The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

TABLE OF CONTENTS

VOLUME 41 NOVEMBER 2003 NUMBER 5

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980) and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

THE r-SUBCOMPLETE PARTITIONS

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1. INTRODUCTION

MacMahon [1], pp. 217-223, studied special kinds of partitions of a positive integer, which he called perfect partitions and subperfect partitions. He defined a perfect partition of a number as "a partition which contains one and only one partition of every lesser number" and a subperfect partition as "a partition which contains one and only one partition of every lesser number if it is permissible to regard the several parts as affected with either the positive or negative sign". For instance, (1 1 3) is a perfect partition of 5 because we can uniquely express each of the numbers 1 through 5 by using the parts of two 1's and a 3. Thus (1), (1 1), (3), and (1 3) are the partitions referred to. The partition (1 3) is a subperfect partition of 4 because 1 is represented by the part 1, 2 by $-1+3$, and 3 3 by the part 3. In [1] MacMahon derived a recurrence relation for the number of such partitions using generating functions and found a nice relation between the number of perfect partitions and the number of ordered factorizations. See [4] for more information.

One way of generalizing MacMahon's idea is to eliminate the uniqueness condition, which was done by the second author [2]. He defines a complete partition of n to be a partition $\lambda = (\lambda_1 \cdots \lambda_l)$ such that every number m with $1 \leq m \leq n$ can be represented by the form of $m = \sum_{i=1}^{l} \alpha_i \lambda_i$, where $\alpha_i \in S = \{0, 1\}$. He also studied the case of the set $S = \{0, 1, \cdots, r\}$ in [3]. In this paper we shall study the r-subcomplete partitions which are complete partitions with the set $S = \{-r, \cdots, -1, 0, 1, \cdots, r\}$, where r is a positive integer.

2. THE r-SUBCOMPLETE PARTITIONS

Even if it is well-known, we start with a definition of partitions. Throughout this paper the number n represents a positive integer.

Definition 2.1: *A partition of n is a finite non-decreasing sequence $\lambda = (\lambda_1 \cdots \lambda_l)$ such that $\sum_{i=1}^{l} \lambda_i = n$ and $\lambda_i > 0$ for all $i = 1, \cdots, l$. The λ_i are called the parts of the partition and the number l is called the length of the partition.*

We sometimes write $\lambda = (1^{m_1} 2^{m_2} \cdots)$, which means there are exactly m_i parts equal to i in the partition λ. For example, we can write 7 partitions of 5 as (5), (1 4), (2 3), (1^2 3), (1 2^2), (1^3 2), and (1^5). The following two concepts are already mentioned, but we formally define them again to see how we can generalize them.

Definition 2.2: *A partition $\lambda = (\lambda_1^{m_1} \cdots \lambda_l^{m_l})$ of n is a perfect partition of n if every integer m with $1 \leq m \leq n$ can be uniquely expressed as $m = \sum_{i=1}^{l} \alpha_i \lambda_i$, where $\alpha_i \in \{0, 1, \cdots, m_i\}$ and repeated parts are regarded as indistinguishable.*

Definition 2.3: *A partition $\lambda = (\lambda_1^{m_1} \cdots \lambda_l^{m_l})$ of n is a subperfect partition if each integer m with $1 \leq m \leq n$ can be uniquely represented as $\sum_{i=1}^{l} \alpha_i \lambda_i$, where $\alpha_i \in \{-m_i, \cdots, -1, 0, 1, \cdots, m_i\}$ and repeated parts are regarded as indistinguishable.*

Definition 2.4: *A partition of n $\lambda = (\lambda_1 \cdots \lambda_l)$ is a complete partition of n if each integer m with $1 \leq m \leq n$ can be represented as $m = \sum_{i=1}^{l} \alpha_i \lambda_i$, where $\alpha_i \in \{0, 1\}$.*

For $n = 6$, partitions $(1^6), (1^4\ 2), (1^3\ 3), (1^2\ 2^2)$ and $(1\ 2\ 3)$ are complete partitions. We refer to the paper [2] for more information on complete partitions. Now we are ready to define our main topic, the r-subcomplete partitions.

Definition 2.5: *A partition of n $\lambda = (\lambda_1 \cdots \lambda_l)$ is an r-subcomplete partition of n if each integer m with $1 \leq m \leq rn$ can be expressed as $m = \sum_{i=1}^{l} \alpha_i \lambda_i$, where $\alpha_i \in \{-r, \cdots, -1, 0, 1, \cdots, r\}$. Such as m is said to be r-representable.*

We also say a partition $\lambda = (\lambda_1 \cdots \lambda_l)$ is r-subcomplete if it is an r-subcomplete partition of the number $\lambda_1 + \cdots + \lambda_l$. We will write $\{0, \pm1, \pm2, \cdots, \pm r\}$ for the set $\{-r, \cdots, -1, 0, 1, \cdots, r\}$ and the letter r represents a positive integer throughout this paper. The r-subcomplete partitions with the set $\{0, 1, \cdots, r\}$ are called r-complete partitions. See [3] for more information.

Example 1: *The partition $(1\ 4)$ is a 2-subcomplete partition of 5. To see this we list 2-representations of numbers from 1 to 10;* $1 = 1$, $2 = 2 \cdot 1 + 0 \cdot 4, 3 = -1 \cdot 1 + 1 \cdot 4, 4 = 0 \cdot 1 + 1 \cdot 4, 5 = 1 \cdot 1 + 1 \cdot 4, 6 = 2 \cdot 1 + 1 \cdot 4, 7 = -1 \cdot 1 + 2 \cdot 4, 8 = 0 \cdot 1 + 2 \cdot 4, 9 = 1 \cdot 1 + 2 \cdot 4,$ *and* $10 = 2 \cdot 1 + 2 \cdot 4$.

It is easy to see that every integer m with $-rn \leq m \leq 0$ can also be expressed in the form $\sum_{i=1}^{l} \alpha_i \lambda_i$ with $\alpha_i \in \{0, \pm1, \pm2, \cdots, \pm r\}$ if $\lambda = (\lambda_1 \cdots \lambda_l)$ is an r-subcomplete partition of n. So one can say if $\lambda = (\lambda_1 \cdots \lambda_l)$ is an r-subcomplete partition of n then each number between $-rn$ and rn can be represented by the form. We will need this simple fact in the proof of Lemma 2.9 and Theorem 2.10. The following Lemma shows that every r-subcomplete partition should have 1 as the first part.

Lemma 2.6: *Let $\lambda = (\lambda_1 \cdots \lambda_l)$ be an r-subcomplete partition of n. Then λ_1 is 1.*

Proof: Suppose not. Then $\lambda_1 > 1$. Since λ is an r-subcomplete partition of n, the numbers 1 and $rn - 1$ are r-representable. Let $1 = \sum_{i=1}^{l} \alpha_i \lambda_i$, where $\alpha_i \in \{0, \pm1, \pm2, \cdots, \pm r\}$. Then there should be at least one $\alpha_j < 0$ for some j since $\lambda_i > 1$ for all i. Then $rn - 1 = r(\sum_{i=1}^{l} \lambda_i) - \sum_{i=1}^{l} \alpha_i \lambda_i = \sum_{i=1}^{l} (r - \alpha_i) \lambda_i$. Then $r - \alpha_j > r$, which means $rn - 1$ is not r-representable, which is a contradiction. □

Theorem 2.7: *Let $\lambda = (\lambda_1 \cdots \lambda_l)$ be an r-subcomplete partition of n. Then $\lambda_i \leq 1 + 2r \sum_{j=1}^{i-1} \lambda_j$ for each $i = 2, \cdots, l$.*

Proof: Suppose not. Then there exists at least one number k such that $\lambda_k > 1 + 2r \sum_{j=1}^{k-1} \lambda_j$, where $2 \leq k \leq l$. Thus,

$$rn > rn - \left(1 + 2r \sum_{j=1}^{k-1} \lambda_j\right) > rn - \lambda_k = r \sum_{j=1}^{l} \lambda_j - \lambda_k$$

$$= r \sum_{j=1}^{k-1} \lambda_j + (r-1)\lambda_k + r \sum_{j=k+1}^{l} \lambda_j.$$

Since the number $rn - \left(1 + 2r\sum_{j=1}^{k-1}\lambda_j\right)$ is r-representable, we can let $rn - \left(1 + 2r\sum_{j=1}^{k-1}\lambda_j\right) = \sum_{i=1}^{l}\alpha_i\lambda_i$, where $\alpha_i \in \{0, \pm 1, \pm 2, \cdots, \pm r\}$. Then $\alpha_k = \alpha_{k+1} = \cdots = \alpha_l = r$ because $rn - \left(1 + 2r\sum_{j=1}^{k-1}\lambda_j\right) > r\sum_{j=1}^{k-1}\lambda_j + (r-1)\lambda_k + r\sum_{j=k+1}^{l}\lambda_j$ and $\lambda_1 \le \cdots \le \lambda_k \le \lambda_{k+1} \le \cdots \le \lambda_l$. Thus,

$$rn - \left(1 + 2r\sum_{j=1}^{k-1}\lambda_j\right) = \sum_{j=1}^{k-1}\alpha_j\lambda_j + r\sum_{j=k}^{l}\lambda_j.$$

So

$$rn = 1 + \sum_{j=1}^{k-1}(2r + \alpha_j)\lambda_j + r\sum_{j=k}^{l}\lambda_j \ge rn + 1,$$

which is a contradiction. □

Corollary 2.8: *Let $\lambda = (\lambda_1 \cdots \lambda_l)$ be an r-subcomplete partition of n. Then $\lambda_i \le (2r+1)^{i-1}$ for each $i = 1, \cdots, l$.*

Proof: For $i = 1$, the result is obvious. Assuming that $\lambda_i \le (2r+1)^{i-1}$ for $i = 1, \cdots, k$,

$$\lambda_{k+1} \le 1 + 2r\sum_{j=1}^{k}\lambda_j \le 1 + 2r \cdot \frac{(2r+1)^k - 1}{(2r+1) - 1} = (2r+1)^k. \quad \square$$

Lemma 2.9: *Let $\lambda = (\lambda_1 \cdots \lambda_l)$ be an r-subcomplete partition of n. Then for $k = 1, \cdots, l$ each partition $(\lambda_1 \cdots \lambda_k)$ is r-subcomplete of the number $\lambda_1 + \cdots + \lambda_k$.*

Proof: Clearly, (1) is an r-subcomplete partition of 1 for all r. Assume that $(\lambda_1 \cdots \lambda_k)$ is an r-subcomplete partition of $\lambda_1 + \cdots + \lambda_k$. We only need to show that for $w = 1, \cdots, r$ each m such that $r(\lambda_1 + \cdots + \lambda_k) + (w-1)\lambda_{k+1} < m \le r(\lambda_1 + \cdots + \lambda_k) + w\lambda_{k+1}$ is r-representable. Since $\lambda_{k+1} \le 1 + 2r\sum_{j=1}^{k}\lambda_j$ from Theorem 2.7,

$$r(\lambda_1 + \cdots + \lambda_k) - \lambda_{k+1} < m - w\lambda_{k+1} \le r(\lambda_1 + \cdots + \lambda_k)$$

$$-\left(1 + r\sum_{j=1}^{k}\lambda_j\right) < m - w\lambda_{k+1} \le r\sum_{j=1}^{k}\lambda_j$$

$$-r\sum_{j=1}^{k}\lambda_j \le m - w\lambda_{k+1} \le r\sum_{j=1}^{k}\lambda_j.$$

Thus by the inductive assumption, the number $m - w\lambda_{k+1}$ is r-representable by $\lambda_1, \cdots, \lambda_k$. So $m - w\lambda_{k+1} = \sum_{j=1}^{k} \alpha_j \lambda_j$, where $\alpha_j \in \{0, \pm 1, \pm 2, \cdots, \pm r\}$. Therefore, $m = \sum_{j=1}^{k} \alpha_j \lambda_j + w\lambda_{k+1}$. This completes the proof. □

The converse of Theorem 2.7 is also true and it gives a criterion to determine r-subcomplete partitions.

Theorem 2.10: *Let $\lambda = (\lambda_1 \cdots \lambda_l)$ be a partition of n with $\lambda_1 = 1$ and $\lambda_i \leq 1 + 2r \sum_{j=1}^{i-1} \lambda_j$ for $i = 2, \cdots, l$. Then λ is an r-subcomplete partition of n.*

Proof: Obviously, (1) is an r-subcomplete partition of 1 for all r. Asusme that $(\lambda_1 \cdots \lambda_k)$ is r-subcomplete. Then by Lemma 2.9, every partition $(\lambda_1 \cdots \lambda_i)$ is r-subcomplete for $i = 2, \cdots, k$. We want to show that $(\lambda_1 \cdots \lambda_k \lambda_{k+1})$ is r-subcomplete. To do this we use similar steps to the proof of Lemma 2.9. Let m satisfy $r(\lambda_1 + \cdots + \lambda_k) + (w-1)\lambda_{k+1} < m \leq r(\lambda_1 + \cdots + \lambda_k) + w\lambda_{k+1}$, where $w = 1, 2, \cdots, r$. Now, since $r(\lambda_1 + \cdots + \lambda_k) - \lambda_{k+1} < m - w\lambda_{k+1} \leq r(\lambda_1 + \cdots + \lambda_k)$ and from the given condition $\lambda_{k+1} \leq 1 + 2r \sum_{j=1}^{k} \lambda_j$, we have $-r(\lambda_1 + \cdots + \lambda_k) \leq m - w\lambda_{k+1} \leq r(\lambda_1 + \cdots + \lambda_k)$. By the inductive assumption and Lemma 2.9 $m = w\lambda_{k+1} + \sum_{i=1}^{k} \alpha_i \lambda_i$, that is, $(\lambda_1 \cdots \lambda_k, \lambda_{k+1})$ is an r-subcomplete partition. Thus, the partition $(\lambda_1 \cdots \lambda_l)$ is an r-subcomplete partition. □

Proposition 2.11: *Let $\lambda = (\lambda_1 \cdots \lambda_l)$ be an r-subcomplete partition of n. Then the minimum possible length l is $\lceil \log_{(2r+1)}(2rn+1) \rceil$, where $\lceil x \rceil$ is the least integer which is greater than or equal to x.*

Proof: By Corollary 2.8, $n = \sum_{j=1}^{l} \lambda_j \leq \sum_{j=1}^{l} (2r+1)^{j-1} = \frac{(2r+1)^l - 1}{2r}$. Therefore, $l \geq \lceil \log_{(2r+1)}(2rn+1) \rceil$. □

Proposition 2.12: *Let $\lambda = (\lambda_1 \cdots \lambda_l)$ be an r-subcomplete partition of n. Then the largest possible part is $\lfloor \frac{2rn+1}{2r+1} \rfloor$, where $\lfloor x \rfloor$ is the largest integer which is less than or equal to x.*

Proof: Let $n = \lambda_1 + \cdots + \lambda_{l-1} + \lambda_l$. Then λ_l is the largest and $n - \lambda_l = \sum_{j=1}^{l-1} \lambda_j$. By Theorem 2.7, $\lambda_l \leq 1 + 2r \sum_{j=1}^{l-1} \lambda_j = 1 + 2r(n - \lambda_l)$. Thus $\lambda_l \leq \lfloor \frac{2rn+1}{2r+1} \rfloor$. □

Now, we try to find two recurrence relations and a generating function for r-subcomplete partitions. Let $S_{r,k}(n)$ be the number of r-subcomplete partitions of n whose largest part is at most k. The set of such partitions can be partitioned into two subsets: one with the largest part at most $k-1$ and the other with the largest part exactly k. The latter type of partitions can be obtained by adding k as the last part of an r-subcomplete partitions of $n-k$ whose largest part is at most k. We know from Proposition 2.12 that the largest possible part k should satisfy $1 \leq k \leq \lfloor \frac{2rn+1}{2r+1} \rfloor$. If $k > \lfloor \frac{2rn+1}{2r+1} \rfloor$, $S_{r,k}(n)$ becomes actually $S_{r, \lfloor \frac{2rn+1}{2r+1} \rfloor}(n)$, which is the number of all r-subcomplete partitions of n. It is easy to see from the definition of $S_{r,k}(n)$ that $S_{r,k}(1) = 1$ for all k and $S_{r,1}(n) = 1$ for all n. Thus we obtain

Theorem 2.13: *Let $S_{r,k}(n)$ be the number of r-subcomplete partitions of n whose largest part is at most k. Then $S_{r,1}(n) = 1$ for all n and for $k \geq 2$*

$$S_{r,k}(n) = \begin{cases} S_{r,k-1}(n) + S_{r,k}(n-k) & \text{if } 1 \leq k \leq \lfloor \frac{2rn+1}{2r+1} \rfloor \\ S_{r, \lfloor \frac{2rn+1}{2r+1} \rfloor}(n) & \text{if } k > \lfloor \frac{2rn+1}{2r+1} \rfloor \end{cases}$$

with initial conditions $S_{r,1}(0) = 1$ *and* $S_{r,0}(n) = 0$ *for all* n.

Example 2:

$$\begin{aligned}
S_{2,6}(7) = S_{2,5}(7) &= S_{2,4}(7) + S_{2,5}(2) = (S_{2,3}(7) + S_{2,4}(3)) + S_{2,1}(2) \\
&= (S_{2,2}(7) + S_{2,3}(4)) + S_{2,2}(3) + 1 \\
&= \{(S_{2,1}(7) + S_{2,2}(5)) + (S_{2,2}(4) + S_{2,3}(1))\} + (S_{2,1}(3) + S_{2,2}(1)) + 1 \\
&= 1 + (S_{2,1}(5) + S_{2,2}(3)) + (S_{2,1}(4) + S_{2,2}(2)) + 1 + (1 + 1) + 1 \\
&= 1 + 1 + (S_{2,1}(3) + S_{2,2}(1)) + 1 + S_{2,1}(2) + 4 \\
&= 2 + (1 + S_{2,1}(1)) + 1 + 1 + 4 = 10
\end{aligned}$$

Now let us count the number of r-subcomplete partitions whose largest part is exactly k and find a generating function for this number. Let $E_{r,k}(n)$ be the number of r-subcomplete partitions of n whose largest part is exactly k. A recurrence relation for $E_{r,k}(n)$ can be obtained by the method we used in deriving $S_{r,k}(n)$ above, but we can use the number $S_{r,k}(n)$ itself as follows. From the recurrence relation for $S_{r,k}(n)$,

$$\begin{aligned}
E_{r,k}(n) &= S_{r,k}(n) - S_{r,k-1}(n) = S_{r,k}(n-k) \\
&= S_{r,k}(n-k) - S_{r,k-1}(n-k) + S_{r,k-1}(n-k) \\
&= E_{r,k}(n-k) + E_{r,k-1}(n-1).
\end{aligned}$$

It is easy to see that $E_{r,1}(n) = 1$ for all n. The numbers $E_{r,k}(n), E_{r,k-1}(n-1)$, and $E_{r,k}(n-k)$ count corresponding r-subcomplete partitions of $n, n-1$, and $n-k$, respectively. So they should satisfy the condition of Proposition 2.12. In other words, each of them must have $k \leq \lfloor \frac{2rn+1}{2r+1} \rfloor, k-1 \leq \lfloor \frac{2r(n-1)+1}{2r+1} \rfloor$, and $k \leq \lfloor \frac{2r(n-k)+1}{2r+1} \rfloor$, respectively. Summarizing these, we obtain

Theorem 2.14: *Let* $E_{r,k}(n)$ *be the number of r-subcomplete partitions of a positive integer n whose largest part is exactly k. Then* $E_{r,1}(n) = 1$ *for all n, and for* $k \geq 2$

$$E_{r,k}(n) = \begin{cases} E_{r,k-1}(n-1) + E_{r,k}(n-k) & \text{if } n \geq 2k + \frac{k-1}{2r} \\ E_{r,k-1}(n-1) & \text{if } k + \frac{k-1}{2r} \leq n < 2k + \frac{k-1}{2r} \\ 0 & \text{if } n < k + \frac{k-1}{2r} \end{cases}$$

with $E_{r,0}(0) = 1, E_{r,0}(n) = 0$ *for all n and* $E_{r,k}(n) = 0$ *for all* $n \leq k$.

Example 3:

1. $E_{2,2}(5) = E_{2,1}(4) + E_{2,2}(3) = 1 + E_{2,1}(2) = 2.$
2. $E_{2,2}(6) = E_{2,1}(5) + E_{2,2}(4) = 1 + E_{2,1}(3) = 2.$
3. $E_{2,3}(9) = E_{2,2}(8) + E_{2,3}(6) = E_{2,1}(7) + E_{2,2}(6) + E_{2,2}(5) = 1 + 2 + 2 = 5.$
4. $E_{2,4}(6) = E_{2,3}(5) + E_{2,2}(4) = E_{2,1}(3) = 1.$
5. $E_{2,4}(5) = E_{2,3}(4) = E_{2,2}(3) = E_{2,1}(2) = 1.$
6. $E_{2,5}(11) = E_{2,4}(10) + E_{2,5}(6) = (E_{2,3}(9) + E_{2,4}(6)) + E_{2,4}(5) = 5 + 1 + 1 = 7.$

The following three tables show the first few values of r-subcomplete partitions with $r = 1, 2$ and 3. We denote $S_r(n)$ as the number of all r-subcomplete partitions of n.

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	2	2	3	3	4	4	5
3				1	1	2	3	4	5	7	8
4						1	2	3	4	6	9
5							1	2	3	4	6
6									2	3	4
7										2	3
$S_1(n)$	1	1	2	3	4	6	10	13	19	27	36

Table I $r = 1$

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	2	2	3	3	4	4	5
3				1	1	2	3	4	5	7	8
4					1	1	2	3	5	6	9
5						1	1	2	3	5	7
6								1	2	3	5
7									1	2	3
8										1	2
9											1
$S_2(n)$	1	1	2	3	5	7	10	14	21	29	41

Table II $r = 2$

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	2	2	3	3	4	4	5
3				1	1	2	3	4	5	7	8
4					1	1	2	3	5	6	9
5						1	1	2	3	5	7
6							1	1	2	3	5
7								1	1	2	3
8										1	2
9											1
$S_3(n)$	1	1	2	3	5	7	11	15	21	29	41

Table III $r = 3$

Now we are ready to find a generating function for $E_{r,k}(n)$. Based on Theorem 2.14 we obtain the following.

Theorem 2.15: *Let $F_{r,k}(q) = \sum_{n=0}^{\infty} E_{r,k}(n)q^n$ for $k \geq 1$. Then*

$$F_{r,k}(q) = \frac{q^k}{(q)_k}\left[1 - \sum_{i=0}^{s-1} q^i (q)_{2ri+1} E_{r,2ri+1}((2r+1)i+1)\right], \tag{2.1}$$

where $s = \lceil \frac{k-1}{2r} \rceil$, $F_{r,0}(q) = 1$, and $(q)_k = (1-q^k)(1-q^{k-1})\cdots(1-q)$.

Proof: By Theorem 2.14,

$$\begin{aligned}
F_{r,k}(q) &= \sum_{n=k+s}^{\infty} E_{r,k}(n)q^n \\
&= \sum_{n=k+s}^{2k+s-1} E_{r,k-1}(n-1)q^n + \sum_{n=2k+s}^{\infty} [E_{r,k-1}(n-1) + E_{r,k}(n-k)]q^n \\
&= \sum_{n=k+s}^{\infty} E_{r,k-1}(n-1)q^n + \sum_{n=2k+s}^{\infty} E_{r,k}(n-k)q^n \\
&= q \sum_{n=k+s-1}^{\infty} E_{r,k-1}(n)q^n + q^k \sum_{n=k+s}^{\infty} E_{r,k}(n)q^n.
\end{aligned} \tag{2.2}$$

Since $s = \lceil \frac{k-1}{2r} \rceil$, its value depends on k and r. Thus from Proposition 2.12, $F_{r,k-1}(q)$ becomes $F_{r,k-1}(q) = \sum_{n=k+s-1}^{\infty} E_{r,k-1}(n)q^n$ or $F_{r,k-1}(q) = \sum_{n=k+s-2}^{\infty} E_{r,k-1}(n)q^n$. Let $2rp+3 \leq k \leq 2r(p+1)+1$ for some $p = 0, 1, 2, \cdots$. Then $2rp+2 \leq k-1 \leq 2r(p+1)$ and $s = p+1$. Consider $E_{r,k-1}(k+s-2)$. This is the number of r-subcomplete partitions of $k+s-2 = k-1+p$ whose largest part is exactly $k-1$. So any number between 1 and $r(k-1+p)$ should be r-representable. But with $k-1 = 2rp+t (2 \leq t \leq 2r)$ fixed as the largest part, the number $rp+1$ can not be r-representable. Thus, $E_{r,k-1}(k+s-2) = 0$ for $2rp+2 \leq k-1 \leq 2r(p+1)$. Thus, we obtain

$$F_{r,k-1}(q) = \begin{cases} \sum_{n=k+s-1}^{\infty} E_{r,k-1}(n)q^n & \text{if } k \not\equiv 2 (\text{mod } 2r) \\ \sum_{n=k+s-2}^{\infty} E_{r,k-1}(n)q^n & \text{if } k \equiv 2 (\text{mod } 2r). \end{cases}$$

For $k \not\equiv 2(\text{mod } 2r)$ equation (2.2) becomes

$$F_{r,k}(q) = q \sum_{n=k+s-1}^{\infty} E_{r,k-1}(n)q^n + q^k \sum_{n=k+s}^{\infty} E_{r,k}(n)q^n = qF_{r,k-1}(q) + q^k F_{r,k}(q).$$

Thus,

$$F_{r,k}(q) = \frac{q}{1-q^k} F_{r,k-1}(q). \tag{2.3}$$

For $k \equiv 2 (\text{mod } 2r)$ equation (2.2) becomes

$$\begin{aligned} F_{r,k}(q) &= q \sum_{n=k+s-1}^{\infty} E_{r,k-1}(n) q^n + q^k \sum_{n=k+s}^{\infty} E_{r,k}(n) q^n \\ &= q\{F_{r,k-1}(q) - E_{k-1}(k+s-2) q^{k+s-2}\} + q^k F_{r,k}(q). \end{aligned}$$

Thus we have

$$F_{r,k}(q) = \frac{q}{1-q^k} F_{r,k-1}(q) - \frac{q^{k+s-1} E_{r,k-1}(k+s-2)}{1-q^k}. \tag{2.4}$$

Now, let $k = 2rp + 2 + t$ for some non-negative integer p with $1 \le t \le 2r-1$. Then $k \not\equiv 2(\text{mod } 2r)$, so we can iterate equation (2.3) t times to get

$$F_{r,k}(q) = \frac{q^t}{(1-q^k)(1-q^{k-1})\cdots(1-q^{k-t+1})} F_{r,k-t}(q). \tag{2.5}$$

Because $k - t = 2rp + 2 \equiv 2(\text{mod } 2r)$, we have to use identity (2.4) to compute $F_{r,k-t}(q)$ which is equal to $F_{r,2r(s-1)+2}(q)$ since $\lceil \frac{2rp+2}{2r} \rceil = p + 1 = s$. We have

$$\begin{aligned} F_{r,2r(s-1)+2}(q) =& \frac{q}{1-q^{2r(s-1)+2}} F_{r,2r(s-1)+1}(q) - \\ & \frac{q^{2r(s-1)+s+1} E_{r,2r(s-1)+1}(2r(s-1)+s)}{1-q^{2r(s-1)+2}}. \end{aligned} \tag{2.6}$$

Thus by applying (2.6) to (2.5),

$$\begin{aligned} F_{r,k}(q) =& \frac{q^{k-2r(s-1)-1}}{(1-q^k)\cdots(1-q^{2r(s-1)+2})} F_{r,2r(s-1)+1}(q) - \\ & \frac{q^{k+s-1} E_{r,2r(s-1)+1}(2r(s-1)+s)}{(1-q^k)\cdots(1-q^{2r(s-1)+2})}. \end{aligned} \tag{2.7}$$

Now $2r(s-1)+1 = 2rp+1 \not\equiv 2 \pmod{2r}$, so by equation (2.3)

$$F_{r,2r(s-1)+1}(q) = \frac{q}{1-q^{2r(s-1)+1}} F_{r,2r(s-1)}(q).$$

Again this can be iterated $2r-1$ times, which gives us

$$F_{r,2r(s-1)+1}(q) = \frac{q^{2r-1}}{(1-q^{2r(s-1)+1})\cdots(1-q^{2r(s-2)+3})} F_{r,2r(s-2)+2}(q). \tag{2.8}$$

By applying (2.8) to (2.7),

$$F_{r,k}(q) = \frac{q^{k-2r(s-2)-2}}{(1-q^k)\cdots(1-q^{2r(s-2)+3})} F_{r,2r(s-2)+2}(q) - \frac{q^{k+s-1} E_{r,2r(s-1)+1}(2r(s-1)+s)}{(1-q^k)\cdots(1-q^{2r(s-1)+2})}. \tag{2.9}$$

The number $2r(s-2)+2 = 2r(p-1)+2 \equiv 2 \pmod{2r}$. So by (2.4) and with $\lceil \frac{2r(s-2)+1}{2r} \rceil = s-1$,

$$F_{r,2r(s-2)+2}(q) = \frac{q}{(1-q^{2r(s-2)+2})} F_{r,2r(s-2)+1}(q) - \frac{q^{2r(s-2)+s} E_{r,2r(s-2)+1}(2r(s-2)+s-1)}{1-q^{2r(s-2)+2}}. \tag{2.10}$$

Thus,

$$F_{r,k}(q) = \frac{q^{k-2r(s-2)-1}}{(1-q^k)\cdots(1-q^{2r(s-2)+2})} F_{r,2r(s-2)+1}(q) - \frac{q^{k+s-2} E_{r,2r(s-2)+1}(2r(s-2)+s-1)}{(1-q^k)\cdots(1-q^{2r(s-2)+2})} - \frac{q^{k+s-1} E_{r,2r(s-1)+1}(2r(s-1)+s)}{(1-q^k)\cdots(1-q^{2r(s-1)+2})}. \tag{2.11}$$

By continuing iteration on $F_{r,2r(s-2)+1}(q)$, we finally obtain the following.

$$F_{r,k}(q) = \frac{q^k}{(q)_k} - \left[\frac{q^k E_{r,1}(1)}{(1-q^k)\cdots(1-q^2)} + \frac{q^{k+1}E_{r,2r+1}(2r+2)}{(1-q^k)\cdots(1-q^{2r+2})} + \cdots\right.$$

$$+ \frac{q^{k+s-2}E_{r,2r(s-2)+1}(2r(s-2)+s-1)}{(1-q^k)\cdots(1-q^{2r(s-2)+2})} +$$

$$\left.\frac{q^{k+s-1}E_{r,2r(s-1)+1}(2r(s-1)+s)}{(1-q^k)\cdots(1-q^{2r(s-1)+2})}\right]. \quad (2.12)$$

One can easily derive our formula (2.1) from this result. □

Example 4: *The following are generating functions for $k = 2, 4, 5, 6$ and $r = 2$.*

$$F_{2,2}(q) = \frac{q^2}{(q)_2} - \frac{q^2 E_{2,1}(1)}{1-q^2} = \frac{q^2}{(q)_2} - \frac{q^2}{1-q^2}.$$

$$F_{2,4}(q) = \frac{q}{(1-q^4)}F_{2,3} = \frac{q^2}{(1-q^4)(1-q^3)}F_{2,2}(q) = \frac{q^5}{(q)_4}.$$

$$F_{2,5}(q) = \frac{q}{(q)_5}F_{2,4}(q) = \frac{q^6}{(q)_5}.$$

$$F_{2,6}(q) = \frac{q^6}{(q)_6} - \left[\frac{q^7 E_{2,5}(6)}{1-q^6} + \frac{q^6 E_{2,1}(1)}{(1-q^6)(1-q^5)(1-q^4)(1-q^3)(1-q^2)}\right]$$

$$= \frac{q^6}{(q)_6} - \left[\frac{q^7}{1-q^6} + \frac{q^6}{(1-q^6)(1-q^5)(1-q^4)(1-q^3)(1-q^2)}\right].$$

Example 5: *By expanding the above, we obtain the following generating functions whose coefficients are expected from* Table II.

$$F_{2,2}(q) = q^3 + q^4 + 2q^5 + 2q^6 + 3q^7 + 3q^8 + 4q^9 + 4q^{10} + 5q^{11} + 5q^{12} + 6q^{13}\cdots,$$
$$F_{2,4}(q) = q^5 + q^6 + 2q^7 + 3q^8 + 5q^9 + 6q^{10} + 9q^{11} + 11q^{12} + 15q^{13} + \cdots,$$
$$F_{2,5}(q) = q^6 + q^7 + 2q^8 + 3q^9 + 5q^{10} + 7q^{11} + 10q^{12} + 13q^{13} + \cdots,$$
$$F_{2,6}(q) = q^8 + 2q^9 + 3q^{10} + 5q^{11} + 7q^{12} + 10q^{13} + \cdots.$$

ACKNOWLEDGMENT

The first author was supported by Brian Korea 21 Project. The second authors research has been supported in part by 96-BSRI-1423 and by Yonsei Research Fund.

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AMS Classification Numbers: 05A17

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LINEAR RECURRING SEQUENCE SUBGROUPS IN THE COMPLEX FIELD

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(Submitted May 2001-Final Revision January 2002)

Let $S = (s_n)_{n\in\mathbb{Z}}$ be a "doubly infinite" recurring sequence in the complex field, $\mathbb{C}$, satisfying the recurrence

$$s_{n+2} = \sigma s_{n+1} + \rho s_n \tag{1}$$

where $\sigma,\ \rho \in \mathbb{C}$ and $\rho \neq 0$. It can happen that the elements of a minimal periodic segment (see below) of S form a subgroup of the multiplicative group $\mathbb{C}^*$ of $\mathbb{C}$ and our purpose here is to investigate this phenomenon. The analogous situation in the context of finite fields seems to have first been investigated by Somer [2], [3]; see also [1].

Write $f(t) = t^2 - \sigma t - \rho \in \mathbb{C}[t], \rho \neq 0$. A sequence of complex numbers $S = (s_n)_{n\in\mathbb{Z}}$ satisfying (1) will be called an *f-sequence* in $\mathbb{C}$; f is the *characteristic polynomial* of S. If there exists $m \in \mathbb{N}$ such that $s_a = s_{a+m}$ for all $a \in \mathbb{Z}$ and if also m is minimal subject to this then S is *periodic* with *least period m*. By a *minimal periodic segment* we understand the whole sequence if S is not periodic, and any segment consisting of m consecutive members of S if S is periodic with least period m.

Definition 1: Let $f(t) = t^2 - \sigma t - \rho \in \mathbb{C}[t],\ \rho \neq 0$. The subgroup $M \leq \mathbb{C}^*$ is said to be an *f-sequence subgroup* if either
(a) M is infinite and the underlying set of M can be written in such an order as to form a doubly infinite f-sequence $(s_n)_{n\in\mathbb{Z}}$ where $s_a \neq s_b$ if $a \neq b$, or
(b) M is finite, of order m, and the underlying set of M can be written in such an order as to coincide with a minimal periodic segment of an f-sequence $(s_n)_{n\in\mathbb{Z}}$, where $s_a = s_b$ if and only if $a \equiv b \pmod{m}$.

We will write $M = (s_n)_{n\in\mathbb{Z}}$ even if M is finite, and will say that $(s_n)_{n\in\mathbb{Z}}$ is *a representation of*, or *represents*, M as an f-sequence.

If $f(t) \in \mathbb{C}[t],\ f(0) \neq 0$, and if $g,\ h \in \mathbb{C}^*$ are roots of f, then

$$< g >= (\ldots, g^{-2}, g^{-1}, 1, g, g^2, \ldots) = (g^n)_{n\in\mathbb{Z}}$$

is an "obvious" representation of $< g > \leq \mathbb{C}^*$ as an f-sequence subgroup; it can happen that $h \neq g$ but $< h >=< g >$, and then $(h^n)_{n\in\mathbb{Z}}$ is a different representation of the same subgroup. This suggests:

Definition 2: Let $f(t) = t^2 - \sigma t - \rho \in \mathbb{C}[t], \rho \neq 0$.
(a) The f-sequence $(s_n)_{n\in\mathbb{Z}}$ in $\mathbb{C}$ is said to be *cyclic* if there exists $g \in \mathbb{C}$ such that $s_{n+1}/s_n = g$ for all $n \in \mathbb{Z}$.
(b) The f-sequence subgroup M of $\mathbb{C}^*$ is said to be *standard* if whenever M is represented as an f-sequence $M = (s_n)_{n\in\mathbb{Z}}$ then $(s_n)_{n\in\mathbb{Z}}$ is necessarily cyclic. Otherwise, M is said to be *nonstandard*.

(c) Suppose that M is a nonstandard f-sequence subgroup. If M admits representation as a cyclic f-sequence then we say that M is *nonstandard of the first type*; otherwise M is said to be *nonstandard of the second type.*

Essentially, M is standard if the "obvious" ways are the only ways of realising it as an f-sequence subgroup. If $M = (g^n)_{n\in\mathbb{Z}}$ is a representation of M as a cyclic f-sequence, then it is clear that g must be both a root of $f(t)$ and a generator of M as a group, whence M is a cyclic group. It is possible to find polynomials $f(t)$ which admit non-cyclic f-sequence subgroups: see Proposition 6(d) below.

Our main results are Propositions 4 and 6. Suppose that $f(t) \in \mathbb{C}[t]$ and that f has roots $g,\ h \in \mathbb{C}^*$. Except in the case

$$|g| = |h| \neq 1 \text{ and } g \neq \pm h,$$

which remains open, we prove that an f-sequence subgroup must be standard unless $g = -h$; when $g = -h$ we classify the nonstandard subgroups.

Observations 3: Suppose $f(t) = t^2 - \sigma t - \rho \in \mathbb{C}[t],\ \rho \neq 0$, with roots $g,\ h \in \mathbb{C}^*$, and let $(s_n)_{n\in\mathbb{Z}}$ be an f-sequence in $\mathbb{C}$.
(a) Suppose firstly that $g \neq h$. By linear algebra, there exist $\alpha, \beta \in \mathbb{C}$ with $s_0 = \alpha + \beta$ and $s_1 = \alpha g + \beta h$. By induction, $s_n = \alpha g^n + \beta h^n$ for all integers $n \geq 0$, and because $\rho \neq 0$ this may be extended to cover the case of negative n.
(b) Suppose next that $g = h$. There exist $\alpha, \beta \in \mathbb{C}$ such that $s_0 = \alpha$ and $s_1 = g(\alpha + \beta)$. Again, we have $s_n = (\alpha + n\beta)g^n$ for all $n \in \mathbb{Z}$.
(c) The reciprocal polynomial of $f(t)$ is $(-\rho)f^*(t)$ where $f^*(t) = t^2 + (\sigma/\rho)t - (1/\rho)$. The roots of $f^*(t)$ are $g^{-1}, h^{-1} \in \mathbb{C}^*$.

If $(s_n)_{n\in\mathbb{Z}}$ is an f-sequence in $\mathbb{C}$ then $(r_n)_{n\in\mathbb{Z}}$ is an f^*-sequence where $r_n = s_{-n}$. If $M = (s_n)_{n\in\mathbb{Z}}$ is an f-sequence subgroup of $\mathbb{C}^*$ then $M = (r_n)_{n\in\mathbb{Z}}$ is also an f^*-sequence subgroup. Thus M is standard as an f-sequence subgroup if and only if it is standard as an f^*-sequence subgroup. Further, if $s_n = \alpha g^n + \beta h^n$ for all $n \in \mathbb{Z}$ then $r_n = \alpha(g^{-1})^n + \beta(h^{-1})^n$ for all n.

Before continuing, we fix some notation. If $z \in \mathbb{C}$ then $|z|$ will always denote the modulus of z. We will use $\operatorname{ord}(z)$ to denote the multiplicative order of $z \in \mathbb{C}^*$, if z is a root of unity, and $\operatorname{ord}(M)$ to denote the order of the group M, if finite.

Proposition 4: *Let $f(t) = t^2 - \sigma t - \rho \in \mathbb{C}[t],\ \rho \neq 0$. Suppose that f has distinct roots $g,\ h \in \mathbb{C}^*$. Let $M = (s_n)_{n\in\mathbb{Z}} \leq \mathbb{C}^*$ be an f-sequence subgroup and write $s_n = \alpha g^n + \beta h^n$ for all n, for suitable $\alpha, \beta \in \mathbb{C}$. Suppose that either*
(1) $|g| \neq |h|$, *or*
(2) $|g| = |h| \neq 1$, *g/h is not a root of unity and* $|\alpha| \neq |\beta|$.
Then $\alpha\beta = 0$. Further, M is standard.

Proof: Suppose for a contradiction that $\alpha\beta \neq 0$. We may assume that $s_0 = 1$, while by Observation 3(c) we may also assume that $|g| \geq |h|$ and that $|g| > 1$. Write $\gamma = h/g$, so $0 < |\gamma| \leq 1$ and $s_m = g^m(\alpha + \beta\gamma^m)$. Suppose m is positive. Then $|(\alpha + \beta\gamma^m)|$ is bounded above by $|\alpha| + |\beta|$. If $|\gamma| < 1$ or if $|\gamma| = 1$ and $|\alpha| \neq |\beta|$ then $|(\alpha + \beta\gamma^m)|$ is bounded below (away from 0).

Now $s_m s_n \in M$ for all $m, n \in \mathbb{Z}$ because M is a group. Thus there exists a function $u : \mathbb{Z}^2 \to \mathbb{Z} : (m,n) \mapsto u(m,n)$ such that $s_m s_n = s_{u(m,n)}$ for all m, n. Thus, for all m, $n \in \mathbb{Z}$,

$$s_m s_n = g^{m+n}(\alpha + \beta\gamma^m)(\alpha + \beta\gamma^n) = g^{u(m,n)}(\alpha + \beta\gamma^{u(m,n)}). \tag{2}$$

The boundedness of $|\alpha+\beta\gamma^m|_{m>0}$ implies that $|g|^{m+n-u(m,n)}$ is bounded above and below whenever $m, n, u(m,n) \geq 0$. But $|g| > 1$ and so there exists a constant K such that

$$|m + n - u(m,n)| < K \tag{3}$$

whenever $m, n, u(m,n) \geq 0$.

Now fix $i \geq 0$ and suppose that $u(n+i, n-i) \geq 0$ for infinitely many n. By (3), there exists a fixed j with $|j| \leq K$ such that $u(n+i, n-i) = 2n + j$ for infinitely many n. Thus

$$s_{n+i}s_{n-i} = g^{2n}(\alpha + \beta\gamma^{n+i})(\alpha + \beta\gamma^{n-i}) = g^{2n+j}(\alpha + \beta\gamma^{2n+j}),$$

or

$$(\alpha^2 - \alpha g^i) + \alpha\beta(\gamma^i + \gamma^{-i})\gamma^n + (\beta^2 - \beta\gamma^j g^j)\gamma^{2n} = 0$$

for infinitely many n. Now $\alpha\beta \neq 0$, while $(\gamma^i + \gamma^{-i}) \neq 0$ because γ is not a root of unity. Thus, for infinitely many n, γ^n is a root of a fixed polynomial, independent of n, of degree either 1 or 2. Thus infinitely many of the γ^n must coincide, which is impossible because γ is neither zero nor a root of unity.

Thus for fixed $i \geq 0$, $u(n+i, n-i) < 0$ for all positive n but a finite number. Now (2) gives

$$g^{2n}(\alpha + \beta\gamma^{n+i})(\alpha + \beta\gamma^{n-i}) = h^{u(n+i,n-i)}(\alpha\gamma^{-u(n+i,n-i)} + \beta)$$

and so $|g|^{2n}|h|^{-u(n+i,n-i)}$ is bounded, independent of i and of n, provided just that $n > i \geq 0$ and $u(n+i, n-i) < 0$. But given $i \geq 0$, these conditions hold for infinitely many $n > i$, and so $|h| < 1$. It then follows that there exists a positive integer K_1 such that whenever $n > i \geq 0$ and $u(n+i, n-i) < 0$ we have

$$\left|\frac{u(n+i,n-i)}{2n} - \frac{\log|g|}{\log|h|}\right| < \frac{K_1}{2n}. \tag{4}$$

Let $\mathcal{R} = \{0, 1, \ldots, 4K_1 + 2\}$. For each $i \geq 0$, $u(n+i, n-i) < 0$ for all but finitely many positive n and so there exists N such that if $n > N$ we have $u(n+i, n-i) < 0$ for all $i \in \mathcal{R}$ simultaneously. Thus for distinct $i_1, i_2 \in \mathcal{R}$, (4) gives

$$|u(n+i_1, n-i_1) - u(n+i_2, n-i_2)| < 2K_1$$

whenever $n > N$. So for fixed $n_0 > N$, all integers $u(n_0+i, n_0-i)$ for $i \in \mathcal{R}$ belong to an interval of length at most $4K_1$ centered on $u(n_0, n_0)$. By the pigeon hole principle, there exist $i_1 \neq i_2$ such that $u(n_0+i_1, n_0-i_1) = u(n_0+i_2, n_0-i_2)$. Thus

$$s_{n_0+i_1}s_{n_0-i_1} = s_{n_0+i_2}s_{n_0-i_2}$$

and so

$$\alpha\beta(\gamma^{i_1}+\gamma^{-i_1})(gh)^{n_0}=\alpha\beta(\gamma^{i_2}+\gamma^{-i_2})(gh)^{n_0}.$$

Since $\alpha\beta gh \neq 0$, it follows that

$$\gamma^{i_1}-\gamma^{i_2}=\frac{\gamma^{i_1}-\gamma^{i_2}}{\gamma^{i_1}\gamma^{i_2}},$$

so that either $\gamma^{i_1}=\gamma^{i_2}$ or $\gamma^{i_1}\gamma^{i_2}=1$, both of which are impossible because γ is neither zero nor a root of unity. We conclude that $\alpha\beta=0$; it follows that M is standard. □

Lemma 5: *Let $f(t)=t^2-\sigma t-\rho\in\mathbb{C}[t]$, where $|\rho|=1$. Suppose that $f(t)$ has roots $g,h\in\mathbb{C}^*$. Let $M=(s_n)_{n\in\mathbb{Z}}\leq\mathbb{C}^*$ be an f-sequence subgroup.*
(a) *If $g\neq h$, then $|g|=|h|=1$ if and only if $|s|=1$ for all $s\in M$.*
(b) *If $g=h$, then $|g|=1$ and $|s|=1$ for all $s\in M$.*

Proof: (a) Suppose $g\neq h$. By Observation 3(a), there exist $\alpha,\beta\in\mathbb{C}$ with $s_n=\alpha g^n+\beta h^n$ for all $n\in\mathbb{Z}$. Now $|gh|=|\rho|=1$, so $|g|=1$ if and only if $|h|=1$. Suppose $|g|=|h|=1$ and that there exists $s\in M$ with $|s|\neq 1$. Then the cyclic subgroup $<s>\leq M$ contains elements of arbitrarily large modulus. But $|s_n|=|\alpha g^n+\beta h^n|\leq|\alpha|+|\beta|$ for all n, a contradiction.

Suppose next that $|s_n|=1$ for all $n\in\mathbb{Z}$. Assume $|g|>1$, so that $|h|<1$. If $\alpha=0$ then $\beta\neq 0$ and $1=|s_n|=|\beta h^n|$ for all n, which is absurd because β is fixed and $|h|<1$. Thus $\alpha\neq 0$. Now $||\alpha g^n|-|\beta h^n||\leq|\alpha g^n+\beta h^n|=|s_n|=1$. But $|\beta h^n|\leq|\beta|$, while $|\alpha g^n|$ is unbounded as n increases, a contradiction.
(b) Suppose $g=h\in\mathbb{C}^*$ is a double root of $f(t)$, so that $|g|=1$. By Observation 3(b), there exist $\alpha,\beta\in\mathbb{C}$ with $s_n=(\alpha+n\beta)g^n$ for all $n\in\mathbb{Z}$. As $0\notin M$ then not both α,β can be zero. Suppose there exists $s\in M$ with $|s|\neq 1$. Then the subgroup $<s>\leq M$ contains elements of arbitrarily small modulus. But $s_n=(\alpha+n\beta)g^n$, whence $|s_n|\geq||\alpha|-|n\beta||$. Since α,β are fixed and not both zero then $||\alpha|-|n\beta||\neq 0$ whenever $n\in\mathbb{Z}$ is such that $n|\beta|\neq|\alpha|$, and then

$$\{||\alpha|-|n\beta||:n\in\mathbb{Z},\ n|\beta|\neq|\alpha|\}$$

is bounded away from 0, a contradiction. □

Proposition 6: *Let $f(t)=t^2-\sigma t-\rho\in\mathbb{C}[t]$, where $\rho\neq 0$, and suppose that f has roots $g,h\in\mathbb{C}^*$. Let $M\leq\mathbb{C}^*$ be an f-sequence subgroup. Then*
(a) *If $|g|=|h|=1$ and $g\neq\pm h$ then M is standard.*
(b) *If $g=h$ then M is standard.*
(c) *If $g=-h$ then M is finite if and only if ρ is a root of unity.*
(d) *If $g=-h$ and if M is infinite then M has one of the forms:*

$$M=(\ldots,\rho^{-1},\varepsilon\rho^{k-1}\sqrt{\rho},1,\varepsilon\rho^k\sqrt{\rho},\rho,\ldots)\quad \textit{or}$$
$$M=(\ldots,\rho^{-1},-\rho^{k-1},1,-\rho^k,\rho,\ldots),$$

where $\varepsilon\in\{1,-1\}$ and $k\in\mathbb{Z}$. In the first case, $M=<\varepsilon\sqrt{\rho}>$ is cyclic and nonstandard of the first type. In the second case, $M=<-1>\times<\rho>$ is non-cyclic and nonstandard of the second type.
(e) *Suppose $g=-h$ and M is finite of order m. Write $r=$*ord(ρ),*by* (c).

If r is even then $m = 2r$ and M is nonstandard of the first type unless $\rho = -1$ when M is standard.

If r is odd then either $m = r$ and

$$M = (\dots, 1, \rho^{(r+1)/2}, \rho, \dots)$$

is standard, or else $m = 2r$ and

$$M = (\dots, 1, -g^j, g^2, \dots).$$

where $g = \rho^{(r+1)/2}$ and $1 \leq j \leq r$. Further, M is nonstandard of the first type unless $\rho = 1$, when M is standard.

Proof: Write $M = (s_n)_{n\in\mathbb{Z}}$. Without loss, suppose $s_0 = 1$.

(a) Suppose $|g| = |h| = 1$ and $g \neq \pm h$; then $\sigma \neq 0$ and $|\rho| = 1$. By Lemma 5, M lies on the unit circle.

Write $\tau = \sigma/2 \neq 0$. Then $\{g, h\} = \{\tau \pm \sqrt{\tau^2 + \rho}\}$ and $\tau^2 + \rho \neq 0$. If $u, v \in \mathbb{C}^*$ are such that $|u + v| = |u - v|$ then the segments $0u$ and $0v$ are perpendicular, whence $|u \pm v| = \sqrt{|u|^2 + |v|^2}$. Here, $|g| = |h| = 1 = \sqrt{|\tau^2| + |\tau^2 + \rho|}$, and so $1 = |\tau^2| + |\tau^2 + \rho|$. Then

$$1 = |\rho| = | -\tau^2 + \tau^2 + \rho| \leq |\tau^2| + |\tau^2 + \rho| = 1,$$

whence $-\tau^2$ and $\tau^2 + \rho$ are parallel; that is, $\rho = k\tau^2$ where $k \in \mathbb{R}$ and $k < -1$. Thus, $|\tau| < 1$, so $0 < |\sigma| < 2$. Now $s_1 = \sigma 1 + \rho s_{-1}$ and because $|s_{-1}| = 1$, then $|s_1 - \sigma| = |\rho s_{-1}| = 1 = |s_1|$. But given a circle of radius 1, a fixed diameter l and $\lambda \in \mathbb{R}$ with $0 < \lambda < 2$, the circle has exactly two chords of length λ parallel to l. Thus, for σ fixed, there are just two $s \in \mathbb{C}$ such that $|s - \sigma| = |s| = 1$. But the roots $g \neq h$ of $f(t)$ satisfy $|g - \sigma| = |g| = |h - \sigma| = |h| = 1$. Thus the only f-sequence subgroups are $(\dots, 1, g, \dots)$ and $(\dots, 1, h, \dots)$, and M is standard in this case.

(b) Suppose that $g = h$. By Observation 3(b), there exist $\alpha, \beta \in \mathbb{C}$ with $s_n = g^n(\alpha + \beta n)$ for $n \in \mathbb{Z}$, while $\alpha = 1$ because $s_0 = 1$.

Suppose firstly that $|g| = 1$. Now $\sigma = 2g$ and $\rho = -g^2$, so $|\rho| = 1$ and then $|s| = 1$ for all $s \in M$ by Lemma 5(b). But $s_1 = 2g - g^2 s_{-1}$ because $s_0 = 1$. Thus, $|s_1 - 2g| = |g^2 s_{-1}| = 1$, so s_1 and $s_1 - 2g$ lie on the unit circle at distance $|2g| = 2$ from each other. Thus $s_1 = g$ and $M = (\dots, 1, g, \dots)$ is standard.

By Observation 3(c) we may now suppose $|g| > 1$. It is easy to check that

$$\lim_{n\to\infty} |s_n| = \infty \text{ and } \lim_{n\to\infty} |1 + \beta n|/|1 + \beta(n+1)| = 1;$$

in the second limit, the denominator is equal to $|s_{n+1}/g^{n+1}|$ and so is non-zero. Therefore there exists $N_1 \in \mathbb{N}$ such that both $|g| > |1+\beta n|/|1+\beta(n+1)|$ and $|s_n| > 1$ whenever $n > N_1$. Thus $|s_{n+1}| > |s_n| > 1$ for $n > N_1$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $|s_{n-1}| < |s_n| < 1$ whenever $n < -N_2$ and so there exists $K \in \mathbb{N}$ with $K > N_1$ such that

$$|s_n| > \max\{|s_j|, 1/|s_j| : -N_2 \leq j \leq N_1\}$$

whenever $n \geq K$, in particular, $|s_K| > |s_j|$ if $j < K$. Thus, $s_K^{-1} = s_L$ for some $L < -N_2$. The monotonicity of $|s_n|$ with respect to n outside the interval $[-N_2, N_1]$ and the fact that M is a group now guarantee that $s_{K+j}^{-1} = s_{L-j}$ for all $j \in \mathbb{N}_0$. It follows that

$$g^{K+j}(\alpha + \beta(K+j))g^{L-j}(\alpha + \beta(L-j)) = 1, \ j = 0, 1, 2.$$

Simplification gives

$$g^{K+L}\beta^2 KL = g^{K+L}\beta^2(K+1)(L-1) = g^{K+L}\beta^2(K+2)(L-2).$$

Now $g \neq 0$ because $\rho \neq 0$. If $\beta \neq 0$ then both $L - K - 1 = 0$ and $2(L-K) - 4 = 0$, which is absurd. Thus $\beta = 0$ and M is standard, proving (b).

We now assume for the rest of the proof that $g = -h$, so that $\sigma = 0$, $f(t) = t^2 - \rho$, $g^2 = \rho$ and $\{g, h\} = \{\sqrt{\rho}, -\sqrt{\rho}\}$. Then $s_{n+2} = \rho s_n$ for all $n \in \mathbb{Z}$, and so $M = (\ldots, 1, x, \rho, x\rho, \ldots)$ where $x = s_1$: we will fix this interpretation for x.

(c) If M is infinite then $\rho^i \neq \rho^j$ whenever $i \neq j$ and so ρ is not a root of unity. If M is finite then the powers of ρ cannot be all distinct, whence ρ is a root of unity.

(d) Suppose that M is infinite. Then the elements ρ^j and $x\rho^j$ are all distinct as j runs over $\mathbb{Z}$. Now $x^2 \in M$ and so either $x^2 = x\rho^j$ or $x^2 = \rho^j$, for suitable j. If $x^2 = x\rho^j$ then $x = \rho^j$, contrary to distinctness; thus $x^2 = \rho^j$. There are two cases:

(1) Suppose $j = 2k+1$ is odd. Then $x = \varepsilon\rho^k\sqrt{\rho}$, where $\varepsilon \in \{1, -1\}$ and

$$M = (\ldots, \rho^{-1}, \varepsilon\rho^{k-1}\sqrt{\rho}, 1, \varepsilon\rho^k\sqrt{\rho}, \rho, \ldots).$$

We may shift the subsequence $(s_n)_{n \text{ odd}}$ relative to $(s_n)_{n \text{ even}}$ any number of places to the left or right and obtain different representations of M as an f-sequence: this corresponds to taking different values of k. With $k = 0$ we obtain a cyclic representation of M as an f-sequence, and so M is nonstandard of the first type.

(2) Suppose $j = 2k$ is even. Then $x \in \{\rho^k, -\rho^k\}$, whence $x = -\rho^k$ by distinctness. Then

$$M = (\ldots, \rho^{-1}, -\rho^{k-1}, 1, -\rho^k, \rho, \ldots),$$

so that $M = < -1 > \times < \rho >$ is a non-cyclic group; thus M is nonstandard of the second type.

(e) Suppose M is finite of order m. We have $\rho = g^2$, while $x^2 = \rho^j$ with $1 \leq j \leq r$ by distinctness. Thus $x = \varepsilon g^j$ where $\varepsilon \in \{-1, 1\}$, and so $s_{2k} = g^{2k}$ and $s_{2k+1} = \varepsilon g^{2k+j}$ for all k. Then

$$M = (\ldots, 1, \varepsilon g^j, g^2, \varepsilon g^{j+2}, \ldots, g^{2k}, \varepsilon g^{2k+j}, \ldots).$$

The distinct elements of M are just the terms from $s_0 = 1$ to s_{m-1}, where s_m is the first occurence of 1 after s_0.

Suppose firstly that r is even. Then $\varepsilon \in < \rho >$, $\text{ord}(g) = 2r$ and $< \rho >$ contains no odd power of g. Thus j is odd as otherwise $s_{2k+1} = \varepsilon g^{2k+j}$ would be an even power of g, against distinctness. But now $s_{2k+1} = \varepsilon g^{2k+j} \neq 1$ for all k, so s_{2r} is the first occurrence of 1 and $m = 2r$; we may shift $(s_n)_{n \text{ odd}}$ to obtain r distinct sequences, with that for $j = 1$ being cyclic. Thus M is nonstandard of the first type unless $r = 2$ when $M = (\ldots, 1, \varepsilon i, -1, -\varepsilon i, 1, \ldots)$ is standard.

Suppose next that r is odd. Then $-1 \notin <\rho>$ and $<\rho>$ contains a unique square-root of ρ, namely $\rho^{(r+1)/2}$. We may suppose that $g = \rho^{(r+1)/2}$; then ord(g) =ord$(\rho) = r$.

Suppose $\varepsilon = 1$. Then j is odd, by distinctness. Write $d = (r-j)/2 \geq 0$. Then $s_{2d+1} = g^{2d+j} = 1$ and this is evidently the first occurrence of 1 after s_0, whence $m = 2d+1$. But now $g^{2d+2} = s_{2d+2} = s_1 = g^j$ and so $r - j + 2 = 2d + 2 \equiv j (\text{mod } r)$. It follows that $j = 1$, $m = r$ and

$$M = (\ldots, 1, g, g^2, \ldots) = (\ldots, 1, \rho^{(r+1)/2}, \rho, \ldots)$$

is standard.

Suppose $\varepsilon = -1$. As $g \in <\rho>$ but $-1 \notin <\rho>$ then no term $s_{2k+1} = -g^{2k+j}$ belongs to $<\rho>$; thus the first occurrence of 1 after s_0 is $s_{2r} = g^{2r} = 1$, and so $m = 2r$. Again we may shift $(s_n)_{n \text{ odd}}$ to obtain r distinct sequences, with that for $j = 1$ being cyclic, so that M is nonstandard of the first type unless $r = 1$ and $M = (\ldots, 1, -1, 1, \ldots)$, which is standard. $\square$

Examples 7: (a) Let $f(t) = t^2 - 2$. As in Proposition 6(d), the following are f-sequence subgroups of $\mathbb{C}^*$, where $\varepsilon \in \{-1, 1\}$ and $k \in \mathbb{Z}$:

$$M_{1,\varepsilon} = (\ldots, 2^{-1}, \varepsilon 2^{k-1}\sqrt{2}, 1, \varepsilon 2^k \sqrt{2}, 2, \ldots) \text{ and}$$
$$M_2 = (\ldots, 2^{-1}, -2^{k-1}, 1, -2^k, 2, \ldots).$$

The groups $M_{1,\varepsilon} = <\varepsilon\sqrt{2}>$ are cyclic and nonstandard of the first type, while $M_2 = <-1> \times <2>$ is non-cyclic and nonstandard of the second type.
(b) Let $f(t) = t^2 - \omega$ where $\omega = e^{2\pi i/3} \in \mathbb{C}$. As in Proposition 6(e), the following are f-sequence subgroups:

$$M_1 = (\ldots, 1, \omega^2, \omega, 1, \ldots), \text{ and}$$
$$M_{-1} = (\ldots, 1, -\omega^j, \omega, -\omega^{j+1}, \omega^2, -\omega^{j+2}, 1, \ldots), \text{ where } 1 \leq j \leq 3.$$

The group M_1, of order 3, is standard, while M_{-1}, of order 6, is nonstandard of the first type (because the sequence with $j = 2$ is cyclic).
(c) Let $f(t) = t^2 - i$. The following are f-sequence subgroups of $\mathbb{C}^*$:

$$M_\varepsilon = (\ldots, 1, \varepsilon i^l \sqrt{i}, i, \varepsilon i^{l+1}\sqrt{i}, -1, \varepsilon i^{l+2}\sqrt{i}, -i, \varepsilon i^{l+3}\sqrt{i}, \ldots),$$

where $\varepsilon \in \{1, -1\}$ and $1 \leq l \leq 4$. The sequences with $l = 4$ are cyclic and so each M_ε is nonstandard of the first type.

Lemma 8: *Let $f(t) = t^2 - \sigma t - \rho \in \mathbb{C}[t]$, where $\rho \neq 0$, and suppose that f has roots $g, h \in \mathbb{C}^*$ with $|g| = |h| \neq 1, g \neq \pm h$. Suppose that $M = (s_n)_{n\in\mathbb{Z}}$ is an f-sequence subgroup of $\mathbb{C}^*$. Then M is infinite.*

Proof: By Observation 3(c), we may suppose that $|g| = |h| > 1$. Write $\gamma = h/g$; then $|\gamma| = 1$ but $\gamma \neq \pm 1$. By Observation 3(a), there exist $\alpha, \beta \in \mathbb{C}$ such that $s_n = g^n(\alpha + \beta\gamma^n)$ for $n \in \mathbb{Z}$. If M were finite then $1 = |s_n| = |g|^n |\alpha + \beta\gamma^n|$ for all n. But $|g|^n$ increases with n, and so $|\alpha + \beta\gamma^n|$ decreases. As n increases, the points $\alpha + \beta\gamma^n$ move (as $\gamma \neq 1$) around the circle with centre α and radius $|\beta|$. Thus $|\alpha + \beta\gamma^n|$ cannot decrease and so M cannot be finite. $\square$

Proposition 9: *Let $f(t) = t^2 - \sigma t - \rho \in \mathbb{C}[t]$, where $\rho \neq 0$. Suppose M is a finite f-sequence subgroup of $\mathbb{C}^*$. Then M is standard unless both $\sigma = 0$ and* $\operatorname{ord}(M)$ *is even and at least* 6, *in which case it is nonstandard of the first type.*

Proof: The result follows from Propositions 4 and 6 together with Lemma 8. □

ACKNOWLEDGMENT

We thank the referee for valuable suggestions, including a simpler proof of Proposition 4.

The first author wishes to acknowledge the partial support of the "Centro de Estruturas Lineares e Combinatórias" and of the Praxis Program (Praxis/2/2.1/mat/73/94).

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AMS Classification Numbers: 11B37, 11B39

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ZECKENDORF INTEGER ARITHMETIC

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(Submitted June 2001)

1. INTRODUCTION

Of the many ways of representing numbers, as described by Fraenkel [3] for example, the most usual and most important represent an integer N as the scalar product

$$N = \boldsymbol{D}.\boldsymbol{W}$$

where $\boldsymbol{D}$ is the *digit vector* (the visible digits of the representation) and $\boldsymbol{W}$ is a *weight vector*. To conform with normal conventions for displaying number representations these vectors are written in the order

$$\ldots w_i, w_{i-1}, \ldots, w_2, w_1, w_0.$$

The weight vector is in turn derived from the *base vector* $\boldsymbol{B}$ by

$$w_k = \prod_{i=0}^{k-1} b_i.$$

In the conventional *uniform base* number systems such as binary or decimal, $b_i = b \;\forall\; i$ and $w_k = b^k$, where the constant b is the base of the number system. So a number such as $40 = 1 \times 2^5 + 1 \times 2^3$ has the binary (base 2) representation 101000. For measurements in a mixed base system, the base vector has an appropriate mixture of values. As an example for {miles, yards, feet, inches}, the base vector is $\boldsymbol{B} = \{1760, 3, 12, 1\}$ and the weight vector is $\boldsymbol{W} = \{63360, 36, 12, 1\}$ to give lengths in inches.

But Zeckendorf has shown [8] that the Fibonacci numbers

$$F_n = \ldots, 34, 21, 13, 8, 5, 3, 2, 1, 1 \quad \text{(least-significant weight on right)}$$

can be used as the weight vector, in conjunction with a digit vector $\boldsymbol{D}$ in which $d_i \in \{0, 1\}$, for a representation which resembles a binary number. This gives what is now called the *Zeckendorf representation* on the integers; we will denote the Zeckendorf representation of N as $\mathcal{Z}(N)$. The Zeckendorf representation usually omits the redundant bit corresponding to $F_1 = 1$, so that the least-significant bit corresponds to F_2 (which is also equal to 1). For example as $30 = 21 + 8 + 1$, $\mathcal{Z}(30) = 1010001$. It has the important property that the Zeckendorf representation of a positive integer will never have two or more adjacent 1s; by the definition of its Fibonacci weights, any bit string such as $\ldots 00110\ldots$ is equivalent to $\ldots 01000\ldots$.

The Zeckendorf representations are of more than just intellectual interest. For example, Apostolico and Fraenkel [1] and Fraenkel and Klein [4] show that the Zeckendorf representations (although they do not call them by that name) are the basis of a "variable length" representation of the integers. These representations are important in coding theory, where a sequence of integers must be represented as a stream of bits, such that the average length of each integer in the bit stream is minimised, and the representations are self-delimiting. By transmitting the Zeckendorf representation least-significant bit first and following its most significant 1 by another 1, we get the illegal sequence $\ldots 011$ which can act as a terminating "comma".

Here though, we are more interested in showing that it is possible to perform arithmetic on integers in the Zeckendorf representation.

2. ARITHMETIC WITH ZECKENDORF INTEGERS

There is a little prior work in this connection. Graham, Knuth and Patashnik [7] discuss the addition of 1 in the Zeckendorf representation, but do not proceed to actual arithmetic. Freitag and Phillips [6] discuss addition and multiplication, and refer to Filiponi [2] and their own earlier paper [5] for subtraction. Thus no previous work discusses arithmetic as a coherent whole, covering all of the major operations, including multiplication and division.

The emphasis of this paper is frankly pragmatic, developing practical algorithms to perform the arithmetic operations. All have been implemented and tested on a computer. Most of the algorithms are developed by analogy with conventional arithmetic methods, supplemented as necessary by the requirements and constraints of the Zeckendorf representation. For example, multiplication will be performed by the addition of suitable multiples of the multiplicand, selected according to the bit pattern of the multiplier. Division will use a sequence of trial subtractions, as in normal long division.

2.1 ADDITION

We start addition by adding each pair of bits as separate numbers, giving an initial sum whose digits are $d_i \in \{0, 1, 2\}$, where each d_i corresponds to its Fibonacci number F_i. We then sweep over the whole representation until there is no further change, applying the following rules to eliminate the 2s (which are illegal digits) and consecutive 1s. The representation must be extended by one place to include as a trailing digit the d_1 term which is usually omitted.

Removal of '2' digits: From the fundamental relation that $F_n = F_{n-2} + F_{n-1}$, it is readily shown that $2F_n = F_{n+1} + F_{n-2}$. In digit patterns, we replace $\ldots 00200 \ldots$ by $\ldots 01001 \ldots$, subtracting the 2 and adding the two 1's to the nearby positions. Equivalently, a digit pattern $x2yz$ transforms to $(1+x)0y(1+z)$. A least-significant digit pattern of $\ldots 20$ clearly overflows beyond the least significant bit. We handle this by temporarily extending the representation by one place to include the d_1 digit so that the original $\ldots 020$ converts to $\ldots 1001$. (This rule does not apply to the d_2 and d_1 terms with weights of 1; this is covered by the special case below.)

Removal of adjacent 1s: Again using the fundamental relation $F_n = F_{n-2} + F_{n-1}$, we can replace two adjacent non-zero digits by a more-significant 1. This step should be performed by the left-to-right scan through the representation to avoid a "piling up" of the left-propagating carry with long runs of 1s.

Least-significant 1s: The first rule fails if we have 2 in the least-significant (F_2) digit, because there is nowhere to receive the rightward carry propagation. In this case we restore the F_1 term and replace the least-significant $\ldots 20$ by $\ldots 11$, which has the same numeric value. If the *extended* bit pattern is now $\ldots 111$, the first two 1s may be eliminated by the "adjacent 1s" rule. If the F_3 bit is a 0, bit pattern $\ldots 011$, we can immediately transform to $\ldots 100$

(still extended), and then eliminate the extension bit. It may in turn be replaced by $\ldots 100$, by the rule for consecutive 1s. (This rule could be eliminated entirely by an extension of the representation to include d_0 with a non-standard weight $w_0 = 1$.)

Remove the temporary d_1 term: If at any stage, $d_2 = 0$ and $d_1 = 1$ we can set $d_2 = 1$ and set $d_1 = 0$ (the two bits have the same weight), which is equivalent to discarding the d_1 term which was introduced. Setting $d_2 = 1$ may force a removal of adjacent 1s.

Addition	Fibonacci weights		F_{i+1}	F_i	F_{i-1}	F_{i-2}
Consecutive 1s			x	y	1	1
	becomes		x	$y+1$	0	0
Eliminate a 2	here $x \geq 2$		w	x	y	z
	becomes		$w+1$	$x-2$	y	$z+1$
Add, right bits			F_3	F_2	F_1	
$d_2 \geq 2$	here $x \geq 2$			x	0	
	becomes			$x-1$	1	
$d_2 \geq 2$ (alternate)			w	x	0	
	becomes		$w+1$	$x-2$	0	
$d_1 = 1$				0	1	
	becomes			1	0	
Subtraction	Fibonacci weights	F_{i+2}	F_{i+1}	F_i	F_{i-1}	F_{i-2}
eliminate -1		1	0	0	0	-1
	becomes	0	1	1	0	-1
	and again	0	1	0	1	0

Table 1: Adjustments and corrections in addition and subtraction

Zeckendorf addition has *two* carries, one going one place left to higher significance and one two places right to lower significance. The first is entirely analogous to the carry of conventional binary arithmetic, while the second reflects the special nature of the Zeckendorf representation.

These adjustments and corrections are summarised in Table 1, which also includes the sign-fill from subtraction (Section 2.2). Note that in all cases which show a 1 being inserted,

augend		1	0	0	0	0	1	0	1	$= 38$
addend			1	0	0	0	0	1	0	$= 23$
initial sum		1	1	0	0	0	1	1	1	$= 61$
consecutive 1s	1	0	0	0	0	1	0	0	1	$= 61$
result	1	0	0	0	0	1	0	0	1	$= 61$
Check - 38+23=61										

augend			1	0	0	0	1	0		$= 15$
addend		1	0	0	0	0	1	0		$= 23$
initial sum		1	1	0	0	0	2	0		$= 38$
carries		1	1	0	0	1	0	0	1	$= 38$
consecutive 1s	1	0	0	0	0	1	0	0	1	$= 38$
remove F_1 bit	1	0	0	0	0	1	0	1		$= 38$
result	1	0	0	0	0	1	0	1		$= 38$
Check - 15+23=38										

Figure 1: Two addition examples, $(38+23)$ and $(15+23)$

the real action is to *add* the 1 to the previous value of that digit; eliminating one 2 may very well change another 1 to a 2, which must in turn be corrected. The removal of a 2 is likewise performed as a *subtraction*, rather than a simple deletion.

The addition may be compared with conventional binary addition. Binary addition (or decimal, or in any other polynomial number system) has a single carry which propagates to more-significant digits. If we start the add from the least-significant end we need only a single pass and the carry management is readily included in the standard simple algorithm. The two carries of Zeckendorf addition make the operation much more complicated and seem to necessitate multiple passes to absorb carries.

To illustrate, Figure 1 shows the addition of $38+23 = 10000101 + 1000010$ and $15+23 = 100010 + 1000010$. Both display the decimal value to the right of each line to emphasise that the correction and redistribution of bits does not affect the value. One example shows the temporary extension of the Zeckendorf representation to include the F_1 term. Each line presents the representation and value after the operation given at the start of the line. The various rules of Table 1 may be applied in any order, possibly changing the intermediate values but not the final result.

2.2 SUBTRACTION

For subtraction say $X - Y \to Z$, where X is the minuend, Y the subtrahend and Z the difference, we start with a digit-wise subtraction $x_i - y_i \to z_i$, giving $z_i \in \{-1, 0, 1\}$. The two values 0 and 1 pose no problem, as they are valid digits in $\mathcal{Z}(Z)$.

The case $z_i = -1$ is rather more difficult. From where $z_i = -1$ we scan to its left looking for the next most-significant 1 bit. Then rewrite this bit by the Fibonacci rule $100\cdots \to 011\ldots$, and then repeat rewriting the rightmost 1-bit of the pair of 1s

$$1000\cdots \to 0110\cdots \to 001011\cdots \to 00101011\ldots$$

until one of the two rightmost 1 bits coincides in position with the -1 of

subtrahend	1	0	1	0	0	0	0	1	= 48
minuend	1	0	0	0	0	1	0	0	= 37
subtract digit-by-digit			1	0	0	-1	0	1	= 11
rewrite 1000				1	1	-1	0	1	= 11
rewrite 0110, cancelling -1				1	0	0	1	1	= 11
rewrite adjacent 1s				1	0	1	0	0	= 11
Result				1	0	1	0	0	= 11

Figure 2: Example of subtraction $-(48 - 37)$

the result and cancels it, leaving a 0 result. (There may of course be *no* more significant 1. This corresponds to a negative result; we introduce a suitable large F_n and proceed from there, producing an "F_n complement" as discussed later.) The scan for digits $z_i = -1$ should be performed from most-significant to least-significant digits. This action is included in Table 1 earlier.

The preceding rule eliminates all of the digits whose value is -1, but often introduces other digits greater than 1, or pairs of adjacent 1s. All of these situations must be handled by the rules already introduced for addition. Subtraction is therefore an extension of addition.

Figure 2 shows an example, subtracting 37 from 48. As with addition, the various rewriting rules may be applied in any order; changing that order will change the finer details of the subtraction.

Note that we cannot easily propagate a borrow left from the place where $z_i = -1$. The rewriting rule steps *two* positions at each step and without knowing the distance to the next-significant 1 we know neither the alignment of the 01 bits which are introduced nor which of the two final 11 bits will be finally cancelled. (Conventional binary subtraction rewrites, for example, 10000 as $01100 + 100 \to 01110 + 10 \to 01111 + 1$; the final $+1$ is cancelled against the 1 of the subtrahend. Each stage proceeds by only one place and there is no ambiguity in reversing the process for the conventional right-to-left borrow propagation.)

2.3 COMPLEMENTING

Subtraction quickly leads to negative numbers and their representations. Computer designers now prefer the 2s complement representation in which a number and its complement, added as unsigned quantities, total 2^n.

By analogy we can represent a negative value by its F_n complement. Also by analogy we say that a value is negative if its representation has its most-significant bit a 1. But this immediately introduces a major problem. An F_n complement representation has F_{n-2} values with a leading 1 and F_{n-1} values with a leading 0; there are about 1.6 times as many positive values as negative and about 38% of all positive values have no complement! (Complementing seemed almost incomprehensible until its asymmetrical range was realised. By analogy with binary numbers it was expected that there would be similar numbers of positive and negative values but there was no simple way of differentiating signed integers.)

N	$\mathcal{Z}(N)$	$F(8)$ comp	$F(9)$ comp	$F(10)$ comp	$F(11)$ comp
1	1	101010	1010101	10101010	101010101
2	10	101001	1010100	10101001	101010100
3	100	101000	1010010	10101000	101010010
4	101	100101	1010001	10100101	101010001
5	1000	100100	1010000	10100100	101010000
6	1001	100010	1001010	10100010	101001010
7	1010	100001	1001001	10100001	101001001
8	10000	100000	1001000	10100000	101001000
9	10001	–	1000101	10010101	101000101
10	10010	–	1000100	10010100	101000100
11	10100	–	1000010	10010010	101000010

Table 2: Illustration of Fibonacci $F(n)$ complements

If a positive number N requires n bits, then $N < F_{n+1}$. The signed number $-N$ requires at least $n+2$ bits and must use at least the F_{n+3} complement. (Numbers of this precision will require at least the original n bits, place space for the sign. The "sign" of a negative number is, by definition, a '1' bit, but this 1 *must* be followed by a 0 for a valid Zeckendorf representation.)

Complementing is most easily handled by subtraction from zero; there seems to be simple complementing rule. In comparison with binary arithmetic it is complicated considerably by the bi-directional carries and by the "sign-fill" pattern of 101010..., whose alignment with respect to the significant bits is not easily decided. Because the sign fill pattern is 1010... an extension from unsigned to signed numbers requires at least 2 extra bits.

Some complements are shown in Table 2. We see that a negative number is characterised by a leading 1010... bit pattern, rather than the 1111... usually associated with binary numbers. The 1010... pattern has two alignments with respect to the bits of the value being complemented; these two alignments and interactions with the "numerically significant" bits lead to two different bit patterns in the complement. In the example, $\mathcal{Z}(7) = 1010$ and the two patterns are ...0001 for n even and ...01001 for n odd.

2.4 MULTIPLICATION

In the introduction we discussed the representation of an integer N as the scalar product

$$N = \boldsymbol{D}.\boldsymbol{W}$$

where $\boldsymbol{D}$ is the *digit vector* (the visible digits of the representation) and $\boldsymbol{W}$ is a *weight vector*. We now develop multiplication by analogy with conventional multiplication, building on this representation. Only positive values will be considered for both multiplication and, later, division.

To calculate the product $Z \leftarrow X \times Y$, we first write X (the multiplier) as $\boldsymbol{X}.\boldsymbol{W}$, giving

$$Z \leftarrow \boldsymbol{X}.\boldsymbol{W} \times Y$$

whence

$$Z \leftarrow \sum x_i.w_i.Y = \sum x_i.(w_i.Y).$$

multiplicand						1	0	0	1	0	1	= 17
multiplier							1	0	1	0	0	= 11
Make Fibonacci Multiples of multiplicand												
F_3 multiple				1	0	0	0	0	0	0	0	= 34
F_4 multiple				1	0	1	0	0	1	0	1	= 51
F_5 multiple			1	0	1	0	1	0	0	0	1	= 85
F_6 multiple		1	0	1	0	1	0	0	0	0	0	= 136
Accumulate appropriate multiples												
add F_4 multiple				1	0	1	0	0	1	0	1	= 51
add F_6 multiple		1	0	1	0	1	0	0	0	0	0	= 136
product	1	0	0	1	0	0	1	0	0	0	1	= 187

Check that $17 \times 11 = 187$

Table 3: Example of Zeckendorf multiplication (17×11)

The product is the sum of appropriately weighted multiples of the multiplicand Y, each multiple in turn multiplied by the multiplier digit x_i. In a uniform base number system the scaling is easily done by "left shifting", or appending 0s to the right of $\boldsymbol{Y}$, as in standard long multiplication.

With Fibonacci arithmetic, the scaling must mirror the generation of the Fibonacci numbers themselves; we can no longer use simple shifts or inclusion of 0s. The weight vector $\boldsymbol{W}$ is now the Fibonacci numbers; given a multiplicand Y, we generate its *Fibonacci multiples* M_n as -

$$M_1 = M_2 = \mathcal{Z}(Y),\ M_3 = M_1 + M_2, \ldots, M_k = M_{k-1} + M_{k-2}, \ldots$$

and then add these weighted by the bits of $\mathcal{Z}(X)$. (All arithmetic is of course done using the Zeckendorf addition of Section 2.1.)

An example of multiplication given in Table 3. It shows first the two factors, then the multiples of the multiplicand, and finally the steps of the multiplication proper.

3. DIVISION

Division is, as might be expected, the reversal of miltiplication. The procedure is precisely that of a conventional "long division", but adapted to use the Fibonacci multiples of multiplication rather than the scaled multiples of conventional arithmetic. Starting with the dividend, we try to subtract successively decreasing Fibonacci multiples of the divisor, entering quotient bits as appropriate, a 0 for an unsuccessful subtraction and a 1 for a successful subtraction. Again, we restrict ourselves to positive inputs.

Table 4 shows an example of Zeckendorf division. We start by building a suitable table of the Fibonacci multiples of the divisor, stopping when the multiple has more bits than the dividend. Note here that the dividend, assumed unsigned, must be extended by at least 2 bits to accommodate the negative values which arise during division.

We then enter the cycle of trial subtractions. At each stage, if the residue[1] is negative ("unsuccessful" subtraction), we enter a 0 bit in the quotient and restore the previous residue. If the residue is positive ("successful" subtraction"), we enter a quotient bit of 1 and use the

dividend		1	0	0	1	0	0	0	1	0	1	0	1	=300
divisor								1	0	0	1	0	1	=17
Make Fibonacci Multiples of divisor														
F_2 multiple								1	0	0	1	0	1	=17
F_3 multiple						1	0	0	0	0	0	0	0	=34
F_4 multiple						1	0	1	0	0	1	0	1	=51
F_5 multiple					1	0	1	0	1	0	0	0	1	=85
F_6 multiple				1	0	1	0	1	0	0	0	0	0	=136
F_7 multiple			1	0	1	0	1	0	0	0	0	0	1	=221
F_8 multiple		1	0	1	0	1	0	0	0	0	0	0	1	=357
F_9 multiple	1	0	1	0	1	0	0	0	0	0	0	1	0	=578
Trial subtractions														
F_9 overdraw =														
F_8 overdraw =														
F_7 residue =					1	0	1	0	0	0	1	0	0	=79
F_5 overdraw =														
F_4 residue =							1	0	0	1	0	1	0	=28
F_2 residue =									1	0	1	0	0	=11
quotient =								1	0	0	1	0	1	=17
remainder =									1	0	1	0	0	=11

Check $-300 \div 17 = 17(+\text{rem} = 11)$

Table 4: Example of Zeckendorf division ($300 \div 17$)

new residue for the next trial subtraction. As an optimisation with a successful subtraction, we can avoid the next multiple completely, going immediately from say M_i to M_{i-2}. The subtraction with the M_{i-1} multiple cannot succeed because that would give two consecutive 1s in the quotient.

4. CONCLUSIONS

Although we have demonstrated the main arithmetic operations on Zeckendorf integers, this arithmetic is unlikely to remain more than a curiosity. It is much more complex than normal binary arithmetic based on powers of 2 and the Zeckendorf representations are themselves more bulky than the corresponding binary representations.

Another problem lies in the representation of fractions. Powers of 2 extend naturally to negative powers and fractional values. Extending the Fibonacci numbers F_n for $n < 0$ repeats the values for positive n, but with alternating sign; they provide no way of representing fractions.

[1] There is no generally accepted term for the working value from which divisor multiples are subtracted during division. While some authors use "partial remainder" or "partial dividend", the preference here is for "residue".

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AMS Classification Numbers: 11B39, 03H15

✠✠✠

HEPTAGONAL NUMBERS IN THE FIBONACCI SEQUENCE AND DIOPHANTINE EQUATIONS $4x^2 = 5y^2(5y-3)^2 \pm 16$

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(Submitted June 2001-Final Revision October 2001)

1. INTRODUCTION

The numbers of the form $\frac{m(5m-3)}{2}$, where m is any positive integer, are called heptagonal numbers. The first few are 1, 7, 18, 34, 55, 81, ..., and are listed in [4] as sequence number A000566. In this paper it is established that 0, 1, 13, 34 and 55 are the only generalized heptagonal numbers (where m is any integer) in the *Fibonacci sequence* $\{F_n\}$. These numbers can also solve the Diophantine equations of the title. Earlier, J.H.E. Cohn [1] has identified the squares and Ming Luo (see [2] and [3]) has identified the triangular, pentagonal numbers in $\{F_n\}$. Furthermore, in [5] it is proved that 1, 4, 7 and 18 are the only generalized heptagonal numbers in the *Lucas sequence* $\{L_n\}$.

2. IDENTITIES AND PRELIMINARY LEMMAS

We have the following well known properties of $\{F_n\}$ and $\{L_n\}$:

$$F_{-n} = (-1)^{n+1}F_n \text{ and } L_{-n} = (-1)^n L_n \tag{1}$$

$$2F_{m+n} = F_m L_n + F_n L_m \text{ and } 2L_{m+n} = 5F_m F_n + L_m L_n \tag{2}$$

$$F_{2n} = F_n L_n \text{ and } L_{2n} = L_n^2 + 2(-1)^{n+1} \tag{3}$$

$$L_n^2 = 5F_n^2 + 4(-1)^n \tag{4}$$

$$2|F_n \text{ iff } 3|n \text{ and } 2|L_n \text{ iff } 3|n \tag{5}$$

$$3|F_n \text{ iff } 4|n \text{ and } 3|L_n \text{ iff } n \equiv 2 (\text{mod } 4) \tag{6}$$

$$9|F_n \text{ iff } 12|n \text{ and } 9|L_n \text{ iff } n \equiv 6 (\text{mod } 12) \tag{7}$$

$$L_{8n} \equiv 2 (\text{mod } 3). \tag{8}$$

If $m \equiv \pm 2 (\text{mod } 6)$, then

$$L_m \equiv 3 (\text{mod } 4) \text{ and } L_{2m} \equiv 7 (\text{mod } 8), \tag{9}$$

$$F_{2mt+n} \equiv (-1)^t F_n (\text{mod } L_m), \tag{10}$$

where n, m, and t denote integers.

Since, N is a generalized heptagonal number if and only if $40N + 9$ is the square of an integer congruent to 7(mod 10), we identify those n for which $40F_n + 9$ is a perfect square. We begin with

Lemma 1: Suppose $n \equiv 0 (\text{mod } 2^4 \cdot 17)$. Then $40F_n + 9$ is a perfect square if and only if $n = 0$.

Proof: If $n = 0$, then $40F_n + 9 = 3^2$.

Conversely, suppose $n \equiv 0 (\text{mod } 2^4 \cdot 17)$ and $n \neq 0$. Then n can be written as $n = 2 \cdot 17 \cdot 2^\theta \cdot g$, where $\theta \geq 3$ and $2 \not| g$. And since for $\theta \geq 3, 2^{\theta+8} \equiv 2^\theta (\text{mod } 680)$, taking $k = 2^\theta$ if $\theta \equiv 0$, 5 or 7 (mod 8) and $k = 17 \cdot 2^\theta$ for the other values of θ, we have

$$k \equiv 32, 128, \pm 136, 256, 272 \text{ or } 408 (\text{mod } 680). \tag{11}$$

Since $k \equiv \pm 2 (\text{mod } 6)$, from (10), we get

$$40F_n + 9 = 40F_{2k(2x+1)} + 9 \equiv 40(-1)^x F_{2k} + 9 (\text{mod } L_{2k}).$$

Therefore, using properties (1) to (9) of $\{F_n\}$ and $\{L_n\}$, the Jacobi symbol

$$\left(\frac{40F_n + 9}{L_{2k}}\right) = \left(\frac{\pm 40F_{2k} + 9}{L_{2k}}\right) = \left(\frac{3}{L_{2k}}\right)\left(\frac{\pm 40\frac{F_{2k}}{3} + 3}{L_{2k}}\right) = -\left(\frac{L_{2k}}{3}\right)\left(\frac{\pm 80\frac{F_k}{3}L_k + 3L_k^2}{L_{2k}}\right).$$

Letting $u_k = \frac{F_k}{3}$ and $v_k = 80u_k \pm 3L_k$ we obtain

$$\left(\frac{40F_n + 9}{L_{2k}}\right) = \pm\left(\frac{80u_kL_k \pm 3L_k^2}{L_{2k}}\right) = -\left(\frac{L_{2k}}{80u_kL_k \pm 3L_k^2}\right) = -\left(\frac{L_{2k}}{L_k}\right)\left(\frac{L_{2k}}{v_k}\right)$$

$$= -\left(\frac{-2}{L_k}\right)\left(\frac{\frac{1}{2}(5F_k^2 + L_k^2)}{v_k}\right) = \left(\frac{2}{L_k \cdot v_k}\right)\left(\frac{720F_k^2 + 144L_k^2}{v_k}\right)$$

Since $v_k = \frac{80F_k}{3} \pm 3L_k$, then $144L_k^2 \equiv \frac{102400F_k^2}{9}$ (mod v_k) and

$$\left(\frac{720F_k^2 + 144L_k^2}{v_k}\right) = \left(\frac{108880U_k^2}{v_k}\right) = \left(\frac{5 \times 1361}{v_k}\right) = \left(\frac{v_k}{5}\right)\left(\frac{v_k}{1361}\right) = \left(\frac{v_k}{1361}\right)$$

$$= -\left(\frac{80F_k \pm 9L_k}{1361}\right).$$

Furthermore, $\left(\frac{2}{L_k \cdot v_k}\right) = -1$, it follows that $\left(\frac{40F_n+9}{L_{2k}}\right) = \left(\frac{80F_k \pm 9L_k}{1361}\right)$.

But modulo 1361, the sequence $\{80F_n \pm 9L_n\}$ is periodic with period 680 and by (11), $\left(\frac{80F_k \pm 9L_k}{1361}\right) = -1$, for all values of k. The lemma follows.

Lemma 2: Suppose $n \equiv \pm 1,\ 2,\ \pm 7,\ \pm 9,\ 10 (\text{mod } 133280)$. Then $40F_n + 9$ is a perfect square if and only if $n = \pm 1,\ 2, \pm 7, \pm 9, 10$.

Proof: To prove this, we adopt the following procedure which enables us to tabulate the corresponding values reducing repetition and space.

Suppose $n \equiv \varepsilon (\text{mod } N)$ and $n \neq \varepsilon$. Then n can be written as $n = 2 \cdot \delta \cdot 2^\theta \cdot g + \varepsilon$, where $\theta \geq \gamma$ and $2 \not| g$. Then, $n = 2km + \varepsilon$, where k is odd, and m is even.

Now, using (10), we choose m such that $m \equiv \pm 2 (\text{mod } 6)$. Thus,

$$40F_n + 9 = 40F_{2km+\varepsilon} + 9 \equiv 40(-1)^k F_\varepsilon + 9 (\text{mod } L_m).$$

Therefore, the Jacobi symbol

$$\left(\frac{40F_n + 9}{L_m}\right) = \left(\frac{-40F_\varepsilon + 9}{L_m}\right) = \left(\frac{L_m}{M}\right). \tag{12}$$

But modulo M, $\{L_n\}$ is periodic with period P. Now, since for $\theta \geq \gamma, 2^{\theta+s} \equiv 2^\theta (\text{mod } P)$, choosing $m = \mu \cdot 2^\theta$ if $\theta \equiv \zeta (\text{mod } s)$ and $m = 2^\theta$ otherwise, we have $m \equiv c (\text{mod } P)$ and $\left(\frac{L_m}{M}\right) = -1$, for all values of m. From (12), it follows that $\left(\frac{40F_n+9}{L_m}\right) = -1$, for $n \neq \varepsilon$. For each value of ε, the corresponding values are tabulated in this way (Table A).

<table>
<tr><th>ε</th><th>N</th><th>δ</th><th>γ</th><th>s</th><th>M</th><th>P</th><th>μ</th><th>ζ(mod s)</th><th>c (mod P)</th></tr>
<tr><td>±1, 2</td><td>$2^2 \cdot 7^2$</td><td>7^2</td><td>1</td><td>4</td><td>31</td><td>30</td><td>7^2</td><td>2, 3.</td><td>2, 16.</td></tr>
<tr><td rowspan="2">±7</td><td rowspan="2">$2^5 \cdot 7^2$</td><td rowspan="2">7^2</td><td rowspan="2">4</td><td rowspan="2">36</td><td rowspan="2">511</td><td rowspan="2">592</td><td>7^2</td><td>13, 31.</td><td rowspan="2">±16, ±32, ±48, ±144, ±160, ±192, ±208, ±240, ±272, ±288.</td></tr>
<tr><td>7</td><td>0, 1, 6, 7, 8, ±9, 16, 18, 19, 24, 25, 26, 34.</td></tr>
<tr><td rowspan="3">±9</td><td rowspan="3">$2^5 \cdot 5 \cdot 7^2$</td><td rowspan="3">$5 \cdot 7^2$</td><td rowspan="3">4</td><td rowspan="3">48</td><td rowspan="3">1351</td><td rowspan="3">1552</td><td>$5 \cdot 7^2$</td><td>2, 20, 26, 44.</td><td rowspan="3">±32, ±48, ±64, ±112, ±208, ±256, ±304, ±352, ±368, ±432, ±464, ±480, ±528, ±560, ±592, ±672, ±688, ±704, ±752, ±768</td></tr>
<tr><td>7^2</td><td>7, 15, 18, 31, 39, 42.</td></tr>
<tr><td>7</td><td>0, 1, 4, 9, 11, 19, 21, 24, 25, 28, 33, 35, 43, 45.</td></tr>
<tr><td rowspan="3">10</td><td rowspan="3">$2^5 \cdot 7^2 \cdot 17$</td><td rowspan="3">$17 \cdot 7^2$</td><td rowspan="3">4</td><td rowspan="3">52</td><td rowspan="3">2191</td><td rowspan="3">2512</td><td>$17 \cdot 7^2$</td><td>0, 8, 26, 34.</td><td rowspan="3">±32, ±48, ±112, ±128, ±224, ±272, ±432, ±448, ±512, ±624, ±1024, ±1040, ±1072, ±1248</td></tr>
<tr><td>7^2</td><td>1, 11, 14, 19, 21, 27, 37, 40, 45, 47.</td></tr>
<tr><td>7</td><td>±4, 6, 12, ±13, 18, ±22, 25, 32, 38, 44, 51.</td></tr>
</table>

Table A.

Since L.C.M. of $(2^2 \cdot 7^2, 2^5 \cdot 7^2, 2^5 \cdot 5 \cdot 7^2, 2^5 \cdot 7^2 \cdot 17) = 133280$, the lemma follows.

As a consequence of Lemma 1 and 2 we have the following.

Corollary 1: Suppose $n \equiv 0, \pm1, 2, \pm7, \pm9, 10 (\text{mod } 133280)$. Then $40F_n+9$ is a perfect square if and only if $n = 0, \pm1, 2, \pm7, \pm9, 10$.

Lemma 3: $40F_n + 9$ is not a perfect square if $n \not\equiv 0, \pm1, 2, \pm7, \pm9, 10 (\text{mod } 133280)$.

Proof: We prove the lemma in different steps eliminating at each stage certain integers n congruent modulo 133280 for which $40F_n + 9$ is not a square. In each step, we choose an integer m such that the period p (of the sequence $\{F_n\}$ mod m) is a divisor of 133280 and thereby eliminate certain residue class modulo p. For example

Mod 29: The sequence $\{F_n\}$ mod 29 has period 14. We can eliminate $n \equiv \pm3, \pm6$ and 12 (mod 14), since $40F_n + 9 \equiv 2, 10, 8$ and $27 (\text{mod } 29)$ respectively and they are quadratic nonresidue modulo 29. There remain $n \equiv 0, \pm1, 2, \pm4, \pm5$ or $7 (\text{mod } 14)$, equivalently, $n \equiv 0, \pm1, 2, \pm4, \pm5, \pm7, \pm9, \pm10, \pm13, 14$ or $16 (\text{mod } 28)$.

Similarly we can eliminate the remaining values of n. After reaching modulus 133280, if there remain any values of n we eliminate them in the higher modulus (that is in the miltiples of 133280). We tablulate them in the following way (Table B).

<table>
<tr><th>Period p</th><th>Modulus m</th><th>Required values of n where $\left(\frac{40F_n+9}{m}\right)=-1$</th><th>Left out values of n (mod k) where k is a positive integer</th></tr>
<tr><td>14</td><td>29</td><td>±3, ±6, 12.</td><td>0, ±1, 2, ±4, ±5, 7 (mod 14)</td></tr>
<tr><td>28</td><td>13</td><td>±13, 16, 18, 24.</td><td>0, ±1, 2, 4, ±5, ±7, ±9, 10, 14 (mod 28)</td></tr>
<tr><td>8</td><td>3</td><td>±3, 6.</td><td rowspan="2">0, ±1, 2, ±7, ±9, 10, ±23, 28, 32 (mod 56)</td></tr>
<tr><td>56</td><td>281</td><td>4, 42.</td></tr>
<tr><td>16</td><td>7</td><td>4.</td><td rowspan="2">0, ±1, 2, ±7, ±9, 10, ±23, 28, ±33, 56 (mod 112)</td></tr>
<tr><td>112</td><td>14503</td><td>32, ±47, ±49, ±55, 58, 66, 88.</td></tr>
<tr><td>32</td><td>47</td><td>12, 24, 28.</td><td>0, ±1, 2, ±7, ±9, 10, ±23, ±33, ±79, ±89, ±103, ±105, ±111, 112, 114, 168 (mod 224)</td></tr>
<tr><td>10</td><td>11</td><td>±4, 8.</td><td rowspan="8">0, ±1, 2, ±7, ±9, 10, ±551, 560, 1010 (mod 1120)</td></tr>
<tr><td>40</td><td>41</td><td>±15, ±17, 32.</td></tr>
<tr><td rowspan="2">70</td><td>71</td><td>±19, ±21, ±23, ±27, ±33.</td></tr>
<tr><td>911</td><td>±41.</td></tr>
<tr><td rowspan="2">160</td><td>1601</td><td>±39, 40, 90, 122, 130.</td></tr>
<tr><td>3041</td><td>±79, ±73, 82.</td></tr>
<tr><td>80</td><td>2161</td><td>±41, 42.</td></tr>
<tr><td>140</td><td>141961</td><td>±61.</td></tr>
<tr><td>196</td><td>97</td><td>±19, ±27, 28, ±29, ±35, 56, ±57, ±65, 66, 86, ±91, 122, 150, 178.</td><td rowspan="5">0, ±1, 2, ±7, ±9, 10, ±3369, ±3911, 3920 (mod 7840)</td></tr>
<tr><td rowspan="2">490</td><td>491</td><td>72, ±77, 100, ±133, ±141, 142, ±147, 170, ±201, ±209, 210, 212, ±219, 310, 352, 430.</td></tr>
<tr><td>1471</td><td>30, 140, ±149, ±217, 240, 280, 290, 422.</td></tr>
<tr><td>392</td><td>5881</td><td>58, ±113, 168.</td></tr>
<tr><td>7840</td><td>54881</td><td>±551.</td></tr>
<tr><td>136</td><td>67</td><td>8, ±17, ±23, ±25, 26, 32, 34, ±39, 40, ±41, 42, 48, ±55, ±56, ±65, 90, 112, 114.</td><td rowspan="10">0, ±1, 2, ±7, ±9, 10, 66640 (mod 133280)</td></tr>
<tr><td>238</td><td>239</td><td>±19, 24, 28, ±35, ±37, ±41, ±43, 44, ±49, ±57, ±69, 70, ±71, ±75, ±77, 86, 100, ±103, ±107, 108, 142, 154, 164, 184, 196, 206.</td></tr>
<tr><td>680</td><td>1361</td><td>±73, ±121, ±151, ±167, ±193, ±319, ±321.</td></tr>
<tr><td>68</td><td>1597</td><td>±5, ±11, ±14, 20, 38, 64.</td></tr>
<tr><td>2380</td><td>2381</td><td>560, ±973, 1962, 2102.</td></tr>
<tr><td>34</td><td>3571</td><td>±4, ±13, 32.</td></tr>
<tr><td>1360</td><td>5441</td><td>160, 322, 970.</td></tr>
<tr><td rowspan="2">8330</td><td>16661</td><td>±919, ±1461, 7360.</td></tr>
<tr><td>124951</td><td>±2389.</td></tr>
<tr><td>26656</td><td>39983</td><td>±13319.</td></tr>
</table>

Table B

We now eliminate $n \equiv 66640 (\text{mod } 133280)$, equivalently, $n \equiv 66640$ or 199920 (mod 266560). Now, modulo 449, the sequence $\{40F_n + 9\}$ is periodic with period 448. Also, $66640 \equiv 336 (\text{mod } 448)$, $\left(\frac{40F_{336}+9}{449}\right) = -1$ and $199920 \equiv 112$ (mod 448), $\left(\frac{40F_{112}+9}{449}\right) = -1$. The lemma follows.

3. MAIN THEOREM

Theorem 1: **(a)** F_n is a generalized heptagonal number only for $n = 0, \pm 1, 2, \pm 7, \pm 9$ or 10; and **(b)** F_n is a heptagonal number only for $n = \pm 1, 2, \pm 9$ or 10.

Proof: Part **(a)** of the theorem follows from Corollary 1 and Lemma 3. For part **(b)**, since, an integer N is heptagonal if and only if $40N + 9 = (10.m - 3)^2$ where m is a positive integer. We have the following table.

n	0	± 1	2	± 7	± 9	10
F_n	0	1	1	13	34	55
$40F_n + 9$	3^2	7^2	7^2	23^2	37^2	47^2
m	0	1	1	-2	4	5
L_n	2	± 1	3	± 29	± 76	123

Table C.

4. SOLUTIONS OF CERTAIN DIOPHANTINE EQUATIONS

It is well known that if $x_1 + y_1\sqrt{D}$ (where D is not a perfect square and x_1, y_1 are least positive integers) is the fundamental solution of Pell's equation $x^2 - Dy^2 = \pm 1$, then the general solution is given by $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$. Therefore, by (4), we have

$$L_{2n} + \sqrt{5}F_{2n} \text{ is a solution of } x^2 - 5y^2 = 4, \tag{13}$$

while

$$L_{2n+1} + \sqrt{5}F_{2n+1} \text{ is a solution of } x^2 - 5y^2 = -4. \tag{14}$$

We have, by (13), (14), Theorem 1, and Table C, the following two corollaries.

Corollary 2: The solution set of the Diophantine equation $4x^2 = 5y^2(5y-3)^2 - 16$ is $\{(\pm 1, 1), (\pm 29, -2), (\pm 76, 4)\}$.

Corollary 3: The solution set of the Diophantine equation $4x^2 = 5y^2(5y-3)^2 + 16$ is $\{(\pm 2, 0), (\pm 3, 1), (\pm 123, 5)\}$.

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AMS Classification Numbers: 11B39, 11D25, 11B37

MAPPED SHUFFLED FIBONACCI LANGUAGES

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1. INTRODUCTION

The purpose of this paper is to study properties of mapped shuffled Fibonacci languages $F_{(a,b)}$ and $F_{(u,v)}$. Let $X = \{a, b\}$ be an alphabet and let X^* be the free monoid generated by X. Let 1 be the empty word and let $X^+ = X^* \setminus \{1\}$. The length of a word u is denoted by $\lg(u)$. Every subset of X^* is called a *language*. For two words $u, v \in X^+$, we consider the following type of Fibonacci sequence of words:

$$w_1 = u, w_2 = v, w_3 = uv, \ldots, w_n = w_{n-2}w_{n-1}, \ldots, n \geq 3.$$

Let $F_{u,v} = \{w_i | i \geq 1\}$. If $u = a$ and $v = b$, then $F_{u,v}$ is denoted by $F_{a,b}$. The sequence of Fibonacci words plays a very important role in the combinatorial theory of free monoids for the recursively defined structure and remarkable combinatorial properties of Fibonacci words can be shown. Some properties concerning the Fiboancci language $F_{u,v}$ have been investigated by De Luca in [2], by Fan and Shyr in [3] and by Knuth, Morris and Pratt in [6].

In [1], properties of Fibonacci words generated through the bicatenation operation, i.e., $F_i = F_{i-1}F_{i-2} \cup F_{i-2}F_{i-1} = \{uv, vu | u \in F_{i-1}, v \in F_{i-2}\}$ where $F_1 = \{a\}$ and $F_2 = \{b\}$, are investigated. Here we consider the shuffle operation. For $u, v \in X^*$, the *shuffle product* of u and v is the set $u \diamond v$ defined by:

$$u \diamond v = \{u_1v_1u_2v_2 \cdots u_nv_n | u_i, v_j \in X^*, 1 \leq i, j \leq n,\ u_1u_2 \cdots u_n = u,\ v_1v_2 \cdots v_n = v\}.$$

For $A, B \subseteq X^*$, the *shuffle product* of A and B is defined as: $A \diamond B = \bigcup_{u \in A, v \in B}(u \diamond v)$. We consider the following type of Fibonacci sequence of sets:

$$F_1 = \{a\},\ F_2 = \{b\},\ F_{n+2} = F_n \diamond F_{n+1} \text{ for } n \geq 1.$$

Let $F_{(a,b)} = \bigcup_{i \geq 1} F_i$. Remark that every word in the same F_i has the same length. For $u, v \in X^+$, let the homomorphism $h : X^* \to X^*$ be defined by $h(a) = u$ and $h(b) = v$. The *mapped shuffled Fibonacci language* $F_{(u,v)}$ is defined to be the language $h(F_{(a,b)}) = \{h(w) | w \in F_{(a,b)}\}$.

Section 2 concerns properties of the mapped shuffled Fibonacci language $F_{(u,v)}$ related to the theory of formal languages. We prove that $F_{(a,b)}$ is equal to the set of all combinations of words in the Fibonacci language $F_{a,b}$. In [3], Fan and Shyr show that $F_{a,b}$ is regular free. Then clearly $F_{a,b}$ is not a regular language. For more complicated cases, we show that $F_{(u,v)}$ is neither dense nor context-free for any $\{u, v\} \neq X$. In Section 2, we also show that $F_{(u,v)}$ is a context-sensitive language.

Section 3 is dedicated to investigate the relationships between Fibonacci words in $F_{(u,v)}$ and primitive words. In [3] and [5], the powers of a word which can be contained as a subword in a Fibonacci word are studied. Here we show that $F_{(a,b)}$ contains only primitive words. Some properties of words u and v such that $F_{(u,v)}$ contains primitive words are investigated in Section 3 too.

In Section 4, some conditions of u and v such that the homomorphism $h : X^* \to X^*$ defined by $h(a) = u$ and $h(b) = v$ is palindrome preserving or d-primitive preserving are studied. We also count the number of palindrome words in each F_i. Codes contained in $F_{(u,v)}$ are investigated in Section 5.

Items not defined here or in the subsequent sections can be found in [4] and [9].

2. THE MAPPED SHUFFLED FIBONACCI LANGUAGE $F_{(u,v)}$

In this paper we let the sequence of Fibonacci numbers m_i be defined by $m_1 = 1$, $m_2 = 1$ and $m_i = m_{i-1} + m_{i-2}$ for $i \geq 3$. We also let $m_0 = 0$. Let the Fibonacci language $F_{a,b}$ be ordered in the lexicographic order as $F_{a,b} = \{w_1, w_2, w_3, \ldots, w_n, \ldots\}$. For $u \in X^+$, $\mathcal{C}(u)$ denotes the set of all combinations of the word u.

Let $F_1 = \{a\}, F_2 = \{b\}$. Then

$$\begin{aligned}
&F_3 = \{ab, ba\} = \mathcal{C}(ab) = \mathcal{C}(w_3),\\
&F_4 = \{bab, abb, bba\} = \mathcal{C}(abb) = \mathcal{C}(w_4),\\
&F_5 = \{abbab, babab, baabb, ababb, aabbb, abbba, babba, bbaba, bbaab, bbbaa\}\\
&\quad = \mathcal{C}(aabbb) = \mathcal{C}(w_5).
\end{aligned}$$

For $u \in X^*$ and $a \in X$, let $n_a(u)$ denote the number of a's in u. We shall show the above observations can be applied to all F_i. That is the following property:

Proposition 2.1: $F_1 = \{a\}$, $F_2 = \{b\}$ and $F_i = \mathcal{C}(a^{m_{i-2}}b^{m_{i-1}}) = \mathcal{C}(w_i)$ for $i \geq 3$.

Proof: From the previous observation, it is true for $i = 1, 2, 3, 4, 5$. Suppose that the hypothesis holds true for $i \leq n$ with an integer $n \geq 5$. Now consider sets F_{n+1} and $\mathcal{C}(a^{m_{n-1}}b^{m_n})$. From the facts that $F_{n-1} = \mathcal{C}(a^{m_{n-3}}b^{m_{n-2}})$ and $F_n = \mathcal{C}(a^{m_{n-2}}b^{m_{n-1}})$, it follows that $F_{n+1} = F_{n-1} \diamond F_n \subseteq \mathcal{C}(a^{m_{n-1}}b^{m_n})$. Next, let $w \in \mathcal{C}(a^{m_{n-1}}b^{m_n})$. Let $u \in \mathcal{C}(a^{m_{n-3}}b^{m_{n-2}}) = F_{n-1}$ be the word arranged in the same order as the first m_{n-3} a's and the first m_{n-2} b's of w. One can take $v \in X^+$ such that $w \in u \diamond v$. Then we get $n_a(v) = n_a(w) - m_{n-3} = m_{n-2}$ and $n_b(v) = n_b(w) - m_{n-2} = m_{n-1}$. Thus $v \in \mathcal{C}(a^{m_{n-2}}b^{m_{n-1}}) = F_n$. Therefore, $w \in u \diamond v \subseteq F_{n-1} \diamond F_n = F_{n+1}$. □.

For $L \subseteq X^*$, let $\mathcal{C}(L) = \bigcup_{u \in L} \mathcal{C}(u)$. Proposition 2.1 derives that $F_{(a,b)} = \mathcal{C}(F_{a,b})$. A language L is said to be *dense* if $L \cap X^*uX^* \neq \emptyset$ for every $u \in X^*$.

Proposition 2.2: The language $F_{(a,b)}$ is dense.

Proof: It is clear that $n_a(w_i) = m_{i-2}$ and $n_b(w_i) = m_{i-1}$ for $i \geq 3$. For every $u \in X^*$, let $k = \lg(u), m = m_{k+2} - n_a(u)$ and $n = m_{k+3} - n_b(u)$. Then $a^m u b^n \in \mathcal{C}(w_{k+4}) \subseteq F_{(a,b)}$. Thus $F_{(a,b)}$ is dense. □

For a given language $L \subseteq X^*$, the *principal congruence* P_L determined by L is defined as follows:

$$u \equiv v(P_L) \Longleftrightarrow (xuy \in L \Longleftrightarrow xvy \in L \ \forall x, y \in X^*).$$

It is well known that the language L is accepted by a finite automaton if and only if L has finite P_L congruence classes, that is P_L is a finite index. A language which is accepted by a finite automaton is called a *regular language* ([4]). We call a language L *disjunctive* if P_L is the equality. Clearly, a disjunctive language is not regular. It is known that every disjunctive language is dense (see [9]).

Corollary 2.3: The language $F_{(a,b)}$ is not disjunctive.

Proof: For any two distinct words $u, v \in X^*$ with $n_a(u) = n_a(v)$ and $n_b(u) = n_b(v)$, in view of Proposition 2.1, we have $xuy \in F_{(a,b)}$ if and only if $xvy \in F_{(a,b)}$ for $x, y \in X^*$. Hence the Fibonacci language $F_{(a,b)}$ is not disjunctive. □

Lemma 2.4: ([13]) Let $h : X^* \to X^*$ be a homomorphism. If $h(L)$ is dense for some $L \subseteq X^*$, then $h(X) = X$.

Corollary 2.5: For $u, v \in X^+$, if $\{u, v\} \neq X$, then $F_{(u,v)}$ is not dense.

Proof: If $\{u, v\} \neq X$, then by Lemma 2.4, $h(F_{(a,b)}) = F_{(u,v)}$ is not dense . □

Corollary 2.3 shows that $F_{(a,b)}$ is not disjunctive. Moreover, Corollary 2.5 shows that $F_{(u,v)}$ is not dense for $\{u, v\} \neq X$. In the following we shall show that $F_{(u,v)}$ is neither regular nor context-free for any $u, v \in X^+$. A language L is said to be *regular free* (*context-free free*) if every infinite subset of L is not a regular (context-free) language. Of course, if a language is context-free free, then it is also regular free. It is known that if L is an infinite context-free language, then there exist $x_1, x_2, x_3, x_4, x_5 \in X^*$ with $\lg(x_2x_4) \geq 1$ such that $\{x_1x_2^nx_3x_4^nx_5 | n \geq 0\} \subseteq L$ (see [4]). The language of the form $\{x_1x_2^nx_3x_4^nx_5 | n \geq 0\}$ is called a *context-free component.*

Proposition 2.6: For any $u, v \in X^+$, $F_{(u,v)}$ is context-free free.

Proof: Suppose on the contrary that $F_{(u,v)}$ is not context-free free. Then there is an infinite context-free subset of $F_{(u,v)}$. That is, there exist $x_1, x_2, x_3, x_4, x_5 \in X^*$ with $\lg(x_2x_4) \geq 1$ such that $\{x_1x_2^nx_3x_4^nx_5 | n \geq 0\} \subseteq F_{(u,v)}$. Remark that $F_1 = \{u\}, F_2 = \{v\}$, $F_i = F_{i-2} \diamond F_{i-1}$ for $i \geq 3$, $F_{(u,v)} = \bigcup_{i \geq 1} F_i$, and $m_i < m_{i+1}$ for every $i \geq 2$. There is $k \geq 3$ such that $x_1x_2^jx_3x_4^jx_5 \in F_k$ for some $j \geq 1$ and $m_{k-1} > \lg(x_2x_4)$. This implies that $m_{k+1} = m_{k-1} + m_k > \lg(x_1x_2^{j+1}x_3x_4^{j+1}x_5)$. Thus $x_1x_2^{j+1}x_3x_4^{j+1}x_5 \notin F_{(u,v)}$, which leads to a contradiction. Therefore, $F_{(u,v)}$ is context-free free. □

Moreover, we shall show that $F_{(u,v)}$ is a context-sensitive language. For definitions and properties of context-sensitive languages and linear bounded automata, one is referred to [4].

Proposition 2.7: For $u, v \in X^+$, $F_{(u,v)}$ is a context-sensitive language.

Proof: Here we consider the language $L = F_{(a,b)} \setminus \{a, b\}$. It is known that if L is context-sensitive, so is $F_{(a,b)}$. By Proposition 2.1, $F_{i+2} = \mathcal{C}(a^{m_i}b^{m_{i+1}})$ for $i \geq 1$. We construct a 5 track linear bounded automation such that the first track stores the input word w, the second track stores the number m_{i-1}, the third and fourth tracks store the number m_i and the fifth track stores the number m_{i+1}. This automation is initialized by $i = 1$, i.e., track 2 stores m_0, track 3 stores m_1, and so on. For any input word w in track 1, we check the number m_i stored in track 4 with a's in w. If $n_a < m_i$, then $w \notin L$. If $n_a > m_i$, then we put m_i from track 4 into track 5, put m_{i-1} from track 2 into track 4, replace the number in track 2 by m_i in track 3, replace the number in track 3 by the number in track 4, and compare the number in track 4 with a's in w again. If the number m_i in track 4 equals $n_a(w)$, then we compare the number m_{i+1} in track 5 with b's in w. If $m_{i+1} = n_b(w)$, then $w \in L$. Otherwise, $w \notin L$. This automation is a linear bounded automation which accepts L. Therefore, L is context-sensitive. As context-sensitive languages are closed under 1-free substitution, $F_{(u,v)}$ is also a context-sensitive language. □

Here, we consider one property of Fibonacci numbers. Then we shall study the difference between the shuffled Fibonacci language $F_{(a,b)}$ and the inserted Fibonacci language $I_{(a,b)}$.

Proposition 2.8: Let $i \geq 10$. Then
(1) $\lfloor m_i/(m_{i-2}+1)\rfloor = 2 = \lfloor m_{i-1}/(m_{i-3}-1)\rfloor$ and
(2) $0 < m_{i-2} - 2(m_{i-4}-1) \leq m_{i-4} - 1$.

Proof: By definition, $m_5 = 5$, $m_i = m_{i-3} + 2m_{i-2}$ and $m_i = m_{i-1} + m_{i-2} \geq m_{i-1} + 5$ for $i \geq 7$. Let $i \geq 10$. Then $m_{i-2} + 1 > m_{i-3} - 2 > 0$ and $m_{i-3} - 1 > m_{i-4} + 2 > 0$. This together with the equalities $m_i/(m_{i-2}+1) = 2 + (m_{i-3}-2)/(m_{i-2}+1)$ and $m_{i-1}/(m_{i-3}-1) = 2 + (m_{i-4}+2)/(m_{i-3}-1)$ imply that $\lfloor m_i/(m_{i-2}+1)\rfloor = 2 = \lfloor m_{i-1}/(m_{i-3}-1)\rfloor$. Moreover, $0 < m_{i-2} - 2(m_{i-4}-1) = m_{i-5} + 2 \leq m_{i-4} - 1$. □

For $A, B \subseteq X^*$, the *insertion* of B into A is defined as:

$$B \xrightarrow{i} A = \{uvw | u, w \in X^*, uw \in A, v \in B\}.$$

Let $I_1 = \{a\}, I_2 = \{b\}$ and $I_i = I_{i-2} \xrightarrow{i} I_{i-1}$ for $i \geq 3$. The *inserted Fibonacci language* $I_{(a,b)}$ is defined by $I_{(a,b)} = \cup_{i\geq 1} I_i$. Clearly, $I_i \subseteq \mathcal{C}(a^{m_{i-2}}b^{m_{i-1}}) = \mathcal{C}(w_i) = F_i$ for $i \geq 3$. By observation, $I_i = F_i$ for $i = 1,2,3,4,5,6,7,8,9$.

Proposition 2.9: $I_i \subset F_i$ for every $i \geq 10$.

Proof: It is clear that $I_i \subseteq F_i$ for $i \geq 1$. Let $w = a^7b^{14}a^7b^{14}a^7b^6$. Then $w \in F_{10}$ but $w \notin (I_8 \xrightarrow{i} I_9) = I_{10}$. Indeed, one can take $r = m_{i-2} - 2(m_{i-4}-1)$ and $s = m_{i-1} - 2(m_{i-3}+1)$ for $i \geq 10$. This is conjunction with Proposition 2.8 yields $0 < r \leq m_{i-4} - 1$ and $0 < s < m_{i-3} + 1$. Let $w = (a^{m_{i-4}-1}b^{m_{i-3}+1})^2 a^r b^s$. Then $w \in \mathcal{C}(a^{m_{i-2}}b^{m_{i-1}}) = F_i$ and $w \notin \mathcal{C}(a^{m_{i-4}}b^{m_{i-3}}) \xrightarrow{i} \mathcal{C}(a^{m_{i-3}}b^{m_{i-2}}) = F_{i-2} \xrightarrow{i} F_{i-1}$. Since $I_i = I_{i-2} \xrightarrow{i} I_{i-1} \subseteq F_{i-2} \xrightarrow{i} F_{i-1}$, we have $w \notin I_i$, which completes the proof. □

3. $F_{(u,v)}$ AND PRIMITIVE WORDS

A word $p \in X^+$ which is not a power of any other word is called a *primitive word.* Let Q be the set of all primitive words over X ([9]). It is known that every word in X^+ can be uniquely expressed as a power of a primitive word ([8]). In [3], Fan and Shyr have proved that the Fibonacci language $F_{a,b}$ is a subset of Q. Here we show that $F_{(a,b)} \subseteq Q$. We also want to find words u, v such that $F_{(u,v)} \subseteq Q$.

Proposition 3.1: $F_{(a,b)} \subseteq Q$.

Proof: We consider $w \in F_i$ for some $i \geq 3$ whenever $a, b \in Q$. By Proposition 2.1, $w \in \mathcal{C}(a^{m_{i-2}}b^{m_{i-1}})$. Since m_{i-2} and m_{i-1} are relatively prime, $w \in Q$. Therefore, $F_{(a,b)} \subseteq Q$. □

For $u \in X^+$, if $u = p^n$ and p is a primitive word, then $\sqrt{u} = p$ is called the *primitive root* of u. For a language $L \subseteq X^+$, let $\sqrt{L} = \{\sqrt{u} | u \in L\}$. A language $L \subseteq X^+$ is called *pure* if for any $u \in L^+$, $\sqrt{u} \in L^+$.

A non-empty language L is a *code* if for $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m \in L, x_1 x_2 \cdots x_n = y_1 y_2 \cdots y_m$ implies that $m = n$ and $x_i = y_i$ for $i = 1, 2, \ldots, n$. Let $\{u, v\}$ be a code and let $h : X^* \to X^*$ be defined by $h(a) = u$ and $h(b) = v$. Then h being injective is derived directly from the definition of codes.

Proposition 3.2: ([10]) Let $h : X^* \to X^*$ be an injective homomorphism. If $h(X)$ is a pure code, then h preserves the primitive words.

Proposition 3.3: For two distinct words $u, v \in X^+$, if $\{u, v\}$ is a pure code, then $F_{(u,v)} \subseteq Q$.

Proof: By Proposition 3.1, $F_{(a,b)} \subseteq Q$. Let $\{u, v\}$ be a pure code. We define the injective homomorphism $h : X^* \to X^*$ as $h(a) = u$ and $h(b) = v$. Also, for a language $L \subseteq X^+$, let $h(L) = \{h(u) | u \in L\}$. Clearly, $F_{(u,v)} = h\left(F_{(a,b)}\right)$. From Proposition 3.2, one has that $F_{(u,v)} \subseteq Q$. □

The definition of pure codes makes checking whether $\{u, v\}$ is a pure code not easy. We are going to find some other properties of u and v related to the primitivity of $F_{(u,v)}$. A word u is a *conjugate* of a word w if there exist $x, y \in X^*$ such that $u = xy$ and $w = yx$. The following lemmas concerning basic properties of decompositions and catenations of words will be needed in the sequel.

Lemma 3.4: ([8]) For $x, y \in X^+, xy = yx$ implies that $\sqrt{x} = \sqrt{y}$.

Remark: In fact that for $x, y \in X^+, xy = yx$ if and only if $\sqrt{x} = \sqrt{y}$.

Lemma 3.5: ([11]) Let $xy = p^i, x, y \in X^+, p \in Q, i \geq 1$. Then $yx = q^i$ for some $q \in Q$.

Lemma 3.6: ([12]) Let $xq^m = g^k$ for some $m, k \geq 1, x \in X^+, q \in Q$ and $g \in Q$, with $x \notin q^+$. Then $q \neq g$ and $\lg(g) > \lg(q^{m-1})$.

If $u = xy$ for $x, y \in X^*$, then x is called a *prefix* of u and it is denoted by $x \leq_p u$; the word y is called a *suffix* of u and denoted by $y \leq_s u$.

Proposition 3.7: Let $u, v \in X^+$ with $\lg(u) = \lg(v)$ and $uv \in Q$, and let $h : X^* \to X^*$ be a homomorphism defined by $h(a) = u$ and $h(b) = v$ where $X = \{a, b\}$. Then h preserves primitive words except a and b. That is, $h(Q) \backslash Q \subseteq \{u, v\}$.

Proof: Let $u, v \in X^+$, $\lg(u)$ =$\lg(v)$ and $uv \in Q$. By Lemma 3.5, $vu \in Q$. As $uv \in Q, u \neq v$. Define $h : X^* \to X^*$ by $h(a) = u$ and $h(b) = v$. Since $\{u, v\}$ is a uniform code, h is an injective homomorphism. We want to show that $h(w) \in Q$ whenever $w \in Q \backslash \{a, b, ab, ba\}$. Suppose on the contrary that there exists $w \in Q \backslash \{a, b, ab, ba\}$ such that $h(w) \notin Q$. As $w \in Q \backslash \{a, b, ab, ba\}, \lg(w) \geq 3$. Let w' be a conjugate of w. From Lemma 3.5, one has that $w \in Q$ if and only if $w' \in Q$. As $\lg(w) \geq 3$ and $w \in Q, n_a(w) \neq 0$ and $n_b(w) \neq 0$. If no conjugate of w contains any one of the following subwords b^2a or a^2b, then $w = (ab)^i$ or $w = (ba)^i$ for some $i \geq 2$. This implies that $w \notin Q$, a contradiction. Thus there is a conjugate of w that contains a subword b^2a or a^2b. In the other word, there exists a conjugate w' of w such that $a \leq_p w'$ and $b^2 \leq_s w'$, or $b \leq_p w'$ and $a^2 \leq_s w'$. Without loss of any generality, we let $a \leq_p w'$ and $b^2 \leq_s w'$. Clearly, $u \leq_p h(w')$ and $v^2 \leq_s h(w')$. Note that $h(w')$ is a conjugate of $h(w)$. This in conjunction with $h(w) \notin Q$ and Lemma 3.5 yields $h(w') \notin Q$. That is, there exist $p \in Q$ and $j \geq 1$ such that $h(w') = p^{j+1}$. Since $\lg(u)$ =$\lg(v)$ and $v^2 \leq_s h(w')$, by Lemma 3.6, we get $\lg(p) > \lg(u)$. Hence there exists $y \in X^+$ such that $p = uy$.

If $y \in \{u, v\}^+$, then $h(w') = (uy)^{j+1}$ and $uy \in \{u, v\}^+$. This implies that $w' = h^{-1}(h(w')) = h^{-1}((uy)^{j+1}) = (h^{-1}(uy))^{j+1} \notin Q$, a contradiction. Hence, $y \notin \{u, v\}^+$. Since $(uy)^{j+1} \in \{u, v\}^+$, we have $y(uy)^j \in \{u, v\}^+$. Hence there exist $y_1 \in \{u, v\}^*$ and $y_2 \in X^+$ such that $y = y_1 y_2$ and $\lg(y_2)$ <$\lg(u)$. The fact $(uy_1y_2)^{j+1} = p^{j+1} = h(w') \in \{u, v\}^+$ implies that $w_1 = y_2(uy_1y_2)^j \in \{u, v\}^+$. Not that $\lg(w_1) = k \lg(v)$ for some positive integer k and that $\lg(w_1) > \lg(v)$. Hence $\lg(w_1) \geq 2\lg(v)$. This in conjunction with $v^2 \leq_s uy_1w_1 = h(w')$ yields $v^2 \leq_s w_1$. We consider the following cases:

(1) $u \leq_s uy_1$. As $v^2 \leq_s w_1$ and $\lg(v) > \lg(y_2)$, there exists $y_3 \in X^+$ such that $v = y_3y_2$. It follows that $w_1 = y_4(y_3y_2)^2$ for some $y_4 \in \{u,v\}^*$. Since $u \leq_s uy_1$, we obtain $u \leq_s y_2(uy_1y_2)^{j-1}uy_1 = y_4(y_3y_2)y_3$. This together with $\lg(u) = \lg(v)$ yields $u = y_2y_3$. Now we consider the following four subcases:

(1-a) $u^2 = y_2y_3y_2y_3 \leq_p w_1 = y_2(uy_1y_2)^j$. Then $y_3y_2 \leq_p uy_1y_2$. As $\lg(u) = \lg(y_2y_3)$, $u = y_3y_2 = v$. This implies that $uv \notin Q$, a contradiction.

(1-b) $uv = y_2y_3y_3y_2 \leq_p w_1 = y_2uy_1y_2(uy_1y_2)^{j-1}$. Then $y_2 \leq_p y_3y_3y_2 \leq_p (uy_1y_2)^j$. There exist $y_4 \leq_p y_3$ and $r \geq 0$ such that $y_2 = y_3^ry_4$. Thus $y_3^{r+1}y_4 \leq_p y_3^ry_4y_3y_1y_2(uy_1y_2)^{j-1}$, i.e., $y_3y_3y_4 \leq_p y_4y_3y_1y_3$. It follows that $y_3 = y_4y_5 = y_5y_4$ for some $y_5 \in X^*$. By Lemma 3.4, we have $\sqrt{y_4} = \sqrt{y_5} = \sqrt{y_3}$. This is conjunction with $y_2 = y_3^ry_4$ and $\lg(u) = \lg(v)$ yields $u = y_2y_3 = y_3y_2 = v$ and $uv \notin Q$; a contradiction.

(1-c) $vu = y_3y_2y_2y_3 \leq_p w_1 = y_2(y_2y_3y_1y_2)^j$. This implies that $y_3y_2y_2 = y_2y_2y_3$. By Lemma 3.4, $\sqrt{y_2} = \sqrt{y_2^2} = \sqrt{y_3}$. Thus $y_2y_3 = y_3y_2$ and $u = v$. Hence, $uv \notin Q$, a contradiction.

(1-d) $v^2 \leq_p w_1$. As $v \leq_p w_1 = y_2(uy_1y_2)^j$ and $v = y_3y_2$, there exists $y_4 \in X^+$ such that $v = y_2y_4$ with $\lg(y_4) = \lg(y_3)$. Since $v^2 = y_2y_4y_2y_4 \leq_p w_1 = y_2(uy_1y_2)^j$ and $\lg(u) = \lg(y_2y_3) = \lg(y_4y_2)$, $u = y_4y_2$. Consider the case that $\lg(y_4) \leq \lg(y_2)$. There exists $y_5 \in X^*$ such that $y_2 = y_4y_5$. Then $v = y_4y_5y_4$ and $u = y_4y_4y_5$. As $v^2 = vy_4y_5y_4 \leq_s w_1 = (y_2uy_1)^jy_2 = (y_2uy_1)^jy_4y_5, y_5y_4 = y_4y_5$ and $u = v$. Hence $uv \notin Q$, a contradiction. Now, let $\lg(y_4) > \lg(y_2)$. There exists $y_5 \in X^+$ such that $y_4 = y_2y_5$. Then $u = y_2y_5y_2$ and $v = y_2y_2y_5$. As $v^2 = vy_2y_2y_5 \leq_s w_1$ and $uy_2 = y_2y_5y_2y_2 \leq_s w_1, y_2y_2y_5 = y_5y_2y_2$. By Lemma 3.4, $\sqrt{y_2} = \sqrt{y_2^2} = \sqrt{y_5}$. This implies that $\sqrt{u} = \sqrt{v}$ and $uv \notin Q$, a contradiction.

(2) $v \leq_s uy_1$. As $v^2 \leq_s w_1$, there exists $y_3 \in X^+$ such that $v = y_3y_2 = y_2y_3$. By Lemma 3.4, $\sqrt{y_2} = \sqrt{y_3}$. That is, there exist $q \in Q$ and $r_1, r_2 \geq 1$ such that $y_2 = q^{r_1}, y_3 = q^{r_2}$ and $v = q^{r_1+r_2}$. We consider the following four subcases:

(2-a) $u^2 \leq_p w_1 = y_2(uy_1y_2)^j$. There exists $y_4 \in X^+$ such that $u = y_2y_4 = y_4y_2$. Thus $\sqrt{y_4} = \sqrt{y_2}$. This in conjunction with $\sqrt{y_2} = \sqrt{y_3}$ yields $u = q^{r_1+r_2} = v$ and $uv \notin Q$, a contradiction.

(2-b) $uv \leq_p w_1 = y_2(uy_1y_2)^j$. There exist $y_4, y_5, y_6 \in X^+$ such that $u = y_2y_4 = y_4y_5, v = y_5y_6, \lg(y_5) = \lg(y_2) = \lg(q^{r_1})$ and $\lg(y_4) = \lg(y_3)$. Thus $y_5 = q^{r_1} = y_2$. As $u = y_2y_4 = y_4y_2$, by Lemma 3.4, $\sqrt{y_4} = \sqrt{y_2}$. Thus $uv = y_2y_4 \notin Q$, a contradiction.

(2-c) $v^2 = y_2y_3y_2y_3 \leq_p w_1 = y_2(uy_1y_2)^j$. The condition $\lg(u) = \lg(v)$ implies that $u = y_3y_2 = v$ and $uv \notin Q$, a contradiction.

(2-d) $vu \leq_p w_1$. As $v = y_2y_3 \leq_p w_1 = y_2(uy_1y_2)^j$, there exist $y_4, y_5 \in X^+$ such that $u = y_3y_4 = y_4y_5$ with $\lg(y_4) = \lg(y_2)$. This implies that $y_4 = (y_3)^{r_3}y_6$ for some $r_3 \geq 0$ and $y_6 \leq_p y_3$. Since $\lg(y_4) = \lg(y_2) = \lg(q^{r_1})$ and $y_3 = q^{r_2}, y_4 = q^{r_1}$.
Thus $u = q^{r_1+r_2} = v$ and $uv \notin Q$, a contradiction. $\square$

From the proof of Proposition 3.1, we have the following result immediately.

Corollary 3.8: Let $A = \{a,b\}$ and B a finite nonempty alphabet. If $h : A^* \to B^*$ is a homomorphism of A^* into B^* defined by $h(a) = u$ and $h(b) = v$ for some primitive words $u, v \in B^+$ such that $\lg(u) = \lg(v)$ and uv is a primitive word, then h preserves primitive words.

Corollary 3.9: $F_{(u,v)} \backslash Q \subseteq \{u,v\}$ for any two words $u, v \in X^+$ with $\lg(u) = \lg(v)$ and $uv \in Q$.

Proof: Let $u, v \in X^+$ with $\lg(u) = \lg(v)$ and $uv \in Q$. By Proposition 3.1, $F_{(a,b)} \subseteq Q$. From Proposition 3.7, one has that $F_{(u,v)} \backslash Q \subseteq \{u,v\}$. $\square$

For $u, v \in X^+$, we conjecture that $\{uv, uv^2\} \subseteq Q$ if and only if $F_{(u,v)} \backslash Q \subseteq \{u, v\}$. This is left for our further research. The partially primitive-preserving homomorphisms is also an interesting research topic for our further work.

4. PALINDROME WORDS AND d-PRIMITIVE WORDS IN $F_{(u,v)}$

If $x = a_1 a_2 \cdots a_n$, where $a_i \in X$, then we define the *reverse* (or *mirror image*) of the word x to be $\hat{x} = a_n \cdots a_2 a_1$. A word x is called *palindromic* if $x = \hat{x}$ ([7]).

Proposition 4.1: Let n_i be the number of palindrome words in $F_i = \mathcal{C}(a^{m_{i-2}} b^{m_{i-1}})$. Then $n_1 = 1, n_2 = 1$, and for $i \geq 3$,

$$n_i = \begin{cases} 0, & \text{if } m_i \text{ is an even number,} \\ \frac{(k_1+k_2)!}{k_1! k_2!}, & \text{if } m_i \text{ is an odd number,} \end{cases}$$

where $k_1 = \lfloor \frac{m_{i-2}}{2} \rfloor$ and $k_2 = \lfloor \frac{m_{i-1}}{2} \rfloor$.

Proof: If w is a palindrome word with $\lg(w) \geq 2$, then there exist $u \in X^+$ and $v \in X \cup \{1\}$ such that $w = \hat{u}vu$. By the definition of reverses, we have $n_a(u) = n_a(\hat{u})$ and $n_b(u) = n_b(\hat{u})$. Thus at most one of $n_a(w)$ and $n_b(w)$ can be odd whenever w is a palindrome word. From definitions: $m_1 = 1, m_2 = 1$ and $m_i = m_{i-1} + m_{i-2}$ for $i \geq 3$, it follows that m_i is an even number if and only if m_{i-1} and m_{i-2} are odd numbers. Consider $i \geq 3$. Then $m_i \geq 2$. If $w \in F_i$ and m_i is an even number, then $\lg(w) = m_i$ and $w \in \mathcal{C}(a^{m_{i-2}} b^{m_{i-1}})$ where both m_{i-1} and m_{i-2} are odd numbers. This implies that there exists no palindrome word in F_i if m_i is an even number. Now we consider the case that m_i is an odd number. Let $w = \hat{u}vu \in F_i$ for some $u \in X^+$ and $v \in X$. Then $u \in \mathcal{C}(a^{k_1} b^{k_2})$, where $k_1 = \lfloor \frac{m_{i-2}}{2} \rfloor$, $k_2 = \lfloor \frac{m_{i-1}}{2} \rfloor$. This implies that $n_i = \frac{(k_1+k_2)!}{k_1! k_2!}$. $\square$

Lemma 4.2: ([7]) Let $u, v \in X^+$ be two distinct words and let $h : X^* \to X^*$ be defined by $h(a) = u$ and $h(b) = v$. Then u and v are palindrome words if and only if h is a palindrome preserving homomorphism.

It is known that $\{u, v\} \subseteq X^+$ is a code if and only if $\sqrt{u} \neq \sqrt{v}$ (see [9]). For two words $u, v \in X^+, \{u, v\}$ being a code implies that h is an injective homomorphism where $h(a) = u$ and $h(b) = v$.

Proposition 4.3: Let $u, v \in X^+$ be two palindrome words. Then $\sqrt{u} \neq \sqrt{v}$ if and only if L and $h(L)$ contain the same number of palindrome words for every $L \subseteq X^+$.

Proof: Let $u, v \in X^+$ be two palindrome words with $\sqrt{u} \neq \sqrt{v}$. For $w \in X^*$, by Lemma 4.2, $h(w)$ is a palindrome word whenever w is a palindrome word. Now, let $w = a_1 a_2 \cdots a_n$ be such that $h(w)$ is a palindrome word, where $a_i \in X, 1 \leq i \leq n$, i.e., $h(w) = \widehat{h(w)}$. Note that $\widehat{h(w)} = \widehat{h(a_n)} \widehat{h(a_{n-1})} \cdots \widehat{h(a_1)} = h(a_n) h(a_{n-1}) \cdots h(a_1) = h(\hat{w})$. This in conjunction with the fact that h is injective whenever $\sqrt{u} \neq \sqrt{v}$ yields $w = \hat{w}$, i.e., w is a palindrome word. Therefore, L and $h(L)$ contain the same number of palindrome words for every $L \subseteq X^+$. Conversely, we assume that for every $L \subseteq X^+, L$ and $h(L)$ contain the same number of palindrome words. Let $L_1 = \{a, b\}$ and $L_2 = \{ab, ba\}$. Then a, b being palindrome words, by Lemma 4.2, implies that both $h(a) = u$ and $h(b) = v$ are also palindrome words. Since ab and ba are not palindrome words, $uv \neq \widehat{uv} = \hat{v}\hat{u} = vu$. By the remark of Lemma 3.4, we obtain $\sqrt{u} \neq \sqrt{v}$. $\square$

Proposition 4.3 derives that for two palindrome words u and v, if $h(a) = u, h(b) = v$ and $\sqrt{u} \neq \sqrt{v}$, then F_i and $h(F_i)$ contain the same number of palindrome words for every $i \geq 1$. A word $d \in X^*$ is said to be a *proper d-factor* of a word $z \in X^+$ if $d \neq z$ and $z = dx = yd$ for some words x, y. The family of words which have i distinct proper d-factors is denoted by $D(i)$. A word $x \in X^+$ is *d-primitive* if $x = dy_1 = y_2d$, where $d \in X^+$ and $y_1, y_2 \in X^*$, implies that $x = d$ and $y_1 = y_2 = 1$. The set $D(1)$ is exactly the family of all d-primitive words. For the properties of $D(i)$, one is referred to [13]. For $u, v \in X^+$, let $d_{u,v}$ denote the maximal word in X^* being such that $u = xd_{u,v}$ and $v = d_{u,v}y$ for some $x, y \in X^*$.

Lemma 4.4: ([7]) Let $u, v \in X^+$ be two distinct d-primitive words such that $d_{u,v} = d_{v,u} = 1$ and let $h(a) = u$ and $h(b) = v$. Then h is d-primitive preserving.

Proposition 4.5: Let $u, v \in D(1)$ with $d_{u,v} = d_{v,u} = 1$ and let $h(a) = u$ and $h(b) = v$. Then the following two statements hold true:
(1) $w \in D(1)$ if and only if $h(w) \in D(1)$;
(2) L and $h(L)$ contain the same number of d-primitive words for any $L \subseteq X^+$.

Proof: By Lemma 4.4, h is d-primitive preserving. If $w \in D(1)$, then $h(w) \in D(1)$. Now assume that $w \in X^+ \backslash D(1)$. That is, there exist $d, x, y \in X^+$ such that $w = xd = dy$. Then $h(x)h(d) = h(xd) = h(w) = h(dy) = h(d)h(y)$. This implies that $h(d)$ is a non-empty d-factor of $h(w)$ and $h(w) \notin D(1)$. Thus statement (1) holds true. For any $L \subseteq X^+$, as h is injective and by (1), L and $h(L)$ contain the same number of d-primitive words. $\square$

Proposition 4.5 derives that for $u, v \in D(1)$ with $d_{u,v} = d_{v,u} = 1$, F_i and $h(F_i)$ contain the same number of d-primitive words where $h(a) = u$ and $h(b) = v$.

5. $F_{(u,v)}$ AND CODES

Proposition 2.1 derives that $F_{(a,b)} \supseteq \{a^{m_i}b^{m_{i+1}} | i \geq 2\}$ which is a bifix code. Let $F_{a,b}$ be ordered in the lexicographic order as $\{w_1, w_2, \ldots, w_n, \ldots\}$. In [3], Fan and Shyr show that languages $\{w_{2n} | n \geq 1\}$ and $\{w_{2n-1} | n \geq 1\}$ are codes. In [14], we show that for $k \geq 2$, $\{w_{nk} | n \geq 1\}$ is a code. Here we are going to find some other codes contained in $F_{(u,v)}$.

Example: For a given integer $k \geq 2$, let $L_n = \mathcal{C}(a^{m_{n-2}}b^{m_{n-1}-m_{n-k}})b^{m_{n-k}}$ for $n > k$. Then $L = \cup_{i \geq 2} L_{ik}$ is a suffix code contained in $F_{(a,b)}$.

Lemma 5.1 ([10]) Let $h : X^* \to X^*$ be a homomorphism. Then the following statements are equivalent:
(1) h is code preserving;
(2) h is injective;
(3) $|h(X)| = |X|$ and $h(X)$ is a code.

Corollary 5.2 For $u, v \in X^+$, let $h(a) = u$ and $h(b) = v$. Let $L \subseteq F_{(a,b)}$ be a code. Then $\{u, v\}$ being a code implies that $h(L)$ is a code.

According to Corollary 5.2, we then consider codes in $F_{(a,b)}$ instead of codes in $F_{(u,v)}$. We quote the following lemma from [14], which is needed in the sequel.

Lemma 5.3: ([14])
(1) For every $i \geq 1, w_i \not\leq_p w_{i+1}$;
(2) $w_i \leq_p w_j$ implies that $j - i$ is an even number;
(3) for $k \geq 5$ and $1 \leq i \leq k-4, w_i \leq_p w_k$ implies that $w_iw_{i+1}w_{i+1}w_iw_{i+1} \leq_p w_k$;
(4) for each $k \geq 2, w_iw_i \not\leq_p w_k$ for every $i < k$.

Proposition 5.4: Let $L_i = w_{i-1}X^{m_{i-2}}$ for $i \geq 3$. For $k \geq 3$, let $L \subseteq \cup_{n\geq 1} L_{nk}$ be such that $|L \cap L_{ik}| = 1$ for each $i \geq 1$. Then L is a code.

Proof: Suppose there exists $k \geq 3$ such that there is $L \subseteq \bigcup_{n\geq 1} L_{nk}$ with $|L \cap L_{ik}| = 1$ for each $i \geq 1$ and that L is not a code. Then there exist $u_1, u_2, \ldots u_n, v_1, v_2, \ldots, v_m \in L$ for some finite integers $m, n \geq 1$ such that $u_1 \neq v_1$ and $u_1u_2\cdots u_n = v_1v_2\cdots v_m$. Since $u_1 \neq v_1$, without loss of generality, let $u_1 <_p v_1$. There exist $i_1 < j_1$ such that $u_1 \in L_{ki_1}$ and $v_1 \in L_{kj_1}$. This implies that $w_{ki_1-1} \leq_p w_{kj_1-1}$. By the definition of L and $i_1 \geq 1$, $kj_1 - 1 \geq ki_1 + k - 1 \geq 2k - 1 \geq 5$. Moreover, $kj_1 - ki_1 \geq 3$ which follows immediately from the inequalities $k \geq 3$ and $j_1 > i_1$. Then apply (2) of Lemma 5.3 to get $kj_1 - ki_1 \geq 4$, i.e., $(ki_1 - 1) \leq (kj_1 - 1) - 4$. This is the case considered in the following:

(*) By (3) of Lemma 5.3, $w_{ki_1-1}w_{ki_1}w_{ki_1}w_{ki_1-1}w_{ki_1} \leq_p w_{kj_1-1} <_p v_1$. This in conjunction with $u_1 \leq_p v_1, u_1 \in w_{ki_1-1}X^{m_{ki_1-2}}$ and $w_{ki_1} = w_{ki_1-2}w_{ki_1-1}$ yields $u_1 = w_{ki_1-1}w_{ki_1-2}$. Thus
$$u_1w_{ki_1+1}w_{ki_1+1} = u_1w_{ki_1-1}w_{ki_1}w_{ki_1-1}w_{ki_1} \leq_p v_1.$$
Let $u_2 \in L_{ki_2}$ for some $i_2 \geq 1$. If $i_2 > i_1$, then $\lg(u_2) = m_{ki_2-2} + m_{ki_2-1} \geq m_{ki_1+1} + m_{ki_1+2} \geq \lg(w_{ki_1+1}w_{ki_1+1})$. This together with $u_1u_2\cdots u_n = v_1v_2\cdots v_m$ and $u_1w_{ki_1+1}w_{ki_1+1} \leq_p v_1$ yields $w_{ki_1+1} \leq_p w_{ki_2-1} \leq_p u_2$. By (1) of Lemma 5.3, $ki_2 - 1 > ki_1 + 2$. This implies that $w_{ki_1+1}w_{ki_1+1} \leq_p w_{ki_2-1}$, in contradiction with (4) of Lemma 5.3. Thus $i_2 \leq i_1$. We consider the following two subcases:

(*1) $i_2 = i_1$. Then $u_2 = u_1 = w_{ki_1-1}w_{ki_1-2}$ and $u_1u_2w_{ki_1-1}w_{ki_1-1}w_{ki_1} \leq_p v_1$. Let $u_3 \in L_{ki_3}$ for some $i_3 \geq 1$. Then again by (4) of Lemma 5.3, $m_{ki_1+1} > 2m_{ki_1-1} = \lg(w_{ki_1-1}w_{ki_1-1}) > \lg(w_{ki_3-1}) = m_{ki_3-1}$. Thus $i_1 \geq i_3$ and $m_{ki_3} \leq 2m_{ki_1-1}$. It follows that $u_3 \leq_p w_{ki_1-1}w_{ki_1-1}w_{ki_1}$. If $i_3 = i_1, u_3 \in w_{ki_1-1}X^{m_{ki_1-2}}$ implies that $u_3 = w_{ki_1-1}w_{ki_1-3}w_{ki_1-4} \neq u_1$. This contradicts the fact that $|L \cap L_{ki_1}| = 1$. Thus one has the following case:

(*1') $i_3 < i_1$. Then we have $ki_1 - 1 \geq ki_3 + k - 1 \geq 5$. Since $ki_1 - ki_3 \geq 3$, by (2) of Lemma 5.3, $ki_1 - ki_3 \geq 4$. Note that $u_1u_2\cdots u_n = v_1v_2\cdots v_m, u_1u_2w_{ki_1-1}w_{ki_1-1}w_{ki_1} \leq_p v$ and $u_3 \in w_{ki_3-1}X^{m_{ki_3-2}}$. Hence $w_{ki_3-1} \leq pw_{ki_1-1} \leq_p v_1$. By (3) of Lemma 5.3, we obtain $w_{ki_3-1}w_{ki_3}w_{ki_3}w_{ki_3-1}w_{ki_3} \leq_p w_{ki_1-1} \leq_p v_1$. This is the same case as the case (*).

(*2) $i_2 < i_1$. This case is analogous to the case (*1') which is the same as the case (*). This implies that $u_1u_2\cdots u_n <_p v_1$, i.e. $u_1u_2\cdots u_n \neq v_1v_2\cdots v_m$, a contradiction, which completes the proof. □

Clearly, $L \subseteq F_{(a,b)} \cap \bigcup_{n\geq 1} L_{nk}$ with $|L \cap L_{nk}| = 1$ is also a code for any $k \geq 3$. Remark that the code L given in Proposition 5.4 can be neither a prefix code nor a suffix code. Furthermore, we conjecture that if we choose a word from each $F_{2n}, n \geq 2$, to form a set L, the L is a code. This is left for our further research.

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AMS Classification Numbers: 68R15

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ON THE NUMBER OF NIVEN NUMBERS UP TO x

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(Submitted June 2001)

1. INTRODUCTION

A positive integer n is said to be a *Niven number* (or a Harshad number) if it is divisible by the sum of its (decimal) digits. For instance, 153 is a Niven number since 9 divides 153, while 154 is not.

Let $N(x)$ denote the number of Niven numbers $\leq x$. Using a computer, one can obtain the following table:

x	$N(x)$
10	10
100	33
1000	213

x	$N(x)$
10^4	1538
10^5	11872
10^6	95428

x	$N(x)$
10^7	806095
10^8	6954793
10^9	61574510

It has been established by R.E. Kennedy & C.N. Cooper [4] that the set of Niven numbers is of zero density, and later by I. Vardi [5] that, given any $\varepsilon > 0$

$$N(x) \ll \frac{x}{(\log x)^{1/2-\varepsilon}}. \tag{1}$$

We have not found in the literature any lower bound for $N(x)$, although I. Vardi [5] has obtained that there exists a positive constant α such that

$$N(x) > \alpha \frac{x}{(\log x)^{11/2}} \tag{2}$$

for infinitely many integers x, namely for all sufficiently large x of the form $x = 10^{10k+n+2}$, k and n being positive integers satisfying $10^n = 45k+10$. Even though inequality (2) most likely holds for all sufficiently large x, it has not yet been proved. More recent results concerning Niven numbers have been obtained (see for instance H.G. Grundman [3] and T. Cai [1]).

Our goal is to provide a non trivial lower bound for $N(x)$ and also to improve on (1). Hence we shall prove the following result.

Theorem: *Given any $\varepsilon > 0$, then*

$$x^{1-\varepsilon} \ll N(x) \ll \frac{x \log\log x}{\log x}. \tag{3}$$

We shall further give a heuristic argument which would lead to an asymptotic formula for $N(x)$, namely $N(x) \sim c\frac{x}{\log x}$, where

[1]Research supported in part by a grant from CRSNG.

$$c = \frac{14}{27}\log 10 \approx 1.1939. \tag{4}$$

2. THE LOWER BOUND FOR $N(x)$

We shall establish that given any $\varepsilon > 0$, there exists a positive real number $x_0 = x_0(\varepsilon)$ such that

$$N(x) > x^{1-\varepsilon} \qquad \text{for all } x \geq x_0. \tag{5}$$

Before we start the proof of this result, we introduce some notation and establish two lemmas.

Given a positive integer $n = [d_1, d_2, \ldots, d_k]$, where $d_1, d_2, \ldots, d_k$ are the (decimal) digits of n, we set $s(n) = \sum_{i=1}^{k} d_i$. Hence n is a Niven number if $s(n)|n$. For convenience we set $s(0) = 0$.

Further let H stand for the set of positive integers h for which there exist two non negative integers a and b such that $h = 2^a \cdot 10^b$. Hence

$$H = \{1, 2, 4, 8, 10, 16, 20, 32, 40, 64, 80, 100, 128, 160, 200, 256, 320, 400, 512, 640, \ldots\}.$$

Now given a positive integer n, define $h(n)$ as the largest integer $h \in H$ such that $h \leq n$. For instance $h(23) = 20$ and $h(189) = 160$.

Lemma 1: *Given $\varepsilon > 0$, there exists a positive integer n_0 such that $\frac{n}{h(n)} < 1+\varepsilon$ for all $n \geq n_0$.*

Proof: Let $\varepsilon > 0$ and assume that $n \geq 2$. First observe that

$$\frac{n}{h(n)} < 1 + \varepsilon \Longleftrightarrow \log n - \log h(n) < \log(1+\varepsilon) := \varepsilon_1,$$

say. It follows from classical results on approximation of real numbers by rational ones that there exist two positive integers p and q such that

$$0 < \delta := p\log 10 - q\log 2 < \varepsilon_1. \tag{6}$$

For each integer $n \geq 2$, define

$$r := \left[\frac{\log n}{\log 2}\right] \qquad \text{and} \qquad t := \left[\frac{\log n - r\log 2}{\delta}\right]. \tag{7}$$

From (6) and (7), it follows that

$$\log n - (r\log 2 + t(p\log 10 - q\log 2)) < \delta < \varepsilon_1,$$

that is

$$\frac{n}{2^{r-qt} \cdot 10^{tp}} < 1 + \varepsilon.$$

In order to complete the proof of Lemma 1, it remains to establish that $2^{r-qt} \cdot 10^{tp} \in H$, that is that $r - qt \geq 0$. But it follows from (7) that

$$t \leq \frac{\log n - r\log 2}{\delta} \leq \frac{\log n}{\delta} - \frac{\log 2}{\delta}\left(\frac{\log n}{\log 2} - 1\right) = \frac{\log 2}{\delta},$$

so that

$$r - qt \geq r - \frac{q\log 2}{\delta} = \left[\frac{\log n}{\log 2}\right] - \frac{q\log 2}{\delta} > \frac{\log n}{\log 2} - \frac{q\log 2}{\delta} - 1,$$

a quantity which will certainly be positive if n is chosen to satisfy

$$\frac{\log n}{\log 2} \geq \frac{q\log 2}{\delta} + 1,$$

that is

$$n \geq n_0 := \left[2^{(q\log 2)/\delta + 1}\right] + 1.$$

Noting that q and δ depend only on ε, the proof of Lemma 1 is complete.

Given two non negative integers r and y, let

$$M(r,y) := \#\{0 \leq n < 10^r : s(n) = y\}. \tag{8}$$

For instance $M(2,9) = 10$. Since the average value of $s(n)$ for $n = 0, 1, 2, \ldots, 10^r - 1$ is $\frac{9}{2}r$, one should expect that, given a positive integer r, the expression $M(r,y)$ attains its maximal value at $y = [\frac{9}{2}r]$. This motivates the following result.

Lemma 2: *Given any positive integer r, one has*

$$M(r, [4.5r]) \geq \frac{10^r}{9r+1}.$$

Proof: As n runs through the integers $0, 1, 2, 3, \ldots, 10^r - 1$, it is clear that $s(n)$ takes on $9r+1$ distinct values, namely $0, 1, 2, 3, \ldots, 9r$. This implies that there exists a number $y = y(r)$ such that $M(r,y) \geq \frac{10^r}{9r+1}$. By showing that the function $M(r,y)$ takes on its maximal value when $y = [4.5r]$, the proof of Lemma 2 will be complete. We first prove:

(a) If r is even, $M(r, 4.5r + y) = M(r, 4.5r - y)$ for $0 \leq y \leq 4.5r$; if r is odd, $M(r, 4.5r + y + 0.5) = M(r, 4.5r - y - 0.5)$ for $0 \leq y < 4.5r$;

(b) if $y < 4.5r$, then $M(r,y) \leq M(r, y+1)$.

To prove (a), let

$$z = \begin{cases} 4.5r + y & \text{if } r \text{ is even,} \\ 4.5r + y + 0.5 & \text{if } r \text{ is odd,} \end{cases} \tag{9}$$

and consider the set K of non negative integers $k < 10^r$ such that $s(k) = z$ and the set L of non negative integers $\ell < 10^r$ such that $s(\ell) = 9r - z$. Observe that the function $\sigma : K \to L$ defined by

$$\sigma(k) = \sigma([d_1, d_2, \ldots, d_r]) = [9 - d_1, 9 - d_2, \ldots, 9 - d_r]$$

is one-to-one. Note that here, for convenience, if n has t digits, $t < r$, we assume that n begins with a string of $r - t$ zeros, thus allowing it to have r digits. It follows from this that $|K| = |L|$ and therefore that

$$M(r, z) = M(r, 9r - z). \tag{10}$$

Combining (9) and (10) establishes (a).

To prove (b), we proceed by induction on r. Since $M(1, y) = 1$ for $0 \leq y \leq 9$, it follows that (b) holds for $r = 1$.

Now given any integer $r \geq 2$, it is clear that

$$M(r, y) = \sum_{i=0}^{9} M(r - 1, y - i),$$

from which it follows immediately that

$$M(r, y + 1) - M(r, y) = M(r - 1, y + 1) - M(r - 1, y - 9). \tag{11}$$

Hence to prove (b) we only need to show that the right hand side of (11) is non negative. Assuming that y is an integer smaller than $4.5r$, we have that $y \leq 4.5r - 0.5 = 4.5(r-1) + 4$ and hence $y = 4.5(r-1) + 4 - j$ for some real number $j \geq 0$ (actually an integer or half an integer). Using (a) and the induction argument, it follows that $M(r - 1, y + 1) - M(r - 1, y - 9) \geq 0$ holds if $|4.5(r - 1) - (y + 1)| \leq |4.5(r - 1) - (y - 9)|$. Replacing y by $4.5(r - 1) + 4 - j$, we obtain that this last inequality is equivalent to $|j - 5| \leq |j + 5|$, which clearly holds for any real number $j \geq 0$, thus proving (b) and completing the proof of Lemma 2.

We are now ready to establish the lower bound (5). In fact, we shall prove that given any $\varepsilon > 0$, there exists an integer r_0 such that

$$N\left(10^{r(1+\varepsilon)}\right) > 10^{r(1-\varepsilon)} \quad \text{for all integers } r \geq r_0. \tag{12}$$

To see that this statement is equivalent to (5), it is sufficient to choose $x_0 > 10^{r_0(1+\varepsilon)}$. Indeed, by doing so, if $x \geq x_0$, then

$$10^{r(1+\varepsilon)} \leq x \leq 10^{(r+1)(1+\varepsilon)} \quad \text{for a certain } r \geq r_0,$$

in which case

$$N(x) \geq N\left(10^{r(1+\varepsilon)}\right) > 10^{r(1-\varepsilon)},$$

and since $x \leq 10^{(r+1)(1+\varepsilon)}$, we have

$$x^{\frac{r(1-\varepsilon)}{(r+1)(1+\varepsilon)}} \leq 10^{r(1-\varepsilon)} < N(x),$$

that is

$$x^{1-\varepsilon_1} \leq 10^{r(1-\varepsilon)} < N(x),$$

for some $\varepsilon_1 = \varepsilon_1(r, \varepsilon)$ which tends to 0 as $\varepsilon \to 0$ and $r \to \infty$.

It is therefore sufficient to prove the existence of a positive integer r_0 for which (12) holds.

First for each integer $r \geq 1$, define the non negative integers $a(r)$ and $b(r)$ implicitly by

$$2^{a(r)} \cdot 10^{b(r)} = h([4.5r]). \tag{13}$$

We shall now construct a set of integers n satisfying certain conditions. First we limit ourselves to those integers n such that $s(n) = 2^{a(r)} \cdot 10^{b(r)}$. Such integers n are divisible by $s(n)$ if and only if their last $a(r) + b(r)$ digits form a number divisible by $2^{a(r)} \cdot 10^{b(r)}$. Hence we further restrict our set of integers n to those for which the (fixed) number v formed by the last $a(r) + b(r)$ digits of n is a multiple of $s(n)$.

Finally for the first digit of n, we choose an integer d, $1 \leq d \leq 9$, in such a manner that

$$2^{a(r)} \cdot 10^{b(r)} - s(v) - d \equiv 0 \pmod 9. \tag{14}$$

Thus let n be written as the concatenation of the digits of d, u and v, which we write as $n = [d, u, v]$, where u is yet to be determined. Clearly such an integer n shall be a Niven number if $d + s(u) + s(v) = s(n) = 2^{a(r)} \cdot 10^{b(r)}$, that is if $s(u) = 2^{a(r)} \cdot 10^{b(r)} - d - s(v)$. We shall now choose u among those integers having at most $\beta := \frac{2^{a(r)} \cdot 10^{b(r)} - d - s(v)}{4.5}$ digits. Note that β is an integer because of condition (14).

Now Lemma 2 guarantees that there are at least $\frac{10^\beta}{9\beta+1}$ possible choices for u.

Let us now find upper and lower bounds for β in terms of r.

On one hand, we have

$$\beta = \frac{h([4.5r]) - d - s(v)}{4.5} < \frac{h([4.5r])}{4.5} \leq r. \tag{15}$$

On the other hand, recalling (13), we have $s(v) < 9(a(r) + b(r)) < 9\frac{\log h([4.5r])}{\log 2}$, and therefore

$$\beta = \frac{h([4.5r]) - d - s(v)}{4.5} > \frac{h([4.5r]) - 9 - 9\frac{\log h([4.5r])}{\log 2}}{4.5}. \tag{16}$$

Using Lemma 1, we have that, if r is large enough, $h([4.5r]) > 4.5r(1 - \varepsilon/2)$. Hence it follows from (16) that

$$\beta > \frac{4.5r(1 - \varepsilon/2) - 9 - 9\frac{\log h([4.5r])}{\log 2}}{4.5} > r(1 - \varepsilon), \tag{17}$$

provided r is sufficiently large, say $r \geq r_1$.

Again using (13), we have that

$$a(r)+b(r)+1<\frac{\log(h[4.5r])}{\log 2}+1.$$

Since $h(n)\leq n$, and choosing r sufficiently large, say $r\geq r_2$, it follows from this last inequality that

$$a(r)+b(r)+1<\frac{\log(4.5r)}{\log 2}+1<r\varepsilon \quad (r\geq r_2).$$

Combining this inequality with (15), we have that

$$\beta+a(r)+b(r)+1<r(1+\varepsilon) \quad (r\geq r_2). \tag{18}$$

Hence, because n has $\beta+a(r)+b(r)+1$ digits, it follows from (18) that

$$n<10^{r(1+\varepsilon)} \quad (r\geq r_2) \tag{19}$$

Since, as we saw above, there are at least $\frac{10^\beta}{9\beta+1}$ ways of choosing u, we may conclude from (19) that there exist at least $\frac{10^\beta}{9\beta+1}$ Niven numbers smaller than $10^{r(1+\varepsilon)}$, that is

$$N\left(10^{r(1+\varepsilon)}\right)>\frac{10^\beta}{9\beta+1}>\frac{10^{r(1-\varepsilon)}}{9r(1-\varepsilon)+1}>10^{r(1-2\varepsilon)},$$

for r sufficiently large, say $r\geq r_3$, where we used (17) and the fact that $\frac{10^\beta}{9\beta+1}$ increases with β.

From this, (12) follows with $r_0=\max(r_1,r_2,r_3)$, and thus the lower bound (5).

3. THE UPPER BOUND FOR $N(x)$

We shall establish that

$$N(x)<330\cdot\log 10\cdot\frac{x}{\log x}+\frac{495}{2}\cdot\log 10\cdot\frac{x}{\log x}\log\left(\frac{5\log x+5\log 10}{\log 10}\right), \tag{20}$$

from which the upper bound of our Theorem will follow immediately.

To establish (20), we first prove that for any positive integer r,

$$N(10^r)<\frac{99\cdot\log(5r)}{4r}\cdot 10^r+\frac{33}{r}\cdot 10^r. \tag{21}$$

Clearly (20) follows from (21) by choosing $r = \left[\frac{\log x}{\log 10}\right] + 1$.

In order to prove (21), we first write

$$N(10^r) = A(r) + B(r) + 1,$$

where

$$A(r) = \#\{1 \le n < 10^r : s(n)|n \text{ and } |s(n) - 4.5r| > 0.5r\}$$

and

$$B(r) = \#\{1 \le n < 10^r : s(n)|n \text{ and } 4r \le s(n) \le 5r\}$$

To estimate $A(r)$, we use the idea introduced by Kennedy & Cooper [4] of considering the value $s(n)$, in the range $0, 1, 2, \ldots, 10^r - 1$ as a random variable of mean $\mu = 4.5r$ and variance $\sigma^2 = 8.25r$. This is justified by considering each digit of n as an independant variable taking each of the values $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ with a probability equal to $\frac{1}{10}$. Thus, according to Chebyshev's inequality (see for instance Galambos [2], p. 23), we have

$$P(|s(n) - \mu| > k) < \frac{\sigma^2}{k^2}, \text{ that is } P(|s(n) - 4.5r| > 0.5r) < \frac{8.25r}{(0.5r)^2} = \frac{33}{r}.$$

Now multiplying out this probability by the length of the interval $[1, 10^r - 1]$, we obtain the estimate

$$A(r) < \frac{33 \cdot 10^r}{r}. \tag{22}$$

The estimation of $B(r)$ requires a little bit more effort.

If we denote by $\alpha = \alpha(s(n))$ the number of digits of $s(n)$, then, since $4r \le s(n) \le 5r$, we have

$$\left[\frac{\log 4r}{\log 10}\right] + 1 \le \alpha \le \left[\frac{\log 5r}{\log 10}\right] + 1. \tag{23}$$

We shall write each integer n counted in $B(r)$ as the concatenation $n = [c, d]$, where $d = d(n)$ is the number formed by the last α digits of n and $c = c(n)$ is the number formed by the first $r - \alpha$ digits of n. Here, again for convenience, we allow c and thus n to begin with a string of 0's. Using this notation, it is clear that $s(n) = s(c) + s(d)$ which means that $s(c) = s(n) - s(d)$. From this, follows the double inequality

$$s(n) - 9\alpha \le s(c) \le s(n).$$

Hence, for any fixed value of $s(n)$, say $a = s(n)$, the number of distinct ways of choosing c is at most

$$\sum_{s(c)=a-9\alpha}^{a} M(r - \alpha, s(c)), \tag{24}$$

where $M(r, y)$ was defined in (8).

For fixed values of $s(n)$ and c, we now count the number of distinct ways of choosing d so that $s(n)|n$. This number is clearly no larger than the number of multiples of $s(n)$ located in the interval $I := [c \cdot 10^\alpha, (c+1) \cdot 10^\alpha]$. Since the length of this interval is 10^α, it follows that I contains at most $L := \left[\frac{10^\alpha}{s(n)} + 1\right]$ multiples of $s(n)$. Since α represents the number of digits of $s(n)$, it is clear that $L \leq 10 + 1 = 11$.

We have thus established that for fixed values of $s(n)$ and c, we have at most 11 different ways of choosing d.

It follows from this that for a fixed value a of $s(n) \in [4r, 5r]$, the number of "c, d combinations" yielding a positive integer $n < 10^r$ such that $s(n)|n$, that is $a|n$, is at most 11 times the quantity (24), that is

$$11 \sum_{s(c)=a-9\alpha}^{a} M(r-\alpha, s(c)). \tag{25}$$

Summing this last quantity in the range $4r \leq a \leq 5r$, we obtain that

$$B(r) \leq 11 \sum_{a=4r}^{5r} \sum_{s(c)=a-9\alpha}^{a} M(r-\alpha, s(c)).$$

Observing that in this double summation, $s(c)$ takes its values in the interval $[4r - 9\alpha, 5r]$ and that $s(c)$ takes each integer value belonging to this interval at most 9α times, we obtain that

$$B(r) \leq 11 \cdot 9\alpha \sum_{s(c)=4r-9\alpha}^{5r} M(r-\alpha, s(c)).$$

By widening our summation bounds and using (23), we have that

$$B(r) \leq 99\alpha \sum_{y=0}^{9r} M(r-\alpha, y) = 99\alpha \cdot 10^{r-\alpha} < 99\left(\frac{\log 5r}{\log 10} + 1\right) \cdot 10^{r-\alpha}.$$

Since by (23), $\alpha > \frac{\log 4r}{\log 10}$, we finally obtain that

$$B(r) \leq \frac{99 \cdot \log(4r) \cdot 10^r}{4r}. \tag{26}$$

Recalling that $N(10^r) = A(r) + B(r) + 1$, (21) follows immediately from (22) and (26), thus completing the proof of the upper bound, and thus of our Theorem.

Remarks:

1. We treated both $r - \alpha$ and $4r - 9\alpha$ as non negative integers without justification. Since it is sufficient to check that $4r > 9\alpha$ and since $\alpha \leq \frac{\log 5r + \log 10}{\log 10}$, it is enough to verify that $4r > \frac{9 \log 5r + 9 \log 10}{\log 10}$, which holds for all integers $r \geq 6$. For each $r \leq 5$, (21) is easily verified by direct computation.
2. Although we used probability theory, there was no breach in rigor. Indeed, this is because it is a fact that for $n < 10^r$, the i^{th} digit of n, for each $i = 1, 2, \ldots, r$ (allowing, as we did above, each number n to begin with a string of 0's so that is has r digits), takes on each integer value in [0,9] exactly one time out of ten.

4. THE SEARCH FOR THE ASYMPTOTIC BEHAVIOUR OF $N(x)$

By examining the table in §1, it is difficult to imagine if $N(x)$ is asymptotic to some expression of the form $x/L(x)$, where $L(x)$ is some slowly oscillating function such as $\log x$.

Nevertheless we believe that, as $x \to \infty$

$$N(x) = (c + o(1)) \frac{x}{\log x}. \tag{27}$$

where c is given in (4). We base our conjecture on a heuristic argument.

Here is how it goes. First we make the reasonable assumption that the probability that $s(n)|n$ is $1/s(n)$, provided that $s(n)$ is not a multiple of 3. On the other hand, since $3|s(n)$ if and only if $3|n$, we assume that, if $3 \parallel s(n)$, then the probability that $s(n)|n$ is $3/s(n)$. In a like manner, we shall assume that, if $9|s(n)$, then $s(n)|n$ with a probability of $9/s(n)$.

Hence using conditional probability, we may write that

$$\begin{aligned} P(s(n)|n) &= P(s(n)|n \text{ assuming that } 3\not|\,(n)) \cdot P(3\not|\, s(n)) \\ &\quad + P(s(n)|n \text{ assuming that } 3 \parallel s(n)) \cdot P(3 \parallel s(n)) \\ &\quad + P(s(n)|n \text{ assuming that } 9|s(n)) \cdot P(9|s(n)) \\ &= \frac{1}{s(n)} \cdot \frac{2}{3} + \frac{3}{s(n)} \cdot \frac{2}{9} + \frac{9}{s(n)} \cdot \frac{1}{9} = \frac{7}{3} \cdot \frac{1}{s(n)}. \end{aligned} \tag{28}$$

As we saw above, the expected value of $s(n)$ for $n \in [0, 10^r - 1]$ is $\frac{9}{2}r$. Combining this observation with (28), we obtain that if n is chosen at random in the interval $[0, 10^r - 1]$, then

$$P(s(n)|n) = \frac{7}{3} \cdot \frac{1}{9r/2} = \frac{14}{27r}.$$

Multiplying this probability by the length of the interval $[0, 10^r - 1]$, it follows that we can expect $\frac{14 \cdot 10^r}{27 \cdot r}$ Niven numbers in the interval $[0, 10^r - 1]$.

Therefore, given a large number x, if we let $r = \left[\frac{\log x}{\log 10}\right]$, we immediately obtain (27).

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AMS Classification Numbers: 11A63, 11A25

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INTERVAL-FILLING SEQUENCES INVOLVING RECIPROCAL FIBONACCI NUMBERS

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1. INTRODUCTION

Let $r > 0$ be a fixed real number. In this paper we will study infinite series of the form:

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n}{(F_n)^r} \; (\epsilon_n = 1 \text{ or } 0)\,, \tag{1}$$

where $x \in [0, I_r]$. I_r signifies the sum of series (1) if $\epsilon_n = 1$ for all $n \in N$. The convergence of the series (1), if $x = I_r$, can be easily proved by the well-known Binet formula! Letting $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ we have

$$F_n = (\alpha^n - \beta^n)/\sqrt{5}. \tag{2}$$

Notice that $0 < \alpha^{-r} < 1$ and that Binet's formula yields $\lim_{n\to\infty}(F_n)^r/\alpha^{rn} = \left(\sqrt{5}\right)^{-r}$. Thus applying the quotient-criterion for infinite series and geometric series proves the convergence of (1). For example: $I_1 = 3,359\ldots$. Furthermore it is easy to see that

$$I_r > I_{r'} \text{ for } r < r',\; I_r \to \infty \text{ for } r \to 0 \text{ and } I_r \to 2 \text{ for } r \to \infty. \tag{3}$$

We begin with certain results due to J.L. Brown in [1] and P. Ribenboim in [8] dealing with the representation of real numbers in the form (1). In [1] J.L. Brown treated the case $r = 1$. In [8] P. Ribenboim proved that for every positive real number x there exists a unique integer $m \geq 1$ such that $I_{1/(m-1)} < x \leq I_{1/m}$ and x is representable in the form (1) with $r = 1/m$, but x is not of the form (1) with $r = 1/(m-1)$ because $x > I_{1/(m-1)}(I_\infty = 0)$. Besides requiring $r > 0$ we do not make any other restrictions on r.

The following theorem is basic for our considerations.

Theorem 1: (S. Kakeya, 1914) Let (λ_n) be a sequence of positive real numbers, such that the series

$$\sum_{n=1}^{\infty} \lambda_n = s \tag{4}$$

is convergent with sum s and the inequalities

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \tag{5}$$

are fulfilled.

Then, each number $x \in [0, s]$ may be written in the form

$$x = \sum_{n=1}^{\infty} \epsilon_n \lambda_n \qquad \epsilon_n \in \{0, 1\} \tag{6}$$

if and only if

$$\lambda_n \leq \lambda_{n+1} + \lambda_{n+2} + \dots \tag{7}$$

for all $n \in N$.

The "digits" ϵ_n of the expansion may be determined recursively by the following algorithm: If $n \geq 1$ and if the digits ϵ_i of the expansion of x are already defined for all $i < n$, then we let

$$\epsilon_n = 1 \text{ if } \sum_{i=1}^{n-1} \epsilon_i \lambda_i + \lambda_n < x. \tag{7a}$$

Otherwise, we set $\epsilon_n = 0$.

Then, each expansion with $x > 0$ is infinite, i.e. there is an infinite set of integers n with $\epsilon_n = 1$.

A proof of Theorem 1 can be found in [1], or in [7, exercise 131] or in [8].

For our purpose it is practical to introduce the following notion (see [6]):

Definition: A sequence (λ_n) satisfying conditions (4) and (5) of Theorem 1 is said to be interval-filling (relating to $[0, s]$) if every number $x \in [0, s]$ can be written in the form (6).

2. THE CASE $0 < r \leq 1$

First we give an example of an application of

Theorem 1: Let $\lambda_n = 1/F_n^r$ for all $n \in N$, where r is a fixed number with $0 < r \leq 1$. As we have mentioned above this sequence satisfies condition (4) of Theorem 1. (5) is also valid. For the proof of (7) we note first that $1/F_n < 2/F_{n+1}$ is valid for all $n \in N$. With $0 < r \leq 1$ we get

$$\frac{1}{F_n} < \frac{2}{F_{n+1}} \leq \frac{2^{1/r}}{F_{n+1}} \text{ which yields } \frac{1}{(F_n)^r} < \frac{2}{(F_{n+1})^r}.$$

From this we obtain by mathematical induction:

$$1/(F_n)^r - 1/(F_{n+k})^r < 1/(F_{n+1})^r + 1/(F_{n+2})^r + \dots + 1/(F_{n+k})^r$$

for all $k \geq 1$.

Now let $k \to \infty$. The limits of the two sides in the preceding inequality exist and we obtain

$$\frac{1}{(F_n)^r} \leq \sum_{k=1}^{\infty} \frac{1}{(F_{n+k})^r}.$$

Condition (7) is thus established. The application of Theorem 1 immediately yields, that each real number x with $0 < x \leq I_r$, where $0 < r \leq 1$, has (at least) one expansion of the form (1). In other words: $(1/F_n^r)_{n=1}^{\infty}$ is interval-filling relating to $[0, I_r]$.

This statement can be extended considerably.

Theorem 2: For each real number x with $0 < x < I_r$ and fixed r with $0 < r \leq 1$ the set C_x, which consists of all different expansions for x of the form (1), is uncountable; it has cardinality c (the power of the continuum).

The proof is based on an idea which is used in [2] and [3] considering the representation of the real number x in the form

$$x = \sum_{n=1}^{\infty} \epsilon_n q^{-n}$$

with non-integral base q. Such an expansion is not unique in general.

Our central point is the construction of a subsequence of $(1/(F_n)^r)_{n=1}^{\infty}$ which also satisfies the conditions of Theorem 1.

Before we give a proof of Theorem 2 we need some results on sums of Fibonacci reciprocals.

Theorem 3: (Jensen's inequality, see [5]). Let $0 < r \leq 1$ and let A be a finite or infinite subset of N. Then, we claim that

$$\sum_{i \in A} 1/F_i \leq \left(\sum_{i \in A} 1/(F_i)^r \right)^{1/r}.$$

Proof: Let us let $a = \left(\sum_{i \in A} 1/(F_i)^r\right)^{1/r}$. Thus, $\sum_{i \in A} 1/(F_i a)^r = 1$ and we get $1/(F_i a) \leq 1$ for all $i \in A$. $1 \geq r$ yields $1/(F_i a) \leq 1/(F_i a)^r$ for $i \in A$. Therefore,

$$\sum_{i \in A} 1/(F_i a) \leq \sum_{i \in A} 1/(F_i a)^r = 1.$$

Multiply by a. From the defintion of a and because of the last inequality we obtain thc assertion. □

Theorem 4: Let $0 < r \leq 1$. Let z denote a positive integer.

(i) If $z = 2k + 1$, then

$$\frac{1}{(F_z)^r} < \frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+2})^r}.$$

(ii) If $z = 2k$, then

$$\frac{1}{(F_z)^r} < \frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+2})^r} + \frac{1}{(F_{z+3})^r}.$$

(iii) If $z = 2k + 1$, then

$$\frac{1}{(F_z)^r} < \frac{1}{(F_{z+2})^r} + \frac{1}{(F_{z+3})^r} + \cdots + \frac{1}{(F_{z+n(z)})^r},$$

with an integer $n(z)$ dependent on the odd integer z, with $n(z) \leq n(z+2)$ and $n(2k+1) \to \infty$, as $k \to \infty$.

(iv) If $z = 2k$, then

$$\frac{1}{(F_z)^r} < \frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+3})^r} + \frac{1}{(F_{z+4})^r} + \cdots + \frac{1}{(F_{z+k})^r}$$

with $k = 7$ if $z = 2$ and $k = 5$ if $z \geq 4$.

Proof: First we treat the case $r = 1$.

(i) $z = 2k + 1$. The assertion is equivalent to $F_{z+1}F_{z+2} < F_z(F_{z+1} + F_{z+2})$ or $F_{z+1}(F_{z+2} - F_z) < F_zF_{z+2}$ or $(F_{z+1})^2 < F_zF_{z+2}$. Then, the well-known formula $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$ with $n = z + 1$ yields $(F_{z+1})^2 = F_{z+2}F_z - 1 < F_{z+2}F_z$. The proof of (i) for $r = 1$ is complete.

(ii) $z = 2k$. Using (i), we get

$$\frac{1}{F_{z+1}} < \frac{1}{F_{z+2}} + \frac{1}{F_{z+3}} \text{ and then}$$

$$\frac{1}{F_z} < \frac{2}{F_{z+1}} < \frac{1}{F_{z+1}} + \frac{1}{F_{z+2}} + \frac{1}{F_{z+3}}.$$

(iii) $z = 2k + 1$. For the purpose of abbreviation let $\delta = \beta/\alpha = \left(\sqrt{5} - 3\right)/2$. Then, $|\delta| < 1$. Using the Binet's formula we have

$$\frac{F_z}{F_{z+2}} + \frac{F_z}{F_{z+3}} + \cdots + \frac{F_z}{F_{z+n}} = \frac{\alpha^z - \beta^z}{\alpha^{z+2} - \beta^{z+2}} + \cdots + \frac{\alpha^z - \beta^z}{\alpha^{z+n} - \beta^{z+n}}$$

$$= \alpha^{-2}\frac{(1 + |\delta|^z)}{1 + |\delta|^{z+2}} + \cdots + \alpha^{-n}\frac{(1 + |\delta|^z)}{1 \mp |\delta|^{z+n}}$$

$$> \frac{(1 + |\delta|^z)(1 - (/\alpha)^{n-1})}{(1 + |\delta|^{z+2})\alpha^2(1 - (1/\alpha))} \quad \text{(Note } \alpha^2(1 - (1/\alpha)) = 1!\text{)}$$

$$= \frac{(1 + |\delta|^z)(1 - (1/\alpha)^{n-1}}{(1 + |\delta|^{z+2})}.$$

Because $|\delta| < 1$ it follows that $(1 + |\delta|^z)/(1 + |\delta|^{z+2}) > 1$.

Further, we notice that the increasing sequence $((1 - (1/\alpha)^{n-1})$ has limit 1, as $n \to \infty$. Therefore, it follows that the inequality

$$\frac{(1 + |\delta|^z)(1 - (1/\alpha)^{n-1})}{(1 + |\delta|^{z+2})} > 1$$

is valid for all sufficient large values of $n \in N$. We denote the minimum of these values by $n(z)$. Thus, we have

$$\frac{F_z}{F_{z+2}} + \frac{F_z}{F_{z+3}} + \cdots + \frac{F_z}{F_{z+n}} > 1$$

for all $n \geq n(z)$. This is equivalent to (iii).

The assertions $n(z) \leq n(z+2)$ and $n(2k+1) \to \infty$ as $k \to \infty$ are easily proved.

(iv) Let $z = 2k$. If $z = 2$ a direct computation leads to the assertion. We observe that for $z \geq 4$, the desired result is equivalent to

$$\frac{F_z F_{z+1}}{F_{z-1}} \left(\frac{1}{F_{z+3}} + \frac{1}{F_{z+4}} + \frac{1}{F_{z+5}} \right) > 1.$$

Applying (i) with the odd integer $z+3$ to the parenthesis on the left hand side, we obtain

$$\frac{F_z F_{z+1}}{F_{z-1}} \left(\frac{1}{F_{z+3}} + \frac{1}{F_{z+4}} + \frac{1}{F_{z+5}} \right) > \frac{2F_z F_{z+1}}{F_{z-1} F_{z+3}}.$$

Therefore, it is enough to establish that $2F_z F_{z+1} > F_{z-1} F_{z+3}$. For that purpose we begin with the well-known equation $F_{n+2}F_{n-1} - F_n F_{n+1} = (-1)^n$ $(n \in N)$. We obtain $F_{z-2}F_{z+1} - F_{z-1}F_z = -1$ and from this $2F_{z+1}F_{z-2} > F_{z-1}F_z$. It follows step-by-step that $2F_{z+1}(F_z - F_{z-1}) > F_{z-1}F_z$; $2F_z F_{z+1} > 2F_{z-1}F_{z+1} + F_{z-1}F_z = F_{z-1}F_{z+3}$. We have therefore proved all parts of the theorem for $r = 1$.

The general assertions for $0 < r \leq 1$ are immediate consequences of Theorem 3. For instance: In the event of (i) the subset A is as follows: $A = \{z+1, z+2\}$. Then, we have by Theorem 3

$$\frac{1}{F_z} < \frac{1}{F_{z+1}} + \frac{1}{F_{z+2}} \leq \left(\frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+2})^r} \right)^{1/r}.$$

Raising both sides to the r^{th} power we have (i).

All other cases follow in a similar way. Therefore, the proof of Theorem 4 is complete. □

Before we continue with the proof of Theorem 2 let us give a simple application of Theorem 4.

r will be chosen with $0 < r \leq 1$. Assume that we have a representation of $x \in [0, I_r]$ in the form (1) with the interval-filling sequence $(1/(F_n)^r)_{n=1}^{\infty}$ on the basis of the algorithm (7a).

Theorem 5: Consider the sequence $(\epsilon_n(x))_{n=1}^{\infty}$ of digits. A chain of consecutive digits "1" following a digit "0" has at most length two.

Proof: Let $\epsilon_n(x) = 0, \epsilon_{n+1}(x) = 1, \epsilon_{n+2}(x) = 1, \ldots, \epsilon_{n+k}(x) = 1$ be a chain of the described kind. Then, we obtain by algorithm (7a)

$$\sum_{i=1}^{n-1} \frac{\epsilon_i}{(F_i)^r} + \frac{1}{(F_{n+1})^r} + \cdots + \frac{1}{(F_{n+k})^r} < x \leq \sum_{i=1}^{n-1} \frac{\epsilon_i}{(F_i)^r} + \frac{1}{(F_n)^r}.$$

Thus,

$$\frac{1}{(F_n)^r} > \frac{1}{(F_{n+1})^r} + \cdots + \frac{1}{(F_{n+k})^r}.$$

We now appeal to Theorem 4. It implies that k must be equal to 1 (at most equal to 2), if n is an odd (even) number since the assumption $k \geq 2$ ($k \geq 3$) leads to a contradiction with Theorem 4(i) or (ii).

The proof is complete. $\square$

Proof of Theorem 2: We choose a sequence of even integers $(z_j)_{j=1}^{\infty} = (2k_j)_{j=1}^{\infty}$ with $z_{j+1} - z_j > \max\{9, n(z_j - 1)\}$ for all $j \in N$. The first member z_1 will be chosen (later) to be sufficiently large. Let $M = N - \{z_j\}_{j=1}^{\infty}$. Consider the set $\{1/(F_m)^r : m \in M\}$ as a non increasing sequence $(\lambda_n)_{n=1}^{\infty}$ of numbers: $\lambda_1 = 1/(F_1)^r, \lambda_2 = 1/(F_2)^r, \ldots, \lambda_{z_1-1} = 1/(F_{z_1-1})^r, \lambda_{z_1} = 1/(F_{z_1+1})^r, \lambda_{z_1+1} = 1/(F_{z_1+2})^r, \ldots$.

Next, we shall show that Theorem 1 is applicable to the sequence $(\lambda_n)_{n=1}^{\infty}$, in particular the validity of (6).

First we determine for each $m \in M$ the unique number $j \in N$ such that the condition $z_{j-1} + 1 \leq m \leq z_j - 1$ is satisfied ($z_o = 0$). Then, we obtain with the help of Theorem 4

$$\sum_{n>m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+1})^r} + \frac{1}{(F_{m+2})^r} + \frac{1}{(F_{m+3})^r} > \frac{1}{(F_m)^r},$$

$n \in M$ if $z_{j-1} + 1 \leq m \leq z_j - 4$ in view of Theorem 4(i), (ii);

$$\sum_{n>m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+1})^r} + \frac{1}{(F_{m+2})^r} > \frac{1}{(F_m)^r},$$

$n \in M$ if $m = z_j - 3$ in view of Theorem 4(i);

$$\sum_{n>m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+1})^r} + \frac{1}{(F_{m+3})^r} + \cdots + \frac{1}{(F_{m+k})^r} > \frac{1}{(F_m)^r},$$

$n \in M$ if $m = z_j - 2$ in view of Theorem 4(iv); and

$$\sum_{n>m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+2})^r} + \frac{1}{(F_{m+3})^r} + \cdots + \frac{1}{(F_{m+n(m)})^r} > \frac{1}{(F_m)^r},$$

$n \in M$ if $m = z_j - 1$ in view of Theorem 4(iii).

So, we obtain for each $m \in M : 1/(F_m)^r < \sum_{n>m, n\in M} 1/(F_n)^r$, that is we proved that condition (7) of Theorem 1 is satisfied. It is clear that (4) and (5) are valid.

Let $0 < x < I_r$. We choose z_1 so that the following conditions are satisfied simultaneously:

$$(*) \quad \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r} < x \text{ and } x + \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r} < I_r.$$

This is possible, because $\lim_{z_1\to\infty}\sum_{n\geq z_1} 1/(F_n)^r = 0$. Let $\triangle$ be any subset of the set $\{z_j\}_{j=1}^{\infty}$. We define now the 0-1-sequence $(\delta_j)_{j=1}^{\infty}$ in the following way: $\delta_j = 1$, if $z_j \in \triangle$, $\delta_j = 0$, if $z_j \notin \triangle$. Consider the number

$$y = x - \sum_{j=1}^{\infty} \frac{\delta_j}{(F_{z_j})^r}.$$

We obtain from the above conditions (*) that

$$y \geq x - \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r} > 0 \text{ and } y \leq x < I_r - \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r}.$$

It follows from this that $0 < y < \sum_{n=1}^{\infty} \lambda_n$.

Now the key point is the application of Theorem 1. For each real number in the interval $[0, \sum_{n=1}^{\infty} \lambda_n]$ there is a series of the form

$$\sum_{n=1}^{\infty} \epsilon_n \lambda_n \quad \epsilon_n \in \{0,1\}.$$

With a view to the definition of y we receive the following representation:

$$x = \sum_{n=1}^{\infty} \epsilon_n \lambda_n + \sum_{j=1}^{\infty} \frac{\delta_j}{(F_{z_j})^r}.$$

We note that a Fibonacci reciprocal contained in the second sum cannot occur in the first, which implies that the representation of x is dependent on the sequence (δ_j). Two different sequence (δ_j) and (δ'_j) lead to different representations of x. It is well-known that the set of all 0-1-sequences has cardinality c (the power of the continuum). Therefore, the set C_x of different representations of x in the form (1) has at least cardinality c. Because the cardinality of the set of 0-1-sequences equals the cardinality of the continuum, the set C_x has cardinality at most c.

Theorem 2 is thus established. $\square$

Next, we will draw a comparison between our Theorem 2 and results in [2] and [3], which are due to P. Erdös, M. Horvath, I. Joó and V. Komornik.

First we make the observation that by Binet's formula $\lim_{n\to\infty} F_n/\alpha^n = 1/\sqrt{5}$, that is F_n and α^n are "almost" proportional as $n \to \infty$. To simplify matters we assume $F_n \sim \alpha^n$. Then it follows that $(F_n)^r \sim \alpha^{nr} = q^n$ with $q = \alpha^r$ and the interval $0 < r \leq 1$ corresponds to the interval $1 < q \leq \alpha$.

Let $q \in (1, \alpha)$. It was proven in [2] (see Theorem 3 in [2]) that for every $x \in (0, 1/q - 1)$ there are c different expansions of the form

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n}{q^n} \quad \epsilon_n \in \{0,1\}. \tag{8}$$

We can say that this result is analogous to our Theorem 2, if we take into consideration the above-mentioned remark on $(F_n)^r$ and q^n.

On the other hand it was shown in [3] (see the proof of Theorem 1 in [3]) that, if we assume in (8) $x = 1$ and $q = \alpha$, there exist precisely countably many expansions of the form (8). It is surprising that we have different cardinal numbers relating the set of representations for $x = 1$ and $r = 1$ according to (1) and the set of representation for $x = 1$ and $q = \alpha$ according to (8).

3. THE CASE $r > 1$

We shall prove two further theorems regarding expansions of the form (1).

Theorem 6: Let r satisfy $1 < r < \log 2/\log\alpha$. Then, there is an even integer $m(r)$ such that the sequence $(1/(F_n)^r)_{n=m(r)-1}^{\infty}$ is interval-filling.

Theorem 7: Let r satisfy $r \geq \log 2/\log\alpha$. Then, there is no integer $m \in N$ such that $(1/(F_n)^r)_m^{\infty}$ is an interval-filling sequence.

Proof of Theorem 6: In view of the equation $\beta/\alpha = -1/\alpha^2$ and with the help of Binet's formula it easily follows that

$$F_{n+1}/F_n = \alpha E(n) \text{ where } E(n) = \frac{1 + (-1)^{n+2}\alpha^{-2n-2}}{1 + (-1)^{n+1}\alpha^{-2n}}.$$

If $1 < r < \log 2/\log\alpha$ holds, then $2 > 2^{1/r} > \alpha$. As soon as n is an odd integer we get $E(n) < 1$. Thus, it follows that $F_{n+1}/F_n < 2^{1/r}$ for an odd integer n. On the other hand, we obtain from the definition of $E(n)$ that for even integers the following statements are valid: $E(n) > 1, E(n) > E(n+2), \lim_{n\to\infty} E(n) = 1$. Hence, there is a smallest even integer $m(r)$ such that $1 < E(m(r)) < 2^{1/r}/\alpha$. Therefore, $F_{n+1}/F_n = \alpha E(n) \leq \alpha E(m(r)) < 2^{1/r}$ for each even $n \geq m(r)$. Summarizing we obtain $F_{n+1}/F_n < 2^{1/r}$ or $(F_{n+1})^r/(F_n)^r < 2$ for all integers $n \geq m(r) - 1$. This implies that Theorem 1 is applicable since the sequence $(1/(F_n)^r)$ with $n \geq m(r) - 1$ meets all the requirements of the theorem, in particular condition (7). Theorem 6 is thus established. □

Proof of Theorem 7: Let $r \geq \log 2/\log\alpha$, equivalent to $\alpha^r \geq 2$. First we shall prove that, for all even integers $n \in N$, we have

$$1/(F_n)^r > 1/(F_{n+1})^r + 2/(F_{n+2})^r. \tag{10}$$

We again use the defintion of $E(n)$ in the proof of Theorem 6. We receive from (10) the equivalent inequality

$$\alpha^r(E(n)E(n+1))^r > (E(n+1))^r + 2/\alpha^r \text{ (n even).} \tag{11}$$

On the other side, we obtain for even $n \in N$:

$$E(n)E(n+1) = 1 + \frac{1 - 1/\alpha^{+4}}{\alpha^{2n} - 1} > 1 \text{ and } E(n+1) < 1.$$

The last two inequalities and $\alpha^r \geq 2$ yield that (11) is valid for all even $n \in N$, because

$$\alpha^r(E(n)E(n+1))^r \geq 2((E(n)E(n+1))^r > 2 > (E(n+1))^r + 1 \geq (E(n+1))^r + 2/\alpha^r.$$

Thus, the equivalent statement (10) follows, from which we obtain by mathematical induction:

$$\frac{1}{(F_n)^r} - \frac{1}{(F_{n+2k})^r} > \sum_{i=1}^{2k} \frac{1}{(F_{n+i})^r} \tag{12}$$

for all $k \geq 1$ and even n.

Then, it follows from (12) as $k \to \infty$:

$$\frac{1}{(F_n)^r} \geq \sum_{i=1}^{\infty} \frac{1}{(F_{n+i})^r} \quad (n \in N,\ n \text{ even}). \tag{13}$$

Now, suppose that in (13) for two consecutive even numbers $n = v$ and $n = v + 2$ the equals sign is valid.

Then, a simple calculation shows that we have a contradiction to (10):

$$\frac{1}{(F_v)^r} = \frac{1}{(F_{v+1})^r} + \frac{2}{(F_{v+2})^r},$$

i.e. from two successive inequalities (13) there is at most one equality. Next, consider the set

$$A(r) - \{n | n \in 2N,\ 1/(F_n)^r > \sum_{i=1}^{\infty} 1/(F_{n+i})^r\}.$$

From the preceding argument it is clear that $A(r)$ is an infinite subset of N, such that condition (7) of Theorem 1 is not true for $n \in A(r)$. We conclude that there is no integer $m \in N$, such that the sequence $(1/(F_n)^r)_{n=m}^{\infty}$ is interval-filling. □

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AMS Classification Numbers: 03E17, 11A17, 11B39

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THE LINEAR ALGEBRA OF THE GENERALIZED FIBONACCI MATRICES

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(Submitted June 2001–Final Revision February 2002)

1. INTRODUCTION

Let x be any nonzero real number. The n by n *generalized Fibonacci matrix of the first kind*, $\mathcal{F}_n[x] - [f_{ij}]$, is defined as

$$f_{ij} = \begin{cases} F_{i-j+1}x^{i-j} & i-j+1 \geq 0, \\ 0 & i-j+1 < 0. \end{cases} \tag{1}$$

We define the n by n *generalized Fibonacci matrix of the second kind*, $\mathcal{R}_n[x] = [r_{ij}]$, as

$$r_{ij} = \begin{cases} F_{i-j+1}x^{i+j-2} & i-j+1 \geq 0, \\ 0 & i-j+1 < 0. \end{cases} \tag{2}$$

Note that $\mathcal{F}_n[1] = \mathcal{R}_n[1]$ and $\mathcal{F}_n[1]$ is called the Fibonacci matrix (see [3]).

The n by n *generalized symmetric Fibonacci matrix*, $\mathcal{Q}_n[x] = [q_{ij}]$, is defined as

$$q_{ij} = q_{ji} = \begin{cases} \sum_{k=1}^{i} F_k^2 x^{2i-2} & i - j, \\ q_{i,j-2}x^2 + q_{i,j-1}x & i+1 \leq j, \end{cases}$$

where $q_{1,0} = 0$. Then we know that for $j \geq 1$, $q_{1j} = q_{j1} = F_j x^{j-1}$ and $q_{2j} = q_{j2} = F_{j+1}x^j$. $\mathcal{Q}_n[1]$ is called the symmetric Fibonacci matrix (see [3]). For example,

$$\mathcal{F}_5[x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ 2x^2 & x & 1 & 0 & 0 \\ 3x^3 & 2x^2 & x & 1 & 0 \\ 5x^4 & 3x^3 & 2x^2 & x & 1 \end{bmatrix}, \qquad \mathcal{R}_5[x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & x^2 & 0 & 0 & 0 \\ 2x^2 & x^3 & x^4 & 0 & 0 \\ 3x^3 & 2x^4 & x^5 & x^6 & 0 \\ 5x^4 & 3x^5 & 2x^6 & x^7 & x^8 \end{bmatrix}$$

$$\mathcal{Q}_5[x] = \begin{bmatrix} 1 & x & 2x^2 & 3x^3 & 5x^4 \\ x & 2x^2 & 3x^3 & 5x^4 & 8x^5 \\ 2x^2 & 3x^3 & 6x^4 & 9x^5 & 15x^6 \\ 3x^3 & 5x^4 & 9x^5 & 15x^6 & 24x^7 \\ 5x^4 & 8x^5 & 15x^6 & 24x^7 & 40x^8 \end{bmatrix}.$$

Let $\mathcal{D} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in R^n : x_1 \geq x_2 \geq \cdots \geq x_n\}$, where R is the set of real numbers. For $\boldsymbol{x},\ \boldsymbol{y} \in \mathcal{D},\ \boldsymbol{x} \prec \boldsymbol{y}$ if $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i,\ k = 1, 2, \ldots, n$ and if $k = n$ then equality holds. When $\boldsymbol{x} \prec \boldsymbol{y}$, $\boldsymbol{x}$ is said to be *majorized* by $\boldsymbol{y}$, or $\boldsymbol{y}$ is said to *majorize* $\boldsymbol{x}$. The condition for majorization can be rewritten as follows: for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}, \boldsymbol{x} \prec \boldsymbol{y}$ if $\sum_{i=0}^{k} x_{n-i} \geq \sum_{i=0}^{k} y_{n-i},\ k = 0, 1, \ldots, n-2$ and if $k = n-1$ then equality holds.

The following is an interesting simple fact.

$$(\overline{x}, \ldots, \overline{x}) \prec (x_1, \ldots, x_n),$$

where $\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$. More interesting facts about majorization can be found in [4].

An $n \times n$ matrix $P = [p_{ij}]$ is *doubly stochastic* if $p_{ij} \geq 0$ for $i, j = 1, 2, \ldots, n, \sum_{i=1}^{n} p_{ij} = 1, j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} p_{ij} = 1, i = 1, 2, \ldots, n$. In 1929, Hardy, Littlewood and Polya proved that a necessary and sufficient condition that $\boldsymbol{x} \prec \boldsymbol{y}$ is that there exist a doubly stochastic matrix P such that $\boldsymbol{x} = \boldsymbol{y}P$.

We know both the eigenvalues and the main diagonal elements of a real symmetric matrix, are real numbers. The precise relationship between the main diagonal elements and the eigenvalues is given by the notion of majorization as follows: the vector of eigenvalues of a symmetric matrix majorize the main diagonal elements of the matrix (see [2]).

In [1] and [5], the authors gave factorizations of the Pascal matrix and generalized Pascal matrix. In [3],the authors gave factorizations of the Fibonacci matrix $\mathcal{F}_n[1]$ and discussed the Cholesky factorization and the eigenvalues of the symmetric Fibonacci matrix $\mathcal{Q}_n[1]$.

In this paper, we consider factorizations of the generalized Fibonacci matrices of the first kind and the second kind, and consider the Cholesky factorization of the generalized symmetric Fibonacci matrix. Also, we consider the eigenvalues of $\mathcal{Q}_n[x]$.

2. FACTORIZATIONS

In this section, we discuss factorizations of $\mathcal{F}_n[x]$, $\mathcal{R}_n[x]$ and $\mathcal{Q}_n[x]$ for any nonzero real number x.

Let I_n be the identity matrix of order n. We define the matrices $S_n[x]$, $\overline{\mathcal{F}}_n[x]$ and $G_k[x]$ by

$$S_0[x] = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^2 & 0 & 1 \end{bmatrix}, S_{-1}[x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{bmatrix},$$

and $S_k[x] = S_0[x] \oplus I_k,\ k = 1, 2, \ldots,\ \overline{\mathcal{F}}_n[x] = [1] \oplus \mathcal{F}_{n-1}[x],\ G_1[x] = I_n,\ G_2[x] = I_{n-3} \oplus S_{-1}[x]$, and, for $k \geq 3,\ G_k[x] = I_{n-k} \oplus S_{k-3}[x]$.

In [3], the authors gave a factorization of the Fibonacci matrix $\mathcal{F}_n[1]$ as follows:

Theorem 2.1: For $n \geq 1$ a positive integer,

$$\mathcal{F}_n[1] = G_1[1]G_2[1] \ldots G_n[1].$$

Now, we consider a factorization of the generalized Fibonacci matrix of the first kind. From the definition of the matrix product and a familiar Fibonacci sequence, we have the following lemma.

Lemma 2.2: For $k \geq 3$,

$$\overline{\mathcal{F}}_k[x]S_{k-3}[x] = \mathcal{F}_k[x].$$

Recall that $G_n[x] = S_{n-3}[x]$, $G_1[x] = I_n$ and $G_2[x] = I_{n-3} \oplus S_{-1}[x]$. As an immediate consequence of lemma 2.2, we have the following theorem.

Theorem 2.3: The n by n generalized Fibonacci matrix of the first kind, $\mathcal{F}_n[x]$, can be factorized by $G_k[x]$'s as follows.

$$\mathcal{F}_n[x] = G_1[x]G_2[x]\dots G_n[x].$$

We consider another factorization of $\mathcal{F}_n[x]$. Then n by n matrix $C_n[x] = [c_{ij}]$ is defined as:

$$c_{ij} = \begin{cases} F_i x^{i-j} & j = 1, \\ 1 & i = j, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{i.e., } C_n[x] = \begin{bmatrix} F_1 & 0 & \dots & 0 \\ F_2 x & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_n x^{n-1} & 0 & \dots & 1 \end{bmatrix}.$$

The next theorem follows, by a simple calculation.

Theorem 2.4: For $n \geq 2$,

$$\mathcal{F}_n[x] = C_n[x](I_1 - \oplus C_{n-1}[x])(I_2 \oplus C_{n-2}[x]) \dots (I_{n-2} \oplus C_2[x]).$$

Also we can easily find the inverse of the generalized Fibonacci matrix of the first kind. We know that

$$S_0[x]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ -x^2 & 0 & 1 \end{bmatrix}, \quad S_{-1}[x]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -x & 1 \end{bmatrix},$$

and $S_k[x]^{-1} = S_0[x]^{-1} \oplus I_k$. Define $H_k[x] = G_k[x]^{-1}$. Then $H_1[x] = G_1[x]^{-1} = I_n$, $H_2[x] = G_2[x]^{-1} = I_{n-3} \oplus S_{-1}[x]^{-1} = I_{n-2} \oplus \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix}$ and $H_n[x] = S_{n-3}[x]^{-1}$. Also, we know that

$$C_n[x]^{-1} = \begin{bmatrix} F_1 & 0 & \dots & 0 \\ -F_2 x & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -F_n x^{n-1} & 0 & \dots & 1 \end{bmatrix} \text{ and } (I_k \oplus C_{n-k}[x])^{-1} = I_k \oplus C_{n-k}[x]^{-1}.$$

So, the following corollary holds.

Corollary 2.5: For $n \geq 2$,

$$\begin{aligned} \mathcal{F}_n[x]^{-1} &= G_n[x]^{-1}G_{n-1}[x]^{-1} \dots G_2[x]^{-1}G_1[x]^{-1} \\ &= H_n[x]H_{n-1}[x] \dots H_2[x]H_1[x] \\ &= (I_{n-2} \oplus C_2[x]^{-1}) \dots (I_1 \oplus C_{n-1}[x]^{-1})C_n[x]^{-1}. \end{aligned}$$

From corollary 2.5, we have

$$\mathcal{F}_n[x]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ -x & 1 & 0 & 0 & \dots & 0 \\ -x^2 & -x & 1 & 0 & \dots & 0 \\ 0 & -x^2 & -x & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -x^2 & -x & 1 \end{bmatrix}. \tag{3}$$

For a factorization of the generalized Fibonacci matrix of the second kind, $\mathcal{R}_n[x]$, we define the matrices $M_n[x]$, $\overline{\mathcal{R}}_n[x]$ and $N_k[x]$ by

$$M_0[x] = \begin{bmatrix} 1 & 0 & 0 \\ x & x^2 & 0 \\ 1 & 0 & x^2 \end{bmatrix}, \quad M_{-1}[x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & x^2 \end{bmatrix},$$

and $M_k[x] = M_0[x] \oplus x^2 I_k, k = 1, 2, \dots, \overline{\mathcal{R}}_n[x] = [1] \oplus \mathcal{R}_{n-1}[x], N_1[x] = I_n, N_2[x] = I_{n-3} \oplus M_{-1}[x]$, and, for $k \geq 3$, $N_k[x] = I_{n-k} \oplus M_{k-3}[x]$. Then we have the following lemma.

Lemma 2.6: For $k \geq 3$,

$$\mathcal{R}_k[x] = \overline{\mathcal{R}}_k[x] M_{k-3}[x].$$

Proof: For $k = 3$, we have $\overline{\mathcal{R}}_3[x] M_0[x] = \mathcal{R}_3[x]$. Let $k > 3$. From the definition of the matrix product and a familiar Fibonacci sequence, the conclusion follows. □

As an immediate consequence of lemma 2.6, we have the following theorem.

Theorem 2.7: The n by n generalized Fibonacci matrix of the second kind, $\mathcal{R}_n[x]$, can be factorized by N_k's as follows.

$$\mathcal{R}_n[x] = N_1[x] N_2[x] \dots N_n[x].$$

Now, we consider another factorization of $\mathcal{R}_n[x]$. The n by n matrix $L_n[x] = [l_{ij}]$ is defined as:

$$l_{ij} = \begin{cases} F_i x^{i-j} & j = 1, \\ x^2 & i = j, j \geq 2 \\ 0 & \text{otherwise,} \end{cases} \qquad \text{i.e., } L_n[x] = \begin{bmatrix} F_1 & 0 & \dots & 0 \\ F_2 x & x^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_n x^{n-1} & 0 & \dots & x^2 \end{bmatrix}.$$

From the definition of the matrix $L_n[x]$, the following theorem holds.

Theorem 2.8: For $n \geq 2$,

$$\mathcal{R}_n[x] = L_n[x](I_1 \oplus L_{n-1}[x])(I_2 \oplus L_{n-2}[x]) \dots (I_{n-2} \oplus L_2[x]).$$

We can easily find the inverse of the generalized Fibonacci matrix of the second kind. We know that

$$M_0^{-1}[x] = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{x} & \frac{1}{x^2} & 0 \\ -\frac{1}{x^2} & 0 & \frac{1}{x^2} \end{bmatrix}, \quad M_{-1}^{-1}[x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{x} & \frac{1}{x^2} \end{bmatrix},$$

and for $k \geq 1$,

$$M_k^{-1}[x] = M_0^{-1}[x] \oplus \frac{1}{x^2} I_k.$$

Define $U_k[x] = N_k^{-1}[x]$. Then $U_1[x] = I_n$, $U_2[x] = N_2^{-1}[x] = I_{n-3} \oplus M_{-1}^{-1}[x]$, and, for $k \geq 3$, $U_k[x] = N_k^{-1}[x] = I_{n-k} \oplus M_{k-3}^{-1}[x]$. Also, we know that

$$L_n[x]^{-1} = \begin{bmatrix} F_1 & 0 & \dots & \dots & \dots & 0 \\ -\frac{F_2}{x} & \frac{1}{x^2} & \dots & \dots & \dots & 0 \\ -F_3 & 0 & \frac{1}{x^2} & \dots & \dots & 0 \\ -F_4 x & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -F_n x^{n-3} & 0 & \dots & 0 & 0 & \frac{1}{x^2} \end{bmatrix}$$

and $(I_k \oplus L_{n-k}[x])^{-1} = I_k \oplus L_{n-k}[x]^{-1}$. Then we have the following corollary.

Corollary 2.9: For $n \geq 2$,

$$\begin{aligned} \mathcal{R}_n[x]^{-1} &= U_n[x] U_{n-1}[x] \dots U_1[x] \\ &= (I_{n-2} \oplus L_2[x]^{-1}) \dots (I_1 \oplus L_{n-1}[x]^{-1}) L_n[x]^{-1}. \end{aligned}$$

From corollary 2.9, we have

$$\mathcal{R}_n[x]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ -\frac{1}{x} & \frac{1}{x^2} & 0 & 0 & \dots & 0 \\ -\frac{1}{x^2} & -\frac{1}{x^3} & \frac{1}{x^4} & 0 & \dots & 0 \\ 0 & -\frac{1}{x^4} & -\frac{1}{x^5} & \frac{1}{x^6} & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\frac{1}{x^{2n-4}} & -\frac{1}{x^{2n-3}} & \frac{1}{x^{2n-2}} \end{bmatrix}. \tag{4}$$

Note that $\mathcal{F}_n[1]^{-1} = \mathcal{R}_n[1]^{-1}$.

Now, we consider a factorization of $\mathcal{Q}_n[x]$. In [3], the authors gave the Cholesky factorization of the symmetric Fibonacci matrix $\mathcal{Q}_n[1]$ as follows:

Theorem 2.10: For $n \geq 1$ a positive integer,

$$\mathcal{Q}_n[1] = \mathcal{F}_n[1] \mathcal{F}_n[1]^T.$$

From the definition of $\mathcal{Q}_n[x]$, we derive the following lemma.

Lemma 2.11: For $n \geq 1$ a positive integer, let $\mathcal{Q}_n[x] = [q_{ij}]$. Then
(i) For $j \geq 3$, $q_{3j} = F_4(F_{j-3} + F_{j-2}F_3)x^{j+1}$.
(ii) For $j \geq 4$, $q_{4j} = F_4(F_{j-4} + F_{j-4}F_3 + F_{j-3}F_5)x^{j+2}$.
(iii) For $j \geq 5$, $q_{5j} = [F_{j-5}F_4(1 + F_3 + F_5) + F_{j-4}F_5F_6]x^{j+3}$.
(iv) For $j \geq i \geq 6$, $q_{ij} = [F_{j-i}F_4(1 + F_3 + F_5) + F_{j-i}F_5F_6 + \cdots + F_{j-i}F_{i-1}F_i + F_{j-i+1}F_iF_{i+1}]x^{i+j-2}$.

Proof: We know that $q_{3,3} = \sum_{k=1}^{3} F_k^2 x^4 = (F_1^2 + F_2^2 + F_3^2)x^4 = F_3F_4x^4$, and hence $q_{3,3} = F_4F_3x^4 = F_4(F_0 + F_1F_3)x^4$ for $F_0 = 0$. By induction, $q_{3j} = F_4(F_{j-3} + F_{j-2}F_3)x^{j+1}$ for $j \geq 3$. Thus, we have (i).

We know that $q_{1,3} = q_{3,1} = F_3x^2$ and $q_{2,3} = q_{3,2} = F_4x^3$. Also, we know that $q_{4,1} = q_{1,4} = F_4x^3, q_{4,2} = q_{2,4} = F_5x^4$ and $q_{3,4} = q_{4,3} = F_4(F_1 + F_2F_3)x^5$ by (i). By induction, we have $q_{4j} = F_4(F_{j-4} + F_{j-4}F_3 + F_{j-3}F_5)x^{j+2}$ for $j \geq 4$. Thus, (ii) holds.

By induction, (iii) and (iv) also hold. $\square$

Now, we have the following theorem.

Theorem 2.12: For $n \geq 1$ a positive integer

$$U_n[x]U_{n-1}[x]\ldots U_1[x]\mathcal{Q}_n[x] = \mathcal{F}_n[x]^T$$

and the Cholesky factorization of $\mathcal{Q}_n[x]$ is given by

$$\mathcal{Q}_n[x] = \mathcal{R}_n[x]\mathcal{F}_n[x]^T.$$

Proof: By corollary 2.9, $U_n[x]U_{n-1}[x]\ldots U_1[x] = \mathcal{R}_n[x]^{-1}$. So, if we have $\mathcal{R}_n[x]^{-1}\mathcal{Q}_n[x] = \mathcal{F}_n[x]^T$ then the theorem holds.

Note that $\mathcal{Q}_n[x]$ is a symmetric matrix. Let $A[x] = [a_{ij}] = \mathcal{R}_n[x]^{-1}\mathcal{Q}_n[x]$. By the definition of $\mathcal{Q}_n[x]$ and (4), $a_{ij} = 0$ for $j + 1 \leq i$.

Now we consider the case $j \geq i$. By (4) and lemma 2.11, we know that $a_{ij} = f_{ji}$ for $i \leq 5$.

We consider $j \geq i \geq 6$. Then, by (4), we have

$$
\begin{aligned}
a_{ij} &= -\frac{1}{x^{2i-4}} q_{i-2,j} - \frac{1}{x^{2i-3}} q_{i-1,j} + \frac{1}{x^{2i-2}} q_{i,j} \\
&= \frac{1}{x^{2i-2}}[F_{j-i}F_4(1+F_3+F_5) + F_{j-1}F_5F_6 + \cdots + F_{j-1}F_{i-1}F_i \\
&\quad + F_{j-i+1}F_iF_{i+1}]x^{i+j-2} \\
&- \frac{1}{x^{2i-3}}[F_{j-i+1}F_4(1+F_3+F_5) + F_{j-i+1}F_5F_6 + \cdots + \\
&\quad F_{j-i+1}F_{i-2}F_{i-1} + F_{j-i+2}F_{i-1}F_i]x^{i+j-3} \\
&- \frac{1}{x^{2i-4}}[F_{j-i+2}F_4(1+F_3+F_5) + F_{j-i+2}F_5F_6 + \cdots + \\
&\quad F_{j-i+2}F_{i-3}F_{i-2} + F_{j-i+3}F_{i-2}F_{i-1}]x^{i+j-4} \\
&= [(F_{j-i} - F_{j-i+1} - F_{j-i+2})F_4(1+F_3+F_5) + (F_{j-i} - F_{j-i+1} \\
&\quad - F_{j-i+2})F_5F_6 + \cdots + (F_{j-i} - F_{j-i+1} - F_{j-i+2})F_{i-3}F_{i-2} \\
&\quad + (F_{j-i} - F_{j-i+1} - F_{j-i+3})F_{i-2}F_{i-1} \\
&\quad + (F_{j-i} - F_{j-i+2})F_{i-1}F_i + F_{j-i+1}F_iF_{i+1}]x^{j-i}.
\end{aligned}
$$

Since $F_{j-i} - F_{j-i+1} - F_{j-i+2} = -2F_{j-i+1}, F_{j-i} - F_{j-i+1} - F_{j-i+3} = -3F_{j-i+1}$, and $F_{j-i} - F_{j-i+2} = -F_{j-i+1}$, we have

$$a_{ij} = F_{j-i+1}[-2F_4 - 2(F_3F_4 + F_4F_5 + \cdots + F_{i-2}F_{i-1}) - F_{i-2}F_{i-1} - F_{i-1}F_i + F_iF_{i+1}]x^{j-i}.$$

Since $F_4 = 3$ and

$$F_1F_2 + F_2F_3 + \cdots + F_{i-1}F_i = \frac{F_{2i-1} + F_iF_{i-1} - 1}{2},$$

we have

$$
\begin{aligned}
a_{ij} &= \left[-6 - 2\left(\frac{F_{2(i-1)-1} + F_{i-1}F_{(i-1)-1} - 1}{2} - F_1F_2 - F_2F_3\right)\right. \\
&- \quad F_{i-2}F_{i-1} - F_{i-1}F_i + F_iF_{i+1}]F_{j-i+1}x^{j-i} \\
&= (1 - 2F_{i-1}F_{i-2} - F_{2i-3} - F_{i-1}F_i + F_iF_{i+1})F_{j-i+1}x^{j-i}.
\end{aligned}
$$

Since $F_{i+1} = F_i + F_{i-1}$ and $F_{i+1}^2 + F_i^2 = F_{2i+1}$,

$$\begin{aligned} a_{ij} &= (1 - 2F_{i-1}F_{i-2} - (F_{i-1}^2 + F_{i-2}^2) + F_i^2) + F_{j-i+1}x^{j-i} \\ &= F_{j-i+1}x^{j-i} \\ &= f_{ji}. \end{aligned}$$

Thus, $A[x] = \mathcal{F}_n[x]^T$ for $1 \leq i,\ j \leq n$.

Therefore, $\mathcal{R}_n[x]^{-1}\mathcal{Q}_n[x] = \mathcal{F}_n[x]^T$, i.e., the Cholesky factorization of $\mathcal{Q}_n[x]$ is given by $\mathcal{Q}_n[x] = \mathcal{R}_n[x]\mathcal{F}_n[x]^T$. $\square$

For example,

$$\begin{aligned} \mathcal{Q}_5[x] &= \begin{bmatrix} 1 & x & 2x^2 & 3x^3 & 5x^4 \\ x & 2x^2 & 3x^3 & 5x^4 & 8x^5 \\ 2x^2 & 3x^3 & 6x^4 & 9x^5 & 15x^6 \\ 3x^3 & 5x^4 & 9x^5 & 15x^6 & 24x^7 \\ 5x^4 & 8x^5 & 15x^6 & 24x^7 & 40x^8 \end{bmatrix} \\ &= \mathcal{R}_5[x]\mathcal{F}_5[x]^T \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & x^2 & 0 & 0 & 0 \\ 2x^2 & x^3 & x^4 & 0 & 0 \\ 3x^3 & 2x^4 & x^5 & x^6 & 0 \\ 5x^4 & 3x^5 & 2x^6 & x^7 & x^8 \end{bmatrix} \begin{bmatrix} 1 & x & 2x^2 & 3x^3 & 5x^4 \\ 0 & 1 & x & 2x^2 & 3x^3 \\ 0 & 0 & 1 & x & 2x^2 \\ 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Since $\mathcal{Q}_n[x]^{-1} = (\mathcal{F}_n[x]^T)^{-1}\mathcal{R}_n[x]^{-1}$, we have

$$\mathcal{Q}_n[x]^{-1} = \begin{bmatrix} 3 & 0 & -\frac{1}{x^2} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{3}{x^2} & 0 & -\frac{1}{x^4} & 0 & 0 & \cdots & 0 \\ -\frac{1}{x^2} & 0 & \frac{3}{x^4} & 0 & -\frac{1}{x^6} & 0 & \cdots & 0 \\ 0 & -\frac{1}{x^4} & 0 & \frac{3}{x^6} & 0 & -\frac{1}{x^8} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{1}{x^{2n-8}} & 0 & \frac{3}{x^{2n-6}} & 0 & -\frac{1}{x^{2n-4}} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{x^{2n-6}} & 0 & \frac{2}{x^{2n-4}} & -\frac{1}{x^{2n-3}} \\ 0 & \cdots & \cdots & \cdots & 0 & -\frac{1}{x^{2n-4}} & -\frac{1}{x^{2n-3}} & \frac{1}{x^{2n-2}} \end{bmatrix}. \tag{5}$$

From theorem 2.12, we have the following corollary.

Corollary 2.13: If k is an odd number, then

$$(F_nF_{n-k} + \cdots + F_{k+1}F_1)x^{2n-k-2} = \begin{cases} (F_nF_{n-(k-1)} - F_k)x^{2n-k-2} & \text{if } n \text{ is odd,} \\ (F_nF_{n-(k-1)})x^{2n-k-2} & \text{if } n \text{ is even.} \end{cases}$$

If k is an even number, then

$$(F_nF_{n-k} + \cdots + F_{k+1}F_1)x^{2n-k-2} = \begin{cases} (F_nF_{n-(k-1)})x^{2n-k-2} & \text{if } n \text{ is odd,} \\ (F_nF_{n-(k-1)} - F_k)x^{2n-k-2} & \text{if } n \text{ is even.} \end{cases}$$

3. EIGENVALUES OF $\mathcal{Q}_n[x]$

Let A be an m by n matrix. For index sets $\alpha \subseteq \{1, 2, \ldots, m\}$ and $\beta \subseteq \{1, 2, \ldots, n\}$, we denote the submatrix that lies in the rows of A indexed by α and the columns indexed by β as $A(\alpha, \beta)$. If $m = n$ and $\alpha = \beta$, the submatrix $A(\alpha, \alpha)$ is a principal submatrix of A and is abbreviated $A(\alpha)$. We denote by A_i the leading principal submatrix of A determined by the first i rows and columns, $A_i \equiv A(\{1, 2, \ldots, i\}), i = 2, \ldots, n$. Note that if A is Hermitian, so is each A_i, and therefore each A_i has a real determinant.

We know that if A is positive definite, then all principal minors of A are positive, and, in fact, the converse is valid when A is Hermitian. However, in [2], we have the following stronger result: If A is an n by n Hermitian matrix, then A is positive definite if and only if $\det A_i > 0$ for $i = 1, 2, \ldots, n$. We know that $\mathcal{Q}_n[x]$ is a Hermitian matrix, $\det \mathcal{R}_n[x] = x^{n(n-1)}$ and $\det \mathcal{F}_n[x] = 1$ for $n \geq 2$. By theorem 2.12, we have $\det \mathcal{Q}_n[x] = \det(\mathcal{R}_n[x]\mathcal{F}_n[x]^T) = x^{n(n-1)}$. Since x is a nonzero real, we have $\det \mathcal{Q}_i[x] > 0, i = 2, 3, \ldots, n$. Thus, the matrix $\mathcal{Q}_n[x]$ is a positive definite matrix, and hence the eigenvalues of $\mathcal{Q}_n[x]$ are all positive.

Let $\lambda_1[x], \lambda_2[x], \ldots, \lambda_n[x]$ be the eigenvalues of $\mathcal{Q}_n[x]$. Since

$$q_{ii} = \sum_{k=1}^{i} F_k^2 x^{2i-2} = F_{i+1}F_i x^{2i-2},$$

we have

$$(F_{n+1}F_n x^{2n-2}, F_nF_{n-1}x^{2n-4}, \ldots, F_3F_2x^2, F_2F_1) \prec (\lambda_1[x], \lambda_2[x], \ldots, \lambda_n[x]).$$

Let $s_n[x] = \sum_{i=1}^{n} \lambda_i[x]$. Then,

$$s_n[x] = F_{n+1}F_n x^{2n-2} + F_nF_{n-1}x^{2n-4} + \cdots + F_3F_2x^2 + F_2F_1.$$

Thus, $\lambda_1[1], \lambda_2[1], \ldots, \lambda_n[1]$ are the eigenvalues of $\mathcal{Q}_n[1]$ and

$$(F_{n+1}F_n, F_nF_{n-1}, \ldots, F_3F_2, F_2F_1) \prec (\lambda_1[1], \lambda_2[1], \ldots, \lambda_n[1]).$$

We know the interesting combinatorial property

$$\sum_{i=0}^{n} \binom{n-i}{i} = F_{n+1}.$$

In [3], the authors gave the following result:

$$\lambda_1[1] + \lambda_2[1] + \cdots + \lambda_n[1] = \begin{cases} \left(\sum_{i=0}^{n} \binom{n-i}{i}\right)^2 - 1 & \text{if } n \text{ is odd,} \\ \left(\sum_{i=0}^{n} \binom{n-i}{i}\right)^2 & \text{if } n \text{ is even.} \end{cases}$$

Also, we have

$$\left(\frac{s_n[1]}{n}, \ldots, \frac{s_n[1]}{n}\right) \prec (\lambda_1[1], \lambda_2[1], \ldots, \lambda_n[1]).$$

So, we have $\lambda_n[1] \leq \frac{s_n[1]}{n} \leq \lambda_1[1]$, i.e., if n is an odd number then

$$n\lambda_n[1] \leq \left(\sum_{i=0}^{n} \binom{n-i}{i}\right)^2 - 1 \leq n\lambda_1[1],$$

if n is an even number then

$$n\lambda_n[1] \leq \left(\sum_{i=0}^{n} \binom{n-i}{i}\right)^2 \leq n\lambda_1[1].$$

Suppose that $x \geq 1$ and $(\lambda_1[x], \lambda_2[x], \ldots, \lambda_n[x]) \in \mathcal{D}$. Then, from (5), we have

$$\left(3, \frac{3}{x^2}, \frac{3}{x^4}, \ldots, \frac{3}{x^{2n-6}}, \frac{2}{x^{2n-4}}, \frac{1}{x^{2n-2}}\right) \prec \left(\frac{1}{\lambda_n[x]}, \frac{1}{\lambda_{n-1}[x]}, \ldots, \frac{1}{\lambda_1[x]}\right). \tag{6}$$

Thus, there exists a doubly stochastic matrix $T = [t_{ij}]$ such that

$$\left(3, \frac{3}{x^2}, \frac{3}{x^4}, \ldots, \frac{3}{x^{2n-6}}, \frac{2}{x^{2n-4}}, \frac{1}{x^{2n-2}}\right)$$

$$= \left(\frac{1}{\lambda_n[x]}, \frac{1}{\lambda_{n-1}[x]}, \ldots, \frac{1}{\lambda_1[x]}\right) \begin{bmatrix} t_{11} & t_{12} & \ldots & t_{1n} \\ t_{21} & t_{22} & \ldots & t_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ t_{n1} & t_{n2} & \ldots & t_{nn} \end{bmatrix}.$$

So, we have

$$3 = \frac{t_{11}}{\lambda_n[x]} + \frac{t_{21}}{\lambda_{n-1}[x]} + \cdots + \frac{t_{n1}}{\lambda_1[x]},$$

i.e.,

$$1 = \frac{t_{11}}{3\lambda_n[x]} + \frac{t_{21}}{3\lambda_{n-1}[x]} + \cdots + \frac{t_{n1}}{3\lambda_1[x]}.$$

Since the matrix T is a doubly stochastic matrix,

$$t_{11} + t_{21} + \cdots + t_{n1} = 1.$$

Lemma 3.1: Suppose that $x \geq 1$. For each $i = 1, 2, \ldots, n, n \geq 2$,

$$t_{n-(i-1),1} \leq \frac{3\lambda_i[x]}{n-1}.$$

Proof: Suppose that $t_{n-(i-1),1} > \frac{3\lambda_i[x]}{n-1}, i = 1, 2, \ldots, n$. Then

$$t_{11} + t_{21} + \cdots + t_{n1} > \frac{3\lambda_1[x]}{n-1} + \frac{3\lambda_2[x]}{n-1} + \cdots + \frac{3\lambda_n[x]}{n-1}$$

$$= \frac{3}{n-1}(\lambda_1[x] + \lambda_2[x] + \cdots + \lambda_n[x]).$$

Since $x \geq 1$ and

$$\lambda_1[x] + \lambda_2[x] + \cdots + \lambda_n[x] = F_{n+1}F_n x^{2n-2} + \cdots + F_3F_2x^2 + F_2F_1 > n,$$

this yields a contradiction.

Therefore, $t_{n-(i-1),1} \leq \frac{3\lambda_i[x]}{n-1}, i = 1, 2, \ldots, n$. $\square$

In [3], the authors found properties of the eigenvalues of $\mathcal{Q}_n[1]$ and proved the following result.

Theorem 3.2: Let $\tau = s_n[1] - (n-1)$. For $(\lambda_1[1], \lambda_2[1], \ldots, \lambda_n[1]) \in \mathcal{D}$,

$$(\tau, 1, 1, \ldots, 1) \prec (\lambda_1[1], \lambda_2[1], \ldots, \lambda_n[1]).$$

Let $\sigma[x] = s_n[x] - \frac{n-1}{3}$. Then, we have $(\sigma[x], \frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}) \in \mathcal{D}$ and $s_n[x] = \sigma[x] + \frac{n-1}{3} = \sum_{i=1}^{n} \lambda_i[x]$. In the next theorem, we have another majorization of the eigenvalues of $\mathcal{Q}_n[x]$.

Theorem 3.3: Suppose that $x \geq 1$. For $(\lambda_1[x], \lambda_2[x], \ldots, \lambda_n[x]) \in \mathcal{D}$, we have

$$\left(\sigma[x], \frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right) \prec (\lambda_1[x], \lambda_2[x], \ldots, \lambda_n[x]).$$

Proof: Let $P = [p_{ij}]$ be an n by n matrix as follows:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{12} \\ p_{21} & p_{22} & \cdots & p_{22} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{n2} \end{bmatrix},$$

where $p_{i2} = \frac{t_{n-(i-1),1}}{3\lambda_i[x]}$ and $p_{i1} = 1-(n-1)p_{i2}, i = 1, 2, \ldots, n$. Since T is doubly stochastic and $\lambda_i[x] > 0, p_{i2} \geq 0; i = 1, 2, \ldots, n$. By lemma 3.1, $p_{i1} \geq 0, i = 1, 2, \ldots, n$. Then

$$p_{12} + p_{22} + \cdots + p_{n2} = \frac{t_{n,1}}{3\lambda_1[x]} + \frac{t_{n-1,1}}{3\lambda_2[x]} + \cdots + \frac{t_{1,1}}{3\lambda_n[x]} = 1,$$

$p_{i1} + (n-1)p_{i2} = 1-(n-1)p_{i2} + (n-1)p_{i2} = 1$, and

$$\begin{aligned} &p_{11} + p_{21} + \cdots + p_{n1} \\ &\quad = 1-(n-1)p_{12} + 1-(n-1)p_{22} + \cdots + 1-(n-1)p_{n2} \\ &\quad = n-(n-1)(p_{12} + p_{22} + \cdots + p_{n2}) = 1. \end{aligned}$$

Thus, P is a doubly stochastic matrix. Furthermore,

$$\begin{aligned} \lambda_1[x]p_{12} + \lambda_2[x]p_{22} + \cdots + \lambda_n[x]p_{n2} &= \frac{\lambda_1[x]t_{n,1}}{3\lambda_1[x]} + \frac{\lambda_2[x]t_{n-1,1}}{3\lambda_2[x]} + \cdots + \frac{\lambda_n[x]t_{1,1}}{3\lambda_n[x]} \\ &= \frac{1}{3}(t_{n,1} + t_{n-1,1} + \cdots + t_{1,1}) = \frac{1}{3}, \end{aligned}$$

and

$$\begin{aligned} &\lambda_1[x]p_{11} + \lambda_2[x]p_{21} + \cdots + \lambda_n[x]p_{n1} \\ &\quad = \lambda_1[x](1-(n-1)p_{12}) + \cdots + \lambda_n[x](1-(n-1)p_{n2}) \\ &\quad = \lambda_1[x] + \lambda_2[x] + \cdots + \lambda_n[x] - (n-1)(\lambda_1[x]p_{12} + \lambda_2[x]p_{22} + \cdots + \lambda_n[x]p_{n2}) \\ &\quad = s_n[x] - (n-1)\frac{1}{3}(t_{n,1} + t_{n-1,1} + \cdots + t_{1,1}) \\ &\quad = \sigma[x]. \end{aligned}$$

Thus, $(\sigma[x], \frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}) = (\lambda_1[x], \lambda_2[x], \ldots, \lambda_n[x])P$.

Therefore,

$$\left(\sigma[x], \frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right) \prec (\lambda_1[x], \lambda_2[x], \ldots, \lambda_n[x]). \quad \square$$

From (6), we have the following lemma.

Lemma 3.4: Suppose that $x \geq 1$. For $k = 2, 3, \ldots, n$,

$$\frac{1}{3(k-1)} \leq \lambda_k[x].$$

Proof: From (6), for $k \geq 2$,

$$\frac{1}{\lambda_1[x]} + \frac{1}{\lambda_2[x]} + \cdots + \frac{1}{\lambda_k[x]} \leq \frac{1}{x^{2n-2}} + \frac{2}{x^{2n-4}} + \frac{3}{x^{2n-6}} + \cdots + \frac{3}{x^{2n-2k}}.$$

Since $x \geq 1$, we have

$$\frac{1}{\lambda_1[x]} + \frac{1}{\lambda_2[x]} + \cdots + \frac{1}{\lambda_k[x]} \leq 1 + 2 + 3 + \cdots + 3 = 3(k-1).$$

Thus,

$$\frac{1}{\lambda_k[x]} \leq 3(k-1) - \left(\frac{1}{\lambda_1[x]} + \frac{1}{\lambda_2[x]} + \cdots + \frac{1}{\lambda_{k-1}[x]}\right) \leq 3(k-1).$$

Therefore, $\frac{1}{3(k-1)} \leq \lambda_k[x]$. $\square$

In [3], the authors gave a bound for the eigenvalues of $\mathcal{Q}_n[1]$ as follows: for $k = 1, 2, \ldots, n-2$,

$$\lambda_{n-k}[1] \leq (k+1) - \frac{n-k}{3(n-1)}. \tag{7}$$

In the next theorem, we have a bound for the eigenvalues of $\mathcal{Q}_n[x]$ that is better than (7).
Theorem 3.5: Suppose that $x \geq 1$. For $k = 2, 3, \ldots, n-2$,

$$\frac{1}{3(n-k-1)} \leq \lambda_{n-k}[x] \leq \frac{1}{3}\left[k + 2 - \ln\left(\frac{n}{n-k-1}\right)\right].$$

In particular,

$$\sigma[x] \leq \lambda_1[x] \leq 3^{n-1}(n-1)!x^{n(n-1)},$$

$$\frac{1}{3(n-2)} \leq \lambda_{n-1}[x] \leq \frac{2n-3}{3(n-1)},$$

and

$$\frac{1}{3(n-1)} \leq \lambda_n[x] \leq \frac{1}{3}.$$

Proof: By theorem 3.3, we have $\sigma[x] \leq \lambda_1[x]$ and $\lambda_n[x] \leq \frac{1}{3}$. By lemma 3.4, we have $\frac{1}{3(n-1)} \leq \lambda_n[x]$. Since

$$\det \mathcal{Q}_n[x] = \det(\mathcal{R}_n[x]\mathcal{F}_n[x]^T) = x^{n(n-1)} = \lambda_1[x]\lambda_2[x]\ldots\lambda_n[x],$$

we have, by lemma 3.4,

$$\frac{1}{3^{n-1}(n-1)!} \leq \lambda_2[x] \ldots \lambda_n[x].$$

Thus, $\lambda_1[x] \leq 3^{n-1}(n-1)! x^{n(n-1)}$.

By lemma 3.4, $\frac{1}{3(n-2)} \leq \lambda_{n-1}[x]$ and $\lambda_n[x] + \lambda_{n-1}[x] \leq \frac{2}{3}$. So,

$$\lambda_{n-1}[x] \leq \frac{2}{3} - \lambda_n[x] \leq \frac{2}{3} - \frac{1}{3(n-1)} = \frac{2n-3}{3(n-1)}.$$

We know that

$$\frac{1}{2} + \cdots + \frac{1}{n} \leq \int_1^n \frac{1}{x} dx \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n-1},$$

i.e., $\frac{1}{2} + \cdots + \frac{1}{n} \leq \ln n \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$. So, we have

$$\begin{aligned} \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{n-k} &\geq \ln n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-k-1}\right) \\ &\geq \ln n - \ln(n-k-1) - 1. \end{aligned} \tag{8}$$

Since, by (8) and

$$\lambda_{n-k}[x] \leq \frac{k+1}{3} - (\lambda_n[x] + \lambda_{n-1}[x] + \cdots + \lambda_{n-k+1}[x]),$$

we have

$$\lambda_{n-k}[x] \leq \frac{1}{3}\left[k + 2 - \ln\left(\frac{n}{n-k-1}\right)\right].$$

Therefore,

$$\frac{1}{3(n-k-1)} \leq \lambda_{n-k}[x] \leq \frac{1}{3}\left[k + 2 - \ln\left(\frac{n}{n-k-1}\right)\right]. \quad \square$$

ACKNOWLEDGMENTS

This paper was supported by grant No. 2001-1-10200-003-1 from the Basic Research Program of the Korea Science and Engineering Foundation.

The second author was supported by the BK21 project for the Korea Education Ministry.

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[2] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.

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AMS Classification Numbers: 11B39, 15A18, 15A23, 15A42

✠ ✠ ✠

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2004. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-966 Proposed by Stanley Rabinowitz, Math Pro, Westford, MA

Find a recurrence relation for $r_n = \frac{1}{1+F_n}$.

B-967 Proposed by Juan Pla, Paris, France

Prove that $\frac{5}{32}F_{6n}^2$ is an integer of the form $\frac{m(m+1)}{2}$.

B-968 Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN

Let $F(n) = \sum_{i=2}^{n} \frac{4+1000F_i}{F_{i-1}F_{i+1}}$ where F_i is the i^{th} Fibonacci number. Find $\lim_{n\to\infty} F(n)$.

B-969 Proposed by José Luis Díaz-Barrero, UPC, Barcelona, Spain

Evaluate the following sum

$$\sum_{n=1}^{\infty} \frac{F_{n+1}[F_{2n+3}+(-1)^{n+1}]F_{n+3}}{F_{n+2}[F_{2n+1}+(-1)^n][F_{2n+5}+(-1)^{n+2}]}.$$

B-970 Proposed by Peter G. Anderson, Rochester Institute of Technology, Rochester, NY

Define a second-order and three third-order recursions by:

$$f_n = f_{n-1} + f_{n-2}, \text{ with } f_0 = 1, f_1 = 1.$$
$$g_n = g_{n-1} + g_{n-3}, \text{ with } g_0 = 1, g_1 = 1, g_2 = 1.$$
$$h_n = h_{n-2} + h_{n-3}, \text{ with } h_0 = 1, h_1 = 0, h_2 = 1.$$

and

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}, \text{ with } t_0 = 1, t_1 = 1, t_2 = 2.$$

Prove:

1. $t_{n+3} = f_{n+3} + \sum_{p+q=n} f_p t_q$.
2. $t_{n+2} = g_{n+2} + \sum_{p+q=n} g_p t_q$.
3. $t_{n+1} = h_{n+1} + \sum_{p+q=n} h_p t_q$.

SOLUTIONS

Another Fibonacci Sequence

B-951 Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA (Vol. 41, no. 1, February 2003)

The sequence $\langle u_n \rangle$ is defined by the recurrence

$$u_{n+1} = \frac{3u_n + 1}{5u_n + 3}$$

with the initial condition $u_1 = 1$. Express u_n in terms of Fibonacci and/or Lucas numbers.

Solution by Carl Libis, University of Rhode Island, Kingston, RI.

We will show by induction that $u_n = F_{2n-1}/L_{2n-1}$. Note that $u_1 = 1 = F_1/L_1$. Assume that $u_n = F_{2n-1}/L_{2n-1}$. Then

$$u_{n+1} = \frac{3u_n+1}{5u_n+3} = \frac{3\frac{F_{2n-1}}{L_{2n-1}}+1}{5\frac{F_{2n-1}}{L_{2n-1}}+3} = \frac{3F_{2n-1}+L_{2n-1}}{5F_{2n-1}+3L_{2n-1}} = \frac{3F_{2n-1}+(2F_{2n-2}+F_{2n-1})}{5F_{2n-1}+3(2F_{2n-2}+F_{2n-1})}$$

$$= \frac{2(2F_{2n-1}+F_{2n-2})}{2(4F_{2n-1}+3F_{2n-2})} = \frac{F_{2n-1}+F_{2n}}{F_{2n-1}+3F_{2n}} = \frac{F_{2n+1}}{F_{2n+1}+2F_{2n}} = \frac{F_{2n+1}}{F_{2n+2}+F_{2n}} = \frac{F_{2n+1}}{L_{2n+1}}$$

This completes the induction.

All the received solutions used a similar argument. A slight generalization was given by H.J. Seifert.
Also solved by Paul S. Bruckman, Mario Catalani, Charles Cook, Kenneth Davenport, Steve Edwards, Sergio Falcón and Angel Plaza (jointly), Ovidiu Furdui, Walther Janous, Emrah Kiliç, Harris Kwong, Kathleen Lewis, Reiner Martin, H.-J. Seiffert, James Sellers, J. Spilker, David Stone, J. Suck, Haixing Zhao, and the proposer.

And ... a Fibonacci Identity

B-952 Proposed by Scott H. Brown, Auburn University, Montgomery, AL (Vol. 41, no. 1, February 2003)

Show that

$$10F_{10n-5}+12F_{10n-10}+F_{10n-15}=25F_{2n}^5+25F_{2n}^3+5F_{2n}$$

for all integers $n \geq 2$.

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY.

Using the following identities from [1]

$$\begin{array}{ll} (I_{16}) & 5F_{2m}^2=L_{4m}-2, \\ (I_{15}) & L_{2m}^2=L_{4m}+2, \\ (I_{24}) & L_mF_p=F_{m+p}-F_{m-p}, \quad p \text{ even}, \end{array}$$

we find

$$\begin{aligned} 25F_{2n}^5+25F_{2n}^3+5F_{2n} &= F_{2n}[(5F_{2n}^2)^2+5\cdot 5F_{2n}^2+5] \\ &= F_{2n}[(L_{4n}-2)^2+5(L_{4n}-2)+5] \\ &= F_{2n}[L_{4n}^2+L_{4n}-1] \\ &= F_{2n}[L_{8n}+L_{4n}+1] \\ &= (F_{10n}-F_{6n})+(F_{6n}-F_{2n})+F_{2n} \\ &= F_{10n}. \end{aligned}$$

Letting $G_n=F_{5n}$, it suffices to prove that

$$G_{2n}=10G_{2n-1}+12G_{2n-2}+G_{2n-3}, \quad n \geq 2. \tag{1}$$

From the generating function

$$\sum_{n=0}^{\infty} G_nx^n=\frac{1}{\alpha-\beta}\left\{\sum_{n=0}^{\infty}(\alpha^5x)^n-\sum_{n=0}^{\infty}(\beta^5x)^n\right\}=\frac{1}{\alpha-\beta}\left\{\frac{1}{1-\alpha^5x}-\frac{1}{1-\beta^5x}\right\}$$

$$=\frac{1}{\alpha-\beta}\cdot\frac{(\alpha^5-\beta^5)x}{1-(\alpha^5+\beta^5)x+(\alpha^5\beta^5)x^2}=\frac{F_5x}{1-L_5x+(-1)^5x^2}=\frac{5x}{1-11x-x^2},$$

we deduce that $q^2 - q - 1 = 0$ is the characteristic equation for G_n. Hence G_n satisifies the recurrence relation

$$G_n = 11G_{n-1} + G_{n-2}, \quad n \geq 2,$$

from which (1) follows immediately.

Reference

1. Verner E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*, pages 56-59, Fibonacci Association, 1969.

1. Walther Janous made the remark that " ... an immeidate consequence of the above identity is that for all $n \geq 2$, $5F_{2n}$ divides $10F_{10n-5} + 12F_{10n-10} + F_{10n-15}$. This suggests the following problem: Determine all 7-tuples (a, b, c, d, A, B, C) where a, b, c, A, B, C are positive integers and d is a non-zero integer such that for all $n \geq \max(0, \frac{-c}{d})$, aF_{bn} divides $AF_{cn+d} + BF_{cn+2d} + CF_{cn+3d}$ and g.c.d$(a, A, B, C) = 1$ and a can not be increased".
2. H.-J. Seiffert prove the identity

$$10F_{k-5} + 12F_{k-10} + F_{k-15} = F_k \text{ for all } K \in Z.$$

Also solved by Paul Bruckman, Mario Catalani, Kenny Davenport, L.A.G. Dresel, Sergio Falcón and Angel Plaza (jointly), Ovidiu Furdui, N. Gauthier, Walther Janous, Emrah Kilis, William Moser, H.-J. Seiffert, J. Suck, Haixing Zhao, and the proposer.

Never Perfect!

B-953 Proposed by Harvey J. Hindin, Huntington Station, NY (Vol. 41, no. 1, February 2003)

Show that

$$(F_n)^4 + (F_{n+1})^4 + (F_{n+2})^4$$

is never a perfect square. Similarly, show that

$$(qW_n)^4 + (pW_{n+1})^4 + (W_{n+2})^4$$

is never a perfect square, when W_n is defined for all integers n by $W_n = pW_{n-1} - qW_{n-2}$ and where $W_0 = a$ and $W_1 = b$.

Solution by H.-J. Seiffert, Berlin, Germany

If $p = q = \sqrt[4]{2}$ and $a = b = 1$, then $W_0 = W_1 = 1$ and $W_2 = 0$, so that $(qW_0)^4 + (pW_1)^4 + (W_2)^4 = 4$ is a perfect square.

Now, suppose that p, q, a, and b are all integers with $pq \neq 0$. Let n be an integer such that the integer $q^2W_n^2 + p^2W_{n+1}^2 + W_{n+2}^2$ is nonzero (this is satisfied for all integers n if $\langle W_n \rangle = \langle F_n \rangle$). Since [1]

$$(qW_n)^4 + (pW_{n+1})^4 + (W_{n+2})^4 = (q^2W_n^2 + p^2W_{n+1}^2 + W_{n+2}^2)^2/2$$

and since $\sqrt{2}$ is irrational, the expression on the left hand side of the above identity then cannot be a perfect square.

Reference

1. R.S. Melham & H. Kwong. "Problem B-927." *The Fibonacci Quarterly* **40.4** (2002): 374-75.

Paul Brukman discussed the Case $pq = 0$ and showed that it lead to trivial solutions. Even when $pq = 0, A = (gW_n)^4 + (pW_{n+1})^4 + (W_{n+2})^4$ may still be zero for some n, if a and b are properly chosen. To avoid much difficulties, he suggested the addition of " ... is never a <u>*non − zero*</u> perfect square ... " in the statement of the problem.

Also solved by Paul Bruckman, Mario Catalani, L.A.G. Dresel, Sergio Falcón and Angel Plaza (jointly), Ovidiu Furdui, Walther Janous, Harris Kwong, Carl Libis (1st part), Reiner Martin, and the proposer.

<u>A Fibonacci floor-and-ceiling Equality</u>

<u>B-954</u> Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 41, no. 1, February 2003)

Let n be a nonnegative integer. Show that

$$\sqrt{(\sqrt{5}+2)(\sqrt{5}F_{2n+1}-2)} = L_{2\lfloor n/2\rfloor+1} + \sqrt{5}F_{2\lceil n/2\rceil},$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor-and ceiling-function, respectively.

Solution by L.A.G. Dresel, Reading, England

Let S_n denote the expression on the left side of the proposed identity. Since $\alpha+\beta = 1, \alpha\beta = -1$ and $\alpha^2 = \alpha + 1$, we have $\sqrt{5}+2 = 2\alpha+1 = \alpha^2+\alpha = \alpha(\alpha+1) = \alpha^3$, so that $S_n = \sqrt{\{\alpha^3(\alpha^{2n+1} - 2 - \beta^{2n+1})\}} = \alpha\sqrt{\{\alpha^{2n+2} - 2\alpha + \beta^{2n}\}} = \alpha\{\alpha^{n+1} - (-1)^n\beta^n\}$, giving $S_n = (1+\alpha)\alpha^n - (-1)^n(1-\beta)\beta^n = \alpha^n + \alpha^{n+1} - (-1)^n(\beta^n - \beta^{n+1})$. Therefore, when n is <u>*even*</u>, we have $S_n = L_{n+1} + \sqrt{5}F_n$, and when n is <u>odd</u>, we have $S_n = L_n + \sqrt{5}F_{n+1}$, which agrees with the given formula.

Also solved by Paul Bruckman, Mario Catalani, Kenny Davenport, Sergio Falcón and Angel Plaza (jointly), Ovidiu Furdui, Walther Janous, Harris Kwong, Haixing Zhao, and the proposer.

<u>A Strict Inequality</u>

<u>B-955</u> Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

(Vol. 41, no. 1, February 2003)

Prove that

$$1 < \frac{F_{2n}}{\sqrt{1+F_{2n}^2}} + \frac{1}{\sqrt{1+F_{2n+1}^2}} + \frac{1}{\sqrt{1+F_{2n+2}^2}} < \frac{3}{2}$$

for all integers $n \geq 0$.

Solution by Paul S. Bruckman, Berkeley, CA

Let $x_1 = F_{2n}\{1 + (F_{2n})^2\}^{-1/2}, x_2 = \{1 + (F_{2n+1})^2\}^{-1/2}, x_3 = \{1 + (F_{2n+2})^2\}^{-1/2}$, for a given $n \geq 0$. Clearly, $x_1 \geq 0, x_2 \geq x_3 > 0$. Moreover, if $n > 0, x_1 > x_2 > x_3 > 0$. Let $S(n) = x_1 + x_2 + x_3$. Note $S(0) = 0 + 2 \cdot 2^{-1/2} = 2^{1/2} \approx 1.41$, hence $1 < S(0) < 1.5$. Also, $S(1) = 2^{-1/2} + 5^{-1/2} + 10^{-1/2} \approx 1.47$, hence $1 < S(1) < 1.5$.

Next, $S(2) = 3 \cdot 10^{-1/2} + 26^{-1/2} + 65^{-1/2} \approx 1.27$. hence $1 < S(2) < 1.5$.

Henceforth, we suppose $n \geq 3$. Then $S(n) < (F_{2n} + 2)\{1 + (F_{2n})^2\}^{-1/2} < 1 + 2/F_{2n} \leq 1 + 2/F_6 = 1.25$. Hence $S(n) < 1.5$ for all $n \geq 0$. In fact, $S(n) \leq S(1)$ for all $n \geq 0$.

On the other hand, if $n \geq 3$,

$$\begin{aligned}
S(n) &= \{1 + (F_{2n})^{-2}\}^{-1/2} + (F_{2n+1})^{-1}\{1 + (F_{2n+1})^{-2}\}^{-1/2} + (F_{2n+2})^{-1}\{1 + (F_{2n+2})^{-2}\}^{-1/2} \\
&> 1 - 1/\{2(F_{2n})^2\} + (F_{2n+1})^{-1}(1 - 1/\{2(F_{2n+1})^2\}) + (F_{2n+2})^{-1}(1 - 1/\{2(F_{2n+2})^2\}) \\
&= 1 + 1/F_{2n+1} + 1/F_{2n+2} - 1/2\{1/(F_{2n})^2 + 1/(F_{2n+1})^3 + 1/(F_{2n+2})^3\} \\
&> 1 + 2/F_{2n+2} - (F_{2n} + 2)/\{2(F_{2n})^3\} > 1 + 2/F_{2n+2} - 1/(F_{2n})^2.
\end{aligned}$$

Note that $2(F_{2n})^2 - F_{2n+2} > 0$ if $n \geq 3$, hence $2/F_{2n+2} - 1/(F_{2n})^2 > 0$. Therefore, $S(n) > 1$ for all $n \geq 3$. From our previous results, $S(n) > 1$ for all $n \geq 0$.
Q.E.D.

We may also note that $\lim\limits_{n \to \infty} S(n) = 1$.

Also solved by Walther Janous, Angel Plaza and Sergio Falcón (jonitly), and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Florian Luca

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-603 Proposed by the E. Herrmann, Siegburg, Germany

Show that if $n \geq 3$ and $n \equiv 1 \pmod 2$, then

$$\frac{1}{F_n} < \sum_{k=0}^{\infty} \frac{1}{F_{n+2k}} < \frac{1}{F_{n-1}}.$$

However, if $n \geq 4$ and $n \equiv 0 \pmod 2$, then

$$\frac{1}{F_{n-1}} < \sum_{k=0}^{\infty} \frac{1}{F_{n+2k}} < \frac{1}{F_{n-2}}.$$

H-604 Proposed by Mario Catalani, Torino, Italy

In **H-592**, the proposers introduced, for $n \geq 2$, a nondiagonal $n \times n$ matrix A such that $A^2 = xA + yI$, where x, y are indeterminates and I is the identity matrix.

a) State the conditions under which all the eigenvalues of A are equal.

b) Assume now that not all the eigenvalues of A are equal. Assume that A is a $2n \times 2n$ matrix, and that $\operatorname{tr}(A) = nx$. Consider the Hamilton-Cayley equation for A

$$\sum_{k=0}^{2n} (-1)^k \lambda_k A^{2n-k} = 0,$$

where $\lambda_0 = 1$. Find $\sum_{k=0}^{2n} \lambda_k$.

H-605 Proposed by José Luis Díaz-Barrero & Juan José Egozcue, Barcelona, Spain

Find the smallest integer k for which $\lambda_0 a_n + \lambda_1 a_{n-1} + \cdots + \lambda_k a_{n+k} = 0$ holds for all $n \geq 1$ with some integers $\lambda_0, \ldots, \lambda_k$ not all zero, where $\{a_n\}_{n \geq 1}$is the integer sequence defined by

$$a_n = \Big(\sum_{\ell=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2\ell+1} 2^{\ell} \Big) \Big(\sum_{\ell=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2^{n-1}} \binom{n}{2\ell+1} 5^{\ell} \Big).$$

SOLUTIONS

Some properties of the number 5

H-591 Proposed by H.-J. Sieffert, Berlin, Germany
(Vol. 40, no. 5, November 2002)

Prove that, for all positive integers n,

(a)

$$5^n F_{2n-1} = \sum_{\substack{k=0 \\ 5 \nmid 2n-k+3}}^{2n} (-1)^{\lfloor (4n+3k)/5 \rfloor} \binom{4n+1}{k},$$

(b)

$$5^n L_{2n} = \sum_{\substack{k=0 \\ 5 \nmid 2n-k+4}}^{2n+1} (-1)^{\lfloor (4n+3k-3)/5 \rfloor} \binom{4n+3}{k},$$

(c)

$$5^{n-1} F_{2n} = \sum_{\substack{k=0 \\ 5 \nmid 2n-k+1}}^{2n-2} (-1)^{\lfloor (8n+k+3)/5 \rfloor} \binom{4n-3}{k},$$

(d)

$$5^{n-1} L_{2n+1} = \sum_{\substack{k=0 \\ 5 \nmid 2n-k+2}}^{2n-1} \binom{4n-1}{k},$$

where $\lfloor\ \rfloor$ denotes the greatest integer function.

Solution by the proposer

Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$, and $F_{k+2}(x) = xF_{k+1}(x) + F_k(x)$ for $k \geq 0$. From **H-492**, we know that, for all complex numbers x and y and all nonegative integers n,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} F_{n-2k}(x) F_{n-2k}(y) = z^{n-1} F_n(xy/z),$$

where $z = \sqrt{x^2 + y^2 + 4}$. Replacing n by $2n+1$ and taking $y = 0$, after a suitable reindexing, we obtain

$$\sum_{k=0}^{n} \binom{2n+1}{n-k} F_{2k+1}(x) = (x^2+4)^n. \tag{1}$$

Let $B_k = (-1)^k F_{2k+1}(\,\mathrm{i}\alpha)$, $k \geq 0$, where $\mathrm{i} = \sqrt{-1}$ and α is the golden section. Then, the sequence $\{B_k\}_{k \geq 0}$ satisfies the recursion $B_{k+2} = -\beta B_{k+1} - B_k$ for $k \geq 0$, where β is the conjugate of α, and a simple induction argument shows that

$$B_k = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 5, \\ \alpha & \text{if } k \equiv 1 \pmod 5, \\ 0 & \text{if } k \equiv 0 \pmod 5, \\ -\alpha & \text{if } k \equiv 0 \pmod 5, \\ -1 & \text{if } k \equiv 0 \pmod 5. \end{cases} \tag{2}$$

Since $4-\alpha^2=-\sqrt{5}\beta$, identity (1) with $x=$ iα gives

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}B_k=(-1)^n5^{n/2}\beta^n. \tag{3}$$

Define the sequences $\{c_k\}_{k\geq 0}$ and $\{d_k\}_{k\geq 0}$ by

$$c_k=\begin{cases}1 & if k\equiv 0 \pmod 5,\\ -1 & if k\equiv 4 \pmod 5,\\ 0 & \text{otherwise,}\end{cases} \quad \text{and} \quad d_k=\begin{cases}1 & if k\equiv 1 \pmod 5,\\ -1 & if k\equiv 3 \pmod 5,\\ 0 & \text{otherwise,}\end{cases}$$

and let

$$S_n=\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}c_k \quad \text{and} \quad T_n=\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}d_k.$$

Then, by (2) and (3), $S_n+\alpha T_n=(-1)^n5^{n/2}\beta^n$. Since $2\beta^n=L_n-\sqrt{5}F_n$, we then have

$$2S_n+2\alpha T_n=\begin{cases}5^{n/2}L_n-5^{(n+1)/2}F_n & \text{if } n \text{ is even,}\\ 5^{(n+1)/2}F_n-5^{n/2}L_n & \text{if } n \text{ is odd.}\end{cases}$$

Using $2\alpha=1+\sqrt{5}$ together with the fact $\sqrt{5}$ is irrational, we then must have

$$2S_n+T_n=\begin{cases}5^{n/2}L_n & \text{if } n \text{ is even,}\\ 5^{(n+1)/2}F_n & \text{if } n \text{ is odd,}\end{cases} \tag{4}$$

and

$$T_n=\begin{cases}-5^{n/2}F_n & \text{if } n \text{ is even,}\\ -5^{(n-1)/2}L_n & \text{if } n \text{ is odd.}\end{cases} \tag{5}$$

Substracting (5) from (4) yields

$$2S_n=\begin{cases}5^{n/2}(F_n+L_n) & \text{if } n \text{ is even,}\\ 5^{(n-1)/2}(5F_n+L_n) & \text{if } n \text{ is odd.}\end{cases} \tag{6}$$

Dividing (6) by 2, subtracting the resulting equation from (4), and noting that $L_n-F_n=2F_{n-1}$ and $5F_n-L_n=2L_{n-1}$, we find

$$S_n+T_n=\begin{cases}5^{n/2}F_{n-1} & \text{if } n \text{ is even,}\\ 5^{(n-1)/2}L_{n-1} & \text{if } n \text{ is odd.}\end{cases} \tag{7}$$

On the other hand, from

$$c_k + d_k = \begin{cases} (-1)^{\lfloor 2k/5 \rfloor} & \text{if } k \not\equiv 2 \pmod 5, \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$S_n + T_n = \sum_{\substack{k=0 \\ 5 \nmid k+3}}^{n} (-1)^{\lfloor 7k/5 \rfloor} \binom{2n+1}{n-k}. \tag{8}$$

The desired identities (a) and (b) now easily follow from (7) and (8) by repacing n by $2n$ (respectively, by $2n+1$), and reindexing.

Multiplying (5) by 3, substracting the resulting equation from (4), and noting that $L_n + 3F_n = 2F_{n+2}$ and $5F_n + 3L_n = 2L_{n+2}$, we obtain

$$S_n - T_n = \begin{cases} 5^{n/2} F_{n+2} & \text{if } n \text{ is even,} \\ \\ 5^{(n-1)/2} L_{n+2} & \text{if } n \text{ is odd.} \end{cases} \tag{9}$$

Since

$$c_k - d_k = \begin{cases} (-1)^{\lfloor (4k+1)/5 \rfloor} & \text{if } k \not\equiv 2 \pmod 5, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$S_n - T_n = \sum_{\substack{k=0 \\ 5 \nmid k+3}}^{n} (-1)^{\lfloor (9k+1)/5 \rfloor} \binom{2n+1}{n-k}. \tag{10}$$

The desired identities (c) and (d) now easily follow from (9) and (10) by repacing n by $2n-2$ (respectively, by $2n-1$), and reindexing.

Also solved by Paul Bruckman and Vincent Mathe.

Matrices satisfying quadratic equations

H-592 Proposed by N. Gautheir & J.B. Gosselin, Royal Military College of Canada (Vol. 40, no. 5, November 2002)

For integers $m \geq 1$, $n \geq 2$, let X be a nontrivial $n \times n$ matrix such that

$$X^2 = xX + yI, \tag{1}$$

where x, y are indeterminates and I is a unit matrix. (By definition, a trivial matrix is diagonal.) Then consider the Fibonacci and Lucas sequences of polynomials, $\{F_l(x,y)\}_{l=0}^{\infty}$ and $\{L_l(x,y)\}_{l=0}^{\infty}$, defined by the recurences

$$F_0(x,y) = 0,\ F_1(x,y) = 1,\ F_{l+2}(x,y) = xF_{l+1}(x,y) + yF_l(x,y), \tag{2}$$

$$L_0(x,y) = 2,\ L_1(x,y) = x,\ L_{l+2}(x,y) = xL_{l+1}(x,y) + yL_l(x,y), \tag{3}$$

respectively.

a. Show that

$$X^m = a_m X + b_m yI \qquad \text{and that} \qquad X^m + (-y)^m X^{-m} = c_m I,$$

where a_m, b_m, and c_m are to be expressed in closed form as functions of the polynomials (2).

b. Now let

$$f(\lambda; x, y) \equiv |\lambda I - X| \equiv \sum_{m=0}^{n} (-1)^{n-m} \lambda_{n-m} \lambda^m$$

be the characteristic (monic) polynomial associated to X, where the set of coefficients,

$$\{\lambda_l \equiv \lambda_l(x, y); 0 \le l \le n\}$$

is entirely determined from the defining relation for $f(\lambda; x, y)$. For example, $\lambda_0 = 1$, $\lambda_1 = \text{tr}(X)$, $\lambda_n = \det(X)$, etc. Show that

$$\sum_{m=1}^{n} (-1)^m \lambda_{n-m} F_m(x, y) = 0 \qquad \text{and that} \qquad y \sum_{m=1}^{n} (-1)^m \lambda_{n-m} F_{m-1}(x, y) + \lambda_n = 0.$$

Solution by the proposers

a. First note that X has an inverse since $X(X - xX) = yI$ implies $\det(X) \neq 0$ (here, y is assumed to be a nonzero indeterminate). We prove by induction on $m \ge 1$ that

$$X^m = F_m(x, y)X + yF_{m-1}(x, y)I, \tag{3}$$

and that

$$X^m + (-y)^m X^{-m} = L_m(x, y)I, \tag{4}$$

so $a_m = F_m(x, y)$, $b_m = F_{m-1}(x, y)$ and $c_m = L_m(x, y)$. It is clear that (3) is true for $m = 1$ and 2. Now assume its validity for an arbitrary value of m and multiply (3) by X to get

$$X^{m+1} = F_m(x, y)X^2 + yF_{m-1}X = F_m(x, y)(xX + yI) + yF_{m-1}(x, y)X$$

$$= (xF_m(x, y) + yF_{m-1}(x, y))X + yF_m(x, y)I = F_{m+1}(x, y)X + yF_m(x, y)I,$$

which is formula (3) for $m + 1$. To prove (4), note that it is true for $m = 1$ since (1) implies $X - yX^{-1} = xI$. Squaring this last result then shows that (4) also holds for $m = 2$. Now assume that (4) holds for $m \ge 2$ and multiply it by $X - yX^{-1} = xI$ to get

$$(X^{m+1} + (-y)^{m+1} X^{-(m+1)}) + (-yX^{m-1} + (-y)^m X^{-(m-1)}) = xL_m(x, y)I,$$

i.e.,

$$X^{m+1} + (-y)^{m+1} X^{-(m+1)} = y(X^{m-1} + (-y)^{m-1} X^{-(m-1)}) + xL_m(x, y)I$$

$$= yL_{m-1}(x, y)I + xL_m(x, y)I = L_{m+1}(x, y)I,$$

which proves that (4) holds for $m + 1$ as well.

b. According to the Hamilton-Cayley theorem, if $f(\lambda; x, y)$ is the characteristic polynomial associated with the matrix X, then $f(X; x, y) = 0$. Consequently, upon cancelling out an overall factor of $(-1)^n$ and upon using (3) for X^m, we find that

$$0 = \sum_{m=0}^{n} (-1)^m \lambda_{n-m} X^m = \sum_{m=1}^{n} (-1)^m \lambda_{n-m} (F_m(x, y)X + yF_{m-1}(x, y)I) + \lambda_n I,$$

which leads to the formulae given in the statement of the problem when X and I are linearly independent, i.e., in nontrivial cases.

Vincent Mathe points out that that in the case $y = 0$ the matrix X is not necessarily invertible; see for example, the Solution of **H-578**, vol. 40, pages 474–476.

Also solved by Paul Bruckman, Mario Catalani, Toufik Mansour and Vincent Mathe.

A Lucas prime congruence

H-593 Proposed by H.-J. Seiffert, Berlin, Germany
(Vol. 41, no. 1, February 2003)

Let $p > 5$ be a prime. Prove the congruence

$$2\sum_{k=0}^{\lfloor (p-5)/10\rfloor} \frac{(-1)^k}{2k+1} \equiv (-1)^{(p-1)/2}\frac{2^{p-1}-L_p}{p} \pmod p.$$

Solution by the proposer

It is wellknown that $L_p \equiv 1 \pmod p$. Since by Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod p$, we see that the expression appearing on the right hand side of the desired congruence is an integer.

From **H-562**, we know that, for all nonnegative integers n,

$$5\sum_{k=0}^{\lfloor (n-2)/5\rfloor} \binom{2n+1}{n-5k-2} = 4^n - L_{2n+1}. \tag{1}$$

If k is an integer such that $0 \le k \le \lfloor (p-5)/10\rfloor$, then

$$p > \frac{p-1}{2} - 5k - 2 \ge \frac{p-1}{2} - 5\left\lfloor\frac{p-5}{10}\right\rfloor - 2 > \frac{p-1}{2} - 5\cdot\frac{p-5}{10} - 2 = 0,$$

because $(p-5)/10$ is not an integer. Since, as is known,

$$\frac{1}{p}\binom{p}{j} \equiv \frac{(-1)^{j-1}}{j} \pmod p \qquad \text{for } j = 1, \ldots, p-1,$$

relation (1) with $n = (p-1)/2$ gives

$$5\sum_{k=0}^{\lfloor (p-5)/10\rfloor} \frac{(-1)^{(p+1)/2+k}}{(p-1)/2-5k-2} \equiv \frac{2^{p-1}-L_p}{p} \pmod p.$$

Multiplying by $(-1)^{(p-1)/2}$ and noting that

$$\frac{-5}{(p-1)/2-5k-2} = \frac{-10}{p-10k-5} \equiv \frac{2}{2k+1} \pmod p$$

gives the desired congruence.

Also solved by Paul Bruckman.

Please Send in Proposals!

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau, Fibonacci Association (FA), 1965. $18.00

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972. $23.00

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