# FIBONACCI NUMBERS AND SOME PRIME RECIPROCALS

R. S. BUCKNELL
Chiangmai, Thailand

The series of Fibonacci numbers has been shown to bear some interesting relationships to the reciprocals of certain prime numbers. For instance, Maxey Brooke and C. R. Wall set up as Problems B-14 (Fibonacci Quarterly, Vol. 1 (1963), No. 2, p. 86) to show

$$\sum_{n=1}^{\infty} F_n 10^{-n} = 10/89 , \tag{1}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} F_n 10^{-n} = 10/109 , \tag{2}$$

where $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$.

The relationship (1) is but a special case of the general property

$$\sum_{n=1}^{\infty} F_n x^{-n-1} = \frac{1}{x^2 - x - 1} \tag{3}$$

where $x$, an integer $>2$, is the radix in terms of which the number $(x^2 - x - 1)$ and the Fibonacci numbers are expressed. Equation (3) is readily proved by considering

$$\begin{aligned}
(x^2 - x - 1) \sum_{n=1}^{\infty} F_n x^{-n-1} &= x^2 \sum_{n=1}^{\infty} F_n x^{-n-1} - x \sum_{n=1}^{\infty} F_n x^{-n-1} - \sum_{n=1}^{\infty} F_n x^{-n-1} \\
&= \sum_{n=1}^{\infty} F_n x^{-n+1} - \sum_{n=1}^{\infty} F_n x^{-n} - \sum_{n=1}^{\infty} F_n x^{-n-1} \\
&= \left[ \sum_{n=1}^{\infty} F_{n+2} x^{-n-1} + F_1 + F_2 x^{-1} \right] - \left[ \sum_{n=1}^{\infty} F_{n+1} x^{-n-1} + F_1 x^{-1} \right]
\end{aligned}$$

$$- \left[ \sum_{n=1}^{\infty} F_n x^{-n-1} \right] = 1$$

since

$$F_1 = F_2 = 1$$

and

$$\sum_{n=1}^{\infty} F_{n+2} x^{-n-1} - \sum_{n=1}^{\infty} F_{n+1} x^{-n-1} - \sum_{n=1}^{\infty} F_n x^{-n-1} = 0 \; .$$

Note that $x^2 - x - 1$ may be composite (e. g., for $x = 8, 13$), but that $x^2 - x - 1$ and $x$ are relatively prime.

Another interesting relationship that has been discovered to exist between the series of Fibonacci numbers and the number $1/N = 1/(x^2 - x - 1)$ is exemplified by the special case where $N = 109$. It is found that the 108-digit period of 1/109 is

0091743119266055045871559633027522935779816513761467889909256880733944-
95412844036697247706422018348623853211 .

A generalization is possible for the number $N - x^2 - x - 1 - y^2 + y - 1$ ($x$ an integer $>2$, $y = x - 1$) when $1/N$ is expanded in terms of radix $y$. It will be shown that if the period of $1/N$ is the $(N - 1)$-digit number

$$P = \frac{y^{N-1} - 1}{N} \; ,$$

then the number

$$\sum_{n=1}^{N-1} F_n y^{n-1}$$

has as its last $N - 1$ digits the number $P$.

Let the residue, $R$, be defined by

$$R = \sum_{n=1}^{N-1} F_n y^{n-1} - P \quad . \tag{4}$$

The expression

$$\sum_{n=1}^{N-1} F_n y^{n-1}$$

will be summed using

$$F_n = \frac{a^n - b^n}{a - b} ,$$

where

$$a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2} \quad .$$

Then

$$\begin{aligned}
\sum_{n=1}^{N-1} F_n y^{n-1} &= \sum_{n=1}^{N-1} \left[\left(\frac{a^n - b^n}{a - b}\right) \cdot y^{n-1}\right] \\
&= \sum_{n=1}^{N-1} \left[(ay)^n - (by)^n\right]/(a - b)y \\
&= \left[\frac{1 - (ay)^N}{1 - ay} - \frac{1 - (by)^N}{1 - by}\right] \Bigg/ (a - b)y \\
&= \frac{y^N F_{N-1} + y^{N-1} F_N - 1}{y^2 + y - 1} - \frac{y^{N-1} - 1}{y^2 + y - 1} \\
&= \frac{y^{N-1}}{y^2 + y - 1}\left[yF_{N-1} + F_N - 1\right] \quad .
\end{aligned}$$

Now consider the term

$$\frac{y^{N-1}}{y^2 + y - 1}$$

Clearly no factor of y will divide $y^2 + y - 1$, hence $y^{N-1}$ and $y^2 + y - 1$ are relatively prime, and since R is an integer, $y^{N-1}$ divides R. Thus R is a number ending in N - 1 zeros when expressed in terms of radix y. Since P contains not more than N - 1 digits, it follows that the number

$$\sum_{n=1}^{N-1} F_n y^{n-1} = P + R$$

has as its last (N - 1) digits the number P.

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# RESIDUES OF FIBONACCI-LIKE SEQUENCES

LAURENCE TAYLOR
Oak Ridge, Tennessee

In the February, 1964, issue of the Fibonacci Quarterly, Brother U. Alfred [1] advanced the conjecture (later proved by J. H. Halton [2]) that, when any Fibonacci number is divided by another Fibonacci number, one or the other of the least positive and negative residues is again a Fibonacci number. The object of this paper is to prove that the only Fibonacci-like sequence for which this is true is the Fibonacci sequence. If zero is excluded as a remainder, then the Lucas sequence has the above property.

The proof falls naturally into two parts. The first part will be to show that every Fibonacci-like sequence, modulo any member of the sequence, is congruent to a sequence made up of a subsequence of the original sequence and the negatives of these values. The second part will be to show that these subsequences are actually remainders of the divisor for only the Fibonacci and Lucas sequences.

Obviously, a sequence has the property described above if and only if any non-zero integral multiple of it does. Since any divisor of two neighboring members of Fibonacci-like sequences divides every member of the sequence, we will consider only sequences with neighboring terms relatively prime. In what follows, $H_i$ will denote the $i^{th}$ member of a general Fibonacci-like sequence defined by $H_{i+1} = H_i + H_{i-1}$, where $H_0$ and $H_1$ are arbitrary. The set of integers will be denoted by I, the set of non-negative integers by P, and the set of natural numbers by N.

## PART I

Since it is easily established by induction that

$$H_{m+k} = F_k H_{m-1} + F_{k+1} H_m ,$$

for all integers m and k, the following two lemmas readily follow.

Lemma 1: $H_{m+i} \equiv F_i H_{m-1} \pmod{H_m}$ for all integers i.

Lemma 2: $H_{m-i} \equiv F_{-i} H_{m-1} \equiv (-1)^{i+1} H_{m+i} \pmod{H_m}$ for all integers i.

It is known that any number must eventually divide one of the Fibonacci numbers, and that $F_{n-1}^2 = F_{n-2}F_n + (-1)^n$ for all integers n. Applying these results and Lemma 1, it is not difficult to prove Lemmas 3 and 4.

Lemma 3: Let n be any integer such that $F_n \equiv 0 \pmod{H_m}$. Then

$$H_{m\pm n} \equiv 0 \pmod{H_m} .$$

Lemma 4: For the n of Lemma 3, $F_{n-1}^2 \equiv (-1)^n \pmod{H_m}$.

Lemma 5: For the n of Lemma 3, and for all integers i,

$$F_{n-1}H_{m-i} \equiv (-1)^{n+i+1}H_{m-n+i} \pmod{H_m} .$$

Proof: The proof is by induction on i. For i = 0, apply Lemma 3. For i = 1, apply Lemma 1. Assume that Lemma 5 holds for i = k - 1 and i = k - 2, or that

$$F_{n-1}H_{m-(k-1)} \equiv (-1)^{n+k}H_{m-n+(k-1)} \pmod{H_m} ,$$

$$F_{n-1}H_{m-(k-2)} \equiv (-1)^{n+k-1}H_{m-n+(k-2)} \pmod{H_m} .$$

Subtracting the first formula from the second yields the expected result for i = k. Hence, the formula is correct for all i ∈ P. Lemma 2 can be used to extend the result to include negative integers.

Lemma 6: Let t = nq + r. Then, if q ∈ N and $F_n \equiv 0 \pmod{H_m}$,

$$H_{m-n+t} \equiv F_r F_{n-1}^{q-1} H_{m-1} \pmod{H_m} .$$

Proof: The proof is once again by induction on q. When q = 1, the expression above becomes identical to Lemma 1. Assume that Lemma 6 holds for q = k - 1, or that

$$H_{m-n+t} \equiv F_{t-(k-1)n} F_{n-1}^{k-2} H_{m-1} \pmod{H_m} .$$

But, $F_{t-(k-1)n} = F_{t-kn}F_{n-1} + F_{t-kn+1}F_n \equiv F_{t-kn}F_{n-1} \pmod{H_m}$, since $H_m$ divides $F_n$ by hypothesis. Substituting back into the formula above,

$$H_{m-n+t} \equiv F_{t-kn}F_{n-1}^{k-1}H_{m-1} \pmod{H_m}.$$

Hence, Lemma 6 is true for all $q \in N$.

Theorem 1: For every $i \in I$, there exists a $k \in I$, $m - n \leq k \leq m$, such that

$$H_i \equiv \pm H_k \pmod{H_m},$$

where $n$ is the smallest natural number such that $F_n \equiv 0 \pmod{H_m}$.

Proof: Let $i = m - n + t$, $k = m - n + r$, and $t = nq + r$, $0 \leq r \leq n$. The case $q = 0$ is trivial, since then $t = p$ and $i = k$. The case $q < 0$ is equivalent to $t < 0$. But, by Lemma 2 and properties of congruences,

$$H_{m-n+t} \equiv (-1)^{t+1}H_{m-n+(-t)} \pmod{H_{m-n}} \equiv (-1)^{t+1}H_{m-n+(-t)} \pmod{H_m}.$$

Since $-t > 0$, we need consider only the case $t > 0$ or $q \in N$. By Lemma 6,

$$H_{m-n+t} \equiv F_rF_{n-1}^{q-1}H_{m-1} \pmod{H_m}.$$

By Lemma 1,

$$F_rH_{m-1} \equiv (-1)^{r+1}H_{m-r} \pmod{H_m}.$$

Substituting,

$$H_{m-n+t} \equiv (-1)^{r+1}F_{n-1}^{q-1}H_{m-r} \pmod{H_m}.$$

By Lemma 4,

$$F_{n-1}^2 \equiv (-1)^n \pmod{H_m}.$$

We must now distinguish two cases.

Case 1: If $q$ is odd,

$$F_{n-1}^{q-1} = (-1)^{n(q-1)/2} \pmod{H_m},$$

leading to $H_{m-n+t} = \pm H_{m-r} \pmod{H_m}$, where $m - n \leq m - r \leq m$.

Case 2: If $q$ is even,

$$F_{n-1}^{q-1} = (-1)^{n(q-2)/2} F_{n-1} \pmod{H_m}.$$

By Lemma 5,

$$F_{n-1} H_{m-r} \equiv (-1)^{n+r+1} H_{m-n+r} \pmod{H_m}.$$

Substituting these two results leads to

$$H_{m-n+t} \equiv (-1)^{nq/2} H_{m-n+r} \pmod{H_m}.$$

where $0 \leq r \leq n$, so $m - n \leq m - n + r \leq m$.

In Theorem 1, if $H_m$ divides $H_i$, we can take $k = m$ or $k = m - n$. While every $H_i$ divides some other member of the sequence (see Lemma 3), it is necessary to notice that zero cannot appear as a member of the subsequence of Theorem 1 unless our Fibonacci-like sequence is the Fibonacci sequence itself. Since zero can occur as a remainder in any Fibonacci-like sequence and since Theorem 1, applied to Fibonacci numbers, leads to the theorem proved by Halton in [2], the only Fibonacci-like sequence which strictly fulfills the requirements of Brother Alfred's conjecture is the Fibonacci sequence.

In Part II, we will investigate Fibonacci-like sequences to determine if any other sequence leaves residues which, in all cases, are either zero or equal in absolute value to members of the original sequence.

## PART II

Now, if our sequence is to have the desired property, there must be a set of elements of the sequence whose absolute values are less than that of $H_m$. The first observation to be made about Fibonacci-like sequences is that far to the right and to the left, the absolute values increase without limit. Hence, we need only examine a small section of the whole sequence to determine if it has the desired property.

There must be at least one $H_i$ with a minimal absolute value, and, because of the divergence of the sequence in both directions, there can be only a finite number of such minima.

Lemma 7: If $H_0$ is a minimum, $|H_0| \geq 2$, then, if $H_1 > 0$, the only possible remainder equal in absolute value to a member of the original sequence upon division by $H_{-2}$ is $\pm H_0$, and if $H_1 < 0$, the only such remainder for $H_2$ is $\pm H_0$.

Proof: If $H_0$ is negative, we will obtain the negative of the sequence for $H_0$ positive. Hence, consider only $H_0 \geq 2$. None of $H_1$, $H_2$, $H_{-1}$, $H_{-2}$ can be a minima, since each of $|H_i| = H_0 \geq 2$, $i = \pm 1, \pm 2$, leads to a contradiction.

If $H_1 > 0$, to avoid $|H_i| < H_0$ for some $i$, for the terms near $H_0$ we can have only the following:

$$
\begin{aligned}
H_{-3} &= 3H_0 + 2\alpha = H_1 + \alpha \\
H_{-2} &= -(H_0 + \alpha) \\
H_{-1} &= 2H_0 + \alpha \\
H_0 &= H_0 \\
H_1 &= 3H_0 + \alpha \\
H_2 &= 4H_0 + \alpha \\
&\dots \\
H_i &= L_{i+1}H_0 + F_i\alpha , \qquad \alpha \geq 1 ,
\end{aligned}
$$

where $L_n$ and $F_n$ are respectively the $n^{th}$ Lucas and Fibonacci numbers.

If $H_1 < 0$, with the conditions above we obtain

$$H_i = (-1)^i(L_{i+1}H_0 + F_i\alpha) ,$$

or a new sequence which, except for changes in sign, is the sequence for $H_1 > 0$ reflected about $H_0$. In particular, $H_2 = -(H_0 + \alpha)$.

Notice that the sequence diverges for $|i| > 2$. From the sequence above, it is easy to see that the only remainder in the sequence for $H_{-2}$ will be $\pm H_0$ when $H_1 > 0$, and when $H_1 < 0$, the only remainder for $H_2$ will be $\pm H_0$.

Lemma 8: If $H_0$ is a minimum, $|H_0| \geqq 1$, and neither $H_2$ nor $H_{-2}$ is a minimum, then the only remainder equal in absolute value to a member of the original sequence upon division by $H_{-2}$ is $\pm H_0$ when $H_1 > 0$, and the only such remainder for $H_2$ is $\pm H_0$ when $H_1 < 0$.

Proof: Avoiding $|H_2| = |H_0|$ and $|H_{-2}| = |H_0|$ as well as $|H_i| < |H_0|$ leads to the formulae of Lemma 7.

Lemma 9: If $H_0$ is a minimum, $|H_0| \geqq 2$, then there exist numbers $H_i$ which leave remainders which are neither zero nor equal in absolute value to a member of the original sequence.

Proof: If any number $H_j$ is divided by $H_0$, the remainder must be less in absolute value than $H_0$, the minimum of the sequence. Thus, if $|H_0| \geqq 2$, all remainders cannot be zero because any two adjacent terms are relatively prime, and any non-zero remainder is a number not equal in absolute value to a member of the original sequence. So $H_0$ is a number $H_i$ for the lemma.

Suppose we exclude division by $H_0$. Since $(H_0, H_1) = 1$, $H_1$ is not a minimum. Either $H_1$ is positive or $H_1$ is negative. Without loss of generality (see proof of Lemma 7), we assume that $H_1$ is negative. By Theorem 1, if $n_2$ is the least natural number such that $F_{n_2} \equiv 0 \pmod{H_2}$, and if $t = qn_2 + r$, $0 \le r < n_2$, for $q$ an odd number,

$$H_{2-n_2+t} \equiv \pm H_{2-r} \pmod{H_2}.$$

Now, $H_{2-r} = H_0$ if and only if $r = 2$. If $|H_0| \ge 2$, $|H_2| \ge 3 = F_4$, so $n_2 \geqq 4$, and at least $0 \leqq r < 4$. Set $t = qn_2 + 3$ for an odd number $q$, say $q = 1$. Substituting, we have $H_5 \equiv \pm H_{-1} \pmod{H_2}$, and $\pm H_{-1} \not\equiv \pm H_0 \pmod{H_2}$ by inspecting the proof of Lemma 7. Thus, we can take $i = 2$.

Lemma 10: If $|H_0| = 1$ is a minimum, and neither $H_2$ nor $H_{-2}$ is a minimum, then there exist numbers $H_i$ which leave remainders which are neither zero nor equal in absolute value to a number in the original sequence.

Proof: Without loss of generality, we assume that $H_1 < 0$. If $|H_2| \ge 3$, so that $n_2 \geqq 4$, by Lemma 8 we can use the same proof as for Lemma 9. Since $H_2$ is not a minimum, $H_2 \ne 1$ and $H_2 \ne -1$. The only remaining case is when $|H_2| = 2$, which leads only to the following sequence,

$$\cdots, -23, 14, -9, 5, -4, 1, -3, -2, -5, -7, -12, -19, -31, -50, \cdots,$$

where $31 \equiv 8 \pmod{-23}$ while neither $\pm 8$ nor $\pm 15$ is in the original sequence.

Theorem 2: The only sequences which possess the property that, upon division by a (non-zero) member of that sequence, the members of the sequence leave least positive or negative residues which are either zero or equal in absolute value to a member of the original sequence are the Fibonacci and Lucas sequences.

Proof: By Lemmas 9 and 10, for a sequence to possess the above property, its minimum must be either $H_0 = 0$ or $|H_0| = 1$ with one of $H_2$ and $H_{-2}$ also a minimum.

If $H_0 = 0$, we can have only the Fibonacci sequence.

Considering the cases $|H_0| = 1$ and $|H_2| = 1$; $|H_0| = 1$ and $|H_{-2}| = 1$, leads to the Lucas sequence and the negative of the Lucas sequence.

It can be shown that, since when Theorem is applied to Lucas numbers, for each $L_k$, $|L_k| < |L_m|$ or $L_k = 0 \pmod{L_m}$, that the Lucas numbers do indeed have the property of Theorem 2. The Fibonacci numbers are known to also have this property, as proved by Halton in [2].

We have used a minimum value greater than 2 as a criterion to determine if there exist numbers $H_i$ which leave remainders which do not satisfy Theorem 2. Another criterion is that such numbers $H_i$ exist if and only if $|H_j| \neq |H_{-j}|$ for any j, where the sequence has been renumbered so that either $H_0$ is the minimum or $H_0$ is between the two minima $H_1$ and $H_{-1}$. This second criterion requires a longer proof, but not a difficult one, done by examining all cases.

Examining several sequences to aid in the formulation of the proofs given here led to an interesting question. If Brother Alfred's conjecture is not true for a whole sequence, can it be true for some elements of the sequence, and if so, which ones?

## REFERENCES

1. Brother U. Alfred, "Exploring Fibonacci Residues," Fibonacci Quarterly, Vol. 2, No. 1, Feb., 1964, p. 42.
2. J. H. Halton, "On Fibonacci Residues," Fibonacci Quarterly, Vol. 2, No. 3, Oct., 1964, pp. 217-218.

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# ON m-TIC RESIDUES MODULO n

JOHN H. E. COHN
London, N. W., England

## 1. INTRODUCTION

The object of this paper is to investigate the values of the residues modulo $n$ of $x^m$, where $0 \le x \le (n-1)$, and in particular for the case $n = m$. We shall define

$$\Sigma(n;m) = \{x^m \pmod{n} \mid 0 \le x \le (n-1)\}$$

and

$$\Phi(n;m) = \{x^m \pmod{n} \mid 1 \le x \le (n-1),\ (n,x) = 1\}$$

Clearly $\Phi(n;m)$ is a subset of $\Sigma(n;m)$. We shall use the symbol $\phi(n;m)$ to denote the number of distinct elements of $\Phi(n;m)$. Also whenever there is no risk of confusion we shall omit the symbol (mod n). We shall prove certain theorems which will enable the work of computing $\Sigma(n;m)$ to be reduced considerably, and conclude with a table of $\Sigma(n;n)$.

## 2. PROPERTIES OF $\Phi(n;m)$

Theorem 1. $\Sigma(n;m) = \{xy^m \mid x \in \Phi(n;m),\ y|n\}$

Proof. Suppose $z \in \Sigma(n;m)$. Then $z \equiv d^m \pmod{n}$. Now let $y = (n,d)$. Then $d = cy$, $(n,c) = 1$, $y\ n$. Hence $s = xy^m$, where $x = c^m \in \Phi(n;m)$. This concludes the proof of the theorem. In view of it, and the fact that for several reasons $\Phi(n;m)$ is rather easier to deal with, we shall first consider the properties of $\Phi(n;m)$.

In the first place, we shall define the integer $1(n)$ for $n \ge 2$, as follows

(i) if $n = p^r$, where $p$ is an odd prime and $r > 1$, then $1(n) = p^{r-1}(p-1)$

(ii) if $n = 2^r$, then $1(n) = 2^{r-1}$ if $r = 1, 2$ and $1(n) = 2^{r-2}$ if $r \ge 3$.

(iii) if



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$$n = \prod_{i=1}^{N} p_i^{r_i} ,$$

then

$$l(n) = \text{l. c. m. } \{l(p_i^{r_i})\}, \quad t = 1, 2, \cdots, N .$$

Then we have

Theorem 2. If

$$k = (m, l(n)),$$

then if $k \neq l(n)$,

$$\Phi(n; m) = \Phi(n; k) ,$$

whereas if $k = l(n)$, then $\Phi(n; m) = \{1\}$.

Proof.

$$\Phi(n; 1) = \{x \mid (n,x) = 1\}$$

is a multiplicative Abelian Group whose structure is known

$$\Phi(n;1) = \begin{matrix} C_{l(p_1^{r_1})} \times C_{l(p_2^{r_2})} \times \cdots \times C_{l(p_n^{r_n})} & \text{if } 8 \nmid n \\ C_{l(p_1^{r_1})} \times C_{l(p_2^{r_2})} \times \cdots \times C_{l(p_n^{r_n})} \quad C_2 & \text{if } 8 \mid n \end{matrix}$$

Now

$$l(n) = \text{l. c. m. } \{l (p_1^{r_1})\}$$

and so

$$\Phi(n; l(n)) = \{1\} ,$$

and clearly $l(n)$ is the least integer for which this is true. Thus we have

$$x^{l(n)} \equiv 1 \pmod n$$

if

$$(n, x) = 1 .$$

Now if

$$k = l(n) = (m, l(n)),$$

then

$$l(n) \mid m ,$$

and so whenever $(n,x) = 1$, $x^m \equiv 1 \pmod{n}$, i.e., $\Phi(n;m) = \{1\}$.

Secondly, if $(m, \lambda(n)) = k$ where $0 \le k \le \lambda(n)$, then there exist integers $a, b, c$ such that

$$m = ak \quad \text{and} \quad k = bm - c\lambda(n) .$$

Hence if $(n,x) = 1$ we have

$$x^m = x^{ak} \equiv (x^a)^k \pmod{n}$$

and so

$$\Phi(n;m) \subset \Phi(n;k)$$

Also,

$$\begin{aligned} x^k = x^{bm-c\lambda(n)} &\equiv x^{bm} \pmod{n} \\ &\equiv (x^b)^m, \pmod{n} \end{aligned}$$

Thus

$$\Phi(n;k) \subset \Phi(n;m) \quad ,$$

and so by our previous result

$$\Phi(n;k) = \Phi(n;m) \quad .$$

Hence in considering $\Phi(n;m)$ we need only consider values of $m$ which are divisors of $\lambda(n)$.

## 3. PROPERTIES OF $\Sigma(n;m)$

Theorem 3. if

$$x \equiv y \pmod{n}$$

and $a|n$, then

$$x^a \equiv y^a \pmod{an} .$$

Proof. Let

$$x = y + cn .$$

Then

$$\begin{aligned} x^a &= (y + cn)^a \\ &= y^a + acny^{a-1} + \cdots + \\ &\quad + a(cn)^{a-1}y + (cn)^a \\ &\equiv y^a \pmod{an} \text{ since } a|n . \end{aligned}$$

This concludes the proof. A simple induction argument now shows that for any $r$, if $x \equiv y \pmod n$ and $a|n$ then

$$x^{a^r} \equiv y^{a^r} \pmod{a^r n}$$

and this gives immediately

Theorem 4.

$$\Sigma(a^r n;\ a^r m) = \{ x^{a^r} \pmod{a^r n} \mid x \in \Sigma(n;m)\}$$

where $a$ is any factor of $n$.

Theorem 5. If $n$ is square-free, and if $\Phi(n;m) = \Phi(n;1)$, then $\Sigma(n;m) = \Sigma(n;1)$, for by Theorem 1,

$$\begin{aligned}\Sigma(n;m) &= \{xy^m \mid x \in \Phi(n;m),\ \ y|n\} \\ &= \{xy^m \mid (n,x) = 1,\ \ y|n\}\end{aligned}$$

Now consider any prime factor $p$ of $n$. Since $n$ is square free $(p^m, n) = p$ and so there exist integers $a, b$ such that

$$\begin{aligned}p &= ap^m + bn \\ &\equiv ap^m \pmod n \text{ and so } (n,a) = 1 \text{ or } p\end{aligned}$$

Now if $(n, a) = p$ then let $a' = a + n/p$. Then $(n, a') = 1$ and $p \equiv a'p^m \pmod n$. Hence $p \in \Sigma(n;m)$, and so every prime factor belongs to $\Sigma(n;m)$. Hence if $m$ is any number between 1 and $(n - 1)$

$$z = c \prod_1^N p_i^{s_i}$$

where $(c, n) = 1$ and the $p_i$ are prime factors of $n$. Hence $z \equiv a^m \pmod n$. This concludes the proof, since clearly $0 \in \Sigma(n;m)$.

Theorem 6. If $k = (m, l(n))$ then if

$$n = \prod_{i=1}^N p_i^{r_i}$$

$$\phi(n;m) = \begin{cases} \prod_{i=1}^{N} \dfrac{l(p_i^{r_i})}{(k, l(p_i^{r_i}))} & \text{unless } 8|n \text{ and } m \text{ is odd} \\ 2\prod_{i=1}^{N} \dfrac{l(p_i^{r_i})}{(k, l(p_i^{r_i}))} & \text{if } 8|n \text{ and } m \text{ is odd}. \end{cases}$$

For, $(n;m) = \phi(n;k)$ and the result follows from the structure of $\Phi(n;1)$, since when s n, k is odd if and only if $m$ is odd.

## 4. PROPERTIES OF $\Sigma(n;n)$

Theorem 7. $\Sigma(n;n) = \{0, 1, 2, \cdots, (n-1)\}$ if and only if $(n, l(n)) = 1$.

Proof. (i) If $\Sigma(n;n) = \{0, 1, 2, \cdots, (n-1)\}$ then $\Phi(n;n) = \Phi(n;1)$ and so by Theorem 6 $(n, l(n)) = 1$.

(ii) If $(n, l(n)) = 1$ then by Theorem 2 $\Phi(n;n) = \Phi(n;1)$ and so by Theorem 5, $\Sigma(n;n) = \Sigma(n;1)$ since $n$ must be square-free to make $(n, l(n)) = 1$.

Theorem 8. If $l(n)|n$, then $\Sigma(n;n) = \{x^n | x|n\}$. This follows immediately from Theorems 1 and 2.

Theorem 9. (i) if $n = 2^r$, then $\Sigma(n;n) = \{0, 1\}$

(ii) if $n = 3^r$, then $\Sigma(n;n) = \{0, 1, n-1\}$

(iii) if $n = p^r$, where $p$ is an odd prime then $\Sigma(n;n)$ consists of the $p$ different elements $0, \pm 1, \pm 2^t, \cdots, \pm\{\frac{1}{2}(p-1)\}^t$ where $t = p^{r-1}$.

Proof. (i) if $n = 2^r$, then since $\Sigma(2;2) = \{0, 1\}$, the result follows by Theorem 4.

(ii) if $n = 3^r$, then since $\Sigma(3;3) = \{0, 1, 2\}$ or equivalently $\{0, 1, -1\}$ it follows by Theorem 4 that $\Sigma(n;n) = \{0, 1, n-1\}$.

(iii) if $n = p^r$, then since $l(p) = p - 1$, $(p, l(p)) = 1$ and so by Theorem 7, $\Sigma(p;p) = \{0, 1, 2, \cdots, (p-1)\}$ or equivalently, $\{0, \pm 1, \pm 2, \cdots, \pm\frac{1}{2}(p-1)\}$. Hence by Theorem 4,

$$\Sigma(n;n) = \{0, \pm 1, \pm 2^t, \cdots, \pm\{\tfrac{1}{2}(p-1)\}^t\} \qquad t = p^{r-1}$$

It merely remains to show that all these $p$ elements are distinct. Now $n = p^r$, $l(n) = p^{r-1}(p-1)$, $k = (n, l(n)) = p^{r-1}$. Hence by Theorem 6, $\phi(n;n) = p - 1$.

Hence the elements $\pm 1, \pm 2^t, \cdots, \pm \frac{1}{2}(p-1)^t$ are all distinct, and clearly they are all distinct from 0. This concludes the proof.

Theorem 10. If $n = 2p$, $p$ an odd prime, then

$$\Sigma(n;n) = \{0, p, q, q+p \mid (q|p) = +1\}$$

Proof. $l(n) = p - 1$, and so $k = (n, l(n)) = 2$. Hence

$$\Phi(n;n) = \Phi(n;2) = \{x^2 \mid (z, x) = 1\},$$

by Theorem 2. Hence by Theorem 1,

$$\Sigma(n;n) = \{ay^n \mid s \in \Phi(n;2), \quad y = 0, 1, 2, p\}$$

Now $y = 0$ gives only the element 0, and since must always be odd, $y = p$ gives only the element $p$. Also,

$$2^p \equiv 2 \pmod 2$$

and

$$2^p \equiv 2 \pmod p$$

hence

$$2^p \equiv 2 \pmod n$$

hence

$$2^n \equiv 4 \pmod n$$

Thus

$$\Sigma(n;n) = \{0, p, z, 4z \mid z \in \Phi(n;2)\}$$

Now

$$z = x^2$$

where

$$(n, x) = (p, x) = 1$$

and

$$4z = (2x)^2 = y^2 \pmod{n}$$

where

$$(y, n) = (2x, 2p) = 2 .$$

Hence

$$\Sigma(n;n) = \{0, p, x^2 \mid (x, p) = 1\}$$

For each element of the form $x^2$ there are now two possibilities.

(i) $0 < x^2 \pmod{n} < p$. Then $x^2 \equiv q$ where $0 < q < p$, $(q|p) = +1$

(ii) $p < x^2 \pmod{n} < 2p$.

Then

$$\begin{aligned}(x + p)^2 &= x^2 + 2px + p^2 \\ &\equiv x^2 - p \pmod{n}\end{aligned}$$

Hence

$$x^2 \equiv p + q \pmod{n}$$

where

$$0 < q < p \text{ and } (q|p) = +1$$

This concludes the proof.

Theorem 11. If $n = 2p^r$ where $p$ is an odd prime, then

$$\Sigma(n;n) = \{0, p^r, q^t, p^r + q^t \mid t = p^{r-1},\ 0 < q < p, (q|p) = +1\}$$

Proof. For each $p$, we shall prove the result by induction on $r$. By the previous theorem, the result is true for $r = 1$. Now suppose that it is true for $r = R$. Thus

$$\Sigma(2p^R;2p^R) = \{0, p^R, q^t, q^t + p^R\} \text{ where } t = p^R$$

Hence by Theorem 4,

$$\Sigma(2p^{R+1};2p^{R+1}) = \{x^p \mid x \in \Sigma(2p^R; 2p^R)\}$$

Now $x = 0$ gives $x^p = 0$ and $x = p^R$ gives

$$\begin{aligned} x^p &= p^{pR} \\ &= p^{R+1}(p^{pR-R-1} - 1) + p^{R+1} \\ &\equiv p^{R+1} \pmod{n} \end{aligned}$$

$x = q^t$ gives

$$x^p = q^{tp} = q^T \quad \text{where } R = pt = p^{R+1}$$

$x = q^t + p^R$ gives

$$\begin{aligned} x^p &= (q^t + p^R)^p \\ &= q^T + q^{t(p-1)}p^{R+1} + q^{t(p-2)}p^{2R+1}\binom{p-1}{2} \\ &\quad + \cdots + q^t p^{R(p-1)+1} + p^{Rp} \\ &\equiv q^T \pmod{p^{R+1}} \\ &\equiv q^T + p^{R+1} \pmod{p^{R+1}} \end{aligned}$$

Also $$x^p \equiv q^T + p^{R+1} \pmod{2}$$

for if $x$ is even, $q$ is odd and vice-versa.

Hence

$$x^p \equiv q^T + p^{R+1} \pmod{n} .$$

This concludes the proof, and gives, for example,

$$\Sigma(2\cdot 3^r; 2\cdot 3^r) = \{0, 1, 3^r, 3^r + 1\}$$

$$\Sigma(2\cdot 5^r; 2\cdot 5^r) = \{0, 1, 5^r - 1, 5^r, 5^r + 1, 2\cdot 5^r - 1\}$$

Theorem 12. If $n = 4p$, where $p$ is an odd prime, then

(i) if $p \equiv 3 \pmod 4$, $\Sigma(n;n) = \{x^2 \mid x = 0, 1, 2, \cdots, p\}$

(ii) if $p \equiv 1 \pmod 4$,

$$\Sigma(n;n) = \{x^2 \mid x = 0,\ p,\ q \text{ where } 0 < q < p,\ (q|p) = +1\}$$

Proof. By Theorem 4,

$$\Sigma(n;n) = \{x^2 \mid x \in \Sigma(2p;2p)\},$$

and so by Theorem 10,

$$\Sigma(n;n) = \{x^2 \mid x = 0,\ p,\ q,\ q + p \ ,$$

where $\qquad 0 \le q \le p$ and $(q|p) = +1\}$

(i) if $p \equiv 3 \pmod 4$ then $(-1|p) = -1$ and so $q$ takes exactly half of the values $1, 2, \cdots, (p - 1)$ and the other half are of the form $p - q$. Now

$$(q + p)^2 - (p - q)^2 = 4pq \equiv 0 \pmod n$$

Hence in this case

$$\Sigma(n;n) = \{x^2 \mid x = 0, 1, 2, \cdots, p\}$$

(ii) if $p \equiv 1 \pmod 4$ then $(-1|p) = +1$ and so $q$ takes half the values $1, 2, \cdots, (p - 1)$, these same values being of the form $(p - q)$ and again

$$(q + p)^2 \equiv (p - q)^2 \pmod n$$

Hence

$$\Sigma(n;n) = \{x^2 \mid x = 0, p, q, \text{ where } 0 \le q \le p \text{ and } (q|p) = +1\}$$

This concludes the proof.

Theorem 13. If $1(n)|n$ and if $n = rs$ where $(r, s) = 1$ and if $R \equiv r^n \pmod n$ and $S \equiv s^n \pmod n$ are elements of $\Sigma(n;n)$ then $R + S \equiv 1 \pmod n$.

Proof. Since $1(n)|n$, it follows from Theorem 2 that $\Phi(n;n) = \{1\}$.

Since $n = rs$ and $(r,s) = 1$, $(n, r+s) = 1$ and so

$$(r+s)^n \equiv 1 \pmod{n} .$$

Now each of $r$ and $s$ is a factor of $(r+s)^n - r^n - s^n$ and so since $r$ and $s$ have no factor in common and $n = rs$,

$$\begin{aligned}(r+s)^n &\equiv r^n + s^n \pmod{n} \\ &\equiv R + s \pmod{n}\end{aligned}$$

Hence by the above remark,

$$R + S \equiv 1 \pmod{n} .$$

## 5. TABLES OF $\Sigma(n;n)$

Our theorems enable us to compute tables of $\Sigma(n;n)$ fairly easily, at least in the cases that $n$ can be factorized into fairly small factors. By Theorem 7, $\Sigma(n;n)$ consists of all the residues when $n$ is a prime, and so there is no need to calculate the residues in this case. Also it is clear that the elements 0 and 1 always belong to $\Sigma(n;n)$. We give a table; giving $\Sigma(n;n)$ for all values other than primes up to $n = 100$ and also for a few easily calculable values between 100 and 1000.

| n | $\Sigma(n;n)$ contains 0, 1, and |
|---|---|
| 4 | no others |
| 6 | 3, 4 |
| 8 | no others |
| 9 | 8 |
| 10 | 4, 5, 6, 9 |
| 12 | 4, 9 |
| 14 | 2, 4, 7, 8, 9, 11 |
| 15 | all residues |
| 16 | no others |

| | |
|---|---|
| 18 | 9, 10 |
| 20 | 5, 16 |
| 21 | 6, 7, 8, 13, 14, 15, 20 |
| 22 | 3, 4, 5, 9, 11, 12, 14, 15, 16, 20 |
| 24 | 9, 16 |
| 25 | 7, 18, 24 |
| 26 | 3, 4, 9, 10, 12, 13, 14, 16, 17, 22, 23, 25 |
| 27 | 26 |
| 28 | 4, 8, 9, 16, 21, 25 |
| 30 | 4, 6, 9, 10, 15, 16, 19, 21, 24, 25 |
| 32 | no others |
| 33 | all residues |
| 34 | 2, 4, 8, 9, 13, 15, 16, 17, 18, 19, 21, 25, 26, 30, 32, 33 |
| 35 | all residues |
| 36 | 9, 28 |
| 38 | 4, 5, 6, 7, 9, 11, 16, 17, 19, 20, 23, 24, 25, 26, 28, 30, 35, 36 |
| 39 | 5, 8, 12, 13, 14, 18, 21, 25, 26, 27, 31, 34, 38 |
| 40 | 16, 25 |
| 42 | 7, 15, 21, 22, 28, 36 |
| 44 | 4, 5, 9, 12, 16, 25, 33, 36, 37 |
| 45 | 8, 9, 10, 17, 18, 19, 26, 27, 28, 35, 37, 37, 44 |
| 46 | 2, 3, 4, 6, 8, 9, 12, 13, 16, 18, 23, 24, 25, 26, 27, 29, 31, 32, 35, 36, 39, 41 |
| 48 | 16, 33 |
| 49 | 18, 19, 30, 31, 48 |
| 50 | 24, 25, 26, 49 |
| 51 | all residues |
| 52 | 9, 13, 16, 29, 40, 48 |
| 54 | 27, 28 |
| 55 | 10, 11, 12, 21, 22, 23, 32, 33, 34, 43, 44, 45, 54 |
| 56 | 8, 9, 16, 25, 32, 49 |
| 57 | 7, 8, 11, 12, 18, 19, 20, 26, 27, 30, 31, 37, 38, 39, 45, 46, 49, 50, 56 |
| 58 | 4, 5, 6, 7, 9, 13, 16, 20, 22, 23, 24, 25, 28, 29, 30, 33, 34, 35, 36, 38, 42, 45, 49, 51, 52, 53, 54, 57 |
| 60 | 16, 21, 25, 36, 40, 45 |

| | |
|---|---|
| 62 | 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28, 31, 32, 33, 35, 36, 38, 39, 40, 41, 45, 47, 49, 50, 51, 56, 59 |
| 63 | 8, 27, 28, 35, 36, 55, 62 |
| 64 | no others |
| 65 | all residues |
| 66 | 3, 4, 9, 12, 15, 16, 22, 25, 27, 31, 33, 34, 36, 37, 42, 45, 48, 49, 55, 58, 60, 64 |
| 68 | 4, 13, 16, 17, 21, 33, 52, 64 |
| 69 | all residues |
| 70 | 4, 9, 11, 14, 15, 16, 21, 25, 29, 30, 35, 36, 39, 44, 46, 49, 50, 51, 56, 60, 64, 65 |
| 72 | 9, 64 |
| 74 | 3, 4, 7, 9, 10, 11, 12, 16, 21, 25, 26, 27, 28, 30, 33, 34, 36, 37, 38, 40, 41, 44, 46 47, 48, 49, 53, 58, 62, 63, 64, 65, 67, 70, 71, 73 |
| 75 | 7, 18, 24, 25, 26, 32, 43, 49, 50, 51, 68, 74 |
| 76 | 4, 5, 9, 16, 17, 20, 24, 25, 28, 36, 44, 45, 49, 57, 61, 64, 68, 73 |
| 77 | all residues |
| 78 | 12, 13, 25, 27, 39, 40, 51, 52, 64, 66 |
| 80 | 16, 65 |
| 81 | 80 |
| 82 | 2, 4, 5, 8, 10, 16, 18, 20, 21, 23, 25, 31, 32, 33, 36, 37, 39, 40, 41, 42, 43, 45, 46, 49, 50, 51, 57, 59, 61, 62, 64, 66, 72, 73, 74, 77, 78, 80, 81 |
| 84 | 21, 28, 36, 49, 57, 64 |
| 85 | all residues |
| 86 | 4, 6, 9, 10, 11, 13, 14, 15, 16, 17, 21, 23, 24, 25, 31, 35, 36, 38, 40, 41, 43, 44, 47, 49, 52, 53, 54, 56, 57, 58, 59, 60, 64, 66, 67, 68, 74, 78, 79, 81, 83, 84 |
| 87 | all residues |
| 88 | 9, 16, 25, 33, 48, 49, 56, 64, 80, 81 |
| 90 | 9, 10, 19, 36, 45, 46, 54, 55, 64, 81 |
| 91 | all residues |
| 92 | 4, 8, 9, 12, 13, 16, 24, 25, 29, 32, 36, 41, 48, 49, 52, 64, 69, 72, 73, 77, 81, 85 |
| 93 | 2, 4, 8, 15, 16, 23, 27, 29, 30, 31, 32, 33, 35, 39, 46, 47, 54, 58, 60, 61, 62, 63, 64 66, 70, 77, 78, 85, 89, 91, 92 |
| 94 | 2, 3, 4, 6, 7, 8, 9, 12, 14, 16, 17, 18, 21, 24, 25, 27, 28, 32, 34, 36, 37, 42, 48, 49, 50, 51, 53, 54, 55, 56, 59, 61, 63, 64, 65, 68, 71, 72, 74, 75, 79, 81, 83, 84, 89 |
| 95 | all residues |
| 96 | 33, 64 |
| 98 | 32, 44, 49, 67, 79, 86 |

| | |
|---|---|
| 99 | 8, 9, 10, 17, 18, 19, 26, 27, 28, 35, 36, 37, 44, 45, 53, 54, 55, 62, 63, 64, 71, 72, 73, 80, 81, 82, 89, 90, 91, 98 |
| 100 | 25, 76 |
| 108 | 28, 81 |
| 120 | 16, 25, 40, 81, 96, 105 |
| 125 | 57, 68, 124 |
| 128 | no others |
| 136 | 16, 17, 33, 120 |
| 144 | 64, 81 |
| 150 | 24, 25, 49, 51, 75, 76, 99, 100, 124, 126 |
| 160 | 65, 96 |
| 162 | 81, 82 |
| 192 | 64, 129 |
| 200 | 25, 176 |
| 216 | 81, 136 |
| 240 | 16, 81, 96, 145, 160, 225 |
| 243 | 242 |
| 250 | 124, 125, 126, 249 |
| 256 | no others |
| 272 | 17, 256 |
| 288 | 64, 225 |
| 300 | 25, 76, 100, 201, 225, 276 |
| 320 | 65, 256 |
| 324 | 81, 244 |
| 360 | 81, 136, 145, 216, 225, 280 |
| 384 | 129, 256 |
| 400 | 176, 225 |
| 432 | 81, 352 |
| 480 | 96, 160, 225, 256, 321, 385 |
| 486 | 243, 244 |
| 500 | 125, 376 |

| | |
|---|---|
| 512 | no others |
| 544 | 256, 289 |
| 576 | 64, 513 |
| 600 | 25, 201, 225, 376, 400, 576 |
| 625 | 182, 443, 624 |
| 640 | 256, 385 |
| 648 | 81, 568 |
| 720 | 81, 145, 225, 496, 576, 640 |
| 729 | 728 |
| 768 | 256, 513 |
| 800 | 225, 576 |
| 864 | 352, 513 |
| 900 | 100, 225, 325, 576, 676, 801 |
| 960 | 256, 321, 385, 576, 640, 705 |
| 972 | 244, 729 |
| 1000 | 376, 625 |

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# CERTAIN LUCAS-LIKE SEQUENCES AND THEIR GENERATION BY PARTITIONS OF NUMBERS

DANIEL C. FIELDER
Georgia Institute of Technology, Atlanta, Georgia

## 1. INTRODUCTION

An interesting paper by S. L. Basin in the April, 1964, issue of this journal [1] develops the $k^{th}$ Lucas number $L_k = S_k$ where $S_k$ is the sum of the $k^{th}$ powers of the roots of

$$f(x) = a_0x^2 + a_1x + a_2 , \tag{1}$$

in which $a_0 = 1$, $a_1 = a_2 = -1$. Although Basin's $S_k$ originated from a demonstration of a property of Waring's formula, it is obvious, as Basin implies, that the same results could be obtained using Newton's formulas for $S_k$ in terms of elementary symmetric functions.

In a previous paper [2], the author tabulated $S_k$ from Newton's formulas for

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n . \tag{2}$$

The values of $S_k$ for $k = 1(1)11$ applicable for $1 \leq n \leq 11$ are reproduced as Table 1* of this paper.

It is proposed to examine the special case of (2),

$$f(x) = x^n - x^{n-1} - x^{n-2} - \cdots - 1 , \tag{3}$$

for $n \geq 2$ and to use Table 1 as a guide in extending the true Lucas sequence found from (1) to Lucas-like sequences. Also, a method by which partitions of numbers can generate terms of the Lucas-like sequences is presented.

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* Reprinted by permission from the American Mathematical Society from Mathematics of Computation, Vol. 12, No. 63, pp. 194-198. Actually, it is $S_n$ which is tabulated in [2] but is presented herein as $S_k$ to be consistent with this paper.

Table 1

$S_k$ for k = 1(1)11

$$
\begin{aligned}
S_1 &= -\, a_1/a_0, \\
S_2 &= a_1^2/a_0^2 - 2a_2/a_0, \\
S_3 &= -\, a_1^3/a_0^3 + 3a_1a_2/a_0^2 - 3a_3/a_0, \\
S_4 &= a_1^4/a_0^4 - 4a_1^2a_2/a_0^3 + (4a_1a_3 + 2a_2^2)/a_0^2 - 4a_4/a_0, \\
S_5 &= -\, a_1^5/a_0^5 + 5a_1^3a_2/a_0^4 - (5a_1^2a_3 + 5a_1a_2^2)/a_0^3 + (5a_1a_4 + 5a_2a_3)/a_0^2 - 5a_5/a_0, \\
S_6 &= a_1^6/a_0^6 - 6a_1^4a_2/a_0^5 + (6a_1^3a_3 + 9a_1^2a_2^2)/a_0^4 - (6a_1^2a_4 + 12a_1a_2a_3 + 2a_2^3)/a_0^3 \\
&\quad + (6a_1a_5 + 6a_2a_4 + 3a_3^2)/a_0^2 - 6a_6/a_0, \\
S_7 &= -\, a_1^7/a_0^7 + 7a_1^5a_2/a_0^6 - (7a_1^4a_3 + 14a_1^3a_2^2)/a_0^5 \\
&\quad + (7a_1^3a_4 + 21a_1^2a_2a_3 + 7a_1a_2^3)/a_0^4 - (7a^2{}_1a_5 + 14a_1a_2a_4 + 7a^2{}_2a_3 + 7a_1a^2{}_3)/a_0^3 \\
&\quad + (7a_1a_6 + 7a_2a_5 + 7a_3a_4)/a_0^2 - 7a_7/a_0, \\
S_8 &= a_1^8/a_0^8 - 8a_1^6a_2/a_0^7 + (8a_1^5a_3 + 20a_1^4a_2^2)/a_0^6 \\
&\quad - (8a_1^4a_4 + 32a_1^3a_2a_3 + 16a_1^2a^3{}_2)/a_0^5 \\
&\quad + (8a_1^3a_5 + 24a_1^2a_2a_4 + 12a_1^2a_3^2 + 24a_1a_2^2a_3 + 2a_2^4)/a_0^4 \\
&\quad - (8a_1^2a_6 + 16a_1a_2a_5 + 16a_1a_3a_4 + 8a_2^2a_4 + 8a_2a_3^2)/a_0^3 \\
&\quad + (8a_1a_7 + 8a_2a_6 + 8a_3a_5 + 4a_4^2)/a_0^2 - 8a_8/a_0, \\
S_9 &= -\, a_1^9/a_0^9 + 9a_1^7a_2/a_0^8 - (9a_1^6a_3 + 27a_1^5a_2^2)/a_0^7 \\
&\quad + (9a_1^5a_4 + 45a_1^4a_2a_3 + 30a_1^3a_2^3)/a_0^6 \\
&\quad - (9a_1^4a_5 + 36a_1^3a_2a_4 + 18a_1^3a_3^2 + 54a_1^2a_2^2a_3 + 9a_1a_2^4)/a_0^5 \\
&\quad + (9a_1^3a_6 + 27a_1^2a_2a_5 + 27a_1^2a_3a_4 + 27a_1a_2^2a_4 + 27a_1a_2a_3^2 + 9a_2^3a_3)/a_0^4 \\
&\quad - (9a_1^2a_7 + 18a_1a_2a_6 + 18a_1a_3a_5 + 9a_1a_4^2 + 9a_2^2a_5 + 18a_2a_3a_4 + 3a_3^3)/a_0^3 \\
&\quad + (9a_1a_8 + 9a_2a_7 + 9a_3a_6 + 9a_4a_5)/a_0^2 - 9a_9/a_0, \\
S_{10} &= a_1^{10}/a_0^{10} - 10a_1^8a_2/a_0^9 + (10a_1^7a_3 + 35a_1^6a_2^2)/a_0^8 \\
&\quad - (10a_1^6a_4 + 60a_1^5a_2a_3 + 50a_1^4a_2^3)/a_0^7 + (10a_1^5a_5 + 50a_1^4a_2a_4 \\
&\quad + 25a_1^4a_3^2 + 100a_1^3a_2^2a_3 + 25a_1^2a_2^4)/a_0^6 - (10a_1^4a_6 + 40a_1^3a_2a_5 \\
&\quad + 40a_1^3a_3a_4 + 60a_1^2a_2^2a_4 + 60a_1^2a_3^2a_2 + 40a_2^3a_3a_1 + 2a_2^5)/a_0^5 \\
&\quad + (10a_1^3a_7 + 30a_1^2a_2a_6 + 30a_1^2a_3a_5 + 15a_1^2a_4^2 + 30a_1a_2^2a_5 \\
&\quad + 60a_1a_2a_3a_4 + 10a_2^3a_4 + 15a_2^2a_3^2 + 10a_1a_3^3)/a_0^4 \\
&\quad - (10a_1^2a_8 + 20a_1a_2a_7 + 20a_1a_3a_6 + 20a_1a_4a_5 + 20a_2a_3a_5 \\
&\quad + 10a_2a_4^2 + 10a_2^2a_6 + 10a_3^2a_4)/a_0^3 + (10a_1a_9 + 10a_2a_8 \\
&\quad + 10a_3a_7 + 10a_4a_6 + 5a_5^2)/a_0^2 - 10a_{10}/a_0, \\
S_{11} &= -\, a_1^{11}/a_0^{11} + 11a_1^9a_2/a_0^{10} - (11a_1^8a_3 + 44a_1^7a_2^2)/a_0^9 \\
&\quad + (11a_1^7a_4 + 77a_1^6a_2a_3 + 77a_1^5a_2^3)/a_0^8 - (11a_1^6a_5 + 66a_1^5a_2a_4 \\
&\quad + 33a_1^5a^2{}_3 + 165a_1^4a_2^2a_3 + 55a_1^3a_2^4)/a_0^7 + (11a_1^5a_6 + 55a_1^4a_2a_5 \\
&\quad + 55a_1^4a_3a_4 + 110a_1^2a_2^3a_3 + 110a_1^3a_2^2a_4 + 110a_1^3a_2a_3^2 + 11a_1a_2^5)/a_0^6 \\
&\quad - (11a_1^4a_7 + 44a_1^3a_2a_6 + 44a_1^3a_3a_5 + 22a_1^3a_4^2 + 66a_1^2a_2^2a_5 \\
&\quad + 132a_1^2a_2a_3a_4 + 44a_1a_2^3a_4 + 66a_1a_2^2a_3^2 + 11a_2^4a_3 + 22a_1^2a_3^3)/a_0^5 \\
&\quad + (11a_1^3a_8 + 33a_1^2a_2a_7 + 33a_1^2a_3a_6 + 33a_1^2a_4a_5 + 33a_1a_2^2a_6 \\
&\quad + 66a_1a_2a_3a_5 + 33a_1a_2a_4^2 + 33a_1a_3^2a_4 + 11a_2^3a_5 + 33a_2^2a_3a_4 + 11a_3^3a_2)/a_0^4 \\
&\quad - (11a_1^2a_9 + 22a_1a_2a_8 + 22a_1a_3a_7 + 22a_1a_4a_6 + 11a_1a_5^2 + 11a_2^2a_7 \\
&\quad + 22a_2a_3a_6 + 22a_2a_4a_5 + 11a_3a_4^2 + 11a_3^2a_5)/a_0^3 + (11a_1a_{10} + 11a_2a_9 \\
&\quad + 11a_3a_8 + 11a_4a_7 + 11a_5a_6)/a_0^2 - 11a_{11}/a_0.
\end{aligned}
$$

The liberty of calling the sequence "Lucas-like" appears justified since (1) (as used by Basin) is a special case of (3) and, moreover, the sequences do indeed share characteristics with the true Lucas sequences.

## 2. OBSERVED BEHAVIOR

To identify terms of a sequence and at the same time to retain a Lucas flavor, the terminology $L_k^{(n)}$ is used to specify the $k^{th}$ term of a Lucas-like sequence obtained from (3) for a given $n \geq 2$. For convenience, $S_k^{(n)} = L_k^{(n)}$. It is noted that $L_k^{(2)}$ is the true $k^{th}$ Lucas number, $L_k$. For any given $k \leq 11$ and $2 \leq n \leq 11$, it is a simple matter to enter Table 1, reject all coefficients of a terms having subscripts greater than n and add the numerical coefficients of the remaining a terms to obtain a $k^{th}$ Lucas-like number. The choice of signs in (3) automatically makes the signs of the numerical coefficients positive. For example:

$$L_4^{(2)} = 1 + 4 + (0 + 2) + 0 = 7 \; . \tag{4}$$

The first seven terms of several Lucas-like sequences obtained in this manner are recorded in Table 2. For later use, a zig-zag line divides the table into two parts. For $n = 2$, it is seen that the difference between the first <u>two</u> terms (those above the zig-zag line) is 2 (i.e., $2^{n-1}$ for $n = 2$), and that the sum of <u>two</u> consecutive terms determines the next term. For $n = 3$, the difference between the two first terms is $2^{n-2}$, and the difference between the second and third terms is $2^{n-1}$. There are <u>three</u> terms above the zig-zag line. For $n = 3$, the sum of <u>three</u> consecutive terms determines the next term.

Table 2
VALUES OF $L_k^{(n)}$

| k \ n | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 3 | 3 | 3 | 3 | 3 |
| 3 | 1 | 4 | 7 | 7 | 7 | 7 |
| 4 | 1 | 7 | 11 | 15 | 15 | 15 |
| 5 | 1 | 11 | 21 | 26 | 31 | 31 |
| 6 | 1 | 18 | 39 | 51 | 57 | 63 |
| 7 | 1 | 29 | 71 | 99 | 113 | 120 |

The obvious pattern is repeated for $n = 4, 5$, etc.

One immediate conclusion is that each Lucas-like sequence is, in reality, the blend of two sequences. The first sequence is $1, 3, 7, \cdots, 2^n - 1$ having $n$ terms and governed within its range by the recursion formula

$$L_{k+1}^{(n)} = L_k^{(n)} + 2^k, \quad (L_1^{(n)} = 1) \ . \tag{5}$$

The second sequence starts with the sum of the $n$ terms of the first sequence. The first term is

$$L_{n+1}^{(n)} = 2^{n+1} - n - 2 \ . \tag{6}$$

Succeeding terms are

$$L_{n+2}^{(n)} = L_{n+1}^{(n)} + (2^{n+1} - n - 2) - 1 \ , \tag{7}$$

$$L_{n+3}^{(n)} = L_{n+2}^{(n)} + (2^{n+1} - n - 2) - (1 + 3) \ , \tag{8}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$L_{2n+1}^{(n)} = L_{2n-1}^{(n)} + \cdots + L_{n+1}^{(n)} \ . \tag{9}$$

In general, the second sequence follows the recursion formula

$$L_k^{(n)} = L_{k-1}^{(n)} + L_{k-2}^{(n)} + \cdots + L_{k-n}^{(n)} \quad (k \geq n + 1) \ . \tag{10}$$

It is interesting to note from (7), (8), and (9) that at least one term of the first sequence appears directly in the summation for $L_k^{(n)}$ for $n \leq k \leq 2n$. After $k > 2n$, the influence of the first sequence is reduced.

## 3. PARTITION CALCULATION OF SEQUENCE TERMS

Several methods are available for finding a particular $L_k^{(n)}$. One method is the direct use of recursion formulas. Another is to solve the $n^{th}$ order difference equation for the second sequence subject to the $n$ conditions (or their equivalents) imposed by the first sequence. A third method, discussed herein, is to assume that desired partitions of $n$ are available and to use them as a combinatorial means of finding the $L_k^{(n)}$.

In Chrystal's [3] notation, $P(k|p|\le q)$ is the number of p-part partitions of k, no member of which exceeds q. If the original value of q exceeds $k+1-p$, it can be replaced by $q = k+1-p$ since there are the same number of partitions for $q = k+1-p$ as for $q \le k+1-p$. However, for $q \le k+1-p$ it is obvious that less than $P(k|p|\le k+1-p)$ partitions exist. Suppose, now, that desired partitions can be called up at will and are available from this point on. The actual set of such partitions which have the same limitations as the enumeration counterpart is given the terminology $PV(k|p|\le q)$.

If any $S_k$ of Table 1 is stripped of all terms except subscripts and superscripts (exponents) of the numerator a's, there remains the conventional representation of all the partitions of k. The partition representation for $k = 6$ is exemplified in Table 3. It is seen that, in general, p ranges from k to 1. The quantity $k \cdot (p-1)!$ divided by the product of the factorials of the exponents of a particular combination yields (neglecting sign) the numerical part of the contribution of that combination to $S_k$. To illustrate, if $k = 6$, $p = 3$, the numerical coefficient associated with the partition $2^3 = 2, 2, 2$, is $(6 \times 2!)/3! = 2$. This well-known result employs much the same reasoning as finding a coefficient of a multinomial expansion. The numerical coefficients for $k = 6$, $n = 6$, and the total $63 = L_6^{(6)}$ are given in Table 3. Thus, once the exponents are found from the available partitions, $L_k^{(n)}$ follows.

Table 3

| Partition Representation | | $PV(k\|p\|\le q)$ | Numerical Coefficient |
|---|---|---|---|
| $1^6$ | 1, 1, 1, 1, 1, 1 | $PV(k\|p\|\le q)$ | $(6 \times 5!)/6! = 1$ |
| $1^4, 2$ | 1, 1, 1, 1, 2 | $PV(6\|5\|\le 2)$ | $(6 \times 4!)/4! = 6$ |
| $1^3, 3$ | 1, 1, 1, 3 | $PV(6\|4\|\le 3)$ | $(6 \times 3!)/3! = 6$ |
| $1^2, 2^2$ | 1, 1, 2, 2 | | $(6 \times 3!)/(2! \times 2!) = 9$ |
| $1^2 4$ | 1, 1, 4 | $PV(6\|3\|\le 4)$ | $(6 \times 2!)/2! = 6$ |
| 1, 2, 3 | 1, 2, 3 | | $(6 \times 2!)/1 = 12$ |
| $2^3$ | 2, 2, 2 | | $(6 \times 2!)/3! = 2$ |
| 1, 5 | 1, 5 | $PV(6\|2\|\le 5)$ | $(6 \times 1!)/1 = 6$ |
| 24 | 2, 4 | | $(6 \times 1!)/1 = 6$ |
| $3^2$ | 3, 3 | | $(6 \times 1!)/2! = 3$ |
| 6 | 6 | $PV(6\|1\|\le 6)$ | $(6 \times 0!)/1 = 6$ |
| | | | Total = 63 |

As long as $k \le n$, the sum of numerical coefficients obtained from the $PV(k\,p \le k + 1 - p)$'s is the desired $L_k^{(n)}$. When $k \ge n$, the a terms with subscripts greater than n are zero. Since the corresponding products with numerical coefficients are zero, these numerical coefficients are not used. The elimination of these numerical coefficients is accomplished by limiting q to $1 \le q \le n$ and using only those partitions which result. Table 4 gives an example of this situation $k = 6$, $n = 2$.

Table 4

| Partition Representation | | $PV(k\|p\|q)$ | Numerical Coefficient |
|---|---|---|---|
| $1^6$ | 1, 1, 1, 1, 1, 1 | $PV(6\|6\|1)$ | 1 |
| $1^4, 2$ | 1, 1, 1, 1, 2 | $PV(6\|5\|2)$ | 6 |
| $1^2, 2^2$ | 1, 1, 2, 2 | $PV(6\|4\|2)$ | 9 |
| $2^3$ | 2, 2, 2 | $PV(6\|3\|2)$ | 2 |
| None | | $PV(6\|2\|2)$ | 0 |
| None | | $PV(6\|1\|2)$ | 0 |
| | | | Total = 18 |

The above methods have been successfully applied to digital computation of electrical network problems [4] in which the a coefficients had values other than $\pm 1$ and in which it was necessary to consider the signs of the resultant numerical coefficients.

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# REMARKS ON TWO RELATED SEQUENCES OF NUMBERS

DANIEL C. FIELDER
Georgia Institute of Technology, Atlanta, Georgia

## 1. INTRODUCTION

The expansion of $(x + y)^n$ usually takes the form

$$(1) \qquad (x + y)^n = \sum_{k=1}^{n+1} C(n,k)x^{n-k+1}y^{k-1} ,$$

where $C(n,k)$ are the well-known binomial coefficients and are sequences of integers generated by the expansion (1). Another device for obtaining the $C(n,k)$ is, of course, Pascal's triangle.

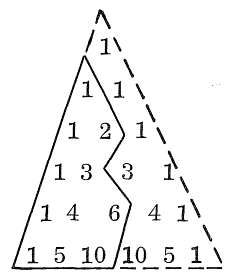
Different sequences of numbers can be obtained from the coefficients resulting from the expansion of $(x + y)^n$ in terms of $(x^k + y^k)(xy)^{n-k}$. Further, a sort of inverse can be obtained by expressing $(x^n + y^n)$ in terms of $(x + y)^k(xy)^{n-k}$. In both cases the coefficients share characteristics with certain binomial coefficients and terms from sums of powers of roots of selected polynomials. In the inverse sequences, except for appropriate changes in sign, the numerical coefficients are those observed in a recently proposed approach to the generation of Lucas numbers from partitions of numbers [1]. The relationship between partitions of numbers and both sequence is outlined briefly.

## 2. SEQUENCE OF THE FIRST KIND

For brevity, let $(x + y) = u = u_1$ (interchangeably), let $(x^k + y^k) = u_k$, and let $(xy) = v$. The numerical coefficients of the resultant direct expansion shown below are called coefficients of the first kind.

$$(2) \qquad \begin{aligned} u &= u_1 \\ u^2 &= u_2 + 2v^2 \\ u^3 &= u_3 + 3u_1v^2 \\ u^4 &= u_4 + 4u_2v^2 + 6v^4 \\ u^5 &= u_5 + 5u_3v^2 + 10u_1v^4 \\ u^6 &= u_6 + 6u_4v^2 + 15u_2v^4 + 20v^6 \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

As might be suspected, the coefficients are binomial coefficients without the symmetrically repeated coefficients of the expansion (1). The coefficients of (2) form the half-Pascal triangle enclosed by solid lines below.



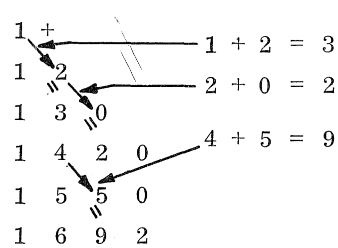

## 3. SEQUENCE OF THE SECOND KIND

A rearrangement of (2) yields

$$
\begin{aligned}
u_1 &= u \\
u_2 &= u^2 - 2v^2 \\
u_3 &= u^3 - 3uv^2 \\
u_4 &= u^4 - 4u^2v^2 + 2v^4 \\
u_5 &= u^5 - 5u^3v^2 + 5uv^4 \\
u_6 &= u^6 - 6u^4v^2 + 9u^2v^4 - 2v^6 \\
&\cdots\cdots\cdots\cdots
\end{aligned} \tag{3}
$$

If the minus signs are temporarily neglected in (3), the diagram below illustrates one of the simple additive methods by which the coefficients can be obtained.

(4)




If signs are neglected, it is interesting to note that the sum of the coefficients for any given index is identically the Lucas number of that index. Additional comments on this will be made later.

## 4. INTERRELATIONS

In [1], the sums of the powers of roots of

$$f(x) = x^n - x^{n-1} - x^{n-2} - \cdots - 1 \tag{5}$$

were obtained from a previously developed tabulation of Newton's formulas for powers of roots. The first few entries of that tabulation are given below without literal coefficients and without negative signs.

$$\begin{aligned}
S_1 &= 1, \\
S_1 &= 1 + 2, \\
S_3 &= 1 + 3 + 3, \\
S_4 &= 1 + 4 + (4 + 2) + 4, \\
S_5 &= 1 + 5 + (5 + 5) + (5 + 5) + 5, \\
S_6 &= 1 + 6 + (6 + 9) + (6 + 12 + 2)(6 + 6 + 3) + 6, \\
S_7 &= 1 + 7 + (7 + 14) + (7 + 21 + 7) + (7 + 14 + 7 + 7) + (7 + 7 + 7) + 7,
\end{aligned} \tag{6}$$

Except for a missing final 1, the numbers as grouped in (6) are complete sets of binomial coefficients; hence, by selecting the appropriate numbers from (6), the coefficients for the first kind of sequence are readily obtained.

The extraction of the coefficients for the sequence of the second kind is more interesting. The same sets of numbers as those for the first kind of coefficients are considered. If in (6) a number is not parenthesized, it is one of the second kind coefficients as well as one of the first kind coefficients. However, it can be observed that whereas a first kind coefficient is equal to the sum of numbers within parentheses, the corresponding second kind coefficient is equal to the last number only of the numbers included within parentheses.

Without repeating the details covered in [1], it can be stated that the second kind coefficients be used to obtain the powers of roots of (5) for the case $n = 2$. Proper choice of sign leads to the ultimate identification.

In the previous paper [1], it was shown that the $k^{th}$ Lucas number can be generated from the two-part partitions of $k$. The sum of the terms resulting from operations on the partitions is equal to the $k^{th}$ Lucas number. The same operation on partitions can be used for finding the second kind coefficients.

However, here the individual terms, not the sum, are used. Proper choice of sign must be made since the partition method generates only positive numbers. It may be added that this latter method is of advantage only if a rapid and convenient means for obtaining partitions is available.

## REFERENCES

1. D. C. Fielder, "Certain Lucas-Like Sequences and their Generation by Partitions of Numbers," Fibonacci Quarterly, Vol. 5, No. 4, pp. 319-324.

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## RECURRING SEQUENCES

Review of Book by Dov Jarden
By Brother Alfred Brousseau

For some time the volume, Recurring Sequences, by Dov Jarden has been unavailable, but now a printing has been made of a revised version. The new book contains articles published by the author on Fibonacci numbers and related matters in Riveon Lematematika and other publications. A number of these articles were originally in Hebrew and hence unavailable to the general reading public. This volume now enables the reader to become acquainted with this extensive material (some thirty articles) in convenient form.

In addition, there is a list of Fibonacci and Lucas numbers as well as their known factorizations up to the 385th number in each case. Many new results in this section are the work of John Brillhart of the University of San Francisco and the University of California.

There is likewise, a Fibonacci bibliography which has been extended to include articles to the year 1962.

This valuable reference for Fibonacci fanciers is now available through the Fibonacci Association for the price of $6.00. All requests for the volume should be sent to Brother Alfred Brousseau, Managing Editor, St. Mary's College, Calif., 94575.

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# BASES FOR INFINITE INTERVALS OF INTEGERS

D. E. DAYKIN and A. J. W. HILTON
University of Malaya, Kuala Lumpur, Malaya
and The University, Reading

## 1. INTRODUCTION

In this paper we discuss the problem of representing uniquely each member of an arbitrary infinite interval of integers. The integers of the interval, and no others, are to be expressed as sums of terms of a sequence $(b_n)$ of integers. We also discuss the problem of representing uniquely each positive integer, and no other integer, as the linear combination of terms of a sequence $(b_n)$ of integers, where the coefficients in the linear combination are prescribed and have the value $+1$ or $-1$. In each problem, roughly speaking, we choose an integer $k \geq 1$ and require that any two terms of $(b_n)$ whose suffixes differ by less than $k$ shall not both be used in the representation of any given integer. The precise definitions and results are in the next section, where we also show the way in which earlier work [1] by one of us (D. E. D.) is related to our definition of an $(h,k)$ base.

In a later paper we will discuss an analogous problem of representing uniquely each real number in the interval $(0, c]$, where $c$ is any positive real number. Finally, we would like to thank Professor R. Rado for his helpful suggestions in the preparation of this paper.

## 2. STATEMENT OF RESULTS

Throughout this paper, $h, k$ and $m$ are integers such that

$$h + 1 \geq k \geq h \geq 0, \quad k \geq 1 \quad \text{and} \quad m \geq 1 .$$

Also, unless we state otherwise for a particular sequence, the subscript of the first term of a finite or infinite sequence is the number 1, e.g.,

$$(a_n) = (a_1, a_2, \cdots) \quad .$$

We denote by $(v_n)$ the $(h,k)^{\text{th}}$ Fibonacci sequence defined by

$$(2.1)\qquad \begin{cases} v_n = n & \text{for } 1 \le n \le k, \\ v_n = v_{n-1} + v_{n-k} + (k-h) & \text{for } n > k\ . \end{cases}$$

An equivalent definition of this sequence was given (for $h \ge 1$) in [1] (p. 144). We denote by $(u_n)$ the $(k,k)^{th}$ Fibonacci sequence and by $(f_n)$ the $(2,2)^{th}$ Fibonacci sequence, which is clearly the original Fibonacci sequence $(1, 2, 3, 5, 8, 13, 21, \cdots)$. Further, we write $[a,b]$ for the interval of integers $x$, $a \le x \le b$, with the obvious interpretation when $a = -\infty$ or $b = +\infty$.

Suppose $(a_n)$, $(k_n)$ is a pair of sequences of positive integers with the following property $\underset{\sim}{P}$.

$\underset{\sim}{P}$. Each integer $N \in [1,\infty]$ has a unique representation

$$N = a_{i_1} + a_{i_2} + \cdots + a_{i_\alpha}$$

where $\alpha = \alpha(N)$ and $i_{\nu+1} - i_\nu \ge k$ for $1 \le \nu < \alpha$.

It is shown in [1] (Theorem D) that if $(a_n)$ is increasing and the pair $(a_n)$, $(k_n)$ have the property $\underset{\sim}{P}$ then $k_1 \le k_2 \le k_1 + 1$, $k_2 = k_\nu$ for $\nu \ge 2$, and $(a_n)$ is the $(k_1, k_2)^{th}$ Fibonacci sequence. This result leads us to make the following definition.

Definition 1. A finite or infinite sequence $(b_n)$ of integers is an $(h,k)$ base for an interval $[a,b]$ if each integer $N \in \{0\} \cup [a,b]$ has a unique representation

$$(2.2)\qquad N = b_{i_1} + b_{i_2} + \cdots + b_{i_\alpha}\ ,$$

where

$$\alpha = \alpha(N),\ i_2 \ge i_1 + h \text{ if } \alpha > 1, \text{ and } i_{\nu+1} \ge i_\nu + k \text{ for } 1 \le \nu < \alpha\ ,$$

and further, if $N$ is an integer which can be expressed in the form (2.2) then

$$N \in \{0\} \cup [a,b]\ .$$

Notice that the representation of $0$ in the form (2.2) is the empty sum.

Theorem 1 is a statement in this notation of another result proved for $h \ge 1$ in the earlier paper ([1], Theorem C). This result can easily be shown to be true for $h = 0$ also.

Theorem 1. The first $n$ terms $(v_1, v_2, \cdots, v_n)$ of the $(h,k)^{th}$ Fibonacci sequence $(v_n)$ form an $(h,k)$ base for $[1, v_{n+1} - 1]$, and $(v_n)$ forms an $(h,k)$ base for $[1, \infty]$.

Our first new results, Theorems 2-6, are concerned with the existence of $(h, k)$ bases for the infinite interval $[m, \infty]$ with $m \neq 1$, and for the infinite intervals $[-m, \infty]$ and $[-\infty, \infty]$. We conjecture that there is no $(h,k)$ base for $[m, \infty]$ when $m \geq 3$, but have only been able to prove the following theorem.

Theorem 2. If $m > 1$ and $(b_n)$ is an increasing sequence of integers, $b_n \neq (2, 3, 4, \cdots, 2^{n-1}, \cdots)$, then $(b_n)$ is not an $(h,k)$ base for $[m,\infty]$. However, $(b_n) = (2, 3, 4, \cdots, 2^{n-1}, \cdots)$ is an $(h,k)$ base for $[m,\infty]$ if and only if $h = k = 1$, and $m = 2$. By the statement that $(b_n)$ is an increasing sequence, we mean that $b_1 \leq b_2 \leq \cdots$ .

It is easier to deal with the intervals $[-m, \infty]$ and $[-\infty,\infty]$, provided that $h = k$. However, we have been unable to settle the question of the existence of $(h,k)$ bases for these intervals when $h \neq k$.

Theorem 3. If $-m$ is a negative integer then there exists a $(k,k)$ base for $[-m,\infty]$.

For the set of all integers, $[-\infty,\infty]$, there are infinitely many $(k,k)$ bases, and in fact we can choose the sign which each term of a $(k,k)$ base is to have, subject to the condition that the signs change infinitely often.

Theorem 4. Let $(s_n)$ be a sequence such that

$$(2.3) \qquad \begin{cases} s_n \in \{-1, 1\} \text{ for } n \geq 1, \quad \text{and} \\ s_n \cdot s_{n-1} = -1 \text{ for infinitely many } n > 1. \end{cases}$$

Then there is a $(k,k)$ base $(b_n)$ for $[-\infty,\infty]$ with $s_n b_n > 0$ for $n \geq 1$.

For $k = 2$, we give an explicit example of a $(k,k)$ base for $[-m,\infty]$ in terms of the Fibonacci sequence $(f_n)$. We first represent $m$ in the form

$$(2.4) \qquad m = f_{i_1} + f_{i_2} + \cdots + f_{i_\alpha},$$

where

$$i_{\nu+1} \geq i_\nu + 2 \text{ for } 1 \leq \nu < \alpha .$$

The existence and uniqueness of this representation is proved by Theorem 1. Next we let $(s_n)$ be the sequence defined in terms of the suffixes $i_\nu$ of (2.4) as follows

$$(2.5)\qquad \begin{aligned} s_{i_{\nu+1}} &= -1 \text{ for } 1 \le \nu \le \alpha \\ s_n &= 1 \text{ otherwise.} \end{aligned}$$

Then an explicit formula for a (2,2) base for $[-m,\infty]$ is given in the following theorem.

Theorem 5. Let $-m$ be a negative integer, let the sequence $(s_n)$ be defined as in (2.5), and let

$$(2.6)\qquad b_1 = s_1 \text{ and } b_n = \begin{Bmatrix} s_n f_n & \text{if } s_n \cdot s_{n-1} = 1 \\ s_n f_{n-1} & \text{if } s_n \cdot s_{n-1} = -1 \end{Bmatrix} \text{ for } n > 1 .$$

Then $(b_n)$ is a (2,2) base for $[-m,\infty]$.

Similarly, we have an explicit formula for a (2,2) base for $[-\infty,\infty]$, in terms of the Fibonacci sequence $(f_n)$. We prescribe the sign of each term of the base, subject to the condition that the signs change infinitely often.

Theorem 6. If the sequence $(s_n)$ satisfies (2.3) and the sequence $(b_n)$ is determined in terms of $(s_n)$ by the relations (2.6), then $(b_n)$ is a (2,2) base for $[-\infty,\infty]$ with $s_n b_n \ge 0$ for $n \ge 1$.

So far we have been concerned with unique representations of integers as sums of terms of a base. It is interesting to consider the problem of uniquely representing integers as linear combinations of terms of a sequence $(b_n)$ of integers, where the coefficients in the linear combination are prescribed and have the value $+1$ or $-1$. We first make the following definition.

Definition 2. Let a sequence $S = (s_n)$, where $s_n \in \{-1,1\}$ for $n \ge 1$, be given. A sequence $(b_n)$ of integers is an $(h+1, k; S)$ base for $[0,\infty]$ if each integer $N \in [0,\infty]$ has a unique representation

$$(2.7)\qquad N = s_\alpha b_{i_1} + s_{\alpha-1} b_{i_2} + \cdots + s_1 b_{i_\alpha} ,$$

where

$$\alpha = \alpha(N), \quad i_2 \ge i_1 + h + 1 \text{ if } \alpha > 1,$$

and

$$i_{\nu+1} \ge i_\nu + k \text{ for } 2 \le \nu < \alpha$$

and further, if $N$ is an integer which can be expressed in the form (2.7) then $N \in [0,\infty]$.

Theorem 1 shows that the $(h,k)^{th}$ Fibonacci sequence $(v_n)$ is an $(h,k)$ base for $[1,\infty]$. It follows that $(v_n)$ is an $(h,k)$ base for the set of all non-negative integers, $[0,\infty]$, and we have been able to determine the conditions under which $(v_n)$ is an $(h+1, k;S)$ base for this same set of integers.

Theorem 7. The $(h,k)^{th}$ Fibonacci sequence $(v_n)$ is an $(h+1, k;S)$ base for $[0,\infty]$ if and only if $s_n = (-1)^{n+1}$ for $n \geq 1$.

In our last theorem we give an explicit formula for the terms of $(v_n)$, the $(h,k)^{th}$ Fibonacci sequence. It is well known that the terms of the Fibonacci sequence $(f_n)$ are sums of the elements in the diagonals of Pascal's triangle, and Theorem 8 extends this result.

Theorem 8.

$$v_n = \sum_{i=k-h}^{\infty} \binom{n-h+(k-1)(2-i)}{i} \quad \text{for } n \geq 1 . \tag{2.8}$$

Here, as usual, $\binom{a}{b}$ denotes the binomial coefficient $a!/(a-b)!(b!)$.

## 3. PROOF OF THEOREM 2

We assume that the sequence $(b_n)$ is increasing and is an $(h,k)$ base for $[m,\infty]$, and in each of the first three cases we deduce a contradiction of definition 1 of an $(h,k)$ base by finding a number which has two representations in the form (2.2).

Lemma 1. $b_n = n + m - 1$ for $1 \leq n \leq m + h$.

Proof. As the sequence $(b_n)$ is increasing, it is strictly increasing, so that $b_1 = m$ and

$$b_n \geq m + n - 1 \quad \text{for } n \geq 1 . \tag{3.1}$$

The smallest number of the form (2.2) with $\alpha > 1$ is $b_1 + b_{1+h}$, and, by (3.1), $b_1 + b_{1+h} \geq 2m + h$. Hence $b_n = m + n - 1$ for all $n \geq 1$ such that $m + n - 1 < 2m + h$; i.e., $n \leq m + h$. This proves Lemma 1.

We consider now the various cases.

Case [1]. $m \geq 3$. Then by Lemma 1,

$$b_1 + b_{h+3} = m + (m + h + 3 - 1) = (m + 1)$$
$$+ (m + h + 2 - 1) = b_2 + b_{h+2} .$$

Case [2]. $m = 2$, $k > 1$. By Lemma 1, $b_n = n + 1$ for $1 \leq n \leq h + 2$, and so $b_1 + b_{1+h} = 4 + h$, $b_1 + b_{2+h} = 5 + h$, and $b_2 + b_{2+h} = 6 + h$. Clearly, $6 + h$ is the largest number which can be represented in the form (2.2) with $i_\alpha \leq 2 + h$ and $\alpha = 2$. However, the smallest number which can be represented with $\alpha = 3$ is

$$b_1 + b_{1+h} + b_{1+h+k} \geq 4 + h + b_{3+h} > 4 + h + 6 + h \geq 10 + h .$$

Therefore $b_{3+h} = 7 + h$. But $b_1 + b_{3+h} = 2 + (7 + h) = 9 + h$, so that $8+h$ has no representation with $i_\alpha \leq 3 + h$. Hence $b_{4+h} = 8 + h$. But then we have

$$b_1 + b_{4+h} = 2 + (8 + h) = 3 + (7 + h) = b_2 + b_{3+h} .$$

Case [3]. $m = 2$, $k = 1$, $h = 0$. Then by Lemma 1, $b_1 = 2$ and $b_2 = 3$. Therefore the representations of 4, 5, 6 and 7 are $b_1 + b_1$, $b_1 + b_2$, $b_2 + b_2$ and $b_1 + b_1 + b_2$ respectively. The number 8 cannot be represented in the form (2.2) with $i_\alpha \leq 2$. Hence $b_3 = 8$. Similarly the number 9 cannot be represented with $i_\alpha \leq 3$. Hence $b_4 = 9$. But then $b_1 + b_4 = 2 + 9 = 3 + 8 = b_2 + b_3$.

We have now only to deal with the cases when $m = 2$, $k = 1$, $h = 1$. It follows, therefore, from the contradictions obtained in the first 3 cases that if $(b_n)$ is an $(h,k)$ base for $[m,\infty]$, then $h = k = 1$ and $m = 2$.

Case [4]. $h = k = 1$, $m = b_1 = 2$, $b_2 = 3$ and $b_n = 2^{n-1}$ for $n \geq 3$. In this case $(b_n)$ is a $(1,1)$ base for $[2,\infty]$.

For let $N \geq 2$ be an integer. If $N$ is even, then its representation in the form (2.2) is the binary representation, which is unique. If N is odd, then $N - 3$ is even, and so the representation of $N$ is the binary representation of $N - 3$ together with $b_2$; hence this representation is also unique.

Notice that, for $p \geq 3$, each of the numbers $2, 3, \cdots, 2^{p-1} - 2, 2^{p-1} - 1, 2^{p-1} + 1$, and no others, can be represented in the form (2.2) using $(b_1, b_2, \cdots, b_{p-1})$. This fact is used in the proof of the next case.

Case [5]. $h = k = 1$, $m = 2$, $(b_n \neq (2, 3, 4, 8, \cdots, 2^{n-1}, \cdots)$. Again we assume that $(b_n)$ is increasing and is an $(h,k)$ base for $[m,\infty]$, so that, by Lemma 1, $b_1 = 2$ and $b_2 = 3$. Let $p \geq 3$ be an integer. Suppose that $b_p \neq 2^{p-1}$, and, if $p \geq 3$, also suppose that $b_3 = 4$, $b_4 = 8, \cdots, b_{p-1} = 2^{p-2}$. Then, by the remark at the end of the last case, $b_p \geq 2^{p-1} + 1$. But then $2^{p-1}$ has no representation in the form (2.2), which contradicts definition 1 of an $(h,k)$ base. This completes the proof of Theorem 2.

## 4. PROOFS OF THEOREMS 3, 4, 5 and 6

Throughout this section, namely Lemmas 2-8 and the proofs of Theorems 3-6, the sequences $(t_n)$, $(a_n)$, $(d_n)$ and $(e_n)$ are as defined immediately below. We let $(t_n)$ be a sequence such that $t_n \in \{-1, 1\}$ for $n \geq 1$. The three sequences $(a_n)$, $(d_n)$ and $(e_n)$ are simultaneously defined by induction in terms of the sequence $(t_n)$. First we put $a_1 = t_1$ and $d_n = e_n = 0$ for $n \leq 0$. If $n > 1$ and we have defined the terms $d_\nu$, $e_\nu$ for $\nu \leq n - 2$, and the terms $a_\nu$ for $1 \leq \nu \leq n - 1$, then we define $d_{n-1}$, $e_{n-1}$ and $a_n$ as follows.

i) $d_{n-1}$ is the largest, and $e_{n-1}$ is the smallest of the number 0 and the numbers representable in the form

$$a_{i_1} + a_{i_2} + \cdots + a_{i_\alpha}, \tag{4.1}$$

where

$$i_\alpha \leq n - 1 \quad \text{and} \quad i_{\nu+1} \geq i_\nu + k \quad \text{for} \quad 1 \leq \nu \leq \alpha .$$

ii)

$$a_n = \begin{cases} d_{n-1} - e_{n-k} + 1 & \text{if} \quad t_n = +1 \\ e_{n-1} - d_{n-k} - 1 & \text{if} \quad t_n = -1 \end{cases} . \tag{4.2}$$

The relation (4.2) is clearly true for $n = 1$ also.

Lemma 2. (i) For all $n$, $0 \leq d_{n-1} \leq d_n$ and $e_n \leq e_{n-1} \leq 0$.

(ii) For $n \geq 1$, $t_n a_n \geq 0$.

Proof. (i) Follows immediately from the definitions of $(d_n)$ and $(e_n)$.

(ii) For $n \geq 1$, if $t_n = +1$, then, by (4.2) and part (i),

$$a_n = d_{n-1} - e_{n-k} + 1 \geq 1 .$$

The proof when $t_n = -1$ is similar and completes the proof of Lemma 2.

Lemma 3. For $n \geq 1$, if $t_n = +1$, then $d_n = d_{n-k} + a_n$ and $e_n = e_{n-1}$, and if $t_n = -1$, then $e_n = e_{n-k} + a_n$ and $d_n = d_{n-1}$.

Proof. (i) We assume that $t_n = +1$ and show that $e_n = e_{n-1}$. Since $t_n = +1$, by Lemma 2(ii), $a_n > 0$. The number $e_{n-1}$ is, by definition, the smallest of the number 0 and the numbers representable in the form (4.1), and since $a_n > 0$, no smaller number can be formed by adding $a_n$. Hence $e_n = e_{n-1}$. Similar reasoning shows that $d_n = d_{n-1}$ if $t_n = -1$.

(ii) We assume that $t_n = +1$ and show that $d_n = d_{n-k} + a_n$. From the definition of $(d_n)$, $d_n \geq d_{n-k} + a_n$. We suppose that $d_n > d_{n-k} + a_n$, so that $d_n = d_{n-r-k} + a_{n-r}$ for some $r > 0$. Hence $d_{n-r-k} + a_{n-r} > d_{n-k} + a_n$. However by Lemma 2(i), $d_{n-r-k} \leq d_{n-k}$, so that $a_{n-r} \geq a_n$. Since $t_n = +1$ it follows from Lemma 2(ii) that $a_n > 0$. Therefore $a_{n-r} > 0$ and so $t_{n-r} = +1$. Therefore, by (4.2),

$$d_{n-r-1} - a_{n-r-k} + 1 > d_{n-1} - e_{n-k} + 1 . \tag{4.3}$$

However, by Lemma 2(i), $d_{n-1} \geq d_{n-r-1}$ and $-e_{n-k} \geq -e_{n-r-k}$, which contradicts (4.3) and so proves that $d_n = d_{n-k} + a_n$. The proof that if $t_n = -1$ then $e_n = e_{n-k} + a_n$ is similar. This completes the proof of Lemma 3.

Lemma 4. For all n, the finite sequence $(a_1, a_2, \cdots, a_n)$ is a (k,k) base for $[e_n, d_n]$.

Proof. We use induction upon n. When $n < 1$, $(a_1, a_2, \cdots, a_n) = \phi$, the empty set. Since $e_n = d_n = 0$, the lemma is true in this case.

Let $m \geq 1$, and suppose the lemma is true for $n < m$. Then $(a_1, a_2, \cdots, a_{m-k})$ is a (k,k) base for $[e_{m-k}, d_{m-k}]$. From (4.2) and Lemma 2(i), if $t_m = +1$ then $a_m + e_{m-k} > d_{m-k}$, and if $t_m = -1$ then $a_m + d_{m-k} < e_{m-k}$. Therefore $(a_1, a_2, \cdots, a_{m-k}, a_m)$ is a (k,k) base for

$$[e_{m-k}, d_{m-k}] \cup [e_{m-k} + a_m, d_{m-k} + a_m].$$

Also by the induction hypothesis, $(a_1, a_2, \cdots, a_{m-1})$ is a base for $[e_{m-1}, d_{m-1}]$. By (4.2), if $t_m = +1$ then $d_{m-1} + 1 = a_m + e_{m-k}$, and if $t_m = -1$ then $e_{m-1} - 1 = a_m + d_{m-k}$. Since also, from Lemma 2(i),

$$[e_{m-1}, d_{m-1}] \supseteq [e_{m-k}, d_{m-k}]$$

it follows that $(a_1, a_2, \cdots, a_m)$ is a $(k,k)$ base for $[e_{m-k}, d_{m-k} + a_m]$ if $t_m = +1$, or for $[e_{m-k} + a_m, d_{m-1}]$ if $t_m = -1$. Hence, by Lemma 3, $(a_1, a_2, \cdots, a_m)$ is a $(k,k)$ base for $[e_m, d_m]$.

Lemma 4 now follows by induction.

Proof of Theorem 4. Suppose $t_n = s_n$ for $n \geq 1$, where $(s_n)$ is the sequence defined in (2.3). Then $(t_n)$ has the additional property that $t_n \cdot t_{n-1} = -1$ for infinitely many $n > 1$. It is then clear from Lemmas 2(ii) and 3 that $d_n \to \infty$ and $e_n \to -\infty$ as $n \to \infty$, so that, by Lemma 4 $(a_n)$ is a $(k,k)$ base for $[-\infty, \infty]$. We have already shown (Lemma 2(ii)) that $a_n t_n > 0$ for $n \geq 1$, so that Theorem 4 is proved.

Only part (i) of the following lemma is needed in this section. Part (ii) is used in Section 5. We let $N_n(\ell)$ be the number of finite sequences $(i_1, i_2, \cdots, i_\alpha)$ of positive integers such that

$$(4.4)\quad 1 \leq i_\alpha \leq n, \quad i_2 \geq i_1 + \ell \quad \text{if } \alpha > 1, \text{ and } i_{\nu+1} \geq i_\nu + k \text{ for } 2 \leq \nu < \alpha;$$

and are only interested in the values $\ell = h$ and $\ell = h + 1$.

Lemma 5. (i) For $n \geq 1$, $N_n(h) = v_{n+1}$,

(ii) For $n \geq 1$, $N_n(h + 1) = v_n + 1$.

Proof. (i) By Theorem 1, for $n \geq 1$, there is a 1:1 correspondence between sums of the form $v_{i_1} + v_{i_2} + \cdots + v_{i_\alpha}$ with condition (4.4) applied with $\ell = h$, and the integers in $[0, v_{n+1} - 1]$. Hence $N_n(h) = v_{n+1}$.

(ii) If each finite sequence $(i_1, i_2, \cdots, i_\alpha)$ of positive integers, with condition (4.4) applied with $\ell = h$, is transformed by putting $i_1 = j_1$ and $i_\nu + 1 = j_\nu$ for $2 \leq \nu \leq \alpha$, then we obtain all but one of the finite sequences $(j_1, j_2, \cdots, j_\alpha)$ of positive integers, where $1 \leq j_\alpha \leq n + 1$, $j_2 \geq j_1 + h + 1$ if $\alpha \geq 1$ and $j_{\nu+1} \geq j_\nu + k$ for $2 \leq \nu < \alpha$. The finite sequence we do not obtain is $(j_1)$, when $j_1 = n + 1$. Therefore, by part (i), $N_{n+1}(h + 1) = v_{n+1} + 1$ for $n \geq 1$. Hence $N_n(h + 1) = v_n + 1$ for $n \geq 2$. As part (ii) is clearly true when $n = 1$, the proof of Lemma 5 is completed.

Lemma 6.

$$(d_n - d_{n-1}) - (e_n - e_{n-1}) = \begin{cases} u_{n-k+1} & \text{for } n \geq k \\ 1 & \text{for } 1 \leq n < k \\ 0 & \text{for } n < 1 . \end{cases}$$

The sequence $(u_n)$ is the $(k,k)^{th}$ Fibonacci sequence.

Proof. By Lemma 5(i), $N_n(k) = u_{n+1}$ for $n \geq 1$. Therefore it follows from Lemma 4 that $d_n - e_n = u_{n+1} - 1$. Hence for $n \geq 2$,

$$\begin{aligned}(d_n - d_{n-1}) - (e_n - e_{n-1}) &= (d_n - e_n) - (d_{n-1} - e_{n-1}) \\ &= (u_{n+1} - 1) - (u_n - 1) \\ &= \begin{cases} u_{n-k+1} & \text{for } n \geq k , \\ 1 & \text{for } 2 \leq n < k . \end{cases}\end{aligned}$$

The result is easily seen to be true for $n = 1$ and is trivially true for $n < 1$. This proves Lemma 6.

Lemma 7. If $k = 2$, then $a_1 = t_1$ and for $n > 1$,

$$a_n = \begin{cases} t_n f_n & \text{if } t_n \cdot t_{n-1} = 1 , \\ t_n f_{n-1} & \text{if } t_n \cdot t_{n-1} = -1 . \end{cases}$$

The sequence $(f_n)$ is the $(2,2)^{th}$ Fibonacci sequence $(1, 2, 3, 5, 8, \cdots)$.

Proof. By definition $a_1 = t_1$.

Let $n \geq 2$ and $t_n \cdot t_{n-1} = 1$. By Lemma 3, if $t_1 = +1$ then $a_n = d_n - d_{n-2}$ and $e_n = e_{n-1} = e_{n-2}$, and if $t_n = -1$ then $a_n = e_n - e_{n-2}$ and $d_n = d_{n-1} = d_{n-2}$. Also, by Lemma 2(i), $0 \leq d_{n-2} \leq d_n$ and $e_n \leq e_{n-2} \leq 0$. Hence

$$\begin{aligned} a_n &= \{t_n (d_n - d_{n-2}) - (e_n - e_{n-2})\} \\ &= t_n\{(d_n - d_{n-1}) - (e_n - e_{n-1}) + (d_{n-1} - d_{n-2}) - (e_{n-1} - e_{n-2})\} \\ &= \begin{Bmatrix} t_n(f_{n-1} + f_{n-2}) & \text{if } n \geq 3 \\ t_n(f_{n-1} + 1) & \text{if } n = 2 \end{Bmatrix}, \text{ by Lemma 6,} \\ &= t_n f_n . \end{aligned}$$

Now let $t_n \cdot t_{n-1} = -1$. Then similarly by Lemmas 2(i) and 3,

$$a_n = \begin{cases} d_n - d_{n-1} & \text{if } t_n = +1 , \\ e_n - e_{n-1} & \text{if } t_n = -1 , \end{cases}$$
$$= t_n\{(d_n - d_{n-1}) - (e_n - e_{n-1})\}$$
$$= t_n f_{n-1}, \text{ by Lemma 6 .}$$

This proves Lemma 7.

Proof of Theorem 6. We take $k = 2$. We suppose that $t_n = s_n$ for $n \geq 1$, where $(s_n)$ is the sequence defined in (2.3). Then, by Theorem 4, $(a_n)$ is a (2,2) base for $[-\infty, \infty]$ with $a_n s_n > 0$ for $n \geq 1$. But by Lemma 7, $a_1 = s_1$ and

$$a_n = \begin{Bmatrix} s_n f_n & \text{if } s_n \cdot s_{n-1} = 1 \\ s_n f_{n-1} & \text{if } s_n \cdot s_{n-1} = -1 \end{Bmatrix} \text{ for } n \geq 1.$$

Therefore the sequence $(a_n)$ is the same as the sequence $(b_n)$ defined in the statement of Theorem 6. Hence $(b_n)$ is a (2,2) base for $[-\infty,\infty]$ with $a_n b_n > 0$ for $n \geq 1$. This proves Theorem 6.

Lemma 8. For $n \geq 1$, if $x$ is an integer such that $-u_{n+1} + 1 \leq x \leq 0$ then there exists a choice of $(t_1, t_2, \cdots, t_n)$ for which $e_n = x$.

Proof. We use induction upon $n$. If $t_1 = +1$ then $e_1 = 0$, while if $t_1 = -1$ then $e_1 = -1$; since $-u_2 + 1 = -1$, the Lemma is true in the case when $n = 1$.

Let $m \geq 2$ be an integer and suppose that the Lemma is true for $1 \leq n < m$. Then if $-u_m + 1 \leq x \leq 0$ there exists a choice of $(t_1, t_2, \cdots, t_{m-1})$, which we denote by $(t'_1, t'_2, \cdots, t'_{m-1})$, for which $e_{m-1} = x$. Hence, if we choose $(t_1, t_2, \cdots, t_m)$ to be $(t'_1, t'_2, \cdots, t'_{m-1}, +1)$, then by Lemma 3, $e_m = x$.

However, suppose that

$$-u_{m+1} \leq x \leq -u_m + 1 . \tag{4.5}$$

Then

$$x + u_m \leq 0 . \tag{4.6}$$

Therefore, by (4.5) and (4.6),

$$(4.7)\quad \begin{cases} -u_{m+1} + 1 + u_{m+1-k} \le x + u_{m+1-k} \le 0 \text{ if } m \ge k, \\ \text{and} \\ -u_{m+1} + 1 + 1 \le x + 1 \le 0 . \end{cases}$$

But, from (2.1),

$$(4.8)\qquad -u_m + 1 = \begin{cases} -u_{m+1} + 1 + u_{m+1-k} & \text{if } m \ge k, \\ -u_{m+1} + 1 + 1 & \text{if } 2 \le m < k. \end{cases}$$

Therefore, by (4.7) and (4.8),

$$(4.9)\qquad \begin{cases} (-u_m + 1) \le x + u_{m+1-k} \le 0 & \text{if } m \ge k, \text{ and} \\ (-u_m + 1) \le x + 1 \le 0 & \text{if } 2 \le m \le k. \end{cases}$$

Therefore, by the induction hypothesis and (4.9), there exists a choice of $(t_1, t_2, \cdots, t_{m-1})$, which we denote by $(t_1', t_2', \cdots, t_{m-1}')$, for which

$$(4.10)\qquad e_{m-1} = \begin{cases} x + u_{m+1-k} & \text{if } m \ge k; \\ x + 1 & \text{if } 2 \le m < k. \end{cases}$$

If we choose $(t_1, t_2, \cdots, t_m)$ to be $(t_1', t_2', \cdots, t_{m-1}', -1)$, then by Lemma 3, $d_m = d_{m-1}$, and so by Lemma 6,

$$(4.11)\qquad e_m = \begin{cases} e_{m-1} - u_{m+1-k} & \text{if } m \ge k, \\ e_{m-1} - 1 & \text{if } 2 \le m < k. \end{cases}$$

Hence, by (4.10) and (4.11), $e_m = x$.

Lemma 8 now follows by induction.

Proof of Theorem 3. If $p$ is an integer such that $-u_{p+1} + 1 \le -m$ then, by Lemmas 4 and 8, there exists a choice of $(t_1, t_2, \cdots, t_p)$, which we denote by $(t_1', t_2', \cdots, t_p')$, such that $(a_1, a_2, \cdots, a_p)$ is a $(k,k)$ base for $[-m, d_p]$.

Then, if $t_n = t'_n$ for $1 \le n \le p$ and $t_n = +1$ for $n > p$, by Lemmas 3 and 4, $(a_1, a_2, \cdots, a_n)$ is a $(k,k)$ base for $[-m, d_n]$ for $n \ge p$. By Lemmas 2(ii) and 3, $d_n \to \infty$ as $n \to \infty$. Hence $(a_n)$ is a $(k,k)$ base for $[-m, \infty]$.

Proof of Theorem 5. We take $k = 2$. We suppose that $t_n = s_n$ for $n \ge 1$, where $(s_n)$ is the sequence defined in (2.5). By (2.4), $i_{\nu+1} \ge i_\nu + 2$ for $1 \le \nu < \alpha$, and so, by (2.5) and Lemma 2(ii), for $n \ge i_\alpha + 1$,

$$e_n = a_{i_1+1} + a_{i_2+1} + \cdots + a_{i_\alpha+1} . \tag{4.12}$$

However, by Lemma 7, $a_1 = s_1$ and

$$a_n = \begin{cases} s_n f_n & \text{if } s_n \cdot s_{n-1} = +1 \\ s_n f_{n-1} & \text{if } s_n \cdot s_{n-1} = -1 \end{cases} \quad \text{for } n \ge 2 \tag{4.13}$$

Hence the sequence $(a_n)$ is the same as the sequence $(b_n)$ defined in (2.6). But by (4.13) and (2.5), $a_{i_\nu+1} = s_{i_\nu+1} f_{i_\nu} = -f_{i_\nu}$ for $1 \le \nu \le \alpha$. Hence, by (4.12) and (2.4), $e_n = -m$ for $n \ge i_\alpha + 1$. From Lemmas 2(ii) and 3, $d_n \to \infty$ as $n \to \infty$, and so, by Lemma 4, $(b_n)$ is a (2,2) base for $[-m, \infty]$.

## 5. PROOF OF THEOREM 7

Let $S = (s_n)$ be a sequence such that $s_n \in \{-1, 1\}$ for $n \ge 1$, let m be a positive integer, and let $(i_1, i_2, \cdots, i_\alpha)$ be a finite sequence of positive integers such that

$$i_2 \ge i_1 + h + 1 \text{ if } \alpha > 1 \text{ and } i_{\nu+1} \ge i_\nu + k \text{ for } 2 \le \nu < \alpha . \tag{5.1}$$

Lemma 9. If $(i_1, i_2, \cdots, i_\alpha) = \phi$, the empty set, then

$$s_i v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} = 0 .$$

Lemma 10. If $s_1 = +1$ then $s_1 v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} \ge 0$.

Proof. Let $s_1 = 1$. If $\alpha = 1$, then $s_i v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} = v_{i_\alpha} \ge 1$. If $\alpha > 1$ then

$$\begin{aligned} s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} &\geq v_{i_\alpha} - (v_{i_{\alpha-1}} + v_{i_{\alpha-2}} + \cdots + v_{i_1}) \\ &\geq v_{i_\alpha} - (v_{i_{\alpha-1}+1} - 1), \text{ by Theorem 1,} \\ &\geq 1. \end{aligned}$$

This, together with Lemma 9, proves Lemma 10.

Proof of Sufficiency. Suppose that $s_n = (-1)^{n+1}$ for $n \geq 1$, and that $i_\alpha \leq m$. Since $(v_n)$ is a strictly increasing sequence, it follows that if $\alpha \geq 1$ then $s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} \leq v_{i_\alpha} \leq v_m$. Hence, and in view of Lemmas 9 and 10,

$$0 \leq s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} \leq v_m . \tag{5.2}$$

We show now that any two distinct finite sequences $(i_1, i_2, \cdots, i_\alpha)$ which satisfy (5.1) yield distinct values of $s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1}$. Suppose therefore that two such distinct finite sequences are $(j_1, j_2, \cdots, j_\beta)$ and $(g_1, g_2, \cdots, g_\gamma)$. We suppose without loss of generality that $v_{j_\beta} \geq v_{g_\gamma}$, and consider three cases.

Case [1]. $\beta = 1$. Then

$$s_1v_{j_\beta} + s_2v_{j_{\beta-1}} + \cdots + s_\beta v_{j_1} = v_{j_\beta} \geq v_{g_\gamma} \geq s_1v_{g_\gamma} + s_2v_{g_{\gamma-1}} + \cdots + s_\gamma v_{g_1}.$$

Case [2]. $\beta = 2$. Then

$$\begin{aligned} s_1v_{j_\beta} + s_2v_{j_{\beta-1}} + \cdots + s_\beta v_{j_1} &= v_{j_\beta} - v_{j_{\beta-1}} \\ &\geq v_{j_\beta} - v_{j_\beta-h-1} \\ &\left.\begin{cases} = v_{j_\beta} - v_{j_\beta-k} \geq v_{j_\beta-1}, & \text{if } k = h+1 \\ \geq v_{j_\beta} - v_{j_\beta-k} = v_{j_\beta-1}, & \text{if } k = h \end{cases}\right\} \text{by (2.1),} \\ &\geq v_{g_\gamma} \geq s_1v_{g_\gamma} + s_2v_{g_{\gamma-1}} + \cdots + s_\gamma v_{g_1} . \end{aligned}$$

Case [3]. $\beta = 2$.
Then

$$
\begin{aligned}
s_1v_{j_\beta} + s_2v_{j_{\beta-1}} + \cdots + s_\beta v_{j_1} &> v_{j_\beta} - v_{j_{\beta-1}} \\
&\geq v_{j_\beta} - v_{j_{\beta-k}} \\
&\geq v_{j_\beta} - v_{j_{\beta-k}} - (k-h) \\
&= v_{j_\beta} - 1, \quad \text{by (2.1)}, \\
&\geq v_{g_\gamma} \geq s_1v_{g_\gamma} + s_2v_{g_{\gamma-1}} + \cdots + s_\gamma v_{g_1}.
\end{aligned}
$$

By Lemma 5(ii), the number of distinct finite sequences $(i_1, i_2, \cdots, i_\alpha)$ with $i_\alpha \leq m$ which satisfy (5.11) is $v_m + 1$. Therefore, since any two such distinct finite sequences yield distinct values of $s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1}$, and in view of (5.2), it follows that $(v_1, v_2, \cdots, v_m)$ is an $(h+1, k; S)$ base for $[0, v_m]$ when $s_n = (-1)^{n+1}$ for $n \geq 1$. The sufficiency of the condition follows.

Proof of Necessity. Suppose that $(v_n)$ is an $(h+1, k; S)$ base for $[0, \infty]$. We show that $s_n = (-1)^{n+1}$ for $n \geq 1$. Clearly $s_1 = +1$, for otherwise $s_1v_1 = -1$, a contradiction. We suppose that $s_n = (-1)^{n+1}$ for $1 \leq n \leq m$ and that $s_{m+1} = (-1)^{m+1}$, and deduce a contradiction in every case.

Case [1]. $m = 1$. Then $s_1 = s_2 = +1$. We write $M = v_{h+2} - v_1$. If $\alpha = 1$ and $i_\alpha \geq h+2$ then $s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} = v_{i_\alpha} > M$, whereas if $\alpha = 1$ and $i_\alpha \leq h+1$ then

$$
\begin{aligned}
M &= v_{h+2} - v_1 \\
&= v_{h+1} + v_{h+2-k} + (k-h) - v_1, \quad \text{by (2.1)} \\
&= \begin{cases} v_{h+1} + v_2 - v_1, & \text{if } h = k, \\ v_{h+1} + v_1 + 1 - v_1, & \text{if } h+1 = k, \end{cases} \\
&> v_{h+1} \geq v_{i_\alpha} = s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1}.
\end{aligned}
$$

On the other hand if $\alpha > 1$ then $i_\alpha \geq h+2$, and so

$$
\begin{aligned}
s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} &\geq v_{i_\alpha} + v_{i_{\alpha-1}} - (v_{i_{\alpha-2}} + v_{i_{\alpha-3}} + \cdots + v_{i_1}) \\
&\geq v_{i_\alpha} + v_{i_{\alpha-1}} - (v_{i_{\alpha-2}+1} - 1), \\
&\qquad \text{by Theorem 1}, \\
&\geq v_{i_\alpha} + 1 \\
&> v_{h+2} > M.
\end{aligned}
$$

Hence $M \neq s_1 v_{i_\alpha} + s_2 v_{i_\alpha} + \cdots + s_\alpha v_{i_1}$ for any finite sequence $(i_1, i_2, \cdots, i_\alpha)$ satisfying (5.1), which contradicts our assumption that $(v_n)$ is an $(h+1, k; S)$ base for $[0, \infty]$.

Case [2]. $m > 1$. We write

$$N = s_1 v_{(m-1)k+h+2} + s_2 v_{(m-2)k+h+2} + \cdots + s_m v_{h+2} + s_{m+1} v_1 . \tag{5.3}$$

It follows from Lemma 10 that $N \geq 0$. If $m = 2$ then $N = v_{k+h-2} - v_{h+2} - v_1 < v_{k+h+2} = v_{(m-1)k+h+2}$, while if $m > 2$ then

$$\begin{aligned} N &\leq v_{(m-1)k+h+2} - (v_{(m-2)k+h+2} - v_{(m-3)k+h+2}) + 1 \\ &= v_{(m-1)k+h+2} - (v_{(m-2)k+h+1} + (k - h)) + 1, \quad \text{by (2.1)}, \\ &< v_{(m-1)k+h+2} . \end{aligned}$$

Hence

$$0 \leq N \leq v_{(m-1)k+h+2} .$$

Now $N$ is the only number of the form $s_1 v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1}$ with $\alpha = m + 1$ and $i_\alpha \leq (m-1)k + h + 2$. Hence, by the proof of the sufficiency,

$$\{n : n = s_1 v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} ; i_\alpha \leq (m-1)k + h + 2 \text{ and } \alpha \leq m\} \cup \{N - 2s_{m+1} v_1\} = \{0, 1, 2, \cdots, v_{(m-1)k+h+2}\} .$$

Therefore, by (5.4), $N$ can be put in the form

$$N = s_1 v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1}$$

with $\alpha \leq m$. Hence, and by (5.3) $N$ has two representations in the form $N = s_1 v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1}$, which contradicts our assumption that $(v_n)$ is an $(h+1, k; S)$ base for $[0, \infty]$.

We conclude therefore that $s_n = (-1)^{n+1}$ for $n \geq 1$. This completes the proof of Theorem 7.

## 6. PROOF OF THEOREM 8

We show that if $(v_n)$ is defined by (2.8) then the defining relations (2.1) of the $(h,k)^{th}$ Fibonacci sequence hold.

If $a < b$ then $\binom{a}{b} = 0$. Hence the infinite sum of (2.8) contains only a finite number of non-zero terms. In fact, for $1 \leq n \leq k$, the relation (2.8) reduces to

$$v_n = \binom{n+k-2}{0} + \binom{n-1}{1}$$

if $k = h$, or $v_n = \binom{n}{1}$ if $k = h + 1$, and so the first of the relations (2.1) holds. On the other hand, if $n > k$, by checking each stage with $h = k$ and $h + 1 = k$, and using the fact that

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1},$$

we have

$$\begin{aligned} v_{n-1} &+ v_{n-k} + (k-h) \\ &= (k-h) + \sum_{i=k-h}^{\infty} \binom{n-1-h+(k-1)(2-i)}{i} \\ &\qquad + \sum_{i=k-h}^{\infty} \binom{n-k-h+(k-1)(2-i)}{i} \\ &= 1 + \sum_{i=1}^{\infty} \binom{n-1-h+(k-1)(2-i)}{i} + \sum_{i=1+k-h}^{\infty} \binom{n-1-h+(k-1)(2-i)}{i-1} \\ &= 1 - (k-h) + \sum_{i=1}^{\infty} \left\{ \binom{n-1-h+(k-1)(2-i)}{i} \right. \\ &\qquad \left. + \binom{n-1-h+(k-1)(2-i)}{i-1} \right\} \end{aligned}$$

$$= \sum_{i=k-h}^{\infty} \binom{n - h + (k-1)(2-i)}{i}$$

$$= v_n \text{ , as required.}$$

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# A NEW IMPORTANT FORMULA FOR LUCAS NUMBERS

Dov Jarden
Jerusalem, Israel

The formula

$$\frac{L_{10n}}{L_{2n}} = (L_{4n} - 3)^2 + (5F_{2n})^2 \tag{1}$$

may be easily verified putting $L_n = \alpha^n + \beta^n$,

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} , \qquad \alpha\beta = -1 ,$$

Since for $n > 0$, (1) gives a decomposition of $L_{10n}/L_{2n}$ into a sum of 2 squares, and since any divisor of a sum of 2 squares is -1 (mod 4), it follows that any primitive divisor of $L_{10n}$, $n > 0$, is -1 (mod 4).

* * * * *

# SOME PROPERTIES ASSOCIATED WITH SQUARE FIBONACCI NUMBERS

JOHN H. HALTON
Brookhaven National Laboratory, Upton, Long Island, New York

## 1. INTRODUCTION

In 1963, both Moser and Carlitz [11] and Rollett [12] posed a problem.

Conjecture 1. The only square Fibonacci numbers are

$$F_0 = 0, \quad F_{-1} = F_1 = F_2 = 1, \quad \text{and} \quad F_{12} = 144 .$$

Wunderlich [14] showed, by an ingenious computational method, that for $3 \le m \le 1000008$, the only square $F_m$ is $F_{12}$; and the conjecture was proved analytically by Cohn [5, 6, 7], Burr [2], and Wyler [15]; while a similar result for Lucas numbers was obtained by Cohn [6] and Brother Alfred [1].

Closely associated with Conjecture 1 is

Conjecture 2. When p is prime, the smallest Fibonacci number divisible by p is not divisible by $p^2$.

It is known (mostly from Wunderlich's computation) that Conjecture 2 holds for the first 3140 primes ($p \le 28837$) and for $p = 135721$, 141961, and 514229. Clearly, Conjecture 2, together with Carmichael's theorem (see [4], Theorem XXIII, and [9], Theorem 6), which asserts that, if $m \ge 0$, with the exception of $m = 1$, 2, 6, and 12, for each $F_m$ there is a prime p, such that $F_m$ is the smallest Fibonacci number divisible by p (whence $F_m$ is not divisible by $p^2$ and so cannot be a square, if Conjecture 2 holds), implies Conjecture 1; but not vice versa. If Conjecture 2 holds, then the divisibility sequence theorem ([9], Theorem 1) can be strengthened to say that, if p is an odd prime and $n \ge 1$, then

$$\alpha(p,n) = p^{n-1}\alpha(p) \quad . \tag{1}$$

In the notation of [9], Conjecture 2 for a given prime p states that $F_{\alpha(p)}$ is not divisible by $p^2$. This, by Lemma 8 and Theorem 1 of [9], is

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equivalent to

$$\alpha(p^2) = p\alpha(p) \quad . \tag{2}$$

Since $\nu(p)$ is the highest power of p dividing $F_{\alpha(p)}$, this is equivalent to:

$$\nu(p) = 1 \, . \tag{3}$$

By Lemma 11 of [9], p divides one and only one of $F_{p-1}$, $F_p$, and $F_{p+1}$, namely $F_{\lambda(p)}$, where $\lambda(p) = p - (5/p)$ and $(5/p)$ is the Legendre index. Thus, if $p \geq 5$, since $\lambda(p)$ is not divisible by p, while it is divisible by $\alpha(p)$, (2) is equivalent to

$$F_{\lambda(p)} \text{ is not divisible by } p^2 \quad , \tag{4}$$

and inspection of the cases $p = 2$, 3, and 5, shows that the equivalence holds for these primes also. Finally, (4) is equivalent to:

$$F_{p-1}F_{p+1} \text{ is not divisible by } p^2 \quad . \tag{5}$$

This paper presents certain results obtained in the course of investigating the two Conjectures, the latter of which is still in doubt.

## 2. A THEOREM OF M. WARD

We begin with a theorem posed as a problem (published posthumously) by Ward [13]. A different proof from that given below was obtained independently by Carlitz [3].

Theorem A. Let

$$\phi_n(x) = \sum_{s=1}^{n} x^s/s \tag{6}$$

and

(7) $$k_p(x) = (x^{p-1} - 1)/p \ ;$$

then, for any prime number $p \geq 5$, $p^2$ divides the smallest Fibonacci number divisible by $p$ if and only if

(8) $$\phi_{\frac{1}{2}(p-1)}\left(\frac{5}{9}\right) \equiv 2k_p\left(\frac{3}{2}\right) \pmod{p} \ .$$

<u>Proof</u>. We shall show that (8) is true if and only if (5) is false. We shall use the congruence (see [10], page 105) that, when $1 \leq t \leq p - 1$,

(9) $$\frac{t}{p}\binom{p}{t} \equiv (-1)^{t-1} \pmod{p} \ ,$$

and Fermat's theorem (see [10], page 63), that

(10) $$\text{if } (a,p) = 1, \ a^{p-1} \equiv 1 \pmod{p} \ .$$

The identities

(11) $$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right\} \ ,$$

(12) $$F_{2n\pm1} = F_n^2 + F_{n\pm1}^2 \ ,$$

(13) $$F_{2n} = F_n(F_{n-1} + F_{n+1}) \ ,$$

(14) $$F_n^2 + (-1)^n = F_{n-1}F_{n+1} \ .$$

and

(15) $$3F_n^2 + 2(-1)^n = F_{n-1}^2 + F_{n+1}^2 \ ,$$

are well known (see [8], equations (3), (5), (64), (65), (67), and (95) with $m = 1$). From then it follows that (since $(1 \pm \sqrt{5})^2 = 2(3 \pm \sqrt{5})$ )

$$\left(\frac{3}{2}\right)^n Q_n = \left(\frac{3}{2}\right)^n \left\{\left(1+\frac{\sqrt{5}}{3}\right)^n + \left(1-\frac{\sqrt{5}}{3}\right)^n\right\} = \left(\frac{1+\sqrt{5}}{2}\right)^{2n} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n}$$

$$(16) \qquad = F_{4n}/F_{2n} = F_{2n-1} + F_{2n+1} = 2F_n^2 + F_{n-1}^2 + F_{n+1}^2$$

$$= 5F_n^2 + 2(-1)^n = 5F_{n-1}F_{n+1} - 3(-1)^n \quad .$$

Now, since $p > 5$, $p$ is odd and $\frac{1}{2}(p-1)$ is an integer. By (6) and (7), the factor $f = 6^p[\frac{1}{2}(p-1)]!$ is prime to $p$ and makes both $\phi_{\frac{1}{2}(p-1)}(5/9)$ and $k_p(3/2)$ into integers. Thus, modulo p, by (6), (9), (16), (7), and (10),

$$f\phi_{\frac{1}{2}(p-1)}\left(\frac{5}{9}\right) = 2f\sum_{s=1}^{\frac{1}{2}(p-1)} \frac{1}{2s}\left(\frac{\sqrt{5}}{3}\right)^{2s} = f\sum_{t=1}^{p-1}\frac{1}{t}\left\{\left(-\frac{\sqrt{5}}{3}\right)^t + \left(\frac{\sqrt{5}}{3}\right)^t\right\}$$

$$\equiv -\frac{f}{p}\sum_{t=1}^{p-1}\binom{p}{t}\left\{\left(\frac{\sqrt{5}}{3}\right)^t + \left(-\frac{\sqrt{5}}{3}\right)^t\right\} = -\frac{f}{p}(Q_p - 2)$$

$$= -\frac{f}{p}\left\{\left(\frac{2}{3}\right)^p (5F_{p-1}F_{p+1} + 3) - 2\right\}$$

$$= -5\cdot 4^p[\tfrac{1}{2}(p-1)]!\,(F_{p-1}F_{p+1}/p) + f\left(\frac{2}{3}\right)^{p-1}\cdot 2k_p\left(\frac{3}{2}\right)$$

$$\equiv f\cdot 2k_p\left(\frac{3}{2}\right) - g(F_{p-1}F_{p+1}/p) \quad ,$$

where $f$ and $g$ are integers prime to $p$, and $F_{p-1}F_{p+1}/p$ is an integer. It follows that (8) is true if and only if $(F_{p-1}F_{p+1}/p) \equiv 0 \pmod p$, and this contradicts (5), proving the theorem.

## 3. ANOTHER CONJECTURE

We end the paper with an examination of a conjecture, which implies the first conjecture (known now to be true), in a rather different way from Conjecture 2. The underlying result is

Theorem B. Let $p$ be a prime, and suppose that there exists a positive integer $M$, such that

(i) for no integer $n$, prime to $p$ and greater than $M$, is $F_m$ a square or $p$ times a square; and

(ii) if $n$ is positive and not greater than $N$, and $F_n$ is a square or $p$ times a square, then $F_k$ is neither a square nor $p$ times a square, when $k$ is the least integer greater than $M$, such that $k/n$ is a power of $p$;

then no $F_m$ at all is a square or $p$ times a square for $m > M$.

Proof. Suppose that (i) and (ii) hold, and that $F_m$ is a square or $p$ times a square. In contradiction of the theorem, let $m > M$. Then, by (i), $m$ is divisible by $p$. Let $m = pm_1$, and write $F_m = AB^2C^2$, $F_{m_1} = BC^2D$, where $D$ divides $A$ and $A$ is $1$ or $p$. This makes $F_m$ a square or $p$ times a square, and divisible by $F_{m_1}$. Now, by the well-known identity (see [8, equation (35)], or [9], equation (8)] )

$$F_m F_{m_1} = \sum_{h=1}^{p} \binom{p}{h} F_{m_1}^{h-1} F_{m_1-1}^{p-h} F_h \ , \tag{17}$$

we get that

$$B(A/D) = BC^2D \sum_{h=2}^{p} \binom{p}{h} F_{m_1}^{h-2} F_{m_1-1}^{p-h} F_h + pF_{m_1-1}^{p-1}$$

Also, $(F_{m_1} - 1, F_{m_1}) = 1$, so $B$ must divide $p$; that is, $B$ is $1$ or $p$; and again $D$ is $1$ or $p$. It follows that $F_{m_1}$, too, is a square or $p$ times a square. Arguing similarly, we see that, if $m = p^r m_r$, then $F_{m_r}$ is a square or $p$ times a square. This will continue until $(m_s, p) = 1$, and then, by (i) $1 \leq m_s \leq M$. But then, by (ii), if $p^t m_s = m_{s-t}$ is the least such number greater than $M$, $s \geq t$, and $F_{m_{s-t}}$ cannot be a square or $p$ times a square. This contradiction shows the correctness of the theorem.

Conjecture 3. There is no odd integer $m > 12$, such that $F_m$ is a square or twice a square.

Theorem C. Conjecture 3 implies Conjecture 1.

Proof. Conjecture 3 states condition (i) of Theorem B, when $p = 2$ and $M = 12$. The only $F_m$, with $1 \leq m \leq 12$, which are squares or twice squares are $F_1 = F_2 = 1$, $F_3 = 2$, $F_6 = 8$, and $F_{12} = 144$. However, the corresponding $F_k$ are $F_{16} = 3 \cdot 7 \cdot 47$ and $F_{24} = 2^5 \cdot 3^2 \cdot 7 \cdot 23$, and neither is a square or twice a square. Thus (ii) holds also, whence the conclusion of Theorem B, which includes Conjecture 1, is established.

## 4. PYTHAGOREAN RELATIONS

We close this paper by a rather closer examination of Conjecture 3, using the identities (12) and (13), with the well-known result, that the relation

$$x^2 + y^2 = z^2 \tag{18}$$

holds between integers if and only if there are integers s and t, mutually prime and of different parities, and an integer u, such that

$$x = (s^2 - t^2)u, \quad y = 2stu, \quad \text{and} \quad z = (s^2 + t^2)u \; . \tag{19}$$

Conjecture 3 leads us to examine the properties of Fibonacci numbers $F_m$, which are squares or twice squares, for odd integers m. We obtain the following rather remarkable results.

Theorem D. If m is odd, $F_m$ is a square if and only if there are integers r, s, and t, such that $m = 12r \pm 1$, $s > t \geq 0$, s is odd, t is even, $(s, t) = 1$, and

$$F_{6r} = 2st, \qquad F_{6r\pm1} = s^2 - t^2 \; . \tag{20}$$

Proof. Since m is odd, put $m = 4n \pm 1$, determining n uniquely. Then, by (12),

$$F_m = F_{4n\pm1} = F_{2n}^2 + F_{2n\pm1}^2 \; . \tag{21}$$

Thus $F_m$ is a square if and only if $F_{2n}$, $F_{2n\pm1}$, and $\sqrt{F_m}$ form a Pythagorean triplet. Since $(F_{2n}, F_{2n\pm1}) = 1$, $u = 1$, and this pair is $(s^2 - t^2)$ and 2st, while $F_{4n\pm1} = (s^2 \pm t^2)^2$. This gives that s and t are mutually prime and of different parities, with $s > t \geq 0$. By (12), $F_{2n\pm1} = F_n^2 + F_{n\pm1}^2$. Since $(F_n, F_{n\pm1}) = 1$, not both numbers are even, whence $F_{2n\pm1}$ is either odd or the sum of two odd squares, which must be of the form $8k + 2$. Since 2st is divisible by 4, it follows that

$$F_{2n} = 2st, \qquad F_{2n\pm1} = s^2 - t^2 \; . \tag{22}$$

Also, by (13), $2st = F_n(F_{n-1} + F_{n+1}) = F_n(2F_{n-1} + F_n)$. Since this must be divisible by 4, and $(F_n, 2F_{n-1} + F_n) = (F_n, 2)$, $F_n$ must be even, so that $n = 3r$ (since $F_3 = 2$); whence $m = 12r \pm 1$, as stated in the theorem, and (22) becomes (20). Finally, $s^2 - t^2 = F_n^2 + F_{n-1}^2$ is of the form $4k + 1$, being the sum of an odd and an even square. Thus $s$ must be odd and $t$ even, as was asserted.

Since Conjecture 1 is valid, it follows from Theorem D that, if $r \geq 2$, the equations (20) are not satisfied by any integers $r$, $s$, and 6.

<u>Theorem E.</u> If $m$ is odd, $F_m$ is twice a square if and only if there are integers $r$, $s$, and $t$, such that $m = 12r \pm 3$, $s \geq t > 0$, $s$ and $t$ are both odd, $(s, t) = 1$, and

$$F_{6r} = s^2 - t^2,\ F_{6r\pm3} = 2\,st\ . \tag{23}$$

<u>Proof.</u> We proceed much as for Theorem D. Let $m = 4n \pm 1$. Then, by (21), $F_{2n}$ and $F_{2n\pm1}$ must both be odd (since they cannot both be even), so that $F_{2n\pm1}$ is even (since one out of every consecutive triplet of Fibonacci numbers, one is even, and its index is a multiple of 3). Thus $2n \pm 1 = 6r \pm 3$, whence $m = 12r \pm 3$, as stated in the theorem. It is easily verified that, since $2n = 6r \pm 2$ and $2n \pm 1 = 6r \pm 1$, and

$$\begin{aligned} F_{6r+2} + F_{6r+1} &= F_{6r+3}, \quad & F_{6r+2} - F_{6r+1} &= F_{6r}\ , \\ F_{6r-2} + F_{6r-1} &= F_{6r}, \quad & F_{6r-2} - F_{6r-1} &= -F_{6r-3}\ . \end{aligned} \tag{24}$$

equation (21) yields that

$$\begin{aligned} 2F_{12r\pm3} &= (F_{6r\pm2} + F_{6r\pm1})^2 + (F_{6r\pm2} - F_{6r\pm1})^2 \\ &= F_{6r}^2 + F_{6r\pm3}^2\ . \end{aligned} \tag{25}$$

Thus $F_m$ is twice a square if and only if $F_{6r}$, $F_{6r\pm3}$, and $\sqrt{2F_{12r\pm3}}$ form a Pythagorean triplet. Clearly, since $F_3 = 2$ and $F_6 = 8$, $F_{6r}$ is divisible by 8, but $F_{12r\pm3}$ and $F_{6r\pm3}$ are divisible by 2, but not by 4. Thus $u = 2$ and $F_{6r}$ and $F_{6r\pm3}$ are of the forms $2(S^2 - T^2)$ and $4ST$, where $S > T \geq 0$, $(S, T) = 1$, and $S$ and $T$ are of opposite parities. In fact,

(26) $$F_{6r} = 4ST \text{ and } F_{6r\pm 3} = 2(S^2 - T^2) ,$$

since 4ST is clearly divisible by 8. Put $S + T = s$ and $S - T = t$; then (23) holds, and clearly $s \geq t \geq 0$, $(s,t) = 1$, and s and t are both odd, as stated in the theorem.

We finally note that Conjecture 3 holds if, for $r \geq 2$, the equations (23) are not satisfied by any integers r, s, and t.

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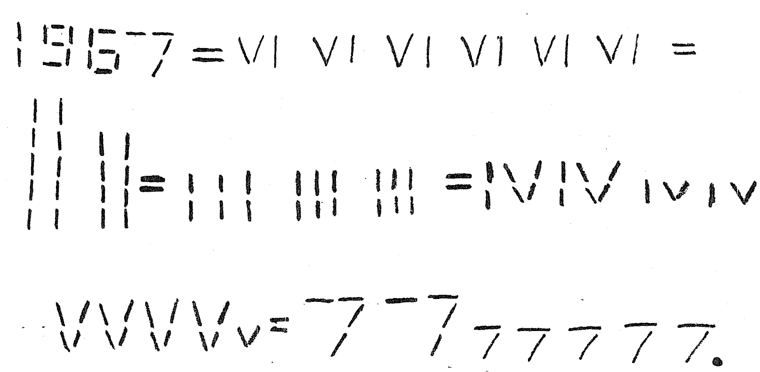
# PICKING AWAY AT 1967

CHARLES W. TRIGG

San Diego, California

1967 can be made from two 2's, three 3's, four 4's, five 5's, six 6's, or seven 7's with the aid of eighteen toothpicks and mixed Arabic and Roman number symbolism. That is,



# A GENERATING FUNCTION ASSOCIATED WITH THE GENERALIZED STIRLING NUMBERS

ROBERT FRAY
Florida State University, Tallahassee, Florida

## 1. INTRODUCTION

E. T. Bell [2] has defined a set of generalized Stirling numbers of the second kind $S_k(n,r)$; the numbers $S_1(n,r)$ are the ordinary Stirling numbers of the second kind. Letting $\lambda(n)$ denote the number of odd $S_1(n+1, 2r+1)$ Carlitz [3] has shown that

$$\sum_{n=0}^{\infty} \lambda(n)x^n = \prod_{n=0}^{\infty} (1 + x^{2^n} + x^{2^{n+1}}) .$$

In Section 3, we shall determine the generating function for the number of odd generalized Stirling numbers $S_2(n,r)$. Indeed we shall prove the following theorem.

Theorem. Let $\omega(n)$ denote the number of odd generalized Stirling numbers $S_2(n + r, 4r)$; then

$$\sum_{n=0}^{\infty} \omega(n)x^n = \prod_{n=0}^{\infty} (1 + x^{3 \cdot 2^n} + x^{2^{n+2}}) .$$

Later Carlitz [4] obtained the generating function for the number of $S_1(n, r)$ that are relatively prime to $p$ for any given prime $p$. It would be of interest to obtain such a generating function for the generalized Stirling numbers $S_k(n, r)$. At present the apparent difficulty with the method used herein is that, except for the case $k = 2$ and $p = 2$, the basic recurrence (2.4) for $S_k(n, r)$ with $k > 1$ is a recurrence of more than three terms, whereas for the cases that have been solved we had a three-term recurrence. In Section 4, we shall discuss this problem for the numbers $S_2(n, r)$ and the prime $p = 3$; several congruences will also be obtained for this case.

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## 2. PRELIMINARIES

The numbers $S_k(n, r)$ may be defined by introducing an operator $\tau$ which transforms $t^n$ into $(e^t - 1)^n$. Powers of $\tau$ are defined recursively as follows:

$$\tau^u t^n = \tau(\tau^{u-1} t^n) , \tag{2.1}$$

where $u$ is a positive integer. We shall also define $\tau^0 t^n = t^n$. The generalized Stirling numbers are then defined by

$$\tau^k t^r = r! \sum_{n=0}^{\infty} S_k(n, r) \frac{t^n}{n!} . \tag{2.2}$$

Hence $S_1(n, r)$ is the ordinary Stirling number of the second kind (see [5, pp. 42-43]) and $S_0(n, r) = \delta(n, r)$, the Kronecker delta. From (2.1) and (2.2) we can readily see [2, p. 93] that

$$S_{j+k}(n, r) = \sum_{i=r}^{n} S_j(n, i) S_k(i, r) . \tag{2.3}$$

Hence the numbers $S_k(n, r)$ can be derived from the ordinary Stirling numbers of the second kind by repeated matrix multiplication (see [5, p. 34]).

Becker and Riordan [1] have studied some of the arithmetic properties of these numbers; in particular, they obtained for $S_k(n, r)$ the period modulo p, a prime. In the same paper they derived the following basic recurrence modulo p (equation (5.4)):

$$\begin{aligned} S_k(n + p^s, r) \equiv & \sum_{j=0}^{k-1} \sum_{i} \binom{s+j-1}{j} S_j(n, i) S_{k-j}(i+1, r) \\ & + \sum_{j=1}^{s} \binom{s+k-1-j}{k-1} S_k(n, r-p^j) \pmod{p} . \end{aligned} \tag{2.4}$$

## 3. PROOF OF THEOREM

For $p = 2$ we have from (2.4) that

$$S_2(n+4, r) \equiv S_2(n+1, r) + S_2(n, r-4) \pmod 2$$

Hence if we let

$$S_n(x) = \sum_{r=0}^{n} S_2(n, r)x^r , \tag{3.1}$$

it follows that

$$S_{n+4}(x) + S_{n+1}(x) + x^4 S_n(x) \equiv 0 \pmod 2 . \tag{3.2}$$

Let $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\alpha_4$ be the roots of the equation

$$y^4 + y + x^4 = 0$$

in $F[y]$, where $F = GF(2, x)$, the function field obtained by adjoining the indeterminate $x$ to the finite field $GF(2)$. Also let

$$\phi_n(x) = \sum_{j=1}^{4} \alpha_j^n . \tag{3.3}$$

Then from the definition of the $\alpha$'s we see that

$$\phi_0(x) = \phi_1(x) = \phi_2(x) = \phi_4(x) = 0, \ \phi_3(x) = 1 .$$

Moreover

$$\phi_{n+4}(x) = \phi_{n+1}(x) + x^4\phi_n(x) ; \tag{3.4}$$

hence

$$\phi_5(x) = 0, \quad \phi_6(x) = 1 .$$

Now put

$$\overline{S}_n(x) = (x^3 + x + 1)\phi_n(x) + x^2\phi_{n+1}(x) + x\phi_{n+2}(x) + \phi_{n+3}(x) . \tag{3.5}$$

Then

$$\overline{S}_0(x) = 1 \qquad \overline{S}_2(x) = x^2$$
$$\overline{S}_1(x) = x \qquad \overline{S}_3(x) = x^3 + x + 2 .$$

Referring to the table at the end of the paper we see that by (3.1)

$$\overline{S}_n(x) \equiv S_n(x) \pmod 2$$

for $n = 0, 1, 2,$ and $3$. Therefore we see from (3.2), (3.4), and (3.5) that

$$\overline{S}_n(x) \equiv S_n(x) \pmod 2 \tag{3.6}$$

for all non-negative integers $n$.

From (3.3) we have with a little calculation that

$$\sum_{n=0}^{\infty} \phi_n(x)t^n = \sum_{j=1}^{4} \frac{1}{1 - \alpha_j t}$$
$$= \frac{t^3}{1 + t^3 + x^4t^4}$$
$$= \sum_{n=0}^{\infty} t^n \sum_{3k+j+3=n} \binom{k}{j} x^{4j} ;$$

therefore

$$\phi_n(x) = \sum_k \binom{k}{n - 3k - 3} x^{4(n-3k-3)} . \tag{3.7}$$

Combining (3.1), (3.5), (3.6) and (3.7) we have

$$\begin{aligned}\sum_{r=0}^{n} S_2(n,r)x^r \equiv \sum_k \binom{k-1}{n-3k-1} x^{4(n-3k)} \\ + x\left\{\sum_k \binom{k}{n-3k-3} x^{4(n-3k-3)} + \sum_k \binom{k}{n-3k-1} x^{4(n-3k-1)}\right\} \\ + x^2 \sum_k \binom{k}{n-3k-2} x^{4(n-3k-2)} + x^3 \sum_k \binom{k}{n-3k-3} x^{4(n-3k-3)} \\ \pmod 2\end{aligned}$$

Comparing coefficients we see that

$$(3.8)\quad \begin{cases} S_2(n,4j) \equiv \binom{r}{j-1} & (j = n-3r-3) \\ S_2(n,4j+1) \equiv \binom{r}{j} & (j = n-3r-3 \text{ or } n-3r-1) \\ S_2(n,4j+2) \equiv \binom{r}{j} & (j = n-3r-2) \\ S_2(n,4j+3) \equiv \binom{r}{j} & (j = n-3r-3)\quad, \end{cases}$$

where the modulus 2 is understood in each congruence.

Let $\theta_j(n)$ denote the number of odd $S_2(n,k)$, $0 \le k \le n$, with

$$k \equiv j \pmod 4 \quad (j = 0,1,2,3)\ .$$

By the first congruence in (3.8) we see that

$$S_2(n+1,\ 4j+4) \equiv \binom{r}{j} \pmod 2 \quad (j = n-3r-3)\ ,$$

and hence

$$(3.9)\qquad \theta_0(n+1) = \theta_3(n)\ .$$

Similarly since

$$S_2(n+2,\ 4j+4) \equiv \binom{r}{j} \pmod 2 \quad (j = n-3r-2)\ ,$$

it follows that

$$\theta_0(n+2) = \theta_2(n) \tag{3.10}$$

In a like manner we obtain

$$\begin{aligned}\theta_1(n) &= \theta_3(n) + \theta_2(n+1) \\ &= \theta_0(n+1) + \theta_0(n+3) \ ;\end{aligned}$$

the second equation follows from (3.9) and (3.10). Since all $\theta_j(n)$ may be expressed in terms of $\theta_0(n)$ it will suffice to determine the generating function for $\theta_0(n)$ alone.

Now by (3.8)

$$S_2(2n,\ 4j) \equiv \binom{r}{j-1} \pmod 2 \qquad (j = 2n - 3r - 3) \ .$$

From this it follows that

$$S_2(2n,\ 4j) \equiv 0 \pmod 2$$

unless

$$j \equiv r + 1 \pmod 2 \ .$$

Hence if we let

$$r = 2r' + s,\ \ j - 1 = 2j' + s \quad (s = 0, 1) \ ,$$

then

$$S_2(2n,\ 4j) \equiv \binom{r'}{j'} \pmod 2 \quad (j' = n - 3r' - 2s - 2) \ ,$$

and therefore

(3.11)
$$\theta_0(2n) = \theta_2(n) + \theta_3(n-1)$$
$$= \theta_0(n+2) + \theta_0(n) .$$

Similarly, since

$$S_2(2n+1), 4j) \equiv \binom{r}{j-1} \pmod 2 \quad (j = 2n - 3r - 2) ,$$

we have

$$S_2(2n+1, 4j) \equiv 0 \pmod 2$$

unless

$$r \equiv j \equiv 1 \pmod 2 .$$

Letting

$$r = 2r' + 1, \quad j = 2j' + 1$$

we get

$$S_2(2n+1, 4j) \equiv \binom{r'}{j'} \pmod 2 \quad (j' = n - 3r' - 3) .$$

Therefore

(3.12)
$$\theta_0(2n+1) = \theta_3(n) = \theta_0(n+1) .$$

If we let

$$\omega(n) = \theta_0(n+4)$$

we obtain from (3.11) and (3.12) that

$$\omega(2n) = \omega(n) + \omega(n-2)$$

and

$$\omega(2n + 1) = \omega(n - 1) .$$

Since $\theta_0(1) = \theta_0(2) = \theta_0(3) = 0$, we have $\omega(n) = 0$ for $n < 0$, and these equations for $\omega(n)$ are valid for all $n = 0, 1, 2, \cdots$. Hence we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \omega(n)x^n &= \sum_{n=0}^{\infty} \omega(2n)x^{2n} + \sum_{n=0}^{\infty} \omega(2n + 1)x^{2n+1} \\
&= \sum_{n=0}^{\infty} \omega(n)x^{2n} + \sum_{n=0}^{\infty} \omega(n - 2)x^{2n} + \sum_{n=0}^{\infty} \omega(n - 1)x^{2n+1} \\
&= (1 + x^3 + x^4) \sum_{n=0}^{\infty} \omega(n)x^{2n} \\
&= \prod_{n=0}^{\infty} (1 + x^{3 \cdot 2^n} + x^{2^{n+2}}) ,
\end{aligned}$$

and the theorem is proved.

From this generating function we see that $\omega(n)$ also denotes the number of partitions

$$n = n_0 + n_1 \cdot 2 + n_2 \cdot 2^2 + n_3 \cdot 2^3 + \cdots \quad (n_j = 0, 3, 4) .$$

## 4. THE CASE $p = 3$

We shall now consider the above problem for the prime $p = 3$. Since the work is similar to that of Section 3, many of the details will be omitted.

From (2.4) we have

$$S_2(n + 9, j) \equiv 2S_2(n + 3, j) + 2S_2(n + 1, j) + S_2(n, j - 9) \pmod 3 . \tag{4.1}$$

Therefore letting

(4.2) $$S_n(x) = \sum_{j=0}^{n} S_2(n, j)x^j ,$$

we have

(4.3) $$S_{n+9}(x) \equiv 2S_{n+3}(x) + 2S_{n+1}(x) + x^9 S_n(x) \pmod 3 .$$

Let $\alpha_1, \alpha_2, \cdots, \alpha_9$ be the roots of the equation

$$y^9 + y^3 + y - x^9 = 0$$

in $F[y]$, where $F = GF(3, x)$. Then if

$$\phi_n(x) = \sum_{j=1}^{9} \alpha_j^n ,$$

we see that

(4.4) $$\phi_0(x) = \phi_1(x) = \cdots = \phi_7(x) = 0, \ \phi_8(x) = 1 .$$

Moreover

(4.5) $$\phi_{n+9}(x) = x^9\phi_n(x) - \phi_{n+1}(x) - \phi_{n+3}(x) ,$$

and hence

(4.6) $$\phi_9(x) = \phi_{10}(x) = \cdots = \phi_{13}(x) = \phi_{15}(x) = 0, \ \phi_{14}(x) = \phi_{16}(x) = -1.$$

If we let

(4.7) $$\begin{cases} f_0(x) = S_0(x) + S_2(x) + S_8(x) \\ f_1(x) = S_1(x) + S_7(x) \\ f_2(x) = S_0(x) + S_6(x) \\ f_j(x) = S_{8-j}(x) \qquad (j = 3, 4, \cdots, 8) \end{cases}$$

and

$$(4.8) \qquad \overline{S}_n(x) = \sum_{j=0}^{8} f_j(x)\,\phi_{n+j}(x) \quad ,$$

it is clear from (4.3), (4.4), $\cdots$, (4.8) that

$$(4.9) \qquad \overline{S}_n(x) \equiv S_n(x) \pmod 3 \quad (n = 0, 1, 2, \cdots) .$$

As in Section 3 we see that

$$\sum_{n=0}^{\infty} \phi_n(x)t^n = \sum_{n=0}^{\infty} t^n \sum_{6k+8+r=n} (-1)^k \sum_{2j+h=r} \binom{k}{j}\binom{j}{h}(-1)^h x^{9h}$$

and hence

$$(4.10) \qquad \phi_n(x) = \sum_k (-1)^{n+k} \sum_j \binom{k}{j}\binom{j}{n-6k-8-2j} x^{9(n-8-6k-2j)} .$$

By expanding (4.8), comparing coefficients and combining terms we have, for instance, from (4.2), (4.9), and (4.10) that

$$S_2(n+9,\ 9h+9) \equiv \sum_{j,k} (-1)^{n+k} \binom{k}{j}\binom{j}{h} \pmod 3$$

and

$$S_2(n+8,\ 9h+8) \equiv \sum_{j,k} (-1)^{n+k} \binom{k}{j}\binom{j}{h} \pmod 3 \quad ,$$

but

$$S_2(n+8,\ 9h+6) \equiv \sum_{j,k} (-1)^{n+k} \left\{ \binom{k}{j}\binom{j}{h} + \binom{k}{j+1}\binom{j+1}{h} \right\} \pmod 3 \quad ,$$

where the summations are over all nonnegative integers j and k such that $h = n - 6k - 2j$. The numbers $S_2(n,\ 9h + j)$ for $j = 0, 1, \cdots, 5$ are more complicated.

At this point the method employed in Section 3 seems to fail. As was mentioned in Section 1, the apparent difficulty in this case is the fact that the recurrence (4.1) is a four-term recurrence. If we consider the generalized Stirling number $S_3(n, r)$ and the prime $p = 2$ we again get a four-term recurrence; the development of the problem in this case is very similar to our work in the present section.

TABLE

Generalized Stirling Numbers of the Second Kind $S_2(n, r)$

| n \ r | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | |
| 2 | 2 | 1 | | | | | | |
| 3 | 5 | 6 | 1 | | | | | |
| 4 | 15 | 32 | 12 | 1 | | | | |
| 5 | 52 | 175 | 110 | 20 | 1 | | | |
| 6 | 203 | 1012 | 945 | 280 | 30 | 1 | | |
| 7 | 877 | 6230 | 8092 | 3465 | 595 | 42 | 1 | |
| 8 | 4140 | 40819 | 70756 | 40992 | 10010 | 1120 | 56 | 1 |

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# IDENTITIES FOR PRODUCTS OF FIBONACCI AND LUCAS NUMBERS

D. E. DAYKIN and L. A. G. DRESEL
University of Reading, Reading, England

The Fibonacci numbers $F_n$ and Lucas numbers $L_n$ may be defined by

(1) $$F_n = (\alpha^n - \beta^n)/(\alpha - \beta) \quad \text{and} \quad L_n = \alpha^n + \beta^n$$

where n is any integer, $\alpha = \frac{1}{2}(1 + \sqrt{5})$ and $\beta = \frac{1}{2}(1 - \sqrt{5})$, so that

(2) $$\alpha - \beta = \sqrt{5} \quad \text{and} \quad \alpha\beta = -1\,.$$

Recently Brother Alfred Brousseau asked for generalizations of the identity

$$F_{2k}F_{2\ell} = F^2_{k+\ell} - F^2_{k-\ell}$$

proved by I. D. Ruggles in [1], and as a result V. E. Hoggatt, D. Lind, C. R. Wall [2], and Sheryl B. Tadlock [3] between them gave seven further identities. In this note we point out that these eight identities belong to the family of sixteen identities given in Theorem 1 below. Furthermore, we show that this theorem can be proved by a very simple method which can be used to generate identities for arbitrary products of Fibonacci and Lucas numbers.

Theorem 1. If i, j, s, and t are integers such that $i + j = 2s$ and $i - j = 2t$, then

(3) $$(5F_iF_j \text{ or } L_iL_j) = (5F^2_s \text{ or } L^2_s) \pm (5F^2_t \text{ or } L^2_t) \pm (0 \text{ or } 4(-1)^t)\,.$$

where either term may be chosen at will in the first three brackets; in addition i, j may be chosen as being either both even or both odd. The choice of term in the last bracket and the sign preceding it depend on the combination of the previously mentioned four choices, but the choice of the first ± sign depends only on the parity of i and on the term chosen from the first bracket. If we fix the two choices to be made on the left side of the identity, then the four identities obtained by varying the choices made on the right side are deducible from each other by application of the well-known identity

$$5F_n^2 = L_n^2 - 4(-1)^n .$$

This fact was used by Sheryl B. Tadlock in [3].

To obtain further identities such as those of Theorem 1, we consider an arbitrary product of Fibonacci and Lucas numbers. In other words we let $m, n, i_0, i_1, \cdots, j_1, j_2, \cdots$ be any integers with $m, n \geq 0$ and put

$$P = 5^m F_{i_1} F_{i_2} \cdots F_{i_{2m}} L_{j_1} L_{j_2} \cdots L_{j_n} , \tag{4}$$

$$Q = 5^m F_{i_0} F_{i_1} \cdots F_{i_{2m}} L_{j_1} L_{j_2} \cdots L_{j_n} . \tag{5}$$

Notice that $P$ contains an even and $Q$ an odd number of Fibonacci numbers $F_i$.

First we discuss $P$. By (2) we have $5^m = (\alpha - \beta)^{2m}$, so substituting from (1) in (4) and expanding we see that $P$ is symmetric in $\alpha, \beta$ and is therefore a sum of $2^{2m+n-1}$ terms of the form $\pm(\alpha^p \beta^q + \alpha^q \beta^p)$. But by (1), (2) we have

$$\alpha^p \beta^q + \alpha^q \beta^p = (\alpha\beta)^q (\alpha^{p-q} + \beta^{p-q}) = (-1)^q L_{p-q} . \tag{6}$$

Hence $P$ can be expressed as the difference of two sums of Lucas numbers.

Now suppose that the sum of the subscripts occurring in (4) is even, so that we have

$$i_1 + i_2 + \cdots + i_{2m} + j_1 + j_2 + \cdots + j_n = 2s \tag{7}$$

for some integer $s$. Then for each of the terms $\pm(\alpha^p\beta^q + \alpha^q\beta^p)$ in the expansion of (4) we have $p + q = 2s$, and so $p - q$ is also even. Putting $p - q = 2t$ in (6) and noting that

$$\begin{aligned} \alpha^{p-q} + \beta^{p-q} = \alpha^{2t} + \beta^{2t} &= (\alpha^t - \beta^t)^2 + 2(\alpha\beta)^t = 5F_t^2 + 2(-1)^t \\ &= (\alpha^t + \beta^t)^2 - 2(\alpha\beta)^t = L_t^2 - 2(-1)^t , \end{aligned}$$

we see that our general term is of the form

$$\begin{aligned}\pm(\alpha^p\beta^q + \alpha^q\beta^p) &= \pm(-1)^q L_{2t} = \pm(-1)^q[5F_t^2 + 2(-1)^t] \\ &= \pm(-1)^q[L_t^2 - 2(-1)^t].\end{aligned}$$

Thus we have shown that the product P given by (4) can be expressed, in a large number of ways, as a sum of terms $\pm(5F_t^2$ or $L_t^2)$ and terms $\pm 2$. As an example we give the identity

$$\begin{aligned}5F_{2i}F_{2j}L_{2k}L_{2h} &= 5F^2_{i+j+k+h} + L^2_{i+j+k-h} + 5F^2_{i+j-k+h} + L^2_{i+j-k-h} \\ &\quad -5F^2_{i-j+k+h} - 5F^2_{i-j+k-h} - L^2_{i-j-k+h} - L^2_{i-j-k-h}\,.\end{aligned}$$

The right-hand side of this identity is of course only one of the $2^8$ possible expressions of this form, though many of these would involve a further term 4C, where C is an integer in the range $-4 \le C \le 4$.

Finally we discuss the product Q given by (5). Substituting from (1) and expanding we see that Q is a sum of $2^{2m+n}$ terms of the form

$$\pm(\alpha^p\beta^q - \alpha^q\beta^p)/(\alpha - \beta) = \pm(-1)^q F_{p-q}\ ,$$

so that Q can be expressed in terms of Fibonacci numbers $F_u$. There is no immediate analogue for Q to the results obtained by (7) for P. However, if each of $i_0, i_1, \cdots, j_1, j_2, \cdots$ is divisible by 3, then in our general term we have $p - q = 3r$ and

$$F_{p-q} = (\alpha^{3r} - \beta^{3r})/(\alpha - \beta) = [5F_r^3 + 3(-1)^r F_r].$$

In this way one can obtain Q as a sum of terms $\pm[5F_r^3 + 3(-1)^r F_r]$, for example (after some re-arranging of terms) we have

$$\begin{aligned}F_{3i}L_{3j}L_{3k} &= 5[F^3_{i+j+k} + (-1)^k F^3_{i+j-k} + (-1)^j F^3_{i-j+k} - (-1)^i F^3_{-i+j+k}] + \\ &\quad + 3(-1)^{i+j+k}[F_{i+j+k} + (-1)^k F_{i+j-k} + (-1)^j F_{i-j+k} - (-1)^i F_{-i+j+k}].\end{aligned}$$

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3. Sheryl B. Tadlock, "Products of Odds," The Fibonacci Quarterly, Vol. 3, No. 1 (1965), pp. 54-56.

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All subscription correspondence should be addressed to Brother Alfred Brousseau, St. Mary's College, Calif. All checks ($1.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Manuscripts intended for publication in the Quarterly should be sent to V. E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, Calif. All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscript sent to the editors.

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To the Editor:

The lemma proven by M. Bicknell and V. E. Hoggatt, Jr. in 'Fibonacci Matrices and Lambda Functions', The Fibonacci Quarterly, Vol. 1 (April 1963), page 49 was essentially established in my note 'Theorem on Determinants', Mathematics Magazine Vol. 34 (September 1961), page 328. Namely, 'If the difference of each pair of corresponding elements of any two columns (rows) of a determinant are equal, then any quantity may be added to each element of the determinant without changing its value.'

Charles W. Trigg

# FIBONACCI FUNCTIONS

MERRITT ELMORE
San Jose City College, San Jose, Calif.

## 1. INTRODUCTION

There is a sequence of continuous functions of one variable having many of the properties of the Fibonacci sequence of numbers, with some intriguing variations. Derivatives and integrals of these functions are easily found, and lead to more relations involving Fibonacci numbers. Other topics of calculus can undoubtedly be applied to these functions with very useful and interesting results.

Let $a_0, a_1, a_2, \cdots$ be a sequence such that $a_{m+1} = a_m + a_{m-1}$. Then the power series

$$y = \sum_{i=0}^{\infty} \frac{a_i x^i}{i!}$$

satisfies the differential equation

$$y'' - y' - y = 0 \tag{1}$$

whose solution is $c_1 e^{\alpha x} + C_2 e^{\beta x}$, where $\alpha$ and $\beta$ are the roots of $u^2 - u - 1 = 0$,

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2} \quad .$$

If the sequence $\{a_m\}$ is the Fibonacci sequence $\{F_m\}$, then $a_0 = 0$ and $a_1 = 1$, so that we get the boundary conditions $x = 0$, $y = 0$, $y' = 1$. This yields (see [1])

$$y = \frac{e^{\alpha x} - e^{\beta x}}{\sqrt{5}} \quad \text{and} \quad F_m = \frac{\alpha^m - \beta^m}{\sqrt{5}} \tag{2}$$

On the other hand, if the sequence $\{a_m\}$ is the Lucas sequence $\{L_m\}$, then $a_0 = 2$ and $a_1 = 1$, so that we get the boundary conditions $x = 0$, $y = 2$, $y' = 1$, yielding

(3) $$y = e^{\alpha x} + e^{\beta x} \quad \text{and} \quad L_m = \alpha^m + \beta^m$$

The writing of (1) in the form $y'' = y' + y$ is very suggestive: the sum of the function and its first derivative is the second derivative. And generally, if $y$ is any solution of (1), we see that

(1a) $$y^{(m+1)} = y^{(m)} + y^{(m-1)}$$

This suggests that we use the notation

$$f_0(x) = \frac{e^{\alpha x} - e^{\beta x}}{\sqrt{5}}, \quad f_1(x) = f_0'(x), \quad f_2(x) = f_0''(x), \quad f_3(x) = f_0^{(3)}(x),$$

and so forth. Thus

(4) $$f_m(x) = f_0^{(m)}(x) = \frac{\alpha^m e^{\alpha x} - \beta^m e^{\beta x}}{\sqrt{5}}$$

giving us the sequence of functions $\{f_m(x)\}$ with the property that

(5) $$f_{m+1}(x) = f_m(x) + f_{m-1}(x)$$

We shall refer to these functions as Fibonacci functions.

Likewise if $l_0(x) = e^{\alpha x} + e^{\beta x}$, $l_1(x) = l_0'(x)$, $l_2(x) = l_0''(x)$, etc., we have

(6) $$l_m(x) = l_0^{(m)}(x) = \alpha^m e^{\alpha x} + \beta^m e^{\beta x}$$

(7) $$l_{m+1}(x) = l_m(x) + l_{m-1}(x)$$

and these functions will be called Lucas functions here.

Evidently, $F_0 = f_0(0) = 0$, $F_1 = f_1(0) = 1$, $F_2 = f_2(0) = 1$, $F_3 = f_3(0) = 2, \cdots$,

$$F_m = f_m(0) = \frac{\alpha^m - \beta^m}{\sqrt{5}} \tag{8}$$

with similar results for Lucas numbers.

Let us define

$$f_{-k}(x) = \frac{\alpha^{-k}e^{\alpha x} - \beta^{-k}e^{\beta x}}{\sqrt{5}}, \qquad l_{-k}(x) = \alpha^{-k}e^{\alpha x} + \beta^{-k}e^{\beta x} \tag{9}$$

With the understanding that $f_0^{(-k)}(x)$ is the $k^{th}$ antiderivative of $f_0(x)$, and similarly for $l_0^{(-k)}(k)$, we can easily verify that all the preceding results (2) through (8) hold for $m$ a negative integer.

## 2. GRAPHS

Elementary notions of calculus regarding intercepts, slope, symmetry, extent, critical points, points of inflection, etc., may be used in plotting the graphs of these functions. Figure 1 shows the graphs of some of the Fibonacci functions $f_m(x)$.

Note first of all that the y-intercept of the curve $y = f_m(x)$ is $F_m$.

Observe also that the functions with even subscripts are monotonic increasing, and extend from $-\infty$ to $+\infty$ both horizontally and vertically. The functions with odd subscripts, however, are never negative (since $\beta < 0$), and each has one relative minimum.

In fact, $f_{2k-1}(x)$, where $k$ is any integer, has its relative minimum at the zero of $f_{2k}(x)$, which is also the $x$ at which $f_{2k-2}(x)$ has its point of inflection.

Let us therefore call these points $x_{2k}$. That is, $x_{2k}$ is such that

$$f_{2k}(x_{2k}) = 0 \tag{10}$$

Let the minima of $f_{2k-1}(x)$ be called $y_{2k}$. Thus

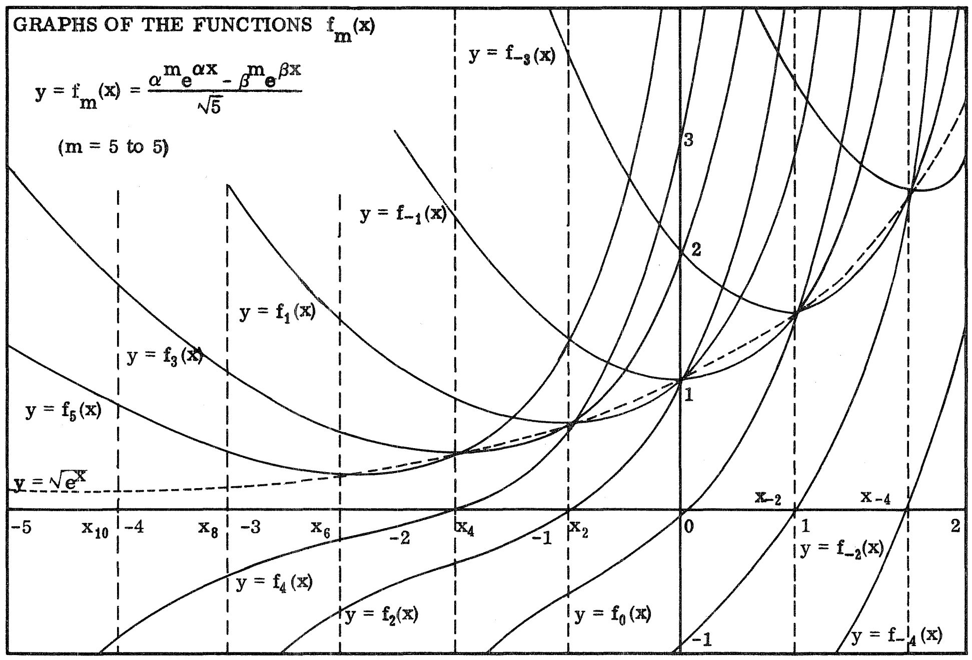


Fig. 1

(11) $$y_{2k} = f_{2k-1}(x_{2k})$$

Some manipulation and calculation result in

(12) $$x_{2k} = \frac{4k}{\sqrt{5}} \log \frac{1}{\alpha} \approx -0.86k$$

(13) $$y_{2k} = \left(\frac{1}{\alpha}\right)^{\frac{2k}{\sqrt{5}}} = e^{x_k} \approx (0.65)^k, \quad \text{where} \quad x_k = \frac{1}{2} x_{2k}$$

Thus the minima of $f_{2k-1}(x)$ occur at points evenly spaced along the negative x-axis and have values in geometric progression, which approach 0 as $x \to -\infty$.

Because

$$y_{2k} = e^{\frac{1}{2}x_{2k}}$$

at these minimum points, they lie on the graph of $y = \sqrt{e^x}$ (see the dotted-line curve in Fig. 1).

Since

$$f_{2k+1}(x_{2k}) = f_{2k}(x_{2k}) + f_{2k-1}(x_{2k}) = 0 + y_{2k} = y_{2k}$$

and since

$$f_{2k+2}(x_{2k}) = f_{2k+1}(x_{2k}) + f_{2k}(x_{2k}) = y_{2k} + 0 = y_{2k}$$

we see that the graphs of

$$f_{2k-1}(x), \quad f_{2k+1}(x), \quad \text{and} \quad f_{2k+2}(x)$$

all intersect at $(x_{2k}, y_{2k})$.

Likewise

$$f_{2k+3}(x_{2k}) = f_{2k+2}(x_{2k}) + f_{2k+1}(x_{2k}) = 2y_{2k} \ ,$$

etc.; and induction leads to

$$(14) \qquad f_{2k+j}(x_{2k}) = F_j y_{2k} = F_j e^{x_k}, \quad \text{or} \quad f_m(x_{2k}) = F_{m-2k} y_{2k} = F_{m-2k} e^{x_k}$$

which is a specialization of the more general relation to be derived in the next section.

## 3. AN IMPORTANT IDENTITY

That the identity

$$F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$$

for Fibonacci numbers has a counterpart for the Fibonacci functions can be investigated by substituting into its right side:

$f_{m-1}(x)f_n(x) + f_m(x)f_n(x) =$

$$\frac{\alpha^{m-1}e^{\alpha x} - \beta^{m-1}e^{\beta x}}{\sqrt{5}} \cdot \frac{\alpha^{n}e^{\alpha x} - \beta^{n}e^{\beta x}}{\sqrt{5}} + \frac{\alpha^{m}e^{\alpha x} - \beta^{m}e^{\beta x}}{\sqrt{5}} \cdot \frac{\alpha^{n+1}e^{\alpha x} - \beta^{n+1}e^{\beta x}}{\sqrt{5}}$$

Multiplying and simplifying using $\alpha\beta = -1$, the terms in $e^{\alpha x+\beta x}$ vanish, giving us

$$\frac{\alpha^{m+n-1}(1 + \alpha^2)e^{2\alpha x} + \beta^{m+n-1}(1 + \beta^2)e^{2\beta x}}{5}$$

whence

$$1 + \alpha^2 = \alpha\sqrt{5},\; 1 + \beta^2 = -\beta\sqrt{5}$$

lead to

$$f_{m-1}(x)f_n(x) + f_m(x)f_{n+1}(x) = f_{m+n}(2x)$$

We see then that the formula is the same except for the important change in the argument. We generalize this by repeating almost exactly the same steps, and obtain

$$f_{m+n}(u + v) = f_{m-1}(u)f_n(v) + f_m(u)f_{n+1}(v) \tag{15}$$

## 4. APPLICATION OF (15) TO GRAPHS

Using the identity (15) with $m = 2k$, $n = 0$, $u = x_{2k}$, and $v = t$, we obtain

$$f_{2k}(x_{2k} + t) = f_{2k-1}(x_{2k})f_0(t) + f_{2k}(x_{2k})\, f_1(t) = y_{2k}f_0(t) + 0 \cdot f_1(t)$$

$$f_{2k}(x_{2k} + t) = e^{x_k}f_0(t) \tag{16}$$

Similarly

$$f_{2k+1}(x_{2k} + t) = e^{x_k} f_1(t) \tag{17}$$

Hence each of the graphs can be obtained from the graph of either $y = f_0(x)$ or $y = f_1(x)$, according to whether $m$ is even or odd, by expanding it by the factor $e^{x_k}$ and translating it $-x_{2k}$ units to the left.

Since $f_0(x)$ and $f_1(x)$ in turn can be written as

$$f_0(x) = \frac{2e^{\frac{x}{2}}}{\sqrt{5}} \sinh \frac{\sqrt{5}}{2} x, \quad f_1(x) = \frac{2e^{\frac{x}{2}}}{\sqrt{5}} \cosh\left(\frac{\sqrt{5}}{2} x + \cosh^{-1}\left(\frac{\sqrt{5}}{2}\right)\right) \tag{18}$$

all of the graphs are distortions of hyperbolic sine or cosine curves through multiplication by $\sqrt{e^x}$.

## 5. INTEGRALS

From the definition of $f_m(x)$, the antiderivative of $f_m(x)$ is $f_{m-1}(x)$. This leads to a wealth of problems involving Fibonacci numbers, two of which follow. (See Fig. 2.)

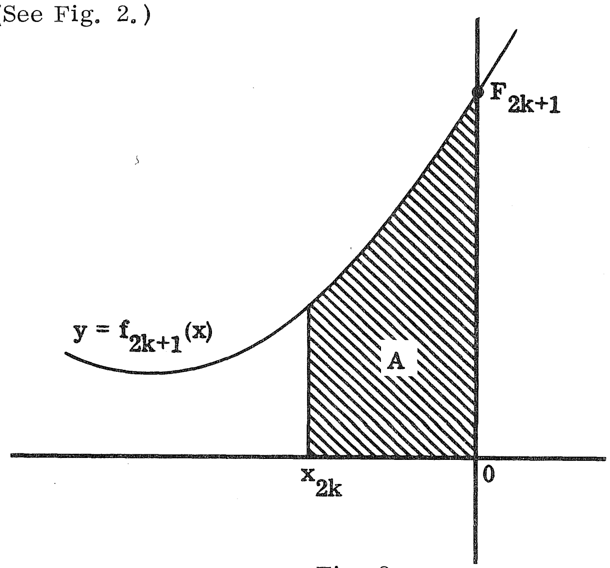


Fig. 2

Let A = the area under the curve $f_{2k+1}(x)$ between $x = x_{2k}$ and $x = 0$, and above the x-axis. Then

$$A = \int_{x_{2k}}^{0} f_{2k+1}(x)dx = f_{2k}(x)\Big]_{x_{2k}}^{0} = f_{2k}(0) - f_{2k}(x_{2k}) = F_{2k} - 0 = F_{2k}$$

More generally,

$$\int_{x_{2k}}^{x_{2n}} f_{m+1}(x)dx = F_{m-2k}e^{x_k} - F_{m-2n}e^{x_n} \tag{19}$$

Use of (15) and formulas for differentiation and integration lead to many others.

## 6. IDENTITIES

Many of the familiar identities for Fibonacci and Lucas numbers, besides (15), also have their counterparts for these Fibonacci and Lucas <u>functions</u>. Obtaining them is often merely a matter of substitution of

$$f_m(x) = \frac{\alpha^m e^{\alpha x} - \beta^m e^{\beta x}}{\sqrt{5}}, \quad l_m(x) = \alpha^m e^{\alpha x} + \beta^m e^{\beta x}$$

into one side of the identity, and the use of such relations as $\alpha + \beta = 1$, $\alpha\beta = -1$, $\alpha^2 + 1 = \alpha\sqrt{5}$, etc.

Thus, for example, one easily obtains

$$f_{m-1}(x)f_{m+1}(x) = f_m^2(x) + (-1)^m e^x \tag{20}$$

$$l_m(x) = f_{m-1}(x) + f_{m+1}(x) \tag{21}$$

$$5f_m^2(x) = l_m^2(x) + (-1)^{m-1}4e^x \tag{22}$$

$$f_{-m}(x) = (-1)^{m+1}e^x f_m(-x) \tag{23}$$

(24) $$\left(\frac{l_m(x) + \sqrt{5}f_m(x)}{2}\right)^k = \frac{l_{km}(kx) + \sqrt{5}f_{km}(kx)}{2}$$

Note that the corresponding identities for the Fibonacci and Lucas <u>numbers</u> emerge immediately when $x = 0$.

From the formula (15) already treated come such familiar-appearing identities as

(25) $$f_{2m-1}(2x) = f^2_{m-1}(x) + f^2_m(x) \quad \text{and} \quad f_{2m}(2x) = f_m(x)l_m(x)$$

(26) $$f_{3m-1}(3x) = f^3_{m-1}(x) + 3f_{m-1}(x)f^2_m(x) + f^3_m(x)\,, \qquad \text{and}$$

$$f_{3m}(3x) = 3f^2_{m-1}(x)f_m(x) + 3f_{m-1}(x)f^2_m(x) + 2f^3_m(x)$$

while a generalization by induction on $k$ and $p$ yields

(27) $$f_{km+p}(kx) = \sum_{i=0}^{k} \binom{k}{i} F_{i+p}f^{k-i}_{m-1}(x)f^i_m(x)$$

By using (15) to write

$$f_m(u)f_n(v) = (-1)\left[f_{m+1}(u)f_{n+1}(v) - f_{m+n+1}(u + v)\right]$$

and

$$(-1)f_{m+1}(u)f_{n+1}(v) = (-1)^2\left[f_{m+2}(u)f_{n+2}(v) - f_{m+n+3}(u + v)\right]$$

and adding, one obtains

$$f_m(u)f_n(v) = (-1)^2\left[f_{m+2}(u)f_{n+2}(v) - f_{m+n+2}(u + v)\right]$$

Repeating the process, and the use of induction lead to

(28) $$f_m(u)f_n(v) = (-1)^r\left[f_{m+r}(u)f_{n+r}(v) - F_rf_{m+n+r}(u + v)\right]$$

Multiplication of (28) by $f_m(u)$ gives

$$f_m^2(u)f_n(v) = (-1)^r\left\{f_{m+r}(u)\left[f_m(u)f_{n+r}(v)\right] - F_r\left[f_m(u)f_{m+n+r}(u+v)\right]\right\}$$

while the use of (28) to expand the expressions in the square brackets here yields

$$(-1)^{2r}\left[f_{m+r}^2(u)f_{n+2r}(v) - 2F_r f_{m+r}(u)f_{m+n+2r}(u+v) + F_r^2\ f_{2m+n+2r}(2u+v)\right]$$

whence induction leads to

$$f_m^k(u)f_n(v) = (-1)^{kr}\sum_{i=0}^{k}\binom{k}{i}(-1)^i F_r^i f_{m+r}^{k-i}(u)f_{n+kr+im}(iu+v) \tag{29}$$

These two formulas are counterparts of two given by Halton [2]. In exactly the same way as he did, (29) can be used to develop a host of identities by choosing particular values of $m$, $n$, $k$, and $r$.

It is interesting to note that

$$F_m f_m(v-u)e^u = (-1)^r\left[f_{m+r}(u)f_{n+r}(v) - f_r(u)f_{m+n+r}(v)\right] \tag{30}$$

is a "sibling" of (28), having been derived by substitution using (4), as a counterpart of the same formula

$$F_mF_n = (-1)^r\left[F_{m+r}F_{n+r} - F_rF_{m+n+r}\right].$$

One is intrigued by the conjecture that they are both special cases of a more general formula in which no capital $F$'s appear.

## 7. FIBONACCI FUNCTIONS OF TWO VARIABLES

Suppose $\{a_m\}$ is replaced by $\{f_m(x)\}$ in the series

$$\sum_{i=0}^{\infty} a_i \cdot \frac{y^i}{i!} \quad ,$$

to give a function of two variables $\phi(x, y)$.

$$\phi(x, y) = f_0(x) + f_1(x)y + f_2(x)\frac{y^2}{2!} + f_3(x)\frac{y^3}{3!} + \cdots \tag{31}$$

Differentiating term-by-term, we obtain

$$\frac{\partial\phi(x, y)}{\partial x} = f_1(x) + f_2(x)y + f_3(x)\frac{y^2}{2!} + f_4(x)\frac{y^3}{3!} + \cdots$$

$$\frac{\partial\phi(x, y)}{\partial y} = 0 + f_1(x) + f_2(x)\frac{2y}{2!} + f_3(x)\frac{3y^2}{3!} + f_4(x)\frac{4y^3}{4!} + \cdots$$

We see that

$$\frac{\partial\phi(x, y)}{\partial x} = \frac{\partial\phi(x, y)}{\partial y} \quad ,$$

and likewise it can be verified that all the second partial derivatives are the same, all the third partial derivatives are the same, etc. Let us therefore adopt the notation

$$\phi_0(x, y) = \phi(x, y), \quad \phi_1(x, y) = \frac{\partial\phi(x, y)}{\partial x} = \frac{\partial\phi(x, y)}{\partial y} \, ,$$

$$\phi_2(x, y) = \frac{\partial^2\phi(x, y)}{\partial x^2} = \frac{\partial^2\phi(x, y)}{\partial x \partial y} = \frac{\partial^2\phi(x, y)}{\partial y^2} \, , \; \cdots$$

so that

$$\phi_m(x, y) = \frac{\partial^m\phi(x, y)}{\partial x^r \partial y^s} = \sum_{i=0}^{\infty} f_{m+i}(x)\frac{y^i}{i!} = \sum_{i=0}^{\infty} f_{m+i}(y)\frac{x^i}{i!} \quad , \tag{32}$$

where r and s are positive integers such that $r + s = m$. Note that

(33) $$\phi_m(x, 0) = f_m(x), \quad \phi_m(0, y) = f_m(y), \quad \text{and} \quad \phi_m(0, 0) = F_m \quad .$$

Expand $\phi_m(x, y)$ into a power series in two variables at $(0, 0)$:

$$\phi_m(x, y) = \phi_m(0, 0) + \left[ x \frac{\partial \phi_m(0, 0)}{\partial x} + y \frac{\partial \phi_m(0, 0)}{\partial y} \right] + \frac{1}{2!}\left[ x^2 \frac{\partial^2 \phi_m(0, 0)}{\partial x^2} + 2xy \frac{\partial^2 \phi_m(0, 0)}{\partial x \partial y} + y^2 \frac{\partial^2 \phi_m(0, 0)}{\partial y^2} \right] + \cdots$$

$$= F_m + \left[ xF_{m+1} + yF_{m+1} \right] + \frac{1}{2!}\left[ x^2 F_{m+2} + 2xyF_{m+2} + y^2 F_{m+2} \right] + \cdots$$

$$= F_m + F_{m+1}\frac{(x + y)}{1!} + F_{m+2}\frac{(x + y)^2}{2!} + F_{m+3}\frac{(x + y)^3}{3!} + \cdots$$

Thus

(34) $$\phi_m(x, y) = f_m(x + y) = \frac{\alpha^m e^{\alpha(x+y)} - \beta^m e^{\beta(x+y)}}{\sqrt{5}}$$

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2. John H. Halton, "On a General Fibonacci Identity," Fibonacci Quarterly, Vol. 3, No. 1, February 1965, pp. 31-43.

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# FIBONACCI NUMBERS AND GENERALIZED BINOMIAL COEFFICIENTS

V. E. HOGGATT, JR.
San Jose State College, San Jose, California

DEDICATED TO THE MEMORY OF MARK FEINBERG

## 1. INTRODUCTION

The first time most students meet the binomial coefficients is in the expansion

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}y^j \quad , \quad n \geq 0 \quad ,$$

where

$$\binom{n}{m} = 0 \text{ for } m > n \, , \qquad \binom{n}{n} = \binom{n}{0} = 1, \text{ and}$$

(1) $$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}, \quad 0 < m < n$$

Consistent with the above definition is

(2) $$\binom{n}{m} = \frac{n(n-1)\cdots 2 \cdot 1}{m(m-1)\cdots 2 \cdot 1 \, (n-m)(n-m-1)\cdots 2 \cdot 1} = \frac{n!}{m!\,(n-m)!} \quad ,$$

where

$$n! = n(n-1)\cdots 2 \cdot 1 \text{ and } 0! = 1 \, .$$

Given the first lines of Pascal's arithmetic triangle one can extend the table to the next line by using directly definition (2) or the recurrence relation (1).

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We now can see just how the ordinary binomial coefficients $\binom{n}{m}$ are related to the sequence of integers $1, 2, 3, \cdots, k, \cdots$. Let us generalize this observation using the Fibonacci sequence.

## II. THE FIBONOMIAL COEFFICIENTS

Let the Fibonomial coefficients (which are a special case of the generalized binomial coefficients) be defined as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{F_n F_{n-1} \cdots F_2 F_1}{(F_m F_{m-1} \cdots F_2 F_1)(F_{m-n} F_{m-n-1} \cdots F_2 F_1)}, \quad 0 < m < n ,$$

and

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1 ,$$

where $F_n$ is the $n^{th}$ Fibonacci number, defined by

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = F_2 = 1 .$$

We next seek a convenient recurrence relation, like (1) for the ordinary binomial coefficients, to get the next line from the first few lines of the Fibonomial triangle, the generalization of which will come shortly.

To find two such recurrence relations we recall the Q-matrix,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} ,$$

for which it is easily established by mathematical induction that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \quad n \geq 0 .$$

The Laws of Exponents hold for the Q-matrix so that

$$Q^n = Q^m Q^{n-m} .$$

Thus

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n-m+1} & F_{n-m} \\ F_{n-m} & F_{n-m-1} \end{pmatrix}$$

$$= \begin{pmatrix} F_{m+1}F_{n-m+1} + F_mF_{n-m} & F_{m+1}F_{n-m} + F_mF_{n-m-1} \\ F_mF_{n-m+1} + F_{m-1}F_{n-m} & F_mF_{n-m} + F_{m-1}F_{n-m-1} \end{pmatrix}$$

yielding, upon equating corresponding elements,

(A) $$F_n = F_{m+1}F_{n-m} + F_mF_{n-m-1} \quad \text{(upper right)} ,$$

(B) $$F_n = F_mF_{n-m+1} + F_{m-1}F_{n-m} \quad \text{(lower left)} .$$

These two identities will be very handy in what follows.

Define C so that

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{F_nF_{n-1} \cdots F_2F_1}{(F_mF_{m-1} \cdots F_2F_1)(F_{n-m}F_{n-m-1} \cdots F_2F_1)} = F_nC .$$

With C defined above, then

$$\begin{bmatrix} n-1 \\ m \end{bmatrix} = F_{n-m}C \text{ and } \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = F_m C \; .$$

Returning now to identity (A),

$$F_n = F_{m+1}F_{n-m} + F_m F_{n-m-1} \, ,$$

we may write for $C \neq 0$ ,

$$F_n C = F_{m+1}(F_{n-m}C) + F_{n-m-1}(F_m C)$$

but since

$$\begin{bmatrix} n \\ m \end{bmatrix} = F_n C \; , \quad \begin{bmatrix} n-1 \\ m \end{bmatrix} = F_{n-m} C \; , \quad \text{and} \quad \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = F_m C \; ,$$

we have derived

$$\text{(D)} \qquad \begin{bmatrix} n \\ m \end{bmatrix} = F_{m+1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m-1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \; .$$

Similarly, using identity (B), one can establish

$$\text{(E)} \qquad \begin{bmatrix} n \\ m \end{bmatrix} = F_{m-1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m+1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \; .$$

It is thus now easy to establish by mathematical induction that if the Fibonomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}$ are integers for an integer n ($m = 0, 1, \cdots, n$), then they are integers for an integer $n + 1$ ($m = 0, 1, 2, \cdots, n + 1$).

Recalling

$$L_m = F_{m+1} + F_{m-1} ,$$

then adding (D) and (E) yields

$$(3) \qquad 2\begin{bmatrix} n \\ m \end{bmatrix} = L_m \begin{bmatrix} n-1 \\ m \end{bmatrix} + L_{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} ,$$

where $L_m$ is the $m^{th}$ Lucas number, a result given in problem H-5, Fibonacci Quarterly Journal, Feb., 1963, page 47. From (3) it is harder to show that $\begin{bmatrix} n \\ m \end{bmatrix}$ is an integer.

With a slight change in notation, let us return to identities (A) and (B),

$$(A) \qquad F_{n'} = F_{m'+1}F_{n'-m'} + F_{m'}F_{n'-m'-1} ,$$

$$(B) \qquad F_{n'} = F_{m'}F_{n'-m'+1} + F_{m'-1}F_{n'-m'} .$$

For $k > 0$, let $n' = nk$ and $m' = mk$, then

$$(A') \qquad F_{nk} = F_{mk+1}F_{k(n-m)} + F_{mk}F_{k(n-m)-1} ,$$

$$(B') \qquad F_{nk} = F_{mk}F_{k(n-m)+1} + F_{mk-1}F_{k(n-m)} .$$

Let $u_n \equiv F_{nk}$. Then one can show, in a manner similar to above, using (A') and (B'), that if

$$\begin{bmatrix} n \\ m \end{bmatrix}_k = \frac{u_n u_{n-1} \cdots u_2 u_1}{(u_m u_{m-1} \cdots u_2 u_1)(u_{n-m} u_{n-m-1} \cdots u_2 u_1)} , \quad 0 < m < n ,$$

and

$$\begin{bmatrix} n \\ n \end{bmatrix}_k = \begin{bmatrix} n \\ 0 \end{bmatrix}_k = 1 \;, \text{ then}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_k = F_{km+1}\begin{bmatrix} n-1 \\ m \end{bmatrix}_k + F_{k(n-m)-1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_k \;,$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix}_k = F_{km-1}\begin{bmatrix} n-1 \\ m \end{bmatrix}_k + F_{k(n-m)+1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_k \;,$$

or, adding these two,

$$2\begin{bmatrix} n \\ m \end{bmatrix}_k = L_{km}\begin{bmatrix} n-1 \\ m \end{bmatrix}_k + L_{k(n-m)}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_k \;,$$

a generalization of (3). We note here each $u_n$ is divisible by $F_k$ and we'd get the same generalized binomial coefficients from

$$u_n \equiv F_{nk} / F_k \;,$$

## III. THE FIBONOMIAL TRIANGLE

Pascal's arithmetic triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$\binom{n}{0} \quad \binom{n}{1} \cdots \binom{n}{m} \cdots \binom{n}{n-1} \quad \binom{n}{n}$$

has been the subject of many studies and has always generated interest. We note here to get the next line we merely use the recurrence relation

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1} ,$$

Here we point out two interpretations, one of which shows a direction for Fibonacci generalization. The usual first meeting with Pascal's triangle lies in the binomial theorem expansion,

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j .$$

However, of much interest to us is the difference equation interpretation. The difference equation satisfied by $n^0$ is

$$(n + 1)^0 - n^0 = 0 ,$$

while the difference equation satisfied by $n$ is

$$(n + 2) - 2(n + 1) + n = 0 .$$

For $n^2$ the difference equation is

$$(n + 3)^2 - 3(n + 2)^2 + 3(n + 1)^2 - n^2 = 0 .$$

Certainly one notices the binomial coefficients with alternating signs appearing here. In fact,

$$\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (n + m + 1 - j)^m = 0 .$$

It is this connection with the difference equations for the powers of the integers that leads us naturally to the Fibonomial triangle.

Similar to the difference equation coefficients array for the powers of the positive integers which results in Pascal's arithmetic triangle with alternating signs, there is the Fibonomial triangle made up of the Fibonomial coefficients, with doubly alternated signs. We first write down the Fibonomial triangle for the first six levels.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 6 & & 3 & & 1 & & \\
 & 1 & & 5 & & 15 & & 15 & & 5 & & 1 & \\
1 & & 8 & & 40 & & 60 & & 40 & & 8 & & 1
\end{array}$$

The top line is the $0^{th}$ row and the coefficients of the difference equation satisfied by $F_n^k$ are the numbers in the $(k + 1)^{st}$ row. Of course, we can get the next line of Fibonomial coefficients by using our recurrence relation (D),

$$\begin{bmatrix} n \\ m \end{bmatrix} = F_{m+1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m-1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}, \quad 0 < m < n .$$

We now rewrite the Fibonomial triangle with appropriate signs so that the rows are properly signed to be the coefficients in the difference equations satisfied by $F_n^k$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | | | | | | |
| $F_n^0$ : | 1 | -1 | | | | | |
| $F_n^1$ : | 1 | -1 | -1 | | | | |
| $F_n^2$ : | 1 | -2 | -2 | +1 | | | |
| $F_n^3$ : | 1 | -3 | -6 | +3 | +1 | | |
| $F_n^4$ : | 1 | -5 | -15 | +15 | +5 | -1 | |
| $F_n^5$ : | 1 | -8 | -40 | +60 | +40 | -8 | -1 |

Thus, from the above we may write

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0$$

and

$$F_{n+5}^4 - 5F_{n+4}^4 - 15F_{n+3}^4 + 15F_{n+2}^4 + 5F_{n+1}^4 - F_n^4 = 0 .$$

In Jarden [1] and Hoggatt and Hillman [2] is given the auxiliary polynomial for the difference equation satisfied by $F_n^m$,

$$\sum_{h=0}^{m+1} \begin{bmatrix} m+1 \\ h \end{bmatrix} (-1)^{h(h+1)/2} x^{m+1-h} ,$$

which shows that the sign pattern of doubly alternating signs persists. For an interesting related problem, see [5] and [6].

## IV. THE GENERALIZED FIBONOMIAL TRIANGLE

If, instead of the Fibonacci Sequence, we consider the sequence

$$u_n \equiv F_{nk} \quad (k = 1, 2, 3, \cdots) ,$$

there results another triangular array for each $k > 0$ which all have integer entries. We illustrate with $F_{2n}$. The recurrence relation is

$$\begin{bmatrix} n \\ m \end{bmatrix}_2 = F_{2m-1}\begin{bmatrix} n-1 \\ m \end{bmatrix}_2 + F_{2(m-n)+1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_2$$

and

$$\begin{bmatrix} n \\ n \end{bmatrix}_2 = \begin{bmatrix} n \\ 0 \end{bmatrix}_2 = 1 .$$

The first few lines, with signs, are given below:

| | | | | | | |
|---|---|---|---|---|---|---|
| | 1 | | | | | |
| $F^0_{2n}$ : | 1 | -1 | | | | |
| $F^1_{2n}$ : | 1 | -3 | +1 | | | |
| $F^2_{2n}$ : | 1 | -8 | +8 | -1 | | |
| $F^3_{2n}$ : | 1 | -21 | +56 | -21 | +1 | |
| $F^4_{2n}$ : | 1 | -55 | 385 | -385 | +55 | -1 |

We are saying that the difference equation satisfied by $F^4_{2n}$ is

$$F^4_{2n+10} - 55F^4_{2n+8} + 385F^4_{2n+6} - 385F^4_{2n+4} + 55F^4_{2n+2} - F^4_{2n} = 0 .$$

The algebraic signs of each triangle (singly alternating or doubly alternating) will be determined by the second row by the auxiliary polynomial of $F_{kn}$ which is

$$x^2 - L_k x + (-1)^k .$$

For the general second-order recurrence relation

$$u_{n+2} = pu_{n+1} + qu_n , \quad q \neq 0 ,$$

the auxiliary polynomial is given in [2] to be

$$\sum_{h=0}^{m+1} (-1)^h \begin{bmatrix} m+1 \\ h \end{bmatrix} (-q)^{h(h-1)/2} x^{m+1-h} ,$$

where

$$\begin{bmatrix} m+1 \\ h \end{bmatrix}$$

is the generalized binomial coefficient which in our case becomes

$$\begin{bmatrix} m+1 \\ h \end{bmatrix}_k .$$

Thus for all generalized Fibonomial triangles the generalized Fibonomial coefficients with appropriate signs present arrays which are the coefficients of the difference equations satisfied by the powers, $F_{kn}^m$, of the Fibonacci sequence.

## V. A GENERAL TECHNIQUE

Three simple pieces of information can be used to directly obtain the auxiliary polynomials for $F_{kn}^m$.

<u>Lemma.</u> If sequence $u_n$ is such that

$$(E^2 + pE + q)u_n \equiv 0$$

and sequence $v_n$ is such that

$$(E^2 + p'E + q')v_n \equiv 0 ,$$

where

$$x^2 + px + q = 0 \quad \text{and} \quad x^2 + p'x + q' = 0$$

have no common roots, then the sequence

$$w_n = Au_n + Bv_n$$

is such that

$$(E^2 + pE + q)(E^2 + p'E + q')w_n = 0 ,$$

for arbitrary constants A and B. See problem B-65, <u>Fibonacci Quarterly Journal,</u> April, 1965, page 153.

The auxiliary polynomial for $F_{nk}$ is

$$x^2 - L_k x + (-1)^k .$$

The Binet Form for

$$F_m = (\alpha^m - \beta^m)/(\alpha - \beta)$$

and

$$L_m = \alpha^m + \beta^m ,$$

where

$$\alpha = (1 + \sqrt{5})/2 \quad \text{and} \quad \beta = (1 - \sqrt{5})/2 .$$

Suppose we wish to find the auxiliary polynomial associated with, say, $F_{2n}^3$.

$$F_{2n}^3 = \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)^3 = \frac{\alpha^{6n} - 3\alpha^{4n}\beta^{2n} + 3\alpha^{2n}\beta^{4n} - \beta^{6n}}{5(\alpha - \beta)}$$

$$= \frac{1}{5}\left\{\frac{\alpha^{6n} - \beta^{6n}}{\alpha = \beta} - 3(\alpha\beta)^{2n}\left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)\right\}$$

$$= \frac{1}{5}(F_{6n} - 3F_{2n}) .$$

Now, the auxiliary polynomial for $\frac{1}{5}F_{6n}$ is

$$x^2 - L_6 x + 1$$

and for $\frac{-3}{5}F_{2n}$ is

$$x^2 - L_2 x + 1 .$$

Thus the auxiliary polynomial associated with $F_{2n}^3$ is

$$(x^2 - 18x + 1)(x^2 - 3x + 1) = x^4 - 21x^3 + 56x^2 - 21x + 1 .$$

We illustrate the technique with $F_n^5$ .

$$F_n^5 = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^5 = \frac{\alpha^{5n} - 5\alpha^{4n}\beta^n + 10\alpha^{3n}\beta^{2n} - 10\alpha^{2n}\beta^{3n} + 5\alpha^n\beta^{4n} - \beta^{5n}}{25(\alpha - \beta)}$$

$$= \frac{1}{25}\left(F_{5n} - 5(\alpha\beta)^n F_{3n} + 10(\alpha\beta)^{2n} F_n\right) .$$

The auxiliary polynomials are

$$\frac{1}{25} F_{5n} \;:\qquad x^2 - L_5 x - 1 = x^2 - 11x - 1$$

$$\frac{1}{5}(-1)^n F_{3n} :\qquad x^2 + L_3 x - 1 = x^2 + 4x - 1$$

$$\frac{10}{25} F_n :\qquad x^2 - L_1 x - 1 = x^2 - x - 1$$

so that the auxiliary polynomial for $F_n^5$ is

$$(x^2 - 11x - 1)(x^2 + 4x - 1)(x^2 - x - 1) = x^6 - 8x^5 - 40x^4 + 60x^3 + 40x^2 - 8x - 1$$

which the reader should check with the array in Section III with the Fibonomial Triangle.

This technique can thus be used to find the factored form or recurrence relationship for the auxiliary polynomials for any $F_{nk}^m$ $(m = 0, 1, 2, \cdots)$. See [1] and [3] and particularly [4].

## VI. THE GENERAL SECOND-ORDER RECURRENCE

Consider the sequence $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = pu_{n+1} + qu_n$, for $n \geq 0$. Define the generalized binomial coefficient

$$\left\{ {n \atop m} \right\} = \frac{u_n u_{n-1} \cdots u_2 u_1}{(u_m u_{m-1} \cdots u_2 u_1)(u_{n-m} u_{n-m-1} \cdots u_2 u_1)}\,, \quad 1 \le m \le n - 1\,,$$

with

$$\left\{ {n \atop 0} \right\} = \left\{ {n \atop n} \right\} = 1\,, \quad \left\{ {n \atop m} \right\} = 0 \text{ for } m > n \ge 0 \quad .$$

Starting with

$$R = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}\,,$$

then

$$R^n = \begin{pmatrix} g_{n+1} & qg_n \\ g_n & qg_{n-1} \end{pmatrix}, \quad n \ge 1\,,$$

can be easily established by mathematical induction. Thus we can easily obtain, as in Section II, that

$$g_n = g_{m+1} g_{n-m} + q g_m g_{n-m-1}$$

$$g_n = g_m g_{n-m+1} + q g_{m-1} g_{n-m}$$

Thus, we can immediately write

$$\text{(F)} \qquad \left\{ {n \atop m} \right\} = g_{m+1} \left\{ {n-1 \atop m} \right\} + q g_{n-m-1} \left\{ {n-1 \atop m-1} \right\}$$

and

$$\left\{ {n \atop m} \right\} = q g_{m-1} \left\{ {n-1 \atop m} \right\} + g_{n-m+1} \left\{ {n-1 \atop m-1} \right\}$$

We can now examine some special cases. If $p = 2$ and $q = -1$, then $g_n = n$. The above identities become ordinary binomial coefficients,

$$\binom{n}{m} = (m+1)\binom{n-1}{m} - (n-m-1)\binom{n-1}{m-1}$$

$$\binom{n}{m} = -(m-1)\binom{n-1}{m} + (n-m+1)\binom{n-1}{m-1}.$$

and adding yields $$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}\,.$$

Thus we can conclude that the binomial coefficients are integers and that the product of any m consecutive positive integers is divisible by m-factorial. Since the Fibonomial coefficients are integers, then the product of any m consecutive Fibonacci numbers (with positive subscripts) is exactly divisible by the product of the first m Fibonacci numbers.

If, on the other hand, $p = x$ and $q = 1$, then $g_n(x) = f_n(x)$, the Fibonacci polynomials, and the rows of the generalized binomial coefficients array are indeed the coefficients, with doubly alternated signs, of the difference equations satisfied by the powers of the Fibonacci polynomials, which are $f_0(x) = 0$, $f_1(x) = 1$, and $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$, $n \geq 0$. The resulting generalized binomial coefficients are monic polynomials with integral coefficients.

## VII. THE FIBONACCI POLYNOMIAL BINOMIAL COEFFICIENT TRIANGLE

The first few Fibonacci polynomials are

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2 + 1, \quad f_4(x) = x^3 + 2x, \quad f_5(x) = x^4 + 3x^2 + 1,$$

and the first few lines of the Fibonacci polynomial triangle are

$$\begin{array}{lccccccccc}
 & & & & & 1 & & & & \\
f_n^0(x): & & & & 1 & & -1 & & & \\
f_n^1(x): & & & 1 & & -x & & -1 & & \\
f_n^2(x): & & 1 & & -(x^2+1) & & -(x^2+1) & & +1 & \\
f_n^3(x): & 1 & & -(x^3+2x) & & -(x^2+1)(x^2+2) & & +(x^3+2x) & & +1 \\
 & \left\{ n \atop 0 \right\} & \left\{ n \atop 1 \right\} & \cdots & & \left\{ n \atop m \right\} & & \cdots & \left\{ n \atop n-1 \right\} & \left\{ n \atop n \right\},
\end{array}$$

where the double signs are to be attached to the $\left\{ n \atop m \right\}$ as required. We are saying

$$f_{n+4}^3(x) - (x^3 + 2x)f_{n+3}^3(x) - (x^2 + 1)(x^2 + 2)f_{n+2}^3(x) + (x^3 + 2x)f_{n+1}^3(x) + f_n^3(x) \equiv 0 .$$

The next line can be obtained by using recurrence relation (F).

$$\text{(F)} \qquad \left\{ {n \atop m} \right\} = f_{m+1}(x) \left\{ {n-1 \atop m} \right\} + f_{n-m-1}(x) \left\{ {n-1 \atop m-1} \right\} ,$$

where

$$\left\{ {n \atop 0} \right\} = 1 = \left\{ {n \atop n} \right\} .$$

This triangular array collapses into the Fibonomial triangle when $x = 1$. From (F) it is easy to establish by induction that $\left\{ {n \atop m} \right\}$ are monic polynomials with integral coefficients. For every integral $x$ we get an array of integers.

## VIII. THE CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

The Chebyshev polynomials of the second kind are

$$u_0(x) = 1, \quad u_1(x) = 2x, \quad \text{and} \quad u_{n+2}(x) = 2x\, u_{n+1}(x) - u_n(x) .$$

If

$$g_n(x) = u_{n-1}(x),$$

then

$$g_0(x) = 0, \qquad g_1(x) = 1 ,$$

and we have the conditions for our Pascal triangle rows to have singly alternating signs to reflect the difference equations for the powers of $g_n(x)$. Since $g_n(x/2)$ also satisfies this, the Fibonacci polynomials and the Chebyshev polynomials yield all possible Pascal triangles with integral coefficients.

## IX. THE FINAL DISCUSSION

In [1] and [2] it is given that the auxiliary polynomial associated with the general second-order recurrence

$$y_{n+2} = py_{n+1} + qy_n, \quad q \neq 0 ,$$

is

$$\sum_{h=0}^{m+1} (-1)^h \begin{Bmatrix} m+1 \\ h \end{Bmatrix} (-q)^{h(h-1)/2} x^{m+1-h}$$

Thus, if the columns of Pascal's generalized binomial coefficient triangle is left justified with the first column on the left being the $0^{th}$ column then multiplying the $h^{th}$ column by $q^{h(h-1)/2}$ yields a modified array whose coefficients along each row (with singly alternating signs if Chebyshev related or doubly alternating if Fibonacci related) are the coefficients of the difference equations satisfied by $u_n^m$ .

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