THE BRACKET FUNCTION, q-BINOMIAL COEFFICIENTS, AND SOME NEW STIRLING NUMBER FORMULAS*

H. W. GOULD
W. Virginia University, Morgantown, W. Va.

TO PROFESSOR LEONARD CARLITZ ON HIS SIXTIETH BIRTHDAY, 26 December, 1967.

In a recent paper [3] the author proved that the binomial coefficient and the bracket function ($[x]$ = greatest integer $\le x$) are related by

$$\binom{n}{k} = \sum_{j=k}^{n} \left[\frac{n}{j}\right] R_k(j) \tag{1}$$

and

$$\left[\frac{n}{k}\right] = \sum_{j=k}^{n} \binom{n}{j} A_k(j) , \tag{2}$$

where

(3) $R_k(j)$ = Number of compositions of j into k relatively prime positive summands,

$$= \sum_{\substack{a_1+\cdots+a_k=j \\ (a_1,\cdots,a_k)=1}} 1 ,$$

$$= \sum_{d|j} \binom{d-1}{k-1} \mu(j/d) ,$$

and

$$A_k(j) = \sum_{d=k}^{j} (-1)^{j-d} \binom{j}{d} \left[\frac{d}{k}\right] = \sum_{1\le d\le j/k} (-1)^{j-kd} \binom{j-1}{kd-1} . \tag{4}$$

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Moreover, the fact that the numbers R and A are orthogonal proved the elegant general result that for any two sequences $f(n, k)$, $g(n, k)$, then

$$f(n, k) = \sum_{j=k}^{n} g(n, j)R_k(j) \tag{5}$$

if and only if

$$g(n, k) = \sum_{j=k}^{n} f(n, j)A_k(j) \quad . \tag{6}$$

Notice that (5) and (6) do not imply (1) and (2); one at least of the special expansions must be proved before the inverse relation follows from (5)-(6).

Finally, it was found that R and A satisfy the congruences

$$\begin{aligned} R_k(j) &\equiv 0 \pmod{k} \\ A_k(j) &\equiv 0 \pmod{k} \end{aligned} \tag{7}$$

for all natural numbers $j \geq k + 1$ if and only if k is a prime.

These congruences, together with the fact that $R_k(k) = A_k(k) = 1$ then showed that either of (1) and (2) implies that

$$\binom{n}{k} \equiv \left[\frac{n}{k}\right] \pmod{k} \quad (k \geq 2) \tag{8}$$

for all natural numbers n if and only if k is a prime.

Naturally, similar congruences are implied for any f and g which satisfy the pair (5)-(6).

Now it is natural to look for an extension of these results to the more general situation where $\binom{n}{k}$ is replaced by the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{j=1}^{k} \frac{q^{n-j+1} - 1}{q^j - 1}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1 \; . \tag{9}$$

In the limiting case $q = 1$ these become ordinary binomial coefficients. This is the motivation for the present paper. Ordinarily we omit the subscript q unless we wish to emphasize the base used.

We follow the terminology in [2] and, since that paper is intimately connected with the results below, the reader is referred there for detailed statements and for further references to the literature. Cf. also [1].

In the present paper we exhibit q-analogs of expansions (1) and (2) in terms of q-extensions of R and A. Moreover, the generating functions for $R_k(j, q)$ and $A_k(j, q)$ prove their orthogonal nature so that we obtain an elegant and direct generalization of the inverse pair (5)-(6) to the q-coefficient case. By consideration of the expressions

$$\sum_{j=k}^{n} R_k(j, q)A_j(n, p) \ , \qquad \sum_{j=k}^{n} A_k(j, q)R_j(n, p) \ , \qquad q \neq p \ ,$$

we are then able to obtain new expressions for q-Stirling numbers of first and second kind, with the ordinary Stirling numbers as limiting cases.

Our emphasis is on the various series expansions involving R and A and a detailed study of arithmetic properties will be left for a separate paper.

The principal results developed here are embodied in Theorems 1-16. Special attention is called to 1, 2, and 6. A few arithmetic results also appear.

We begin by generalizing (2). Put

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q A_k(j, q), \qquad k \geq 1 .$$

Now, inverse relations (7.3)-(7.4) in [2] may be stated in the form

$$F(n) = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2} f(j) \tag{10}$$

if and only if

$$f(n) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} F(j) . \tag{11}$$

Thus

$$A_k(n, q) = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2} \begin{bmatrix} j \\ k \end{bmatrix} .$$

In this sum the bracketed term is zero for $0 \leq j < k$ so that the index j need range only from k to n, and it is then also clear that $A_k(n, q) = 0$ for $n < k$. Moreover $A_k(k, q) = 1$ for all $k \geq 1$ and any q. Evidently we have proved

Theorem 1. The q-binomial coefficient expansion of the bracket function is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=k}^{n} \begin{bmatrix} n \\ j \end{bmatrix} A_k(j, q) = \begin{bmatrix} n \\ k \end{bmatrix} + \sum_{j=k+1}^{n} \begin{bmatrix} n \\ j \end{bmatrix} A_k(j, q) , \tag{12}$$

where

$$\begin{aligned} A_k(j, q) &= \sum_{d=k}^{j} (-1)^{j-d} \begin{bmatrix} j \\ d \end{bmatrix} q^{(j-d)(j-d-1)/2} \begin{bmatrix} d \\ k \end{bmatrix} \\ &= q^{j(j-1)/2} \sum_{d=k}^{j} (-1)^{j-d} \begin{bmatrix} j \\ d \end{bmatrix}_p p^{d(d-1)/2} \begin{bmatrix} d \\ k \end{bmatrix} \end{aligned} \tag{13}$$

with $pq = 1$. Cf. also Theorem 15.

The indicated second form of (13) follows from the reciprocal transformation [2]

$$\begin{bmatrix} n \\ k \end{bmatrix}_p = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{for} \quad pq = 1 .$$

The sum may be written also as in the second form of (4) above. See Theorem 15.

The ease of finding (12) suggests that it should not be difficult to invert the formula. To do this, i. e., to derive a q-analog of (1), we shall proceed exactly as in the proof of Theorem 7 in [3]. We need a q-analog of the relation

$$\sum_{d=k}^{n} \binom{d-1}{k-1} = \binom{n}{k}$$

which was exploited in [3] in the proof of Theorem 7 as well as in the study of the combinatorial meaning of $R_k(j)$.

The q-binomial coefficient satisfies [2] the recurrence relations

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = q^k \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix},$$

and the second of these gives

$$q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} d \\ k \end{bmatrix} - \begin{bmatrix} d-1 \\ k \end{bmatrix},$$

so that by summing both sides we have the desired q-analog

$$\sum_{d=k}^{n} q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}. \tag{14}$$

We also recall the formula of Meissel [3]

$$\sum_{m \le x} \left[\frac{x}{m} \right] \mu(m) = 1 , \tag{15}$$

where μ is the familiar Moebius function in number theory.

We are now in a position to prove

Theorem 2. The bracket function expansion of the q-binomial coefficient is given by

$$(16)\qquad \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=k}^{n} \left[\frac{n}{j}\right] R_k(j, q) = \left[\frac{n}{k}\right] + \sum_{j=k+1}^{n} \left[\frac{n}{j}\right] R_k(j, q) \quad ,$$

where

$$(17)\qquad R_k(j, q) = \sum_{\substack{d \mid j \\ d \geq k}} q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} \mu(j/d) \ .$$

Proof. As in [3, p. 248] we have

$$\sum_{j \leq n} \left[\frac{n}{j}\right] \sum_{d \mid j} q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} \mu(j/d)$$

$$= \sum_{d \leq n} q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} \sum_{m \leq n/d} \left[\frac{n/d}{m}\right] \mu(m)$$

$$= \sum_{d \leq n} q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix},$$

by (15), then (14).

This completes the proof since it is evident that $R_k(j, q) = 0$ for $j < k$ and $R_k(k, q) = 1$ for all $k \geq 1$ and any q.

We next obtain a Lambert series expansion having $R_k(j, q)$ as coefficient. We need a q-analog of the formula

$$(18)\qquad \sum_{n=k}^{\infty} \binom{n}{k} x^n = x^k (1 - x)^{-k-1}, \quad k \geq 0 \ ,$$

which was used in [3, p. 246].

By using (14), it easily follows that

$$S(k,x) = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} x^n = \sum_{n=k}^{\infty} x^n \sum_{j=k}^{n} q^{j-k} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}$$

$$= q^{1-k} x(1-x)^{-1} S(k-1, qx) ,$$

with

$$S(0,qx) = (1-qx)^{-1} ,$$

so that iteration yields the desired formula

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} x^n = x^k \prod_{j=0}^{k} (1-q^j x)^{-1}, \quad k \geq 0 . \tag{19}$$

We also recall [3, (3)]

$$\sum_{n=k}^{\infty} \left[\frac{n}{k} \right] x^n = x^k (1-x)^{-1} (1-x^k)^{-1}, \quad k \geq 1 . \tag{19'}$$

We may now state

<u>Theorem 3.</u> The number-theoretic function $R_k(j,q)$ is the coefficient in the Lambert series

$$\sum_{j=k}^{\infty} R_k(j,q) \frac{x^j}{1-x^j} = x^k \prod_{j=1}^{k} (1-q^j x)^{-1} = \sum_{n=k}^{\infty} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} q^{n-k} x^n . \tag{20}$$

Indeed, the same steps used in [3, p. 246] apply here. One substitutes in (19) by means of (16), rearranges the series, and then uses (19'). Since we are only concerned with the coefficients in formal generating functions no problem about convergence arises at this point. Later, in Theorem 16, we expand (20) as a power series in a variant form. The right-hand summation in (20) follows easily from (19).

The expansion inverse to (20) is just as easily found, and we state

Theorem 4. The number-theoretic function $A_k(j, q)$ is the coefficient in the expansion

$$\sum_{j=k}^{\infty} A_k(j,q)x^j \prod_{i=1}^{j} (1 - q^i x)^{-1} = \frac{x^k}{1 - x^k} . \tag{21}$$

Indeed, the proof parallels that in [3, 252] in that one starts with (19'), substitutes by means of (12), rearranges, and applies (19).

Now it is evident that the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial of degree $k(n - k)$ in q. Thus it is evident from (4) and (17) that $A_k(j,q)$ and $R_k(j, q)$ are each polynomials in q. In terms of the formal algebra of generating functions we may then equate corresponding coefficients in series to derive identities. Substitution of (20) into (21), and conversely, yields the following orthogonality relations which we state as

Theorem 5. The numbers $A_k(j,q)$ and $R_k(j,q)$ are orthogonal in the sense that

$$\sum_{j=k}^{n} R_k(j,q)A_j(n,q) = \delta_k^n , \tag{22}$$

and

$$\sum_{j=k}^{n} A_k(j,q)R_j(n,q) = \delta_k^n . \tag{23}$$

Thus we have evidently also proved the quite general inversion

Theorem 6. For two sequences $F(n,k,q)$, $G(n,k,q)$, then

$$F(n,k,q) = \sum_{j=k}^{n} G(n,j,q)R_k(j,q) \tag{24}$$

if and only if

$$G(n, k, q) = \sum_{j=k}^{n} F(n, j, q) A_k(j, q) \quad . \tag{25}$$

Again we note that Theorem 6 does not immediately imply Theorem 1 or Theorem 2, as one at least of these must be proved before Theorem 6 yields the other. The expansion and inversion theories are quite separate ideas.

It was seen in [3, p. 247] that the number of compositions of n into k positive summands, $C_k(n)$, is related to $R_k(j)$ by the formula

$$C_k(n) = \binom{n-1}{k-1} = \sum_{d|n} R_k(d) \quad , \tag{26}$$

which was then inverted by the Moebius inversion theorem to get that part of (3) above involving the Moebius function. Since that paper started from the number-theoretic interpretation of $R_k(j)$ and only later used the formula of Meissel to obtain the expansion without starting from the theory of compositions, it is of interest in the present paper to proceed in reverse. The Moebius inversion theorem applied to (17) above gives us at once

Theorem 7. The function $R_k(j, q)$ satisfies the q-analog of (26).

$$q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \sum_{d|n} R_k(d, q) \quad . \tag{27}$$

We now turn to the connections between $R_k(j, q)$ and $A_k(j, q)$ and the Stirling numbers. A formula due to Carlitz was stated in [1] in the form

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{s=k}^{n} \binom{n}{s} (q-1)^{s-k} S_2(k, s-k, q) \ , \tag{28}$$

where $S_2(n, k, q)$ is a q-Stirling number of the second kind and, explicitly,

$$(29) \qquad S_2(n, k, q) = (q-1)^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k+n}{k-j} \begin{bmatrix} j+n \\ j \end{bmatrix} .$$

It is evident from the expansions which we have examined here that we may obtain formula (28) in quite a different manner.

Indeed, substitution of (2) into (16) above gives us at once

$$(30) \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{s=k}^{n} \binom{n}{s} \sum_{j=k}^{s} R_k(j, q) A_j(s) ,$$

and this must agree with (28), so that we are left to assert

Theorem 8. The q-Stirling number of the second kind as defined by (29) may be expressed as

$$(31) \qquad (q-1)^{s-k} S_2(k, s-k, q) = \sum_{j=k}^{s} R_k(j, q) A_j(s) .$$

This is an interesting result, because when $q = 1$ the left-hand member is zero $(k \neq s)$, and the right-hand member is zero because of the fact of orthogonality of $R_k(j)$ and $A_j(s)$. As a corollary to this theorem we have

Theorem 9. The ordinary Stirling numbers of the second kind (in the author's notation [1]) are given by

$$(32) \qquad S_2(k, n-k) = \lim_{q \to 1} (q-1)^{k-n} \sum_{j=k}^{n} R_k(j, q) A_j(n) ,$$

where $R_k(j, q)$ is given by (17) and $A_j(n) = A_j(n, 1)$ is given by (4).

It is natural to request a companion formula for the Stirling numbers of the first kind. To attempt this we next need a formula inverse to (28), as the formula inverse to (30) is apparent. We proceed by making use of the q-inversion theorem expressed in relations (10)-(11) above.

Put

$$\binom{n}{k} = \sum_{s=0}^{n} \binom{n}{s} f(s, k, q) \ . \tag{33}$$

then by (10)-(11) this inverts to yield

$$f(n, k, q) = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2} \binom{j}{k} \tag{34}$$

It was found in [1, (3.19)] that the q-Stirling numbers of the first kind as there defined could be expressed in the form

$$S_1(n, k, q) = (q - 1)^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)/2} , \tag{35}$$

which may be rewritten as follows:

$$\begin{aligned} S_1(n, n-k, q) &= (q-1)^{k-n} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-j}{n-k-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)/2} , \\ &= (q-1)^{k-n} \sum_{j=0}^{n} (-1)^{n-k-j} \binom{n-j}{k} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)/2} , \\ &= (q-1)^{k-n} \sum_{j=0}^{n} (-1)^{k-j} \binom{j}{k} \begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j+1)/2} , \end{aligned}$$

so that we may write

$$(36)\qquad S_1(n, n-k, q) = (1-q)^{k-n}\sum_{j=0}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix}\binom{j}{k} q^{(n-j)(n-j-1)/2} q^{n-j} .$$

This looks somewhat like $f(n, k, q)$ as given by (34), but with an important difference: the factor q^{n-j}. It seems rather difficult to modify the work so as to remove this factor and express $f(n, k, q)$ easily in terms of $S_1(n, k, q)$. We could call $f(n, k, q)$ a modified Stirling number of the first kind. We illustrate further the difficulty involved. Instead of (33) let us put

$$(37)\qquad q^{-n}\binom{n}{k} = \sum_{s=0}^{n}\begin{bmatrix} n \\ s \end{bmatrix} g(s, k, q) .$$

This inverts by (10)-(11) to give

$$g(n, k, q) = \sum_{j=0}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2}\binom{j}{k} q^{-j} ,$$

and comparison of this with (36) yields at once

$$(38)\qquad g(n, k, q) = q^{-n}(1-q)^{n-k} S_1(n, n-k, q) .$$

This, however, leads to difficulty when we examine the analog of (30). Indeed, substitution of (12) into (1) gives us at once

$$(39)\qquad \binom{n}{k} = \sum_{s=k}^{n}\begin{bmatrix} n \\ s \end{bmatrix}\sum_{j=k}^{s} R_k(j) A_j(s, q) .$$

However, expansion (37) gives us

$$(40)\qquad \binom{n}{k} = \sum_{s=0}^{n}\begin{bmatrix} n \\ s \end{bmatrix} q^n g(s, k, q) ,$$

and we may not equate coefficients since (39) requires the coefficient of the q-binomial coefficient to be independent of n, but in (40) it is not.

Of course, by (33) and (39) we do have

$$\sum_{j=k}^{s} R_k(j)A_j(s, q) = \sum_{j=k}^{s} (-1)^{s-j} \begin{bmatrix} s \\ j \end{bmatrix} \binom{j}{k} q^{(s-j)(s-j-1)/2}, \tag{41}$$

which is the best companion to (31) noted at this time.

Another approach would be to develop a q-bracket function (q-greatest integer function) and proceed in a manner similar to the above by expanding the binomial coefficient $\binom{n}{j}$ in terms of a q-bracket function and using this in relation (2) just as we here used relation (2) in (16) to get (30) and then (31). The development of the q-analog of the greatest integer function will be left for a separate account.

It seems not without interest to exhibit a numerical example of (32). From definition, $S_2(2, 3) = 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 2 = 15$, being the sum of the 4 possible products, each with 3 factors (repetition allowed), which may be formed from the first 2 natural numbers. The table of values of $A_j(n)$ in [3, p. 254] and the formula (17) may be used. We find that

$$\begin{aligned} S_2(2, 3) = S_2(2, 5-2) &= \lim_{q \to 1} (q-1)^{-3} \sum_{j=2}^{5} R_2(j, q)A_j(5) \\ &= \lim_{q \to 1} (q-1)^{-3}(-8 + 6q(q+1) - 4(-1+q^2+q^3+q^4) + (q^3+q^4+q^5+q^6)) \\ &= \lim_{q \to 1} (q-1)^{-3} (-4 + 6q + 2q^2 - 3q^3 - 3q^4 - q^5 + q^6) = 15, \end{aligned}$$

the limit being easily found by l'Hospital's theorem.

We should remark for the convenience of the reader that the Stirling numbers appear in various forms of notation and the notations of Riordan [5], Jordan [4], and the author [1] are related as follows:

$$s(n, k) = S_n^k = (-1)^{n-k} S_1(n-1, n-k), \tag{42}$$

and

$$(43) \qquad S(n,k) = \mathfrak{S}_n^k = S_2(k, n-k) = \frac{1}{k!}\Delta^k 0^n .$$

The S_1 and S_2 notations are convenient because of the generating functions

$$(44) \qquad \prod_{k=0}^{n} (1 + kx) = \sum_{k=0}^{n} S_1(n,k)x^k, \quad \prod_{k=0}^{n} (1 - kx)^{-1} = \sum_{k=0}^{\infty} S_2(n,k)x^k .$$

Also, in [1] will be found a discussion of the interesting continuation formulas

$$(45) \qquad S_2(-n-1, k, 1/q) = q^k S_1(n,k,q), \quad S_1(-n-1,k,1/q) = q^k S_2(n,k,q) .$$

A q-polynomial was suggested in [1] which would include both S_1 and S_2 as instances. The q-Stirling numbers as defined in [1] satisfy the generating relations

$$(46) \qquad \prod_{k=0}^{n} (1 + [k]x) = \sum_{k=0}^{n} S_1(n,k,q)x^k, \quad \prod_{k=0}^{n} (1 - [k]x)^{-1} = \sum_{k=0}^{\infty} S_2(n,k,q)x^k,$$

in analogy to (44). Here $[k]$ is called a q-number and is defined by

$$[k]_q = [k] = \frac{q^k - 1}{q - 1} ,$$

so that

$$\lim_{q \to 1} [k] = k .$$

The notation $[k]$ must not be confused with that for the bracket function.

Relations (31) and (41) suggest that we consider the following. By using Theorem 3 with base q, and substituting with Theorem 4 and base p, we find the identity

$$(47)\qquad x^k \prod_{j=1}^{k} (1 - q^j x)^{-1} = \sum_{n=k}^{\infty} x^n \prod_{i=1}^{n} (1 - p^i x)^{-1} \sum_{j=k}^{n} R_k(j, q)A_j(n, p) .$$

It will be recalled from [3] that for $p = q$ the inner sum is merely a Kronecker delta. In view of Theorem 8, we may look on the sum

$$(48)\qquad \sum_{j=k}^{n} R_k(j, q)A_j(n, p) = f(n, k, p, q)$$

as a kind of generalized Stirling number.

Some of the results already found extend to real numbers instead of natural numbers only. The product definition (9) holds for $n = x =$ real number. We may also extend the range of validity of (16) just as was done in the proof of Theorem 7 in [3]. Indeed we have

Theorem 10. For two sequences $F(x, k, q)$, $G(x, k, q)$, then for real x and all natural numbers k

$$(49)\qquad F(x, k, q) = \sum_{k \le j \le x} G(x, j, q)R_k(j, q)$$

if and only if

$$(50)\qquad G(x, k, q) = \sum_{k \le j \le x} G(x, j, q)A_k(j, q) ,$$

where R and A are defined by (17) and (13).

The proof uses nothing more than Theorem 5.

The real-number extension of Theorem 1 most readily found is as follows.

Theorem 11. For real x and natural numbers k

$$(51)\qquad \begin{bmatrix} x \\ k \end{bmatrix} = \sum_{k \le j \le x} \begin{bmatrix} [x] \\ j \end{bmatrix}_q A_k(j, q) .$$

The proof parallels that of Theorem 7 in [3]. Note that the 'expansion'

(52) $$\left[\frac{x}{k}\right] = \sum_{k \leq j \leq x} \left[\frac{x}{j}\right]_q A_k(j, q)$$

is incorrect. What is really expanded in (51) is

$$\left[\frac{[x]}{k}\right], \text{ however in fact } \left[\frac{[x]}{k}\right] = \left[\frac{x}{k}\right],$$

so that what one might first try from (50) does not hold.

Similarly, a correct generalization of Theorem 2, by inversion of (51), is

Theorem 12. For real x and natural numbers k

(53) $$\left[\frac{[x]}{k}\right]_q = \sum_{k \leq j \leq x} \left[\frac{x}{j}\right] R_k(j, q) .$$

The failure of (52) suggests two new procedures. First, we may define a kind of q-greatest integer function (not the only possible definition) by

(54) $$\left[\frac{x}{k}, q\right] = \sum_{k \leq j \leq x} \left[\frac{x}{j}\right]_q A_k(j, q) ,$$

and secondly, we may introduce new coefficients such that

(55) $$\left[\frac{x}{k}\right] = \sum_{k \leq j \leq x} \left[\frac{x}{j}\right]_q B_k(j, q) ,$$

but these are not easily determined. We shall leave a detailed discussion of such extensions for another paper.

Although we omit a detailed study of the arithmetical properties of the functions $R_k(j, q)$ and $A_k(j, q)$, we remark that such a study makes use of arithmetical properties of the q-binomial coefficients. Fray [6] has recently announced some results in that direction. In particular he announces the following theorem. Let q be rational and $q \not\equiv 0 \pmod p$, and let e = exponent to which q belongs (mod p). Let $n = a_0 + ea$, $0 \le a_0 < e$, and $k = b_0 + eb$, $0 \le b_0 < e$. Then

$$\begin{bmatrix} n \\ k \end{bmatrix} \equiv \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \binom{a}{b} \pmod p \quad . \tag{56}$$

We do explore certain arithmetical properties which are of a different nature. First of all, (17) gives

$$qR_1(n, q) = \sum_{d \mid n} q^d \mu(n/d) \quad ,$$

and by a theorem of Gegenbauer [3, p. 256] this sum is always divisible by n for any natural number q. Thus we have the congruence

$$qR_1(n, q) \equiv 0 \pmod n \tag{57}$$

for all integers n, q. This is trivial for $R_1(n, 1) = R_1(n) = 0$ for $n \ge 2$.

On the other hand, let $n = p$ be a prime. Then we have for integers q

$$R_1(p, q) = q^{p-1} - 1 \equiv 0 \pmod p, \quad \text{for } (p, q) = 1 \ , \tag{58}$$

this following from the Fermat congruence. Again this is trivial when $q = 1$.

It is possible to obtain various identical congruences for the functions studied in this paper. If $f(q)$ and $g(q)$ are two polynomials in q with integer coefficients, we recall that $f(q) \equiv g(q) \pmod m$ is an identical congruence (mod m) provided that respective coefficients of powers of q are congruent. We shall call such congruences <u>identical q-congruences</u>. Thus we have

Theorem 13. The functions defined by (13) and (34) satisfy the identical q-congruence

$$A_k(n, q) \equiv f(n, k, q) \pmod{k} \qquad (k \geq 2,\ n = 1, 2, 3, \cdots)$$

if and only if k is prime.

Proof. Apply (8) to (13) and (34).

Another way of seeing this is to note that (33) and (39) imply

$$f(n, k, q) = \sum_{j=k}^{n} R_k(j) A_j(n, q) = A_k(n, q) + \sum_{j=k+1}^{n} R_k(j) A_j(n, q) ,$$

and recall (7), whence the result follows.

In similar fashion one can obtain various congruences involving the q-Stirling numbers.

As a final remark about identical congruences we wish to note the following q-criterion for a prime.

Theorem 14. The identical q-congruence (for $k \geq 2$)

$$(1 - q)^{k-1} \equiv [k]_q \pmod{k} \tag{59}$$

is true if and only if k is a prime. Here, the q-number

$$[k]_q = (q^k - 1)/(q - 1) .$$

Proof. We shall use the easily established q-analog identity:

$$(q - 1)^{k-1} = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} [j]_q . \tag{60}$$

From this we have

$$(q - 1)^{k-1} - [k]_q = \sum_{j=1}^{k-1} (-1)^{k-j} \binom{k}{j} [j]_q . \tag{61}$$

Now it is easily seen that

$$k \,\Big|\, \binom{k}{j} \quad \text{if } k = \text{prime and } 1 \le j \le k-1 .$$

Hence it is trivial that (59) holds when k = prime.

Assume then that (59) holds for a composite k. Then we have (61) so that

$$k \,\Big|\, \binom{k}{j}, \quad 1 \le j \le k-1 .$$

Let p be a prime divisor of k. Then for some value of j, $1 \le j \le k-1$, $j = p$, whence $k \Big| \binom{k}{p}$. Considering this in the form

$$k \,\Big|\, \frac{k(k-1)\cdots(k-p+1)}{p(p-1)!}$$

we have $(k, j) = 1$, whence k is relatively prime to every factor $k - j$ in the numerator and we have $p(p-1)! \,\big|\, (k-1)(k-2)\cdots(k-p+1)$. This implies that $p|(k-j)$ for some j with $1 \le j \le k-1$, or since $p|k$ (by hypothesis), therefore $p|j$ which is impossible. Thus the only possibility is that k is prime itself.

If we write out the congruence as

$$(1-q)^{k-1} \equiv \frac{1-q^k}{1-q} \pmod{k} ,$$

and multiply through by $1 - q$ we have the equivalent identical congruence

$$(62) \qquad (1-q)^k \equiv 1 - q^k \pmod{k}$$

if and only if k = prime $(k \ge 2)$.

It was noted in [3] that E. M. Wright's proof of (8) was to show that (8) is equivalent to the identical q-congruence (62). We note a typographical mistake in [3, p. 241] in that the identical congruence there should read

(63) $$(1 - x)^p \equiv 1 - x^p \pmod p$$

if and only if p is prime.

The proof above for (59) is equivalent to Wright's proof of (62), however it is felt to be of interest to present it by way of the q-identity (61). Of course, the generating functions (1) and (2) show that (8) and (63) are equivalent.

Since [3] was concerned with compositions and partitions, it is of interest to recall a theorem of Cayley to the effect that the number of partitions of n into j or fewer parts, each summand $\le i$, is the coefficient of q^n in the series expansion of the q-binomial coefficient

$$\begin{bmatrix} j+i \\ j \end{bmatrix} = \prod_{k=1}^{j} \frac{1 - q^{k+i}}{1 - q^k} .$$

When $|q| < 1$ and $i \to \infty$, $j \to \infty$, this reduces to Euler's formula for the partition of n into any number of parts at all,

$$\prod_{k=1}^{\infty} (1 - q^k)^{-1} = 1 + \sum_{n=1}^{\infty} p(n) q^n .$$

It is expected that the q-identities derived here have further implications for partitions and compositions.

As another result we show that $A_k(n, q)$ may be written in such a way that the greatest integer function does not explicitly appear. This is analogous to relation (41) in [3]. We have

Theorem 15. For the numbers defined by (13) we have

(64) $$A_k(n, q) = \sum_{1 \le m \le n/k} (-1)^{n-mk} \begin{bmatrix} n-1 \\ mk-1 \end{bmatrix} q^{(n-mk)(n-mk+1)/2} .$$

Proof. Recall that

$$\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j \end{bmatrix} + \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{n-j} .$$

Then by (13) and this we have

$$\begin{aligned} A_k(n, q) &= \sum_{j=0}^{n-1} (-1)^{n-j} \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2} \begin{bmatrix} j \\ k \end{bmatrix} \\ &\quad + \sum_{j=1}^{n} (-1)^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{n-j} q^{(n-j)(n-j-1)/2} \begin{bmatrix} j \\ k \end{bmatrix} \\ &= \sum_{j=1}^{n} (-1)^{n-j+1} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{(n-j)(n-j+1)/2} \begin{bmatrix} j-1 \\ k \end{bmatrix} \\ &\quad + \sum_{j=1}^{n} (-1)^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{(n-j)(n-j+1)/2} \begin{bmatrix} j \\ k \end{bmatrix} \\ &= \sum_{j=1}^{n} (-1)^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{(n-j)(n-j+1)/2} \left\{ \begin{bmatrix} j \\ k \end{bmatrix} - \begin{bmatrix} j-1 \\ k \end{bmatrix} \right\} \\ &= \sum_{\substack{k \le j \le n \\ k \mid j}} (-1)^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{(n-j)(n-j+1)/2} , \end{aligned}$$

which may then be written as we indicate, letting $j = mk$ in the summation.

An alternative form of the power series expansion for (20) is easily found. Indeed, the product on the right side of (20) may be written as follows:

$$\prod_{j=1}^{k} (1 - q^j x)^{-1} = \prod_{j=0}^{k-1} (1 - q^j qx)^{-1} = \prod_{j=0}^{\infty} \frac{1 - xq q^{j+k}}{1 - xq q^j} .$$

However, Carlitz [7, p. 525] has noted the expansion (due to Cauchy [8])

$$\prod_{j=0}^{\infty} \frac{1 - atq^j}{1 - btq^j} = \sum_{n=0}^{\infty} \frac{(b-a)_n}{(q)_n} t^n ,$$

where

$$(b-a)_n = \prod_{j=0}^{n-1} (b - q^j a) ,$$

and

$$(q)_n = \prod_{j=1}^{n} (1 - q^j) .$$

Setting $a = q^k$, $b = 1$, $t = qx$, we can obtain the desired expansion. We state the result as

Theorem 16. The Lambert series for $R_k(j, q)$ may be written as a power series in the form

$$\sum_{j=k}^{\infty} R_k(j,q) \frac{x^j}{1 - x^j} = \sum_{n=0}^{\infty} \frac{(1 - q^k)_n}{(q)_n} q^n x^{n+k} . \tag{65}$$

Further results relating to compositions and partitions will be left for a future paper.

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SPECIAL PROPERTIES OF THE SEQUENCE $W_n(a,b;p,q)$

A. F. HORADAM
University of New England, Armidale, N.S.W., Australia

1. INTRODUCTION

Elsewhere in this journal [1] the sequence $\{w_n(a,b;p,q)\}$ has been introduced and its basic properties exhibited. Here we investigate three special properties of the sequence, namely, the "Pythagorean" property (2), the geometrical-paradox property (3), and the complex case (4). These are generalizations of results earlier published for the sequence $\{h_n(r,s)\} \equiv \{w_n(r, r+s;1,-1)\}$ which may be consulted in [3], [4], [5] respectively.

But observe that with reference to $\{h_n(r,s)\}$ the notation in this paper varies slightly from that used in [2], [3], [4] and [5]. Our properties in this paper form the second of the proposed series of articles envisaged in [1]. Notation and content of [1] are assumed, when required.

Some interesting special cases of $\{w_n(a,b;p,q)\}$ occur which we record for later reference (2):

		a	b	p	q
(1.1)	integers	a = 1,	b = 2,	p = 2,	q = 1
(1.2)	odd numbers	1	3	2	1
(1.3)	arithmetic progression (common difference)	a	a + d	2	1
(1.4)	geometric progression (common ratio q)	a	q	q + 1	q
(1.5)	Fermat's sequence u_n (3, 2)	1	3	3	2
(1.6)	Fermat's sequence v_n (3, 2)	2	3	3	2
(1.7)	Pell's sequence u_n (2, -1)	1	2	2	-1
(1.8)	Pell's sequence v_n (2, -1)	2	2	2	-1

Sequence (1.1) has already been noted in [1], while sequences (1.5) — (1.8) were mentioned in [6]. However, sequences (1.2) — (1.4) have not been previously recorded in this series of papers.

2. THE "PYTHAGOREAN" PROPERTY

Any w_n at all may be substituted in the known formula for Pythagorean triples: $(u^2 - v^2)^2 + (2uv)^2 = (u^2+v^2)^2$. Writing $u = w_{n+2}$, $v = w_{n+1}$, we obtain

$$(2.1) \qquad (w_{n+2}^2 - w_{n+1}^2)^2 + (2w_{n+2}w_{n+1})^2 = (w_{n+2}^2 + w_{n+1}^2)^2 \ .$$

Next, using the recurrence relation $w_{n+2} = pw_{n+1} - qw_n$ [1], we may express (2.1) in a variety of ways, some of them quite complicated. Generally, we have

$$(2.2) \qquad \left[(pw_{n+1} - qw_n)^2 - w_{n+1}^2\right]^2 + \left[2w_{n+1}(pw_{n+1} - qw_n)\right]^2 = \left[(pw_{n+1} - qw_n)^2 + w_{n+1}^2\right]^2 \ .$$

Assigned values of n, p, q (and a, b) may be inserted in this formula to yield various Pythagorean triples. For example, $n = 0$ with $a = 1$ $(=w_0)$, $b = 2$ $(=w_1)$, $p = 5$, $q = -1$ (a fairly random choice) produces the Pythagorean set 117, 4 4, 125.

More particularly, for the special sequences described in paragraph 1, we deduce, with $n = 0$ for simplicity, the following Pythagorean triples:

(1.1)	5	12	13
(1.2)	16	30	34
(1.3)	$2ad + 3d^2$	$2a^2 + 6ad + 4d^2$	$2a^2 + 6ad + 5d^2$
(1.4)	$a^2q^2(q^2 - 1)$	$2a^2q^3$	$a^2q^2(q^2 + 1)$
(1.5)	40	4 2	58
(1.6)	16	30	34
(1.7)	21	20	29
(1.8)	32	24	40

Triples for (1.2) and (1.6) just happen to coincide with $n = 0$ since $w_1 = 3$, $w_2 = 5$ for both sequences. No other values of n reproduce this coincidence for these two sequences.

Our concern here is not so much with the general Pythagorean formula (2.2) as with the cases arising when $p = 1$, $q = -1$ since these restrictions lead to $\{h_n(r, s)\}$, $\{f_n\}$ and $\{a_n\}$. In this respect, observe that, in (2.1), $w_{n+2}^2 - w_{n+1}^2 = (w_{n+2} + w_{n+1})(w_{n+2} - w_{n+1})$.

Substitution of $p = 1$, $q = -1$ in (2.2) yields

$$(2.2)' \qquad (w_n w_{n+3})^2 + (2w_{n+2}w_{n+1})^2 = (w_{n+2}^2 + w_{n+1}^2)^2$$

with a similar result for the case $p = -1$, $q = -1$. No other values of p, q produce the term $(w_n w_{n+3})^2$.

Thus we have the sequences whose n^{th} terms are

$$w_n(a,b;1,-1) \equiv af_{n-2} + bf_{n-1} \equiv h_n(a, b-a) \tag{2.3}$$

and

$$w_n(a,b;-1,-1) \equiv (-1)^n(af_{n-2} - bf_{n-1}) \equiv g_n(a, b-a) \quad \text{(say)} \tag{2.4}$$

where the g- and h-notation are introduced for convenience.

Putting $a = r$, $b = r + s$ in (2.2)', we derive the Pythagorean generalization for $\{h_n(r,s)\}$ determined in [2] and [3], namely,

$$(h_n h_{n+3})^2 + (2h_{n+1} h_{n+2})^2 = (2h_{n+1} h_{n+2} + h_n^2)^2 \tag{2.5}$$

in which the right-hand side is merely an alternative expression for the sum of the squares in the right-hand side of (2.2)'.

Examples of (2.2)' are, with (say) $n = 0$, $a = 5$, $b = 2$, from (2.3), $45^2 + 28^2 = 55^2$, and, from (2.4), $5^2 + 12^2 = 13^2$. Illustrations of the Pythagorean formula (2.5) have been given in [3]. More especially, for the Fibonacci and Lucas sequences $\{f_n\}$, $\{a_n\}$ the Pythagorean triples are, for $n = 0$, 3, 4, 5 and 8, 6, 10, respectively, while for $n = 1$ (say) they are 5, 12, 13 and 7, 24, 25, respectively.

As the properties of $\{h_n(r,s)\}$ have been developed in [2], it is thought worthwhile to examine some similar properties of the companion g-sequence relating to Pythagorean number triples. To this purpose we now direct our attention.

Just as it was shown in [3], with reference to (2.3), that all Pythagorean number triples are Fibonacci number triples, so may we likewise demonstrate the same for (2.4). Instead of putting

$$a = x - y, \quad b = y \tag{2.6}$$

in (2.3), we substitute

$$a = x + y, \quad b = y \tag{2.7}$$

in (2.4). In some of the concrete cases of (2.3) and (2.4), some part of the number triples will be negative; for instance, in the second case quoted above, the actual triple is $-5,\ -12,\ 13$.

Many different, but related, sequences give the same triple, but for different values of n. First, take the case $p = 1$, $q = -1$. Write $x = w_{n+2}$, $y = w_{n+1}$ as in [3]. Then by (2.3)

$$\begin{cases} x = af_n + bf_{n+1} \\ y = af_{n-1} + bf_n \end{cases} \tag{2.8}$$

Solve (2.6). Hence

$$\begin{cases} a = (-1)^n(xf_n - yf_{n+1}) \\ b = (-1)^{n+1}(xf_{n-1} - yf_n) \end{cases} , \tag{2.9}$$

where we have used the fundamental Fibonacci formula [2]

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^{n+1} .$$

Giving n all possible integral values, we obtain an infinite sequence of sequences of which a selected few are

$$\begin{cases} h_n(y, x - y), \quad h_n(x - y, -x + 2y) , \\ h_n(-x + 2y, 2x - 3y), \quad h_n(2x - 3y, -3x + 5y) , \end{cases} \tag{2.10}$$

corresponding to $n = -1, 0, 1, 2,$ respectively.

The second of the sequences (2.10) already occurs in (2.6). A given Pythagorean triple may be derived from any of these sequences if the correct value of n is associated with it (since we are operating on the same 4 numbers $x - y$, y, x, $x + y$ in each sequence). Examples are (i), if $x = 3$, $y = 2$, the triple 5, 12, 13 is obtained from the sequences $h_n(2, 1)$, $h_n(1, 1)$, $h_n(1, 0)$ and $h_n(0, 1)$ when $n = -1, 0, 1, 2$ respectively: (ii) if $x = 4$, $y = 3$, the triple

7, 24, 25 is obtained from the sequences $h_n(3,1)$, $h_n(1,2)$, $h_n(2,-1)$, $h_n(-1,3)$ when $n = -1, 0, 1, 2$ respectively.

Correspondingly, in the case $p = -1$, $q = -1$, write $x = w_{n+2}$, $y = -w_{n+1}$ so that by (2.4)

$$\text{(2.11)} \qquad \begin{cases} x = (-1)^n (af_n - bf_{n+1}) \\ y = (-1)^n (-af_{n-1} + bf_n) \end{cases}$$

whence, solving with the aid of the fundamental Fibonacci formula quoted above, we have

$$\text{(2.12)} \qquad \begin{cases} a = xf_n + yf_{n+1} \\ b = xf_{n+1} + yf_n \end{cases}$$

leading to an infinite sequence of sequences of which a selected few are, for $n = -1, 0, 1, 2$,

$$\text{(2.13)} \qquad \begin{cases} g_n(y,\ x-y), & g_n(x+y,\ -x)\ , \\ g_n(x+2y,\ -y), & g_n(2x+3y,\ -x-y)\ , \end{cases}$$

respectively. With $x = 3$, $y = 2$, for instance, the triple $-5, -12, 13$ arises from $g_n(2,1)$, $g_n(5,-3)$, $g_n(7,-2)$, $g_n(12,-5)$ when $n = -1, 0, 1, 2$ respectively. Observe that the second sequence in (2.13) already occurs in (2.7). Had we written $x = -w_{n+2}$, $y = w_{n+1}$ above, then of course we would have obtained the negatives of the values of a, b given in (2.12).

Remarks similar to the other remarks for $h_n(a,b,-a)$ in [3] may be paralleled for $g_n(a,b-a)$.

3. THE GEOMETRICAL PARADOX

A well-known geometrical problem requires a given square to be sub-divided in a specified manner and re-arranged so as to form a rectangle of certain dimensions. In the process of re-arrangement, it appears as though a small area of one square unit has been gained or lost. This illusion is due to inaccurate re-assembling of the sub-divided parts. Precise re-arrangement reveals the existance of a very small parallelogram of unit area included in the rectangle. Mathematically, the secret of the paradox lies with the Fibonacci formula quoted in Section 2.

Previously in [4] I generalized this paradox to the sequence $\{h_n(r,s)\}$. Our basic generalized formula now is 1 , with n replaced by n + 1, $w_n w_{n+2} - w_{n+1}^2 = eq^n$. As in [4], the construction guarantees two cases, n even and n odd. See Figs. 1, 2, 3. Clearly, the spirit of the standard construction is preserved only if $q < 0$. Write $q_1 = -q$ $(q_1 > 0)$. From the figures, we see that the exigencies of the constructions impose the restriction $p = q_1 = 1$, so that the defining recurrence relation [1] is now $w_{n+2} = w_{n+1} + w_n$, the fundamental formula [1] is $w_n w_{n+2} - w_{n+1}^2 = (-1)^n e$, and the area of the parallelogram [4] is e. Consequently, the only sequences for which the standard construction is applicable are $w_n(a,b;1,-1) = h_n(a,b-a)$ by (2.3).

Briefly repeating the basic results proved in [4], we have, after calculations:

$$\lambda_n = \sqrt{w_{n+1}^2 + w_{n-1}^2}, \quad \mu_n = \sqrt{w_n^2 + w_{n-2}^2} \quad ; \tag{3.1}$$

$$\lim_{n\to\infty} \left(\frac{\lambda_n}{\mu_n}\right) = \alpha_1 \tag{3.2}$$

$$\left\{\begin{aligned} \tan\theta_n &= \tan\left(\frac{\pi}{2} - \gamma_n - \delta_n\right), \left[\tan\gamma_n = \frac{w_{n-1}}{w_{n+1}},\ \tan\delta_n = \frac{w_n}{w_{n-2}}\right] \\ &= \frac{e_1}{e_1 + 3w_n w_{n-1}} = t_n \end{aligned}\right. \tag{3.3}$$

$$\lim_{n\to\infty}\left(\frac{t_n}{t_{n+1}}\right) = \alpha_1^2 = 1 + \alpha_1 , \tag{3.4}$$

where in (3.3) we have set

$$e_1 = ab + a^2 - b^2 . \tag{3.5}$$

Initially, in Fig. 3 we have

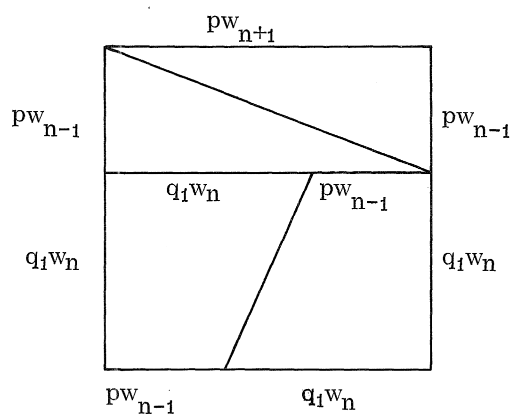

Fig. 1

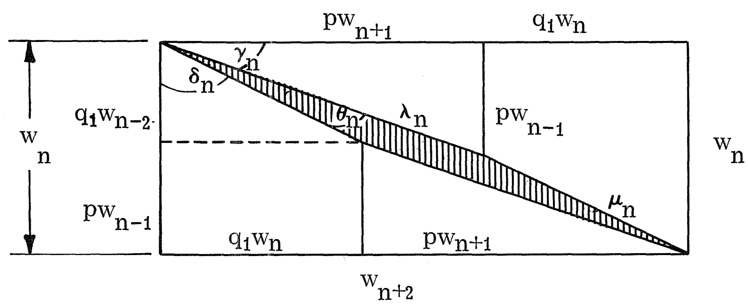

Fig. 2 (n even)

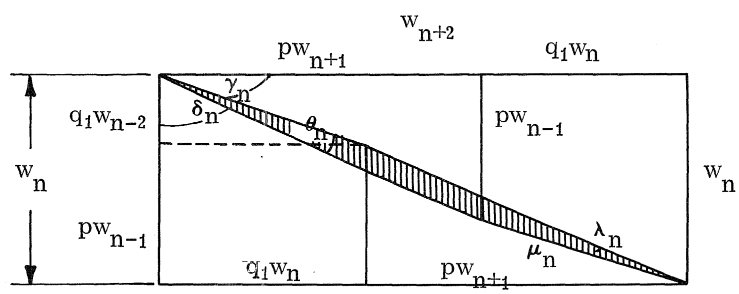

Fig. 3 (n odd)

($p = q_1 = 1$ in Figs. 1-3)

$$\tan\theta_n = \tan(\gamma_n + \delta_n - \pi/2) \quad . \tag{3.6}$$

Eventually, after calculation this leads back to (3.3).

Worth noting is the fact that (3.3) is a considerable simplification of the form for $\tan\theta_n$ given in [4].

Concrete instances of the paradox, with details of specific values for θ_n, λ_n, μ_n, are to be found in [4].

4. THE COMPLEX CASE

Label each of the fundamental constants a, b, p, q, e associated with a sequence differen t from $\{w_n\}$ by a subscript symbolic of that sequence; that is, for the sequence $\{h_n\}$, for instance, express these constants as a_h, b_h, p_h, q_h, e_h.

Define

$$\begin{cases} d_n = w_n + iw_{n+1} \qquad (i^2 = -1) \\ \quad\;\; = bu_{n-1} - qau_{n-2} + i(bu_n - qau_{n-1}) \end{cases} \tag{4.1}$$

using a known expression [1] for w_n. Hence

$$\begin{cases} d_0 = a_d = a + ib \\ d_1 = b_d = b + i(pb - qa) \end{cases} \tag{4.2}$$

After substituting $u_n = pu_{n-1} - qu_{n-2}$, we deduce from (4.1), (4.2) that

$$d_n = pd_{n-1} - qd_{n-2} \tag{4.3}$$

and

$$\begin{cases} d_n = \{b + i(pb - qa)\}\, u_{n-1} - q(a + ib)u_{n-2} \\ \quad\;\; = (w_1 + iw_2)u_{n-1} - q(w_0 + iw_1)u_{n-2} \\ \quad\;\; = d_1\, u_{n-1} - qd_0\, u_{n-2} \\ \quad\;\; = b_d\, u_{n-1} - qa_d\, u_{n-2} \end{cases} \tag{4.4}$$

from (4.1), which is a form we could anticipate. Of course, we could have substituted $w_n = au_n + (b - pa)u_{n-1}$ and obtained an equivalent result. Thus

(4.5) $$\{d_n\} \equiv \{w_n\,(a + ib,\ b + i(pb - qa)\,;\ p, q)\}.$$

Moreover,

(4.6) $$\begin{cases} e_d = p\,a_d\,b_d - q\,a_d^2 - b_d^2 \\ \quad = (1 - q + ip)e \end{cases}$$

after calculation.

Fundamental properties of d_n are deducible in an analogous way to those of w_n [1]. Only the three most interesting general properties are stated for the record:

(4.7) $$d_{n-1}\,d_{n+1} - d_n^2 = e_d\,q^{n-1}$$

(4.8) $$(d_n\,d_{n+3})^2 + (-2pq\,d_{n+1}\,d_{n+2})^2 = (-2pq\,d_{n+1}\,d_{n+2} + d_n^2)^2 + 2c_1c_2d_n^2$$

(4.9) $$\frac{d_{n+r} + q^r d_{n-r}}{d_n} = v_r$$

(that is, the right-hand side of (4.9) is independent of a, b, n). In the Pythagorean result (4.8), we have written

(4.10) $$\begin{cases} c_1 = pd_{n+2} - qd_{n+1} - d_n \\ c_2 = c_1 + 2d_n \end{cases}$$

All these results are easy to verify using as appropriate (4.3) or (4.1) with $w_n = A\alpha^n + B\beta^n$ [1] being a convenient substitution on (4.7) and (4.9). Be it noted that with this approach we may need to use $w_{n-1}w_{n+2} - w_nw_{n+1} = epq^{n-1}$, which is a special case of [1] (4.18) for which $r = t = 1$.

Particular cases of the above theoretical results lead back to those in [5]. For example $p = -q = 1$ implies $w_n(a, b; 1, -1) = h_n(a, b - a)$ by (2.3)

Under these conditions, replace d_n by k_n. Then (4.6), for instance, gives [5].

$$e_k = e_c e_h , \tag{4.11}$$

where c is the complex Fibonacci sequence for which $a = b = 1$ and [5], (3.5),

$$e_c = 2 + i, \quad e_h = ab + a^2 - b^2 . \tag{4.12}$$

Extending [5] we may define a generalized quaternion as:

$$q_n = w_n + iw_{n+1} + jw_{n+2} + kw_{n+3} \tag{4.13}$$

with conjugate quaternion

$$\overline{q}_n = w_n - iw_{n+1} - jw_{n+2} - kw_{n+3} , \tag{4.14}$$

where $i^2 = j^2 = k^2 = -1$, $ij = -ji$, $jk = -kj$, $ki = -ik$.

From (4.13), (4.14),

$$w_n = \frac{q_n + \overline{q}_n}{2} . \tag{4.15}$$

Finally, for the conjugate $\overline{d}_n$ it follows that

$$\begin{cases} a_{\overline{d}} = \overline{a_d} \\ b_{\overline{d}} = \overline{b_d} \\ e_{\overline{d}} = \overline{e_d} \end{cases} \tag{4.16}$$

(Note: Helpful advice from the referee has been incorporated into the early part of Section 2 and is hereby acknowledged.)

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All subscription correspondence should be addressed to Bro. A. Brousseau, St. Mary's College, Calif. All checks ($4.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Manuscripts intended for publication in the Quarterly should be sent to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, Calif. All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscripts sent to the editors.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

Notice: If any of the readers have not received acknowledgment for their solutions in previous issues, the editor will acknowledge them after receipt of notification of such omissions.

H-123 Proposed by D. Lind, University of Virginia, Charlottesville, Virginia.

Prove

$$F_n = \sum_{m=0}^{n}\sum_{k=0}^{m} \mathcal{S}_n^{(m)} S_m^{(k)} F_k ,$$

where $S_r^{(s)}$ and $\mathcal{S}_r^{(s)}$ are Stirling numbers of the first and second kinds, respectively, and F_n is the n^{th} Fibonacci number.

H-124 Proposed by J. A. H. Hunter, Toronto, Canada

Prove the following identity:

$$F_{m+n}^2 L_{m+n}^2 - F_m^2 L_m^2 = F_{2n} F_{2(m+n)}$$

where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively.

H-125 Proposed by Stanley Rabinowitz, Far Rockaway, New York

Define a sequence of positive integers to be left-normal if given any string of digits, there exists a member of the given sequence beginning with this string of digits, and define the sequence to be right-normal if there exists a member of the sequence ending with this string of digits.

Show that the sequences whose n^{th} terms are given by the following are left-normal but not right-normal.

a) $P(n)$, where $P(x)$ is a polynomial function with integral coefficients.

b) P_n, where P_n is the n^{th} prime.

c) $n!$

d) F_n, where F_n is the n^{th} Fibonacci number.

SOLUTIONS

EUREKA!

H-59 Proposed by D. W. Robinson, Brigham Young University, Provo, Utah.

Show that if $m > 2$, then the period of the Fibonacci sequence $0, 1, 2, 3, \cdots, F_n, \cdots$ reduced modulo m is twice the least positive integer, n, such that

$$F_{n+1} \equiv (-1)^n F_{n-1} \pmod{m}$$

Solution by James E. Desmond, Tallahassee, Florida.

Let s be the period of the Fibonacci sequence modulo m. Then by defintion, s is the least positive integer such that

$$(1) \qquad F_{s-1} \equiv 1 \pmod{m} \text{ and } F_s \equiv 0 \pmod{m}.$$

By the well-known formula

$$F_{s+1}F_{s-1} - F_s^2 = (-1)^s$$

We find that $1 \equiv (-1)^s \pmod{m}$ which implies, since $m > 2$, that $s = 2t$ for some positive integer t. It is easily verified that

(2) $$F_{2t-1} = F_tL_{t-1} + (-1)^t = F_{t-1}L_t + (-1)^{t+1} .$$

Since $s = 2t$ we have by (1) and (2) that

(3) $$F_tL_{t-1} \equiv 0 \pmod{m} \text{ if } t \text{ is even, and}$$

(4) $$F_{t-1}L_t \equiv 0 \pmod{m} \text{ if } t \text{ is odd.}$$

It is well known that

(5) $$F_{2t} = F_tL_t , \text{ and}$$

(6) $$(L_{t-1}, L_t) = (F_{t-1}, F_t) = 1 .$$

Thus by (1), (3), (4), (5), and (6) we have

$$\begin{aligned} &F_t \equiv 0 \pmod{m} \text{ if } t \text{ is even, and} \\ &L_t \equiv 0 \pmod{m} \text{ if } t \text{ is odd, i.e.,} \\ &F_{t+1} + (-1)^{t+1}F_{t-1} \equiv 0 \pmod{m} . \end{aligned}$$

Now, let n be the least positive integer such that $F_{n+1} + (-1)^{n+1}F_{n-1} \equiv 0 \pmod{m}$ and we obtain $n \le t$. We also find that $F_n \equiv 0 \pmod{m}$ if n is even, and $L_n \equiv 0 \pmod{m}$ if n is odd. Thus by (2) we have, $F_{2n-1} \equiv 1 \pmod{m}$ and by (5), $F_{2n} \equiv 0 \pmod{m}$. Since s is the period modulo m, **it** follows by definition that $2t = s \le 2n$. Hence $n = t$.

RESTRICTED UNFRIENDLY SUBSETS

H-75 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia.

Show that the number of distinct integers with one element n, all other elements less than n and not less than k, and such that no two consecutive integers appear in the set is F_{n-k+1}.

Solution by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pa.

Since each admissible set of integers must contain n, any given admissible set is uniquely determined by specifying which of the remaining $n - k - 1$ integers $(k, k+1, k+2, \cdots, n-2)$ are included in the set. (Note that the integer $n - 1$ cannot be included since n is in each set and consecutive integers are not permitted.) For each set, this information can be given concisely by a sequence of $n - k - 1$ binary digits, using a 1 in the m^{th} place ($m = 1, 2, \cdots, n-k-1$) if the integer $k + m - 1$ is included in the set and 0 in the m^{th} place otherwise.

If we require additionally that the terms of each such binary sequence $(\alpha_1, \alpha_2, \cdots, \alpha_{n-k-1})$ satisfy $\alpha_i\alpha_{i+1} = 0$ for $i = 1, 2, \cdots, n-k-2$, then this requirement is equivalent to the condition that no two consecutive integers appear in the corresponding set. But the number of distinct binary sequences of length $n - k - 1$ satisfying $\alpha_i\alpha_{i+1} = 0$ for $i \geq 1$ is known to be $F_{(n-k-1)+2} = F_{n-k+1}$ as required. [See The Fibonacci Quarterly, Vol. 2, No. 3, pp. 166-167 for a proof using Zeckendorf's Theorem.]

FIBONOMIAL COEFFICIENT GENERATORS

H-78 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

(i) Show

$$\frac{x^{n-1}}{(1-x)^n} = \sum_{m=0}^{\infty} \binom{m}{n-1} x^m, \quad (n \geq 1)$$

where $\binom{m}{n}$ are the binomial coefficients.

(ii) Show

$$\frac{x}{(1 - x - x^2)} = \sum_{m=0}^{\infty} \begin{bmatrix} m \\ 1 \end{bmatrix} x^m ,$$

$$\frac{x^2}{(1 - 2x - 2x^2 + x^3)} = \sum_{m=0}^{\infty} \begin{bmatrix} m \\ 2 \end{bmatrix} x^m ,$$

$$\frac{x^3}{(1 - 3x - 6x^2 + 3x^3 + x^4)} = \sum_{m=0}^{\infty} \begin{bmatrix} m \\ 3 \end{bmatrix} x^m ,$$

where $\begin{bmatrix} m \\ n \end{bmatrix}$ are the Fibonomial coefficients as in H-63, April 1965, <u>Fibonacci Quarterly</u> and H-72 of Dec. 1965, <u>Fibonacci Quarterly</u>.

The generalization is: Let

$$f(x) = \sum_{h=0}^{k} (-1)^{h(h+1)/2} \begin{bmatrix} k \\ h \end{bmatrix} x^h ,$$

then show

$$\frac{x^{k-1}}{f(x)} = \sum_{m=0}^{\infty} \begin{bmatrix} m \\ k-1 \end{bmatrix} x^m , \quad (k \geq 1) .$$

Solution by L. Carlitz, Duke University.

(i) This is a special case of the binomial theorem.

(ii) The general resultscan be viewed as the q-analog of (i), namely

$$\prod_{j=0}^{k-1} (1 - q^j x)^{-1} = \sum_{j-0}^{\infty} \begin{Bmatrix} k+j-1 \\ j \end{Bmatrix} x^j ,$$

where

$$\begin{Bmatrix} k+j-1 \\ j \end{Bmatrix} = \frac{(1 - q^k)(1 - q^{k+1}) \cdots (1 - q^{k+j-1})}{(1 - q)(1 - q^2) \cdots (1 - q^j)} .$$

We shall also need

$$\prod_{j=0}^{k-1} (1 - q^j x) = \sum_{j=0}^{k} (-1)^j q^{\frac{1}{2}j(j-1)} \left\{ {k \atop j} \right\} x^j$$

Now take $q = \beta/\alpha$, $\alpha = \frac{1}{2}(1 + \sqrt{5})$, $\beta = \frac{1}{2}(1 - \sqrt{5})$. Then

$$\left\{ {k \atop j} \right\} \longrightarrow \alpha^{-j(k-j)} \frac{F_k F_{k-1} \cdots F_{k-j+1}}{F_1 F_2 \cdots F_j} = \alpha^{-(k-1)j} \begin{bmatrix} k \\ j \end{bmatrix}.$$

(Compare "Generating Functions for Powers of Certain Sequences of Numbers," Duke Mathematical Journal, Vol. 29 (1962), pp. 521-538, particularly p. 530.)

Since

$$\begin{aligned} (-1)^j \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}j(j-1)} \alpha^{-j(k-j)} &= (-1)^j (-\alpha^{-2})^{\frac{1}{2}j(j-1)} \alpha^{-j(k-j)} \\ &= (-1)^{\frac{1}{2}j(j+1)} \alpha^{-j(k-1)} , \end{aligned}$$

we get, after replacing x by $\alpha^{k-1}x$, the identity

$$\left\{ \sum_{j=0}^{k-1} (-1)^{\frac{1}{2}j(j+1)} \begin{bmatrix} k \\ j \end{bmatrix} x^j \right\}^{-1} = \sum_{j=0}^{\infty} \begin{bmatrix} k + j - 1 \\ j \end{bmatrix} x^j = \sum_{j=0}^{\infty} \begin{bmatrix} k + j - 1 \\ k - 1 \end{bmatrix} x^j$$

A FOURTH-POWER FORMULA

H-79 Proposed by J. A. H. Hunter, Toronto, Ontario, Canada.

Show

$$F_{n+1}^4 + F_n^4 + F_{n-1}^4 = 2\left[2F_n^2 + (-1)^n\right]^2 .$$

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

From the well-known identity,

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

we have,

$$\begin{aligned}
2\left[2F_n^2 + (-1)^n\right]^2 &= 2\left[F_{n-1}F_{n+1} + F_n^2\right]^2 \\
&= F_n^4 + F_n^4 + 2F_{n-1}^2F_{n+1}^2 + 4F_n^2F_{n-1}F_{n+1} \\
&= F_n^4 + F_n^2(F_n^2 + 4F_{n-1}F_{n+1}) + 2F_{n-1}^2F_{n+1}^2 \\
&= F_n^4 + F_n^2\left[(F_{n+1} - F_{n-1})^2 + 4F_{n-1}F_{n+1}\right] + 2F_{n-1}^2F_{n+1}^2 \\
&= F_n^4 + (F_{n+1} - F_{n-1})^2(F_{n+1} + F_{n-1})^2 + 2F_{n-1}^2F_{n+1}^2 \\
&= F_n^4 + (F_{n+1}^2 - F_{n-1}^2)^2 + 2F_{n-1}^2F_{n+1}^2 \\
&= F_n^4 + F_{n+1}^4 + F_{n-1}^4
\end{aligned}$$

Hence,

$$F_{n+1}^4 + F_n^4 + F_{n-1}^4 = 2\left[2F_n^2 + (-1)^n\right]^2$$

Also solved by Thomas Dence, F. D. Parker, and L. Carlitz.

A PLEASANT SURPRISE

H-80 Proposed by J. A. H. Hunter, Toronto, Canada, and Max Rumney, London, England (corrected).

Show

$$\sum_{r=0}^{n} \binom{n}{r} F_{r+2}^2 = \sum_{r=0}^{n} \binom{n-1}{r} F_{2r+5} .$$

Solution by L. Carlitz, Duke University

This is correct for $n = 0$, so we assume that $n > 0$. Since

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \alpha = \tfrac{1}{2}(1 + \sqrt{5}), \ \beta = \tfrac{1}{2}(1 - \sqrt{5}),$$

we have

$$5\sum_{r=0}^{n}\binom{n}{r} F_{r+2}^2 = \sum_{r=0}^{n}\binom{n}{r}\left[\alpha^{2r+4} - 2(-1)^r + \beta^{2r+4}\right]$$

$$= \alpha^4(\alpha^2 + 1)^n + \beta^4(\beta^2 + 1)^n .$$

On the other hand

$$\sum_{r=0}^{n-1}\binom{n-1}{r} F_{2r+5} = \frac{1}{\alpha - \beta}\sum_{r=0}^{n-1}\binom{n-1}{r}(\alpha^{2r+5} - \beta^{2r+5})$$

$$= \frac{\alpha^5(\alpha^2 + 1)^{n-1} - \beta^5(\beta^2 + 1)^{n-1}}{\alpha - \beta}$$

Thus it suffices to show that

$$\alpha^4(\alpha^2 + 1)^{n-1} + \beta^4(\beta^2 + 1)^{n-1} = (\alpha - \beta)\left[\alpha^5(\alpha^2 + 1)^{n-1} - \beta^5(\beta^2 + 1)^{n-1}\right].$$

The right side is equal to

$$\alpha^6(\alpha^2 + 1)^{n-1} + \beta^6(\beta^2 + 1)^{n-1} + \alpha^4(\alpha^2 + 1)^{n-1} + \beta^4(\beta^2 + 1)^{n-1}$$

$$= \alpha^4(\alpha^2 + 1)^n + \beta^4(\beta^2 + 1)^n .$$

<u>Remark.</u> More generally we have

$$\sum_{r=0}^{n} \binom{n}{r} F^2_{kr+2k} = F_k \sum_{r=0}^{n-1} \binom{n-1}{r} F_{2kr+5k}$$

for k odd and $n > 0$.

Also solved by M. N. S. Swamy, F. D. Parker, and Douglas Lind.

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A NOTE OF JOY

We have received with great pleasure the announcement of the forthcoming Journal of Recreational Mathematics under the editorship of Joseph S. Madachy. Volume 1, Number 1 is to appear in January, 1968. The journal "will deal with the lighter side of mathematics, that side devoted to the enjoyment of mathematics; it will depart radically from textbook problems and discussions and will present original, thought-provoking, lucid and exciting articles which will appeal to both students and teachers in the field of mathematics." The journal will feature authoritative articles concerning number theory, geometric constructions, dissections, paper folding, magic squares, and other number phenomena. There will be problems and puzzles, mathematical biographies and histories. Subscriptions for the Journal of Recreational Mathematics are handled by Greenwood Periodicals, Inc., 211 East 43rd St., New York, N. Y. 10017. We wish this valuable and important journal all possible success. H.W.E.

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—RECREATION CORNER—

POPULATION EXPLOSION

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

It appears that the time has come for the puzzlist to update some of the old-time conundrums. Take, for example, the type of problem in which some forty or fifty people are lined up in a circle and then beginning at a certain one, the group is decimated until there are only nine people left. The problem would be to choose one of the safe spots.

Now with the population explosion and the advent of the computer, the puzzle of the future may run like this. At 3:52 P.M. of December 2nd, all citizens are warned that there is to be a selection of numbers for the purpose of determining who will not pay an extra 5 percent income tax. The executives being very fair-minded make plain what the plan is: Starting with 1,000,000 and working backward every third number is to be selected in cyclic fashion until there are only two numbers left. Those persons who select one of these two numbers will not have to pay the extra 5 percent income tax.

Evidently, a very fair plan and extremely educational. Anybody who is stupid enough to select a number congruent to 1 modulo 3 deserves to be penalized for his lack of mathematical ability. But of course there are 666,665 other numbers that have to be dodged. The bureaucrats allow twenty-four hours for choosing a number. Here is where the Ancient Order of Puzzlists comes in. By having ready at hand some quick and efficient method for finding the two favored numbers, they can render a distinct service to their fellow citizens.

What are the two favored numbers and is there some reasonably simple method of finding them?

The answer and the method of arriving at it will be published in February, 1968.

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A PRIMER FOR THE FIBONACCI NUMBERS: PART VI

V. E. HOGGATT, JR., AND D. A. LIND
San Jose State College, San Jose, California, and University of Virginia, Charlottesville, Va.

1. INTRODUCTION

We shall devote this part of the primer to the topic of generating functions. These play an important role both in the general theory of recurring sequences and in combinatorial analysis. They provide a tool with which every Fibonacci enthusiast should be familiar.

2. GENERAL THEORY OF GENERATING FUNCTIONS

Let $a_0, a_1, a_2, \cdots$ be a sequence of real numbers. The ordinary generating function of the sequence $\{a_n\}$ is the series

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n .$$

Another type of generating function of great use in combinatorial problems involving permutations is the exponential generating function of $\{a_n\}$, namely

$$E(x) = a_0 + a_1 x/1! + a_2 x^2/2! + \cdots = \sum_{n=0}^{\infty} a_n x^n /n! .$$

For some examples of the two types of generating functions, first let $a_n = a^n$. The ordinary generating function of $\{a_n\}$ is then the geometric series

$$A(x) = \frac{1}{1 - ax} = \sum_{n=0}^{\infty} a^n x^n , \tag{2.1}$$

while the exponential generating function is

$$E(x) = e^{ax} = \sum_{n=0}^{\infty} a^n x^n / n! \quad .$$

Similarly, if $a_n = na^n$, then

$$(2.2)\qquad \begin{aligned} A(x) &= \frac{ax}{(1 - ax)^2} = \sum_{n=0}^{\infty} na^n x^n \quad , \\ E(x) &= axe^{ax} = \sum_{n=0}^{\infty} na^n x^n / n! \quad , \end{aligned}$$

each of these being obtained from the preceding one of the same type by differentiation and multiplication by x. A good exercise for the reader to check his understanding is to verify that if $a_n = n^2$, then

$$A(x) = \frac{x(x+1)}{(1 - x)^3} = \sum_{n=0}^{\infty} n^2 x^n \quad ,$$

$$E(x) = x(x + 1)e^x = \sum_{n=0}^{\infty} n^2 x^n / n! \quad .$$

(Hint: Differentiate the previous results again.)

For the rest of the time, however, we will deal exclusively with ordinary generating functions.

We adopt the point of view here that x is an indeterminant, a means of distinguishing the elements of the sequence through its powers. Used in this context, the generating function becomes a tool in an algebra of these sequences (see [3]). Then formal operations, such as addition, multiplication, differentiation with respect to x, and so forth, and equating equations of like powers of x after these operations merely express relations in this algebra, so that convergence of the series is irrelevant.

The basic rules of manipulation in this algebra are analogous to those for handling polynomials. If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are real sequences with (ordinary) generating functions $A(x)$, $B(x)$, $C(x)$ respectively, then $A(x) + B(x) = C(x)$ if and only if $a_n + b_n = c_n$, and $A(x)B(x) = C(x)$ if and only if

$$c_n = a_n b_0 + a_{n-1}b_1 + \cdots + a_1 b_{n-1} + a_0 b_n .$$

Both results are obtained by expanding the indicated sum or product of generating functions and comparing coefficients of like powers of x. The product here is called the Cauchy product of the sequences $\{a_n\}$ and $\{b_n\}$, and the sequence $\{c_n\}$ is called the convolution of the two sequences $\{a_n\}$ and $\{b_n\}$.

To give an example of the usefulness and convenience of generating functions, we shall derive a well-known but nontrivial binomial identity. First note that for a fixed real number k the generating function for the sequence

$$a_n = \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$

is

$$A_k(x) = (1 + x)^k$$

by the binomial theorem. If k is a nonnegative integer, the generating function is finite since

$$\binom{k}{n} = 0 \quad \text{if} \quad n > k \geq 0 \quad \text{or} \quad n < 0 \tag{2.3}$$

by definition. Then

$$A_k(x) = (1 + x)^k = (1 + x)^{k-m}(1 + x)^m = A_{k-m}(x)A_m(x) .$$

Using the product rule gives

$$\sum_{n=0}^{k}\binom{k}{n}x^n = \sum_{n=0}^{\infty}\binom{k}{n}x^n = \left(\sum_{n=0}^{\infty}\binom{k-m}{n}x^n\right)\left(\sum_{n=0}^{\infty}\binom{m}{n}x^n\right)$$

$$= \sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}\binom{k-m}{j}\binom{m}{n-j}\right]x^n ,$$

so that equating coefficients of x^n shows

$$\binom{k}{n} = \sum_{j=0}^{n}\binom{k-m}{j}\binom{m}{n-j} .$$

This can be found in Chapter 1 of [8].

If the generating function for $\{a_n\}$ is known, it is sometimes desirable to convert it to the generating function for $\{a_{n+k}\}$ as follows. If

$$A(x) = \sum_{n=0}^{\infty} a_n x^n ,$$

then

$$\frac{A(x) - a_0}{x} = \sum_{n=0}^{\infty} a_{n+1}x^n .$$

This can be repeated as often as needed to obtain the generating function for $\{a_{n+k}\}$.

Generating functions are a powerful tool in the theory of linear recurring sequences and the solution of linear difference equations. As an example, we shall solve completely a second-order linear difference equation using the technique of generating functions. Let $\{c_n\}$ be a sequence of real numbers which obey

$$c_{n+2} - pc_{n+1} + qc_n = 0, \quad n \geq 0,$$

where c_0 and c_1 are arbitrary. Then by using the Cauchy product we find

$$\begin{aligned}(1 - px + qx^2) \sum_{n=0}^{\infty} c_n x^n &= c_0 + (c_1 - pc_0)x + 0 \cdot x^2 + \cdots \\ &= c_0 + (c_1 - pc_0)x = r(x) \quad ,\end{aligned}$$

so that

$$\sum_{n=0}^{\infty} c_n x^n = \frac{r(x)}{1 - px + qx^2} \quad . \tag{2.4}$$

Suppose a and b are the roots of the auxiliary polynomial $x^2 - px + q$, so the denominator of the generating function factors as $(1 - ax)(1 - bx)$. We divide the treatment into two cases, namely, $a \neq b$ and $a = b$.

If a and b are distinct (i.e., $p^2 - 4q \neq 0$), we may split the generating function into partial functions, giving

$$\frac{r(x)}{1 - px + qx^2} = \frac{r(x)}{(1 - ax)(1 - bx)} = \frac{A}{1 - ax} + \frac{B}{1 - bx} \tag{2.5}$$

for some constants A and B. Then using (2.1) we find

$$\sum_{n=0}^{\infty} c_n x^n = A \sum_{n=0}^{\infty} a^n x^n + B \sum_{n=0}^{\infty} b^n x^n = \sum_{n=0}^{\infty} (Aa^n + Bb^n) x^n \quad ,$$

so that an explicit formula for c_n is

$$c_n = Aa^n + Bb^n \quad . \tag{2.6}$$

Here A and B can be determined from the initial conditions resulting from assigning values to c_0 and c_1.

On the other hand, if the roots are equal (i.e., $p^2 - 4q = 0$), the situation is somewhat different because the partial fraction expansion (2.5) is not valid. Letting $r(x) = r + sx$, we may use (2.2), however, to find

$$\sum_{n=0}^{\infty} c_n x^n = \frac{r + sx}{(1 - ax)^2} = (r + sx) \sum_{n=0}^{\infty} (n + 1)a^n x^n$$

$$= \sum_{n=0}^{\infty} (r(n + 1)a^n + sna^{n-1})x^n = \sum_{n=0}^{\infty} ((r + s/a)n + r)a^n x^n ,$$

showing that

$$c_n = (An + B)a^n ,$$

where

$$A = r + s/a, \qquad B = r$$

are constants which again can be determined from the initial values c_0 and c_1.

This technique can be easily extended to recurring sequences of higher order. For further developments, the reader is referred to Jeske [6], where a generalized version of the above is derived in another way. For a discussion of the general theory of generating functions, see Chapter 2 of [8] and Chapter 3 of [2].

3. APPLICATIONS TO FIBONACCI NUMBERS

The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} - F_{n+1} - F_n = 0$, $n \geq 0$. Using the general solution of the second-order difference equation given above, where $p = 1$, $q = -1$, $r(x) = x$, we find that the generating function for the Fibonacci numbers is

$$F(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n \quad . \tag{3.1}$$

The reader should actually divide out the middle part of (3.1) by long division to see that Fibonacci numbers really do appear as coefficients.

Since the roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ of the auxiliary polynomial $x^2 - x - 1$ are distinct, we see from (2.6) that

$$F_n = A\alpha^n + B\beta^n \quad . \tag{3.2}$$

Putting $n = 0, 1$ and solving the resulting system of equations shows that

$$A = 1/\sqrt{5} = 1/(\alpha - \beta), \quad B = -1/\sqrt{5} \ ,$$

establishing the familiar Binet form,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad . \tag{3.3}$$

We shall now turn around and use this form to derive the original generating function (3.1) by using a technique first exploited by H. W. Gould [5]. Suppose that some sequence $\{a_n\}$ has the generating function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \quad .$$

Then

$$\frac{A(\alpha x) - A(\beta x)}{\alpha - \beta} = \sum_{n=0}^{\infty} a_n \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) x^n = \sum_{n=0}^{\infty} a_n F_n x^n \quad . \tag{3.4}$$

In particular, if $a_n = 1$, then $A(x) = 1/(1 - x)$, so that

$$F(x) = \frac{1}{\alpha - \beta}\left(\frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x}\right) = \frac{x}{1 - x - x^2} .$$

Next we use (3.1) to prove that the Fibonacci numbers are the sums of terms along the rising diagonals of Pascal's Triangle. We write

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2} = \frac{x}{1 - (x + x^2)} = x \sum_{n=0}^{\infty} x^n (1 + x)^n$$

$$= \sum_{n=0}^{\infty} x^{n+1} \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^{n+k+1}$$

$$= \sum_{m=1}^{\infty} \sum_{j=0}^{[(m-1)/2]} \binom{m - j - 1}{j} x^m ,$$

where $[m]$ denotes the greatest integer contained in m. The inner sum is the sum of coefficients of x^m in the preceding sum, and the upper limit of summation is determined by the inequality $m - j - 1 < j$, recalling (2.3). The reader is urged to carry through the details of this typical generating function calculation. Equating coefficients x^n shows that

$$F_n = \sum_{j=0}^{[(n-1)/2]} \binom{n - j - 1}{j} \tag{3.5}$$

linking the Fibonacci numbers to the binomial coefficients.

It follows from (3.1) upon division by x that

$$G(x) = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n . \tag{3.6}$$

Differentiating this yields

$$G'(x) = \frac{2x+1}{(1-x-x^2)^2} = \left(\frac{1}{1-x-x^2}\right)\left(\frac{1+2x}{1-x-x^2}\right) = \sum_{n=0}^{\infty}(n+1)F_{n+2}x^n.$$

Now

$$\frac{1+2x}{1-x-x^2} = \sum_{n=0}^{\infty} L_{n+1}\, x^n \quad ,$$

where the L_n are the Lucas numbers defined by $L_1 = 1$,

$$L_1 = 1, \quad L_2 = 3, \quad L_{n+2} = L_{n+1} + L_n, \quad n \geq 0.$$

Hence

$$G'(x) = \left(\sum_{n=0}^{\infty} F_{n+1}\, x^n\right)\left(\sum_{n=0}^{\infty} L_{n+1} x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} F_{n-k+1}L_{k+1}\right) x^n ,$$

so that

$$\sum_{k=0}^{n} F_{n-k+1}L_{k+1} = (n+1)F_{n+2} \quad ,$$

a convolution of the Fibonacci and Lucas sequences.

We leave it to the reader to verify that

$$\frac{x}{(1-x)(1-x-x^2)} = \frac{x}{1-2x+x^3} = \sum_{n=0}^{\infty}(F_{n+2} - 1)x^n \quad .$$

Also

$$\frac{x}{(1-x)(1-x-x^2)} = \frac{1}{1-x}\,\frac{x}{1-x-x^2} = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} F_n x^n\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} F_j\right) x^n .$$

Equating coefficients shows

$$\sum_{j=0}^{n} F_j = F_{n+2} - 1 ,$$

which is really the convolution of the Fibonacci sequence with the constant sequence $\{1, 1, 1, \cdots\}$.

Consider the sequence $\{F_{kn}\}_{n=0}^{\infty}$, where $k \neq 0$ is an arbitrary but fixed integer. Since

$$F_{kn} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta}$$

we have

$$\sum_{n=0}^{\infty} F_{kn} x^n = \frac{1}{\alpha - \beta}\left(\sum_{n=0}^{\infty} \alpha^{kn} x^n - \sum_{n=0}^{\infty} \beta^{kn} x^n\right)$$

$$(3.7) \qquad = \frac{1}{\alpha - \beta}\left(\frac{1}{1 - \alpha^k x} - \frac{1}{1 - \beta^k x}\right) = \frac{1}{\alpha - \beta}\left(\frac{(\alpha^k - \beta^k)\, x}{1 - (\alpha^k + \beta^k)x + (\alpha^k \beta^k)x^2}\right)$$

$$= \frac{F_k\, x}{1 - L_k x + (-1)^k x^2} ,$$

where we have used $\alpha\beta = -1$ and the Binet form $L_n = \alpha^n + \beta^n$ for the Lucas numbers. Incidentally, since here the integer in the numerator must divide all coefficients in the expansion, we have a quick proof that F_k divides F_{nk} for all n. A generalization of (3.7) is given in equation (4.18) of Section 4.

We turn to generating functions for powers of the Fibonacci numbers. First we expand

$$F_n^2 = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 = \frac{1}{(\alpha - \beta)^2}\left(\alpha^{2n} - 2(\alpha\beta)^n + \beta^{2n}\right) .$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^2 x^n &= \frac{1}{(\alpha - \beta)^2}\left(\sum_{n=0}^{\infty}\alpha^{2n}x^n - 2\sum_{n=0}^{\infty}(\alpha\beta)^n x^n + \sum_{n=0}^{\infty}\beta^{2n}x^n\right) \\ &= \frac{1}{(\alpha - \beta)^2}\left(\frac{1}{1 - \alpha^2 x} - \frac{2}{1 - \alpha\beta x} + \frac{1}{1 - \beta^2 x}\right) \\ &= \frac{x - x^2}{(1 - \alpha^2 x)(1 - \alpha\beta x)(1 - \beta^2 x)} = \frac{x - x^2}{1 - 2x - 2x^2 + x^3}\end{aligned}$$

This also shows that $\left\{F_n^2\right\}$ obeys

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0 .$$

We remark that Gould's technique (3.3) may be applied to $F(x)$, and leads to exactly the same result.

In general, to find the generating function for the p^{th} power of the Fibonacci numbers, first expand F_n^p by the binomial theorem. This gives F_n^p as a linear combination of α^{np}, $\alpha^{n(p-1)}\beta^n, \cdots,$ $\alpha^n\beta^{n(p-1)}$, β^{np} so that as above the generating function will have the denominator

$$(1 - \alpha^p x)(1 - \alpha^{p-1}\beta x) \cdots (1 - \alpha\beta^{p-1}x)(1 - \beta^p x) .$$

Fortunately, this product can be expressed in a better way. Define the Fibonomial coefficients $\begin{bmatrix} k \\ r \end{bmatrix}$ by

$$\begin{bmatrix} k \\ r \end{bmatrix} = \frac{F_k F_{k-1} \cdots F_{k-r+1}}{F_1 F_2 \cdots F_r} \quad (r > 0); \quad \begin{bmatrix} k \\ 0 \end{bmatrix} = 1 .$$

Then it has been shown [7] that

$$Q_p(x) = \prod_{j=0}^{p} (1 - \alpha^{p-j}\beta^j x) = \sum_{j=0}^{p+1} (-1)^{j(j+1)/2} \begin{bmatrix} p+1 \\ j \end{bmatrix} x^j .$$

For example,

$$\begin{aligned} Q_1(x) &= 1 - x - x^2 \\ Q_2(x) &= 1 - 2x - 2x^2 + x^3 \\ Q_3(x) &= 1 - 3x - 6x^2 + 3x^3 + x^4 \\ Q_4(x) &= 1 - 5x - 15x + 15x^3 + 5x^4 - x^5 . \end{aligned}$$

Since any sequence obeying the Fibonacci recurrence relation can be written in the form $A\alpha^n + B\beta^n$, $Q_p(x)$ is the denominator of the generating function of the p^{th} power of any such sequence. The numerators of the generating functions can be found by simply multiplying through $Q_p(x)$. For example, to find the generating function of $\{F^2_{n+2}\}$, we have

$$\sum_{n=0}^{\infty} F^2_{n+2} x^n = \frac{r(x)}{1 - 2x - 2x^2 + x^3} .$$

Then $r(x)$ can be found by multiplying $Q_2(x)$, giving

$$\begin{aligned} r(x) &= (1 - 2x - 2x^2 + x^3)(1 + 4x + 9x^2 + 25x^4 + \cdots) \\ &= 1 + 2x - x^2 + 0 \cdot x^3 + \cdots = 1 + 2x - x^2 . \end{aligned}$$

This is (4.7) of Section 4. However, for fixed p, once we have obtained the generating functions for $\{F^p_n\}$, $\{F^p_{n+1}\}$, $\cdots$, $\{F^p_{n+p}\}$, the one for $\{F^p_{n+k}\}$ follows directly from the identity of Hoggatt and Lind [4]

$$F^p_{n+k} = \sum_{j=0}^{p} (-1)^{(p-j)(p-j+3)/2} \begin{bmatrix} k \\ p \end{bmatrix} \begin{bmatrix} p \\ j \end{bmatrix} \left(\frac{F_{k-p}}{F_{k-j}} \right) F^p_{n+j} , \tag{3.5}$$

where we use the convention $F_0/F_0 = 1$. For example, for $p = 1$ this gives

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n .$$

Using the generating function for $\{F_{n+1}\}$ in (3.4) and $\{F_n\}$ in (3.1), we get

$$\sum_{n=0}^{\infty} F_{n+k} x^n = F_k \sum_{n=0}^{\infty} F_{n+1} x^n + F_{k-1} \sum_{n=0}^{\infty} F_n x^n$$

$$= \frac{F_k + F_{k-1} x}{1 - x - x^2} .$$

In fact, one of the main purposes for deriving (3.5) was to express the generating function of $\{F^p_{n+k}\}$ as a linear combination of those of $\{F^p_n\}, \cdots, \{F^p_{n+p}\}$.

Alternatively, to obtain the generating function of $\{F^p_{n+k}\}$ from that of $\{F^p_n\}$, we could apply k times in succession the technique mentioned in Section 2 of finding the generating function of $\{a_{n+1}\}$ from that of $\{a_n\}$.

The generating function of powers of the Fibonacci numbers have been investigated by several authors (see [3], [5], and [7]).

4. SOME STANDARD GENERATING FUNCTIONS

We list here for reference some of the generating functions we have already derived along with others which can be established in the same way.

$$\frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n \tag{4.1}$$

(4.2)
$$\frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1}\, x^n$$

(4.3)
$$\frac{2 - x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_n\, x^n$$

(4.4)
$$\frac{1 + 2x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_{n+1}\, x^n$$

(4.5)
$$\frac{x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_n^2 x^n$$

(4.6)
$$\frac{1 - x}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_{n+1}^2\, x^n$$

(4.7)
$$\frac{1 + 2x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_{n+2}^2\, x^n$$

(4.8)
$$\frac{x}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_n F_{n+1}\, x^n$$

(4.9)
$$\frac{4 - 7x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} L_n^2\, x^n$$

(4.10)
$$\frac{1 + 7x - 4x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} L_{n+1}^2\, x^n$$

$$\frac{9 - 2x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} L_{n+2}^2 x^n \tag{4.11}$$

$$\frac{x - 2x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_n^3 x^n \tag{4.12}$$

$$\frac{1 - 2x - x^2}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_{n+1}^3 x^n \tag{4.13}$$

$$\frac{1 + 5x - 3x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_{n+2}^3 x^n \tag{4.14}$$

$$\frac{8 + 3x - 4x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_{n+3}^3 x^n \tag{4.15}$$

$$\frac{2x}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_n F_{n+1} F_{n+2} x^n \tag{4.16}$$

$$\frac{F_k x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn} x^n \tag{4.17}$$

$$\frac{F_r + (-1)^r F_{k-r} x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn+r} x^n \tag{4.18}$$

Many thanks to Kathleen Weland and Allan Scott.

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A SHIFT FORMULA FOR RECURRENCE RELATIONS OF ORDER m

GARY G. FORD
Student, University of Santa Clara, Santa Clara, California

It is well known that if F_i is the i^{th} Fibonacci number, then

$$F_{n+k+1} = F_{n+1}F_{k+1} + F_nF_k$$

for all integers n, k. A generalization of this identity to recurrence relations of any order m is given here.

Let m be a positive integer and let $p_1, p_2, \cdots, p_m$ $(p_m \neq 0)$ be m elements of a field F. Furthermore, let $\{y_i\}$ and $\{U_i\}$ be two sequences in F obeying the recurrence relation whose auxiliary polynomial is

$$P(x) = x^m - \sum_{j=0}^{m-1} p_{m-j}x^j ,$$

and let $\{U_i\}$ have the initial values

$$U_0 = U_1 = \cdots = U_{m-2} = 0$$

and

$$U_{m-1} = 1 .$$

Then,

$$(1) \qquad y_{n+k} = \sum_{j=0}^{m-1}\sum_{i=0}^{j} p_{m-i}U_{k+i-j-1}y_{n+j}$$

for all integers n and k.

The proof of (1) is by induction on k. Let n be fixed. For $0 \le k < m$ it is clear that

$$(2)\qquad \sum_{i=0}^{j} p_{m-i}U_{k+i-j-1} = \begin{cases} 0 & \text{if } j < k \\ p_m U_{-1} = 1 & \text{if } j = k \\ \displaystyle\sum_{i=0}^{m-1} p_{m-i}U_{k+i-j-1} = U_{k+m-j-1} = 0 & \text{if } k < j < m\,. \end{cases}$$

From (2) it immediately follows that (1) holds for $k = 0, 1, \cdots, m - 1$. From here, applications of the recurrence relation (corresponding to $P(x)$) for $\{y_i\}$ and $\{U_i\}$, in both the forward and backward directions, easily prove that if (1) holds for $k = h, h + 1, \cdots, h + m - 1$, then (1) holds for $k = h - 1, h, \cdots, h + m$. By application of finite induction, it follows that (1) holds for all integers n, k.

Let $P(x) = (x - r_1)(x - r_2)\cdots(x - r_m)$ in an extension G of F and suppose that G is of characteristic zero. Further suppose that the r_j are pairwise distinct. Define D_k as the determinant produced by the process of substituting the vector $(r_1^k, r_2^k, \cdots, r_m^k)$ for the m^{th} row $(r_1^{m-1}, r_2^{m-1}, \cdots, r_m^{m-1})$ in the Vandermonde determinant of $r_1, r_2, \cdots, r_m$. It is proven in [1] that for every integer k,

$$(3)\qquad U_k = \frac{D_k}{D_{m-1}}\ .$$

The case for repetitions among the r_j is handled in the following way: Start with the form for U_k in (3) and, pretending that the r_j are real, apply L'Hospital's Rule successively as $r_I \to r_J$ for all repetitions $r_I = r_J$ among the r_j.

A combination of (1) and (3) now comes with ease. Still taking the r_j to be pairwise distinct, define E_k as the determinant produced by the process of replacing the element r_h^k of the m^{th} row of D_k by

$$\sum_{j=0}^{m-1}\sum_{i=0}^{j} p_m y_j r_h^{k+i-j-1}\ ,$$

and this for $h = 1, 2, \cdots, m$. Then combination of (1) with (3) yields: For every integer k,

$$y_k = \frac{E_k}{D_{m-1}} \quad . \tag{4}$$

The case for repeated roots is handled as with (3). In [2] identities akin to (4) are developed.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-124 Proposed by J. H. Butchart, Northern Arizona University, Flagstaff, Ariz.

Show that

$$\sum_{i=0}^{\infty}(a_i / 2^i) = 4 ,$$

where

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \cdots$$

are the Fibonacci numbers.

B-125 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Is

$$\sum_{k=3}^{n} \frac{1}{F_k}$$

ever an integer for $n \geq 3$? Explain.

B-126 Proposed by J. A. H. Hunter, Toronto, Canada

Each distinct letter in this alphametic stands, of course, for a particular and different digit. The advice is sound, for our FQ is truly prime. What do you make of it all?

$$\begin{array}{rrrr} R & E & A & D \\ & & F & Q \\ R & E & A & D \\ & & F & Q \\ \hline D & E & A & R \end{array}$$

B-127 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Show that

$$2^n L_n \equiv 2 \pmod 5,$$
$$2^n F_n \equiv 2n \pmod 5.$$

B-128 Proposed by M. N. S. Swamy, Nova Scotia Tech. College, Halifax, Canada.

Let f_n be the generalized Fibonacci sequence with $f_1 = a$, $f_2 = b$, and $f_{n+1} = f_n + f_{n-1}$. Let g_n be the associated generalized Lucas sequence defined by $g_n = f_{n-1} + f_{n+1}$. Also let $S_n = f_1 + f_2 + \cdots + f_n$. It is true that $S_4 = g_4$ and $S_8 = 3g_6$. Generalize these formulas.

B-129 Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.

For a given positive integer, k, find

$$\lim_{n \to \infty} (F_{n+k}/L_n) .$$

B-130 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let coefficients $c_j(n)$ be defined by

$$(1 + x + x^2)^n = c_0(n) + c_1(n)x + c_2(n)x^2 + \cdots + c_{2n}(n)x^{2n}$$

and show that

$$\sum_{j=0}^{2n} \left[c_j(n)\right]^2 = c_{2n}(2n) .$$

Generalize to

$$(1 + x + x^2 + \cdots + x^k)^n .$$

B-131 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Prove that for m odd

$$\frac{L_{n-m} + L_{n+m}}{F_{n-m} + F_{n+m}} = \frac{5F_n}{L_n}$$

and for m even

$$\frac{F_{n-m} + F_{n+m}}{L_{n-m} + L_{n+m}} = \frac{F_n}{L_n} .$$

SOLUTIONS

Note: In the last issue, we inadvertently omitted M. N. S. Swamy from the solvers of B-100, B-101, and B-104.

FIBONACCI-LUCAS ADDITION FORMULAS

B-106 Proposed by H. H. Ferns, Victoria, B.C., Canada.

Prove the following identities:

$$2F_{i+j} = F_iL_j + F_jL_i \quad ,$$
$$2L_{i+j} = L_iL_j + 5F_iF_j \quad .$$

Solution by Charles R. Wall, University of Tennessee, Knoxville, Tennessee.

From the Binet formulas we have

$$F_iL_j + F_jL_i = \frac{1}{\sqrt{5}}\left\{(\alpha^i - \beta^i)(\alpha^j + \beta^j) + (\alpha^j - \beta^j)(\alpha^i + \beta^i)\right\}$$
$$= \frac{2}{\sqrt{5}}(\alpha^{i+j} - \beta^{i+j}) = 2F_{i+j} \quad ,$$

and

$$L_iL_j + 5F_iF_j = (\alpha^i + \beta^i)(\alpha^j + \beta^j) + (\alpha^i - \beta^i)(\alpha^j - \beta^j)$$
$$= 2(\alpha^{i+j} + \beta^{i+j}) = 2L_{i+j} \quad .$$

Also solved by John H. Biggs, Douglas Lind, William C. Lombard, C. B. A. Peck, A. G. Shannon, M. N. S. Swamy, John Wessner, David Zeitlin, and the proposer.

AN APPROXIMATION

B-107 Proposed by Robert S. Seamons, Yakima Valley College, Yakima, Wash.

Let M_n and G_n be respectively the n^{th} terms of the sequences (of Lucas and Fibonacci) for which $M_n = M_{n-1}^2 - 2$, $M_1 = 3$, and $G_n = G_{n-1} + G_{n-2}$, $G_1 = 1$, $G_2 = 2$. Prove that

$$M_n = 1 + \left[\sqrt{5}\,G_m\right] \quad ,$$

where $m = 2^n - 1$ and $[x]$ is the greatest integer function.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

In standard notation we have $M_n = L_{2^n}$ and $G_n = F_{n+1}$, where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively. The problem then becomes to show

$$L_{2^n} = \left[1 + \sqrt{5}\, F_{2^n}\right] \quad ,$$

which follows immediately from Problem B-89.

Also solved by William C. Lombard, C. B. A. Peck, A. G. Shannon, David Zeitlin, and the proposer.

GENERALIZED FIBONACCI NUMBERS

B-108 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let $u_1 = p$, $y_2 = q$, and $u_{n+2} = u_{n+1} + u_n$. Also let $S_n = u_1 + u_2 + \cdots + u_n$. It is true that $S_6 = 4u_4$ and $S_{10} = 11u_7$. Generalize these formulas.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

The problem should read $S_6 = 4u_5$. The fact that

$$\sum_{i=1}^{4k-2} u_i = L_{2k-1}u_{2k+1}$$

where L_n is the n^{th} Lucas number, appears in the solution of Problem 4272, American Math. Monthly, Vol. 56 (1949), p. 421.

Also solved by William C. Lombard, F. D. Parker, C. B. A. Peck, A. G. Shannon, M. N. S. Swamy, Charles R. Wall, David Zeitlin, and the proposer.

SECOND-ORDER DIFFERENCE EQUATION

B-109 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let r and s be the roots of the quadratic equation $x^2 - px - q = 0$, $(r \neq s)$. Let $U_n = (r^n - s^n)/(r - s)$ and $V_n = r^n + s^n$. Show that

$$V_n = U_{n+1} + qU_{n-1} .$$

Solution by Charles W. Trigg, San Diego, California.

$$q = -rs,$$

so

$$\begin{aligned} U_{n+1} + qU_{n-1} &= (r^{n+1} - s^{n+1})/(r - s) + (-rs)(r^{n-1} - s^{n-1})/(r - s) \\ &= [r^n(r - s) + s^n(r - s)]/(r - s) \\ &= V_n . \end{aligned}$$

Also solved by Harold Don Allen, J. H. Biggs, Douglas Lind, William C. Lombard, F. D. Parker, C. B. A. Peck, M. N. S. Swamy, Charles R. Wall, John Wessner, David Zeitlin, and the proposer.

AN INFINITE SERIES EQUALITY

B-110 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2n+1}}$$

Solution by the proposer.

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \; L_n = \alpha^n + \beta^n, \; \alpha = \tfrac{1}{2}(1 + \sqrt{5}), \; \beta = \tfrac{1}{2}(1 - \sqrt{5}) .$$

Then

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}} = (\alpha-\beta)\sum_{n=0}^{\infty}\frac{1}{\alpha^{2n+1}-\beta^{2n+1}}$$

$$= (\alpha-\beta)\sum_{n=0}^{\infty}\frac{1}{\alpha^{2n+1}}\,\frac{1}{1+\alpha^{-2(2n+1)}}$$

$$= (\alpha-\beta)\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r\alpha^{-(2r+1)(2n+1)}$$

$$= (\alpha-\beta)\sum_{r=0}^{\infty}\frac{(-1)^r\alpha^{-2r-1}}{1-\alpha^{-2(2r+1)}}$$

$$= (\alpha-\beta)\sum_{r=0}^{\infty}\frac{(-1)^r}{\alpha^{2r+1}-\alpha^{-2r-1}}$$

$$= (\alpha-\beta)\sum_{r=0}^{\infty}\frac{(-1)^r}{\alpha^{2r+1}+\beta^{2r+1}}$$

$$= \sqrt{5}\sum_{r=0}^{\infty}\frac{(-1)^r}{L_{2r+1}}\ .$$

ANOTHER SERIES EQUALITY

B-111 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Show that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{F_{4n+2}} = \sqrt{5}\sum_{n=0}^{\infty}\frac{1}{L_{4n+2}}\ .$$

Solution by the proposer.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{F_{2(2n+1)}} = (\alpha - \beta) \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha^{2(2n+1)} - \beta^{2(2n+1)}}$$

$$= (\alpha - \beta) \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha^{2(2n+1)}} \frac{1}{1 - \alpha^{-4(2n+1)}}$$

$$= (\alpha - \beta) \sum_{n=0}^{\infty} (-1)^n \sum_{r=0}^{\infty} \alpha^{-2(2r+1)(2n+1)}$$

$$= (\alpha - \beta) \sum_{r=0}^{\infty} \frac{\alpha^{-2(2r+1)}}{1 + \alpha^{-4(2r+1)}}$$

$$= (\alpha - \beta) \sum_{r=0}^{\infty} \frac{1}{\alpha^{2(2r+1)} + \beta^{2(2r+1)}}$$

$$= \sqrt{5} \sum_{r=0}^{\infty} \frac{1}{L_{2(2r+1)}} \quad .$$

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GETTING PRIMED FOR 1967

CHARLES W. TRIGG
San Diego, California

(A)

$$\begin{aligned} 1967 &= 7(281) = 7(17 + 67 + 197) \\ &= (2 + 5)(2 + 19 + 17 + 67 + 79 + 97) \\ &= (2 + 5)(5 + 107 + 109) \end{aligned}$$

(B)

$$\begin{aligned} 1967 &= 7 + 977 + 983 \\ &= 11 + 479 + 487 + 491 + 499 \\ &= 11 + 311 + 313 + 317 + 331 + 337 + 347 \\ &= 67 + 223 + 227 + 229 + 233 + 239 + 241 + 251 + 257 \\ &= 53 + 167 + 173 + 179 + 181 + 191 + 193 + 197 + 199 + 211 + 223 \\ &= 11 + 83 + 89 + 97 + 101 + 103 + 107 + 109 + 113 + 127 + 131 \\ &\quad + 137 + 139 + 149 + 151 + 157 + 163 \\ &= 19 + 53 + 59 + 61 + 67 + 71 + 73 + 79 + 83 + 89 + 97 + 101 \\ &\quad + 103 + 107 + 109 + 113 + 127 + 131 + 137 + 139 + 149 \\ &= 7 + 13 + 17 + 19 + 23 + 29 + 31 + 37 + 41 + 43 + 47 + 53 + 59 \\ &\quad + 61 + 67 + 71 + 73 + 79 + 83 + 89 + 97 + 101 + 103 + 107 \\ &\quad + 109 + 113 + 127 + 131 + 137 . \end{aligned}$$

(C)

$$\begin{aligned} 1 + 9 + 6 + 7 &= 23 \\ 1^2 + 9^2 + 6^2 + 7^2 &= 167 \\ 1^3 + 9^3 + 6^3 + 7^3 &= 1289 \\ 1^4 + 9^4 + 6^4 + 7^4 &= 10259 \\ 1^1 + 9^2 + 6^3 + 7^4 &= 2669 \\ 1^4 + 9^1 + 6^2 + 7^3 &= 389 \\ 1^4 + 9^3 + 6^2 + 7^1 &= 773 \\ 1^1 + 9^4 + 6^3 + 7^2 &= 6827 \\ 76^2 + 91^2 &= 14057 \end{aligned}$$

(D)

$$\begin{aligned} 2 &= (\sqrt{196})/7 \\ 3 &= -1 + \sqrt{9} - 6 + 7 \\ 5 &= 1 - 9 + 6 + 7 \end{aligned}$$

$$\begin{aligned}
7 &= -1 + 9 + 6 - 7 \\
11 &= 1 + 9 - 6 + 7 \\
13 &= 1^9(6 + 7) \\
17 &= 1 + \sqrt{9} + 6 + 7 \\
19 &= 19(-6 + 7) \\
23 &= 1 + 9 + 6 + 7 \\
29 &= 1(\sqrt{9}!)(6) - 7 \\
31 &= (1 + 9 - 6)! + 7 \\
37 &= 1 - (\sqrt{9})! + 6(7) \\
41 &= -1^9 + 6(7) \\
43 &= 1(\sqrt{9}!)(6) + 7 \\
47 &= 1(9)(6) - 7 \\
53 &= (1 + 9)(6) - 7 \\
59 &= 1 - 9 + 67 \\
61 &= 1(9)(6) + 7 \\
67 &= (1 + 9)(6) + 7 \\
71 &= 1 + \sqrt{9} + 67 \\
73 &= 1(\ 9!) + 67 \\
79 &= 1 + (\sqrt{9}!)(6 + 7) \\
83 &= -1 + (\sqrt{9}! + 6)(7) \\
89 &= 1(96) - 7
\end{aligned}$$

In every case above, the expression for the prime has the digits of 1967 in that order.

(E) Of the twelve two-digit numbers that can be written with the digits of 1967, there are seven primes, including two palindromic pairs:

17, 71; 79, 97; 19, 61, and 67.

Of the twenty-four three-digit numbers that can be written with the digits of 1967, eleven are prime; including three palindromic pairs:

167, 761; 179, 971; 769, 967; 197, 617, 619, 691, and 719.

(Article continued on p. 476)

CURIOSA IN 1967

CHARLES W. TRIGG
San Diego, California

(A)
$$\begin{aligned}1967 &= (-1+9+6-7)(196-7+91-6+7)\\ &= -12+(34)(56)+78-\sqrt{9}\\ &= 0!+1!+2(3!)(4!)+(5)(6)(7)(8)-\sqrt{9}\\ &= 2^0+2^1+2^2+2^3+2^5+2^7+2^8+2^9+2^{10}\end{aligned}$$

(B)
$$\begin{aligned}1967_{10} &= 117E_{12} = 1529_{11} = 2625_9 = 3657_8 = 5510_7 = 13035_6\\ &= 30332_5 = 132233_4 = 2200212_3\\ &= 11110101111_2, \quad \text{a palindrome.}\end{aligned}$$

(C)
$$(1!\,9!\,6!\,7!)(!1!\,9!\,6!\,7!) = 0, \quad \text{where } !x \text{ is subfactorial } x.$$

(D) Expressed in Fibonacci numbers:
$$\begin{aligned}1967 &= 1-8+377+1597\\ &= 1597+377-5-2\\ &= 1+13+34+89+233+1597\\ &= 1+2+3+8+13+21+34+55+89+144+233+377+987\end{aligned}$$

(E) Four squares can be formed from the digits of 1967, namely: 196, 169, 961, and 16, which latter also is a fourth power.
$$\begin{aligned}1967 &= (4^2-3^2)(5^2+16^2)\\ &= 144^2-137^2 = 984^2-983^2\\ &= 1^2+2^2+3^2+4^2+5^2+6^2+8^2+9^2+10^2+11^2+12^2\\ &\quad +14^2+15^2+16^2+17^2+20^2\end{aligned}$$

(F)
$$\begin{aligned}1967 &= (1111-111-11)(1+1)-11\\ &= 22^2\cdot 2^2+2^{2+2+2/2}-2/2\\ &= (3+3)(333)-33+3-3/3\\ &= 44^{\sqrt{4}}+4(4+\sqrt{4})-4/4\\ &= (555-5/5)(5-5/5)-5(55-5)+5/5\\ &= (666+6/6)(6+6+6)/6-6(6)+(6+6)/6\end{aligned}$$

$$= 777 + 77(7 + 7) + (777 + 7)/7$$
$$= (888 + 88 + 8 - 8/8)(8 + 8)/8 + 8/8$$
$$= (999 - \sqrt{9})(9 + 9)/9 - 9\sqrt{9} + (9 + 9)/9$$

(G) Here are several ways in which 1967 can be written using conventional mathematical symbols and one 1, nine 9's, six 6's, and seven 7's in order from left to right.

$$\begin{aligned} 1967 &= 19(99 + 999/999) + 66 + 66/66 + 7(777 - 777) \\ &= 19(99 + 999/999) + 66(66/66) + 7^{(777-777)} \\ &= 19(99 + 999/999) + 6(66/6 - 6/6) + 7(777/777) \\ &= 1(999 + 9/9) + 9(99 + \sqrt{9} - 6/6 - 6/6) \\ &\quad + 6(6 - 7 + 77/7) + 7(7/7) \end{aligned}$$

(H) If to 1967 its reversal is added, and the process repeated several times, a palindromic number is produced in five operations.

$$\begin{array}{r} 1967 \\ \underline{7691} \\ 9658 \\ \underline{8569} \\ 18227 \\ \underline{72281} \\ 90508 \\ \underline{80509} \\ 171017 \\ \underline{710171} \\ 881188 \end{array}$$

(I) 7691 - 1967 = 5724, 5724 - 4275 = 1449, 9441 - 1449 = 7992, 7992 - 2997 = 4995, 5994 - 4995 = 99, a palindromic number after five subtractions.

(J) If the digits of 1967 be written in descending order before reversal and subtraction and the process be repeated continuously:

9761 - 1679 = 8082, 8820 - 0288 = 8532,

8532 - 2358 = 6174, 7641 - 1467 = 6174,

Thus Kaprekar's constant 6174 is reached in three operations.

(K)

$$\text{The circulant}\quad \begin{vmatrix} 1 & 9 & 6 & 7 \\ 7 & 1 & 9 & 6 \\ 6 & 7 & 1 & 9 \\ 9 & 6 & 7 & 1 \end{vmatrix} = -3^2(23)(29)\,.$$

$$\begin{vmatrix} 1 & 7 \\ 6 & 9 \end{vmatrix} \text{ divides } \begin{vmatrix} 1 & 9 & 6 & 7 \\ 9 & 6 & 7 & 6 \\ 6 & 7 & 6 & 9 \\ 7 & 6 & 9 & 1 \end{vmatrix}, \text{ that is, } \frac{3(11)^2}{-3(11)} = -11\,.$$

$$\begin{vmatrix} 1 & 9 \\ 7 & 6 \end{vmatrix} \text{ divides } \begin{vmatrix} 1 & 9 & 6 & 7 \\ 9 & 1 & 1 & 6 \\ 6 & 1 & 1 & 9 \\ 7 & 6 & 9 & 1 \end{vmatrix}, \text{ that is, } \frac{9(11)(19)}{-3(19)} = -33\,.$$

$$\begin{vmatrix} 1 & 9 & 6 & 7 \\ 9 & 9 & 9 & 6 \\ 6 & 9 & 9 & 9 \\ 7 & 6 & 9 & 1 \end{vmatrix} = 9^3. \quad \begin{vmatrix} 1 & 9 & 6 & 7 \\ 9 & 6 & 6 & 6 \\ 6 & 6 & 6 & 9 \\ 7 & 6 & 9 & 1 \end{vmatrix} = 3^3(43)\,. \quad \begin{vmatrix} 1 & 9 & 6 & 7 \\ 9 & 7 & 7 & 6 \\ 6 & 7 & 7 & 9 \\ 7 & 6 & 9 & 1 \end{vmatrix} = 3^2(113)\,.$$

$$\begin{vmatrix} 1 & 9 & 6 & 7 \\ 9 & 0 & 0 & 6 \\ 6 & 0 & 0 & 9 \\ 7 & 6 & 9 & 1 \end{vmatrix} = 45^2\,. \quad \begin{vmatrix} 1 & 9 & 6 & 7 \\ 9 & 6 & 7 & 0 \\ 6 & 7 & 0 & 0 \\ 7 & 0 & 0 & 0 \end{vmatrix} = 7^4\,.$$

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(Continued from p. 473.)

Of the twenty-four four-digit numbers that can be written with the digits of 1967, seven are prime:

1697, 6197, 6719, 6791, 6917, 6971, and 7691.

(F)

$$-\begin{vmatrix} 1 & 9 \\ 6 & 7 \end{vmatrix} = 47\,.$$

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A DIGITAL BRACELET FOR 1967

CHARLES W. TRIGG
San Diego, California

A bracelet is one period of a simply periodic series considered as a closed sequence with terms equally spaced around a circle. Thus distances between terms may be measured in degrees. A bracelet may be regenerated by starting at any arbitrary point to apply the generating law. A bracelet may be cut at any arbitrary point for straight line representation without loss of any properties.

A digital bracelet may be constructed by starting with a sequence of four digits, affixing the units' digit of their sum, again affixing the units' digit of the sum of the last four digits, and continuing the process.

Starting with 1967 this process will generate the sequence

1 9 6 7 3 5 1 6 5 7 9 7 8 1 5 ···

in which four odd digits and one even digit alternate throughout. Since there are only 5^4 sets of four ordered odd digits, the sequence must repeat in not over $5(5^4)$ or 3125 operations. In fact, it does repeat after 1560 operations producing a bracelet of 1560 digits. The complete bracelet is given on page 480.

This bracelet could be said to belong to 1967, but 1560 years have an equal claim to it, for example, the following from the twentieth century:

1901	1923	1935	1949	1957	1978	1991	1999
1903	1929	1937	1951	1958	1979	1992	
1907	1930	1938	1952	1967	1983	1994	
1912	1932	1941	1953	1973	1985	1996	
1917	1933	1947	1956	1974	1987	1997	

By retaining only the units' digits in the generation of the series we actually reduced each sum modulo 10. To be consistent we will reduce modulo 10 the results of all operations (such as multiplication) to which the elements of the bracelet are subjected. Thus we deal only with digits in a modular arithmetic wherein 3, 9, 7, 1 is a cyclic geometric progression.

In order to establish relationships between equidistant digits the bracelet may be written in several rows of various but equal lengths so that each digit column consists of equidistant digits.

Digits 180^0 apart may be written in two rows:

19673 51657 97815 15231 17211 15859 79051 51297 97253 77295 39631 ···
91437 59453 13295 95879 93899 95251 31059 59813 13857 33815 71479 ···

So each pair of diametrically opposite digits sum to zero, and the sum of all the digits in the bracelet is zero.

All the digits 120^0 apart in the bracelet may be exhibited in three rows:

19673 51657 97815 15231 17211 15859 79051 51297 97253 77295 39631 ···
97411 39473 37033 39833 37695 77879 15275 93417 57091 77497 77015 ···
59071 75035 31217 11091 11259 73437 71839 11451 11811 11473 59419 ···

Each column of pentads is composed of two odd-digit, one even-digit, and two odd-digit columns. Each pentad column sums to 55055. The digit columns encompass all the sets of three odd integers that sum to 5 except 5, 5, 5 and all the sets of three even integers, other than 0, 0, 0, which sum to zero.

When the digits of the bracelet are written in four equal rows the digits in each column are 90^0 apart. Thus

19673 51657 97815 15231 17211 15859 79051 51297 97253 77295 39631 ···
37819 53851 71435 35693 31633 35457 17053 53671 71659 11675 97893 ···
91437 59453 13295 95879 93899 95251 31059 59813 13857 33815 71479 ···
73291 57259 39675 75417 79477 75653 93057 57439 39451 99435 13217 ···

Each column of digits is a cyclic permutation of 3, 9, 7, 1; 6, 8, 4, 2; 5, 5, 5, 5; or 0, 0, 0, 0. Hence each column is in geometric progression with $r = 3$. So successive multiplication by 3 will rotate the bracelet counterclockwise in 90^0 jumps. The same result is obtained by multiplying the bracelet by 3, 9, 7, 1 in order. The sums of the pentads form the array

6	4	0	2	2	8	2	4	6	0	2
8	2	0	6	6	4	6	2	8	0	6
4	6	0	8	8	2	8	6	4	0	8
2	8	0	4	4	6	4	8	2	0	4

Each of these columns is in G. P. with $r = 3$.

When the digits of the bracelet are written in five equal rows the digits in each column are 72^0 apart. Thus

19673	51657	97815	15231	17211	15859	79051	51297	97253	77295	39631 ···
69173	01157	47315	65731	67711	65359	29551	01797	47753	27795	89131 ···
19123	51107	97365	15781	17761	15309	79501	51747	97703	77745	39181 ···
14173	56157	92315	10731	12711	10359	74551	56797	92753	72795	34131 ···
19178	51152	97310	15736	17716	15354	79556	51792	97758	77790	39136 ···

Each column is a cyclic permutation of 0, 5, 5, 5, 5; 2, 7, 7, 7, 7; 4, 9, 9, 9, 9; 6, 1, 1, 1, 1; or 8, 3, 3, 3, 3. Each of these sets derives from the first set by addition of an even digit. The sum of the digits in every pentad is even, and all five pentads in a column have the same sum.

When the digits of the bracelet are written in six equal rows, the digits in each column are 60^0 apart. Thus

19673	51657	97815	15231	17211	15859	79051	51297	97253	77295	39631 ···
51039	35075	79893	99019	99851	37673	39271	99659	99299	99637	51691 ···
97411	39473	37033	39833	37695	77879	15275	93417	57091	77497	77015 ···
91437	59453	13295	95879	93899	95251	31059	59813	13857	33815	71479 ···
59071	75035	31217	11091	11259	73437	71839	11451	11811	11473	59419 ···
13699	71637	73077	71277	73415	33231	95835	17693	53019	33613	33095 ···

The digit columns are cyclic permutations of one of the four palindromes 159951, 208802, 357753, or 406604, all of which are multiples of the first one; or of the bracelets symmetrical about a diameter 193917, 286824, 37931, or 462648, all of which are multiples of the first one. The sum of the digits in each of the pentads is even, and the sums of the pentad-digits in each column of pentads form a cyclic permutation of one of the four even sequences listed above.

The Complete 1560-Digit Bracelet

19673	51657	97815	15231	17211	15859	79051	51297	97253	77295	39631
99211	37235	77217	77239	15837	31453	35671	93035	19831	13835	95217
55853	17671	15411	17097	39877	13891	19011	13611	19235	99693	75495
31879	59037	99839	99075	13655	95431	31835	73831	57697	91639	97837
53839	33837	19077	37415	77093	91257	59677	99277	51039	35075	79893
99019	99851	37673	39271	99659	99299	99637	51691	73011	57473	15657
31677	11653	59295	51017	97477	53277	95891	31497	11877	35277	17277
39653	37819	53851	71435	35693	31633	35457	17053	53671	71659	11675
97893	77633	91695	11631	11697	35491	93259	95813	79095	37493	39495
75631	55459	31813	35233	31071	97411	39473	37033	39833	37695	77879
15275	93417	57091	77497	77015	39855	75293	93495	19493	51871	73897
71491	59497	99491	37011	91235	11079	73651	57811	77611	53097	95015
17479	77037	77453	91819	97613	77857	77677	77891	53873	19033	51219
35851	93811	33859	57675	53031	71211	59611	75473	93271	33411	95611
31611	97859	91437	59453	13295	95879	93899	95251	31059	59813	13857
33815	71479	11899	73875	33893	33871	95273	79657	75439	17075	91279
97275	15893	55257	93439	95699	93013	71233	97219	91099	97499	91875
11417	35615	79231	51073	11271	11035	97455	15679	79275	37279	53413
19471	13273	57271	77273	91033	73695	33017	19853	51433	11833	59071
75035	31217	11091	11259	73437	71839	11451	11811	11473	59419	37099
53637	95453	79433	99457	51815	59093	13633	57833	15219	79613	99233
75833	93833	71457	73291	57259	39675	75417	79477	75653	93057	57439
39451	99435	13217	33477	19415	99479	99413	75619	17851	15297	31015
73617	71615	35479	55651	79297	75877	79039	13699	71637	73077	71277
73415	33231	95835	17693	53019	33613	33095	71255	35817	17615	91617
59239	37213	39619	51613	11619	73099	19875	99031	37459	53299	33499
57013	15095	93631	33073	33657	19291	13497	33253	33433	33219	57237
91077	59891	75259	17299	77251	53435	57079	39899	51499	35637	17839
77699	15499	79499	13251'1967							

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A GENERAL FIBONACCI FUNCTION

RICHARD L. HEIMER
Airborne Instruments Laboratory, Deerpark, L.I., N.Y.

Probably many of us who have an interest in Fibonacci series have plotted F_n as a function of n on graph paper. If we connect the points with straight line segments on cartesian coordinate paper, we achieve a continuous piecewise linear Fibonacci Function (see Fig. 1).

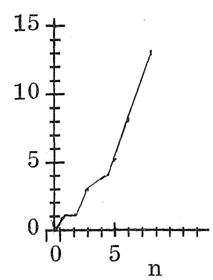

Fig. 1 The Fibonacci Function

This Fibonacci Function has many interesting properties other than at the integral values of the n. In fact, this function gives rise to the concept of F_x, where x is any real number.

If we tabulate the function, it becomes easier to discern the relationships involved.

PARTIAL TABLE OF THE FIBONACCI FUNCTION
F_x Versus x (tenths)

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Δ
0	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	.1
1	1	1	1	1	1	1	1	1	1	1	0
2	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	.1
3	2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	.1
4	3	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	.2
5	5	5.3	5.6	5.9	6.2	6.5	6.8	7.1	7.4	7.7	.3
6	8	8.5	9.0	9.5	10.0	10.5	11.0	11.5	12.0	12.5	.5
7	13	13.8	14.6	15.4	16.2	17.0	17.8	18.6	19.4	20.2	.8

(Example: $F_{6.3} = 9.5$)

One immediately notes that between $x = 0$ and $x = 1$, $F_x = x$. Because of this, it is convenient to set

$$x = n + r ,$$

where n is an integer and r is the balance less than unity. Thus:

$$F_r = r$$
$$F_{1+r} = 1$$
$$F_{2+r} = 1 + r$$
$$F_{3+r} = 2 + r$$
$$F_{4+r} = 3 + 2r$$
$$F_{n+r} = F_n + F_{n-1}r$$
$$F_x = F_n + F_{n-1}\, r$$

One may also observe in any column in the table, that any particular entry is the sum of the preceding two entries, i.e.,

$$F_{x+1} = F_x + F_{x-1}$$

Other interesting properties that are obvious by inspection include:

$$2\,F_{n+0.5} = F_{n+2}$$
$$3\,F_{n+0.333} = L_{n+1} ,$$ where L is the Lucas number.

Not so obvious is the fact that there are relationships between the squares and the products of the entries in any column of the table. In fact

$$F_x^2 = \left[(F_{x-1})(F_{x+1}) - 1\right] + r^2 + r$$

and when r is the golden ratio (0.618034)

$$F^2_{x\ \text{golden}} = (F_{x-1})(F_{x+1})$$

The proof is left to the reader.

Note also that this function allows any Fibonacci-type sequence to be normalized into the r, 1, $1 + r$ form. For example, a 2, 10, 12, 22 ⋯ sequence converts to a 0.2, 1 ⋯ general type sequence by dividing by 10.

CONCLUSION

In general, this particular method of expressing the Fibonacci Function has the potential of being a rich area of Fibonacci discovery. Possibilities include verification and reformulation of all Fibonacci formulae. Also an inverse table of F_x's versus all the real numbers may be formed and investigated.

Because this function represents the normalization of all Fibonacci-type sequences, any results should demonstrate broad fulfillment of the goals of the investigator.

REFERENCE

1. Dewey C. Duncan, "Chains of Fibonacci-Wise Triangles," Fibonacci Quarterly Journal, Feb. 1967.

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All subscription correspondence should be addressed to Bro. A. Brousseau, St. Mary's College, Calif. All checks ($4.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Manuscripts intended for publication in the Quarterly should be sent to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, Calif. All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscripts sent to the editors.

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A RESULT FOR HERONIAN TRIANGLES

J. A. H. HUNTER
Toronto, Ontario, Canada

Arising from some particular solutions communicated to me by Mr. W. W. Horner, I developed what seemed to be a new approach to the general problem of Heronian triangles. The results were interesting.

In such a triangle all three sides, and also the area, must be integral. Hence all three altitudes must be rational, as must be the sines of all three angles. It can be shown that the sides of such a triangle are divided into rational segments by the altitudes so that the cosines are also rational.

Now consider a Heronian triangle with sides a, b, c, with angle C contained by sides a and b.

Say, $\sin C = 2xy/(x^2 + y^2)$, $\cos C = (x^2 - y^2)/(x^2 + y^2)$, where x and y are positive integers, $x > y$.

Using the cosine formula:

$$\cos C = (a^2 + b^2 - c^2)/2ab = (x^2 - y^2)/(x^2 + y^2)$$

So,

$$\begin{aligned}
(x^2 + y^2)c^2 &= (x^2 + y^2)(a^2 + b^2) - 2(x^2 - y^2)ab \,. \\
\left[(x^2 + y^2)c\right]^2 &= (x^2 + y^2)^2a^2 - 2(x^4 - y^4)ab + (x^2 + y^2)^2b^2 \\
&= \left[(x^2 + y^2)a\right]^2 - 2(x^4 - y^4)ab + (x^2 - y^2)^2b^2 + \\
&\qquad + 4x^2y^2b^2 \\
&= \left[(x^2 + y^2)a - (x^2 - y^2)b\right]^2 + (2xyb)^2 \,,
\end{aligned}$$

which has the fully general integral solution:

$$\left.\begin{aligned}
(x^2 + y^2)c &= (m^2 + n^2)t \\
(x^2 + y^2)a - (x^2 - y^2)b &= (m^2 - n^2)t \\
xyb &= mnt
\end{aligned}\right\} \begin{array}{l} m \text{ and } n \text{ any positive integers,} \\ m > n. \text{ And } t \text{ a common ra-} \\ \text{tional divisor or multiplier.} \end{array}$$

Then

$$\left.\begin{aligned} xy(x^2 + y^2)a &= \left[xy(m^2 - n^2) + (x^2 - y^2)mn\right]t \\ xy(x^2 + y^2)b &= (x^2 + y^2)mnt \\ xy(x^2 + y^2)c &= xy(m^2 + n^2)t \end{aligned}\right\}$$

Without loss of generality, say $t = xy(x^2 + y^2)k$, then:

$$\left.\begin{aligned} a &= \left[xy(m^2 - n^2) + (x^2 - y^2)mn\right]k \\ b &= (x^2 + y^2)mnk \\ c &= xy(m^2 + n^2)k \end{aligned}\right\} \begin{aligned} &\text{where } k \text{ is any} \\ &\text{rational common} \\ &\text{divisor or multiplier.} \end{aligned}$$

The Heronian formula for area of a triangle is:

$$\Delta = \sqrt{s(s - a)(s - b)(s - c)} \,,$$

where

$$2s = s + b + c\,.$$

Hence, substituting for a, b, c, we have:

$$\text{Area} = xymn(xm - yn)(xn + ym)k^2\,.$$

The results cover all Heronian triangles.

* * * * *

A NOTE OF SADNESS

Mark Feinberg, a sophomore at the University of Pennsylvania, died Oct. 29, 1967, from injuries sustained in an automobile-motorcycle collision. It is a tragic loss to the Editorial Staff of the Fibonacci Quarterly Journal, as Mark had already published two articles in our pages. Included in this issue is a paper he last submitted.

This young scholar, Mark Feinberg, was both a brilliant young student and a winner of many prizes and scholarships. (Continued on page 490)

A LUCAS TRIANGLE

MARK FEINBERG
Student, University of Pennsylvania, Philadelphia, Pennsylvania

It is well known that the Fibonacci Sequence can be derived by summing diagonals of Pascal's Triangle. How about the Lucas Sequence? Is there an arithmetical triangle whose diagonals sum to give the Lucas Sequence?

One such triangle is generated by the coefficients of the expansion $(a + b)^{n-1}(a + 2b)$:

1 3 4 7 11 18 29 47 76 123 ...

row n											
1	1	2									
2	1	3	2								
3	1	4	5	2							
4	1	5	9	7	2						
5	1	6	14	16	9	2					
6	1	7	20	30	25	11	2				
7	1	8	27	50	55	36	13	2			
8	1	9	35	77	105	91	49	15	2		
9	1	10	44	112	182	196	140	64	17	2	
10	1	11	54	156	294	378	336	204	81	19	2
column r	1	2	3	4	5	6	7	8	9	10	11

The sum of the numbers on row n is $3 \times 2^{n-1}$. For row $n = 5$:

$$1 + 6 + 14 + 16 + 9 + 2 = 48 = 3 \times 2^4 .$$

The n^{th} row has $n + 1$ terms. Each number of this Lucas Triangle is the sum of the number above it and the number to the left of that one. Except for the first column, the sum of the first R numbers in column r equals the R^{th} number of column $r + 1$:

$$2 + 5 + 9 + 14 = 30$$

For $r > 1$, any number of the triangle can be expressed as

$$\frac{n!}{(r-1)!\,(n-r+1)!} + \frac{(n-1)!}{(r-2)!\,(n-r+1)!}$$

For example, the 5th number of row 7 is 55:

$$\frac{7!}{4!\,3!} + \frac{6!}{3!\,3!} = 35 + 20 = 55 .$$

Actually, one can find row n of this Lucas Triangle by adding row n of Pascal's Triangle to row $n - 1$ of Pascal's Triangle:

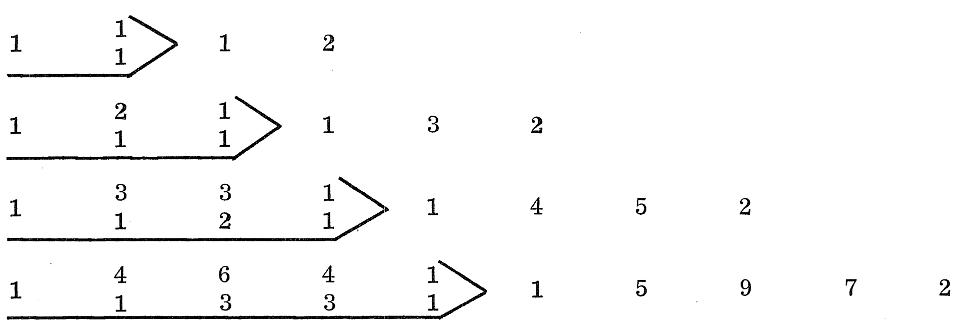

Figure 1

This fact is not extremely surprising. By summing the n^{th} diagonal of the Lucas Triangle, one is actually simultaneously adding the $(n + 1)^{st}$ and the $(n - 1)^{st}$ diagonals of Pascal's Triangle. The $(n + 1)^{st}$ diagonal of Pascal's Triangle adds up to F_{n+1}; the $(n - 1)^{st}$ diagonal sums to F_{n-1}, and one Fibonacci-Lucas identity is:

$$F_{n+1} + F_{n-1} = L_n .$$

The vertical columns of the Lucas Triangle are of interest. Notice that the second column, $r = 2$, is equivalent to enumeration. From the general expression, any number in this column is

$$\frac{n!}{1!\,(n-1)!} + \frac{(n-1)!}{0!\,(n-1)!}$$

The R^{th} number of this column is on row $n = R$. Thus the R^{th} number can be expressed as

$$\frac{R!}{1!\,(R-1)!} + \frac{(R-1)!}{0!\,(R-1)!} = R + 1$$

Any number in the third column, $r = 3$, is given by

$$\frac{n!}{2!\,(n-2)!} + \frac{(n-1)!}{1!\,(n-2)!}$$

The R^{th} number of this column is on row $n = R + 1$. The R^{th} number is then given by

$$\frac{(R+1)!}{2!\,(R-1)!} + \frac{R!}{1!\,(R-1)!}$$

$$= \frac{(R+1)! + 2(R!)}{2!\,(R-1)!} = \frac{R!\,(R+1+2)}{2!\,(R-1)!} = \frac{R(R+3)}{2} = \frac{R^2+3R}{2}$$

The 6th number of the column is 27:

$$\frac{36 + 18}{2} = 27\ .$$

One can generalize to say that the R^{th} number of the column which begins on row $n = N$ is given by

$$\frac{(R+N-1)!}{N!\,(R-1)!} + \frac{(R+N-2)!}{(N-1)!\,(R-1)!}$$

The 4th number of the column which starts on row $n = 6$ is 140:

$$\frac{(4+6-1)!}{6!\,(4-1)!} + \frac{(4+6-2)!}{5!\,(4-1)!} = \frac{9!}{6!\,3!} + \frac{8!}{5!\,3!} = 84 + 56 = 140\ .$$

Pascal's Triangle is symmetric. Flipping the Triangle around doesn't change it. Not so with the Lucas Triangle. Rotating the Lucas Triangle 180^0 gives:

2 3 5 8 13 21 34 55 89 ...

row n										
1	2	1								
2	2	3	1							
3	2	5	4	1						
4	2	7	9	5	1					
5	2	9	16	14	6	1				
6	2	11	25	30	20	7	1			
7	2	13	36	55	50	27	8	1		
8	2	15	49	91	105	77	35	9	1	
9	2	17	64	140	196	182	112	44	10	1
column r	1	2	3	4	5	6	7	8	9	10

Summing diagonals of this arrangement gives the Fibonacci Sequence. This can be explained by referring to Figure 1. The n^{th} diagonal of the rotated Lucas Triangle is the sum of the n^{th} and $n+1^{th}$ Pascal diagonals. The n^{th} Pascal diagonal of the rotated Lucas Triangle sums to

$$F_n + F_{n+1} = F_{n+2} .$$

The second column of this rotated triangle is composed of the odd numbers. Any number in this column can be expressed as

$$\frac{n!}{(n-1)!\,1!} + \frac{(n-1)!}{(n-2)!\,1!} .$$

Since the R^{th} number of the column is on row $n = R$, the above expression is equivalent to

$$\frac{R!}{(R-1)!\,1!} + \frac{(R-1)!}{(R-2)!\,1!} = R + R - 1 = 2R - 1 .$$

Perhaps the most interesting of all the Lucas Triangle's vertical columns is the third column of the rotated arrangement. Here, the R^{th} number is R^2. The expression for any number of this column is

$$\frac{n!}{(n-2)!\,2!} + \frac{(n-1)!}{(n-3)!\,2!} .$$

Since the R^{th} number of the column is on row $n = R + 1$, the R^{th} number is given by

$$\frac{(R+1)!}{(R-1)!\,2!} + \frac{R!}{(R-2)!\,2!}$$

$$= \frac{(R+1)! + (R-1)(R!)}{(R-1)!\,2!} = \frac{R!\,(R+1+R-1)}{(R-1)!\,2!}$$

$$= \frac{R(2R)}{2} = R^2 .$$

In general, the R^{th} number of the column which begins on row $n = N$ of the rotated triangle is

$$\frac{(R+N-1)!}{(R-1)!\,N!} + \frac{(R+N-2)!}{(R-2)!\,N!}$$

For example, the 5th number of the column beginning on row $n = 4$ is 105:

$$\frac{(5+4-1)!}{(5-1)!\,4!} + \frac{(5+4-2)!}{(5-2)!\,4!} = \frac{8!}{4!\,4!} + \frac{7!}{3!\,4!} = 70 + 35 = 105 .$$

In conclusion, the coefficients of the expansion $(a + b)^{n-1}(a + 2b)$ produce an interesting Lucas Triangle. This triangle is not, however, unique. Quite conceivably, utilization of various other Fibonacci-Lucas identities will lead to different and, perhaps, even more interesting Lucas Triangles.

* * * * *

Mark's younger brother, Andrew, is also a Science Fair Champion, and we hope soon we'll have the privilege of publishing his first mathematics paper.

The following are Mark's <u>Fibonacci Quarterly</u> papers:

1. Fibonacci-Tribonacci — Oct. 1963
2. New Slants — Oct. 1964
3. Lucas Triangle — Dec. 1967

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VOLUME INDEX

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