, an equation $f(x)=0$ has multiple roots if and only if $R\left(f(x), f^{\prime}(x)\right)=0$.

Now, we define a sequence. For fixed $k, k \geq 2$, and a complex number $a$, a sequence of $k$-Fibonacci functions, $\left\{G_{n}(k, a)\right\}$, is defined recursively as follows:

$$
\begin{equation*}
G_{0}(k, a)=s_{0}, G_{1}(k, a)=s_{1}, \ldots, G_{k-1}(k, a)=s_{k-1}, \tag{6}
\end{equation*}
$$

$$
\begin{equation*}
G_{n}(k, a)=p_{1} G_{n-1}(k, a)+p_{2} G_{n-2}(k, a)+\cdots+p_{k} G_{n-k}(k, a), \quad n \geq k, \tag{7}
\end{equation*}
$$

where $s_{0}, s_{1}, \ldots, s_{k-1}, p_{1}, p_{2}, \ldots, p_{k}$ are complex numbers.
Our natural question now becomes, for $k \geq 2$, what is an explicit expression for $G_{n}(k, a)$ is terms of $s_{0}, s_{1}, \ldots, s_{k-1}, p_{1}, \ldots, p_{k}$ ? If $s_{0}=\cdots=s_{k-2}=0, s_{k-1}=s_{k}=1, p_{1}=\cdots=$ $p_{k}=1$ and $a=0$, then by theorem 2.2 we have $G_{n}(k, 0)=g_{n}$. In [8], Rosenbaum gave the explicit expression for $k=2$.

In this section, we give an explicit expression for $G_{n}(k, a)=p_{1} G_{n-1}(k, a)+p_{2} G_{n-2}(k, a)+$ $\cdots+p_{k} G_{n-k}(k, a), n \geq k$ in terms of initial conditions $G_{0}(k, a)=s_{0}, G_{1}(k, a)=$ $s_{1}, \ldots, G_{k-1}(k, a)=s_{k-1}, k \geq 2$.

Let $\tilde{\mathbf{G}}_{n}(k)=\left(G_{n}(k, a), \ldots, G_{n-k+1}(k, a)\right)^{T}$ for $k \geq 2$. The fundamental recurrence relation (7) can be defined by the vector recurrence relation $\tilde{\mathbf{G}}_{n}(k)=\tilde{Q}_{k} \tilde{\mathbf{G}}_{n-1}(k)$, where

$$
\tilde{Q}_{k}=\left[\begin{array}{cc}
\mathrm{p} & p_{k} \\
I_{k-1} & 0
\end{array}\right] \text { and } \mathrm{p}=\left[p_{1}, p_{2}, \ldots, p_{k-1}\right] .
$$

Let $\mathbf{s}=\left(s_{k-1}, \ldots, s_{0}\right)^{T}$. Then, we have, for $n \geq 0, \tilde{\mathbf{G}}_{n+k-1}(k)=\tilde{Q}_{k}^{n} \mathbf{s}$, and the characteristic equation of $\tilde{Q}_{k}$ is

$$
f(\lambda)=\lambda^{k}-p_{1} \lambda^{k-1}-\cdots-p_{k-1} \lambda-p_{k}=0 .
$$

If $R\left(f(\lambda), f^{\prime}(\lambda)\right) \neq 0$, then the equation $f(\lambda)=0$ has distinct $k$ roots.
Theorem 3.1: Let $f(\lambda)$ be the characteristic equation of the matrix $\tilde{Q}_{k}$. If $R\left(f(\lambda), f^{\prime}(\lambda)\right) \neq 0$, then $G_{n}(k, a)=p_{1} G_{n-1}(k, a)+p_{2} G_{n-2}(k, a)+\cdots+p_{k} G_{n-k}(k, a)$ has an explicit expression in terms of $s_{0}, \ldots, s_{k-1}$.

Proof: If $R\left(f(\lambda), f^{\prime}(\lambda)\right) \neq 0$, then the characteristic equation of $\tilde{Q}_{k}$ has $k$ distinct roots, say $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}$. Since the matrix $\tilde{Q}_{k}$ is diagonalizable, there exists a matrix $\Lambda$ such that $\Lambda^{-1} \tilde{Q}_{k} \Lambda=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}\right)$. Then $\tilde{\mathbf{G}}_{n+k-1}(k)=\Lambda \operatorname{diag}\left(\lambda_{1}^{n}, \lambda_{2}^{n}, \ldots, \lambda_{k}^{n}\right) \Lambda^{-1} \mathbf{s}$, and hence we have

$$
G_{n}(k, a)=d_{1} \lambda_{1}^{n}+d_{2} \lambda_{2}^{n}+\cdots+d_{k} \lambda_{k}^{n}=\sum_{i=1}^{k} d_{i} \lambda_{i}^{n},
$$

where $d_{1}, d_{2}, \ldots, d_{k}$ are complex numbers independent of $n$. We can determine the values of $d_{1}, d_{2}, \ldots, d_{k}$ by Cramer's rule. That is, by setting $n=0,1, \ldots, k-1$, we have

$$
\begin{aligned}
G_{0}(k, a) & =d_{1}+d_{2}+\cdots+d_{k} \\
G_{1}(k, a) & =d_{1} \lambda_{1}+d_{2} \lambda_{2}+\cdots+d_{k} \lambda_{k} \\
& \vdots \\
G_{k-1}(k, a) & =d_{1} \lambda_{1}^{k-1}+d_{2} \lambda_{2}^{k-1}+\cdots+d_{k} \lambda_{k}^{k-1},
\end{aligned}
$$

and hence

$$
\begin{equation*}
V \mathbf{d}=\mathbf{s}, \quad \mathbf{d}=\left(d_{1}, d_{2}, \ldots, d_{k}\right)^{T} . \tag{8}
\end{equation*}
$$

Therefore, we now have the desired result from (8).
Recall that

$$
\tilde{Q}_{k}=\left[\begin{array}{cc}
\mathbf{p} & p_{k} \\
I_{k-1} & \mathbf{0}
\end{array}\right]
$$

where $\left[\mathrm{p}=p_{1}, p_{2}, \ldots, p_{k-1}\right]$. Then, in [1], we have the following theorem.
Theorem 3.2 [1]: The $(i, j)$ entry $q_{i j}^{(n)}\left(p_{1}, p_{2}, \ldots, p_{k}\right)$ in $\tilde{Q}_{k}^{n}$ is given by the following formula:

$$
\begin{align*}
q_{i j}^{(n)}\left(p_{1}, p_{2}, \ldots, p_{k}\right)= & \sum_{\left(m_{1}, \ldots, m_{k}\right)} \frac{m_{j}+m_{j+1}+\cdots+m_{k}}{m_{1}+\cdots+m_{k}} \\
& \times\binom{ m_{1}+\cdots+m_{k}}{m_{1}, m_{2}, \ldots, m_{k}} p_{1}^{m_{1}} \ldots p_{k}^{m_{k}} \tag{9}
\end{align*}
$$

where the summation is over nonnegative integers satisfying $m_{1}+2 m_{2}+\cdots+k m_{k}=n-i+j$, and the coefficient in (9) is defined to be 1 if $n=i-j$.

Applying the $\tilde{\mathbf{G}}_{n+k-1}(k)=\tilde{Q}_{k}^{n}$ s to the above theorem, we have

$$
\begin{align*}
G_{n}(k, a)= & q_{k 1}^{(n)}\left(p_{1}, \ldots, p_{k}\right) s_{k-1}+q_{k 2}^{(n)}\left(p_{1}, \ldots, p_{k}\right) s_{k-2}+ \\
& \cdots+q_{k k}^{(n)}\left(p_{1}, \ldots, p_{k}\right) s_{0} \\
= & \sum_{j=1}^{k} q_{k j}^{(n)}\left(p_{1}, \ldots, p_{k}\right) s_{k-j} . \tag{10}
\end{align*}
$$

From (9), we have

$$
\begin{aligned}
q_{k j}^{(n)}\left(p_{1}, \ldots, p_{k}\right)= & \sum_{\left(m_{1}, \ldots, m_{k}\right)} \frac{m_{j}+m_{j+1}+\cdots+m_{k}}{m_{1}+\cdots+m_{k}} \\
& \times\binom{ m_{1}+\cdots+m_{k}}{m_{1}, m_{2}, \ldots, m_{k}} p_{1}^{m_{1}} \ldots p_{k}^{m_{k}},
\end{aligned}
$$

where the summation is over nonnegative integers satisfying $m_{1}+2 m_{2}+\cdots+k m_{k}=n-k+j$, and the coefficient in (10) is defined to be 1 if $n=k-j$.

Hence, from theorem 3.1 and (10),

$$
\begin{aligned}
G_{n}(k, a) & =\sum_{j=1}^{k} q_{k j}^{(n)}\left(p_{1}, \ldots, p_{k}\right) s_{k-j} \\
& =\sum_{i=1}^{k} d_{i} \lambda_{i}^{n}
\end{aligned}
$$

Example: In (6) and (7), if we take $a=0, s_{0}=s_{1}=\cdots=s_{k-3}=0, s_{k-2}=s_{k-1}=1$ and $p_{1}=\cdots=p_{k}=1$, then

$$
G_{0}(k, 0)=\cdots=G_{k-3}(k, 0)=0, G_{k-2}(k, 0)=G_{k-1}(k, 0)=1
$$

and for $n \geq k \geq 2$,

$$
\begin{aligned}
G_{n}(k, 0) & =G_{n-1}(k, 0)+G_{n-2}(k, 0)+\cdots+G_{n-k}(k, 0) \\
& =g_{n}=g_{n-1}+g_{n-2}+\cdots+g_{n-k}
\end{aligned}
$$

Let $\tilde{\mathrm{g}}_{n}^{(k)}=\left(g_{n}^{(k)}, \ldots, g_{n-k+1}^{(k)}\right)^{T}$. For any $k \geq 2$, the fundamental recurrence relation $g_{n}^{(k)}=g_{n-1}^{(k)}+g_{n-2}^{(k)}+\cdots+g_{n-k}^{(k)}$ can be defined by the vector recurrence relation $\tilde{\mathbf{g}}_{n}^{(k)}=\tilde{Q}_{k} \tilde{\mathbf{g}}_{n-1}^{(k)}$. Then, we have $\tilde{\mathbf{g}}_{n}^{(k)}=\tilde{Q}_{k}^{n} \tilde{\mathbf{g}}_{0}^{(k)}=\tilde{Q}_{k}^{n}(1,1,0, \ldots, 0)^{T}$. Since $\tilde{Q}_{k}$ has $k$ distinct eigenvalues (see [5]),

$$
g_{n}^{(k)}=d_{1} \lambda_{1}^{n}+\cdots+d_{k} \lambda_{k}^{n}
$$

Hence, we can determine $d_{1}, d_{2}, \ldots, d_{k}$ from (8).
For example, if $k=3$, then the characteristic equation of $\tilde{Q}_{3}$ is $f(\lambda)=\lambda^{3}-\lambda^{2}-\lambda-1=0$, and hence

$$
R\left(f(\lambda), f^{\prime}(\lambda)\right)=\left|\begin{array}{ccccc}
1 & -1 & -1 & -1 & 0 \\
0 & 1 & -1 & -1 & -1 \\
3 & -2 & -1 & 0 & 0 \\
0 & 3 & -2 & -1 & 0 \\
0 & 0 & 3 & -2 & -1
\end{array}\right|=44 \neq 0
$$

Thus $f(\lambda)=0$ has 3 distinct roots. Suppose $\alpha, \beta$ and $\gamma$ are the distinct roots of $f(\lambda)=0$. Then we have

$$
\begin{aligned}
& \alpha=\frac{1}{3}(u+v)+\frac{1}{3} \\
& \beta=-\frac{1}{6}(u+v)+\frac{i \sqrt{3}}{6}(u-v)+\frac{1}{3} \\
& \gamma=-\frac{1}{6}(u+v)-\frac{i \sqrt{3}}{6}(u-v)+\frac{1}{3}
\end{aligned}
$$

where

$$
i=\sqrt{-1}, \quad u=\sqrt[3]{19+3 \sqrt{33}} \text { and } \quad v=\sqrt[3]{19-3 \sqrt{33}}
$$

So, we have

$$
\begin{equation*}
g_{n}^{(3)}=d_{1} \alpha^{n}+d_{2} \beta^{n}+d_{3} \gamma^{n} \tag{11}
\end{equation*}
$$

and hence

$$
\left[\begin{array}{ccc}
1 & 1 & 1 \\
\alpha & \beta & \gamma \\
\alpha^{2} & \beta^{2} & \gamma^{2}
\end{array}\right]\left[\begin{array}{l}
d_{1} \\
d_{2} \\
d_{3}
\end{array}\right]=\left[\begin{array}{l}
0 \\
1 \\
1
\end{array}\right]
$$

Set

$$
\delta=\operatorname{det}\left[\begin{array}{ccc}
1 & 1 & 1 \\
\alpha & \beta & \gamma \\
\alpha^{2} & \beta^{2} & \gamma^{2}
\end{array}\right], \delta_{\alpha}=\operatorname{det}\left[\begin{array}{ccc}
0 & 1 & 1 \\
1 & \beta & \gamma \\
1 & \beta^{2} & \gamma^{2}
\end{array}\right], \delta_{\beta}=\operatorname{det}\left[\begin{array}{ccc}
1 & 0 & 1 \\
\alpha & 1 & \gamma \\
\alpha^{2} & 1 & \gamma^{2}
\end{array}\right]
$$

and

$$
\delta_{\lambda}=\operatorname{det}\left[\begin{array}{ccc}
1 & 1 & 0 \\
\alpha & \beta & 1 \\
\alpha^{2} & \beta^{2} & 1
\end{array}\right]
$$

Then we have

$$
d_{1}=\frac{\delta_{\alpha}}{\delta}, d_{2}=\frac{\delta_{\beta}}{\delta}, \quad \text { and } d_{3}=\frac{\delta_{\gamma}}{\delta}
$$

As we know, the complex numbers $d_{1}, d_{2}$, and $d_{3}$ are independent of $n$.
We can also find an expression for $g_{n}^{(3)}$ in [6] follows:

$$
\begin{equation*}
g_{n}^{(3)}=\frac{\left(g_{n-1}^{(3)}+g_{n-2}^{(3)}\right)(\beta-\gamma)-\left(\beta^{n}-\alpha^{n}\right)}{(\alpha-1)(\beta-\gamma)} \tag{12}
\end{equation*}
$$

So, by (11) and (12),

$$
\frac{\delta_{\alpha} \alpha^{n}+\delta_{\beta} \beta^{n}+\delta_{\gamma} \gamma^{n}}{\delta}=\frac{\left(g_{n-1}^{(3)}+g_{n-2}^{(3)}\right)(\beta-\gamma)-\left(\beta^{n}-\alpha^{n}\right)}{(\alpha-1)(\beta-\gamma)}
$$

Similarly, if $k=2$, then

$$
\begin{equation*}
g_{n}^{(2)}=F_{n}=\frac{1}{\lambda_{1}-\lambda_{2}}\left(\lambda_{1}^{n}-\lambda_{2}^{n}\right), \tag{13}
\end{equation*}
$$

where $\lambda_{1}$ and $\lambda_{2}$ are the eigenvalues of $Q_{2}$. Actually

$$
\lambda_{1}=\frac{1+\sqrt{5}}{2} \text { and } \lambda_{2}=\frac{1-\sqrt{5}}{2} .
$$

In this case,

$$
d_{1}=\frac{1}{\lambda_{1}-\lambda_{2}}=\frac{1}{\sqrt{5}}, \quad d_{2}=\frac{1}{\lambda_{2}-\lambda_{1}}=-\frac{1}{\sqrt{5}}
$$

and (13) is Binet's formula for the $n$th Fibonacci number $F_{n}$.

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# THE MULTIPLICATIVE GROUP GENERATED BY THE LEHMER NUMBERS 

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## 1. INTRODUCTION

Let $\left(u_{n}\right)_{n \geq 0}$ be a sequence of positive integers. We denote by $G(u)$ the multiplicative subgroup of $\mathbf{Q}_{+}^{*}$ generated by all the members of $\left(u_{n}\right)_{n \geq 0}$. That is,

$$
\begin{equation*}
G(u)=\left\{u_{n_{1}}^{\alpha_{1}} \cdot u_{n_{2}}^{\alpha_{2}} \cdot \ldots \cdot u_{n_{s}}^{\alpha_{s}} \mid s \geq 0,0 \leq n_{1}<n_{2}<\cdots<n_{s}, \text { and } \alpha_{i} \in \mathbb{Z}^{*} \text { for } i=1,2, \ldots, s\right\} \tag{1.1}
\end{equation*}
$$

In some cases, the group $G(u)$ is very easy to understand. For example, if $\left(u_{n}\right)_{n \geq 0}$ is a geometrical progression of first term $u_{0}$ and ratio $r=u_{1} / u_{0}$, then

$$
\begin{equation*}
G(u)=\left\{u_{0}^{\alpha} r^{\beta} \mid \text { for some } \alpha, \beta \in \mathbb{Z}\right\} \tag{1.2}
\end{equation*}
$$

For a sequence $\left(u_{n}\right)_{n \geq 0}$ we also denote by

$$
\begin{equation*}
U=\left\{m \in \mathbf{N} \mid m=u_{n} \text { for some } n \geq 0\right\} \tag{1.3}
\end{equation*}
$$

That is, $U$ is the range of the sequence $\left(u_{n}\right)_{n \geq 0}$. In this paper, we look at the set $G(u) \cap \mathbf{N}$. Certainly, $U \subseteq G(u) \cap \mathbf{N} \subseteq \mathbf{N}$. It is easy to see that the extreme cases of the above inclusions can occur in some non-trivial instances. For example, if $u_{n}=n$ ! for all $n \geq 0$, then $m=$ $u_{m} / u_{m-1}$ for all $m \geq 1$, therefore $G(u)=\mathbf{N}$. However, if $\left(u_{n}\right)_{n \geq 0}$ is an arithmetical progression of first term 1 and difference $k>1$, then $G(u) \cap \mathbf{N}=U$. Indeed, notice that $1=u_{0} \in U$, and that if we write some $m \in G(u) \cap \mathbf{N}, m \neq 1$ as

$$
\begin{equation*}
m=\prod_{i=1}^{s} u_{n_{i}}^{\alpha_{i}}, \text { for some } s \geq 1 \text { and } \alpha_{i} \in \mathbf{Z}^{*} \text { for } i=1,2, \ldots, s \tag{1.4}
\end{equation*}
$$

then we can rearrange equation (1.4) as

$$
\begin{equation*}
m \prod_{\substack{1 \leq i \leq s \\ \alpha_{i} \leq 0}} u_{n_{i}}^{-\alpha_{i}}=\prod_{\substack{1 \leq i \leq s \\ \alpha_{i}>0}} u_{n_{i}}^{\alpha_{i}} \tag{1.5}
\end{equation*}
$$

We may now reduce equation (1.5) modulo $k$ and get $m \equiv 1(\bmod k)$, therefore $m \in U$. While both the group $G(u)$ and the semigroup $G(u) \cap \mathbf{N}$ are very easy to understand for the above mentioned sequences $\left(u_{n}\right)_{n \geq 0}$, not the same is true when $\left(u_{n}\right)_{n \geq 0}$ is a non-degenerate linearly recurrent sequence. In this note, we investigate the group $G \overline{(u)}$ and the semigroup $G(u) \cap \mathbf{N}$ when $\left(u_{n}\right)_{n \geq 0}$ is a Lehmer sequence.

Recall that if $L$ and $M$ are two non-zero coprime integers with $L-4 M \neq 0$, then the $n^{\text {th }}$ Lehmer number corresponding to the pair $(L, M)$ and denoted by $P_{n}$ is defined as

$$
P_{n}= \begin{cases}\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta} & \text { for } n \equiv 1(\bmod 2)  \tag{1.6}\\ \frac{\alpha^{n}-\beta^{n}}{\alpha^{2}-\beta^{2}} & \text { otherwise }\end{cases}
$$

where $\alpha$ and $\beta$ are the two roots of the characteristic equation

$$
\begin{equation*}
x^{2}-\sqrt{L} x+M=0 \tag{1.7}
\end{equation*}
$$

To avoid degenerate cases, we assume that $\alpha / \beta$ is not a root of 1 . In what follows, we denote by $u_{n}=\left|P_{n}\right|$ and by $G=G(u)$. Our main results say that though the set $G$ is topologically dense in the set of non-negative real numbers, its asymptotic density in the set of positive integers is zero. Before stating it, we introduce one more notation.

For every positive real number $x$ let $G(x)=G \cap \mathbf{N} \cap(0, x)$. For every finite set of prime numbers $\mathcal{P}$, let $G_{\mathcal{P}}$ be the subgroup of $\mathbb{Q}_{+}^{*}$ generated by $G$ and $\mathcal{P}$. If $x$ is a positive real number, we denote $G_{\mathcal{P}}(x)=G_{\mathcal{P}} \cap \mathbb{N} \cap(0, x)$.

We have the following results.
Theorem 1: The set $G$ is dense in the set of non-negative real numbers.
Theorem 2: For any positive number $\delta$ there exists a computable constant $C$ depending on $\delta, \mathcal{P}, L$ and $M$ such that

$$
\begin{equation*}
\# G_{\mathcal{P}}(x)<\frac{x}{(\log x)^{\delta}} \tag{1.8}
\end{equation*}
$$

holds for all $x>C$.
The above Theorem 2 has the following immediate consequence.
Corollary 1:
(i) Both the group $G$ and the factor group $Q_{+}^{*} / G$ are infinitely generated.
(ii) There exist infinitely many prime numbers $p$ which do not belong to $G$.
(iii) There exist infintely many n's such that $n$ ! does not belong to $G$.

Since the group $G$ is a subgroup of $\mathrm{Q}_{+}^{*}$, we know that $G$ contains no torsion elements.
However, this is not necessarily the case for the factor group $\mathrm{Q}_{+}^{*} / G$. Let $\bar{G}=\mathrm{Q}_{+}^{*} / G$. Since $\bar{G}$ is abelian, it follows that $\bar{G}$ has a torsion part, let's call it $T(\bar{G})$, and

$$
\begin{equation*}
F(\bar{G}):=\frac{\bar{G}}{T(\bar{G})} \tag{1.9}
\end{equation*}
$$

is torsion free. The following result is slightly stronger version of the above Corollary.
Proposition: The group $F(\bar{G})$ contains a free subgroup of infinite rank.
The following Corollary follows from the above Proposition and is a stronger version of Corollary 1 (ii).
Corollary 2: There exist infinitely many prime numbers $p$ such that $p^{k} \notin G$ for any positive integer $k$.

## 3. THE PROOFS

The Proof of Theorem 1: It is proved in Lemma 2 of [3] that if $p$ and $q$ are two coprime integers with $1<p<q$, then each non-negative real number is a limit-point of the set of all fractions of the form $p^{m} q^{-n}$, where $m$ and $n$ are positive integers. Since for all positive integers $k$ and $s$ we have $\left(u_{k}, u_{s}\right)=u_{(k, s)}$, the above result applied to positive integers $u_{s} / u_{(k, s)}$ and $u_{k} / u_{(k, s)}$ proves Theorem 1.

We now proceed to the proof of Theorem 2.
In what follows, we recall the definition of a primitive prime divisor of a term of a Lehmer sequence. It is well known that $u_{n} \mid u_{m}$ whenever $n \mid m$. A primitive prime divisor of $u_{m}$ is defined to be a prime number $p \mid u_{m}$ such that $p \nmid u_{n}$ for any $n<m$. Moreover, an intrinsic primitive prime divisor of $u_{m}$ is defined to be a primitive prime divisor $p$ of $u_{m}$ such that $p$ does not divide the discriminant $\Delta=L-4 M$ of $\left(u_{n}\right)_{n \geq 0}$. In order not to complicate the terminology, in what follows we will refer to an intrinsic primitive prime divisor of $u_{m}$ as simply a primitive divisor of $u_{m}$. By results of Ward [5] for the case in which $\left(u_{n}\right)_{n \geq 0}$ has positive discriminant, and Bilu, Hanrot and Voutier [1] for the general case, we know that $u_{m}$ has a primitive divisor for all $m>30$. It is also well known that any primitive divisor $p$ of $u_{m}$ satisifies $p \equiv \pm 1(\bmod m)$.

For every finite set of prime numbers $\mathcal{P}$ we denote by

$$
\begin{equation*}
M_{\mathcal{P}}=\max (30, p+1 \mid p \in \mathcal{P}) . \tag{2.1}
\end{equation*}
$$

When $\mathcal{P}$ is empty, we simply set $M=M_{\emptyset}=30$. From the above remarks, it follows that whenever $n>M_{\mathcal{P}}, u_{n}$ has primitive divisors and none of them belongs to $\mathcal{P}$.

We begin by pointing out a large free subgroup of $G$.

## Lemma 1:

(i) Let $G_{1}$ be the subgroup of $G$ generated by the set $\left\{u_{n}\right\}_{1 \leq n \leq 30}$ and $G_{2}$ be the subgroup of $G$ generated by the set $\left\{u_{n}\right\}_{n>30}$. Then, $G_{2}$ is free on the set of generators $\left\{u_{n}\right\}_{n>30}$ and $G$ is the direct product of $G_{1}$ and $G_{2}$.
(ii) Let $G_{1, \mathcal{P}}$ be the subgroup of $G_{\mathcal{P}}$ generated by the set $\mathcal{P} \cup\left\{u_{n}\right\}_{n \leq M_{\mathcal{P}}}$ let $G_{2, \mathcal{P}}$ be the subgroup of $G_{\mathcal{P}}$ generated by the set $\left\{u_{n}\right\}_{n>M_{\mathcal{P}}}$. Then, $G_{2, \mathcal{P}}$ is free on the set of generators $\left\{u_{n}\right\}_{n>M_{\mathcal{P}}}$ and $G_{\mathcal{P}}$ is the direct product of $G_{1, \mathcal{P}}$ and $G_{2, \mathcal{P}}$.
The Proof of Lemma 1: We prove only (i) as the proof of (ii) is entirely similar. It is clear that $G$ is the product of $G_{1}$ and $G_{2}$. In order to prove that this product is direct and that $G_{2}$ is indeed free on the indicated set of generators, it suffices to show that if

$$
\begin{equation*}
\prod_{i=1}^{s} u_{n_{i}}^{\alpha_{i}}=1, \quad \text { for some } s \geq 1, \quad \alpha_{i} \in \mathbf{Z}^{*} \text { and } n_{1}<n_{2}<\cdots<n_{s} \tag{2.2}
\end{equation*}
$$

then $n_{s} \leq 30$. But this follows right away because $u_{n}$ has a primitive divisor of $n>30$.
Let $g \in G_{\mathcal{P}} \backslash G_{1, \mathcal{P}}$. By the definition of $G_{\mathcal{P}} \backslash G_{1, \mathcal{P}}$, it follows that one may write

$$
\begin{equation*}
g=\prod_{p \in \mathcal{P}} p^{\beta_{p}} \prod_{i=1}^{s} u_{n_{i}}^{\alpha_{i}} \tag{2.3}
\end{equation*}
$$

where $\beta_{p} \in \mathbb{Z}$ for all $p \in \mathcal{P}, s \geq 1, \alpha_{i} \in \mathbb{Z}^{*}$ for $i=1,2, \ldots, s$ and $n_{1}<n_{2}<\cdots<n_{s}$ with $n_{s}>M_{\mathcal{P}}$. Of course, the above representation (2.3) for $g$ need not be unique. However, by Lemma 1 above, we get that both the index $n_{s}$ and the exponent $\alpha_{s}$ of $u_{n_{s}}$ do not depend on the representation of $g$ of the form (2.3). Thus, we may define two functions $f, h: G_{\mathcal{P}} \backslash G_{1, \mathcal{P}} \rightarrow \mathbb{Z}$ by $f(g)=n_{s}$ and $h(g)=\alpha_{s}$. We also extend the function $f$ to the whole $G_{\mathcal{P}}$ by simply setting $f(g)=M_{\mathcal{P}}$ when $g \in G_{1, \mathcal{P}}$.

The following observation is relevant in what follows.
Lemma 2: Assume that $g \in G_{\mathcal{P}} \backslash G_{1, \mathcal{P}}$. If $g \in \mathbf{N}$, then $h(g)>0$.
The Proof of Lemma 2: This is almost obvious. Indeed, assume that $g$ is given by formula (2.3) and that $\alpha_{s}<0$. Since $n_{s}>M_{\mathcal{P}}$, it follows that $u_{n_{s}}$ has primitive divisors. Pick a primitive divisor $q$ of $u_{n_{s}}$. By the remarks preceeding Lemma 1, we know that $q \notin \mathcal{P}$. Since $g \in \mathbf{N}$ and $\alpha_{s}<0$, formula (2.3) implies that

$$
\begin{equation*}
q \mid \prod_{p \in \mathcal{P}} p \prod_{1 \leq j<n_{s}} u_{j} \tag{2.4}
\end{equation*}
$$

which is obviously impossible.
The Proof of Theorem 2: We assume that $|\alpha| \geq|\beta|$. Notice that $|\alpha|>1$. For any $n>30$, we denote by $\operatorname{Pr}(n)$ the primitive part of $u_{n}$. That is, $\operatorname{Pr}(n)$ is the product of all the primitive prime divisors of $u_{n}$ at the powers at which they appear in the prime factor decomposition of $u_{n}$. It is well known (see [4]), that if we denote by $\zeta_{i}$ all the primitive roots of unity of order $n$ for $i=1,2, \ldots, \phi(n)$, then

$$
\begin{equation*}
\operatorname{Pr}(n)=\frac{\left|\Phi_{n}(\alpha, \beta)\right|}{q(n)} \tag{2.5}
\end{equation*}
$$

where

$$
\Phi_{n}(X, Y)=\prod_{i=1}^{\phi(n)}\left(X-\zeta_{i} Y\right) \in \mathbb{Z}[X, Y]
$$

is the homogenized version of the $n^{\text {th }}$ cyclotomic polynomial and $q(n)$ is either 1 or the largest prime factor of $n$. We also denote by $\operatorname{Pr}_{\mathcal{P}}(n)$ the primitive part of $u_{n}$ which is coprime to all the prime numbers $p \in \mathcal{P}$. By using linear forms in logarithms, both complex and $p$-adic
with respect to the primes $p \in \mathcal{P}$ (see [4]), it follows easily that there exist two effectively computable constants $c_{1}$ and $c_{2}$ depending on $L, M$ and $\mathcal{P}$ such that

$$
\begin{equation*}
\operatorname{Pr}_{\mathcal{P}}(n)>|\alpha|^{\phi(n)-c_{1} d(n) \log n}, \quad \text { whenever } n>c_{2} \tag{2.6}
\end{equation*}
$$

where $d(n)$ is the number of divisors of $n$. Since $d(n)<n^{\epsilon}$ for every $\epsilon>0$ provided that $n$ is large enough (with respect to $\epsilon$ ) and since

$$
\begin{equation*}
\phi(n)>\frac{c_{3} n}{\log \log n}, \quad \text { whenever } n>c_{4} \tag{2.7}
\end{equation*}
$$

for some absolute constants $c_{3}$ and $c_{4}$, it follows that there exists a constant $c_{5}$ (depending on $L, M$ and $\mathcal{P}$ ) such that

$$
\begin{equation*}
\operatorname{Pr} \operatorname{P}(n)>e^{\sqrt{n}}, \quad \text { whenever } n>c_{5} \tag{2.8}
\end{equation*}
$$

We may assume that $c_{5}>30$.
We now look at the elements $g \in G_{\mathcal{P}} \cap \mathbf{N}$. Let $y$ be a very large positive real number $(y>30)$, and set

$$
\begin{equation*}
A(y)=\left\{g \in G_{\mathcal{P}} \cap \mathbf{N} \mid f(g)<y\right\} \tag{2.9}
\end{equation*}
$$

and

$$
\begin{equation*}
B(y)=\left\{g \in G_{\mathcal{P}} \cap \mathbf{N} \mid f(g) \geq y\right\} \tag{2.10}
\end{equation*}
$$

Certainly, $G_{\mathcal{P}} \cap \mathbf{N}=A(y) \cup B(y)$ holds for every $y$. For a real number $x$ set $A(x, y)=$ $A(y) \cap(0, x)$ and $B(x, y)=B(y) \cap(0, x)$. Thus, in order to bound the cardinality of $G_{\mathcal{P}}(x)$, it suffices to bound both the cardinality of $A(x, y)$ and $B(x, y)$.

We start by bounding the cardinality of $A(x, y)$. Assume that $q_{1}<q_{2}<\cdots<q_{k}$ are all the possible prime factors of an integer $g \in A(x, y)$. Then,

$$
\begin{equation*}
\prod_{i=1}^{k} q_{i} \mid \prod_{p \in \mathcal{P}} p \cdot \prod_{j \leq y} u_{j} \tag{2.11}
\end{equation*}
$$

Since $\mathcal{P}$ is fixed and since $u_{n}<(2|\alpha|)^{n}$ holds for all $n \geq 1$, it follows that there exists a constant $c_{6}$ (depending on $L, M$ and $\mathcal{P}$ ) such that

$$
\begin{equation*}
\prod_{i=1}^{k} q_{i}<e^{c_{6} y^{2}} \tag{2.12}
\end{equation*}
$$

From the Prime Number Theorem, we know that there exists an absolute constant $c_{7}>0$ such that

$$
\begin{equation*}
e^{c_{7} k}<\prod_{i=1}^{k} q_{i} \tag{2.13}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
k<c_{8} y^{2} \tag{2.14}
\end{equation*}
$$

where $c_{8}=c_{6} / c_{7}$. Assume now that $g \in A(x, y)$ has the prime factor decomposition

$$
\begin{equation*}
g=\prod_{i=1}^{k} q_{i}^{\mu_{i}}, \quad \text { where } \mu_{i} \geq 0 \text { for } i=1,2, \ldots, k \tag{2.15}
\end{equation*}
$$

Since $g \leq x$, it follows that

$$
\begin{equation*}
\mu_{i} \leq \frac{\log x}{\log q_{i}} \leq \frac{\log x}{\log 2} \quad \text { for all } i=1,2, \ldots, k \tag{2.16}
\end{equation*}
$$

From inequalities (2.14) and (2.15), it follows that there exists a constant $c_{9}$ such that

$$
\begin{equation*}
\# A(x, y)<(\log x)^{c_{9} y^{2}}, \quad \text { for all } x \geq 3 \tag{2.17}
\end{equation*}
$$

The above inequality (2.17) holds for all $y>30$.
We now bound the cardinality of $B(x, y)$ for $y$ large enough.
Assume that $y>M_{\mathcal{P}}$ and assume that $g \in B(x, y)$. From the definition of $B(x, y)$, it follows that $f(g) \geq y$. Moreover, from Lemma 2, it follows that $h(g)>0$. By writing

$$
\begin{equation*}
g=\prod_{p \in \mathcal{P}} p^{\beta_{p}} \prod_{i=1}^{s} u_{n_{i}}^{\alpha_{i}} \tag{2.18}
\end{equation*}
$$

where $\beta_{p} \in \mathbb{Z}, s \geq 1, \alpha_{i} \in \mathbb{Z}^{*}$ for $i=1,2, \ldots, s$ and $n_{1}<n_{2}<\cdots<n_{s}$, with $n_{s}=f(g)>y$ and $\alpha_{s}=h(g)>0$, we get that the positive integer $g$ is a multiple of $\operatorname{Pr}(f(g))$. There are at most

$$
\frac{x}{\operatorname{Pr}(f(g))}
$$

positive integers less than $x$ which are multiples of $\operatorname{Prp}_{\mathcal{P}}(f(g))$. Hence, this argument shows that the cardinality of $B(x, y)$ is bounded above by

$$
\begin{equation*}
\# B(x, y) \leq \sum_{t \geq y} \frac{x}{\operatorname{Pr}(\mathcal{P}(t)} \tag{2.19}
\end{equation*}
$$

We now assume that $y>c_{5}$ and use the lower bound (2.8) on $\operatorname{PrP}_{\mathcal{P}}(t)$ for $t \geq y>c_{5}$ to infer that

$$
\begin{equation*}
\# B(x, y) \leq \sum_{t \geq y} \frac{x}{e^{\sqrt{t}}} \tag{2.20}
\end{equation*}
$$

By inequality (2.20), it follows that there exists an absolute constant $c_{10}$ such that

$$
\begin{equation*}
\# B(x, y)<\frac{c_{10} \sqrt{y}}{e^{\sqrt{y}}} \cdot x \tag{2.21}
\end{equation*}
$$

Combining inequalities (2.17) and (2.21), we get that

$$
\begin{equation*}
G_{\mathcal{P}}(x)<(\log x)^{c_{9} y^{2}}+\frac{c_{10} \sqrt{y}}{e^{\sqrt{y}}} \cdot x, \quad \text { provided that } x \geq 3 \text { and } y \geq c_{5} \tag{2.22}
\end{equation*}
$$

All it remains to show is that one may choose $y$ (depending on $x$ ) such that

$$
\begin{equation*}
(\log x)^{c_{9} y^{2}}+\frac{c_{10} \sqrt{y}}{e^{\sqrt{y}}} \cdot x<\frac{x}{(\log x)^{\delta}} \tag{2.23}
\end{equation*}
$$

To see how (2.23) holds, we choose any $\epsilon>0$ small enough and set

$$
\begin{equation*}
y=(\log x)^{\frac{1}{2}-\epsilon} \tag{2.24}
\end{equation*}
$$

Clearly, $y \geq c_{5}$ when $x$ is large enough. Moreover, the inequality

$$
\begin{equation*}
(\log x)^{c_{9} y^{2}}<\frac{x}{2(\log x)^{\delta}} \tag{2.25}
\end{equation*}
$$

is equivalent to

$$
\left(c_{9} y^{2}+\delta\right) \log \log x<\log x-\log 2
$$

or

$$
\left(c_{9}(\log x)^{1-2 \epsilon}+\delta\right) \log \log x<\log x-\log 2
$$

which certainly holds for $x$ large enough. Finally, the inequality

$$
\begin{equation*}
\frac{c_{10} \sqrt{y}}{e^{\sqrt{y}}} \cdot x<\frac{x}{2(\log x)^{\delta}} \tag{2.26}
\end{equation*}
$$

is equivalent to

$$
\log 2 c_{10}+\frac{1}{2} \log y+\delta \log \log x<\sqrt{y}
$$

or

$$
\log 2 c_{10}+\left(\frac{1}{2}\left(\frac{1}{2}-\epsilon\right)+\delta\right) \log \log x<(\log x)^{\frac{1}{4}-\frac{\epsilon}{2}}
$$

which is again satisfied for $x$ large enough. Inequalities (2.25) and (2.26) imply inequality (2.23).

Theorem 2 is therefore proved.
The Proof of Corollary 1:
(i). The fact that $G$ is infinitely generated follows from Lemma 1. Assume now that $\mathbf{Q}_{+}^{*} / G$ is finitely generated. It now follows that there exists a finite set of prime numbers, call it $\mathcal{P}$, such that $G_{\mathcal{P}}=\mathbf{Q}_{+}^{*}$. It now follows that $G_{\mathcal{P}} \cap \mathbf{N}=\mathbf{N}$, which contradicts the Theorem 2.
(ii). If there are only finitely many prime numbers $p$ not belonging to $G$, then $\mathrm{Q}_{+}^{*} / G$ is finitely generated, which contradicts (i).
(iii). Assume that there exists $n_{0} \in \mathbb{N}$ such that $n!\in G$ for all $n>n_{0}$. Since $n=n!/(n-1)!\in$ $G$, whenever $n>n_{0}+1$, it follows that $\mathrm{Q}_{+}^{*} / G$ is finitely generated, which contradicts (i).

We now give the proof of the Proposition. This proof is based on the following Lemma due to Schinzel (see [2]).
Lemma 3: There exists a strictly increasing sequence of integers $\left(m_{i}\right)_{i \geq 1}$ with $m_{1}>30$ such that $u_{m_{i}}$, has at least two primitive divisors.

Using Lemma 3 above and the Axiom of Choice, it follows that one may select an infinite set of prime numbers $\mathcal{Q}=\left\{q_{i}\right\}_{i \geq 1}$ such that $q_{i}$ is a primitive divisor of $u_{m_{i}}$ for all $i \geq 1$. We introduce on $\mathcal{Q}$ the order relation induced by the natural ordering of the orders of apparition $m_{i}$ 's of the $q_{i}$ 's and denote this by $q_{i} \prec q_{i+1}$ for all $i \geq 1$. Based on Lemma 3 above, we infer the following auxiliary result.
Lemma 4: With the above notations, let $G_{3}$ be the subgroup of $\mathbb{Q}_{+}^{*}$ generated by the set $\mathcal{Q}$. Then, $G \cap G_{3}=\{1\}$.
The Proof of Lemma 4: Assume that this is not so and let $g \in G \cap G_{3} \backslash\{1\}$. It follows that

$$
\begin{equation*}
g=\prod_{i=1}^{s} u_{n_{i}}^{\alpha_{i}}=\prod_{j=1}^{t} q_{k_{j}}^{\beta_{j}} \tag{2.27}
\end{equation*}
$$

where $s \geq 1, t \geq 1, n_{1}<n_{2}<\cdots<n_{s}, k_{1}<k_{2}<\cdots<k_{t}$ and $\alpha_{i}, \beta_{j} \in \mathbb{Z}^{*}$ for $i=1,2, \ldots, s$ and $j=1,2, \ldots, t$. We first show that $n_{s}=m_{k_{t}}$. Indeed, since $q_{k_{t}} \mid \prod_{i=1}^{s} u_{n_{i}}$, and $q_{k_{t}}$ is a primitive divisor of $u_{m_{k_{t}}}$, it follows that there exists some $i$ with $1 \leq i \leq s$ such that $m_{k_{t}} \mid n_{i}$. In particular, $n_{s} \geq m_{k_{t}}$. Assume that $n_{s}>m_{k_{t}}$. Since $m_{k_{t}} \geq m_{1}>30$, it follows that $u_{n_{s}}$ has a primitive divisor, call it $q$. Since $q$ is a primitive divisor of $u_{n_{s}}$ and $n_{s}>n_{i}$ for all $i<s$, it follows that $q=q_{k_{j}}$ for some $j \leq t$. But this impossible because $q_{k_{j}}$ is a primitive divisor of $u_{m_{k_{j}}}$ and $m_{k_{j}} \leq m_{k_{t}}<n_{s}$. Thus, $n_{s}=m_{k_{t}}$. Now $u_{n_{s}}$ has at least two primitive prime divisors. Pick a primitive prime divisor $q$ of $u_{n_{s}}$ different than $q_{k_{t}}$. Arguments similar to the preceeding ones show that $q \nmid u_{n_{i}}$ for $i<s$ and $q \neq q_{k_{j}}$ for any $j \leq t$. This contradicts formula (2.27).

The Proof of the Proposition: The proof of the Proposition is contained in Lemma 3. Indeed, by Lemma 3 , it follows easily that the factor group $\bar{G}=\mathrm{Q}_{+}^{*} / G$ contains the subgroup
$G G_{3} / G \cong G_{3}$. This subgroup is free on the basis $\{q G \mid q \in \mathcal{Q}\}$. Thus, this subgroup can be identified with a subgroup of $F(\bar{G})$ and therefore $F(\bar{G})$ has a free subgroup of infinite rank.
The Proof of Corollary 2: This follows from the Proposition. Indeed, assume that there exist only finitely many prime numbers, call them $p_{1}, p_{2}, \ldots, p_{s}$, such that whenever $q$ is a prime number with $q \neq p_{i}$ for any $i=1,2, \ldots, s$, there exists $k>0$ (depending on $q$ ) such that $q^{k} \in G$. Since $q^{k} \in G$ is equivalent to the fact that the coset $q G$ has exponent $k$ in the factor group $\bar{G}=\mathrm{Q}_{+}^{*} / G$, it follows that $q G \in T(\bar{G})$, whenever $q \neq p_{i}$ for $i=1,2, \ldots, s$. Hence, $F(\bar{G})$ is finitely generated, which contradicts the Proposition.

## 3. AN EXAMPLE

The well known Fibonacci sequence $\left(F_{n}\right)_{n>0}$ is given by $F_{0}=0, F_{1}=1$ and $F_{n+2}=$ $F_{n+1}+F_{n}$ for all $n \geq 0$. The set of its terms $U=\left\{F_{n}\right\}_{n>0}$ coincides with the set of terms of the Lehmer sequence corresponding to the pair $(L, M)=(1,1)$. For this sequence, the only $n$ 's for which $F_{n}$ does not have a primitive divisor are $n=1,2,5,6,12$. Since $F_{1}=F_{2}=1, F_{5}=$ $5, F_{6}=F_{3}^{3}$ and $F_{12}=F_{3}^{4} F_{4}^{2}$, it follows, by Lemma 1 from the previous section, that the group $G$ for the Fibonacci sequence is free having the set $\left\{F_{n}\right\}_{n \neq 1,2,6,12}$ as basis. Since we know that $G \cap \mathbf{N}$ has density zero, it follows that $G$ does not contain all the positive integers. An easy computation shows that the first positive integer in $\mathbf{N} \backslash G$ is 37.

For this sequence, one can point out a nice structure by means of a trace map. That is, let $g \in G \backslash\{1\}$ and write $g$ as

$$
\begin{equation*}
g=\prod_{i=1}^{s} u_{n_{i}}^{\alpha_{i}} \tag{3.1}
\end{equation*}
$$

for some $s \geq 1$, where $\alpha_{i} \in \mathbb{Z}^{*}$ and $3 \leq n_{1}<n_{2}<\ldots n_{s}$ are such that $n_{i} \neq 6$ or 12 for any $i=1,2, \ldots, s$. From the above arguments, we know that every $g \in G \backslash\{1\}$ can be represented in this way and that such a representation is unique. Thus, we may define the trace of $g$ as

$$
\begin{equation*}
I(g)=\sum_{i=1}^{s} \alpha_{i} n_{i} \tag{3.2}
\end{equation*}
$$

When $g=1$, we simply set $I(1)=0$. It is easy to see that $I: G \rightarrow \mathbb{Z}$ is a group homomorphism whose kernel is $G_{0}=\{g \in G \mid I(g)=0\}$. Moreover, $G / G_{0} \cong \mathbb{Z}$. The subgroup $G_{0}$ of $G$ has a topological interpretation in the sense that it contains elements which are arbitrarily close to the identity 1 of $G$.

## 4. COMMENTS AND PROBLEMS

While our Theorem 2 guarantees that the density of the set $G \cap \mathbf{N}$ is zero, it seems reasonable to conjecture that, in fact, a much better upper bound for cardinality of the set $G_{\mathcal{P}}(x)$ than the one asserted at (1.8) holds. Thus, we propose the following problem.
Problem 1: Prove that for every $\epsilon>0$, there exists a computable constant $C$ depending only on $\epsilon, \mathcal{P}, L$ and $M$ such that

$$
\#\left\{m \in G_{\mathcal{P}} \cap \mathbf{N} \mid m \leq x\right\}<x^{\epsilon}
$$

holds for all $x>C$.
Assume that $m_{1}<m_{2}<\cdots<m_{n}<\ldots$ are all the elements of $G \cap \mathbf{N}$. Our result shows that for every $k$, there exists a computable constant $C_{k}$ such that $m_{n}>n(\log n)^{k}$ holds for all $n>C_{k}$. In particular, the series

$$
\begin{equation*}
\sum_{k \geq 1} \frac{1}{m_{k}} \tag{4.1}
\end{equation*}
$$

is convergent. It is certainly a very difficult problem to decide whether or not the number given by (4.1) is rational or irrational (or algebraic, respectively, transcendental).

Another interesting question to investigate would be the distribution of the positive integers $\left(m_{i}\right)_{i \geq 1}$. By Theorem 2, we know that the set of those integers has density zero. One may ask how fast does the sequence $\left(m_{i}\right)_{i \geq 1}$ grow. For example, if it were true that the sequence of differences $m_{i+1}-m_{i}$ diverges to infinity with $i$, then we would get an alternative proof for the fact that $G \cap \mathbf{N}$ has density zero. Unfortunately, such a statement need not be true in general. Indeed, let $\left(F_{n}\right)_{n \geq 0}$ be the Fibonacci sequence mentioned above and let $\left(L_{n}\right)_{n \geq 0}$ be its Lucas companion sequence. Then the identity

$$
\begin{equation*}
F_{n}^{2}-F_{n+1} F_{n-1}=(-1)^{n+1}, \quad \text { for all } n=0,1, \ldots \tag{4.2}
\end{equation*}
$$

provides infinitely many examples of positive integers $i$ for which $m_{i+1}-m_{i}=1$. Moreover, either one of the identities

$$
L_{n}^{2}-5 F_{n}^{2}=4 \cdot(-1)^{n}
$$

or

$$
L_{2 n}-L_{n}^{2}=2 \cdot(-1)^{n+1}
$$

which hold for all $n=0,1, \ldots$, together with the fact that $L_{n}=F_{2 n} / F_{n} \in G$ for all $n \geq 1$, provides infinitely many examples of positive integers $i$ for which $m_{i+1}-m_{i} \leq 4$. In our Proposition, we pointed out that the group $F(\bar{G})$ contains a free subgroup of infinite rank but we said nothing about the subgroup $T(\bar{G})$. Concerning the subgroup $T(\bar{G})$, we propose the following conjecture.
Problem 2: Prove that $T(\bar{G})$ is finite.
Finally, it could be of interest to analyze the dependence of the group $G$ of the starting Lehmer sequence $\left(P_{n}\right)_{n \geq 0}$. More precisely, assume that $\left(P_{n}\right)_{n \geq 0}$ and $\left(P_{n}^{\prime}\right)_{n \geq 0}$ are two Lehmer sequences. Let $u_{n}=\left|P_{n}\right|$ and $u_{n}^{\prime}=\left|P_{n}^{\prime}\right|$ and define $G, G^{\prime}$ and $U, U^{\prime}$ as before. We offer the following conjecture.
Problem 3: Prove that if $G \cap G^{\prime}$ is infinitely generated, then $U \cap U^{\prime}$ is infinite.
It is well-known, and it follows from the theory of linear forms in logarithms, that if $U \cap U^{\prime}$ is infinite, then there exist two arithmetical progressions $(a n+b)_{n \geq 0}$ and $(c n+d)_{n \geq 0}$ with $a b \neq 0$ such that $\left|P_{a n+b}=\left|P_{c n+d}^{\prime}\right|\right.$ holds for all $n \geq 0$. Thus, Problem 3 above is just a generalization of this well known result from diophantine equations.

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# ROUNDING THE SOLUTIONS OF FIBONACCI-LIKE DIFFERENCE EQUATIONS 

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## 1. INTRODUCTION

It is well known that the Fibonacci numbers can be expressed in the form

$$
\text { Round }\left\{\frac{1}{\sqrt{5}}[(1+\sqrt{5}) / 2]^{n}\right\} .
$$

[5] We look at integer sequences which are solutions to non-negative difference equations and show that if the equation is $\mathbf{1}$-Bounded then the solution can be expressed as Round $\left\{\alpha \lambda_{0}^{n}\right\}$ where $\alpha$ is a constant and $\lambda_{0}$ is the unique positive real root of the characteristic polynomial. We also give an easy to test sufficient condition which uses monotonicity of the coefficients of the polynomial and one evaluation of the polynomial at an integer point. We use our theorems to show that the generalized Fibonacci numbers [6] can be expressed in this rounded form.

In simple examples, the solution to a recurrence relation is often a constant times a power of an eigenvalue. For example, $x_{n}=2 x_{n-1}$, with $x_{0}=3$ has the solution $x_{n}=3 \cdot 2^{n}$. Somewhat surprisingly even when we have irrational eigenvalues, the same form of solution may obtain, but with the extra complication of a rounding operation. For example, for the Fibonacci difference equation $F_{n}=F_{n-1}+F_{n-2}$ with $F_{0}=0$ and $F_{1}=1$, we have the solution

$$
F_{n}=\frac{1}{\sqrt{5}}\left(\alpha^{n}-\beta^{n}\right)
$$

where $\alpha=(1+\sqrt{5}) / 2$ and $\beta=(1-\sqrt{5}) / 2$ and both $\alpha$ and $\beta$ are irrational numbers. But pleasantly,

$$
\begin{equation*}
F_{n}=\operatorname{Round}\left(\frac{\alpha^{n}}{\sqrt{5}}\right) \tag{1}
\end{equation*}
$$

where Round $(X)$ returns the integer nearest to $X$. (This leaves Round ( $\frac{1}{2}$ ) undefined.) A simple explanation for this ability to use Round is that $|\beta|<1$ and $1 / \sqrt{5}<1 / 2$, and so $\left|\beta^{n} / \sqrt{5}\right|<1 / 2$ for $n \geq 0$. (One should note that this rounding only works for $n \geq 0$. For $n<0$, this formula is incorrect, but other simple formulas with Round are possible.)

[^0]This simple example might suggest that roundability would follow from some simple conditions on the eigenvalues. A possible conjecture might be that: if every eigenvalue, except for the largest, were small, and if the initial deviations were small, then the deviations would stay small, and the integer sequence could be computed by rounding. When we are speaking about deviations here, we mean the difference between the sequence value and the approximation, e.g.,

$$
d_{n}=F_{n}-\frac{1}{\sqrt{5}} \alpha^{n}
$$

would be the deviation for the Fibonacci sequence. More generally, we would have a sequence $s_{n}$ and an approximation of the form $\alpha \lambda_{0}^{n}$ where $\lambda_{0}$ is an eigenvalue of the operator for the sequence, $\alpha$ is a constant which depends on the initial values, and the deviation would be

$$
d_{n}=s_{n}-\alpha \lambda_{0}^{n}
$$

(Occasionally in the following, we may also say deviation when we mean the absolute value of the deviation.)

So to maintain roundability, we would like the absolute value of the deviations to start small and stay small. We might wish that the deviations were always decreasing in absolute value, but that may not be the case.

Consider a sequence $s_{n}$ defined by a $k^{t h}$ order difference equation. If the $k$ eigenvalues are distinct, $s_{n}$ can be written as

$$
s_{n}=\sum_{i=0}^{k-1} \alpha_{i} \lambda_{i}^{n}
$$

and if $\lambda_{0}$ is the largest positive eigenvalue, we can write the deviation as

$$
d_{n}=s_{n}-\alpha_{0} \lambda_{0}^{n}=\sum_{i=1}^{k-1} \alpha_{i} \lambda_{i}^{n}
$$

and by the familar absolute value inequality

$$
\left|d_{n}\right| \leq \sum_{i=1}^{k-1}\left|\alpha_{i}\right|\left|\lambda_{i}\right|^{n}
$$

So, if $\left|\lambda_{i}\right|<1$ for each $i \in\{1,2, \ldots, k-1\}$ then

$$
\left|d_{n}\right| \leq \sum_{i=1}^{k-1}\left|\alpha_{i}\right|
$$

and if $\sum_{i=1}^{k-1}\left|\alpha_{i}\right|<1 / 2$ then $\left|d_{n}\right|<1 / 2$, for all $n \geq 0$.
So it seems that we have found the desired result. We have a result that takes care of the Fibonacci sequence, but this result will be difficult to apply to more general sequences since it seems to require us to calculate each of the $\alpha_{i}$. Notice that we can easily compute

$$
d_{n}=s_{n}-\alpha_{0} \lambda_{0}^{n}
$$

but this really tells us little about $\sum_{i=1}^{k-1}\left|\alpha_{i}\right|$.
Even though we can bound the absolute value of the deviations, the bound may have to be a severe overestimate to handle the possibly irregular behavior of the deviations. What sort of irregular behavior is possible? One possibility is spiking behavior, that is, the deviations may be nearly 0 , say for $n \in\{1,2,3,4,5\}$, but then be relatively large for $n=6$. Such spiking could occur if $\lambda_{1}=(1-\epsilon) \omega$ where $\epsilon$ is a small positive number and $\omega$ is a $6^{t h}$ root of unity. Longer period spiking could be possible if, say, $\lambda_{1}=\left(1-\epsilon_{1}\right) \omega_{3}$ and $\lambda_{2}=\left(1-\epsilon_{2}\right) \omega_{5}$, then spiking with period 15 would be possible because the period 3 spike and the period 5 spike could add to give a large spike of period 15. Obviously, even longer periods are possible because a number of short periods could multiply together to give a long period. The simple absolute value bound produces an upper envelope for the deviations which can dance around rather erratically beneath this envelope. In general, this envelope may be the best easy estimate that one can find. As in other situations, restricting our difference equations to non-negative equations can help. But, we will need more than non-negativity for a strong result.

## 2. A ROUNDING THEOREM

Definition 2.1: A difference equation $x_{n}=c_{1} x_{n-1}+\cdots+c_{k} x_{n-k}$ is $\mathbb{1}$-bounded iff

- $\forall i c_{i} \in \mathbb{N}$ and $c_{k} \in \mathbb{N}^{+}$
- $\frac{\lambda-1}{\lambda-\lambda_{0}} \operatorname{ch}(\lambda)$ is a non-negative polynomial
where $\operatorname{ch}(\lambda)=\lambda^{k}-c_{1} \lambda^{k-1}-\cdots-c_{k}$ is the characteristic polynomial of the difference equation, and $\lambda_{0}$ is the unique positive root of $\operatorname{ch}(\lambda)$. If, in addition, $\frac{\lambda-1}{\lambda-\lambda_{0}} \operatorname{ch}(\lambda)$ is primitive (aperiodic), that is, $g c d\left\{i \mid c_{i}>0\right\}=1$, the difference equation is strongly $\mathbf{1}$-bounded.

We want to use this definition to show:
Theorem 2.1: If $x_{n}$ is an integer sequence which is a solution to a 1-bounded difference equation, then there is an $\alpha$ so that
a)

$$
\forall n \geq 0 \quad\left|x_{n}-\alpha \lambda_{0}^{n}\right| \leq \max _{0 \leq j \leq k-1}\left\{\left|x_{j}-\alpha \lambda_{0}^{j}\right|\right\}
$$

b) If

$$
\max _{0 \leq j \leq k-1}\left\{\left|x_{j}-\alpha \lambda_{0}^{j}\right|\right\}<1 / 2
$$

then $\forall n \geq 0, x_{n}=\operatorname{Round}\left(\alpha \lambda_{0}^{n}\right)$.
c) If the difference equation is strongly $\mathbb{1}$-bounded

$$
\exists n_{0} \forall n \geq n_{0} \quad x_{n}=\operatorname{Round}\left(\alpha \lambda_{0}^{n}\right)
$$

We approach this theorem via a simple lemma.
Lemma 2.2: If $y_{n}$ is a solution of $y_{n}=a_{1} y_{n-1}+\cdots+a_{k} y_{n-k}$ and $\sum_{i=1}^{k}\left|a_{i}\right| \leq 1$, then $\left|y_{n}\right| \leq M=\max \left\{\left|y_{0}\right|,\left|y_{1}\right|, \ldots\left|y_{k-1}\right|\right\}$.

Proof: Clearly the conclusion follows for all $n \in\{0, \ldots, k-1\}$. For larger $n$,

$$
y_{n}=a_{1} y_{n-1}+\cdots+a_{k} y_{n-k}=\sum_{i=1}^{k} a_{i} y_{n-i}
$$

and so

$$
\left|y_{n}\right| \leq \sum_{i=1}^{k}\left|a_{i}\right|\left|y_{n-i}\right| \leq M \sum_{i=1}^{k}\left|a_{i}\right| \leq M
$$

where the first $\leq$ is the absolute value inequality, the second $\leq$ comes from the inductive hypothesis that each $\left|y_{n-i}\right| \leq M$, and the third $\leq$ is from the assumption that $\sum\left|a_{i}\right| \leq 1$.

Next consider the polynomial $\frac{\lambda-1}{\lambda-\lambda_{0}} \operatorname{ch}(\lambda)$. If this polynomial is non-negative then it has the form $\lambda^{k}-b_{1} \lambda^{k-1}-\cdots-b_{k}$ with each $b_{i} \geq 0$. Since substituting 1 for $\lambda$ must give 0 , we have $1-b_{1}-b_{2}-\cdots-b_{k}=0$ and hence $\sum b_{i}=\sum\left|b_{i}\right|=1 \leq 1$.

Now if $d_{n}$ is any solution to $x_{n}=c_{1} x_{n-1}+\cdots+c_{k} x_{n-k}$ and $d_{n}$ has no $\lambda_{0}^{n}$ component, then $d_{n}$ is a solution to the difference equation which has $\frac{c h(\lambda)}{\lambda-\lambda_{0}}$ as its characteristic polynomial, and $d_{n}$ is also a solution to the difference equation whose characteristic polynomial is $\frac{(\lambda-1) c h(\lambda)}{\lambda-\lambda_{0}}$. So by the previous remarks and the lemma, $\left|d_{n}\right| \leq \max \left\{\left|d_{0}\right|,\left|d_{1}\right|, \ldots,\left|d_{k-1}\right|\right\}=M$. Since $x_{n}-\alpha \lambda_{0}^{n}$ meets the assumptions for $d_{n}$ when $\alpha$ is chosen to exactly cancel the $\lambda_{0}^{n}$ component in $x_{n}$, we have also proved part (a) of the theorem.

For part (b), if $M<1 / 2$ then since $\left|x_{n}-\alpha \lambda_{0}^{n}\right| \leq M<1 / 2$, we have $\alpha \lambda_{0}^{n}-1 / 2<x_{n}<$ $\alpha \lambda_{0}^{n}+1 / 2$ and clearly $x_{n}=\operatorname{Round}\left(\alpha \lambda_{0}^{n}\right)$.

For part (c), strongly 1-bounded implies that all of the eigenvalues $\lambda_{i}$ used in the expansion of $d_{n}$ have absolute value strictly less than 1 . Hence,

$$
d_{n}=\sum_{i=1}^{k-1} \alpha_{i} D^{m_{i}}\left[\lambda_{i}^{n}\right]
$$

and if $\left|\lambda_{1}\right| \geq\left|\lambda_{2}\right| \geq \cdots \geq\left|\lambda_{k-1}\right|$ then

$$
\left|d_{n}\right| \leq \sum_{i=1}^{k-1}\left|\alpha_{i}\right|\left|D^{m_{i}}\left[\lambda_{i}^{n}\right]\right| \leq \sum_{i=1}^{k-1}\left|\alpha_{i}\right| \cdot c \cdot n^{k}\left|\lambda_{1}^{n}\right| \leq \hat{c} n^{k}\left|\lambda_{1}^{n}\right|
$$

which will be $<1 / 2$ for large enough $n$ because $\left|\lambda_{1}\right|^{n}$ is exponentially decreasing to 0 while $n^{k}$ is growing at only a polynomial rate. In these inequalities we may want to recall that $D^{m_{i}}$ is
the $m_{i}$ fold derivative operator, and that a polynomial can always be bounded from above by a constant times the highest power of the variable in the polynomial.

### 2.1. AN EASY TO CHECK SUFFICIENT CONDITION

It might seem relatively difficult to show that $\frac{\lambda-1}{\lambda-\lambda_{0}} \operatorname{ch}(\lambda)$ is non-negative. After all, it seems that at least one would have to calculate $\lambda_{0}$. Fortunately, there is a relatively easy to check sufficient condition.

Let us first look at computing $\frac{\operatorname{ch}(\lambda)}{\lambda-\lambda_{0}}$. It is easy to check that

$$
\frac{\operatorname{ch}(\lambda)}{\lambda-\lambda_{0}}=g_{0} \lambda^{k-1}+g_{1} \lambda^{k-2}+\cdots+g_{k-1}
$$

where $g_{0}=1$ and $g_{i+1}=\lambda_{0} g_{i}-c_{i+1}$. So carrying out this division is the same work as evaluating $\operatorname{ch}\left(\lambda_{0}\right)$ stage by stage. In fact,

$$
g_{k}=\lambda_{0} g_{k-1}-c_{k}=\lambda_{0}^{k}-c_{1} \lambda_{0}^{k-1}-\cdots-c_{k}=\operatorname{ch}\left(\lambda_{0}\right)=0
$$

that is, the division is possible because $\lambda-\lambda_{0}$ divides $c h(\lambda)$ is equivalent to $\lambda_{0}$ being a root of $c h(\lambda)$.

Since we want to know if $\frac{\lambda-1}{\lambda-\lambda_{0}} \operatorname{ch}(\lambda)$ is non-negative, we can compute

$$
\frac{\lambda-1}{\lambda-\lambda_{0}} \operatorname{ch}(\lambda)=\lambda^{k}-\left(1-g_{1}\right) \lambda^{k-1}-\left(g_{1}-g_{2}\right) \lambda^{k-2}-\cdots-\left(g_{k-2}-g_{k-1}\right) \lambda-g_{k-1}
$$

and we want $1 \geq g_{1} \geq g_{2} \geq \cdots \geq g_{k-1}>0$. We have the condition $g_{k-1}>0$ for free because $g_{k}=0=\lambda_{0} g_{k-1}-c_{k}$, and so $g_{k-1}=c_{k} / \lambda_{0}$, and by assumption $c_{k} \neq 0$.

Now for the condition $g_{i} \geq g_{i+1}$, we would need

$$
\lambda_{0}^{i}-c_{1} \lambda_{0}^{i-1}-\cdots-c_{i} \geq \lambda_{0}^{i+1}-c_{1} \lambda_{0}^{i}-\cdots-c_{i+1}
$$

or equivalently

$$
0 \geq \lambda_{0}^{i+1}-\left(c_{i}+1\right) \lambda_{0}^{i}-\left(c_{2}-c_{1}\right) \lambda_{0}^{i-1}-\cdots-\left(c_{i+1}-c_{i}\right)
$$

for $i \in\{0,1, \ldots, k-2\}$. These inequalities give the necessary condition that

$$
\lambda_{0} \leq c_{1}+1
$$

and the rest of the inequalities are implied by the sufficient conditions

$$
c_{k-1} \geq c_{k-2} \geq \cdots \geq c_{2} \geq c_{1}
$$

Obviously these sufficient conditions are easy to test by looking at the coefficients of the original polynomial $c h(\lambda)$. It might seem that testing $\lambda_{0} \leq c_{1}+1$ would require one to know the value
of $\lambda_{0}$, but as one can show, $c_{1}+1 \geq \lambda_{0}$ iff $\operatorname{ch}\left(c_{1}+1\right) \geq 0$. So testing this condition can be done using only $k$ integer multiplications.

One minor problem remains. Although the conditions force the polynomial $\frac{\lambda-1}{\lambda-\lambda_{0}} \operatorname{ch}(\lambda)$ to be non-negative, they do not force this polynomial to be primitive. That is, it is still possible for some of the eigenvalues to have absolute value equal to 1 . The simplest way to force primitivity is to require $c_{1}+1>\lambda_{0}$ because this will force the second coefficient in the polynomial to be strictly positive.

We collect these observations in the following theorem.
Theorem 2.3: Assume $x_{n}$ is an integer sequence which is a solution of the non-negative difference equation $x_{n}=c_{1} x_{n-1}+\cdots+c_{k} x_{n-k}$, so that $x_{n}=\alpha \lambda_{0}^{n}+d_{n}$ where $\lambda_{0}$ is the positive eigenvalue of the difference equation and $d_{n}$ has no $\lambda_{0}^{n}$ component. If

- $c_{k-1} \geq \cdots \geq c_{1}$
- and $c_{1}+1 \geq \lambda_{0}$
- and $\max \left\{\left|\bar{d}_{0}\right|,\left|d_{1}\right|, \ldots,\left|d_{k-1}\right|\right\}<1 / 2$
then $x_{n}=\operatorname{Round}\left(\alpha \lambda_{0}^{n}\right)$ for all $n \geq 0$.
If
- $c_{k-1} \geq \cdots \geq c_{1}$
- and $c_{1}+1>\lambda_{0}$
then there is an $n_{0}$ so that $x_{n}=\operatorname{Round}\left(\alpha \lambda_{0}^{n}\right)$ for all $n \geq n_{0}$, and $n_{0}$ is the least integer so that $\max \left\{\left|d_{n_{0}}\right|,\left|d_{n_{0}+1}\right|, \ldots,\left|d_{n_{0}+k-1}\right|\right\}<1 / 2$.


## 3. USING THE ROUNDING THEOREM

Consider the generalization of the Fibonacci difference equation from order 2 to order $k$, that is,

$$
f_{n}=f_{n-1}+f_{n-2}+\cdots+f_{n-k} .
$$

These numbers have been studied by many authors [3] [4] [6] [7]. Here the coefficients are all 1 , that is, $1=c_{1}=c_{2}=\cdots=c_{k}$. So the first condition of the theorem is satisfied. Although we do not know the value of $\lambda_{0}$, we do know that $c_{1}+1=2$. To show that $2>\lambda_{0}$, all we have to do is evaluate the polynomial $\lambda^{k}-\lambda^{k-1}-\cdots-\lambda-1$ at $\lambda=2$ and show that the value of the polynomial is positive. But $2^{k}-2^{k-1}-\cdots-2-1=2^{k}-\left(2^{k}-1\right)=1$, and so $2>\lambda_{0}$. Hence the theorem assures us that there is some $n_{0}$ so that $f_{n}=\operatorname{Round}\left(\alpha \lambda_{0}^{n}\right)$ for $n \geq n_{0}$. Notice that we have said nothing about initial conditions. We know that the value of $n_{0}$ will depend on the initial conditions.

The generalized Fibonacci numbers satisfy the $k^{\text {th }}$ order Fibonacci difference equation and the initial conditions $f_{0}=0, f_{1}=1, f_{2}=1, f_{3}=2, \ldots, f_{k-1}=2^{k-3}$. Working the difference equation backward we can show that an equivalent set of initial conditions is $f_{-(k-2)}=f_{-(k-3)}=\cdots=f_{-1}=f_{0}=0$ and $f_{1}=1$. Then using standard methods (e.g. bi-orthogonal bases), we can show that

$$
\alpha=\frac{\lambda_{0}-1}{\lambda_{0}\left[(k+1) \lambda_{0}-2 k\right]} .
$$

The corresponding deviations are

$$
\begin{gathered}
d_{-(k-2)}=0-\alpha \lambda^{-(k-2)} \\
d_{-(k-3)}=0-\alpha \lambda^{-(k-3)} \\
\vdots \\
d_{0}=0-\alpha \\
d_{1}=1-\alpha .
\end{gathered}
$$

Notice that $\max \left\{\left|d_{-(k-2)}\right|,\left|d_{-(k-3)}\right|, \ldots,\left|d_{0}\right|\right\}=\left|d_{0}\right|=\alpha$ because $\lambda_{0}>1$. The fact that $\lambda_{0}>1$ can be easily shown by evaluating $\operatorname{ch}(\lambda)$ at $\lambda=1$, which gives $-(k-1)$, a negative value, and so $\lambda_{0}>1$. We note that $d_{1}>0$ because otherwise $d_{n}$ would always be negative and would therefore have a $\lambda_{0}^{n}$ component. If we can show that both $\alpha<1 / 2$ and $1-\alpha \lambda_{0}<1 / 2$, then we can take $n_{0}=-(k-2)$, and the generalized Fibonacci numbers can be calculated by $f_{n}=\operatorname{Round}\left(\alpha \lambda_{0}^{n}\right)$ for all $n \geq-(k-2)$.

To show that $1-\alpha \lambda_{0}<1 / 2$, we only need $\frac{2\left(\lambda_{0}-1\right)}{(k+1) \lambda_{0}-2 k}>1$, but this can be written as $0>(k-1)\left(\lambda_{0}-2\right)$ which is true because $2>\lambda_{0}$.

To show $1 / 2>\alpha$, we need $\frac{1}{2}>\underline{\lambda_{0}-1} \lambda_{0}\left[(k+1) \lambda_{0}-2 k\right]$ which can be rewritten as $2>(k+1) \lambda_{0}\left(2-\lambda_{0}\right)$ and using the fact that $2-\lambda_{0}=\lambda_{0}^{-k}$, this can be written as $2 \lambda_{0}^{k-1}>$ $k+1$. For $k=2$, this reduces to $\lambda_{0}>3 / 2$ which is easy to verify. For $k>2$, we use $\lambda_{0}^{k-1}=\lambda_{0}^{k-2}+\cdots+1+\frac{1}{\lambda_{0}}$ to get $2 \lambda_{0}^{k-1}=2\left(\lambda_{0}^{k-2}+\cdots+\frac{1}{\lambda_{0}}>2(k-1)\right.$ using the fact that $\lambda_{0}>1$. Finally, $2(k-1) \geq k+1$ if $k \geq 3$, and $1 / 2>\alpha$ is established.

We had previously established this result by a more complicated argument [2]. Some of the applications of generalized Fibonacci numbers are described by Capocelli [1].

As another example, let us consider

$$
x_{n}=2 x_{n-1}+2 x_{n-2}+3 x_{n-3}
$$

The characteristic polynomial is $\lambda^{3}-2 \lambda^{2}-2 \lambda-3$ which has the dominant root $\lambda_{0}=3$. Here $k=3$, and $c_{k-1}=c_{2}=2 \geq c_{1}$, and $c_{1}+1=2+1=3 \geq \lambda_{0}$. So the first and second conditions of the theorem are satisfied. But, as yet we do not have initial conditions which are needed to specify the deviations. It is easy to check that $\alpha=\frac{1}{13}\left(x_{0}+x_{1}+x_{2}\right)$. So, for example, if we choose the initial conditions $x_{0}=1, x_{1}=3, x_{2}=9$, then $\alpha=1$ and $x_{n}=1 \times 3^{n}$. For these initial conditions

$$
\begin{aligned}
& d_{0}=x_{0}-\alpha \times 3^{0}=1-1=0 \\
& d_{1}=x_{1}-\alpha \times 3^{1}=3-3=0 \\
& d_{2}=x_{2}-\alpha \times 3^{2}=9-9=0
\end{aligned}
$$

So the third condition of the theorem is satisfied and $x_{n}=\operatorname{Round}\left(\alpha \lambda_{0}^{n}\right)=\operatorname{Round}\left(1 \times 3^{n}=3^{n}\right)$. Of course, this result could have been found directly without using the rounding theorem.

Let us consider a different set of initial conditions. For example, $x_{0}=0, x_{1}=0, x_{2}=1$. Now $\alpha=1 / 13$ and the deviations are

$$
\begin{aligned}
& d_{0}=0-\frac{1}{13}=-\frac{1}{13} . \\
& d_{1}=0-\frac{3}{13}=-\frac{3}{13} \\
& d_{2}=1-\frac{9}{13}=+\frac{4}{13} .
\end{aligned}
$$

So the absolute values of these deviations are all $<1 / 2$ and the third condition of the theorem is satisfied. Thus, $x_{n}=\operatorname{Round}\left(\frac{1}{13} \lambda_{0}^{n}\right)$. In this example, the theorem tells us that the solution can be obtained by rounding, and this result was not obvious without the theorem.

Let us consider one more example of initial conditions for this difference equation, namely, $x_{0}=0, x_{1}=3, x_{2}=9$. Here, $\alpha=12 / 13$ and the deviations are:

$$
\begin{aligned}
& d_{0}=0-\frac{12}{13}=-\frac{12}{13} \\
& d_{1}=3-\frac{36}{13}=+\frac{3}{13} \\
& d_{2}=9-\frac{108}{13}=\frac{9}{13} .
\end{aligned}
$$

In this case, the deviations are not all less than $1 / 2$ in absolute value. Further, $c_{1}+1=\lambda_{0}$, so neither immediate rounding nor eventual rounding is promised by the theorem. It is easy to calculate that

$$
\begin{aligned}
& d_{3}=24-\frac{12}{13} \times 3^{3}=-\frac{12}{13} \\
& d_{4}=75-\frac{12}{13} \times 3^{4}=+\frac{3}{13} \\
& d_{5}=225-\frac{12}{13} \times 3^{5}=\frac{9}{13} .
\end{aligned}
$$

So in this example, the deviations are periodic with period 3, and the deviations do not decrease. The theorem does not say that rounding is possible and, in fact, rounding is not possible.

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# A FIBONACCI IDENTITY IN THE SPIRIT OF SIMSON AND GELIN-CESARO 

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The identities

$$
\begin{equation*}
F_{n-1} F_{n+1}-F_{n}^{2}=(-1)^{n} \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{n-2} F_{n-1} F_{n+1} F_{n+2}-F_{n}^{4}=-1 \tag{2}
\end{equation*}
$$

are very old, dating back to 1753 and 1880, respectively. According to Dickson [1], pages 393 and 401, the first was proved by Robert Simson and the second was stated by E. Gelin and proved by E. Cesàro. Simson's identity and the Gelin-Cesàro identity have been generalized many times. For details and references we refer the reader to [3] and [4].

We began to wonder if there was a pleasing identity involving the difference of third-order products, but a search of the literature revealed nothing to match the beauty of the identities above. We offer the following:

$$
\begin{equation*}
F_{n+1} F_{n+2} F_{n+6}-F_{n+3}^{3}=(-1)^{n} F_{n} \tag{3}
\end{equation*}
$$

One method of proof is to use (1) to substitute for $(-1)^{n}$, then express each of $F_{n-1}, F_{n}, F_{n+3}$, and $F_{n+6}$ in terms of $F_{n+1}$ and $F_{n+2}$, and expand both sides. We prefer this method of proof since it carries over nicely to our generalization of (3), which we give next.

Our generalization is stated for the sequence $\left\{W_{n}\right\}=\left\{W_{n}(a, b ; p, q)\right\}$ defined by

$$
W_{n}=p W_{n-1}-q W_{n-2}, \quad W_{0}=a, \quad W_{1}=b
$$

where $a, b, p$, and $q$ are taken to be arbitrary complex numbers with $q \neq 0$. Since $q \neq 0,\left\{W_{n}\right\}$ is defined for all integers $n$. Put $e=p a b-q a^{2}-b^{2}$. Then

$$
\begin{equation*}
W_{n+1} W_{n+2} W_{n+6}-W_{n+3}^{3}=e q^{n+1}\left(p^{3} W_{n+2}-q^{2} W_{n+1}\right) \tag{4}
\end{equation*}
$$

which clearly generalizes (3). Generalizations often suffer through a loss of elegance, but this is not the case here, adding testimony to the charm of (3).

To prove (4) we require the identity

$$
\begin{equation*}
W_{n+1} W_{n+3}-W_{n+2}^{2}=e q^{n+1} \tag{5}
\end{equation*}
$$

which generalizes (1) and is a variant of (4.3) in [2]. In addition, we require

$$
\left\{\begin{array}{l}
W_{n+3}=p W_{n+2}-q W_{n+1}  \tag{6}\\
W_{n+6}=\left(p^{4}-3 p^{2} q+q^{2}\right) W_{n+2}-\left(p^{3} q-2 p q^{2}\right) W_{n+1}
\end{array}\right.
$$

## A FIBONACCI IDENTITY IN THE SPIRIT OF SIMSON AND GELIN-CESARO

where each identity in (6) is obtained with the use of the recurrence for $\left\{W_{n}\right\}$. Now, using (5) and (6) we express (4) in terms of $p, q, W_{n+1}$, and $W_{n+2}$, and thus verify equality of the left and right sides.

Finally, we remark that Waddill (see (18) in [5]) proved the equivalent of (5) with an elegant use of matrices, which means that our proof of (4) does not rely upon the use of the Binet form ([2]) for $W_{n}$.

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# COMPUTATIONAL FORMULAS FOR CONVOLUTED GENERALIZED FIBONACCI AND LUCAS NUMBERS 

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## 1. INTRODUCTION

In the notation of Horadam [2], let $W_{n}=W_{n}(a, b ; p, q)$, where

$$
\begin{gather*}
W_{n}=p W_{n-1}-q W_{n-2} \quad(n \geq 2)  \tag{1}\\
W_{0}=a, \quad W_{1}=b
\end{gather*}
$$

If $\alpha$ and $\beta$, assumed distinct, are the roots of

$$
\lambda^{2}-p \lambda+q=0
$$

we have the Binet form

$$
\begin{equation*}
W_{n}=\frac{A \alpha^{n}-B \beta^{n}}{\alpha-\beta} \tag{2}
\end{equation*}
$$

in which $A=b-a \beta$ and $B=b-a \alpha$.
The $n^{t h}$ terms of the well-known Fibonacci and Lucas numbers can be denoted by $F_{n}=$ $W_{n}(0,1 ; 1,-1)$ and $L_{n}=W_{n}(2,1 ; 1,-1)$, respectively.

We also denote

$$
\begin{equation*}
U_{n}=W_{n}(0,1 ; p, q)=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}, V_{n}=W_{n}(2, p ; p, q)=\alpha^{n}+\beta^{n} \tag{3}
\end{equation*}
$$

By simple computing, we have

$$
\begin{equation*}
\sum_{n=0}^{\infty} W_{m n} x^{n}=\frac{a+\left(b U_{m}-a U_{m+1}\right) x}{1-V_{m} x+q^{m} x^{2}} \tag{4}
\end{equation*}
$$

Let $W=\left\{W_{n}\right\}$ be defined as above, with $W_{0}=0$. For any positive integer $k \geq 2, W$. Zhang [3] obtained the following summation:

$$
\sum_{a_{1}+a_{2}+\cdots+a_{k}=n} W_{a_{1}} W_{a_{2}} \ldots W_{a_{k}}=\frac{b^{k-1}}{\left(p^{2}-4 q\right)^{k-1}(k-1)!}\left[g_{k-1}(n) W_{n-k+1}+h_{k-1}(n) W_{n-k}\right]
$$

where the summation is taken over all $n$-tuples with positive integer coordinates ( $a_{1}, a_{2}, \ldots, a_{k}$ ) such that $a_{1}+a_{2}+\cdots+a_{k}=n$. Moreover, $g_{k-1}(x)$ and $h_{k-1}(x)$ are two effectively computable polynomials of degree $k-1$ with coefficients only depending on $p, q$ and $k$.

Recently, Z. Zhang and X. Wang [4] gave explicit expressions for $g_{k-1}(x)$ and $h_{k-1}(x)$. P. He and Z. Zhang [1] discussed generalized Lucas numbers. The purpose of this paper is to generalize the above results, i.e., to evaluate the following summation:

$$
\sum_{a_{1}+a_{2}+\cdots+a_{k}=n} W_{m a_{1}} W_{m a_{2}} \ldots W_{m a_{k}}
$$

## 2. THE CONVOLUTED FORMULA OF GENERALIZED FIBONACCI NUMBERS

Throughout this section, with $W_{0}=0$, if we let

$$
\begin{equation*}
G_{k}(x)=\left(\frac{b U_{m}}{1-V_{m} x+q^{m} x^{2}}\right)^{k}=\sum_{n=0}^{\infty} W_{m n}^{(k)} x^{n-1} \tag{5}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\sum_{a_{1}+a_{2}+\cdots+a_{k}=n} W_{m a_{1}} W_{m a_{2}} \ldots W_{m a_{k}}=W_{m(n-k+1)}^{(k)} \tag{6}
\end{equation*}
$$

## Theorem 2.1:

$$
\begin{equation*}
W_{m n}^{(k+1)}=\frac{b U_{m}}{k\left(V_{m}^{2}-4 q^{m}\right)}\left\{n V_{M} W_{m(n+1)}^{(k)}-2 q^{m}(n+2 k-1) W_{m n}^{(k)}\right\} . \tag{7}
\end{equation*}
$$

Proof: Noting that

$$
\frac{d}{d x}\left(G_{k}(x)\left(V_{m}-2 q^{m} x\right)^{k}\right)=G_{k}^{\prime}(x)\left(V_{m}-2 q^{m} x\right)^{k}+G_{k}(x) k\left(V_{m}-2 q^{m} x\right)^{k-1}\left(-2 q^{m}\right)
$$

and

$$
\begin{gathered}
\frac{d}{d x}\left(G_{k}(x)\left(V_{m}-2 q^{m} x\right)^{k}\right)=\frac{d}{d x}\left(\frac{b U_{m}\left(V_{m}-2 q^{m} x\right.}{1-V_{m} x+q^{m} x^{2}}\right)^{k} \\
=k\left(\frac{b U_{m}\left(V_{m}-2 q^{m} x\right)}{1-V_{m} x+q^{m} x^{2}}\right)^{k-1} b U_{m} \frac{2 q^{m}\left(1-V_{m} x+q^{m} x^{2}\right)+V_{m}^{2}-4 q^{m}}{\left(1-V_{m} x+q^{m} x^{2}\right)^{2}},
\end{gathered}
$$

we have

$$
G_{k}^{\prime}(x) b U_{m}\left(V_{m}-2 q^{m} x\right)-2 b k U_{m} q^{m} G_{k}(x)=2 b k U_{m} q^{m} G_{k}(x)+k\left(V_{m}^{2}-4 q^{m}\right) G_{k+1}(x) .
$$

Comparing the coefficients of both sides of the equation, our theorem holds.
We denote by $\sigma_{i}(n, k)$ the summation of all products of choosing $i$ elements from $n+k-$ $i+1, n+k-i+2, \ldots, n+2 k-1$ but not containing any two consecutive elements, i.e.,

$$
\begin{equation*}
\sigma_{i}(n, k)=\sum \prod_{t=1}^{i}\left(n+k-i+j_{t}\right) \tag{8}
\end{equation*}
$$

where the summation is taken over all $i$-tuples with positive integer coordinates ( $j_{1}, j_{2}, \ldots, j_{i}$ ) such that $1 \leq j_{1}<j_{2}<\cdots<j_{i} \leq k+i-1$ and $\left|j_{r}-j_{s}\right| \geq 2$ for $1 \leq r \neq s \leq i$.

We note that $\sigma_{0}(n, k)=1$. It is easy to prove that

$$
\begin{equation*}
(n+2 k-1) \sigma_{k-1}(n, k-1)=\sigma_{k}(n, k), \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
(n+2 k-1) \sigma_{i-1}(n, k-1)+\sigma_{i}(n+1, k-1)=\sigma_{i}(n, k), \tag{10}
\end{equation*}
$$

which yields
Theorem 2.2:

$$
\begin{equation*}
W_{m n}^{(k+1)}=\frac{\left(b U_{m}\right)^{k}}{k!\left(V_{m}^{2}-4 q^{m}\right)^{k}} \sum_{i=0}^{k}\left(-2 q^{m}\right)^{i} V_{m}^{k-i}\langle n\rangle_{k-i} \sigma_{i}(n, k) W_{m(n+k-i)}, \tag{11}
\end{equation*}
$$

where $\langle n\rangle_{k}=n(n+1)(n+2) \ldots(n+k-1)$.

Proof: We prove the theorem by induction on $k$. When $k=0,1$, the theorem is true by applying Theorem 2.1. Assume the theorem is true for a positive integer $k-1$. Then

$$
\begin{gathered}
W_{m n}^{(k+1)}=\frac{b U_{m}}{k\left(V_{m}^{2}-4 q^{m}\right)}\left\{n V_{m} W_{m(n+1)}^{(k)}-2 q^{m}(n+2 k-1) W_{m n}^{(k)}\right\} \\
=\frac{b U_{m}}{k\left(V_{m}^{2}-4 q^{m}\right)}\left\{n V_{m} \frac{\left(b U_{m}\right)^{k-1}}{(k-1)!\left(V_{m}^{2}-4 q^{m}\right)^{k-1}} \sum_{i=0}^{k-1}\left(-2 q^{m}\right)^{i} V_{m}^{k-i-1}\langle n+1\rangle_{k-i-1} \times\right. \\
\sigma_{i}(n+1, k-1) W_{m(n+k-i)}+\left(-2 q^{m}\right)(n-1+2 k) \frac{\left(b U_{m}\right)^{k-1}}{(k-1)!\left(V_{m}^{2}-4 q^{m}\right)^{k-1}} \times \\
\left.\sum_{i=0}^{k-1}\left(-2 q^{m}\right)^{i} V_{m}^{k-i-1}\langle n\rangle_{k-i-1} \sigma_{i}(n, k-1) W_{m(n+k-i-1)}\right\} \\
\frac{\left(b U_{m}\right)^{k}}{\left(k!\left(V_{m}^{2}-4 q^{m}\right)^{k}\right.}\left\{V_{m}^{k} n\langle n+1\rangle_{k-1} \sigma_{0}(n+1, k-1) W_{m(n+k)}\right. \\
+\sum_{i=1}^{k-1}\left(-2 q^{m}\right) V_{m}^{k-i} n\langle n+1\rangle_{k-i} \sigma_{i}(n+1, k-1) W_{m(n+k-i)} \\
\left.+\sum_{i=1}^{k}\left(-2 q^{m}\right)^{i} V_{m}^{k-i}(n+2 k-1)\langle n\rangle_{k-i} \sigma_{i-1}(n, k-1) W_{m(n+k-i)}\right\} \\
=\frac{\left(b U_{m}\right)^{k}}{k!\left(V_{m}^{2}-4 q^{m}\right)^{k}}\left\{V_{m}^{k}\langle n\rangle_{k} \sigma_{0}(n, k) W_{m(n+k)}\right. \\
\left.+\left(-2 q^{m}\right)^{k}(n+2 k-1) \sigma_{k-1}(n, k-1) W_{m n}\right\}
\end{gathered}
$$

$$
\begin{gathered}
=\frac{\left(b U_{m}\right)^{k}}{k!\left(V_{m}^{2}-4 q^{m}\right)^{k}}\left\{V_{m}^{k}\langle n\rangle_{k} \sigma_{0}(n, k) W_{m(n+k)}\right. \\
\left.+\sum_{i=1}^{k-1}\left(-2 q^{m}\right)^{i} V_{m}^{k-i}\langle n\rangle_{k-i} \sigma_{i}(n, k) W_{m(n+k-i)}+\left(-2 q^{m}\right)^{k} \sigma_{k}(n, k) W_{m n}\right\} \\
=\frac{\left(b U_{m}\right)^{k}}{k!\left(V_{m}^{2}-4 q^{m}\right)^{k}} \sum_{i=0}^{k}\left(-2 q^{m}\right)^{i} V_{m}^{k-i}\langle n\rangle_{k-i} \sigma_{i}(n, k) W_{m(n+k-i)} .
\end{gathered}
$$

Hence the theorem is also true for $k$. This completes the proof.
Theorem 2.3:

$$
\begin{gather*}
\sum_{a_{1}+a_{2}+\cdots+a_{k}=n} W_{m a_{1}} W_{m a_{2}} \ldots W_{m a_{k}} \\
=\frac{\left(b U_{m}\right)^{k-1}}{(k-1)!\left(V_{m}^{2}-4 q^{m}\right)^{k-1}} \sum_{i=0}^{k-1}\left(-2 q^{m}\right)^{i} V_{m}^{k-1-i}\langle n-k+1\rangle_{k-1-i} \sigma_{i}(n-k+1, k-1) W_{m(n-i)} . \tag{12}
\end{gather*}
$$

Proof: Use (6) and Theorem 2.2.
Lemma 2.4:

$$
\begin{equation*}
U_{m} W_{m(k+n)}=U_{m n} W_{m(k+1)}-q^{m} U_{m(n-1)} W_{m k} \tag{13}
\end{equation*}
$$

Proof: Use (2), (3).
Let

$$
\begin{equation*}
g_{k-1}^{(m)}(n)=\sum_{i=0}^{k-1}\left(-2 q^{m}\right)^{i} V_{m}^{k-1-i}\langle n-k+1\rangle_{k-1-i} \sigma_{i}(n-k+1, k-1) U_{m(k-i)} \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
h_{k-1}^{(m)}(n)=-q M \sum_{i=0}^{k-1}\left(-2 q^{m}\right)^{i} V_{m}^{k-1-i}\langle n-k+1\rangle_{k-1-i} \sigma_{i}(n-k+1, k) U_{m(k-1-i)} \tag{15}
\end{equation*}
$$

Then we have the following theorem.
Theorem 2.5:

$$
\sum_{a_{1}+a_{2}+\cdots+a_{k}=n} W_{m a_{1}} W_{m a_{2}} \ldots W_{m a_{k}}
$$

COMPUTATIONAL FORMULAS FOR CONVOLUTED GENERALIZED FIBONACCI AND LUCAS NUMBERS

$$
\begin{equation*}
=\frac{\left(b^{k-1} U_{m}^{k-2}\right.}{(k-1)!\left(V_{m}^{2}-4 q^{m}\right)^{k-1}}\left\{g_{k-1}^{(m)}(n) W_{m(n-k+1)}+h_{k-1}^{(m)}(n) W_{m(n-k)}\right\} \tag{16}
\end{equation*}
$$

Proof: Use Theorem 2.3 and Lemma 2.4.
Corollary 2.6:

$$
\begin{gathered}
\sum_{a+b=n} W_{m a} W_{m b} \\
=\frac{b}{V_{m}^{2}-4 q^{m}}\left\{\left[(n-1) V_{m} U_{2 m}-2 q_{m} n U_{m}\right] W_{m(n-1)}-q^{m}(n-1) V_{m} U_{m} W_{m(n-2)}\right\} \\
\sum_{a+b+c=n} W_{m a} W_{m b} W_{m c} \\
\frac{b^{2} U_{m}}{2\left(V_{m}^{2}-4 q^{m}\right)^{2}}\left\{(n-2)(n-1) V_{m}^{2} U_{3 m}-2 q_{m}(n-2)(2 n+1) V_{m} U_{2 m}+4 q^{2 m}(n-1)\right. \\
\left.\left.(n+1) U_{m}\right] W_{m(n-2)}-q^{m}\left[(n-2)(n-1) V_{m}^{2} U_{2 m}-2 q^{m}(n-2)(2 n+1) V_{m} U_{m}\right] W_{m(n-3)}\right\} .
\end{gathered}
$$

Proof: Take $k=2,3$ in Theorem 2.5.
From (16), we have

## Corollary 2.7:

$$
\begin{equation*}
b^{k-1} U_{m}^{k-2}\left\{g_{k-1}^{(m)}(n) W_{m(n-k+1)}+h_{k-1}^{(m)}(n) W_{m(n-k)}\right\} \equiv 0 \quad\left(\quad \bmod (k-1)!\left(V_{m}^{2}-4 q^{m}\right)^{k-1}\right) \tag{17}
\end{equation*}
$$

## 3. THE CONVOLUTED FORMULA OF GENERALIZED LUCAS NUMBERS

Let $\Delta=2 U_{m+1}-p U_{m}$. Taking $a=2, b=p$ and using (4) we have

$$
\sum_{n=0}^{\infty} V_{m n} x^{n}=\frac{2-\Delta x}{1-V_{m} x+q^{m} x^{2}}
$$

Let

$$
H_{k}(x)=\sum_{n=0}^{\infty} V_{m n}^{(k)} x^{n}=\left(\frac{2-\Delta x}{1-V_{m} x+q^{m} x^{2}}\right)^{k}
$$

Obviously, $V_{m n}^{(1)}=V_{m n}$. From these, we have

$$
\begin{equation*}
\sum_{a_{1}+a_{2}+\cdots+a_{k}=n} V_{m a_{1}} V_{m a_{2}} \ldots V_{m a_{k}}=V_{m n}^{(k)} \tag{18}
\end{equation*}
$$

## Theorem 3.1:

$$
\begin{equation*}
k\left(\Delta V_{m}-4 q^{m}\right) V_{m n}^{(k+1)}=4(n+2) V_{m(n+2)}^{(k)}-2(2 n+k+2) \Delta V_{m(n+1)}^{(k)}+(n+k) \Delta^{2} V_{m n}^{(k)} \tag{19}
\end{equation*}
$$

Proof: Noting that

$$
\begin{gathered}
\frac{d}{d x}\left(H_{k}(x)=\frac{d}{d x}\left(\frac{2-\Delta x}{1-V_{m} x+q^{m} x^{2}}\right)^{k}\right. \\
=k\left(\frac{2-\Delta x}{1-V_{m} x+q^{m} x^{2}}\right)^{k-1} \frac{-\Delta+2 V_{m}-4 q^{m} x+q^{m} \Delta x^{2}}{\left(1-V_{m} x+q^{m} x^{2}\right)^{2}} \\
=k\left(\frac{2-\Delta x}{1-V_{m} x+q^{m} x^{2}}\right)^{k-1} \frac{\Delta\left[1-V_{m} x+q^{m} x^{2}\right]+\left[\Delta V_{m}-4 q^{m}\right.}{\left(1-V_{m} x+q^{m} x^{2}\right)^{2}} \\
=k \frac{\Delta}{2-\Delta x}\left(\frac{2-\Delta x}{1-V_{m} x+q^{m} x^{2}}\right)^{k}+k \frac{\Delta V_{m}-4 q^{m}}{(2-\Delta x)^{2}} x\left(\frac{2-\Delta x}{\left(1-V_{m} x+q^{m} x^{2}\right)}\right)^{k+1},
\end{gathered}
$$

we have

$$
(2-\Delta x)^{2} \frac{d}{d x} H_{k}(x)=k\left[\Delta V_{m}-4 q^{m}\right] x H_{k+1}(x)+k \Delta(2-\Delta x) H_{k}(x)
$$

This implies

$$
k\left[\Delta V_{m}-4 q^{m}\right] x H_{k+1}(x)=\left(4-4 \Delta x+\Delta^{2} x^{2}\right) \frac{d}{d x} H_{k}(x)-k \Delta(2-\Delta x) H_{k}(x)
$$

Comparing the coefficients of both sides in the above equation, the theorem holds.

## Theorem 3.2:

$$
\begin{gather*}
\sum_{a+b=n} V_{m a} V_{m b} \\
=\frac{1}{\Delta V_{m}-4 q^{m}}\left\{4(n+2) V_{m(n+2)}+2(2 n+3) \Delta V_{m(n+1)}+(n+1) \Delta^{2} V_{m n}\right\} . \tag{20}
\end{gather*}
$$

$$
\begin{gather*}
\sum_{a+b+c=n} V_{m a} V_{m b} V_{m c}=\frac{1}{2\left(\Delta V_{m}-4 q^{m}\right)^{2}}\left\{16(n+4)\left(n_{+} 2\right) V_{m(n+4)}\right. \\
+8((n+2)(2 n+7)+(n+3)(2 n+4)) \Delta V_{m(n+3)}+4((n+2)(n+3)+(2 n+4)(2 n+5) \\
\left.\left.+(n+2)^{2}\right) \Delta^{2} V_{m(n+2)}+2(n+2)(4 n+7) \Delta^{3} V_{m(n+1)}+(n+1)(n+2) \Delta^{4} V_{m n}\right\} \tag{21}
\end{gather*}
$$

Proof: Use (18) and Theorem 3.1.
In Theorem 3.1, taking $m=1,2$ gives the main results of paper [1].

## Corollary 3.3:

$$
\begin{gather*}
k\left(p^{2}-4 q\right) V_{n}^{(k+1)}=4(n+2) V_{n+2}^{(k)}-2 p(2 n+k+2) V_{n+1}^{(k)}+p^{2}(n+k) V_{n}^{(k)}  \tag{22}\\
k p^{2}\left(p^{2}-4 q\right) V_{2 n}^{(k+1)}=4(n+2) V_{2(n+2)}^{(k)}-2(2 n+k+2)\left(p^{2}-2 q\right) V_{2(n+1)}^{(k)}+(n+k)\left(p^{2}-2 q\right)^{2} V_{2 n}^{(k)} . \tag{23}
\end{gather*}
$$

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# A NON-INTEGER PROPERTY OF ELEMENTARY SYMMETRIC FUNCTIONS IN RECIPROCALS OF GENERALISED FIBONACCI NUMBERS 

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## 1. INTRODUCTION AND MAIN RESULT

A well-known but classicial result concerning the harmonic series is that the sequence of partial sums $\sum_{r=1}^{n} \frac{1}{r}$ can never be an integer for $n>1$. More generally, Nagell [3] showed that $\sum_{r=1}^{n} \frac{1}{m+r d}$ cannot be an integer for any positive integers $m, n$ and $d$. As an extension of these results the author, in a recent paper [4], constructed further examples of positive rational termed series having non-integer partial sums. These partial sums were of the form $\sum_{r=1}^{n} \frac{1}{U_{r}}$, where $\left\{U_{n}\right\}$ are the sequence of generalised Fibonacci numbers generated, for $n \geq 2$, via the recurrence relation

$$
\begin{equation*}
U_{n}=P U_{n-1}-Q U_{n-2} \tag{1}
\end{equation*}
$$

with $U_{0}=0, U_{1}=1$ and $(P, Q)$ a relatively prime pair of integers satisfying $|P|>Q>0$ or $P \neq 0, Q<0$. (Note when $(P, Q)=(2,1)$ one has $U_{n}=n$ ). By viewing these partial sums as the symmetric function formed from summing the products of the terms $\frac{1}{U_{1}}, \frac{1}{U_{2}}, \ldots \frac{1}{U_{n}}$ taken one at a time, one may naturally ask whether all other symmetric fucntions in the reciprocals of such generalised Fibonacci numbers can be non-integer. In this paper we will show that for sequences $\left\{U_{n}\right\}$ generated via (1), with $P \geq 2$ and $Q<0$, there can in fact be at most finitely many $n$ such that one or more of the elementary symmetric functions in $\frac{1}{U_{1}}, \frac{1}{U_{2}}, \ldots \frac{1}{U_{n}}$ is an integer. To establish this result we will require two preliminary Lemmas, the first of which is a refinement of Bertrand's postulate due to Ingham [2].

Lemma 1.1: For any real number $x>1$ there always exists a prime in the interval $\left(x, x+x^{\frac{5}{8}}\right)$.
The second lemma is a standard result of generalised Fibonacci sequences, a proof of which can be found in [1].
Lemma 1.2: For any sequence $\left\{U_{n}\right\}$ generated with respect to a relatively prime pair of integers $(P, Q)$ via (1) then $\left(U_{m}, U_{n}\right)=U_{(m, n)}$.

We now can prove the following theorem:
Theorem 1.1: Suppose the sequence $\left\{U_{n}\right\}$ is generated via (1) with respect to the relatively prime pair $(P, Q)$ such that $P \geq 2$ and $Q<0$. Denote the $k^{\text {th }}$ elementary symmetric function in $\frac{1}{U_{1}}, \frac{1}{U_{2}}, \ldots, \frac{1}{U_{n}}$ by $\phi(n, k)$, then for this family of functions there exists a uniform lower bound $N$ on $n$, such that $\phi(n, k)$ is non-integer for $n \geq N$ and $1 \leq k \leq n$.

Proof: To establish the non-integer status of $\phi(n, k)$ it will suffice to consider the two separate cases of $k>3 \log n$ and $k<3 \log n$, noting here that it is sufficient to take only strict inequalities as $\log n$ can never be an integer for integer $n>1$. In both cases we will demonstrate the existence of the lower bounds given by $N_{1}=\min \left\{s \in \mathbb{N}: \log n \geq \frac{e}{3-e}\right.$ for
$n \geq s\}=\left\lceil e^{\frac{e}{3-e}}\right\rceil$ and $N_{2}=\min \left\{s \in \mathbb{N}: \frac{9(\log n)^{2}}{n}+\frac{3 \log n}{n}<\frac{1}{2}, \frac{n^{3}}{(3 \log n+1)^{11}}>2^{8}(1+\log 3)^{5}\right.$ for
all $n \geq s\}$ respectively on $n$, for which $\phi(n, k)$ is non-integer. As $N_{1}$ and $N_{2}$ are constructed independently of $k$, one can then set $N=\max \left\{N_{1}, N_{2}\right\}$ from which it is immeidate that $\phi(n, k)$ must be non-integer for all $n \geq N$ and $1 \leq k \leq n$. Furthermore, as $N_{1}$ and $N_{2}$ are not dependent on the specific choice of the sequence $\left\{U_{n}\right\}$, one sees that the lower bound $N$ must hold uniformly over the family of generalised Fibonacci sequences as specified in the theorem statement. We now proceed with the following two cases.
Case 1: $k>3 \log n$
First note for the prescribed values of $(P, Q)$ it can be shown, via an easy induction on $n$, that $U_{n} \geq n$. Now, as $\phi(n, k)$ is formed from summing the terms $\frac{1}{U_{1}}, \frac{1}{U_{2}}, \ldots, \frac{1}{U_{n}}$ taken $k$ at a time, we observe that $\phi(n, k)$ must occur $k!$ times in the multinomial expan$\operatorname{sion}\left(\frac{1}{U_{1}}+\frac{1}{U_{2}}+\cdots+\frac{1}{U_{n}}\right)^{k}$. Hence, using the usual comparison of $\log n$ with the terms of the harmonic series, we obtain that

$$
\begin{align*}
\phi(n, k)<\frac{1}{k!}\left(\frac{1}{U_{1}}+\frac{1}{U_{2}}+\cdots+\frac{1}{U_{n}}\right)^{k} & <\frac{1}{k!}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)^{k} \\
& <\frac{1}{k!}(1+\log n)^{k} \tag{2}
\end{align*}
$$

Now by definition of $N_{1}$ if $n \geq N_{1}$ then $\log n>\frac{e}{3-e}$ and so $k>\frac{3 e}{3-e}$. Consequently

$$
\frac{1}{k!}(1+\log n)^{k}<\frac{1}{k!}\left(1+\frac{k}{3}\right)^{k}=\frac{k^{k}}{k!}\left(\frac{1}{k}+\frac{1}{3}\right)^{k}<\left(\frac{e}{k}+\frac{e}{3}\right)^{k}<1
$$

noting here that the second last inequality follows from the fact that $\frac{k^{k}}{k!}<e^{k}$. Hence, we deduce from the previous inequality and (2) that $0<\phi(n, k)<1$ for any $n \geq N_{1}$ as required. Case 2: $k<3 \log n$

In this case it first will be necessary to show that for $n \geq N_{2}$

$$
\begin{equation*}
\left(\frac{n}{k(k+1)}-1\right)^{8}>\left(\frac{n}{k}+1\right)^{5} \tag{3}
\end{equation*}
$$

Upon factoring out $\frac{n}{k}$ and $\frac{n}{k(k+1)}$ from the right and left hand side respectively of the conjectured inequality in (3) one finds that

$$
\begin{equation*}
\frac{n^{3}}{k^{3}(k+1)^{8}}\left(1-\frac{k(k+1)}{n}\right)^{8}>\left(1+\frac{k}{n}\right)^{5} \tag{4}
\end{equation*}
$$

Now, as $k<3 \log n$ and so $\frac{k}{n}<\frac{3 \log n}{n} \rightarrow 0$ monotonically for $n>e$, it is clear the term $\left(1+\frac{k}{n}\right)^{5}$ can be bounded above by $(1+\log 3)^{5}$ for $n \geq 3$ say. Similarly, as $\frac{k(k+1)}{n}<$ $\frac{9(\log n)^{2}}{n}+\frac{3 \log n}{n} \rightarrow 0$ and $\frac{n^{3}}{k^{3}(k+1)^{8}}>\frac{n^{3}}{(3 \log n+1)^{11}} \rightarrow \infty$ as $n \rightarrow \infty$, one can choose $n$ sufficiently large but finite and independent of $k$, such that $\frac{k(k+1)}{n}<\frac{1}{2}$ and $\frac{n^{3}}{k^{3}(k+1)^{8}}>2^{8}(1+\log 3)^{5}$. Consequently by definition of $N_{2}$ one has for $n \geq N_{2}$

$$
\frac{n^{3}}{k^{3}(k+1)^{8}}\left(1-\frac{k(k+1)}{n}\right)^{8}>(1+\log 3)^{5}
$$

and so one concludes that (3) must hold for all $n \geq N_{2}$. Now raising both sides of (3) to the power $\frac{1}{8}$ one finds upon rearrangement that

$$
\frac{n}{k}>\left(1+\frac{n}{k+1}\right)+\left(1+\frac{n}{k+1}\right)^{\frac{5}{8}}
$$

Hence for $n \geq N_{2}$ there must exist, by Lemma 1.1, a prime $p$ in the open interval $\left(1+\frac{n}{k+1}, \frac{n}{k}\right)$. By construction $p$ must be such that $1<m p<n$ for $m=1,2, \ldots, k$ but $(k+1) p>n$. Considering again $\phi(n, k)$ as a sum of the product of the terms $\frac{1}{U_{1}}, \frac{1}{U_{2}}, \ldots, \frac{1}{U_{n}}$ taken $k$ at a time we can write

$$
\phi(n, k)=\sum_{i=1}^{\binom{n}{k}} \frac{1}{c_{i}}=\frac{b_{1}+b_{2}+\cdots+b_{\binom{n}{k}}}{U_{1} U_{2} \ldots U_{n}}=\frac{B}{C}
$$

where $c_{i}$ is one of the possible $\binom{n}{k}$ products of the terms $U_{1}, U_{2}, \ldots, U_{n}$ taken $k$ at a time and

$$
b_{i}=\frac{U_{1} U_{2} \ldots U_{n}}{c_{i}}
$$

By the above $U_{p} U_{2 p} \ldots U_{k p}=c_{s}$, for some $s \in\left\{1,2, \ldots,\binom{n}{k}\right\}$, and as $(k+1) p>n$, no other of the remaining $\binom{n}{k}-1$ products $c_{i}$ can contain generalised Fibonacci numbers in which all of the corresponding $k$ subscripts are a multiple of $p$. Consequently, by construction each $b_{i}$, with
$i \neq s$, must contain at least one of the terms in the set $A=\left\{U_{p}, U_{2 p}, \ldots, U_{k p}\right\}$ while $b_{s}$ will contain none of the terms in $A$. Now by Lemma 1.2 as $p$ is prime $\left(U_{p}, U_{m p}\right)=U_{(p, m p)}=U_{p}$, for each $m=1,2, \ldots, k$, and so $U_{p} \mid b_{i}$ for every $i \neq s$. Also for $(r, p)=1$ one has $\left(U_{p}, U_{r}\right)=U_{1}=1$ but as $b_{s}$ contains only those terms $U_{r}$ for which $(r, p)=1$, we conclude that $U_{p}$ must be relatively prime to $b_{s}$, and so $U_{p} / b_{s}$, which in turn implies that $U_{p} / B$. Thus $\phi(n, k)=\frac{B}{C}$ where $U_{p} \mid C$ but $U_{p} \gamma B$, that is $\phi(n, k)$ cannot be an integer for any $n \geq N_{2}$ as required.
Remark 1.1: It is clear that the above argument could easily be applied to higher order recurrences $\left\{U_{n}\right\}$ with $U_{n} \geq n$ if an analogous result in Lemma 1.2 could be found.

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# THE EXISTENCE OF SPECIAL MULTIPLIERS OF SECOND-ORDER RECURRENCE SEQUENCES 

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## 1. INTRODUCTION

One approach to the study of the distributions of residues of second-order recurrence sequences $\left(w_{n}\right)$ modulo powers of a prime $p$ is to identify and examine subsequences $w_{t}^{*}=w_{n+t m}$, that are themselves first-order recurrence sequences. In particular, the restricted period, $h=h\left(p^{r}\right)$, and the multiplier, $M=M\left(p^{r}\right)$, satisfy $w_{n+t h} \equiv M^{t} w_{n}\left(\bmod p^{r}\right)$ for all $t$ and all $n$, and are independent of the initial terms of the sequence (see, e.g., [1]). In [1], we generalized the notion of the restricted period and multiplier to that of the special restricted period and special multiplier. Theorem 3.5 of [1] shows that if the sequence $w$ is $p$-regular for an odd prime $p, r$ is sufficiently large, and $w_{n}$ is not divisible by $p$, then $w_{n+t h\left(p^{\left.r^{*}\right)}\right.} \equiv\left(M^{*}\left(n, p^{r}\right)\right)^{t} w_{n}$ $\left(\bmod p^{r}\right)$ for all $t$, where $r^{*}=\lceil r / 2\rceil$, and the integer $M^{*}\left(n, p^{r}\right)$, which is defined up to congruence modulo $p^{r}$ and depends upon $n$, is called the special multiplier of $w$ with respect to $n$.

In this article, we examine the residues $d$ that actually occur as special multipliers of a second-order recurrence sequence. We show that if there exists a $p$-regular sequence satisfying a given second-order recursion and $r$ is sufficiently large, then every conceivable special multiplier actually exists modulo $p^{r}$. Since the special multiplier $M^{*}\left(n, p^{r}\right)$ must satisfy the congruence

$$
M^{*}\left(n, p^{r}\right) \equiv M\left(p^{r^{*}}\right) \quad\left(\bmod p^{r^{*}}\right)
$$

this amounts to showing that if $d \equiv M\left(p^{r^{*}}\right)\left(\bmod p^{r^{*}}\right)$, then there exists a sequence $w$ that satisfies the given recursion, and an index $n$, such that $d$ actually occurs as the special multiplier $M^{*}\left(n, p^{r}\right)$ of that sequence.

The proof of the theorem is broken into three cases depending upon whether $\left(\frac{D}{p}\right)=-1,1$, or 0 , where $D$ is the discriminant of the sequence $w$.

## 2. PRELIMINARIES

We employ the standard notation of [1]. In particular, $w(a, b)$ represents a second-order sequence that satisfies the recursion

$$
\begin{equation*}
w_{n+2}=a w_{n+1}-b w_{n} \tag{2.1}
\end{equation*}
$$

and, for a given odd prime $p, \mathcal{F}(a, b)$ denotes the family of sequences $(w)$ that satisfy (2.1) and for which $p \gamma\left(w_{0}, w_{1}\right)$. We let $\lambda_{w}\left(p^{r}\right)$ denote the period of $w(a, b)$ modulo $p^{r}$, i.e., the least positive integer $\lambda$ such that for all $n$

$$
w_{n+\lambda} \equiv w_{n} \quad\left(\bmod p^{r}\right)
$$

and similarly, we let $h_{w}\left(p^{r}\right)$ denote the restricted period of $w(a, b)$ modulo $p^{r}$, i.e., the least positive integer $h$ such that for some integer $M$ and for all $n$

$$
w_{n+h} \equiv M w_{n} \quad\left(\bmod p^{r}\right)
$$

The integer $M=M_{w}\left(p^{r}\right)$, defined up to congruence modulo $p^{r}$, is called the multiplier of $w(a, b)$ modulo $p^{r}$. Since they are critical to the present study, we also remind the reader of the definitions of the special restricted period and special multipliers of a sequence $w \in \mathcal{F}(a, b)$.
Definition 2.1: For fixed $n$ and $r$, let $h_{w}^{*}\left(n, p^{r}\right)$ be the least integer $m$ of the set $\left\{h_{w}\left(p^{c}\right) \mid 1 \leq\right.$ $c \leq r\}$ for which the sequence $w_{t}^{*}=w_{n+t m}$ satisfies a first-order recurrence relation $w_{t+1}^{*} \equiv$ $M^{*} w_{t}^{*}\left(\bmod p^{r}\right)$ for some integer $M^{*}$. The integer $h^{*}=h_{w}^{*}\left(n, p^{r}\right)$ is called the special restricted period and $M^{*}=M_{w}^{*}\left(n, p^{r}\right)$ (defined up to congruence modulo $p^{r}$ ) the special multiplier with respect to $w_{n}$ modulo $p^{r}$.

Finally, we let $f(x)=x^{2}-a x+b$ be the characteristic polynomial of $(w)$ and $D=$ $D(a, b)=a^{2}-4 b$ the discriminant of $(w)$.

In general, when studying recursive sequences $w(a, b)$ modulo powers of a prime $p$, elements $w_{n}$ for which $p \mid w_{n}$ behave quite differently from elements for which $p \mid w_{n}$. It is convenient to refer to a term $w_{n}$ for which $p / w_{n}$ as $p$-regular and a term $w_{n}$ for which $p \mid w_{n}$ as $p$-singular. Analogously, we call an integer $d p$-singular if $p \mid d$ and $p$-regular if $p / d$.

The sequences of $\mathcal{F}(a, b)$ are partitioned into equivalence classes, usually called $m$-blocks, by the equivalence relation mot, which relates two sequences if one is equivalent modulo $m$ to a multiple of a translation of the other. We are interested here in $p^{r}$-blocks, where $p$ is an odd prime.

A recurrence $w(a, b)$ is $p$-regular if $w_{0} w_{2}-w_{1}^{2} \not \equiv 0(\bmod p)$, and $p$-irregular (or simply irregular, if the prime $p$ is evident) otherwise. It is well known that either every sequence in a block of $\mathcal{F}(a, b)$ is $p$-regular, or none of them are, and hence, the blocks of $\mathcal{F}(a, b)$ are divided into $p$-regular and $p$-irregular blocks. It is also easy to see that all $p$-regular sequences in $\mathcal{F}(a, b)$ have the same period, restricted period, and multiplier. Consequently, the period, restricted period, and multiplier of a regular sequence in $\mathcal{F}(a, b)$ are independent of the initial terms of the sequence, and are global parameters of the family $\mathcal{F}(a, b)$. We denote these global parameters by $\lambda\left(p^{r}\right), h\left(p^{r}\right)$, and $M\left(p^{r}\right)$, respectively. If $u(a, b) \in \mathcal{F}(a, b)$ denotes the generalized Fibonacci sequence, i.e., the sequence in $\mathcal{F}(a, b)$ with initial terms 0 and 1 , then $u(a, b)$ is $p$-regular and therefore can be used to determine the global parameters of $\mathcal{F}(a, b)$. In particular, $h\left(p^{r}\right)=h_{u}\left(p^{r}\right)$.

For most second-order sequences $w(a, b)$, the restricted period modulo $p^{r+1}$ is $p$ times the restricted period of $w(a, b)$ modulo $p^{r}$ when the exponent $r$ is sufficiently large. The precise value of $r$ that constitutes sufficiently large in this sense is denoted by the critical parameter $e(w)$, as defined below.
Definition 2.2: If $w(a, b) \in \mathcal{F}(a, b)$, then we define $e=e(w)$ to be the largest integer, if it exists, such that $h_{w}\left(p^{e}\right)=h_{w}(p)$.

The period of a second-order recurrence manifests a similar behavior and we define the corresponding parameter $f(w)$.
Definintion 2.3: If $w(a, b) \in \mathcal{F}(a, b)$, then we define $f=f(w)$ to be the largest integer, if it exists, such that $\lambda_{w}\left(p^{f}\right)=\lambda_{w}(p)$.

The sequence $w(a, b)$ is said to be nondegenerate if the parameter $e(w)$ exists, and degenerate otherwise. If $w$ is $p$-regular and $e(w)$ does not exist, then $h_{u}(p)=0$, and all of
the $p$-regular sequences in $\mathcal{F}(a, b)$ are degenerate. Our main theorem, Theorem 5.1 , concerns families $\mathcal{F}(a, b)$ that contain a nondegenerate $p$-regular sequence. It follows that all of the $p$-regular sequences examined in this paper are nondegenerate.

On the other hand, we must take into account degenerate $p$-irregular sequences in $\mathcal{F}(a, b)$. For notational convenience we adopt the convention that $e(w)=\infty$ when $w$ is degenerate, and consider the statement $r<e(w)$ to be true when $e(w)=\infty$. Note that a degenerate $p$-irregular second-order recurrence satisfies a first-order recurrence modulo $p^{r}$ for all positive integers $r$.

The restricted periods of $p$-regular sequences are given by the following important theorem.
Theorem 2.4 (Theorem 2.11 of [1]): Suppose that $w(a, b) \in \mathcal{F}(a, b)$ is p-regular and that $e=e(w)$ and $f=f(w)$ both exist. Let $e^{*}=\min (r, e), f^{*}=\min (r, f)$, and $s=\lambda(p) / h(p)$. Then, for all positive integers $r$,

$$
\begin{align*}
& h\left(p^{r}\right)=p^{r-e^{*}} h\left(p^{e}\right)  \tag{2.2}\\
& \lambda\left(p^{r}\right)=p^{r-f^{*}} \lambda\left(p^{f}\right) \quad \text { and }  \tag{2.3}\\
& E\left(p^{r}\right)=\operatorname{ord}_{p^{r}}\left(M\left(p^{r}\right)\right)=\frac{\lambda\left(p^{r}\right)}{h\left(p^{r}\right)}=\frac{p^{r-f^{*}} \lambda(p)}{p^{r-e^{*}} h(p)}=p^{e^{*}-f^{*}} s . \tag{2.4}
\end{align*}
$$

The following theorem is analogue for $p$-irregular sequences. We note that both Theorem 2.5 and Corollary 2.6 remain true in the case that $w$ is degenerate.

Theorem 2.5: Let $w(a, b) \in \mathcal{F}(a, b)$ be a $p$-irregular recurrence and set $h^{\prime}\left(p^{r}\right)=h_{w}\left(p^{r}\right)$, $e=$ $e(u)$, and $e^{\prime}=e(w)$. Let $\hat{r}=\max \left(r-e^{\prime}, 0\right)$. Then

$$
h^{\prime}\left(p^{r}\right)=h_{w}\left(p^{r}\right)=h\left(p^{\hat{r}}\right)= \begin{cases}1 & \text { if } r \leq e^{\prime} \\ h\left(p^{r-e^{\prime}}\right)=h\left(p^{e}\right)=h(p) & \text { if } e^{\prime}<r<e^{\prime}+e \\ h\left(p^{r-e^{\prime}}\right)=p^{r-e-e^{\prime}} h(p) & \text { if } e^{\prime}+e \leq r\end{cases}
$$

Theorem 2.5 has an important corollary that we require below.
Corollary 2.6: If $w, w^{\prime} \in \mathcal{F}(a, b)$ are $p$-irregular and satisfy $e(w)=e\left(w^{\prime}\right)$, then $h_{w}\left(p^{r}\right)=$ $h_{w^{\prime}}\left(p^{r}\right)$.

Proof: It is clear from Theorem 2.5 that the restricted period depends only on $e(w)$ and the global parameters $h(p)$ and $e$.

The ratios of terms of recurrences $(w)$ modulo $p^{r}$ are closely related to multipliers and play a key role in our study. If $a, b, c$, and $d$ are integers, with $p \nmid b$ and $p \gamma d$, then the quotients $a / b$ and $c / d$ may be viewed as elements of $\mathbb{Z}_{p}$, the localization of the integers at the prime ideal $(p)$. It is then natural to define

$$
a / b \equiv c / d\left(\bmod p^{r}\right) \quad \text { if and only if } a d-b c \equiv 0\left(\bmod p^{r}\right)
$$

In [1], the notation $\rho_{w}(n, m)$ was introduced to represent the ratio of elements $w_{n+m}$ and $w_{n}$ of a second-order recurrence sequence $(w)$ when $w_{n}$ was not divisible by $p$. We extend that
notation to include the situation when the $p$-power dividing $w_{n}$ does not exceed the $p$-power dividing $w_{n+m}$.
Definition 2.7: If $w(a, b) \in \mathcal{F}(a, b)$ and $m$ and $n$ are nonnegative integers such that $p^{k} \| w_{n}$ and $p^{k} \| w_{n+m}$, then we define $\rho(n, m)=\rho_{w}(n, m)$ to be the element $\left(w_{n+m} / p^{k}\right) /\left(w_{n} / p^{k}\right) \in$ $\mathbb{Z}_{p}$.

Note, in particular, that if $w_{n}$ is $p$-regular, then the multiplier and special multiplier modulo $p^{r}$ can be expressed in terms of ratios:

$$
\begin{aligned}
& M_{w}\left(p^{r}\right) \equiv \rho\left(n, h_{w}\left(p^{r}\right)\right) \quad\left(\bmod p^{r}\right) \\
& M_{w}^{*}\left(p^{r}\right) \equiv \rho\left(n, h_{w}^{*}\left(p^{r}\right)\right) \quad\left(\bmod p^{r}\right)
\end{aligned}
$$

To make it convenient to refer to elements congruent to ratios modulo $p^{r}$, we introduce the mapping $\pi_{r}: \mathbb{Z}_{p} \rightarrow \mathbb{Z} / p^{r} \mathbb{Z}$, the canonical extension to $\mathbb{Z}_{p}$ of the quotient $\operatorname{map} \pi: \mathbb{Z} \rightarrow \mathbb{Z} / p^{\prime} \mathbb{Z}$.

We require the following three basic lemmas from [1] in our analysis below.
Lemma 2.8 (Lemma 3.3 of [1]): Let $w(a, b) \in \mathcal{F}(a, b)$ and fix a positive integer $c$. Let $i$ and $j$ be two integers such that $i<j$. Let $\ell$ be the largest integer (possibly zero) such that $h\left(p^{\ell}\right) \mid c$ and $m$ the largest integer (possibly zero) such that $h_{w}\left(p^{m}\right) \mid j-i$. Then

$$
w_{i+c} w_{j}-w_{j+c} w_{i} \equiv 0 \quad\left(\bmod p^{r}\right)
$$

if and only if $\ell+m \geq r$. In particular, if $w_{i}$ and $w_{j}$ are $p$-regular, then $\rho_{w}(i, c) \equiv \rho_{w}(j, c)$ $\left(\bmod p^{r}\right)$ if and only if $\ell+m \geq r$.
Lemma 2.9 (Lemma 3.4) of [1]): Let $w(a, b) \in \mathcal{F}(a, b)$ and $w^{\prime}(a, b) \in \mathcal{F}(a, b)$ and fix a positive integer $c$. Let $\ell$ be the largest integer such that $h\left(p^{\ell}\right) \mid c$ and assume that $\ell<r$. If, for integers $n$ and $i$,

$$
\begin{equation*}
w_{n+c}^{\prime} w_{n+i}-w_{n+i+c} w_{n}^{\prime} \equiv 0 \quad\left(\bmod p^{r}\right) \tag{2.5}
\end{equation*}
$$

then $w^{\prime}(a, b)$ is a mot of $w(a, b)$ modulo $p^{r-\ell}$.
Conversely, if $w^{\prime}(a, b)$ is a mot of $w(a, b)$ modulo $p^{r-\ell}$, then there exists an $i$ such that (2.5) holds for all $n$.

Lemma 2. 10 (Lemma 2.13 of [1]): Let $\mathcal{B}$ be a $p^{r}$-block of $\mathcal{F}(a, b)$ containing the sequence $w$. Then, up to congruence modulo $p^{r}, \mathcal{B}$ contains $p^{r-1}(p-1) h_{w}\left(p^{r}\right)$ distinct sequences.

Finally, we require two tools to "lift" roots modulo $p$ of the characteristic polynomial to roots modulo higher powers of $p$. When $\left(\frac{D}{p}\right)=1$, the characteristic polynomial $f(x)$ has nonsingular roots, that is, roots that are not simultaneously roots of $f^{\prime}(x)$. In this situation, each of the roots modulo $p$ lifts to a unique root modulo each higher power of $p$. The required lifting theorem is Hensel's lemma, which we state here for reference.
Theorem 2.11 (Hensel's lemma): Suppose that $f(x)$ is a polynomial with integral coefficients. If $f(m) \equiv 0\left(\bmod p^{i}\right)$ and $f^{\prime}(m) \not \equiv 0(\bmod p)$, then there is a unique $t$ modulo $p$ such that $f\left(m+t p^{i}\right) \equiv 0\left(\bmod p^{i+1}\right)$.

Proof: See Theorem 2.23, p. 87 of [2].
When $\left(\frac{D}{p}\right)=0$, the characteristic polynomial $f(x)$ has only one singular root modulo $p$, that is, the single root of $f(x)$ modulo $p$ is simultaneously a root of $f^{\prime}(x)$. In this case, the lifting of roots is governed by the following theorem.

Theorem 2.12: Suppose that $f(x)$ is a polynomial with integral coefficients. If $f(m) \equiv$ $0\left(\bmod p^{i}\right)$ and $f^{\prime}(m) \equiv 0(\bmod p)$, then $f\left(m+t p^{i}\right) \equiv f(m)\left(\bmod p^{i+1}\right)$. Furthermore, one of the following occurs:
(a) Each of the $p$ distinct residues $m+t p^{i}\left(\bmod p^{i+1}\right)$, for $0 \leq t<p$, satisfy $f\left(m+t p^{i}\right) \equiv$ $0\left(\bmod p^{i+1}\right)$.
(b) None of the residues $m+t p^{i}\left(\bmod p^{i+1}\right)$, for $0 \leq t<p$, satisfy $f\left(m+t p^{i}\right) \equiv 0\left(\bmod p^{i+1}\right)$. Proof: See p. 88 of [2].
The lifting of singular roots of a polynomial $f(x)$ is more complicated than that of nonsingular roots and is best described by the modulo $p$ root tree of $f(x)$.
Definition 2.13: The modulo $p$ root tree of $f(x)$ is a tree whose nodes at the $k$-th level are labelled by the roots of $f(x)$ modulo $p^{k}$. The nodes at level $k+1$ are connected to the nodes at level $k$ corresponding to the roots from which they are lifted. A terminal node of the root tree at the $k$-th level corresponds to a root modulo $p^{k}$ that does not lift to a root modulo $p^{k+1}$.

For use below, we denote by $n_{k}$ the number of nonterminal nodes of the modulo $p$ root tree of $f(x)$ at the $k$-th level. In other words, $f(x)$ has exactly $n_{k}$ roots modulo $p^{k}$ that lift to roots modulo $p^{k+1}$.

The root tree may be finite or infinite: in the first case, all the nodes at some level of the root tree are terminal; in the second case, one of the roots modulo $p$ lifts to a root modulo $p^{k}$ for all $k$. The polynomials that concern us in this paper, the quadratic characteristic polynomials $f(x)=x^{2}-a x+b$, have at most one singular root when $\left(\frac{D}{p}\right)=0$, and consequently the root tree is connected with a single base node. We illustrate the root tree with two examples.
Example 2.14: Let $f(x)=x^{2}-x-1$, the characteristic polynomial of the Fibonacci family $\mathcal{F}(1,-1)$. Since $D=a^{2}-4 b=1+4=5 \equiv 0(\bmod 5)$, we see that $\left(\frac{D}{5}\right)=0$ and $f(x)$ has a unique, singular root modulo 5 , namely $m=3$. However, since $f(3)=5 \not \equiv 0(\bmod 25)$, this root does not lift to any root of $f(x)$ modulo 25 . It follows that the root tree of $f(x)$ modulo 5 consists of only the base node.


FIGURE 1. The Modulo 3 Root Tree of $f(x)=x^{2}+x+61$

Example 2.15: Let $f(x)=x^{2}+x+61$, the characteristic polynomial of the family $\mathcal{F}(-1,61)$. Since $D=1-244=-243 \equiv 0(\bmod 3)$, we see that $f(x)$ has a unique singular root modulo 3 , namely $m=1$. Since $f(1)=63 \equiv 0(\bmod 9)$, Theorem 2.12 implies that the root 1 lifts to three distinct roots modulo 9 , namely 1,4 , and 7. Thus the root tree of $f(x)$ modulo 3 has three nodes on the second level.

Since $f(1)=63 \not \equiv 0(\bmod 27)$ and $f(7)=117 \not \equiv 0(\bmod 27)$, neither 1 nor 7 lifts to a root of $f(x)$ modulo 27 . However, $f(4)=81 \equiv 0(\bmod 27)$, so Theorem 2.12 implies that the root 4 lifts to three roots of $f(x)$ modulo 27 , namely 4,13 , and 22 . We conclude that the root tree of $f(x)$ modulo 3 has three nodes on the third level.

Next, we observe that $f(4) \equiv f(13) \equiv f(22) \equiv 0(\bmod 81)$, so each of these roots lifts to three roots modulo 81 . Clearly the root 4 lifts to 4 , 31 , and $58 ; 13$ lifts to 13,40 , and 67 ; and 22 lifts to 22,49 , and 76 . Therefore the root tree of $f(x)$ modulo 3 has nine nodes on the fourth level.

To compute the fifth level of the root tree, we observe that $f(x) \not \equiv 0(\bmod 243)$ when $x \in\{4,31,58,22,49,76\}$ while $f(x) \equiv 0(\bmod 243)$ when $x \in\{13,40,67\}$. Therefore the roots 13,40 , and 67 each lift to three roots modulo 243 , namely $13,94,175,40,121,202,67,148$, and 229. Thus the fifth level of the root tree has nine nodes.

Finally, it is easy to check that none of the nine roots of $f(x)$ modulo 243 lifts to a root modulo 729. Thus the root tree of $f(x)$ modulo 3 is finite with five levels. (See Figure 1.)

## 3. $p$-REGULAR BLOCKS

Our analysis of special multipliers requries a careful accounting of the number of $p^{r}$-blocks in $\mathcal{F}(a, b)$ having certain properties. For $p$-regular blocks, this accounting was performed in [1].
Theorenn 3.1 (Corollary 2.17 of [1]): Let $T_{\text {sing }}\left(p^{r}\right)$ and $T_{\mathrm{reg}}\left(p^{r}\right)$ denote, respectively, the number of $p$-regular blocks in $\mathcal{F}(a, b)$ with and without $p$-singular terms. Then

$$
\begin{equation*}
T_{\mathrm{sing}}\left(p^{r}\right)=\frac{p^{r-1} h(p)}{h\left(p^{r}\right)} \quad \text { and } \quad T_{\mathrm{reg}}\left(p^{r}\right)=\frac{p^{r-1}\left(p-\left(\frac{D}{p}\right)-h(p)\right)}{h\left(p^{r}\right)} \tag{3.1}
\end{equation*}
$$

## 4. $p$-IRREGULAR BLOCKS

Counting the number of $p$-irregular blocks in $\mathcal{F}(a, b)$ is somewhat more complicated, and requires examination of several cases. Note that the $p$-irregular sequences $w$ in this section may be degenerate, in which case $e(w)=\infty$. By convention, the assertion that $e(w)>r$ includes the possibility that $e(w)=\infty$.
Lemma 4.1: If $w \in \mathcal{F}(a, b)$ is $p$-irregular and $r \leq e(w)$, then $w$ lies in the same $p^{r}$-block as a sequence in $\mathcal{F}(a, b)$ that has initial terms $1, \gamma$, where $\gamma$ is congruent modulo $p^{r}$ to a root of the characteristic polynomial $f(x)=x^{2}-a x+b$.

Proof: Since $w$ is $p$-irregular and $r \leq e(w)$, the sequence $w$ is first-order modulo $p^{r}$. Moreover, since $w \in \mathcal{F}(a, b)$, we know that $p \\left(w_{0}, w_{1}\right)$, and therefore $\left.p\right\rangle w_{0}$. Choose $\zeta$ and $\gamma \in \mathbb{Z}$ to satisfy $\zeta \equiv w_{0}^{-1}\left(\bmod p^{r}\right)$ and $\gamma \equiv w_{0}^{-1} w_{1}\left(\bmod p^{r}\right)$. Then the multiple $\zeta w$ of the
sequence $w$ is first-order modulo $p^{r}$, satisfies the recurrence relation (2.1), and has initial terms $1, \gamma$. It follows that $\gamma^{2} \equiv a \gamma-b\left(\bmod p^{r}\right)$, and we see that $f(\gamma) \equiv 0\left(\bmod p^{r}\right)$, as desired.
Theorem 4.2: Suppose $\mathcal{B}$ is a p-irregular $p^{r}$-block of $\mathcal{F}(a, b)$ and $w, w^{\prime} \in \mathcal{B}$. Then either $e(w)=e\left(w^{\prime}\right)<r$ or $e(w), e\left(w^{\prime}\right) \geq r$.

Proof: First suppose that $w, w^{\prime} \in \mathcal{B} \subseteq \mathcal{F}(a, b)$ and $r \leq e(w)$. Since $r \leq e(w)$ and $w$ is $p$-irregular, it follows that $w$ satisfies a first-order recurrence modulo $p^{r}$. Since $w^{\prime}$ lies in the same $p^{r}$-block as $w$, and, obviously, any multiple of a translation of a first-order recurrence is also a first-order recurrence, it is clear that $w^{\prime}$ is also first-order modulo $p^{r}$, and hence $r \leq e\left(w^{\prime}\right)$.

Next, suppose that $w, w^{\prime} \in \mathcal{B} \subseteq \mathcal{F}$ and $r>e(w)$. Without loss of generality, we may assume that $e\left(w^{\prime}\right) \geq e(w)$. If $e\left(w^{\prime}\right) \geq r$, then $w^{\prime}$ is a first-order recurrence modulo $p^{r}$, and, since $w$ belongs to the same $p^{r}$-block, $w$ must also be first-order modulo $p^{r}$. But then $e(w) \geq r$, a contradiction. Thus, $r>e\left(w^{\prime}\right) \geq e(w)$.

Since $r>e\left(w^{\prime}\right)$, it is now clear that $w$ and $w^{\prime}$ belong to the same $p^{e\left(w^{\prime}\right)}$-block. By definition of $e\left(w^{\prime}\right)$ and the fact that $w^{\prime}$ is $p$-irregular, we know that $w^{\prime}$ is first-order modulo $p^{e\left(w^{\prime}\right)}$. Therefore $w$ is also first-order modulo $p^{e\left(w^{\prime}\right)}$, and hence $e(w) \geq e\left(w^{\prime}\right)$. We now conclude that $e(w)=e\left(w^{\prime}\right)$, as desired.

The next two theorems provide an accounting of $p$-irregular $p^{r}$-blocks when $\left(\frac{D}{p}\right)=1$, making extensive use of Hensel's lemma.
Theorem 4.3: If $\left(\frac{D}{p}\right)=1$, then there are exactly two distinct p-irregular $p^{r}$-blocks in $\mathcal{F}(a, b)$ that contain a sequence $w$ with the property that $r \leq e_{w}$.

Proof: Since $\left(\frac{D}{p}\right)=1$, the characteristic polynomial $f(x)=x^{2}-a x+b$ has two distinct roots in $\mathbf{Z} / p \mathbf{Z}$. Suppose that $\alpha, \beta \in \mathbb{Z}$ project onto these distinct roots. It is easy to verify that $f^{\prime}(\alpha) \not \equiv 0(\bmod p)$ and $f^{\prime}(\beta) \not \equiv 0(\bmod p)$, as otherwise $\alpha \equiv \beta \equiv a / 2(\bmod p)$, and the roots are not distinct. By Hensel's lemma, applied repeatedly, the polynomial $f(x)$ has exactly two distinct roots modulo $p^{r}$. If we suppose now that $\alpha$ and $\beta \in \mathbf{Z}$ were chosen to project onto these distinct roots modulo $p^{r}$, then the two sequences $w_{\alpha}$ and $w_{\beta}$ that satisfy the recursion (2.1) and have initial terms $1, \alpha$ and $1, \beta$, respectively, are $p$-irregular and first-order modulo $p^{r}$. Hence $e\left(w_{\alpha}\right) \geq r$ and $e\left(w_{\beta}\right) \geq r$. Moreover, it is clear that $w_{\alpha}$ and $w_{\beta}$ lie in different $p^{r}$-blocks.

Conversely, if $w$ is $p$-irregular with $e(w) \geq r$, then, by Lemma 4.1, $w$ lies in the same $p^{r}$-block as $w_{\alpha}$ or $\left(w_{\beta}\right)$, as desired.
Theorem 4.4: If $\left(\frac{D}{p}\right)=1$ and $k<r$, then there are exactly $2 p^{r-k-1}(p-1) / h_{w}\left(p^{r}\right)$ distinct $p$-irregular $p^{r}$-blocks in $\mathcal{F}(a, b)$ that contain a sequence $w$ with the property that $e_{w}=k$.

Proof: First we note that, by Corollary 2.6, $h_{w}\left(p^{r}\right)$ is independent of the choice of the sequence $w$.

Next, we count the number of sequences, up to congruence modulo $p^{r}$, in the set

$$
\Omega_{k}=\left\{w \in \mathcal{F}(a, b) \mid w \text { is } p \text {-irregular, } e(w)=k, \text { and } w_{0} \equiv 1\left(\bmod p^{r}\right)\right\} .
$$

## THE EXISTENCE OF SPECIAL MULTIPLIERS OF SECOND-ORDER RECURRENCE SEQUENCES

If $w \in \Omega_{k}$, then $w$ is first-order modulo $p^{k}$, but is not first order modulo $p^{k+1}$. Consequently, if $f(x)=x^{2}-a x+b$ is the characteristic polynomial of $w$ and $w_{1}=\gamma$, then

$$
\begin{align*}
& f(\gamma) \equiv 0 \quad\left(\bmod p^{k}\right) \quad \text { and }  \tag{4.1}\\
& f(\gamma) \not \equiv 0 \quad\left(\bmod p^{k+1}\right) . \tag{4.2}
\end{align*}
$$

Since $\left(\frac{D}{p}\right)=1$, the polynomial $f(x)$ has two distinct roots modulo $p$. By Hensel's lemma, there are exactly two residues modulo $p^{k}$ that satisfy (4.1), and, again by Hensel's lemma, exactly $2(p-1)$ residues modulo $p^{k+1}$ that satisfy both (4.1) and (4.2). It follows that there are $2 p^{r-(k+1)}(p-1)$ residues modulo $p^{r}$ that satisfy both (4.1) and (4.2).

On the other hand, if $w \in \mathcal{F}(a, b)$ has initial terms $1, \gamma$, where $\gamma$ is congruent modulo $p^{r}$ to one of the $2 p^{r-(k+1)}(p-1)$ residues that satisfy both (4.1) and (4.2), then $w$ satisfies (2.1) and is first-order modulo $p^{k}$, but is not first order modulo $p^{k+1}$. Thus $e(w)=k$ and $w \in \Omega_{k}$. It follows that $\Omega_{k}$ contains exactly $2 p^{r-(k+1)}(p-1)$ sequences that are distinct modulo $p^{r}$.

Since the initial term $w_{0}$ of a $p$-irregular sequence in $\mathcal{F}(a, b)$ is invertible, it is clear that each $p$-irregular sequence in $\mathcal{F}(a, b)$ for which $e(w)=k$ is equivalent modulo $p^{r}$ to one of the sequences in $\Omega_{k}$. Moreover, the $\phi\left(p^{r}\right)$ multiples by an invertible element of $\mathbb{Z} / p^{r} \mathbb{Z}$ of each of the $2 p^{r-(k+1)}(p-1)$ sequences in $\Omega_{k}$ are distinct modulo $p^{r}$. Thus there are exactly $2 \phi\left(p^{r}\right) p^{r-(k+1)}(p-1)=2 p^{2 r-k-2}(p-1)^{2} p$-irregular sequences $w \in \mathcal{F}(a, b)$ that are distinct modulo $p^{r}$ and satisfy $e(w)=k$.

Finally, by Lemma 2.10 and Corollary 2.6, every $p^{r}$-block of $\mathcal{F}(a, b)$ that contains a sequence $w$ that is $p$-irregular and satisfies $e(w)=k$ contains $p^{r-1}(p-1) h_{w}\left(p^{r}\right)$ distinct sequences modulo $p^{r}$, and hence there are $2 p^{r-(k+1)}(p-1) / h_{w}\left(p^{r}\right)$ such $p^{r}$-blocks.

Finally, we examine the situation when $\left(\frac{D}{p}\right)=0$. Again, our objective is to count the number of $p$-irregular $p^{r}$-blocks and the primary technique is to lift the roots of the characteristic polynomial. In this situation, however, the roots are singular, and the primary tool is Theorem 2.12 rather than Hensel's lemma.

As in the analysis when $\left(\frac{D}{p}\right)=1$, we wish to count separately the $p$-irregular blocks that which contain a sequence $w$ for which $e_{w}<r$ and those that which contain a sequence $w$ for which $r \leq e_{w}$. However, the computation here depends heavily on the parameters $a$ and $b$. Consequently, our results will depend upon the structure of the modulo $p$ root tree of the characteristic polynomial $f(x)=x^{2}-a x+b$. In particular, the next two results depend upon the number of nonterminal nodes $n_{k}$ at the $k$-th level of the root tree.

Theorem 4.5: If $\left(\frac{D}{p}\right)=0$ and $n_{r-1}$ is the number of nonterminal nodes at level $r-1$ of the modulo $p$ root tree of $f(x)=x^{2}-a x+b$, then there are exactly $p n_{r-1}$ distinct $p$-irregular $p^{r}$-blocks in $\mathcal{F}(a, b)$ that contain a sequence $w$ with the property that $r \leq e_{w}$.

Proof: As in the proof of Theorem 4.3, the $p$-irregular $p^{r}$-blocks that contain a sequence $w$ for which $r \leq e_{w}$ correspond to the sequences $w_{\alpha}$ that satisfy the recursion (2.1) and have initial terms $1, \alpha$, where $\alpha \in \mathbb{Z}$ projects onto a root of $f(x)$ modulo $p^{r}$. The roots of $f(x)$ modulo $p^{r}$ correspond to the nodes at the $r$-th level of the modulo $p$ root tree.

By Theorem 2.12 each root of $f(x)$ modulo $p^{r-1}$ either fails to lift to any root modulo $p^{r}$, or lifts to $p$ distinct roots modulo $p^{r}$. By definition of $n_{r-1}$, the characteristic polynomial
$f(x)$ has exactly $n_{r-1}$ roots modulo $p^{r-1}$ that lift to roots modulo $p^{r}$. It follows that there are exactly $p n_{r-1}$ distinct roots of $f(x)$ modulo $p^{r}$, and consequently, there are exactly $p n_{r-1}$ distinct $p$-irregular $p^{r}$-blocks that contain a sequence $w$ for which $r \leq e_{w}$.

Theorem 4.6: Suppose that $\left(\frac{D}{p}\right)=0$ and $k<r$. Let $n_{k}$ denote the number of nonterminal nodes at the $k$-th level of the modulo $p$ root tree of $f(x)=x^{2}-a x+b$. Then the number of distinct p-irregular $p^{r}$-blocks in $\mathcal{F}(a, b)$ that contain a sequence $w$ with the property that $e_{w}=k$ is exactly
(a) $\left(1-n_{1}\right) p^{r-1} / h_{w}\left(p^{r}\right)$, if $k=1$, and
(b) $\left(p n_{k-1}-n_{k}\right) p^{r-k} / h_{w}\left(p^{r}\right)$, if $1<k<r$.

Proof: As in the proof of Theorem 4.4, we first observe that, by Corollary 2.6, $h_{w}\left(p^{r}\right)$ is independent of the choice of the sequence $w$.

For each $k<r$, we let

$$
\Omega_{k}=\left\{w \in \mathcal{F}(a, b) \mid w \text { is } p \text {-irregular, } e(w)=k, \text { and } w_{0} \equiv 1\left(\bmod p^{r}\right)\right\}
$$

As in the proof of Theorem 4.4, if $w \in \Omega_{k}$, then $w$ is first-order modulo $p^{k}$, but is not first order modulo $p^{k+1}$. Consequently, if $f(x)=x^{2}-a x+b$ is the characteristic polynomial of $w$ and $w_{1}=\gamma$, then

$$
\begin{array}{ll}
f(\gamma) \equiv 0 & \left(\bmod p^{k}\right) \quad \text { and } \\
f(\gamma) \not \equiv 0 & \left(\bmod p^{k+1}\right) \tag{4.4}
\end{array}
$$

Therefore, $\gamma$ corresponds to a node on level $k$ of the modulo $p$ root tree of $f(x)$, but not on level $k+1$, i.e., a terminal node on the $k$-th level of the root tree.

Suppose that $k=1$. We know that there is a unique node at the first level of the root tree, corresponding to the unique root modulo $p$ of the characteristic polynomial $f(x)$. Since $n_{1}$, which must be either 0 or 1 , is the number of nodes that lift, there remain ( $1-n_{1}$ ) terminal nodes, that is, $\left(1-n_{1}\right)$ roots modulo $p$ that satisfy both (4.3) and (4.4). It follows that there are $\left(1-n_{1}\right) p^{r-1}$ residues modulo $p^{r}$ that satisfy both (4.3) and (4.4), and hence $\left|\Omega_{1}\right|=\left(1-n_{1}\right) p^{r-1}$.

Now suppose that $1<k<r$. By Theorem 2.12 there are $p n_{k-1}$ nodes at the $k$-th level of the modulo $p$ root tree, and $n_{k}$ of these lift. It follows that the $k$-th level of the root tree contains $\left(p n_{k-1}-n_{k}\right)$ terminal nodes. These nodes correspond to roots $\gamma$ of $f(x)$ modulo $p^{k}$ that satisfy both (4.3) and (4.4). Therefore there are $\left(p n_{k-1}-n_{k}\right) p^{r-k}$ distinct residues modulo $p^{r}$ that satisfy both (4.3) and (4.4), and hence $\left|\Omega_{k}\right|=\left(p n_{k-1}-n_{k}\right) p^{r-k}$.

Since the initial term $w_{0}$ of a $p$-irregular sequence in $\mathcal{F}(a, b)$ is invertible, it is clear that each $p$-irregular sequence in $\mathcal{F}(a, b)$ for which $e(w)=k$ is equivalent modulo $p^{r}$ to one of the sequences in $\Omega_{k}$. Moreover, the $\phi\left(p^{r}\right)$ multiples by an invertible element of $\mathbf{Z} / p^{r} \mathbf{Z}$ of each of the sequences in $\Omega_{k}$ are distinct modulo $p^{r}$. Thus there are exactly $\phi\left(p^{r}\right)\left|\Omega_{k}\right| p$-irregular sequences $w \in \mathcal{F}(a, b)$ that are distinct modulo $p^{r}$ and satisfy $e(w)=k$.

Finally, by Lemma 2.10 and Corollary 2.6, every $p^{r}$-block of $\mathcal{F}(a, b)$ that contains a sequence $w$ that is $p$-irregular and satisfies $e(w)=k$ contains $p^{r-1}(p-1) h_{w}\left(p^{r}\right)=\phi\left(p^{r}\right) h_{w}\left(p^{r}\right)$ distinct sequences modulo $p^{r}$, and hence there are $\left|\Omega_{k}\right| / h_{w}\left(p^{r}\right)$ such $p^{r}$-blocks.

By substituting in the computed values of $\left|\Omega_{k}\right|$ for $k=1$ and $1<k<r$, we obtain the conclusion of the theorem.

## 5. THE MAIN THEOREM

Theorem 5.1: Suppose that $\mathcal{F}(a, b)$ contains a nondegenerate $p$-regular sequence and $r>e$. Let $d$ be a nonnegative integer such that $d \equiv M\left(p^{r^{*}}\right)\left(\bmod p^{r^{*}}\right)$. Then there exists a recurrence $w(a, b) \in \mathcal{F}(a, b)$ and an index $n$ such that $0 \leq n<\hbar\left(p^{r-r^{*}}\right)$ and

$$
d \equiv \rho_{w}\left(n, h^{*}\right) \quad\left(\bmod p^{r}\right)
$$

Proof: If $w(a, b) \in \mathcal{F}(a, b)$, then for each $p$-regular term $w_{n}$, the ratio $\rho_{w}\left(n, h^{*}\right)$ satisfies

$$
\rho_{w}\left(n, h^{*}\right) \equiv M\left(p^{r^{*}}\right) \quad\left(\bmod p^{r^{*}}\right)
$$

There are exactly $p^{r-r^{*}}$ residues $t$ modulo $p^{r}$ with the property that $t \equiv d\left(\bmod p^{r^{*}}\right)$. Consequently, if we can show that the residues $\pi_{r}\left(\rho_{w}\left(n, h^{*}\right)\right) \in \mathbb{Z} / p^{r} \mathbb{Z}$, corresponding to the ratios $\rho_{w}\left(n, h^{*}\right)$ arising from every $p$-regular term $w_{n}$ of every sequence $w(a, b) \in \mathcal{F}(a, b)$, account for $p^{r-r^{*}}$ distinct residues modulo $p^{r}$, then one ratio must satisfy the required congruence $\rho_{w}\left(n, h^{*}\right) \equiv d\left(\bmod p^{r}\right)$. To this end, we carefully enumerate the distinct residues modulo $p^{r}$ that appear as ratios $\rho_{w}\left(n, h^{*}\right)\left(\bmod p^{r}\right)$ for sequences $w(a, b) \in \mathcal{F}$.

First observe that, by Lemma 2.8, the ratios $\rho_{w}\left(n, h^{*}\right)$ are distinct modulo $p^{r}$ for $0 \leq n<$ $h\left(p^{r-r^{*}}\right)$. Second, by Lemma 2.9,

$$
\left\{\pi_{r}\left(\rho_{w}\left(n, h^{*}\right)\right) \mid 0 \leq n<\left(p^{r-r^{*}}\right)\right\}=\left\{\pi_{r}\left(\rho_{w^{\prime}}\left(n, h^{*}\right)\right) \mid 0 \leq n<h\left(p^{r-r^{*}}\right)\right\}
$$

when $w$ and $w^{\prime}$ lie in the same block modulo $p^{r-r^{*}}$, while

$$
\left\{\pi_{r}\left(\rho_{w}\left(n, h^{*}\right)\right) \mid 0 \leq n<\left(p^{r-r^{*}}\right)\right\} \cap\left\{\pi_{r}\left(\rho_{w \prime}\left(n, h^{*}\right)\right) \mid 0 \leq n<h\left(p^{r-r^{*}}\right)\right\}=\phi
$$

when $w(a, b)$ and $w^{\prime}(a, b)$ lie in different blocks modulo $p^{r-r^{*}}$. Thus we may narrow our analysis to one sequence from each $p^{r-r^{*}}$-block of $\mathcal{F}(a, b)$.

If $w(a, b)$ contains no $p$-singular elements, then the ratios $\left\{\rho_{w}\left(n, h^{*}\right) \mid 0 \leq n<h\left(p^{r-r^{*}}\right)\right\}$ account for $h\left(p^{r-r^{*}}\right)$ distinct residues modulo $p^{r}$. On the other hand, suppose that $w(a, b)$ contains $p$-singular terms. Clearly every cycle in the same block as $w(a, b)$ has the same number of $p$-singular terms, and without loss of generality, we may assume that $w_{0}$ is $p$-singular. Then $w_{m}$ is $p$-singular if and only if $h(p) \mid m$. Consequently, one restricted period of $w(a, b)$ contains $h\left(p^{r-r^{*}}\right) / h(p) p$-singular terms and $h\left(p^{r-r^{*}}\right)-h\left(p^{r-r^{*}}\right) / h(p) p$-regular terms. As noted above, these $p$ regular terms $w_{m}$ give rise to distinct ratios $\rho_{w}\left(m, h^{*}\right)$ modulo $p^{r}$, and hence the block of $w(a, b)$ contributes $h\left(p^{r-r^{*}}\right)-h\left(p^{r-r^{*}}\right) / h(p)$ ratios modulo $p^{r}$.

We can now apply Theorem 3.1 to count the distinct special multipliers that arise from sequences in the $p$-regular $p^{r-r^{*}}$-blocks of $\mathcal{F}(a, b)$. The number of distinct ratios $\rho_{w}\left(n, h^{*}\right)$ modulo $p^{i r}$ is:

$$
\begin{align*}
T_{\mathrm{reg}}\left(p^{r-r^{*}}\right) & \cdot h\left(p^{r-r^{*}}\right)+T_{\mathrm{sing}}\left(p^{r-r^{*}}\right) \cdot\left(h\left(p^{r-r^{*}}\right)-\frac{h\left(p^{r-r^{*}}\right)}{h(p)}\right) \\
& =\left(T_{\mathrm{reg}}\left(p^{r-r^{*}}\right)+T_{\mathrm{sing}}\left(p^{r-r^{*}}\right)\right) h\left(p^{r-r^{*}}\right)-T_{\operatorname{sing}}\left(p^{r-r^{*}}\right)-\left(\frac{h\left(p^{r-r^{*}}\right)}{h(p)}\right)  \tag{5.1}\\
& =p^{r-r^{*}-1}\left(p-\left(\frac{D}{p}\right)\right)-p^{r-r^{*}-1}
\end{align*}
$$

To complete the proof, we count the number of distinct special multipliers that arise from sequences in the $p$-irregular $p^{r-r^{*}}$-blocks. We break the analysis into three cases corresponding to $\left(\frac{D}{p}\right)=-1,1$, and 0 .
Case 1: $\left(\frac{D}{p}\right)=-1$.
If $\left(\frac{D}{p}\right)=-1,(5.1)$ yields $p^{r-r^{*}}$ distinct ratios, and the argument is complete.
Case 2: $\left(\frac{D}{p}\right)=1$.
Assume that $\left(\frac{D}{p}\right)=1$. Then (5.1) yields $p^{r-r^{*}}-2 p^{r-r^{*}-1}$ distinct ratios arising from the $p$-regular $p^{r-r^{*}}$-blocks. To complete the argument, we counnt the distinct ratios arising from sequences in the $p$-irregular $p^{r-r^{*}}$-blocks.

If $\mathcal{B}$ is a $p$-irregular $p^{r-r^{*}}$-block of $\mathcal{F}(a, b)$ that contains a sequence $w$ for which $e(w) \geq$ $r-r^{*}$ then, by Theorem $2.5, h_{w}\left(p^{r-r^{*}}\right)=1$ and the block $\mathcal{B}$ contributes only one additional ratio, $\rho(0,1)$. Since, by Theorem 4.3 , there are exactly two such blocks, these blocks contribute two additional ratios.

If $\mathcal{B}$ is a $p^{r-r^{*}}$-block containing a sequence $w$ for which $e(w)=k<r-r^{*}$, then $w$ contributes $h_{w}\left(p^{r-r^{*}}\right)$ additional ratios. By Theorem 4.4, there are exactly $2 p^{r-r^{*}-k-1}(p-$ $1) / h_{w}\left(p^{r-r^{*}}\right)$ such $p^{r-r^{*}}$-blocks, and therefore these blocks contribute

$$
\frac{2 p^{r-r^{*}-k-1}(p-1)}{h_{w}\left(p^{r-r^{*}}\right)} \cdot h_{w}\left(p^{r-r^{*}}\right)=2 p^{r-r^{*}-k-1}(p-1)
$$

additional ratios. If we sum over all possible values of $k$, i.e., $1 \leq k<r-r^{*}$, we obtain

$$
\begin{align*}
\sum_{k=1}^{r-r^{*}-1} 2 p^{r-r^{*}-k-1}(p-1) & =2(p-1) \sum_{k=1}^{r-r^{*}-1} p^{r-r^{*}-1-k} \\
& =2(p-1) \frac{p^{r-r^{*}-1}-1}{p-1}=2 p^{r-r^{*}-1}-2 \tag{5.2}
\end{align*}
$$

additional ratios.
Combining the new ratios obtained from the $p$-irregular $p^{r-r^{*}}$-blocks with those obtained from the $p$-regular $p^{r-r^{*}}$-blocks yields $p^{r-r^{*}}-2 p^{r-r^{*}-1}+2 p^{r-r^{*}-1}-2+2=p^{r-r^{*}}$ ratios, as desired.
Case 3: $\left(\frac{D}{p}\right)=0$.

Assume that $\left(\frac{D}{p}\right)=0$. Then (5.1) yields $p^{r-r^{*}}-p^{r-r^{*}-1}$ distinct ratios arising from the $p$-regular $p^{r-r^{*}}$-blocks. As in the previous case, we complete the argument by counting the distinct ratios arising from sequences in the $p$-irregular $p^{r-r^{*}}$-blocks.

As usual, for each $k$ satisfying $1 \leq k \leq r$, let $n_{k}$ represent the number of nonterminal nodes at the $k$-th level of the modulo $p$ root tree of the characteristic polynomial $f(x)=x^{2}-a x+b$.

If $\mathcal{B}$ is a $p$-irregular $p^{r-r^{*}}$-block of $\mathcal{F}(a, b)$ that contains a sequence $w$ for which $e(w) \geq$ $r-r^{*}$ then, by Theorem $2.5, h_{w} p^{\left(r-r^{*}\right)}=1$ and the block $\mathcal{B}$ contributes only one additional ratio, $\rho(0,1)$. Since, by Theorem 4.5, there are exactly $p n_{r-r^{*}-1}$ such blocks, these blocks contribute $p n_{r-r^{*}-1}$ ratios.

If $\mathcal{B}$ is a $p$-irregular $p^{r-r^{*}}$-block that contains a sequence $w$ for which $e(w)=k<r-r^{*}$, then $w$ contributes $h_{w} p^{\left(r-r^{*}\right)}$ additional ratios. Theorem 4.6 implies that there are exactly $\left(1-n_{1}\right) p^{r-r^{*}-1} / h_{w}\left(p^{r-r^{*}}\right)$ such $p^{r-r^{*}}$-blocks when $k=1$ and $\left(p n_{k-1}-n_{k}\right) p^{r-r^{*}-k} / h_{w}\left(p^{r-r^{*}}\right)$ such $p^{r-r^{*}}$-blocks when $1<k<r-r^{*}$. Therefore the number of additional ratios contributed by these blocks is

$$
\begin{array}{cc}
\frac{\left(1-n_{1}\right) p^{r-r^{*}-1}}{h_{w}\left(p^{r-r^{*}}\right)} \cdot h_{w}\left(p^{r-r^{*}}\right)=p\left(1-n_{1}\right) p^{r-r^{*}-1}, & \text { when } k=1 \\
\frac{\left(p n_{k-1}-n_{k}\right) p^{r-r^{*}-k}}{h_{w}\left(p^{r-r^{*}}\right)} \cdot h_{w}\left(p^{r-r^{*}}\right)=\left(p n_{k-1}-n_{k}\right) p^{r-r^{*}-k}, \quad \text { when } k>1
\end{array}
$$

If we sum over all possible values of $k$, i.e., $1 \leq k<r-r^{*}$, we obtain a simple telescoping sum:

$$
\begin{aligned}
& \left(1-n_{1}\right) p^{r-r^{*}-1}+\sum_{k=2}^{r-r^{*}-1}\left(p n_{k-1}-n_{k}\right) p^{r-r^{*}-k} \\
& \quad=\left(1-n_{1}\right) p^{r-r^{*}-1}+\left(p n_{1}-n_{2}\right) p^{r-r^{*}-2}+\cdots+\left(p n_{r-r^{*}-2}-n_{r-r^{*}-1}\right) p \\
& \quad=p^{r-r^{*}-1}-p n_{r-r^{*}-1}
\end{aligned}
$$

Adding the count of new ratios obtained from the $p$ irregular $p^{r-r^{*}}$-blocks to that obtained from the $p$-regular $p^{r-r^{*}}$-blocks yields $p^{r-r^{*}}-p^{r-r^{*}-1}+p n_{r-r^{*}-1}+p^{r-r^{*}-1}-p n_{r-r^{*}-1}=p^{r-r^{*}}$ ratios, as desired.

## ACKNOWLEDGMENTS

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## 国国

# STERN'S DIATOMIC ARRAY APPLIED TO FIBONACCI REPRESENTATIONS 

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## 1. STERN'S DIATOMIC ARRAY

Each row of Pascal's triangle is formed by addition of adjacent elements of the preceding row, producing binomial coefficients and counting combinations. Each row of Stern's diatomic array is formed by addition of adjacent elements of the preceding row, but interspersed with elements of the preceding row. In this case, the rows of the table will be shown to count certain Fibonacci representations.

Starting with 1 and 1 , form a table in which each line is formed by copying the preceding line, and inserting the sum of consecutive elements: 1,$1 ; 1,1+1,1 ; 1,1+2,2,2+1,1 ; \ldots$ The body of the table contains Stern's diatomic array, sequence A049456 in [10]. Actually, this array has been called Stern's diatomic series in the literature [9], [11], but it should have been called the Eisenstein-Stern diatomic series by earlier authors because Stern's introduction refers back to Eisenstein.


If $a_{n, k}$ is the $k^{t h}$ term in the $n^{t h}$ line, $k=1,2, \ldots, n=0,1,2, \ldots$,

$$
\begin{equation*}
a_{n, 2 m}=a_{n-1, m}+a_{n-1, m+1} \text { and } a_{n, 2 m-1}=a_{n-1, m} \tag{1.1}
\end{equation*}
$$

Lehmer [9] reports that Stern took the initial line 1, 1 as the zero ${ }^{\text {th }}$ line and proved, among others, the following properties:

1. The number of terms in the $n^{t h}$ line is $2^{n}+1$, and their sum is $3^{n}+1$.
2. The table is symmetric; in the $n^{t h}$ line the $k^{t h}$ term equals the $\left(2^{n}+2-k\right)^{\text {th }}$ term.
3. Terms appearing in the $n^{t h}$ line as sums of their two adjacent terms are called dyads. There are $2^{n-1}$ dyads and $2^{n-1}+1$ non-dyads on the $n^{t h}$ line. The dyads a occupy positions of even number $k$ (called rank) on the line.
4. Two consecutive terms, $a$ and $b$, have no common factor.
5. Every ordered pair ( $a, b$ ) occurs exactly once as consecutive terms in some line of the table.
6. If $a$ and $b$ are relatively prime, the pair of consecutive terms ( $a, b$ ) appears in the line whose number is one less than the sum of the quotients appearing in the expansion of $a / b$ in a regular continued fraction.

## STERN'S DIATOMIC ARRAY APPLIED TO FIBONACCI REPRESENTATIONS

Lehmer [9] then uses the quotients of the continued fraction expansion of $r_{1} / r_{2}$ to place the consecutive terms $r_{1}$ and $r_{2}$ into the table by computing both the line number and the rank of $r_{1}$. Further, he shows that the largest dyads in the $n^{t h}$ line, $n \geq 2$, have the value $F_{n+2}$, the $(n+2)^{n d}$ Fibonacci number. Lehmer's results for the line number and rank are summarized in Theorem 1.1.
Theorem 1.1: If consecutive terms $r_{1}$ and $r_{2}$ occur on the $n^{\text {th }}$ line of Stern's diatomic array and if the continued fraction for $r_{1} / r_{2}$ is $\left[q_{1} ; q_{2}, q_{3}, \ldots, q_{m-2}, r_{m-1}\right]$, then

$$
\begin{equation*}
n=q_{1}+q_{2}+q_{3}+\cdots+q_{m-2}+r_{m-1}-1 \tag{1.2}
\end{equation*}
$$

and if $m$ is odd (even), $r_{1}$ is the left (right) neighbor of $r_{2}$ in the first (second) half of line $n$. If $m$ is odd, the position number $k$ (rank) of $r_{1}$ in the first half of line $n$ is

$$
\begin{equation*}
k=2^{q_{1}+q_{2}+\cdots+q_{m-2}}-2^{q_{1}+q_{2}+\cdots+q_{m-3}}+\cdots-2^{q_{1}+q_{2}}+2^{q_{1}} \tag{1.3}
\end{equation*}
$$

More recently, Calkin and Wilf [6] use Stern's diatomic array to explicitly describe a sequence $b(n)$ (sequence A002487 in [10]) such that every positive rational appears exactly once as $b(n) / b(n+1)$,

$$
\begin{equation*}
\{b(n)\}=\{1,1,2,1,3,2,3,1,4,3,5,2,5,3,4,1,5,4,7, \ldots\} \tag{1.4}
\end{equation*}
$$

It is shown in [6] that $b(n)$ counts the number of hyperbinary representations of the integer $n, n \geq 1$; that is, the number of ways of writing $n$ as a sum of powers of 2 , each power being used at most twice, $b(0)=1$. Here, we apply Stern's diatomic array to counting Fibonacci representations.

## 2. FIBONACCI REPRESENTATIONS

Let $R(N)$ denote the number of Fibonacci representations [4] of the positive integer $N$; that is, the number of representations of $N$ as sums of distinct Fibonacci numbers $F_{k}$, (or as a single Fibonacci number $F_{k}$ ), $k \geq 2$, written in descending order. We define $R(0)=1$. The Zeckendorf representation of $N$, denoted Zeck $N$, is the unique representation of $N$ using only non-consecutive Fibonacci numbers $F_{k}, k \geq 2$. The largest Fibonacci number contained in $N$ will be listed first in Zeck $N$. Whenever $R(N)$ is prime, Zeck $N$ uses only Fibonacci numbers whose subscripts have the same parity [3], [5]. For that reason, we are interested in integers $N$ whose Zeckendorf representation uses only even-subscripted Fibonacci numbers; we call such $N$ an even-Zeck integer, denoted $\tilde{N}$, sequence A054204 in [10]. The $j^{t h}$ even-Zeck integer $\tilde{N}=\tilde{N}(j)$ can be written immediately when $j$ is known.

We list early values $R(\tilde{N})$ for consecutive even-Zeck integers $\tilde{N}$, augmented with $R(0)=1$, sequence A002487 in [10]:

$$
\begin{array}{cccccccccccccccccc}
j & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \ldots  \tag{2.1}\\
\tilde{N}(j) & 0 & 1 & 3 & 4 & 8 & 9 & 11 & 12 & 21 & 22 & 24 & 25 & 29 & 30 & 32 & 33 & \ldots \\
R(\tilde{N}(j)) & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 1 & \ldots
\end{array}
$$

The even-Zeck integers $\tilde{N}(j)$ are enumerated below for $j=1,2, \ldots$; we define $\tilde{N}(0)=0$.

| $j$ | binary | powers of 2 | Zeck $\tilde{N}(j)$ | $\tilde{N}(j)$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | $2^{0}$ | $F_{2}$ | 1 |
| 2 | 10 | $2^{1}$ | $F_{4}$ | 3 |
| 3 | 11 | $2^{1}+2^{0}$ | $F_{4}+F_{2}$ | 4 |
| 4 | 100 | $2^{2}$ | $F_{6}$ | 8 |
| 5 | 101 | $2^{2}+2^{0}$ | $F_{6}+F_{2}$ | 9 |
| 6 | 110 | $2^{2}+2^{1}$ | $F_{6}+F_{4}$ | 11 |
| 7 | 111 | $2^{2}+2^{1}+2^{0}$ | $F_{6}+F_{4}+F_{2}$ | 12 |
| 8 | 1000 | $2^{3}$ | $F_{8}$ | 21 |
| 9 | 1001 | $2^{3}+2^{0}$ | $F_{8}+F_{2}$ | 22 |
| 10 | 1010 | $2^{3}+2^{1}$ | $F_{8}+F_{4}$ | 24 |
| $\cdots$ | $\cdots$ | $\ldots$ | $\cdots$ | $\cdots$ |

Lemma 2.1: If $j$ is represented as the sum of distinct powers of 2 in descending order, $j_{\tilde{N}}=2^{r}+2^{s}+\cdots+2^{w}, r>s>w$, then the $j^{\text {th }}$ even-Zeck integer $\tilde{N}=\tilde{N}(j)$ is given by Zeck $\tilde{N}(j)=F_{2(r+1)}+F_{2(s+1)}+\cdots+F_{2(w+1)}$. In short, replace the binary representation of $j$ in the powers $2^{p}, p=0,1, \ldots$, by $F_{2(p+1)}$ to find $\tilde{N}=\tilde{N}(j)$.

Proof: The short table displays Lemma 2.1 for $j=1,2, \ldots, 10$. The next even-Zeck integer $\tilde{N}(j+1)$ will be formed from the binary representation of $(j+1)$.
Lemma 2.2. (i) If Zeck $\tilde{N}=\tilde{N}(j), j \geq 2$, has $F_{2}$ for its smallest term, then $\tilde{N}-1=\tilde{N}(j-1)$, but $\tilde{N}+1$ is not an even-Zeck integer.
(ii) If Zeck $\tilde{N}=\tilde{N}(j), j \geq 2$, has $F_{2 c}, c \geq 2$, for its smallest term, then $\tilde{N}+1=\tilde{N}(j+1)$, but $N-1$ is not an even-Zeck integer.
(iii) The even-Zeck integer $\tilde{N}^{*}$ preceding $\tilde{N}=\tilde{N}(j), j \geq 2$, with $F_{2 c}, c \geq 1$, for its smallest term, is $\tilde{N}(j-1)=\tilde{N}^{*}=\tilde{N}-F_{2 c-2}-1$.

Proof: Let $c=1$, and take $\tilde{N}(j)=F_{2 n}+\cdots+F_{2 p}+F_{2}, p \geq 2, n \geq 3$. Then $\tilde{N}-1=\tilde{N}(j-1)$, but $\tilde{N}+1=F_{2 n}+\cdots+F_{2 p}+F_{3}$ is not an even-Zeck integer, illustrating (i). Further, $\tilde{N}(j-1)=\tilde{N}-F_{0}-1=\tilde{N}-F_{2 c-2}-1, c=1$, satifying (iii).

Let $\tilde{N}(j)=F_{2 n}+\cdots+F_{2 c+2 p}+F_{2 c}, c \geq 2, p \geq 1, n \geq 3$. Then $\tilde{N}(j)+1=F_{2 n}+\cdots+$ $F_{2 c+2}+F_{2 c}+F_{2}=\tilde{N}(j+1)$, but $\tilde{N}(j)-1=F_{2 n}+\cdots+F_{2 c+2 p}+F_{2 c}-1=F_{2 n}+\cdots+$ $F_{2 c+2 p}+\left(F_{2 c-1}+\cdots+F_{7}+F_{5}+F_{3}\right)$, not an even-Zeck integer, as in (ii). Part (iii) follows from

$$
\begin{aligned}
\tilde{N}(j)-F_{2 c-2}-1 & =F_{2 n}+\cdots+F_{2 c+2 p}+F_{2 c}-F_{2 c-2}=1 \\
& =F_{2 n}+\cdots+F_{2 c+2}+\left(F_{2 c-1}-1\right) \\
& =F_{2 n}+\cdots+F_{2 c+2 p}+\left(F_{2 c-2}+\cdots+F_{6}+F_{4}+F_{2}\right)=\tilde{N}(j-1)=\tilde{N}^{*} .
\end{aligned}
$$

Cut the list from $R(\tilde{N})$ given earlier in (2.1) at the boundary 1's to form rows

$$
\left(\begin{array}{ll}
1 & 1
\end{array}\right),\left(\begin{array}{lll}
1 & 2 & 1
\end{array}\right),\left(\begin{array}{lllll}
1 & 3 & 2 & 3 & 1
\end{array}\right),\left(\begin{array}{lllllllll}
1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 1
\end{array}\right), \ldots,
$$

where we keep the leftmost 1 for symmetry. Each row, after the first, includes the list of $R(\tilde{N})$ for the preceding row, interspersed with sums of successive pairs of adjacent terms from the preceding row:
$(1,1),(1,1+1,1),(1,2+1,2,2+1,1),(1,1+3,3,3+2,2,2+3,3,3+1,1), \ldots$
We recognize the first four lines of Stern's Diatomic array. Our $n^{\text {th }}$ row, $1, n,(n-1), \ldots$, contains 1 followed by the number of Fibonacci representations $R(\tilde{N})$ for consecutive even-Zeck integers $\tilde{N}, F_{2 n} \leq \tilde{N} \leq F_{2 n+1}-1$ where $R\left(F_{2 n}\right)=n, n \geq 1$. We next prove that this array is indeed the same as Stern's diatomic array. Lemma 2.3, which allows us to shift subscripts, was Hoggatt's conjecture and was proved by Klarner [8, Thm. 4]. Lemma 2.4 is part of Lemma 11 from [4].
Lemma 2.3: If sequence $\left\{b_{n}\right\}$ satisfies the Fibonacci recurrence $b_{n+2}=b_{n+1}+b_{n}$, then $R\left(b_{k}-1\right)=R\left(b_{k+1}-1\right)$ for $k$ sufficiently large.
Lemma 2.4: Let $N$ be an integer whose Zeckendorf representation has $F_{2 c}, c \geq 2$, as its smallest term. Then $R(N)=R(N-1)+R(N+1)$.
Theorem 2.1: Let the $n^{\text {th }}$ row of an array list the number of Fibonacci representations $R(\tilde{N})$ for consecutive even-Zeck Integers $\tilde{N}, F_{2 n} \leq \tilde{N} \leq F_{2 n+1}-1$. Let $b_{n, k}$ denote the $k^{\text {th }}$ term of the $n^{\text {th }}$ row, $n=1,2,3, \ldots$, where $b_{n, 1} \equiv 1$, and $b_{n, k}=R\left(\tilde{N}\left(j_{n, k}\right)\right)$ for $j_{n, k}=2^{n-1}+k-2, k=2,3, \ldots, 2^{n-1}+1$. Then $b_{n, k}=a_{n-1, k}$, the $k^{t h}$ term in the $(n-1)^{s t}$ line in Stern's diatomic array, $n=1,2, \ldots$, and $k=1,2, \ldots, 2^{n-1}+1$.

Proof: Interpret the leftmost column $(k=1)$ of 1's as $R\left(F_{2 n-1}-1\right)=1$, where $F_{2 n-1}-1$ is the even-Zeck integer preceding $F_{2 n}$ according to Lemma 2.2 (iii) with $\tilde{N}=F_{2 n}, n \geq 1$. In particular, $b_{1,1}=1=a_{0,1}$, and $b_{1,2}=1=a_{0,2}$. We show that the two arrays have the same rule of formation by establishing

$$
\begin{equation*}
b_{n, 2 m}=b_{n-1, m}+b_{n-1, m+1} \text { and } b_{n, 2 m-1}=b_{n-1, m}, n \geq 2 \tag{2.2}
\end{equation*}
$$

(a) We first prove $b_{n, 2 m}=b_{n-1, m}+b_{n-1, m+1}$ for $n \geq 2, m=1, \ldots, 2^{n-2}$. The case $m=1$ is satisfied because $b_{n-1,1}=1$ by definition, and $b_{n-1,2}=R\left(\tilde{N}\left(2^{n-2}\right)\right)=R\left(F_{2(n-1)}\right)=n-1$ from above. For $m=2, \ldots, 2^{n-2}, \tilde{N}(j) \equiv \tilde{N}\left(j_{n, 2 m}\right)=F_{2 n}+\cdots+F_{2 c+2 p}+F_{2 c}, c \geq 2,1 \leq$ $p \leq n-3$, when $n \geq 3$ because $c=1\left(F_{2}\right)$ is not present for even $k \leq 2 m$ in row $n$. Also, $n \geq c+p$; hence, $1 \leq p \leq n-c \leq n-2$ for $n \geq 3$; if $n=2, p=0$ and $c=2$. From Lemma 2.2, $\tilde{N}^{*}=\tilde{N}(j-1), \tilde{N}=\tilde{N}(j) \equiv \tilde{N}\left(j_{n, 2 m}\right)$, and $(\tilde{N}+1)=\tilde{N}(j+1)$ are consecutive even-Zeck integers. Hence $b_{n, 2 m-1}=R\left(\tilde{N}^{*}\right), b_{n, 2 m}=R(\tilde{N})$, and $b_{n, 2 m+1}=R(\tilde{N}+1)$ are consecutive entries in the $n^{\text {th }}$ row. Since $\tilde{N}(j-1)$ and $\tilde{N}(j+1)$ are each a term in some Fibonacci sequence, apply Lemma 2.3 to shift subscripts down 2 in the expressions for $R(\tilde{N}(j+1))$ and $R(\tilde{N}(j-1))$.

$$
\begin{align*}
R(\tilde{N}(j+1)) & =R(\tilde{N}+1)=R\left(F_{2 n}+\cdots+F_{2 c+2 p}+F_{2 c}+\left(F_{3}-1\right)\right) \\
& =R\left(\left(F_{2 n-2}+\cdots+F_{2 c+2 p-2}+F_{2 c-2}+F_{1}\right)-1\right) \\
& =R\left(F_{2(n-1)}+\cdots+F_{2 c+2 p-2}+F_{2 c-2}\right)=R(\tilde{M}), \tag{2.3}
\end{align*}
$$

which is in the $(n-1)^{s t}$ row, and $\tilde{M}$, the argument given above, is an even-Zeck integer. The binary representation of $\tilde{M}=F_{2 n-2}+\cdots+F_{2 c+2 p-2}+F_{2 c-2}, c \geq 2$, is obtained from the binary representation of $\tilde{N}$ by a right-shift by one position (see Lemma 2.1). Because $\tilde{N} \equiv \tilde{N}\left(j_{n, 2 m}\right)$, one therefore finds $\tilde{M} \equiv \tilde{N}\left(j_{n-1, m+1}\right)$; hence by definition $R(\tilde{M})=b_{n-1, m+1}$. From Lemma 2.2 (iii) (with $c \rightarrow c-1 \geq 1$ ), and Lemma 2.3,

$$
\begin{align*}
R(\tilde{N}(j-1)) & =R\left(\tilde{N}^{*}\right)=R\left(\tilde{N}-F_{2 c-2}-1\right) \\
& =R\left(F_{2 n-2}+\cdots+F_{2 c+2 p-2}+F_{2 c-2}-F_{2 c-4}-1\right)=R\left(\tilde{M}^{*}\right) \tag{2.4}
\end{align*}
$$

where $\tilde{M}^{*}$, defined as the argument of the last $R$, is the even-Zeck preceding $\tilde{M}$. Hence, $\tilde{M}^{*}=\tilde{N}\left(j_{n-1, m}\right)$, and by definition, $R\left(\tilde{M}^{*}\right)=b_{n-1, m}$. What we have to prove now is $R(\tilde{N})=R\left(\tilde{M}^{*}\right)+R(\tilde{M})$. For this we want to use Lemma 2.4 with $N \rightarrow \tilde{N}$. We know already that $R(\tilde{N}+1)=R(\tilde{M})$ but $\tilde{N}-1$ is not an even-Zeck integer for $c \geq 2$. However, we now show that $R(\tilde{N}-1)=R\left(\tilde{M}^{*}\right)$.

$$
\begin{aligned}
R(\tilde{N}-1) & =R\left(F_{2 n}+\cdots+F_{2 c+2 p}+F_{2 c}-1\right) \\
& =R\left(F_{2 n-2 c+2}+\cdots+F_{2 p+2}+F_{2}-1\right)=R(\tilde{K})
\end{aligned}
$$

by shifting subscripts down $(2 c-2)$. Recalculate $R\left(\tilde{N}^{*}\right)$ as

$$
\begin{aligned}
R(\tilde{N}(j-1)) & =R\left(F_{2 n}+\cdots+F_{2 c+2 p}+F_{2 c}-F_{2 c-2}-1\right) \\
& =R\left(\left(F_{2 n}+\cdots+F_{2 c+2 p}+F_{2 c-1}-1\right)\right. \\
& =R\left(F_{2 n-2 c+2}+\cdots+F_{2 p+2}+F_{1}-1\right)=R(\tilde{K}),
\end{aligned}
$$

by shifting subscripts down $(2 c-2)$. Thus,

$$
\begin{equation*}
R(\tilde{N}-1)=R(\tilde{N}(j-1))=R\left(\tilde{M}^{*}\right) \tag{2.5}
\end{equation*}
$$

Therefore, $b_{n, 2 m}=b_{n-1, m}+b_{n-1, m+1}$, and part (a) of the proof is finished.
(b) We prove $b_{n, 2 m-1}=b_{n-1, m}$ for $n \geq 2, m=1, \ldots, 2^{n-2}$. For $m=1, b_{n, 1}=1=b_{n-1,1}$ by definition. If $m=2, \ldots, 2^{n-2}, b_{n, 2 m-1}=R\left(\tilde{N}\left(j_{n, 2 m-1}\right)\right)=R\left(\tilde{N}^{*}\right)$ if we use the same notation as in part (a) of the proof. There we have already shown $R\left(\tilde{N}^{*}\right)=R\left(\tilde{M}^{*}\right)=b_{n-1, m}$, which finishes part (b) of the proof. Together with the input $b_{1,1}=1=b_{1,2}$ we have shown that $b_{n, k}=a_{n-1, k}, n=1,2, \ldots$, and $k=1,2, \ldots, 2^{n-1}+1$.

Corollary 2.1.1: If $\tilde{N}$ is an even-Zeck integer such that Zeck $\tilde{N}$ ends in $F_{2 c}, c \geq 2$, and if $\tilde{N}^{*}$ is the preceding even-Zeck integer, then $R\left(\tilde{N}^{*}\right)=R(\tilde{N}-1)$. Also, $R(\tilde{N}(j-1))=R(\tilde{N}(j)-1)$ with $\tilde{N}=\tilde{N}(j)$.

Proof: See equations (2.4) and (2.5).
Theorem 2.2: Let $\tilde{N}=\tilde{N}(j)$ be the $j^{\text {th }}$ even-Zeck integer, $j=0,1,2, \ldots$, with $\tilde{N}(0)=0$. If $R(\tilde{N})=b_{n, k}$ with $b_{n, k}$ defined in Theorem 2.1, then $\tilde{N}=\tilde{N}\left(j_{n, k}\right)$ with $j_{n, k}=2^{n-1}+k-2, k=$ $1,2, \ldots, 2^{n-1}+1$, for $n=1,2, \ldots \tilde{N}(j), j \geq 1$, is obtained by replacing powers $2^{p}$ in the dual representation of $j_{n, k}$ by $F_{2(p+1)}$; if $j=0$, then $\tilde{N}=0$. Alternately, $\tilde{N}=F_{2 n}+\tilde{K}(k-2)$, where $\tilde{K}(k-2)$ is the $(k-2)^{n d}$ even-Zeck integer.

Proof: Apply Lemma 2.1 to Theorem 2.1.
To illustrate Theorem 2.2, $R(\tilde{N})=7=b_{5,8}$ appears as the $8^{\text {th }}$ term in the $5^{\text {th }}$ row; $n-2=8-2=6=2^{2}+2^{1}$, yielding $\tilde{N}=F_{2 \cdot 5}+F_{2(2+1)}+F_{2(1+1)}=F_{10}+F_{6}+F_{4}=66$, and $R(66)=7$. The earlier $R(\tilde{N})=7=b_{5,4}$ in that row occurs for $\tilde{N}=F_{10}+F_{4}=58$.

Since the $n^{\text {th }}$ row of the array for $R(\tilde{N})$ is the $(n-1)^{s t}$ line of Stern's array, several properties of Fibonacci representations of even-Zeck integers $\tilde{N}$ correspond directly to properties given for elements of Stern's diatomic array from Section 1.

1. There are $2^{n-1}$ even-Zeck integers $\tilde{N}$ in the interval $F_{2 n} \leq \tilde{N} \leq F_{2 n+1}-1$. There are $2^{n-1}+1$ terms $R(\tilde{N})$ in the $n^{\text {th }}$ row, whose sum is $3^{n-1}+1$.
2. The table of $R(\tilde{N})$ values is symmetric; in the $n^{\text {th }}$ row, the $k^{\text {th }}$ term equals the $\left(2^{n-1}+\right.$ $2-k)^{t h}$ term. Compare with $R\left(F_{2 n}+M\right)=R\left(F_{2 n+1}-2-M\right), 0 \leq M \leq F_{2 n-1}, n \geq 2$, formed from Theorem 1 of [4] by replacing $n$ with $2 n$.
3. Dyads $R(\tilde{N})$ correspond to Zeck $\tilde{N}$ ending in $F_{2 c}, c \geq 2$; excepting the first column, non-dyads $R(\tilde{N})$ have Zeck $\tilde{N}$ ending in $F_{2}=1$. The dyads have even term numbers.
4. For even-Zeck $\tilde{N}$, consecutive values for $R(\tilde{N})$ are relatively prime. Consecutive values for even-Zeck integers $\tilde{N}$ appear in relatively prime pairs, $(3,4),(8,9),(11,12),(21,22)$, $(24,25), \ldots$
The largest value [2] for $R(\tilde{N})$ in row $n$ is $F_{n+1}$, corresponding to $F_{n+2}$ as the largest dyad in the $n^{\text {th }}$ line as given by Lehmer [9]. Notice that Lemma 2.3 appears in the table as the columns of constants, and the central term in each row is 2. Properties 5 and 6 are explored in the next section.

## 3. STERN'S DIATOMIC ARRAY APPLIED TO FIBONACCI REPRESENTATIONS

We can find many even-Zeck integers $\tilde{N}$ having a specified value for $R(\tilde{N})$ by applying Theorem 1.1. According to Lehmer [9], Stern gives Euler's $\Phi(m)$ as the number of times that an element $m$ appears in the $(m-1)^{s t}$ and all succeeding lines of Stern's diatomic array; this,
of course, is our $m^{t h}$ row, where values for $R(\tilde{N})$ are the elements, and Euler's $\Phi(m)$ is the number of integers not exceeding $m$ and prime to $m$. We express $R(\tilde{N})$ as the sum of a pair of relatively prime integers $r_{1}$ and $r_{2}$, and then use the Euclidean algorithm to write quotients used in the continued fraction for $r_{1} / r_{2}$. The row and column numbers for $R(\tilde{N})=b_{n, k}$, as well as the Zeckendorf representation of $\tilde{N}$, can be written from those same quotients.
Theorem 3.1: Let $R(\tilde{N})=b_{n, k}$ as in Theorem 2.1. Let $R(\tilde{N})=r_{1}+r_{2}, r_{1}$ and $r_{2}$ relatively prime. Use the Euclidean algorithm to write $r_{1}=q_{1} r_{2}+r_{3}, r_{2}=q_{2} r_{3}+r_{4}, r_{3}=q_{3} r_{4}+$ $r_{5}, \ldots, r_{m-2}=q_{m-2}+r_{m-1}+r_{m}, r_{m}=1$. Then $r_{1} / r_{2}=\left[q_{1} ; q_{2}, q_{3}, \ldots, q_{m-2}, r_{m-1}\right]$, a regular continued fraction. The dyad value $R(\tilde{N})$ occurs in row $n$, where

$$
\begin{equation*}
n=q_{1}+q_{2}+q_{3}+\cdots+q_{m-2}+r_{m-1}+1 \tag{3.1}
\end{equation*}
$$

$R(\tilde{N})$ occurs between $r_{1}$ and $r_{2}$, in columns $k$ and $\left(2^{n-1}+2-k\right), k \geq 2$, where

$$
\begin{gather*}
k=2^{q_{1}+q_{2}+\cdots+q_{m-2}+1}-2^{q_{1}+q_{2}+\cdots+q_{m-3}+1}+\cdots-2^{q_{1}+q_{2}+1}+2^{q_{1}+1}, m \text { odd }  \tag{3.2a}\\
\text { or } k=2^{q_{1}+q_{2}+\cdots+q_{m-2}+1}-2^{q_{1}+q_{2}+\cdots+q_{m-3}+1}+\cdots-2^{q_{1}+1}+2, m \text { even } \tag{3.2b}
\end{gather*}
$$

Proof: Equation (3.1) is (1.2), adjusted by adding 2, since our row numbers are one more than Stern's line numbers, and we are one row farther out. Equation (3.2a) is (1.3) when $m$ is odd, taken one row farther out; $k$ is twice the column number of $r_{1}$ in the $(n-1)^{s t}$ row. If $r_{1}$ is a dyad and thus has an even column number, let $r_{2}=b_{n-1,2 w+1}$. If $r_{1}$ is the left neighbor of $r_{2}$, then $r_{1}=b_{n-1,2 w}$ and $b_{n, k}=r_{1}+r_{2}=b_{n, 2(2 w)} ; k$ is twice the column number of $r_{1}$ as (3.2a). If $r_{1}$ is the right neighbor of $r_{2}$, then $r_{1}=b_{n-1,2 w+2}$, and $b_{n, k}=r_{2}+r_{1}=b_{n-1,2 w+1}+b_{n-1,2 w+2}=b_{2(2 w+1)}=b_{n, 4 w+2}$, so that $k$ is 2 more than twice the column number of $r_{1}$ as in (3.2b).
Lemma 3.1: Let $b_{n, k}$ be the $k^{\text {th }}$ term of the $n^{\text {th }}$ row of the array of Theorem 2.1. The term directly below $b_{n, k}$ in the $(n+p)^{t h}$ row is $b_{n, k}=b_{n+p, 2^{p}(k-1)+1}$. In particular

$$
\begin{gather*}
b_{1,1}=b_{1+(n-1), 2^{n-1}(1-1)+1}=b_{n, 1}=1, n \geq 1 \\
b_{1,2}=b_{1+(n-1), 2^{n-1}(2-1)+1}=b_{n, 2^{n-1}+1}=1, n \geq 1 \\
b_{p, 2}=b_{p+(n-p), 2^{n-p}(2-1)+1}=b_{n, 2^{n-p}+1}=p, n \geq p, p=1,2, \ldots \tag{3.3}
\end{gather*}
$$

Proof: Lemma 3.1 restates Theorem 1 from [9]: If $N$ has rank $R_{n}$ in the $n^{\text {th }}$ line, it appears directly below in the $(n+k)^{t h}$ line with rank $R_{n+k}=2^{k}\left(R_{n}-1\right)+1$.

Define a zigzag path through the array of Theorem 2.1 as movement down and right alternating with movement down and left. Define $Z R(y)$ as a movement down $y$ rows and right 1 term; $Z L(x)$, down $x$ rows and left 1 term. From Lemma 3.1,

$$
\begin{align*}
& Z R(y): b_{w, t} \rightarrow b_{w+y,\left[2^{y}(t-1)+1\right]+1}=b_{w+y, 2^{y}(t-1)+2} \\
& Z L(x): b_{w, t} \rightarrow b_{w+x,\left[2^{x}(t-1)+1\right]-1}=b_{w+x, 2^{x}(t-1)} \tag{3.4}
\end{align*}
$$

Lemma 3.2: Let $R(\tilde{N})=b_{n, k}=r_{1}+r_{2}, r_{1} / r_{2}=\left[q_{1} ; q_{2}, q_{3}, \ldots, q_{m-2}, r_{m-1}\right], r_{m-1} \geq 2$, where $r_{1}$ is a dyad, $r_{1}>r_{2}$. If $b_{n, k}$ is on the left side of the table, the zigzag path from $b_{1,1}$ to $b_{n, k}$ is $Z R\left(r_{m-1}-1\right) Z L\left(q_{m-2}\right) \ldots Z R\left(q_{2}\right) Z L\left(q_{1}\right) Z R(1)$, where $r_{1}$ is on the left of $r_{2}, m$ is odd, and $k$ is given by (3.2a); or, $Z R\left(r_{m-1}-1\right) Z L\left(q_{m-2}\right) \ldots Z R\left(q_{1}\right) Z L(1)$, where $r_{1}$ is on the right of $r_{2}, m$ is even, and $k$ is given by (3.2b).

If $b_{n, k}$ is on the right side of the table, the zigzag path from $b_{1,2}$ to $b_{n, k}$ is $Z L\left(r_{m-1}-1\right) Z R\left(q_{m-2}\right) \ldots Z L\left(q_{2}\right) Z R\left(q_{1}\right) Z L(1)$, for $r_{1}$ on the right of $r_{2}, m$ odd; or, $Z L\left(r_{m-1}-1\right) Z R\left(q_{m-2}\right) \ldots Z L\left(q_{1}\right) Z R(1), r_{1}$ on the left of $r_{2}, m$ even. $R(\tilde{N})=b_{n, 2^{n-1}+2-k}$ for $k$ as in (3.2a) or (3.2b) as $m$ is odd or even.

Proof: On the left side of the table, the path from $b_{1,1}$ begins $Z R\left(r_{m-1}-1\right)$ to $b_{r_{m-1}, 2}$ followed by $Z L\left(q_{m-2}\right)$. If $r_{1}$ is on the left of $r_{2}$, the path from $b_{1,1}$ will end with a move $Z R(1)$ to $R(\tilde{N})$, preceded by $Z L\left(a_{1}\right)$ to $r_{1} ; m$ is odd. If $r_{1}$ is on the right of $r_{2}$, the path from $b_{1,1}$ to $b_{n, k}$ ends $\ldots Z R\left(q_{1}\right) Z L(1)$, so that $m$ is even. Suppose $r_{1} / r_{2}=\left[a_{1} ; a_{2}, a_{3}, r_{m-1}\right]$. The zigzag path from $b_{1,1}$ to $b_{n, k}$ is $Z R\left(r_{m-1}-1\right) Z L\left(a_{3}\right) Z R\left(a_{2}\right) Z L\left(a_{1}\right) Z R(1)$ :

$$
\begin{aligned}
b_{1,1} \rightarrow b_{r_{m-1}, 2} & \rightarrow b_{a_{3}+r_{m-1}, 2^{a_{3}}(2-1)+0} \rightarrow b_{a_{2}+a_{3}+r_{m-1}, 2^{a_{2}}\left(2^{a_{3}}-1\right)+2} \\
& \rightarrow b_{a_{1}+a_{2}+a_{3}+r_{m-1}, 2^{a_{1}}\left(2^{\left.a_{2}+a_{3}-2^{a_{2}}+2-1\right)+0}\right.} \\
& \rightarrow b_{a_{1}+a_{2}+a_{3}+r_{m-1}+1,2\left(2^{a_{1}+a_{2}+a_{3}}-2^{a_{1}+a_{2}}+2^{a_{1}}-1\right)+2} \\
& =b_{n, 2^{a_{1}+a_{2}+a_{3}+1}-2^{a_{1}+a_{2}+1}+2^{a_{1}+1}}
\end{aligned}
$$

$k$ is given by (3.2a), $a_{1}=q_{i}, i=1,2,3$. This pattern continues for $m$ odd. Suppose $r_{1} / r_{2}=$ $\left[a_{1} ; a_{2}, r_{m-1}\right]$. The zigzag path from $b_{1,1}$ to $b_{n, k}$ is $Z R\left(r_{m-1}-1\right) Z L\left(a_{2}\right) Z R\left(a_{1}\right) Z L(1)$ :

$$
\begin{aligned}
b_{1,1} \rightarrow b_{r_{m-1}, 2} & \rightarrow b_{a_{2}+r_{m-1}, 2^{a_{2}}(2-1)+0} \rightarrow b_{a_{1}+a_{2}+r_{m-1}, 2^{a_{1}}\left(2^{a_{2}-1}\right)+2} \\
& \rightarrow b_{a_{1}+a_{2}+r_{m-1}+1,2\left(2^{a_{1}+a_{2}}-2^{a_{1}}+2-1\right)+0}=b_{n, 2^{a_{1}+a_{2}+1}-2^{a_{1}+1}+2}
\end{aligned}
$$

$k$ is given by (3.2b), $a_{i}=q_{i}, i=1,2$. The pattern continues for $m$ even.
The situation on the right side of the table is similar. The path from $b_{1,2}$ to $b_{n, k}$ on the right side is the mirror image of the path from $b_{1,1}$ to $b_{n, k}$ on the left.

Lemma 3.3: Let $R(\tilde{N})=b_{n, k}=r_{1}+r_{2}, r_{1} / r_{2}=\left[q_{1} ; q_{2}, q_{3}, \ldots, q_{m-2}, r_{m-1}\right], r_{m-1} \geq 2$. If $r_{1}$ is the left neighbor or $r_{2}$ in the $(n-1)^{s t}$ row, and $m$ is odd (even), the ordered sequence, $r_{1}, R(\tilde{N}), r_{2}$, appears in the $n^{t h}$ row on the left (right) side of the table.

Theorem 3.2 generalizes the zigzag paths of Lemma 3.2 to $\mathfrak{R L R} \mathfrak{L} .$. patterns, where $\mathfrak{R}(q)$ means to write the next $(q)$ even-subscripted Fibonacci numbers; $\mathfrak{L}(q)$, omit the next $(q)$ even-subscripts. Note that $r_{1}$ and $r_{2}$ are not ordered.
Theorem 3.2: Let the dyad $R(\tilde{N})=r_{1}+r_{2}, r_{1}$ and $r_{2}$ relatively prime, appear in the $n^{t h}$ row as in Theorem 3.1. If $R(\tilde{N})$ is between $r_{1}$ and $r_{2}$ on the left side of the table, Zeck $\tilde{N}$ is given from $r_{1} / r_{2}=\left[q_{1} ; q_{2}, q_{3}, \ldots, q_{m-2}, r_{m-1}\right], r_{m-1} \geq 2$, by the $\mathfrak{R L R L} \ldots$ pattern

$$
\begin{equation*}
\mathfrak{R}(1) \mathfrak{L}\left(r_{m-1}-1\right) \mathfrak{R}\left(q_{m-2}\right) \mathfrak{L}\left(q_{m-3}\right) \ldots \mathfrak{R}\left(q_{1}\right) \mathfrak{L}(1), m \text { odd } \tag{3.5}
\end{equation*}
$$

$\ldots \mathfrak{R}\left(q_{2}\right) \mathfrak{L}\left(q_{1}\right) \mathfrak{L}(1), m$ even. The first Fibonacci number written is $F_{2 n}$.

Proof: Let $2^{q}$ correspond to $F_{2(p+1)}$ as in Lemma 2.1; $R(\tilde{N})=b_{n, k}$ is the term appearing $(k-2)$ entries to the right of $b_{n, 2}=R\left(F_{2 n}\right)$ where $F_{2 n}=\tilde{N}\left(2^{n-1}\right)$. From (3.1) with (3.2a) or (3.2b), the highest power of 2 in $k$ has exponent $\left(q_{1}+q_{2}+\cdots+q_{m-2}\right)=\left(n-r_{m-1}-1\right)$. From (3.2a),

$$
\begin{aligned}
k-2= & \left(2^{q_{1}+q_{2}+\cdots+q_{m-2}+1}-2^{q_{1}+q_{2}+\cdots+q_{m-3}+1}+\cdots+\left(2^{q_{1}+q_{2}+q_{3}+1}-2^{q_{1}+q_{2}+1}\right)+\left(2^{q_{1}+1}-2\right)\right. \\
= & 2^{q_{1}+q_{2}+\cdots+q_{m-3}+1}\left(2^{q_{m-2}}-1\right)+\cdots+2^{q_{1}+q_{2}+1}\left(2^{q_{3}}-1\right)+2\left(2^{q_{1}}-1\right) \\
= & 2^{q_{1}+q_{2}+\cdots+q_{m-3}+1}\left(2^{q_{m-2}-1}+\cdots+2+1\right)+\cdots+2^{q_{1}+q_{2}+1}\left(2^{q_{3}-1}+\cdots+2+1\right) \\
& \quad+2\left(2^{q_{1}-1}+\cdots+1\right)
\end{aligned}
$$

which contains $q_{m-2}$ consecutive powers of 2 beginning with $2^{q_{1}+q_{2}+\cdots+q_{m-2}}$, followed by $q_{m-3}$ consecutive missing powers of 2 , followed by $q_{m-4}$ consecutive powers of $2, \ldots$, ending with $q_{1}$ consecutive powers of 2 , with the one final term $2^{0}$ missing. (Recall that $k$ is even, since $R(\tilde{N})$ is a dyad.) In the sum ( $2^{n-1}+(k-2)$ ), the leading exponent in each block of consecutive powers of 2 results from successively subtracting $r_{m-1}, q_{m-2}, q_{m-3}$, from $(n-1)$. If $m$ is even, $(k-2)$ as calculated from (3.2b) ends with $\cdots+\left(2^{q_{1}}+2\right)-2$, or $\left(q_{1}+1\right)$ missing powers of 2 ; note that $2^{0}$ is always missing. The pattern of (3.5) follows from Theorems 2.2 and 3.1, and Lemma 3.2.
Corollary 3.2.1: The zigzag path in which all quotients are 1 leads to $b_{n, k}=R(\tilde{N})=F_{n+1}$, for $\tilde{N}=F_{2 n}+F_{2 n-4}+F_{2 n-8}+\ldots$, with smallest term $F_{6}$ or $F_{4}$, as $n$ is odd or even.

Proof: Rewrite $[1 ; 1,1, \ldots, 1,1,1]$ as $[1 ; 1,1, \ldots, 1,2]$ and use Theorem 3.2. On the right side, $\tilde{N}=F_{2 n}+F_{2 n-2}+F_{2 n-6}+\ldots$, which results from (3.5) if $r_{m-1}=1$.
Corollary 3.2.2: If $R(\underset{\sim}{N})$ from Theorem 3.2 is between $r_{1}$ and $r_{2}$ on the right side of the table, Zeck $\tilde{N}$ is written from the $\mathfrak{R} \mathfrak{A} \mathfrak{R L} .$. 1) $\mathfrak{L}\left(q_{m-2}\right) \mathfrak{R}\left(q_{m-3}\right) \ldots \mathfrak{R}\left(q_{1}\right) \mathfrak{L}(1), m$ even; or, ending $\ldots \mathfrak{L}\left(q_{1}\right) \mathfrak{L}(1), m$ odd; $r_{m-1} \geq 2$ 。

Proof: The zigzag path from $b_{1,2}$ to $b_{n, k}$ on the right side is the mirror image of that from $b_{1,1}$ to $b_{n, k}$ on the left side. Recall that $b_{n, k}=b_{n, 2^{n-1}+2-k}$ by symmetry.

To illustrate, compute $\tilde{N}$ from $R(\tilde{N})=27=19+8 . \quad 19 / 8=[2 ; 2,1,2] ; n=(2+2+$ $1+2)+1=8, m=3$. We are on the left side, and Zeck $\tilde{N}$ begins $F_{16} ; b_{8,2}=8$. Interpret the pattern $\mathfrak{R}(1) \mathfrak{L}(2-1) \mathfrak{R}(1) \mathfrak{L}(2) \mathfrak{R}(2) \mathfrak{L}(1)$ as use 16 ; omit 14 ; use 12 ; omit 10 and 8 ; use 6 and 4; omit 2. Thus, Zeck $\tilde{N}=F_{16}+F_{12}+F_{6}+F_{4}=1142 ; R(1142)=27$. The sequence $19,27,8$, occurs with $27=b_{8,40}=R\left(\tilde{N}\left(j_{n, k}\right)\right)$ for $j_{n, k}=2^{8-1}+\left(2^{5}+2^{2}+2^{1}\right)-2$, verifying
$\tilde{N}=F_{2(7+1)}+F_{2(5+1)}+F_{2(2+1)}+F_{2(1+1)}$. On the right side, Corollary 3.2 .2 gives the associated solution $\tilde{N}^{\prime}$ from $\mathfrak{A}(1) \mathfrak{R}(2-1) \mathfrak{L}(1) \mathfrak{R}(2) \mathfrak{L}(2) \mathfrak{L}(1)$ as $\tilde{N}^{\prime}=F_{16}+F_{14}+F_{10}+F_{8}=1440$, where $R(1440)=27=b_{8,90}, 8=b_{8,89}$ and $19=b_{8,91}$.

The symmetries of the array for $R(\tilde{N})$ let us find other even-Zeck integers $\tilde{M}$ such that $R(\tilde{M})=R(\tilde{N})$, with $R(\tilde{M})$ and $R(\tilde{N})$ both appearing in the $n^{\text {th }}$ row. Theorem 3.3 gives a special solution for $\tilde{M}$.

Theorem 3.3: Let $R(\tilde{N})=r_{1}+r_{2}, r_{1} / r_{2}=\left[q_{1} ; q_{2}, q_{3}, \ldots, q_{m-2}, r_{m-1}\right]$, as in Theorem 3.2; $q_{1} \geq 1, r_{m-1} \geq 2$. Let Zeck $\tilde{M}$ be written from the $\mathfrak{R L R L} \ldots$ pattern of (3.5), adjusted by taking the quotients of $r_{1} / r_{2}$ in ascending order: $\mathfrak{R}(1) \mathfrak{L}\left(q_{1}\right) \mathfrak{R}\left(q_{2}\right) \mathfrak{L}\left(q_{3}\right) \ldots \mathfrak{R}\left(r_{m-1}-1\right) \mathfrak{L}(1)$, $m$ odd; $\ldots \mathfrak{R}\left(q_{m-2}\right) \mathfrak{L}\left(r_{m-1}-1\right) \mathfrak{L}(1), m$ even. Then $R(\tilde{M})=R(\tilde{N})$, both appearing in row $n$.

Proof: A reversal identity for continued fractions appears as Theorem 1 in [1]: if $\left[a_{0}, a_{1}, \ldots, a_{n-1}, a_{n}\right]=p_{n} / q_{n}$, then $\left[a_{n}, a_{n-1}, \ldots, a_{1}, a_{0}\right\}=p_{n} / p_{n-1}$. Here, $R(M)=p_{n}=$ $R(\tilde{N})$.

Theorem 3.3 applied to the preceding example gives $\mathfrak{R}(1) \mathfrak{L}(2) \mathfrak{R}(2) \mathfrak{L}(1) \mathfrak{R}(2-1) \mathfrak{L}(1)$ or $\tilde{M}=F_{16}+F_{10}+F_{8}+F_{4}=1066 ; R(1066)=27$, but $\tilde{N}=1142$.

The Calkin and Wilf [6] sequence (1.4) is the same as our sequence (2.1); that is, $b(j)=$ $R(\tilde{N}(j))$, where $b(j) / b(j+1)$ is the $j^{\text {th }}$ rational number, $j=0,1,2, \ldots$ Thus, the results of the present paper allow us to write the $j^{\text {th }}$ rational number. Given $j$, by Lemma 2.1, we can write Zeck $\tilde{N}(j)$, the Zeckendorf representation of the $j^{\text {th }}$ even-Zeck integer; there are several ways [4] to compute $R(\tilde{N}(j))$ and $R(\tilde{N}(j+1))$. Given any rational number $a / b$, Theorem 3.2 can be adapted to find $\tilde{N}(j)$ such that $a / b=R(\tilde{N}(j)) / R(\tilde{N}(j+1))$. For example, to answer at which position the rational number $13 / 8$ appears, place 13 between 5 and 8 in the $n^{\text {th }}$ row, $5,13,8 ; r_{1} / r_{2}=5 / 8=[0 ; 1,1,1,2], n=6, m$ is even. Since $R(\tilde{N}(j))=13$ is on the right side of the table, Corollary 3.2 .2 gives $\tilde{N}(j)=F_{12}+F_{10}+F_{6}=207$, and $\tilde{N}(j+1)=$ 208 , where $R(207)=13, R(208)=8$. From Zeck $\tilde{N}(j), j=2^{12 / 2-1}+2^{10 / 2-1}+2^{6 / 2-1}=$ $2^{5}+2^{4}+2^{2}=52$; thus, $13 / 8$ is the $52^{\text {nd }}$ rational number. Another example: to find $5 / 12$, use the sequence $5,12,7 ; 5 / 7=[0 ; 1,2,2], n=6, m=3$. We are on the left side; Theorem 3.2 gives $\tilde{N}(j+1)=F_{12}+F_{8}+F_{6}=173 ; R(173)=12$. The preceding even-Zeck integer $\tilde{N}(j)=F_{12}+F_{8}+F_{4}+F_{2}=169, R(169)=5 ; j=2^{12 / 2-1}+2^{8 / 2-1}+2^{4 / 2-1}+2^{2 / 2-1}=43$. Thus, $5 / 12$ is the $43^{r d}$ rational number. We note that $R(\tilde{N}(j))$ is another function $f(j)$ such that $f(j) / f(j+1)$ takes every rational value exactly once, answering a question posed in [6].


Figure 1. The Calkin-Wilf version of the tree of fractions

Further, we can write the address of the rational number $r_{1} / r_{2}$ appearing in Calkin and Wilf's tree of fractions, which is a variant of the Stern-Brocot tree [7]. The tree of fractions (Figure 1) has $1 / 1$ at the top of the tree. Each vertex $r_{1} / r_{2}$ has two children; its left child is $r_{1} /\left(r_{1}+r_{2}\right)$, and its right child is $\left(r_{1}+r_{2}\right) / r_{2}$; each fraction is $R(\tilde{N}(j)) / R(\tilde{N}(j+1))$ for some $j$. In the $n^{\text {th }}$ row of the tree, the numerators are the first $2^{n-1}$ terms of our $n^{\text {th }}$ row. Let $r_{1} / r_{2}=\left[q_{1} ; q_{2}, q_{3}, \ldots, q_{m-2}, r_{m-1}\right], r_{m}=1, q_{1} \geq 0, r_{m-1} \geq 2$; if $m$ is odd (even), $r_{1} / r_{2}$ appears on the left (right) side of the tree, and $r_{1}$ is on the left (right) of $r_{2}$ in the table. Starting from $1 / 1$, if $m$ is odd, the vertex $r_{1} / r_{2}$ has the address $L^{r_{m-1}-1} R^{q_{m-2}} \ldots L^{q_{2}} R^{q_{1}}$; if $m$ is even, $R^{r_{m-1}-1} L^{q_{m-2}} \ldots R^{q_{2}} L^{q_{1}}$; where $L^{q}$ means to move $q$ vertices left; $R^{q}$, move $q$ vertices right; $L^{0}$ and $R^{0}$ are not written. If $r_{1}$ is the left neighbor of $r_{2}$ in the table and $R(\tilde{N}(j))=r_{1}+r_{2}$, then $R(\tilde{N}(j)) / R(\tilde{N}(j+1))$ is the right child of $r_{1} / r_{2}$; if instead $R(\tilde{N}(j+1))=r_{1}+r_{2}$, then $R(\tilde{N}(j)) / R(\tilde{N}(j+1))$ is the left child of $r_{1} / r_{2}$.

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AMS Classification Numbers: 11B39, 11B37, 11Y55
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Tribute


JoAnn Vine

JoAnn Vine, Fibonacci Quarterly typist for 25 years, is retiring. She never missed a deadline and hates to give it up, but it is time to retire.

JoAnn sang with the San Francisco Opera before she married Richard Vine (FQ Subscription Manager for 17 years). She started her statistical typing business in 1964, typing theses for students at Stanford and San Jose State.

Thank you, JoAnn, for your years of dedicated service to the Fibonacci Association!

# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by<br>Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, 'Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a selfaddressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2003. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

## BASIC FORMULAS

The Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ satisfy

$$
\begin{aligned}
& F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1 \\
& L_{n+2}=L_{n+1}+L_{n}, L_{0}=2, L_{1}=1
\end{aligned}
$$

Also, $\alpha=(1+\sqrt{5}) / 2, \beta=(1-\sqrt{5}) / 2, F_{n}=\left(\alpha^{n}-\beta^{n}\right) / \sqrt{5}$, and $\mathbb{\Psi}_{n}=\alpha^{n}+\beta^{n}$.

## PROBLEMS PROPOSED IN THIS ISSUE

B-956 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI
Prove that

$$
\frac{1+\sqrt{5}}{4} \leq \sum_{n=0}^{\infty} \frac{1}{L_{2 n}} \leq \frac{3}{2}
$$

## B-957 Proposed by Muneer Jebreel, Jerusalem, Israel

For $n \geq 1$, prove that
(a) $L_{2^{n}+3}^{2}+4=4 L_{2^{n+1}+3}+L_{2^{n}}^{2}$
and
(b) $L_{2^{n}+6}^{2}=4+4 L_{2^{n+1}+9}+L_{2^{n}+3}^{2}$.

B-958 Proposed by José Luis Diaz-Barrero \& Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain

Find the greatest common divisor of

$$
2+\sum_{k=1}^{n} L_{k}^{2} \text { and } 3+\sum_{k=1}^{n} L_{k}
$$

B-959 Proposed by John H. Jaroma, Austin College, Sherman, TX
Find the sum of the infinite series
$1+\frac{1}{2}+\frac{1}{4}+\frac{3}{8}+\frac{3}{16}+\frac{8}{32}+\frac{7}{64}+\frac{21}{128}+\frac{15}{256}+\frac{55}{512}+\frac{31}{1024}+\frac{144}{2048}+\frac{63}{4096}+\frac{377}{8192}+\frac{127}{16384}+\ldots$

B-960 Proposed by Bob Johnson, Durham University, Durham, England
If $a+b=c+d$, prove that

$$
F_{a} F_{b}-F_{c} F_{d}=(-1)^{r}\left[F_{a-r} F_{b-r}-F_{c-r} F_{d-r}\right]
$$

for all integers $a, b, c, d$ and $r$.

## SOLUTIONS

## Circle the Squares

## B-940 Proposed by Gabriela Stănică \& Pantelimon Stănică,

 Auborn Univ. Montgomery, Montgomery AL.(Vol. 40, no. 4, August 2002)
How many perfect squares are in the sequence

$$
x_{n}=1+\sum_{k=0}^{n} F_{k}!\quad \text { for } n \geq 0 ?
$$

## Solution by Martin Reiner, New York, NY.

We claim that the only square in this sequence is $x_{2}=4$.
Note that $x_{0}, \ldots, x_{4}=2,3,4,6,12$. For $k \geq 5$ we have $F_{k} \geq 5$, and so $F_{k}!\equiv 0(\bmod 5)$. Thus $x_{k} \equiv x_{k-1} \equiv \cdots \equiv x_{4} \equiv 2(\bmod 5)$. But modulo 5 any square is either congruent to 0 , 1 , or 4 .

## Also solved by Scott Brown, Paul Bruckman, Ovidiu Furdiu, Walther Janous, Jaroslaze Seibert, H.-J. Seiffert, and the proposer.

All solutions received follow more or less the same method as the featured one.

## It is Always Negative

## B-941 Proposed by Walther Janous, Innsbruck, Austria

 (Vol. 40, no. 4, August 2002)Show that

$$
\frac{n F_{n+6}}{2^{n+1}}+\frac{F_{n+8}}{2^{n}}-F_{8}<0 \text { for } n \geq 1
$$

Solution by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.

First, we observe that the given statement is equivalent to

$$
\begin{equation*}
n F_{n+6}+2 F_{n+8}<2^{n+1} F_{8} . \tag{1}
\end{equation*}
$$

In order to prove the preceding inequality we will argue by induction. The case when $n=1$ trivially holds. Assume that (1) holds for $n=1,2, \ldots, n$ and we should prove that

$$
(n+1) F_{n+7}+2 F_{n+9}<2^{n+2} F_{8}
$$

also holds. In fact,

$$
\begin{gathered}
(n+1) F_{n+7}+2 F_{n+9}=(n+1)\left(F_{n+6}+F_{n+5}\right)+2\left(F_{n+7}+F_{n+8}\right) \\
=\left(n F_{n+6}+2 F_{n+8}\right)+\left[(n-1) F_{n+5}+2 F_{n+7}\right]+2 F_{n+5}+F_{n+6} \\
<2^{n+1} F_{8}+2^{n} F_{8}+(n-1) F_{n+5}+2\left(F_{n+5}+F_{n+6}\right)<2^{n+2} F_{8}
\end{gathered}
$$

and the result stated in (1) follows by strong mathematical induction.
H.-J. Seiffert showed that $\frac{n F_{n+6}}{2^{n+1}}+\frac{F_{n+8}}{2^{n}}-F_{8} \leq-\frac{3}{4}$ for $n \geq 1$ and L.A.G. Dresel generalized the inequality to $\frac{n F_{n+t}}{2^{n+1}}+\frac{2 F_{n+t+2}}{2^{n+1}}-F_{n+t+2}<0$ for $n \geq 1$ and $t \geq 3$.

The condition $n \geq 1$ was inadvertantly left out by the editor.
Also solved by Charles Ashbacker, Gurdial Arora and Donna Stutson, Paul Bruckman, Mario Catalani, Kenneth Davenport, L.A.G. Dresel, Ovidiu Furdui, John Jaroma, Gerald A. Heuer, Jaroslav Seibert, H.J. Seiffert, Adam Stinchcomb, and the proposer.

## As Close As It Gets

B-942 Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA
(Vol. 40, no. 4, August 2002)
(a) For $n>3$, find the Fibonacci number closest to $L_{n}$.
(b) For $n>3$, find the Fibonacci number closest to $L_{n}^{2}$.

Solution by L.A.G. Dresel, Reading England.
(a) The identity $L_{n}=F_{n+1}+F_{n-1}$ gives $L_{n}=\left(F_{n+1}+F_{n}\right)+\left(F_{n-1}-F_{n}\right)=F_{n+2}-F_{n-2}$. Therefore $F_{n+1}<L_{n}<F_{n+2}$, and as we have $F_{n-2}<F_{n-1}$ for $n \geq 4$, it follows that $F_{n+2}$ is the Fibonacci number closest to $L_{n}$.
(b) Since $L_{n}=\alpha^{n}+\beta^{n}$ we have $\left(L_{n}\right)^{2}=L_{2 n}+2(-1)^{n}$. As before, we have

$$
\begin{gathered}
L_{2 n}=F_{2 n+1}+F_{2 n-1}=F_{2 n+2}-F_{2 n-2}, \text { so that } \\
\left(L_{n}\right)^{2}=F_{2 n+1}+F_{2 n-1}+2(-1)^{n}=F_{2 n+2}-\left\{F_{2 n-2}-2(-1)^{n}\right\} .
\end{gathered}
$$

It follows that $\left(L_{n}\right)^{2}$ lies between $F_{2 n+1}$ and $F_{2 n+2}$, and is closest to $F_{2 n+2}$ provided that $F_{2 n-1}+2(-1)^{n}>F_{2 n-2}-2(-1)^{n}$, giving $F_{2 n-3}+4(-1)^{n}>0$. This condition is satisfied for $n \geq 4$. Therefore $F_{2 n+2}$ is closest.

Also solved by Scott Brown (part (a)), Paul Bruckman, Mario Catalani, Charles Cook, Ovidiu Furdui, Walther Janous, John Jaroma, Harris Kwong, Reiner Martin, Jaroslav Seibert, H.-J. Seiffert and the proposer.

## Inequality, Equality Matters

B-943 Proposed by José Luis Diaz \& Juan J. Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain
(Vol. 40, no. 4, August 2002)
Let $n$ be a positive integer. Prove that

$$
\sum_{k=1}^{n} \frac{L_{k}^{2}}{F_{k}} \geq \frac{\left.\left(L_{n+2}-3\right)^{2}\right)}{F_{n+2}-1}
$$

When does equality occur?

## Solution by Graham Lord, Princeton, NJ

We shall use the known identities that the sum of the first $n$ Fibonacci numbers is $F_{n+2}-1$, and the sum of the first $n$ Lucas numbers is $L_{n+2}-3$. (See, for example, pages 52 and 54 of Fibonacci and Lucas Numbers by V.E. Hoggatt, 1969.)

Then, by appealing to the Cauchy-Schwarz inequality $(\Sigma a b)^{2} \leq\left(\Sigma a^{2}\right) \cdot\left(\Sigma b^{2}\right)$, we have: (all sums are over $k$ from $i$ to $n$ )

$$
\begin{aligned}
& \left(L_{n+2}-3\right)^{2}=\left(\sum L_{k}\right)^{2}=\left(\sum \frac{L_{k}}{\sqrt{F_{k}}} \cdot \sqrt{F_{k j}}\right)^{2} \leq \\
& \left(\sum \frac{L_{k}^{2}}{F_{j}}\right)\left(\sum F_{k}\right)=\left(\sum \frac{L_{k}^{2}}{F_{k}}\right)\left(F_{n+2}-1\right)
\end{aligned}
$$

We will have equality iff the two sets of numbers $L_{k}^{2} / F_{k}$ and $F_{k}$ are proportional, that is, iff $L_{1} / F_{1}=L_{2} / F_{2}=\cdots=L_{n} / F_{n}$. The latter condition is only true iff $n=1$.

Walther Janous proved a more general inequality that may appear as a separate proposal.

## A Prime Congruence

## B-944 Proposed by Paul S. Bruckman, Berkeley, CA

(Vol. 40, no. 4, August 2002)
For all odd primes $p$, prove that

$$
L_{p} \equiv 1-\frac{p}{2} \sum_{k=1}^{p-1} \frac{L_{k}}{k}\left(\bmod p^{2}\right)
$$

where $\frac{1}{k}$ represents the residue $k^{-1}(\bmod p)$.

## Solution by H.-J. Seiffert, Berlin, Germany

Let $p$ be an odd prime. From

$$
k(p-1)(p-2) \ldots(p-k+1) \equiv(-1)^{k-1} k!(\bmod p)
$$

we obtain the well-known congruence

$$
(-1)^{k}\binom{p}{k} \equiv-\frac{p}{k}\left(\bmod p^{2}\right), k=1,2, \ldots, p-1 .
$$

Since (see, for example, P. Haukkanen. "On a Binomial Sum for the Fibonacci and Related Numbers." The Fibonacci Quarterly 34.4 (1996): 326-31, Corollary 2)

$$
L_{p}=\sum_{k=0}^{p}(-1)^{k}\binom{p}{k} L_{k},
$$

modulo $p^{2}$, we then have

$$
L_{p} \equiv L_{0}+(-1)^{p} L_{p}=p \sum_{k=1}^{p-1} \frac{L_{k}}{k}\left(\bmod p^{2}\right) .
$$

The desired congruence now easily follows by observing that $L_{0}=2$ and that $p$ is odd.
Also solved by L.A.G. Dresel and the proposer.

## A Simpler Expression

## B-945 Proposed by N. Gauthier, Royal Military College of Canada

(Vol. 40, no. 4, August 2002)
For $n \geq 0, q>0, s$ integers, show that

$$
\sum_{l=0}^{n}\binom{n}{l} F_{q-1}^{l} F_{(q+1)(n-l)+s}=F_{q+1}^{n} F_{2 n+s}
$$

## Solution I by Paul S. Bruckman, Berkeley, CA

Denote the given sum as $S(n ; q, s)$.
Then $S(n ; q, s)=5^{-1 / 2} \sum_{k=0}^{n}+n C_{k}\left(F_{q-1}\right)^{k}\left\{\alpha^{s+(q+1)(n-k)}-\beta^{s+(q+1)(n-k)}\right\}=5^{-1 / 2}\left\{\alpha^{s}\left(F_{q-}\right.\right.$ $\left.\left.\alpha^{q+1}\right)^{n}-\beta^{s}\left(F_{q-1}+\beta^{q+1}\right)^{n}\right\}$. Now $F_{q-1}+\alpha^{q+1}=F_{q-1}+\alpha F_{q+1}+F_{q}=F_{q+1}(1+\alpha)=\alpha^{2} F_{q+1} ;$ likewise, $F_{q-1}+\beta^{q+1}=\beta^{2} F_{q+1}$. Therefore, $S(n ; q, s)=5^{-1 / 2}\left(F_{q+1}\right)^{n}\left\{\alpha^{s+2 n}-\beta^{s+2 n}\right\}=$ $\left(F_{q+1}\right)^{n} F_{2 n+s}$.

## Solution II by Pentti Haukkanen, University of Tampere, Tampere, Finland

Problem B-945 is a special case of Problem H-121. In fact, according to Problem H-121

$$
\sum_{l=0}^{n}\binom{n}{l}\left(\frac{F_{2 k}}{F_{m-2 k}}\right)^{l} F_{m l+s}=\left(\frac{F_{m}}{F_{m-2 k}}\right)^{n} F_{2 n k+s}
$$

Replacing $l$ with $n-1$ we obtain

$$
\sum_{l=0}^{n}\binom{n}{l}\left(\frac{F_{2 k}}{F_{m-2 k}}\right)^{n-l} F_{m(n-l)+s}=\left(\frac{F_{m}}{F_{m-2 k}}\right)^{n} F_{2 n k+s} .
$$

Writing $m=q+1, k=1$ and multiplying with $F_{q-1}^{n}$ we arrive at the proposed identity

$$
\sum_{l=0}^{n}\binom{n}{l} F_{q-1}^{l} F_{(q+1)(n-l)+s}=F_{q+1}^{n} F_{2 n+s}
$$

Also solved by Mario Catalani, Kenneth B. Davenport, Ovidiu Furdui, and the proposer.

## NOTES

1. We would like to belatedly acknowledge the receipt of a solution to problem B-938 by Jereme Jarome. Also, Kenneth Davenport submitted a late solution to the same problem.
2. Solution I by Pantelimon Stanica to problem B-933 contains a fatal error. In fact, the first inequality in the proof, $F_{n+1} F_{n+1}>F_{n+1} F_{n}$ should be reversed. We would like to apologize for the oversight.

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by<br>Florian Luca

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI),CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

## PROBLEMS PROPOSED IN THIS ISSUE

## H-596 Proposed by the Editor

A beautiful result of McDaniel (The Fibonacci Quarterly 40.1, 2002) says that $F_{n}$ has a prime divisor $p \equiv 1(\bmod 4)$ for all but finitely many positive integers $n$. Show that the asymptotic density of the set of positive integers $n$ for which $F_{n}$ has a prime divisor $p \equiv 3(\bmod 4)$ is $1 / 2$. Recall that a subset $\mathcal{N}$ of all the positive integers is said to have an asymptotic density $\lambda$ if the limit

$$
\lim _{x \rightarrow \infty} \frac{\#\{1 \leq n<x \mid n \in \mathcal{N}\}}{x}
$$

exists and equals $\lambda$.

## H-597 Proposed by Mario Catalani, University of Torino, Torino, Italy

Let $\alpha, \beta, \gamma$ be the roots of the trinomial $x^{3}-x^{2}-x-1=0$. Express

$$
U_{n}=\sum_{i=1}^{n} \sum_{j=0}^{n-i} \alpha^{i} \beta^{j} \gamma^{n-i-j}
$$

interms of the Tribonacci sequence $\left\{T_{n}\right\}$ given by $T_{0}=0, T_{1}=1, T_{2}=1$ and $T_{n}=T_{n-1}+$ $T_{n-2}+T_{n-3}$ for $n \geq 3$.

H-598 Proposed by José Díaz-Barrero \& Juan Egozcue, Barcelona, Spain Show that all the roots of the equation

$$
\left|\begin{array}{ccccc}
F_{1} F_{n} & \cdots & F_{1} F_{3} & F_{1} F_{2} & F_{1}^{2}-x \\
F_{2} F_{n} & \cdots & F_{2} F_{3} & F_{2}^{2}-x & F_{2} F_{1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
F_{n}^{2}-x & \cdots & F_{n} F_{3} & F_{n} F_{2} & F_{n} F_{1}
\end{array}\right|=0
$$

are integers.

## SOLUTIONS

## Representing reals in Fibonacci series

## H-582 Proposed by Ernst Herrmann, Siegburg, Germany

a) Let $A$ denote the set $\left\{2,3,5,8, \ldots, F_{m+2}\right\}$ of $m$ succesive Fibonacci numbers, where $m \geq 4$. Prove that each real number $x$ of the interval $I=\left[\left(F_{m+2}-1\right)^{-1}, 1\right]$ has a series representation of the form

$$
\begin{equation*}
x=\sum_{i=1}^{\infty} \frac{1}{F_{k_{1}} F_{k_{2}} \ldots F_{k_{i}}}, \tag{1}
\end{equation*}
$$

where $F_{k_{i}} \in A$ for all $i \in N$.
b) It is impossible to change the assumption $m \geq 4$ into $m \geq 3$, that is, if $A=\{2,3,5\}$ and $I=[1 / 4,1]$, then there are real numbers with no representation of the form (1), where $F_{k_{i}} \in A$. Find such a number.

## Solution by Paul Bruckman, Sacramento, CA

Given an infinite sequence $\left\{c_{n}\right\}$ of real numbers with $c_{n} \geq 2$ write $S\left(c_{1}, c_{2}, \ldots\right)$ for the value of the series $1 / c_{1}+1 /\left(c_{1} c_{2}\right)+\ldots$. Note that $S\left(c_{1}, c_{2}, \ldots\right)$ is well defined and is in the interval $(0,1]$.

Now consider the series $S\left(F_{k_{1}}, F_{k_{2}}, \ldots\right)$, where $k_{i} \geq 3$ for $i \in N$. For notational convenience write $U\left(k_{1}, k_{2}, \ldots\right)=S\left(F_{k_{1}}, F_{k_{2}}, \ldots\right)$. We first show that for all $x$ with $0<x \leq 1, x$ has an $U$-series with no restriction of the subscripts $k_{i}$ other than $k_{i} \geq 3$ for $i \in N$. To see why this is so, observe that for all real numbers $A \geq 1$, there is always a Fibonacci number $F_{j} \geq 2$, such that $A<F_{j} \leq 2 A$. In particular, given $x_{1}$ with $0<x_{1} \leq 1$, we may choose $k_{1} \geq 3$ such that $1 / x_{1}<F_{k_{1}} \leq 2 / x_{1}$. Let $x_{2}=x_{1} F_{k_{1}}-1$. Clearly, $0<x_{2} \leq 1$, and therefore we may repeat the above algorithm. In other words, there exists $k_{2} \geq 3$ such that if we write $x_{3}=x_{2} F_{k_{2}}-1$, then $0<x_{3} \leq 1$. Continuing in this fashion, we generate an infinite sequence $\left\{k_{1}, k_{2}, \ldots\right\}$ such that $x_{1}=U\left(k_{1}, k_{2}, \ldots\right)$. Note that, in general, such a sequence is not uniquely determined.

We now prove a). Given $m \geq 4$, let $I_{m}=\left[\left(F_{m+2}-1\right)^{-1}, 1\right]$, write $A_{m}=$ $\left\{F_{3}, F_{4}, \ldots, F_{m+2}\right\}$, and consider a given $x_{1}$ in $I_{m}$. As we showed above, there exists a sequence $\left\{k_{i}\right\}$ with $k_{i} \geq 3$ for $i \in N$ such that $x_{1}=U\left(k_{1}, k_{2}, \ldots\right)$. We prove that among all such sequences there exists one which satisfies the additional constraint that $k_{i} \in A_{m}$ for all $i \in N$. To achieve this, we partition $I_{m}$ into $m$ disjoint intervals as follows:
a) suppose first that $\left(F_{m+2}-1\right)^{-1} \leq x_{1} \leq 2 / F_{m+2}$. Then $x_{1} \leq x_{1} F_{m+2}-1 \leq 1$. Thus, we may choose $k_{1}=m+2$, put $x_{2}=x_{1} F_{k_{1}}-1$, and then $x_{2} \in I_{m}$ and we may continue the process. Note that there might be other values of $k_{1}$ for which $x_{2}$ is in $I_{m}$.
b) suppose now that $2 / F_{k+1}<x_{1} \leq 2 / F_{k}$ for some $k=3,4, \ldots, m+1$. Then, $F_{k} \leq 2 / x_{1}$ and $F_{k+1}>2 / x_{1}$. Since $F_{k+1}<2 F_{k}$, it follows that $1 / x_{1}<F_{k} \leq 2 / x_{1}$. Note that there might be other values of $k$ for which this last inequality is satisfied. Choose $k_{1}=k$ and write $x_{2}=x_{1} F_{k_{1}}-1$. Then $x_{2} \leq 1$, and

$$
x_{2}>\frac{2 F_{k}}{F_{k+1}}-1=\frac{F_{k-2}}{F_{k+1}}
$$

## ADVANCED PROBLEMS AND SOLUTIONS

Now note that the right hand side of the above inequality is larger than or equal to $\left(F_{m+2}-\right.$ $1)^{-1}$ when $m \geq 4$. Indeed, the inequality

$$
\frac{F_{k-2}}{F_{k+1}} \geq \frac{1}{F_{m+2}-1}
$$

is equivalent to

$$
F_{k-2} F_{m+2} \geq F_{k+1}+F_{k-2}
$$

The above inequality holds for all $m \geq 4$ and $k \in\{3,4, \ldots, m+1\}$, but fails at $m=3$ and $k=m+1$. Thus, when $m \geq 4$, the number $x_{2} \in I_{m}$ and we may continue the process. This proves part a).

For part b), consider the interval $J_{3}=(2 / 5,5 / 12) \subset I_{3}$. If we take $x_{1} \in J_{3}$, we see that $k_{1}=4$ is the only possibility. We then obtain $x_{2}=x_{1} F_{k_{1}}-1=3 x_{1}-1$, hence $1 / 5<x_{2}<1 / 4$, and it is now clear that it is not possible that $k_{i} \in\{3,4,5\}$ for all $i \geq 2$. This argument shows that all values of $x_{1} \in J_{3}$ have the property that they do not have a representation of the form (1) with $k_{i} \in A_{3}$ for all $i \in N$, which, in particular, answers both questions from part $\mathbf{b}$ ).

Bruckman also attaches some examples of specific representations of the form (1) for some numbers stressing on the fact that such representations are, in general, not unique. A nice one is

$$
\begin{gathered}
0.41=U(4,5,6 ; \overline{6,3})=U(4,5,7,3,3,3 ; \overline{3,6})= \\
U(4,6,3,3 ; \overline{3,4})=U(4,6,3,3,4,8,3 ; \overline{4,7,9})=U(4,6,3,3 ; \overline{4,7,9})
\end{gathered}
$$

where the bar notation above has the same meaning as the one from the theory of periodic continued fractions. Note that $0.41 \in J_{3}$ so no such representation of it exists with $k_{i} \in A_{3}$ for all $i \in N$.
Also solved by the proposer.

## Identities with Fibonacci polynomials

## H-586 Proposed by H.-J. Seiffert, Berlin, Germany

Define the sequence of Fibonacci and Lucas polynomials by

$$
\begin{aligned}
& F_{0}(x)=0, F_{1}(x)=1, F_{n+1}(x)=x F_{n}(x)+F_{n-1}(x), n \in N \\
& L_{0}(x)=2, L_{1}(x)=x, L_{n+1}(x)=x L_{n}(x)+L_{n-1}(x), n \in N
\end{aligned}
$$

respectively. Show that, for all complex numbers $x$ and all positive integers $n$,

$$
\sum_{k=0}^{[n / 2]}\binom{n-k}{k} x^{k} F_{3 k}(x)=\frac{x F_{2 n+1}(x)-F_{2 n}(x)+(-x)^{n+2} F_{n}(x)+(-x)^{n+1} F_{n-1}(x)}{2 x^{2}-1}
$$

## ADVANCED PROBLEMS AND SOLUTIONS

and

$$
\sum_{k=0}^{[n / 2]}\binom{n-k}{k} x^{k} L_{3 k}(x)=\frac{x L_{2 n+1}(x)-L_{2 n}(x)+(-x)^{n+2} L_{n}(x)+(-x)^{n+1} L_{n-1}(x)}{2 x^{2}-1} .
$$

## Solution by the proposer

It is well known that

$$
\begin{equation*}
F_{n+1}(x)=\frac{\alpha(x)^{n+1}-\beta(x)^{n+1}}{\sqrt{x^{2}+4}} \tag{1}
\end{equation*}
$$

where $\alpha(x)=\left(x+\sqrt{x^{2}+4}\right) / 2$ and $\beta(x)=\left(x-\sqrt{x^{2}+4}\right) / 2$, and that

$$
\begin{equation*}
F_{n+1}(x)=\sum_{k=0}^{[n / 2]}\binom{n-k}{k} x^{n-2 k} . \tag{2}
\end{equation*}
$$

Each of the sides of the desired identities becomes a polynomial in $x$ when multiplied by $2 x^{2}-1$. Thus, it suffices to prove these identities for real numbers $x>1$. For such $x$, let $y=\sqrt{\alpha(x) / x}-\sqrt{-x \beta(x)}$. Since $\alpha(x) \beta(x)=-1$, we have

$$
\begin{gathered}
\sqrt{y^{2}+4}=\sqrt{\alpha(x) / x}+\sqrt{-x \beta(x)}=\frac{\alpha(x)+x}{\sqrt{x \alpha(x)}} \\
y+\sqrt{y^{2}+4}=2 \sqrt{\alpha(x) / x} \quad \text { and } \quad y-\sqrt{y^{2}+4}=-2 \sqrt{-x \beta(x)} .
\end{gathered}
$$

Noting that $(\alpha(x)+x)(\beta(x)+x)=2 x^{2}-1$, from (1), it now easily follows that

$$
x^{n / 2} \alpha(x)^{3 n / 2} F_{n+1}(y)=\frac{\beta(x)+x}{2 x^{2}-1} \cdot\left(\alpha(x)^{2 n+1}-(-x)^{n+1} \alpha(x)^{n}\right),
$$

or, since $\alpha(x) \beta(x)=-1$,

$$
\begin{equation*}
x^{n / 2} \alpha(x)^{3 n / 2} F_{n+1}(y)=\frac{x \alpha(x)^{2 n+1}-\alpha(x)^{2 n}+(-x)^{n+2} \alpha(x)^{n}+(-x)^{n+1} \alpha(x)^{n-1}}{2 x^{2}-1} . \tag{3}
\end{equation*}
$$

## ADVANCED PROBLEMS AND SOLUTIONS

From $\beta(x)^{3}=\left(x^{2}+1\right) \beta(x)+x=x^{2} \beta(x)-\alpha(x)+2 x$, we obtain $-\beta^{3}(x) / x=\alpha(x) / x-x \beta(x)-2$, giving $y=\sqrt{-\beta^{3}(x) / x}$. Since $\alpha(x) \beta(x)=-1$, from (2), it follows that

$$
\begin{equation*}
x^{n / 2} \alpha(x)^{3 n / 2} F_{n+1}(y)=\sum_{k=0}^{[n / 2]}\binom{n-k}{k} x^{k} \alpha(x)^{3 k} \tag{4}
\end{equation*}
$$

Combining (3) and (4) and using the known relations $2 \alpha(x)^{j}=L_{j}(x)+\sqrt{x^{2}+4} F_{j}(x)$, we obtain the desired identities.
Also solved by P. Bruckman, M. Catalani, K. Davenport, and V. Mathe.

## Matrices with Fibonacci Polynomials

## H-587 Proposed by N. Gauthier \& J.R. Gosselin, Royal Military College of Canada

Let $x$ and $y$ be indeterminates and let

$$
\alpha \equiv \alpha(x, y)=\frac{1}{2}\left(x+\sqrt{x^{2}+4 y}\right), \beta \equiv \beta(x, y)=\frac{1}{2}\left(x-\sqrt{x^{2}+4 y}\right)
$$

be the distinct roots of the characteristic equation for the generalized Fibonacci sequence $\left\{H_{n}(x, y)\right\}_{n=0}^{n=0}$, where

$$
H_{n+2}(x, y)=x H_{n+1}(x, y)+y H_{n}(x, y)
$$

If the initial conditions are taken as $H_{0}(x, y)=0, H_{1}(x, y)=1$, then the sequence gives the generalized Fibonacci polynomials $\left\{F_{n}(x, y)\right\}_{n=0}^{n=\infty}$. On the other hand, if $H_{0}(x, y)=$ $2, H_{1}(x, y)=x$, then the sequence gives the generalized Lucas polynomials $\left\{L_{n}(x, y)\right\}_{n=0}^{n=\infty}$.

Consider the following $2 \times 2$ matrices,

$$
A=\left(\begin{array}{cc}
\alpha & 1 \\
0 & \alpha
\end{array}\right), \quad B=\left(\begin{array}{cc}
\beta & 1 \\
0 & \beta
\end{array}\right), \quad C=\left(\begin{array}{cc}
\alpha & 1 \\
0 & \beta
\end{array}\right), \quad D=\left(\begin{array}{cc}
\beta & 1 \\
0 & \alpha
\end{array}\right), \quad I=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)
$$

and let $n$ and $m$ be nonnegative integers. [By definition, a matrix raised to the power 0 is equal to the unitmatrix $I$.]
a. Express $f_{n, m}(x, y) \equiv\left[(A-B)^{-1}\left(A^{n}-B^{n}\right)\right]^{m}$ in closed form, in terms of the Fibonacci polynomials.
b. Express $g_{n, m}(x, y) \equiv\left[A^{n}+B^{n}\right]^{m}$ in closed form, in terms of the Lucas polynomials.
c. Express $h_{n, m}(x, y) \equiv\left[C^{n}+D^{n}\right]^{m}$ in closed form, in terms of the Fibonacci and Lucas polynomials.
Combined solution by Paul Bruckman, Sacramento, CA and Mario Catalani, Torino, Italy

To simplify notations, we write $\alpha=\alpha(x, y), \beta=\beta(x, y), F_{n}=F_{n}(x, y)$, and $L_{n}=$ $L_{n}(x, y)$. The Binet formulas for the Fibonacci and Lucas polynomialsare

$$
F_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta} \quad \text { and } \quad L_{n}=\alpha^{n}+\beta^{n}
$$

Clearly,

$$
(A-B)^{-1}=\frac{I}{\alpha-\beta}
$$

By an easy induction on $n$ one provesthat
$A^{n}=\left(\begin{array}{cc}\alpha^{n} & n \alpha^{n-1} \\ 0 & \alpha^{n}\end{array}\right), \quad B^{n}=\left(\begin{array}{cc}\beta^{n} & n \beta^{n-1} \\ 0 & \beta^{n}\end{array}\right), \quad A^{n}-B^{n}=\left(\begin{array}{cc}\alpha^{n}-\beta^{n} & n\left(\alpha^{n-1}-\beta^{n-1}\right) \\ 0 & \alpha^{n}-\beta^{n}\end{array}\right.$
and

$$
C^{n}=\left(\begin{array}{cc}
\alpha^{n} & F_{n} \\
0 & \beta^{n}
\end{array}\right), \quad D^{n}=\left(\begin{array}{cc}
\beta^{n} & F_{n} \\
0 & \alpha^{n}
\end{array}\right), C^{n}+D^{n}=\left(\begin{array}{cc}
\alpha^{n}+\beta^{n} & 2 F_{n} \\
0 & \alpha^{n}-\beta^{n}
\end{array}\right)
$$

By induction on $m$ when $n$ is fixed, it now follows that

$$
\begin{gathered}
{\left[(A-B)^{-1}\left(A^{n}-B^{n}\right)\right]^{m}=\left(\begin{array}{cc}
F_{n} & n F_{n-1} \\
0 & F_{n}
\end{array}\right)^{m}=\left(\begin{array}{cc}
F_{n}^{m} & n m F_{n}^{m-1} F_{n-1} \\
0 & F_{n}^{m}
\end{array}\right)} \\
{\left[A^{n}+B^{n}\right]^{m}=\left(\begin{array}{cc}
L_{n} & n L_{n-1} \\
0 & L_{n}
\end{array}\right)^{m}=\left(\begin{array}{cc}
L_{n}^{m} & n m L_{n}^{m-1} L_{n-1} \\
0 & L_{n}^{m}
\end{array}\right)}
\end{gathered}
$$

and

$$
\left[C^{n}+D^{n}\right]^{m}=\left(\begin{array}{cc}
L_{n} & 2 F_{n} \\
0 & L_{n}
\end{array}\right)^{m}=\left(\begin{array}{cc}
L_{n}^{m} & 2 m L_{n}^{m-1} F_{n} \\
0 & L_{n}^{m}
\end{array}\right)
$$

## Also solved by the proposers.

Errata: In the displayed formula in Proposed Problem H-595 (volume 41.1) the equal sign "=" should have been " $\leq$ ".

## Please Send in Proposals!

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## BOOKS AVAILABLE THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau, Fibonacci Association (FA), 1965. \$18.00

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972. \$23.00
A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972. $\$ 32.00$

Fibonacci's Problem Book, Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974. $\$ 19.00$

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969. \$6.00

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971. \$6.00
Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972. $\$ 30.00$

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973. \$39.00
Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965. $\$ 14.00$

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965. $\$ 14.00$

A Collection of Manuscripts Related to the Fibonacci Sequence-18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980. \$38.00

Applications of Fibonacci Numbers, Volumes 1-7. Edited by G.E. Bergum, A.F. Horadam and A.N. Philippou. Contact Kluwer Academic Publishers for price.

Applications of Fibonacci Numbers, Volume 8. Edited by F.T. Howard. Contact Kluwer Academic Publishers for price.

Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger. FA, 1993. $\$ 37.00$

Fibonacci Entry Points and Periods for Primes 100,003 through 415,993 by Daniel C. Fielder and Paul S. Bruckman. $\$ 20.00$

Shipping and handling charges will be $\$ 4.00$ for each book in the United States and Canada. For Foreign orders, the shipping and handling charge will be $\$ 9.00$ for each book.
Please write to the Fibonacci Association, P.O. Box 320, Aurora, S.D. 57002-0320, U.S.A., for more information.


[^0]:    ${ }^{\text {* }}$ Professor Renato Capocelli passed away a few years ago while he was still a young man. This paper is the culmination of some work we started together when he visited Oregon State University.

