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ON SUMMATION FORMULAS AND IDENTITIES FOR FIBONACCI NUMBERS

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1. REMARKS ON THE PAPER OF BROTHER U. ALFRED

Alfred [1] has shown that

$$(1.1)\qquad \sum_{k=0}^{n-1} k^m F_{k+r} = \sum_{i=0}^{m} (-1)^i F_{2i+n+r+1}\Delta^i(n^m) + C_1 \quad ,$$

where C_1 is a constant independent of n and $\Delta g(n) = g(n+1) - g(n)$, with $\Delta^i g(n) = \Delta(\Delta^{i-1}g(n))$. The following result yields (1.1) as a special case:

<u>Theorem 1.</u> Let $H_{n+2} = H_{n+1} + H_n$, $n = 0, 1, \cdots$, with $H_0 = \rho$ and $H_1 = \sigma$. Then for $n = 1, 2, \cdots$, we have

$$(1.2)\qquad \sum_{k=0}^{n-1} k^m H_{k+r} = H_{n+r}\sum_{s=0}^{m}\left[\binom{m}{s}\sum_{i=0}^{m}(-1)^i (i!) F_{2i} G^i_{m-s}\right] n^s$$

$$+ H_{n+r+1}\sum_{s=0}^{m}\left[\binom{m}{s}\sum_{i=0}^{m}(-1)^i (i!) F_{2i+1} G^i_{m-s}\right] n^s + C_2$$

$$(r, m = 0, 1, \cdots),$$

where

$$(1.3)\qquad C_2 = -H_r\sum_{i=0}^{m}(-1)^i (i!) F_{2i} G^i_m - H_{r+1}\sum_{i=0}^{m}(-1)^i (i!) F_{2i+1} G^i_m$$

$$(r, m = 0, 1, \cdots),$$

and G^i_m (see [2]) are Stirling numbers of the second kind with the properties that $G^i_0 = 0$ if $i \neq 0$, $G^i_i = 1$, $i = 0, 1, \cdots$, $G^0_i = 0$ if $i \neq 0$, and $G^i_s = 0$ if $i > s$.

Proof of Theorem 1. We assert that

$$(1.4)\qquad \sum_{k=0}^{n-1} k^m H_{k+r} = \sum_{i=0}^{m} (-1)^i H_{2i+n+r+1}\Delta^i (n^m) + C_2 \quad ,$$

We note that if $\Delta g(n) = \Delta h(n)$, then $g(n) = h(n) + C_2$. Thus, using the Δ operator on both sides of (1.4), we obtain

$$(1.5)\qquad n^m H_{n+r} = \sum_{i=0}^{m} (-1)^i H_{2i+n+r+2}\Delta^i (n+1)^m - \sum_{i=0}^{m} (-1)^i H_{2i+n+r+1}\Delta^i (n^m).$$

Since $(n+1)^m - n^m = \Delta(n^m)$, we have $\Delta^i (n+1)^m = \Delta^i(n^m) + \Delta^{i+1}(n^m)$. Thus, since $H_{n+2} = H_{n+1} + H_n$, (1.5) simplifies to

$$(1.6)\qquad n^m H_{n+r} = \sum_{j=0}^{m} (-1)^j H_{2j+n+r+2}\Delta^{j+1}(n^m) + \sum_{i=0}^{m} (-1)^i H_{2i+n+r}\Delta^i (n^m).$$

Let $j + 1 = i$ in the first sum of (1.6). Since $\Delta^{m+1}(n^m) = 0$, the right-hand side sums cancel, except the term for $i = 0$, which yields $n^m H_{n+r}$.

We proceed now to simplify (1.4). Since [2, p. 9]

$$(1.7)\qquad \Delta^i g(n) = (-1)^i \sum_{k=0}^{i} (-1)^k \binom{i}{k} g(n+k) \qquad (i = 0, 1, \cdots) \quad ,$$

we have for $g(n) = n^m$

$$(1.8)\qquad \begin{aligned}\Delta^i(n^m) &= (-1)^i \sum_{k=0}^{i} (-1)^k \binom{i}{k} (n+k)^m \\ &= (-1)^i \sum_{k=0}^{i} (-1)^k \binom{i}{k} \sum_{s=0}^{m} \binom{m}{s} k^{m-s} n^s \\ &= \sum_{s=0}^{m} \binom{m}{s} n^s (-1)^i \sum_{k=0}^{i} (-1)^k \binom{i}{k} k^{m-s} \\ &= i! \sum_{s=0}^{m} \binom{m}{s} G^i_{m-s} n^s \quad ,\end{aligned}$$

since [2, p. 169, (3)]

$$(1.9)\qquad (-1)^i (i!) G^i_m = \sum_{k=0}^{i} (-1)^k \binom{i}{k} k^m \qquad (i = 0, 1, \cdots, m) \quad .$$

Buschman [3, p. 6, (12)] showed that

$$(1.10)\qquad H_{p+s} = F_s H_{p-1} + F_{s+1} H_p \quad ,$$

and thus from (1.10), with $s = 2i$ and $p = n + r + 1$, we obtain

$$(1.11)\qquad H_{2i+n+r+1} = F_{2i} H_{n+r} + F_{2i+1} H_{n+r+1} \quad .$$

Using (1.11), we obtain from (1.4)

$$(1.12)\qquad \sum_{k=0}^{n-1} k^m H_{k+r} = H_{n+r} \sum_{i=0}^{m} (-1)^i F_{2i} \Delta^i(n^m) + H_{n+r+1} \sum_{i=0}^{m} (-1)^i F_{2i+1} \Delta^i(n^m) + C_2 \quad .$$

If we substitute for $\Delta^i(n^m)$ in (1.12) by (1.8), we obtain, upon interchanging summations, (1.2). Add $n^m H_{n+r}$ to both sides of (1.2). Then, for $n = 0$, all terms in the sums are 0 except for $s = 0$, and so we obtain C_2 as given by (1.3).

If $p = 0$ and $q = 1$, then $H_n \equiv F_n$, and C_2 (1.3) yields C_1 in (1.1). For calculation purposes, (1.2) is more suitable than (1.1), since Stirling numbers are tabulated. Moreover, (1.2) and (1.3) are in the simplest form possible. Using the properties of F_n and G_n^i, we note that the coefficient of H_{n+r} in (1.2) is a polynomial in n of degree $m - 1$, while the coefficient of H_{n+r+1} is a polynomial in n of degree m.

The following result is a generalization of Theorem 1:

Theorem 2. Let

$$P(x) = \sum_{j=0}^{m} a_j x^j , \qquad a_m \neq 0 \quad ,$$

where a_j, $j = 0, 1, \cdots, m$, are constants. Then for $n = 1, 2, \cdots$, we have

$$(1.13) \quad \sum_{k=0}^{n-1} P(k) H_{k+r} = H_{n+r} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!) F_{2i} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^i \right\} \right] n^s$$

$$+ H_{n+r+1} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!) F_{2i+1} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^i \right\} \right] n^s + C_3$$

$$(r, m = 0, 1, \cdots),$$

where

$$(1.14) \qquad C_3 = -H_r \sum_{i=0}^{m} (-1)^i (i!) F_{2i} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\}$$

$$- H_{r+1} \sum_{i=0}^{m} (-1)^i (i!) F_{2i+1} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\}$$

$$(r, m = 0, 1, \cdots).$$

Comments. If $a_j = 0$, $j = 0, 1, \cdots, m - 1$, and $a_m = 1$, then (1.13) and (1.14) reduce to (1.2) and (1.3), respectively. A special case of (1.13) occurs when

$$P(k) \equiv k^{(m)} \equiv k(k-1)\cdots(k-m+1) = \sum_{j=1}^{m} S_m^j k^j ,$$

where (see [2, p. 142]) S_m^j are Stirling numbers of the first kind. Then, since $k^{(m)} = m! \binom{k}{m}$, we have

$$\sum_{k=0}^{n-1} k^{(m)} H_{k+r} = m! \sum_{k=m}^{n-1} \binom{k}{m} H_{k+r} \qquad (n = m+1, m+2, \cdots) .$$

Moreover, since $a_j = S_m^j$, $j = 0, 1, \cdots, m$, we have

$$\sum_{j=i}^{m} a_j G_j^i = \sum_{j=i}^{m} S_m^j G_j^i = \binom{0}{m-i} = \begin{bmatrix} 1 & \text{if } i = m \\ 0 & \text{if } i \neq m \end{bmatrix}$$

(see [2, p. 182, (1)]). Using (1.10), we obtain from (1.14)

$$(1.15) \qquad C_3 = (-1)^{m+1}(m!)(F_{2m}H_r + F_{2m+1}H_{r+1}) = (-1)^{m+1}(m!)H_{2m+r+1} .$$

It should be noted that C_3 in (1.14) was obtained from (1.13) for $n = 0$. For $P(k) \equiv k^{(m)}$, the same value of C_3 (1.15) is also obtained from (1.13) for $n = 0, 1, \cdots, m-1$ $(m \geq 1)$. Let $P(k) \equiv k^{(m)}$ in (1.13), where $a_j = S_m^j$, and let (1.13) be written as follows:

$$(1.16) \qquad \sum_{k=0}^{n} k^{(m)} H_{k+r} - n^{(m)} H_{n+r}$$

$$= L_1(m,n)H_{n+r} + L_2(m,n)H_{n+r+1} - (-1)^m (m!)H_{2m+r+1} .$$

We obtain from (1.16)

$$(1.17)\qquad (-1)^m(m!)H_{2m+r+1} = L_1(m,n)H_{n+r} + L_2(m,n)H_{n+r+1}$$
$$(n = 0,1,\cdots,m-1).$$

From (1.10) with $p = n + r + 1$ and $s = 2m - n$, we obtain

$$(1.18)\qquad H_{2m+r+1} = F_{2m-n}H_{n+r} + F_{2m+1-n}H_{n+r+1} \; .$$

If we substitute for H_{2m+r+1} in (1.17) by (1.18) and then equate coefficients of H_{n+r} and H_{n+r+1} in (1.17), we obtain the following identities:

$$(1.19)\qquad (-1)^m(m!)F_{2m-n} = \sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)F_{2i}\left\{\sum_{j=s+i}^{m}\binom{j}{s} S_m^j G_{j-s}^i\right\}\right]n^s$$
$$(n = 0,1,\cdots, m-1;\; m = 1,2,\cdots)\,,$$

$$(1.20)\qquad (-1)^m (m!)F_{2m+1-n} = \sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)F_{2i+1}\left\{\sum_{j=s+i}^{m}\binom{j}{s} S_m^j G_{j-s}^i\right\}\right]n^s$$
$$(n = 0,1,\cdots,m-1;\; m = 1,2,\cdots)\,.$$

By repeated additions, one obtains (interchanging summations in the final result)

$$(1.21)\qquad (-1)^m(m!)F_{2m+k-n} = \sum_{i=0}^{m}(-1)^i(i!)F_{2i+k}\sum_{s=0}^{m}\left\{\sum_{j=s+i}^{m}\binom{j}{s}S_m^j G_{j-s}^i\right\}n^s$$
$$(k = 0,1,\cdots;\; n = 0,1,\cdots,m-1;\; m = 1,2,\cdots).$$

<u>Proof of Theorem 2.</u> Noting that $\Delta^{m+1}P(n) = 0$, we find, imitating the proof of Theorem 1, that

$$\sum_{k=0}^{n-1} P(k)H_{k+r} = \sum_{i=0}^{m}(-1)^i H_{2i+n+r+1}\Delta^i P(n) + C_3$$

$$= H_{n+r}\sum_{i=0}^{m}(-1)^i F_{2i}\Delta^i P(n) + H_{n+r+1}\sum_{i=0}^{m}(-1)^i F_{2i+1}\Delta^i P(n) + C_3$$

Since

$$P(n) = \sum_{j=0}^{m} a_j n^j, \quad \Delta^i P(n) = \sum_{j=0}^{m} a_j \Delta^i(n^j)$$

and using (1.8), we have

$$\sum_{i=0}^{m}(-1)^i F_{2i}\Delta^i P(n) = \sum_{i=0}^{m}(-1)^i F_{2i}\sum_{j=0}^{m} a_j \Delta^i(n^j)$$

$$= \sum_{i=0}^{m}(-1)^i F_{2i}\sum_{j=0}^{m} a_j (i!)\sum_{s=0}^{j}\binom{j}{s} G_{j-s}^i n^s$$

$$= \sum_{i=0}^{m}(-1)^i (i!) F_{2i}\sum_{s=0}^{m}\left\{\sum_{j=s}^{m} a_j\binom{j}{s} G_{j-s}^i\right\} n^s$$

$$= \sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i (i!) F_{2i}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s} G_{j-s}^i\right\}\right] n^s ,$$

since

$$\sum_{j=0}^{m}\sum_{s=0}^{j} f(s,j) = \sum_{s=0}^{m}\sum_{j=s}^{m} f(s,j) \text{ and } G_{j-s}^i = 0 \text{ if } j - s < i .$$

The value of C_3 is obtained from (1.13) for $n = 0$.

2. REMARKS ON THE PAPER BY R. REICHMAN

The operator Δ, where $\Delta g(n) = g(n+1) - g(n)$, is referred to as the forward difference operator, while the operator ∇, where $\nabla g(n) = g(n) - g(n-1)$, is referred to as the backward difference operator. Indeed,

$$(2.1) \quad \nabla^i g(n) = \sum_{s=0}^{i}(-1)^s\binom{i}{s} g(n-s) = (-1)^i\sum_{k=0}^{i}(-1)^k\binom{i}{k} g(n-i+k) .$$

If we compare (2.1) and (1.7), we note that

$$\nabla^i g(n) \equiv \Delta^i g(n - i) \qquad (i = 0, 1, \cdots) ; \tag{2.2}$$

and if $g(n) \equiv n^m$, we have

$$\nabla^i (n^m) \equiv \Delta^i (n - i)^m \qquad (i = 0, 1, \cdots, m + 1) . \tag{2.3}$$

Reichman [4] gave the following results:

$$\sum_{k=0}^{n} k^m F_k = \sum_{i=0}^{m} (-1)^i F_{n+2+i} \nabla^i (n^m) + C_4 , \tag{2.4}$$

$$\sum_{k=0}^{n} k^m F_{2k} = \sum_{i=0}^{m} (-1)^i F_{2n+1-i} \nabla^i (n^m) + C_5 , \tag{2.5}$$

$$\sum_{k=0}^{n} k^m F_{2k-1} = \sum_{i=0}^{m} (-1)^i F_{2n-i} \nabla^i (n^m) + C_6 . \tag{2.6}$$

Rao [5] generalized (2.4) and gave

$$\sum_{k=0}^{n} k^m H_k = \sum_{i=0}^{m} (-1)^i H_{n+2+i} \nabla^i (n^m) + C_4^{\star} . \tag{2.7}$$

The following results contain (2.4), (2.5), (2.6) and (2.7) as special cases. The notation is consistent with Theorems 1 and 2.

Theorem 3. For $n = 0, 1, \cdots ; r = 0, \pm 1, \pm 2, \cdots,$ we have

$$(2.8)\quad \sum_{k=0}^{n} P(k)H_{2k+r}$$

$$= -H_{2n+r}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m}(-1)^i(i!)F_i\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ H_{2n+r+1}\sum_{s=0}^{m}(-1)^s\left[(-1)^s a_s + \sum_{i=1}^{m}(-1)^i(i!)F_{i-1}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ C_7 \qquad (m = 0, 1, \cdots),$$

where

$$(2.9)\quad C_7 = H_r\left[a_0 + \sum_{i=0}^{m}(-1)^i(i!)F_i\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}\right]$$

$$- H_{r+1}\left[a_0 + \sum_{i=1}^{m}(-1)^i(i!)F_{i-1}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}\right].$$

Proof of Theorem 3. Since $\nabla P(n+1) = P(n+1) - P(n)$, we have $\nabla^i P(n) = \nabla^i P(n+1) - \nabla^{i+1}P(n+1)$, and $\nabla^{m+1}P(n+1) = 0$. Thus, imitating the proof of Theorem 1, we find that

$$\sum_{k=0}^{n} P(k)H_{2k+r} = \sum_{i=0}^{m}(-1)^i H_{2n+r+1-i}\nabla^i P(n) + C_7$$

$$= H_{2n+r}\sum_{i=0}^{m}(-1)^i F_{-i}\nabla^i P(n) + H_{2n+r+1}\sum_{i=0}^{m}(-1)^i F_{1-i}\nabla^i P(n) + C_7,$$

since $H_{2n+r+1-i} = F_{-i}H_{2n+r} + F_{1-i}H_{2n+r+1}$, which is obtained from (1.10) where $S = -i$ and $p = 2n + r + 1$. Using (2.1) and (1.9), we obtain

$$\nabla^i n^j = (-1)^{j+i}(i!)\sum_{s=0}^{j}(-1)^s\binom{j}{s}G^i_{j-s}n^s$$

Since

$$\nabla^i P(n) = \sum_{j=0}^{m} a_j \nabla^i n^j ,$$

we have

$$\begin{aligned}\sum_{i=0}^{m}(-1)^i F_{-i}\nabla^i P(n) &= \sum_{i=0}^{m} i!\, F_{-i}\sum_{j=0}^{m}\sum_{s=0}^{j}(-1)^{s+j}a_j\binom{j}{s}G^i_{j-s}n^s \\ &= \sum_{i=0}^{m} i!\, F_{-i}\sum_{s=0}^{m}(-1)^s n^s\sum_{j=s}^{m}(-1)^j a_j\binom{j}{s}G^i_{j-s} \\ &= \sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\, F_{-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G^i_{j-s}\right\}\right]n^s .\end{aligned}$$

Additional simplifications are obtained by noting that $F_{-i} = (-1)^{i+1}F_i$ and $F_{-(i-1)} = (-1)^i F_{i-1}$. The value of C_7 is obtained from (2.8) for $n = 0$.

<u>Comments.</u> We note that (2.5) and (2.6) are special cases of (2.8). Suppose now

$$P(k) \equiv (-k)^{(m)} = \sum_{j=1}^{m}(-1)^j S^j_m k^j .$$

Since $(-k)^{(m)} = (-k)(-k-1)\cdots(-k-m+1) = (-1)^m k(k+1)\cdots(k+m-1)$, we have

$$\sum_{k=0}^{n}(-k)^{(m)}H_{2k+r} = (-1)^m(m!)\sum_{k=1}^{n}\binom{k+m-1}{m}H_{2k+r} ,$$

and

$$\sum_{j=i}^{m}(-1)^j a_j G_j^i = \sum_{j=i}^{m} S_m^j G_j^i = \begin{bmatrix} 1 & \text{if} & i = m \\ 0 & \text{if} & i \neq m \end{bmatrix} . \tag{2.10}$$

Thus, from (2.9), with $a_0 = 0$ and $a_j = (-1)^j S_m^j$, $j = 1, \cdots, m$, we obtain (using (1.10))

$$\begin{aligned} C_7 &= (-1)^m (m!)(F_m H_r - F_{m-1} H_{r+1}) \\ &= -(m!)(F_{-m} H_r + F_{1-m} H_{r+1}) = -(m!) H_{r+1-m} . \end{aligned} \tag{2.11}$$

The following result, derived via forward differences, is an alternate form of Theorem 3, which was derived via backward differences.

Theorem 4. For $n = 0, 1, \cdots; r = 0, \pm 1, \pm 2, \cdots$, we have

$$\begin{aligned} \sum_{k=0}^{n} P(k) H_{2k+r} &= H_{2n+r} \sum_{s=0}^{m} \left[\sum_{i=1}^{m} (-1)^i (i!) F_{i-2} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^i \right\} \right] n^s \\ &+ H_{2n+r+1} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!) F_{i-1} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^i \right\} \right] n^s + C_7 \\ &\qquad (m = 0, 1, \cdots) , \end{aligned} \tag{2.12}$$

where

$$C_7 = H_r \left[a_0 - \sum_{i=1}^{m} (-1)^i (i!) F_{i-2} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\} \right] - H_{r+1} \sum_{i=0}^{m} (-1)^i (i!) F_{i-1} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\} . \tag{2.13}$$

Comments. If we compare (2.8) with (2.12), we conclude that for arbitrary a_j, $j = 0, 1, \cdots, m$,

$$(2.14)\qquad (-1)^{s+1}\sum_{i=0}^{m}(-1)^i(i!)F_i\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^i\right\} = \sum_{i=1}^{m}(-1)^i(i!)F_{i-2}\left\{\sum_{j=s+i}^{m}a_j\binom{j}{s}G_{j-s}^i\right\}$$

$$(s = 0, 1, \cdots, m-1);$$

$$(2.15)\qquad (-1)^{s}\sum_{i=0}^{m}(-1)^i(i!)F_{i-1}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^i\right\} = \sum_{i=0}^{m}(-1)^i(i!)F_{i-1}\left\{\sum_{j=s+i}^{m}a_j\binom{j}{s}G_{j-s}^i\right\}$$

$$(s = 0, 1, \cdots, m).$$

For $a_j = S_m^j$, $j = 0, 1, \cdots, m$, (2.14) and (2.15) with $s = 0$, yield (noting (2.10)), respectively,

$$(2.16)\qquad (-1)^{m-1}(m!)F_{m-2} = \sum_{i=0}^{m}(-1)^i(i!)F_i\sum_{j=i}^{m}(-1)^j S_m^j G_j^i \qquad (m = 1, 2, \cdots);$$

$$(2.17)\qquad (-1)^{m}(m!)F_{m-1} = \sum_{i=0}^{m}(-1)^i(i!)F_{i-1}\sum_{j=i}^{m}(-1)^j S_m^j G_j^i \qquad (m = 0, 1, \cdots).$$

Addition of (2.16) and (2.17) gives

$$(2.18)\qquad (-1)^{m}(m!)F_{m-3} = \sum_{i=0}^{m}(-1)^i(i!)F_{i+1}\sum_{j=i}^{m}(-1)^j S_m^j G_j^i \qquad (m = 1, 2, \cdots).$$

Since $L_n = F_{n+1} + F_{n-1}$, addition of (2.17) and (2.18) gives

$$(2.19) \qquad (-1)^m (m!) L_{m-2} = \sum_{i=0}^{m} (-1)^i (i!) L_i \sum_{j=i}^{m} (-1)^j S_m^j G_j^i \qquad (m = 1, 2, \cdots) .$$

We note that (2.17) may be written as

$$(2.20) \qquad (m!) F_{m-1} [-1 + (-1)^m] = \sum_{i=0}^{m-1} (-1)^i (i!) F_{i-1} \sum_{j=i}^{m} (-1)^j S_m^j G_j^i$$

$$(m = 1, 2, \cdots) .$$

Thus, for $m = 2n$, $n = 1, 2, \cdots$, (2.20) gives

$$(2.21) \qquad \sum_{i=0}^{2n-1} (-1)^i (i!) F_{i-1} \sum_{j=i}^{2n} (-1)^j S_{2n}^j G_j^i = 0 \quad (n = 1, 2, \cdots) .$$

Since ([2, pp. 149, 171])

$$S_{2n}^{2n-1} = -\binom{2n}{2} = -G_{2n}^{2n-1} ,$$

(2.21) may be written as

$$(2.22) \qquad (2n)!\,(2n - 1) F_{2n-2} = \sum_{i=0}^{2n-2} (-1)^i (i!) F_{i-1} \sum_{j=i}^{2n} (-1)^j S_{2n}^j G_j^i$$

$$(n = 1, 2, \cdots) .$$

Suppose now

$$P(k) \equiv k^{(m)} = \sum_{j=1}^{m} S_m^j k^j$$

in (2.12). Noting (2.10), we obtain from (2.13)

$$(2.23) \qquad C_7 = (-1)^{m+1}(m!)(F_{m-2}H_r + F_{m-1}H_{r+1}) = (-1)^{m+1}(m!)H_{r+m-1} \quad .$$

If we rewrite (2.12) as

$$(2.24) \qquad \sum_{k=0}^{n} k^{(m)} H_{2k+r} = L_1^\star(m,n)H_{2n+r} + L_2^\star(m,n)H_{2n+r+1} + C_7 \quad ,$$

we obtain from (2.24)

$$(2.25) \qquad (-1)^m(m!)H_{r+m-1} = L_1^\star(m,n)H_{2n+r} + L_2^\star(m,n)H_{2n+r+1} \quad (n = 0, 1, \cdots, m-1) \; .$$

From (1.10) with $p = 2n + r + 1$ and $s = m - 2 - 2n$, we obtain

$$(2.26) \qquad H_{r+m-1} = F_{m-2-2n}H_{2n+r} + F_{m-1-2n}H_{2n+r+1} \quad .$$

If we substitute for H_{r+m-1} in (2.25) by (2.26) and then equate coefficients of H_{2n+r} and H_{2n+r+1} in (2.25), we obtain the following identities:

$$(2.27) \qquad (-1)^m(m!)F_{m-2-2n} = \sum_{s=0}^{m}\left[\sum_{i=1}^{m}(-1)^i(i!)F_{i-2}\left\{\sum_{j=s+i}^{m}\binom{j}{s} S_m^j G_{j-s}^i\right\}\right]n^s$$

$$(n = 0, 1, \cdots, m-1; \; m = 1, 2, \cdots) \; ,$$

$$(2.28) \qquad (-1)^m(m!)F_{m-1-2n} = \sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)F_{i-1}\left\{\sum_{j=s+i}^{m}\binom{j}{s} S_m^j G_{j-s}^i\right\}\right]n^s$$

$$(n = 0, 1, \cdots, m-1; \; m = 1, 2, \cdots) \; .$$

<u>Proof of Theorem 4.</u> It is readily verified that

$$(2.29)\quad \sum_{k=0}^{n-1} P(k)H_{2k+r} = \sum_{i=0}^{m} (-1)^i H_{2n+r-1+i}\, \Delta^i P(n) + C_7$$

$$= H_{2n+r}\left[-P(n) + \sum_{i=1}^{m} (-1)^i F_{i-2}\, \Delta^i P(n)\right]$$

$$+ H_{2n+r+1} \sum_{i=0}^{m} (-1)^i F_{i-1}\, \Delta^i P(n) + C_7 \quad ,$$

since $H_{2n+r-1+i} = F_{i-2}H_{2n+r} + F_{i-1}H_{2n+r+1}$, which is obtained from (1.10) where $s = i - 2$ and $p = 2n + r + 1$. The simplification of (2.29) to the form (2.12) proceeds in the same manner as in the proof of Theorem 2. The value of C_7 (2.13) is obtained from (2.12) for $n = 0$.

The following result, derived via backward differences, is an alternate form of Theorem 2, which was derived via forward differences. Since

$$(2.30)\quad \sum_{k=0}^{n} P(k)H_{k+r} = \sum_{i=0}^{m} (-1)^i H_{n+r+2+i}\, \nabla^i P(n) + C_3$$

$$= H_{n+r}\left[P(n) + \sum_{i=1}^{m} (-1)^i F_{i+1}\, \nabla^i P(n)\right]$$

$$+ H_{n+r+1} \sum_{i=0}^{m} (-1)^i F_{i+2}\, \nabla^i P(n) + C_3 \quad ,$$

we may now state

<u>Theorem 5.</u> For $m = 0, 1, \cdots; n = 1, 2, \cdots$,

$$(2.31)\quad \sum_{k=0}^{n-1} P(k)H_{k+r} = H_{n+r} \sum_{s=0}^{m} (-1)^s \left[\sum_{i=1}^{m} i!\, F_{i+1} \left\{ \sum_{j=s+i}^{m} (-1)^j a_j \binom{j}{s} G_{j-s}^{i} \right\}\right] n^s$$

$$+ H_{n+r+1} \sum_{s=0}^{m} (-1)^s \left[\sum_{i=0}^{m} i!\, F_{i+2} \left\{ \sum_{j=s+i}^{m} (-1)^j a_j \binom{j}{s} G_{j-s}^{i} \right\}\right] n^s$$

$$+ C_3 \quad ,$$

where

$$(2.32)\qquad C_3 = -H_r\sum_{i=1}^{m} i!\,F_{i+1}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\} - H_{r+1}\sum_{i=0}^{m} i!\,F_{i+2}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}$$

$$(r, m = 0, 1, \cdots) \quad .$$

Comments. If we compare (1.13) with (2.31), we conclude that for arbitrary a_j, $j = 0, 1, \cdots, m$,

$$(2.33)\qquad \sum_{i=1}^{m}(-1)^i(i!)F_{2i}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^i\right\} = (-1)^s\sum_{i=1}^{m} i!\,F_{i+1}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^i\right\}$$

$$(s = 0, 1, \cdots, m-1) \;;$$

$$(2.34)\qquad \sum_{i=0}^{m}(-1)^i(i!)F_{2i+1}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^i\right\} = (-1)^s\sum_{i=0}^{m} i!\,F_{i+2}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^i\right\}$$

$$(s = 0, 1, \cdots, m) \quad .$$

For $a_j = (-1)^j S_m^j$, $j = 0, 1, \cdots, m$, (2.33) and (2.34) with $s = 0$, yield (noting (2.10)), respectively

$$(2.35)\qquad m!\,F_{m+1} = \sum_{i=1}^{m}(-1)^i(i!)F_{2i}\sum_{j=i}^{m}(-1)^j S_m^j G_j^i$$

$$(m = 1, 2, \cdots) \;;$$

$$(2.36)\qquad m!\,F_{m+2} = \sum_{i=0}^{m} (-1)^i (i!) F_{2i+1} \sum_{j=i}^{m} (-1)^j S_m^j G_j^i \qquad (m = 0, 1, \cdots) .$$

Suppose now

$$P(k) \equiv (-k)^{(m)} = \sum_{j=1}^{m} (-1)^j S_m^j k^j$$

in (2.31). Then

$$\sum_{k=0}^{n-1} (-k)^{(m)} H_{k+r} = (-1)^m (m!) \sum_{k=1}^{n-1} \binom{k+m-1}{m} H_{k+r} ,$$

and from (2.32) we obtain

$$(2.37)\qquad C_3 = -(m!)(F_{m+1}H_r + F_{m+2}H_{r+1}) = -(m!)H_{m+r+2} .$$

We note that (2.4) and (2.7) are special cases of (2.30).

3. ADDITIONAL RESULTS

In terms of forward differences it is readily verified that

$$(3.1)\qquad \sum_{k=0}^{n-1} P(k)H_{3k+r} = \sum_{i=0}^{m} (-1)^i 2^{-i-1} H_{3n+r-1+2i}\, \Delta^i P(n) + C_8$$

$$= H_{3n+r} \sum_{i=0}^{m} (-1)^i 2^{-i-1} F_{2i-2}\, \Delta^i P(n)$$

$$+ H_{3n+r+1} \sum_{i=0}^{m} (-1)^i 2^{-i-1} F_{2i-1}\, \Delta^i P(n) + C_8 .$$

Moreover, in terms of backward differences, it is readily verified that

$$(3.2)\qquad \sum_{k=0}^{n} P(k)H_{3k+r} = \sum_{i=0}^{m} (-1)^i 2^{-i-1} H_{3n+r+2-i} \nabla^i P(n) + C_8$$

$$= H_{3n+r} \sum_{i=0}^{m} (-1)^i 2^{-i-1} F_{1-i} \nabla^i P(n)$$

$$+ H_{3n+r+1} \sum_{i=0}^{m} (-1)^i 2^{-i-1} F_{2-i} \nabla^i P(n) + C_8 .$$

The following result is a restatement of (3.1) and (3.2):

Theorem 6. For $n = 1, 2, \cdots;\ r = 0, \pm 1, \pm 2, \cdots,$ we have

$$(3.3)\qquad \sum_{k=0}^{n-1} P(k)H_{3k+r} = H_{3n+r} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!) 2^{-i-1} F_{2i-2} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^{i} \right\} \right] n^s$$

$$+ H_{3n+r+1} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!) 2^{-i-1} F_{2i-1} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^{i} \right\} \right] n^s$$

$$+ C_8 \qquad\qquad (m = 0, 1, \cdots) ,$$

where

$$(3.4)\qquad C_8 = -H_r \sum_{i=0}^{m} (-1)^i (i!) 2^{-i-1} F_{2i-2} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\}$$

$$- H_{r+1} \sum_{i=0}^{m} (-1)^i (i!) 2^{-i-1} F_{2i-1} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\} .$$

For $n = 0, 1, \cdots;\ r = 0, \pm 1, \pm 2, \cdots,$ we have

$$(3.5)\qquad \sum_{k=0}^{n} P(k)H_{3k+r} = H_{3n+r}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\,2^{-i-1}F_{1-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+\ H_{3n+r+1}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\,2^{-i-1}F_{2-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+\ C_8 \qquad (m = 0, 1, \cdots)\quad ,$$

where

$$(3.6)\qquad C_8 = H_r\left[a_0 - \sum_{i=0}^{m} i!\,2^{-i-1}F_{1-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^{i}\right\}\right]$$

$$-\ H_{r+1}\sum_{i=0}^{m} i!\,2^{-i-1}F_{2-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^{i}\right\} \quad .$$

<u>Comments</u>. Add $P(n)H_{3n+r}$ to both sides of (3.3). Then, comparing (3.3) and (3.5), we conclude that for arbitrary a_j, $j = 0, 1, \cdots, m$,

$$(3.7)\qquad a_s + \sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-2}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^{i}\right\}$$

$$= (-1)^s\sum_{i=0}^{m} i!\,2^{-i-1}F_{1-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}$$

$$(s = 0, 1, \cdots, m)\ ;$$

$$(3.8)\qquad \sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-1}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^{i}\right\}$$

$$= (-1)^s\sum_{i=0}^{m} i!\,2^{-i-1}F_{2-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\} \qquad (s = 0, 1, \cdots, m)\ .$$

For $a_j = S_m^j$, $j = 0, 1, \cdots, m$, (3.7) and (3.8) with $s = 0$, yield (noting (2.10)), respectively

$$(3.9) \qquad m!\,2^{-m-1}F_{2m-2} = \sum_{i=0}^{m} i!\,2^{-i-1}F_{1-i}\left\{\sum_{j=i}^{m}(-1)^j S_m^j G_j^i\right\} \quad (m = 1, 2, \cdots),$$

$$(3.10) \qquad m!\,2^{-m-1}F_{2m-1} = \sum_{i=0}^{m} i!\,2^{-i-1}F_{2-i}\left\{\sum_{j=i}^{m}(-1)^j S_m^j G_j^i\right\} \quad (m = 0, 1, \cdots),$$

which may be simplified by noting that $F_{1-i} = (-1)^i F_{i-1}$ and $F_{2-i} = (-1)^{i+1}F_{i-2}$.

If $a_j = (-1)^j S_m^j$, $j = 0, 1, \cdots, m$, (3.7) and (3.8) with $s = 0$ yield, respectively,

$$(3.11) \qquad (m!)2^{-m-1}F_{m-1} = \sum_{i=0}^{m}(-1)^i (i!)2^{-i-1}F_{2i-2}\left\{\sum_{j=i}^{m}(-1)^j S_m^j G_j^i\right\} \quad (m = 1, 2, \cdots),$$

$$(3.12) \qquad -(m!)2^{-m-1}F_{m-2} = \sum_{i=0}^{m}(-1)^i (i!)2^{-i-1}F_{2i-1}\left\{\sum_{j=i}^{m}(-1)^j S_m^j G_j^i\right\} \quad (m = 0, 1, \cdots).$$

By repeated additions, (3.9) and (3.10), as well as (3.11) and (3.12), give similar identities for Lucas numbers, L_n.

Suppose now

$$P(k) \equiv k^{(m)} = \sum_{j=1}^{m} S_m^j k^j$$

in (3.3). Then, from (3.4), we obtain

$$\begin{aligned} C_8 &= (-1)^{m+1}(m!)2^{-m-1}(F_{2m-2}H_r + F_{2m-1}H_{r+1}) \\ &= (-1)^{m+1}2^{-m-1}H_{2m+r-1}(m!) . \end{aligned}$$

From (1.10) with $p = 3n + r + 1$ and $s = 2m - 2 - 3n$, we obtain

$$H_{2m+r-1} = F_{2m-2-3n}H_{3n+r} + F_{2m-1-3n}H_{3n+r+1} .$$

If we substitute for C_8 in (3.3) and then equate coefficients of H_{3n+r} and H_{3n+r+1}, we obtain the following identities:

$$(3.13)\quad (-1)^m(m!)2^{-m-1}F_{2m-2-3n} = \sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-2}\left\{\sum_{j=s+i}^{m}\binom{j}{s}S_m^jG_{j-s}^i\right\}\right]n^s$$

$$(n = 0, 1, \cdots, m-1;\ m = 1, 2, \cdots) ,$$

$$(3.14)\quad (-1)^m(m!)2^{-m-1}F_{2m-1-3n} = \sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-1}\left\{\sum_{j=s+i}^{m}\binom{j}{s}S_m^jG_{j-s}^i\right\}\right]n^s$$

$$(n = 0, 1, \cdots, m-1; m = 1, 2, \cdots) .$$

Suppose now

$$P(k) \equiv (-k)^{(m)} = \sum_{j=1}^{m}(-1)^jS_m^jk^j$$

in (3.5). Then from (3.6), we obtain

$$C_8 = -(m!)2^{-m-1}(F_{1-m}H_r + F_{2-m}H_{r+1}) = -(m!)2^{-m-1}H_{r+2-m} .$$

4. GENERALIZATIONS

Let a, b, U_0 and U_1 be arbitrary real numbers, and consider the following three sequences:

$$(4.1) \qquad U_{n+2} = aU_{n+1} + bU_n , \quad ab = 1, \quad a \neq -1, \quad (n = 0, 1, \cdots) ,$$

$$(4.2) \qquad U_{n+2} = aU_{n+1} + U_n , \quad a \neq 0, \quad (n = 0, 1, \cdots) ,$$

$$(4.3) \qquad U_{n+2} = U_{n+1} + bU_n , \quad b = 0, \quad (n = 0, 1, \cdots) , .$$

We note that (4.1), (4.2), and (4.3) reduce to the Fibonacci sequence for the proper choices of a and b. We shall obtain summation formulas, using both forward and backward differences, for each of the three sequences, as defined by (4.1), (4.2), and (4.3), which yield the previous results, i.e., Theorems 2, 3, 4, 5, and 6, as special cases for the proper choices of a and b. We have already seen how certain procedures may be used to obtain various identities from our Theorems 2, ⋯, 6. In view of space limitations, no attempt will be made to use these procedures to fully exploit the general results obtained in this section. Identities given in the proofs of Theorems 2 and 3 will be used to obtain the explicit formulas cited in our general theorems, whose proofs are similar to that used for Theorem 2 (if forward differences are involved) or to that used for Theorem 3 (if backward differences are involved). We shall use repeatedly the following identity [3, p. 6, 12]

$$U_{p+s} = b\phi_s U_{p-1} + \phi_{s+1} U_p \tag{4.4}$$

where $\phi_0 = 0$, $\phi_1 = 1$, and $\phi_{n+2} = a\phi_{n+1} + b\phi_n$, $n = 0, 1, \cdots$. We note that (4.4) yields (1.10) for $a = b = 1$. All results in this section are valid for the parameter range, $r = 0, \pm 1, \pm 2, \cdots$. $P(k)$ (see Theorem 2) is defined as before. For negative subscripts, we define

$$U_{-n} = (U_0 V_n - U_n)/(-b)^n \qquad (n = 1, 2, \cdots) \quad , \tag{4.5}$$

where $V_0 = 2$, $V_1 = a$, and $V_{n+2} = aV_{n+1} + bV_n$, $n = 0, 1, \cdots$. We note that $\phi_{-n} = -\phi_n/(-b)^n$, $n = 1, 2, \cdots$.

(i) Let U_n satisfy (4.1). Since

$$\begin{aligned} \sum_{k=0}^{n-1} P(k) U_{3k+r} &= \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} U_{3n+r-1+2i}\, \Delta^i P(n) + C_8^* \\ &= bU_{3n+r} \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} \phi_{2i-2}\, \Delta^i P(n) \\ &\quad + U_{3n+r+1} \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} \phi_{2i-1}\, \Delta^i P(n) + C_8^* \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \sum_{k=0}^{n} P(k) U_{3k+r} &= \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} U_{3n+r+2-i}\, \nabla^i P(n) + C_8^* \\ &= bU_{3n+r} \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} \phi_{1-i}\, \nabla^i P(n) \\ &\quad + U_{3n+r+1} \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} \phi_{2-i}\, \nabla^i P(n) + C_8^* \;, \end{aligned} \tag{4.7}$$

We may now state

Theorem 7. Let U_n satisfy (4.1). For $n = 1, 2, \cdots$, and $m = 0, 1, \cdots$, we have

$$(4.8) \qquad \sum_{k=0}^{n-1} P(k) U_{3k+r}$$

$$= bU_{3n+r} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!)(a^2 + b)^{-i-1} \phi_{2i-2} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G^i_{j-s} \right\} \right] n^s$$

$$+ U_{3n+r+1} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!)(a^2 + b)^{-i-1} \phi_{2i-1} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G^i_{j-s} \right\} \right] n^s + C_8^\star ,$$

where

$$(4.9) \qquad C_8^\star = -bU_r \sum_{i=0}^{m} (-1)^i (i!)(a^2 + b)^{-i-1} \phi_{2i-2} \left\{ \sum_{j=i}^{m} a_j G^i_j \right\}$$

$$- U_{r+1} \sum_{i=0}^{m} (-1)^i (i!)(a^2 + b)^{-i-1} \phi_{2i-1} \left\{ \sum_{j=i}^{m} a_j G^i_j \right\} .$$

For $n = 0, 1, \cdots$, and $m = 0, 1, \cdots$, we have

$$(4.10) \qquad \sum_{k=0}^{n} P(k) U_{3k+r}$$

$$= bU_{3n+r} \sum_{s=0}^{m} (-1)^s \left[\sum_{i=0}^{m} i!\, (a^2 + b)^{-i-1} \phi_{1-i} \left\{ \sum_{j=s+i}^{m} (-1)^j a_j \binom{j}{s} G^i_{j-s} \right\} \right] n^s$$

$$+ U_{3n+r+1} \sum_{s=0}^{m} (-1)^s \left[\sum_{i=0}^{m} i!\, (a^2 + b)^{-i-1} \phi_{2-i} \left\{ \sum_{j=s+i}^{m} (-1)^j a_j \binom{j}{s} G^i_{j-s} \right\} \right] n^s + C_8^\star ,$$

where

$$(4.11)\qquad C_8^\star = U_r\left[a_0 - b\sum_{i=0}^{m} i!\,(a^2+b)^{-i-1}\phi_{1-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}\right]$$

$$- U_{r+1}\sum_{i=0}^{m} i!\,(a^2+b)^{-i-1}\phi_{2-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\} .$$

For $a = b = 1$, Theorem 7 yields Theorem 6.

(ii) Let U_n satisfy (4.2). Since

$$(4.12)\qquad \sum_{k=0}^{n-1} P(k)U_{2k+r} = \sum_{i=0}^{m}(-1)^i a^{-i-1} U_{2n+r-1+i}\,\Delta^i P(n) + C_7^\star$$

$$= U_{2n+r}\left[-P(n) + \sum_{i=1}^{m}(-1)^i a^{-i-1}\phi_{i-2}\,\Delta^i P(n)\right]$$

$$+ U_{2n+r+1}\sum_{i=0}^{m}(-1)^i a^{-i-1}\phi_{i-1}\,\Delta^i P(n) + C_7^\star$$

and

$$(4.13)\qquad \sum_{k=0}^{n} P(k)U_{2k+r} = \sum_{i=0}^{m}(-1)^i a^{-i-1} U_{2n+r+1-i}\,\nabla^i P(n) + C_7^\star$$

$$= U_{2n+r}\sum_{i=0}^{m}(-1)^i a^{-i-1}\phi_{-i}\,\nabla^i P(n)$$

$$+ U_{2n+r+1}\sum_{i=0}^{m}(-1)^i a^{-i-1}\,\phi_{1-i}\,\nabla^i P(n) + C_7^\star ,$$

we may now state

Theorem 8. Let U_n satisfy (4.2). For $n, m = 0, 1, \cdots,$ we have

$$(4.14)\qquad \sum_{k=0}^{n} P(k)U_{2k+r}$$

$$= U_{2n+r}\sum_{s=0}^{m}\left[\sum_{i=1}^{m}(-1)^i(i!)a^{-i-1}\phi_{i-2}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ U_{2n+r+1}\sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)a^{-i-1}\phi_{i-1}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s + C_7^\star,$$

where

$$(4.15)\qquad C_7^\star = U_r\left[a_0 - \sum_{i=1}^{m}(-1)^i(i!)a^{-i-1}\phi_{i-2}\sum_{j=i}^{m}a_jG_j^i\right]$$

$$- U_{r+1}\sum_{i=0}^{m}(-1)^i(i!)a^{-i-1}\phi_{i-1}\sum_{j=i}^{m}a_jG_j^i .$$

For $n, m = 0, 1, \cdots,$ we have

$$(4.16)\qquad \sum_{k=0}^{n} P(k)U_{2k+r}$$

$$= U_{2n+r}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\,a^{-i-1}\phi_{-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ U_{2n+r+1}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\,a^{-i-1}\phi_{1-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ C_7^\star ,$$

where

$$(4.17)\qquad C_7^\star = U_r\left[a_0 - \sum_{i=0}^{m} i!\, a^{-i-1}\phi_{-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}\right] - U_{r+1}\sum_{i=0}^{m} i!\, a^{-i-1}\phi_{1-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\} .$$

For $a = 1$, (4.14) and (4.15) yield Theorem 4; and (4.16) and (4.17) yield Theorem 3.

(iii) Let U_n satisfy (4.3). Since

$$(4.18)\qquad \sum_{k=0}^{n-1} P(k)U_{k+r} = \sum_{i=0}^{m}(-1)^i b^{-i-1} U_{n+r+1+2i}\, \Delta^i P(n) + C_3^\star$$

$$= bU_{n+r}\sum_{i=0}^{m}(-1)^i b^{-i-1}\phi_{2i}\, \Delta^i P(n)$$

$$+ U_{n+r+1}\sum_{i=0}^{m}(-1)^i b^{-i-1}\phi_{2i+1}\, \Delta^i P(n) + C_3^\star$$

and

$$(4.19)\qquad \sum_{k=0}^{n} P(k)U_{k+r} = \sum_{i=0}^{m}(-1)^i b^{-i-1} U_{n+r+2+i}\nabla^i P(n) + C_3^\star$$

$$= bU_{n+r}\left[b^{-1}P(n) + \sum_{i=1}^{m}(-1)^i b^{-i-1}\phi_{i+1}\nabla^i P(n)\right]$$

$$+ U_{n+r+1}\sum_{i=0}^{m}(-1)^i b^{-i-1}\phi_{i+2}\, \nabla^i P(n) + C_3^\star \quad ,$$

We may now state

Theorem 9. Let U_n satisfy (4.3). For $m = 0, 1, \cdots$; $n = 1, 2, \cdots$, we have

$$(4.20) \quad \sum_{k=0}^{n-1} P(k)U_{k+r}$$
$$= bU_{n+r} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!) b^{-i-1} \phi_{2i} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^{i} \right\} \right] n^s$$
$$+ U_{n+r+1} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!) b^{-i-1} \phi_{2i+1} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^{i} \right\} \right] n^s + C_3^\star ,$$

where

$$(4.21) \quad C_3^\star = -bU_r \sum_{i=0}^{m} (-1)^i (i!) b^{-i-1} \phi_{2i} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\}$$
$$- U_{r+1} \sum_{i=0}^{m} (-1)^i (i!) b^{-i-1} \phi_{2i+1} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\} .$$

For $m = 0, 1, \cdots$; $n = 1, 2, \cdots$, we have

$$(4.22) \quad \sum_{k=0}^{n-1} P(k)U_{k+r} = bU_{n+r} \sum_{s=0}^{m} (-1)^s \left[\sum_{i=1}^{m} i!\, b^{-i-1} \phi_{i+1} \left\{ \sum_{j=s+i}^{m} (-1)^j a_j \binom{j}{s} G_{j-s}^{i} \right\} \right] r$$
$$+ U_{n+r+1} \sum_{s=0}^{m} (-1)^s \left[\sum_{i=0}^{m} i!\, b^{-i-1} \phi_{i+2} \left\{ \sum_{j=s+i}^{m} (-1)^j a_j \binom{j}{s} G_{j-s}^{i} \right. \right.$$
$$+ C_3^\star \quad ,$$

where

$$(4.23)\qquad C_3^{\star} = -bU_r \sum_{i=1}^{m} i!\, b^{-i-1}\phi_{i+1}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}$$

$$- U_{r+1}\sum_{i=0}^{m} i!\, b^{-i-1}\phi_{i+2}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}.$$

For $b = 1$, (4.20) and (4.21) yield Theorem 2; and (4.22) and (4.23) yield Theorem 5.

5. APPLICATIONS FOR A SUMMATION FORMULA

Recently, the author [6] proved the following result:

Lemma 1. Let u_i, $i = 0, 1, \cdots, p-1$, be arbitrary real numbers, and let u_n, $n = 0, 1, \cdots$, satisfy a homogeneous, linear difference equation of order p with real, constant coefficients.

$$(5.1)\qquad b_0u_{n+p} + b_1u_{n+p-1} + \cdots + b_pu_n = 0 \qquad (b_0b_p \neq 0)\ .$$

Let x be a real number. Then

$$(5.2)\qquad -\left[\sum_{i=0}^{p} b_i x^i\right]\left[\sum_{k=0}^{n} u_k x^k\right] = \sum_{k=0}^{p-1}\left[\sum_{j=0}^{k} b_j u_{n+1+k-j}\right] x^{n+1+k}$$

$$- \sum_{k=0}^{p-1}\left[\sum_{j=0}^{k} b_j u_{k-j}\right] x^k \ ;$$

$$\sum_{k=0}^{\infty} u_k x^k = \frac{\sum_{k=0}^{p-1}\left[\sum_{j=0}^{k} b_j u_{k-j}\right] x^k}{\sum_{i=0}^{p} b_i x^i} \tag{5.3}$$

The series in (5.3) converges for $|x| < |\lambda|$, where λ is the root of $b_p x^p + \cdots + b_1 x + b_0 = 0$ with the smallest absolute value.

In [6], (5.2) was used to obtain a closed form for

$$\sum_{k=0}^{n} k^p x^k .$$

If x_0 is a value of x such that

$$\sum_{i=0}^{p} b_i x_0^i = 0 ,$$

then

$$\sum_{k=0}^{n} u_k x_0^k$$

is obtained from (5.2) by applying L'Hospital's rule.

As before, let

$$P(k) = \sum_{j=0}^{m} a_j k^j , \qquad a_m \neq 0 ,$$

and consider $u_k \equiv P(k) w_{qk+r}$, $k = 0, 1, \cdots$, where $q = 1, 2, \cdots$; $r = 0, \pm 1, \pm 2, \cdots$, and

(5.4) $$w_{n+2} + d_1 w_{n+1} + d_2 w_n = 0, \quad d_1 d_2 \neq 0, \quad d_1^2 - 4d_2 \neq 0, \quad (n = 0, 1, \cdots).$$

If α and β are the roots of $x^2 + d_1 x + d_2 = 0$, then $U_k \equiv w_{qk+r}$ satisfies

(5.5) $$U_{k+2} - V_q U_{k+1} + d_2^q U_k = 0 \quad (k = 0, 1, \cdots) ,$$

since $(x - \alpha^q)(x - \beta^q) = x^2 - V_q x + d_2^q$, where $V_n = \alpha^n + \beta^n$, $n = 0, 1, \cdots$, with $V_0 = 2$, $V_1 = -d_1$, satisfies (5.4). We note that $P(k) w_{qk+r}$ is a solution of a homogeneous, linear difference equation of order $2m + 2$ with real, constant coefficients whose characteristic equation is given by

(5.6) $$\left[(x - \alpha^q)(x - \beta^q)\right]^{m+1} \equiv (x^2 - V_q x + d_2^q)^{m+1} = 0 .$$

Since

$$(x^2 - V_q x + d_2^q)^{m+1} = \sum_{s=0}^{2m+2} b_{2m+2-s} x^s ,$$

we have that

$$(1 - V_q x + d_2^q x^2)^{m+1} = \sum_{j=0}^{2m+2} b_j x^j .$$

In [2, p. 30, example 3], it is shown that

(5.7) $$b_j = (-1)^j \sum_{i=0}^{m+1} \binom{m+1}{i} \binom{i}{j-i} V_q^{2i-j} d_2^{q(j-i)} \quad (j = 0, 1, \cdots, 2m+2).$$

Thus, (5.2), in which $p = 2m + 2$ and b_j defined by (5.7), yields a closed form for

$$\sum_{k=0}^{n} P(k) w_{qk+r} x^k .$$

If $w_k \equiv H_k$, then $d_1 = d_2 = -1$, $V_q \equiv L_q$, and (5.2) yields

$$-(1 - L_q x + (-1)^q x^2)^{m+1} \sum_{k=0}^{n} P(k) H_{qk+r} x^k \tag{5.8}$$
$$= \sum_{k=0}^{2m+1} \left[\sum_{j=0}^{k} b_j P(n + 1 + k - j) H_{q(n+1+k-j)+r} \right] x^{n+1+k}$$
$$- \sum_{k=0}^{2m+1} \left[\sum_{j=0}^{k} b_j P(k - j) H_{q(k-j)+r} \right] x^k \qquad (n = 0, 1, \cdots) ,$$

where (see (5.7))

$$b_j = (-1)^{j(q+1)} \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} \binom{i}{j-i} L_q^{2i-j} \qquad (j = 0, 1, \cdots, 2m + 2). \tag{5.9}$$

If $P(k) = k^{(m)} = m!\binom{k}{m}$ in (5.8), we conclude that for arbitrary x,

$$\sum_{k=0}^{2m+1} \left[\sum_{j=0}^{k} b_j \binom{n+1+k-j}{m} H_{q(n+1+k-j)+r} \right] x^{n+1+k} \tag{5.10}$$
$$= \sum_{k=0}^{2m+1} \left[\sum_{j=0}^{k} b_j \binom{k-j}{m} H_{q(k-j)+r} \right] x^k$$
$$(n = 0, 1, \cdots, m - 1;\ m = 1, 2, \cdots) .$$

If $n = 0$ in (5.10), the coefficient of x^{2m+2} must be 0, i.e.,

$$\sum_{j=0}^{2m+1} b_j \binom{2m+2-j}{m} H_{q(2m+2-j)+r} = 0 \qquad (m = 1, 2, \cdots) . \tag{5.11}$$

If $P(k) = (-k)^{(m)} = (-1)^m (m!)\binom{k+m-1}{m}$ in (5.8), we conclude that for arbitrary x and $n = 0$,

$$(5.12)\qquad \sum_{k=0}^{2m+1}\left[\sum_{j=0}^{k} b_j \binom{k-j+m}{m} H_{q(1+k-j)+r}\right] x^{1+k}$$

$$= \sum_{k=0}^{2m+1}\left[\sum_{j=0}^{k} b_j \binom{k-j+m-1}{m} H_{q(k-j)+r}\right] x^{k} \qquad (m = 1, 2, \cdots).$$

In (5.12), the coefficient of x^{2m+2} must be 0, i.e.,

$$(5.13)\qquad \sum_{j=0}^{2m+1} b_j \binom{3m+1-j}{m} H_{q(2m+2-j)+r} = 0 \qquad (m = 1, 2, \cdots).$$

If $P(k) \equiv 1$, then (5.8) yields a result which has already been proved by the author [7, p. 105, (5)], using a different procedure.

Noting that $w_m = \cos m\theta$ and $w_n = \sin n\theta$ satisfy $w_{n+2} - 2\cos\theta w_{n+1} + w_n = 0$, $n = 0, 1, \cdots$, with $V_n = 2\cos n\theta$, where $\theta \neq 0, \pi$, $0 < \theta < 2\pi$, we obtain from (5.2) the following two identities:

$$(5.14)\qquad -[1 - 2(\cos q\theta)x + x^2]^{m+1} \sum_{k=0}^{n} P(k) \begin{Bmatrix} \cos\,(qk+r)\theta \\ \sin\,(qk+r)\theta \end{Bmatrix} x^k$$

$$= \sum_{k=0}^{2m+1}\left[\sum_{j=0}^{k} b_j P(n+1+k-j) \begin{Bmatrix} \cos\,[q(n+1+k-j)+r]\theta \\ \sin\,[q(n+1+k-j)+r]\theta \end{Bmatrix}\right] x^{n+1+k}$$

$$- \sum_{k=0}^{2m+1}\left[\sum_{j=0}^{k} b_j P(k-j) \begin{Bmatrix} \cos\,[q(k-j)+r]\theta \\ \sin\,[q(k-j)+r]\theta \end{Bmatrix}\right] x^{k} \qquad (n = 0, 1, \cdots),$$

where (see (5.7))

$$(5.15)\qquad b_j = (-1)^j \sum_{i=0}^{m+1} \binom{m+1}{i} \binom{i}{j-i} (2\cos q\theta)^{2i-j} \qquad (j = 0, 1, \cdots, 2m+2).$$

The relative simplicity of our results, (5.14) and (5.15), may be compared with the less general (as well as less elegant) results obtained by Schwatt [8, pp. 217-219], who used the differential operator, $(xd/dx)^m$.

For choices of $P(k) \equiv k^{(m)}$ or $(-k)^{(m)}$, we obtain (in the same manner as (5.11) and (5.13)) the identities (pairwise)

$$(5.16)\qquad \sum_{j=0}^{2m+1} b_j \binom{2m+2-j}{m} \begin{Bmatrix} \cos\left[q(2m+2-j)+r\right]\theta \\ \sin\left[q(2m+2-j)+r\right]\theta \end{Bmatrix} = 0 \qquad (m = 1, 2, \cdots),$$

$$(5.17)\qquad \sum_{j=0}^{2m+1} b_j \binom{3m+1-j}{m} \begin{Bmatrix} \cos\left[q(2m+2-j)+r\right]\theta \\ \sin\left[q(2m+2-j)+r\right]\theta \end{Bmatrix} = 0 \qquad (m = 1, 2, \cdots).$$

Identities (5.16) and (5.17) may be transformed to hold for hyperbolic functions by recalling that $\cosh(i\theta) = \cos\theta$ and $\sinh(i\theta) = i\sin\theta$.

As an application of (5.3), we have

$$(5.18)\qquad (1 - V_q x + d_2^q x^2)^{m+1} \sum_{k=0}^{\infty} P(k) w_{qk+r} x^k = \sum_{k=0}^{2m+1} \left[\sum_{j=0}^{k} b_j P(k-j) w_{q(k-j)+r} \right] x^k \quad ,$$

where b_j is defined by (5.7).

It is desirable to have check formulas for the computed values of b_j. In our discussion, consider b_j, as given by (5.7), where

$$(5.19)\qquad (1 - V_q x + d_2^q x^2)^{m+1} = \sum_{j=0}^{2m+2} b_j x^j \qquad (m = 0, 1, \cdots).$$

We may set $x = \pm 1$ in (5.19). A substantial reduction in the effort required to evaluate all the b_j, $j = 0, 1, \cdots, 2m+2$, is afforded by noting that

(5.20) $$b_{2m+2-j} = d_2^{q(m+1-j)} b_j \qquad (j = 0, 1, \cdots, m+1) .$$

To prove (5.20), multiply both sides of (5.19) by $d_2^{q(m+1)}$, and so

(5.21) $$(d_2^q - d_2 V_q x + d_2^{2q} x^2)^{m+1} = \sum_{j=0}^{2m+2} b_j d_2^{q(m+1)} x^j .$$

Replacing x in (5.21) by x/d_2^q, we obtain (in reverse order)

(5.22) $$(x^2 - V_q x + d_2^q)^{m+1} = \sum_{j=0}^{2m+2} b_j d_2^{q(m+1-j)} x^j = \sum_{j=0}^{2m+2} b_{2m+2-j} x^j ;$$

and thus (5.20) is obtained by comparing the coefficients of x^j in the sums in (5.22).

Let $t = 1, 2, \cdots$, and let $g_{t+1}(x) = 0$ (where $g_{t+1}(x)$ is a polynomial in x of degree $t + 1$) be the characteristic equation determined by H^t_{qk+r}. Then the characteristic equation determined by $u_k \equiv P(k)H^t_{qk+r}$ is given by $[g_{t+1}(x)]^{m+1} = 0$. Since

$$[x^{t+1} g_{t+1}(1/x)]^{m+1} = \sum_{j=0}^{(t+1)(m+1)} b_j x^j ,$$

(5.2) may be applied to yield a closed form for

$$\sum_{k=0}^{n} P(k) H^t_{qk+r} x^k .$$

A formidable obstacle in this procedure is the complex nature of the b_j, which involve multiple summations.

As a simple example, consider H^2_{n+r}, where $H^2_{n+3+r} - 2H^2_{n+2+r} - 2H^2_{n+1+r} + H^2_{n+r} = 0$, and $g_3(x) \equiv x^3 - 2x^2 - 2x + 1$. Then $x^3g(1/x) \equiv 1 - 2x - 2x^2 + x^3$ and

$$(1 - 2x - 2x^2 + x^3)^{m+1} \equiv \sum_{i=0}^{3(m+1)} b_j x^j \quad .$$

Using the binomial theorem and then applying (5.7) (with the proper change of notation for the coefficients), we obtain

$$\begin{aligned}(1 - 2x - 2x^2 + x^3)^{m+1} &= \sum_{i=0}^{m+1} \binom{m+1}{i} (-2x)^i \left[1 + x - (x^2/2)\right]^i \\ &= \sum_{i=0}^{m+1} \binom{m+1}{i} (-2)^i \sum_{k=0}^{2i} c_k x^{k+i} = \sum_{j=0}^{3m+3} b_j x^j\end{aligned}$$

where

$$c_k = \sum_{s=0}^{i} \binom{i}{s} \binom{s}{k-s} (-1/2)^{k-s} \qquad (k = 0, 1, \cdots, 2i) \ ,$$

and

$$\begin{aligned}(5.23) \qquad b_j &= \sum_{i=0}^{m+1} \binom{m+1}{i} (-2)^i c_{j-i} \\ &= (-2)^{-j} \sum_{i=0}^{m+1} 2^{2i} \binom{m+1}{i} \sum_{s=0}^{i} (-2)^s \binom{i}{s} \binom{s}{j-i-s} \\ &\qquad (j = 0, 1, \cdots, 3m+3) \ .\end{aligned}$$

Thus, from (5.2) with $p = 3m + 3$ and $u_k = P(k)H^2_{k+r}$, we obtain (where b_j is defined by (5.23)),

$$(5.24)\qquad -(1 - 2x - 2x^2 + x^3)^{m+1}\sum_{k=0}^{n} P(k)H_{k+r}^2 x^k$$

$$= \sum_{k=0}^{3m+2}\left[\sum_{j=0}^{k} b_j P(n + 1 + k - j)H_{n+1+k-j+r}^2\right]x^{n+1+k} - \sum_{k=0}^{3m+2}\left[\sum_{j=0}^{k} b_j P(k - j)H_{k-j+r}^2\right]x^k.$$

Recalling the manner by which (5.11), (5.13), (5.16), and (5.17) were obtained, we may now state the following result:

Theorem 10. Let

$$(5.25)\qquad [x^{t+1}g_{t+1}(1/x)]^{m+1} = \sum_{j=0}^{(t+1)(m+1)} b_j x^j \qquad (m = 1, 2, \cdots) .$$

Then

$$(5.26)\qquad \sum_{j=0}^{(t+1)(m+1)-1} b_j \binom{(t + 1)(m + 1) - j}{m} H^t_{q(tm+t+m+1-j)+r} = 0$$

$$(q, t, m = 1, 2, \cdots; \; r = 0, \pm 1, \pm 2, \cdots) ;$$

$$(5.27)\qquad \sum_{j=0}^{(t+1)(m+1)-1} b_j \binom{(t + 1)(m + 1) - j - 1 + m}{m} H^t_{q(tm+t+m+1-j)+r} = 0$$

$$(q, t, m = 1, 2, \cdots; \; r = 0, \pm 1, \pm 2, \cdots) .$$

We note that (5.26) and (5.27) are identical for $m = 1$.

6. REMARKS ON THE PAPER BY LEDIN [9]

From our (2.31) with $r = 0$, $H_k = F_k$, and $P(k) = k^m$ (so that $a_m = 1$, $a_j = 0$, $j = 0, 1, \cdots, m-1$), we conclude (see 9, (3a), (3b) for notation) that

$$(6.1) \qquad M_{1,j} = \sum_{k=0}^{j} k!\, F_{k+1} G_j^k \qquad (j = 0, 1, \cdots),$$

$$(6.2) \qquad M_{2,j} = \sum_{k=0}^{j} k!\, F_{k+2} G_j^k \qquad (j = 0, 1, \cdots).$$

From [9, (6a)], we obtain for $i = 3$

$$(6.3) \qquad M_{3,j} = \sum_{k=0}^{j} k!\, F_{k+3} G_j^k - 0^j \qquad (j = 0, 1, \cdots).$$

Thus, the assertion [9, (6e)] is valid only for $i = 1$,) (with $j = 0, 1, \cdots$) and $i = 3$ ($j = 1, 2, \cdots$). Since $F_{k+i} = F_{i-1}F_{k+2} + F_{i-2}F_{k+1}$ (see (1.10)), we obtain from [9, (6b)], using (6.1) and (6.2) above, that

$$(6.4) \qquad M_{i,j} = \sum_{k=0}^{j} k!\, F_{k+i} G_j^k - \sum_{k=0}^{i-4} (k+1)^j F_{i-3-k} \qquad (j = 1, 2, \cdots).$$

Noting (6.1), (6.2), and (6.4), we are tempted to define

$$M_{0,j} = \sum_{k=0}^{j} k!\, F_k G_j^k \qquad (j = 0, 1, \cdots).$$

It should be noted that (6.1) and (6.2) are not uniquely defined. In the notation of [9, (8)], our (1.2) (with $r = 0$ and $H_k \equiv F_k$) can be written as

$$S(m, n-1) = F_n P_3(m,n) + F_{n-1} P_2(m,n) + C(m) \quad , \tag{6.5}$$

where (using 9, (2b), (3b))

$$C(m) = (-1)^{m+1} M_{2,m} \qquad (m = 0, 1, \cdots) \ . \tag{6.6}$$

Thus, from (1.2), we obtain

$$M_{3,j} = (-1)^j \sum_{k=0}^{j} (-1)^k (k!) F_{2k+2} G_j^k \qquad (j = 0, 1, \cdots) \quad , \tag{6.7}$$

$$M_{2,j} = (-1)^j \sum_{k=0}^{j} (-1)^k (k!) F_{2k+1} G_j^k \qquad (j = 0, 1, \cdots) \ . \tag{6.8}$$

Since $M_{3,j} = M_{2,j} + M_{1,j}$ for $j = 1, 2, \cdots$, we obtain from (6.7) and (6.8) that

$$M_{1,j} = (-1)^j \sum_{k=0}^{j} (-1)^k (k!) F_{2k} G_j^k \qquad (j = 1, 2, \cdots) \ . \tag{6.9}$$

Since $F_{2k+i-1} = F_{i-1}F_{2k+1} + F_{i-2}F_{2k}$ (see (1.10)), we obtain from [9, (6b)], using (6.8) and (6.9), that

$$M_{i,j} = (-1)^j \sum_{k=0}^{j} (-1)^k (k!) F_{2k+i-1} G_j^k - \sum_{k=0}^{i-4} (k+1)^j F_{i-3-k} \qquad (j = 1, 2, \cdots) \ . \tag{6.10}$$

From (6.4) and (6.10), we conclude that

$$(-1)^j \sum_{k=0}^{j} (-1)^k (k!) F_{2k+i-1} G_j^k = \sum_{k=0}^{j} k! F_{k+i} G_j^k \qquad (j = 1, 2, \cdots; \ i = 0, 1, \cdots). \tag{6.11}$$

It should be noted that [9, (7c)] was obtained from [9, (6a)], using [9, (7a)]. Since [9, (7c)] is a linear difference equation of second order in i, its solution is

$$(6.12)\quad P_i(m,n) = F_{i-1}P_2(m,n) + F_{i-2}P_1(m,n) - \sum_{k=0}^{i-3}(n-k)^m F_{i-1-k} \qquad (i = 3,4,\cdots)\ .$$

Using (6.12) and (1.10), [9, (8)] can be simplified to

$$(6.13)\quad S(m, n-h) = F_n P_1(m,n) + F_{n+1}P_2(m,n) + (-1)^{m+1}M_{2,m} - \sum_{k=0}^{h-2}(n-k)^m F_{n-k+1} - (n+1-h)^m F_{n+1-h} \quad (h = 2,3,\cdots)\ .$$

Since $P_3^\star(m,n) = (-1)^m P_3(m,-n)$ [9, (9)] can be simplified (using [9, (6a), (7c)]) to

$$(6.14)\quad \sum_{k=1}^{n}(n-k+1)^m F_k = M_{1,m}F_{n+1} + M_{2,m}F_{n+2} + n^m + (-1)^{m+1}(P_2(m,-n) + P_1(m,-n)) \quad (m = 1,2,\cdots)\ .$$

Since (see [9, (11)]) $P_i(m,n) = (-1)^m Q(m, -n+i-1)$, where $Q(m,n)$ are the Weinshenk polynomials in n of degree m (see reference [8] cited in [9]), it follows that

$$(6.15)\qquad Q(m,n) = (-1)^m P_1(m,n) = \sum_{j=0}^{m}\binom{m}{j} M_{1,j}n^{m-j}\ .$$

Thus (6.15), where $M_{1,k}$ is defined by (6.1), affords a closed form for the coefficients of $Q(m,n)$. From (6.12), with n replaced by -n, we obtain the following recursion relation for the Weinshenk polynomials:

$$(6.16)\qquad Q(m, n + i - 1) = F_{i-1}Q(m, n + 1) + F_{i-2}Q(m, n) - \sum_{k=0}^{i-3}(n + k)^m F_{i-1-k} \qquad (i = 3, 4, \cdots) .$$

In [9, (7a)] there is defined

$$(6.17)\qquad P_i(m, n) = \sum_{j=0}^{m}(-1)^j \binom{m}{j} M_{i,j}\, n^{m-j} \qquad (m = 0, 1, \cdots) .$$

If we apply the well-known inverse pair relations,

$$(6.18)\qquad A_m = \sum_{k=0}^{m}(-1)^k \binom{m}{k} B_k , \qquad B_m = \sum_{k=0}^{m}(-1)^k \binom{m}{k} A_k$$

to (6.17), we obtain as its inverse

$$(6.19)\qquad M_{i,m} = \sum_{j=0}^{m}(-1)^j \binom{m}{j} P_i(j, n)n^{m-j} \qquad (m = 0, 1, \cdots) .$$

Since $P_i(j, n) = (-1)^j Q(j, -n + i - 1)$, we obtain from (6.19)

$$(6.20)\qquad M_{i,m} = \sum_{j=0}^{m}\binom{m}{j} Q(j, -n + i - 1)n^{m-j} .$$

From (1.19), we obtain for $n = 0$, recalling (6.9),

$$(6.21)\qquad (-1)^m(m!)F_{2m} = \sum_{j=1}^{m}(-1)^j S_m^j M_{1,j} \qquad (m = 1, 2, \cdots) .$$

From (1.20), we obtain for $n = 0$, recalling (6.8),

$$(-1)^m (m!) F_{2m+1} = \sum_{j=0}^{m} (-1)^j S_m^j M_{2,j} \quad (m = 0, 1, \cdots) . \tag{6.22}$$

From (2.35), we obtain, recalling (6.9),

$$m!\, F_{m+1} = \sum_{j=1}^{m} S_m^j M_{1,j} \quad (m = 1, 2, \cdots) . \tag{6.23}$$

From (2.36), we obtain, recalling (6.8),

$$m!\, F_{m+2} = \sum_{j=0}^{m} S_m^j M_{2,j} \quad (m = 0, 1, \cdots) . \tag{6.24}$$

If we set $b = 2$ in (4.3), then $U_n = (-1)^n$ is a solution of (4.3). In (4.20), set $P(k) = k^m$ so that $a_m = 1$, $a_j = 0$, $j = 0, 1, \cdots, m - 1$. Thus, (4.20), with $b = 2$ and $r = 0$, gives a closed form for

$$\sum_{k=0}^{n-1} (-1)^k k^m .$$

ACKNOWLEDGEMENT

This paper, consisting of the first five sections, was submitted to this Quarterly on September 21, 1964, after the author reviewed the paper by Alfred [1]. Section 6 of this paper was written, in essence, after October, 1966, after having read the paper by Ledin [9]. I wish to thank Dr. Hoggatt for the opportunity of reading the papers [1] and [9] before their publication. As a result, my present paper is more complete.

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* * * * *

THE Q MATRIX AS A COUNTEREXAMPLE IN GROUP THEORY

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If g is an element of a group G, then $o(g)$, the order of g, is defined to be the number of distinct elements of G in the set $\{e, g^{\pm 1}, g^{\pm 2}, \cdots\}$, where e is the identity of G. This is equivalent to defining $o(g)$ to be the number of elements in the cyclic subgroup of G generated by g. It is an easy consequence that the order of g equals the least positive integer n such that $g^n = e$. If no such integer exists, g is said to be of infinite order.

In an abelian group H (i.e., $ab = ba$ for all $a, b \in H$) it is easy to show that the product of two elements of finite order must again be of finite order. Indeed, if $o(a) = m$, $o(b) = n$ for some $a, b \in H$, then $(ab)^{mn} = (a^m)^n (b^n)^m = e^n e^m = e$, so $o(ab) \leq mn$. However, this does not necessarily hold in general, as shown in the following counterexample involving the Q matrix.

Let G be the multiplicative group of all nonsingular 2x2 matrices, and let

$$R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

be elements of G. One can check that $R^2 = S^3 = I$, the identity matrix, so that R and S are of finite order. But

$$RS = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = Q ,$$

the Q matrix. Now Basin and Hoggatt [1] have shown that

$$(RS)^n = Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \neq I$$

for any $n > 0$. Thus RS has infinite order.

(See page 80 for reference.)

ON A CERTAIN KIND OF FIBONACCI SUMS*

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INTRODUCTION

The sum

$$S(m,n) = \sum_{k=1}^{n} k^m F_k$$

(where F_k is the k^{th} Fibonacci number) has been studied for particular values of m. The cases $m = 0$ and $m = 1$ are well known [1, 2]. The case $m = 3$ was proposed as a problem [3] by Brother U. Alfred of St. Mary's College, California; this problem was later solved [4] by means of translational operator techniques and linear recurrence relations [5]. This method of solution [4] can be generalized for arbitrary positive integral values of m, but it usually will involve the time-consuming, error-inviting procedure of solving $2m + 2$ simultaneous equations in $2m + 2$ variables, which is already a complicated task for $m = 3$.

The method outlined in this paper is much more elementary, and the work required in finding a particular sum is reduced to several simple integrations. The procedure discussed below not only facilitates the computation of these sums, but it is also a useful tool in the solution of other problems, such as the problem of Fibonacci "centroids" proposed by the author [6], certain aspects of Fibonacci convolutions, and the like.

THEORY

Consider the sum

$$(1) \qquad \sum_{k=1}^{n} k^m F_k = S(m,n) = F_{n+1}P_2(m,n) + F_nP_1(m,n) + C(m)$$

*This paper was originally presented at the Fibonacci Association Meeting of 21 May 1966.

where F_k denotes the k^{th} Fibonacci number ($F_0 = 0$, $F_1 = 1$, $F_{k+2} = F_{k+1} + F_k$), $P_1(m,n)$ and $P_2(m,n)$ are polynomials in n of degree m, and $C(m)$ is a constant depending only on the degree m.

Thus we can write

$$P_1(m,n) = a_m n^m + a_{m-1} n^{m-1} + \cdots + a_1 n + a_0 \tag{2a}$$

$$P_2(m,n) = b_m n^m + b_{m-1} n^{m-1} + \cdots + b_1 n + b_0 \tag{2b}$$

Theorem 1.

$$C(m) = -b_0$$

Proof.

Take $S(m,0) = F_1P_2(m,0) + F_0P_1(m,0) + C(m)$ from (1))

$0 = P_2(m,0) + C(m)$ but $P_2(m,0) = b_0$ from (2b)) .

Inspection of the first few values of m (see Table I) leads us to the following determination of the polynomials (2a) and (2b).

$$P_1(m,n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} M_{1,j} n^{m-j} \tag{3a}$$

$$P_2(m,n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} M_{2,j} n^{m-j} \tag{3b}$$

where $\binom{m}{j}$ are the binomial coefficients, and $M_{1,j}$ and $M_{2,j}$ are certain numbers, the law of formation of which is yet to be determined (refer to Table II).

Theorem 2.

$$P_1(m+1,n) = (m+1)\int_0^n P_1(m,n)dx + a_0' \tag{4a}$$

$$P_2(m+1,n) = (m+1)\int_0^n P_2(m,x)dx + b_0' \tag{4b}$$

Table I

LIST OF FIBONACCI SUMS OF THE TYPE

$$S(m,n) = \sum_{k=1}^{n} k^m F_k = F_{n+1}P_2(m,n) + F_nP_1(m,n) + C(m)$$

m = 0 $S(0,n) = F_{n+1}(1) + F_n(1) - 1$

m = 1 $S(1,n) = F_{n+1}(n - 2) + F_n(n - 1) + 2$

m = 2 $S(2,n) = F_{n+1}(n^2 - 4n + 8) + F_n(n^2 - 2n + 5) - 8$

m = 3 $S(3,n) = F_{n+1}(n^3 - 6n^2 + 24n - 50) + F_n(n^3 - 3n^2 + 15n - 31) + 50$

m = 4 $S(4,n) = F_{n+1}(n^4 - 8n^3 + 48n^2 - 200n + 416) + F_n(n^4 - 4n^3 + 30n^2 - 124n + 257) - 416$

m = 5 $S(5,n) = F_{n+1}(n^5 - 10n^4 + 80n^3 - 500n^2 + 2080n - 4322) + F_n(n^5 - 5n^4 + 50n^3 - 310n^2 + 1285n - 2671) + 4322$

m = 6 $S(6,n) = F_{n+1}(n^6 - 12n^5 + 120n^4 - 1000n^3 + 6240n^2 - 25932n + 53888) + F_n(n^6 - 6n^5 + 75n^4 - 620n^3 + 3855n^2 - 16026n + 33305) - 53888$

m = 7 $S(7,n) = F_{n+1}(n^7 - 14n^6 + 168n^5 - 1750n^4 + 14560n^3 - 90762n^2 + 377216n - 783890) + F_n(n^7 - 7n^6 + 105n^5 - 1085n^4 + 8995n^3 - 56091n^2 + 233135n - 484471) + 783890$

m = 8 $S(8,n) = F_{n+1}(n^8 - 16n^7 + 224n^6 - 2800n^5 + 29120n^4 - 242032n^3 + 1508864n^2 - 6271120n + 13031936) + F_n(n^8 - 8n^7 + 140n^6 - 1736n^5 + 17990n^4 - 149576n^3 + 932540n^2 - 3875768n + 8054177) - 13031936$

m = 9 $S(9,n) = F_{n+1}(n^9 - 18n^8 + 288n^7 - 4200n^6 + 52416n^5 - 544572n^4 + 4526592n^3 - 28220040n^2 + 117287424n - 243733442) + F_n(n^9 - 9n^8 + 180n^7 - 2604n^6 + 32382n^5 - 336546n^4 + 2797620n^3 - 17440956n^2 + 72487593n - 150635551) + 243733442$

m = 10 $S(10,n) = F_{n+1}(n^{10} - 20n^9 + 360n^8 - 6000n^7 + 87360n^6 - 1089144n^5 + 11316480n^4 - 94066800n^3 + 586487120n^2 - 2437334420n + 5064892768) + F_n(n^{10} - 10n^9 + 225n^8 - 3720n^7 + 53970n^6 - 673092n^5 + 6994050n^4 - 58136520n^3 + 362437965n^2 - 1506355510n + 3130287705) - 5064892768$

Table II

LIST OF THE $M_{1,j}$ AND $M_{2,j}$ NUMBERS

j	$M_{1,j}$	$M_{2,j}$
0	1	1
1	1	2
2	5	8
3	31	50
4	257	416
5	2671	4322
6	33305	53888
7	484471	783890
8	8054177	13031936
9	150635551	243733442
10	3130287705	5064892768

(5a) $$a_0' = 1 - (m+1)\int_0^1 (P_1(m,x) + P_2(m,x))dx$$

(5b) $$b_0' = 1 - (m+1)\int_0^1 (P_1(m,x) + 2P_2(m,x))dx$$

Proof.

Prove (4a) first. Using (3a) we have

$$(m+1)\int_0^n P_1(m,x)dx = (m+1)\int_0^n \sum_{j=0}^{m} (-1)^j \binom{m}{j} M_{1,j}x^{m-j}dx =$$

$$= (m+1)\sum_{j=0}^{m} (-1)^j M_{1,j}\binom{m}{j}\int_0^n x^{m-j}dx =$$

$$= (m+1)\sum_{j=0}^{m} (-1)^j M_{1,j}\binom{m}{j}\frac{n^{m+1-j}}{m+1-j} = \sum_{j=0}^{m}(-1)^j M_{1,j}\binom{m+1}{j} n^{m+1-j}$$

$$= P_1(m+1,n) - a_0'$$

(a_0' is determined for $j = m + 1$, a value which is missing from the summation sign.) A similar proof establishes (4b).

Now,

$$a_0' = P_1(m + 1, 0) = P_1(m + 1, 1) - (m + 1)\int_0^1 P_1(m, x)dx$$

and

$$b_0' = P_2(m + 1, 0) = P_2(m + 1, 1) - (m + 1)\int_0^1 P_2(m,x)dx$$

and since $S(m + 1, 1) = 1 = P_2(m + 1, 1) + P_1(m + 1, 1) + C(m + 1)$ ($C(m + 1) = -b_0'$ by Theorem 1) then

$$1 = (m + 1)\int_0^1 P_1(m, x)dx + a_0' + (m + 1)\int_0^1 P_2(m, x)dx$$

and the value of a_0' follows. A similar manipulation yields the required value of b_0'.

Corollary 1

$$\frac{dP_1(m + 1, n)}{dn} = (m + 1)\, P_1(m, n); \frac{dP_2(m + 1, n)}{dn^r} = m(m + 1)P_2(m, n) \quad .$$

Corollary 2

$$\frac{d^r P_1(m, n)}{dn^r} = m(m - 1)\cdots(m - r + 1)P_1(m - r, n); \frac{d^r P_2(m, n)}{dn^r} = m(m - 1)\cdots$$
$$\cdots(m - r + 1)P_2(m - r, n) \quad .$$

Corollary 3

$$P_2(m, 1) = a_0 \qquad \text{(refer to (2a, 2b))}.$$

Example 1

Problem. Obtain the sum $\sum_{k=1}^{n} kF_k$.

Solution. We know

$$\sum_{k=1}^{n} F_k = F_{n+1} + F_n - 1 \quad (m = 0) .$$

So the polynomials are $P_1(0,n) = 1$, $P_2(0,n) = 1$. Now, applying Theorem 2,

$$P_1(1,n) = \int_0^n 1dx + a_0' = n + a_0' \quad \text{and} \quad P_2(1,n) = \int_0^n 1dx + b_0' = n + b_0'$$

$$a_0' = 1 - \int_0^1 (1 + 1)dx = 1 - 2 = -1 \quad \text{and} \quad b_0' = 1 - \int_0^1 (1 + 2)dx = 1 - 3 = -2$$

Thus, the required sum is equal to $F_{n+1}(n - 2) + F_n(n - 1) + 2$.

Example 2

Problem. Obtain the sum

$$\sum_{k=1}^{n} k^2 F_k .$$

Solution. From Example 1, we know

$$\sum_{k=1}^{n} kF_k = F_{n+1}(n - 2) + F_n(n - 1) + 2$$

So the polynomials are $P_1(1,n) = n - 1$, $P_2(1,n) = n - 2$. Now, applying Theorem 2

$$P_1(2,n) = 2\int_0^n (x - 1)dx + a_0' = n^2 - 2n + a_0' \quad \text{and} \quad P_2(2,n) = 2\int_0^n (x - 2)dx + b_0' = n^2 - 4n + b_0'$$

$$a_0' = 1 - 2\int_0^1 (x - 1 + x - 2)dx = 1 - 2\int_0^1 (2x - 3)dx = 1 - 2(1 - 3) = 1 + 4 = 5$$

$$b_0' = 1 - 2\int_0^1 (x - 1 + 2x - 4)dx = 1 - 2\int_0^1 (3x - 5)dx = 1 - (3 - 10) = 1 + 7 = 8$$

Thus, the required sum is equal to $F_{n+1}(n^2 - 4n + 8) + F_n(n^2 - 2n + 5) - 8.$

Theorem 3.

If u_k are the "generalized" Fibonacci numbers (i.e., numbers obeying the Fibonacci recurrence relation, but with different initial conditions) with the properties $u_{k+2} = u_{k+1} + u_k$, $u_0 = q$, $u_1 = p$, [7], then

$$\sum_{k=1}^{n} k^m u_k = u_{n+1} P_2(m, n) + u_n P_1(m, n) + K(m),$$

where P_2 and P_1 are polynomials defined as above (3a, 3b) and $K(m) = -(pb_0 + qa_0)$.

In Theorem 3 we have stated a simple and useful result. The proof of this theorem is trivial, since $u_k = pF_k + qF_{k-1}$ [7]. Two particular cases are most interesting. The Fibonacci case ($p = 1$, $q = 0$) has been discussed above; the Lucas case ($p = 1$, $q = 2$) is also quite simple (refer to Table III).

At this stage it seems clear that a study of the polynomials $P_1(m, n)$ and $P_2(m, n)$ and of the numbers $M_{1,j}$ and $M_{2,j}$ pose by themselves an interesting problem. The intuitive bounds

$$M_{1,j+1} \geq 2(j + 1)M_{1,j} \qquad M_{2,j+1} \geq 2(j + 1)M_{2,j} \qquad (j \geq 1)$$

hold for all cases shown on Table II and can be proven by total induction using the formulas developed for a_0' and b_0'. A very curious relationship exists between these numbers; this relationship, and the fact that these numbers are members of a whole class of numbers $M_{1,j}$ can be appreciated effectively in Table IV. Horizontal addition of two consecutive $M_{1,j}$ numbers is the basic

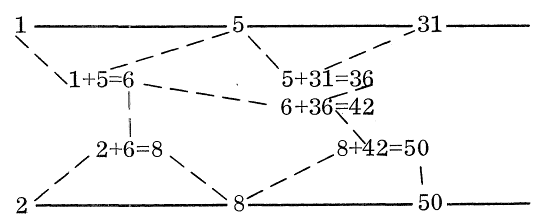
Table III

LIST OF LUCAS SUMS OF THE TYPE

$$T(m,n) = \sum_{k=1}^{n} k^m L_k = L_{n+1}P_2(m,n) + L_n P_1(m,n) + K(m)$$

$m = 0$	$T(0,n) = L_{n+1}(1) + L_n(1) - 3$
$m = 1$	$T(1,n) = L_{n+1}(n - 2) + L_n(n - 1) + 4$
$m = 2$	$T(2,n) = L_{n+1}(n^2 - 4n + 8) + L_n(n^2 - 2n + 5) - 18$
$m = 3$	$T(3,n) = L_{n+1}(n^3 - 6n^2 + 24n - 50) + L_n(n^3 - 3n^2 + 15n - 31) + 112$
$m = 4$	$T(4,n) = L_{n+1}(n^4 - 8n^3 + 48n^2 - 200n + 416) + L_n(n^4 - 4n^3 + 30n^2 - 124n + 257) - 930$
$m = 5$	$T(5,n) = L_{n+1}(n^5 - 10n^4 + 80n^3 - 500n^2 + 2080n - 4322) + L_n(n^5 - 5n^4 + 50n^3 - 310n^2 + 1285n - 2671) + 9664$

principle in the construction of Table IV; the results of successive horizontal additions can be followed with the aid of the broken lines. The following illustration should clarify the process:



These zig-zag relationships imply the second-order linear difference equation

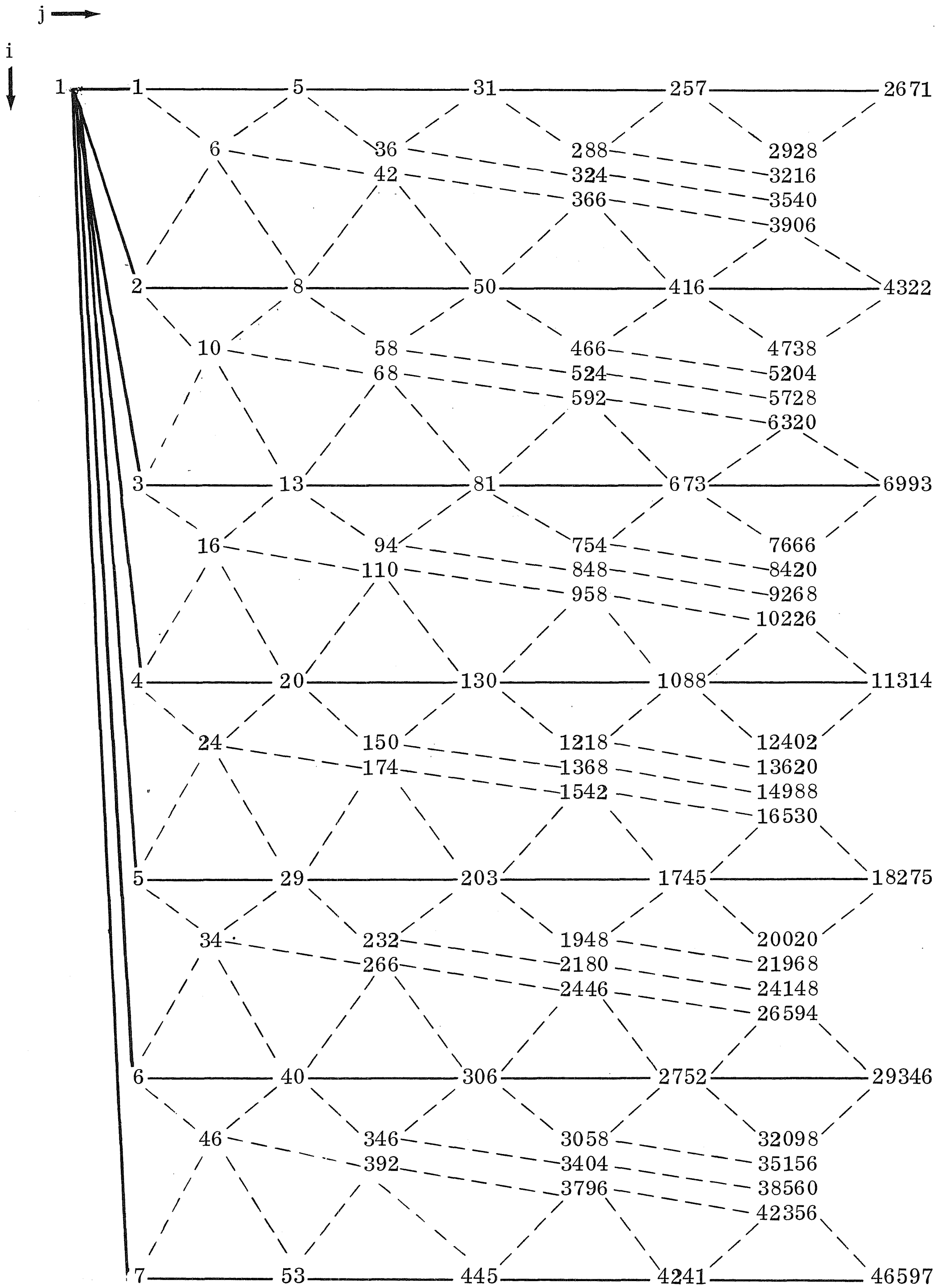
(6a) $$M_{i,j} = M_{i-1,j} + M_{i-2,j} - (i-3)^j$$

$$(i = 3, 4, 5, \cdots;\ j = 0, 1, 2, \cdots)$$

the solution of which is shown in Eq. (6b).

Table IV

INTERDEPENDENCE CHART FOR THE $M_{i,j}$ NUMBERS



$$M_{i,j} = F_{i-1}M_{2,j} + F_{i-2}M_{1,j} - \sum_{k=0}^{i-4} (k+1)^j F_{i-3-k} \tag{6b}$$

where F_i represents the i^{th} Fibonacci number.

The interdependence of the fundamental set of numbers $M_{1,j}$ and $M_{2,j}$ is noted from the formulas

$$M_{1,j} = \sum_{h=0}^{j} (-1)^h \binom{j}{h} M_{2,j-h} \quad \text{and} \quad M_{2,j} = \sum_{h=0}^{j} \binom{j}{h} M_{1,j-h} \tag{6c}$$

The interdependence of the complete set of numbers $M_{i,j}$ is evidenced with the formula[1]:

$$M_{i,j} = (i-1)^j + \sum_{h=0}^{j-1} (2^{j-h} - 1) \binom{j}{h} M_{i,h} \tag{6d}$$

with $j \geq 0$, $M_{i,0} = 1$, $M_{i,1} = i \geq 1$.

David Zeitlin, in a paper to be published in the Fibonacci Quarterly,[2] has shown that the following relationship holds:

$$M_{i,j} = \sum_{h=0}^{j} h!\, \mathcal{S}_j^h F_{h+i} \tag{6e}$$

where $\mathcal{S}_j^h$ are the Stirling numbers of the second kind.

The polynomials P_1 and P_2 are, similarly, special cases of a more general case of polynomials.

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[1]The author is indebted to Dr. Verner E. Hoggatt, Jr. for pointing out this relationship through personal correspondence.

[2]The author acknowledges the referee for this interesting remark.

(7a) $$P_i(m,n) = \sum_{j=0}^{m} (-1)^j M_{i,j} \binom{m}{j} n^{m-j}$$

which are interrelated in the following ways:

(7b) $$P_{i+h}(m,n) = P_i(m,n-h)$$

(7c) $$P_i(m,n) = P_{i-1}(m,n) + P_{i-2}(m,n) - (n+3-i)^m$$

$$(i = 3, 4, 5, \cdots)$$

These properties (7) enable us to obtain the following formula, thus generalize (1):

(8) $$S(m, n-h) = F_{n-h+1}P_{2+h}(m,n) + F_{n-h}P_{1+h}(m,n) + C(m)$$

We have investigated sums of the form

$$F_1 + 2^mF_2 + 3^mF_3 + \cdots + (n-1)^mF_{n-1} + n^mF_n$$

and it seems quite natural* that we apply our results to the "convolution type" sums of the form

$$n^mF_1 + (n-1)^mF_2 + (n-2)^mF_3 + \cdots + 2^mF_{n-1} + F_n \quad .$$

Theorem 4.

(9) $$\sum_{k=1}^{n} (n-k+1)^mF_k = R(m,n) = M_{3,m}F_{n+1} + M_{2,m}F_n - P_3^*(m,n)$$

*Mathematicians' beloved excuse.

where $M_{3,m}$ and $M_{2,m}$ are particular cases of the $M_{i,j}$ numbers (see Table IV) and $P_3^*(m,n)$ (the "conjugate" of the polynomial $P_3(m,n)$) is defined as follows

$$P_3^{\star}(m,n) = \sum_{j=0}^{m} M_{3,j} \binom{m}{j} n^{m-j} \tag{10}$$

A list of these "convolution-type" sums is provided in Table V.

Table V

LIST OF SUMS OF THE TYPE

$$\sum_{k=1}^{n} (n - k + 1)^m F_k = R(m,n) = M_{3,m} F_{n+1} + M_{2,m} F_n - P_3(m,n)$$

$m = 0$	$R(0,n) = F_{n+1} + F_n - 1$
$m = 1$	$R(1,n) = 3F_{n+1} + 2F_n - (n + 3)$
$m = 2$	$R(2,n) = 13F_{n+1} + 8F_n - (n^2 + 6n + 13)$
$m = 3$	$R(3,n) = 81F_{n+1} + 50F_n - (n^3 + 9n^2 + 39n + 81)$
$m = 4$	$R(4,n) = 673F_{n+1} + 416F_n - (n^4 + 12n^3 + 78n^2 + 324n + 673)$
$m = 5$	$R(5,n) = 6993F_{n+1} + 4322F_n - (n^5 + 15n^4 + 130n^3 + 810n^2 + 3365n + 6993)$

If $Q(m,n)$ are the Weinshenk polynomials in n of degree m [8], then

$$P_i^*(m,n) = Q(m, n + i - 1) \quad \text{and} \quad P_i(m,n) = (-1)^m Q(m, -n + i - 1) \tag{11}$$

The above relationships (11) follow from the fact that $P_i^*(m,n) = (-1)^m P_i(m,-n)$. The constant term is then $C(m) = P_1^*(m,1) = Q(m,1)$, and the original sum (1) can be further written as follows:

$$S(m,n) = (-1)^m \left\{ F_{n+1} Q(m, -n + 1) + F_n Q(m, -n) - Q(m, 1) \right\} \tag{12}$$

The theoretical interest that these sums arouse is beyond doubt the primary motive for their scrutiny. Weinshenk [8] has applied some of these results to a problem of reflection of light. The problem of centroids [6] can be dealt in a more general manner with the aid of an auxiliary function defined by

$$G(r, s, n) = \frac{\sum_{k=1}^{n} k^r F_k}{\sum_{k=1}^{n} k^s F_k} \tag{13}$$

In particular, $G(1, 0, n) = G_n$ has the following limiting behavior:

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \lim_{n\to\infty} (G_{n+1} - G_n) = 1 .$$

The problems investigated in this paper are far from being completely solved. Although we could have generalized the subscripts in all our sums [9], we purposely avoided this. However, some questions of importance have not been answered. Some of these questions are:

1. Could the theory of $S(m,n)$ be extended to negative m? (All we need to study is $m = -1$, since the rest of the sums can be obtained with the aid of the algorithms developed in this paper; notice that

$$P_i(-1, n) = \lim_{m\to 0} \frac{\partial^2 P_i(m, n)}{\partial n \partial m} \quad . \quad)$$

2. Could the theory of $S(m, n)$ be extended to rational (and to real) [10] m? If this is possible, what can be said about complex m?

3. What is the possibility of studying sums of the type

$$S(r, s, n) = \sum_{k=1}^{n} k^r F_k^s$$

with the aid of standard techniques?

ACKNOWLEDGEMENTS

The author wishes to express his gratitide to Dr. Verner E. Hoggatt, Jr. and the referee, who had very useful and constructive comments on several aspects of this paper.

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LATTICE POINT SOLUTION OF THE GENERALIZED PROBLEM OF TERQUEM AND AN EXTENSION OF FIBONACCI NUMBERS

C. A. CHURCH, Jr. and H. W. GOULD, W. Virginia University, Morgantown, W. Va.

In this paper we give a simple lattice point solution to a generalized permutation problem of Terquem and develop some elementary results for the extended Fibonacci numbers associated with the permutation problem.

The classical permutation problem of Terquem [12] has been stated by Riordan [10, p. 17, ex. 15] in the following manner. Consider combinations of n numbered things in natural (rising) order, with f(n,r) the number of r-combinations with odd elements in odd position and even elements in even positions, or, what is equivalent, with f(n,r) the number of combinations with an equal number of odd and even elements for r even and with the number of odd elements one greater than the number of even for r odd.

It is easy to show that $f(n,r) = f(n-1, r-1) + f(n-2, r)$, with $f(n, 0) = 1$, and explicitly

$$f(n, r) = \binom{\left[\frac{n+r}{2}\right]}{r}. \tag{1}$$

Moreover,

$$f(n) = \sum_{r=0}^{n} f(n, r) = f(n-1) + f(n-2) \tag{2}$$

so that f(n) is an ordinary Fibonacci number with $f(0) = 1$ and $f(1) = 2$.

A detailed discussion of Terquem's problem is given by Netto [8, pp. 84-87] and Thoralf Skolem [8, pp. 313-314] has appended notes on an extension of the problem in which the even and odd question is replaced by the more general question of what happens when one uses a modulus m to determine the position of an element in the permutation.

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More precisely, for a modulus $m \geq 2$, Skolem's generalization may be stated as follows. From among the first n natural numbers let $f(n,r;m)$ denote the number of combinations in natural order of r of these numbers such that the j^{th} element in the combination is congruent to j modulo m. That is,

$$(3)\qquad f(n,r;m) = N\left\{a_1 a_2 \cdots a_r :\ 1 \leq a_1 < a_2 < \cdots < a_r \leq n,\ \ a_j \equiv j \ (\mathrm{mod}\ m)\right\}$$

$$= \sum_{\substack{1 \leq a_1 < a_2 < \cdots < a_r \leq n \\ a_j \equiv j \ (\mathrm{mod}\ m)}} 1 \quad .$$

Consider the array in Fig. 1, where the last entry is $r + km$, with

$$k = \left[\frac{n - r}{m}\right]$$

since $r + km \leq n$ implies that the largest integral value of k cannot exceed $(n - r)/m$. This array contains those, and only those, elements from among $1, 2, \cdots, n$ which may appear in a combination. That is, the j^{th} column consists of all those elements $\leq n$ in the same congruence class (mod m) which may appear in the j^{th} position.

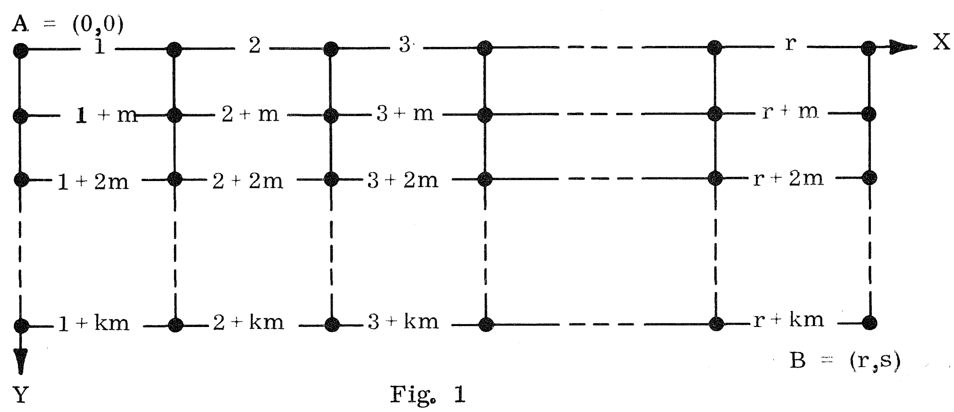


Fig. 1

From the lattice appended to the array in Fig. 1, we can systematically write out the desired combinations, and evaluate $f(n, r; m)$.

To get the desired result, let "a path from A to B" mean a path along the vertical and horizontal segments of the lattice, always moving downward or from left to right (we take the positive x-axis to the right, the positive y-axis downward, thus agreeing with the informal way of writing down the permutations). Each such path will generate a combination of the desired type, and conversely, as follows: Starting at A each horizontal step picks up an entry and vertical steps line up entries. Now, it is well known how many lattice paths there are from a = (0,0) to B = (r,s). MacMahon [7, Vol. I, p. 167] shows that this number is precisely

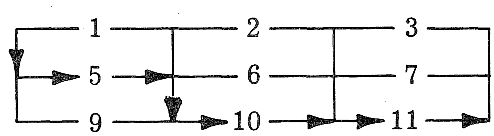
$$\binom{r+s}{r} .$$

In our case $s = [(n-r)/m]$. Thus we have at once that

$$f(n, r; m) = \binom{r + \left[\frac{n-r}{m}\right]}{r} = \binom{\left[\frac{n+(m-1)r}{m}\right]}{r} \tag{4}$$

as found by Skolem. Terquem's (1) follows when $m = 2$. To illustrate, we consider some examples.

Example 1. Let $n = 12$, $r = 3$, $m = 4$. Then the corresponding array is

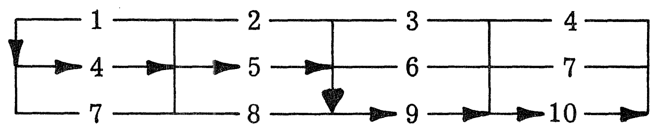


and the ten combinations are

1	2	3	1	6	7	5	6	7	9	10	11
1	2	7	1	6	11	5	6	11			
1	2	11	1	10	11	5	10	11			

and the particular combination 5, 10, 11 corresponds to the path indicated by arrows. Informally, one writes out the combinations by paths from the left column to the right column, moving horizontally and/or diagonally. The clue to a systematic count is found by superimposing the rectangular grid.

Example 2. Let $n = 12$, $r = 4$, $m = 3$. Then the corresponding array is



and the fifteen combinations are

1	2	3	4	1	2	9	10	4	5	6	7
1	2	3	7	1	5	6	7	4	5	6	10
1	2	3	10	1	5	6	10	4	5	9	10
1	2	6	7	1	5	9	10	4	8	9	10
1	2	6	10	1	8	9	10	7	8	9	10

and the combination 4, 5, 9, 10 corresponds to the path indicated by arrows.

It is felt that our proof shows atruism of mathematics: one may often find a simpler proof by embedding a given problem (Terquem's) in a more general setting. The lattice point enumeration we used is well known, but may not be apparent in the original problem because of its specialized form.

The extended Fibonacci numbers, in analogy to (2), are now defined by

$$f(n) = f_m(n) = \sum_{r=0}^{n} \binom{\left[\frac{n + (m-1)r}{m}\right]}{r} , \tag{5}$$

and it is not difficult to verify that they satisfy the recurrence relation

$$f_m(n) = f_m(n-1) + f_m(n-m) . \tag{6}$$

For example, with $m = 3$ we have the sequence $1, 2, 3, 4, 6, 9, 13, 19, 28, \cdots$. By well-known theorems in the theory of linear difference equations, if the distinct roots of the equation

$$t^m - t^{m-1} - 1 = 0 \tag{7}$$

are $t_1, t_2, \cdots t_m$, then there exist constants C_r such that

$$f_m(n) = \sum_{r=1}^{m} C_r t_r^n \quad . \tag{8}$$

This generalizes the familiar formulas

$$F_n = \frac{a^n - b^n}{a - b}, \qquad L_n = a^n + b^n \quad ,$$

for the Fibonacci-Lucas numbers. The constants C_r may be determined from the system of m linear equations in C_r:

$$\sum_{r=1}^{m} C_r t_r^j = j + 1, \qquad \text{for} \qquad j = 0, 1, 2, \cdots, m - 1 \quad . \tag{9}$$

For example, when $m = 3$, an approximate solution of the equation (7) is given by

$$\begin{cases} t_1 = 1.4655 \ , \\ t_2 = -0.23275 + 0.79255i \ , \\ t_3 = -0.23275 - 0.79255i \ , \end{cases} \tag{10}$$

where $i^2 = -1$. Relations (5) through (9) are given by Skolem [8, 313-314].

When $m = 3$ the exact solution of (7) is given by

$$\begin{aligned} t_1 &= A + B + \frac{1}{3} \ , \\ t_2 &= \frac{1}{3} - \frac{A + B}{2} + \frac{A - B}{2}\sqrt{-3} \ , \\ t_3 &= \frac{1}{3} - \frac{A + B}{2} - \frac{A - B}{2}\sqrt{-3} \ , \end{aligned} \tag{11}$$

where

$$A = \frac{1}{3}\sqrt{\frac{29 + \sqrt{837}}{2}} = 1.0237 \text{ approx.}$$

$$B = \frac{1}{3}\sqrt{\frac{29 - \sqrt{837}}{2}} = 0.10854 \text{ approx.}$$

As a partial check on the values of the roots, we note the following theorem from the theory of equations. Let

$$\prod_{j=1}^{m} (t - t_j) = t^m - t^{m-1} - z \quad . \tag{12}$$

Then

$$\sum_{j=1}^{m} t_j^n = \sum_{k=0}^{\left[\frac{n}{m}\right]} A_k(n, 1 - m)\, z^k \quad , \qquad n \geq 1 \ , \tag{13}$$

where

$$A_k(a, b) = \frac{a}{a + bk} \binom{a + bk}{k}.$$

This may be compared with the well-known [2, 3, 4, 5] expansion

$$x^a = \sum_{k=0}^{\infty} A_k(a, b) z^k \quad , \qquad \text{with } z = \frac{x - 1}{x^b} \quad , \tag{14}$$

which was actually found by Lagrange in his great memoir of 1770 (Vol. 24 of Proc. of the Berlin Academy of Sciences) and which leads at once to the general addition theorem discussed in [2, 3, 4, 5] as first noted by H. A. Rothe. See relation (20), this paper.

For the equation $t^m - t^{m-1} - z = 0$, we define the power sums of the roots t_j by

$$S(n) = \sum_{j=1}^{m} t_j^n \quad . \tag{15}$$

Since $t_j^{m-1} + z = t_j^m$, we find that

$$S(n-1) + zS(n-m) = \sum_{j=1}^{m} \left\{ t_j^{n-1} + zt_j^{n-m} \right\} = \sum_{j=1}^{m} t_j^{n-m} \left(t_j^{m-1} + z \right) = \sum_{j=1}^{m} t_j^{n-m} t_j^m ,$$

so that $S(n)$ itself also satisfies a Fibonacci-type recurrence

$$S(n) = S(n-1) + zS(n-m) . \tag{16}$$

Using the values $z = 1$, $m = 3$, the previous roots (10) yield the approximate values (by log tables): $S(1) = 1$, $S(2) = 0.9998$, $S(3) = 3.9995$, and $S(4) = 5$ very nearly. This gives a partial check on (10).

In any event, we may consider the sequence defined by (13), (15), (16) as a kind of extended Fibonacci sequence. In particular,

$$S(n) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{n}{n-(m-1)k} \binom{n-(m-1)k}{k} z^k , \qquad n \geq 1 , \tag{17}$$

satisfies (16) just as (5) satisfies (6). There are similarities and contrasts if we compare (17) and (5). We also call attention to another such result given recently by J. A. Raab [9], who found that the sequence defined by

$$x_n = \sum_{k=0}^{\left[\frac{n}{r+1}\right]} \binom{n-rk}{k} a^{n-k(r+1)} b^k \tag{18}$$

satisfies

$$x_n = ax_{n-1} + bx_{n-r-1} . \tag{19}$$

Formula (13) is substantially that given by Arthur Cayley [1]. The classical Lagrange inversion formula for series is inherent in all these formulas. One should also compare the Fibonacci-type relations here with the expansions given in [5]. For $m = 3$, (17) gives the sequence $1, 1, 4, 5, 6, 10, 15, 21, 31, \cdots$.

We also call attention to the two well-known special cases

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} 2^{n-2k} z^k = \frac{x^{n+1} - y^{n+1}}{x - y}$$

and

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} \frac{n}{n-k} 2^{n-2k-1} z^k = \frac{x^n + y^n}{x + y}$$

where $x = 1 + \sqrt{z+1}$, $y = 1 - \sqrt{z+1}$. F_n and L_n occur when $z = 4$.

Relations (17) and (5) differ because the initial conditions differ. For $z = 1$, (17) satisfies precisely the same recurrence as (5). If the initial values were the same then we would have found a formula for the permutation problem not unlike (17). There are many papers (too numerous to mention) in which complicated binomial sums are found by lattice point enumerations. The convolutions in [2, 3, 4, 5] may mostly be found by such counting methods. We also note the recent papers of Greenwood [6] and Stocks [11] wherein the Fibonacci numbers occur.

The convolution addition theorem [2, 3, 4, 5] of H. A. Rothe (1793)

$$\sum_{k=0}^{n} A_k(a, b) A_{n-k}(c, b) = A_n(a + c, b) , \tag{20}$$

valid for all real or complex a, b, c (being a polynomial identity in these), has been derived several times by lattice point methods. We mention only a novel derivation by Lyness [13]. Relation (20) has been rediscovered dozens of times since 1793, and its application in probability, graph theory, analysis, and the enumeration of flexagons, etc., shows that the theorem is very useful. In fact, it is a natural source of binomial identities. We should like to raise the question here as to whether any analogous relation involving the generalized Terquem coefficients (4) exists. It seems appropriate to study the generating function defined by

$$T(x;a,b) = \sum_{n=0}^{\infty} \binom{\left[\frac{a + (b-1)n}{b}\right]}{n} x^n \tag{21}$$

for as general a and b as possible. If b is a natural number and a is an integer ≥ 0, the series terminates with that term where $n = a$, as is evident from the fact that $a + (b-1)n < bn$ for $n > a$ and the fact that $\binom{k}{n} = 0$ for $k < n$ when $n \geq 0$, provided $k \geq 0$. We also note that for arbitrary complex a and $|x| < 1$

$$T(x;a,1) = \sum_{n=0}^{\infty} \binom{a}{n} x^n = (1+x)^a ,$$

so that in this case we do have an addition theorem:

$$T(x;a,1)T(x;c,1) = T(x;a+c,1) .$$

This, of course, corresponds to the case $b = 0$ in formula (20); the relation implies the familiar Vandermonde convolution or addition theorem.

There does not seem to be any especially simple closed sum for the series

$$C_n(a,c,b) = \sum_{k=0}^{n} \binom{\left[\frac{a + (b-1)k}{b}\right]}{k} \binom{\left[\frac{c + (b-1)(n-k)}{b}\right]}{n-k} \tag{22}$$

which occurs in

$$T(x;a,b)T(x,c,b) = \sum_{n=0}^{\infty} x^n C_n(a,c,b) \quad ,$$

for arbitrary b.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by V. E. HOGGATT, JR., San Jose State College, San Jose, Calif.

Send all communications concerning Advanced Problems and Solutions to Raymond Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within three months after publication of the problems.

NOTICE: PLEASE SEND ALL SOLUTIONS AND NEW PROPOSALS TO PROFESSOR RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA.

H-103 **Proposed by David Zeitlin, Minneapolis, Minnesota.**

Show that

$$8\sum_{k=0}^{n} F_{3k+1}F_{3k+2}F_{6k+3} = F_{3n+3}^4 .$$

H-104 **Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.**

Show

$$\frac{L_m X}{1 - 5F_m X + (-1)^{m+1}5X^2} = \sum_{k=0}^{\infty} 5^k (F_{2mk} + XL_{(2k+1)m})X^{2k} ,$$

where L_m and F_m are the m^{th} Lucas and Fibonacci numbers, respectively.

H-105 **Proposed by Edgar Karst, Norman, Oklahoma, and S. O. Rorem, Davenport, Iowa.**

Show for all positive integral n and primes $p > 2$ that

$$(n + 1)^p - n^p = 6N + 1 ,$$

where N is a positive integer. Generalize.

H-106 **Proposed by L. Carlitz, Duke University, Durham, N. Carolina.**

Show that

a) $$\sum_{k=0}^{n} \binom{n}{k}^2 L_{2k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} L_{n-k}$$

b) $$\sum_{k=0}^{n} \binom{n}{k}^2 F_{2k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} F_{n-k}$$

H-107 **Proposed by Vladimir Ivanoff, San Carlos, California.**

Show that

$$\begin{vmatrix} F_{p+2n} & F_{p+n} & F_p \\ F_{q+2n} & F_{q+n} & F_q \\ F_{r+2n} & F_{r+n} & F_r \end{vmatrix} = 0$$

for all integers p, q, r, and n.

H-108 **Proposed by H. E. Huntley, Hutton, Somerset, U.K.**

Find the sides of a tetrahedron, the faces of which are all scalene triangles similar to each other, and having sides of integral lengths.

H-109 **Proposed by George Ledin, Jr., San Francisco, California.**

Solve

$$X^2 + Y^2 + 1 = 3XY$$

for all integral solutions and consequently derive the identity:

$$F_{6k+7}^2 + F_{6k+5}^2 + 1 = 3F_{6k+7}F_{6k+5} .$$

H-110 **Proposed by George Ledin, Jr., San Francisco, California**

Evaluate the double sum

$$S_n = \sum_{m=1}^{n} \sum_{k=1}^{\infty} F_{\left[\frac{m}{k}\right]}$$

where [x] is the greatest integer in x.

H-111 Proposed by John L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania.

Show that

$$L_n = \prod_{k=1}^{\left[\frac{n}{2}\right]} \left\{1 + 4\cos^2 \frac{2k-1}{n}\left(\frac{\pi}{2}\right)\right\} \quad \text{for } n \geq 1 .$$

H-112 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that, for $n \geq 1$,

a) $L_{n+1}^5 - L_n^5 - L_{n-1}^5 = 5L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)$

b) $F_{n+1}^5 - F_n^5 - F_{n-1}^5 = 5F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)$

c) $L_{n+1}^7 - L_n^7 - L_{n-1}^7 = 7L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)^2$

d) $F_{n+1}^7 - F_n^7 - F_{n-1}^7 = 7F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)^2$.

SOLUTIONS

NO SOLUTIONS RECEIVED

H-59 Proposed by D. W. Robinson, Brigham Young University, Provo, Utah.

Show that, if $m > 2$, then the period of the Fibonacci sequence 0, 1, 1, 2, 3, ···, F_n, ··· reduced modulo m is twice the least positive integer n such that $F_{n+1} \equiv (-1)^n F_{n-1}$ (mod m).

H-60 Proposed by Verner E. Hoggatt, San Jose State College, San Jose, Calif.

It is well known that if p_k is the least integer such that $F_{n+pk} \equiv F_n$ mod 10^k, then $p_1 = 60$, $p_2 = 300$ and $p_k = 1.5 \times 10^k$ for $k \geq 3$. If Q(n, k) is the k^{th} digit of the n^{th} Fibonacci, then for fixed k, Q(n, k) is periodic, that is q_k is the least integer such that $Q(n + q_k, k) \equiv Q(n, k)$ mod 10. Find an explicit expression for q_k.

H-62 Proposed by H. W. Gould, W. Virginia University, Morgantown, West Virginia (corrected).

Find all polynomials f(x) and g(x), of the form

$$f(x+1) = \sum_{j=0}^{r} a_j x^j, \quad a_j \text{ an integer}$$

$$g(x) = \sum_{j=0}^{s} b_j x^j, \quad b_j \text{ an integer}$$

such that

$$2\left\{x^2 f^3(x+1) - (x+1)^2 g^3(x)\right\} + 3\left\{x^2 f^2(x+1) - (x+1)^2 g^2(x)\right\}$$
$$+ (2x+1)\left\{x f(x+1) - (x+1) g(x)\right\} = 0 .$$

LIMIT OF LIMITS

H-61 Proposed by P. F. Byrd, San Jose State College, San Jose, Calif. (corrected)

Let

$$f_{n,k} = 0 \quad \text{for } 0 \le n \le k-2, \quad f_{k-1,k} = 1 \quad \text{and}$$

$$f_{n,k} = \sum_{j=1}^{k} f_{n-j,k} \qquad \text{for } n \ge k .$$

Show that

$$\frac{1}{2} < \frac{f_{n,k}}{f_{n+1,k}} < \frac{1}{2} + \frac{1}{2k} \quad \text{as } n \to \infty .$$

Hence

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{f_{n,k}}{f_{n+1,k}} = \frac{1}{2} .$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

The sequence $\{f_{n,k}\}_{n=0}^{\infty}$ obeys a recurrence whose auxiliary polynomial is

$$f(x) = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 .$$

Let $r_{1,k}, r_{2,k}, \cdots, r_{k,k}$ be the k roots of $f(x) = 0$. The k initial conditions given determine constants $b_{1,k}, b_{2,k}, \cdots, b_{k,k}$ such that

$$f_{n,k} = \sum_{j=1}^{k} b_{j,k} r_{j,k}^n \quad .$$

Now Miles ["Generalized Fibonacci Numbers and Associated Matrices," Amer. Math. Monthly, Vol. 67, pp. 745-57] has shown that all but one of the roots $r_{j,k}$ lie within the unit circle, so that $|r_{j,k}| < 1$ $(1 \le j < k)$. Note that $f(1) = 1 - k < 0$, $f(2) = 1$, and since f is continuous, the remaining root $r_{k,k}$ must be a real number between 1 and 2. Then $b_{k,k} \ne 0$, because $\lim_{n\to\infty} r_{j,k}^n = 0$ $(1 \le j < k)$ while $\lim_{n\to\infty} f_{n,k} = \infty$. We also have

$$\lim_{n\to\infty} \frac{r_{j,k}^n}{r_{k,k}^n} = 0 \qquad (1 \le j < k) \quad ,$$

so that

$$\lim_{n\to\infty} f_{n,k}/f_{n+1,k} = \lim_{n\to\infty} \left(\sum_{j=1}^{k} b_{j,k} r_{j,k}^n\right) \Big/ \left(\sum_{j=1}^{k} b_{j,k} r_{j,k}^{n+1}\right) = 1/r_{k,k} \, .$$

We have already shown $r_{k,k} < 2$. Now

$$\begin{aligned} &(2k)^k - (2k)^{k-1}(k+1) - (2k)^{k-2}(k+1)^2 - \cdots - (2k)(k+1)^{k-1} - (k+1)^k \\ &< (2k)^k - (2k)^{k-1}k - (2k)^{k-2}k - \cdots - (2k)k^{k-1} - k^k - k^k \\ &= 2^k k^k - k^k(2^{k-1} + 2^{k-2} + \cdots + 2 + 1 + 1) = 0 \quad , \end{aligned}$$

and division by $(k+1)^k$ shows

$$f\left(\frac{2k}{k+1}\right) < 0 \quad .$$

Since

$$1 < \frac{2k}{k+1} < 2 \quad ,$$

we have $2 > r_{k,k} > 2k/(k+1)$, and inversion gives the first result of the problem. The second result follows by taking limits as $k \to \infty$.

ODD ROW SUMS OF FIBONOMIAL COEFFICIENTS

H-63 Proposed by Stephen Jerbic, San Jose State College, San Jose, California.

Let

$$F(m,0) = 1 \text{ and } F(m,n) = \frac{F_m F_{m-1} \cdots F_{m-n+1}}{F_n F_{n-1} \cdots F_1} \qquad 0 < n \le m \quad ,$$

be the Fibonomial coefficients, where F_n is the n^{th} Fibonacci number. Show

$$\sum_{n=0}^{2m-1} F(2m-1, n) = \prod_{i=0}^{m-1} L_{2i}, \quad m \geq 1 .$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

Put
$$S_n = \sum_{r=0}^{n} F(n, r)$$
and

(1)
$$f_n(x) = \sum_{r=0}^{n} (-1)^{r(r+1)/2} F(n, r) x^{n-r} .$$

Brennan ("Fibonacci Powers and Pascal's Triangle in a Matrix," Fibonacci Quarterly, Vol. 2, No. 2, pp. 93-103) has shown

$$f_n(x) = (-1)^{n-2}(x^2 - L_{n-1}x + (-1)^{n-1})f_{n-2}(-x)$$

where L_n is the n^{th} Lucas number. Setting $x = \sqrt{-1}$, $n = 2m + 1$, we find

$$f_{2m+1}(i) = iL_{2m} f_{2m-1}(-i) .$$

Using (1) this becomes

$$\sum_{r=0}^{m} F(2m+1, 2r) + i \sum_{r=0}^{m} F(2m+1, 2r+1)$$
$$= L_{2m} \sum_{r=0}^{m-1} F(2m-1, 2r+1) + iL_{2m} \sum_{r=0}^{m-1} F(2m-1, 2r) ,$$

and so equating real and imaginary parts, taking absolute values, and adding we get $S_{2m+1} = L_{2m} S_{2m-1}$ which, with $S_1 = 2 = L_0$, proves the proposition.

ONE OF MANY FORMS

H-64 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia.

Show
$$F_{n+1} = \prod_{j=1}^{n} \left(1 - 2i \cos \frac{j\Pi}{n+1}\right) .$$
where F_n is the n^{th} Fibonacci number.

Solution by David Zeitlin, Minneapolis, Minnesota.

For a generalization, let W_0, W_1, $C \neq 0$, and $d \neq 0$ be arbitrary real numbers, and define $W_{n+2} = dW_{n+1} - cW_n$, $n = 0, 1, \cdots$, with $d^2 - 4c \neq 0$. We define $V_n \equiv W_n$, $n = 0, 1, \cdots$, when $W_0 = 1$ and $W_1 = d$; and set Z_n

$= W_n$, $n = 0, 1, \cdots$, when $W_0 = 0$ and $W_1 = 1$. In terms of Chebyshev polynomials of the first kind, $T_n(x)$, and of the second kind $U_n(x)$, it is readily verified that

$$(1) \qquad Z_{n+1} \equiv c^{n/2} U_n\left(\frac{d}{2\sqrt{c}}\right) \quad ; \qquad V_n \equiv 2c^{n/2} T_n\left(\frac{d}{2\sqrt{c}}\right)$$

Since

$$U_n(x) = 2^n \prod_{i=1}^{n} \left(x - \cos \frac{j\Pi}{n+1}\right), \quad T_n(x) = 2^{n-1} \prod_{j=1}^{n} \left(x - \cos \frac{(2j-1)\Pi}{2n}\right),$$

we obtain from (1)

$$(2) \qquad Z_{n+1} = c^{n/2} \prod_{j=1}^{n} \left(\frac{d}{\sqrt{c}} - 2 \cos \frac{j\Pi}{n+1}\right),$$

$$(3) \qquad V_n = c^{n/2} \prod_{j=1}^{n} \left(\frac{d}{\sqrt{c}} - 2 \cos \frac{(2j-1)\Pi}{2n}\right)$$

If $d = 1$ and $c = -1$, then $Z_n = F_n$ and $V_n = L_n$. Since $-1 = i^2$, we obtain from (2) and (3), respectively,

$$(4) \qquad F_{n+1} = \prod_{j=1}^{n} \left(1 - 2i \cos \frac{j\Pi}{n+1}\right),$$

$$(5) \qquad L_n = \prod_{j=1}^{n} \left(1 - 2i \cos \frac{(2j-1)\Pi}{2n}\right).$$

Also solved by F. D. Parker, John L. Brown, Jr., and the proposer.

FIBONACCI RELATED NUMBER

H-65 Proposed by J. Wlodarski, Porz-Westhoven, Federal Republic of Germany.

The units digit of a positive integer, M, is 9. Take the 9 and put it on the left of the remaining digits of M forming a new integer, N, such that $N = 9M$. Find the smallest M for which this is possible.

Solution by Robert H. Anglin, Danville, Va., and Murray Berg, Oakland, Calif.

$$M = 9 + \sum_{i=1}^{n} x_i 10^i$$

$$N = \sum_{i=1}^{n} x_i 10^{i-1} + 9 \cdot 10^n = 9M$$

$$10N = 90M = \sum_{i=1}^{n} x_i 10^i + 90 \cdot 10^n = M - 9 + 90 \cdot 10^n$$

$$89M = 9(10^{n+1} - 1)$$

$$M = \frac{9(10^{n+1} - 1)}{89} = \frac{89999\cdots 991}{89}$$

By performing the actual division the first zero occurs when the quotient is

M = 1011 23595 50561 79775 28089 88764 04494 38202 24719

Wlodarski notes

$$M = 10^{43} \div \left[10^{41} \sum_{m=1}^{\infty} \frac{F_m}{10^m} \right]$$

where $[x]$ is the greatest integer function in x.

Also solved by Marjorie Bicknell, James Desmond, A. B. Western, Jr., C.B.A. Peck, and the proposer.

A STIRLING NUMBER SOLUTION

H-66 **Proposed by Douglas Lind, University of Virginia, Charlottesville, Va., and Raymond E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.**

Let

$$\sum_{j=0}^{k} a_j y_{n+j} = 0$$

be a linear homogeneous recurrence relation with constant coefficients a_j. Let the roots of the auxiliary polynomial

$$\sum_{j=0}^{k} a_j x^j = 0$$

be $r_1, r_2, \cdots, r_m$ and each root r_i be of multiplicity m_i $(i = 1, 2, \cdots, m)$.

Jeske (Linear Recurrence Relations — Part I, Fibonacci Quarterly, Vol. No. 2, pp. 69-74) showed that

$$\sum_{n=0}^{\infty} y_n \frac{t^n}{n!} = \sum_{i=1}^{m} e^{r_i t} \sum_{j=0}^{m_i - 1} b_{ij} t^j .$$

He also stated that from this we may obtain

(*) $$y_n = \sum_{i=1}^{m} r_i^n \sum_{j=0}^{m_i-1} b_{ij}n^j .$$

(i) Show that (*) is in general incorrect, (ii) state under what conditions it yields the correct result, and (iii) give the correct formulation.

Solution by the proposers.

Let $s_i = m_i - 1$, and put

$$Y(t) = \sum_{n=0}^{\infty} y_n \frac{t^n}{n!} = \sum_{i=1}^{m} e^{r_i t} \sum_{j=0}^{s_i} b_{ij}t^j .$$

Now define $n_{(s)} = n(n-1)\cdots(n-s+1)$, $n_{(0)} = 1$, and for $k = 1, 2, \cdots, m$ let

$$Y_k(t) = e^{r_k t} \sum_{j=0}^{s_k} b_{kj}t^j$$

so that

$$Y_k(t) = \sum_{v=0}^{\infty} \sum_{j=0}^{s_k} b_{kj} \frac{r_k^v t^{v+j}}{v!}$$

$$= \sum_{v=0}^{\infty} \sum_{j=0}^{s_k} b_{kj} r_k^v (v+j)_{(j)} \frac{t^{v+j}}{(v+j)!} .$$

For $p = 0, 1, \cdots, s_k$ put

$$Y_{k,p}(t) = \sum_{v=0}^{\infty} b_{kp} r_k^v (v+p)_{(p)} \frac{t^{v+p}}{(v+p)!}$$

Differentiating this n times and setting $t = 0$,

$$Y_{k,p}^{(n)}(0) = \sum_{v=0}^{\infty} b_{kp} r_k^v (v+p)_{(p)} \left. \frac{t^{v+p-n}}{(v+p-n)!} \right|_{t=0} = b_{kp} r_k^{n-p} n_{(p)} = r_k^n (b_{kp} n_{(p)} r_k^{-p}) .$$

Thus applying the inverse transform (3.3), we find

$$y_n = Y^{(n)}(0) = \sum_{k=1}^{m} \sum_{p=0}^{s_k} Y_{k,p}^{(n)}(0) = \sum_{i=1}^{m} r_i^n \sum_{j=0}^{m_i-1} b_{ij} n_{(j)} r_i^{-j}$$

which is the correct form.

(ii) If $m_i = 1$ $(i = 1, 2, \cdots, m)$, then since $n_{(0)} = n^0$, Jeske's form gives the correct result. Also, since $n_{(1)} = n^1$, his result will be correct if all roots of multiplicity two are one, and there are no roots of greater multiplicity. For higher multiplicities his form almost never gives the correct result.

(i) We need only take a recurrence whose auxiliary equation does not satisfy the conditions of (ii) to form a counterexample to (*).

Also solved by P. F. Byrd and D. Zeitlin.

Editorial Comment: The b_{ij} in the first displayed equation above are arbitrary constants. The b_{ij} in the second displayed equation are also arbitrary constants. In this sense Jeske is correct. However, most readers would probably incorrectly infer that after you have determined the specific constants for a given problem one can then use these in the second displayed equation which, of course, is not true in all cases. V. E. H.

AN INTERESTING ANGLE

H-67 Proposed by J. W. Gootherts, Sunnyvale, California.

Let $B = (B_0, B_1, \cdots, B_n)$ and $V = (F_m, F_{m+1}, \cdots, F_{m+n})$ be two vectors in Euclidian $n + 1$ space. The B_i's are binomial coefficients of degree n and the F_{m+i}'s are consecutive Fibonacci numbers starting at any integer m.

Find the limit of the angle between these vectors as n approaches infinity.

Solution by F. D. Parker, Sony at Buffalo, N.Y.

We start with the formula

$$\cos^2\theta = \frac{(B \cdot V)^2}{|B|^2|V|^2},$$

where $B \cdot V$ is the scalar product of B and V, and $|B|$, $|V|$ represent the magnitudes of B and V, respectively.

The following results are easy to verify by mathematical induction:

(1) $$B \cdot V = F_{m+2n}$$

(2) $$|B| = \sqrt{\binom{2n}{n}}$$

(3) $$|V| = \sqrt{F_{m+n}F_{m+n+1} - F_{m-1}F_m}$$

Thus

$$\frac{|B\cdot V|^2}{|B|^2|V|^2} = \frac{(F_{m+2n})^2}{\binom{2n}{n}(F_{m+n}F_{m+n+1} - F_{m-1}F_m)} .$$

But

$$\lim_{n\to\infty} \frac{(F_{m+2n})^2}{F_{m+n}F_{m+n+1} - F_{m-1}F_m} = 0 , \qquad \text{and}$$

$$\lim_{n\to\infty}\binom{2n}{n} = \infty, \quad \text{and hence} \quad \lim_{n\to\infty} \cos\theta = 0, \quad \text{and}$$

$$\lim_{n\to\infty} \theta = \pi/2 .$$

Also solved by the proposer.

MANY ROADS TO MORGANTOWN

H-68 Proposed by H. W. Gould, W. Virginia University, Morgantown, W. Va.

Prove that

$$\sum_{k=1}^{n} \frac{1}{F_k} \ge \frac{n^2}{F_{n+2} - 1} , \qquad n \ge 1$$

with equality only for $n = 1, 2$.

Solution by the proposer.

The well-known identity

$$\sum_{i=1}^{n} A_i \sum_{j=1}^{n} B_j = n \sum_{i=1}^{n} A_iB_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(A_i - A_j)(B_j - B_i)$$

yields the special case

$$\sum_{i=1}^{n} A_i \sum_{j=1}^{n} \frac{1}{A_j} = n^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{(A_i - A_j)^2}{A_iA_j} ,$$

whence it is evident that for positive A's we have the inequality

$$\sum_{i=1}^{n} A_i \sum_{j=1}^{n} \frac{1}{A_j} \ge n^2$$

with equality only when $A_i = A_j$ for all $1 \le i \le n$, $1 \le j \le n$. The application to the Fibonacci numbers F_n (with $F_{n+1} = F_n + F_{n-1}$ and $F_1 = 1$, $F_2 = 1$) is evident from the formula

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 ,$$

so that we find

$$\sum_{i=1}^{n} \frac{1}{F_i} \ge \frac{n^2}{F_{n+2} - 1} ,$$

with equality only for $n = 1, 2$.

Zeitlin and Desmond used the Arithmetic-Harmonic mean inequality. Brown used the Schwarz inequality.

Further results are:

$$\sum_{k=1}^{n} \frac{1}{H_k} \ge \frac{n^2}{H_{n+2} - H_2} , \quad n \ge 1 \qquad \text{(Zeitlin)}$$

$$\sum_{k=1}^{n} \frac{1}{F_k^2} \ge \frac{n^2}{F_n F_{n+1}} , \quad n \ge 1 \qquad \text{(Hoggatt)}$$

Also solved by D. Zeitlin, John L. Brown, Jr., M.N.S. Swamy, D. Lind, C.B.A. Peck, and John Wessner.

SOME BELATED SOLVERS' CREDITS

H-37 Dermott A. Breault

H-48 John L. Brown, Jr., and Charles R. Wall

H-52 C.B.A. Peck, F. D. Parker, and D. Lind

H-57 John L. Brown, Jr., Charles R. Wall, Marjorie Bicknell, F.D. Parker, and M.N.S. Swamy

H-58 David Klarner

H-74 John L. Brown, Jr.

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Continued from page 44.

REFERENCE

1. S. L. Basin and V. E. Hoggatt, Jr., "A Primer on the Fibonacci Sequence — Part II," Fibonacci Quarterly, Vol. 1 (1963), No. 2, 61-68.

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RELATIONS INVOLVING LATTICE PATHS AND CERTAIN SEQUENCES OF INTEGERS

DAVID R. STOCKS, JR., Arlington State College, Arlington, Texas

Relations involving certain special planar lattice paths and certain sequences of integers have been studied previously [1], [2]. We will state certain basic definitions which pertain to these studies, develop additional results involving other planar lattice paths, and finally, indicate generalizations of these results for lattice paths in k dimensional space. For convenience of reference some of the definitions are collected together and presented in Part 1. The remaining material will be found in Part 2.

Part 1

In Euclidean k-dimensional space the set X of points such that p belongs to X if and only if each coordinate of p is an integer is called the unit lattice of that space.

The statement that P is a lattice path in a certain space means that P is a sequence such that

1) each term of P is a member of the unit lattice of that space, and
2) if X is a term of P and Y is the next term of P and x_i and y_i are the i^{th} coordinates of X and Y respectively, then $|x_i - y_i| = 1$ or 0 and for some j, $|x_j - y_j| = 1$.

If each of X and Y is a point of the unit lattice in Euclidean k-dimensional space, then the statement that the lattice path P is a path from X to Y means that P is finite, X is the first term of P, and Y is the last term of P. If P is a lattice path, X is a term of P, and Y is the next term of P, then by the step $[X, Y]$ of P is meant the line interval whose end points are X and Y.

A lattice path P in Euclidean 2 or 3-space is said to be symmetric with respect to the line k if and only if it is true that if X is a point of some step of P, then either X is a point of k or there exists a point Y of some step of P such that k is the perpendicular bisector of the line interval $[X, Y]$.

Suppose that $S = [(x_1,y_1), (x_2,y_2)]$ is a step of some lattice path P in Euclidean 2-space. S is said to be x-increasing if $x_2 - x_1 = 1$ and x-decreasing if $x_2 - x_1 = -1$. The terms y-increasing and y-decreasing are similarly defined. A step is said to be <u>xy-increasing</u> if it is both x-increasing and y-increasing. To say that S is <u>x-increasing only</u> means that S is x-increasing but neither y-increasing nor y-decreasing. P is said to be <u>x-monotonically increasing</u> if and only if it is true that if Σ is a step of P, then Σ is not x-decreasing. The term y-monotonically increasing is similarly defined. A step Σ is said to be <u>vertical</u> if it is neither x-increasing nor x-decreasing. A step Σ is said to be <u>horizontal</u> if it is neither y-increasing nor y-decreasing. The statement that the path P is <u>duotonically increasing</u> means that P is both x-monotonically increasing and y-monotonically increasing.

Part 2

In Euclidean 2-space a path from (0,0) to (n,n) is said to have property G if and only if:

1) it is duotonically increasing,
2) it is symmetric with respect to the line $x + y = n$, and
3) no step of it which contains a point below the line $x + y = n$ is vertical.

A path having property G will be called a <u>G-path.</u>

<u>Theorem 1</u> (Greenwood)

Let $g(0) = 1$ and $g(1) = 1$. For each positive integer $n \geq 2$, let $g(n)$ denote the number of G-paths from (0,0) to $(n - 1, n - 1)$. The sequence $\{g(0), g(1), \cdots, g(n), \cdots\}$ is the Fibonacci sequence.

<u>Proof.</u> By definition $g(0) = g(1) = 1$. Suppose $n = 2$. The only G-paths from (0,0) to (1,1) are $\{(0,0), (1,0), (1,1)\}$ and $\{(0,0), (1,1)\}$, thus $g(2) = 2$. For $n = 3$, the G-paths from (0,0) to (2,2) are $\{(0,0), (1,0), (2,0), (2,1), (2,2)\}$, $\{(0,0), (1,0), (2,1), (2,2)\}$ and $\{(0,0), (1,1), (2,2)\}$, so that $g(3) = 3$.

Suppose $n \geq 4$. Each G-path from (0,0) to $(n - 1, n - 1)$ has as its initial step either $[(0,0), (1,0)]$ or $[(0,0), (1,1)]$. If a G-path has as its initial step $[(0,0), (1,0)]$, then, because of symmetry, its terminal step is $[(n - 1, n - 2), (n - 1, n - 1)]$; and thus it contains as a subsequence a G-path from (1,0) to $(n - 1, n - 2)$. But the number of G-paths from (1,0) to $(n - 1, n - 2)$ is the number of G-paths from (0,0) to $(n - 2, n - 2)$, i.e., $g(n - 1)$.

Likewise, if a G-path has as its initial step $[(0,0), (1,1)]$, then its terminal step is $[(n - 2, n - 2), (n - 1, n - 1)]$, and it contains as a subsequence a G-path from (1,1) to $(n - 2, n - 2)$. The number of such G-paths is the number of G-paths from (0,0) to $(n - 3, n - 3)$, which is $g(n - 2)$. Thus $g(n) = g(n - 1) + g(n - 2)$.

The statement that a path in Euclidean 2-space has property H means that it has property G and is such that one of its terms belongs to the line $x + y = n$. A path having property H will be called an <u>H-path</u>.

Obviously, if n is a positive integer, then the set of all H-paths from (0,0) to (n,n) is a proper subset of the set of all G-paths from (0,0) to (n,n); yet, using an argument similar to the above, we may establish the following.

<u>Theorem 2.</u>

Let $h(0) = 1$ and, for each positive integer n, let $h(n)$ denote the number of H-paths from (0,0) to (n,n). The sequence $\{h(0), h(1), \cdots, h(n), \cdots\}$ is the Fibonacci sequence.

An obvious but interesting corollary is that the number of H-paths from (0,0) to (n,n) is the number of G-paths from (0,0) to $(n - 1, n - 1)$.

Greenwood has discussed G-paths [1]. A method of enumeration different from that used by Greenwood leads to the following [2].

<u>Theorem 3.</u>

Let

$$z(1,i) = 1,$$

$$z(2,i) = \left[\frac{i-1}{2}\right], \text{ where } [\,] \text{ denotes the greatest integer function,}$$

$$z(3,i) = z(3,i-1) + z(2,i-1),$$

$$z(4,i) = z(4,i-2) + z(3,i-2),$$

$$\cdots$$

$$z(2n,i) = z(2n,i-2) + z(2n-1,i-2),$$

$$z(2n+1,i) = z(2n+1,i-1) + z(2n,i-1),$$

$$\cdots$$

with the restriction that $z(k,i) = 0$ if $k > i$. For each positive integer i, let

$$f(i) = \sum_{k=1}^{i} z(k,i) \quad .$$

The sequence $\{f(i) | i = 1, 2, \cdots\}$ is the Fibonacci sequence.

The proof is direct and is omitted. A geometric interpretation of the numbers $z(k,i)$ and $f(i)$ is given in [2].

It is interesting to note the sequence obtained by considering paths in 3-space that are analogous to H-paths in 2-space. In Euclidean 3-space, a path from $(0,0,0)$ to (n,n,n) is said to have property F if and only if it is such that:

1) it is symmetric with respect to the line $z = (n/2)$ in the plane $x + y = n$,
2) if the step $[P_1, P_2]$ of it is z-increasing only, then P_1 belongs to the plane $x + y = n$,
3) if S is a step of it which is not z-increasing only, then either S is x-increasing only, y-increasing only, or xyz-increasing, and
4) some term of it belongs to the plane $x + y = n$.

We will call a path an F-path if it has a property F.

We define $f(0) = 1$; and, for each positive integer n, let $f(n)$ denote the number of F-paths from $(0,0,0)$ to (n,n,n). We note that $f(1) = 2$ and $f(2) = 5$. If $n > 2$, then each F-path has as its second term either $(1,0,0)$, $(0,1,0)$, or $(1,1,1)$. If an F-path from $(0,0,0)$ to (n,n,n) has as its second term $(1,0,0)$ or $(0,1,0)$, then it has as its next to last term $(n, n-1, n)$ or $(n-1, n, n)$ respectively. The number of F-paths from $(0,0,0)$ to (n,n,n) which have as their second term either $(0,1,0)$ or $(1,0,0)$ is the number of F-paths from $(0,0,0)$ to $(n-1, n-1, n-1)$. Hence, the number of F-paths from $(0,0,0)$ to (n,n,n) whose second term is either $(1,0,0)$ or $(0,1,0)$ is $2f(n-1)$. Similarly, the number of F-paths from $(0,0,0)$ to (n,n,n) whose second term is $(1,1,1)$ is $f(n-2)$. Hence, if $n > 2$, then $f(n) = 2f(n-1) + f(n-2)$.

It is noted that the expression $f(n) = 2f(n-1) + f(n-2)$ is the special case of the Fibonacci polynomial $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$ for $f_0(x) = 0$, $f_1(x) = 1$, and $x = 2$.

Using the methods of finite difference equations we may obtain an expression for calculating $f(n)$ directly. Consider again the recursion relation $f(n) = 2f(n-1) + f(n-2)$ in the form of the second order homogeneous difference equation

$$f(n+2) - 2f(n+1) - f(n) = 0 .$$

The corresponding characteristic equation

$$r^2 - 2r - 1 = 0$$

has roots

$$r_1 = 1 + \sqrt{2} \quad \text{and} \quad r_2 = 1 - \sqrt{2} .$$

The general solution of the above difference equation is

$$f(n) = C_1(1 + \sqrt{2})^n + C_2(1 - \sqrt{2})^n .$$

Using the initial conditions of $f(0) = 1$ and $f(1) = 2$, the constants C_1 and C_2 are found to be

$$(\sqrt{2} + 1)/2\sqrt{2} \quad \text{and} \quad (\sqrt{2} - 1)/2\sqrt{2}$$

respectively, so that we have finally

$$f(n) = \frac{(1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1}}{2\sqrt{2}}$$

An analysis similar to that used to obtain the recursion relation for F-paths in 3-space suffices to show that in k-dimensional space the number of paths from $(0,0,0,\cdots,0)$ to $(n,n,n,\cdots,n)$ that are analogous to F paths in 3-space satisfies the recursion relation $f(n) = (k-1)f(n-1) + f(n-k+1)$.

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CHAINS OF EQUIVALENT FIBONACCI-WISE TRIANGLES

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Consider the infinite set of ordered and equally dispersed Fibonacci numbers, F_{n+ih}, $i = 0, 1, 2, \cdots, n, h$, arbitrary positive integers. The triangle having vertices at the points designated by the rectangular cartesian coordinates

(F_n, F_{n+h}), (F_{n+2h}, F_{n+3h}), (F_{n+4h}, F_{n+5h}) has the area

$$F_{2h}F_{5h} + F_hF_{4h} - F_{3h}F_{4h} - F_hF_{2h} ,$$

which is noted to be independent of n and depends only upon the dispersion of the Fibonacci numbers used for coordinates of the vertices.

PROOF

Twice the area of the specified triangle is equal to the absolute value of the determinant

$$\begin{vmatrix} F_n & F_{n+h} & 1 \\ F_{n+2h} & F_{n+3h} & 1 \\ F_{n+4h} & F_{n+5h} & 1 \end{vmatrix}$$

whose expanded form, simplified by the identity

$$F_{a+b} = F_bF_{a+1} + F_{b-1}F_a ,$$

reduces to

$$AF_{n+1}^2 + BF_{n+1}F_n + CF_n^2 ,$$

wherein

$$A = F_{2h}F_{5h} + F_hF_{4h} - F_{3h}F_{4h} - F_hF_{2h} ,$$

$$B = F_{2h}F_{5h-1} + F_{5h}F_{2h-1} + F_{h-1}F_{4h} + F_{3h} - F_{3h}F_{4h-1} - F_{4h}F_{3h-1} - F_hF_{2h-1} - F_{5h} ,$$

$$C = F_{2h-1}F_{5h-1} + F_{h-1}F_{4h-1} + F_{3h-1} - F_{3h-1}F_{4h-1} - F_{h-1}F_{2h-1} - F_{5h-1} .$$

By use of the identity cited above, the fundamental relationship $F_n + F_{n+1} = F_{n+2}$, one may easily prove that $A = -B = -C$. Furthermore, since

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = \pm 1 ,$$

the area of the triangle is observed to be half the value of

$$F_{2h}F_{5h} + F_hF_{4h} - F_hF_{2h} - F_{3h}F_{4h} \text{ .} \qquad \text{Q. E. D.}$$

COROLLARIES

1. For any positive integral value of h there are 2h chains of Fibonacci-wise triangles; i.e., triangles of equal area extending along the two series of vertex points whose rectangular cartesian coordinates are equally dispersed Fibonacci numbers. In each chain consecutive triangles have two vertices in common.

2. By exhibiting the fundamental relationship of Fibonacci numbers as $F_{n+1} - F_n = F_{n-1}$, one may define the Fibonacci numbers for zero and negative indices, to wit, $F_0 = 0$, $F_{-1} = 1$, $F_{-2} = -1$, $F_{-3} = 2$, and quite generally, $F_{-n} = (-1)^{n+1}F_n$. Accordingly, the 2h chains of Fibonacci-wise triangles extend indefinitely in both directions.

3. Again, the Fibonacci relationship

$$F_k + F_{k+1} = F_{k+2}$$

is observed to be valid for all real values of k for the added two compatible definitions

$$F_k = k \quad \text{for } 0 \leq k \leq 1, \quad \text{and} \quad F_k = 1 \quad \text{for } 1 \leq k \leq 2 \text{ .}$$

Hence one obtains a non-denumerably infinite set of Fibonacci-wise chains of triangles for any prescribed positive integral value of h, wherein individual triangles of neighboring chains extend continuously along the sets of real Fibonacci numbers employed as rectangular cartesian coordinates of vertices.

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ITERATED FIBONACCI AND LUCAS SUBSCRIPTS

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Raymond Whitney [3] has proposed the problem of finding recurrence relations for the sequences $U_n = F_{F_n}$, $V_n = F_{L_n}$, $W_n = L_{L_n}$, and $X_n = L_{F_n}$, where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively. In this note we give the required recurrence relations for more general sequences of the form $Y_n = F_{H_n}$, $Z_n = L_{H_n}$, where the H_n are generalized Fibonacci numbers introduced by Horadam.

We will make use of several identities. It follows from the Binet forms for Fibonacci and Lucas numbers that

$$2F_{n+1} = F_n + L_n \quad , \tag{1}$$

$$F_{n-1} = \tfrac{1}{2}(L_n - F_n) \quad , \tag{2}$$

$$L_n^2 - 5F_n^2 = 4(-1)^n \quad , \tag{3}$$

$$2L_{n+1} = 5F_n + L_n \quad . \tag{4}$$

From these H. H. Ferns [1] has shown

$$F_{n+1} = \tfrac{1}{2}\left(\sqrt{5F_n^2 + 4(-1)^n} + F_n\right) \quad , \tag{5}$$

$$L_{n+1} = \tfrac{1}{2}\left(\sqrt{5L_n^2 - 20(-1)^n} + L_n\right) \quad . \tag{6}$$

Equation (5) implies

$$F_{n-1} = \tfrac{1}{2}\left(\sqrt{5F_n^2 + 4(-1)^n} - F_n\right) \quad . \tag{7}$$

We shall also require

$$F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1} \quad , \tag{8}$$

$$L_{m+n+1} = F_m L_n + F_{m+1} L_{n+1} \quad , \tag{9}$$

which are found in [2; Section 5]. Finally, it is convenient to define $s(n) = n^2 - 3[n^2/3]$, where [ ] denotes the greatest integer function. Since $s(n) = 1$ if $3 \nmid n$ while $s(n) = 0$ if $3 \mid n$, it follows that

$$(-1)^{s(n)} = (-1)^{F_n} = (-1)^{L_n} .$$

First consider the sequence $Y_n = F_{H_n}$, where H_n obeys $H_{n+2} = H_{n+1} + H_n$. Then using (8), (7), and (5), we find

$$Y_{n+2} = F_{H_{n+2}} = F_{H_{n+1}+H_n} = F_{H_{n+1}-1}F_{H_n} + F_{H_{n+1}}F_{H_n+1}$$

$$= \tfrac{1}{2}F_{H_n}\left(\sqrt{5F^2_{H_{n+1}} + 4(-1)^{H_{n+1}}} - F_{H_{n+1}}\right) + \tfrac{1}{2}F_{H_{n+1}}\left(\sqrt{5F^2_{H_n} + 4(-1)^{H_n}} + F_{H_n}\right)$$

$$= \tfrac{1}{2}\left[Y_n\sqrt{5Y^2_{n+1}\ \ 4(-1)^{H_{n+1}}} + Y_{n+1}\sqrt{5Y^2_n + 4(-1)^{H_n}}\ \right].$$

If $H_n = F_n$, then $Y_n = U_n$ and we have

$$U_{n+2} = \tfrac{1}{2}\left[U_n\sqrt{5U^2_{n+1}+4(-1)^{s(n+1)}} + U_{n+1}\sqrt{5U^2_n + 4(-1)^{s(n)}}\ \right] \quad (n > 0)$$

while if $H_n = L_n$, then $Y_n = V_n$ and we find

$$V_{n+2} = \tfrac{1}{2}\left[V_n\sqrt{5V^2_{n+1}+4(-1)^{s(n+1)}} + V_{n+1}\sqrt{5V^2_n + 4(-1)^{s(n)}}\ \right] \quad (n > 0)$$

Now consider the sequence $Z_n = L_{H_n}$, where H_n is as before. Using (9), (2), (3), and (6), we see

$$Z_{n+2} = L_{H_{n+2}} = L_{H_{n+1}+H_n} = F_{H_{n+1}-1}L_{H_n} + F_{H_{n+1}}L_{H_n+1}$$

$$= \tfrac{1}{2}L_{H_n}L_{H_{n+1}} - \tfrac{1}{2}L_{H_n}F_{H_{n+1}} + F_{H_{n+1}}L_{H_n+1}$$

$$= \tfrac{1}{2}L_{H_n}L_{H_{n+1}} + \tfrac{1}{2}\sqrt{(L^2_{H_{n+1}} - 4(-1)^{H_{n+1}})/5}\sqrt{5(L^2_{H_n} - 4(-1)^{H_n})}$$

$$= \tfrac{1}{2}\left[Z_{n+1}Z_n + \sqrt{(Z^2_{n+1} - 4(-1)^{H_{n+1}})(Z^2_n - 4(-1)^{H_n})}\ \right]$$

Now if $H_n = F_n$, then $Z_n = X_n$ and we get

$$X_{n+2} = \tfrac{1}{2}\left[X_{n+1}X_n + \sqrt{(X^2_{n+1} - 4(-1)^{s(n+1)})(X^2_n - 4(-1)^{s(n)})}\ \right]$$

and if $H_n = L_n$, we have $Z_n = W_n$ and

$$W_{n+2} = \tfrac{1}{2}\left[W_{n+1}W_n + \sqrt{(W^2_{n+1} - 4(-1)^{s(n+1)})(W^2_n - 4(-1)^{s(n)})}\ \right].$$

See page 86 for References.

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SUMMATION OF $\sum_{k=1}^{n} k^m F_{k+r}$ FINITE DIFFERENCE APPROACH

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Let it be proposed to discover an expression for the summation

$$\sum_{k=1}^{n} k^m F_k$$

or more generally

$$\sum_{k=1}^{n} k^m F_{k+r}$$

where m and r are positive integers. One possible approach is a modified version of finite differences. Given an expression f(n) where n is a positive integer, the usual finite difference relation is

$$\Delta f(n) = f(n+1) - f(n)$$

The adapted finite difference pertains to a quantity of the form

$$f\left[n,\ F_{(n)}\right]$$

where f is a function of n and Fibonacci numbers involving n in their subscripts. We shall define

$$\Delta f\left[n,\ F_{(n)}\right] = f\left[(n+1),\ F_{(n+1)}\right] - f\left[n, F_{(n)}\right]$$

For example,

$$\begin{aligned}\Delta(n^2F_n) &= (n+1)^2F_{n+1} - n^2F_n \\ &= n^2F_{n-1} + (2n+1)F_{n+1}\end{aligned}$$

Likewise we define Δ^{-1} to be the inverse of Δ so that

$$\Delta^{-1}[n^2F_{n-1} + (2n + 1)F_{n+1}] = n^2F_n + C$$

where there is an arbitrary summation constant C which may involve Fibonacci numbers but these as well as other constituent elements must be free of n.

For our purposes it turns out to be more convenient to seek the value of

$$\sum_{k=1}^{n-1} k^m F_{k+r}$$

Let this summation be denoted by $\phi[n, F_{(n)}]$. Then

$$\Delta\phi[n, F_{(n)}] = \sum_{k=1}^{n} k^m F_{k+r} - \sum_{k=1}^{n-1} k^m F_{k+r} = n^m F_{n+r}$$

Thus

$$\phi[n, F_{(n)}] = \sum_{k=1}^{n-1} k^m F_{k+r} = \Delta^{-1}(n^m F_{n+r})$$

We need then simply to evaluate this inverse finite difference in order to obtain an expression for the summation.

We develop certain relations for this purpose.

(1) $\Delta(n F_{n+r+1}) = (n + 1)F_{n+r+2} - n F_{n+r+1} = n F_{n+r} + F_{n+r+2}$

(2) $\Delta(n^2F_{n+r+1}) = n^2F_{n+r} + (2n + 1)F_{n+r+2} = n^2F_{n+r} + \Delta(n^2)F_{n+r+2}$

and in general

(3) $\Delta(n^mF_{n+r+1}) = n^mF_{n+r} + \Delta(n^m)F_{n+r+2}$

Using formula (1)

$$(4) \qquad \Delta^{-1}(nF_{n+r}) = nF_{n+r+1} - \Delta^{-1}(F_{n+r+2}) = nF_{n+r+1} - F_{n+r+3} + C .$$

Then from this result and (2)

$$\Delta^{-1}(n^2F_{n+r}) = n^2F_{n+r+1} - (2n + 1)F_{n+r+3} + 2F_{n+r+5} + C$$

$$= n^2F_{n+r+1} - \Delta(n^2)F_{n+r+3} + \Delta^2(n^2)F_{n+r+5} + C$$

The general formula that suggests itself is

$$(6) \qquad \Delta^{-1}(n^mF_{n+r}) = n^mF_{n+r+1} - \Delta(n^m)F_{n+r+3} + \Delta^2(n^m)F_{n+r+5} + \cdots$$

$$= \sum_{t=0}^{m} (-1)^t\Delta^t(n^m)F_{n+r+2t+1} + C$$

That this result is correct may be shown by calculating

$$\Delta[\Delta^{-1}(n^mF_{n+r})]$$

from the summation in (6). The result is n^mF_{n+r} as can be readily seen from the fact that apart from the first term in the expansion all succeeding terms cancel in pairs. The results for the first two terms will show the pattern.

$$\Delta(n^mF_{n+r+1}) = n^mF_{n+r} + \Delta(n^m)F_{n+r+2} \qquad \text{by (3)}$$

$$\begin{aligned} \Delta[-\Delta(n^m)F_{n+r+3}] &= -\Delta(n + 1)^mF_{n+r+4} + \Delta(n^m)F_{n+r+3} \\ &= -\Delta(n^m)F_{n+r+4} - \Delta^2(n^m)F_{n+r+4} + \Delta(n^m)F_{n+r+3} \\ &= -\Delta(n^m)F_{n+r+2} - \Delta^2(n^m)F_{n+r+4} \quad . \end{aligned}$$

Hence (6) provides the required formula apart from making explicit the coefficients in terms of n and calculating the undetermined constant. The former are given subsequently in tables; the latter may be obtained as shown below for the particular case in which $m = 5$.

We set $n = 2$ in (6) so that

$$F_{r+1} = 32F_{r+3} - 211F_{r+5} + 570F_{r+7} - 750F_{r+9} + 480F_{r+11} - 120F_{r+13} + C$$

Using the formulas

$$F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1}$$

and

$$F_n = (-1)^{k-1}(F_kF_{n+k+1} - F_{k+1}F_{n+k})$$

C is found to be $16679 F_{r+9} + 10324 F_{r+8}$.

Table 1
COEFFICIENTS OF $\Delta(n^m)$

m	1	n	n^2	n^3	n^4	n^5	n^6	n^7	n^8	n^9
1	1									
2	1	2								
3	1	3	3							
4	1	4	6	4						
5	1	5	10	10	5					
6	1	6	15	20	15	6				
7	1	7	21	35	35	21	7			
8	1	8	28	56	70	56	28	8		
9	1	9	36	84	126	126	84	36	9	
10	1	10	45	120	210	252	210	120	45	10

Table 2
COEFFICIENTS OF $\Delta^2(n^m)$

m	1	n	n^2	n^3	n^4	n^5	n^6	n^7	n^8
2	2								
3	6	6							
4	14	24	12						
5	30	70	60	20					
6	62	180	210	120	30				
7	126	434	630	490	210	42			
8	254	1008	1736	1680	980	336	56		
9	510	2286	4536	5208	3780	1764	504	72	
10	1022	5100	11430	15120	13020	7560	2940	720	90

Table 3

COEFFICIENTS OF $\Delta^3(n^m)$

m	1	n	n^2	n^3	n^4	n^5	n^6	n^7
3	6							
4	36	24						
5	150	180	60					
6	540	900	540	120				
7	1806	3780	3150	1260	210			
8	5796	14448	15120	8400	2520	336		
9	18150	52164	65016	45360	18900	4536	504	
10	55980	181500	260820	216720	113400	37800	7560	720

Table 4

COEFFICIENTS OF $\Delta^4(n^m)$

m	1	n	n^2	n^3	n^4	n^5	n^6
4	24						
5	240	120					
6	1560	1440	360				
7	8400	10920	5040	840			
8	40824	67200	43680	13440	1680		
9	186480	367416	302400	131040	30240	3024	
10	818520	1864800	1837080	1008000	327600	60480	5040

Table 5

COEFFICIENTS OF $\Delta^5(n^m)$

m	1	n	n^2	n^3	n^4	n^5
5	120					
6	1800	720				
7	16800	12600	2520			
8	126000	134400	50400	6720		
9	834120	1134000	604800	151200	15120	
10	5103000	8341200	5670000	2016000	378000	30240

Table 6
COEFFICIENTS OF $\Delta^6(n^m)$

m	1	n	n^2	n^3	n^4
6	720				
7	15120	5040			
8	191520	120960	20160		
9	1905120	1723680	544320	60480	
10	16435440	19051200	8618400	1814400	151200

Table 7
COEFFICIENTS OF $\Delta^7(n^m)$

m	1	n	n^2	n^3
7	5040			
8	141120	40320		
9	2328480	1270080	181440	
10	29635200	23284800	6350400	604800

Table 8
COEFFICIENTS OF $\Delta^8(n^m)$

m	1	n	n^2
8	40320		
9	1451520	362880	
10	30240000	14515200	1814400

Table 9
COEFFICIENTS OF $\Delta^9(n^m)$

m	1	n
9	362880	
10	16329600	3628800

Table 10
COEFFICIENTS OF $\Delta^{10}(n^m)$

m	1
10	3628800

Table 11
SUMMATION CONSTANTS

m	Summation Constants
1	F_{r+3}
2	$-F_{r+6}$
3	$7F_{r+5} + 5F_{r+4}$
4	$-37F_{r+6} - 24F_{r+5}$
5	$242F_{r+7} + 147F_{r+6}$
6	$-1861F_{r+8} - 1139F_{r+7}$
7	$16679F_{r+9} + 10324F_{r+8}$
8	$-171362F_{r+10} - 106089F_{r+9}$
9	$1981723F_{r+11} + 1224729F_{r+10}$
10	$-25453505F_{r+12} - 15726832F_{r+11}$

To be able to write out a complete formula one uses formula (6) and the various tables. The case $m = 7$ is given below.

$$\begin{aligned}\sum_{k=1}^{n} k^7 F_{k+r} = {} & n^7 F_{n+r+1} - (7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n + 1)F_{n+r+3} \\ & + (42n^5 + 210n^4 + 490n^3 + 630n^2 + 434n + 126)F_{n+r+5} \\ & - (210n^4 + 1260n^3 + 3150n^2 + 3780n + 1806)F_{n+r+7} \\ & + (840n^3 + 5040n^2 + 10920n + 8400)F_{n+r+9} \\ & - (2520n^2 + 12600n + 16800)F_{n+r+11} + (5040n + 15120)F_{n+r+13} \\ & - 5040F_{n+r+15}\end{aligned}$$

CALCULATION BY FINITE DIFFERENCES

Except for the smaller values of m, the explicit formulas given above in terms of n are apt to involve undue calculations. These can be obviated by going directly to finite differences and using formula (6).

For example, to calculate

$$\sum_{k=1}^{49} k^5 F_{k+7}$$

we would first write down the values of k^5 for $k = 50, 51$, etc.

FINITE DIFFERENCE APPROACH

k	k^5
50	312500000
51	345025251
52	380204032
53	418195493
54	459165024
55	503284375

Then

$$\Delta\,[k^5]_{k=50} = 34502521 - 312500000 = 32525251$$

$$\Delta^2[k^5]_{k=50} = 380204032 - 2 \cdot 34502521 + 312500000 = 2653530$$

$$\Delta^3[k^5]_{k=50} = 418195493 - 3 \cdot 380204032 + 3 \cdot 345025251 - 312500000$$
$$= 159150$$

$$\Delta^4[k^5]_{k=50} = 459165024 - 4 \cdot 418195493 + 6 \cdot 380204032 - 4 \cdot 345025251$$
$$+ 312500000 = 6240$$

$$\Delta^5[k^5]_{k=50} = 120$$

The value of the summation is:

$$312500000\, F_{58} - 32525251\, F_{60} + 2653530\, F_{62} - 159150\, F_{64} + 6240\, F_{66} - 120\, F_{68}$$
$$+ 242\, F_{14} + 147 F_{13}$$

which can either be calculated directly or sum of the terms can be unified and the number of multiplications of large numbers can be decreased.

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ON RATIOS OF FIBONACCI AND LUCAS NUMBERS

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Recently the author has conducted in-service training sessions in mathematics for the elementary school teachers of the Williamstown, Massachusetts public schools. During a session on the lowest common multiple and greatest common divisor of two positive integers, two teachers observed that if the two numbers are in the ratio 2:3, then the sum of the numbers is equal to the difference between their lowest common multiple and their greatest common divisor. It is shown in [2] that this is the only ratio for which this relation holds.

Of course, one gets similar relations for other ratios. For example, if the two numbers are in the ratio 3:5, then twice their sum is equal to the sum of their lowest common multiple and their greatest common divisor. Again it is shown in [2] that this is the only ratio for which this relation holds. This is not always the case since, for example, both ratios 5:7 and 4:11 yield the result that three times the sum of the numbers is equal to the sum of their lowest common multiple and their greatest common divisor.

If one specializes to the Fibonacci and the Lucas sequences, one gets theorems of the type given below, in which families of such relations are exhibited and formulas for finding all ratios satisfying these relations are obtained.

Let $\{F_n\}$ be the sequence of Fibonacci numbers, where $F_1 = 1, F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for $n \geq 1$.

Let $\{L_n\}$ be the sequence of Lucas numbers, where $L_1 = 1, L_2 = 3$ and $L_{n+2} = L_n + L_{n+1}$ for $n \geq 1$.

The following known results are assumed. (See [1] or [3].)

(i) Neighboring Fibonacci numbers are relatively prime.

(ii) $F_{n+1}^2 = F_n F_{n+2} + (-1)^n$.

(iii) $F_{2n-1} = F_n F_{n+1} - F_{n-1} F_{n-2}$

(iv) Neighboring Lucas numbers are relatively prime.

(v) $F_{2n} = F_n L_n$

(vi) $L_{n+1} = F_n + F_{n+2}$.

For the remainder of the article, let a and b be natural numbers. Denote by $[a,b]$ the lowest common multiple of a and b and by (a,b) the greatest common divisor of a and b.

Theorem 1: (1) If a and b are in the ratio $F_n:F_{n+1}$, then

$$F_{n-1}(a+b) = [a,b] + (-1)^n(a,b) \qquad \text{for } n \geq 2.$$

(2) Let c and d be relatively prime natural numbers with $b = (c/d)a$. If $F_{n-1}(a+b) = [a,b] + (-1)^n(a,b)$ for $n \geq 3$, then the number of solutions for the ratio $c:d$ is one-half the number of divisors of $F_{n-2}F_n$, and among the solutions is the ratio $F_n:F_{n+1}$.

Proof: (1) Suppose $b = (F_n/F_{n+1})a$. Then $a = F_{n+1}k$, $b = F_nk$, $(a,b) = k$ and $[a,b] = F_nF_{n+1}k$, for k a natural number. Then

$$\begin{aligned} F_{n+1}(a+b) &= F_{n-1}(F_n + F_{n+1})k = F_{n-1}F_{n+2}k = (F_{n+1} - F_n)F_{n+2}k \\ &= F_{n+1}(F_n + F_{n+1})k - F_nF_{n+2}k \\ &= F_nF_{n+1}k + (F_{n+1}^2 - F_nF_{n+2})k \\ &= [a,b] + (-1)^n(a,b), \qquad \text{for } n \geq 2. \end{aligned}$$

(2) If $b = (c/d)a$, where c and d are relatively prime, then $a = dk$, $b = ck$, $(a,b) = k$ and $[a,b] = cdk$, for k a natural number. Then

$$F_{n-1}(a+b) = [a,b] + (-1)^n(a,b) \qquad \text{for } n \geq 3$$

implies

$$F_{n-1}(c+d) = cd + (-1)^n,$$

for which we wish to find all positive integral solutions. Solving for c, we get

$$c = \frac{F_{n-1}d - (-1)^n}{d - F_{n-1}} = F_{n-1} + \frac{F_{n-1}^2 - (-1)^n}{d - F_{n-1}}$$

so that by (ii),

$$c = F_{n-1} + \frac{F_{n-2}F_n}{d - F_{n-1}}$$

We need only consider the case $d > F_{n-1}$, for if $0 < d < F_{n-1}$, then $c < 0$. The total number of integral solutions for c and d is given by the number of divisors of $F_{n-2}F_n$. But there is an obvious symmetry in these solutions so that if $c = A$, $d = B$ is a solution, then so is $c = B$, $d = A$. Thus the number of distinct solutions for the ratio $c:d$ is one-half the number of divisors of $F_{n-2}F_n$.

Finally, if $d = F_{n+1}$, then

$$c = F_{n-1} + \frac{F_{n-2}F_n}{F_n} = F_{n-1} + F_{n-2} = F_n ,$$

and the ratio $F_n:F_{n+1}$ is among the solutions. This completes the proof.

Example: If $n = 8$, then $F_{n-2} = 8$, $F_{n-1} = 13$, $F_n = 21$, and $F_{n+1} = 34$.

(1) If a and b are in the ratio 21:34, then

$$13(a + b) = [a,b] + (a,b) .$$

(2) If $b = (c/d)a$, then $13(a + b) = [a,b] + (a,b)$ implies that

$$c = 13 + \frac{168}{d - 13}$$

168 has 16 divisors, so there are 8 distinct solutions. They are: 14:181, 15:97, 16:69, 17:55, 19:41, 20:37, 21:34, and 25:27, among which is the Fibonacci pair 21:34.

The following lemma is needed for the proof of the second theorem.

Lemma: $F_{2n-1} = F_{n+1}L_{n+2} - L_nL_{n+1}$ for $n \geq 2$.

Proof: The proof is by induction. The identity is easily verified for $n = 2$. Assume it is true for $n = k$, so that

$$F_{2k-1} = F_{k+1}L_{k+2} - L_kL_{k+1} \quad .$$

Then

$$\begin{aligned} F_{2k+1} &= F_{2k} + F_{2k-1} = F_kL_k + F_{k+1}L_{k+2} - L_kL_{k+1} \\ &= F_kL_k + (F_{k+2} - F_k)L_{k+2} - L_kL_{k+1} \\ &= F_kL_k + F_{k+2}(L_{k+3} - L_{k+1}) - F_kL_{k+2} - L_kL_{k+1} \\ &= F_{k+2}L_{k+3} - F_{k+2}L_{k+1} - F_kL_{k+1} - (L_{k+2} - L_{k+1})L_{k+1} \\ &= F_{k+2}L_{k+3} - L_{k+1}L_{k+2} + L_{k+1}(L_{k+1} - F_{k+2} - F_k) \quad . \end{aligned}$$

But

$$L_{k+1} - F_{k+2} - F_k = 0$$

by (vi), so that

$$F_{2k+1} = F_{k+2}L_{k+3} - L_{k+1}L_{k+2} \quad ,$$

completing the induction step and the proof.

Theorem 2: (1) If a and b are in the ratio $L_n:L_{n+1}$, then

$$F_{n+1}(a + b) = [a,b] + F_{2n-1}(a,b) \quad \text{for } n \geq 2 \; .$$

(2) If a and b are in the ratio $F_{n-2}:F_{n-1}$, then

$$F_{n+1}(a + b) = [a,b] + F_{2n-1}(a,b) \quad \text{for } n \geq 3 \; .$$

(3) Let c and d be relatively prime natural numbers with

$b = (c/d)a$. If $F_{n+1}(a + b) = [a,b] + F_{2n-1}(a,b)$ for $n \geq 2$, then the possible ratios $c:d$ are determined from the divisors of $(F_{n+1}^2 - F_{2n-1})$. Among these ratios is $L_n:L_{n+1}$. For $n \geq 3$, $F_{n-2}:F_{n-1}$ is also a solution.

Proof: (1) Suppose

$$b = \frac{L_n}{L_{n+1}} a \quad .$$

Then

$$a = L_{n+1}k, \qquad b = L_n k, \qquad (a,b) = k \quad \text{and} \quad [a,b] = L_n L_{n+1} k \quad ,$$

for k a natural number. Then

$$F_{n+1}(a,b) = F_{n+1}(L_n + L_{n+1})k = F_{n+1}L_{n+2}k \quad .$$

Using the lemma, we get

$$F_{n+1}(a + b) = (F_{2n-1} + L_n L_{n+1})k \quad ,$$

so that

$$F_{n+1}(a + b) = [a,b] + F_{2n-1}(a,b) \quad ,$$

as required.

(2) If

$$b = \frac{F_{n-2}}{F_{n-1}} a \quad ,$$

then

$$a = F_{n-1}k \; , \qquad b = F_{n-2}k \; , \qquad (a,b) = k \quad \text{and} \quad [a,b] = F_{n-1}F_{n-2}k$$

for k a natural number. Then, using (iii),

$$F_{n+1}(a + b) = F_{n+1}(F_{n-1} + F_{n-2})k = F_{n+1}F_nk$$
$$= (F_{2n-1} + F_{n-1}F_{n-2})k$$
$$= [a,b] + F_{2n-1}(a,b) \quad ,$$

as required.

(3) If $b = (c/d)a$, where c and d are relatively prime, then, once again

$$a = dk, \quad b = ck, \quad (a,b) = k \quad \text{and} \quad [a,b] = cdk \quad ,$$

for k a natural number. The relation

$$F_{n+1}(a + b) = [a,b] + F_{2n-1}(a,b)$$

implies

$$F_{n+1}(c + d) = cd + F_{2n-1}$$

Solving this equation for c, we get

$$c = \frac{F_{n+1}d - F_{2n-1}}{d - F_{n+1}} = F_{n+1} + \frac{F_{n+1}^2 - F_{2n-1}}{d - F_{n+1}}$$

We seek positive integral solutions for c and d. The possible ratios $c:d$ are determined from the divisors of $(F_{n+1}^2 - F_{2n-1})$.

Using the lemma, we show that $c = L_n$, $d = L_{n+1}$ is a solution. By symmetry, $c = L_{n+1}$, $d = L_n$ is also a solution. So let $d = L_{n+1}$. Then

$$c = \frac{F_{n+1}L_{n+1} - F_{2n-1}}{L_{n+1} - F_{n+1}} = \frac{F_{n+1}L_{n+1} - F_{n+1}L_{n+2} + L_nL_{n+1}}{L_{n+1} - F_{n+1}}$$

$$= \frac{F_{n+1}(L_{n+1} - L_{n+2}) + L_nL_{n+1}}{L_{n+1} - F_{n+1}} = \frac{-L_nF_{n+1} + L_nL_{n+1}}{L_{n+1} - F_{n+1}} = L_n \quad .$$

The situation here differs from that in the second part of Theorem 1, for not all solutions are obtained by considering the case $d > F_{n+1}$. For example, let $d = F_{n-1}$. Then, using (iii),

$$c = \frac{F_{n+1}F_{n-1} - F_{2n-1}}{F_{n-1} - F_{n+1}} = \frac{F_{n+1}F_{n-1} - F_nF_{n+1} + F_{n-1}F_{n-2}}{F_{n-1} - F_{n+1}}$$

$$= \frac{-F_{n+1}F_{n-2} + F_{n-1}F_{n-2}}{F_{n-1} - F_{n+1}} = F_{n-2} \quad .$$

Thus the ratio $F_{n-2}:F_{n-1}$ is a solution. This completes the proof of the theorem.

<u>Example:</u> If $n = 7$, then $L_n = 29$, $L_{n+1} = 47$, $F_{n-2} = 5$, $F_{n-1} = 8$, $F_{n+1} = 21$ and $F_{2n-1} = 233$.

(1) and (2): If a and b are in the ratio 29:47 or 5:8, then

$$21(a + b) = [a,b] + 233(a,b) \quad .$$

(3): If $b = (c/d)a$, then

$$21(a + b) = [a,b] + 233(a,b)$$

implies that

$$c = 21 + \frac{441 - 233}{d - 21} = 21 + \frac{208}{d - 21}$$

The divisors of 208 are 1, 2, 4, 8, 13, 16, 26, 52, 104 and 208. The solutions are 22:229, 23:125, 25:73, 29:47, 34:37 and 5:8. Among these ratios are $29:47 = L_n:L_{n+1}$ and $5:8 = F_{n-2}:F_{n-1}$.

ACKNOWLEDGEMENTS

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2. G. Cross and H. Renzi, "Teachers Discover New Math Theorem," The Arithmetic Teacher, Vol. 12 No. 8 (1965), pp 625-626.
3. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, London, 1954, pp 148-150.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A. P. HILLMAN, University of New Mexico, Albuquerque, New Mex.

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-106 Proposed by H. H. Ferns, Victoria, B.C., Canada.

Prove the following identities:

$$2F_{i+j} = F_iL_j + F_jL_i ,$$

$$2L_{i+j} = L_iL_j + 5F_iF_j .$$

B-107 Proposed by Robert S. Seamons, Yakima Valley College, Yakima, Wash.

Let M_n and G_n be respectively the n^{th} terms of the sequences (of Lucas and Fibonacci) for which $M_n = M_{n-1}^2 - 2$, $M_1 = 3$, and $G_n = G_{n-1} + G_{n-2}$, $G_1 = 1$, $G_2 = 2$. Prove that

$$M_n = 1 + [\sqrt{5}\,G_m] ,$$

where $m = 2^n - 1$ and $[x]$ is the greatest integer function.

B-108 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let $u_1 = p$, $y_2 = q$, and $u_{n+2} = u_{n+1} + u_n$. Also let $S_n = u_1 + u_2 + \cdots + u_n$. It is true that $S_6 = 4u_4$ and $S_{10} = 11u_7$. Generalize these formulas.

B-109 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let r and s be the roots of the quadratic equation $x^2 - px - q = 0$, $(r \neq s)$. Let $U_n = (r^n - s^n)/(r - s)$ and $V_n = r^n + s^n$. Show that

$$V_n = U_{n+1} + qU_{n-1} .$$

B-110 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2n+1}}$$

B-111 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{F_{4n+2}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{1}{L_{4n+2}}$$

SOLUTIONS

LUCAS NUMBERS MODULO 5

B-88 Proposed by John Wessner, Melbourne, Florida.

Let L_0, L_2, L_4, $L_6, \cdots$ be the Lucas numbers 2, 3, 7, $18, \cdots$. Show that

$$L_{2k} \equiv 2(-1)^k \pmod 5 .$$

Solution by J. A. H. Hunter, Toronto, Canada

All (mod 5) we have: $L_1 \equiv 1$, $L_2 \equiv -2$, $L_3 \equiv -1$, $L_4 \equiv 2$, $L_5 \equiv 1$, $L_6 \equiv -2$, etc., so it follows that $L_{4t+2} \equiv -2$ and $L_{4t} \equiv +2$. Hence $L_{2k} \equiv 2(-1)^k \pmod 5$.

Also solved by James E. Desmond, H. H. Ferns, Joseph D. E. Konhauser, Douglas Lind, F. D. Parker, C.B.A. Peck, Jeremy C. Pond, David Zeitlin, and the proposer.

A CLOSE APPROXIMATION

B-89 Proposed by Robert S. Seamons, Yakima Valley College, Yakima, Wash.

Let F_n and L_n be the n^{th} Fibonacci and n^{th} Lucas numbers, respectively. Let $[x]$ be the greatest integer function. Show that $L_{2m} = 1 + [\sqrt{5}F_{2m}]$ for all positive integers m.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

From the Binet forms for F_n and L_n, the statement is equivalent to $\alpha^{2m} + \beta^{2m} = [1 + \alpha^{2m} - \beta^{2m}]$, where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$. But $1/2 > \beta^{2m} > 0$ for $m > 0$, so we have

$$\alpha^{2m} + \beta^{2m} \le \alpha^{2m} - \beta^{2m} + 1 < \alpha^{2m} + \beta^{2m} + 1$$

which implies $\alpha^{2m} + \beta^{2m} = [1 + \alpha^{2m} - \beta^{2m}]$, as desired.

Also solved by James E. Desmond, H. H. Ferns, C.B.A. Peck, Jeremy C. Pond, David Zeitlin, and the proposer.

B-90 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let $b_1, b_2, \cdots$ be the sequence $3, 7, 47, \cdots$ with recurrence relation $b_{n+1} = b_n^2 - 2$. Show that the roots of

$$x^2 - 2b_n x + 4 = 0$$

are expressible in the form $c + d\sqrt{5}$, where c and d are integers.

Solution by David Zeitlin, Minneapolis, Minnesota.

The roots are $b_n \pm \sqrt{b_n^2 - 4} = b_n \pm \sqrt{b_{n+1} - 2}$. The recursion relation may be written as $U_{n+1} = (b_n + 2)U_n$, where $U_n = b_n - 2$, $U_1 = 1$. Thus,

$$\frac{U_{n+1}}{U_1} = \prod_{k=1}^{n} \frac{U_{k+1}}{U_k} = \prod_{k=1}^{n} (b_k + 2) = 5 \prod_{k=2}^{n} (b_k + 2)$$

$$= 5 \prod_{j=1}^{n-1} (b_{j+1} + 2) = 5 \prod_{j=1}^{n-1} b_j^2 ,$$

or

$$b_{n+1} - 2 = 5 \prod_{j=1}^{n-1} b_j^2 .$$

Thus,

$$c = b_n, \quad \text{and} \quad d = \pm \prod_{j=1}^{n-1} b_j , \quad n = 2, 3, \cdots .$$

Also solved by James E. Desmond, H. H. Ferns, Douglas Lind, C.B.A. Peck, Jeremy C. Pond, John Wessner, and the proposer.

CONVERGENCE OF SERIES

B-91 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

If F_n is the n^{th} Fibonacci number, show that

$$\sum_{j=1}^{\infty} (1/F_j)$$

converges while

$$\sum_{j=3}^{\infty} (1/\ln F_j)$$

diverges.

Solution by Jeremy C. Pond, Sussex, England.

$$\left(\frac{1}{F_n}\right)\bigg/\left(\frac{1}{F_{n-1}}\right) = \frac{F_{n+1}}{F_n} \rightarrow \frac{1+\sqrt{5}}{2} > 1 \quad \text{as} \quad n \rightarrow \infty$$

and so

$$\sum_{j=1}^{\infty} (1/F_j)$$

converges by d'Alembert's test. Also,

$$(1/\ln F_n)/(1/n) \rightarrow 1/\ln\left(\frac{1+\sqrt{5}}{2}\right) > 0$$

and so $1/\ln F_j$ and $1/n$ diverge together.

Also solved by C. B. A. Peck and the proposer.

GREATEST COMMON DIVISOR

B-92 Proposed by J. L. Brown, Jr., The Pennsylvania State University.

Let (x,y) denote the g.c.d of positive integers x and y. Show that $(F_m, F_n) = (F_m, F_{m+n}) = (F_n, F_{m+n})$ for all positive integers m and n.

I. Solution by Joseph D. E. Konhauser, Univ. of Minnesota, Minneapolis, Minn.

We use the well-known identity

$$F_{m+n} = F_{n-1}F_m + F_nF_{m+1}$$

and the fact that two consecutive Fibonacci numbers are relatively prime.

Let $d = (F_m, F_n)$ then, from (1) $d|F_{m+n}$. Let $e = (F_{m+n}, F_m)$ then, from (1), $e|F_n$, since $(F_m, F_{m+1}) = 1$. On the one hand, $e|d$ (since $e|F_m$ and $e|F_n$). On the other hand, $d|e$ (since $d|F_m$ and $d|F_{m+n}$). Therefore, $d = e$; that is, $(F_m, F_n) = (F_m, F_{m+n})$. In like manner, it follows that $(F_m, F_n) = (F_n, F_{m+n})$.

II. Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

It is well known [N. N. Vorobyov, The Fibonacci Numbers, page 23, Theorem 4] that $(F_m, F_n) = F_{(m,n)}$. The desired result then follows immediately from the easily established fact that $(m,n) = (m,m+n) = (n,m+n)$.

Also solved by Thomas P. Dence, James E. Desmond, John E. Homer, Jr., C.B.A. Peck, Jeremy C. Pond, David Zeitlin, and the proposer.

L_n MODULO n

B-93 Proposed by Martin Pettet, Toronto, Ontario, Canada

Show that if n is a positive prime, $L_n \equiv 1 \pmod n$. Is the converse true?

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

From the Binet form we have

$$L_n = 2^{-n}\{(1+\sqrt{5})^n + (1-\sqrt{5})^n\} = 2^{-n}\left(\sum_{j=0}^{n} \binom{n}{j} 5^{j/2}\{1 + (-1)^j\}\right)$$
$$= 2^{-n+1}\sum_{j=0}^{[n/2]} \binom{n}{2j} 5^j ,$$

where $[x]$ denotes the greatest integer contained in x. Now if n is prime,

$$\binom{n}{2j} \equiv 0 \pmod{n} \quad (j = 1, 2, \cdots, n/2),$$

so that

$$L_n \equiv \binom{n}{0} \Big/ 2^{n-1} \equiv 1/2^{n-1} \pmod{n}.$$

By Fermat's Lesser Theorem, $2^{n-1} \equiv 1 \pmod{n}$, so that $L_n \equiv 1 \pmod{n}$ if n is prime.

I have not been able to prove or disprove the converse of this statement. A calculation by computer indicates that the converse is true for $n < 700$.

Also solved by the proposer who stated that the converse is false and gave 705, 2465, and 2737 as the first few composite values of n.

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