A LIMITED ARITHMETIC ON SIMPLE CONTINUED FRACTIONS

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1. Introduction. As is well known, a number of remarkable and interesting relationships exist between the golden ratio of the Greeks and the numbers in the Fibonacci sequence. Binet's formula is one example of such a relationship and another is the familiar equation

$$\alpha = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} = \lim_{n \to \infty} \frac{F_{n+1}}{F_n}$$

where $\alpha = (1 + \sqrt{5})/2$ and F_n denotes the n^{th} Fibonacci number. In this paper, we derive other interesting relationships involving the Fibonacci numbers and the simple continued fraction expansions of multiples of the golden ratio. We also extend these results to obtain more general theorems about a certain class of quadratic surds.

Specifically we establish necessary and sufficient conditions for integral multiples of the golden ratio to be of period one, obtain sufficient conditions for these multiples to be of period two and establish some partial converses for those of period two. We then generalize by replacing the golden ratio by arbitrary simple continued fractions of period one and then by arbitrary simple continued fractions of period two. Some results are exactly analogous while others are only partial. Some curious side results are also established.

2. Results involving the golden ratio. We begin by considering the following table of simple continued fraction expansions of positive integral multiples of α. Of course, these expansions are periodic and the repeating part of the expansion is indicated by dots in the style of Hardy and Wright [1].

Careful scrutiny of the table reveals a variety of patterns. Some of the patterns are only apparent but others, as indicated by the theorems following the table are generally true.

Note that small Latin letters will always be used to denote positive integers. Also, L_n will always denote the n^{th} Lucas number.

n	Expansion of $n\alpha$
1	$[1, \dot{1}]$
2	$[3, \dot{4}]$
3	$[4, \dot{1}, \dot{5}]$
4	$[6, \dot{2}, \dot{8}]$
5	$[8, \dot{11}]$
6	$[9, \dot{1}, 2, 2, 2, 1, \dot{12}]$
7	$[11, \dot{3}, \dot{15}]$
8	$[12, \dot{1}, \dot{16}]$
9	$[14, 1, 1, 3, 1, 1, \dot{19}]$
10	$[16, \dot{5}, 1, 1, 5, \dot{22}]$
11	$[17, \dot{1}, 3, 1, \dot{23}]$
12	$[19, \dot{2}, 2, 2, \dot{26}]$
13	$[21, \dot{29}]$
14	$[22, \dot{1}, 1, 1, 7, 6, 7, 1, 1, 1, \dot{30}]$
15	$[24, \dot{3}, 1, 2, 3, 2, 1, 3, \dot{33}]$
16	$[25, \dot{1}, 7, 1, \dot{34}]$
17	$[27, \dot{1}, 1, \dot{37}]$
18	$[29, \dot{8}, \dot{40}]$
19	$[30, \dot{1}, 2, 1, 7, 1, 2, 1, \dot{41}]$
20	$[32, \dot{2}, 1, 3, 2, 1, 1, 10, 1, 1, 2, 3, 1, 2, \dot{44}]$
21	$[33, \dot{1}, \dot{45}]$
22	$[35, \dot{1}, 1, 2, 11, 1, 8, 1, 11, 2, 1, 1, \dot{48}]$
29	$[46, \dot{1}, 11, 1, \dot{63}]$
34	$[55, \dot{76}]$
36	$[58, \dot{4}, \dot{80}]$
47	$[76, \dot{21}, \dot{105}]$
55	$[88, \dot{1}, \dot{121}]$
89	$[144, \dot{199}]$

<u>Theorem 1.</u> Let n be a positive integer. Then $n\alpha = [a, \dot{b}]$ if and only if $n = F_{2m-1}$, $a = F_{2m}$, and $b = L_{2m-1}$ for some $m \geq 1$.

<u>Theorem 2.</u> Let n be a positive integer. Then $n\alpha = [a, \dot{1}, \dot{c}]$ if and only if $n = F_{2m}$, $a = F_{2m+1}$, and $c = L_{2m} - 2$ for some $m \geq 1$.

Theorem 3. If we admit the expansions $\alpha = [2, \dot{1}, -1, 1, \dot{3}]$ and $4\alpha = [6, \dot{1}, 0, 1, \dot{8}]$, then for every integer $r \geq 1$, we have

a) $$L_{2r}\alpha = [L_{2r+1}, \dot{F}_{2r}, 5\dot{F}_{2r}],$$

and

b) $$L_{2r-1}\alpha = [L_{2r} - 1, \dot{1}, F_{2r-1} - 2, 1, 5\dot{F}_{2r-1} - 2] .$$

Unlike Theorems 1 and 2, the converse of Theorem 3 is not true as is easily seen by considering the expansions of 4α, 16α, and 36α. The following theorem, however, provides a partial converse of the first assertion of Theorem 3.

Theorem 4. Let n be a positive integer. Then $n\alpha = [a, \dot{b}, \dot{c}]$ if and only if $nb = F_{2m}$, $ab = F_{2m+1} - 1$, and $bc = L_{2m} - 2$ for some $m \geq 1$.

Before proving these results we derive two lemmas which incidentally provide unusual characterizations of the Fibonacci and Lucas numbers.

Lemma 1. The Pell equation $x^2 - 5y^2 = -4$ is solvable in positive integers if and only if $x = L_{2n-1}$ and $y = F_{2n-1}$ for $n \geq 1$.

Proof. Since $x = y = 1$ is the least positive solution of the given equation, it is well known [2] that every positive solution is given by

$$\begin{aligned} x + y\sqrt{5} &= 2\left(\frac{1 + \sqrt{5}}{2}\right)^{2n-1} \\ &= \frac{1}{2^{2n-2}} \cdot \sum_{k=0}^{n-1} \binom{2n-1}{2k} 5^k + \frac{\sqrt{5}}{2^{2n-2}} \cdot \sum_{j=1}^{n} \binom{2n-1}{2j-1} 5^{j-1} \end{aligned}$$

for $n \geq 1$. On the other hand, by Binet's formula,

$$\begin{aligned} F_{2n-1} &= \frac{1}{\sqrt{5}} \left\{\left(\frac{1 + \sqrt{5}}{2}\right)^{2n-1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{2n-1}\right\} \\ &= \frac{1}{2^{2n-2}} \sum_{j=1}^{n} \binom{2n-1}{2j-1} 5^{j-1} , \end{aligned}$$

and

$$L_{2n-1} = \left(\frac{1+\sqrt{5}}{2}\right)^{2n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n-1}$$

$$= \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{2k} 5^k \quad .$$

Combining these three results we have that all positive integral solutions to $x^2 - 5y^2 = -4$ are given by $x = L_{2n-1}$ and $y = F_{2n-1}$ for $n \geq 1$ as claimed.

Lemma 2. The Pell equation $x^2 - 5y^2 = 4$ is solvable in positive integers if and only if $x = L_{2n}$ and $y = F_{2n}$ for $n \geq 1$.

Proof. As in the proof of Lemma 1, it is easy to show that

$$\frac{1}{2^{k-1}} (1 + \sqrt{5})^k = L_k + \sqrt{5}\, F_k , \quad k \geq 0$$

where we take $L_0 = 2$. Therefore, since $x = 3$, $y = 1$ is the least positive integral solution of the given equation, every solution in positive integers is given by

$$x + y\sqrt{5} = 2\left(\frac{3+\sqrt{5}}{2}\right)^n$$

$$= 2\left\{\frac{2 + (1+\sqrt{5})}{2}\right\}^n$$

$$= \sum_{k=0}^{n} \frac{1}{2^{k-1}} (1+\sqrt{5})^k \binom{n}{k}$$

$$= \sum_{k=0}^{n} L_k \binom{n}{k} + \sqrt{5} \sum_{k=0}^{n} F_k \binom{n}{k}$$

$$= L_{2n} + \sqrt{5}\, F_{2n} \quad .$$

where the last equality is a result of Lucas [3, p. 191]. Thus, all solutions in positive integers are given by $x = L_{2n}$, $y = F_{2n}$ for $n \geq 1$ as claimed.

We note in passing that Lucas [3, p. 199] observes that $L_n^2 - 5F_n^2 = \pm 4$ and that Wasteels [4] proved that if $5x^2 \pm 4$ is the square of an integer then x is a Fibonacci number.

Proof of Theorem 1. By direct calculation we obtain

$$[a, \dot{b}] = \frac{2a - b + \sqrt{b^2 + 4}}{2} .$$

Therefore, $n\alpha = [a, \dot{b}]$ if and only if

$$(1) \qquad n = 2a - b \quad \text{and} \quad n\sqrt{5} = \sqrt{b^2 + 4} .$$

The second of these equations is equivalent to

$$b^2 - 5n^2 = -4$$

and, by Lemma 1, this is solvable in positive integers if and only if $n = F_{2m-1}$ and $b = L_{2m-1}$. Finally, since $F_m + L_m = 2F_{m+1}$ for every m, it follows from (1) that

$$a = \frac{n + b}{2} = \frac{F_{2m-1} + L_{2m-1}}{2} = F_{2m}$$

and the proof is complete.

The proofs of Theorems 2 and 4, which depend on Lemma 2, are exactly analogous to the proof of Theorem 1 and will therefore be omitted. Of course, Theorem 2 is the special case of Theorem 4 with $b = 1$.

Proof of Theorem 3. Part (a) follows directly from Theorem 4 with $n = L_{2r}$, $a = L_{2r+1}$, $b = F_{2r}$, $c = 5F_{2r}$, and $m = 2r$ since it is easily shown that $L_{2r}F_{2r} = F_{4r}$, $L_{2r+1}F_{2r} = F_{4r+1} - 1$, and $5F_{2r}^2 = L_{4r} - 2$.

To obtain Part (b) we define the sequence β_i for $c' \geq 1$ by the following series of calculations which depend on Lemma 1: Let

$$\beta_1 = L_{2r-1}\alpha - L_{2r} + 1$$
$$= \frac{\sqrt{5}\, L_{2r-1} - 5F_{2r-1}}{2} + 1$$
$$= \frac{-10}{\sqrt{5}\, L_{2r-1} + 5F_{2r-1}} + 1 \ .$$

Then

$$\frac{1}{\beta_1} = 1 + \frac{10}{\sqrt{5}L_{2r-1} + 5F_{2r-1} - 10}$$
$$= 1 + \beta_2 \ ,$$
$$\frac{1}{\beta_2} = \frac{\sqrt{5}\, L_{2r-1} + 5F_{2r-1} - 10}{10}$$
$$= F_{2r-1} - 2 + \frac{\sqrt{5}\, L_{2r-1} - 5F_{2r-1} + 10}{10}$$
$$= F_{2r-1} - 2 + \beta_3 \ ,$$
$$\frac{1}{\beta_3} = \frac{10}{\sqrt{5}\, L_{2r-1} - 5F_{2r-1} + 10}$$
$$= \frac{10(\sqrt{5}\, L_{2r-1} + 5F_{2r-1})}{-20 + 10(\sqrt{5}\, L_{2r-1} + 5F_{2r-1})}$$
$$= 1 + \frac{2}{\sqrt{5}\, L_{2r-1} + 5F_{2r-1} - 2}$$
$$= 1 + \beta_4 \ ,$$

and

$$\frac{1}{\beta_4} = \frac{\sqrt{5}\, L_{2r-1} + 5F_{2r-1} - 2}{2}$$
$$= 5F_{2r-1} - 2 + \frac{\sqrt{5}\, L_{2r-1} - 5F_{2r-1}}{2} + 1$$
$$= 5F_{2r-1} - 2 + \beta_5 \ .$$

Since $\beta_5 = \beta_1$, the sequence now repeats and it follows that

$$
\begin{aligned}
L_{2r-1}\alpha &= L_{2r} - 1 + \beta_1 \\
&= L_{2r} - 1 + \frac{1}{1 + \beta_2} \\
&= L_{2r} - 1 + \cfrac{1}{1 + \cfrac{1}{F_{2r-1} - 2 + \beta_3}} \\
&= \cdots \\
&= [L_{2r-1} - 1, \dot{1}, F_{2r-1} - 2, 1, 5\dot{F}_{2r-1} - 2]
\end{aligned}
$$

as claimed.

3. <u>More general results.</u> Since F_{n+1}/F_n is a convergent in the simple continued fraction expansion of $(1 + \sqrt{5})/2$, the results of the preceding section suggest that one ask if there is any interesting connection between the simple continued fraction expansion of a quadratic surd ξ and the simple continued fraction expansion of $q_n\xi$ where p_n/q_n is the n^{th} convergent to ξ. The following theorems, which generalize those of Section 2, answer this question in the affirmative for surds of the form $\xi = [a, \dot{b}]$ or $\xi = [a, \dot{b}, \dot{c}]$.

<u>Theorem 5.</u> Let $\xi = [a, \dot{b}]$, let n be a positive integer, let p_k/q_k denote the k^{th} convergent to ξ and let $t_k = q_{k-1} + q_{k+1}$. Then $n\xi = [r, \dot{s}]$ if and only if $n = q_{2m-2}$, $r = p_{2m-2}$, and $s = t_{2m-2}$ for some integer $m \geq 1$.

<u>Theorem 6.</u> Let $\xi, n, p_k/q_k$ and t_k be as in Theorem 5. Then $n\xi = [u, \dot{v}, \dot{w}]$ if and only if $vn = q_{2m-1}$, $vu = p_{2m-1} - 1$, and $vw = t_{2m-1} - 2$ for some integer $m \geq 1$.

<u>Theorem 7.</u> Let $\xi = [a, \dot{b}, \dot{c}]$, let p_k/q_k be the k^{th} convergent to ξ, let $t_k = q_{k-1} + q_{k+1}$, and let $s_k = p_{k-1} + p_{k+1}$. Then, for every integer $r \geq 1$, we have

a) $$q_{2r}\xi = \left[p_{2r}, \dot{t}_{2r}, \frac{c}{b}\dot{t}_{2r}\right],$$

b) $$q_{2r-1}\xi = [p_{2r-1} - 1, \dot{1}, \dot{t}_{2r} - 2],$$

c) $$t_{2r-1}\xi = \left[s_{2r-1},\ \dot{q}_{2r-1},\ \left(c^2 + \frac{4c}{b}\right)\dot{q}_{2r}\right],$$

and

d) $$t_{2r}\xi = [s_{2r} - 1,\ \dot{1},\ q_{2r} - 2,\ 1,\ (bc + 4)\dot{q}_{2r} - 2]$$

Proof of Theorem 5. The convergents to $\xi = [a, \dot{b}]$ are given by the difference equations

$$q_n = bq_{n-1} + q_{n-2}$$

$$p_n = bp_{n-1} + p_{n-2}$$

with the initial conditions $q_0 = 1$, $q_1 = b$, $p_0 = a$, and $p_1 = ab + 1$. These are easily solved to obtain

$$q_n = \frac{1}{\sqrt{b^2 + 4}}\left\{\left(\frac{b + \sqrt{b^2 + 4}}{2}\right)^{n+1} - \left(\frac{b - \sqrt{b^2 + 4}}{2}\right)^{n+1}\right\},$$

(2) $$p_n = \frac{a}{2} \cdot t_{n-1} + \frac{ab + 2}{2} \cdot q_{n-1},$$

where

$$t_{n-1} = \left(\frac{b + \sqrt{b^2 + 4}}{2}\right)^n + \left(\frac{b - \sqrt{b^2 + 4}}{2}\right)^n$$

and it is easily shown by induction that $t_n = q_{n-1} + q_{n+1}$ for $n \geq 0$.

Moreover, since $[a, \dot{b}] = (2a - b + \sqrt{b^2 + 4})/2$, it follows that $n\xi = [r, \dot{s}]$ if and only if the equations

(3) $$n(2a - b) = 2r - s,$$ $$n\sqrt{b^2 + 4} = \sqrt{s^2 + 4}$$

simultaneously hold. The second of these equations is equivalent to

$$s^2 - n^2(b^2 + 4) = -4$$

and $s = b$, $n = 1$ is clearly the minimal positive solution. Therefore, every solution (s, n) in positive integers is given by the equation

$$s + n\sqrt{b^2 + 4} = 2\left(\frac{s + \sqrt{b^2 + 4}}{2}\right)^{2m-1}, \quad m = 1, 2, \cdots,$$

and it is easily shown by expanding the powers here and in (2) that this reduces to

$$s + n\sqrt{b^2 + 4} = t_{2m-2} + q_{2m-2}\sqrt{b^2 + 4} .$$

Also, from the second equation in (2), we have

$$\begin{aligned} r &= \frac{n(2a - b) + s}{2} \\ &= \frac{(2a - b)q_{2m-2} + t_{2m-2}}{2} \\ &= aq_{2m-2} + \frac{q_{2m-3} + q_{2m-1} - bq_{2m-2}}{2} \\ &= aq_{2m-2} + q_{2m-3} \\ &= p_{2m-2} \end{aligned}$$

since it is easily proved by induction that $aq_n + q_{n-1} = p_n$ for all n. This completes the proof.

Proof of Theorem 6. Note in particular that the preceding argument essentially shows that

$$\frac{(b + \sqrt{b^2 + 4})^k}{2^{k-1}} = t_{k-1} + q_{k-1}\sqrt{b^2 + 4}, \quad k \geq 1 . \tag{4}$$

Also, it is easily shown by induction that

$$\sum_{k=0}^{m} \binom{m}{k} b^k q_{k-1} = q_{2m-1} \quad ,$$

$$\sum_{k=0}^{m+1} \binom{m}{k-1} b^k q_{k-1} = bq_{2m} \quad ,$$

$$\sum_{k=0}^{m} \binom{m}{k} b^k t_{k-1} = q_{2m-1} \quad ,$$

and

$$\sum_{k=0}^{m} \binom{m}{k} b^k q_{k-2} = q_{2m-2} \quad .$$

Now, as in the preceding proof, one can show that $n\xi = [u, \dot{v}, \dot{w}]$ if and only if $vw + 2$ and vn are simultaneously positive integral solutions of the Pell equation

$$(vw + 2)^2 - (vn)^2(b^2 + 4) = 4 \tag{6}$$

and of $n(2a - b) = 2u - w$. Also, the general solution of (6) is given by

$$(vw + 2) + vn\sqrt{b^2 + 4} = 2\left\{\frac{b^2 + 2 + b\sqrt{b^2 + 4}}{2}\right\}^m, \quad m = 1, 2, \cdots .$$

Using the equalities in (4) and (5) this may be simplified to give

$$2\left\{\frac{(b^2 + 2) + b\sqrt{b^2 + 4}}{2}\right\}^m = 2\left\{\frac{2 + b(b + \sqrt{b^2 + 4})}{2}\right\}^m$$

$$= \sum_{k=0}^{m} \binom{m}{k} \frac{(b + \sqrt{b^2 + 4})^k b^k}{2^{k-1}}$$

$$= \sum_{k=0}^{m} \binom{m}{k} (t_{k-1} + q_{k-1}\sqrt{b^2 + 4})b^k$$

$$= \sum_{k=0}^{m} \binom{m}{k} t_{k-1} + \sqrt{b^2 + 4}\sum_{k=0}^{m} \binom{m}{k} b^k q_{k-1}$$

$$= t_{2m-1} + \sqrt{b^2 + 4} \cdot q_{2m-1} \quad .$$

Thus, $vw + 2 = t_{2m-1}$, $vn = q_{2m-1}$, and

$$vu = \frac{vn(2a - b) + vw}{2}$$

$$= \frac{(2a - b)q_{2m-1} + t_{2m-1} - 2}{2}$$

$$= p_{2m-1} - 1$$

as in the preceding proof.

Proof of Theorem 7. For

$$\xi = [a, \dot{b}, \dot{c}] = a + \frac{-bc + \sqrt{b^2c^2 + 4bc}}{2b} \quad ,$$

define

$$A = \xi - a, \quad B = A + c, \quad C = bA/c,$$

and

$$D = bB/c = C + b \; .$$

The following identities are useful:

a) $$q_{2k}^2 = 1 + cq_{2k+1}q_{2k-1}/b$$

b) $$q_{2k+1}^2 = b(q_{2k+2}q_{2k} + 1)/c$$

c) $$p_{2k} = aq_{2k} + cq_{2k-1}/b$$

d) $$p_{2k+1} = aq_{2k+1} + q_{2k}$$

e) $$(q_{2k}B - cq_{2k+1}/b)(q_{2k}A + cq_{2k+1}/b) = c/b$$

f) $$(q_{2k}D - q_{2k+1})(q_{2k}C + q_{2k+1}) = b/c$$

g) $$(q_{2k+1}B - q_{2k+2} + 1)(q_{2k+1}A + q_{2k+2} - 1) = q_{2k+2} + q_{2k} - 2 = t_{2k+1} - 2$$

h) $$t_{2r} = bt_{2r-1} + t_{2r-2}$$

i) $$t_{2r+1} = ct_{2r} + t_{2r-1}$$

j) $$s_{2k} = at_{2k} + t_{2k-1}$$

k) $$s_{2k+1} = at_{2k+1} + ct_{2k}/b$$

m) $$ct_{2k}^2 - bt_{2k+1}t_{2k-1} = ct_{2k}t_{2k+2} - bt_{2k+1}^2 = -b(bc + 4)$$

n) $$(t_{2r-1}B - ct_{2r}/b)(t_{2r-1}A + ct_{2r}/b) = c(bc + 4)/b$$

o) $$(t_{2r-1}D - t_{2r})(t_{2r-1}C + t_{2r}) = b(bc + 4)/c \quad .$$

These are proved in a straightforward manner.

To prove 7a, we have by identity c) and the definition of B that

$$\begin{aligned} q_{2k}\xi &= q_{2k}a + q_{2k}A \\ &= q_{2k}a + cq_{2k-1}/b - cq_{2k-1}/b + q_{2k}A \\ &= p_{2k} + Bq_{2k} - cq_{2k+1}/b\,, \qquad \text{definition of } B, \\ &= p_{2k} + 1/\beta_1 \quad . \end{aligned}$$

Using identity e) and the definitions of C, D and t_{2k}, we have

$$
\begin{aligned}
\beta_1 &= b(q_{2k}A + cq_{2k+1}/b)/c \\
&= q_{2k}C + q_{2k+1}\,, \qquad\qquad \text{definition of } C, \\
&= q_{2k+1} + q_{2k-1} + q_{2k}C - q_{2k-1} \\
&= t_{2k} + q_{2k}D - q_{2k+1} \\
&= t_{2k} + 1/\beta_2 \quad .
\end{aligned}
$$

Using identity f) and the definitions of C and t_{2k}, we have

$$
\begin{aligned}
\beta_2 &= c(q_{2k}C + q_{2k+1})/b \\
&= q_{2k}A + cq_{2k+1}/b \\
&= c(q_{2k+1} + q_{2k-1})/b + q_{2k}A - cq_{2k-1}/b \\
&= ct_{2k}/b + 1/\beta_1 \ .
\end{aligned}
$$

We therefore have

$$
q_{2k}\xi = [p_{2k},\ \dot{t}_{2k},\ c\dot{t}_{2k}/b] \ ,
$$

proving 7a.

The proof of 7b is similar and uses identity g) at a key point in the argument. The argument will not be presented here.

To prove 7c, we note that

$$
\begin{aligned}
t_{2r-1}\xi &= t_{2r-1}a + t_{2r-1}A \\
&= s_{2r-1} + t_{2r-1}A - ct_{2r-2}/b \\
&= s_{2r-1} + (t_{2r-1}B - ct_{2r}/b) \\
&= s_{2r-1} + \frac{c(bc + 4)/b}{t_{2r-1}A + ct_{2r}/b} \\
&= s_{2r-1} + \frac{1}{\beta_1} \quad ,
\end{aligned}
$$

$$\begin{aligned}
\beta_1 &= (bt_{2r-1}A + ct_{2r})/c(bc+4) \\
&= q_{2r-1} + (bt_{2r-1}A + ct_{2r} - c(bc+4)q_{2r-1})/c(bc+4) \\
&= q_{2r-1} + (bt_{2r-1}A + cq_{2r+1} - c(bc+3)q_{2r-1})/c(bc+4) \\
&= q_{2r-1} + (bt_{2r}A + cbq_{2r} - c(bc+2)q_{2r-1})\,/c(bc+4) \\
&= q_{2r-1} + (bt_{2r-1}A + cbq_{2r} - (bc+2)(q_{2r} - q_{2r-2}))/c(bc+4) \\
&= q_{2r-1} + (bt_{2r-1}A - 2q_{2r} + (bc+2)q_{2r-2})/c(bc+4) \\
&= q_{2r-1} + (bt_{2r-1}A - 2(cq_{2r-1} + q_{2r-2}) + (bc+2)q_{2r-2})/c(bc+4) \\
&= q_{2r-1} + (bt_{2r-1}A - 2cq_{2r-1} + bcq_{2r-2})/c(bc+4) \\
&= q_{2r-1} + (bt_{2r-1}A - c(q_{2r-1} + q_{2r-3}))/c(bc+4) \\
&= q_{2r-1} + (bt_{2r-1}A - c(t_{2r-2}))/c(bc+4) \\
&= q_{2r-1} + (t_{2r-1}C - t_{2r-2})/(bc+4) \\
&= q_{2r-1} + (t_{2r-1}D - t_{2r})/(bc+4) \\
&= q_{2r-1} + \frac{b(bc+4)/c}{(bc+4)(t_{2r-1}C + t_{2r})} \\
&= q_{2r-1} + \frac{1}{t_{2r-1}A + ct_{2r}/b} \\
&= q_{2r-1} + \frac{1}{\beta_2} = q_{2r-1} + \frac{1}{\beta_1((bc^2+4c)/b)}
\end{aligned}$$

and

$$\begin{aligned}
\beta_2 &= (bc^2 + 4c)\beta_1/b = ((bc^2+4c)/b)(q_{2r-1} + \frac{1}{\beta_2}) \\
&= ((bc^2+4c)/b)q_{2r-1} + \frac{1}{\beta_1} \quad .
\end{aligned}$$

Hence we obtain

$$t_{2k-1}\xi = [s_{2k-1},\ \dot{q}_{2k-1},\ ((bc^2+4c)\dot{/}b)q_{2k-1}] \quad .$$

The proof of part d) is similar and the argument is omitted.

The following theorem, which is stated without proof, is a partial converse of Theorem 7.

Theorem 8. Let ξ, p_k, q_k, s_k, and t_k be as in Theorem 7 and let n, u, and v be positive integers.

a) If v is such that b divides cv and $n\xi = [u, \dot{v}, c\dot{v}/b]$, then $n = q_{2r}$, $u = t_{2r}$, and $v = p_{2r}$ for some positive integer r.

b) If $n\xi = [u, \dot{1}, \dot{v}]$, then $n = q_{2r-1}$, $u = p_{2r-1} - 1$, and $v = t_{2r} - 2$ for some positive integer r.

Remark. When a simple continued fraction has a partial quotient 1 the corresponding approximation of the convergent to the number in question is not as good as when other integers are partial quotients. The 1's can be eliminated as all but the first partial quotient if it is permitted to have -1's as numerators. The corresponding convergents would then be better approximations than the original ones.

Setting about to purge the 1's from the expressions obtained in Theorems 2, 3b, 7b and 7d we ran across an interesting pattern that allowed us to simplify the notation. Let us define the symbol $-[a_0, a_1, a_2, \cdots]$ to be the expression

$$a_0 + \cfrac{-1}{a_1 + \cfrac{-1}{a_2 + \cdots}}$$

Although this expression might not always be meaningful, it is in the cases we consider here.

With the new notation we are able to restate a few of the theorems as Theorem 9:

a) $$F_k\alpha = (-1)^{k+1}\left[F_{k+1}, \dot{L}_k\right].$$

b) $$L_k\alpha = (-1)^k\left[L_{k+1}, \dot{F}_k, 5\dot{F}_k\right].$$

c) If $\xi = [a, \dot{b}]$ then $q_k\xi = (-1)^k[p_k, \dot{t}_k]$.

d) If $\xi = [a, \dot{b}, \dot{c}]$ and k odd, then $q_k\xi = -[p_k, \dot{t}_k]$.

e) If $\xi = [a, \dot{b}, \dot{c}]$ and k even, then $t_k \xi = -[s_k, \dot{q}_k, (bc + \dot{4})q_k]$.

The proofs are quite similar to the original proofs and are omitted.

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RECURRENCE RELATIONS FOR SEQUENCES LIKE $\{F_{F_n}\}$

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In [1] the problem of finding recurrence relations for the sequences $\{F_{F_n}\}$, $\{F_{L_n}\}$, $\{L_{F_n}\}$, $\{L_{L_n}\}$ — where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively — is proposed. What follows is an investigation of this problem and some of its generalizations.

Let r and s be any two nonzero elements of a field $F^* = (F, +, \cdot)$ in which r^n is defined in the usual way with the field operations, $+, \cdot$. Define $\{U_n\}$ and $\{V_n\}$ by $U_n = (r^n - s^n)/(r - s)$ and $V_n = r^n + s^n$ for all integers n. Furthermore, let $\{H_n\}$ be any generalized Fibonacci sequence consisting of integers — that is H_0 and H_1 are integers and $H_{n+2} = H_{n+1} + H_n$ for all integers n. Some recurrence relations for sequences such as $\{U_{H_n}\}$ and $\{V_{H_n}\}$ will be derived here.

Let $\{g_n\}$ be any sequence in n obeying the recurrence relation $g_{n+2} = (r + s)g_{n+1} - rsg_n$ for all integers n. Then there are constants C_1 and C_2 in F^* such that $g_n = C_1r^n + C_2s^n$ for all integers n. Define $\{X_n\}$, $\{Y_n\}$ and $\{G_n\}$ by $X_n = U_{H_n}$, $Y_n = V_{H_n}$ and $G_n = g_{H_n}$ for all integers n. From here on, when n is written, understand that n can take on all integer values unless otherwise indicated. For convenience write $R_n = r^{H_n}$ and $S_n = s^{H_n}$.

Consider the product $G_{n+2}Y_{n+1}$.

$$\begin{aligned} G_{n+2}Y_{n+1} &= (C_1R_{n+2} + C_2S_{n+2})(R_{n+1} + S_{n+1}) \\ &= C_1R_{n+2}R_{n+1} + C_2S_{n+2}S_{n+1} + C_1R_{n+2}S_{n+1} + C_2R_{n+1}S_{n+2} \\ &= C_1R_{n+3} + C_2S_{n+3} + R_{n+1}S_{n+1}(C_1R_n + C_2S_n) \\ &= G_{n+3} + (rs)^{H_{n+1}} \cdot G_n \end{aligned}$$

Thus,

$$G_{n+3} = G_{n+2}Y_{n+1} - (rs)^{H_{n+1}}G_n \quad . \tag{1}$$

*Student

A corollary to (1) is the relatively simple recurrence relation for $\{Y_n\}$.

(2) $$Y_{n+3} = Y_{n+2}Y_{n+1} - (rs)^{H_{n+1}} \cdot Y_n \quad .$$

When $rs = \pm 1$, (2) is especially simple;

$$r = \frac{1}{2}(1 + \sqrt{5}) \quad \text{and} \quad s = \frac{1}{2}(1 - \sqrt{5})$$

gives

(3) $$L_{H_{n+3}} = L_{H_{n+2}}L_{H_{n+1}} - (-1)^{H_{n+1}}L_{H_n} \quad ,$$

where L_n is the n^{th} Lucas number.

Consider the product $Y_{n+2}G_{n+1}$.

$$\begin{aligned} Y_{n+2}G_{n+1} &= C_1R_{n+2}R_{n+1} + C_2S_{n+2}S_{n+1} + C_1R_{n+1}S_{n+2} + C_2R_{n+2}S_{n+1} \\ &= G_{n+3} + R_{n+1}S_{n+1}(C_1S_n + C_2R_n) \quad . \end{aligned}$$

But

$$C_1s^n + C_2r^n = (C_1 + C_2)V_n - (C_1r^n + C_2s^n) = g_0V_n - g_n \quad .$$

Thus

$$C_1S_n + C_2R_n = g_0Y_n - G_n \quad ,$$

and

$$Y_{n+2}\,G_{n+1} = G_{n+3} + (rs)^{H_{n+1}}(g_0\,Y_n - G_n) \quad .$$

That is,

(4) $$G_{n+3} = Y_{n+2}G_{n+1} + (rs)^{H_{n+1}}(G_n - g_0Y_n) \quad .$$

Add (1) and (4) to get

$$(5) \qquad 2G_{n+3} = G_{n+1}Y_{n+2} + G_{n+2}Y_{n+1} - (rs)^{H_{n+1}}g_0Y_n$$

Now consider the product $(r - s)^2X_{n+2}Y_{n+1}$.

$$\begin{aligned}(r - s)^2X_{n+1}X_{n+2} &= (R_{n+2} - S_{n+2})(R_{n+1} - S_{n+1}) \\ &= R_{n+3} + S_{n+3} - R_{n+1}S_{n+1}(R_n + S_n) \\ &= Y_{n+3} - (rs)^{H_{n+1}}Y_n \ .\end{aligned}$$

Thus,

$$(6) \qquad Y_{n+3} = (r - s)^2X_{n+2}X_{n+1} + (rs)^{H_{n+1}}Y_n \ .$$

Some second-order recurrence relations can be obtained by using the following simple and easily verified identities — which hold for all integers a and b — by putting $a = H_n$ and $b = H_{n+1}$ or $a = H_{n+1}$ and $b = H_n$.

$$U_{a+b} = r^aU_b + s^bU_a$$
$$V_{a+b} = r^aV_b - (r - s)s^bU_a = s^aV_b + (r - s)r^bU_a$$
$$(r - s)U_{a+b} = r^aV_b - s^bV_a$$

Some of the recurrence relations are

$$(7) \qquad \begin{aligned} X_{n+2} &= R_nX_{n+1} + S_{n+1}X_n = S_nX_{n+1} + R_{n+1}X_n \\ Y_{n+2} &= R_nY_{n+1} - (r - s)S_{n+1}X_n \\ &= S_nY_{n+1} + (r - s)R_{n+1}X_n \\ &= (r - s)R_nX_{n+1} + S_{n+1}Y_n \\ &= -(r - s)S_nX_{n+1} + R_{n+1}Y_n \end{aligned}$$

From (7) it immediately follows that

$$(7') \qquad \begin{aligned} 2X_{n+2} &= X_{n+1}Y_n - X_nY_{n+1} \\ 2Y_{n+2} &= (r - s)^2X_{n+1}X_n - Y_{n+1}Y_n \end{aligned}$$

For a fixed integer j define $\{Z_n\}$ and $\{W_n\}$ by $Z_n = U_{H_n+j}$ and $W_n = V_{H_n+j}$. Thus,

$$(r - s)Z_n = r^jR_n - s^jS_n \quad \text{and} \quad W_n = r^jR_n + s^jS_n \quad .$$

Now,

$$\begin{aligned} (r - s)Z_{n+2} &= r^jR_{n+1}R_n - s^jS_{n+1}S_n \\ &= R_n(r^jR_{n+1} - s^jS_{n+1}) + S_{n+1}(r^jR_n - s^jS_n) \\ &\qquad - R_nS_{n+1}(r^j - s^j) \end{aligned}$$

so that

$$(8) \qquad Z_{n+2} = R_nZ_{n+1} + S_{n+1}Z_n - (rs)^{H_n}S_{n-1}U_j$$

Similarly,

$$(9) \qquad Z_{n+2} = S_nZ_{n+1} + R_{n+1}Z_n - (rs)^{H_n}R_{n-1}U_j$$

Add (8) and (9) to get

$$(10) \qquad 2Z_{n+2} = Y_nZ_{n+1} + Y_{n+1}Z_n - (rs)^{H_n}Y_{n-1}U_j$$

Also,

$$\begin{aligned} W_{n+2} &= r^jR_{n+1}R_n + s^jS_{n+1}S_n \\ &= R_n(r^jR_{n+1} - s^jS_{n+1}) + S_{n+1}(r^jR_n + s^jS_n) \\ &\qquad - R_nS_{n+1}(r^j - s^j) \end{aligned}$$

and

$$(11) \qquad W_{n+2} = (r - s)R_n Z_{n+1} + S_{n+1}W_n - (r - s)(rs)^{H_n} S_{n-1}U_j \quad ;$$

Similarly,

$$(12) \qquad W_{n+2} = (s - r)S_n Z_{n+1} + R_{n+1}W_n - (s - r)(rs)^{H_n} R_{n-1}U_j$$

Add (11) and (12) to get

$$(13) \qquad 2W_{n+2} = (r - s)^2 X_n Z_{n+1} + Y_{n+1}W_n + (r - s)^2(rs)^{H_n} X_{n-1}U_j \; .$$

When $r = (1 + \sqrt{5})/2$ and $s = (1 - \sqrt{5})/2$, (10) and (13) become

$$(14) \qquad \begin{aligned} 2F_{H_{n+2}+j} &= L_{H_n} F_{H_{n+1}+j} + L_{H_{n+1}} F_{H_n+j} - (-1)^{H_n} L_{H_{n-1}} F_j \\ 2L_{H_{n+2}+j} &= 5F_{H_n} F_{H_{n+1}+j} + L_{H_{n+1}} L_{H_n+j} + 5(-1)^{H_n} F_{H_{n-1}} F_j \end{aligned}$$

where F_n is the n^{th} Fibonacci number and L_n is the n^{th} Lucas number.

The techniques used above in deriving recurrence relations are not entirely inhibited when sequences of the type $\{U_{K_n}\}$ and $\{V_{K_n}\}$, where $\{K_n\}$ is a sequence of integers obeying a linear, homogeneous recurrence relation with constant coefficients, are considered. Let $\{K_n\}$ obey the recurrence relation

$$K_{n+m} = \sum_{j=0}^{m} p_j K_{n+m-j} \quad ,$$

where m is a fixed integer, and with p_j, K_n being integers when n is nonnegative. Then $\{V_{K_n}\}$ and $\{U_{K_n}\}$ are defined for n nonnegative; if $p_m = \pm 1$, then the definition applies for all n. Repeated application of the identity

$U_{a+b} = r^a U_b + s^b U_a$ gives $U_{a_1+a_2+\cdots a_m}$ as a linear combination of U_{aj}, $j = 1, 2, \cdots, m$, with the coefficients being products of powers of r and s. By putting $a_j = p_j K_{n+m-j}$, when $n + m - j$ is nonnegative, and by utilizing repeatedly the identities

$$U_{-n} = -(rs)^{-n} U_n$$

$$U_{2n} = U_n V_n$$

$$U_{(2k+1)n} = U_n \left[(rs)^{kn} + \sum_{j=0}^{k-1} (rs)^{jn} V_{2(k-j)n} \right], \quad k \geq 1,$$

m^{th} order recurrence relations are easily produced for $\{U_{K_n}\}$. $\{V_{K_n}\}$ may be treated similarly by repeated application of the identity $V_{a+b} = r^a V_b - (r - s)s^b U_a$ and by utilization of the identities

$$V_{-n} = (rs)^{-n} V_n$$

$$V_{2kn} = V_{kn}^2 - 2(rs)^{kn}$$

$$V_{(2k+1)n} = V_n (-r^n s^n)^k + \sum_{j=0}^{k-1} (-r^n s^n)^j V_{2(k-j)n}, \quad k \geq 1$$

$$(r - s)U_{a+b} = r^a V_b - s^b V_a .$$

A special case of interest occurs when $m = 3$ and $p_j = 1$, $j = 1, 2, 3$. Letting $A_n = r^{K_n}$ and $B_n = s^{K_n}$, $D_n = U_{K_n}$ and $E_n = V_{K_n}$, then $U_{a+b} = r^a U_b - s^b U_a$ gives

$$\text{(15)} \qquad D_{n+3} = A_n A_{n+1} D_{n+2} + A_n B_{n+2} D_{n+1} + B_{n+1} B_{n+2} D_n$$
$$= A_n A_{n+1} D_{n+2} + B_n B_{n+2} D_{n+1} + A_{n+1} B_{n+2} D_n$$

and

$$2D_{n+3} = 2A_n A_{n+1} D_{n+2} + B_{n+2} E_n D_{n+1} + B_{n+2} E_{n+1} D_n .$$

Similarly,

$$2D_{n+3} = 2B_nB_{n+1}D_{n+2} + A_{n+2}E_nD_{n+1} + A_{n+2}E_{n+1}D_n .$$

Thus,

$$4D_{n+3} = 2(A_nA_{n+1} + B_nB_{n+1})D_{n+2} + E_nE_{n+2}D_{n+1} + E_{n+1}E_{n+2}D_n .$$

But

$$\begin{aligned} A_nA_{n+1} + B_nB_{n+1} &= A_n(A_{n+1} + B_{n+1}) - B_{n+1}(A_n - B_n) \\ &= A_nE_{n+1} - B_{n+1}(r - s)D_n \\ &= B_nE_{n+1} + A_{n+1}(r - s)D_n , \end{aligned}$$

so that

$$2(A_nA_{n+1} + B_nB_{n+1}) = E_nE_{n+1} + (r - s)^2D_nD_{n+1} ,$$

and

$$(16) \quad 4D_{n+3} = (E_nE_{n+1} + (r - s)^2D_nD_{n+1})D_{n+2} + E_nE_{n+2}D_{n+1} + E_{n+1}E_{n+2}D_n .$$

Also, $V_{a+b} = r^aV_b - (r - s)s^bU_a$ and $(r - s)U_{a+b} = r^aV_b - s^bV_a$ give

$$(17) \quad \begin{aligned} E_{n+3} &= A_nA_{n+1}E_{n+2} - A_nB_{n+2}E_{n+1} + B_{n+1}B_{n+2}E_n \\ &= A_nA_{n+1}E_{n+2} + B_nB_{n+2}E_{n+1} - A_{n+1}B_{n+2}E_n \end{aligned}$$

and

$$2E_{n+3} = 2A_nA_{n+1}E_{n+2} - (r - s)B_{n+2}D_nE_{n+1} - (r - s)B_{n+2}D_{n+1}E_n .$$

Similarly,

$$2E_{n+3} = 2B_nB_{n+1}E_{n+2} + (r - s)A_{n+2}D_nE_{n+1} + (r - s)A_{n+2}D_{n+1}E_n .$$

Thus,

4E $$4E_{n+3} = 2(A_nA_{n+1} + B_nB_{n+1})E_{n+2} + (r - s)^2D_nD_{n+2}E_{n+1} + (r - s)^2D_{n+1}D_{n+2}E_n$$

and

$$4E_{n+3} = (E_nE_{n+1} + (r - s)^2D_nD_{n+1})E_{n+2} + (r - s)^2D_nD_{n+2}E_{n+1} + (r - s)^2D_{n+1}D_{n+2}E_n \ . \tag{18}$$

Given D_0, D_1, D_2, E_0, E_1 and E_2, (16) and (18) completely determine $\{D_n\}$ and $\{E_n\}$, for $n \geq 0$.

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AN APPLICATION OF UNIFORM DISTRIBUTIONS TO THE FIBONACCI NUMBERS

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Let $\mu_1 = 1$, $\mu_2 = 2$ and $\mu_n = \mu_{n-1} + \mu_{n-2}$ $(n \geq 3)$ be the Fibonacci numbers and let p_n be the number of digits in μ_n. It is known [1] that the number of divisions required to determine (μ_{n+1}, μ_n) by the Euclidean Algorithm is n. Also, it is shown in the proof of Lamé's theorem [2] that

$$n < \frac{p_n}{\log \xi} + 1, \quad \text{where} \quad \xi = \frac{1 + \sqrt{5}}{2} \quad .$$

A similar argument [1] shows that

$$n > \frac{p_n - 1}{\log \xi} \quad .$$

Combining these results, we have

$$\left[\frac{p_n - 1}{\log \xi}\right] \leq n - 1 \leq \left[\frac{p_n}{\log \xi}\right] \quad . \tag{1}$$

It has been shown by Brown [3] that the upper bound in (1) is attained for infinitely many n. The object of this note is to show that both the upper and lower bounds in (1) are attained for sets of values of n having positive density.

Let ϕ_n be the fractional part (mantissa) of $\log \mu_n$. Then, since $p_n = 1 + [\log \mu_n]$, we have $p_n = 1 + \log \mu_n - \phi_n$. Also, since

$$\mu_n \sim \frac{\xi^{n+1}}{\sqrt{5}} \quad ,$$

we have

$$P_n = 1 + (n + 1) \log \xi - \log \sqrt{5} - \phi_n + o(1). \tag{2}$$

Hence

$$n - 1 = \frac{p_n - 1}{\log \xi} - 2 + \frac{\log \sqrt{5}}{\log \xi} + \frac{\phi_n}{\log \xi} + o(1)$$

and it follows that

$$n - 1 < \frac{p_n - 1}{\log \xi} - \frac{1}{4} + 5\phi_n$$

for all sufficiently large n. Thus

$$n - 1 \leq \left[\frac{p_n - 1}{\log \xi}\right]$$

and

$$n - 1 = \left[\frac{p_n - 1}{\log \xi}\right]$$

if

$$\phi_n \leq \frac{1}{20}$$

and n is sufficiently large.

It also follows from (2) that

$$p_n \leq (n + 1) \log \xi - \log \sqrt{5} + \frac{1}{10} + o(1)$$

or

$$n - 1 \geq \frac{p_n}{\log \xi} + \frac{\log \sqrt{5} - \frac{1}{10}}{\log \xi} - 2 + o(1)$$

when

$$\phi_n \geq \frac{9}{10} .$$

Hence

$$n - 1 > \frac{p_n}{\log \xi} - 1$$

and it follows that

$$n - 1 \geq \left[\frac{p_n}{\log \xi} \right]$$

and

$$n - 1 = \left[\frac{p_n}{\log \xi} \right]$$

when

$$\phi_n \geq \frac{9}{10}$$

and n is sufficiently large.

The desired result will follow when we show that the sequence $\{\log \mu_n\}$ is uniformly distributed modulo one [4]. By (2) we have

$$\log \mu_n = (n + 1) \log \xi - \log \sqrt{5} + o(1) .$$

Thus, for every positive integer h,

$$\begin{aligned} \exp (2\pi ih \log \mu_n) &= \exp (-2\pi ih \log \sqrt{5}) \exp (o(1)) \exp (2\pi ih (n + 1) \log \xi) \\ &= c(1 + o(1)) \exp (2\pi ih (n + 1) \log \xi) , \end{aligned}$$

where c is a constant.

Hence

$$\sum_{n=1}^{m} \exp(2\pi ih \log \mu_n) = c \sum_{n=1}^{m} \exp(2\pi ih(n+1) \log \xi) + o(m) .$$

since $\log \xi$ is irrational, the sequence $\{(n+1)\log \xi\}$ is uniformly distributed modulo one and it follows from Weyl's criterion that the sequence $\{\log \mu_n\}$ is uniformly distributed modulo one.

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THE HEIGHTS OF FIBONACCI POLYNOMIALS AND AN ASSOCIATED FUNCTION

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Define the sequence of Fibonacci polynomials $\{f_n(x)\}$ by

$$f_1(x) = 1, \quad f_2(x) = x; \quad f_n(x) = xf_{n-1}(x) + f_{n-2}(x) \quad (n \geq 3) .$$

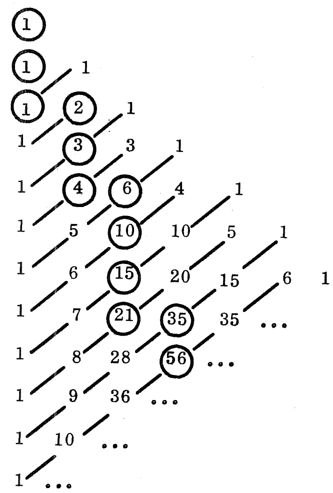
Then it has been shown [3] that

$$(1) \qquad f_n(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-j-1}{j} x^{n-2j-1} ,$$

where $[x]$ represents the greatest integer contained in x. Since $f_{n+1}(x) = i^{-n}U_n(ix/2)$, where the $U_n(x)$ are the Chebyshev polynomials of the second kind, we note that the Fibonacci polynomials are essentially the Chebyshev polynomials. Define the Fibonacci sequence $\{F_n\}$ by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ $(n \geq 3)$, and the Lucas sequence $\{L_n\}$ by $L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$ $(n \geq 3)$. It then follows from these recurrence relations that $f_n(1) = F_n$. By the height of $f_n(x)$, denoted by $m(n)$, we mean the greatest coefficient of $f_n(x)$, that is

$$m(n) = \max \left\{ \binom{n-j-1}{j} \right\} \quad (j = 0, 1, \cdots, [(n-1)/2]) .$$

Since the coefficients in (1) are diagonals of Pascal's Triangle, the $m(n)$ are the maximum entries along these diagonals, and they form the pattern exhibited below. Interest was first aroused in these numbers when it was observed that if the heights $m(n)$ and $m(n+1)$ were in adjacent columns, then they were in the ratio of consecutive Fibonacci numbers (e.g., 1:2, 4:6 = 2:3, 21:35 = 3:5). Although this is not true in general, some interesting properties derived from these ratios were found.



In order to define these cross-over ratios, we must first verify that this initial pattern continues, so that the only changes in the column pattern are lateral jumps of one column. We do this in the following:

Theorem 1. Denote logical implication by "$\Rightarrow$". Then

(i) $$\binom{n}{k} \ge \binom{n-1}{k+1} \Rightarrow \binom{n-1}{k+1} \ge \binom{n-2}{k+2},$$

(ii) $$\binom{n-1}{k+1} > \binom{n}{k} \Rightarrow \binom{n}{k+1} > \binom{n+1}{k},$$

(iii) $$\binom{n}{k} > \binom{n+1}{k-1} \Rightarrow \binom{n+1}{k=1} > \binom{n+2}{k-1}$$

(iv) $$\binom{n}{k} > \binom{n-1}{k+1} \text{ and } \binom{n+1}{k} \le \binom{n}{k+1} \Longrightarrow \binom{n}{k+1} > \binom{n-1}{k+2}$$

Proof. We prove (i), the other parts using similar techniques. (i) is trivial for $n \le k+1$. Assume $n - k \ge 2$, so that denoting logical equivalence by "$\Leftrightarrow$" we have

$$\binom{n}{k} \ge \binom{n-1}{k+1} \Leftrightarrow n(k+1) \ge (n-k)(n-k-1) \Rightarrow n(k+1)+n-k-2 = (n-1)(k+2)$$

$$\ge (n-k)^2 - 2 + 8 - 5(n-k) = (n-k-2)(n-k-3) \Longleftrightarrow \binom{n-1}{k+1} \ge \binom{n-2}{k+2}.$$

A little reflection will show that these results imply that if

$$m(n) = \binom{u}{v}, \quad \text{then } m(n+1) = \binom{u+1}{v} \text{ or } \binom{u}{v+1} .$$

Call the column of ones the 0^{th} column, and label the other columns of Pascal's Triangle consecutively. Choose n such that $m(n)$ appears in the $(k-1)^{th}$ column and $m(n+1)$ appears in the k^{th} column. Then $r_k = m(n)/m(n+1)$ is called the k^{th} cross-over ratio. By Theorem 1, r_k is well-defined and unique.

Theorem 2. For r_k as defined above we have

$$r_k = k/[\tfrac{1}{2}(k + 1 + \sqrt{5k^2 - 2k + 1})] .$$

where $[x]$ denotes the greatest integer contained in x.

Proof. If n is the greatest integer for which

$$\binom{n}{k-1} > \binom{n-1}{k} ,$$

then clearly

$$\binom{n}{k-1}$$

is the greatest height in the $(k-1)^{th}$ column. This criterion is equivalent to

$$nk > (n - k - 1)(n - k) \Longleftrightarrow n^2 - (3k-1)n + k^2 - k < 0 .$$

The greatest n for which this holds is the greatest integer contained in the largest root of

$$n^2 - (3k - 1)n + k^2 - k = 0 .$$

so that

$$n = [\tfrac{1}{2}(3k - 1 + \sqrt{5k^2 - 2k + 1})] .$$

Thus

$$r_k = \binom{n}{k-1} \bigg/ \binom{n}{k} = \frac{k}{n-k+1} = k/[\tfrac{1}{2}(k + 1 + \sqrt{5k^2 - 2k + 1})].$$

This result makes computation of the cross-over ratio for a given column simple. A limited evaluation of the expression in the denominator is given later in the paper. From Theorem 2 we may conclude

Theorem 3. For $\alpha = (1 + \sqrt{5})/2$, we have

$$\lim_{n \to \infty} r_n = 1/\alpha .$$

Proof. Since

$$n/\{\tfrac{1}{2}(n + 1 + \sqrt{5n^2 - 2n + 1})\} \le r_n < n/\{\tfrac{1}{2}(n - 1 + \sqrt{5n^2 - 2n + 1})\} ,$$

the result follows from

$$\lim_{n \to \infty} n/\{\tfrac{1}{2}(n + 1 + \sqrt{5n^2 - 2n + 1})\} = \lim_{n \to \infty} n/\{\tfrac{1}{2}(n - 1 + \sqrt{5n^2 - 2n + 1})\} = 1/\alpha$$

It has been shown [4] that

$$\lim_{n \to \infty} F_n/F_{n+1} = \lim_{n \to \infty} L_n/L_{n+1} = 1/\alpha ,$$

so it is not surprising to observe that F_n/F_{n+1} is a cross-over ratio for $n \ge 2$, and L_n/L_{n+1} is one for $n \ge 4$. It is our aim to prove this holds in general.

Theorem 4. Let $h(k) = [\frac{1}{2}(k + 1 + \sqrt{5k^2 - 2k + 1})]$. Then $h(F_n) = F_{n+1}$ for $n \ge 2$, and $h(L_n) = L_{n+1}$ for $n \ge 4$.

Proof. We will prove $h(F_n) = F_{n+1}$, the proof for Lucas numbers involving no new ideas. Since $x - 1 < [x] \le x$, the assertion is equivalent to

$$\tfrac{1}{2}\left(F_n - 1 + \sqrt{5F_n^2 - 2F_n + 1}\right) < F_{n+1} \le \tfrac{1}{2}\left(F_n + 1 + \sqrt{5F_n^2 - 2F_n + 1}\right).$$

The right side is equivalent to

$$(2F_{n+1} - F_n - 1)^2 = (L_n - 1)^2 \le 5F_n^2 - 2F_n + 1$$

$$\Longleftrightarrow L_n^2 - 5F_n^2 \le 2(L_n - F_n) = 4F_{n-1}.$$

Now it is known [2; Identity XI] that $L_n^2 - 5F_n^2 = 4(-1)^n$, so the last inequality is valid for $n > 2$. The case $n = 2$ is verified directly. The left side is equivalent to

$$(2F_{n+1} - F_n + 1)^2 = (L_n + 1)^2 > 5F_n^2 - 2F_n + 1$$

$$\Longleftrightarrow L_n^2 - 5F_n^2 > -2(L_n + F_n)$$

$$\Longleftrightarrow 4(-1)^n > -4F_{n+1}$$

which is valid for $n > 1$, completing the proof.

Theorem 5. F_n/F_{n+1} is a cross-over ratio for $n \ge 2$, and L_n/L_{n+1} is a cross-over ratio for $n \ge 4$.

Proof. By Theorems 2 and 4 we have $r_{F_n} = F_n/F_{n+1}$ for $n \ge 2$, and $r_{L_n} = L_n/L_{n+1}$ for $n \ge 4$.

We mention in passing that the results of Theorem 4,

$$F_{n+1} = \left[\tfrac{1}{2}\left(F_n + 1 + \sqrt{5F_n^2 - 2F_n + 1}\right)\right] \quad (n \ge 2)$$

$$L_{n+1} = \left[\tfrac{1}{2}\left(L_n + 1 + \sqrt{5L_n^2 - 2L_n + 1}\right)\right] \quad (n \ge 4),$$

form an essentially different solution to Problem B-42 [1], perhaps an improvement over the published solution since the value of n is not required.

We shall apply these results in a test to determine whether a given integer is a Fibonacci number, but we need first to establish a certain property of Fibonacci-type sequences.

Theorem 6. Define a Fibonacci sequence $\{f_n\}$ by specifying two integers f_0 and f_1, along with the recurrence relation $f_n = f_{n-1} + f_{n-2}$. Then

$$\left| f_{n+1}^2 - f_{n+1}f_n - f_n^2 \right| = \left| f_1^2 - f_1 f_0 - f_0^2 \right| = D$$

for all $n \geq 0$.

Proof. We proceed by induction. The statement is true for $n = 0$. Assume it is true for $n = k \geq 0$. Then

$$D = \left| f_{k+1}^2 - f_{k+1}f_k - f_k^2 \right| = \left| -f_{k+1}^2 + f_{k+1}f_k + f_k^2 + f_{k+1}^2 + f_{k+1}f_k - f_{k+1}^2 - f_{k+1}f_k \right|$$
$$= \left| (f_{k+1} + f_k)^2 - f_{k+1}(f_{k+1} + f_k) - f_{k+1}^2 \right| = \left| f_{k+2}^2 - f_{k+2}f_{k+1} - f_{k+1}^2 \right| ,$$

so the assertion is true for $n = k + 1$, completing the induction step and the proof.

Now for the Fibonacci sequence $D = |F_1^2 - F_1F_0 - F_0^2| = 1$. Since $h(F_n) = F_{n+1}$, all Fibonacci numbers F_n satisfy

$$\left| h^2(F_n) - F_n h(F_n) - F_n^2 \right| = 1 .$$

We shall show that only Fibonacci numbers satisfy this equation, thus providing a necessary and sufficient condition for an integer to be a Fibonacci number.

Theorem 7. Let m be a positive integer, and $g(m) = |h^2(m) - mh(m) - m^2|$. Then m is a Fibonacci number if and only if $g(m) = 1$. Also, $m \geq 7$ is a Lucas number if and only if $g(m) = 5$.

Proof. We have shown above that if m is a Fibonacci number then $g(m) = 1$. Now assume $g(m) = 1$, and we wish to show m is a Fibonacci number. Since $h(m) > m$, we may form a decreasing Fibonacci sequence

(2) $$h(m),\ m,\ h(m) - m,\ 2m - h(m), \cdots,\ f_1,\ f_0 ,$$

where f_0 is the least nonnegative term of this sequence. Then $f_0 < \frac{1}{2}f_1$, for if $f_1 > f_0 > \frac{1}{2}f_1$, then there is another term of the sequence f_{-1} such that

$0 \le f_{-1} = f_1 - f_0 < f_0$ contradicting the definition of f_0, while if $f_0 = \frac{1}{2}f_1, f_0 - (f_1 - f_0) = 0$ is another term of the sequence $<f_0$, again contradicting the definition of f_0. Thus $f_1 = 2f_0 + a$ where $a > 0$. But by Theorem 6, $1 = g(m) = \left|(2f_0 + a)^2 - (2f_0 + a)f_0 - f_0^2\right| = \left|f_0^2 + 3af_0 + a^2\right|$, and since f_0 and a are nonnegative integers, we must have $f_0 = 0$, $a = 1$, so that $f_1 = 1$. Hence m is a member of a Fibonacci sequence which begins with $f_0 = 0$ and $f_1 = 1$; that is, m is a Fibonacci number.

We now prove the latter half of the theorem. Suppose $m \ge 7$ is a Lucas number L_n. For the Lucas sequence $D = 5$, and so by Theorems 4 and 5 we have $g(m) = 5$. Now assume $g(m) = 5$ where $m \ge 7$, and as above let f_0 be the least nonnegative term in a decreasing Fibonacci sequence defined in (2). Clearly $f_0 > 0$, for $f_0 = 0$ implies $5 = g(m) = \left|f_1^2 - f_1f_0 - f_0^2\right| = \left|f_1^2\right.$. Also, as in the first section, $f_0 < \frac{1}{2}f_1$, so $f_1 = 2f_0 + a$ where $a > 0$. Then $5 = g(m) = \left|(2f_0 + a)^2 - (2f_0 + a)f_0 - f_0^2\right| = \left|f_0^2 + 3af_0 + a^2\right|$, and since f_0 and a are positive integers, we must have $f_0 = a = 1$, so that $f_1 = 3$. Thus m belongs to a Fibonacci sequence with $f_0 = 1$, $f_1 = 3$; that is, m is a Lucas number.

We note that Theorem 6 is also implied by the result of Long and Jordan [5] that the only solutions of the diophantine equation $\left|x^2 - 5y^2\right| = 4$ are $x = L_n$, $y = F_n$.

Define

$$h_n(k) \quad \text{for} \quad n \ge 0$$

by

$$h_0(k) = k \qquad \text{and} \qquad h_n(k) = h\{h_{n-1}(k)\}$$

for $n \ge 0$. Then

$$\{h_n(k)\}_{n=0}^{\infty}$$

is a sequence of integers for each choice of k. Values of $h_n(k)$ for $1 \le k \le 10$ and $0 \le n \le 9$, which were computed by Terry Brennan, are given in Table 1. It appears from the Table that

$$\{h_n(k)\}$$

obeys either a homogeneous or nonhomogeneous Fibonacci recurrence relation. We prove this holds in general.

Table 1

Values of $h_n(k)$

k \ n	0	1	2	3	4	5	6	7	8	9
1	1	2	3	5	8	13	21	34	55	89
2	2	3	5	8	13	21	34	55	89	144
3	3	5	8	13	21	34	55	89	144	233
4	5	6	10	16	26	42	68	110	178	288
5	5	8	13	21	34	55	89	144	233	377
6	6	10	16	26	42	68	110	178	288	466
7	7	11	18	29	47	76	123	199	322	521
8	8	13	21	34	55	89	144	233	377	610
9	9	14	22	35	56	90	145	234	378	611
10	10	16	26	42	68	110	178	288	466	754

Theorem 8. For each choice of k the sequence $\{h_n(k)\}$ obeys one of the following recurrence relations:

$$h_{n+2}(k) = h_{n+1}(k) + h_n(k), \quad n = 0, 1, \cdots \quad \text{(Fibonacci homogeneous)}$$

$$h_{n+2}(k) = h_{n+1}(k) + h_n(k) - 1, \quad n = 0, 1, \cdots \quad \text{(Fibonacci nonhomogeneous)}$$

Proof. The assertion is true for $k = 1$ since $h_n(1) = F_{n+2}$ obeys the first relation. We thus consider $k \geq 1$. We shall use the property that $x - 1 \leq [x] \leq x$ to show that $h_2(k) = h_0(k) + h_1(k)$ or $h_0(k) + h_1(k) - 1$. We shall then use induction to prove this initial recurrence continues to hold throughout the sequence. For sake of brevity we let throughout the rest of the paper

$$h_n(k) \equiv h_n, \quad h_0(k) \equiv k, \quad h_1(k) \equiv h(k) \equiv h, \quad \text{and } "\Leftarrow"$$

mean "if". From the definition we have

$$\tfrac{1}{2}\left(h_n - 1 + \sqrt{5h_n^2 - 2h_n + 1}\right) \leq h_{n+1} \leq \tfrac{1}{2}\left(h_n + 1 + \sqrt{5h_n^2 - 2h_n + 1}\right) .$$

Then

$$\begin{aligned}
h_2(k) \geq k + h(k) - 1 &\Leftarrow \tfrac{1}{2}(h - 1 + \sqrt{5h^2 - 2h + 1}) \geq k + h - 2 \\
&\Longleftrightarrow 5h^2 - 2h + 1 \geq (2k + h - 3)^2 \\
&\Longleftrightarrow h^2 \geq k^2 + kh - h - 3k + 2 \\
&\Leftarrow k^2 + \tfrac{1}{2}(k^2 + k + k\sqrt{5k^2 - 2k + 1}) - \tfrac{1}{2}(k - 1 + \\
&\qquad \sqrt{5k^2 - 2k + 1}) - 2k + 2 \leq \tfrac{1}{4}(k - 1 + \\
&\qquad \sqrt{5k^2 - 2k + 1})^2 \\
&\Longleftrightarrow 8 \leq 8k
\end{aligned}$$

which is valid. Also

$$\begin{aligned}
h_2(k) \leq k + h(k) &\Longleftrightarrow \tfrac{1}{2}(h + 1 + \sqrt{5h^2 - 2h + 1}) \leq k + h + 1 \\
&\Longleftrightarrow 5h^2 - 2h + 1 \leq (2k + h + 1)^2 \\
&\Longleftrightarrow 4h^2 \leq 4(k^2 + kh + h + k) \\
&\Leftarrow (k + 1 + \sqrt{5k^2 - 2k + 1})^2 \leq 4k^2 + \\
&\qquad 2(k + 1)(k - 1 + \sqrt{5k^2 - 2k + 1}) + 4k \\
&\Longleftrightarrow 4 \leq 4k
\end{aligned}$$

which is true. Together these imply

$$h_2(k) = k + h(k) \qquad \text{or} \qquad k + h(k) - 1 \ .$$

We now show in the homogeneous case that this recurrence continues. Assume

$$h_{i-1} + h_i = h_{i+1} \quad \text{for} \quad i = 1, 2, \cdots, n \text{ where } n \geq 2.$$

We will prove that

$$h_n + h_{n+1} = h_{n+2} \ ,$$

that is

$$\tfrac{1}{2}(h_{n+1} - 1 + \sqrt{5h_{n+1}^2 - 2h_{n+1} + 1}) < h_n + h_{n+1} \le \tfrac{1}{2}(h_{n+1} + 1 + \sqrt{5h_{n+1}^2 - 2h_{n+1} + 1}).$$

The right side is equivalent to

$$
\begin{aligned}
&(2h_n + h_{n+1} - 1)^2 \le 5h_{n+1}^2 - 2h_{n+1} + 1 \\
&\quad \Leftrightarrow h_n^2 + h_n h_{n+1} - h_n \le h_{n+1}^2 \\
&\quad \Leftrightarrow h_n^2 - h_n h_{n-1} - h_{n-1}^2 \le h_n \\
&\quad \Leftarrow |h_1^2 - h_1 h_0 - h_0^2| \le h_2 \le h_n \\
&\quad \Leftarrow \left|\tfrac{1}{4}(h_0 + 1 + \sqrt{5h_0^2 - 2h_0 + 1})^2 - \tfrac{1}{2}h_0(h_0 - 1 + \sqrt{5h^2 - 2h_0 + 1}) - h_0^2\right| \le h_2 \qquad (3) \\
&\quad \Leftrightarrow \left|\tfrac{1}{2}(h_0 + 1 + \sqrt{5h_0^2 - 2h_0 + 1})\right| \le h_2 \\
&\quad \Leftarrow \left|\tfrac{1}{2}(h_0 + 1 + \sqrt{5h_0^2 - 2h_0 + 1})\right| \le h_1 + 1 \le h_1 + h_0 = h_2 .
\end{aligned}
$$

But this last statement is true, verifying the right side. The left side is equivalent to

$$
\begin{aligned}
&5h_{n+1}^2 - 2h_{n+1} + 1 < (2h_n + h_{n+1} + 1)^2 \\
&\quad \Leftrightarrow h_{n+1}^2 < h_n^2 + h_n h_{n+1} + h_{n+1} + h_n \\
&\quad \Leftrightarrow -(h_n^2 - h_n h_{n-1} - h_{n-1}^2) < 2h_n + h_{n-1} = h_{n+2}
\end{aligned}
$$

which is certainly true in light of (3) and Theorem 6. Proof of the nonhomogeneous case uses essentially the same techniques, although it is more complicated and is therefore omitted.

It is natural to ask for which integers k the sequence $\{h_n(k)\}$ is homogeneous and for which it is nonhomogeneous.

Theorem 9. The sequence $\{h_n(k)\}$ is nonhomogeneous if and only if

$$h^2(k) < k^2 - k + kh(k) \quad . \tag{4}$$

Proof. Using Theorem 8 it follows that $\{h_n(k)\}$ is nonhomogeneous if and only if

$$\begin{aligned} k + h(k) &= h_2(k) - 1 \\ &\Longleftrightarrow \tfrac{1}{2}(h + 1 + \sqrt{5h^2 - 2h + 1}) < k + h \\ &\Longleftrightarrow 5h^2 - 2h + 1 < (2k + h - 1)^2 \\ &\Longleftrightarrow h^2 < k^2 - k + kh \quad . \end{aligned}$$

However the characterization of the k which obey (4) seems difficult. It appears that numbers of the form $k = F_m + 1$ $(m > 5)$ satisfy (4), but there are others.

From the recurrence relations of Theorem 8 we may establish the following generating functions using standard techniques. If $\{h_n(k)\}$ is homogeneous,

$$\frac{p(x)}{1 - x - x^2} = \sum_{n=0}^{\infty} h_n(k)\, x^n \quad ,$$

where $p(x) = \{h(k) - k\}x + k$; if $\{h_n(k)\}$ is nonhomogeneous,

$$\frac{q(x)}{(1 - x)(1 - x - x^2)} = \sum_{n=0}^{\infty} h_n(k)\, x^n \quad ,$$

where $q(x) = \{h_2(k) - 2h(k)\}x^2 + \{h(k) - 2k\}x + k.$

Finally we show an interweaving of the numbers $h_n(k)$ in Table 1.

Theorem 10. For $r \geq 0$ let $M_r(n)$ be the number of integers not greater than n which do not appear in the sequence $\{h_r(j)\}_{j=1}^{\infty}$. Then for $n \geq r$,

$$M_r(h_n(k)) = h_n(k) - h_{n-r}(k) .$$

Proof. We begin by observing that if $s > t$ then $h(s) > h(t)$. First assume $n = r$, so that $h_{n-r}(k) = h_0(k) = k$. The k distinct integers $h_r(1), \cdots, h_r(k)$ are the only members of $\{h_r(j)\}_{j=1}^{\infty}$ not greater than $h_r(k)$, so that $M_r(h_r(k)) = h_r(k) - k = h_r(k) - h_{r-r}(k)$, as required. Now assume $n \geq r$, and let $h_{n-r}(k) = m$. Then $h_n(k) = h_r(m)$, so by the above

$$M_r(h_r(m)) = h_r(m) - h_0(m)$$

which implies

$$M_r(h_n(k)) = h_n(k) - h_{n-r}(k) ,$$

the desired result.

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ON LAMÉ'S THEOREM

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We define the Fibonacci numbers $\{u_i\}$ as follows:

$$u_1 = 1,\ u_2 = 2,\ u_{n+2} = u_{n+1} + u_n \text{ for } n \geq 1 .$$

In a recent note, R. L. Duncan has shown [1] that the determination of the greatest common divisor, (u_{n+1}, u_n), for any $n \geq 1$ by means of the Euclidean Algorithm always requires a number of divisions n satisfying the inequality,

$$n > \frac{p_n}{\log \xi} - 5 ,$$

where p_n is the number of digits in u_n and

$$\xi = \frac{1 + \sqrt{5}}{2} .$$

Duncan then contrasts the classical Lamé result [2] for this case, namely

$$n < \frac{p_n}{\log \xi} + 1 ,$$

and concludes that Lamé's theorem is virtually the best possible. [Recall Lamé's theorem asserts that if a and b are positive integers, then the number of divisions, n, required to determine (a, b) by the Euclidean Algorithm satisfies the inequality,

$$n < \frac{p}{\log \xi} + 1 ,$$

where p is the number of digits in the smaller of the two integers a and b.]

Our purpose here is to show that when Lamé's bound for the number of divisions n in the algorithmic determination of (u_{n+1}, u_n) is written in the equivalent form

$$n \le \left[\frac{p_n}{\log \xi}\right] + 1$$

then there exist infinitely many pairs of consecutive Fibonacci numbers u_n and u_{n+1} such that the determination of (u_{n+1}, u_n) by the Euclidean Algorithm requires exactly

$$\left[\frac{p_n}{\log \xi}\right] + 1$$

divisions. Thus the integer 1 which appears in Lame's bound

$$\left[\frac{p_n}{\log \xi}\right] + 1 ,$$

cannot be reduced, and in this sense, Lamé's theorem cannot be improved.

From consideration of tables, we find that to determine the g. c. d. of each of the pairs, $u_5 = 8$ and $u_6 = 13$, $u_{10} = 89$ and $u_{11} = 144$, $u_{15} = 987$ and $u_{16} = 1597$, a number of divisions is required that is equal to the Lamé bound. Note that the smaller number in each pair contains exactly one less digit than the larger number; this property will also be imposed in the general analysis. It is not clear a priori that there are infinitely many such pairs for which the Lamé bound is realized. For example, the next logical pair, $u_{19} = 6765$ and $u_{20} = 10946$, requires only 19 divisions but the Lamé result gives an upper bound of 20.

THEOREM 1: There exist an infinite number of distinct positive integers n such that the determination of (u_{n+1}, u_n) by the Euclidean Algorithm requires exactly n divisions with n satisfying

(1)
$$n > \frac{p_n}{\log \xi} - \frac{1}{2} \ .$$

PROOF: It is known [1] that the algorithmic determination of (u_{n+1}, u_n) requires exactly n divisions; it remains to prove (1) holds for infinitely many values of n.

For $n \geq 1$, Binet's formula [3] states

$$u_n = \frac{\xi^{n+1} - \zeta^{n+1}}{\sqrt{5}} \ ,$$

where

$$\xi = \frac{1 + \sqrt{5}}{2}$$

and

$$\zeta = \frac{1 - \sqrt{5}}{2} \ .$$

Thus,

(2)
$$\left| u_n - \frac{\xi^{n+1}}{\sqrt{5}} \right| = \left| \frac{\zeta^{n+1}}{\sqrt{5}} \right| \ .$$

Since

$$|\zeta| < 1 \ , \quad \lim_{n \to \infty} \frac{\xi^{n+1}}{\sqrt{5}} = u_n \ .$$

Choose $\epsilon > 0$ such that

(3)
$$\frac{\log \sqrt{5}}{\log \xi} - \frac{2\epsilon}{\log \xi} > 1.5 \ .$$

[This is possible since

$$\log \sqrt{5} = 0.350$$

and

$$\log \xi = 0.208] \ .$$

Corresponding to this value of ϵ, $\exists$ a positive integer n_o such that

$$\left|\log u_n - \log \frac{\xi^{n+1}}{\sqrt{5}}\right| < \epsilon \quad \text{for } n > n_o \ ,$$

or, equivalently,

$$(4) \qquad \left|\log u_n - (n+1) \log \xi + \log \sqrt{5}\right| < \epsilon \quad \text{for } n > n_o \ .$$

Now, p_n, the number of digits in u_n, is given by $p_n = \left[\log u_n\right] + 1$, where the square brackets denote the greatest integer contained in the bracketed quantity.

Clearly,

$$(5) \qquad \log u_n = p_n - 1 + \theta_n \quad \text{where } 0 < \theta_n < 1 \ ,$$

and (4) becomes

$$(6) \qquad \left|p_n - (1-\theta_n) - (n+1) \log \xi + \log \sqrt{5}\right| < \epsilon \quad (n > n_o) \ .$$

Since (5) holds for arbitrary n, we also have

$$(7) \qquad \log u_{n+1} = p_{n+1} - 1 + \theta_{n+1} \quad \text{with } 0 < \theta_{n+1} < 1 \ ,$$

where p_{n+1} is the number of digits in u_{n+1}.

Subtracting (5) from (7), we find

$$(8) \qquad \log \frac{u_{n+1}}{u_n} = (p_{n+1} - p_n) + (\theta_{n+1} - \theta_n) \ .$$

But it is well-known that

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \xi \ ;$$

therefore, for the previosuly chosen $\epsilon > 0$, $\exists$ a positive integer n_o' such that for $n > n_o'$,

(9) $$\left|(p_{n+1} - p_n) + (\theta_{n+1} - \theta_n) - \log \xi\right| < \epsilon \ .$$

We further restrict n so that

(10) $$p_{n+1} - p_n = 1$$

is satisfied; that is, u_{n+1} is required to have exactly one more digit than u_n. Since

$$\lim_{n \to \infty} u_n = +\infty ,$$

it is clear that (10) is satisfied for infinitely many values of n. With this additional restriction on n, equation (9) yields

$$\theta_n > \theta_{n+1} + (1 - \log \xi) - \epsilon ,$$

or noting $\theta_{n+1} > 0$,

(11) $$\theta_n > (1 - \log \xi) - \epsilon .$$

From (6),

(12) $$p_n - (1 - \theta_n) - (n+1) \log \xi + \log \sqrt{5} < \epsilon \ (n > n_o) \ .$$

If we now choose $n > \max(n_o, n_o')$ and such that (10) is satisfied, then using (11) in (12),

(13) $$p_n - \log \xi - (n+1) \log \xi + \log \sqrt{5} - \epsilon < \epsilon ,$$

or

(14) $$n > \frac{p_n}{\log \xi} - 2 + \frac{\log \sqrt{5}}{\log \xi} - \frac{2\epsilon}{\log \xi} .$$

Using (3), we conclude that for $n > \max(n_o, n_o')$ and satisfying (10),

(15) $$n > \frac{p_n}{\log \xi} - \frac{1}{2} \quad \text{as asserted.}$$ q. e. d.

According to Lamé's theorem [2], the number of divisions n required to determine (u_{n+1}, u_n) is bounded above (strong inequality) by

$$\frac{p_n}{\log \xi} + 1 ,$$

or equivalently

(16) $$n \le \left[\frac{p_n}{\log \xi}\right] + 1 .$$

On the other hand, (15) asserts that

(17) $$n \ge \left[\frac{p_n}{\log \xi} - \frac{1}{2}\right] + 1 = \left[\frac{p_n}{\log \xi} + \frac{1}{2}\right]$$

for infinitely many values of n. Under certain circumstances, the bounds in (16) and (17) are equal. We first prove a simple lemma: (cf. [4], Theorem 6.3, p. 72):

<u>LEMMA</u>: Given α irrational, $\exists$ infinitely many integers n such that $n\alpha - [n\alpha] > \frac{1}{2}$.

<u>PROOF</u>: If $\exists\, n_0$ such that $n\alpha - [n\alpha] > \frac{1}{2}$ for all $n > n_0$, the proposition is proved; otherwise, for given $n_0 > 0$, $\exists$ n with $n > n_0$ such that $n\alpha = [n\alpha] + \beta$ with $0 < \beta < \frac{1}{2}$ (β irrational).

Choose k such that

$$\frac{1}{2^{k+1}} < \beta < \frac{1}{2^k} \quad \text{or} \quad \frac{1}{2} < 2^k\beta < 1 .$$

Then, letting $N = 2^k n$, we have

$$N\alpha = 2^k[n\alpha] + 2^k\beta .$$

Since $2^k\beta < 1$, $[N\alpha] = 2^k[n\alpha]$ and $N\alpha - [N\alpha] = 2^k\beta > \frac{1}{2}$.

Thus, $\exists$ arbitrarily large integers n with $n\alpha - [n\alpha] > \frac{1}{2}$ as asserted.

The following theorem shows that Lamé's result

$$n \leq \left[\frac{p_n}{\log \xi}\right] + 1$$

is the best possible.

THEOREM 2: There exist infinitely many distinct values of n such that the determination (u_{n+1}, u_n) by the Euclidean Algorithm requires exactly n divisions where n is given by

$$n = \left[\frac{p_n}{\log \xi}\right] + 1 \tag{18}$$

PROOF: From Theorem 1, - infinitely many values of n such that

$$n \geq \left[\frac{p_n}{\log \xi} + \frac{1}{2}\right]. \tag{19}$$

The proof of Theorem 1 shows that if p_M is the number of digits in u_M where $M = \max(n_o, n_o') + 1$, than an n can be found such that p_n assumes any integer value $> p_M$ and such that (19) is satisfied.

The Lemma assures us that there are infinitely many values of $p_n > p_M$ such that

$$\frac{p_n}{\log \xi} - \left[\frac{p_n}{\log \xi}\right] > \frac{1}{2} \tag{20}$$

and each of these values of p_n can be combined with an appropriate value of n such that (19) is satisfied. But (20) implies

$$\left[\frac{p_n}{\log \xi} + \frac{1}{2}\right] = \left[\frac{p_n}{\log \xi} + 1\right] = \left[\frac{p_n}{\log \xi}\right] + 1 .$$

Thus (19) in combination with Lamé's bound $n \leq \left[\frac{p_n}{\log \xi}\right] + 1$ shows

$$n = \left[\frac{p_n}{\log \xi}\right] + 1, \text{ proving the theorem.} \qquad \underline{\text{q. e. d.}}$$

The above results have been proved using only elementary techniques. A more concise proof can be obtained using some theorems on the uniform distribution (mod 1) of sequences; this will be the subject of a forthcoming note by R. L. Duncan.

REFERENCES

1. R. L. Duncan, Note on the Euclidean Algorithm, The Fibonacci Quarterly. Vol. 4, No. 4, pp. 367-68.
2. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill 1939, pp. 43-45.
3. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell Pub. Co., 1961, p. 20.
4. I. Niven, Irrational Numbers, Carus Math. Monograph, No. 11, Math. Ass'n. of America, 1956.

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CORRECTION

Please correct the last phrase of "A Recursive Generation on Two-Digit Integers," appearing on page 90 of the April 1965 issue of the Fibonacci Quarterly to read: "so that it takes the five odd digits to generate the set."

Edward Rayher points out that there are only nine two-digit generators. Eliminated from the published set should be "24" which obviously comes from the 21 at the end of the line preceding it in group (4), and "47" which follows 37 in the sequence of the same group.

D. R. Kapreker calls these generators "self-numbers" in his 21-page pamphlet, "The Mathematics of the New Self Numbers," personally published by him in Devlali, India in 1963. He lets the generated sequences run to infinity rather than reducing the numbers modulo 100 so that they lead to loops.

C. W. TRIGG

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ADVANCED PROBLEMS AND SOLUTIONS

Edited By
V. E. HOGGATT, JR.
San Jose State College, San Jose, Calif.

Send all communications concerning Advanced Problems and Solutions to Raymond Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within three months after publication of the problems.

NOTICE: PLEASE SEND ALL SOLUTIONS AND NEW PROPOSALS TO PROFESSOR RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA.

H-113 Proposed by V. E. Hoggatt, Jr., San Jose State, College, San Jose, Calif.

If

$$\sum_{n=0}^{\infty} R(n)X^n = \prod_{j=1}^{\infty}(1 + X^{F_j}) ,$$

then show

i) $$R(L_{2n} - 1) = R(L_{2n+1} - 1) = 2n \qquad n \ge 2$$

ii) $$R(L_{n+3} + 1) = 2n \qquad n \ge 2$$

(In "Representations by Complete Sequences," Oct. 1963 Fibonacci Quarterly, Theorem 3 states

$$R(L_{2n-1}) = R(L_{2n}) = 2n - 1 \quad (n \ge 1)$$

This should read $n \ge 2$.)

H-114 Proposed by William C. Lombard and V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show that the sequence $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ $(n \ge 0)$ is complete.

Show that, if any L_k $(k > 2)$ is deleted, then the deleted sequence is still complete.

Show that, if L_0 or L_1 or (L_j and L_k; $k > j \geq 2$) is (are) deleted, then the deleted (doubly deleted) sequence is incomplete.

(See H-53, Vol. 3, No. 1, page 45, Fibonacci Quarterly.)

H-115 Proposed by Stephen Headley, San Jose State College, San Jose, Calif.

If

$$\sum_{n=0}^{\infty} R(n)X^n = \prod_{i=0}^{\infty}(1 + X^{L_i}) \quad ,$$

where L_i is the i^{th} Lucas number, show $R(L_{2n}) = R(L_{2n+1}) = n + 1$.

H-116 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

If

$$\sum_{n=0}^{\infty} R(n)X^n = \prod_{j=0}^{\infty}(1 + X^{L_j}) \quad ,$$

then for $n \geq 0$ show

i) $$R(F_{4n}) = R(F_{4n+1}) = R(F_{4n+2}) = F_{2n+1}$$

ii) $$R(F_{4n+3}) = F_{2n+2}$$

H-117 Proposed by George Ledin, Jr., San Francisco, Calif.

Prove

$$\begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_n \\ F_{n+2} & F_{n+3} & F_n & F_{n+1} \\ F_{n+1} & F_n & F_{n+3} & F_{n+2} \\ F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{vmatrix} = F_{2n+6}F_{2n}$$

H-118 Proposed by George Ledin, Jr., San Francisco, Calif.

Solve the difference equation

$$C_{n+2} = F_{n+2}C_{n+1} + C_n$$

with $C_1 = a$, $C_2 = b$, and F_n is the n^{th} Fibonacci number.

SOLUTIONS

A MANY SPLENDORED THING

H-69 Proposed by M. N. S. Swamy, University of Saskatchewan, Regina, Canada

Given the polynomials $B_n(x)$ and $b_n(x)$ defined by,

$$\begin{aligned} b_n(x) &= xB_{n-1}(x) + b_{n-1}(x) \qquad (n > 0) \\ B_n(x) &= (x+1)B_{n-1}(x) + b_{n-1}(x) \qquad (n > 0) \\ b_0(x) &= B_0(x) = 1 \end{aligned}$$

It is possible to show that

$$B_n(x) = \sum_{r=0}^{n} \binom{n+r+1}{n-r} x^r ,$$

and

$$b_n(x) = \sum_{r=0}^{n} \binom{n+r}{n-r} x^r .$$

It can also be shown that the zeros of $B_n(x)$ or $b_n(x)$ are all real, negative and distinct. The problem is whether it is possible to factorize $B_n(x)$ and $b_n(x)$. I have found that for the first few values of n, the result

$$B_n(x) = \prod_{r=1}^{n} \left[x + 4\cos^2\left(\frac{r}{n+1}\right) \cdot \frac{\pi}{2} \right]$$

holds. Does this result hold good for all n? Is it possible to find a similar result for $b_n(x)$?

Solution by John C. Sjoherg, Carlisle, Pennsylvania

Let

$$\begin{aligned} f_{2n}(x) &= b_n(x^2) \\ f_{2n+1}(x) &= xB_n(x^2) \qquad n \geq 0 \end{aligned}$$

then

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x) ,$$

with

$$f_0(x) = 1 \quad \text{and} \quad f_1(x) = x$$

and

$$f_n(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n-r}{r} x^{n-2r} .$$

Thus

$$f_n(2i \cos y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n-r}{r} (2i)^{n-2r}(\cos y)^{n-2r} .$$

We have by definition that

$$f_n(2i \cos y) = i^n \frac{\sin (n+1)y}{\sin y}$$

The zeros of $f_n(2i \cos y)$ are then

$$y = \frac{r\pi}{n+1} \quad \text{for} \quad r = 1, 2, 3, \cdots, n$$

and the zeros of $f_n(x)$ are then

$$x = 2i \cos \frac{r\pi}{n+1} \quad \text{for} \quad r = 1, 2, 3, \cdots, n$$

We have therefore that

$$b_n(x^2) = \prod_{r=1}^{2n} \left(x - 2i \cos \frac{r\pi}{2n+1} \right)$$

$$b_n(x) = \prod_{r=1}^{n}\left(x + 4\cos^2\frac{r\pi}{2n+1}\right)$$

Similarly

$$B_n(x) = \prod_{r=1}^{n}\left(x + 4\cos^2\frac{r\pi}{2n+2}\right)$$

Since

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$$

with

$$f_0(x) = 1 \quad \text{and} \quad f_1(x) = x \;,$$

we have

$$f_n(1) = F_{n+1} \;.$$

Also solved by the Proposer.

NO SOLUTION

H-70 Proposed by C.A. Church, Jr., West Virginia University, Morgantown, West Va.

For $n = 2m$ show that the total number of k-combinations of the first n natural numbers such that no two elements i and $i + 2$ appear together in the same selection is F^2_{m+2}, and if $n = 2m + 1$, the total is $F_{m+2}F_{m+3}$.

Additional Comment by the Proposer.

A corresponding problem for circular permutations may also be posed using Kaplansky's second lemma [same reference] which leads in this case to Lucas numbers. That is, the number of ways of selecting k objects, from n arrayed on a circle, with no two consecutive is

$$\frac{n}{n-k}\binom{n-k}{k} \;.$$

Another solution to the problem of choosing k elements from among $1, 2, \cdots, n$ such that i and $i + 2$ do not both occur, is given by M. Abramson [Explicit Expressions for a Class of Permutation Problems, Canadian Mathematical Bulletin, 7(1964), 349]. Namely, there are

$$\sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{n - 2k + 2 + i}{k - i} \binom{k - i}{i}$$

ways. This, of course, suggests a couple of binomial identities, when his answer is compared with mine.

A VERY PRETTY RESULT

H-71 Proposed by John L. Brown, Jr., Penn. State University, State College, Pa.

Show

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^{k-1} L_k = 5^n$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^{k-1} F_k = 0$$

See also H-77.

Solution by the Proposer.

Let

$$a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2} ,$$

so that

$$F_k = \frac{a^k - b^k}{\sqrt{5}} \quad \text{and} \quad L_k = a^k + b^k .$$

Hence

$$5^n = (2a - 1)^{2n} = \sum_{k=0}^{2n} (2a)^k (-1)^{2n-k} \binom{2n}{k}$$

and

$$5^n = (1 - 2b)^{2n} = \sum_{k=0}^{2n} (-1)^k (2b)^k \binom{2n}{k} \quad .$$

Add to get

$$2 \cdot 5^n = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^k (a^k + b^k)$$

<u>or</u>

$$\boxed{5^n = \sum_{k=0}^{2n} (-1)^k 2^{k-1} \binom{2n}{k} L_k}$$

Subtract to get

$$0 = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^k (a^k - b^k)$$

<u>or</u>

$$\boxed{0 = \sum_{k=0}^{2n} (-1)^k 2^{k-1} \binom{2n}{k} F_k}$$

GENERALIZED FIBONOMIAL COEFFICIENTS

H-72 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let $u_n = F_{nk}$, where F_m is the m^{th} Fibonacci number, and k is any positive integer; and let

$$\begin{bmatrix} m \\ 0 \end{bmatrix} = \begin{bmatrix} m \\ m \end{bmatrix} = 1, \begin{bmatrix} m \\ n \end{bmatrix} = \frac{u_m \cdots u_1}{u_n u_{n-1} \cdots u_1 u_{m-n} u_{m-n-1} \cdots u_1}$$

then show

$$2\begin{bmatrix} m \\ n \end{bmatrix} = L_{nk}\begin{bmatrix} m-1 \\ n \end{bmatrix} + L_{(m-n)k}\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}$$

This problem and many others related are thoroughly discussed in a paper, "Fibonacci Numbers and Generalized Binomial Coefficients," to appear soon in the Fibonacci Quarterly.

* * * * *

CORRECTIONS

Please make the following corrections on the paper, "On a Certain Kind of Fibonacci Sums," Vol. 5, No. 1, pp. 45-58, Fibonacci Quarterly:

Page 46: In Eq. (4a), change $P_1(m,n)dx$ to $P_1(m,x)dx$

Page 49: In Corollary 1, the denominator of the second fraction should be dn instead of dn^r. Delete the first m following second $=$ sign.

Page 51: Change the first part of the last paragraph to read:

At this stage it seems clear that a study of the polynomials $P_1(m,n)$ and $P_2(m,n)$ and of the numbers $M_{1,j}$ and $M_{2,j}$ is of **basic** importance to the development of any further theory. The numbers $M_{1,j}$ and $M_{2,j}$ pose by themselves an interesting problem. The intuitive bounds...

In the last two lines, change $M_{1,j}$ to $M_{i,j}$.

Page 54: In the last line, change case to class.

Page 56: In the table title, add an asterisk to P_3, i.e., $P_3^\star(m,n)$

In the last line before Eq. (12), change written to rewritten.

Page 58: Delete the extra with in Reference 8. G. L. Jr.

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EDITORIAL

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

The Fibonacci Quarterly and the Fibonacci Association are now in their fifth year. In retrospect two things stand out strongly: one, the decision to start such a specialized magazine; two, the inclusion of two sections, advanced and elementary. The first was evidently a risk, both from the financial standpoint and, in the minds of some, from a consideration of the possibility of exhausting the field. Neither of these eventualities has proved substantial. Support in the form of subscriptions, memberships, sustaining memberships and more recently page charges has been sufficient to provide for continued publication and even a bit of expansion. As to the second danger, the flood of articles after the first initial steps has continued to grow until now it has become something of a problem.

The other risk, the establishment of two sections, was probably in the minds of some not desirable, since they may have thought that this would tend to lower the academic level of the magazine. Apparently, our readers, including the libraries of major colleges and universities, have not reacted adversely. However, in the course of time, with the pressure of articles demanding publication and the tendency to publish what was on hand, the elementary section has become more and more advanced. The result is that some of our readers have begun to lose heart.

With this situation in mind, action has been set afoot to revitalize the elementary portion of the Fibonacci Quarterly by the appointment of a special editor with the thought of promoting material of value to the readers of this section of our magazine. Already, it has been decided to have a Recreation Corner and another entitled "Let's Do Some Research." The purpose of the present editorial is to point out that while there is a superabundance of articles on hand, there are not sufficiently many articles of good quality suitable for the elementary section. More importantly, the type of article desired will be made explicit.

For the average reader of mathematical articles, I would presume that the major objection to what is available on the market is that it is NOT

READABLE. Usually, an article will start off with three, four or five references, assume everything that is in them and then launch quickly into further developments. Anybody who would want to follow the discussion intelligently would have to go back and work through the articles listed before being able to do anything with the most recent addition. Usually, none of the articles is READABLE. They are compressed, cryptic, truncated in their manner of presentation, so that in effect anyone who desires to find out what is being said has to start with paper and pencil and work through everything step by step. Given the general lack of time available, most people simply stop at the first roadblock, get some idea of what the article is about and pass on.

In the elementary section, we need articles that can be READ. This means that they should start with well-known ideas, should provide examples of the basic patterns which lead to conjectures, should give the proofs of the main theorems clearly, simply and completely, should go on to numerical illustrations of the theorems, and in general, should be sufficiently explicit so that a very good high school student, a good college student, a high school teacher, an interested amateur can READ the material without having to go through all the development on paper while still possibly remaining in doubt as to whether he has the idea or not.

Writers of articles for this section should be warned of the twin evils of compression and impression: compression, to put the ideas in as small a compass as possible; impression, to work up such a framework as to overawe the uninitiated. On the contrary, there should be a definite attempt to keep the number of definitions to a minimum, to use as little special notation as possible, to attempt to express ideas in simple direct English as well as in formulas.

There is a great wealth of material in the past issues of the Fibonacci Quarterly awaiting the deft touch of able expository writers. At present, this valuable material is a closed book to all but a relatively small number of specialists. Writers are also encouraged to consider the sub-title of the Fibonacci Quarterly: "A Journal devoted to the study of integers with special properties." While we certainly favor articles on Fibonacci numbers, a little variety especially in the elementary section would be very welcome indeed.

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A FIBONACCI GENERALIZATION

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

It is well known that the sum of any ten numbers of a Fibonacci sequence is divisible by 11. For example, starting with 11, 15 and proceeding to 26, 41, 67, 108, 175, 283, 458, 741, the sum of these ten terms is 1925 which is divisible by 11, the quotient being 175, the seventh member of the set of ten successive terms. This can be proved to hold in general, for if we start with any two numbers a, b, the successive terms are $a + b$, $a + 2b$, $2a + 3b$, $3a + 5b$, $5a + 8b$, $8a + 13b$, $13a + 21b$, $21a + 34b$, the sum of which is $55a + 88b$ which on being divided by 11 gives a quotient of $5a + 8b$, the seventh term of the set of ten terms.

This curious property might lead one to speculate on the possibility of having various sets of successive terms of a generalized Fibonacci sequence divisible by some common quantity. To analyze the situation let us start with terms $T_1 = a$, $T_2 = b$, the usual Fibonacci relation.

$$T_{n+1} = T_n + T_{n-1}$$

Thus $T_3 = a + b$, $T_4 = a + 2b$, $T_5 = 2a + 3b$, $T_6 = 3a + 5b$, $T_7 = 5a + 8b$, etc. It appears that the coefficients of a and b are Fibonacci numbers from the sequence $(1, 1, 2, 3, 5, 8, 13, \cdots)$ with the general formula being

$$T_n = F_{n-2}a + F_{n-1}b$$

However, what we are considering is the sum of a certain number of terms of this sequence. We want to find what:

$$\sum_{k=1}^{n} T_k = T_1 + T_2 + T_3 + \cdots + T_n$$

equals in terms of a, b, and the Fibonacci numbers. Now it will be observed that the coefficients of a in the summation are the Fibonacci numbers

$1, 1, 2, 3, \cdots, F_{n-3}, F_{n-2}$ with an extra 1, so that by the usual formula for the sum of the Fibonacci numbers, the coefficient of a in the sum must be F_n; the coefficient of b is simply the sum of the Fibonacci numbers up to and including F_{n-1}, so that this coefficient is $F_{n+1} - 1$. Thus

$$\sum_{k=1}^{n} T_k = F_n a + (F_{n+1} - 1)b .$$

We are looking for a common factor of this sum, no matter what values a and b may have. Thus, we seek common factors of F_n and $F_{n+1} - 1$. In the case $n = 10$, $F_{10} = 55$, $F_{11} - 1 = 88$, so that the common factor is 11. A little experimentation leads to the following table. We begin with 10 and go to higher values.

55	$11 \cdot 5$
89-1	$11 \cdot 8$
144	$8 \cdot 18$
233-1	$8 \cdot 29$
377	$29 \cdot 13$
610-1	$29 \cdot 21$
987	$21 \cdot 47$
1597-1	$21 \cdot 76$
2584	$76 \cdot 34$
4181-1	$76 \cdot 55$
6765	$55 \cdot 123$
10946-1	$55 \cdot 199$
17711	$199 \cdot 89$
28657-1	$199 \cdot 144$

It appears that we have two cases. When n is of the form $4k + 2$, the common factor is a Lucas number and the quotients are successive Fibonacci numbers; while if n is of the form $4k$, the common factor is a Fibonacci number and the quotients successive Lucas numbers. More precisely, the intuitive relations would seem to be as follows:

$$\sum_{j=1}^{4k+2} T_j = F_{4k+2}a + (F_{4k+3} - 1)b$$
$$= L_{2k+1}F_{2k+1}a + L_{2k+1}F_{2k+2}b$$
$$= L_{2k+1}(F_{2k+1}a + F_{2k+2}b)$$
$$= L_{2k+1}T_{2k+3}$$

and for the other case:

$$\sum_{j=1}^{4k} T_j = F_{4k}a + (F_{4k+1} - 1)b$$
$$= L_{2k}F_{2k}a + F_{2k}L_{2k+1}b$$
$$= F_{2k}(L_{2k}a + L_{2k+1}b)$$
$$= F_{2k}(F_{2k-1}a + F_{2k}b + F_{2k+1}a + F_{2k+2}b)$$
$$= F_{2k}(T_{2k+1} + T_{2k+3})$$

The formula $F_{2n} = F_nL_n$ is well known. There are two other formulas $F_{4n+3} - 1 = L_{2n+1}F_{2n+2}$ and $F_{4n+1} - 1 = L_{2n+1}F_{2n}$ which need to be justified. We first verify them for small values of n. For $n = 0$, the first formula gives $F_3 - 1 = 2 - 1$ or 1 and $L_1F_2 = 1 \cdot 1$ and hence is 1 as well. For $n = 1$, the first formula has on the left $F_7 - 1 = 13 - 1$ or 12 and on the right, $L_3F_4 = 4 \cdot 3$ or 12. Thus the first formula holds for small values of n. Similarly, the second can be verified for these small values. We now assume that the various formulas hold up to F_{4n+2} and that $F_{2n} = F_nL_n$ holds in general. Then

$$\begin{array}{rl}
 & F_{4n+1} - 1 = L_{2n+1}F_n \\
 & \underline{F_{4n+1} \quad\;\; = L_{2n+1}F_{2n+1}} \\
\text{Adding} & F_{4n+3} - 1 = L_{2n+1}F_{2n+2} \\
\text{Then} & \underline{F_{4n+4} \quad\;\; = F_{2n+2}L_{2n+2}} \\
\text{Adding} & F_{4n+5} - 1 = F_{2n+2}L_{2n+3}
\end{array}$$

Hence, since these formulas hold for small values of n as shown, it follows that they can be proved to hold for any value of n by reason of mathematical induction. Thus the intuitive formulas are seen to hold in general.

As a numerical illustration of these formulas, consider the series starting with $a = 8$, $b = 11$.

k	T_k	ΣT_k	Factorization	In Symbols
1	8	8		
2	11	19	$1 \cdot 19$	L_1T_3
3	19	38		
4	30	68	$1 \cdot 68$	$F_2(T_3 + T_5)$
5	49	117		
6	79	196	$4 \cdot 49$	L_3T_5
7	128	324		
8	207	531	$3 \cdot 177$	$F_4(T_5 + T_7)$
9	335	866		
10	542	1408	$11 \cdot 128$	L_5T_7
11	877	2285		
12	1419	3704	$8 \cdot 463$	$F_6(T_7 + T_9)$
13	2296	6000		
14	3715	9715	$29 \cdot 335$	L_7T_9
15	6011	15726		
16	9726	25452	$21 \cdot 1212$	$F_8(T_9 + T_{11})$
17	15737	41189		
18	25463	66652	$76 \cdot 877$	L_9T_{11}
19	41200	107852		
20	66663	174515	$55 \cdot 3173$	$F_{10}(T_{11} + T_{13})$
21	107863	282378		
22	174526	456904	$199 \cdot 2296$	$L_{11}T_{13}$
23	282389	739293		
24	456915	1196208	$144 \cdot 8307$	$F_{12}(T_{13} + T_{15})$
25	739304	1935512		
26	1196219	3131731	$521 \cdot 6011$	$L_{13}T_{15}$
27	1935523	5067254		
28	3131742	8198996	$377 \cdot 21748$	$F_{14}(T_{15} + T_{17})$

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FIBONACCI AND LUCAS NUMBERS IN THE SEQUENCE OF GOLDEN NUMBERS

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Beginning with the golden rectangle with base 2 and altitude $\sqrt{5} - 1$, one may proceed to construct a sequence of numbers which represent altitudes (shortest sides) of the nested golden rectangles.

$$(1) \qquad \sqrt{5} - 1,\ 3 - \sqrt{5},\ 2\sqrt{5} - 4,\ 7 - 3\sqrt{5},\ 5\sqrt{5} - 11,\ 18 - 8\sqrt{5},\ \cdots$$

We shall call this the sequence of golden numbers. These numbers, as one may suspect, are closely related to Fibonacci numbers, as is suggested by Theorem 2 below. First, however, we need to observe that the n^{th} golden number may be expressed by the following recursive formula:

Theorem 1. If g_n denotes the n^{th} golden number, then $g_n = 1/2\, g_1 \cdot g_{n-1}$.

Proof. This follows immediately from the method of finding the altitude of a golden rectangle given its base (details left for the reader).

As an immediate consequence we have a corollary:

$$g_n = \frac{(\sqrt{5} - 1)^n}{2^{n-1}} \quad .$$

We next observe after considering the first few golden numbers that

Theorem 2. $\quad g_n = g_{n-2} - g_{n-1}$

Proof. Using the form for g_n given in the Corollary to Theorem 1, we have

$$g_{n-2} - g_{n-1} = \frac{(\sqrt{5}-1)^{n-2}}{2^{n-3}} - \frac{(\sqrt{5}-1)^{n-1}}{2^{n-2}}$$

$$= \frac{2^2 \cdot (\sqrt{5}-1)^{n-2}}{2^{n-1}} - \frac{2 \cdot (\sqrt{5}-1)^{n-1}}{2^{n-1}} = \frac{2(\sqrt{5}-1)^{n-2}\left[2 - \sqrt{5} + 1\right]}{2^{n-1}}$$

$$= \frac{2(\sqrt{5}-1)^{n-2} \cdot (3 - \sqrt{5})}{2^{n-1}} = \frac{2(\sqrt{5}-1)^{n-2} \cdot \frac{(\sqrt{5}-1)^2}{2}}{2^{n-1}} = \frac{(\sqrt{5}-1)^n}{2^{n-1}}$$

$$= g_n \ .$$

Another rather interesting observation is that the coefficients of radical 5 appear to be the sequence of Fibonacci numbers with alternating signs. We may formalize the conjecture after observing that as a result of a multiplication by $(\sqrt{5}-1)/2$, the signs of each term of the golden numbers alternate and the n^{th} golden number may be expressed in the form

$$g_n = (-1)^{n-1}\ a_n \cdot \sqrt{5} - b_n$$

where a_n and b_n are positive integers.

Theorem 3. If

$$g_n = (-1)^{n-1}[\,a_n \cdot \sqrt{5} - b_n\,]$$

represents the n^{th} golden number, then a_n is the n^{th} Fibonacci number, F_n.

Proof.

$$g_{n+1} = g_n \cdot \frac{\sqrt{5}-1}{2} = \frac{(-1)^{n-1}}{2}\,[\,5a_n - \sqrt{5}b_n - \sqrt{5}a_n + b_n\,]$$

$$= (-1)^n \left[\frac{(a_n + b_n)}{2}\sqrt{5} - \frac{(5a_n + b_n)}{2}\right]$$

$$\therefore\ a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \frac{5a_n + b_n}{2}$$

Then

$$a_{n-1} + a_n = a_{n-1} + \frac{a_{n-1} + b_{n-1}}{2} = \frac{3a_{n-1} + b_{n-1}}{2}$$

and

$$a_{n+1} = \frac{a_n + b_n}{2} = \frac{\frac{a_{n-1} + b_{n-1}}{2} + \frac{5a_{n-1} + b_{n-1}}{2}}{2} = \frac{3a_{n-1} + b_{n-1}}{2}$$

$$= a_{n-1} + a_n \longrightarrow a_n = F_n .$$

Yet another observation may be made from the sequence (1). It is stated:

Theorem 4. If g_n and g_{n+1} are any two successive golden numbers, then $F_{n+1} \cdot g_n + F_n \cdot g_{n+1} = 2$.

Proof. Using the representation for F_{n+1} developed in the proof of Theorem 3, we write

$$F_{n+1} = \frac{F_n + b_n}{2} \rightarrow b_n = 2F_{n+1} - F_n$$

Therefore, we may express g_n and g_{n+1} in terms of Fibonacci numbers only:

$$g_n = (-1)^{n-1}(F_n\sqrt{5} + F_n - 2F_{n+1})$$

and

$$g_{n+1} = (-1)^n(F_{n+1}\sqrt{5} + F_{n+1} - 2F_{n+2}) .$$

Thus we obtain:

$$F_{n+1} \cdot g_n + F_n \cdot g_{n+1} = (-1)^{n-1}[F_n \cdot F_{n+1}\sqrt{5} + F_n \cdot F_{n+1} - 2F^2_{n+1}$$

$$- F_n \cdot F_{n+1}\sqrt{5} - F_n \cdot F_{n+1} + 2F_n \cdot F_{n+2}]$$

Recalling the fundamental identity

$$F_{n-1} \cdot F_{n+1} = F_n^2 + (-1)^n, \qquad n \geq 2,$$

it follows that

$$F_{n+1} \cdot g_n + F_n \cdot g_{n+1} = (-1)^{n-1}[-2F_{n+1}^2 + 2(F_{n+1}^2 + (-1)^{n+1})] = 2$$

Recalling the representation for g_n used in the proof of Theorem 4,

$$g_n = (-1)^{n-1}(F_n\sqrt{5} + F_n - 2F_{n+1})$$

we observe that

$$F_n - 2F_{n+1} = F_n - 2[F_{n-1} + F_n] = -F_n - 2F_{n-1}$$

which gives us the following alternate forms for the n^{th} golden number:

$$g_n = (-1)^{n-1}(F_n \cdot g_1 - 2F_{n-1})$$

or

$$g_n = (-1)^{n-1}(\sqrt{5}F_n - L_n)$$

where L_n is the n^{th} Lucas number. We now state our final result.

Theorem 5. $g_n = (-1)^{n-1}(\sqrt{5}F_n - L_n)$

Proof. Follows from the identity

$$L_n = F_{n-1} + F_{n+1} \quad .$$

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A MARKOV LIMIT PROCESS INVOLVING FIBONACCI NUMBERS

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Consider the two-state Markov chain with transition matrix

$$P = \begin{array}{c} \\ s_1 \\ s_2 \end{array}\begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \end{array} ,$$

where $0 < a \le 1$, $0 < b \le 1$ and $0 < a + b < 2$. The branch probabilities may be displayed (Fig. 1) on a tree diagram for this chain.

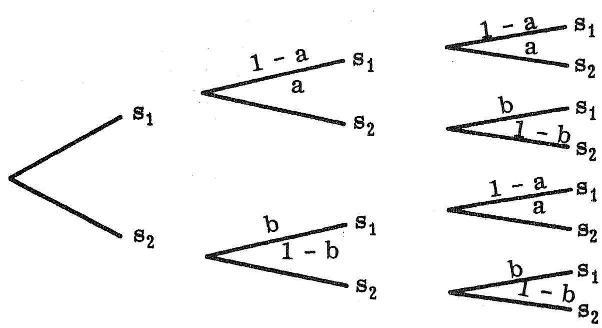

Figure 1

The above matrix P has a fixed vector $\alpha = (\alpha_1, 1 - \alpha_1)$ and limiting matrix

$$A = \lim_{n \to \infty} P^n = \begin{pmatrix} \alpha_1 & 1 - \alpha_1 \\ \alpha_1 & 1 - \alpha_1 \end{pmatrix}$$

for $0 < \alpha_1 < 1$. The entries α_1 and $(1 - \alpha_1)$ may be interpreted as the limiting proportion of times that the process is in state s_1 and s_2, respectively, as the number of steps, n, increases indefinitely.

For example, the Markov chain transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{pmatrix} \end{array}$$

has fixed vector

$$\alpha = \left(\frac{3}{11}, \frac{8}{11} \right)$$

and the process would be expected to be in states s_1 and s_2 in a ratio of 3:8 as the number of steps increases indefinitely.

For the special case

$$P = \begin{array}{c c} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 0 & 1 \\ b & 1 - b \end{pmatrix} \end{array}$$

$(0 \le b \le 1)$, the fixed vector α is

$$\alpha = \left(\frac{b}{1 + b}, \frac{1}{1 + b} \right),$$

using the indicated branch probabilities from the matrix P. However, we delete the branch probabilities from the tree and display only the "bare" branches of the tree (Fig. 2).

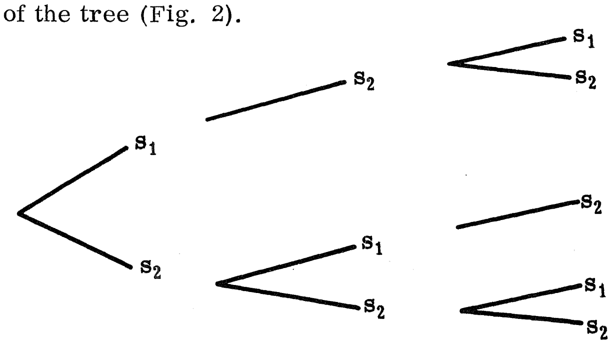

Figure 2

Instead of reading "out" the tree (i.e., left to right), we read "across" the tree (i.e., from top to bottom), so that in the first step, s_1 appears once and s_2 appears twice; in the second step s_1 appears twice and s_2 appears three times, and so on.

Denote by $N(1,n)$ the number of times that s_1 appears in the n^{th} stage, with $N(2,n)$ defined similarly. $(n = 0, 1, 2, 3, \cdots)$. It is apparent that $N(1,n) = u_n$, where u_n is the $(n+1)^{st}$ Fibonacci number in the sequence $u_0 = 1$, $u_1 = 1$, $u_2 = 2$, $u_3 = 3$, $u_4 = 5$, etc. Similarly, the total number of entries in the n^{th} stage is given by $N(1,n) + N(2,n) = u_{n+2}$, and also read $N(2,n) = u_{n+2} - N(1,n) = u_{n+2} - u_n = u_{n+1}$. Thus, continuing to read "across" the tree, the proportion of times that s_2 appears in the n^{th} stage is

$$\frac{N(2,n)}{N(1,n) + N(2,n)} = \frac{u_{n+1}}{u_{n+2}} ,$$

and this proportion has a well-known limiting value

$$\alpha = \frac{\sqrt{5} - 1}{2} = \lim_{n \to \infty} \frac{u_{n+1}}{u_{n+2}} .$$

Thus, consider the case

$$P = \begin{array}{c c} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{\sqrt{5} - 1}{2} & \frac{3 - \sqrt{5}}{2} \end{pmatrix} \end{array} .$$

The fixed vector is

$$\alpha = \left(\frac{3 - \sqrt{5}}{2} , \frac{\sqrt{5} - 1}{2} \right) = (.382\cdots, .618\cdots) ,$$

and so the process would be expected to be in state s_2 approximately 61.8% of the time, as n increases indefinitely, using the indicated branch probabilities. On the other hand, using only the corresponding "bare" tree, the limiting proportion of times that s_2 will appear in the n^{th} "column" (stage) will be the same proportion, as the number of steps increases indefinitely.

REFERENCES

1. J. G. Kemeny and J. L. Snell, Finite Markov Chains, Princeton, D. van Nostrand Co., Inc., 1960.

2. N. N. Vorob'ev, Fibonacci Numbers, New York, Blaisdell Publishing Company, 1961.

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CORRECTIONS

Please make the following corrections on the paper, "On Summation Formulas and Identities for Fibonacci Numbers," Vol. 5, No. 1, pp. 1-43, Fibonacci Quarterly:

Page 22: In Eq. (4.3), change $b = 0$ to read $b \neq 0$.
In the reference on the last line, add parentheses around 12.

Page 36: In the first line of Eq. (5.23), change c_{j-1} to read c_{j-i}.

Page 38: In the second line, insert brackets around the reference.
In the first line following Eq. (6.3), change i = 1) to i = 1,2

Page 39: In the first line following Eq. (6.5), add brackets around reference.

Page 40: In Eq. (6.15), change $P_1(m,n)$ to read $P_1(m,-n)$. D. Z.

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NOTICE

George Ledin, Jr. has been appointed by The Fibonacci Association to collect and classify all existing Fibonacci Identities, Lucas Identities, and Hybrid Identities. We request that readers send copies of their private lists (with possible reference sources) to

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for inclusion in the planned booklet.

Verner E. Hoggatt, Jr.,
Director

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EXTENDED COMPUTATIONS OF TERMINAL DIGIT COINCIDENCES

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In [1] Brother U. Alfred asked the following question: What Fibonacci numbers of index less than 10,000 have terminal digits coincident with the index? Recently in this Quarterly, Gerald R. Deily [2] gave an answer by directly computing these coincidences with the aid of a computer. We note that in Table III of Mr. Deily's article the digits "65" should be added to the last number in each of the lines 14 to 21. Here we extend these computations to indices less than 100,000. The results, given in the table below, were obtained on an IBM 1620 computer using a FORTRAN program logically similar to Mr. Deily's. As a point of observation, we note that all entries are of the form 480n + 5 or 480n - 95, with the four exceptions 60,001, 61,249, 62,501, and 63,749.

The author expresses his appreciation to the Air Force Office of Scientific Research and to the Applied Mathematics Laboratory of the Aerospace Research Laboratory for the use of the computer.

TERMINAL DIGIT COINCIDENCES WITH INDEX
BETWEEN 10,000 AND 100,000

10945	18725	27745	35525
11045	19105	27845	35925
11425	19205	28225	36005
11525	20545	28325	37345
11905	20645	28705	37445
12005	21025	28805	37825
13345	21125	30145	37925
13445	21505	30245	38305
13825	21605	30625	38405
13925	22945	30725	39745
14305	23045	31105	39845
14405	23425	31205	40225
15745	23525	32545	40325
15845	23905	32645	40705
16225	24005	33025	40805
16325	25345	33125	42145
16705	25445	33505	42245
16805	25825	33605	42625
18145	25925	34945	42725
18245	26305	35045	43105
18625	26405	35425	43205
			44545

44645	60001	74405	90725
45025	60005	75745	91105
45125	61249	75845	91205
45505	61345	76225	92545
45605	61445	76325	92645
46945	61825	76705	93025
47045	61925	76805	93125
47425	62305	78145	93505
47525	62405	78245	93605
47905	62501	78625	94945
48005	63745	78725	95045
49345	63749	79105	95425
49445	63845	79205	95525
49825	64225	80545	95905
49925	64325	80645	96005
50305	64705	81025	97345
50405	64805	81125	97445
51745	66145	81505	97825
51845	66245	81605	97925
52225	66625	82945	98305
52325	66725	83045	98405
52705	67105	83425	99745
52805	67205	83525	99845
54145	68545	83905	
54245	68645	84005	
54625	69025	85345	
54725	69125	85445	
55105	69505	85825	
55205	69605	85925	
56545	70945	86305	
56645	71045	86405	
57025	71425	87745	
57125	71525	87845	
57505	71905	88225	
57605	72005	88325	
58945	73345	88705	
59045	73445	88805	
59425	73825	90145	
59525	73925	90245	
59905	74305	90625	

REFERENCES

1. Brother U. Alfred, "Exploring Fibonacci Numbers with a Calculator," Fibonacci Quarterly, 2(1964), No. 2, p. 138.

2. Gerard R. Deily, "Terminal Digit Coincidences Between Fibonacci Numbers and Their Indices," Fibonacci Quarterly, 4(1966), Vol. 2, No. 1, pp 151-156.

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PYTHAGOREAN TRIANGLES AND RELATED CONCEPTS

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INTRODUCTION

The familiar Pythagorean theorem

$$a^2 + b^2 = c^2 \tag{1}$$

(a, b) = length of two sides of a right triangle
c = length of the hypotenuse

has an infinite number of integer solutions, e. g.

a	3	5	8	7
b	4	12	15	24
c	5	13	17	25

as Diophantus of Alexandria first demonstrated and tabulated in the third century [1]. Many of his tabulated entries, however, produce right triangles which differ only in scale, representing redundant or reducible solutions. This paper presents a method for generating only irreducible-integer ("fundamental") solutions and studies some of their common properties:

1. The hypotenuse length is always an odd number.
2. One side is always odd, one side always even.
3. The even length is always divisible by four.
4. Hypotenuse ± even side is always a perfect square.
   Hypotenuse ± odd side is always twice a perfect square.
5. Taking m and n as any distinctly odd and even numbers with no common factor (m   n), the complex quantity,

$$(m + jn)^2 = (\text{one leg}) + j\ (\text{other leg})$$

and its modulus

$$|m + jn|^2 = \text{(hypotenuse)}$$

always generate a fundamental triangle and conversely. Equivalently, the acute angles always correspond to

$$\arg.\ (m + jn)^2 = 2\tan^{-1}\left(\frac{n}{m}\right) = 2\tan^{-1} p$$

and its complement: p = twice or half of some rational number < 1.

6. Any line segment of length $(2k + 1)$ or $4k$, $k = 1, 2, 3, \cdots$, constitutes a leg of at least one fundamental triangle; more in many cases.
7. The necessarily non-integer nature of solutions to $a^n + b^n = c^n$, $n > 2$, (Fermat's Last Theorem) can be proved for $n = 4k$, $k = 1, 2, 3, \cdots$.
8. In a rectangular parallelopiped of integer dimensions and integer length diagonal, two of the dimensions must be even while the third dimension and the diagonal itself must be odd.

GENERATION OF FUNDAMENTAL SOLUTIONS

One method for generating fundamental solutions rewrites (1) as

$$b = \sqrt{(c + a)(c - a)} \tag{2}$$

suggesting the special case: $(c + a) = m^2$, $(c - a) = n^2$. More generally we might set $(c + a) = r_1m^2$, $(c - a) = r_2n^2$, where neither integer r contains any repeated factors. A necessary condition for an integer solution then is $r_1 = r_2$ or equivalently, $a = r(m^2 - n^2)/2$, $b = rmn$, $c = r(m^2 + n^2)/2$.

Substitutions $r' = (r/2)$, $m' = (m + n)$ and $n' = (m - n)$ will yield equivalent expressions, except for (trivially) interchanging the roles of a and b. This equivalence helps to explain why choices of m (or m') and n (or n') as

(i) both odd numbers
(ii) both even numbers
(iii) one odd and one even

are all redundant [2]. We will choose (iii) for the most compact expressions:

(3) $$a = m^2 - n^2 \quad b = 2mn \quad c = m^2 + n^2$$

subject to further condition that m and n possess no common factors. None of the b-factors can then divide evenly into either a or c; the solution is irreducible. Appropriate choices of m and n will thus generate all fundamental solutions.

INITIAL PROPERTIES OF FUNDAMENTAL TRIANGLES

Properties 1 through 5 follow directly from Equation (3) and condition (iii). Thus,

Property 1 $\quad c = \text{hypotenuse} = m^2 + n^2 = (\text{odd}) + (\text{even}) = \text{odd}$

Property 2 $\quad a = m^2 - n^2 = (\text{odd})$

Property 2 and 3 $\quad b = 2mn = 2\,(\text{even}) = 4\,\dfrac{mn}{2} = 4\,(\text{integer})$

Property 4 $\quad c \pm b = (m \pm n)^2; \quad c \pm a = (2m^2; 2n^2)$

Property 5 $\quad (m + jn)^2 = (m^2 - n^2) + j2mn = a + jb$

FUNDAMENTAL TRIANGLES WITH A COMMON SIDE

In Property 6, a chosen value of

(4) $$b = 2mn = 4k(\ (k = 1, 2, 3, \cdots)$$

will provide one or more permissible combinations of m and n, leading to as many fundamental triangles with the same even side. Resolution of $a = m^2 - n^2$ into any distinct odd factors $(m + n)(m - n)$ will likewise provide choices of

(5)
$$c = \frac{(m + n)^2 + (m - n)^2}{2}$$

(6)
$$b = \frac{(m + n)^2 - (m - n)^2}{2}$$

yielding fundamental triangles with the same odd side.

Regarding the number of such triangles, we may express

(7)
$$mn = \frac{b}{2} = 2^x \cdot r_1^{\alpha_1} \cdot r_2^{\alpha_2} \cdot r_3^{\alpha_3} \cdots r_N^{\alpha_N} ,$$

where the $r_1^{\alpha_1}$ represent distinct odd prime factors raised to an integer power. Since m and n contain no common multiple, $r_i^{\alpha_i}$ can be associated with either m or n but not both (e.g., $\alpha_i = 5$, $m \sim r_i^2$, $n \sim r_i^3$ is forbidden), giving two possible choices. The N + 1 factors (counting 2^x) will likewise give 2^{N+1} possible ways of expressing m and n, except that m must always identify as the larger of the two. Half of these ways, however, have simply exchanged the roles of m and n with the other half. We therefore obtain $2^{N+1}/2 = 2^N$ permissible paris of m and n, and 2^N fundamental triangles with the same even-length side.

Again,

(8)
$$a = S_1^{\beta_1} \cdot S_2^{\beta_2} \cdots S_N^{\beta_N} = C \cdot D$$

where the S_i are odd, prime factors. We can similarly associate each $S_i^{\beta_i}$ with either C (odd) or D (odd) in a total of 2^N different ways. Should we specify C = (m + n) as the larger and D = (m - n) as the smaller, the number of distinct possibilities reduces to $(2^N)/2 = 2^{N-1}$ [3] and indicates as many fundamental triangles with the same odd side. Since a-values (odd numbers) occur twice as frequently as b-values (multiples of 4) in an ordered sequence of integers, they may quite reasonably exhibit only half the potency of b-values in generating fundamental triangles.

ORDERED TABULATIONS OF FUNDAMENTAL TRIANGLES

The a- and b-schemes in fact provide two means for ordered tabulations, viz:

b	4	8	12		16	20			60				
mn	2	4	6		8	10			30				
m	2	4	6	3	8	10	5		30	15	10	6	
n	1	1	1	2	1	1	2		1	2	3	5	
a	3	15	35	5	63	99	21		899	221	91	11	
c	5	17	37	13	65	101	29		901	229	109	61	

Table I. Illustration of b-Scheme of Tabulation

a	3	5	7	9	11	13	15			105			
m + n	3	5	7	9	11	13	15	5		105	35	21	15
m - n	1	1	1	1	1	1	1	3		1	3	5	7
b	4	12	24	40	60	84	112	8		5512	608	208	88
c	5	13	25	41	61	85	113	17		5513	617	233	137

Table II. Illustration of a-Scheme of Tabulation

These tables help to illustrate the self-evident conclusions:

1) For any specified even side, there is always a fundamental triangle whose hypotenuse and odd side differ by 2 (corresponds to $n = 1$).
2) For any specified odd side there is always a fundamental triangle whose hypotenuse and even side differ by 1 (corresponds to $m - n = 1$).

FERMAT'S THEOREM

The preceding analysis has applied the identity $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$ which might be rewritten as

(9) $$d^4 + e^4 = f^4 \quad ,$$

where $d = \sqrt{m^2 - n^2}$, $e = \sqrt{2mn}$ and $f = \sqrt{m^2 + n^2}$ corresponds to line segments as in Fig. 1.

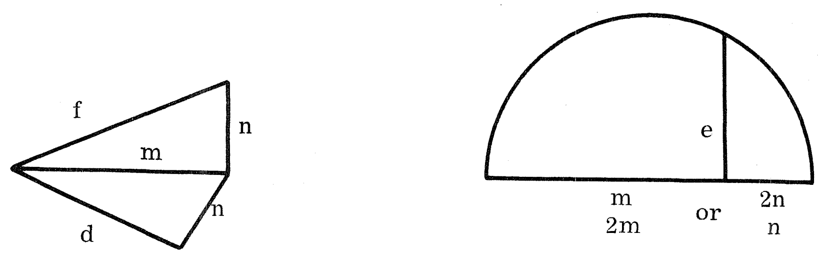

Figure 1. Graphic Constructions Expressing $d^4 + e^4 = f^4$

Thus,

(10) $$d = \sqrt{f^2 - 2n^2} = \sqrt{(f + n\sqrt{2})(f - n\sqrt{2})}$$

cannot assume integer values unless f contains a factor of $\sqrt{2}$, i.e., (9) has no integer solutions. Similarly,

(11) $$(d')^{4k} + (e')^{4k} = (f')^{4k}; \quad k = 1, 2, 3, \cdots$$

finds no integer solutions since we may set $d' = \sqrt[k]{d}$, $e' = \sqrt[k]{e}$ and $f' = \sqrt[k]{f}$.

RECTANGULAR PARALLELOPIPEDS

Some of these results apply directly to integer-sided rectangular parallelopipeds; we shall refer to Fig. 2. According to this figure,

(12) $$a^2 + b^2 = d^2 - c^2 = (d + c)(d - c)$$

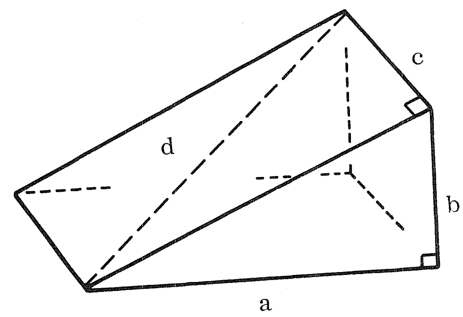

Figure 2. Diagonally Cut Half of Rectangular Parallelopiped

Suggesting that $d + c - (a^2 + b^2)/r$ and $d - c = r =$ some factor of $a^2 + b^2$, Thus,

$$(2d - r)r = (2c + r)r = a^2 + b^2 \tag{13}$$

where non-fractional values of c impose the condition

$$r_{max} \leq \sqrt{a^2 + b^2 + 1} - 1 \ .$$

Even values of r imply even values of a and b, while odd r demand mixed odd/even values for a and b. At least one dimension in any pair of dimensions must therefore be even; i. e., two of the three dimensions must be even. The third dimension must be odd (to prevent reducibility) while

$$d^2 = a^2 + b^2 + c^2 = (\text{even})^2 + (\text{even})^2 + (\text{odd})^2 \tag{14}$$

further requires an odd-length diagonal.

One particular scheme for generating parallelopipeds might thus begin by choosing the odd-length dimension and one of the even ones; call them and b. Evaluate $a^2 + b^2$ (now always odd) and determine the upper bound on r_{max}. Below this bound, suitable choices of r must qualify as factors of $a^2 + b^2$ and are now, likewise, always odd [4]. These choices give values of c and d via Eq. (13) and suggest the following tabulation.

Quantity	Comments	Sample Solutions						
a	One Odd, One Even	1	1		2		5	
b		2	8		9		10	
$a^2 + b^2$	Always Odd	5	65		85		125	
$r_{max} \leq$		1	7		8		10	
r	Always Odd	1	5	1	5	1	5	1
c	$= (a^2 + b^2 - r^2)/2r$	2	4	32	6	42	10	62
d	$= (a^2 + b^2 + r^2)/2r$	3	9	33	11	43	15	63

Table III. Scheme and Solutions for Fundamental Rectangular Parallelopipeds

CONCLUSION

Properties 2, 3, and 8 are the most useful since they lead directly to the generation and tabulation of fundamental solutions. The remaining properties have no such direct application but may represent areas of further study.

BIBLIOGRAPHY AND NOTES

1. G. Gamow, One, Two, Three· · · Infinity, Viking Press, New York, 1958, p. 30, presents these solutions as $a = r + \sqrt{2rs}$, $b = a + \sqrt{2rs}$, $c = r + s + \sqrt{2rs}$.
2. Choice (i) requires $r = 1$ for a fundamental solution. Choice (ii), in prime notation, requires $r' = 1/2$ and contradicts the assumption of integral r. Expressed in terms of unprimed quantities moreover it becomes indistinguishable from (i). Choice (iii) similarly requires $r = 2$. In terms of primed quantities, it also reduces to the form of (i).
3. The original 2^N ways can be grouped into pairs, indistinguishable except that one chooses C as the larger while the other chooses C as the smaller of the two factors. Specifying $C > D$ validates only one member from each pair, having the 2^N original possibilities.
4. An alternative scheme might have started with both even-length dimensions, allowing both odd and even values for r.

* * * * *

ACHIEVING THE "GOLDEN RATIO" BY GROUPING THE "ELEMENTARY" PARTICLES

J. WLODARSKI
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"The mystery presented by the multiplicity of 'elementary' particles seems to be rapidly reaching a climax and perhaps even a solution. The idea is gaining ground that all the known particles can be grouped into a few large families and that within each of these 'supermultiplets' all the particles can be regarded as the mathematical equivalents of one another."

This has recently been published in a scientific magazine [1].

As a matter of fact, supermultiplets of 35 or 56 members can accommodate most of the well-established particles.

A 35-member family can be formed by grouping 17 of the known mesons that have negative parity. Eight of these particles: the pion (π) triplet, the kaon (κ) quartet and the eta (η) singlet have a spin of zero, and therefore only one spin state each (0).

The following nine mesons: the rho (ρ) triplet, the phi (φ) singlet, the omega (ω) singlet and another kaon quartet have a spin of one, or three spin states each (-1, 0 +1).

The total is $8 \times 1 + 9 \times 3 = \underline{35}$ spin states.

A 56-member family consists of the known 56 baryons or 56 antibaryons. They have positive parity. Eight of these particles: the proton-neutron (or nucleon) doublet, the lambda (Λ) singlet, the sigma (Σ) triplet and the xi (Ξ) doublet have a spin of 1/2, and therefore two spin states each (+1/2, -1/2).

The following ten particles: the delta (Δ) quartet, another sigma triplet, another xi doublet and the omega (Ω) singlet have a spin of 3/2, or four spin states each (+3/2, +1/2, -1/2, -3/2).

The total is $8 \times 2 + 10 \times 4 = \underline{56}$ spin states.

It has already been reported that as well as in the world of plants, some ratios in the world of atoms yield approximately the value of the "golden ratio" [2,3].

Now it has turned out that by grouping the "elementary" particles, the ratio of two "magic" numbers – spin states of the "elementary" particles also yields a near value of the "golden ratio."

As a matter of fact, the ratio of 35/56 is 0.625 and differs from the "golden ratio" value by 0.007 only.

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1. "Extended Symmetry," Scientific American, Vol. 212 (March 1965), pp. 52-54.
2. J. Wlodarski, "The 'Golden Ratio' and the Fibonacci Numbers in the World of Atoms," The Fibonacci Quarterly, Vol. 1, Number 4 (1963), pp. 61-63.
3. J. Wlodarski, "The Fibonacci Numbers and the 'Magic' Numbers," The Fibonacci Quarterly, Vol. 3, Number 3 (1965), p. 208.

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CORRECTION

On "Relations Involving Lattice Paths and Certain Sequences of Integers," Vol. 5, No. 1, pp. 81-86, Fibonacci Quarterly, please add the following:
"Work on this paper was supported in part by the Coordinating Board of the Texas College and University System."
Also, please change the author's name on p. 81 from David to Douglas.

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SOME RABBIT PRODUCTION RESULTS INVOLVING FIBONACCI NUMBERS

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University of Santa Clara, Santa Clara, California

Let us consider a pair of rabbits born in the 0-th month which produce B_1 offspring pairs when they are one month old, B_2 offspring pairs when they are two months old and so on. The sequence of numbers

$$B_1,\ B_2,\ B_3,\ \ldots\ B_n,\ \ldots$$

is called the birth sequence, and let its generating function be

$$B(x) = \sum_{n=0}^{\infty} B_n x^n ,$$

where $B_0 = 0$.

Suppose each pair of offspring also produces B_n offspring pairs when it is n months old. Let the number of new arrivals at the n-th month be Rn, and let

$$R(x) = \sum_{n=0}^{\infty} R_n x^n$$

where $R_0 = 1$. Let the total number of rabbits alive at the end of the n-th month be T_n, and let

$$T(x) = \sum_{n=0}^{\infty} T_n x^n$$

where $T_0 = 1$. We will assume that there are no deaths.

*Student

It has been shown (see [1], [2]) that

$$R(x) = \frac{1}{1-B(x)}$$

and

$$T(x) = \frac{1}{(1-x)(1-B(x))} .$$

The purpose of this paper is to show some particular cases in which there are interesting relationships between $B(x)$, $R(x)$, and $T(x)$.

When

$$B(x) = \sum_{n=0}^{\infty} x^{n+2} ,$$

then

$$T(x) = \sum_{n=0}^{\infty} F_{n+1} x^n$$

When

$$B(x) = \sum_{n=2}^{\infty} (2n-1)x^n ,$$

then

$$T(x) = \sum_{n=0}^{\infty} F_{n+1}^2 x^n .$$

When

$$B(x) = \sum_{n=2}^{\infty} (6C_{n-1} + 1)x^n ,$$

where the C_n are terms of the Pell sequence defined by $C_0 = 0$, $C_1 = 1$, $C_{n+2} = 2C_{n+1} + C_n$, then

$$T(x) = \sum_{n=0}^{\infty} F_{n+1}^3 x^n .$$

It is conjectured that when

$$T(x) = \sum_{n=0}^{\infty} F_{n+1}^p x^n ,$$

the corresponding $B(x)$ will have $B_n \geq 0$ for all n. This has been demonstrated for $p \leq 7$.

Hoggatt showed in [1], section 4, that when

(1) $$B(x) = \frac{F_{k+1}x-(-1)^k x^2}{1-F_{k-1}x} ,$$

then

$$R(x) = \sum_{n=0}^{\infty} F_{kn+1}x^n ,$$

and similarly when

(2) $$B(x) = \frac{F_{k-1}x-(-1)^k x^2}{1-F_{k+1}x}$$

then

$$R(x) = \sum_{n=0}^{\infty} F_{kn-1}x^n .$$

By merely changing the sign of the second term of the numerator of equations (1) and (2), we obtain the following results, which depend on the parity of k. When

$$B(x) = \frac{F_{k+1}x+(-1)^k x^2}{1-F_{k-1}x} \quad ,$$

then for k odd we have

$$(3) \qquad R(x) = 1+ \sum_{n=0}^{\infty} \left[U_{n+1}(L_k/2)-F_{k-1}U_n L_k/2)\right] x^{n+1} \quad ,$$

where the $U_n(x)$ are Chebyshev polynomials of the second kind defined by $U_0(x) = 0$, $U_1(x) = 1$, $U_{n+2}(x) = 2xU_{n+1}(x) + U_n(x)$.

For k even we have

$$(4) \qquad R(x) = \sum_{n=0}^{\infty} \left[f_{n+1}(L_k)-F_{k-1}f_n(L_k)\right]x^n \quad ,$$

where the $f_n(x)$ are the Fibonacci polynomials defined by $f_0 = 0$, $f_1 = 1$, $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$. Similarly when

$$B(x) = \frac{F_{k-1}x+(-1)^k x^2}{1-F_{k+1}x}$$

then for k odd we get

$$(5) \qquad R(x) = 1+ \sum_{n=0}^{\infty} \left[U_{n+1}(L_k/2)-F_{k+1}U_n(L_k/2)\right]x^{n+1}$$

while for k even we find

(6) $$R(x) = \sum_{n=0}^{\infty} \left[f_{n+1}(L_k) - F_{k+1}f_n(L_k)\right]x^n \quad ,$$

where $U_n(x)$ and $f_n(x)$ are defined above.

Two other possibilities occur when L_k is substituted for F_k in equations (1) and (2). When

$$B(x) = \frac{L_{k+1}x + (-1)^k x^2}{1 - L_{k-1}x}$$

then for k odd

(7) $$R(x) = 1 + \sum_{n=0}^{\infty} \left[U_{n+1}(5/2\ F_k) - L_{k-1}U_n(5/2\ F_k)\right]x^{n+1} \quad .$$

For k even,

(8) $$R(x) = \sum_{n=0}^{\infty} \left[f_{n+1}(5F_k) - L_{k-1}f_n(5F_k)\right]x^n \quad .$$

Similarly, when

$$B(x) = \frac{L_{k-1}x + (-1)^k x^2}{1 - L_{k+1}x}$$

then for k odd,

$$R(x) = 1+ \sum_{n=0}^{\infty} \left[U_{n+1}(5/2\ F_k) - L_{k+1} U_n(5/2\ F_k)\right] x^{n+1}$$

and for k even,

$$R(x) = \sum_{n=0}^{\infty} \left[f_{n+1}(5F_k) - L_{k+1} f_n(5F_k)\right] x^n \ .$$

Note that equations (7) through (10) are the Lucas duals to equations (3) through (6).

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1. V. E. Hoggatt, Jr., Generalized Rabbits for Generalized Fibonacci Numbers, to appear, The Fibonacci Quarterly.
2. V. E. Hoggatt, Jr. and D. A. Lind, The Dying Rabbit Problem, to appear The Fibonacci Quarterly.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

B-112 Proposed by Gerald Edgar, Boulder, Colorado

Let f_n be the generalized Fibonacci sequence (a, b), i.e., $f_1 = a$, $f_2 = b$, and $f_{n+1} = f_n + f_{n-1}$. Let g_n be the associated generalized Lucas sequence defined by $g_n = f_{n-1} + f_{n+1}$. Prove that $f_n g_n = bf_{2n-1} + af_{2n-2}$.

B-113 Proposed by Douglas Lind, Univ. of Virginia, Charlottesville, Va.

Let (x) denote the fractional part of x, so that if $[x]$ is the greatest integer in x, $(x) = x - [x]$. Let $a = (1 + \sqrt{5})/2$ and let A be the set $\{(a), (a^2), (a^3), \ldots\}$. Find all the cluster points of A.

B-114 Proposed by Gloria C. Padilla, Univ. of New Mexico, Albuquerque, New Mexico

Solve the division alphametic

$$\begin{array}{r|l} & \text{PISA} \\ \hline \text{FIB} & \text{ONACCI} \end{array}$$

where each letter is one of the digits 1, 2, ..., 9 and two letters may represent the same digit. (This is suggested by Maxey Brooke's B-80.)

B-115 Proposed by H. H. Ferns, Victoria, B.C., Canada

From the formulas of B-106:

$$2F_{i+j} = F_iL_j + F_jL_i$$

$$2L_{i+j} = 5F_iF_j + L_iL_j$$

one has

$$F_{2n} = F_nL_n$$

$$F_{3n} = (5F_n^3 + 3F_nL_n^2)/4$$

$$L_{2n} = (5F_n^2 + L_n^2)/2$$

$$L_{3n} = (15F_n^2L_n + L_n^3)/4$$

Find and prove the general formulas of these types.

B-116 Proposed by L. Carlitz, Duke University, Durham, N. Carolina

Find a compact sum for the series

$$\sum_{m,n=0}^{\infty} F_{2m-2n}x^my^n$$

B-117 Proposed by L. Carlitz, Duke University, Durham, N. Carolina

Find a compact sum for the series

$$\sum_{m,n=0}^{\infty} F_{2m-2n+1}x^my^n .$$

SOLUTIONS

TERMS OF A DETERMINANT

B-94 Proposed by Clyde A. Bridger, Springfield Jr. College, Springfield, Ill.

Show that the number N_n of non-zero terms in the expansion of

$$K_n = \begin{vmatrix} a_1 & b_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & a_2 & b_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & a_3 & b_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots \\ \cdots \\ \cdots \\ 0 & 0 & 0 & \cdots & & 0 & -1 & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & & 0 & 0 & -1 & a_n \end{vmatrix}$$

is obtained by replacing each a_i and each b_i by 1 and evaluating K_n. Show further that $N_n = F_{n+1}$, the $(n + 1)$st Fibonacci number.

Solution by F. D. Parker, St. Lawrence University, Canton, N.Y.

Expanding by the last column, we have $K_n = a_nK_{n-1} + b_{n-1}K_{n-2}$. Hence, if N_n is the number of non-zero terms in the expansion, we have $N_n = N_{n-1} + N_{n-2}$. But $N_1 = 1$, $N_2 = 2$, so that $N_n = F_{n+1}$.

Also solved by M.N.S. Swamy and the proposer.

A FIBONACCI FACTORIAL

B-95 Proposed by Brother U. Alfred, St. Mary's College, California.

What is the highest power of 2 that exactly divides

$$F_1F_2F_3 \ldots F_{100} \quad ?$$

Solution by Charles W. Trigg, San Diego, California.

For $n \geq 3$, F_k is divisible by 2^n if k is of the form $2^{n-2} \cdot 3(1+2m)$, F_k is divisible by 2 but by no higher power of 2. Hence, the highest power of 2 that exactly divides $F_1F_2F_3 \ldots F_{100}$ is

$$[(100 - 3(/6 + 1] + 3[(100 + 6(/12] + 4[112/24] + 5[124/48]$$
$$+ 6[148/96] + 7[196/192] \text{ or } 80.$$

As usual, $[x]$ indicates the largest integer in x.

Also solved by Sidney Kravitz, Dewey C. Duncan, and the proposer.

Editorial note: The results in the above solution indicate that the answer may also be expressed as

$$[100/3] + 2[100/6] + [100/12] + [100/24] + [100/48]$$
$$+ [100/96] = 33 + 32 + 8 + 4 + 2 + 1 = 80.$$

LIMITED PARTITIONS

B-96 Proposed by Phil Mana, Univ. of New Mexico, Albuquerque, New Mex.

Let G_n be the number of ways of expressing the positive integer n as an ordered sum $a_1 + a_2 + \ldots + a_s$ with each a_i in the set 1, 2, 3. (For example, $G_3 = 4$ since 3 has just the expressions $3, 2 + 1, 1 + 2, 1 + 1 + 1$.) Find and prove the lowest order linear homogeneous recursion relation satisfied by the G_n.

Solution by the proposer.

Removing the forst term a_1 (which is 1, 2, or 3) from all allowable sums for an $n > 3$ gives all allowable sums for n-1, n-2, and n-3 in unique fashion. Hence $G_n = G_{n-1} + G_{n-2} + G_{n-3}$ for $n > 3$. There is no lower order linear homogeneous recursion relation for the G_n since

$$\begin{vmatrix} G_1 & G_2 & G_3 \\ G_2 & G_3 & G_4 \\ G_3 & G_4 & G_5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 4 & 7 \\ 4 & 7 & 13 \end{vmatrix} \neq 0.$$

DENSITY OF THE FIBONACCI NUMBERS

B-97 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let $A = \{a_k\}_{k=1}^{\infty}$ be an increasing sequence of numbers and let $A(n)$ denote the number of terms of A not greater than n. The Schnirelmann density of A is defined as the greatest lower bound of the ratios $A(n)/n$ for $n=1, 2, \ldots$. Show that the Fibonacci sequence has density zero.

Solution by the proposer.

Let $a = (1 + \sqrt{5})/2$, and $F = \{F_n\}_{n=2}^{\infty}$ be the Fibonacci sequence. It is easy to show by indiction that $a^{n-2} < F_n$ for $n > 0$, so that $F(n) < \log_a(n+2)$. Then since $0 \leq F(n)/n$,

$$0 \leq \lim_{n\to\infty} \frac{F(n)}{n} \leq \lim_{n\to\infty} \frac{\log_a(n+2)}{n} = 0,$$

so that the density of F is 0.

Also solved by C.B.A. Peck.

A COMPACT FINITE GENERATING FUNCTION

B-98 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let F_n be the n^{th} Fibonacci number and find a compact expression for the sum

$$S_n(x) = F_1x^2 + F_3x^3 + \ldots + F_nx^n.$$

Solution by Gloria C. Padilla, University of New Mexico, Albuquerque, N.M.

One easily sees that

$$(x^2 + x - 1)\, S_n(x) = -x + (F_{n-1} + F_n)\, x^{n+1} + F_nx^{n+2}.$$

Hence

$$S_n(x) = (-x + F_{n+1}x^{n+1} + F_nx^{n+2})/(x^2 + x - 1).$$

Also solved by L. Carlitz, Dewey C. Duncan, F. D. Parker, M. N. S. Swamy, Howard L. Walton, David Zeitlin (who pointed out that the result is a special case of formula (5) of his paper "On summation formula for Fibonacci and Lucas numbers" this Quarterly, Vol. 2, No. 2, 1964, p. 105), and the proposer.

COMPACT INFINITE SUM

B-99 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

$$T(x) = S_1(x) + \frac{S_2(x)}{2\,!} + \frac{S_2(x)}{3\,!} + \ldots,$$

where $S_n(x)$ is as defined in B-98.

Solution by David Zeitlin, Minneapolis, Minnesota.

From B-98, we obtain

$$(1 - x - x^2)\,T(x) = -x^2 \sum_{n=0}^{\infty} \frac{F_n x^n}{n!}$$

$$- \sum_{n=0}^{\infty} \frac{(n+1)\,F_{n+1} x^{n+1}}{(n+1)\,!} + ex.$$

Let a and b be the roots of

$$x^2 - x - 1 = 0\,.$$

Then

$$\frac{e^{ax} - e^{bx}}{a-b} = \sum_{n=0}^{\infty} \frac{F_n x^n}{n\,!}\,,$$

and

$$\frac{x(ae^{ax} - be^{bx})}{a-b} = \sum_{n=0}^{\infty} \frac{(n+1)\,F_{n+1} x^{n+1}}{(n+1)\,!}\,,$$

Thus,

$$(1 - x - x^2)\,T(x) = -x^2\left(\frac{e^{ax} - e^{bx}}{a-b}\right) - x\left(\frac{ae^{ax} - be^{bx}}{a-b}\right) + ex.$$

Also solved by L. Carlitz, Dewey C. Duncan, M.N.S. Swamy, and the proposer.

* * * * *