The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

TABLE OF CONTENTS

Characterizations of α-Words, Moments, and Determinants .... *Wai-fong Chuan* 194

On Some Classes of Effectively Integrable Differential Equations and Functional Recurrences .... *Krystyna Grytczuk* 209

A Three-Variable Identity Involving Cubes of Fibonacci Numbers .... *R. S. Melham* 220

An Elementary Proof of Jacobi's Four-Square Theorem .... *John A. Ewell* 224

Rises, Levels, Drops and "+" Signs in Compositions: Extensions of a Paper by Alladi and Hoggatt .... *S. Heubach, P.Z. Chinn and R.P. Grimaldi* 229

Vieta Convolutions and Diagonal Polynomials .... *A.F. Horadam* 240

Dynamic One-Pile Nim .... *Arthur Holshouser, Harold Reiter and James Rudzinski* 253

Fibonacci Numbers and Partitions .... *José Plínio O. Santos and Miloŝ Ivković* 263

A Class of Fibonacci Ideal Lattices in $\mathbb{Z}[\zeta_{12}]$ .... *Michele Elia and J. Carmelo Interlando* 279

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CHARACTERIZATIONS OF α-WORDS, MOMENTS, AND DETERMINANTS

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1. INTRODUCTION

Throughout this paper we consider binary words. All results can easily be stated for words over other two-letter alphabets. For any word w, let $|w|$ denote the *length* of w and let $|w|_1$, called the *height* of w, denote the number of occurrences of the letter 1 in w. For $n \geq 1$ and $c_1, c_2, \ldots, c_n \in \{0, 1\}$, define operators T and $^\sim$ by

$$T(c_1c_2\ldots c_n) = c_2\ldots c_nc_1,$$

$$(c_1c_2\ldots c_n)^\sim = c_n\ldots c_2c_1.$$

For each integer j, let T^j have the obvious meaning. The operator T is called the *cyclic shift* (or *rotation*) *operator*. A word u is called a *conjugate* of a word w if $u = T^j(w)$ for some integer j. The set of all distinct conjugates of w is called the *conjugate class* of w and is denoted by $[w]$. The word $\tilde{w}$ is called the *reversal* of the word w.

A word w is said to be a *palindrome* if either w is the empty word or $\tilde{w} = w$. w is said to be *primitive* if it is not a power of another word. w is said to be a *Lyndon* (resp. *anti-Lyndon word* if it is the smallest (resp., largest) in the lexicographic order in the conjugate class of w. w is said to be *bordered* if there are words x and y with x nonempty such that $w = xyx$; otherwise, w is said to be *unbordered.*

For $w = c_1c_2\ldots c_q$, where each c_i is either 0 or 1, define $M(w) = \sum_{i=1}^{q}(q+1-i)c_i$. $M(w)$ is called the *moment* of w. Define

$$M([w]) = \{M(u) : u \in [w]\},$$
$$\delta(w) = \max\{M(u) - M(v) : u, v \in [w]\}.$$

One way to define α-words is to make use of T and the words $u\left(\frac{p}{q}\right)$ define below. (See [13] for the original definition and basic properties of α-words.)

Let p and q be two relatively prime positive integers with $p < q$. Let $[0, a_1 + 1, a_2, \ldots, a_n]$ be the continued fraction expansion of $\frac{p}{q}$. Define a sequence of words $u_{-1}, u_0, u_1, \ldots, u_n$ recursively as follows: Let $u_{-1} = 1,\ u_0 = 0$, and for $1 \leq k \leq n$, let

$$u_k = \begin{cases} u_{k-2}u_{k-1}^{a_k} & (k \text{ is even}) \\ u_{k-1}^{a_k}u_{k-2} & (k \text{ is odd}). \end{cases}$$

It is know that the word u_n depends on $\frac{p}{q}$, but not the continued fraction expansion [1, 2].

Denote u_n by $u\left(\frac{p}{q}\right)$. Clearly, its first (resp., last) letter is 0 (resp., 1).

A word w is said to be an *α-word* if either $w \in \{0,1\}$ or there are two relatively prime positive integers p and q with $p < q$ such that w is a conjugate of $u\left(\frac{p}{q}\right)$. Conjugates of $u\left(\frac{F_{n-1}}{F_n}\right)$ (resp., $u\left(\frac{F_{n-2}}{F_n}\right)$) are known as *binary Fibonacci words* (see [6]).

We first report briefly some known results about the word $u = u\left(\frac{p}{q}\right)$ and its reversal. The conjugates $u, T(u), \ldots, T^{q-1}(u)$ of u are exactly the distinct α-words with length q and height p. Thus each α-word is primitive. The word u (resp., $\tilde{u}$) is a Lyndon (resp., anti-Lyndon) α-word (see [1,11]). The word u is the only binary word which has two factorizations of the form $u = xy = 0zl$, where x, y, z are palindromes, $|z| = q-2, |y| = s$ and $1 \leq s < q$ is such that $ps \equiv 1 (\bmod\ q)$ (see [20]). The conjugate class $[u]$ of u is closed under taking reversals. Clearly $\tilde{u} = T^{-s}(u)$. Both u and $\tilde{u}$ are unbordered. Furthermore, the set of Lyndon α-words and their reversals are the only unbordered finite Sturmian words (a *finite Sturmian word* is any finite factor (or segment) of any characteristic word (see section 5)) [14]. The set of Lyndon α-words coincides with the set of Christoffel primitives (see [1,2] for the definition of Christoffel primitive).

Let $[0, a_1+1, a_2, \ldots, a_n]$ be the continued fraction expansion of $\frac{p}{q}$. In [13], it was shown that a word w is a conjugate of u if and only if there are integers $r_1, \ldots, r_n$ with $0 \leq r_i \leq a_i, 1 \leq i \leq n$, and words $w_{-1}, w_0, w_1, \ldots, w_n$ such that

$$w_{-1} = 1,\ w_0 = 0,\ w_n = w,$$

$$w_i = w_{i-1}^{a_i - r_i} w_{i-2} w_{i-1}^{r_i}, \quad 1 \leq i \leq n.$$

In fact, each conjugate $T^k(u)$ of u corresponds to those n-tuples $(r_1, \ldots, r_n)$ of integers with $0 \leq r_i \leq a_i,\ 1 \leq i \leq n$ and $k \equiv \sum_{i=1}^{n} r_i q_{i-1} (\bmod\ q)$, where $q_{-1} = q_0 = 1,\ q_i = a_i q_{i-1} + q_{i-2},\ 1 \leq i \leq n$. Thus, each α-word can be obtained recursively by concatenation. The words, having length q and height p, obtained with $r_1 = \cdots = r_n = 0$ or $r_1 = \cdots = r_{n-1} = 1 - r_n = 0$ are called standard Sturmian words (see [1]). It is not hard to see that a word w having length q and height p is a standard Sturmian word if and only if $w = T(u)$ or $w = T(\tilde{u})$.

Let $u\left(\frac{0}{1}\right) = 0$ and $u\left(\frac{1}{1}\right) = 1$. If $\frac{t}{s}$ and $\frac{t'}{s'}$ are consecutive fractions in the Farey sequence of any order with $\frac{t}{s} < \frac{t'}{s'}$, then $u\left(\frac{t+t'}{s+s'}\right) = u\left(\frac{t}{s}\right) u\left(\frac{t'}{s'}\right)$. Also the mapping $r \mapsto u(r)$ is an increasing function from the set of all reduced fractions in [0,1] onto the set of all Lyndon α-words. In other words, if $r < r'$ then $u(r) < u(r')$ in the lexicographic order (see [2]).

More results - both old and new - about $u\left(\frac{p}{q}\right)$ will be presented below.

In an earlier paper, the present author proved that if w is an α-word having length q, then $M([w])$ is a set of q consecutive positive integers and $\delta(w) = q-1$. Each of these properties actually characterizes α-words (Theorem 4.4). The result used to prove this characterization is itself a characterization of α-words (Lemma 2.1) with other interesting consequences besides Theorem 4.4. In section 3, we obtain characterization of elements of the set PER and standard Sturmian words (Corollary 3.2), and we identify those α-words that are palindromes (Corollary 3.4). In section 5, we compute the determinants of a class of matrices involving α-words (Theorem 5.1). As a special case, we obtain a sequence of (0,1)-matrices $A_1, A_2 \ldots$ such that A_n is an $F_n \times F_n$ matrix whose rows are precisely the Fibonacci words having length F_n, height F_{n-1} (resp., F_{n-2}), and $det(A_n) = F_{n-1}$ (resp., F_{n-2}).

2. A LEMMA

[11,14,16,18] present some characterizations of α-words. The characterization proved in [11] is restated in Lemma 2.1 below. With this result, we know exactly where the ones in each α-word are located and so each α-word can be generated directly without using α-words of shorter lengths. Corollary 2.2 shows how all α-words having the same length q and height p may be ordered in such a way that consecutive pairs differ in exactly two adjacent letters. Sections 3-5 present some interesting consequences of Lemma 2.1 and Corollary 2.2.

Lemma 2.1: Let p and q be relatively prime positive integers with $p < q$. Define s as the unique integer with

$$sp \equiv 1 (\text{mod } q) \text{ and } 1 \leq s < q. \tag{1}$$

Let $u = u\left(\frac{p}{q}\right)$. Then for $0 \leq j \leq q-1$,

$$\begin{aligned} &\text{the } k^{th} \text{ letter of } T^{js}(u) \text{ is } 1 \\ \Longleftrightarrow\; & k \equiv (r-j)s \ (\text{mod } q) \text{ for some } r \text{ with } 0 \leq r \leq p-1, \\ \Longleftrightarrow\; & k \equiv 1 + (r+j)(q-s) (\text{mod } q) \text{ for some } r \text{ with } 1 \leq r \leq p. \end{aligned}$$

A proof of Lemma 2.1 appears in the Appendix (see also [11]).

Corollary 2.2: Let p, q, s, and u be as in Lemma 2.1. Let $0 \leq j \leq q-1$. The words $T^{js}(u)$ and $T^{(j+1)s}(u)$ differ by exactly two adjacent letters. If $i \equiv (p-1-j)s$ (mod q) and $1 \leq i \leq q$, then the $(i-1)^{th}$ and the i^{th} letters in $T^{js}(u)$ and $T^{(j+1)s}(u)$ are 01 and 10 respectively.

Proof: Let $0 \leq j \leq q-1$. The positions of 1 in $T^{js}(u)$ and $T^{(j+1)s}(u)$ are respectively

$$-js, (1-j)s, \ldots, (p-2-j)s, (p-1-j)s,$$

and

$$(-j-1)s, -js, (1-j)s, \ldots, (p-2-j)s$$

(mod q). If $(p-j-1)s \equiv i \pmod q$ where $1 \leq i \leq q$, then clearly $i \neq 1$ and $(-j-1)s \equiv i-1 \pmod q$. Hence the words $T^{js}(u)$ and $T^{(j+1)s}(u)$ differ by exactly two letters. The $(i-1)^{th}$ and the i^{th} letters in $T^{js}(u)$ and $T^{(j+1)s}(u)$ are 01 and 10 respectively. □

We remark that when

$$q = F_n \text{ and } p = F_{n-1},\ s = \begin{cases} F_{n-1} & (n \text{ even }) \\ F_{n-2} & (n \text{ odd}) \end{cases}, n \geq 3.$$

Then Lemma 2.1 and Corollary 2.2 reduce to Theorem 2 (or Corollary 12(i) of [6]) and Theorem 3 of [10] respectively.

3. IMMEDIATE CONSEQUENCES

Throughout this section, let p, q, s, and u be as in Lemma 2.1. We shall show how Lemma 2.1 yeilds new and old results on factorization, PER, standard Sturmian words, lexicographic order, reversals and moments.

Corollary 3.1:

(a) $u = xy$, where x and y are palindromes with $|y| = s$ and $|x| = q - s$.

(b) $u = 0zl$, where z is a palindrome.

Note that, by taking reversals, we immediately derive from (a) and (b) respectively that $\tilde{u} = yx$ and $\tilde{u} = lz0$.

Proof: The proofs of (a) and (b) are almost identical so we suffice with the proof of (b). Let $2 \leq k \leq q-1$.

The k^{th} letter of u is 1

$\Longleftrightarrow k \equiv rs \pmod q$ for some r with $1 \leq r \leq p-1$ (by Lemma 2.1 with $j = 0$)

$\Longleftrightarrow q+1-k \equiv (p-r)s \pmod q$ for some $1 \leq r \leq p-1$ (by equation (1))

$\Longleftrightarrow$ the $(q+1-k)^{th}$ letter of u is 1.

Therefore the result follows. □

Let PER= $\{0,1\} \cup \{z : 0z1$ is a Lyndon α-word$\}$. Note that the empty word belongs to PER. Let PER01= $\{z01 : z \in \text{PER}\}$. The set PER10 is defined similarly. The set of standard Sturmian words equals $\{0,1\}\cup$ PER01$\cup$PER10. Elements of PER and standard Sturmian words have been recently studied extensively (see [1]). The following corollary provides characterizations of these words.

Corollary 3.2:

(a) Let $z \in \text{PER}$ with $|z| = q-2$ and $|z|_1 = p-1 \geq 1$. Then

the k^{th} letter of z is 1

$\Longleftrightarrow k \equiv rs - 1 \pmod q$ for some r with $1 \le r \le p-1$

$\Longleftrightarrow k \equiv r(q-s) \pmod q$ for some r with $1 \le r \le p-1$.

(b) Let $w \in$ PER01 and $w' \in$ PER10 with $|w| = |w'| = q$ and $|w|_1 = |w'|_1 = p$. Then the k^{th} letter of w is 1

$\Longleftrightarrow k \equiv rs - 1 \pmod q$ for some r with $1 \le r \le p$;

the k^{th} letter of w' is 1

$\Longleftrightarrow k \equiv r(q-s) \pmod q$ for some r with $1 \le r \le p$.

Proof: Part (a) follows from Lemma 2.1 and the fact that $0z1 = u$. Part (b) follows from the fact that $w = T(\tilde{u})$ and $w' = T(u)$. □

When the conjugates of u are listed as in (2) below, we observe some interesting phenomena.

Corollary 3.3 (see [11]):

(a) The sequence of words

$$u, T^s(u), T^{2s}(u), \ldots, T^{(q-1)s}(u) = \tilde{u} \tag{2}$$

is increasing in lexicographic order.

(b) $T^{js}(u)$ have increasing moments with $M(T^{js}(u)) = \frac{(p-1)(q+1)}{2} + j + 1$ $(0 \le j \le q-1)$.

Proof: Part (a) and the recurrence relation $M(T^{(j+1)s}(u)) = M(T^{js}(u)) + 1, 0 \le j \le q-2$, follow immediately from Corollary 2.2 and the definition of M. Thus $M(T^{js}(u)) = M(u) + j, 0 \le j \le q-1$. We have

$$M(u) = \sum_{h=1}^{p-1}\left(q + 1 - \left(\left[\frac{hq}{p}\right] + 1\right)\right) + 1 \text{ (by definition of } M \text{ and Lemma A3 of Appendix)}$$

$$= q(p-1) - \sum_{h=1}^{p-1}\left[\frac{hq}{p}\right] + 1 \text{ (by rearrangement)}$$

$$= q(p-1) - \frac{(q-1)(p-1)}{2} + 1 \text{ (by e.g. [5])}$$

$$= \frac{(q+1)(p-1)}{2} + 1,$$

proving (b). □

The above corollary generlizes Corollaries 2 and 3 of [10]. The following corollary generalizes Lemmas 6 and 7 of [7].

Corollary 3.4:

(a) $T^{(q-1-j)s}(u) = \left(T^{js}(u)\right)^{\sim}$, $0 \le j \le q-1$.

(b) If q is odd, then $[u]$ contains exactly one palindrome, namely $T^{\left(\frac{q-1}{2}\right)s}(u)$; if q is even, $[u]$ contains no palindrome.

Note, letting $j = 0$ in (a) yields $\tilde{u} = T^{-s}(u)$.

Proof:

Let $0 \le j \le q-1$. By repeated use of Lemma 2.1, for $1 \le k \le q$,

the $(q+1-k)^{th}$ letter of $T^{(q-1-j)s}(u)$ is 1

$\iff q+1-k \equiv 1 + (r + (q-1-j))(q-s) \pmod q$ for some $1 \le r \le p$

$\iff k \equiv (r' - j)s \pmod q$ for some $0 \le r' \le p-1$

$\iff$ the k^{th} letter of $T^{js}(u)$ is 1.

This proves (a). Part (b) follows immediately from part (a) and the distinctness of the $T^j(u)$. □

4. MOMENTS OF α-WORDS

For any binary word w, let $\delta(w) = max\{M(u) - M(v) : u, v \in [w]\}$. The following lemma summarizing the properties of moments of α-words is an immediate consequence of part (b) of Corollary 3.3.

Lemma 4.1: Let w be an α-word with $|w| = q \ge 2$ and $|w|_1 = p$. Let $u = u\left(\frac{p}{q}\right)$. Then

(a) $M(u) = minM([w]) = \frac{(p-1)(q+1)}{2} + 1$, $M(\tilde{u}) = maxM([w]) = \frac{(p+1)(q+1)}{2} - 1$.

(b) $\delta(w) = q - 1$.

(c) $M([w])$ is a set of q consecutive positive integers.

We shall prove in Theorem 4.4 below that each of the conditions (b) and (c) is equivalent to saying that w is an α-word. We need the following lemma which is useful when studying moments of binary words.

Lemma 4.2: Let w be a binary word with $|w| = q$ and $|w|_1 = p$. Let $M_k = M(T^k(w))$, $0 \le k < q$. Let $w = c_1c_2\dots c_q$ where each c_i is either 0 or 1. Define $c_{q+j} = c_j$ for $1 \le j \le q$. Then for $0 \le r < k < q$, we have

$$M_k - M_r = p(k-r) - q\sum_{i=r+1}^{k} c_i.$$

In particular, $M_k - M_0 = pk - q\sum_{i=1}^{k} c_i$ if $k > 0$.

Proof: For each k with $0 \leq k \leq q-1$, since $T^k(w) = c_{k+1}c_{k+2}\dots c_{k+q}$, we have

$$M_k = \sum_{j=1}^{q}(q+1-j)c_{k+j} = \sum_{i=k+1}^{k+q}(k+q+1-i)c_i = p(k+q+1) - \sum_{i=k+1}^{k+q} ic_i.$$

If $r < k$, then

$$M_k - M_r = p(k+q+1) - \sum_{j=k+1}^{k+q} jc_j - p(r+q+1) + \sum_{i=r+1}^{r+q} ic_i$$

$$= p(k-r) + \sum_{i=r+1}^{k} ic_i - \sum_{j=r+q+1}^{k+q} jc_j$$

$$= p(k-r) - q\sum_{i=r+1}^{k} c_i. \ \square$$

Lemma 4.3: Let w be a binary word with $|w| = q \geq 2$ and $|w|_1 = p$. If $\delta(w) = q-1$ then q and p are relatively prime positive integers and w is an α-word conjugate to $u\left(\frac{p}{q}\right)$.

Proof: Let $u \in [w]$ with $M(u) = minM([w])$. Let $k_1, k_2, \dots, k_q$ be a permutation of $0, 1, \dots, q-1$ such that $k_1 = 0$ and $M_{k_1} \leq M_{k_2} \leq \dots \leq M_{k_q}$. Let $u = c_1c_2\dots c_q$ where each c_i is either 0 or 1. Define $c_{q+j} = c_j$ for $1 \leq j \leq q$. By the assumption and Lemma 4.2, we have

$$q-1 = M_{k_q} - M_{k_1} = pk_q - q\sum_{i=1}^{k_q} c_i,$$

and so q and p are relatively prime positive integers. Again by Lemma 4.2, the moments $M_{k_1}, M_{k_2}, \dots, M_{k_q}$ are all distinct and therefore $M_{k_{m+1}} - M_{k_m} = 1$, for $1 \leq m \leq q-1$.

Let $1 \leq m \leq q-1$. Lemma 4.2 also implies that

$$1 = M_{k_{m+1}} - M_{k_m} = \begin{cases} p(k_{m+1} - k_m) - q\sum_{i=k_m+1}^{k_{m+1}} c_i & (\text{if } k_m < k_{m+1}), \\ q\sum_{i=k_{m+1}+1}^{k_m} c_i - p(k_m - k_{m+1}) & (\text{if } k_{m+1} < k_m). \end{cases}$$

Define s by equation (1). Then

$$k_{m+1} - k_m = \begin{cases} s & (k_m < k_{m+1}) \\ s - q & (k_{m+1} < k_m) \end{cases}$$

$$\equiv s \pmod{q}$$

and therefore $k_m \equiv (m-1)s \pmod{q}$.

We claim that $c_{k_r} = 0$ for $p+1 \le r \le q$. To show this, let $1 \le m \le q-p$. Since $k_{m+p} - k_m \equiv (m+p-1)s - (m-1)s = ps \equiv 1 (\mathrm{mod}\ q)$ and $-q+1 \le k_{m+p} - k_m \le q-1$, it follows that $k_{m+p} - k_m$ equals either $-q+1$ or 1. If $k_{m+p} - k_m = -q+1$, then $k_{m+p} = 0$ (and $k_m = q-1$). But then $m+p = 1$, a contradiction. Therefore $k_{m+p} = k_m + 1$. According to Lemma 4.2, we have

$$p = M_{k_{m+p}} - M_{k_m} = p(k_{m+p} - k_m) - q \sum_{i=k_m+1}^{k_{m+p}} c_i = p - qc_{k_{m+p}};$$

so $c_{k_{m+p}} = 0$, proving our claim.

Since $|u|_0 = q - p$, we see that

$$\begin{aligned} c_k = 1 &\iff k = q \text{ or } k_r \text{ for some } r \text{ with } 2 \le r \le p \\ &\iff k \equiv rs \pmod{q} \text{ for some } r \text{ with } 0 \le r \le p-1. \end{aligned}$$

It follows from Lemma 2.1 that $u = u\left(\frac{p}{q}\right)$. Consequently w is an α-word. □

Combining Lemma 4.1 and 4.3, we have the following characterization of α-words.

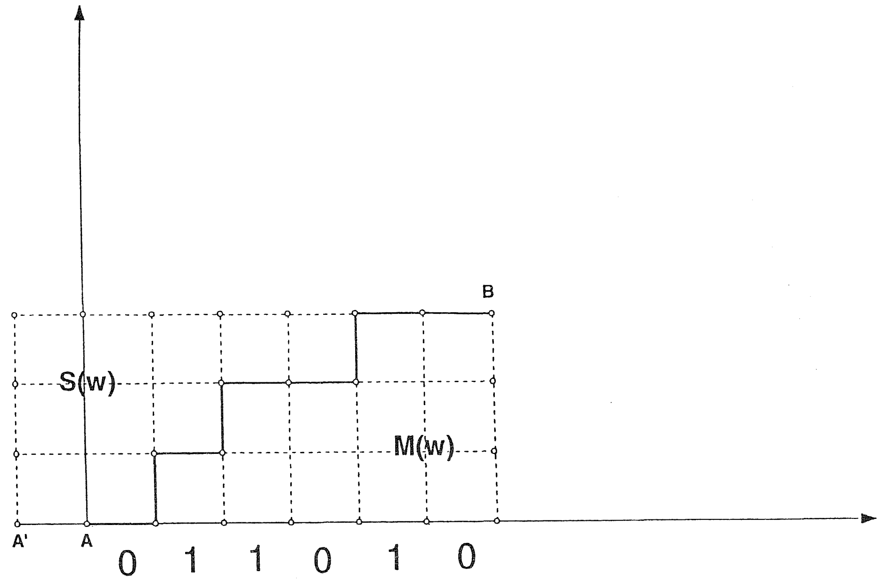
Theorem 4.4: Let w be a binary word with $|w| = q \ge 2$. Then the following statements are equivalent:

(a) $\delta(w) = q - 1$,

(b) w is an α-word,

(c) $M([w])$ is a set of q consecutive positive integers.

Remark 4.5: For $w = c_1c_2\ldots c_q$ where each c_i is either 0 or 1, define $S(w) = \sum_{i=1}^{q} ic_i$. The results about moments can easily be reformulated using $S(w)$ instead of $M(w)$. Plainly $S(w) = M(\tilde{w})$, and $S(w) + M(w) = (|w|+1)|w|_1$. Graphically, a word w is represented by a polygonal path from $A(0,0)$ to $B(|w|, |w|_1)$ as follows: starting from the origin A, represent a 0 (resp., 1) in w by a horizontal unit segment going to the right (resp., a vertical unit segment going upward, followed by a horizontal unit segment going to the right). This polygonal path divides the rectangular region having opposite vertexes $A'(-1,0)$ and B into two subregions. The one below (resp., above) the polygonal path has area $M(w)$ (resp., $S(w)$) (see Figure).



5. DETERMINANTS OF MATRICES INVOLVING α-WORDS

Throughout this section, let q and p be relatively prime positive integers with $p < q$. Let $u = u\left(\frac{p}{q}\right)$. Regarding each binary word as a vector, we consider the $q \times q$ $(0,1)$-matrix whose j^{th} row is the α-word $T^{-(j-1)}(\tilde{u})$, $1 \leq j \leq q$. It is easy to see that this matrix is a circulant matrix, that is, a matrix of the form

$$\begin{bmatrix} c_1 & c_2 & \dots & c_{q-1} & c_q \\ c_q & c_1 & \dots & c_{q-2} & c_{q-1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_2 & c_3 & \dots & c_q & c_1 \end{bmatrix}$$

where c_k is the k^{th} digit of $\tilde{u}$. We denote this matrix by $circ(\tilde{u})$ (see [19]).

Among all the matrices obtained from $circ(\tilde{u})$ by permuting its rows, the matrix $circ(\tilde{u})$ is of particular interest for the following reasons.

Let α be any irrational number between 0 and 1 such that $\frac{p}{q}$ is a convergent of the continued fraction expansion of α. The *characteristic word* $f(\alpha)$ is an infinite binary word whose k^{th} letter is $[(k+1)\alpha] - [k\alpha]$, $k \geq 1$ (see, for example, [3, 13-15, 21, 23]). When $\alpha = \frac{\sqrt{5}-1}{2}, f(\alpha)$ is called the *golden sequence* (see, for example, [4, 8, 9, 12, 17, 24, 25]).

Golden sequence turns out to be the Fibonacci binary word pattern $F(1,01)$ (an infinite word $w_1w_2w_3\dots$, where $w_1 = x$ and $w_2 = y$ are binary words, and $w_n = w_{n-2}w_{n-1}$, $n \geq 3$, is called a *Fibonacci binary word pattern* and is denoted by $F(x,y)$ (see [17, 25])).

It is well-known that for each $k \geq 1$, there are exactly $k+1$ distinct factors (or segments) of $f(\alpha)$ (see [23]). Let y denote the palindrome that differs from u only by the last (resp., first) letter if the q^{th} letter of $f(\alpha)$ is 1 (resp., 0). It was proved in [13] that for $1 \leq k \leq q$, the rows of the upper left $(k+1) \times k$ submatrix of the $(q+1) \times q$ matrix

$$\begin{bmatrix} circ(\tilde{u}) \\ y \end{bmatrix} \left(\text{resp., } \begin{bmatrix} circ(u) \\ y \end{bmatrix}\right)$$

are precisely the $k+1$ distinct factors of $f(\alpha)$ of length k.

Another interesting fact about $circ(\tilde{u})$ is contained in the following theorem.

Theorem 5.1: $\det(circ(\tilde{u})) = p$, if $q \geq 1$. Here $u\left(\frac{0}{1}\right) = 0$ and $u\left(\frac{1}{1}\right) = 1$.

Since the matrices under consideration are circulant matrices, their eigenvalues and hence their determinants can be computed using the q^{th} roots of unity. However the following row rule proof based on the combinatoric properties of Corollary 2.2 is more elegant.

Proof: Let $\tilde{u} = c_1c_2\dots,c_q$ where $c_1,\dots c_q \in \{0,1\}$. Clearly the result holds for $q \leq 2$. Now let $q \geq 3$. Using (1), for $1 \leq t \leq q$, define $1 \leq i_t \leq q$ such that $i_t \equiv 1 + (t-1)s \pmod{q}$. Denote $circ(\tilde{u})$ by A and its (i,k)-entry by $A(i,k)$. For $2 \leq t \leq q$, since row i_t (resp., i_{t-1}) of A is $T^{-i_t+1}(\tilde{u}) = T^{(q-t)s}(u)$ (resp., $T^{(q-t+1)s}(u)$), Corollary 2.2 implies that

$$\begin{aligned} &A(i_{t-1}, i_t - 1) = 1,\ A(i_{t-1}, i_t) = 0, \\ &A(i_t, i_t - 1) = 0,\ A(i_t, i_t) = 1, \\ &A(i_t, k) = A(i_{t-1}, k) \text{ for } k \neq i_t \text{ and } k \neq i_t - 1. \end{aligned}$$

Let B be the matrix obtained from A by adding (-1) times row i_{t-1} to row i_t, for each $t = q, q-1, \dots, 2$, in the order given. Then

$$\begin{aligned} B(1,k) &= A(1,k) = c_k, \\ B(i_t,k) &= (-1)A(i_{t-1},k) + A(i_t,k) \\ &= \begin{cases} -1 \ (k = i_t - 1) \\ 1 \ (k = i_t) \\ 0 \ (\text{otherwise}), \end{cases} \end{aligned}$$

where $2 \leq t \leq q$, and $1 \leq k \leq q$. Since $i_2, i_3, \ldots, i_q$ is a permutation of $2, 3, \ldots, q$, it follows that B is the matrix

$$\begin{bmatrix} c_1 & c_2 & c_3 & \ldots & c_{q-1} & c_q \\ -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{bmatrix}$$

Clearly,

$$det(circ(\tilde{u})) = det(B) = \sum_{k=1}^{q} c_k = p. \quad \square$$

Here is a special case of Theorem 5.1. Let $\{v_n\}$ and $\{z_n\}$ be sequences of Fibonacci words given recursively by

$$v_0 = 1, v_1 = 0, v_2 = 1, v_n = \begin{cases} v_{n-1}v_{n-2} & (n \text{ is odd}) \\ v_{n-2}v_{n-1} & (n \text{ is even}), \end{cases}$$

$$z_1 = 1, z_2 = 0, z_n = \begin{cases} z_{n-2}z_{n-1} & (n \text{ is odd}) \\ z_{n-1}z_{n-2} & (n \text{ is even}), \end{cases}$$

Let $A_n = circ(v_n)$ (resp., $circ(z_n)$), $n \geq 1$. Since $\frac{F_{n-1}}{F_n} = [0, 1, 1, \ldots, 1]$ ($n-1$ ones) (resp., $\frac{F_{n-2}}{F_n} = [0, 2, 1, \ldots, 1]$ ($n-3$ ones)), $n \geq 3$, we see that $v_n = \left(u\left(\frac{F_{n-1}}{F_n}\right)\right)^{\sim}$ $\left(\text{resp., } z_n = \left(u\left(\frac{F_{n-2}}{F_n}\right)\right)^{\sim}\right)$, $n \geq 1$. It follows from Theorem 5.1 that each A_n is an $F_n \times F_n$ $(0,1)$ - matrix whose rows are precisely the Fibonacci words having length F_n and height F_{n-1} (resp., F_{n-2}) and $det(A_n) = F_{n-1}$ (resp., F_{n-2}).

APPENDIX. A PROOF OF LEMMA 2.1

For each real number θ, the infinite binary word $f(\theta)$ whose k^{th} letter is $[(k+1)\theta] - [k\theta]$, $k \geq 1$, is called the *characteristic word* of θ.

Lemma A1 (see [21]): Let $0 < \theta < 1$.

(a) If θ is irrational and $k \geq 1$, then

$$\text{the } k^{th} \text{ letter of } f(\theta) \text{ is } 1$$

$$\Longleftrightarrow k = \left[\frac{h}{\theta}\right] \text{ for some } h \geq 1.$$

(b) If $\theta = \frac{p}{q}$ is rational, where p, q are relatively prime positive integers, and $k \geq 1, k \not\equiv 0$ and $k \not\equiv -1 \pmod q$, then

$$\text{the } k^{th} \text{ letter of } f(\theta) \text{ is } 1$$

$$\Longleftrightarrow k = \left[\frac{h}{\theta}\right] \text{ for some } h \geq 1,\ h \not\equiv 0 \pmod p.$$

Throughout the rest of this section, let p and q be relatively prime positive integers with $p < q$. Let $1 \leq s < q, 1 \leq t < p$, and $ps = qt + 1$. Let $u = u\left(\frac{p}{q}\right)$. If w is a word and $w = xy$ where y is nonempty, we write $x = wy^{-1}$.

Lemma A2: Let θ be a real number between 0 and 1 such that $\frac{p}{q}$ is a convergent of the continued fraction expansion of θ. Let z be a palindrome such that $u = 0z1$.

(a) (see [1,3,21]) z is a prefix of $f(\theta)$.

(b) If $\frac{p}{q} > \theta$, then $u1^{-1}$ (resp., $\tilde{u}$) is a prefix of $0f(\theta)$ (resp., $1f(\theta)$), but u is not a prefix of $0f(\theta)$.

(c) If $\frac{p}{q} \leq \theta$, then u (resp., $\tilde{u}0^{-1}$) is a prefix of $0f(\theta)$ (resp., $1f(\theta)$), but $\tilde{u}$ is not a prefix of $1f(\theta)$.

(d) $0f\left(\frac{p}{q}\right) = u^{\infty}$.

Proof: Part (b) and (c) follow from (a) and the fact that $[(q-1)\theta] = p-1, [(q+1)\theta] = p$, and

$$[q\theta] = \begin{cases} p-1 & \left(\frac{p}{q} > \theta\right) \\ p & \left(\frac{p}{q} \leq \theta\right). \end{cases}$$

Part (d) follows from (b). □

The following lemma follows from Lemmas A1 and A2.

Lemma A3: The first (resp., last) letter of u is 0 (resp., 1). For $1 < k < q$,

$$\text{the } k^{th} \text{ letter of } u \text{ is } 1$$

$$\Longleftrightarrow k-1 = \left[\frac{hq}{p}\right] \text{ for some } 1 \le h \le p-1.$$

Lemma A4: For each h with $1 \le h \le p$, there is a unique r with $1 \le r \le p$ such that $\left[\frac{hq}{p}\right] \equiv rs - 1 \pmod q$. The mapping $h \longmapsto r$ is a bijection from $\{1, 2, \ldots, p\}$ onto itself. Furthermore,

(a) $h \equiv rt$ and $r \equiv h(p-m) \pmod p$, where $1 \le m \le p$, and $q \equiv m \pmod p$.

(b) $h = p \Longleftrightarrow r = p$.

Proof: Let $1 \le h \le p$. Since s and q are relatively prime, there is a unique integer r, $1 \le r \le q$ such that

$$\left[\frac{hq}{p}\right] \equiv rs - 1 \pmod q.$$

Clearly (b) holds. Let n be an integer such that $\left[\frac{hq}{p}\right] = rs - 1 - nq$. Then

$$p\left[\frac{hq}{p}\right] = rps - p - nqp$$

$$= r(qt+1) - p - nqp$$
$$= q(rt - np) + r - p.$$

Since $p\left[\frac{hq}{p}\right] \le hq < p\left[\frac{hq}{p}\right] + p$, we have

$$(rt - np) + \frac{r}{q} - \frac{p}{q} \le h < rt - np + \frac{r}{q},$$

that is,

$$h + np - rt < \frac{r}{q} \le h + np - rt + \frac{p}{q}.$$

Therefore $h + np - rt = \left[\frac{r}{q}\right] = 0$ and $r - p \le q(h + np - rt) = 0$; so $h \equiv rt \pmod p$ and $1 \le r \le p$. The second part of (a) follows immediately from the first part.

It remains to show that if $1 \le h_1 < h_2 \le p$, then $\left[\frac{h_1 q}{p}\right] \not\equiv \left[\frac{h_2 q}{p}\right] \pmod{q}$. Let $k = h_2 - h_1$, where $1 \le h_1 < h_2 \le p$, i.e., $1 \le k \le p-1$. Then

$$\begin{aligned}\left[\frac{h_1 q}{p}\right] + 1 &< \left[\frac{h_1 q}{p}\right] + k\frac{q}{p} \le \frac{h_1 q}{p} + \frac{kq}{p} = \frac{h_2 q}{p} \\ &\le \frac{h_1 q}{p} + \frac{p-1}{p} q < \frac{h_1 q}{p} + q - 1 \\ &< \left[\frac{h_1 q}{p}\right] + q;\end{aligned}$$

so the result follows. □

Lemma 2.1 now follows immediately from Lemmas A3 and A4.

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ON SOME CLASSES OF EFFECTIVELY INTEGRABLE DIFFERENTIAL EQUATIONS AND FUNCTIONAL RECURRENCES

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1. INTRODUCTION

It is well-known (see, [4] p. 411) that the general solution of the differential equation $(x^2-1)y''+xy'-n^2y=0$ is of the form:

$$y = C_1\left(\frac{x+\sqrt{x^2-1}}{2}\right)^n + C_2\left(\frac{x-\sqrt{x^2-1}}{2}\right)^n, \tag{1}$$

where C_1 and C_2 are arbitrary constants and $n \in N$.

For $C_1 = C_2 = 1$ from (1) we get that

$$T_n(x) = \left(\frac{x+\sqrt{x^2-1}}{2}\right)^n + \left(\frac{x-\sqrt{x^2-1}}{2}\right)^n, \tag{2}$$

is the Chebyshev polynomial of the first kind.

In [2] the author has considered a more general class of polynomials, namely:

$$W_n(x;c) = \left(\frac{x+\sqrt{x^2+c}}{2}\right)^n + \left(\frac{x-\sqrt{x^2+c}}{2}\right)^n, \tag{3}$$

where c is a parameter and where $n \geq 1$ is the degree of the polynomial $W_n(x;c)$. Moreover, it has been proved in [2] that the function:

$$y = C_1\left(\frac{x+\sqrt{x^2+c}}{2}\right)^n + C_2\left(\frac{x-\sqrt{x^2+c}}{2}\right)^n, \tag{4}$$

is the general solution of the differential equation:

$$(x^2+c)y''+xy'-n^2y=0, x^2+c>0,\ n \in N. \tag{*}$$

The polynomial $W_n(x;c)$ given by (3) contains the well-known Pell polynomial when $c = 1$ and the Fibonacci polynomial when $c = 4$.

In this paper we give further extensions of this result.

2. BASIC LEMMAS

Lemma 1: Let $s_0,\ u \in C^2(J)$ be real-valued functions of x, where $J = (x_1, x_2) \subset R$ and $u \neq 0$ on J. The function $y_1 = s_0 u^\lambda$, with non-zero real constant λ, is the particular solution of the differential equation:

$$D_0 y'' + D_1 y' + D_2 y = 0 \tag{2.1}$$

if and only if there exist the functions $s_1, s_2 \in C^2(J)$ such that

$$D_0 s_2 + D_1 s_1 + D_2 s_0 = 0. \tag{2.2}$$

Proof: Suppose that the function $y_1 = s_0 u^\lambda$ is the particular solution of (2.1). Then we have $D_0 y_1'' + D_1 y_1' + D_2 y_1 = 0$ and by the assumption on the functions s_0 and u it follows that

$$y_1' = s_0' u^\lambda + s_0 \lambda u^{\lambda-1} u' = u^\lambda \left(s_0' + \lambda s_0 \frac{u'}{u} \right). \tag{2.3}$$

Putting

$$s_1 = s_0' + \lambda s_0 \frac{u'}{u} \tag{2.4}$$

in (2.3) we have $y_1' = s_1 u^\lambda$. In a similar manner we obtain

$$y_1'' = \left(s_1 u^\lambda\right)' = s_1' u^\lambda + \lambda s_1 u^{\lambda-1} u' = u^\lambda \left(s_1' + \lambda s_1 \frac{u'}{u} \right). \tag{2.5}$$

Putting

$$s_2 = s_1' + \lambda s_1 \frac{u'}{u} \tag{2.6}$$

in (2.5) we have $y_1'' = s_2 u^\lambda$, and therefore we obtain $D_0 y_1'' + D_1 y_1' + D_2 y_1 = D_0 s_2 u^\lambda + D_1 s_1 u^\lambda + D_2 s_0 u^\lambda = u^\lambda (D_0 s_2 + D_1 s_1 + D_2 s_0) = 0$.

Since $u \neq 0$ on J then (2.2) follows from the last equality. Now, we suppose that (2.2) is satisfied by some functions $s_0, s_1, s_2 \in C^2(J)$. Then we have

$$D_0 s_2 u^\lambda + D_1 s_1 u^\lambda + D_2 s_0 u^\lambda = 0. \tag{2.7}$$

Putting $y_1 = s_0 u^\lambda$ in (2.7) we obtain $y_1' = s_1 u^\lambda$ and $y_1'' = s_2 u^\lambda$, where the functions s_1 and s_2 are defined by the formulas (2.4) and (2.6), respectively. Hence, $D_0 y_1'' + D_1 y_1' + D_2 y_1 = 0$, and the proof of Lemma 1 is complete. □

Lemma 2: Let $s_0, t_0, u, v \in C^2(J)$ be real-valued functions of x and let $u \neq 0,\ v \neq 0$ on J. Then the functions

$$y_1 = s_0 u^\lambda \quad \text{and} \quad y_2 = t_0 v^\lambda \tag{2.8}$$

are particular solutions of the differential equation:

$$D_0 y'' + D_1 y' + D_2 y = 0, \tag{2.9}$$

if and only if the functions s_1, t_1, s_2, and t_2 are given by the formulas:

$$s_1 = s_0' + \lambda s_0 \frac{u'}{u}, t_1 = t_0' + \lambda t_0 \frac{v'}{v}, s_2 = s_1' + \lambda s_1 \frac{u'}{u}, t_2 = t_1' + \lambda t_1 \frac{v'}{v}, \tag{2.10}$$

and

$$D_0 = \det \begin{pmatrix} s_0 & s_1 \\ t_0 & t_1 \end{pmatrix}, D_1 = \det \begin{pmatrix} s_2 & s_0 \\ t_2 & t_0 \end{pmatrix}, D_2 = \det \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix}. \tag{2.11}$$

Proof: From Lemma 1 it follows that the functions $y_1 = s_0 u^\lambda$ and $y_2 = t_0 v^\lambda$ are particular solutions of the equation (2.9) if and only if

$$D_0 s_2 + D_1 s_1 + D_2 s_0 = 0 \text{ and } D_0 t_2 + D_1 t_1 + D_2 t_0 = 0, \tag{2.12}$$

where the functions s_1, s_2, t_1, and t_2 are defined by the formulas in (2.10). Now, we consider the determinant:

$$W_1 = \det \begin{pmatrix} s_0 & s_1 & s_2 \\ s_0 & s_1 & s_2 \\ t_0 & t_1 & t_2 \end{pmatrix}. \tag{2.13}$$

It is easy to see that $W_1 = 0$, and by Laplace's theorem we obtain

$$s_0 \det \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix} + s_1 \det \begin{pmatrix} s_2 & s_0 \\ t_2 & t_0 \end{pmatrix} + s_2 \det \begin{pmatrix} s_0 & s_2 \\ t_0 & t_1 \end{pmatrix} = 0. \tag{2.14}$$

Denoting $D_0 = \det\begin{pmatrix} s_0 & s_1 \\ t_0 & t_1 \end{pmatrix}, D_1 = \det\begin{pmatrix} s_2 & s_0 \\ t_2 & t_0 \end{pmatrix}, D_2 = \det\begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix}$, in (2.14) we obtain $D_0 s_2 + D_1 s_1 + D_2 s_0 = 0$. In a similar manner we consider the determinant:

$$W_2 = \det\begin{pmatrix} t_0 & t_1 & t_2 \\ t_0 & t_1 & t_2 \\ s_0 & s_1 & s_2 \end{pmatrix}. \tag{2.15}$$

As in the previous case we obtain that $D_0 t_2 + D_1 t_1 + D_2 t_0 = 0$ and the proof of Lemma 2 is complete. □

From Lemma 1 and Lemma 2 we deduce the following lemma:

Lemma 3: Let λ be a non-zero real constant and let $u, v \in C^2(J)$ be a non-zero real-valued functions, linearly independent over R, where $J = (x_1, x_2) \subset R$. Then the general solution of the differential equation:

$$\det\begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix} y'' + \det\begin{pmatrix} g & 1 \\ h & 1 \end{pmatrix} y' + \lambda \det\begin{pmatrix} \frac{u'}{u} & g \\ \frac{v'}{v} & h \end{pmatrix} y = 0, \tag{**}$$

where $g = \frac{u''}{u} - (1-\lambda)\left(\frac{u'}{u}\right)^2$ and $h = \frac{v''}{v} - (1-\lambda)\left(\frac{v'}{v}\right)^2$ is of the form

$$y = C_1 u^\lambda + C_2 v^\lambda, \tag{2.16}$$

where C_1 and C_2 are arbitrary constants.

Proof: Putting $s_0 = t_0 = 1$ in Lemma 1 and Lemma 2, we obtain $s_1 = \lambda\frac{u'}{u}, t_1 = \lambda\frac{v'}{v}$ and $s_2 = s_1' + \lambda s_1 \frac{u'}{u} = \lambda\left(\frac{u''}{u} - (1-\lambda)\left(\frac{u'}{u}\right)^2\right) = \lambda g$, $t_2 = t_1' + \lambda t_1 \frac{v'}{v} = \lambda\left(\frac{v''}{v} - (1-\lambda)\left(\frac{v'}{v}\right)^2\right) = \lambda h$. Hence, we have

$$D_0 = \det\begin{pmatrix} 1 & \lambda\frac{u'}{u} \\ 1 & \lambda\frac{v'}{v} \end{pmatrix} = \lambda \det\begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix} \tag{2.17}$$

$$D_1 = \det\begin{pmatrix} s_2 & 1 \\ t_2 & 1 \end{pmatrix} = \det\begin{pmatrix} \lambda g & 1 \\ \lambda h & 1 \end{pmatrix} = \lambda \det\begin{pmatrix} g & 1 \\ h & 1 \end{pmatrix} \tag{2.18}$$

$$D_2 = \det \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix} = \det \begin{pmatrix} \lambda \frac{u'}{u} & \lambda g \\ \lambda \frac{v'}{v} & \lambda h \end{pmatrix} = \lambda^2 \det \begin{pmatrix} \frac{u'}{u} & g \\ \frac{v'}{v} & h \end{pmatrix}. \tag{2.19}$$

From (2.17)-(2.19) it follows that equation (2.9) reduces to (**), hence by Lemma 2 it follows that the functions $y_1 = u^\lambda$, and $y_2 = v^\lambda$ are particular solutions of (**). It suffices to prove that the functions y_1 and y_2 are linearly independent over R. To this end consider the Wronskian of these functions

$$W(y_1, y_2) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det \begin{pmatrix} u^\lambda & v^\lambda \\ \lambda u^{\lambda-1} u' & \lambda v^{\lambda-1} v' \end{pmatrix} = \lambda u^{\lambda-1} v^{\lambda-1} \det \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}. \tag{2.20}$$

By the assumptions that $u \neq 0$, $v \neq 0$ it follows that $\det \begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \neq 0$ on J and consequently from (2.20) we see that $W(y_1, y_2) \neq 0$ on J. Therefore the function

$$y = C_1 y_1 + C_2 y_2 = C_1 u^\lambda + C_2 v^\lambda$$

is the general solution of the differential equation (**). The proof of Lemma 3 is complete. □

3. THE RESULTS

In this part of our paper we obtain some new classes of second order differential equations which are effectively integrable and with general solutions given in explicit form (Cf. [4]). Namely, we prove of the following theorem.

Theorem 1: Let the functions $a, b \in C^2(J)$, $J = (x_1, x_2) \subset R$ be real-valued and non-zero in x such that $ax \neq \pm bx$ on J, and let a, b be linearly independent over R. Then the function

$$y = C_1(a(x) + b(x))^n + C_2(a(x) - b(x))^n \tag{3.1}$$

where C_1 and C_2 are arbitrary constants and $n \in N$ is a general solution of the differential equation:

$$P_0(x)y'' + P_1(x)y' + nP_2(x)y = 0, \tag{***}$$

where

$$P_0(x) = (a(x)^2 - b(x)^2)(a'(x)b(x) - b'(x)a(x)) = F(x)G(x) \tag{3.2}$$

$$P_1(x) = (a''(x)b(x) - b''(x)a(x))F(x) + 2(n-1)G(x)(a'(x)a(x) - b'(x)b(x)) \tag{3.3}$$

$$P_2(x) = (b''(x)a'(x) - a''(x)b'(x))F(x) - (n-1)\left((a'(x))^2 - (b'(x))^2\right)G(x) \tag{3.4}$$

Proof: Let $u = a(x) - b(x)$, $v = a(x) + b(x)$ and let $y_1 = u^n$ and $y_2 = v^n$, where $n \in N$. Then by Lemma 3 it follows that

$$\det\begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix} = -2\frac{a'(x)b(x) - b'(x)a(x)}{a(x)^2 - b(x)^2} = -2\frac{G(x)}{F(x)} \tag{3.5}$$

$$\det\begin{pmatrix} g & 1 \\ h & 1 \end{pmatrix} = \frac{2(a''(x)b(x) - b''(x)a(x))}{F(x)} + 4(n-1) = \frac{a'(x)a(x) - b'(x)b(x)}{F^2(x)}G(x) \tag{3.6}$$

$$\det\begin{pmatrix} \frac{u'}{u} & g \\ \frac{v'}{v} & h \end{pmatrix} = \frac{2(b''(x)a(x)' - a''(x)b'(x))}{F(x)} - 2(n-1) = \frac{((a'(x))^2 - (b'(x))^2)}{F(x)}G(x). \tag{3.7}$$

Substituting (3.5)-(3.7) in (**) of Lemma 3 we obtain, after some calculation, that (**) reduces to the equation $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$ with the functional coefficients $P_0(x), P_1(x)$, and $P_2(x)$ as given by the formulas (3.2)-(3.4). It remains to prove that the functions $u = a(x) - b(x)$ and $v = a(x) + b(x)$ are linearly independent over R under the assumption that the functions $a(x)$ and $b(x)$ are linearly independent over R. To this end we consider the Wronskian

$$W(u,v) = \det\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} = \det\begin{pmatrix} a(x) - b(x) & a(x) + b(x) \\ a'(x) - b'(x) & a'(x) + b'(x) \end{pmatrix}.$$

From the well-known properties of determinants it follows that

$$W(u,v) = 2\det\begin{pmatrix} a(x) & b(x) \\ a'(x) & b'(x) \end{pmatrix}. \tag{3.8}$$

From (3.8) and by the assumptions of the theorem about the functions a and b it folllows that $W(u,v) \neq 0$ on J and the proof of Theorem 1 is complete. □

Using Theorem 1 we obtain the following:

Theorem 2: The general solution of the differential equation

$$F_0(x)y'' + F_1(x)y' + F_2(x)y = 0 \tag{I}$$

with coefficients $F_0(x), F_1(x)$, and $F_2(x)$ given by the formulas

$$F_0(x) = 2(bx+c)(bx+2c)(x^2+bx+c) \tag{II}$$
$$F_1(x) = \Delta x(bx+c) + 2(n-1)b(bx+2c)(x^2+bx+c)$$

$$F_2(x) = \frac{1}{2}(2\Delta(bx+c) + \Delta(n-1)(bx+2c))$$

where $\Delta = b^2 - 4c$ is the discriminant of the polynomial $f(x) = x^2+bx+c$ and $bx+c \neq 0$ and $bx+2c \neq 0$ on $J = (x_1, x_2) \subset R$ is of the form

$$y = C_1\left(\frac{x+\sqrt{x^2+bx+c}}{2}\right)^n + C_2\left(\frac{x-\sqrt{x^2+bx+c}}{2}\right)^n, \tag{III}$$

where C_1 and C_2 are arbitrary constants and $n \in N$.

Proof: Let $a(x) = \frac{x}{2}$ and $b(x) = \frac{1}{2}\sqrt{x^2+bx+c}$. Then we have $a'(x) = \frac{1}{2}$ and

$$b'(x) = \frac{2x+b}{4\sqrt{x^2+bx+c}}, \text{ so } a''(x) = 0 \text{ and } b''(x) = -\frac{\Delta}{8(x^2+bx+c)\sqrt{x^2+bx+c}}.$$

Using formulas (3.2)-(3.4) from Theorem 1 we obtain

$$P_0(x) = -\frac{(bx+c)(bx+2c)}{32\sqrt{x^2+bx+c}},$$

$$P_1(x) = -\frac{\Delta x(bx+c) + 2(n-1)b(bx+2c)(x^2+bx+c)}{64(x^2+bx+c)\sqrt{x^2+bx+c}},$$

$$P_2(x) = \frac{2\Delta(bx+c) + \Delta(n-1)(bx+2c)}{128(x^2+bx+c)\sqrt{x^2+bx+c}}.$$

From the last formulas it is easy to see that the equation reduces to the equation (I) with the coefficients given by (II). Therefore, it remains to prove that the functions $a(x) = \frac{x}{2}$ and

$b(x) = \frac{1}{2}\sqrt{x^2+bx+c}$ are linearly independent over R, if $bx+2c \neq 0$ on J. Let $W(a,b)$ denotes the Wronskian of the functions a and b. Then we have

$$W(a,b) = \det\begin{pmatrix} a(x) & b(x) \\ a'(x) & b'(x) \end{pmatrix} = \det\begin{pmatrix} \frac{x}{2} & \frac{1}{2}\sqrt{x^2+bx+c} \\ \frac{1}{2} & \frac{2x+b}{4\sqrt{x^2+bx+c}} \end{pmatrix} = -\frac{bx+2c}{8\sqrt{x^2+bx+c}}.$$

From the last equality it follows that $W(a,b) \neq 0$ on J, because $bx+2c \neq 0$ on J.

The proof of Theorem 2 is complete. □

Now, we observe that the result described in Introduction follows immediately from Theorem 2 in the particular case where $b = 0$.

4. FUNCTIONAL RECURRENCES AND GENERALIZED HORADAM-MAHON FORMULA FOR PELL POLYNOMIALS

In [3], Horadam and Mahon consider a matrix method in the investigation of some classes of polynomials such as the Pell polynomials $P_n(x)$. They proved that for every natural number n, we have

$$P_{n-1}(x)P_{n+1}(x) - P_n^2(x) = (-1)^n, \tag{4.1}$$

where $P_n(x)$ is defined by the recurrence formula:

$$P_0(x) = 0,\ P_1(x) = 1,\ P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x). \tag{4.2}$$

In [1], the authors have considered the functional matrix

$$A = A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}.$$

Let $TrA(x) \neq 0$ or $\det A(x) \neq 0$ on $J = (x_1, x_2) \subset R$ and let

$$r = r(x) = TrA(x) = a(x) + d(x), s = s(x) = -\det A(x), \tag{4.3}$$

and

$$u_0 = u_0(x) = r, \quad u_1 = u_1(x) = ru_0(x) + s. \tag{4.4}$$

Let

$$u_n(x) = ru_{n-1}(x) + su_{n-2}(x), \quad \text{for } n \geq 2, \tag{4.5}$$

be a functional recurrence sequence associated with the matrix $A = A(x)$. Then for every natural number $n \geq 2$, we have, in [1],

$$A^n(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}^n = \begin{pmatrix} a(x)u_{n-2}(x) + v_{n-2}(x) & b(x)u_{n-2}(x) \\ c(x)u_{n-2}(x) & d(x)u_{n-2}(x) + v_{n-2}(x) \end{pmatrix}, \tag{4.6}$$

where

$$v_{n-2}(x) = su_{n-3}(x) \text{ for } n \geq 3 \text{ and } u_{-1}(x) = 1 \text{ for } n = 2. \tag{4.7}$$

From (4.6) and (4.7) it follows that the formula (4.8) holds for the recurrence sequence $u_n(x)$ defined by (4.4) and (4.5):

$$u_{n-1}^2(x) - u_n(x)u_{n-2}(x) = (\det A(x))^n \tag{4.8}$$

for every natural number $n \geq 2$. Now, we deduce from (4.8) the Horadam-Mahon formula for Pell polynomials. Indeed, let $a(x) = d(x) = x$ and $b(x) = c(x) = \sqrt{x^2+1}$. Then the matrix $A(x) = P(x)$ has the form

$$P(x) = \begin{pmatrix} x & \sqrt{x^2+1} \\ \sqrt{x^2+1} & x \end{pmatrix}, \tag{4.9}$$

and the recurrence sequence $P_n(x)$ associated with the matrix $P(x)$ satisfies the following conditions:

$$r = TrP(x) = 2x, \;\; s = -\det P(x) = 1, \tag{4.10}$$

and

$$P_n(x) = rP_{n-1}(x) + sP_{n-2}(x) = 2xP_{n-1}(x) + P_{n-2}(x). \tag{4.11}$$

Here, $P_n(x)$ denotes the Pell polynomial. Replacing $u_n(x)$ by $P_n(x)$ in the formula (4.8) we obtain the Horadam-Mahon formula for Pell polynomials.

In the same way we produce more general formulas connected with classes of polynomials $W_n(x; b, c)$ considered in Theorem 2. Namely, we have the following:

Proposition 1: Let $W(x; b, c) = \begin{pmatrix} x & \sqrt{x^2+bx+c} \\ \sqrt{x^2+bx+c} & x \end{pmatrix}$ be a 2×2 functional matrix and let $W_n(x; b, c)$ be the functional recurrence sequence associated with the matrix $W(x; b, c)$ defined by the formulas:

$$\begin{aligned} r &= TrW(x; b, c) = 2x, \;\; s = -\det W(x; b, c) \\ &= -(x^2 - (x^2 + bx + c)) = bx + c \end{aligned}$$

and

$$W_0(x;b,c) = r = 2x, \ W_1(x;b,c) = rW_0(x;b,c) + s = 4x^2 + bx + c$$

and for $n \geq 2$

$$W_n(x;b,c) = rW_{n-1}(x;b,c) + sW_{n-2}(x;b,c) = 2xW_{n-1}(x;b,c) + (bx+c)W_{n-2}(x;b,c).$$

Then for every natural number $n \geq 2$ we have

$$W_{n-1}^2(x;b,c) - W_{n-2}(x;b,c)W_n(x;b,c) = (\det W(x;b,c))^n = (-1)^n(bx+c)^n.$$

Proof: In the first step, by inductive manner as in [1], (pages 116-117), we obtain an analog of formula (4.6) for the powers of the matrix $W(x;b,c)$, using the recurrence sequence $W_n(x;b,c)$. The final step relies on applying Cauchy's theorem on product of determinants. $\square$

In a similar way as in [1], (pages 118-119) we obtain the following:

Proposition 2: Let k be a non-zero constant and let $a = a(x)$ and $b = b(x)$ be given functions of the variable x. Then for every natural number n we have

$$\begin{pmatrix} a(x) & b(x) \\ kb(x) & a(x) \end{pmatrix}^n = \begin{pmatrix} R_n(x) & S_n(x) \\ kS_n(x) & R_n(x) \end{pmatrix},$$

where

$$R_n(x) = \frac{1}{2}\left(\left(a(x) + b(x)\sqrt{k}\right)^n + \left(a(x) - b(x)\sqrt{k}\right)^n\right)$$

and

$$S_n(x) = \frac{1}{2\sqrt{k}}\left(\left(a(x) + b(x)\sqrt{k}\right)^n - \left(a(x) - b(x)\sqrt{k}\right)^n\right). \quad \square$$

Putting $k = 1$ in the last equalities we obtain an explicit connection between the functions $u(x) = a(x) - b(x)$ and $v(x) = a(x) + b(x)$ considered in Theorem 2 with powers of the functional matrices and the corresponding functional recurrences.

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A THREE-VARIABLE IDENTITY INVOLVING CUBES OF FIBONACCI NUMBERS

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1. INTRODUCTION

The identities

$$F_{n+1}^2 + F_n^2 = F_{2n+1} \tag{1.1}$$

and

$$F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n} \tag{1.2}$$

are special cases of identity (5) of Torretto and Fuchs [7]. Interestingly, (1.2) is the only identity involving cubes of Fibonacci numbers that appears in Dickson's *History of the Theory of Numbers* [1, p. 395], and Dickson attributes it to Lucas.

In [6], the following generalizations of (1.1) and (1.2), together with their Lucas counterparts, were given.

$$F_{n+k+1}^2 + F_{n-k}^2 = F_{2k+1}F_{2n+1}; \tag{1.3}$$

$$F_{3k+1}F_{n+k+1}^3 + F_{3k+2}F_{n+k}^3 - F_{n-2k-1}^3 = F_{3k+1}F_{3k+2}F_{3n}. \tag{1.4}$$

In fact, as was proved by Howard [5], (1.3) is equivalent to

$$F_n^2 + (-1)^{n+k+1}F_k^2 = F_{n-k}F_{n+k}, \tag{1.5}$$

occurring as I_{19} on page 59 in [4]. In (1.5), replacing n by $n+k$, and k by n yields

$$F_{n+k}^2 + (-1)^{k+1}F_n^2 = F_kF_{2n+k}, \tag{1.6}$$

equivalent to (1.5), and which we require in the sequel.

Recently, we were made aware of the identity

$$F_{n+2}^3 - 3F_n^3 + F_{n-2}^3 = 3F_{3n} \tag{1.7}$$

due to Ginsburg [3], and this prompted us to search for a more general identity that yields (1.2), (1.4), and (1.7) as special cases. This identity is stated in the next section, and our proof of it relies on a powerful method given recently by Dresel [2]. For instance, in the terminology of Dresel, (1.1) is *homogeneous* of degree 2 in the variable n. As such, to prove it we need only verify its validity for 3 distinct values of n.

Quite often, after discovering a new Fibonacci identity, we expend energy trying to discover its Lucas counterpart. Dresel's *duality theorem* provides us with a way of achieving this quickly. Indeed, the duality theorem produces a *dual* identity for *any* homogeneous Fibonacci-Lucas (FL) identity.

The Duality Theorem (Dresel): Given a homogeneous FL-identity in the variable n, we can arrive at a new dual identity with respect to the variable n by making the following changes throughout:

(i) when j is odd, F_{jn+k} is replaced by $L_{jn+k}/\sqrt{5}$,
(ii) when j is odd, L_{jn+k} is replaced by $\sqrt{5}F_{jn+k}$,
(iii) when j is odd, $(-1)^{jn}$ is replaced by $-(-1)^{jn}$.

The justification for each step in the theorem is easily seen if we refer to the Binet forms. For example, the dual of (1.1) is $L_{n+1}^2 + L_n^2 = 5F_{2n+1}$. We give further illustrations after the proof of our main result, when we employ the duality theorem to produce seven additional identities.

2. THE MAIN RESULT

We make use of the following identities.

$$F_{-n} = (-1)^{n+1}F_n, \tag{2.1}$$

$$F_{n+k} + F_{n-k} = L_n F_k, \quad k \text{ odd}, \tag{2.2}$$

$$F_{n+k} - F_{n-k} = L_n F_k, \quad k \text{ even}, \tag{2.3}$$

$$F_{2n} = F_n L_n, \tag{2.4}$$

$$(-1)^{k+1}F_k F_{n+k}^3 - F_k F_{n-k}^3 + F_{2k}F_n^3 = (-1)^{k+1}F_k^2 F_{2k} F_{3n}. \tag{2.5}$$

Identities (2.1) and (2.4) are well known, while identities (2.2) and (2.3) occur as I_{22} and I_{24}, respectively, on page 59 in [4]. Identity (2.5), which appears as (5.2) in [2], can be expressed more simply if we factor out F_k. However, in its present form, its relationship with our main result is more transparent. Our main result follows.

Theorem: Let k, m, and n be any integers. Then

$$F_m F_{n+k}^3 + (-1)^{k+m+1}F_k F_{n+m}^3 + (-1)^{k+m}F_{k-m}F_n^3 = F_{k-m}F_k F_m F_{3n+k+m}. \tag{2.6}$$

Proof: Since (2.6) is homogeneous of degree 3 in the variable n, we need only verify its validity for four distinct values of n. If $k = m$, or if one of k or m is zero, then (2.6) follows immediately. Furthermore, if $k + m = 0$, then (2.6) follows from (2.5). So we may assume that $km(k-m)(k+m) \neq 0$. But then $0, -k, -m$, and $-k-m$ are distinct, and so we need only verify (2.6) for these four values of n. We perform the verifications for $n = -k$ and $n = -k - m$, and leave the remaining verifications to the reader.

Using (2.1), we find that $F^3_{-k+m} = (-1)^{k-m+1}F^3_{k-m}$, and $F^3_{-k} = (-1)^{k+1}F^3_k$. Then, for $n = -k$,

$$\begin{aligned}
LHS &= (-1)^{k+m+1}F_kF^3_{-k+m} + (-1)^{k+m}F_{k-m}F^3_{-k} \\
&= F_kF^3_{k-m} + (-1)^{m+1}F_{k-m}F^3_k \\
&= F_{k-m}F_k\left[F^2_{k-m} + (-1)^{m+1}F^2_k\right] \\
&= F_{k-m}F_k\left[F^2_{k-m} + (-1)^{-m+1}F^2_k\right] \\
&= F_{k-m}F_kF_{-m}F_{2k-m} \qquad \text{(using (1.6))} \\
&= F_{k-m}F_kF_{-m}F_{-(-2k+m)} \\
&= F_{k-m}F_k(-1)^{m+1}F_m(-1)^{-2k+m+1}F_{-2k+m} \qquad \text{(using (2.1))} \\
&= F_{k-m}F_kF_mF_{-2k+m} \\
&= RHS.
\end{aligned}$$

For $n = -k - m$ we have

$$\begin{aligned}
LHS &= F_mF^3_{-m} + (-1)^{k+m+1}F_kF^3_{-k} + (-1)^{k+m}F_{k-m}F^3_{-k-m} \\
&= (-1)^{m+1}F^4_m + (-1)^mF^4_k - F_{k-m}F^3_{k+m} \qquad \text{(using (2.1))} \\
&= (-1)^m\left[F^4_k - F^4_m\right] - F_{k-m}F^3_{k+m} \\
&= (-1)^m\left[F^2_k + (-1)^{k+m+1}F^2_m\right]\left[F^2_k + (-1)^{k+m}F^2_m\right] - F_{k-m}F^3_{k+m} \\
&= (-1)^m\left[F^2_{m+(k-m)} + (-1)^{k-m+1}F^2_m\right]\left[F^2_k + (-1)^{k+m}F^2_m\right] - F_{k-m}F^3_{k+m} \\
&= (-1)^mF_{k-m}F_{k+m}\left[F^2_k + (-1)^{k+m}F^2_m\right] - F_{k-m}F^3_{k+m} \qquad \text{(using (1.6))} \\
&= F_{k-m}F_{k+m}\left[(-1)^mF^2_k - \left[F^2_{m+k} + (-1)^{k+1}F^2_m\right]\right] \\
&= F_{k-m}F_{k+m}\left[(-1)^mF^2_k - F_kF_{2m+k}\right] \qquad \text{(using (1.6))} \\
&= -F_{k-m}F_{k+m}F_k\left[F_{(m+k)+m} + (-1)^{m+1}F_{(m+k)-m}\right] \\
&= -F_{k-m}F_{k+m}F_kL_{k+m}F_m \qquad \text{(using (2.2) and (2.3))} \\
&= -F_{k-m}F_kF_mF_{2k+2m} \qquad \text{(using (2.4))} \\
&= RHS, \text{ using (2.1).}
\end{aligned}$$

This completes the proof of the Theorem. □

Now, since (2.6) is homogeneous of degree 3 in the variable n, its dual identity, with respect to n is

$$F_m L_{n+k}^3 + (-1)^{k+m+1} F_k L_{n+m}^3 + (-1)^{k+m} F_{k-m} L_n^3 = 5F_{k-m}F_kF_mL_{3n+k+m}. \tag{2.7}$$

Before proceeding we note that, since $(-1)^k = (\alpha\beta)^k, (-1)^k F_k$ has degree 3 with respect to the variable k. Hence (2.6) and (2.7) are each homogeneous of degree 3 in k, and their duals with respect to k are, respectively,

$$F_m L_{n+k}^3 + 5(-1)^{k+m} L_k F_{n+m}^3 + 5(-1)^{k+m+1} L_{k-m} F_n^3 = L_{k-m}L_kF_mL_{3n+k+m}, \tag{2.8}$$

and

$$25F_m F_{n+k}^3 + (-1)^{k+m} L_k L_{n+m}^3 + (-1)^{k+m+1} L_{k-m} L_n^3 = 5L_{k-m}L_kF_mF_{3n+k+m}. \tag{2.9}$$

Finally, since $F_m = (-1)^{2m}F_m, F_{k-m} = (-1)^{m-k+1}F_{m-k}$, and $L_{k-m} = (-1)^{m-k}L_{m-k}$, we see that (2.6)-(2.9) are each homogeneous of degree 5 in m. Accordingly, we find that their duals in the variable m are, respectively,

$$5L_m F_{n+k}^3 + (-1)^{k+m} F_k L_{n+m}^3 + 5(-1)^{k+m+1} L_{k-m} F_n^3 = L_{k-m}F_kL_mL_{3n+k+m}, \tag{2.10}$$

$$L_m L_{n+k}^3 + 25(-1)^{k+m} F_k F_{n+m}^3 + (-1)^{k+m+1} L_{k-m} L_n^3 = 5L_{k-m}F_kL_mF_{3n+k+m}, \tag{2.11}$$

$$L_m L_{n+k}^3 + (-1)^{k+m+1} L_k L_{n+m}^3 + 25(-1)^{k+m} F_{k-m} F_n^3 = 5F_{k-m}L_kL_mF_{3n+k+m}, \tag{2.12}$$

$$25L_m F_{n+k}^3 + 25(-1)^{k+m+1} L_k F_{n+m}^3 + 5(-1)^{k+m} F_{k-m} L_n^3 = 5F_{k-m}L_kL_mL_{3n+k+m}. \tag{2.13}$$

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AN ELEMENTARY PROOF OF JACOBI'S FOUR-SQUARE THEOREM

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1. INTRODUCTION

Recall that $\mathbb{P} := \{1, 2, 3, \ldots\}, \mathbb{N} := \mathbb{P} \cup \{0\}$ and $\mathbb{Z} := \{0 \pm 1, \pm 2, \ldots\}$. Then, for each $n \in \mathbb{N}$,

$$r_4(n) := |\{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 | n = m_1^2 + m_2^2 + m_3^2 + m_4^2\}|.$$

For each $n \in \mathbb{P}, \sigma(n)$ denotes the sum of all positive divisors of n, $b(n)$ denotes the exponent of the largest power of 2 dividing n, and then $Od(n) := n2^{-b(n)}$. (Quite properly, $b(n)$ (or $2^{b(n)}$) is called the binary part of n and $Od(n)$ is called the odd part of n.) In this note we give a simple proof of the following elegant result first stated and proved by Jacobi [1, p. 285].

Theorem 1: *For each* $n \in \mathbb{P}$,

$$r_4(n) = 8(2 + (-1)^n)\sigma(Od(n)).$$

(Of course, $r_4(0) = 1$.*)*

Our proof depends on several immediate consequences of the celebrated Gauss-Jacobi triple-product identity

$$\prod_{n=1}^{\infty}(1 - x^{2n})(1 + tx^{2n-1})(1 + t^{-1}x^{2n-1}) = \sum_{-\infty}^{\infty} x^{n^2} t^n, \tag{1}$$

which is valid for each pair of complex numbers t, x such that $t \neq 0$ and $|x| < 1$. For a proof see [2, pp. 282-283].

2. PROOF OF THEOREM 1

We begin with Jacobi's triangular-number identity [2, p. 285]

$$2\prod_{n=1}^{\infty}(1 - x^n)^3 = \sum_{-\infty}^{\infty}(-1)^k(2k+1)x^{k(k+1)/2}, \tag{2}$$

valid for each x such that $|x| < 1$. In (2) we first let $x \to x^8$, and then multiply the resulting identity by x to get

$$2x\prod_{1}^{\infty}(1 - x^{8n})^3 = \sum_{-\infty}^{\infty}(-1)^k(2k+1)x^{(2k+1)^2}. \tag{3}$$

Next, we square both sides of (3), and appeal to the elementary identity

$$u^2 + v^2 = \frac{1}{2}\{(u+v)^2 + (u-v)^2\}$$

to get

$$4x^2 \prod_1^\infty (1-x^{8n})^6 = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (-1)^{j+k}(2j+1)(2k+1)x^{(2j+1)^2+(2k+1)^2}$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (-1)^{j+k}(2j+1)(2k+1)x^{2[(j+k+1)^2+(j-k)^2]}.$$

Now, with

$$E := \{(r,s) \in \mathbb{Z}^2 | r \text{ and } s \text{ have the same parity}\},$$

it follows easily that the function $F : \mathbb{Z}^2 \to \mathbb{Z}^2$, defined by

$$F(j,k) := (j+k, j-k), \text{ for each } (j,k) \in \mathbb{Z}^2,$$

is one-to-one from $\mathbb{Z}^2$ onto E. Hence, in the foregoing identity let $r = j+k, s = j-k$, so that $j = (1/2)(r+s),\ k = (1/2)(r-s)$, and let $x \to x^{1/2}$ to get

$$4x \prod_1^\infty (1-x^{4n})^6 = \sum_{(r,s)\in E} (-1)^r (r+1+s)(r+1-s)x^{(r+1)^2+s^2}$$

$$= \sum_{(r,s)\in E} (-1)^r \{(r+1)^2 - s^2\} x^{(r+1)^2+s^2}$$

$$= \sum_{-\infty}^{\infty} (2m+1)^2 x^{(2m+1)^2} \sum_{-\infty}^{\infty} x^{(2n)^2} - \sum_{-\infty}^{\infty} x^{(2m+1)^2} \sum_{-\infty}^{\infty} (2n)^2 x^{(2n)^2}$$

$$- \sum_{-\infty}^{\infty} (2m+2)^2 x^{(2m+2)^2} \sum_{-\infty}^{\infty} x^{(2n+1)^2} + \sum_{-\infty}^{\infty} x^{(2m+2)^2} \sum_{-\infty}^{\infty} (2n+1)^2 x^{(2n+1)^2}$$

$$= 2\left\{ \sum_{-\infty}^{\infty} (2m+1)^2 x^{(2m+1)^2} \sum_{-\infty}^{\infty} x^{(2n)^2} - \sum_{-\infty}^{\infty} x^{(2m+1)^2} \sum_{-\infty}^{\infty} (2n)^2 x^{(2n)^2} \right\},$$

since $m \in \mathbb{Z} \Longleftrightarrow m+1 \in \mathbb{Z}$. We cancel a factor of 2 and put

$$f(x) := \sum_{-\infty}^{\infty} x^{(2m+1)^2}, \; g(x) := \sum_{-\infty}^{\infty} x^{(2n)^2}$$

to get

$$2x \prod_{1}^{\infty} (1 - x^{4n})^6 = g(x)^2 \frac{\theta_x f(x) \cdot g(x) - f(x) \cdot \theta_x g(x)}{g(x)^2} \tag{4}$$

where $\theta_x := xD_x, D_x$ denoting differentiation with respect to x. But, with the help of (1), we get

$$f(x) = 2x \prod_{1}^{\infty} (1 - x^{8n})(1 + x^{8n})^2,$$

$$g(x) = \prod_{1}^{\infty} (1 - x^{8n})(1 + x^{8n-4})^2,$$

so that

$$\frac{f(x)}{g(x)} = 2x \prod_{1}^{\infty} \frac{(1 + x^{8n})^2}{(1 + x^{8n-4})^2}.$$

Hence,

$$\theta_x \{f(x)/g(x)\} = \frac{f(x)}{g(x)} \left\{ 1 + 16 \sum_{k=1}^{\infty} \frac{kx^{8k}}{1 + x^{8k}} - 8 \sum_{k=1}^{\infty} \frac{(2k-1)x^{8k-4}}{1 + x^{8k-4}} \right\}.$$

Now,

$$g(x)^2 \frac{f(x)}{g(x)} = f(x)g(x) = 2x \prod_{1}^{\infty} (1 - x^{8n})^2 (1 + x^{4n})^2.$$

With the help of Euler's identity [2, p. 277]

$$\prod_1^{\infty}(1+x^n)(1-x^{2n-1})=1,$$

which is valid for each complex number x such that $|x|<1$, we substitute the foregoing evaluations into (4), cancel $2x$, let $x \longrightarrow x^{1/4}$ and divide both sides of the resulting identity by $\prod(1-x^{2n})^2(1+x^n)^2$ to get

$$\prod_1^{\infty}\frac{(1-x^{2n})^6(1-x^{2n-1})^6}{(1-x^{2n})^2(1+x^n)^2}=\prod_1^{\infty}(1-x^{2n})^4(1-x^{2n-1})^8$$

$$=1+16\sum_{k=1}^{\infty}\frac{kx^{2k}}{1+x^{2k}}-8\sum_{k=1}^{\infty}\frac{(2k-1)x^{2k-1}}{1+x^{2k-1}}. \tag{5}$$

We now digress momentarily to make a couple of key observations. First, we let $t=1$ in (1), and observe that the fourth power of the right-hand side of the resulting identity generates the sequence $r_4(n)$, $n \in \mathbb{N}$. In other words,

$$\prod_1^{\infty}(1-x^{2n})^4(1+x^{2n-1})^8=\left\{\sum_{-\infty}^{\infty}x^{n^2}\right\}^4=\sum_{n=0}^{\infty}r_4(n)x^n.$$

Next, we observe that the composite function $\sigma \circ Od$ arises quite naturally in the expansion:

$$\sum_{k=1}^{\infty}\frac{(2k-1)x^{2k-1}}{1-x^{2k-1}}=\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}(2k-1)x^{2k-1}\cdot x^{j(2k-1)}$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(2k-1)x^{j(2k-1)}$$

$$=\sum_{n=1}^{\infty}x^n\sum_{\substack{d|n\\ d|\mathrm{odd}}}d$$

$$=\sum_{n=1}^{\infty}\sigma(Od(n))x^n.$$

Returning to the proof of our theorem, we appeal to [2, p. 312], and in (5) let $x \to -x$ to get

$$\sum_{n=0}^{\infty} r_4(n)x^n = \prod_{1}^{\infty}(1-x^{2n})^4(1+x^{2n-1})^8$$

$$= 1 + 16\sum_{k=1}^{\infty}\frac{kx^{2k}}{1+x^{2k}} + 8\sum_{k=1}^{\infty}\frac{(2k-1)x^{2k-1}}{1-x^{2k-1}}$$

$$= 1 + 16\sum_{n=1}^{\infty}\frac{(2n-1)x^{4n-2}}{1-x^{4n-2}} + 8\sum_{k=1}^{\infty}\frac{(2k-1)x^{2k-1}}{1-x^{2k-1}}$$

$$= 1 + 16\sum_{n=1}^{\infty}\sigma(Od(n))x^{2n} + 8\sum_{n=1}^{\infty}\sigma(Od(n))x^{n}$$

$$= 1 + 16\sum_{n=1}^{\infty}\sigma(Od(2n))x^{2n} + 8\sum_{n=1}^{\infty}\sigma(Od(2n))x^{2n} + 8\sum_{n=1}^{\infty}\sigma(2n-1)x^{2n-1}$$

$$= 1 + 24\sum_{n=1}^{\infty}\sigma(Od(2n))x^{2n} + 8\sum_{n=1}^{\infty}\sigma(2n-1)x^{2n-1}.$$

Here, we've made use of the obvious facts: $Od(2n) = Od(n)$ and $Od(2n-1) = 2n-1$, for each $n \in \mathbb{P}$. Finally, we equate coefficients of like powers of x to get

$$r_4(0) = 1$$

and for each $n \in \mathbb{P}$,

$$r_4(2n) = 24\sigma(Od(2n)),\ r_4(2n-1) = 8\sigma(2n-1).$$

This completes the proof of theorem 1.

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✠✠✠

RISES, LEVELS, DROPS AND "+" SIGNS IN COMPOSITIONS: EXTENSIONS OF A PAPER BY ALLADI AND HOGGATT

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1. INTRODUCTION

A composition of n consists of an ordered sequence of positive integers whose sum is n. A palindromic composition (or palindrome) is one for which the sequence reads the same forwards and backwards. We derive results for the number of "+" signs, summands, levels (a summand followed by itself), rises (a summand followed by a larger on), and drops (a summand followed by a smaller one) for both compositions and palindromes of n. This generalizes a paper by Alladi and Hoggatt [1], where summands were restricted to be only 1s and 2s.

Some results by Alladi and Hoggatt can be generalized to compositions with summands of all possible sizes, but the connections with the Fibonacci sequence are specific to compositions with 1s and 2s. However, we will establish a connection to the Jacobsthal sequence [8], which arises in many contexts: tilings of a $3 \times n$ board [7], meets between subsets of a lattice [3], and alternating sign matrices [4], to name just a few. Alladi and Hoggatt also derived results about the number of times a particular summand occurs in all compositions and palindromes of n, respectively. Generalizations of these results are given in [2].

In Section 2 we introduce the notation that will be used, methods to generate compositions and palindromes, as well as some easy results on the total numbers of compositions and palindromes, the number of "+" signs and the numbers of summands for both compositions and palindromes. We also derive the number of palindromes into i parts, which form an "enlarged" Pascal's triangle.

Section 3 contains the harder and more interesting results on the numbers of levels, rises and drops for compositions, as well as interesting connections between these quantities. In Section 4 we derive the corresponding results for palindromes. Unlike the case of compositions, we now have to distinguish between odd and even n. The final section contains generating functions for all quantities of interest.

2. NOTATION AND GENERAL RESULTS

We start with some notation and general results. Let

$$
\begin{aligned}
C_n, P_n &= \text{the number of compositions and palindromes of } n\text{, respectively}\\
C_n^+, P_n^+ &= \text{the number of “+” signs in all compositions and palindromes of } n\text{, respectively}\\
C_n^S, P_n^S &= \text{the number of summands in all compositions and palindromes of } n\text{, respectively}\\
C_n(x) &= \text{the number of compositions of } n \text{ ending in } x\\
C_n(x,y) &= \text{the number of compositions of } n \text{ ending in } x+y\\
r_n, l_n, d_n &= \text{the number of rises, levels, and drops in all compositions of } n\text{, respectively}\\
\tilde{r}_n, \tilde{l}_n, \tilde{d}_n &= \text{the number of rises, levels, and drops in all palindromes of } n\text{, respectively.}
\end{aligned}
$$

We now look at ways of creating compositions and palindromes of n. Compositions of $n+1$ can be created from those of n by either appending '+1' to the right end of the composition or by increasing the rightmost summand by 1. This process is reversible and creates no duplicates, hence creates all compositions of $n+1$. To create all palindromes of n, combine a middle summand of size m (with the same parity as n, $0 \le m \le n$) with a composition of $\frac{n-m}{2}$ on the left and its mirror image on the right. Again, the process is reversible and creates no duplicates (see Lemma 2 of [2]). We will refer to these two methods as the *Composition Creation Method (CCM)* and the *Palindrome Creation Method (PCM)*, respectively. Figure 1 illustrates the PCM

$$
\begin{array}{c}
6\\
1\ \ 4\ \ 1\\
\\
2\ \ 2\ \ 2\\
1\ \ 1\ \ 2\ \ 1\ \ 1\\
\\
3\ \ 3\\
1\ \ 2\ \ 2\ \ 1\\
2\ \ 1\ \ 1\ \ 2\\
1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1
\end{array}
\qquad\qquad
\begin{array}{c}
7\\
1\ \ 5\ \ 1\\
\\
2\ \ 3\ \ 2\\
1\ \ 1\ \ 3\ \ 1\ \ 1\\
\\
3\ \ 1\ \ 3\\
1\ \ 2\ \ 1\ \ 2\ \ 1\\
2\ \ 1\ \ 1\ \ 1\ \ 2\\
1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1
\end{array}
$$

Figure 1: Creating palindromes of $n = 6$ and $n = 7$

We can now state some basic results for the number of compositions, palindromes, "+" signs and summands.

Theorem 1:

1. $C_n = 2^{n-1}$ *for* $n \ge 1, C_0 := 1$.
2. $P_{2k} = P_{2k+1} = 2^k$ *for* $k \ge 1$.

3. $C_n^+ = (n-1)2^{n-2}$ *for* $n \geq 1, C_0^+ := 0$.
4. $P_{2k+1}^+ = k2^k$ *for* $k \geq 0, P_{2k}^+ = (2k-1)2^{k-1}$ *for* $k \geq 1, P_0^+ := 0$.
5. $C_n^S = (n+1)2^{n-2}$ *for* $n \geq 1, C_0^S := 1$.
6. $P_{2k+1}^S = (k+1)2^k$ *for* $k \geq 0, P_{2k}^S = (2k+1)2^{k-1}$ *for* $k \geq 1, P_0^S := 1$.

Proof: 1. The number of compositions of n into i parts is $\binom{n-1}{i-1}$ (see Section 1.4 in [5]). Thus, for $n \geq 1$,

$$C_n = \sum_{i=1}^{n} \binom{n-1}{i-1} = 2^{n-1}.$$

2. Using the PCM as illustrated in Figure 1, it is easy to see that

$$P_{2k} = P_{2k+1} = \sum_{i=0}^{k} C_i = 1 + (1 + 2 + \cdots + 2^{k-1}) = 2^k.$$

3. A composition of n with i summands has $i-1$ "+" signs. Thus, the number of "+" signs can be obtained by summing according to the number of summands in the composition:

$$C_n^+ = \sum_{i=1}^{n} (i-1) \cdot \binom{n-1}{i-1} = \sum_{i=2}^{n} (i-1) \cdot \frac{(n-1)!}{(i-1)!(n-i)!}$$

$$= (n-1) \sum_{i=2}^{n} \binom{n-2}{i-2} = (n-1) \cdot 2^{n-2}. \tag{1}$$

4. The number of "+" signs in a palindrome of $2k+1$ is twice the number of "+" signs in the associated composition, plus two "+" signs connecting the two compositions with the middle summand.

$$P_{2k+1}^+ = \sum_{i=1}^{k} (2C_i + 2C_i^+) = \sum_{i=1}^{k} (2 \cdot 2^{i-1} + 2(i-1)2^{i-2})$$

$$= \sum_{i=1}^{k} (i+1)2^{i-1} = k2^k,$$

where the last equality is easily proved by induction. For palindromes of $2k$, the same reasoning applies, except that there is only one "+" sign when a composition of k is combined with its mirror image. Thus,

$$P_{2k}^{+} = \sum_{i=1}^{k-1}(2C_i + 2C_i^{+}) + (C_k + 2C_k^{+}) = \sum_{i=1}^{k}(2C_i + 2C_i^{+}) - C_k$$

$$= k2^k - 2^{k-1} = (2k-1)2^{k-1}.$$

5. & 6. The number of summands in a composition or palindrome is one more than the number of "+" signs, and the results follows by substituting the previous results into $C_n^S = C_n^+ + C_n$ and $P_n^S = P_n^+ + P_n$. □

Part 4 of Theorem 4 could have been proved similarly to part 1, using the number of palindromes of n into i parts, denoted by P_n^i. These numbers exhibit an interesting pattern which will be proved in Lemma 2.

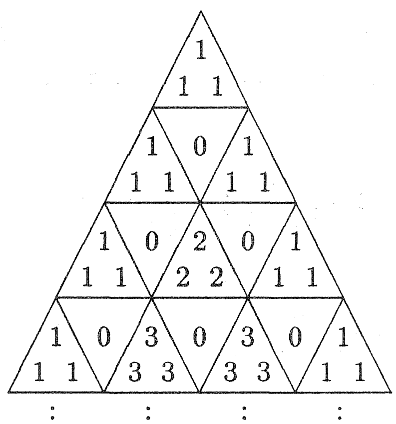

Figure 2: Palindromes with i parts

Lemma 2: $P_{2k-1}^{2j} = 0$ and $P_{2k-1}^{2j-1} = P_{2k}^{2j-1} = P_{2k}^{2j} = \binom{k-1}{j-1}$ for $j = 1, \ldots, k, k \geq 1$.

Proof: The first equality follows from the fact that a palindrome of an odd number n has to have an odd number of summands. For the other cases we will interpret the palindrome as a tiling where cuts are placed to create the parts. Since we want to create a palindrome, we look only at one of the two halves of the tiling and finish the other half as the mirror image. If $n = 2k - 1$, to create $2j - 1$ parts we select $\frac{(2j-1)-1}{2} = j - 1$ positions out of the possible $\frac{(2k-1)-1}{2} = k - 1$ cutting positions. If $n = 2k$, then we need to distinguish between palindromes having an odd or even number of summands. If the number of summands is $2j-1$, then there cannot be a cut directly in the middle, so only $\frac{2k-2}{2} = k-1$ cutting positions are available, out of which we select $\frac{(2j-1)-1}{2} = j-1$. If the number of summands is $2j$, then the number of palindromes corresponds to the number of compositions of k, with half the number of summands ($= j$), which equals $\binom{k-1}{j-1}$. □

3. LEVELS, RISES AND DROPS FOR COMPOSITIONS

We now turn our attention to the harder and more interesting results for the numbers of levels, rises and drops in all compositions of n.

Theorem 3:

1. $l_n = \frac{1}{36}((3n+1)2^n + 8(-1)^n)$ *for* $n \geq 1$ *and* $l_0 = 0$.
2. $r_n = d_n = \frac{1}{9}((3n-5)2^{n-2} - (-1)^n)$ *for* $n \geq 3$ *and* $r_0 = r_1 = r_2 = 0$.

Proof: 1. In order to obtain a recursion for the number of levels in the compositions of n, we look at the right end of the compositions, as this is where the CCM creates changes. Applying the CCM, the levels in the compositions of $n+1$ are twice those in the compositions of n, modified by any changes in the number of levels that occur at the right end. If a 1 is added, an additional level is created in all the compositions of n that end in 1, i.e., a total of $C_n(1) = \frac{1}{2}C_{n-1}$ additional levels. If the rightmost summand is increased by 1, one level is lost if the composition of n ends in $x+x$, and one additional level is created if the composition of n ends in $x+(x-1)$. Thus,

$$l_{2k+1} = 2l_{2k} + \frac{1}{2}C_{2k} - \sum_{x=1}^{k} C_{2k}(x,x) + \sum_{x=2}^{k} C_{2k}(x,x-1)$$

$$= 2l_{2k} + 2^{2k-2} - \sum_{x=1}^{k} C_{2k-2x} + \sum_{x=2}^{k} C_{2k-(2x-1)}$$

$$= 2l_{2k} + 2^{2k-2} - (2^{2k-3} + 2^{2k-5} + \cdots + 2^1 + 1) + (2^{2k-4} + \cdots + 1)$$
$$= 2l_{2k} + (2^{2k-2} - 2^{2k-3} + 2^{2k-4} - \cdots - 2 + 1) - 1$$

$$= 2l_{2k} + \frac{2^{2k-1} - 2}{3},$$

while

$$l_{2k} = 2l_{2k-1} + \frac{1}{2}C_{2k-1} - \sum_{x=1}^{k-1} C_{2k-1}(x,x) + \sum_{x=2}^{k} C_{2k-1}(x,x-1)$$

$$= 2l_{2k-1} + 2^{2k-3} - (2^{2k-4} + 2^{2k-6} + \cdots + 2^2 + 1) + (2^{2k-5} + \cdots + 2^1 + 1)$$
$$= 2l_{2k-1} + (2^{2k-3} - 2^{2k-4} + 2^{2k-5} - \cdots + 2 - 1) + 1$$

$$= 2l_{2k-1} + \frac{2^{2k-2}+2}{3}.$$

Altogether, for all $n \geq 2$,

$$l_n = 2l_{n-1} + \frac{2^{n-2} + 2(-1)^n}{3}. \tag{2}$$

The homogeneous and particular solutions, $l_n^{(h)}$ and $l_n^{(p)}$, respectively, are given by

$$l_n^{(h)} = c \cdot 2^n \text{ and } l_n^{(p)} = A \cdot (-1)^n + B \cdot n2^n.$$

Substituting $l_n^{(p)}$ into Eq. (2) and comparing the coefficients for powers of 2 and -1, respectively, yields $A = \frac{2}{9}$ and $B = \frac{1}{12}$. Substituting $l_n = l_n^{(h)} + l_n^{(p)} = c \cdot 2^n + \frac{2}{9}(-1)^n + \frac{1}{12} \cdot n \cdot 2^n$ into Eq. (2) and using the initial condition $l_2 = 1$ yields $c = \frac{1}{36}$, giving the equation for l_n for $n \geq 3$. (Actually, the formula also holds for $n \geq 1$).

2. It is easy to see that $r_n = d_n$, since for each nonpalindromic composition there is one which has the summands in reverse order. For palindromic compositions, the symmetry matches each rise in the first half with a drop in the second half and vice versa. Since $C_n^+ = r_n + l_n + d_n$, it follows that $r_n = \frac{C_n^+ - l_n}{2}$. □

Table 1 shows values for the quantities of interest. In Theorem 4 we will establish the patterns suggested in this table.

n	1	2	3	4	5	6	7	8	9	10	11	12
C_n^+	0	1	4	12	32	80	192	448	1024	2304	5120	11264
l_n	0	1	2	6	14	34	78	178	398	882	1934	4210
$r_n = d_n$	0	0	1	3	9	23	57	135	313	711	1593	3527

Table 1: Values for C_n^+, l_n and r_n

Theorem 4:

1. $r_{n+1} = r_n + l_n$ *and more generally,* $r_n = \sum_{i=2}^{n-1} l_i$ *for* $n \geq 3$.
2. $C_n^+ = r_n + r_{n+1}$.
3. $C_n^+ = 4 \cdot (l_{n-1} + l_{n-2}) = 4 \cdot (r_n - r_{n-2})$.
4. $l_n - r_n = a_{n-1}$, *where* a_n *is the* n^{th} *term of the Jacobsthal sequence.*

Proof: 1. The first equation follows by substituting the formulas of Theorem 3 for r_n and l_n and collecting terms. The general formula follows by induction.
2. This follows from part 1, since $C_n^+ = r_n + l_n + d_n$ and $r_n = d_n$.
3. The first equality follows by substituting the formula in Theorem 3 for l_{n-1} and l_{n-2}. The second equality follows from part 1.
4. The sequence of values for $f_n = l_n - r_n$ is given by 1,1,3,5,11,21,43, ... This sequence satisfies several recurrence relations, for example $f_n = 2f_{n-1} + (-1)^n$ or $f_n = 2^n - f_{n-1}$, both of which can be verified by substituting the formulas given in Theorem 3. These recursions define the Jacobsthal sequence (A001045 in [8]), and comparison of the intitial values shows that $f_n = a_{n-1}$. □

4. LEVELS, RISES AND DROPS FOR PALINDROMES

We now look at the numbers of levels, rises and drops for palindromes. Unlike the case for compositions, there is no single formula for the number of levels, rises and drops, respectively. Here we have to distinguish between odd and even values of n, as well as look at the remainder of k when divided by 3.

Theorem 5: *For* $k \geq 1$,

$$1. \quad \tilde{l}_{2k} = \frac{2}{9}(-1)^k + 2^k\left(\frac{53}{126} + \frac{k}{3}\right) + \begin{cases} \frac{6}{7} & k \equiv 0 \mod (3) \\ \frac{-2}{7} & k \equiv 1 \mod (3) \\ \frac{-4}{7} & k \equiv 2 \mod (3) \end{cases}$$

$$\tilde{l}_{2k+1} = \frac{2}{9}(-1)^k + 2^k\left(\frac{22}{63} + \frac{k}{3}\right) + \begin{cases} \frac{-4}{7} & k \equiv 0 \mod (3) \\ \frac{6}{7} & k \equiv 1 \mod (3) \\ \frac{-2}{7} & k \equiv 2 \mod (3) \end{cases}$$

2. $$\tilde{r}_{2k} = \tilde{d}_{2k} = -\frac{1}{9}(-1)^k - 2^{k-1}\left(\frac{58}{63} - \frac{2k}{3}\right) + \begin{cases} \frac{-3}{7} & k \equiv 0 \mod (3) \\ \frac{1}{7} & k \equiv 1 \mod (3) \\ \frac{2}{7} & k \equiv 2 \mod (3) \end{cases}$$

$$\tilde{r}_{2k-1} = \tilde{d}_{2k+1} = -\frac{1}{9}(-1)^k - 2^{k-1}\left(\frac{22}{63} - \frac{2k}{3}\right) + \begin{cases} \frac{2}{7} & k \equiv 0 \mod (3) \\ \frac{-3}{7} & k \equiv 1 \mod (3) \\ \frac{1}{7} & k \equiv 2 \mod (3) \end{cases}$$

Proof: We use the PCM, where a middle summand $m = 2l$ or $m = 2l + 1$ $(l \geq 0)$ is combined with a composition of $k - l$ and its mirror image, to create a palindrome of $n = 2k$ or $n = 2k + 1$, respectively. The number of levels in the palindrome is twice the number of levels of the composition, plus any additional levels created when the compositions are joined with the middle summand.

We will first look at the case where n (and thus m) is even. If $l = m = 0$, a composition of k is joined with its mirror image, and we get only one additional level. If $l > 0$, then we get two additional levels for a composition ending in m, for $m = 2l \leq k - l$. Thus,

$$\tilde{l}_{2k} = 2 \cdot \sum_{l=0}^{k} l_{k-l} + C_k + 2 \cdot \sum_{l=1}^{\lfloor k/3 \rfloor} C_{k-l}(2l) = s_1 + 2^{k-1} + s_2. \tag{3}$$

Since $l_0 = l_1 = 0$, the first summand reduces to

$$s_1 = \frac{1}{18} \cdot \sum_{i=2}^{k} \{(3i+1)2^i + 8(-1)^i\} = \frac{2}{9}\sum_{i=2}^{k} 2^{i-2} + \frac{1}{3}\sum_{i=2}^{k} i \cdot 2^{i-1} + \frac{4}{9}\sum_{i=0}^{k}(-1)^i$$

$$= \frac{2}{9} \cdot (2^{k-1} - 1) + \frac{1}{3}\left(\frac{d}{dx}\sum_{i=2}^{k} x^i\right)\Bigg|_{x=2} + \frac{2}{9}((-1)^k + 1)$$

$$= \frac{1}{9}2^k + \frac{1}{3}\{(k+1)2^k - 2^{k+1}\} + \frac{2}{9}(-1)^k = \frac{2}{9}(-1)^k + \left(\frac{k}{3} - \frac{2}{9}\right)2^k. \tag{4}$$

To compute s_2, note that $C_n(i) = C_{n-1}(i-1) = \cdots = C_{n-i+1}(1) = \frac{1}{2}C_{n-i+1} = 2^{n-i-1}$ for $i < n$ and $C_n(n) = 1$. The latter case only occurs when $k = 3l$. Let $k := 3j + r$, where

$r = 1, 2, 3$. (This somewhat unconventional definition allows for a unified proof.) Thus, with $\mathcal{I}_A$ denoting the indicator function of A,

$$s_2 = 2 \cdot \sum_{l=1}^{\lfloor k/3 \rfloor} C_{k-l}(2l) = 2 \cdot \sum_{l=1}^{j} 2^{3j+r-l-2l-1} + 2 \cdot \mathcal{I}_{\{r=3\}}$$

$$= 2^r \cdot \sum_{l=1}^{j} (2^3)^{j-l} + 2 \cdot \mathcal{I}_{\{r=3\}} = 2^r \left(\frac{(2^3)^j - 1}{7} \right) + 2 \cdot \mathcal{I}_{\{r=3\}}$$

$$= \frac{2^k - 2^r}{7} + 2 \cdot \mathcal{I}_{\{r=3\}} = \begin{cases} \frac{2^k+6}{7} & k \equiv 0 \mod (3) \\ \frac{2^k-2^r}{7} & k \equiv r \mod (3), \text{ for } r = 1, 2. \end{cases} \tag{5}$$

Combining Equations (3), (4) and (5) and simplifying gives the result for $\tilde{l}_{2k}$.

For $n = 2k + 1$, we make a similar argument. Again, each palindrome has twice the number of levels of the associated composition, and we get two additional levels whenever the composition ends in m, for $m = 2l + 1 \leq k - l$. Thus,

$$\tilde{l}_{2k+1} = 2 \cdot \sum_{l=0}^{k} l_{k-l} + 2 \cdot \sum_{l=0}^{\lfloor (k-1)/3 \rfloor} C_{k-l}(2l+1) =: s_1 + s_3.$$

With an argument similar to that for s_2, we derive

$$s_3 = \begin{cases} \frac{2^{k+2}-4}{7} & k \equiv 0 \mod (3) \\ \frac{2^{k+2}+6}{7} & k \equiv 1 \mod (3) \\ \frac{2^{k+2}-2}{7} & k \equiv 2 \mod (3). \end{cases} \tag{6}$$

Combing Equations (4) and (6) and simplifying gives the result for $\tilde{l}_{2k+1}$. Finally, the results for $\tilde{r}_n$ and $\tilde{d}_n$ follow from the fact that $\tilde{r}_n = \tilde{d}_n = \frac{P_n^+ - \tilde{l}_n}{2}$. $\square$

5. GENERATING FUNCTIONS

Let $G_{a_n}(x) = \sum_{k=0}^{\infty} a_k x^k$ be the generating function of the sequence $\{a_n\}_0^\infty$. We will give the generating functions for all the quantities of interest.

Theorem 6

1. $G_{C_n}(x) = \frac{1-x}{1-2x}$ *and* $G_{P_n}(x) = \frac{1+x}{1-2x^2}$.
2. $G_{C_n^+}(x) = \frac{x^2}{(1-2x)^2}$ *and* $G_{P_n^+}(x) = \frac{x^2+2x^3+2x^4}{(1-2x^2)^2}$.
3. $G_{C_n^S}(x) = \frac{1-3x+3x^2}{(1-2x)^2}$ *and* $G_{P_n^S}(x) = \frac{1+x-x^2+2x^4}{(1-2x^2)^2}$.
4. $G_{l_n}(x) = \frac{x^2(1-x)}{(1+x)(1-2x)^2}$ *and* $G_{r_n}(x) = G_{d_n}(x) = \frac{x^3}{(1+x)(1-2x)^2}$.
5. $G_{\tilde{l}_n}(x) = \frac{x^2(1+3x+4x^2+x^3-x^4-4x^5-6x^6)}{(1+x^2)(1+x+x^2)(1-2x^2)^2}$ *and* $G_{\tilde{r}_n}(x) = G_{\tilde{d}_n}(x) = \frac{x^4(1+3x+4x^2+4x^3+4x^4)}{(1+x^2)(1+x+x^2)(1-2x^2)^2}$.

Proof: 1. & 2. The generating functions for $\{C_n\}_0^\infty, \{P_n\}_0^\infty$ and $\{C_n^+\}_0^\infty$ are straightforward using the definition and the formulas of Theorem 1. We derive $G_{P_n^+}(x)$, as it needs to take into account the two different formulas for odd and even n. From Theorem 1, we get

$$\begin{aligned} G_{P_n^+}(x) &= \sum_{k=1}^{\infty} P_{2k-1}^+ x^{2k-1} + \sum_{k=1}^{\infty} P_{2k}^+ x^{2k} \\ &= \sum_{k=1}^{\infty}(k-1)2^{k-1}x^{2k-1} + \sum_{k=1}^{\infty}(2k-1)2^{k-1}x^{2k}. \end{aligned} \tag{7}$$

Separating each sum in Eq. (7) into terms with and without a factor of k, and recombining like terms across sums leads to

$$\begin{aligned} G_{P_n^+}(x) &= \frac{1+2x}{4}\sum_{k=1}^{\infty} 4xk(2x^2)^{k-1} - (x+x^2)\sum_{k=1}^{\infty}(2x^2)^{k-1} \\ &= \frac{1+2x}{4}\cdot\frac{d}{dx}\left(\frac{1}{1-2x^2}\right) - \frac{x+x^2}{1-2x^2} = \frac{x^2+2x^3+2x^4}{(1-2x^2)^2}. \end{aligned}$$

3. Since $C_n^S = C_n + C_n^+, G_{C_n^S}(x) = G_{C_n}(x) + G_{C_n^+}(x)$; likewise for $G_{P_n^S}(x)$.
4. The generating function for l_n can be easily computed using Mathematica or Maple, using either the recursive or the explicit description. The relevant Mathematica commands are

```
<<DiscreteMath `RSolve`
GeneratingFunction[{a[n + 1] == 2a[n] + (2/3) * 2^(n - 2) + (-2/3) * (-1)^(n - 2),
  a[0] == 0, a[1] == 0}, a[n], n, z][[1, 1]]
PowerSum[((1/36) + (n/12)) * 2^n + (2/9) * (-1)^n, {z, n, 1}]
```

Furthermore, $G_{r_n}(x) = G_{d_n}(x) = \frac{1}{2}\left(G_{C_n^+}(x) - G_{l_n}(x)\right)$, since $r_n = d_n = \frac{C_n^+ - l_n}{2}$.

5. In this case we have six different formulas for $\tilde{l}_n$, depending on the remainder of n with respect to 6. Let $G_i(x)$ denote the generating function of $\{\tilde{l}_{6k+i}\}_{k=0}^{\infty}$. Then, using the definition of the generating function and separating the sum according to the remainder (similar to the computation in part 2), we get

$$G_{\tilde{l}_n}(x) = G_0(x^6) + x \cdot G_1(x^6) + x^2 \cdot G_2(x^6) + \cdots + x^5 \cdot G_5(x^6).$$

The functions $G_i(x)$ and the resulting generating function $G_{\tilde{l}_n}(x)$ are derived using the following Mathematica commands:

```
<<DiscreteMath `RSolve`
g0[z_] =PowerSum[(1/126)((126(n) + 53) * 2^(3n) + 108 + 28(-1)^(n)), {z, n, 1}]
g1[z_] =PowerSum[(1/63)((63n + 22) * 2^(3n) - 36 + 14(-1)^n), {z, n, 1}]
g2[z_] =PowerSum[(1/63)((126n + 95) * 2^(3n) - 18 - 14(-1)^n), {z, n, 0}]
g3[z_] =PowerSum[(1/63)((126n + 86) * 2^(3n) + 54 - 14(-1)^n), {z, n, 0}]
g4[z_] =PowerSum[(1/63)((252n + 274) * 2^(3n) - 36 + 14(-1)^n), {z, n, 0}]
g5[z_] =PowerSum[(1/63)((252n + 256) * 2^(3n) - 18 + 14(-1)^n), {z, n, 0}]
genfun[z] := g0[z^6] + z g1[z^6] + z^2 g2[z^6] + z^3 g3[z^6] + z^4 g4[z^6] + z^5 g5[z^6]
```

Finally, $G_{\tilde{r}_n}(x) = G_{\tilde{d}_n}(x) = \frac{1}{2}\left(G_{P_n^+}(x) - G_{\tilde{l}_n}(x)\right)$, since $\tilde{r}_n = \tilde{d}_n = \frac{P_n^+ - \tilde{l}_n}{2}$. $\square$

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✠ ✠ ✠

VIETA CONVOLUTIONS AND DIAGONAL POLYNOMIALS

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1. INTRODUCTION. VIETA DIPTYCH

Two separate, but related, matters are discussed in this communication. One presents a few basic properties of Vieta convolutions, the other offers an outline of the main features of rising and falling diagonal polynomial functions for the Vieta polynomials.

Vieta polynomials are of two kinds [7], the *Vieta-Fibonacci polynomials* $V_n(x)$ and the *Vieta-Lucas polynomials* $v_n(x)$, defined for our purposes by generating functions as, respectively,

$$\sum_{n=1}^{\infty} V_n(x)y^{n-1} = [1 - xy + y^2]^{-1}, \quad V_0(x) = 0, \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} v_n(x)y^n = (2 - xy)[1 - xy + y^2]^{-1}. \tag{1.2}$$

Combinatorial, Binet form and recurrence definitions of $V_n(x)$ and $v_n(x)$, along with many detailed properties of these polynomials, are provided in [7]. One might also consult [14] for other facets of $V_n(x)$. Vieta polynomials are so named to honour the French mathematician Vieta (Francois Viète, 1540-1603.)

A Value of Convolutions

Why do we give emphasis to a study of convolutions defined in terms of generating functions?

Looking at (1.1) and (2.1), we see immediately that $V_n(x)$ is a special case of $V_n^{(k)}(x)$ when $k = 0$. Viewed reversely, $V_n^{(k)}(x)$ is a generalization of $V_n(x)$. For the author, the importance of a study of convolutions lies in this dual perspective.

Similar comments apply to $v_n(x)$ (1.2) and $v_n^{(k)}(x)$ (2.8).

2. VIETA CONVOLUTIONS

Vieta-Fibonacci Convolutions

Definition: The k^{th} *Vieta-Fibonacci convolution* $V_n^{(k)}(x)$ of $V_n(x)$ is generated by

$$\sum_{n=1}^{\infty} V_n^{(k)}(x) y^{n-1} = [1 - xy + y^2]^{-(k+1)}, \quad V_0^{(k)}(x) = 0. \tag{2.1}$$

For the explicit representation of the polynomials $V_n^{(k)}(x)$ see Theorem 2 (2.17) and Theorem 1 (2.16) when $k = 1$.

Examples:

$$\begin{aligned} &V_1^{(1)}(x) = 1, V_2^{(1)}(x) = 2x, V_3^{(1)}(x) = 3x^2 - 2, V_4^{(1)}(x) = 4x^3 - 6x, \\ &V_5^{(1)}(x) = 5x^4 - 12x^2 + 3, V_6^{(1)}(x) = 6x^5 - 20x^3 + 12x, \ldots . \end{aligned} \tag{2.2}$$

Evaluation of higher order convolutions ($k \geq 2$) is left to the inclination of the reader. Note that $V_n^{(0)}(x) = V_n(x)$ by (1.1), (2.1).

Basic Properties of $V_n^{(k)}(x)$

Immediately from (2.1)

$$V_n^{(k-1)}(x) = V_n^{(k)}(x) - xV_{n-1}^{(k)}(x) + V_{n-2}^{(k)}(x) \quad (k \geq 1, \ n \geq 2). \tag{2.3}$$

Differentiate (2.1) partially with respect to y after replacing k by $k - 1$. Then

$$(n-1)V_n^{(k-1)}(x) = k\left(xV_{n-1}^{(k)}(x) - 2V_{n-2}^{(k)}(x)\right). \tag{2.4}$$

Eliminate $V_n^{(k-1)}(x)$ from (2.3) and (2.4) to derive

$$(n-1)V_n^{(k)}(x) = (n+k-1)xV_{n-1}^{(k)}(x) - (n+2k-1)V_{n-2}^{(k)}(x). \tag{2.5}$$

Now write

$$\frac{\partial}{\partial x}V_n(x) \equiv V_n'(x), \frac{\partial^2}{\partial x^2}V_n(x) \equiv V_n''(x), \ldots, \frac{\partial^k}{\partial x^k}V_n(x) \equiv V_n^k(x). \tag{2.6}$$

Differentiating (1.1) k-times with respect to y, we arrive at the neat result

$$V_n^k(x) = k! V_{n-k}^{(k)}(x). \tag{2.7}$$

Vieta-Lucas Convolutions

Definition: The k^{th} *Vieta-Lucas convolution* $v_n^{(k)}(x)$ of $v_n(x)$ is generated by

$$\sum_{n=0}^{\infty} v_n^{(k)}(x) y^n = (2 - xy)^{k+1}[1 - xy + y^2]^{-(k+1)} \tag{2.8}$$

so that $v_n^{(0)}(x) = v_n(x)$ by (1.2), (2.8).

For the explicit representation of the polynomials $v_n^{(k)}(x)$ see Theorem 3 (2.19).

Examples:

$$\begin{aligned} &v_0^{(1)}(x) = 4, v_1^{(1)}(x) = 4x, v_2^{(1)}(x) = 5x^2 - 8, v_3^{(1)}(x) = 6x^3 - 16x, \\ &v_4^{(1)}(x) = 7x^4 - 26x^2 + 12, v_5^{(1)}(x) = 8x^5 - 38x^3 + 36x, \ldots . \end{aligned} \tag{2.9}$$

Because of the nature of the complicated algebra involved (unappetizing mental pabulum!), we restrict our treatment to the simplest case $k = 1$.

Basic Properties of $v_n^{(k)}(x)(k = 1)$

Proceeding similary as in (2.3)-(2.5) for $V_n(x)$, we extract the following essential relationships:

$$v_{n-1}^{(1)}(x) = 4V_n^{(1)}(x) - 4xV_{n-1}^{(1)}(x) + x^2 V_{n-2}^{(1)}(x), \tag{2.10}$$

$$nv_n(x) = xV_n^{(1)}(x) - 4V_{n-1}^{(1)}(x) + xV_{n-2}^{(1)}(x), \tag{2.11}$$

$$nxv_n(x) = (x^2 - 4)V_n^{(1)}(x) + v_{n-1}^{(1)}(x). \tag{2.12}$$

Observe the rather different sorts of equations (2.10)-(2.12) here compared with those in (2.3)-(2.5), as a consequence of the primacy and simplicity of the generating function for $V_n^{(1)}(x)$.

Lastly, if we multiply numerator and denominator of (2.8) when $k = 0$ by $1 - xy + y^2$, then the ensuing algebra reduces to

$$v_{n-1}(x) = 2V_n^{(1)}(x) - 3xV_{n-1}^{(1)}(x) + (2 + x^2)V_{n-2}^{(1)}(x) - xV_{n-3}^{(1)}(x). \tag{2.13}$$

Closed Forms

Lemma 1:

$$\binom{N-r}{1}\binom{N-r-1}{r}+2\binom{N-r}{1}\binom{N-r-1}{r-1}=N\binom{N-r}{r}. \tag{2.14}$$

Lemma 2:

$$k\left\{\binom{N+k-1-r}{k}\binom{N-r-1}{r}+2\binom{N+k-1-r}{k}\binom{N-r-1}{r-1}\right\}$$

$$=N\binom{N+k-1-r}{k-1}\binom{N-r}{r}. \tag{2.15}$$

Both lemmas are readily established by routine combinatorial calculation. Clearly, Lemma 1 is a special case of Lemma 2 occurring when $k=1$. Observe that in (2.15), the factor k is absorbed into the product and N emerges as a factor. (See also [8, (2.11a), (4.12a)] where the same two formulas (2.14) and (2.15) appear.)

Theorem 1:

$$V_n^{(1)}(x)=\sum_{r=0}^{[\frac{n-1}{2}]}(-1)^r\binom{n-r}{1}\binom{n-r-1}{r}x^{n-2r-1}. \tag{2.16}$$

Proof (by induction):

The theorem is verifiably valid for $n=1,2,3$ (say). Assume that it is true for $n=N$, that is, assume

$$V_N^{(1)}(x)=\sum_{r=0}^{[\frac{N-1}{2}]}(-1)^r\binom{N-r}{1}\binom{N-r-1}{r}x^{N-2r-1}. \quad (A)$$

Then, with $n \to n+1$, the right-hand side of (2.5) transforms to

$$N\left[xV_N^{(1)}(x) - V_{N-1}^{(1)}(x)\right] + \left[xV_N^{(1)}(x) = 2V_{N-1}^{(1)}(x)\right]$$

$$= N\sum_{r=0}^{[\frac{N}{2}]}(-1)^r\binom{N-r}{1}\binom{N-r}{r}x^{N-2r} + N\sum_{r=0}^{[\frac{N}{2}]}\binom{N-r}{r}x^{N-2r} \qquad \text{by } (A), \text{Lemma 1}$$

$$= N\sum_{r=0}^{[\frac{N}{2}]}(-1)^r\binom{N-r+1}{1}\binom{N-r}{r}x^{N-2r} \qquad (B)$$

$$= NV_{N+1}^{(1)}(x) \qquad (C)$$

which must be the left-hand side of (2.5).

Consequently, (B) and (C) together with (A) reveal that (2.16) is true for all values of n. Accordingly, Theorem 1 is fully established.

Theorem 2:

$$V_n^{(k)}(x) = \sum_{r=0}^{[\frac{n-1}{2}]}(-1)^r\binom{n+k-r-1}{k}\binom{n-r-1}{r}x^{n-2r-1}. \tag{2.17}$$

Proof (by Induction): Follow the procedures in the proof of Theorem 1 while utilizing Lemma 2. (Pascal's Formula is needed in both Theorems 1 and 2.)

Examples:

$$V_1^{(k)} = 1, \quad V_2^{(k)}(x) = \binom{k+1}{1}x, \quad V_3^{(k)}(x) = \binom{k+2}{2}x^2 - \binom{k+1}{1},$$

$$V_4^{(k)}x = \binom{k+3}{3}x^3 - 2\binom{k+2}{2}x, \ldots, \tag{2.18}$$

as may be checked by (2.1).

By virtue of the generating functions (2.1) and (2.8) for $V_n^{(k)}(x)$ and $v_n^{(k)}(x)$ respectively, and in view of Theorem 2, it is clear that $v_n^{(k)}(x)$ may be expressed combinatorially in summation form involving the Vieta convolutions.

Theorem 3:

$$v_n^{(k)}(x) = \sum_{r=0}^{k+1}(-1)^r\binom{k+1}{r}2^{k+1-r}x^r V_{n-r+1}^{(k)}(x), \tag{2.19}$$

where $V_{n-r+1}^{(k)}(x)$ are given in (2.17).

Proof: Expand $(2-xy)^{k+1}$ in conjunction with (2.1) and (2.8). Theorem 3, as enunciated, then follows.

Examples:

$$v_0^{(k)}(x) = 2^{k+1}, \quad v_1^{(k)}(x) = 2^k\binom{k+1}{1}x,$$

$$v_2^{(k)}(x) = 2^{k-1}\left[4\binom{k+2}{2} - 2\binom{k+1}{1}^2 + \binom{k+1}{2}\right]x^2 - 2^{k+1}\binom{k+1}{1}, \ldots \tag{2.20}$$

Putting $k=1$ in (2.20) reduces these expressions to those in (2.9). Theorem 1 corresponds to Theorem 3 when $k=1$.

A Question Answered.

In [7], some elegant results connecting Vieta, Jacobsthal, and Morgan-Voyce polynomials with special arguments $\frac{1}{x}$, $-x^2$, $-\frac{1}{x^2}$ were revealed. Note that in the definitions of Jacobsthal polynomials $J_n(x)$ and Jacobsthal-Lucas polynomials $j_n(x)$ given in [6] and [8], *the factor* $2x$ *is here replaced by* x as in [7].

At the Luxembourg International Fibonacci Conference (July, 2000) the question was asked:

Can these special results be cárried over to convolution theory?

Sadly, the answer is: generally, NO!

Happily, however, there is one positive instance, namely,

Theorem 4:

$$V_n^{(k)}(x) = x^{n-1}J_n^{(k)}\left(-\frac{1}{x^2}\right). \tag{2.21}$$

Proof:

(a) By Theorem 2 and [7, Theorem 1], both expressions are equal to the combinatorial summation

$$\sum_{r=0}^{[\frac{n-1}{2}]}(-1)^r\binom{n+k-r-1}{k}\binom{n-r-1}{r}x^{n-2r-1},$$

with the same initial values 0 and 1 for $n=0,1$.

(b) Working from the recurrence relation [8, (4.13)] for $x^{n-1}J_n^{(k)}\left(-\frac{1}{x^2}\right)$ we quickly have, on multiplying throughout by x^{n-1},

$$x^{n+1}J_{n+2}^{(k)}\left(-\frac{1}{x^2}\right)-x\cdot x^nJ_{n+1}^{(k)}\left(-\frac{1}{x^2}\right)+x^{n-1}J_n^{(k)}\left(-\frac{1}{x^2}\right)=x^{n+1}J_{n+2}^{(k)}\left(-\frac{1}{x^2}\right)$$

which is identical with (2.3) for $x^{n-1}J_n^{(k)}\left(-\frac{1}{x^2}\right)=V_n^{(k)}(x)$, both of which have initial values 0 and 1 for $n=0,1$.

Note:

(i) An analysis of the expansion of the generating function $\left[1-y-\frac{y^2}{x^2}\right]^{-2}$ for $J_{n+1}^{(1)}\left(-\frac{1}{x^2}\right)$

leads us to a verification of Theorem 3 for $V_n^{(1)}(x)$, for small values of n.

(ii) No such joys as in Theorem 4 await us when we turn to $v_n^{(k)}(x)$ and $j_n^{(k)}\left(-\frac{1}{x^2}\right)$, as is evident from the more complicated forms of their generating functions.

Coming to Morgan-Voyce convolutions, we find there is no connection with Vieta and Jacobsthal convolutions for the above special arguments, since the essential provisos in the Proofs in Theorem 4 do not pertain. [Parenthetically, we remark that even the beautiful Cinderella had less attractive sisters!]

Cauchy Product

Convolution polynomials $V_n^{(i)}(x)(i=1,\ldots,k)$ may also be defined by means of summations of **Cauchy products**, thus:

Definition:

$$\begin{aligned} V_n^{(1)}(x) &= \sum_{r=1}^{n} V_r(x)V_{n+1-r}(x), \\ V_n^{(2)}(x) &= \sum_{r=1}^{n} V_r^{(1)}(x)V_{n+1-r}(x), \\ \dots & \quad \dots \\ V_n^{(k)}(x) &= \sum_{r-1}^{n} V_r^{k-1}(x)V_{n+1-r}(x). \end{aligned} \tag{2.22}$$

Examples:

$$\begin{aligned} V_5^{(1)}(x) &= 2V_1(x)V_5(x) + 2V_2V_4(x) + (V_3(x))^2 = 5x^4 - 12x^2 + 3 \text{ as in (2.2), Theorem 1.} \\ V_4^{(2)}(x) &= V_1^{(1)}(x)V_4(x) + V_2^{(1)}(x)V_3(x) + V_3^{(1)}(x)V_2(x) + V_4^{(1)}(x)V_1(x) \\ &= 10x^3 - 12x \quad \text{as from (2.18), } k = 2. \end{aligned}$$

Cauchy products may likewise define the Vieta-Lucas convolution polynomials $v_n^{(i)}(x)(i = 1, \dots, k)$.

Definition:

$$\begin{aligned} v_n^{(1)}(x) &= \sum_{r=0}^{n} v_r(x)v_{n-r}(x), \\ v_n^{(2)}(x) &= \sum_{r=0}^{n} v_r^{(1)}(x)v_{n-r}(x), \\ \dots & \quad \dots \\ v_n^{(k)}(x) &= \sum_{r=0}^{n} v_r^{k-1}(x)v_{n-r}(x). \end{aligned} \tag{2.23}$$

Examples:

$$v_5^{(1)}(x) = 2v_0(x)v_4(x) + 2v_1(x)v_3(x) + (v_2(x))^2 = 7x^4 - 26x^2 + 12$$

as in (2.9)

$$\begin{aligned} v_3^{(2)}(x) &= v_0^{(1)}(x)v_3(x) + v_1^{(1)}(x)v_2(x) + v_2^{(1)}(x)v_1(x) + v_3^{(1)}(x)v_0(x) \\ &= 25x^3 - 60, \quad \text{as from (2.20)}, k = 2. \end{aligned}$$

Thus, there exist three ways of calculating, say, $v_2^{(2)}(x) = 18x^2 - 24$, namely: (i) directly from (2.8), $k = 2, n = 2$ (ii) by substituting $k = 2, n = 2$ in (2.20) [equivalent really to (i)], and (iii) by using the Cauchy product (2.23).

Remarks:

(a) Generally, we may extend (2.22) to

$$V_n^{(k)}(x) = \sum_{r=1}^{n} V_r^{(m)}(x) V_{n+1-r}^{(k-1-m)}(x) \ (m = 0, 1, \dots, k-1). \tag{2.24}$$

Likewise for $v_n^{(k)}(x)$.

(b) Cauchy products as in (2.22-2.24) are applicable analogously to Jacobsthal-type polynomials [8], Morgan-Voyce polynomials [9], Fermat-type polynomials [10], and to Pell and Pell-Lucas polynomials (for which see A.F. Horadam and Bro. J.M. Mahon: "Convolutions for Pell Polynomials." *Fibonacci Numbers and Their Applications* (Eds. A.N. Philippou, G.E. Bergum, and A.F. Horadam), Kluwer Academic Publishers, Dordrecht, The Netherlands (1986): 55-80).

Variation on a Theme

Suppose we replace $+y^2$ by $-y^2$ in (2.1) and (2.8). Designate the ensuing modified polynomials by $*V_n^{(k)}(x)$ and $*v_n^{(k)}(x)$ respectively. Of course, it then transpires that

$$*V_n^{(k)}(x) = F_n^{(k)}(x), \ *v_n^{(k)}(x) = L_n^{(k)}(x), \tag{2.25}$$

where $F_n^{(k)}(x)$ and $L_n^{(k)}(x)$ are the generalized Fibonacci and Lucas k^{th} convolution polynomials, respectively. In fact, for example, $*V_6^{(1)}(x) = 6x^5 + 20x^3 + 12x$.

Mindful that $*V_n^{(0)}(1) = F_n$, the n^{th} Fibonacci number, we may build up the Fibonacci convolution sequences as, e.g.,

$$\begin{aligned} \{*V_n^{(1)}(1)\} &= \{F_n^{(1)}\} = \{1, 2, 5, 10, 20, 38, 71, 130, \dots\}, \\ \{*V_n^{(2)}(1)\} &= \{F_n^{(2)}\} = \{1, 3, 9, 22, 51, 111, 233, \dots\}, \\ \{*V_n^{(3)}(1)\} &= \{F_n^{(3)}\} = \{1, 4, 14, 40, 105, 246, 594, \dots\}, \end{aligned} \tag{2.26}$$

which may, for visual convenience, be expressed in tabular form. Calculations in (2.26) have involved (2.5), (2.17), and (2.18). Verfications may be obtained by recourse to V.E. Hoggatt, Jr. and G.E. Bergum, "Generalized Convolution Arrays", *The Fibonacci Quarterly* 13.3 (1975): 193-196. Sequences occurring in (2.26) appear in the table on page 118 of V.E. Hoggatt, Jr. and Marjorie Bicknell-Johnson, "Fibonacci Convolution Sequences", *The Fibonacci Quarterly* 15.2 (1977): 117-122.

3. VIETA DIAGONAL POLYNOMIALS

Preamble

While sorting out ideas on rising and falling diagonal functions for $V_n(x)$ and $v_n(x)$, the author became aware of the generalized survey in [15] covering similar work already done for Fibonacci, Lucas, Chebyshev [1], [3], [12], Fermat [3], and Jacobsthal [6] polynomials.

To these polynomials we specifically add the earlier study of Pell polynomials [13] and Gegenbauer polynomials [11] (rising diagonals) and [5] (descending diagonals). Work on Morgan-Voyce rising and descending diagonal polynomials is under investigation.

Each polynomial has an individual essence distinguishing it from others. Our justification for treating Vieta diagonal polynomials as separate entities and not just as particular instances of a general situation is that it preserves the distinguishing features of these polynomials and so it enhances our knowledge of Vieta polynomials *per se*.

The slanting criss-cross pattern of rising and falling parallel diagonal "lines" is visually apparent for the polynomials displayed in [2], [3], [4], and [11]. Incidentally, both kinds of Chebyshev polynomials are special cases of Gegenbauer polynomials [5, p. 294], [11, p. 394].

Rising Vieta Diagonal Polynomials

Represent these polynomials for $V_n(x)$ and $v_n(x)$ by $R_n(x)$ and $r_n(x)$ respectively. Then the following fundamental conclusions are relatively easy to establish.

Generating Functions

$$\sum_{n=1}^{\infty} R_n(x)y^{n-1} = [1 - y(x - y^2)]^{-1}, \quad R_0(x) = 0. \tag{3.1}$$

$$\sum_{n=3}^{\infty} r_n(x)y^{n-1} = (1 - y^3)[1 - y(x - y^2)]^{-1}, \quad r_0(x) = 2. \tag{3.2}$$

Recurrence Relations

$$R_n(x) = xR_{n-1}(x) - R_{n-3}(x). \tag{3.3}$$

$$r_n(x) = xr_{n-1}(x) - r_{n-3}(x). \tag{3.4}$$

$$r_n(x) = R_n(x) - R_{n-3}(x). \tag{3.5}$$

Computation of the $R_n(x)$ and $r_n(x)$ in (3.1) and (3.2) is left to the dedication of the reader.

Descending Vieta Diagonal Polynomials

Designate these polynomials for $V_n(x)$ and $v_n(x)$ by $D_n(x)$ and $d_n(x)$ respectively. Analogues of the generating functions and recurrence relations for $R_n(x)$ and $r_n(x)$ are straightforward to discover.

Generating Functions

$$D_n(x) = (x-1)^{n-1}, \quad D_0(x) = 0, \tag{3.6}$$

$$d_n(x) = (x-2)(x-1)^{n-1}, \quad d_0(x) = 2, \tag{3.7}$$

whence

$$\frac{d_n(x)}{D_n(x)} = x - 2. \tag{3.8}$$

Recurrence Relations

$$\frac{d_n(x)}{d_{n-1}(x)} = \frac{D_n(x)}{D_{n-1}(x)} = x - 1. \tag{3.9}$$

Partial Differentiation

Suppose now that we use the generating function symbolism

$$G \equiv G(x,y) = [1-(x-1)y]^{-1} = \sum_{n=1}^{\infty} D_n(x)y^{n-1}. \tag{3.10}$$

An immediate outcome is that

$$(x-1)\frac{\partial G}{\partial x} = y\frac{\partial G}{\partial y}. \tag{3.11}$$

Setting

$$H \equiv H(x,y) = (x-2)[1-(x-1)y]^{-1} = \sum_{n=1}^{\infty} d_n(x)y^n. \tag{3.12}$$

we come to

$$(x-1)(x-2)\frac{\partial H}{\partial x} = (1-y)\frac{\partial H}{\partial y}. \tag{3.13}$$

Partial differentiation along the procedures of (3.10) - (3.13) for $R_n(x)$ and $r_n(x)$ is a suggested exercise.

4. CONCLUSION

In passing, we mention that the 1969 formula occurring in [7, reference [1], p. 14],

$$v_n(p,q) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} p^{n-2k} q^k,$$

and surely of an earlier origin, is equivalent to the 1999 formula [15, (2.22)] when $x = p,\ y = -q$. Attention to the valuable material in [15] is strongly recommended.

Attention might also be directed to the related study of convolutions for generalized Fibonacci and Lucas Polynomials in [10].

The purpose of this paper has been to give a skeletal framework to the theory which, hopefully, could be fleshed out to a more robust body of knowledge.

Finally, the author wishes to thank the anonymous referee for the careful assessment of this submission.

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DYNAMIC ONE-PILE NIM

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1. INTRODUCTION

The purpose of this paper is to solve a class of combinatorial games consisting of one-pile counter pickup games for which the maximum number of counters that can be removed on each successive move changes during the play of the game. Two players alternate removing a positive number of counters from the pile. An ordered pair (N, x) of positive integers is called a *position.* The number N represents the size of the pile of counters and x represents the greatest number of counters that can be removed on the next move. A function $f: Z^+ \to Z^+$ is given which determines the maximum size of the next move in terms of the current move size. Thus a *move* in a game is an ordered pair of positions $(N, x) \longmapsto (N-k, f(k))$, where $1 \leq k \leq \min\{N, x\}$. The game ends when there are no counters left, and the winner is the last player to move in the game. This paper extends two papers, one by Epp and Ferguson [2], and the other by Schwenk [6].

In order to introduce the concepts in this paper, we initially assume that f satisfies

$$(*) \qquad f(n+1) - f(n) \geq -1.$$

Later in the paper we prove the necessary and sufficient conditions on f so that our strategy is effective. In the appendix, we discuss the Epp, Ferguson paper. The authors are grateful to the referee for pointing out the possibility of finding both *necessary* and sufficient conditions on the function f so that the solution is effective.

The game of 'static' one-pile nim is well understood. These are called *subtraction games.* A pile of n counters and a constant k are given. Two players alternately take from 1 up to k counters from the pile. The winner is the last player to remove a counter. The theory of these games is complete. See [1, p. 83].

Before discussing the strategy for playing dynamic one-pile nim, we prove four lemmas. These lemmas appear to have nothing in common with our games, but once they are proved, the strategy for playing will be easily understood.

GENERALIZED BASES

An infinite increasing sequence $B = (b_0 = 1, b_1, b_2, \ldots)$ of positive integers is called an *infinite g-base* if for each $k \geq 0, b_{k+1} \leq 2b_k$. This 'slow growth' of B's members guarantees Lemma 1.

Finite g**-bases**: A finite increasing sequence $B = (b_0 = 1, b_1, b_2, \ldots, b_t)$ of positive integers is called a *finite g-base* if for each $0 \leq k < t, b_{k+1} \leq 2b_k$.

Lemma 1: Let B be an infinite g-base. Then each positive integer N can be represented as $N = b_{i_1} + b_{i_2} + \cdots + b_{i_t}$ where $b_{i_1} < b_{i_2} < \cdots < b_{i_t}$ and each b_{i_j} belongs to B.

Proof: The proof is given by the following recursive algorithm. Note first that $b_0 = 1 \in B$. Suppose all the integers $1, 2, 3, \ldots, m-1$ have been represented as a sum of distinct members of B. Let b_k denote the largest element of B not exceeding m. That is, $b_k \leq m < b_{k+1}$. Then $m = (m - b_k) + b_k$. Now $m - b_k < b_k$, for otherwise $2b_k \leq m$. But $b_{k+1} < 2b_k$, contradicting the definition of b_k. Since $m - b_k$ is less than m, it follows that $m - b_k$ has been represented as a sum of distinct members of B that are less than b_k. Thus we may suppose that $m - b_k = b_{i_1} + b_{i_2} + \cdots + b_{i_{t-1}}$ where $b_{i_1} < b_{i_2} < \cdots < b_{i_{t-1}}$ and each b_{i_j} belongs to B. Then $m = b_{i_1} + b_{i_2} + \cdots + b_{i_t}$, where $b_{i_t} = b_k, b_{i_1} < b_{i_2} < \cdots < b_{i_t}$ and each b_{i_j} belongs to B. $\square$

Note that in general it may be possible to represent an integer N as a sum of distinct members of B in more than one way. We now define a stable representation.

Definition: Let $B = (b_0 = 1, b_1, \ldots)$ be an infinite g-base. Suppose $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$. We say that this representation of N is *stable* if for every $t, 1 \leq t \leq k$,

$$\sum_{\theta=1}^{t} b_{i_\theta} < b_{i_t+1}.$$

Thus, in a stable representation of N, each member b_k of B is greater than the sum of all the summands b_{i_k} of N that are less than b_k.

Lemma 2: Let $B = (b_0 = 1, b_1, \ldots)$ be an infinite g-base. Then each positive integer N has exactly one stable representation. It is generated by the algorithm used in the proof of Lemma 1.

Proof: Let us first suppose that $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$ where $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$ is a stable representation of N. We show that this representation is unique and is generated by the algorithm of Lemma 1. The proof is by mathematical induction on N. For $N = 1$, the representation is certainly unique and generated by the algorithm. Next we show that b_{i_k} is uniquely generated by the algorithm. Let $b_s \leq N < b_{s+1}$. Then $b_{i_k} \leq N < b_{s+1}$. If $b_{i_k} < b_s$, then $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} < b_{i_k+1} \leq b_s$, contradicting the assumption that $b_s \leq N < b_{s+1}$. Therefore, $b_{i_k} \in B, b_{i_k} \geq b_s$, and $b_{i_k} < b_{s+1}$ which together imply that $b_{i_k} = b_s$. This means that b_{i_k} is unique and is computed by the algorithm. Now since $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$ is a stable representation of N, it follows from the definition of stable representation that $N - b_{i_k} = b_{i_1} + b_{i_2} + \cdots + b_{i_{k-1}}$ is a stable representation of $N - b_{i_k}$. Therefore, by induction we see that each of $b_{i_1}, b_{i_2}, \ldots, b_{i_{k-1}}$ is also unique and generated by the algorithm. We next show that any number N has at least one stable representation. To do this, let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$, be generated by the algorithm. We prove by induction on N that this representation is stable. Again the case $N = 1$ is trivial. Suppose $b_s \leq N < b_{s+1}$. Then by definition of the algorithm, $b_{i_k} = b_s$ and

$$N = \sum_{\theta=1}^{k} b_{i_\theta} < b_{s+1} = b_{i_k+1}.$$

Note that $N - b_{i_k} = b_{i_1} + b_{i_2} + \cdots + b_{i_{k-1}}$. Also, by definition of the algorithm, we see that each of $b_{i_1}, b_{i_2}, \ldots, b_{i_{k-1}}$ is generated by the algorithm using the number $N - b_{i_k}$. Therefore, by induction on $N - b_{i_k}$, we know that $b_{i_1} + b_{i_2} + \cdots + b_{i_{k-1}}$ is a stable representation of $N - b_{i_k}$. Therefore, by the definition of stable representation, we know that for every $1 \leq t \leq k-1$,

$$\sum_{\theta=1}^{t} b_{i_\theta} < b_{i_t+1}.$$

Therefore, for every $1 \leq t \leq k$,

$$\sum_{\theta=1}^{t} b_{i_\theta} < b_{i_t+1}. \quad \square$$

Generating g-bases: For every function $f : Z^+ \to Z^+$ satisfying

$$(*) \qquad f(n+1) - f(n) \geq -1,$$

we generate a g-base B_f as follows:
Let $b_0 = 1$. Suppose $(b_0, b_1, \ldots, b_k)$ have been generated. Then $b_{k+1} = b_k + b_i$, where b_i is the smallest member of $\{b_0, b_1, \ldots, b_k\}$ such that $f(b_i) \geq b_k$, if such a b_i exists. If no such b_i exists for some k, the base B_f is finite. In this part of the paper we assume that B_f is infinite. As an example, if $f(n) = 2n$, then $B_f = \{1, 2, 3, 5, 8 \ldots\}$ and we have what is call *Fibonacci Nim*.

For Lemmas 3 and 4 we assume that $B_f = (b_0 = 1, b_1, \dots)$ is the infinite g-base generated by a function f satisfying the inequality $(*)$, and that the positive integer N has stable representation $N = b_{i_1} + b_{i_2} + \dots + b_{i_k}$ with $b_{i_1} < b_{i_2} < \dots < b_{i_k}$.

Lemma 3: $f(b_{i_1}) < b_{i_2}$.

Proof: Because the representation is stable, $b_{i_1} + b_{i_2} < b_{i_2+1} \le b_{i_3}$. Now $b_{i_2+1} = b_{i_2} + b_i$ where b_i is the smallest member of $b_0, b_1, \dots, b_{i_2}$ such that $f(b_i) \ge b_{i_2}$. Since $b_{i_2} + b_{i_1} < b_{i_2+1}$, it follows that b_i is larger than b_{i_1}. Since b_i is the smallest member of $\{b_0, b_1, \dots, b_{i_2}\}$ such that $f(b_i) \ge b_{i_2}$, it follows that $f(b_{i_1}) < b_{i_2}$. □

Lemma 4: Suppose integer x satisfies $1 \le x < b_{i_1}$. Let $b_{i_1} - x = b_{j_1} + b_{j_2} + \dots + b_{j_t}$ be the stable representation of $b_{i_1} - x$ in B_f, where $b_{j_1} < b_{j_2} < \dots < b_{j_t}$. Then

(1) $N - x = b_{j_1} + b_{j_2} + \dots + b_{j_t} + b_{i_2} + b_{i_3} + \dots + b_{i_k}$ is the stable representation of $N - x$ in B_f and

(2) $b_{j_1} \le f(x)$.

Proof: The proof of (1) is trivial. The proof of (2) is by mathematical induction on t. We consider below two cases, the first of which takes care of $t = 1$.

Case (a): $1 \le b_{i_1} - x \le b_{i_1 - 1}$.

Case (b): $b_{i_1 - 1} < b_{i_1} - x < b_{i_1}$.

In case (a), we show that $f(x) \ge b_{i_1} - x$. Since $b_{i_1} - x = b_{j_1} + b_{j_2} + \dots + b_{j_t}$, it follows that $f(x) \ge b_{j_1}$. Now $b_{i_1} = b_{i_1 - 1} + b_i$ where b_i is the smallest member of $b_0, b_1, \dots, b_{i_1 - 1}$ such that $f(b_i) \ge b_{i_1 - 1}$. Therefore, $f(b_i) = f(b_{i_1} - b_{i_1 - 1} \ge b_{i_1 - 1}$. Note that the condition $f(n+1) - f(n) \ge -1$ can be used repeatedly to see that $f(n + N) - f(n) \ge -N$. Thus $f(n + N) \ge f(n) - N$ and

$$\begin{aligned} f(x) &= f(b_{i_1} - b_{i_1 - 1} + [b_{i_1 - 1} - (b_{i_1} - x)]) \\ &\ge f(b_{i_1} - b_{i_1 - 1}) - [b_{i_1 - 1} - (b_{i_1} - x)] \\ &= f(b_i) + b_{i_1} - b_{i_1 - 1} - x \\ &\ge b_{i_1 - 1} + b_{i_1} - b_{i_1 - 1} - x = b_{i_1} - x, \end{aligned}$$

since $f(b_i) \ge b_{i_1 - 1}$. That is, $f(x) \ge b_{i_1} - x$. Note that case (a) completely takes care of Lemma 4 when $t = 1$ and starts the mathematical induction on t.

Case (b) Since $b_{i_1} - x = b_{j_1} + b_{j_2} + \dots + b_{j_t}$ is stable where $b_{j_1} < b_{j_2} < \dots < b_{j_t}$ and since $b_{i_1 - 1} < b_{i_1} - x = b_{j_1} + b_{j_2} + \dots + b_{j_t} < b_{i_1}$, we know from Lemma 2 (or directly from the definition of stable itself) that $b_{j_t} = b_{i_1 - 1}$. Therefore,

$$\begin{aligned} x &= b_{i_1} - (b_{j_1} + b_{j_2} + \dots + b_{j_t}) \\ &= (b_{i_1} - b_{i_1 - 1}) - (b_{j_1} + b_{j_2} + \dots + b_{j_{t-1}}). \end{aligned}$$

Now $b_{i_1} = b_{i_1-1} + b_i$ where b_i is the smallest member of $b_0, b_1, \ldots, b_{i_1-1}$ such that $f(b_i) \geq b_{i_1-1}$. Therefore $x = b_i - (b_{j_1} + b_{j_2} + \cdots + b_{j_{t-1}})$; that is, $b_{j_1} + b_{j_2} + \cdots + b_{j_{t-1}} = b_i - x$. Of course, $b_{j_1} + b_{j_2} + \cdots + b_{j_{t-1}}$ is stable. Therefore, by mathematical induction, $f(x) \geq b_{j_1}$. □

Theorem 1 puts these four lemmas together to establish a strategy for playing dynamic one-pile nim optimally when B_f is infinite.

Theorem 1: Suppose the dynamic one-pile nim game with initial position (N, x) and move function f satisfying $(*)$ is given, and the g-base B_f is infinite. Also, let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$ be the stable representation of N in B_f, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$. Then the first player can win if $x \geq b_{i_1}$ and the second player can win if $x < b_{i_1}$.

Proof: Assuming $x \geq b_{i_1}$, the first player removes b_{i_1} counters. This move results in the position $(N - b_{i_1}, f(b_{i_1})) = (b_{i_2} + b_{i_3} + \cdots + b_{i_k}, f(b_{i_1}))$. Note that the number of summands in the stable representation of the pile size N of the position has been reduced. Also, the representation of $N - b_{i_1}$ is stable and, by Lemma 3, $f(b_{i_1}) < b_{i_2}$.

Thus the second player must remove fewer than b_{i_2} counters. Suppose the second player removes x' counters, where $1 \leq x' < b_{i_2}$. Thus the second player has moved to position $(N - b_{i_1} - x', f(x')) = (b_{i_2} + b_{i_3} + \cdots + b_{i_k} - x', f(x')) = (b_{j_1} + b_{j_2} + \cdots + b_{j_t} + b_{i_3} + \cdots + b_{i_k}, f(x'))$, where $b_{j_1} + b_{j_2} + \cdots + b_{j_t}$ is the stable representation of $b_{i_2} - x'$. By Lemma 4, parts 1 and 2, $b_{j_1} + b_{j_2} + \cdots + b_{j_t} + b_{i_3} + \cdots + b_{i_k}$ is the stable representation of $b_{i_2} + b_{i_3} + \cdots + b_{i_k} - x'$ and $b_{j_1} \leq f(x')$.

Note that the second player has not reduced the number of summands, and after his move, $b_{j_1} \leq f(x')$. The first player is therefore in a position analogous to the initial position, since $b_{j_1} \leq f(x')$. The first player can now reduce the pile by b_{j_1} counters, which again reduces the number of summands. Thus the first player can reduce the number of summands and the second player cannot. This means that the first player will eventually reduce the number of summands to zero, thereby winning.

When the initial position satisfies $x < b_{i_1}$, the second player wins by using the first player's strategy in the case above, that is, by reducing the pile size by the smallest number b_{i_1} that appear in the stable representation of the pile size. □

Next we discuss the case in which the g-base B_f is finite. Note that when f is bounded, B_f is finite. However, a finite g-base is possible even when f is unbounded. As an example consider $f : Z^+ \to Z^+$ defined by $f(1) = f(2) = f(3) = 2$ and $f(n) = n$ for all $n \geq 4$. This function satisfies the unit jump condition $f(n+1) = f(n) \geq -1$. Its g-base is $b_0 = 1, b_1 = b_0 + b_0 = 2, b_2 = b_1 + b_0 = 3$. Of course, b_3 does not exist because there is no member $b_i \in \{b_0, b_1, b_2\} = \{1, 2, 3\}$ such that $f(b_i) \geq b_2 = 3$. Thus the g-base is finite. The proofs of the following four lemmas and the theorem parallel very closely the proofs of the corresponding four lemmas and the theorem for infinite g-bases.

Lemma $1'$: Let $B = (b_0 = 1, b_1, b_2, \ldots, b_t)$ be a finite g-base. Then each positive integer N can be represented as a sum of distinct members of B allowing multiple copies of the largest element of B:

$$N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} + \theta b_t,$$

where $b_{i_1} < b_{i_2} < \cdots < b_{i_k} < b_t$ for some integer $\theta \geq 0$.

As we noted in the case for infinite g-bases, there may be multiple representations. Thus we have the following definition of stable representation.

Definition: Let $B = (b_0 = 1, b_1, \ldots, b_t)$ be a finite g-base. Suppose $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} + \theta b_t$, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k} < b_t$ and θ is a nonnegative integer. We say that this representation of N is *stable* if for every $h, 1 \leq h \leq k$,

$$\sum_{\phi=1}^{h} b_{i_\phi} < b_{i_h+1}.$$

Lemma $2'$: Let $B = (b_0 = 1, b_1, \ldots, b_t)$ be a finite g-base. Then each positive integer N has exactly one stable representation.

For Lemmas $3'$ and $4'$ we assume that $B_f = (b_0 = 1, b_1, \ldots, b_t)$ is the finite g-base generated by a function $f : Z^+ \to Z^+$ satisfying the inequality $(*)$, and that the positive integer N has stable representation $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} + \theta b_t$ with $b_{i_1} < b_{i_2} < \cdots < b_{i_k} < b_t$ and θ is a nonnegative integer.

Lemma $3'$: $f(b_{i_1}) < b_{i_2}$.

Note that for all $b_i \in B_f, f(b_i) < b_t$. This is why B_f is finite.

Lemma $4'$: Suppose integer x satisfies $1 \leq x < b_{i_1}$. Let $b_{i_1} - x = b_{j_1} + b_{j_2} + \cdots + b_{j_h}$ be the stable representation in B_f, where $b_{j_1} < b_{j_2} < \cdots < b_{j_h}$. Then

1. $N - x = b_{j_1} + b_{j_2} + \cdots + b_{j_h} + b_{i_2} + b_{i_3} + \cdots + b_{i_k} + \theta b_t$ is the stable representation of $N - x$ in B_f and
2. $b_{j_1} \leq f(x)$.

Theorem $1'$: Suppose the dynamic one-pile nim game with initial position (N, x) and move function f satisfying $(*)$ is given, and the g-base $B_f = (b_0 = 1, b_1, b_2, \ldots, b_t)$ is finite. Also, let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} + \theta b_t$ be the stable representation of N in B_f, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k} < b_t$. Then the first player can win if $x \geq b_{i_1}$ and the second player can win if $x < b_{i_1}$. In the special case where $N = \theta b_t$, the first player can win if $x \geq b_t$, and the second player can win if $x < b_t$.

We now turn our attention to the converse problem. Let $f : Z^+ \to Z^+$ be any function. We find necessary and sufficient conditions on f so that theorem 1 is true. We generate a

g-base $B_f = (b_0 = 1, b_1, \dots)$ from f just as before. For convenience, we assume B_f is infinite. Lemmas 1-3 remain true since they do not depend on the condition

$$(*) \qquad f(n+1) - f(n) \geq -1.$$

Also, in the proof of Lemma 4, only case (a) of property (2) used property $*$ on f.

Definition: For any positive integer N, let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$ be the stable representation of N in B_f, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$. Then we define $g(N) = b_{i_1}$. Also, $g(0) = g(-N) = 0$.

Lemma 5: Given $f : Z^+ \to Z^+$, theorem 1 is true for f if and only if Lemma 4 is true for f.

Proof: Obviously Lemma 4 implies Theorem 1. We now show that if Lemma 4 is false, then Theorem 1 is false. Since part (1) of Lemma 4 is trivial, we can use the definition of g to see that Lemma 4 is equivalent to the statement for all $b_\theta \in B_f$, and for all $1 \leq x < b_\theta$,

$$g(b_\theta - x) \leq f(x).$$

No matter what f is, $b_0 = 1, b_1 = 2, g(1) = 1, g(2) = 2$ holds. Therefore, $g(b_\theta - x) \leq f(x)$ holds when $b_\theta \in \{b_0, b_1\}$ and $1 \leq x < b_\theta$ for all f. Define b_ϕ to be the smallest member of $\{b_2, b_3, b_4, \dots\}$ such that $g(b_\phi - x) > f(x)$ for some $1 \leq x < b_\phi$. By definition of b_ϕ, this means that Lemma 4 is true for all $b_\theta \in \{b_0, b_1, \dots, b_{\phi-1}\}$ and all $1 \leq x < b_\theta$. This means that Theorem 1 holds for all positions (N, x) when $1 \leq N < b_\phi$ since the base members $b_\phi, b_{\phi+1}, b_{\phi+2}, \dots$ do not come into play when $N < b_\phi$. Next consider the position (b_ϕ, x) as described above. Of course, $1 \leq x < b_\phi$ and $g(b_\phi - x) > f(x)$. We will show that (b_ϕ, x) is an unsafe position, which contradicts Theorem 1. Let the first player remove x counters so that $(b_\phi, x) \mapsto (b_\phi - x, f(x))$. Since $b_\phi - x < b_\phi$, Theorem 1 correctly tells us whether $(b_\phi - x, f(x))$ is safe or unsafe. Because $f(x) < g(b_\phi - x)$, Theorem 1, along with the definition of g tells us that $(b_\phi - x, f(x))$ is a safe position. This means that (b_ϕ, x) is an unsafe position. □

Lemma 6: The necessary and sufficient conditions on f so that Lemma 4 holds is that for all $b_{i_1} \in \{b_1, b_2, \dots\}$, and for all $1 \leq b_{i_1} - x \leq b_{i_1 - 1}$, $g(b_{i_1} - x) \leq f(x)$.

Proof: First note that part (1) of Lemma 4 is a trivial statement and can be ignored. So what we are saying here is that Lemma 4 is true if and only if Lemma 4 is true for part (2), case (a). Note in part (2) that $b_{j_1} = g(b_{i_1} - x)$, from the definition of g, since $b_{i_1} - x = b_{j_1} + b_{j_2} + \cdots + b_{j_t}$ is the stable representation of $b_{i_1} - x$ in B_f and $b_{j_1} < b_{j_2} < \dots b_{j_t}$.

The reason Lemma 4 is true if and only if Lemma 4 is true for part (2), case (a) is that the only place in the proof of Lemma 4 where the property $*$ is used is in proving part (2), case (a). Since we have dropped the condition $*$ of f, the only way that we can now deal with part (2), case (a) is just to assume that Lemma 4 is always true for part (2), case (a). Thus part (2), case (a) becomes the necessary and sufficient condition on f for Lemma 4 to hold. □

Definition: For all nonnegative integers k, let

$$b_{\theta(k)} = b_{k+1} - b_k,$$

where $b_{\theta(k)} \in \{b_0, b_1, b_2, \ldots, b_k\}$.

Lemma 7: The following two conditions are equivalent.

1. For all $b_{k+1} \in \{b_1, b_2, \ldots\}$ and for all $1 \leq b_{k+1} - x \leq b_k$,

$$g(b_{k+1} - x) \leq f(x).$$

2. For all nonnegative integer k and for all nonnegative integer $\overline{x}$,

$$g(b_k - \overline{x}) \leq f(b_{\theta(k)} + \overline{x}).$$

Note that (1) is a restatement of the condition in Lemma 6. Also, (2) uses $g(0) = g(-N) = 0$.

Proof: We first show that (1) implies (2). Since $g(0) = g(-N) = 0$, let us assume $1 \leq b_k - \overline{x}$. Let $x = b_{\theta(k)} + \overline{x}$. Thus, $x = b_{k+1} - b_k + \overline{x}$. Therefore $1 \leq b_{k+1} - x = b_k - \overline{x} \leq b_k$. Hence from (1), $g(b_{k+1} - x) = g(b_k - \overline{x}) \leq f(x) = f(b_{\theta(k)} + \overline{x})$. That is $g(b_k - \overline{x}) \leq f(b_{\theta(k)} + \overline{x})$. We now show that (2) implies (1). Since $b_{k+1} - x \leq b_k$, define $\overline{x}$ by $b_{k+1} - x + \overline{x} = b_k$, where $\overline{x} \geq 0$. Therefore, $b_k - \overline{x} = b_{k+1} - x$. Also, $x = b_{k+1} - b_k + \overline{x} = b_{\theta(k)} + \overline{x}$. Therefore from (2), $g(b_k - \overline{x}) = g(b_{k+1} - x) \leq f(b_{\theta(k)} + \overline{x}) = f(x)$. That is, $g(b_{k+1} - x) \leq f(x)$. □

Main Theorem: Given $f : Z^+ \to Z^+$ with an infinite B_f, the necessary and sufficient conditions on f so that Theorem 1 holds for f is that for all nonnegative k and $\overline{x}$

$$g(b_k - \overline{x}) \leq f(b_{\theta(k)} + \overline{x}).$$

Since $g(N) \leq N$ observe that the following are sufficient but not necessary conditions on f for theorem 1 to hold: for all nonnegative integers k and $\overline{x}$, $f(b_{\theta(k)} + \overline{x}) \geq b_k - \overline{x}$. Recall that $f(b_{\theta(k)}) \geq b_k$ from the definition of B_f. From this it is easy to see that the original restriction (*) on f implies $f(b_{\theta(k)} + \overline{x}) \geq b_k - \overline{x}$.

The following theorem allows the Main Theorem to be used more efficiently since we only have to worry about $f(x)$ when x is not in the base B_f.

Theorem 2: Suppose that $f : Z^+ \to Z^+$ generates the infinite g-base $B_f = \{b_0 = 1, b_1, b_2, \ldots\}$, and f is non-decreasing on B_f. Then f satisfies the hypothesis of the main theorem if and only if the following is true for each x not in B_f. Suppose $b_t < x < b_{t+1}$. Also, suppose $b_{\theta(k)} < x < b_{k+1}$ if and only if $k \in \{t, t+1, t+2, \ldots, t+\overline{t}\}$. Then for this x, we require $g(b_{t+i} - x) \leq f(x)$ for $i = 1, 2, 3, \ldots, \overline{t} + 1$.

The proof of this, which uses part 1 of Lemma 7 is left to the reader. Using this theorem, we see that f generates the Fibonacci base $B_f = \{1, 2, 3, 5, 8, 13, \ldots\}$ and the main theorem is effective for f if and only if the following two conditions hold: a. for every $b_t \in B_f, b_{t+1} \leq f(b_t) < b_{t+2}$ and b. for all nonnegative integers t, and all x satisfying $b_t < x < b_{t+1}, g(b_{t+1} - x) \leq f(x)$. Note that $g(b_{t+1} - x) = g(b_{t+2} - x)$ when $b_t < x < b_{t+1}$, so $g(b_{t+2} - x) \leq f(x)$ is redundant.

THE MISERE VERSION

To win at the misere version (N, x) of dynamic nim, simply use the theory to win the game $(N - 1, x)$, so that your opponent is forced to take the last counter.

APPENDIX

We now discuss Theorem 2.1 of the Epp Ferguson paper. Let $f : Z^+ \to Z^+$ be an arbitrary function defining our one pile dynamic nim game. If a player is confronted with a pile size of $n \geq 1$, let $L(n)$ denote the smallest possible winning move. Of course, $L(n) \leq n$ and equality might hold. Note also that removing k counters from a pile of n is a winning move if and only if $f(k) < L(n-k)$, where $L(0) = \infty$. Theorem 2.1 (Epp, Ferguson): Suppose $f(k) < L(n - k)$. Then $k = L(n)$ if and only if $L(k) = k$. Epp and Ferguson prove this when f is non-decreasing. The reader can easily show that if f satisfies the condition of our main theorem, then $L(L(n)) = L(n)$ for all positive integers n. The following example shows that Theorem 2.1 breaks down when f is not non-decreasing.

Example: There exists f satisfying $f(n + 1) - f(n) \geq -1$ such that there exists $k < n$ with $f(k) < L(n - k), L(k) = k$, and $k \neq L(n)$.

Proof: Consider f defined by $f(n) = 8 - n$ when $1 \leq n \leq 7$ and $f(n) = n$ when $8 \leq n$. Then $B_f = \{1, 2, 3, 4, 5, 6, 7, 8, 16, 32, 64, 128, 256, \ldots\}$. Since $9 = 8 + 1$, we see that $L(9) = 1$. Consider the position (9,8). We see that the following are all winning moves:

$$\begin{aligned}
&(9,8) \mapsto (9-7, f(7)) = (2,1), L(7) = 7 \neq L(9) = 1, \\
&(9,8) \mapsto (9-6, f(6)) = (3,2), L(6) = 6 \neq L(9), \\
&\vdots \\
&(9,8) \mapsto (9-2, f(2)) = (7,6), L(2) = 2 \neq L(9). \quad \square
\end{aligned}$$

The reader might like to show that for the following $f, L(16) = 10$, and $L(10) \neq 10 : f(n) = n, n \neq 10$, and $f(10) = 1$. Of course this f does not satisfy the conditions of our main theorem.

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FIBONACCI NUMBERS AND PARTITIONS

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1. INTRODUCTION

In a series of two papers [6] and [7] Slater gave a list of 130 identities of the Rogers-Ramanujan type. In [2] Andrews has introduced a two variable function in order to look for combinatorial interpretations for those identities. In [5] one of us, Santos, gave conjectures for explicit formulas for families of polynomial that can be obtained using Andrews method for 74 identities of Slater's list.

In this paper we are going to prove the conjectures given by Santos in [5] for identities 94 and 99.

We show, also that the family of polynomials $P_n(q)$ related to identity 94 given by

$$\begin{aligned} &P_0(q)=1,\ P_1(q)=1+q+q^2\\ &P_n(q)=(1+q+q^{2n})P_{n-1}(q)-qP_{n-2}(q) \end{aligned} \tag{1.1}$$

is the generating function for partitions into at most n parts in which every even smaller than the largest part appears at least once and that the family $T_n(q)$ related to identity 99 given by

$$\begin{aligned} &T_0(q)=1,\ T_1(q)=1+q^2\\ &T_n(q)=(1+q+q^{2n})T_{n-1}(q)-qT_{n-2}(q) \end{aligned} \tag{1.2}$$

is the generating function for partitions into at most n parts in which the largest part is even and every even smaller than the largest appears at least once.

In what follows we denote the Fibonacci numbers by F_n where $F_0=0; F_1=1$ and $F_n=F_{n-1}+F_{n-2}, n\geq 2$, and use the standard notation

$$(A;q)_n=(1-A)(1-Aq)\dots(1-Aq^{n-1})$$

and

$$(A;q)_\infty = \prod_{n=0}^{\infty}(1-Aq^n), \quad |q|<1.$$

We need also the following identities for the Gaussian polynomials

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix} \tag{1.3}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \tag{1.4}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^{m}\begin{bmatrix} n-1 \\ m \end{bmatrix} \tag{1.5}$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q;q)_m}{q;q)_m(q;q)_{n-m}}, \text{ for } 0 \le m \le n, \tag{1.6}$$

$$0 \quad \text{otherwise}$$

2. THE FIRST FAMILY OF POLYNOMIALS

We consider now the two variable function associated to identity 94 of Slater [7] which is:

$$f_{94}(q,t) = \sum_{n=0}^{\infty} \frac{t^n q^{n^2+n}}{(t;q^2)_{n+1}(tq;q^2)_{n+1}}. \tag{2.1}$$

From this we have that

$$(1-t)(1-tq)f_{94}(q,t) = 1 + tq^2 f_{94}(q;tq^2)$$

and in order to obtain a recurrence relation from this functional equation we make the following substitution

$$f_{94}(q,t) = \sum_{n=0}^{\infty} P_n t^n.$$

Now we have:

$$(1-t)(1-tq)\sum_{n=0}^{\infty} P_n t^n = 1 + tq^2 \sum_{n=0}^{\infty} P_n (tq^2)^n$$

which implies

$$\sum_{n=0}^{\infty} P_n t^n - \sum_{n=1}^{\infty} P_{n-1} t^n - \sum_{n=1}^{\infty} qP_{n-1} t^n + \sum_{n=2}^{\infty} qP_{n-2} t^n = 1 + \sum_{n=1}^{\infty} q^{2n} P_{n-1} t^n.$$

From this last equation it is easy to see that

$$\begin{aligned} &P_0(q) = 1;\ P_1(q) = 1 + q + q^2 \\ &P_n(q) = (1 + q + q^{2n})P_{n-1}(q) - qP_{n-2}(q). \end{aligned} \tag{2.2}$$

Santos gave in [5] a conjecture $C_n(q)$, for an explicity formula for this family of polynomials:

$$C_n(q) = \sum_{j=-\infty}^{\infty} q^{q^{15j^2+4j}} \begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum q^{15j^2+14j+3} \begin{bmatrix} 2n+1 \\ n-5j-2 \end{bmatrix}. \tag{2.3}$$

In our next theorem we prove that this conjecture is true.

Theorem 2.1: The family $P_n(q)$ given in (2.2) is equal to $C_n(q)$ given in (2.3).

Proof: Considering that $C_0(q) = 1$ and $C_1(q) = 1 + q + q^2$ we have to show that

$C_n(q) = (1 + q + q^{2n})C_{n-1}(q) - qC_{n-2}(q)$ that is:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j} \begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3} \begin{bmatrix} 2n+1 \\ n-5j-2 \end{bmatrix}$$

$$= (1+q+q^{2n}) \left(\sum_{j=-\infty}^{\infty} q^{15j^2+4j} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3} \begin{bmatrix} 2n-1 \\ n-5j-3 \end{bmatrix} \right)$$

$$- q \left(\sum_{j=-\infty}^{\infty} q^{15j^2+4j} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3} \begin{bmatrix} 2n-3 \\ n-5j-4 \end{bmatrix} \right) \tag{2.4}$$

If we apply (1.4) in each expression on the left side of (2.4) we get

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j} \begin{bmatrix} 2n \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1} \begin{bmatrix} 2n \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3} \begin{bmatrix} 2n \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n} \begin{bmatrix} 2n \\ n-5j-3 \end{bmatrix}$$

Applying now (1.5) to each sum in the expression above and replacing it in (2.4) we get after some cancellations

$$\sum_{j=-\infty}^{\infty} q^{15j^2+j+n} \begin{bmatrix} 2n-1 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+4} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n} \begin{bmatrix} 2n-1 \\ n-5j-4 \end{bmatrix}$$

$$= \sum_{j=-\infty}^{\infty} q^{15j^2+4j+1} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4} \begin{bmatrix} 2n-1 \\ n-5j-3 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+4j+1} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4} \begin{bmatrix} 2n-3 \\ n-5j-4 \end{bmatrix}. \tag{2.5}$$

Considering the right side of the last expression and applying (1.4) on the first two sums we get

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j+1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+9j+1+n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+4j+1} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4} \begin{bmatrix} 2n-3 \\ n-5j-4 \end{bmatrix}$$

Applying now (1.5) on the first and third sums on this last expression and making some cancellations we have that the right side of (2.5) is equal to:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-j+n}\begin{bmatrix}2n-3\\n-5j-1\end{bmatrix}+\sum_{j=-\infty}^{\infty} q^{15j^2+9j+1+n}\begin{bmatrix}2n-2\\n-5j-2\end{bmatrix}$$

$$-\sum_{j=-\infty}^{\infty} q^{15j^2+9j+j+1+n}\begin{bmatrix}2n-3\\n-5j-3\end{bmatrix}-\sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n}\begin{bmatrix}2n-2\\n-5j-4\end{bmatrix}$$

If we take now the left side of (2.5) and apply (1.4) to all sums we get:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-j+n}\begin{bmatrix}2n-2\\n-5j\end{bmatrix}+\sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1}\begin{bmatrix}2n-2\\n-5j-1\end{bmatrix}$$

$$+\sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1}\begin{bmatrix}2n-2\\n-5j-2\end{bmatrix}+\sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2}\begin{bmatrix}2n-2\\n-5j-3\end{bmatrix}$$

$$-\sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1}\begin{bmatrix}2n-2\\n-5j-2\end{bmatrix}-\sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2}\begin{bmatrix}2n-2\\n-5j-3\end{bmatrix}$$

$$-\sum_{j=-\infty}^{\infty} q^{15j^2+14j+n+6}\begin{bmatrix}2n-2\\n-5j-4\end{bmatrix}-\sum_{j=-\infty}^{\infty} q^{15j^2+24j+2n+9}\begin{bmatrix}2n-2\\n-5j-5\end{bmatrix}. \tag{2.6}$$

Applying now (1.5) on the first and fifth sums of this last expression and making cancellations with the sums from the right side given in (2.6) we are left with:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-6j+2n}\begin{bmatrix}2n-3\\n-5j\end{bmatrix}+\sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1}\begin{bmatrix}2n-2\\n-5j-1\end{bmatrix}$$

$$+\sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2}\begin{bmatrix}2n-2\\n-5j-3\end{bmatrix}-\sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1}\begin{bmatrix}2n-3\\n-5j-2\end{bmatrix}$$

$$-\sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2}\begin{bmatrix}2n-2\\n-5j-3\end{bmatrix}-\sum_{j=-\infty}^{\infty} q^{15j^2+24j+2n+9}\begin{bmatrix}2n-2\\n-5j-5\end{bmatrix}.$$

Observing that the third sum cancels the fifth and replacing j by $j+1$ in the last sum we get after using (1.4)

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n+1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2-j+3n-2} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix}$$

which is identically zero by (1.5) completing the proof. □

Next we make a few observations regarding the combinatorics of $P_N(q)$ given in (2.2). Knowing that $P_n(q)$ is the coefficient of t^N in (2.1) that is:

$$\sum_{n=0}^{\infty} \frac{t^n q^{n^2+n}}{(1-t)(tq^2;q^2)_n (tq;q^2)_{n+1}}$$

and considering that $n^2+n = 2+4+\cdots+2n$ we can see that the coefficient of t^N in

$$\frac{t^n q^{n^2+n}}{(tq^2;q^2)_n (tq;q^2)_{n+1}}$$

is the generating function for partitions into exactly N parts in which every even smaller than the largest part appears at least once. Because of the factor $(1-t)$ in the denominator we have proved the following theorem:

Theorem 2.2: $P_n(q)$ is the generating function for partitions into at most N parts in which every even smaller than the largest part appears at least once.

To see, now, the connection between the family of polynomials $P_N(q)$ and the Fibonacci numbers we observe first that if we replace q by 1 in (2.2) we have

$$P_0(1) = 1; P_1(1) = 3$$
$$P_n(1) = 3P_{n-1}(1) - P_{n-2}(1)$$

and that for the Fibonacci sequence F_n we have also that $F_2 = 1;\ F_4 = 3$ and

$$F_{2n+2} = 3F_{2n} - F_{2n-2}$$

which allow us to conclude that

$$C_n(1) = P_n(1) = F_{2n+2}$$

and from these considerations we have proved the following:

Theorem 2.3: The total number of partitions into at most N parts in which every even smaller than the largest part appears at least once is equal to F_{2N+2}.

The family given in (2.2) has also an interesting property at $q = -1$. At this point we have

$$P_0(-1) = 1;\ P_1(-1) = 1$$
$$P_n(-1) = P_{n-1}(-1) + P_{n-2}(-1)$$

which tell us that for $q = -1$ we have all the Fibonacci numbers, i.e. $P_n(-1) = F_{n+1}$. In order to be able to see what happens combinatorially at -1 we have to observe that when we change q by $-q$ in (2.1) the only term that changes is $(tq; q^2)_{n+1}$ and that now the coefficient of t^N is going to be just the number of partitions as described in Theorem 2.3 having an even number of odd parts minus the number of partions of that type with an odd number of odd parts. We state this in our next theorem.

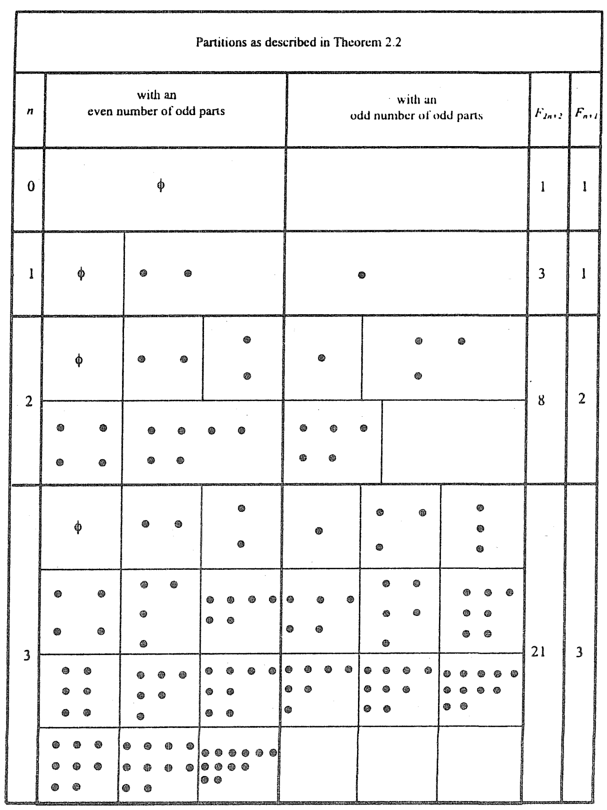

Table 2.1

Theorem 2.4: The total number of partitions into at most N parts in which every even smaller than the largest part appears at least once and having an even number of odd parts minus the number of those with an odd number of odd parts is equal to F_{N+1}.

In the table (2.1) we present, for a few values of n, all the results proved so far. The first column has n, the second the partitions described in theorem 2.4 with an even number of odd parts and the third column those with an odd number of odd parts. The fourth column has F_{2n+2} which is the total number of partitions in columns 2 and 3 and the fifth column has the difference between the number of partitions on the second and third column which is F_{n+1}.

3. THE SECOND FAMILY OF POLYNOMIALS

Now we consider the two variable function given in Santos [5] associated to identity 99 of Slater [7] which is:

$$f_{99}(q,t) = \sum_{n=0}^{\infty} \frac{t^n q^{n^2+n}}{(t;q^2)_{n+1}(tq;q^2)_n} \tag{3.1}$$

From this we can get

$$(1-t)(1-tq)f_{99}(q,t) = 1 - tq + tq^2 f_{99}(q,tq^2)$$

from which we obtain in a way similar to the one used to get (2.2) the following family of polynomials

$$\begin{aligned} &T_0(q) = 1;\ T_1(q) = 1 + q^2 \\ &T_n(q) = (1 + q + q^{2n})T_{n-1}(q) - qT_{n-2}(q) \end{aligned} \tag{3.2}$$

As for the family (2.2) Santos gave in [5] a conjecture for an explicity formula for (3.2) which is

$$B_n(q) = \sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n+1 \\ n-5j-1 \end{bmatrix} \tag{3.3}$$

The proof for this conjecture is given in the next theorem.

Theorem 3.1: The family $T_n(q)$ given in (3.2) is equal to $B_n(q)$ given in (3.3).

Proof: Considering that $B_0(q) = 1$ and $B_1(q) = 1 + q^2$ we have to show that $B_n(q) = (1 + q + q^{2n})B_{n-1}(q) - qB_{n-2}(q)$ which is:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n+1 \\ n-5j-1 \end{bmatrix}$$

$$= (1+q+q^{2n}) \left(\sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} \right)$$

$$- q \left(\sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n-3 \\ n-5j-3 \end{bmatrix} \right) \tag{3.4}$$

We apply (1.4) on each sum on the left to get

$$\sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1} \begin{bmatrix} 2n \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+2} \begin{bmatrix} 2n \\ n-5j-2 \end{bmatrix}$$

Applying now, (1.5) in all sums we obtain:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+3j} \begin{bmatrix} 2n-1 \\ n-5j \end{bmatrix}$$

$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+2j+2n} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+3j+n} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+2} \begin{bmatrix} 2n-1 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix}$$

Replacing this in (3.4) and making cancellations we are left with:

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} q^{15j^2-3j+n}\begin{bmatrix}2n-1\\n-5j\end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1}\begin{bmatrix}2n-1\\n-5j-2\end{bmatrix}\\
&-\sum_{j=-\infty}^{\infty} q^{15j^2+3j+n}\begin{bmatrix}2n-1\\n-5j-1\end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+3}\begin{bmatrix}2n-1\\n-5j-3\end{bmatrix}\\
&=\sum_{j=-\infty}^{\infty} q^{15j^2+2j+1}\begin{bmatrix}2n-1\\n-5j-1\end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2}\begin{bmatrix}2n-1\\n-5j-2\end{bmatrix}\\
&-\sum_{j=-\infty}^{\infty} q^{15j^2+2j+1}\begin{bmatrix}2n-3\\n-5j-2\end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2}\begin{bmatrix}2n-3\\n-5j-3\end{bmatrix}
\end{aligned}\tag{3.5}$$

Applying (1.4) on the first two sums on the right side of this last expression we get for that side:

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} q^{15j^2+2j+1}\begin{bmatrix}2n-2\\n-5j-1\end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1}\begin{bmatrix}2n-2\\n-5j-2\end{bmatrix}\\
&-\sum_{j=-\infty}^{\infty} q^{15j^2+8j}\begin{bmatrix}2n-2\\n-5j-2\end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+1}\begin{bmatrix}2n-2\\n-5j-3\end{bmatrix}\\
&-\sum_{j=-\infty}^{\infty} q^{15j^2+2j+1}\begin{bmatrix}2n-3\\n-5j-2\end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2}\begin{bmatrix}2n-3\\n-5j-3\end{bmatrix}
\end{aligned}$$

Using (1.5) on the first and third sums we get after cancellations

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} q^{15j^2-3j+n}\begin{bmatrix}2n-3\\n-5j-1\end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1}\begin{bmatrix}2n-2\\n-5j-2\end{bmatrix}\\
&-\sum_{j=-\infty}^{\infty} q^{15j^2+3j+n}\begin{bmatrix}2n-3\\n-5j-2\end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+2+n}\begin{bmatrix}2n-2\\n-5j-3\end{bmatrix}
\end{aligned}$$

Applying (1.4) in all sums on the left side of (3.5) and making cancellations with the corresponding sums on the right we get:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-3j+n} \begin{bmatrix} 2n-2 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+2j+2n-1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$

$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+12j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+3j+n} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+18j+2n+4} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix}$$

$$= \sum_{j=-\infty}^{\infty} q^{15j^2-3j+n} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+3j+n} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix}$$

Using (1.5) on the first and fourth sums on the LHS we get:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-8j+2n} \begin{bmatrix} 2n-3 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+2j+2n-1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$

$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+12j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2-2j+2n-1} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+18j+5} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix} = 0.$$

Replacing j by $j-1$ in the last sum and using (1.3) that sum cancels with the third.

If we replace j by $-j$ in the fourth sum using (1.3) and subtract from the second by (1.4) we get finally:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-8j+2n} \begin{bmatrix} 2n-3 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2-3j+3n-2} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} = 0.$$

To see that this expression is, in fact, identically zero we apply (1.4) on the first two sums replacing j by $-j$ and using (1.3) on the result which completes the proof.

Considering that $T_N(q)$ is the coefficient of t^N in the sum

$$\sum_{n=0}^{\infty} \frac{t^n q^{n^2+n}}{(1-t)(tq^2;q^2)_n(tq;q^2)_n}$$

and observing again that $n^2+n = 2+4+\cdots+2n$ we see that the coefficient of t^N in

$$\frac{t^n q^{n^2+n}}{(tq^2;q^2)_n(tq;q^2)_n}$$

is the generating function for partitions into exactly N parts in which the largest part is even and every even smaller the largest part appears at least once. From the presence of the factor $(1-t)$ in the denominator we have proved the following theorem:

Theorem 3.2: $T_n(q)$ is the generating function for partitions into at most N parts in which the largest part is even and every even smaller than the largest appears at least once.

Replacing now q by 1 in (3.2) we get

$$T_0(1) = 1;\ T_1(1) = 2$$
$$T_n(1) = 3T_{n-1}(1) - T_{n-2}(1).$$

But for F_n we have

$$F_1 = 1;\ F_3 = 2$$
$$F_{2n+1} = 3F_{2n-1} - F_{2n-3}$$

which allow us to conclude that

$$B_n(1) = T_n(1) = F_{2n+1}$$

and by these results we have proved.

Theorem 3.3: The total number of partitions into at most N parts in which the largest part is even and every even smaller than the largest part appears at least once is equal to F_{2n+1}.

For family (3.2) we have also that, at $q=-1$, we get all the Fibonacci numbers F_n, $n \geq 2$.

$$T_0(-1) = 1;\ T_1(-1) = 2$$
$$T_n(-1) = T_{n-1}(-1) + T_{n-2}(-1)$$

i.e., $T_n(-1) = F_{n+2}, n \geq 0$.

If we make the same observation that have made for the first family of polynomials regarding the combinatorial interpretation at $q = -1$ we have proved the following result:

Theorem 3.4: The total number of partitions into at most N parts in which the largest part is even and every even smaller than the largest part appears at least once and having an even number of odd parts minus the number of those with an odd number of odd parts is equal to F_{N+2}.

Partitions as described in Theorem 3.3				
n	with an even number of odd parts	with an odd number of odd parts	F_{2n+1}	F_{n+2}
0	ϕ		1	1
1	ϕ ; • •		2	2
2	ϕ ; • • ; • • • • ; • • • • • •	• • •	5	3
3	ϕ ; • • ; • • • • ; • • • • ; • • • • • • ; • • • • • • ; • • • • • • • • ; • • • • • • • • • • ; • • • • • • • • • • • •	• • • ; • • • • • ; • • • • • • • ; • • • • • • • • •	13	5

Table 3.1

In the table (3.1) we present, for a few values of n, all the results proved in this section. The first column has n, the second the partitions described in Theorem 3.3 with an even number of odd parts and the third column those with an odd number of odd parts. The fourth column has F_{2n+1} which is the total number of partitions in columns 2 and 3 and the fifth column has the difference between the number of partitions on the second and third column which is F_{n+2}.

4. A FORMULA FOR F_n

Using the fact that the Gaussian polynomials given in (1.6) are q-analogue of the binomial coefficient, i.e., that

$$\lim_{q\to 1}\begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m}$$

we may take the limits as q approaches 1 in (2.3) and (3.3) to get

$$\lim_{q\to 1} C_n(q) = \lim_{q\to 1}\left(\sum_{j=-\infty}^{\infty} q^{15j^2+4j}\begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3}\begin{bmatrix} 2n+1 \\ n-5j-2 \end{bmatrix}\right)$$

$$= \sum_{j=-\infty}^{\infty}\left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-2}\right] = C_n(1)$$

and

$$\lim_{q\to 1} B_n(q) = \lim_{q\to 1}\left(\sum_{j=-\infty}^{\infty} q^{15j^2+2j}\begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1}\begin{bmatrix} 2n+1 \\ n-5j-1 \end{bmatrix}\right)$$

$$= \sum_{j=-\infty}^{\infty}\left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-1}\right] = B_n(1)$$

But as we have observed

$$C_n(1) = F_{2n+2} \text{ and } B_n(1) = F_{2n+1}$$

which tell us that

$$F_{2n+2} = \sum_{j=-\infty}^{\infty} \left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-2} \right] \tag{4.1}$$

and

$$F_{2n+1} = \sum_{j=-\infty}^{\infty} \left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-1} \right] \tag{4.2}$$

5. LATTICE PATHS AND FIBONACCI NUMBERS

In this section we are going to show how to express the Fibonacci numbers in terms of lattice path.

In Narayana [4], Lemma 4A one can find the following formula

$$|L(m,n;t,s)| \sum_{j=-\infty}^{\infty} \left[\binom{m+n}{m-k(t+s)} - \binom{m+n}{n+k(t+s)+t} \right] \tag{5.1}$$

which give the total number of lattice paths from the origin to (m,n) not touching the lines $y = x - t$ and $y = x + s$.

But considering that we can write (4.1) and (4.2) as follows

$$F_{2n+2} = \sum_{j=-\infty}^{\infty} \left[\binom{n+(n+1)}{n-j(2+3)} - \binom{n+(n+1)}{n+1+j(2+3)+2} \right] \tag{5.2}$$

$$F_{2n+1} = \sum_{j=-\infty}^{\infty} \left[\binom{n+(n+1)}{n-j(1+4)} - \binom{n+(n+1)}{(n+1)+j(1+4)+1} \right] \tag{5.3}$$

we can conclude just by comparing (4.4) and (4.5) with (4.3) that the following theorem holds:

Theorem 5.1: F_{2n+i} is the number of lattice paths from the origin to $(n, n+1)$ not touching the line $y = x - i$ and $y = x + 5 - i$, where $i = 1, 2$.

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A CLASS OF FIBONACCI IDEAL LATTICES IN $\mathbb{Z}[\zeta_{12}]$

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1. INTRODUCTION

Lattices occur in many different areas of science and engineering. They are used to define dense sphere packings in n-dimensional spaces [5], and direct applications of them are found in number theory, in particular, to solve Diophantine equations [1]. There are further applications found in numerical analysis, for example, when evaluating n-dimensional integrals [5, p. 11-12]. In modern digital communication systems, lattice constellations are used to send encoded information through noisy channels, [3, 5, 10]. In this application, lattices with dense sphere packings are desirable. Recently it has been shown that algebraic lattices (those originating from rings of integers via canonical embedding of number fields) can be linearly labeled by elements of a finite field, facilitating the encoding and decoding processes [6]. From the mid-nineties on, concrete applications of lattices began appearing in cryptography [9]. In particular, the NP-hardness of the famous lattices shortest vector problem, namely the problem of finding a lattice point nearest to the origin, was proved by Ajtai [2] in 1997. Similar tools were used to study the hardness of the most significant bits of the secret keys in the Diffie-Hellman and related schemes in prime fields [9, p. 14]. Recall the Diffie-Hellman key exchange protocol: Alice and Bob fix a finite cyclic group G and a generator g. They respectively pick random $a, b \in [1, |G|]$ and exchange g^a and g^b. The secret key is g^{ab}. An interesting realization of this public key exchange is based on quadratic number fields with large class number [8, p. 261] where the cyclic group is provided by the class groups. Proving the security of the Diffie-Hellman protocal has been a challenging problem in cryptography.

It has long been known that several dense lattices are algebraic and, in particular, originate from ideals in rings of integers. We refer to these lattices as *ideal lattices*. Remarkably, the densest four-dimensional lattice, namely D_4, is generated by the ideal $(1-\zeta_8)\mathbb{Z}[\zeta_8]$ where ζ_8 is a primitive eighth root of unity. A good measure of packing density is the center density, defined as the ratio between the lattice density (the proportion of the maximum space that is occupied by nonoverlapping spheres centered in lattice points) and the volume of a sphere of radius one [5, p. 13].

Let $\mathbb{F}$ be an algebraic number field generated by a root of $m(x)$, an irreducible polynomial of degree n over $\mathbb{Q}$. Let us assume that $m(x)$ has r_1 real roots and $2r_2$ complex roots. The center density γ of an ideal lattice of $\mathbb{F}$ is given by

$$\gamma = \left(\frac{d_m}{2}\right)^n \frac{1}{N_{\mathbb{F}}(\mathcal{J})\sqrt{d(\mathbb{F})/2^{2r_2}}}$$

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where $d(\mathbb{F})$ is the field discriminant, $N_{\mathbb{F}}(\mathcal{J})$ is the ideal norm, and d_m^2 is the minimum square Euclidean distance between lattice points, see [5, p. 10] or [7, Exercise 2.43]. D_4 is the only four-dimensional lattice possessing a center density equal to $\frac{1}{8}$, [5, p.9], the maximum achievable in that dimension. On the other hand, in this paper we exhibit a sequence of lattice (Λ_n) generated by principal ideals $(F_n - \zeta_{12}F_{n+2})\mathbb{Z}[\zeta_{12}]$ in $\mathbb{Z}[\zeta_{12}]$ whose center densities approach $\frac{1}{8}$ asymptotically. The sequence (z_n) of complex numbers where $z_n = F_n - \zeta_{12}F_{n+2},\ z_0 = -\zeta_{12}$, and $z_1 = 1 - 2\zeta_{12}$ satisfies Fibonacci's recurrence (see [4]), and so we refer to Λ_n as *Fibonacci ideal lattices*. We show that the center density γ_n of Λ_n is a rational number $\frac{\delta_n^2}{48\cdot\Delta_n}$ which approaches $\frac{1}{8}$ asymptotically as n goes to infinity. The integers δ_n and Δ_n satisfy two linear recurring sequences related to Fibonacci and Lucas numbers. The theta series [5, p.45] $\Theta_{\Lambda_n}(z) = \sum_{x\in\Lambda_n} q^{x\cdot x}$, where z is a complex variable and $q = e^{\pi i z}$, is an expression made of Jacobi theta functions. The Λ_n are definitively different from D_4 because the respective kissing numbers are 12 and 24. The kissing number of a sphere packing in any dimension is defined as the number of spheres that touch one sphere [5]. Given a lattice Λ in $\mathbb{R}^N$ with minimum distance d_m, we can think of the points of Λ as being centers of equal nonoverlapping N-spheres of radius $d_m/2$. Then the kissing number of Λ is the kissing number of this packing just described. Notice that the theta series of Λ provides us with the kissing number τ of Λ, since $\Theta(z) = 1 + \tau q^{d_m^2} + \ldots$ [5].

The following sequences related to Fibonacci and Lucas numbers will be used in the proofs:

$$a_n = F_n^2 + F_{n+2}^2 = \frac{1}{5}(3L_{2n+2} + 4(-1)^{n+1}); \tag{1}$$

$$b_n = F_nF_{n+2} = F_{n+1}^2 + (-1)^{n+1} = \frac{1}{5}(L_{2n+2} + 3(-1)^{n+1}); \tag{2}$$

$$a_n - 3b_n = (-1)^n. \tag{3}$$

The golden section $\omega = \frac{1+\sqrt{5}}{2}$ and $\overline{\omega} = 1 - w$ are the roots of $x^2 - x - 1$ [11].

2. CENTER DENSITY

An integral basis for the ring $\mathbb{Z}[\zeta_{12}]$ is $\mathcal{B} = \{1, \zeta_{12}, \zeta_{12}^2, \zeta_{12}^3\}$ where ζ_{12} is a root of the cyclotomic polynomial $x^4 - x^2 + 1$. A real embedding σ yields the generator matrix of Λ_0

$$B_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

A generator matrix B_n of Λ_n is obtained as the product $B_0 M(z_n)$, where $M(z_n)$ belongs to an integral matrix representation of $\mathbb{Z}[\zeta_{12}]$ with respect to basis $\mathcal{B}$. We have

$$M(\zeta_{12}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \qquad M(z_n) = \begin{bmatrix} F_n & -F_{n+2} & 0 & 0 \\ 0 & F_n & -F_{n+2} & 0 \\ 0 & 0 & F_n & -F_{n+2} \\ F_{n+2} & 0 & -F_{n+2} & F_n \end{bmatrix},$$

and

$$B_n = B_0 M(z_n) = \begin{bmatrix} F_n & -F_{n+2} & F_n & -F_{n+2} \\ \frac{\sqrt{3}}{2}F_n + \frac{1}{2}F_{n+2} & -\frac{\sqrt{3}}{2}F_{n+2} + \frac{1}{2}F_n & -F_{n-2} - \frac{\sqrt{3}}{2}F_n & \frac{\sqrt{3}}{2}F_{n+2} + \frac{1}{2}F_n \\ -\frac{\sqrt{3}}{2}F_{n+2} + \frac{1}{2}F_n & \frac{\sqrt{3}}{2}F_n - \frac{1}{2}F_{n+2} & \frac{1}{2}F_n & -\frac{\sqrt{3}}{2}F_n - \frac{1}{2}F_{n+2} \\ F_{n+2} & F_n & -2F_{n+2} & F_n \end{bmatrix}$$

The squared Euclidean norm in Λ_n is given by the quadratic form $Q(x) = x^T B_n^T B_n x$ with $x \in \mathbb{Z}^4$. The positive definite symmetric matrix of this quadratic form results in

$$A_n = B_n^T B_n = \begin{bmatrix} 2(F_n^2 + F_{n+2}^2) & -3F_nF_{n+2} & F_n^2 + F_{n+2}^2 & 0 \\ -3F_nF_{n+2} & 2(F_n^2 + F_{n+2}^2) & -3F_nF_{n+2} & F_n^2 + F_{n+2}^2 \\ F_n^2 + F_{n+2}^2 & -3F_nF_{n+2} & 2(F_n^2 + F_{n+2}^2) & -3F_nF_{n+2} \\ 0 & F_n^2 + F_{n+2}^2 & -3F_nF_{n+2} & 2(F_n^2 + F_{n+2}^2) \end{bmatrix}.$$

Writing $Q(x) = x^T A_n x = x^T (U^{-1})^T U^T A_n U U^{-1} x = x^T (U^{-1})^T C_n U^{-1} x$, we consider the transformation of A_n by the matrices

$$U = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \qquad \text{and } U^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix},$$

to produce a block diagonal matrix

$$C_n = \begin{bmatrix} 2(F_n^2 + F_{n+2}^2) & -3F_nF_{n+2} & 0 & 0 \\ -3F_nF_{n+2} & \frac{3}{2}(F_n^2 + F_{n+2}^2) & 0 & 0 \\ 0 & 0 & \frac{3}{2}(F_n^2 + F_{n+2}^2) & -3F_nF_{n+2} \\ 0 & 0 & -3F_nF_{n+2} & 2(F_n^2 F_{n+2}) \end{bmatrix}.$$

Thus, setting $g_n = 2a_n^2 + 2(-1)^n a_n - 1$ and making use of the identity (3), $Q(x)$ is written as a linear combination of four squares

$$Q(x) = \frac{1}{2a_n}\{[a_n(2x_1+x_3-x_2)+(-1)^n x_2]^2+g_n x_2^2+g_n x_3^2+[a_n(2x_4+x_2-x_3)+(-1)^n x_3]^2\}. \quad (4)$$

This expression is conveniently written as $Q(x) = Q(x_1, x_3, x_2) + Q(x_4, x_2, x_3)$, by defining

$$Q(u_1, u_2, u_3) = \frac{1}{2a_n}\{[a_n(2u_1 + u_2 - u_3) + (-1)^n u_3]^2 + g_n u_3^2\}.$$

The center density γ_n of a Fibonacci ideal lattice is $\gamma_n = \frac{d_m^4}{4N_{\mathbb{F}}(z_n)\sqrt{d(\mathbb{F})}}$, where $d(\mathbb{F}) = 144$, the norm of the principal ideal $z_n\mathbb{Z}[\zeta_{12}]$ is the field norm of z_n

$$N_{\mathbb{F}}(z_n) = \Delta_n = F_n^4 - F_n^2 F_{n+2}^2 + F_{n+2}^4 = a_n^2 - 3b_n^2,$$

and, given (4), the squared minimum distance is

$$d_m^2 = \delta_n = 2(F_n^2 + F_{n+2}^2) - (1 - (-1)^n) = 2a_n - 1 + (-1)^n. \quad (5)$$

Therefore,

$$\gamma_n = \frac{[2a_n - (1-(-1)^n)]^2}{16 \cdot [3a_n^2 - 9b_n^2]} = \frac{[2a_n - (1-(-1)^n)]^2}{48 \cdot [2a_n^2 + 2(-1)^n a_n - 1]} = \frac{\delta_n^2}{48 \cdot \Delta_n} \asymp \frac{1}{8} + O\left(\frac{1}{a_n}\right),$$

where the asymptotic expression shows that the convergence is exponential as n goes to infinity. Some initial terms are

$$\gamma_0 = \frac{1}{12}, \gamma_1 = \frac{4}{39}, \gamma_2 = \frac{25}{219}, \gamma_3 = \frac{196}{1623}, \gamma_4 = \frac{5329}{43212}, \gamma_5 = \frac{37249}{299532}, \gamma_6 = \frac{255025}{2044236}.$$

Sequence Δ_n. The sequence $\Delta_n = F_n^4 - F_n^2 F_{n+2}^2 + F_{n+2}^4 = (F_n^2 + F_{n+2}^2)^2 - 3F_n^2 F_{n+2}^2$ satisfies a fifth order linear recurrence

$$\Delta_{n+5} = 5\Delta_{n+4} + 15\Delta_{n+3} - 15\Delta_{n+2} - 5\Delta_{n+1} + \Delta_n,$$

with initial values $\Delta_0 = 1$, $\Delta_1 = 13$, $\Delta_2 = 73$, $\Delta_3 = 541$, and $\Delta_4 = 3601$. In fact, the equation

$$\Delta_n = \frac{1}{25}[6L_{2n+2}^2 + 6L_{2n+2}(-1)^{n+1} + 25]$$

$$= \frac{1}{25}[6(\omega^4)^{n+1} + 6(\overline{\omega}^{\,4})^{n+1} + 6(-\omega^2)^{n+1} + 6(-\overline{\omega}^{\,2})^{n+1} + 27]$$

shows that $\omega^4, \overline{\omega}^{\,4}, -\omega^2, -\overline{\omega}^{\,2}$, and 1 are the roots of

$$g_\Delta(x) = (x^2 - L_4x + 1)(x^2 + L_2x + 1)(x - 1) = x^5 - 5x^4 - 15x^3 + 15x^2 + 5x - 1,$$

which is a characteristic polynomial of a fifth order linear recurrence.

Sequence δ_n. The squared minimum distance $d_m^2(n) = \delta_n = 2a_n - (1 - (-1)^n)$ satisfies a fourth order recurrence

$$\delta_{n+4} = 3\delta_{n+3} - 3\delta_{n+1} + \delta_n,$$

with initial values $\delta_0 = 2, \delta_1 = 8, \delta_2 = 20$, and $\delta_3 = 56$. In fact, the equation

$$\delta_n = \frac{6}{5}L_{2n+2} - \frac{3}{5}(-1)^n - 1 = \frac{1}{5}[6(\omega^2)^{n+1} + 6(\overline{\omega}^{\,2})^{n+1} - 3(-1)^n - 5]$$

shows that $\omega^2, \overline{\omega}^{\,2}, -1$, and 1 are the roots of

$$g_\delta(x) = (x^2 - L_2x + 1)(x + 1)(x - 1) = (x^2 - 3x + 1)(x + 1)(x - 1) = x^4 - 3x^3 + 3x - 1,$$

which is a characteristic polynomial of a fourth order linear recurrence.

3. THETA SERIES

In Chapter 4 of [5], Conway and Sloane describe basic techniques for theta series manipulations. Their enlightening example of the hexagonal lattice [5, p. 110] helps us to study Λ_0. This lattice has the following theta series

$$\Theta_{\Lambda_0}(q) = 1 + 12q^2 + 36q^4 + 12q^6 + 84q^8 + 72q^{10} + 36q^{12} + \dots$$

which is obtained using the quadratic form with symmetric matrix

$$A_0 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

A direct computation yields

$$\Theta_{\Lambda_0}(q) = \sum_{x \in \Lambda_0} q^{x \cdot x} = \sum_{x \in \Lambda_0} q^{2(x_1^2 + x_3^2 + x_1x_3 + x_2^2 + x_4^2 + x_2x_4)} = \left(\sum_{x_1, x_3 \in \mathbb{Z}} q^{2(x_1^2 + x_3^2 + x_1x_3)} \right)^2 = \Theta^2_{\Lambda_{exa}}(q^2).$$

Furthermore, it is known [5, p. 111] that

$$\Theta_{\Lambda_{exa}}(q) = \theta_2(z)\theta_2(3z) + \theta_3(z)\theta_3(3z), \tag{6}$$

where $\theta_i(z) = \theta_i(0|z), i = 2, 3$, are Jacobi theta functions with $q = e^{\pi i z}$.

Theorem 1 *For every n, the theta series of Λ_n*

$$\Theta_{\Lambda_n}(q) = \sum_{x \in \mathbb{Z}^4} q^{x^T U^{-1} C_n (U^T)^{-1} x} = \sum_{x_1, z_2, x_3, x_4 \in \mathbb{Z}} q^{Q(x_1, x_3, x_2) + Q(x_4, x_2, x_3)}$$

can be written in the following form

$$\Theta_{\Lambda_n}(q) = \Theta_{00}(n, q)^2 + \Theta_{01}(n, q)^2 + 2\Theta_{11}(n, q) \cdot \Theta_{10}(n, q), \tag{7}$$

where $\Theta_{r_3 r_2}(n, q), r_2, r_3 \in \{0, 1\}$ can be expressed in terms of Jacobi theta functions

$$\theta_1(\xi|z) = \theta_2(\xi - \frac{\pi}{2}|z), \quad \theta_2(\xi|z) = \sum_{m=-\infty}^{\infty} e^{(2m+1)i\xi + i\pi z(m+\frac{1}{2})^2}$$

$$\theta_3(\xi|z) = \sum_{m=-\infty}^{\infty} e^{2im\xi + i\pi z m^2}, \textit{ and } \theta_4(\xi|z) = \theta_3(\xi + \frac{\pi}{2}|z).$$

Proof: In $Q(x_1, x_3, x_2)$ and $Q(x_4, x_2, x_3)$, the expressions $2x_1 + x_3 - x_2$ and $2x_4 - x_3 + x_2$ are even numbers if x_3 and x_2 have the same parity, otherwise they are odd. Setting $x_2 = 2z_2 + r_2$ and $x_3 = 2z_3 + r_3$, where $r_2, r_3 \in \{0, 1\}$ and $z_2, z_3 \in \mathbb{Z}$, we have

$$Q(x_1, x_3, x_2) = \frac{1}{2a_n}[(a_n(2[x_1 + z_3 - z_2] + r_3) + 2z_2 + r_2)^2 + g_n(2z_2 + r_2)^2]$$

$$Q(x_4, x_2, x_3) = \frac{1}{2a_n}[(a_n(2[x_4 + z_2 - z_3] + r_2) + 2z_3 + r_3)^2 + g_n(2z_3 + r_3)^2].$$

The transformation $m_1 = x_1 + z_3 - z_2, m_2 = z_2, m_3 = z_3$, and $m_4 = x_4 + z_2 - z_3$ is unimodular, thus for r_3 and r_2 fixed in

$$Q(x_1, x_3, x_2) = \frac{1}{2a_n}[(a_n(2m_1 + r_3 - r_2) + 2m_2 + r_2)^2 + g_n(2m_2 + r_2)^2]$$

$$Q(x_4, x_2, x_3) = \frac{1}{2a_n}[(a_n(2m_4 + r_2 - r_3) + 2m_3 + r_3)^2 + g_n(2m_3 + r_3)^2],$$

the four variables m_1, m_2, m_3, m_4 range independently over $\mathbb{Z}$. Therefore (7) is obtained defining

$$\Theta_{r_3 r_2}(n,q) = \sum_{m_1,m_2\in\mathbb{Z}} q^{\frac{1}{2a_n}[(a_n(2m_1+r_3-r_2)+2m_2+r_2)^2+g_n(2m_2+r_2)^2]} \quad r_2, r_3 = 0, 1.$$

Now, setting $m_2 = a_n m + r$ and $\ell = m_1 + m$, with $r \in \{0, 1, \ldots, a_n - 1\}$, we obtain

$$\Theta_{r_3 r_2}(n,q) = \sum_{r=0}^{a_n-1} \sum_{m_1,\ell\in\mathbb{Z}} q^{\frac{1}{2a_n}[(a_n(2\ell+r_3-r_2)+2r+r_2)^2+g_n(2a_n m+2r+r_2)^2]} \quad r_2, r_3 = 0, 1.$$

The infinite sums

$$\sum_{m_1,\ell\in\mathbb{Z}} q^{\frac{1}{2a_n}[(a_n(2\ell+r_3-r_2)+2r+r_2)^2+g_n(2a_n m+2r+r_2)^2]} \quad r_2, r_3 = 0, 1, \quad r = 0, \ldots, a_n - 1$$

are actually products of Jacobi theta functions. This will be proved considering the exponent of q as a sum of three terms

$$\begin{aligned} E_1 &= 2a_n\ell^2 + 2(a_n r_3 + 2r + r_2)\ell \\ E_2 &= 2a_n g_n m^2 + 2g_n(2r + r_2)m \\ E_3 &= \frac{a_n r_3^2}{2} + (a_n + (-1)^n)(2r + r_2)^2 + r_3(2r + r_2). \end{aligned}$$

Assuming $q = e^{\pi i z}$, from [5, p. 103] we have

$$\sum_{m=-\infty}^{\infty} q^{2mB+Am^2} = \sum_{m=-\infty}^{\infty} e^{2\pi i m B z + \pi i z A m^2} = \theta_3(\pi B z | A z) = (-iAz)^{-1/2} e^{\frac{\pi B^2 z}{iA}} \theta_3\left(\frac{\pi B}{A} | -\frac{1}{Az}\right).$$

Therefore, two forms for $\Theta_{r_3 r_2}(n,q)$ are possible, based on either of the two forms occurring in Poisson-Jacobi identity, that is,

$$\frac{-1}{2a_n\sqrt{g_n}z} \sum_{r=0}^{a_n-1} \theta_3\left(\pi\frac{a_n r_3 + 2r + r_2}{2a_n} | \frac{-1}{2a_n z}\right) \theta_3\left(\pi\frac{2r + r_2}{2a_n} | \frac{-1}{2a_n g_n z}\right)$$

and

$$\sum_{r=0}^{a_n-1} \theta_3(\pi z(a_n r_3 + 2r + r_2) | 2a_n z)\theta_3(\pi g_n z(2r + r_2) | 2a_n g_n z) e^{\pi i z(\frac{a_n r_3^2}{2} + (a_n + (-1)^n)(2r + r_2)^2 + r_3(2r + r_2))}. \ \square$$

For example, taking $n \equiv 0, 1 \bmod 3$ we get four fairly symmetric expressions for $\Theta_{ij}(n,q)$ in terms of Jacobi theta functions. With the restriction on n, a_n is odd, therefore $-(2r+1)[(a_n-1)/2]$ runs over a full remainder set along with r. Thus, using the properties $\theta_4(\xi|z) = \theta_3(\xi+\frac{\pi}{2}|z)$ and $\theta_3(\xi+\pi|z) = \theta_3(\xi|z)$ [12], we obtain

$$\Theta_{00}(n,q) = \frac{-1}{2a_n\sqrt{g_n}z}\sum_{r=0}^{a_n-1}\theta_3\left(\pi\frac{r}{a_n}|\frac{-1}{2a_nz}\right)\theta_3\left(\pi\frac{r}{a_n}|\frac{-1}{2a_ng_nz}\right)$$

$$\Theta_{01}(n,q) = \frac{-1}{2a_n\sqrt{g_n}z}\sum_{r=0}^{a_n-1}\theta_4\left(\pi\frac{r}{a_n}|\frac{-1}{2a_nz}\right)\theta_4\left(\pi\frac{r}{a_n}|\frac{-1}{2a_ng_nz}\right)$$

$$\Theta_{10}(n,q) = \frac{-1}{2a_n\sqrt{g_n}z}\sum_{r=0}^{a_n-1}\theta_4\left(\pi\frac{r}{a_n}|\frac{-1}{2a_nz}\right)\theta_3\left(\pi\frac{r}{a_n}|\frac{-1}{2a_ng_nz}\right)$$

$$\Theta_{11}(n,q) = \frac{-1}{2a_n\sqrt{g_n}z}\sum_{r=0}^{a_n-1}\theta_3\left(\pi\frac{r}{a_n}|\frac{-1}{2a_nz}\right)\theta_4\left(\pi\frac{r}{a_n}|\frac{-1}{2a_ng_nz}\right).$$

4. CONCLUDING REMARKS

We conclude with an example and a few remarks on open problems related to the construction of n-dimensional lattices with maximum center density.

Fibonacci ideal lattices have been used to design good signal constellations for sending information over communication channels [6]. The goal is to choose a constellation of M points in a space of dimension n with maximum normalized minimum squared distance $\kappa = \frac{d^2_{\text{min}}}{E_{\text{av}}}\log_2 M$, where E_{av} is the average squared norm of the points of the constellation, and d^2_{min} is the minimum squared distance between points of the constellation. For example, the ideal $(2-5\zeta_{12})\mathbb{Z}[\zeta_{12}]$ may be used to construct a constellation of 37 points. A basis for Λ, the lattice generated by $\mathbb{Z}[\zeta_{12}]$, is given by the rows of the following matrix:

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

whereas a basis for the Fibonacci ideal lattice Λ_3 is obtained by left multiplying B by the matrix associated to the ideal $(2 - 5\zeta_{12})$

$$\begin{bmatrix} 2 & -5 & 0 & 0 \\ 0 & 2 & -5 & 0 \\ 0 & 0 & 2 & -5 \\ 5 & 0 & -5 & 2 \end{bmatrix}.$$

The center densities of Λ and Λ_3 are $\gamma = 0.0833$ and $\gamma_3 = 0.1207$ respectively.

The rational prime 37 splits in $\mathbb{Z}[\zeta_{12}]$ as $37 = p_1p_2p_3p_4$, where $p_1 = \langle -1 + 2\zeta_{12} + 2\zeta_{12}^2 \rangle$, and the other primes p_2, p_3, and p_4 are obtained by conjugation, namely, substituting ζ_{12} with $\zeta_{12}^5, \zeta_{12}^7$, and ζ_{12}^{11} respectively. Thus, the set of 37 elements modulo p_1 is $\mathbb{Z}[\zeta_{12}]$ is a field isomorphic to $\mathbb{Z}_{37}$ the set of remainders modulo 37. The following table

ℓ	x_1	x_2	x_3	x_4	ℓ	x_1	x_2	x_3	x_4	ℓ	x_1	x_2	x_3	x_4
0	0	0	0	0	1	1	0	0	0	2	−2	−1	1	1
3	−1	−1	1	1	4	0	0	−1	−1	5	1	0	−1	−1
6	0	0	0	1	7	−1	−1	0	0	8	0	−1	0	0
9	−1	−1	1	2	10	0	0	−1	0	11	1	0	−1	0
12	0	−1	−1	−1	13	−1	−1	0	1	14	0	−1	0	1
15	1	0	−2	−1	16	1	2	0	−1	17	−1	−1	−1	0
18	0	−1	−1	0	19	0	1	1	0	20	1	1	1	0
21	−1	−2	0	1	22	−1	0	2	1	23	0	1	0	−1
24	1	1	0	−1	25	0	1	1	1	26	−1	0	1	0
27	0	0	1	0	28	1	1	−1	−2	29	0	1	0	0
30	1	1	0	0	31	0	0	0	−1	32	−1	0	1	1
33	0	0	1	1	34	1	1	−1	−1	35	2	1	−1	−1
36	−1	0	0	0										

identitifies the constellations where a point with coordinates (x_1, x_2, x_3, x_4) in different bases, namely, $\mathcal{B}_1 = \{1, \zeta_{12}\zeta_{12}^2, \zeta_{12}^3\}$ and $\mathcal{B}_2 = \{-1 + 2\zeta_{12} + 2\zeta_{12}^2, -\zeta_{12} + 2\zeta_{12}^2 - 2\zeta_{12}^2 + 2\zeta_{12}^3, -2 + \zeta_{12}^2 + 2\zeta_{12}^3, -2 - 2\zeta_{12} + 2\zeta_{12}^2 + \zeta_{12}^3\}$, receives the same label $\ell = x_1 - 8x_2 + (-8)^2x_2 - 8^3x_3 = x_1 + 29x_2 + 27x_2 + 6x_3 \mod 37$. The maximum normalized minimum squared distances of constellations with 37 points in Λ and Λ_3 are $\kappa = 3.21$ and $\kappa_3 = 3.98$ respectively.

In dimension four, we have seen that an ideal lattice with maximum center density exists along with a class of ideal lattices achieving the same maximal density asymptotically. For a given m-dimensional space, it would be interesting to ascertain whether the maximum center density is achievable finitely or asymptotically. The theta series $\Theta_{\Lambda_n}(q)$ of a Fibonacci ideal lattice can be expressed by means of Jacobi theta functions. It is also of interest to know whether $\Theta_{\Lambda_n}(q)$ can be expressed in terms of a finite initial set of theta series $\Theta_{\Lambda_0}, \ldots, \Theta_{\Lambda_s}$.

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