1967 AS THE SUM OF SQUARES

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With the coming of a new year, there is always a tendency to find out whether the number that indicates it has some special mathematical properties. 1967 is, of course, no square. But there is a theorem that states that any integer can be represented as the sum of at most four squares. Let us investigate the minimum number of squares that will add up to 1967.

First, we note that the square of every even number is divisible by 4 and the square of every odd number on being divided by 4 gives a remainder of one. If 1967 is to be the sum of the squares of two numbers, one must be odd and one even, otherwise there could not be an odd sum. But the sum of two such squares on being divided by 4 would give a remainder of one, while 1967 on being divided by 4 gives a remainder of 3. Thus 1967 cannot be the sum of two squares.

For three squares, 1967 would have to be the sum of the squares of three odd numbers since the remainder on division by 4 is 3. Now the squares of the odd numbers ending in 1, 3, 5, 7, 9 end respectively in 1, 9, 5, 9, and 1. Taking these endings three at a time, it can be easily shown that only the combinations 1, 5, 1 and 9, 9, 9 give a last digit of 7. So one systematic way to proceed is to consider the various cases corresponding to numbers ending first of all in 5, namely: 5, 15, 25, 35. Subtracting out 25, 225, 625, and 1225 gives remainders of 1942, 1742, 1342, 742. We can proceed by table as follows:

	742	1342	1742	1942
1	741	1341	1741	1941
9	661	1261	1661	1861
11	621	1221	1621	1821
19	381	981	1381	1581
21	301	901	1301	1501
29		501	901	1101
31		381	781	981

We can stop at 31 since this brings us to the halfway point with the largest number 1942. Since no squares appear in the table, this disposes of the possibility 1, 5, 1 as endings. Similar considerations apply for 9, 9, 9. Thus, the only possibility is four squares. One such representation is:

$$1967 = 6^2 + 9^2 + 25^2 + 35^2 \quad .$$

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ANOTHER GENERALIZED FIBONACCI SEQUENCE

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1. INTRODUCTION

Recent issues of numerous periodicals have given indication of a renewed interest in the well-known Fibonacci sequence, namely

$$(1) \qquad 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \cdots,\ C_n,\ \cdots,$$

where

$$C_n = C_{n-1} + C_{n-2},\quad n \geq 3,\quad \text{with } C_1 = C_2 = 1 .$$

Some recent generalizations have produced a variety of new and extended results.

A search of the literature seems to reveal that efforts to generalize the Fibonacci sequence have consisted of either (a) changing the recurrence relation while preserving the initial terms, or (b) altering the initial terms but maintaining the recurrence relation. A combination of these two techniques will be employed here.

Heretofore, all generalizations of the Fibonacci sequence appear to have restricted any given term to being a function (usually sum) of the two preceding terms. In this paper we shall extend this by considering sequences in which any given term is the sum of the <u>three</u> preceding it.

Since the set of all algebraic integers, i.e., all y such that y satisfies some monic polynomial equation,

$$p(x) \equiv x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0 ,$$

with integral coefficients and of degree greater than zero, is an integral domain under the operations of addition and multiplication, it was considered worthwhile to examine sequences in which the initial terms (hence all succeeding terms) are algebraic integers. It will be shown that certain special cases of such sequences are especially useful in the examination of the more general case.

2. THE GENERALIZED SEQUENCE $\{P_n\}$

Specifically we consider the sequence

$$\{P_n\} \equiv P_0,\ P_1,\ P_2,\ \cdots,\ P_n,\ \cdots \quad , \tag{2}$$

where P_0, P_1, P_2 are given, arbitrary algebraic integers, not all zero, and

$$P_n = P_{n-1} + P_{n-2} + P_{n-3},\quad n \geq 3\,. \tag{3}$$

It will also be convenient to consider a companion sequence, so to speak,

$$\{R_n\} \equiv R_0,\ R_1,\ R_2,\ \cdots,\ R_n,\ \cdots \quad , \tag{4}$$

where

$$R_0 = P_1 - P_0,\quad R_1 = P_2 - P_1, \qquad \text{and for } n \geq 2,$$

$$R_n = P_{n-1} + P_{n-2} \tag{5}$$

From (5) and (3), when $n \geq 5$, we have

$$\begin{aligned} R_n = P_{n-1} + P_{n-2} &= (P_{n-2} + P_{n-3}) + (P_{n-4} + P_{n-3}) + (P_{n-4} + P_{n-5}) \\ &= R_{n-1} + R_{n-2} + R_{n-3} \quad . \end{aligned}$$

Using (5) and (3) further, we have

$$\begin{aligned} R_4 &= R_3 + R_2 + R_1 \ , \\ R_3 &= R_2 + R_1 + R_0 \ . \end{aligned}$$

Hence for $n \geq 3$,

$$R_n = R_{n-1} + R_{n-2} + R_{n-3}\ . \tag{6}$$

Thus $\{R_n\}$ is actually the special case of (2) in which $R_0 = P_1 - P_0$, $R_1 = P_2 - P_1$, $R_2 = P_1 + P_0$. The usefulness of the sequence $\{R_n\}$ will be evident in the development of $\{P_n\}$ that follows.

Two other special cases of (2) should be mentioned at this time; namely the cases in which $P_0 = 0$, $P_1 = P_2 = 1$ and $P_0 = 1$, $P_1 = 0$, $P_2 = 1$ respectively, to give the sequences

$$0,\ 1,\ 1,\ 2,\ 4,\ 7,\ 13,\ 24,\ 44,\ \cdots,\ K_n,\ \cdots\ , \tag{7}$$

and

$$1,\ 0,\ 1,\ 2,\ 3,\ 6,\ 11,\ 20,\ 37,\ \cdots,\ L_n,\ \cdots\ . \tag{8}$$

We see immediately that $L_0 = K_1 - K_0$, $L_1 = K_2 - K_1$, and for $n \geq 2$,

$$L_n = K_{n-1} + K_{n-2}\ . \tag{9}$$

Hence we might call $\{K_n\}$ a P_n-type sequence and $\{L_n\}$ an R_n-type sequence.

The sequence $\{K_n\}$ was defined and discussed briefly by M. Agronomoff [1]. The following three relations involving various terms of this sequence were discovered and proved by him:

$$K_{n+p} = K_{p+1}K_n + (K_{p-1} + K_p)K_{n-1} + K_pK_{n-2}\ , \tag{10}$$

$$K_{2n} = K_{n-1}^2 + K_n(K_{n+1} + K_{n-1} + K_{n-2})\ , \tag{11}$$

$$K_{2n-1} = K_n^2 + K_{n-1}^2 + 2K_{n-1}K_{n-2}\ . \tag{12}$$

There is only one basic identity here because the latter two are evidently special cases of the first one upon setting $p = n$ and $p = n - 1$ respectively.

Further, it was conjectured in [1] that even though the sequence (7) was a Fibonacci-type sequence, it quite possibly would possess few of the interesting properties which the Fibonacci sequence has, and even if it should, such properties would be much more difficult to find due to the more complex nature of the recurrence relation determining the sequence.

We turn now to an investigation of the sequence (2) and consider, among other facts, how (10), (11), and (12) occur as special cases of more general relations.

Paralleling the usual treatment of the Fibonacci sequence, we obtain a closed expression for P_n since $\{P_n\}$ satisfies a difference equation. Thus

$$P_n = B_1x_1^n + B_2x_2^n + B_3x_3^n \tag{13}$$

where x_1, x_2, x_3 are the three distinct roots of the equation

$$x^3 - x^2 - x - 1 = 0,$$

and B_1, B_2, B_3 are constants depending on these roots as well as P_0, P_1, P_2, and are determined by the system

$$\begin{cases} B_1 + B_2 + B_3 = P_0 \\ B_1x_1 + B_2x_2 + B_3x_3 = P_1 \\ B_1x_1^2 + B_2x_2^2 + B_3x_3^2 = P_2 \end{cases} .$$

The values of x_1, x_2, x_3, B_1, B_2, B_3 are such as to make (12) too cumbersome to be of any further practical use in the succeeding development and hence will not be written here.

A much more useful way of representing the recurrence relation for $\{P_n\}$ may be found as follows: In the notation of vectors and matrices, we have by (3),

$$\begin{bmatrix} P_3 \\ P_2 \\ P_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_2 \\ P_1 \\ P_0 \end{bmatrix} ,$$

$$\begin{bmatrix} P_4 \\ P_3 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_3 \\ P_2 \\ P_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^2 \begin{bmatrix} P_2 \\ P_1 \\ P_0 \end{bmatrix} ,$$

and by finite induction

$$\text{(14)} \qquad \begin{bmatrix} P_n \\ P_{n-1} \\ P_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} P_2 \\ P_1 \\ P_0 \end{bmatrix}$$

Further, a simple induction proof gives

$$\text{(15)} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n} = \begin{bmatrix} K_{n+1} & L_{n+1} & K_n \\ K_n & L_n & K_{n-1} \\ K_{n-1} & L_{n-1} & K_{n-2} \end{bmatrix};$$

so it might be said that $\{K_n\}$ and $\{L_n\}$ arise "naturally" in the investigation of $\{P_n\}$.

Using (14) and (15), we find for n, p positive integers that

$$\text{(16)} \qquad \begin{bmatrix} P_{n+p} \\ P_{n+p-1} \\ P_{n+p-2} \end{bmatrix} = \begin{bmatrix} K_{p+1} & L_{p+1} & K_p \\ K_p & L_p & K_{p-1} \\ K_{p-1} & L_{p-1} & K_{p-2} \end{bmatrix} \begin{bmatrix} P_n \\ P_{n-1} \\ P_{n-2} \end{bmatrix},$$

from which we immediately see that

$$\text{(17)} \qquad P_{n+p} = K_{p+1}P_n + L_{p+1}P_{n-1} + K_pP_{n-2} ,$$

$$\text{(18)} \qquad \begin{aligned} P_{2n} &= K_{n+1}P_n + L_{n+1}P_{n-1} + K_nP_{n-2} \\ &= K_{n+1}P_n + (K_n + K_{n-1})P_{n-1} + K_nP_{n-2} , \end{aligned}$$

$$\text{(19)} \qquad P_{2n-1} = K_nP_n + (K_{n-1} + K_{n-2})P_{n-1} + K_{n-1}P_{n-2} .$$

Now setting $P_0 = 0$, $P_1 = P_2 = 1$, we have (10), (11), (12) as special cases of (17), (18), and (19), respectively.

Since

$$\begin{bmatrix} P_{n+p} \\ P_{n+p-1} \\ P_{n+p-2} \end{bmatrix} = \begin{bmatrix} K_{p+r+1} & L_{p+r+1} & K_{p+r} \\ K_{p+r} & L_{p+r} & K_{p+r-1} \\ K_{p+r-1} & L_{p+r-1} & K_{p+r-2} \end{bmatrix} \begin{bmatrix} P_{n-r} \\ P_{n-r-1} \\ P_{n-r-2} \end{bmatrix},$$

we also have

(20) $$P_{n+p} = K_{p+r+1}P_{n-r} + L_{p+r+1}P_{n-r-1} + K_{p+r}P_{n-r-2}$$

for n, p, r positive integers, $r \le n - 2$.

Similarly for n, h, k positive integers, we can show that

(21) $$P_{n+h+k} = K_{h+k+1}P_n + L_{h+k+1}P_{n-1} + K_{n+k}P_{n-1} .$$

Using (20) and (21), we have the following useful expression:

(22) $$\begin{bmatrix} P_{n+h+k} \\ P_{n+h} \\ P_n \end{bmatrix} = \begin{bmatrix} K_{h+k+1} & L_{h+k+1} & K_{h+k} \\ K_{h+1} & L_{h+1} & K_h \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} P_2 \\ P_1 \\ P_0 \end{bmatrix}$$

It can be shown quite easily that the sequence

(23) $$P_1,\ R_2,\ P_2,\ R_3,\ P_3,\ \cdots,\ P_n,\ R_{n+1},\ \cdots$$

is generated by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} ,$$

that is,

(24) $$\begin{bmatrix} P_n \\ R_n \\ P_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} P_2 \\ R_2 \\ P_1 \end{bmatrix} .$$

It is an interesting and useful fact that this matrix is the transpose of the generating matrix for $\{P_n\}$.

Using (24) in a way analogous to that in which we established (21), we prove that

$$(25) \qquad P_{n+h+k} = K_{h+k+1}P_n + K_{h+k}R_n + K_{h+k-1}P_{n-1} ,$$

$$(26) \qquad R_{n+h+k} = L_{h+k+1}P_n + L_{h+k}R_n + L_{h+k-1}P_{n-1} ,$$

two relations which are not only interesting in themselves but which also give

$$(27) \qquad \begin{bmatrix} P_{n+h+k} \\ R_{n+h} \\ P_n \end{bmatrix} = \begin{bmatrix} K_{h+k+1} & K_{h+k} & K_{h+k-1} \\ L_{h+1} & L_h & L_{h-1} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} P_2 \\ R_2 \\ P_1 \end{bmatrix} .$$

In order to define P_n for negative n, we use (14) for $n > 0$ written in the form

$$(28) \qquad \begin{bmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{n} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} .$$

Replacing n by $-n$ in (28), we have for $n > 0$,

$$(29) \qquad \begin{bmatrix} P_{-n} \\ P_{-n+1} \\ P_{-n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-n} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} ,$$

which together with (14) determines P_n for all n since P_0, P_1, P_2 are given. The same result is obtained upon replacing n by $-n$ in (3) to get

$$(30) \qquad P_{-n} = P_{-n+3} - P_{-n+2} - P_{-n+1}, \quad n > 0 .$$

R_n is also defined for negative n by (29) and (30) since

$$R_n = P_{n-1} + P_{n-2} .$$

This allows us to remove the restriction placed on n, p, r, h, k above.

3. LINEAR SUMS

A large number of what we shall call linear sum relations on terms of the sequences $\{R_n\}$ and $\{P_n\}$ were found and proved. Since an exhaustive list is not our aim, only a few of the more interesting ones are listed. No proofs will be given here since the proofs may all be made rather easily by finite induction.

$$\sum_{i=0}^{n} P_i = \frac{1}{2}(P_{n+2} + P_n + P_0 - P_2) , \tag{31}$$

$$\sum_{i=1}^{n} P_{3i} = \sum_{i=0}^{3n-1} P_i + P_0 , \tag{32}$$

$$\sum_{i=1}^{n} R_{3i} = P_{3n} - P_0 , \tag{33}$$

$$\sum_{i=1}^{n} R_{3i+1} = P_{3n+1} - P_1 . \tag{34}$$

These relations obviously have special cases for the sequences K_n and L_n. For example (33) becomes

$$\sum_{i=1}^{n} L_{3i} = K_{3n} . \tag{33'}$$

4. QUADRATIC AND CUBIC RELATIONS

An attempt to parallel the quadratic relations of the Fibonacci sequence failed. A different approach was necessary and this was found in the use of the vector-matrix representation of P_n. We have the following interesting quadratic form:

$$(35) \quad P_n^2 + P_{n-1}^2 + 2P_{n-1}P_{n-2} = P_2P_{2n-2} + R_2P_{2n-3} + P_1P_{2n-4} \quad .$$

The proof of (35) follows by considering the left side of the relation as the scalar product of the vectors $[P_n, \; R_n, \; P_{n-1}]$ and $[P_n, \; P_{n-1}, P_{n-2}]$ (recall $R_n = P_{n-1} + P_{n-2}$), and then using (14) and (24), we have

$$P_n^2 + P_{n-1}^2 + 2P_{n-1}P_{n-2} = \begin{bmatrix} P_n, & R_n, & P_{n-1} \end{bmatrix} \begin{bmatrix} P_n \\ P_{n-1} \\ P_{n-2} \end{bmatrix}$$

$$= \begin{bmatrix} P_2, & R_2, & P_1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{2n-4} \begin{bmatrix} P_2 \\ P_1 \\ P_0 \end{bmatrix} = \begin{bmatrix} P_2, & R_2, & P_1 \end{bmatrix} \begin{bmatrix} P_{2n-2} \\ P_{2n-3} \\ P_{2n-4} \end{bmatrix}$$

$$= P_2P_{2n-2} + R_2P_{2n-3} + P_1P_{2n-4} \quad .$$

For $P_0 = 0, \; P_1 = P_2 = 1$, (35) becomes

$$(35') \qquad K_n^2 + K_{n-1}^2 + 2K_{n-1}K_{n-2} = K_{2n-1} \quad ,$$

which is (12). It was shown that (12) is also a special case of (19), but (35) is not obtainable from (19) nor vice versa.

One of the most interesting relations involving terms of the Fibonacci sequence is the one

$$C_{n-1}C_{n+1} - C_n^2 = (-1)^n \quad .$$

There is a relation of this nature for the sequence $\{P_n\}$; however as may have been suspected, it has a cubic rather than a quadratic form. The desired relation is

$$(36)\quad P_n^2 P_{n-3} + P_{n-1}^3 + P_{n-2}^2 P_{n+1} - P_{n+1}P_{n-1}P_{n-3} - 2P_nP_{n-1}P_{n-2} = P_0^3 + 2P_1^3 + P_2^3 + 2P_0^2P_1 + 2P_0P_1^2 + P_0^2P_2 - 2P_1P_2^2 - 2P_0P_1P_2 - P_0P_2^2 \;.$$

Before proving (36), we note that for $P_0 = 0$, $P_1 = P_2 = 1$, (36) becomes

$$(37)\quad K_n^2K_{n-3} + K_{n-1}^3 + K_{n-2}^2K_{n+1} - K_{n+1}K_{n-1}K_{n-3} - 2K_nK_{n-1}K_{n-2} = 1 \;.$$

The proof of (37) follows from (9) and (15) by the use of determinants since

$$K_n^2K_{n-3} + K_{n-1}^3 + K_{n-2}^2K_{n+1} - K_{n+1}K_{n-1}K_{n-3} - 2K_nK_{n-1}K_{n-2} =$$

$$= \begin{vmatrix} K_{n+1} & K_{n-1} & K_n \\ K_n & K_{n-2} & K_{n-1} \\ K_{n-1} & K_{n-3} & K_{n-2} \end{vmatrix} = \begin{vmatrix} K_{n+1} & L_{n+1} & K_n \\ K_n & L_n & K_{n-1} \\ K_{n-1} & L_{n-1} & K_{n-2} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}^n = 1 \;.$$

Proof of (36): Even though (36) may be verified in very much the same manner as (37), we adopt a different method of proof since this is more easily used in a generalized version of (36). First, we state the following lemma whose proof the reader can readily supply.

Lemma: Let A be any 3×3 matrix and let $\bar{x}$ and $\bar{y}$ be three-dimensional vectors; then the cross product $(A\bar{x}) \times (A\bar{y})$ is equal to the cofactor matrix of A multiplied by $\bar{x} \times \bar{y}$; i.e.,

$$(A\bar{x}) \times (A\bar{y}) = (\text{cofactor } A)(\bar{x} \times \bar{y}).$$

Now the left side of (36) can be considered as the triple scalar product of the three vectors $\left[P_{n+1}, P_n, P_{n-1}\right]$, $\left[P_{n-1}, P_{n-2}, P_{n-3}\right]$, and $\left[P_n, P_{n-1}, P_{n-2}\right]$. By (14) and the lemma,

$$\begin{bmatrix} P_{n-1} \\ P_{n-2} \\ P_{n-3} \end{bmatrix} \times \begin{bmatrix} P_n \\ P_{n-1} \\ P_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} P_2 \\ P_1 \\ P_0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} P_3 \\ P_2 \\ P_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}^{n-3} \begin{bmatrix} P_1^2 - P_2P_0 \\ P_3P_0 - P_1P_2 \\ P_2^2 - P_3P_1 \end{bmatrix}$$

Therefore

$$P_n^2P_{n-3} + P_{n-1}^3 + P_{n-2}^2P_{n+1} - P_{n+1}P_{n-1}P_{n-3} - 2P_nP_{n-1}P_{n-2}$$

$$= \begin{bmatrix} P_{n+1} \\ P_n \\ P_{n-1} \end{bmatrix} \cdot \begin{bmatrix} P_{n-1} \\ P_{n-2} \\ P_{n-3} \end{bmatrix} \times \begin{bmatrix} P_n \\ P_{n-1} \\ P_{n-2} \end{bmatrix}$$

$$= \begin{bmatrix} P_4, & P_3, & P_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}^{n-3} \begin{bmatrix} P_1^2 - P_2P_0 \\ P_3P_0 - P_1P_2 \\ P_2^2 - P_3P_1 \end{bmatrix}$$

$$= P_4(P_1^2 - P_2P_0) + P_3(P_3P_0 - P_2P_1) + P_2(P_2^2 - P_3P_1) ,$$

which reduces to the right side of (36).

Example: Suppose we let $P_0 = 0$, $P_1 = 1$; then the right side of (36) becomes

$$P_2^3 - 2P_2 + 2 .$$

Setting this expression equal to zero and solving for P_2, we see that there exist algebraic integers, say $\bar{P}_0$, $\bar{P}_1$, $\bar{P}_2$, such that for the sequence $\{\bar{P}_n\}$,

$$\bar{P}_n^2\bar{P}_{n-3} + \bar{P}_{n-1}^3 + \bar{P}_{n-2}^2\bar{P}_{n+1} = \bar{P}_{n+1}\bar{P}_{n-1}\bar{P}_{n-3} + 2\bar{P}_n\bar{P}_{n-1}\bar{P}_{n-2} .$$

The lemma and (22) may be used as in the previous method of proof to show that for h, k, n, m, t integers

$$(38)\quad \begin{aligned} &P_{n+h}P_{n+m}P_{n+h+k+t} + P_nP_{n+h+t}P_{n+h+k+m} + P_{n+t}P_{n+h+k}P_{n+h+m} \\ &\quad - P_nP_{n+h+m}P_{n+h+k+t} - P_{n+m}P_{n+h+k}P_{n+h+t} - P_{n+h}P_{n+t}P_{n+h+k+m} \\ &= (K_hL_{h+k+1} - K_{h+k}L_{h+1})\Big[P_{t+2}(P_1P_m - P_0P_{m+1}) + P_{t+1}(P_0P_{m+2} - P_2P_m) \\ &\qquad + P_t(P_2P_{m+1} - P_1P_{m+2})\Big] \,. \end{aligned}$$

There are many interesting special cases of this relation. We mention a few. If $P_0 = 0$, $P_1 = P_2 = 1$, (38) becomes

$$(39)\quad \begin{aligned} &K_{n+h}K_{n+m}K_{n+h+k+t} + K_nK_{n+h+t}K_{n+h+k+m} + K_{n+t}K_{n+h+k}K_{n+h+m} \\ &\quad - K_nK_{n+h+m}K_{n+h+k+t} - K_{n+m}K_{n+h+k}K_{n+h+t} \\ &\quad - K_{n+h}K_{n+t}K_{n+h+k+m} \\ &= (K_hK_{h+k-1} - K_{h+k}K_{h-1})(K_{t-1}K_m - K_tK_{m-1}) \,. \end{aligned}$$

If $k = h = t$, $m = 1$, (39) becomes

$$(40)\quad \begin{aligned} &K_{n+1}K_{n+h}K_{n+3h} + K_nK_{n+2h}K_{n+2h+1} + K_{n+h}K_{n+h+1}K_{n+2h} - K_nK_{n+h+1}K_{n+3h} \\ &\quad - K_{n+1}K^2_{n+2h} - K^2_{n+h}K_{n+2h+1} = K_{h-1}K_hK_{2h-1} - K^2_{h-1}K_{2h} \,; \end{aligned}$$

and if $t = h$, $k = m - h$, (39) reduces to

$$(41)\quad \begin{aligned} &K_nK_{n+2h}K_{n+2m} + 2K_{n+h}K_{n+m}K_{n+h+m} - K_nK^2_{n+h+m} - K^2_{n+m}K_{n+2h} \\ &\quad - K^2_{n+h}K_{n+2m} = -(K_hK_{m-1} - K_mK_{n-1})^2 \end{aligned}$$

In order that the above results be valid, we must choose h and k so that

$$K_nK_{n+k-1} - K_{h+k}K_{h-1} = K_hL_{h+k+1} - K_{h+k}L_{h+1} \neq 0 \,,$$

for in the proof of (38), we assume that the matrix

$$\begin{bmatrix} K_{h+k+1} & L_{h+k+1} & K_{h+k} \\ K_{h+1} & L_{h+1} & K_h \\ 1 & 0 & 0 \end{bmatrix}$$

is non-singular.

Using (27) we can find relationships involving terms of both the sequences $\{R_n\}$ and $\{P_n\}$ which reduce to an expression independent of n. For example, it may be proved that

$$\text{(42)} \quad P_{n+h+k+t}(R_{n+h}P_{n+m} - P_nP_{n+h+m}) + R_{n+h+t}(P_nP_{n+h+k+m} - P_{n+m}P_{n+h+k})$$
$$+ P_{n+t}(P_{n+h+k}R_{n+h+m} - R_{n+h}P_{n+h+k+m}) = (K_{h+k-1}L_h - K_{n+k}L_{h-1}) \cdot$$
$$\left[P_{t+2}(P_1P_m - P_0P_{m+1}) + P_{t+1}(P_0P_{m+2} - P_2P_m) + P_t(P_2P_{m+1} - P_1P_{m+2})\right]$$

It should be noted that no terms of the sequence $\{R_n\}$ appear on the right side of (42) and also that the second factor on the right side of the equality sign in (42) is the same as the second factor on the right side of (38).

5. MISCELLANEOUS RESULTS

We conclude with some miscellaneous results. The following limiting relations may be established using (13) and the fact that r_1, r_2, the two complex roots of

$$x^3 - x^2 - x - 1 = 0 ,$$

are such that

$$|r_1| = |r_2| < 1 .$$

$$\text{(43)} \quad \lim_{n \to \infty} \frac{P_{n+1}}{P_n} = \frac{1 - \sqrt[3]{3\sqrt{33} - 19} + \sqrt[3]{19 + 3\sqrt{33}}}{3} ,$$

$$\text{(44)} \quad \lim_{n \to \infty} \frac{P_{n+h}}{P_n} = \left(\frac{1 - \sqrt[3]{3\sqrt{33} - 19} + \sqrt[3]{19 + 3\sqrt{33}}}{3}\right)^h$$

By induction the following theorem may be established:

Theorem: For every positive n,

$$K_{4n} \equiv K_{4n-1} \equiv 0 \pmod 2 ,$$

$$K_{4n-2} \equiv K_{4n-3} \equiv 1 \pmod 2 ,$$

$$K_{4n} \equiv 0 \pmod 4 .$$

If we let $D(P_0, P_1, P_2, \cdots, P_n)$ be the determinant

$$\begin{vmatrix} P_0 & P_1 & P_2 & \cdots & P_n \\ P_1 & P_2 & P_3 & \cdots & P_{n+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ P_n & P_{n+1} & P_{n+2} & \cdots & P_{2n} \end{vmatrix} ,$$

it can be shown that for $n \geq 3$,

$$D(P_0, P_1, P_2, \cdots, P_n) = 0 .$$

This material is taken from Some Generalizations and Extensions of the Fibonacci Sequence, a thesis submitted to the University of Pittsburgh by the first author in partial fulfillment of requirements for the Ph. D. degree.

REFERENCE

1. M. Agronomoff, "Une série récurrente," Mathesis, ser. 4, vol. 4 (1914), p. 126.

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RESTRICTED COMPOSITIONS

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As a continuation of [6] and [7], this paper deals with a restricted set of compositions of an integer (to be defined below) and presents extensions of some results of Gould [2], [3], [4], by interpreting the compositions through the corresponding lattice paths.

By the definition in [7], a $(k + 1)$-composition $(t_1, t_2, \cdots, t_{k+1})$ of an integer n (i. e.,

$$\sum_{i=1}^{k+1} t_i = n \quad \text{and} \quad t_i \geq 1$$

for every i) dominates another $(k + 1)$-composition $(t_1', t_2', \cdots, t_{k+1}')$ of n if and only if

$$\sum_{i=1}^{j} t_i \geq \sum_{i=1}^{j} t_i' \quad \text{for} \quad j = 1, 2, \cdots, k+1 \quad .$$

Using the 1:1 correspondence in [6], we associate with each $(k + 1)$-composition of n a minimal lattice path (onward and upward path through lattice points) from $(0, 0)$ to $(n - k - 1, k)$ such that the directed distance measured along the positive direction of x-axis, of the point $(n - k - 1, k - j)$, $j = 1, 2, \cdots,$ k from the path is

$$\sum_{i=1}^{j} t_i - j \quad .$$

Without any ambiguity, denote this path by

$$\left[t_1 - 1,\ t_1 + t_2 - 2, \cdots, \sum_{i=1}^{k} t_i - k\right].$$

Thus, it is evident that to the set $C(n; a_1, a_2, \cdots, a_k)$ of $(k+1)$-compositions of n, dominated by the $(k+1$-composition $(a_1, a_2, \cdots, a_{k+1})$ of n corresponds the set $L(A_1, A_2, \cdots, A_k)$ of lattice paths which do not cross to the left or above the path

$$\left[A_1, A_2, \cdots, A_k\right] = \left[a_1 - 1,\ a_1 + a_2 - 2, \cdots, \sum_{i=1}^{k} a_i - k\right] \quad .$$

Let the number in the set C (equivalently in L) be represented by $N(n; a_1, a_2, \cdots, a_k)$ for $k \geq 1$, and by $N(n)$ for $k = 0$. Trivially,

(1) $$N(n) = 1 \quad ,$$

(2) $$N(n; a, \underbrace{1, 1, \cdots, 1}_{k-1}) = \binom{a + k - 1}{k} \quad ,$$

and

(3) $$N(n; a_1, a_2, \cdots, a_k) = 0 \quad ,$$

if any a_i is either zero or negative.

Now consider the path

$$\left[A_1', A_2', \cdots, A_k'\right]$$

such that $A_i' \leq A_i$ for all i. Every path in L passes through one of the points $(n - k - A_{i+1}' - 2,\ k - i)$, $i = 0, 1, 2, \cdots, k$, $(A_{k+1}' = A_k')$ before moving to $(n - k - A_{i+1}' - 1,\ k - i)$ and then reaches $(n - k - 1, k)$ not crossing $[A_1', A_2', \cdots, A_k']$. Therefore,

$$(4)\qquad \dot{N}(n;a_1,a_2,\cdots,a_k) = N(n;a_1 - a_1', a_2,\cdots,a_k)N(n)$$
$$+ N(n;a_1 + a_2 - a_1' - a_2', a_3,\cdots,a_k)N(n;a_1')$$
$$+ N(n;a_1 + a_2 + a_3 - a_1' - a_2' - a_3', a_4,\cdots,a_k)N(n;a_1',a_2')$$
$$+\cdots + N\left(n;\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} a_i'\right)N(n;a_1',a_2',\cdots,a_{k-1}')$$
$$+ N(n)N(n;a_1',a_2',\cdots,a_k') \quad .$$

We note that whenever $A_i' = A_i$,

$$N(n;a_1 + \cdots + a_i - a_1' - \cdots - a_i', a_{i+1},\cdots,a_k) = 0 \quad .$$

It may be pointed out that relation (4) in some sense is a generalization of Vandermonde's convolution

$$\sum_{i=0}^{k} \binom{x}{i}\binom{y}{k-i} = \binom{x+y}{k} ,$$

a further discussion of which is given later.

By setting $a_1 = A_k' + 1$ and $a_2 = a_3 = \cdots = a_k = 1$ in (4) and using (2), we get the recursive formula

$$(5)\qquad N(n;a_1',a_2',\cdots,a_k') =$$
$$\binom{A_k' + k}{k} - \sum_{i=1}^{k-1}\binom{A_k' - A_i' + k - i}{k - i + 1} N(n;a_1',a_2',\cdots,a_{i-1}')$$

which is the same as (9) in [1] and (2) in [8]. The solution of (5) is stated in the following theorem.

Theorem 1:

(6) $N(n; a_1, a_2, \cdots, a_k)$

$$= \begin{vmatrix} \binom{A_k + k}{k} & \binom{A_k - A_{k-1} + 1}{2} & \binom{A_k - A_{k-2} + 2}{3} & \cdots & \binom{A_k - A_1 + k - 1}{k} \\ \binom{A_{k-2} + k - 2}{k - 2} & 1 & \binom{A_{k-2} - A_{k-2}}{1} & \cdots & \binom{A_{k-2} - A_1 + k - 3}{k - 2} \\ \binom{A_{k-3} + k - 3}{k - 3} & 0 & 1 & \cdots & \binom{A_{k-3} - A_1 + k - 4}{k - 3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{A_1 + 1}{1} & 0 & 0 & \cdots & \binom{A_1 - A_1}{1} \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

Another way of expressing the number in L leads to

$$N(n; a_1, a_2, \cdots, a_k) = \sum_{x_1=0}^{A_1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_k=x_{k-1}}^{A_k} 1 \tag{7}$$

$$= \sum_{x_1=0}^{\alpha-1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_k=x_{k-1}}^{A_k} 1$$

$$+ \sum_{x_1=\alpha}^{A_1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_k=x_{k-1}}^{A_k} 1, 0 \le \alpha \le A_1 + 1 .$$

Substituting $x_i - \alpha = x_i'$ for $i = 1, 2, \cdots, k$, the second term on the right hand side becomes

$$\sum_{x_1'=0}^{A_1-\alpha} \sum_{x_2'=x_1'}^{A_2-\alpha} \cdots \sum_{x_k'=x_{k-1}'}^{A_k-\alpha} 1 = N(n; a_1 - \alpha, a_2, \cdots, a_k) . \tag{8}$$

On the other hand, the first term can be written as

$$(9) \quad \sum_{x_1=0}^{\alpha-1}\sum_{x_2=x_1}^{A_2}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1 = \sum_{x_1=0}^{\alpha-1}\sum_{x_2=0}^{A_2}\sum_{x_3=x_1}^{A_3}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1 - \sum_{x_1=1}^{\alpha-1}\sum_{x_2=0}^{x_1-1}\sum_{x_3=x_1}^{A_3}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1 ,$$

whereas the last term in (9) can again be expressed as

$$-\sum_{x_1=1}^{\alpha-1}\sum_{x_2=0}^{x_1-1}\sum_{x_3=0}^{A_3}\sum_{x_4=x_3}^{A_4}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1 + \sum_{x_1=2}^{\alpha-1}\sum_{x_2=1}^{x_1-1}\sum_{x_1=0}^{x_2-1}\sum_{x_4=x_3}^{A_4}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1.$$

When we proceed in the above manner, the final expression for (9) is

$$(10) \quad \sum_{i=1}^{k} (-1)^{i+1}\binom{\alpha}{i} N(n;A_{i+1} + 1,\ a_{i+2},\cdots,a_k) ,$$

by noting that

$$\sum_{x_1=i-1}^{\alpha-1}\sum_{x_2=i-2}^{x_1-1}\cdots\sum_{x_i=0}^{x_{i-1}-1} 1 = \sum_{x_1=0}^{\alpha-i}\sum_{x_2=0}^{x_1}\cdots\sum_{x_i=0}^{x_{i-1}} 1 = \binom{\alpha}{i}$$

and

$$\sum_{x_{i+1}=0}^{A_{i+1}}\sum_{x_{i+2}=x_{i+1}}^{A_{i+2}}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1 = N(n;A_{i+1} + 1,\ a_{i+2},\cdots,a_k).$$

Thus it follows from (7), (8) and (10) that

$$(11)\qquad \sum_{i=0}^{k} (-1)^i \binom{\alpha}{i} N(n; A_{i+1} + 1, a_{i+2}, \cdots, a_k) = N(n; a_1 - \alpha,\ a_2, \cdots, a_k)\ .$$

An alternative way of simplifying the first term on the right of (9) is

$$\sum_{x_1=0}^{\alpha-1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_k=0}^{A_k} 1 - \sum_{x_1=0}^{\alpha-1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_k=0}^{x_{k-1}-1} 1\ ,$$

where the sums in the last term for which $x_{k-1} - 1$ is negative are zero. Repetition of this process yields

$$(12)\qquad \sum_{i=1}^{k} (-1)^{i+1} \binom{A_{k+1-i} + 1}{i} N(n; a_1, a_2, \cdots, a_{k-i}) = N(n; a_1, a_2, \cdots, a_k)$$

for $c = a_1$. Relation (12) has been obtained earlier in [7], which is equivalent to (3) in [1].

When $c = a_1$, the solution of either (11) or (12) is stated as Theorem 2, for which a direct elementary proof is provided below.

Theorem 2:

$$(13)\qquad N(n; a_1, a_2, \cdots, a_k)$$

$$= \begin{vmatrix} \binom{A_k + 1}{1} & \binom{A_{k-1} + 1}{2} & \cdots & \binom{A_1 + 1}{k} \\ 1 & \binom{A_{k-1} + 1}{1} & \cdots & \binom{A_1 + 1}{k - 1} \\ 0 & 1 & \cdots & \binom{A_1 + 1}{k - 2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 \end{vmatrix}$$

Proof: Obviously

$$\begin{vmatrix} \binom{x_k}{0} & \binom{x_{k-1}}{1} & \binom{x_{k-2}}{2} & \cdots & \binom{x_1}{k} \\ 0 & 1 & \binom{x_{k-1}}{1} & \cdots & \binom{x_1}{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{x_1}{0} \end{vmatrix} = 1 \quad .$$

Using this in (7), we see that

$$N(n; a_1, a_2, \cdots, a_k) = \sum_{x_1=0}^{A_1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_{k-1}=x_{k-2}}^{A_{k-1}} \begin{vmatrix} \binom{A_k+1}{1} - \binom{x_{k-1}}{1} & \binom{x_{k-1}}{1} & \cdots & \binom{x_1}{k} \\ \binom{A_k+1}{0} - \binom{x_{k-1}}{0} & 1 & \cdots & \binom{x_1}{k-1} \\ 0 & 0 & \cdots & \binom{x_1}{k-2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{x_1}{0} \end{vmatrix}$$

$$= \sum_{x_1=0}^{A_1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_{k-1}=x_{k-2}}^{A_{k-1}} \begin{vmatrix} \binom{A_k+1}{1} & \binom{x_{k-1}}{1} & \cdots & \binom{x_1}{k} \\ 1 & 1 & \cdots & \binom{x_1}{k-1} \\ 0 & 0 & \cdots & \binom{x_1}{k-2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{x_1}{0} \end{vmatrix}$$

The proof is complete when the summation is continued to the end.

Theorems 1 and 2 give rise to an interesting combinational identity on determinants, the direct proof of which is not obvious.

We check either from the theorems or directly that

$$(14)\qquad N(n;a_1, a_2, \cdots, a_k) + N(n;a_1 + a_2, a_3, \cdots, a_k) = N(n;a_1 + 1, a_2, \cdots, a_k) \ ,$$

$$(15)\qquad N(n;1, a_2, \cdots, a_k) = N(n;a_2, a_3, \cdots, a_k) \ ,$$

and

$$(16)\qquad N\left(\sum_{i=1}^{k} a_i + j;\ a_1, a_2, \cdots, a_k\right) = N\left(\sum_{i=1}^{k} a_i + 1;\ a_1, a_2, \cdots, a_k\right) \quad j = 1, 2, \cdots$$

A few important special cases are considered below.

Corollary 1.

$$(17)\qquad N(n;a, \underbrace{b, \cdots, b}_{k-1}) = \frac{a}{a + kb}\binom{a + kb}{k} \ .$$

This is directly verifiable from either one of the theorems. (Also see Theorem 1 in [6]).

In the next, we evaluate

$$N_{p,q}(a, b; c, d) = N(n;a, \underbrace{b, \cdots, b}_{p-1}, c, \underbrace{d, \cdots, d}_{q-1})$$

which has been obtained by a different method as Theorem 3 in [6].

Corollary 2.

$$(18)\qquad N_{p,q}(a, b;c, d) = \sum_{i=0}^{q} (-1)^i \frac{a}{a + (p + q - i)b}\binom{a + (p + q - i)b}{p + q - i}$$

$$\times \frac{(q - i + 1)b - c - (q - i)d}{(q - i + 1)b - c - qd + i}\binom{(q - i + 1)b - c - qd + i}{i} \ .$$

Proof: For $c + q(d - 1) \geq qb$, the result is immediate, by taking $A_i' = (a - 1) + (i - 1)(b - 1)$, $i = 1, 2, \cdots, p + q$ in (4) and applying Corollary 1. When $c + q(d - 1) < qb$, let $s (p < s \leq p + q)$ be the largest integer so that $c + s(d - 1) \geq sb$. $N_{p,q}(a, b; c, d)$ and $N(n; a, \underbrace{b, \cdots, b}_{p+q-1})$, expressed with the help of (4), where

$$A_i' = \begin{cases} (a - 1) + (i - 1)(b - 1) & i = 1, 2, \cdots, s, \\ (a - 1) + (p - 1)(b - 1) + (c - 1) + (i - p - 1)(d - 1) & i = s + 1, \\ & \quad s + 2, \cdots, p + q, \end{cases}$$

lead to (18), after some simplification.

For completeness, we present two more special cases which are known and can easily be derived.

Corollary 3:

$$(19) \quad N_{p,q}(a, b; c, 1) = \binom{a + c - 2 + (p - 1)(b - 1) + p + q}{p + q} - \sum_{i=q+1}^{p+q} \frac{a}{a + (p + q - i)b} \binom{a + (p+q-i)b}{p+q-i} \binom{c + q - (q - i + 1)b - 1}{i} .$$

Corollary 4:

$$(20) \qquad N_{p,q}(a, 2; 2, 1) = \binom{a + 2p + q - 1}{p + q} - \binom{a + 2p + q - 1}{p - 1}$$

In his paper [2], Gould has defined

$$A_k(\beta, \gamma) = \frac{\beta}{\beta + \gamma k} \binom{\beta + \gamma k}{k}$$

and has shown that $A_k(\beta, \gamma)$ satisfies the relation

$$(21) \qquad \sum_{i=0}^{k} A_i(\beta, \gamma) A_{k-i}(\delta, \gamma) = A_k(\beta + \delta, \gamma) .$$

Suppose that β, γ and δ are non-negative integers. Then (21) immediately follows from (4) and (17) by putting $a_1 = \beta + \delta$, $a_1 = a_3 = \cdots = a_k = \gamma$, $a'_1 = \beta$, and $a'_2 = a'_3 = \cdots = a'_k = \gamma$ in (4). Relation (11) in [2] can similarly be verified. Also, the convolution (5.5) in [3] for $t = 0$ can be compared with (11) and their equivalence is easily established.

In what follows, the results on restricted compositions are analogous to those on unrestricted compositions in Gould's paper [4] (Theorems 1 and 5 or equivalently Theorem 6). Fix $a_2, a_3, \cdots, a_k$ and let

$$m = \sum_{i=2}^{k} a_i \ .$$

From (14), (15) and (16) we infer that

$$\text{(22)} \quad N(m + a_1 + 1;\ a_1, a_2, \cdots, a_k) = \sum_{i=1}^{a_1} N(m + i;\ a_2 - 1 + i, a_3, \cdots, a_k)$$

$$= \sum_{i=1}^{a_1} \left[\frac{m + a_1}{m + i}\right] N\{m + i;\ a_2 - 1 + i, a_3, \cdots, a_k\}$$

where $[z]$ is the greatest integer less than or equal to z and $N\{m + i; a_2 - 1 + i,\ a_3, \cdots, a_k\}$ is the number of compositions in the set $S(m + i; a_2 - 1 + i, a_3, \cdots, a_k)$ which is defined as follows: For i negative or equal to zero,

$$S(m + i;\ a_2 - 1 + i, a_3, \cdots, a_k) \text{ is empty;}$$
$$S(m + 1;\ a_2, a_3, \cdots, a_k) = C(m + 1;\ a_2, a_3, \cdots, a_k) \ ;$$

For $i \geq 2$, $S(m + i; a_2 - 1 + i, a_3, \cdots, a_k)$ is the subset of $C(m + i; a_2 - 1 + i, a_3, \cdots, a_k)$ with the property that if $(x_1, x_2, \cdots, x_k) \in S(m + u; a_2 - 1 + u, a_3, \cdots, a_k)$, $u = 1, 2, \cdots, i - 1$, then for r a positive integer $(rx_1, rx_2, \cdots, rx_k) \notin S(m + i; a_2 - 1 + i, a_3, \cdots, a_k)$. Expression (22) corresponds to Theorem 1 in [4].

$$\sum_{j=1}^{\infty} N(m + j + 1;\ j, a_2, \cdots, a_k)\, x^{m+j} = \sum_{j=1}^{\infty} x^{m+j}$$

$$\times \sum_{i=1}^{m} \left[\frac{m + j}{m + i}\right] N\{m + i;\ a_2 - 1 + i,\ a_3, \cdots, a_k\}$$

$$= \sum_{i=1}^{\infty} N\{m + i;\ a_2 - 1 + i, a_3, \cdots, a_k\} \sum_{j=i}^{\infty} \left[\frac{m + j}{m + i}\right] x^{m+j}$$

$$= \sum_{i=1}^{\infty} N\{m + i;\ a_2 - 1 + i, a_3, \cdots, a_k\} \frac{x^{m+i}}{(1 - x)(1 - x^{m+i})}$$

by (3) in [4]. Therefore,

$$(23)\quad \sum_{i=1}^{\infty} N\{m + i;\ a_2 - 1 + i, a_3, \cdots, a_k\} \frac{x^{m+i}}{(1 - x^{m+i})}$$

$$= \sum_{i=1}^{\infty} N(m + i + 1; i, a_2, \cdots, a_k) x^{m+i}(1-x)$$

$$= \sum_{i=1}^{\infty} N(m + i;\ a_2 - 1 + i,\ a_3, \cdots, a_k) x^{m+i}$$

by (14), (15) and (16). But (23) can be written as

$$(24)\quad \sum_{i=m+1}^{\infty} N\{i;\ a_2 - m - 1 + i, a_3, \cdots, a_k\} \frac{x^i}{1 - x^i}$$

$$= \sum_{i=m+1}^{\infty} N(i; a_2 - m - 1 + i, a_3, \cdots, a_k) x^i \quad .$$

In order to extend the summation to $i = 1, 2, \cdots, m$ in (24), define

$$N^{\star}(i; a_2 - m - 1 + i, a_3, \cdots, a_k) = \begin{cases} 0 \quad \text{for} \quad i = 1, 2, \cdots, m \\ N(i; a_2 - m - 1 + i, a_3, \cdots, a_k) \ \text{for}\ i = m+1, m+2 \cdots \end{cases}$$

Thus, following the procedure in [4],

$$(25)\quad N\{n; n + a_2 - m - 1, a_3, \cdots, a_k\} = \sum_{i|n} N^{\star}(n; n + a_2 - m - 1, a_3, \cdots, a_k)\, \mu\left(\frac{n}{i}\right) ,$$

which is similar to that of Theorem 5 in [4].

We finally remark that such results can also be obtained for the number of lattice paths in the set $L_i(A_1, A_2, \cdots, A_k)$ defined as follows:

$L_0(A_1, A_2, \cdots, A_k) = L(A_1, A_2, \cdots, A_k)$; $L_i(A_1, A_2, \cdots, A_k)$ is the subset of $L(A_1 + i, A_2 + i, \cdots, A_k + i)$ such that if $[x_1, x_2, \cdots, x_k] \in L_u(A_1, A_2, \cdots, A_k)$, $u = 0, 1, \cdots, i - 1$, then $[rx_1, rx_2, \cdots, rx_k] \notin L_i(A_1, A_2, \cdots, A_k)$.

REFERENCES

1. F. Gobel, "Some Remarks on Ballot Problems," Report S321a, Math Centrum, Amsterdam, 1964.
2. H. W. Gould, "Some Generalizations of Vandermonde's Convolutions," Amer. Math. Monthly, Vol. 63, 1956, pp. 84-91.
3. H. W. Gould, "A New Convolution Formula and Some New Orthogonal Relations for Inversion of Series," Duke Math. Journal, Vol. 29, 1962,
4. H. W. Gould, "Binomial Coefficients, The Bracket Function, and Compositions with Relative Prime Summands," Fibonacci Quarterly, Vol. 2, 1964, pp. 241-260.
5. S. G. Mohanty, "Some Properties of Compositions and Their Application to the Ballot Problem," Can. Math. Bull. Vol. 8, 1965, pp. 359-372.
6. S. G. Mohanty and T. V. Narayana, "Some Properties of Compositions and Their Application to Probability and Statistics I," Biometrische Zeitschrift, Vol. 3, 1961, pp. 252-258.
7. T. V. Narayana and G. E. Fulton, "A Note on the Compositions of an Integer," Can. Math. Bull., Vol. 1, 1958, pp. 169-173.
8. P. Switzer, "Significance Probability Bounds for Rank Orderings," Ann. Math. Statist., Vol. 35, 1964, pp. 891-894.

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ENUMERATION OF CERTAIN TRIANGULAR ARRAYS

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1. INTRODUCTION

Let k be a positive integer. We define the numbers $F_n(k)$ and $N_n(k)$ by means of the recurrences

$$F_n(k) = F_{n-1}(k) + F_{n-k}(k) \quad (n \geq k) , \tag{1.1}$$

$$N_{n-k}(k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} N_{n-j}(k) \quad (n \geq k) , \tag{1.2}$$

with the initial conditions

$$F_n(k) = n + 1 \quad (0 \leq n < k) , \tag{1.3}$$

$$N_n(k) = \binom{k+n}{n} \quad (0 \leq n < k) , \tag{1.4}$$

Note that

$$F_n(1) = N_n(1) = 2^n , \tag{1.5}$$

$$F_n(2) = F_{n+2} , \tag{1.6}$$

$$N_n(2) = 3.2^{n-1} , \tag{1.7}$$

where F_j denotes the usual Fibonacci number $(F_0 = 0, \; F_1 = 1)$.

Given positive integers m and k, put $m = pk + r$ $(1 \leq r \leq k)$ and let $T(k, m)$ denote the number of arrays

$$\begin{array}{l} n_{11} \cdots n_{1k} n_{1,k+1} \cdots n_{1,pk} n_{1,pk+1} \cdots n_{1m} \\ \qquad\qquad n_{2,k+1} \cdots n_{2,pk} n_{2,pk+1} \cdots n_{2m} \\ \qquad\qquad\qquad\qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ \qquad\qquad\qquad\qquad n_{p,pk} n_{p,pk+1} \cdots n_{pm} \\ \qquad\qquad\qquad\qquad\qquad n_{p+1,pk+1} \cdots n_{p+1,m} \end{array} , \tag{1.8}$$

where n_{11} is either 0 or 1 and

(1.9) $$n_{ij} \geq n_{i,j+1} \geq 0; \quad n_{ij} \geq n_{i+1,j} \geq 0 \quad ,$$

For example, $T(2,5)$ and $T(2,6)$ are the number of arrays

$$\begin{array}{ccccc} x & x & x & x & x \\ & & x & x & x \\ & & & & x \end{array} \;, \qquad \begin{array}{cccccc} x & x & x & x & x & x \\ & & x & x & x & x \\ & & & & x & x \end{array} \;,$$

where each x is either 0 or 1 subject only to the conditions (1.9). As a further example, we have $T(2,3) = 5$, the arrays being

$$\begin{array}{ccc} 1 & 1 & 1 \\ & & 1 \end{array}\;, \quad \begin{array}{ccc} 1 & 1 & 1 \\ & & 0 \end{array}\;, \quad \begin{array}{ccc} 1 & 1 & 0 \\ & & 0 \end{array}\;, \quad \begin{array}{ccc} 1 & 0 & 0 \\ & & 0 \end{array}\;, \quad \begin{array}{ccc} 0 & 0 & 0 \\ & & 0 \end{array}\;.$$

Indeed, we find that

(1.10) $$T(k,n) = F_m(k) \quad (m, k = 1, 2, 3, \cdots) \quad .$$

The numbers $N_n(k)$ also occur in connection with triangular arrays of zeros and ones. We prove that

(1.11) $$N_n(k) = \frac{1}{k} \sum_{j=0}^{k-1} \left[(\rho^{-j} + 1)^k - 1 \right] (\rho^j + 1)^n \quad ,$$

(1.12) $$\sum_{n=0}^{\infty} N_n(k) x^n = \frac{x^k - 1}{x^k - (1 - x)^k} \quad ,$$

where ρ denotes a primitive k^{th} root of unity.

Finally, we have included some one-line arrays which can be enumerated in terms of the numbers $F_n(k)$ and $N_n(k)$.

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2. THE NUMBERS $F_n(k)$

For given positive integers m and k, let $T(k,m)$ denote the number of arrays (1.8) subject to the conditions (1.9).

To evaluate $T(k,m)$, we first note that if $n_{1m} = 1$ in (1.8), then $n_{11} = \cdots = n_{1,m-1} = 1$ and there are $T(k,m-k)$ arrangements of the resulting matrix. On the other hand, if $n_{1m} = 0$, then $n_{2m} = \cdots = n_{p+1,m} = 0$ and there are $T(k,m-1)$ arrays possible. This evidently yields

$$T(k,m) = T(k,m-1) + T(k,m-k) \quad (m > k) \; . \tag{2.1}$$

In the next place, it follows at once from (1.8) and (1.9) that

$$T(k,m) = m + 1 \quad (1 \leq m \leq k) \; . \tag{2.2}$$

This evidently completes the proof of

Theorem 1. The number of arrays (1.8) subject to the conditions (1.9) is given by

$$T(k,m) = F_m(k) \quad (m,k = 1,2,3,\cdots) \; . \tag{2.3}$$

As an immediate corollary of (2.3) we have

Theorem 2. Let $q_k(n;p)$ denote the number of partitions of n into at most p parts, successive parts differing by at least k. Then

$$\sum_{n=0}^{M} q_k(n;p+1) = F_m(k) \; , \tag{2.4}$$

where $m = kp + r$ $(1 \leq r \leq k)$ and $M = m(p+1) - k\binom{p+1}{2}$.

Indeed, using the generating function [2]

$$\sum_{n=0}^{\frac{1}{2}m(m+1)} q_1(n;m)x^n = \prod_{j=1}^{m} (1 + x^j) \; ,$$

we easily verify that, for $k = 1$, (2.4) reduces to (1.5). However, Chaundy [1] has noted that, for $k > 1$, the generating function for $q_k(n;p)$ is not known.

3. THE NUMBERS $N_n(k)$

Given positive integers m and k, put $m = pk + r (1 \leqq r \leqq k)$ and, for $1 \leqq j \leqq k$, let $N_j(m,k)$ denote the number of arrays

$$
\begin{array}{cccccccc}
n_{11} \cdots & n_{1,k+1} \cdots & n_{1,pk+1} \cdots & n_{1m} & & & \\
\vdots & \vdots & \vdots & \vdots & & & \\
n_{j1} \cdots & n_{j,k+1} \cdots & n_{j,pk+1} \cdots & n_{jm} & & & \\
 & \vdots & \vdots & \vdots & & & \\
 & n_{j+k,k+1} \cdots & n_{j+k,pk+1} \cdots & n_{j+k,m} & & & \\
 & & \vdots & \vdots & & & \\
 & & n_{j+pk,pk+1} \cdots & n_{j+pk,m} & & &
\end{array} \tag{3.1}
$$

where n_{11} is either 0 or 1 and the n_{ij} are non-negative integers subject to the conditions (1.9). For example, $N_1(5,2)$ and $N_2(5,2)$ are the number of arrays

$$
\begin{array}{ccccc}
x & x & x & x & x \\
 & & x & x & x \\
 & & x & x & x \\
 & & & & x \\
 & & & & x
\end{array}, \qquad
\begin{array}{ccccc}
x & x & x & x & x \\
x & x & x & x & x \\
 & & x & x & x \\
 & & x & x & x \\
 & & & & x
\end{array},
$$

respectively.

It follows from (3.1) and (1.9) that

$$N_j(m,k) = \binom{m+j}{j} \qquad (1 \leqq m \leqq k), \tag{3.2}$$

$$N_j(m,k) = N_j(m-1,k) + N_{j-1}(m,k) \quad (2 \leqq j \leqq k;\ m > k), \tag{3.3}$$

$$N_1(m,k) = N_1(m-1,k) + N_k(m-k,k) \qquad (m > k). \tag{3.4}$$

The proof of (3.2) is not difficult; (3.3) and (3.4) are proved in exactly the same way as (2.1).

Using (3.3) with $j = k$, we see that

$$N_{k-1}(m,k) = N_k(m,k) - N_k(m-1,k) \quad ,$$

and, in general,

$$(3.5) \qquad N_{k-j}(m,k) = \sum_{r=0}^{j} (-1)^r \binom{j}{r} N_k(m-r,k) \quad (1 \leq j \leq k-1) \; .$$

Comparing (3.4) and (3.5), we obtain the recurrence

$$(3.6) \qquad N_k(m-k,k) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} N_k(m-r,k) \quad ,$$

which should be compared with (1.2).

For $k = 1, 2$ the recurrence (3.6) is easily handled. Indeed, it follows from (3.2) that (3.6) is in agreement with (1.5) and (1.7). Note that (1.7) and (3.5) imply

$$(3.7) \qquad N_1(m,2) = 3 \cdot 2^{m-2} \quad (m \geq 2) \; .$$

To solve the recurrence (3.6) for general k, we make use of some results from the calculus of finite differences [3]. Let ρ denote a primitive k^{th} root of unity and note that the characteristic polynomial of the recurrence is

$$(x-1)^k - 1 \quad ,$$

whose roots are $\rho^j - 1$ $(j = 0, 1, 2, \cdots, k-1)$. Thus there are constants A_0, $A_1, \cdots, A_{k-1}$ such that

$$(3.8) \qquad N_k(n,k) = \sum_{j=0}^{k-1} A_j (\rho^j - 1)^n \; .$$

We show that

$$(3.9)\qquad A_j = \frac{1}{k}\left[(\rho^{-j} + 1)^k - 1\right] \quad (0 \le j \le k - 1) ,$$

first noting that we may extend the recurrence (3.6) and define $N_k(0,k) = 1$.

To prove (3.9), we have, for $0 \leqq r \leqq k - 1$,

$$\sum_{j=0}^{k-1}\left[(\rho^{-j} + 1)^k - 1\right](\rho^j + 1)^r = \sum_{s=0}^{k-1}\binom{k}{s}\sum_{t=0}^{r}\binom{r}{t}\sum_{j=0}^{k-1}\rho^{j(t-s)}$$

$$= k\sum_{s=0}^{r}\binom{k}{s}\binom{s}{t} = k\binom{k+r}{r} ,$$

which, using (3.2), implies (3.9).

It follows from (3.8) and (3.9) that

$$(3.10)\qquad N_k(n,k) = \frac{1}{k}\sum_{j=0}^{k-1}\left[(\rho^{-j} + 1)^k - 1\right](\rho^j - 1)^n ,$$

so that

$$(3.11)\qquad N_k(n,k) = \sum_{s=0}^{k-1}\binom{k}{s}\sum_{r\equiv s\,(\mathrm{mod}\ k)}\binom{n}{r} .$$

If we define generating functions

$$(3.12)\qquad F_{kj}(x) = \sum_{n=0}^{\infty} N_j(n,k)x^n \quad (1 \le j \le k) ,$$

then it is clear from (3.2) and (3.3) that

$$(3.13)\qquad (1 - x)^{j-1}F_{kj}(x) = F_{k1}(x) \quad (j = 2,3,\cdots,k) .$$

Moreover, using (3.4), we have

$$F_{kk}(x) = x^{-k}(1 - x)F_{k1}(x) - \frac{x^{-k}(1 - x^k)}{1 - x} .$$

Comparison with (3.13) then yields

$$(3.14) \qquad F_{k1}(x) = \frac{(x^k - 1)(1 - x)^{k-1}}{x^k - (1 - x)} ,$$

$$(3.15) \qquad F_{kj}(x) = \frac{(x^k - 1)(1 - x)^{k-j}}{x^k - (1 - x)^k} \qquad (1 \leq j \leq k) .$$

We summarize the results of this section by stating

Theorem 3. Let $N_j(n,k)$ denote the number of arrays (3.1) subject to the conditions (1.9). Then $N_j(n,k)$ satisfies (3.6), (3.10), and has generating function (3.15).

4. SOME ONE-LINE ARRAYS

Let $S_k(n_1)$ denote the number of one-line arrays

$$(4.1) \qquad n_1 n_2 n_3 n_4 \cdots ,$$

where the n_j are non-negative integers, subject to the conditions

$$(4.2) \qquad n_j \geqq n_{j+1} + k \quad (j = 1, 2, 3, \cdots) .$$

It is clear from (4.1) and (4.2) that

$$S_k(n) = 1 \qquad (n \leq k) ,$$

$$S_k(n) = \sum_{r=0}^{n-k} S_k(r) \qquad (n > k) ,$$

which implies

$$S_k(n) = S_k(n-1) + S_k(n-k) \qquad (n > k) .$$

Thus an easy induction establishes

Theorem 4. The number of arrays (4.1) subject to the conditions (4.2) is given by

$$S_k(n) = 1 \qquad (1 \leqq n \leqq k) , \tag{4.3}$$

$$S_k(n) = F_{n-k}(k) \qquad (n > k) . \tag{4.4}$$

In particular note that (4.3) and (4.4) yield

$$S_2(n) = F_n \qquad (n = 1, 2, 3, \cdots) . \tag{4.5}$$

Returning to the numbers $F_n(k)$, we see from (1.1) and (1.3) that

$$F_{nk+j}(k) - 1 = \sum_{r=0}^{n} F_{rk+j-1}(k) \qquad (1 \leqq j \leqq k) . \tag{4.6}$$

In the next place, for $1 \leq j \leq k$, let $S_{kj}(n_1)$ denote the number of arrays (4.1), where the n_r are non-negative integers subject to the conditions

$$\begin{aligned} n_r &\geqq n_{r+1} \qquad (r \not\equiv j \pmod{k}) , \\ n_r &> n_{r+1} \qquad (r \equiv j \pmod{k}) . \end{aligned} \tag{4.7}$$

It is immediate from (3.7) that

$$S_{kj}(1) = j \qquad (1 \leqq j \leqq k) , \tag{4.8}$$

$$S_{k,j+1}(n) = 1 + \sum_{r=1}^{n} S_{kj}(r) \qquad (1 \leqq j \leqq k-1) , \tag{4.9}$$

(4.10) $$S_{k1}(n) = 1 + \sum_{r=1}^{n-1} S_{kk}(r) .$$

We shall show that

(4.11) $$S_{kj}(r + 1) = F_{rk+j-1}(k) \quad (1 \le j \le k) .$$

The proof of (4.11) is by induction, the case $r = 0$ being in agreement with (4.8).

Assuming (4.11) for $r \le n - 1$, we see from (4.10) that

$$S_{k1}(n + 1) = F_{(n-1)k}(k) + F_{nk-1}(k) ,$$

which implies

(4.1) $$S_{k1}(n + 1) = F_{nk}(k) .$$

Using (4.6), (4.9), and (4.12), we obtain successively

$$S_{k,j+1}(n + 1) = 1 + \sum_{r=0}^{n} F_{rk+j-1}(k) = F_{nk+j}(k) ,$$

which proves

Theorem 5. The number of arrays (4.1) subject to the conditions (4.7) is given by (4.11).

Finally, we can use the numbers $N_j(n,k)$ to enumerate certain one-line arrays. For $1 \le j \le k$, let $R_{kj}(n)$ denote the number of arrays

(4.13) $$n \ n_1 \ n_2 \ n_3 \ \cdots ,$$

where

(4.14) $$\begin{aligned} n_r &\ge n_{r+1} \quad (r \not\equiv j \pmod{k}) , \\ n_r &\ge k + n_{r+1} \quad (r \equiv j \pmod{k}) . \end{aligned}$$

It follows that

$$R_{kj}(n) = \binom{n+j}{j} \qquad (0 \leqq n \leqq k) , \tag{4.15}$$

$$R_{kj}(n) = \sum_{s=0}^{n} R_{k,j-1}(s) \qquad (2 \leqq j \leqq k) , \tag{4.16}$$

$$R_{k1}(n) = \sum_{s=0}^{n-k} R_{kk}(s) \qquad (n > k) . \tag{4.17}$$

and we deduce

Theorem 6. The number of arrays (4.13) subject to the conditions (4.14) is given by

$$R_{kj}(n) = N_j(n,k) \qquad (1 \leq j \leq k) . \tag{4.18}$$

For convenience of reference, we give the following tables of $F_{n+k}(k)$ and $N_j(n,k)$.

$F_{n+k}(k)$:

k \ n	1	2	3	4	5	6	7
1	4	8	16	32	64	128	256
2	5	8	13	21	34	55	89
3	6	9	13	19	28	41	60
4	7	10	14	19	26	36	50
5	8	11	15	20	26	34	45
6	9	12	16	21	27	34	43
7	10	13	17	22	28	35	43

$N_j(n,k)$:

j	k \ n	1	2	3	4	5	6	7	8
1	1	2	4	8	16	32	64	128	256
1	2	2	3	6	12	24	48	96	192
2	2	3	6	12	24	48	96	192	384
1	3	2	3	4	8	18	38	76	150
2	3	3	6	10	18	36	74	150	300
3	3	4	10	20	38	74	148	298	598
1	4	2	3	4	5	10	25	60	130
2	4	3	6	10	15	25	50	110	240
3	4	4	10	20	35	60	110	220	460
4	4	5	15	35	70	130	240	460	920

5. ADDITIONAL PROPERTIES

The above table of values for $N_j(n,k)$ suggests the formulas

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} N_{k-j}(n,k) = \begin{cases} 0 & (2 \mid k, \quad n > k) , \\ 2N_k(n - k, k) & (2 \nmid k, \quad n > k) , \end{cases} \tag{5.1}$$

$$N_{n+r}(n + km - r, k) = N_{n-r}(n + km + r, k) \qquad (n \geq r) , \tag{5.2}$$

$$N_r(km + r, k) = 2N_{r-1}(km + r, k) . \tag{5.3}$$

To prove (5.1), we have, using (3.5) and (3.6),

$$\begin{aligned} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} N_{k-j}(n, k) &= \sum_{r=0}^{k-1} (-1)^r \binom{k}{r} N_k(n - r, k) \sum_{j=0}^{k-r-1} (-1)^{r+j} \binom{k - r}{j} \\ &= (-1)^{k+1} \sum_{r=0}^{k-1} (-1)^r \binom{k}{r} N_k(n - r, k) \\ &= (1 + (-1)^{k+1}) N_k(n - k, r) , \end{aligned}$$

which implies (5.1).

In the next place, it follows from (3.5) and (3.10) that

$$(5.4)\qquad N_j(n,k) = \sum_{r=0}^{k-1} \left[(\rho^r + 1)^k - 1\right] (\rho^r + 1)^{n+j-k}\rho^{-jr} \quad (1 \leqq j \leqq k;\ n \geqq k)$$

so that

$$(5.5)\qquad N_{k-j}(n,k) = \sum_{s=0}^{k-1} \binom{k}{s} \sum_{r \equiv s+j \,(\mathrm{mod}\ k)} \binom{n-j}{r} \quad (1 \leqq j \leqq k)\,.$$

It is clear from (5.4) that

$$\begin{aligned} N_{n+r}(n \pm km - r, k) &= \sum_{s=0}^{k-1} \left[(\rho^s + 1)^{2n+km} - (\rho^s + 1)^{2n+km-r}\right]\rho^{-s(n+r)} \\ &= \sum_{s=0}^{k-1} \left[(\rho^{-s} + 1)^{2n+km} - (\rho^{-s} + 1)^{2n+km-r}\right]\rho^{s(n+r)} \\ &= \sum_{s=0}^{k-1} \left[(\rho^s + 1)^{2n+km} - (\rho^s + 1)^{2n+km-r}\right]\rho^{-s(n-r)} \end{aligned}$$

which completes the proof of (5.2). We remark that (5.3) is an immediate corollary of (5.2).

Note that (5.2) requires only that $n \geqq r$. This follows because (5.4) is valid for all non-negative j.

REFERENCES

1. T. W. Chaundy, "Partition-Generating Functions," Quarterly Journal of Mathematics (Oxford), Vol. 2 (1931), pp. 234-40.
2. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1954.
3. Charles Jordan, Calculus of Finite Differences, Chelsea, New York, 1947.

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NOTE ON A COMBINATORIAL IDENTITY IN THE THEORY OF BI-COLORED GRAPHS

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In connection with an enumeration problem arising in the theory of labelled bi-colored graphs, C. Y. Lee [2] has obtained the following identities. Defining N(a, b;n) by the expansion

$$\sum_{n=0}^{ab} N(a,b;n)x^n = \sum_{k=0}^{a} (-1)^{a+b+k} \binom{a}{k} \left\{1 - (1+x)^k\right\}^b \tag{1}$$

and noting the lemma

$$\sum_{j=0}^{b} \sum_{i=0}^{kj} f(i,j) = \sum_{i=0}^{kb} \sum_{j=\langle \frac{i}{k} \rangle}^{b} f(i,j) \quad , \tag{2}$$

where $\langle x \rangle$ = the smallest integer $\geq x$, Lee was able to show that

$$N(a,b;n) = \sum_{k=\langle \frac{n}{b} \rangle}^{a} \sum_{j=\langle \frac{n}{k} \rangle}^{b} (-1)^{a+b+k+j} \binom{a}{k}\binom{b}{j}\binom{kj}{n} \quad , \tag{3}$$

from which as a special case he deduced the apparently novel formula

$$\sum_{k=1}^{n} \sum_{j=\langle \frac{n}{k} \rangle}^{n} (-1)^{k+j} \binom{n}{k}\binom{n}{j}\binom{kj}{n} = n! \quad . \tag{4}$$

It may be of interest therefore, to point out that the formulas may be written in much simpler form inasmuch as the introduction of $\langle x \rangle$ leads to

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unnecessarily complicated relations. Indeed it will be shown that relation (4) is essentially trivial and may be generalized by methods of finite differences. The simple nature of relation (4) was also missed in reviews of the paper [4], [5].

In order to determine N from expansion (1) it is not necessary to invoke (2) and instead we shall merely make use of the fact that

$$\binom{m}{p} = 0 \quad \text{for} \quad m < p$$

when m and p are integers with $p \geq 0$, $m \geq 0$.

From (1) we have in fact

$$\begin{aligned}\sum_{n=0}^{ab} N(a,b;n)x^n &= \sum_{k=0}^{a} (-1)^{a+b+k} \binom{a}{k} \sum_{j=0}^{b} (-1)^j \binom{b}{j} (1+x)^{kj} \\ &= \sum_{k=0}^{a}\sum_{j=0}^{b} (-1)^{a+b+k+j} \binom{a}{k}\binom{b}{j} \sum_{n=0}^{m} \binom{kj}{n} x^n \\ &= \sum_{n=0}^{m} x^n \sum_{k=0}^{a}\sum_{j=0}^{b} (-1)^{a+b+k+j} \binom{a}{k}\binom{b}{j}\binom{kj}{n}\end{aligned}$$

provided only that $m \geqslant ab$.

Consequently we have

$$N(a,b;n) = \sum_{k=0}^{a}\sum_{j=0}^{b} (-1)^{a+b+k+j} \binom{a}{k}\binom{b}{j}\binom{kj}{n} \qquad , \tag{5}$$

without any essential need for the restriction on range of summation in (3). Of course, some terms are zero, but it is convenient to allow these to stand in the indicated formula.

Then when we choose $a = b = n$ in order to obtain the identity (4) found by Lee, we see that this would appear more elegantly in the form

$$(6) \qquad \sum_{k=0}^{n} \sum_{j=0}^{n} (-1)^{k+j} \binom{n}{k} \binom{n}{j} \binom{kj}{n} = n!$$

and we shall show in a simple way that this generalizes to give the relation

$$(7) \qquad \sum_{k=0}^{n} \sum_{j=0}^{n} (-1)^{k+j} \binom{n}{k} \binom{n}{j} \binom{c + kj}{n} = n!$$

for any real value of c.

As for the proof, this comes from the familiar fact that when $f(x)$ is a polynomial of degree $\leq n$ in x, say

$$f(x) = \sum_{i=0}^{n} a_i x^i ,$$

then

$$(8) \qquad \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(k) = \begin{cases} 0, & n < r , \\ (-1)^n n!\, a_n, & r = n . \end{cases}$$

Since $\binom{c + dx}{n}$ is a polynomial of degree n in x we have

$$(9) \qquad \sum_{k=0}^{r} (-1)^k \binom{r}{k} \binom{c + dk}{n} = \begin{cases} 0, & n < r , \\ (-d)^n, & r = n , \end{cases}$$

this being true for all real values of c and d. The identity is not new, and appears for example in Schwatt [3, 104] and has been used by the writer [1] in another connection.

Thus

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{c+kj}{n} = (-1)^n k^n \quad ,$$

so that

$$\sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^{k+j}\binom{n}{k}\binom{n}{j}\binom{c+kj}{n} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^n$$

but this is clearly n! by the familiar Euler theorem about n^{th} differences of n^{th} powers of the natural numbers, or we may again apply (8).

If we define

$$N(a,b,c,n) = \sum_{k=0}^{a}\sum_{j=0}^{b}(-1)^{a+b+k+j}\binom{a}{k}\binom{b}{j}\binom{c+kj}{n} \quad , \tag{10}$$

with $N(a,b,0,n) = N(a,b;n)$, then we have the extension of (1) as

$$\sum_{n=0}^{\infty} N(a,b,c,n)x^n = (1+x)^c\sum_{k=0}^{a}(-1)^{a+b+k}\binom{a}{k}\left\{1-(1+x)^k\right\}^b \quad , \tag{11}$$

and it would be of interest to know whether this yields any interesting result about labelled bi-colored graphs.

REFERENCES

1. H. W. Gould, "Some Generalizations of Vandermonde's Convolution," Amer. Math. Monthly, 63(1956), pp. 84-91.
2. C. Y. Lee, "An Enumeration Problem Related to the Number of Labelled Bi-Coloured Graphs," Canadian J. Math., 13(1961), pp. 217-220.
3. I. J. Schwatt, An Introduction to the Operations with Series, Univ. of Pennsylvania Press, 1924; Chelsea Reprint, 1962.
4. H. Künneth, Review in Zentralblatt für Mathematik, 97(1962), p. 391.
5. G. R. Livesay, Review in Math. Reviews, 25(1963), No. 1112.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited By

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Send all communications concerning Advanced Problems and Solutions to Raymond Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania, 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within three months after publication of the problems.

H-119 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Put

$$\overline{H}(m,n,p) = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}(-1)^{i+j+k}\binom{i+j}{j}\binom{j+k}{k}\binom{k+m-i}{m-i}\binom{m-i+n-j}{n-j}$$

$$\cdot \binom{n-j+p-k}{p-k}\binom{p-k+i}{i}$$

Show that $\overline{H}(m,n,p) = 0$ unless m,n,p are all even and that

$$\overline{H}(2m, 2n, 2p) = \sum_{r=0}^{\min(m,n,p)} (-1)^r \frac{(m+n+p-r)!}{r!\,r!\,(m-r)!\,(n-r)!\,(p-r)!} =$$

(The formula

$$\overline{H}(2m, 2n) = \binom{m+n}{m}^2 ,$$

where

$$\overline{H}(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{i+j}{j}\binom{m-i+j}{j}\binom{i+n-j}{n-j}\binom{m-i+n-j}{n-j}$$

is proved in the Fibonacci Quarterly, Vol. 4 (1966), pp. 323-325.)

H-120 Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.

The Fibonacci polynomials are defined by

$$f_{n+1}(x) = x \cdot f_n(x) + f_{n-1}(x)$$
$$f_1(x) = 1, \quad f_2(x) = x .$$

If $z_r = f_r(x) \cdot f_r(y)$ then show that

(i) z_r satisfies the recurrence relation,

$$z_{n+4} - xy \cdot z_{n+3} - (x^2 + y^2 + 2)z_{n+2} - xy \cdot z_{n+1} + z_n = 0.$$

(ii) $(x + y)^2 \cdot \sum_1^n z_r = (z_{n+2} - z_{n-1}) - (xy - 1)(z_{n+1} - z_n).$

H-121 Proposed by H. H. Ferns, University of Victoria, Victoria, B.C., Canada.

Prove the following identity.

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{F_k}{F_{m-k}} \right)^i F_{mi+\lambda} = \left(\frac{F_m}{F_{m-k}} \right)^n F_{nk+\lambda} - F_\lambda \quad (m \neq k) ,$$

where F_n is the n^{th} Fibonacci number, m, λ are any integers or zero and k is an even integer or zero.

Write the form the identity takes if k is an odd integer.

Find an analogous identity involving Lucas numbers.

H-122 Proposed by R. E. Whitney, Lock Haven State College, Lock Haven, Pa.

Let F_n denote the n^{th} Fibonacci number expressed in base 2. Consider the ordered array $F_1F_2F_3\cdots$. Let g_n denote the n^{th} digit of this array. Find a formula for g_n. If possible, generalize for any base.

H-70 Proposed by C. A. Church, Jr., W. Virginia Univ., Morgantown, W. Virginia.

For $n = 2m$, show that the total number of k-combinations of the first n natural numbers such that no two elements i and $i + 2$ appear together in the same selection is F^2_{m+2} and if $n = 2m + 1$, the total is $F_{m+2}F_{m+3}$.

Solution and comments by the proposer.

For his quick solution of the "problème des ménages" Kaplansky [2] gives two results for combinations with restricted positions. We state them in the following form:

The number of k-combinations of the first n natural numbers, on a line, with no two consecutive is

$$\binom{n - k + 1}{k} \quad ; \quad 0 \le k \le \frac{n+1}{2} , \tag{1}$$

if arranged on a circle, so that 1 and n are consecutive, the number is

$$\frac{n}{n - k}\binom{n - k}{k} , \qquad 0 \le k \le \frac{n}{2} . \tag{2}$$

See also [4, p. 198]. Summed over k, (1) and (2) give the Fibonacci and Lucas numbers, respectively.

For the problem as stated we use (1).

The restriction that i and $i + 2$ cannot appear in any selection can be stated as (a) no two consecutive even integers appear and (b) no two consecutive odd integers appear.

If $n = 2m$, a k-combination with the stated restrictions will be made up of s integers from among the m even, no two consecutive, and $k - s$ from among the m odd, no two consecutive. Thus there are

$$\sum_{s=0}^{k} \binom{m - s + 1}{s}\binom{m - (k - s) + 1}{k - s} \tag{3}$$

k-combinations of the first 2m natural numbers such that i and $i + 2$ do not appear.

Summing (3) over k we get the total number

$$F^2_{m+2} = \sum_{k=0}^{2\left[\frac{m+1}{2}\right]} \sum_{s=0}^{k} \binom{m-s+1}{s}\binom{m-(k-s)+1}{k-s}$$

with the usual condition that

$$\binom{a}{b} = 0 \quad \text{for} \quad b > a \geq 0 .$$

For $n = 2m + 1$ we choose s from among the m even integers, no two consecutive, and $k - s$ from among the $m + 1$ odd integers, no two consecutive, to get that there are

$$\sum_{s=0}^{k} \binom{m-s+1}{s}\binom{m-(k-s)+2}{k-s} \tag{4}$$

k-combinations of the first $2m + 1$ natural numbers such that i and $i + 2$ do not appear.

Summed over k, (4) gives the total number

$$F_{m+2}F_{m+3} = \sum_{k=0}^{m+1} \sum_{s=0}^{k} \binom{m-s+1}{s}\binom{m-(k-s)+2}{k-s} .$$

It is also of interest to consider the circular analog of this problem by way of (2).

For $n = 2m$, 2 and 2m are taken to be consecutive as are 1 and $2m - 1$. By the same argument as before we find that there are

$$\sum_{s=0}^{k} \frac{m}{m-s}\binom{m-s}{s} \frac{m}{m-(k-s)}\binom{m-(k-s)}{k-s}$$

circular k-combinations such that i and $i + 2$ do not appear and a total of

$$L_m^2 = \sum_{k=0}^{2\left[\frac{m}{2}\right]} \sum_{s=0}^{k} \frac{m}{m-s}\binom{m-s}{s}\frac{m}{m-(k-s)}\binom{m-(k-s)}{k-s} .$$

For $n \doteq 2m+1$, 2 and $2m$ are consecutive as are 1 and $2m+1$ and we have the total

$$L_m L_{m+1} = \sum_{k=0}^{m}\sum_{s=0}^{k} \frac{m}{m-s}\binom{m-s}{s}\frac{m+1}{m-(k-s)+1}\binom{m-(k-s)+1}{k-s} .$$

Mixed results can also be obtained using both (1) and (2). For example, one can take linear combinations on the evens and circular combinations on the odds.

Remarks. The problem posed in H-70 first appeared in the literature in a paper by N. S. Mendelsohn [3]; an explicit formula was not obtained. The first explicit formula was given by M. Abramson [1, lemma 3]. Abramson's solution for the number of k-combinations such that i and i + 2 do not appear together is

$$\sum_{s=0}^{\left[\frac{k}{2}\right]}\binom{n-2k+s+2}{k-s}\binom{k-s}{s} .$$

REFERENCES

1. M. Abramson, "Explicit Expressions for a Class of Permutation Problems," Canad. Math. Bull., 7(1964), pp. 345-350.
2. I. Kaplansky, Solution of the "Problème des Ménages," Bull. Amer. Math Soc., 49(1943), 784-785.
3. N. S. Mendelsohn, "The Asymptotic Series for a Certain Class of Permutation Problems, Canad. J. of Math., 8(1956), pp. 234-244.
4. J. Riordan, An Introduction to Combinatorial Analysis, New York, 1958.

H-73 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let $f_0(x) = 0, \quad f_1(x) = 1$

and
$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x), \qquad n \ge 0$$

and let $b_n(x)$ and $B_n(x)$ be the polynomials in H-69; show

$$f_{2n+2}(x) = xB_n(x^2),$$

and

$$f_{2n+1}(x) = b_n(x^2).$$

Thus there is an intimate relationship between the Fibonacci polynomials, $f_n(x)$ and the Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$.

Solution by Douglas Lind, Univ. of Virginia, Charlottesville, Virginia.

Using the explicit representations of $B_n(x)$ and $b_n(x)$ given in H-69, and of $f_n(x)$ given in B-74, we find

$$xB_n(x^2) = \sum_{r=0}^{n} \binom{n+r+1}{n-r} x^{2r+1} = \sum_{r=0}^{n} \binom{2n-r+1}{r} x^{2n-2r+1} = f_{2n+2}(x),$$

$$b_n(x^2) = \sum_{r=0}^{n} \binom{n+r}{n-r} x^{2r} = \sum_{r=0}^{n} \binom{2n-r}{r} x^{2n-2r} = f_{2n+1}(x).$$

These relations have been given by R. A. Hayes ["Fibonacci and Lucas Polynomials," (Master's Thesis); equations (3.4-1) and (3.4-2)].

H-77 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{2k+2j+1} = 5^n L_{2n+2k+2}$$

for all integers k. Set $k = -(n+1)$ and derive

$$\sum_{j=0}^{n} \binom{2n+1}{n-j} F_{2j+1} = 5^n,$$

a result of S. G. Guba Problem No. 174, Issue No. 4, July-August 1965, p. 73 of Matematika V. Skŏle.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Since

$$L_n = \alpha^n + \beta^n, \quad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where

$$\alpha = \frac{1}{2}(1 + \sqrt{5}), \quad \beta = \frac{1}{2}(1 - \sqrt{5}),$$

$$1 + \alpha^2 = \alpha\sqrt{5}, \quad 1 + \beta^2 = -\beta\sqrt{5},$$

it follows that

$$\sum_{j=0}^{n} \binom{n}{j} F_{k+2j} = \sum_{j=0}^{n} \binom{n}{j} \frac{\alpha^{k+2j} - \beta^{k+2j}}{\alpha - \beta}$$

$$= \frac{\alpha^k (1 + \alpha^2)^n - \beta^k (1 + \beta^2)^n}{\alpha - \beta}$$

$$= \frac{(\alpha^{k+n} - (-1)^n \beta^{k+n})(\sqrt{5})^n}{\sqrt{5}}$$

$$= \begin{cases} 5^{(n-1)/2} L_{k+n} & (n \text{ odd}) \\ 5^{n/2} F_{k+n} & (n \text{ even}), \end{cases}$$

thus generalizing the stated result. In particular, for $k = -n$, we get

$$\sum_{j=0}^{n} \binom{n}{j} F_{-n+2j} = \begin{cases} 2 \cdot 5^{(n-1)/2} & (n \text{ odd}) \\ 0 & (n \text{ even}). \end{cases}$$

Note that

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{-2n-1+2j} = \sum_{j=0}^{n} \binom{2n+1}{j} F_{-2n-1+2j} + \sum_{j=n+1}^{2n+1} \binom{2n+1}{j} F_{-2n-1+2j}$$

$$= \sum_{j=0}^{n} \binom{2n+1}{j} F_{2n+1-2j} + \sum_{j=0}^{n} \binom{2n+1}{j} F_{2n+1-2j}$$

$$= 2 \sum_{j=0}^{n} \binom{2n+1}{n-j} F_{2j+1} \quad ,$$

so that

$$\sum_{j=0}^{n} \binom{2n+1}{n-j} F_{2j+1} = 5^n \quad .$$

Similarly, we have

$$\sum_{j=0}^{n} \binom{n}{j} L_{k+2j} = \alpha^k (1 + \alpha^2)^n + \beta^k (1 + \beta^2)^n$$

$$= (\alpha^{k+n} + (-1)^n \beta^{k+n}) (\sqrt{5})^n$$

$$= \begin{cases} 5^{n/2} L_{k+n} & (n \text{ even}) \\ 5^{(n+1)/2} F_{k+n} & (n \text{ odd}) \end{cases} \quad .$$

In particular, since

$$\sum_{j=0}^{2n} \binom{2n}{j} L_{-2n+2j} = -2 \binom{2n}{n} + \sum_{j=0}^{n} \binom{2n}{j} L_{-2n+2j} = \sum_{n}^{2n} \binom{2n}{j} L_{-2n+2j}$$

$$= -2 \binom{2n}{n} + 2 \sum_{j=0}^{n} \binom{2n}{n-j} L_{2j} \quad ,$$

so that

$$\sum_{j=0}^{n} \binom{2n}{n-j} L_{2j} = \binom{2n}{n} + 5^n \quad .$$

LATE ACKNOWLEDGEMENT: Problems H-64, H-71, H-72, H-73, and H-77 were also solved by M. N. S. Swamy.

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DIRECT CALCULATION OF k-GENERALIZED FIBONACCI NUMBERS

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SUMMARY

A formula is developed for direct calculation of any k-generalized Fibonacci number $u_{j,k}$ without iteration.

DEFINITIONS

The ordinary Fibonacci number $u_{j,2}$ is defined by

(1) $$u_{j,2} = u_{j-1,2} + u_{j-2,2} \quad (j \geq 2)$$

with the additional conditions usually imposed

(2) $$u_{0,2} = 0; \quad u_{1,2} = 1.$$

The k-generalized Fibonacci number $u_{j,k}$ is defined as the sum of its k predecessors

(3) $$u_{j,k} = u_{j-1,k} + u_{j-2,k} + \cdots + u_{j-k,k} = \sum_{i=j-k}^{j-1} u_{i,k}$$

together with the initial conditions

(4) $$u_{j,k} = 0 \quad (0 \leq j \leq k - 2); \quad u_{k-1,k} = 1.$$

A table of $u_{j,k}$ from $k = 1$ to 7 and $j = 0$ to 15 is found in Table 1.

Table 1
Fibonacci Numbers $u_{j,k}$ for Various Values of j and k

k/j																	
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	
3	0	0	1	1	2	4	7	13	24	44	81	149	274	504	927	1705	
4	0	0	0	1	1	2	4	8	15	29	56	108	208	401	773	1490	
5	0	0	0	0	1	1	2	4	8	16	31	61	120	236	464	912	
6	0	0	0	0	0	1	1	2	4	8	16	32	63	125	248	492	
7	0	0	0	0	0	0	1	1	2	4	8	16	32	64	127	253	

TERM RATIO

The key to direct calculation is the existence of a fixed ratio r_k between successive $u_{j,k}$'s so that in the limit we have

$$(5) \qquad \lim_{n \to \infty} \frac{u_{j+1,k}}{u_{j,k}} = r_k .$$

If such a ratio can be found, an approximate calculation is simple.

Vorob'ev [6] has shown that for $k = 2$, this requires the solution of

$$(6) \qquad q^j = q^{j-1} + q^{j-2}$$

which for $q \neq 0$ reduces to

$$(7) \qquad q^2 - q - 1 = 0$$

for which the roots are

$$(8) \qquad r_1 = \frac{1-\sqrt{5}}{2} \approx -0.6180 \quad \text{and} \quad r_2 = \frac{1+\sqrt{5}}{2} \approx 1.6180 ,$$

where $\approx$ means "approximately equal to."

If f_n is any Fibonacci sequence obeying the difference equation $f_{n+1} - f_n - f_{n-1} = 0$, then f_n has the form (see [4])

$$(9) \qquad f_n = b_1 r_1^n + b_2 r_2^n .$$

Since $|r_1| < 0.7$, $|r_1^2| < \frac{1}{2}$, so that $|r_1^{2n}| < 1/2^n$. Hence there exists an N such that for all $n > N$, $u_{n,2}$ is the greatest integer to $b_2r_2^n$, and we write

$$u_{j,2} \approx b_2r_2^j \quad (j > N) \tag{10}$$

To evaluate the constants b_1 and b_2, we use the initial conditions

$$\begin{aligned} b_1 + b_2 &= u_{0,2} = 0 , \\ b_1r_1 + b_2r_2 &= u_{1,2} = 1 , \end{aligned} \tag{11}$$

which yield

$$b_1 = \frac{-1}{\sqrt{5}}, \quad b_2 = \frac{1}{\sqrt{5}} . \tag{12}$$

An exact expression for $u_{j,2}$ is hence the familiar Binet form

$$u_{j,2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^j - \left(\frac{1-\sqrt{5}}{2}\right)^j\right] \tag{13}$$

GENERALIZATION

To find an expression for the k-generalized Fibonacci numbers, let us first seek solutions to (3) which form a geometric progression, say aq^j. Then (3) leads to a general form of (6),

$$aq^j = aq^{j-1} + aq^{j-2} + \cdots + aq^{j-k} . \tag{14}$$

Thus

$$aq^{j-k}(q^k - q^{k-1} - \cdots - q - 1) = 0 . \tag{15}$$

Since we are looking for solutions which are not identically zero, we can assume $a \neq 0$ and $q \neq 0$. Therefore we see

$$q^k - q^{k-1} - \cdots - q - 1 = 0 . \tag{16}$$

This k^{th} degree equation has k complex roots, say $r_{1,k}$, $r_{2,k}$, $\cdots$, $r_{k,k}$. Now Miles [5] has shown that these roots are distinct, that all but one of them lie within the unit circle in the complex plane, and that the remaining root is real and lies between 1 and 2. Hence with a suitable choice of subscripts we may write

$$|r_{i,k}| < 1 \quad (1 \le i \le k-1) \quad , \tag{17}$$

$$1 \le r_{k,k} \le 2 \quad . \tag{18}$$

Since the roots are distinct, the Vandermond determinant

$$\begin{vmatrix} 1 & r_{1,k} & r_{1,k}^2 & \cdots & r_{1,k}^{k-1} \\ 1 & r_{2,k} & r_{2,k}^2 & \cdots & r_{2,k}^{k-1} \\ \vdots & & & & \vdots \\ 1 & r_{k,k} & r_{k,k}^2 & \cdots & r_{k,k}^{k-1} \end{vmatrix} \ne 0 \quad , \tag{19}$$

and Jeske [4] has shown that the general solution can be written

$$u_{j,k} = \sum_{i=1}^{k} b_i r_{i,k}^j \quad . \tag{20}$$

To evaluate the constants b_i, we use the initial conditions

$$\begin{aligned} &\sum_{i=1}^{k} b_i r_{i,k}^m = 0 \quad (m = 0, 1, \cdots, k-2) \quad , \\ &\sum_{i=1}^{k} b_i r_{i,k}^{k-1} = 1 \quad . \end{aligned} \tag{21}$$

This system has a unique solution by (19) which can be found using Cramer's rule. This yields

(22) $$b_i = \prod_{\substack{\alpha=1 \\ \alpha \neq i}}^{k} (r_{i,k} - r_{\alpha,k})^{-1} ,$$

so that (20) becomes

(23) $$u_{j,k} = \sum_{i=1}^{k} \left(\prod_{\substack{\alpha=1 \\ \alpha \neq i}}^{k} (r_{i,k} - r_{\alpha,k})^{-1} \right) r_{i,k}^{j} .$$

Recalling (17) and (18), we remark that as j becomes large $r_{k,k}^{j}$ becomes the dominant term in (23), so that as before there exists an N such that for all $j > N$, $u_{j,k}$ is the nearest integer to $b_k r_{k,k}^{j}$. We may therefore write

(24) $$u_{j,k} \approx b_k r_{k,k}^{j} \quad (j > N) .$$

It follows from (24) that

(25) $$\lim_{j \to \infty} \frac{u_{j+1,k}}{u_{j,k}} = r_{k,k} ,$$

and more generally

(26) $$\lim_{j \to \infty} \frac{u_{j+m,k}}{u_{j,k}} = r_{k,k}^{m} .$$

APPROXIMATIONS

We first note that as $k \to \infty$ the sequence $u_{j-k,k}$ approaches the geometric progression of powers of two,

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ \cdots,$$

as can be seen from Table 1. It follows that

(27)
$$\lim_{k\to\infty} r_{k,k} = 2 \ .$$

See Table 2 for calculated values of the principal root $r_{k,k}$ for $k = 2$ to 19, which gives striking verification of (27).

Table 2
Fibonacci Roots

k	r_k
2	1.6180340
3	1.8392868
4	1.9275621
5	1.9659483
6	1.9835829
7	1.9919642
8	1.9960312
9	1.9980295
10	1.9990187
11	1.9995105
12	1.9997556
13	1.9998779
14	1.9999390
15	1.9999695
16	1.9999845
17	1.9999925
18	1.9999962
19	1.9999981

From (25) we get then

(28)
$$\lim_{k\to\infty} \lim_{j\to\infty} \frac{u_{j+1,k}}{u_{j,k}} = 2 \ ,$$

which was stated in an equivalent form by P. F. Byrd [1]. We shall now show that b_k is approximately $r_{k,k}^{-k}$ in the sense that

(29) $$\lim_{k\to\infty} b_k \big/ r_{k,k}^{-k} = 1 .$$

To prove (29), first recall that

$$b_k = \frac{1}{(r_{k,k} - r_{1,k}) \cdots (r_{k,k} - r_{k-1,k})} .$$

Since

$$x^k - x^{k-1} - \cdots - x - 1 = (x - r_{1,k}) \cdots (x - r_{k,k}) ,$$

and

$$\begin{aligned} f(x) = (x-1)(x^k - x^{k-1} - \cdots - x - 1) &= x^{k+1} - 2x^k + 1 \\ &= (x-1)(x - r_{1,k}) \cdots (x - r_{k,k}) , \end{aligned}$$

we find

$$f'(r_{k,k}) = (k+1)r_{k,k}^k - 2kr_{k,k}^{k-1} = (r_{k,k} - 1)(r_{k,k} - r_{1,k}) \cdots (r_{k,k} - r_{k-1,k}) .$$

Hence

$$b_k = \frac{r_{k,k} - 1}{(k+1)r_{k,k}^k - 2kr_{k,k}^{k-1}} ,$$

from which (29) follows, since $r_{k,k}^{k+1} - 2r_{k,k}^k = -1$. Then for sufficiently large j and k we may write

(30) $$u_{j,k} \approx r_{k,k}^{j-k} .$$

Call the approximation for $u_{j,k}$ in (24) $u'_{j,k}$. Then using (20), the error committed by this approximation is

(31) $$w_{j,k} = \left| u_{j,k} - u'_{j,k} \right| = \left| \sum_{i=1}^{k-1} b_i r_{i,k}^j \right| .$$

By (17) $|r_{i,k}| < 1$ for $1 \leq i \leq k - 1$, so the triangle inequality shows

(32)
$$w_{j,k} \leq \sum_{i=1}^{k-1} |b_i| \, |r_{i,k}|^j < \sum_{i=1}^{k-1} |b_i| \, .$$

Note that the first inequality in (32) shows that

(33)
$$\lim_{j \to \infty} w_{j,k} = 0 \, ,$$

so that for fixed k the error tends to zero as j becomes large, giving formal justification to (24).

EXTENSION

In the near future tables of b_k will be prepared by computers. These, together with r_k, will provide an excellent approximation for $u_{j,k}$ using an analytic procedure.

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REFERENCES

1. P. F. Byrd, Problem H-61, Fibonacci Quarterly, 3(1965), 201. Solution, 5(1967).
2. Mark Feinberg, "New Slants," Fibonacci Quarterly, 2(1964), 223-227.
3. Mark Feinberg, "Fibonacci-Tribonacci," Fibonacci Quarterly, 1(1963), No. 3, 71-74.
4. James A. Jeske, "Linear Recurrence Relations, Part I," Fibonacci Quarterly, 1(1963), No. 2, 69-74.
5. E. P. Miles, Jr., "Generalized Fibonacci Numbers and Associated Matrices," American Math. Monthly, 67(1960), 745-57.
6. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell, New York, 1961.

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EQUATIONS WHOSE ROOTS ARE THE nth POWERS OF THE ROOTS OF A GIVEN CUBIC EQUATION

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Given the cubic equation

$$x^3 - c_1x^2 + c_2x - c_3 = 0 \quad ,$$

with roots r_1, r_2, r_3, the problem of this paper is to write the equation

$$(1) \quad x^3 - (r_1^n + r_2^n + r_3^n)x^2 + (r_1^n r_2^n + r_1^n r_3^n + r_2^n r_3^n)x - r_1^n r_2^n r_3^n$$
$$= x^3 - c_{(1,n)}x^2 + c_{(2,n)}x - c_{(3,n)} = 0$$

whose roots are r_1^n, r_2^n, r_3^n, and whose coefficients are expressed in terms of the coefficients c_1, c_2, c_3, of the given equation.

This paper extends to the cubic equation a study initiated by the solution of a similar problem for the quadratic by the same authors [1]. Just as a special quadratic equation leads to a relationship between the n^{th} Fibonacci number and a sum of binomial coefficients, so does a special cubic equation relate the n^{th} member of a Tribonacci sequence to a sum of products of binomial coefficients. Some Lucas identities also follow.

The summation for initial values of powers of roots by elementary theory yields the first five entries in the table below. Examination of this sequence reveals an iterative pattern; namely, that if

$$c_{(1,n)} = r_1^n + r_2^n + r_3^n, \qquad n \geq 0 \ ,$$

then

$$c_{(1,n)} = c_1c_{(1,n-1)} - c_2c_{(1,n-2)} + c_3c_{(1,n-3)} \quad ,$$

which is easily proved, since each root, and hence sums of the roots, satisfies the original equation.

Sums through eighth powers appear in the table below. The right-hand column gives the sum of the absolute values of the coefficients for each value of n.

n	$c_{(1,n)} = r_1^n + r_2^n + r_3^n$	Coefficient Sums for n
0	3	3
1	c_1	1
2	$c_1^2 - 2c_2$	3
3	$c_1^3 - 3c_1c_2 + 3c_3$	7
4	$c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3$	11
5	$c_1^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_1^2c_3 - 5c_2c_3$	21
6	$c_1^6 - 6c_1^4c_2 + 9c_1^2c_2^2 - 2c_2^3 + 6c_1^3c_3 - 12c_1c_2c_3 + 3c_3^2$	39
7	$c_1^7 - 7c_1^5c_2 + 14c_1^3c_2^2 - 7c_1c_2^3 + 7c_1^4c_3 - 21c_1^2c_2c_3 + 7c_2^2c_3 + 7c_1c_3^2$	71
8	$c_1^8 - 8c_1^6c_2 + 20c_1^4c_2^2 - 16c_1^2c_2^3 + 2c_2^4$	131
	$+ 8c_1^5c_3 - 32c_1^3c_2c_3 + 24c_1c_2^2c_3 + 12c_1^2c_3^2 - 8c_2c_3^2$	

It is possible to perceive the generalized number pattern for sums of n^{th} powers of the roots by extending the table above and breaking down the sum in terms of coefficients of powers of c_3. If ψ_n is the coefficient of $c_3^n/n!$ in the sum

$$r_1^n + r_2^n + r_3^n \quad ,$$

then

$$\psi_0 = c_1^n - nc_1^{n-2}c_2 + n(n-3)c_1^{n-4}c_2^2/2! - n(n-4)(n-5)c_1^{n-6}c_2^3/3! + \cdots$$

$$\psi_1 = nc_1^{n-3} - n(n-4)c_1^{n-5}c_2 + n(n-5)(n-6)c_1^{n-7}c_2^2/2!$$
$$- n(n-6)(n-7)(n-8)c_1^{n-9}c_2^3/3! + \cdots$$

$$\psi_2 = n(n-5)c_1^{n-6} - n(n-6)(n-7)c_1^{n-8}c_2 + n(n-7)(n-8)(n-9)c_1^{n-10}c_2^2/2!$$
$$- n(n-8)(n-9)(n-10)(n-11)c_1^{n-12}c_2^3/3! + \cdots$$

leading to the three equivalent expressions below. For $c_1c_2c_3 \neq 0$,

$$(2)\qquad r_1^n + r_2^n + r_3^n = \sum_{k=0}^{[n/3]} c_3^k \psi_k / k! \quad ,$$

$$\psi_k = \sum_{m=0}^{[(n-3k)/2]} (-1)^m c_1^{n-2m-3k} c_2^m n(n-m-2k-1)!/(n-2m-3k)!\, m! \ ,$$

$$(3)\qquad r_1^n + r_2^n + r_3^n = \sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^m n(n-m-2k-1)!}{(n-2m-3k)!\, m!\, k!} c_1^{n-2m-3k} c_2^m c_3^k \ ,$$

$$(4)\qquad r_1^n + r_2^n + r_3^n = \sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^m n}{n-m-3k} \binom{n-m-2k-1}{k} \binom{n-m-3k}{m} \times c_1^{n-2m-3k} c_2^m c_3^k \ ,$$

where $[n]$ is the greatest integer $\leq n$ and $\binom{m}{n}$ is a binomial coefficient. Notice that the coefficients of c_3^0 are the same as the coefficients which arose in studying the roots of the quadratic in [1]. The reiterative relationship of the terms $c_{(1,n)}$ suggests a proof by mathematical induction for the three formulae listed, and such a proof has been written by the authors. For the sake of brevity, the proof is omitted. A derivation of the above formulas could also be done using Waring's formula and Newton's identities (see [2]).

Thus far, we have found a way to express the coefficient for x^2 in our general problem. The coefficient for x,

$$c_{(2,n)} = r_1^n r_2^n + r_1^n r_3^n + r_2^n r_3^n \ ,$$

has a similar computation. In the auxiliary cubic equation

$$(x - r_1^n r_2^n)(x - r_1^n r_3^n)(x - r_2^n r_3^n) = 0$$

notice that $c_{(2,n)}$ is the coefficient of x^2. When $n = 1$, the above cubic becomes, upon multiplication,

$$x^3 - (r_1r_2 + r_1r_3 + r_2r_3)x^2 + (r_1^2r_2r_3 + r_1r_2^2r_3 + r_1r_2r_3^2)x - r_1^2r_2^2r_3^2$$
$$= x^3 - c_2x^2 + c_1c_3x - c_3^2 = 0.$$

Comparing this equation with the equations of our original problem, we can apply the three formulae already derived for $c_{(1,n)}$ to find $c_{(2,n)}$ if we replace c_1 by c_2, c_2 by c_1c_3, and c_3 by c_3^2. For example, from Equation (4), if $c_1c_2c_3 \neq 0$, our formula for $c_{(2,n)}$ becomes

$$(5) \qquad r_1^nr_2^n + r_1^nr_3^n + r_2^nr_3^n = \sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^m n(n - m - 2k - 1)!}{(n - 2m - 3k)!\, m!\, k!} \times c_1^m c_2^{n-2m-3k} c_3^{2k+m}$$

In practice, when raising the roots of a given equation, it is simpler to utilize the method of iterating functions than to substitute into the formulae, especially as n becomes larger. An example worked by each method follows.

Given the equation

$$x^3 - 6x^2 + 11x - 6 = 0 \ ,$$

write the cubic whose roots are the fourth powers of the roots of the given equation, without solving for the roots.

(A) By substitution: From the table given earlier or from Equations (3) and (5), the desired cubic is

$$x^3 - (c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3)x^2 + (c_2^4 - 4c_2^2c_1c_3 + 2c_1^2c_3^2 + 4c_2c_3^2)x - c_3^4 = 0.$$

Substituting

$$c_1 = 6, \quad c_2 = 11, \quad c_3 = 6$$

yields

$$x^3 - 98x^2 + 1393x - 6^4 = 0$$

with roots 1^4, 2^4, and 3^4. As a check, the roots of the given equation are 1, 2, and 3.

(B) By iteration: To get $c_{(1,4)}$, we wish to write the sequence

$$c_{(1,0)},\ c_{(1,1)},\ c_{(1,2)},\ c_{(1,3)},\ c_{(1,4)}.$$

Now

$$c_{(1,0)} = 3,\quad c_{(1,1)} = c_1 = 6,\quad c_{(1,2)} = c_1^2 - 2c_2 = 36 - 22 = 14\,.$$

By the iteration relationship,

$$c_{(1,3)} = c_1c_{(1,2)} - c_2c_{(1,1)} + c_3c_{(1,0)} = 6(3) - 11(6) + 6(14) = 36\ ;$$

$$c_{(1,4)} = 6(6) - 11(14) + 6(36) = 98\ .$$

Similarly,

$$c_{(2,0)} = 3,\quad c_{(2,1)} = 11,\quad c_{(2,2)} = c_2^2 - 2c_1c_3 = 49.$$

Since

$$c_{(n,2)} = c_2c_{(2,n-1)} - c_1c_3c_{(2,n-2)} + c_3^2c_{(2,n-3)}$$

substitution yields

$$c_{(2,3)} = 251 \text{ and } c_{(2,4)} = 1393\ ,$$

yielding the same cubic as in (A).

Next let us turn our attention to several special cubic equations. First, for the cubic

$$x^3 - x^2 - x - 1 = 0,\qquad c_{(1,0)} = 3,\quad c_{(1,1)} = 1,\quad c_{(1,2)} = 3,$$

and our repeating multipliers in the iterative relationship are 1, 1, 1. Then,

$$c_{(1,3)} = 1(3) + 1(1) + 1(3) = 7, \quad c_{(1,4)} = 7 + 3 + 1 = 11,$$

$$\ldots, \quad c_{(1,n)} = c_{(1,n-1)} + c_{(1,n-2)} + c_{(1,n-3)} .$$

For this particular equation we have a species of Tribonacci numbers, any term after the third being the sum of the three preceding terms, with the entry terms 3, 1, 3. By Equation (4), the n^{th} term T_n in this Tribonacci sequence is

$$T_n = \sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^m n}{n - m - 3k} \binom{n - m - 2k - 1}{k} \binom{n - m - 3k}{m}$$

Notice that the sums of the coefficients in the table given for $c_{(1,n)}$ are these same numbers. It is interesting to recall that the special equation

$$x^2 - x - 1 = 0$$

led to a formula relating the n^{th} member of the Fibonacci sequence to a sum of binomial coefficients in the earlier study of the quadratic equation [1].

Considering the special equation

$$x^3 - x^2 + x - 1 = 0$$

with roots $1, \pm\sqrt{-1}$, we can write the following from Equation (4):

$$\sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^m n}{n - m - 3k} \binom{n - m - 2k - 1}{k} \binom{n - m - 3k}{m} = \begin{cases} 1 \text{ if } n \text{ is odd} \\ -1 \text{ if } n = 4s+2 \\ 3 \text{ if } n = 4s \end{cases}$$

Of more interest, however, are the following identities for the n^{th} Lucas number L_n, defined by

$$L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2} \quad .$$

We substitute in Equation (3), using

$$r_1 = \alpha = (1 + \sqrt{5})/2, \quad r_2 = \beta = (1 - \sqrt{5})/2 \quad ,$$

and letting r_3 vary. If $r_3 = 1$, Equation (3) cannot be used directly because

$$c_2 = \alpha\beta + \beta + \alpha = 0,$$

and 0^0 is not defined, But, by following the derivation for Equation (3), it is seen that, if $c_2 = 0$, $c_1c_3 \neq 0$,

$$\text{(3')} \qquad r_1^n + r_2^n + r_3^n = \sum_{k=0}^{[n/3]} \frac{n(n - 2k - 1)!}{(n - 3k)!\,k!} c_1^{n-3k} c_3^k$$

Since

$$c_1 = \alpha + \beta + 1 = 2, \quad c_3 = \alpha\beta = -1,$$

and

$$L_n = \alpha^n + \beta^n \quad ,$$

substitution gives

$$L_n + 1 = \sum_{k=0}^{[n/3]} \frac{(-1)^k 2^{n-3k} n(n - 2k - 1)!}{(n - 3k)!\ k!}$$

In general, if

$$r_3 = p, \quad p \neq 1, \quad p \neq -1, \quad p \neq 0 \ ,$$

Equation (3) gives

$$L_n + p^n = \sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^{m+k} n(n - m - 2k - 1)!\,(p + 1)^{n-2m-3k}(p-1)^m p^k}{(n - 2m - 3k)!\; m!\; k!}$$

Similarly, Equation (3') gives the following two identities using

$$r_1 = \alpha, \quad r_2 = \beta, \quad r_3 = -1/\sqrt{5}\ ,$$

and the known relationship for Fibonacci numbers,

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} \quad .$$

Below, n is taken to be $2s + 1$ and $2s$ respectively.

$$F_{2s+1} - 1/5^{s+1} = \sum_{k=0}^{[(2s+1)/3]} \frac{(2s + 1)\,(2s - 2k)!\,(-1)^k 4^{2s-3k+1}}{(2s - 3k + 1)!\,k!\,5^{s-k+1}}$$

$$L_{2s} + 1/5^s = \sum_{k=0}^{[2s/3]} \frac{2s(2s - 2k - 1)!\,(-1)^k\, 4^{2s-3k}}{(2s - 3k)!\; k!\; 5^{s-k}}$$

REFERENCES

1. N. A. Draim and M. Bicknell, "Sums of the n^{th} Powers of the Roots of a Given Quadratic Equation," The Fibonacci Quarterly, 4:2, April, 1966, pp. 170-178.
2. W. S. Burnside and A. W. Panton, An Introduction to the Theory of Binary Algebraic Forms, Dover, New York, 1960, Chapter 15.

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AMATEUR INTERESTS IN THE FIBONACCI SERIES II
CALCULATION OF FIBONACCI NUMBERS AND SUMS FROM THE BINOMIAL

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As mentioned in an earlier paper, my interest in the Fibonacci series stemmed from the observation (in 1959) that the preferred ratios developed in the research of my colleague, H. Ellner, and later included in Department of Defense Handbook H109 [1], were 1, 2, 3, 5, 8. When the supposition was tested, that all preferred ratios would come from the Fibonacci series, the next ratio was calculated and was found to be 13. Then it was noted that the sample sizes, Acceptable Quality Levels (AQL's), and lot size ranges of all sampling standards since the original work in this field by Dodge and Romig [2] were series approximately of the type:

$$u_{n+2} = u_{n+1} + u_n \tag{1}$$

It seemed self-evident that, in some way, the Fibonacci series must be intimately connected with some probability distribution such as the Binomial expansion. Through a little algebraic juggling such a connection was quickly established as follows:

The method of finite differences, described in Chrystal [3] yields interesting results. If successive differences be taken between adjacent Fibonacci numbers:

$$d_{1,n} = u_{n+1} - u_n \tag{2}$$

a series of first-order differences $d_{1,n}$ is generated. In the same way, a series of second-order differences may be generated:

$$d_{2,n} = d_{1,n+1} - d_{1,n} \tag{3}$$

Higher-order differences may be generated in accordance with the general relationship:

(4) $$d_{k,n} = d_{(k-1),(n+1)} - d_{(k-1),n}$$

Taking the Fibonacci series itself, u_n, to constitute the zero order of differences $d_{0,n}$ and if j is some given value of n, and k is the order of differences, we get the following table:

k	j = 1	2	3	4	5	6	7	8	9	10	11	12	13	14. . . etc.	
d_0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	u_n
d_1	0	1	1	2	3	5	8	13	21	34	55	89	144	233	
d_2	1	0	1	1	2	3	5	8	13	21	34	55	89	144	
d_3	-1	1	0	1	1	2	3	5	8	13	21	34	55	89	
d_4	2	-1	1	0	1	1	2	3	5	8	13	21	34	55	
d_5	-3	2	-1	1	0	1	1	2	3	5	8	13	21	34	
d_6	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21	
d_7	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	
d_8	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	
d_9	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	
d_{10}	34	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	
•															
•															
•															
d_k															

It may be seen that as k, the order of differences, and j are increased without limit, the table of d_k and j forms, both horizontally and vertically, four Fibonacci series centering on each zero, such that the two series, one above and one to the right of each zero are positive in all their terms while the series to the left and below each zero have alternate negative terms. In essence, the latter series constitute the negative branch of the Fibonacci series, u_{-n}.

We can calculate u_n, $n = k + j$, from the differences $d_{k,j}$ as follows:

(5) $$u_{k+j} = d_{0,j} + kd_{1,j} + \frac{k(k-1)}{2!} d_{2,j} + \frac{k(k-1)(k-2)}{3!} d_{3,j} + \cdots + \frac{k!}{k!} d_{k,j}$$

where $d_{k,j}$ is d_j of the order k as shown in the table and $u_j = d_{0,j}$. The coefficients of the $d_{k,j}$ terms represent those of the Binomial Expansion, $(a + b)^k$.

Example 1.

Calculate u_{k+j} when $k = 7$, and $j = 3$ $(u_{k+j} = u_{10})$

$$
\begin{aligned}
u_{7+3} &= d_{0,3} + 7d_{1,3} + \frac{7(6)}{2!} d_{2,3} + \frac{7(6)(5)}{3!} d_{3,3} + \cdots + \frac{7!}{7!} d_{7,3} \\
&= 2 + 7(1) + \frac{7(6)}{2}(1) + \frac{7(6)(5)}{3(2)}(0) + \frac{7(6)(5)(4)}{4(3)(2)}(1) \\
&\qquad + \frac{7(6)(5)(4)(3)}{5(4)(3)(2)}(-1) + \frac{7(6)(5)(4)(3)(2)}{(6)(5)(4)(3)(2)}(2) + \frac{7!}{7!}(-3) \\
&= 2 + 7 + \frac{42}{2}(1) + \frac{210}{6}(0) + \frac{840}{24}(1) + \frac{2520}{120}(-1) + \frac{5040}{720}(2) + 1(-3) \\
&= 2 + 7 + 21 + 0 + 35 - 21 + 14 - 3 = 55 = u_{10}\,.
\end{aligned}
$$

The sum of consecutive terms of the Fibonacci series is given by:

$$
\sum_{n=j}^{n=j+k-1} u_n = kd_{0,j} + \frac{k(k-1)}{2!} d_{1,j} + \frac{k(k-1)(k-2)}{3!} d_{2,j} + \cdots + \frac{k!}{k!} d_{k-1,j} \tag{6}
$$

Example 2.

Calculate the sum of $k = 9$ consecutive Fibonacci numbers starting with u_j $(j = 4)$.

$$
\begin{aligned}
\sum_{n=j}^{n=j+k-1} u_n = \sum_{n=4}^{n=9+4-1=12} u_n &= 9(3) + \frac{9 \times 8}{2}(2) + \frac{9 \times 8 \times 7}{3 \times 2}(1) + \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2}(1) \\
&\quad + \frac{9 \times 8 \times 7 \times 6 \times 5}{5 \times 4 \times 3 \times 2}(0) + \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4}{6 \times 5 \times 4 \times 3 \times 2}(1) + \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3}{7 \times 6 \times 5 \times 4 \times 3 \times 2}(-1) \\
&\quad + \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}(2) + \frac{9!}{9!}(-3) \\
&= 27 + 72 + 84 + 126 + 0 + 84 - 36 + 18 - 3 = 372
\end{aligned}
$$

$$
\sum_{n=4}^{n=12} u_n = 3 + 5 + 8 + 13 + 21 + 34 + 55 + 89 + 144 = 372
$$

It is noted that $d_{k,j} \equiv d_{0,k-j}$ such that when $j - k = \pm p$, d_0 is the same numerically and is positive when p is positive or when p is negative and odd. However, d_0 is negative when p is negative and even.

Since d_0, the zero-order of difference, is the same as the Fibonacci series, u_n, equations 5 and 6 may be written in terms of u_n provided reference is made to the proper sign of u_{j-k} when $j - k$ is negative. Thus, examining the table of d_k and j forms, it may be seen that $d_{0,j} = u_j$, $d_{1,j} = d_{0,j-1} = u_{j-1}$, etc. Hence, equations 5 and 6 may be recast as follows:

$$(7)^{\star} \quad u_{k+j} = u_j + ku_{j-1} + \frac{k(k-1)}{2!}u_{j-2} + \frac{k(k-1)(k-2)}{3!}u_{j-3} + \cdots + \frac{k!}{k!}u_{j-k}$$

and

$$(8)^{\star} \quad \sum_{n=j}^{n=j+(k-1)} u_n = ku_j + \frac{k(k-1)}{2!}u_{j-1} + \frac{k(k-1)(k-2)}{3!}u_{j-2} + \cdots + \frac{k!}{k!}u_{j-(k-1)}$$

*Provided the sign of u_{j-k} is:

Positive when $j - k$ is positive

Positive when $j - k$ is negative and odd

Negative when $j - k$ is negative and even.

Also, $u_0 = 0$.

Example 3. Let $j = 3$ and $k = 7$. Calculate $u_{k+j} = u_{10}$

$$u_{10} = u_3 + 7u_2 + \frac{7 \times 6}{2}u_1 + \frac{7 \times 6 \times 5}{3 \times 2}u_0 + \frac{7 \times 6 \times 5 \times 4}{4 \times 3 \times 2}u_{-1} + \frac{7 \times 6 \times 5 \times 4 \times 3}{5 \times 4 \times 3 \times 2}u_{-2}$$

$$+ \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2}{6 \times 5 \times 4 \times 3 \times 2}u_{-3} + \frac{7!}{7!}u_{-4}$$

$$= 2 + 7(1) + 21(1) + 35(0) + 35(1) + 21(-1) + 7(2) + 1(-3)$$

$$= 2 + 7 + 21 + 0 + 35 - 21 + 14 - 3 = 55 = u_{10}$$

Example 4. Let $j = 3$ and $k = 7$. Calculate

$$\sum_{n=j}^{n=j+(k-1)} u_n = \sum_{n=3}^{n=3+7-1=9} u_n$$

$$\sum_{n=3}^{n=9} u_n = 7u_3 + \frac{7 \times 6}{2} u_2 + \frac{7 \times 6 \times 5}{3 \times 2} u_1 + \frac{7 \times 6 \times 5 \times 4}{4 \times 3 \times 2} u_0 + \frac{7 \times 6 \times 5 \times 4 \times 3}{5 \times 4 \times 3 \times 2} u_{-1}$$

$$+ \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2}{6 \times 5 \times 4 \times 3 \times 2} u_{-2} + \frac{7!}{7!} u_{-3}$$

$$= 7(2) + 21(1) + 35(1) + 35(0) + 21(1) + 7(-1) + 1(2)$$

$$= 14 + 21 + 35 + 0 + 21 - 7 + 2 = 86$$

$$= u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + u_9$$

$$= 2 + 3 + 5 + 8 + 13 + 21 + 34 = 86$$

NB. The numbers in parentheses in Examples 3 and 4 are the numerical values appropriate for u_{j-k} with signs as provided above.

It is agreed that the above equations do not provide the least laborious way of calculating u_n or Σu_n but they do show that there is a relation between the Fibonacci series and the Binomial.

REFERENCES

1. Department of Defense Handbook H109, Statistical Procedures for Determining Validity of Suppliers' Attributes Inspection, 6 May 1960.
2. Dodge and Romig, Sampling Inspection Tables, Second Edition, 1959, John Wiley and Sons, Inc.
3. G. Chrystal, Textbook of Algebra, Second Edition, Chelsea Publishing House, Vol. II, p. 398.

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EXISTENCE OF ARBITRARILY LONG SEQUENCES OF CONSECUTIVE MEMBERS IN ARITHMETIC PROGRESSIONS DIVISIBLE BY ARBITRARILY MANY DIFFERENT PRIMES

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It is well known that there exist arbitrarily long sequences of consecutive positive integers that are all composite, e.g., $(n+1)!+2$, $(n+1)!+3, \cdots,$ $(n+1)!+(n+1)$. This statement can also be formulated thus: for any given positive integer n there exist n consecutive composite positive integers each of which has at least one prime divisor. The following is a twofold generalization of the last statement.

Theorem. In any infinite arithmetic progression

$$(1) \qquad ax+b, \quad a, b \text{ integers}, \quad a \neq 0, \quad x = 1, 2, 3, \cdots$$

and for any two positive integers, n, ν, there exist n consecutive members each of which is divisible by at least ν different primes.

Proof. (By induction on ν). Since $a \neq 0$, we have $a < 1$ or $a \leqq 1$. We may suppose, without loss of generality, $a \geq 1$, since if $a < 1$ we can consider the progression $-ax - b$, the members of which have the same absolute values as the corresponding members of (1). Thus for $x > (1-b)/a$, (1) is an increasing sequence of positive integers >1. Since any integer >1 is divisible by at least one prime, our statement is valid for $\nu = 1$. From the validity of the statement for ν we shall prove its validity for $\nu + 1$. As a matter of fact, let $2 \leq a_1 < a_2 < \cdots < a_n$ be n consecutive members of (1) each of which is divisible by at least ν different primes. Consider the sequence of n consecutive positive integers $(a_n)!^2a+a_1, (a_n)!^2a+a_2, \cdots, (a_n)!^2a+a_n$. For $2 \leqq a_1 \leqq a_k \leqq a_n$ we have

$$(a_n)!^2a + a_k = a_k\left[\frac{(a_n)!^2a}{a_k} + 1\right] = a_k\left[\frac{(2\cdot 3\cdot 4\cdots a_{k-1}\cdot a_k\cdot a_{k+1}\cdots a_n)(a_n)!\,a}{a_k} + 1\right]$$

$$= a_k\left[(2\cdot 2\cdot 4\ldots a_{k-1}\cdot a_{k+1}\cdots a_n)(2\cdot 3\cdot 4\cdots a_k\cdots a_n)a+\right.$$

The sum in brackets is composed of two terms, one divisible by a_k, the other being 1. Thus, this sum is coprime with a_k, and since it is greater than 1, it is divisible by a prime not dividing a_k. Hence $(a_n)!^2a + a_k$ is divisible by $\nu + 1$ different primes, for any $1 \leq k \leq n$. On the other hand, since a_k is a member of (1), thus of the form $ax + b$, we have $(a_n)!^2a + a_k \equiv b \pmod a$, thus $(a_n)!^2a + a_k$ is a member of (1), which completes the proof of the theorem.

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A PROPERTY OF LINEAR RECURSION RELATIONS

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If one selects the basic Fibonacci recursion relation,

(1) $$U_{n+2} - U_{n+1} - U_n = 0; \quad (n \geq 0) \quad U_0 = 0, \quad U_1 = 1 ,$$

and applies the well-known series transformation [1],

(2) $$y(t) = \sum_{0}^{\infty} U_k t^k / k! \quad ,$$

one obtains a linear differential equation with a characteristic equation,

(3) $$V_n \equiv n^2 - n - 1 = 0 .$$

It is easily verified that the recursion relation satisfied by $\{V_n\}$ is

(4) $$V_{n+2} - 2V_{n+1} + V_n = 2 .$$

It seemed reasonable to consider the relationship between (1) and (4). When the relationship was investigated, a rather unusual result was obtained. A characterization of recursion relations of polynomials was also obtained from the results. To carry out the investigation, it was expedient to introduce some terminology.

Let a linear recursion relation of order p with constant coefficients be denoted by

$$L_1(U) \equiv \sum_{i=n}^{n+p} a_{i-n} U_i = b \text{ for all } n \geq 1; \; a_p = 1 .$$

When we apply the series transform (2) to the above, we obtain

$$\sum_0^p a_i y^{(i)} = be^t \quad .$$

The characteristic equation of this differential equation is,

$$\sum_0^p a_i n^i = 0 \quad .$$

Let

$$V_n \equiv \sum_0^p a_i n^i ; \quad (n \geq 1) \quad V_0 = a_0 \quad .$$

Define

$$L_2(V) = \sum_{i=n}^{n+p} d_{i-n} V_i = C ; \quad d_p = 1 \quad ,$$

as the conjugate recursion relation of $L_1(U)$. The d's and C will shortly be determined explicitly. Since $d_p = 1$, it may be shown, as follows, that for fixed order p, $L_2(V)$ is unique.

Clearly for any particular recurrence relation, $L_1(U) = b$, $\{V_n\}$ is unique. Suppose $L_2(V)$ were not unique. Then

$$L_2'(V) = \sum_{i=n}^{n+p} d'_{i-n} V_i = C' \tag{5}$$

and

$$L_2''(V) = \sum_{i=n}^{n+p} d''_{i-n} V_i = C'' ; \quad d'_p = d''_p = 1 \quad , \tag{6}$$

are two distinct normalized recurrence relations for $\{V_n\}$. For any fixed p, the series transformed differential equations for (5) and (6), effected by (2), would be distinct p^{th} order linear differential equations with identical boundary conditions and the same solution,

$$y(t) = \sum_0^\infty V_i t^i / i! \quad .$$

Note that the series for $y(t)$ converges for all t, since $|V_n|$ is dominated by

$$\operatorname*{Max}_{i=0,\cdots,p} |a_i| \sum_0^p n^i$$

and thus $|V_n| = O(n^p)$.

By linearity properties of the solutions of linear differential equations with constant coefficients, this is impossible. Hence (5) and (6) are identical.

Thus $L_2(V)$ is the normalized recursion relation of order p, satisfied by $\{V_n\}$. We shall say that $L_1(U)$ is self-conjugate if and only if

$$L_1(U) = L_2(U) \quad .$$

Before we state and prove the central theorem, we shall need a lemma and two preliminary theorems.

<u>Lemma:</u>

$$\sum_{j=0}^p (-1)^j (n + p - j)^{p-k} \binom{p}{j} = \begin{cases} 0 & \text{if } k = 1, 2, \cdots, p \\ p! & \text{if } k = 0 \ . \end{cases}$$

The proof of the above is an elementary albeit tedious exercise in induction and is given as elementary problem, E1253, in [2].

Theorem 1:

$$V_n = \sum_0^p a_i n^i \; ; \quad a_p = 1$$

implies

$$J \equiv \sum_{j=0}^{p} (-1)^j \binom{p}{j} V_{n+p-j} = p!$$

Proof. If we use the polynomial expression for V_n, in J, the coefficient of a_{p-k} in J is

$$\sum_{j=0}^{p} (-1)^j (n + p - j)^{p-k} \binom{p}{j}$$

By the lemma,

$$J = p!\, a_p = p! \qquad \text{QED.}$$

Theorem 2:

$$L_1(U) = b; \quad a_p = 1$$

implies

$$L_2(V) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} V_{n+p-j} = p!$$

Proof. The characteristic equation of the series transform of $L_1(U)$ is

$$\sum_{0}^{p} a_i n^i = 0 .$$

Thus

$$V_n = \sum_{0}^{p} a_i n^i \qquad \text{QED.}$$

and by Theorem 1, the result follows.

Theorem 3: If $L_1(U)$ is of order p, then a necessary and sufficient condition that $L_1(U)$ be self-conjugate is that

$$L_1(U) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} V_{n+p-j} = p!$$

The proof follows from Theorem 2 and the uniqueness of $L_2(V)$.

In the light of the above we have

Corollary: Every polynomial of degree p has the same recursion relation and a recursion relation of the type given in Theorem 3 yields a polynomial expression in closed form.

If we choose $p = 2$, $a_1 = a_0 = -1$, $b = 0$ we obtain (1) above. The conjugate of (1) is (4). Thus the Fibonacci relation is not self-conjugate.

It would be interesting to see if there are other classes of functions which yield a fixed recursion relation for all members of some subclasses of the class of functions. Since the types of solutions of linear recursion formulae with constant coefficients are quite restricted, one would have to consider more general relations to obtain results of much consequence.

REFERENCES

1. James A. Jeske, "Linear Recurrence Relations — Part I," Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 69-74.
2. American Math. Monthly, Vol. 64, No. 8, October, 1957, p. 594.

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SIMULTANEOUS PRIME AND COMPOSITE MEMBERS IN TWO ARITHMETIC PROGRESSIONS

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Theorem. Any one of two non-identical infinite reduced arithmetic progressions has an infinitude of prime members the corresponding members of which in the other arithmetic progression are composite.

Proof. Be

(1) $\quad ax + b, \quad (a,b) = 1, \quad x = 1, 2, 3, \cdots$

(2) $\quad cx + d, \quad (c,d) = 1, \quad x = 1, 2, 3, \cdots \qquad a \neq c$ or $b \neq d$

two non-identical infinite arithmetic progressions. We may suppose, without loss of generality, $a \geq 1$, $c \geq 1$, since if, say, $a < 1$, we can consider the progression $-ax - b$, the members of which have the same absolute values as the corresponding members of (1). Suppose, contrary to the assertion of the theorem, that one of the progressions, say (1), has only a finite number of prime members the corresponding members of which in the other progression are composite. Thus, there is a positive integer N such that $N > |d|$ in case $a = c$, and

$$N > \max\left(\frac{d - b}{a - c}, |d|\right)$$

in case $a \neq c$, and such that, for any positive integer $x > N$, $cx + d$ is a prime if $ax + b$ is a prime. By Dirichlet, (1) has an infinitude of prime members. Hence, there is a positive integer $x_0 > N$ such that $ax_0 + b$ is a prime, whence, by the assumption, also $cx_0 + d$ is a prime. If $a = c$, then $b \neq d$, and $ax + b \neq cx + d$ for any x. If $a \neq c$, then $ax + b - cx + d$ only for $x = (d - b)/a - c)$. Since $x_0 > N > (d - b)/(a - c)$ we have $ax_0 + b \neq cx_0 + d$. Thus the arithmetic progression $a(cx_0 + d)x + (ax_0 + b)$, $x = 1, 2, 3, \cdots$, is reduced. Hence, by Dirichlet, there is a positive integer x_1 such that $a(cx_0 + d)x_1 + (ax_0 + b)$ is a prime. Now put $x_2 = (cx_0 + d)x_1 + x_0$. Since $c \geq 1$, $x_0 > N > |d|$, $x_1 \geq 1$, we have $x_2 > x_0 > N$. Thus $ax_2 + b = a(cx_0 + d)x_1 + (ax_0 + b)$ is a prime with $x_2 > N$, while $cx_2 + d = c(cx_0 + d)x_1 + (cx_0 + d) = (cx_0 + d)(cx_1 + 1)$ is evidently composite with $x_2 > N$, since both $cx_0 + d$, being a prime, and $cx_1 + 1$, with $c \geq 1$, $x_1 \geq 1$, are integers > 1. The contradiction to the assumption thus obtained proves the theorem.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-118 Proposed by J. L. Brown, Jr., Pennsylvania State University, State College, Pa.

Let $F_1 = 1 = F_2$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. Show for all $n \geq 1$ that

$$\sum_{k=1}^{n} (F_k/2^k) < 2 .$$

B-119 Proposed by Jim Woolum, Clayton Valley High School, Concord, Calif.

What is the area of an equilateral trapezoid whose bases are F_{n-1} and F_{n+1} and whose lateral side is F_n ?

B-120 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Find a simple function g such that $g(n)$ is an integer when n is an integer and $g(m + n) - g(m) - g(n) = mn$.

B-121 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let $n, q, d,$ and r be integers with $n \geq 0$, $d > 0$, $n = qd + r$, and $0 \leq r < d$. Prove that

$$F_n \equiv (F_{d+1})^q F_r \pmod{F_d} .$$

B-122 Proposed by A.J. Montleaf, Univ. of New Mex., Albuquerque, N. Mex.

Show that

$$\sin\left[(2k+1)\theta\right]\Big/\sin\theta = 2\cos\left[2k\theta\right] + 2\cos\left[2(k-1)\theta\right] + 2\cos\left[2(k-2)\theta\right] + \cdots + 2\cos\left[2\theta\right] + 1$$

and obtain the analogous formula for $F_{(2k+1)m}/F_m$ in terms of Lucas numbers.

B-123 (From B-102, Proposed by G. L. Alexanderson, Univ. of Santa Clara, Santa Clara, California.

Show that all the positive integral solutions of $x^2 + (x \pm 1)^2 = z^2$ are given by

$$x_n = (P_{n+1})^2 - (P_n)^2\,;\quad z_n = (P_{n+1})^2 + (P_n)^2\,;\quad n = 1, 2, \cdots\,;$$

where P_n is the Pell number defined by $P_1 = 1$, $P_2 = 2$, and $P_{n+2} = 2P_{n+1} + P_n$.

SOLUTIONS

A NON-HOMOGENEOUS DIFFERENCE EQUATION

B-100 Proposed by J. A. H. Hunter, Toronto, Canada.

Let $u_{n+2} = u_{n+1} + u_n - 1$, with $u_1 = 1$ and $u_2 = 3$. Find the general solution for u_n.

Solution by F. D. Parker, St. Lawrence University, Canton, N.Y.

The general solution of the difference equation is $u_n = c_1a^n + c_2b^n + 1$, where c_1 and c_2 are arbitrary constants, $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$. Since $u_1 = 1$ and $u_2 = 3$, we find the particular solution to be

$$u_n = \frac{2}{\sqrt{5}}\,a^{n-1} - \frac{2}{\sqrt{5}}\,b^{n-1} + 1 = 2F_{n-1} + 1\ .$$

Also solved by L. Carlitz; Herta T. Freitag; William T. Jackson; Douglas Lind; William C. Lombard; C.B.A. Peck; Lt. A.G. Shannon, R.A.N; David Zeitlin; and the proposer.

A SEQUENCE OF SEQUENCES

B-101 Proposed by Thomas P. Dence, Bowling Green State Univ., Bowling Green, Ohio.

Let $x_{i,n}$ be defined by $x_{1,n} = 1$, $x_{2,n} = n$, and $x_{i+2,n} = x_{i+1,n} + x_{i,n}$. Express $x_{i,n}$ as a function of F_n and n.

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia

We claim $x_{i,n} = F_i + (n-1)F_{i-1}$. This is clearly true for $i = 1, 2$ and all n. Since both expressions obey the same second-order recurrence relation in i and agree in the first two values, they must coincide for all i and n.

Also solved by Gerald Edgar, Herta T. Freitag, William C. Lombard, John W. Milson, F. D. Parker, David Zeitlin, and the proposer.

<u>NOTE</u>: The problem editor misstated the problem as "Express $x_{i,n}$ in terms of F_n and n" instead of "Express $x_{i,n}$ in terms of n and F_i." The proposer intended that F_{i-1} in the solution printed above be expressed in terms of F_i, as one might do, for example, using the result of B-42.

PELL-PYTHAGOREAN TRIPLES

B-102 Proposed by Gerald L. Alexanderson, Univ. of Santa Clara, Santa Clara, Calif.

The Pell sequence $1, 2, 5, 12, 29, \cdots$ is defined by $P_1 = 1$, $P_2 = 2$ and $P_{n+2} = 2P_{n+1} + P_n$. Let $(P_{n+1} + iP_n)^2 = x_n + iy_n$, with x_n and y_n real and let $z_n = |x_n + iy_n|$. Prove that the numbers x_n, y_n, and z_n are the lengths of the sides of a right triangle and that x_n and y_n are consecutive integers for every positive integer n. Are there any other positive integral solutions of $x^2 + (x \pm 1)^2 = z^2$ than $(x, z) = (x_n, z_n)$?

Solution by Herta T. Freitag, Hollins, Virginia.

(A) $z_n = |x_n + iy_n| = \sqrt{x_n^2 + y_n^2}$; hence x_n, y_n, and z_n may be interpreted as lengths of the sides of a right triangle.

(B) To show that $y_n - x_n = 1$:
Since $x_n = P_{n+1}^2 - P_n^2$ and $y_n = 2P_{n+1}P_n$, we need to show that

$$\left|2P_{n+1}P_n - P_{n+1}^2 + P_n^2\right| = 1 .$$

Proof by mathematical induction:

(1) $\left|2P_2P_1 - P_2^2 + P_1^2\right| = 1$, hence the statement is correct for $n = 1$.

(2) Assume the formula correct for $n = k$, i.e., assume that:

$$\left|2P_{k+1}P_k - P_{k+1}^2 + P_k^2\right| = 1 .$$

Then,

$$\begin{aligned}\left|2P_{k+2}P_{k+1} - P_{k+2}^2 + P_{k+1}^2\right| &= \left|2(2P_{k+1} + P_k)P_{k+1} - (2P_{k+1} + P_k)^2 + P_{k+1}^2\right| \\ &= \left|P_{k+1}^2 - 2P_kP_{k+1} - P_k^2\right| = 1 .\end{aligned}$$

This, however, means that correctness of the statement for $n = k$ causes its correctness for $n = k + 1$, and the query is settled.

(C) No, there are no other positive integral solutions of $x^2 + (x \pm 1)^2 = z^2$ than $(x, z) = (x_n, z_n)$. This, however, is only a hunch; I was unable to establish the unicity.

Also solved by the proposer.

NOTE: See proposed problem B-123.

AN INCREASING SEQUENCE

B-103 Proposed by Douglas Lind, Univ. of Virginia, Charlottesville, Va.

Let

$$a_n = \sum_{d|n} F_d \quad (n \geq 1),$$

where the sum is over all divisors d of n. Prove that $\{a_n\}$ is a strictly increasing sequence. Also show that

$$\sum_{n=1}^{\infty} \frac{F_n x^n}{1 - x^n} = \sum_{n=1}^{\infty} a_n x^n .$$

Solution by Gerald Edgar, Boulder, Colorado.

For $n \geq 1$, we have

$$a_{n+1} = \sum_{d|(n+1)} F_d \geq F_1 + F_{n+1} > F_{n+1} .$$

Observe that

$$\begin{aligned} a_1 &= 1 = F_2 , \\ a_2 &= 2 = F_3 , \\ a_3 &= 3 = F_4 , \end{aligned}$$

and that for $n > 3$, since $(n - 1) \nmid n$ and $(n - 2) \nmid n$,

$$a_n = \sum_{d|n} F_d \leq F_n + \sum_{i=1}^{n-3} F_i = F_n + F_{n-1} - 1 < F_{n+1}$$

so that in all cases for $n \geq 1$, we have $a_n \leq F_{n+1}$.

Therefore, for all $n \geq 1$, $a_n < a_{n+1}$, so that $\{a_n\}$ is a strictly increasing sequence. Also, we have

$$\begin{aligned} \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} \left(\sum_{d|n} F_d \right) x^n \\ &= \sum_{d=1}^{\infty} \left(\sum_{d|n}^{\infty} x^n \right) F_d \quad \text{(rearranging terms)} \\ &= \sum_{d=1}^{\infty} \left(\sum_{i=1}^{\infty} x^{id} \right) F_d \\ &= \sum_{d=1}^{\infty} \left(\frac{x^d}{1 - x^d} \right) F_d \\ &= \sum_{d=1}^{\infty} \frac{F_d x^d}{1 - x^d} \end{aligned}$$

Also solved by the proposer.

TELESCOPING SERIES

B-104 Proposed by H. H. Ferns, Victoria, British Columbia.

Show that

$$\sum_{n=1}^{\infty} \frac{F_{2n+1}}{L_n L_{n+1} L_{n+2}} = \frac{1}{3} \quad ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively.

Solution by L. Carlitz, Duke University, Durham, N.C.

It is easily verified that

$$F_{n+1}L_{n+2} - F_{n+2}L_n = F_{2n+1} \; .$$

Thus

$$\sum_{n=1}^{N} \frac{F_{2n+1}}{L_n L_{n+1} L_{n+2}} = \sum_{n=1}^{N} \left(\frac{F_{n+1}}{L_n L_{n+1}} - \frac{F_{n+2}}{L_{n+1} L_{n+2}} \right) = \frac{F_2}{L_1 L_2} - \frac{F_{N+2}}{L_{N+1} L_{N+2}}$$

and therefore

$$\sum_{n=1}^{\infty} \frac{F_{2n+1}}{L_n L_{n+1} L_{n+2}} = \frac{1}{3} \; .$$

Also solved by Douglas Lind, F. D. Parker, Lt. A. G. Shannon, David Zeitlin, and the proposer.

A PERIODIC SEQUENCE

B-105 Proposed by Phil Mana, University of New Mex., Albuquerque, New Mex.

Let g_n be the number of finite sequences $c_1, c_2, \cdots, c_n$, with $c_1 = 1$, each c_i in $\{0,1\}$, (c_i, c_{i+1}) never $(0,0)$, and (c_i, c_{i+1}, c_{i+2}) never $(0,1,0)$.

Prove that for every integer $s > 1$ there is an integer t with $t \leq s^3 - 3$ and g_t an integral multiple of s.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

Acceptable sequences of length n can be produced by appending a "1" to all sequences of length $n - 1$, and a "110" to those of length $n - 3$. Then all n-sequences not included are not acceptable since they violate the given restraints. It follows that $g_n = g_{n-1} + g_{n-3}$. Put $I_k = (g_k, g_{k+1}, g_{k+2})$. Each I_k determines the entire sequence g_n by using the above recurrence relation. Thus modulo $s > 1$, if $I_j \equiv I_k$, then $\{g_n\}$ is periodic with period $\leq |j - k|$. Now there are $(s - 1)^3$ possible distinct triplets (a, b, c) modulo s such that $a, b, c \not\equiv 0 \pmod{s}$. Also $(s - 1)^3 < s^3 - 5$ for $s > 1$. Thus either one of $I_1, I_2, \cdots, I_{s-5}$ contains a 0, in which case there is a $t \leq s^3 - 3$ such that $g_t \equiv 0 \pmod{s}$, or $I_j \equiv I_k \pmod{s}$ for some $j, k \leq s^3 - 3$ with $j \neq k$. But then $\{g_n\}$ has period $t = |j - k| > 0$, so $g_t \equiv g_0 = 0 \pmod{s}$, where here $t < s^3 - 3$.

Also solved by Robert L. Mercer and the proposer.

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