THE FIBONACCI QUARTERLY

THE OFFICIAL JOURNAL OF

THE FIBONACCI ASSOCIATION

VOLUME 6 NUMBER 1

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FEBRUARY 1968

THE FIBONACCI QUARTERLY

OFFICIAL ORGAN OF THE FIBONACCI ASSOCIATION

A JOURNAL DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

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The Quarterly is entered as third-class mail at the St. Mary's College Post Office, California, as an official publication of the Fibonacci Association.

A FIBONACCI FUNCTION

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Eric Halsey [1] has invented an ingenious method for defining the Fibonacci numbers $F(x)$, when x is a rational number. In addition to the restriction that x must be rational, his calculations yield

$$F(4.1) = 3.155, \quad F(3.1) = 2.1, \quad F(1.1) = 1.1 ,$$

so that the Fibonacci identity

$$F(x) = F(x-1) + F(x-2)$$

is destroyed.

Fortunately, both of these defects can be remedied, and we can establish a function $F(z)$ which (a) coincides with the usual Fibonacci numbers when Z is an integer, (b) is defined for any complex number z (c) is differentiable everywhere in the complex plane, and (d) is a real number when z is real.

The construction is not difficult. Let λ be the larger of the two roots of

$$y^2 - y - 1 = 0 .$$

Then the Fibonacci formula

$$F(n) = \frac{\lambda^n - (-1)^n \lambda^{-n}}{\sqrt{5}}$$

could be applied directly for n a real number, but would be complex at, for example, $x = 1/2$. By replacing $(-1)^n$ by a real function which takes on the value -1 for n odd and 1 for n even, we can extend $F(n)$ to all real values of n. Such a function is $\cos \pi n$. Hence a Fibonacci function can be written as

$$F(z) = \frac{\lambda^z - (\cos \pi z)\lambda^{-z}}{\sqrt{5}} .$$

An examination of this function shows easily that the stated properties are indeed satisfied.

It is possible to take this one step further. Any solution to the Fibonacci difference equation

$$F(n) = F(n-1) + F(n-2)$$

can be similarly treated to yield the equation

$$F(z) = C_1\lambda^z + C_2 (\cos \pi z)\lambda^{-z}$$

where C_1 and C_2 are determined from initial conditions.

In fact, it is possible to generalize even further and produce a similar formula for the solution of the difference equation

$$f(n) = af(n-1) + bf(n-2).$$

Such a formula is

$$f(z) = C_1\lambda^z + C_2 b^z (\cos \pi z)\lambda^{-z} ,$$

where λ is a solution to the quadratic equation

$$y^2 - ay - b = 0 .$$

In case these roots are equal, this formula takes the form

$$f(z) = C_1\lambda^z + C_2 z\lambda^z .$$

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1. Eric Halsey, "The Fibonacci Number Fu, where u is not an Integer," The Fibonacci Quarterly, Vol. 3, No. 2, pp. 147-152.

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FACTORIZATION OF 2 X 2 INTEGRAL MATRICES WITH DETERMINANT ±1

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1. CANONICAL PRODUCTS AND CANONICAL REPRESENTATIVES

Let Z denote the integers and $M_2(Z)$

$$M_2(Z) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z \right\},$$

the set of 2×2 integral matrices. The matrices of $M_2(Z)$ which have inverses in $M_2(Z)$ are denoted by $GL(2, Z)$, i.e.,

$$GL(2, Z) = \left\{ x \in M_2(Z) : \det x = \pm 1 \right\} .$$

We shall develop an algorithm which uses various properties of the Fibonacci numbers for expressing any element of $GL(2, Z)$ as a product of powers of the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} .$$

This of course implies that A and B generate $GL(2, Z)$, a result which has been noted elsewhere [3]. The algorithm forms part of the author's B. A. thesis written under the direction of B. Hunt at Reed College in 1957.

<u>1.1 Definition:</u> A "canonical product" is any product of the form

$$U = A^{a_n} B^{b_n} A^{a_{n-1}} B^{b_{n-1}} \cdots A^{a_2} B^{b_2} A^{a_1} B^{b_1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

where $n \geq 1$ and $a_i \geq 0$, $b_i \geq 0$ where we assume strict inequality except possibly at $i = n$ and $i = 1$ respectively.

We note that

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix},$$

where F_n is the n^{th} Fibonacci number of the following sequence:

n:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
F_n:	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584

1.2 Theorem. If

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is any canonical product then $a \geq c \geq 0$, $b \geq d \geq 0$.

Proof. The theorem is true for A and B the products with one factor. Suppose the theorem true for any product

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of k factors. Then

$$AT = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} \text{ and } \begin{matrix} a+c \geq c \geq 0 \\ b+d \geq d \geq 0 \end{matrix}.$$

$$BT = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a & b \end{pmatrix} \text{ and } \begin{matrix} a+c \geq a \geq 0 \\ b+d \geq b \geq 0 \end{matrix}.$$

hence the throrem holds for any product of $k+1$ factors and hence, by induction, for any U.

1.3 Corollary: Not both c and d are zero and

i) $a > c \geq 0$ unless $U = \begin{pmatrix} 1 & n+1 \\ 1 & n \end{pmatrix} = BA^n$, $n \geq 0$

ii) $b > d \geq 0$ unless $U = \begin{pmatrix} n+1 & 1 \\ n & 1 \end{pmatrix} = BA^{n-1}B$, $n \geq 0$

iii) $c \neq d$ unless $U = \begin{pmatrix} n+1 & n \\ 1 & 1 \end{pmatrix} = A^{n-1}B^2, \quad n \geq 1$

or

$U = \begin{pmatrix} n & n+1 \\ 1 & 1 \end{pmatrix} = A^{n-1}BA, \quad n \geq 1$

iv) $a \neq b$ unless $U = A$ or B.

v) $c > 0$ unless $U = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = A^n$

vi) $d < 0$ unless $U = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix} = A^{n-1}B$.

<u>Proof.</u> These are immediate consequences of the theorem and the fact that $\det U = ad - bc = \pm 1$.

<u>1.4 Corollary:</u> If

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

i) $a - b > 0$ implies $c - d \geq 0$

ii) $a - b < 0$ implies $c - d \leq 0$.

<u>Proof.</u> i) $a > b$, $c \leq d$ implies

$$ad - bc > bd - bc > bc - bc = 0$$

Hence $ad - bc \geq 2$, which is impossible.

ii) $a < b$, $c > d$ implies $b > a \geq c > d \geq 0$ and so $b \geq 2$. Hence $ad - bc < bd - bc = b(d - c) \leq 2(-1) = -2$ which is impossible.

<u>1.5 Theorem.</u> Let

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be a canonical product. Then

1. $a - b = 0$ implies $U = A, B$.
2. $a - b < 0$ implies U ends in A and $a - b \leq c - d \leq 0$
3. $a - b > 0$ implies U ends in B and $a - b \geq c - d \geq 0$.

Proof. 1 follows from Corollary 1.3.

2 and 3. The theorem is immediately verified for products of two factors

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad AB = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } BA = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Suppose the theorem holds for products with k or fewer factors and

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a product with k factors where $k \geq 2$. Then

$$UB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & a \\ c+d & c \end{pmatrix}$$

and

$$(a + b) - a = b \geq 1 > 0 \quad \text{and} \quad b \geq (c + d) - c = d \geq 0 .$$

Note $b = 0$ implies $d = 0$ which contradicts $\det U = \pm 1$. Likewise,

$$UA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$$

hence

$$a - (a + b) = -b < 0 \text{ and } c - (c + d) = -d \text{ and } -b \leq -d \leq 0.$$

Thus the theorem holds for products of $k + 1$ factors and hence for all canonical products by induction.

Theorem 1.2 says that if

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a canonical product then $a \geq c \geq 0$, and $b \geq d \geq 0$ while Theorem 1.5 allows one to decide if the canonical product ends in an A or B. Not every unimodular matrix satisfies the conditions of Theorem 1.2 but the following theorem characterizes the situation.

1.6 Theorem. (See [2].) Any matrix

$$R = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in GL(2, Z)$$

different from I, A, B can, by suitable multiplications by powers of A and B, be brought to the form

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where (a, b, c, d) is some permutation of $(|r|, |s|, |t|, |u|)$ and $a \geq c \geq 0$, $b \geq d \geq 0$. U is called a canonical representative of R.

Proof. From the condition $ru - st = \pm 1$ we can conclude that no three of the quantities r, s, t, u can be negative and the remaining one positive or three of them positive and one negative. There are therefore three remaining cases:

1. r, s, t, u are all non-negative,
2. r, s, t, u are all non-positive,
3. Two of r, s, t, u are negative, two are non-negative.

In case 2 we note

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} -r & -s \\ -t & -u \end{pmatrix}.$$

In case 3 we note

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} -r & -s \\ t & u \end{pmatrix}$$

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -r & s \\ -t & u \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -r & s \\ t & u \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -r & s \\ t & -u \end{pmatrix}$$

While

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} s & r \\ u & t \end{pmatrix}.$$

The multipliers can be expressed as follows:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A^{-1}B, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A^{-1}B^{-1}ABA^{-1},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus we have that

$$M\begin{pmatrix} r & s \\ t & u \end{pmatrix}N = \begin{pmatrix} |r| & |s| \\ |t| & |u| \end{pmatrix},$$

where M and N are suitable products of powers of A and B. Note $|r||u| - |s||t| = |ru| - |st| = \pm 1$ since $1 = |ru - st| \geq ||ru| - |st||$ and $|ru| = \pm st$ is not possible. Also operating with

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we can bring any element, in particular the largest, to the upper left position. If

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq A, B$$

is in this form, i. e., $a \geq b$, $cd \geq 0$ then we may assume that $b \geq d$. For $b < d$ implies $ad - bc \geq ad - cd = (a - c)d \geq 0$ unless $a = c$. If $a = c$ then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \text{ has the property } \begin{matrix} c \geq a \geq 0 \\ d \geq b \geq 0 \end{matrix} \quad .$$

Every unimodular matrix has 2 canonical representatives depending on whether a maximal element is brought to the upper left or upper right-hand corner. We now prove the converse of Theorem 1.2.

1.7 Theorem. Every canonical representative of a unimodular 2×2 matrix is a canonical product.

Proof. Let

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a canonical representative, i. e., $a \geq c \geq 0$, $b \geq d \geq 0$. If the largest element of $U = 1$ then $U = A, B$ and the theorem holds. Assume the theorem is true for $\max(a, b) \leq r$, where $r \geq 1$. Then there are two possibilities for $\max(a, b) = r + 1$:

Case 1.

$$Y = \begin{pmatrix} r+1 & b \\ c & d \end{pmatrix};$$

Case 2.

$$Z = \begin{pmatrix} a & r+1 \\ c & w \end{pmatrix} \quad .$$

We now analyze Case 1. We have that $0 \le c, b, d < r + 1$ otherwise $r + 1 \ge 2$ divides det $Y = \pm 1$.

$$YB^{-1} = \begin{pmatrix} r+1 & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} b & r+1-b \\ d & c-d \end{pmatrix}.$$

If $d = 0$ then $b = c = 1$ and

$$YB^{-1} = \begin{pmatrix} b & r \\ d & c \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = A^r$$

Hence $Y = A^rB$ is a canonical product. If $d \neq 0$ then $c - d \ge 0$ for otherwise

$$\left|\det (YB^{-1})\right| \ge 2 .$$

We need only establish that $r + 1 - b \ge c - d$ to show that YB^{-1} is a canonical representative. If

$$\det Y = (r + 1)d - bc = 1$$

then

$$(r + 1)d - bd = 1 + bc - bd$$

and so

$$(r + 1) - b = \frac{1}{d} + \frac{b}{d}(c - d) \ge c - d$$

since $b \ge d$. If

$$\det Y = (r + 1)d - bc = -1$$

then $b, c > d$ since $b, c < r + 1$.

Hence

$$(r+1) - b = \frac{b}{d}(c-d) - \frac{1}{d} = (c-d) + \left(\frac{b}{d} - 1\right)(c-d) - \frac{1}{d}$$

$$= (c-d) + \frac{1}{d}\left[(b-d)(c-d) - 1\right] \geq c - d.$$

Thus YB^{-1} is a canonical product by induction hypothesis. This implies that Y must also be a canonical product.

Case 2 is analyzed by an analogous treatment of ZA^{-1}. The theorem then follows by induction.

2. THE ALGORITHM

Let $\overline{U} \in GL(2, Z)$. Theorem 1.6 describes how to obtain a canonical representative U for $\overline{U}$. Theorem 1.7 asserts that U is a canonical product and Theorem 1.5 establishes whether U ends in an A or B. The following theorems provide a quantitative counterpart for Theorem 1.5.

2.1 Theorem. If

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a canonical product and $a - b < 0$ then $U = U_1 A^n$ where

$$N = \left[\frac{b}{a}\right]$$

and U_1 ends in B.*

Proof. If $a = 1$ we consult Corollary 1.3. If $a \neq 1$ we note:

1. $b \neq na$ since $b = na$ implies $a | \det U = \pm 1$ and $a = 1$,
2. $a > c$ since $a = c$ implies $a = 1$. Then

*$[\alpha]$ is the greatest integer less than or equal to α

$$UA^{-n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -na+b \\ c & -nc+d \end{pmatrix} = U_1 \,.$$

Since

$$N = \left[\frac{b}{a}\right]$$

$$a - (-na + b) > 0 \,.$$

Hence if U_1 is a canonical product it ends in a B. If $-nc + d < 0$ then, recalling that $a \geq 2$,

$$\det U_1 = a(-nc + d) - c(-na + b) < -2 - c$$

which is impossible. Hence $-nc + d \geq 0$. Further

$$r = -nc + d \leq -na + b$$

since

$$r > -na + b \geq 0$$

implies

$$\begin{aligned} \det U_1 &= ar - c(-na + b) \geq ar = (a - 1)(r - 1) \\ &= ar - ar + (a + r) - 1 \geq 2 + 1 - 1 = 2 \end{aligned}$$

which is impossible.

2.2 Lemma.

1. $\dfrac{F_n}{F_{n-1}} < \dfrac{a}{b} < \dfrac{F_{n+2}}{F_{n+1}} < \dfrac{\sqrt{5}+1}{2}$ implies $b > F_{n+1}$

   if n is even,

2. $\dfrac{\sqrt{5}+1}{2} < \dfrac{F_{n+2}}{F_{n+1}} < \dfrac{a}{b} < \dfrac{F_n}{F_{n-1}}$ implies $b > F_{n+1}$

if n is odd.

Proof. We prove 1 using continued fraction notation [1, Chapter X]. In the following the initial block of ones in the continued fraction symbols will be of length n - 1. Also we note

$$[x_0, x_1, \cdots, x_n, 1] = [x_0, x_1, \cdots, x_n + 1] \quad .$$

$$\frac{F_n}{F_{n-1}} = [1, 1, \cdots, 1]$$

is a convergent to

$$\frac{\sqrt{5}+1}{2} = [1, 1, \cdots] \quad .$$

If we express

$$\frac{a}{b} = [a_0, a_1, \cdots, a_m]$$

as a continued fraction then by the continued fraction algorithm [1, p. 140]

$$\frac{a}{b} = [1, 1, \cdots, 1, a_{n-1}, a_n, a_m] \text{ since } \frac{F_n}{F_{n-1}} < \frac{a}{b} < \frac{F_{n+2}}{F_{n+1}} \ ,$$

i.e., $m \geq n - 1$ and $a_0 = a_1 = \cdots = a_{n-2} = 1$.

Letting

$$[a_{n-1}, a_n, \cdots, a_m] = a'_{n-1} = \frac{r}{s} \quad ,$$

where $(r, s) = 1$ we have

$$\frac{F_n}{F_{n-1}} = [1, 1, \cdots, 1] < \left[1, 1, \cdots, 1, \frac{r}{s}\right] < [1, 1, \cdots, 1, 2] = \frac{F_{n+2}}{F_{n+1}} .$$

Hence

$$\frac{a}{b} = \frac{rF_n + sF_{n-1}}{rF_{n-1} + sF_{n-2}} = \frac{\frac{r}{s}F_n + F_{n-1}}{\frac{r}{s}F_{n-1} + F_{n-2}} < \frac{2F_n + F_{n-1}}{2F_{n-1} + F_{n-2}} = \frac{F_{n+2}}{F_{n+1}} .$$

By the continued fraction algorithm

$$a'_{n-1} = \frac{r}{s} \geq 1 .$$

Now $r/s = 1$ implies

$$\frac{a}{b} = \frac{F_{n+1}}{F_n} > \frac{\sqrt{5}+1}{2} .$$

Hence $r/s > 1$ and $r \geq 2$. Likewise, $r = 2$ implies $s = 1$ and

$$\frac{a}{b} = \frac{F_{n+2}}{F_{n+1}} .$$

Hence $r \geq 2$. If

$$(rF_n + sF_{n-1}, rF_{n-1} + sF_{n-2}) = 1$$

then

$$b \geq rF_{n-1} + sF_{n-2} > F_{n+1} = 2F_{n-1} + F_{n-2} .$$

But

$$d \,\big|\, (rF_n + sF_{n-1}), \quad d \,\big|\, (rF_{n-1} + sF_{n-2})$$

implies

$$d \,\Big|\, \{r(F_n - F_{n-1}) + s(F_{n-1} - F_{n-2})\} = rF_{n-2} + sF_{n-3},$$

Hence

$$d \,\big|\, (rF_1 + F_0) = r \quad ,$$

Also

$$d \,|\, r, \quad d\,(rF_k + sF_{k-1}), \quad 1 \le k \le n$$

implies $d \,|\, sF_{k-1}$ and hence

$$d \,|\, F_{k-1}, \quad 1 \le k \le n \quad ,$$

since $(r, s) = 1$. Thus $d = 1$, since the F_k are relatively prime, a fact which can be established in the same recursive manner.

The same type of argument is used in proving 2.

2.3 Theorem. If

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a - b > 0, \quad b > d > 0$$

then $U = U_1B^n$, where n is determined by locating a/b with respect to the sequence of points

$$\frac{F_2}{F_1} < \frac{F_4}{F_3} < \frac{F_6}{F_5} < \cdots < \frac{\sqrt{5}+1}{2} < \cdots < \frac{F_7}{F_6} < \frac{F_5}{F_4} < \frac{F_3}{F_2}$$

and U_1 is a canonical product ending in A or $U_1 = A, B$. More precisely,

(1) $$\frac{F_n}{F_{n-1}} < \frac{a}{b} \le \frac{F_{n+2}}{F_{n+1}} \quad \text{if } n \text{ is even ;}$$

(2) $$\frac{F_{n+2}}{F_{n+1}} \le \frac{a}{b} < \frac{F_n}{F_{n-1}}, \text{ if } n \text{ is odd};$$

(3) $$2 = \frac{F_3}{F_2} \le \frac{a}{b}, \text{when } n = 1 .$$

Finally, if $b = d = 1$ or $d = 0$ we consult Corollary 1.3.

Proof. We prove 1 and note that the proofs to 2 and 3 are analogous. Suppose

$$\frac{F_n}{F_{n-1}} < \frac{a}{b} \le \frac{F_{n+2}}{F_{n+1}} < \frac{\sqrt{5}+1}{2}$$

Then

$$U_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} B^{-n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (-1)^n \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} a\ F_{n-1} - b\ F_n & -aF_n + b\ F_{n+1} \\ c\ F_{n-1} - d\ F_n & -cF_n + d\ F_{n+1} \end{pmatrix} .$$

We first note that

$$(aF_{n-1} - bF_n) - (-aF_n + bF_{n+1}) = aF_{n+1} - bF_{n+2} \le 0 .$$

Hence, if we establish that U_1 is a canonical product, then U_1 ends in A or $U_1 = A_2B$. To show that U_1 is a canonical product we note that $aF_{n-1} - bF_n \ge 0$ and $-aF_n + bF_{n+1} > 0$, since

$$\frac{F_n}{F_{n-1}} < \frac{a}{b} \le \frac{F_{n+2}}{F_{n+1}} < \frac{F_{n+1}}{F_n} .$$

This shows that the top row has positive entries. If we show

(i) $$\frac{F_n}{F_{n-1}} \le \frac{c}{d} \le \frac{F_{n+1}}{F_n}$$

(ii) $$\frac{F_n}{F_{n-1}} \le \frac{a-c}{b-d} \le \frac{F_{n+1}}{F_n}$$

then the bottom row is positive and the columns decrease since the two equations

$$(aF_{n-1} - bF_n) - (cF_{n-1} - dF_n) = (a - c)F_{n-1} - (b - d)F_n \ge 0$$

$$(-aF_n + bF_{n+1}) - (-cF_n + dF_{n+1}) = -(a - c)F_n + (b - d)F_{n+1} \ge 0$$

are equivalent to (ii). We now note that

$$\left|\frac{a}{b} - \frac{c}{d}\right| = \left|\frac{ab - bc}{bd}\right| = \frac{1}{bd}$$

for use in proving (i),

$$\left|\frac{a}{b} - \frac{a-c}{b-d}\right| = \left|\frac{ab - ad - ab + bc}{b(b-d)}\right| = \frac{1}{b(b-d)}$$

for proving (ii). We conclude by proving (i) since (ii) is similar.

Since

$$\frac{F_n}{F_{n-1}} < \frac{a}{b} \le \frac{F_{n+2}}{F_{n+1}}, \quad b \ge F_{n+1}$$

by Lemma 2.2. If

$$\frac{c}{d} < \frac{F_n}{F_{n-1}}$$

then

$$\frac{1}{dF_{n-1}} > \frac{1}{dF_{n-1}} \geq \frac{1}{bd} = \left|\frac{a}{b} - \frac{c}{d}\right| > \left|\frac{F_n}{F_{n-1}} - \frac{c}{d}\right| > \frac{1}{dF_{n-1}}$$

which is impossible. Hence

$$\frac{F_n}{F_{n-1}} \leq \frac{c}{d} \; .$$

Likewise

$$\frac{c}{d} > \frac{F_{n+1}}{F_n}$$

implies

$$\frac{1}{dF_n} > \frac{1}{dF_{n+1}} \geq \frac{1}{bd} = \frac{c}{d} - \frac{a}{b} \geq \frac{c}{d} - \frac{F_{n+1}}{F_n} \geq \frac{F_{n+1}}{F_n} \geq \frac{F_{n+2}}{F_{n+1}}$$

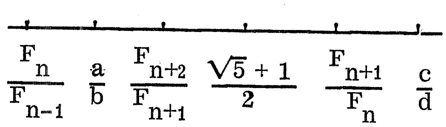
$$\geq \frac{1}{dF_n} + \frac{1}{F_nF_{n+1}}$$

which is impossible since

$$\frac{1}{F_nF_{n+1}} > 0 \; .$$

Hence

$$\frac{c}{d} \leq \frac{F_{n+1}}{F_n} \; .$$



3. EXAMPLE

$$U = \begin{pmatrix} 206 & 1575 \\ 79 & 604 \end{pmatrix}$$

is a canonical product ending in A^7, since $a - b = 206 - 1575 < 0$ and

$$\left[\frac{1575}{206}\right] = 7 .$$

So that

$$\begin{pmatrix} 206 & 1575 \\ 79 & 604 \end{pmatrix}\begin{pmatrix} 1 & -7 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 206 & 133 \\ 79 & 51 \end{pmatrix} = U_1$$

Then,

$$\frac{F_4}{F_3} = 1.5 < \frac{a}{b} = \frac{206}{133} = 1.55 < 1.6 = \frac{F_6}{F_5}$$

hence U_1 ends in B^4. We note that

$$\begin{pmatrix} 206 & 133 \\ 79 & 51 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 13 & 47 \\ 5 & 18 \end{pmatrix}$$

ends in A^3. Since

$$\left[\frac{47}{13}\right] = 3 .$$

$$\begin{pmatrix} 13 & 47 \\ 5 & 18 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 5 & 3 \end{pmatrix} = U_3$$

$$\frac{F_7}{F_6} = \frac{a}{b} = \frac{13}{8} = 1.625 < 1.6667 = \frac{F_5}{F_4}$$

hence U_3 ends in B^5.

$$\begin{pmatrix} 13 & 8 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} -3 & 5 \\ 5 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A .$$

Therefore

$$U = AB^5A^3B^4A^7 .$$

4. TABLE

$\frac{F_3}{F_2} = \frac{2}{1} = 2.0000000$	$\frac{F_{18}}{F_{17}} = \frac{2584}{1597} = 1.6180338$
$\frac{F_5}{F_4} = \frac{5}{3} = 1.6666667$	$\frac{F_{16}}{F_{15}} = \frac{987}{610} = 1.6180328$
$\frac{F_7}{F_6} = \frac{13}{8} = 1.6250000$	$\frac{F_{14}}{F_{13}} = \frac{377}{233} = 1.6180258$
$\frac{F_9}{F_8} = \frac{34}{21} = 1.6190476$	$\frac{F_{12}}{F_{11}} = \frac{144}{89} = 1.6179775$
$\frac{F_{11}}{F_{10}} = \frac{89}{55} = 1.6181818$	$\frac{F_{10}}{F_9} = \frac{55}{34} = 1.6176471$
$\frac{F_{13}}{F_{12}} = \frac{233}{144} = 1.6180556$	$\frac{F_8}{F_7} = \frac{21}{13} = 1.6153846$
$\frac{F_{15}}{F_{14}} = \frac{610}{377} = 1.6180371$	$\frac{F_6}{F_5} = \frac{8}{5} = 1.6000000$
$\frac{F_{17}}{F_{16}} \quad \frac{1597}{987} = 1.6180344$	$\frac{F_4}{F_3} = \frac{3}{2} = 1.5000000$
$\frac{\sqrt{5}+1}{2} = 1.6180340$	$\frac{F_2}{F_1} = \frac{1}{1} = 1.0000000$

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1. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Clarendon, Oxford, 1938.

2. Hunt Burrowes, "Continued Fractions and Indefinite Forms," unpublished article, Notice in Bull. Am. Math. Soc. 62 (1956), 553.

3. George K. White, "On Generators and Defining Relations for the Unimodular Group M_2," Amer. Math. Month. 71 (1964), 743-748.

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ERRATA

Please make the following corrections in "Recurrence Relations for Sequences Like $\{F_{F_n}\}$," The Fibonacci Quarterly, April, 1967, Vol. 5, No. 2, pp. 129-136:

1. Replace "n" by "F" in the first line of the third paragraph on p. 129.
2. Replace the equations of (7') on page 132 by

$$2X_{n+2} = X_{n+1}Y_n + X_nY_{n+1}$$
$$2Y_{n+2} = (r - s)^2X_{n+1}X_n + Y_{n+1}Y_n .$$

3. Replace the "aj" in the first line of p. 134 by "a_j".
4. Replace the minus sign in the line preceding (15) on p. 134 by a plus.
5. Delete the first "4E" on the first line of page 136.

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Please also correct "A Shift Formula for Recurrence Relations of Order m," The Fibonacci Quarterly, December 1967, Vol. 5, No. 5, pp. 461-465, by replacing the "p_m" in the sum on the last line of p. 462 by "p_{m-i}".

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Please make the following correction in

" The Fibonacci Quarterly, November, 1967, Vol. 5, No. 4, p. 370:

In the fourth line from the bottom, replace "difference of each pair" with "differences of the pairs."

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ON A PARTITION OF GENERALIZED FIBONACCI NUMBERS

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As a continuation of results in [4], this paper deals with the concept of minimal and maximal representations of positive integers as sums of generalized Fibonacci numbers (G. F. N.) defined below and presents a partition of the G. F. N. in relation to either minimal or maximal representation.

Consider the sequence $\{F_t\}$, where

$$(1) \qquad \begin{aligned} &F_1 = F_2 = \cdots = F_r = 1, \quad r \geq 2 \\ \text{and} \\ &F_t = F_{t-1} + F_{t-r}, \qquad t \geq r . \end{aligned}$$

Obviously, the sequence gives rise to the sequence of Fibonacci numbers for $r = 2$. For this reason, we call $\{F_t\}$ a sequence of G. F. N. Clearly $\{F_t\}$ is a special case of Daykin's Fibonacci sequence [3], as well as of Harris and Styles' sequence [6].

We remark that it is possible to express any positive integer N as a sum of distinct F_i's, subject to the condition that $F_1, F_2, \cdots, F_{r-1}$ are not used in any sum (reference: Daykin's paper [3]). In other words, we can have

$$(2) \qquad N = \sum_{i=r}^{s} a_i F_i$$

with $a_s = 1$ and $a_i = 1$ or 0, $r \leq i < s$. Here s is the largest integer such that F_s is involved in the sum.

Definition 1: In case (2) is satisfied, the vector $(a_r, a_{r+1}, \cdots, a_s)$ of elements 1 or 0 with $a_s = 1$, is called a representation of N in $\{F_t\}$, having its index as s.

Definition 2: A representation of N in $\{F_t\}$ is said to be minimal or maximal according as $a_i a_{i+j} = 0$ or $a_i + a_{i+j} \geq 1$, for all $i \geq r$ and $j = 1, 2, \cdots, r - 1$.

Definition 2 is just an extension of that by Ferns [4] to $r > 2$.

Now, we state some results in the forms of lemmas, to be used subsequently.

Lemma 1:

(i) Every positive integer N has a unique minimal representation;

(ii) The index of the minimal representation of N ($F_n \le N < F_{n+1}$, $n \ge r$) is n. If $N = F_n$, then $a_r = a_{r+1} = \cdots = a_{n-1} = 0$.

Lemma 2:

(i) Every positive integer N has a unique maximal representation:

(ii) The index of the maximal representation of N ($F_{n+r-1} - r < N \le F_{n+r} - r$, $n \ge r$) is n. If $N = F_{n+r} - r$, then $a_r = a_{r+1} = \cdots = a_{n-1} = 1$.

Lemma 3. If the minimal representation of

$$N' = F_{n+r} - r - N \qquad (F_u \le N' < F_{u+1}, \quad n \ge r, \; r \le u \le n-1)$$

is

$$(a_r, a_{r+1}, \cdots, a_u),$$

then the maximal representation of N

$$(F_{n+r} - F_{u+1} - r < N \le F_{n+r} - F_u - r)$$

is

$$(1 - a_r, 1 - a_{r+1}, \cdots, 1 - a_u, \underbrace{1, 1, \cdots, 1}_{n-u})$$

and conversely.

Note that Lemma 3 provides a method of construction of the maximal (minimal) representation, given the minimal (maximal) representation of integers. Furthermore, the last time a zero occurs in the maximal representation of N ($F_{n+r} - F_{u+1} - r < N \le F_{n+r} - F_u - r$), is at the $(u - r + 1)$st position, that is, $1 - a_u = 0$.

Proof. The proof of Lemma 1 is given in [3] as Theorem C. (Also see Brown's paper [1].) A generalized argument similar to that in the proof of

Theorem 1 in [2] would lead us to Lemma 2 and Lemma 3. However, the basic steps of the proof are indicated.

First, we assert that

$$\sum_{i=r}^{n} F_i + r = F_{r+n} \quad . \tag{3}$$

When $n \geq 2r - 1$,

$$\sum_{i=r}^{n} F_i + r = F_r + \cdots + F_{2r-2} + F_{2r-1} + F_{2r} + \cdots + F_n$$

(since $r = F_{2r-1}$)

$$= F_{r+1} + \cdots + F_{2r-1} + 2F_{2r} + F_{2r+1} + \cdots + F_n$$

$$\cdots$$

$$= F_{r+n} \; .$$

When $n < 2r - 1$, (3) can also be checked.

Next, we develop a method to construct a maximal representation from the system of minimal representation, and finally show that this representation is unique.

When

$$F_{n+r-1} - r < N \leq F_{n+r} - r \;,$$

i. e.,

$$\sum_{i=r}^{n-1} F_i < N \leq \sum_{i=r}^{n} F_i, \quad \text{(by (3))}$$

we get

$$N' = F_{n+r} - r - N = \sum_{i=r}^{n} F_i - N < \sum_{i=r}^{n} F_i - \sum_{i=1}^{n-1} F_i = F_n \,. \tag{4}$$

Because of (4) and Lemma 1, let us assume that

$$F_u \le N' < F_{u+1}, \quad r \le u \le n-1 \;,$$

and that N' has the minimal representation $(a_r, a_{r+1}, \cdots, a_u)$. Thus, $(b_r, b_{r+1}, \cdots, b_n)$, where

$$b_i = \begin{cases} 1 - a_i \;, & i = r, r+1, \cdots, u, \\ 1 \;, & i = u+1, u+2, \cdots, n, \end{cases}$$

is a maximal representation of N as we can show that $b_i + b_{i+j} \ge 1$ from $a_i a_{i+j} = 0$ for all $i \ge r$ and $j = 1, 2, \cdots, r-1$.

Suppose that two maximal representations of N are given by

$$N = \sum_{i=r}^{n} a_i F_i = \sum_{i=r}^{n} a'_i F_i \;, \quad a_n = a'_{n'} = 1 \;,$$

with $n > n'$. Letting $n = cr + d$, we obtain

$$\begin{aligned} \sum_{i=r}^{n} a_i F_i \ge F_n &+ F_{n-2} + F_{n-3} + \cdots + F_{n-r} \\ &+ F_{n-r-2} + F_{n-r-3} + \cdots + F_{n-2r} \\ &+ \cdots \\ &+ F_{n-(c-2)r-2} + \cdots + F_{n-(c-1)r} \end{aligned} \tag{5}$$

$$= F_{n+1} + F_{n-1} + F_{n-2} + \cdots + F_{n+2-r} - (r-1)$$

$$= F_{n+r-1} - (r-1) \,.$$

But

$$\sum_{i=r}^{n'} a'_i F_i \le \sum_{i=r}^{n'} F_i \le \sum_{i=r}^{n-1} F_i = F_{n+r-1} - r\,, \text{ by (3)}\,.$$

This is a contradiction of (5) and therefore $n = n'$.

From

$$N = \sum_{i=r}^{n} (1 - a_i)F_i = \sum_{i=r}^{n} (1 - a'_i)F_i$$

it follows that

$$N^{\star} = \sum_{i=r}^{n} (1 - a_i)F_i = \sum_{i=r}^{n} (1 - a'_i)F_i$$

which corresponds to two admissible minimal representations of $N^{\star}$. The proof is complete, due to Lemma 1.

Definition 3: Define $U(n; m_1, m_2, \cdots, m_r)$ as the number of positive integers N satisfying the following: (the definition arises as a natural consequence of Lemma 1)

(i) $F_n \le N < F_{n+1}$, $n \ge r$;

(ii) In the minimal representation $(a_r, a_{r+1}, \cdots, a_n)$ of N, there are exactly $m_i a'_{i\alpha}$'s among non-zero a's except a_n, such that $\alpha \equiv i - 1 \pmod r$ $i = 1, 2, \cdots, r$.

An illustration of the definition for $r = 3$, might serve a useful purpose. Consider all integers N, $F_{10} \le N < F_{11}$ and their respective minimal representations are:

$19 = F_{10}$, $20 = F_3 + F_{10}$, $21 = F_4 + F_{10}$, $22 = F_5 + F_{10}$, $23 = F_6 + F_{10}$,
$24 = F_3 + F_6 + F_{10}$, $25 = F_7 + F_{10}$, $26 = F_3 + F_7 + F_{10}$, $27 = F_4 + F_7 + F_{10}$.

Then we have

$$U(10;\ 0,\ 0,\ 0) = 1,\quad U(10,\ 1,\ 0,\ 0) = 2,\quad U(10;\ 0,\ 1,\ 0) = 2,$$
$$U(10;\ 0,\ 0,\ 1) = 1,\quad U(10;\ 2,\ 0,\ 0) = 1,\quad U(10,\ 1,\ 1,\ 0) = 1,$$
$$U(10,\ 0,\ 2,\ 0) = 1\ .$$

It may be observed that a_n is omitted in the definition without any ambiguity, as it is present in every representation. Furthermore, it is significant to note that Definition 3 gives rise to a partition of the G. F. N.

Following the procedure in [4] on pages 23 and 24, we can show that either by replacing F_{n-1} by F_n in the minimal representation of every N_1, $F_{n-1} \le N_1 < F_n$, or by adding F_n in the minimal representation of every N_2, $F_{n-r} \le N_2 < F_{n-r+1}$, we get the minimal representation of every N, $F_n \le N < F_{n+1}$. Therefore, $U(n; m_1, m_2, \cdots, m_r)$ satisfies the following difference equations:

For $m > 1$,

$$(6)\qquad \left\{\begin{aligned} U(rm; m_1, m_2, \cdots, m_r) &= U(rm-1; m_1, m_2, \cdots, m_r) \\ &\quad + U(r(m-1); m_1 - 1, m_2, \cdots, m_r) \\ U(rm+1; m_1, m_2, \cdots, m_r) &= U(rm; m_1, m_2, \cdots, m_r) \\ &\quad + U(r(m-1)+1; m_1, m_2 - 1, \cdots, m_r) \\ &\ \vdots \\ U(rm+r-1; m_1, m_2, \cdots, m_r) &= U(rm+r-2; m_1, m_2, \cdots, m_r) \\ &\quad + U(r(m-1)+r-1; m_1, m_2, \cdots, m_r - 1)\ . \end{aligned}\right.$$

Obviously, the boundary conditions given below can easily be checked. These are:

$$(7)\qquad \left\{\begin{aligned} &\text{For } n < r \text{ or for any } m_i < 0, \\ &\qquad U(n; m_1, m_2, \cdots, m_r) = 0\ ; \\ &\text{for } r \le n < 2r \text{ (i.e., for } m = 1), \\ &\qquad U(n; m_1, m_2, \cdots, m_r) = \begin{cases} 1 & \text{when } m_1 = m_2 = \cdots = m_r \\ 0 & \text{otherwise} \end{cases} \end{aligned}\right.$$

Theorem 1.

$$(8)\qquad U(n;m_1, m_2, \cdots, m_r) = \begin{cases} \prod_{k=1}^{r} \binom{M + m_k}{m_k} & \text{if } n = rm + (r-1), \\ \prod_{k=1}^{j+1} \binom{M + m_k}{m_k} \left[\prod_{k=j+2}^{r} \binom{M + m_k - 1}{m_k} \right] & \text{if } n=rm+j \text{ and } 0 \le j \le r-2, \end{cases}$$

where

$$M = m - 1 - \sum_{i=1}^{r} m_i ,$$

and

$$\binom{x}{y}$$

has the usual meaning with

$$\binom{x}{y} = 1 \quad \text{and} \quad \binom{x}{y} = 0$$

when $y < 0$ or when $y > x$.

Proof. Trivially, the results are true when $m = 1$. We can also verify these expressions for $m = 2$. Assume that these are valid for $m \le m'$. Now,

$$\begin{aligned} &U(r(m' + 1); m_1, m_2, \cdots, m_r) \\ &\quad = U(rm' + r - 1; m_1, m_2, \cdots, m_r) + U(rm'; m_1 - 1, m_2, \cdots, m_r) \\ &\quad = \prod_{k=1}^{r} \binom{M' + m_k}{m_k} + \binom{M' + m_1}{m_1 - 1} \prod_{k=2}^{r} \binom{M' + m_k}{m_k} , \end{aligned}$$

$$\text{where } M' = m' - 1 - \sum_{i=1}^{r} m_i$$

$$\quad = \binom{M' + m_1 + 1}{m_1} \prod_{k=2}^{r} \binom{M' + m_k}{m_k}$$

which establishes (8) for $m = m' + 1$ and $j = 0$. Similar verifications for $m' + 1$ and $0 < j \leqq r - 1$ complete the proof of Theorem 1.

Denoting $\Sigma'_{\mu,r}$ as the summation over $m_1, m_2, \cdots, m_r$ with the restriction

$$\sum_{i=1}^{r} m_i = \mu ,$$

we get the following:

Corollary 1.

$$\Sigma'_{\mu,r} U(n; m_1, m_2, \cdots, m_r) = \binom{n - (r-1)\mu - r}{\mu} . \tag{9}$$

Proof. By induction, we shall prove the result, which is seen to be true for small values of n.

$$\sum_{\mu,r} U(rm + j;\, m_1, m_2, \cdots, m_r) \tag{10}$$

$$= \sum_{\nu=0}^{\mu} \binom{m-\nu-2}{\mu-\nu} \Sigma'_{\nu,r-1} \prod_{k=1}^{j+1} \binom{m-\mu+m_k-1}{m_k} \left[\prod_{k=j+2}^{r-1} \binom{m-\mu+m_k-2}{m_k} \right]$$

$$= \sum_{\nu=0}^{\mu} \binom{m-\nu-2}{\mu-\nu} \Sigma'_{\nu,r-1} U((r-1)(m-\mu+\nu) + j; m_1, m_2, \cdots, m_{r-1})$$

$$= \sum_{\nu=0}^{\mu} \binom{m-\nu-2}{\mu-\nu} \binom{(r-1)(m-\mu) + j + \nu - r + 1}{\nu} , \text{ by induction hypothesis,}$$

$$= \sum_{\nu=0}^{\mu} \binom{rm + j - (r-1)\mu - r - \nu - 1}{\mu - \nu} \qquad \text{by (1.13) of [5] ,}$$

$$= \binom{rm + j - (r-1)\mu - r}{\mu} .$$

In addition to (10), a check for $j = r - 1$ establishes (9).

Corollary 1 implies that the number of integers N, $F_n \le N < F_{n+1}$, which require $\mu + 1$ G. F. N. for minimal representation is the right-hand expression in (9), and this is in agreement with the value in [4] for $r = 2$.

Similar to

$$U(n; m_1, m_2, \cdots, m_r) ,$$

we introduce in the next definition

$$V(n; m_1, m_2, \cdots, m_r)$$

which corresponds to the maximal representation.

Definition 4. Define

$$V(n; m_1, m_2, \cdots, m_r)$$

as the number of positive integers N with the following properties:

(i) $$F_{n+r-1} - r < N \le F_{n+r} - r, \quad n \ge r;$$

(ii) In the maximal representation $(a_r, a_{r+1}, \cdots, a_n)$ of N, there are exactly $m_i a'_\alpha$ s among a's which are equal to zero, such that $\alpha \equiv i - 1$ (mod r), $i = 1, 2, \cdots, r$.

The definition is not vacuous, because of Lemma 2. As an illustration for $r = 3$, consider all N, $F_{10} - 3 < N \le F_{11} - 3$. The maximal representations of these integers are:

$$17 = F_3 + F_5 + F_6 + F_8, \quad 18 = F_4 + F_5 + F_6 + F_8, \quad 19 = F_3 + F_4 + F_5 + F_6 + F_8 ,$$
$$20 = F_4 + F_5 + F_7 + F_8, \quad 21 = F_3 + F_4 + F_5 + F_7 + F_8, \quad 22 = F_3 + F_4 + F_6 + F_7 + F_8,$$
$$23 = F_3 + F_5 + F_6 + F_7 + F_8, \quad 24 = F_4 + F_5 + F_6 + F_7 + F_8,$$
$$25 = F_3 + F_4 + F_5 + F_6 + F_7 + F_8 .$$

Thus,

$$V(8;0,0,0) = 1, \quad V(8;1,0,0) = 2, \quad V(8;0,1,0) = 2$$

$$V(8;0,0,1) = 1, \quad V(8;2,0,0) = 1, \quad V(8;1,1,0) = 1, \quad V(8;0,2,0) = 1.$$

Compare these with $U(10;m_1, m_2, m_3)$ and observe the correspondence, which is essentially the result in the theorem given below.

Theorem 2.

$$V(n;m_1, m_2, \cdots, m_r) = \begin{cases} 0 & \text{when } n < r \\ U(n + r - 1;m_1, m_2, \cdots, m_r) & \text{otherwise} \end{cases} \tag{10}$$

Proof. It is readily checked from the last part of Lemma 2(ii) that $V(n;m_1, m_2, \cdots, m_r) = 1$ for every $n \geq r$, when $m_1 = m_2 = \cdots = m_r = 0$. Therefore, we shall discuss the proof when m_i's are not simultaneously equal to zero.

Let $n = rm + j$, $m \geq 0$ and $j = 1, 2, \cdots, r$. A direct verification of the theorem for $m = 0, 1$ is simple. Then, assume that it is true for $m \leq m'$. By induction, we have to show that it holds good for $m = m' + 1$.

Putting $j = 1$, the set of integers counted in

$$V(r(m' + 1) + 1;m_1, m_2, \cdots, m_r)$$

can be partitioned into two sets,

$$\{N_1\},\ F_{r(m'+2)+1} - F_{r(m'+1)} - r < N_1 \leq F_{r(m'+2)+1} - F_r - r$$

and

$$\{N_2\},\ F_{r(m'+2)+1} - F_{r(m'+1)+1} - r < N_2 \leq F_{r(m'+2)+1} - F_{r(m'+1)} - r,$$

each having property (ii) of Definition 4. By Lemma 3, we see that the maximal representation of every N_1 has $a_{r(m'+1)} = 1$ and $a_{r(m'+1)+1} = 1$. Therefore,

$$N_1^\star = N_1 - F_{r(m'+1)+1}\ , \qquad F_{r(m'+2)-1} - r < N_1^\star \leq F_{r(m'+2)} - F_r - r$$

has the maximal representation as that of N_1 without the last element

$$a_{r(m'+1)+1}\ ,$$

whereas m_i's corresponding to $N_1^\star$ have not changed from those corresponding to N_1. Due to this 1:1 correspondence, the number in $\{N_1\}$ is the same as that in $\{N_1^\star\}$ which is equal to

$$V(r(m' + 1); m_1, m_2, \cdots, m_r) \quad .$$

Using Lemma 3 again, we see that $\{N_2\}$ is in 1:1 correspondence with the set

$$\{N_2^\star\}, \quad F_{r(m'+1)} \leq N_2^\star < F_{r(m'+1)+1} \quad ,$$

such that in the minimal representation of $N_2^\star$, there are exactly m_i a_α's among non-zero a's including the last one, with $\alpha \equiv i - 1 \pmod r$, $i = 1, 2, \cdots, r$. The number in $\{N_2^\star\}$ is then equal to

$$U(r(m' + 1); m_1 - 1, m_2, \cdots, m_r) \quad .$$

Hence,

$$\begin{aligned} &V(r(m' + 1) + 1; \; m_1, m_2, \cdots, m_r) \\ &= V(r(m' + 1); \; m_1, m_2, \cdots, m_r) + U(r(m' + 1); m_1 - 1, m_2, \cdots, m_r) \\ &= U(r(m' + 2) - 1; m_1, m_2, \cdots, m_r) + U(r(m' + 1); m_1 - 1, m_2, \cdots, m_r) \end{aligned}$$

by induction hypothesis,

$$= U(r(m' + 2); \; m_1, m_2, \cdots, m_r)$$

by (6).

The cases for $1 < j \leq r$ can be treated analogously and thus the theorem is proved.

As a concluding remark we say that V's define a partition of the G.F.N., which in view of Theorem 2, is the same as given by U's. An application of partition is discussed elsewhere by the author.

I express my sincere appreciation to the referee and to Professor V. E. Hoggatt, Jr., for their suggestions and comments.

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ERRATA

Please make the following changes in articles by C. W. Trigg, appearing in the December, 1967, Vol. 5, No. 5, issue of the Quarterly:

"Getting Primed for 1967" — p. 472: In the fifth line, replace "2669" with "2699."

"Curiosa in 1967" — pp. 473-476: On p. 473, place a square root sign over the 9 in "73 = ⋯ ."

On p. 474, (C), delete the "!" after the second 7.

On p. 474, (F), delete the "+" inside the parentheses.

On p. 475, (I), the last difference equals "999."

"A Digital Bracelet for 1967" — pp. 477-480: On page 478, line 7, replace the first "sum" with "sums."

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A COMBINATORIAL PROBLEM INVOLVING FIBONACCI NUMBERS

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In Advanced Problem H-70 (this Quarterly, Vol. 3, No. 4, p. 299), C. A. Church, Jr. proposed the following combinatorial result:

"For $n = 2m$, show that the total number of k-combinations of the first n natural numbers such that no two elements i and $i + 2$ appear together in the same selection is F^2_{m+2} and if $n = 2m + 1$, the total is $F_{m+2}F_{m+3}$." (Solution appears in [1].)

The purpose of this note is to consider by a different method a more general combinatorial problem which includes Church's problem as a special case. As in the latter problem, the explicit solution will be seen to be expressible entirely in terms of Fibonacci numbers.

<u>PROBLEM:</u> Given the set S consisting of the first n positive integers and a fixed integer ν satisfying $0 < \nu \leq n$, how many different subsets A of S (including the empty subset) can be formed with the property that $a' - a'' \neq \nu$ for any two elements a', a'' of A (that is, subsets A such that integers i and $i + \nu$ do not both appear in A for any $i = 1, 2, \cdots, n - \nu$)?

Church's problem is then recovered from the above formulation on taking $\nu = 2$.

For the solution of the general problem, we let $n = m\ \ + r$ with m an integer and $0 \leq r \leq \nu$, so that $n = r \pmod{\nu}$. Each subset A of S can be made to correspond to an ordered binary sequence of n terms, $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, by the rule that $\alpha_i = 1$ if $i \in A$ and $\alpha_i = 0$ if $i \notin A$. For a given subset A and its corresponding binary sequence $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, we define ν ordered binary sequences $A_1, A_2, \cdots, A_\nu$ as follows: For $1 \leq j \leq r$,

$$A_j = (\alpha_j, \alpha_{j+\nu}, \alpha_{j+2\nu}, \cdots, \alpha_{j+m\nu})$$

and for $r < j \leq \nu$

$$A_j = (\alpha_j, \alpha_{j+\nu}, \alpha_{j+2\nu}, \cdots, \alpha_{j+(m-1)\nu}) .$$

Note that each of the terms $\alpha_1, \alpha_2, \cdots, \alpha_n$ is included in one and only one of these sequences, since for $j = 1, 2, \cdots, \nu - 1$, the sequence A_j contains all α_i's with $i \equiv j \pmod \nu$ while A_ν contains all α_i's with $i \equiv 0 \pmod \nu$.

Now if the subset A under consideration satisfies the problem constraint, then clearly none of the sequences $\{A_j\}_1^\nu$ can contain two consecutive ones; conversely, if A contains both i and $i + \nu$ for some i_0 satisfying $1 \le i_0 \le n - \nu$, then the sequence A_k, where $k \equiv i_0 \pmod \nu$ will contain two successive ones. Thus the subset A under consideration will satisfy the given constraint if and only if each A_j $(j = 1, 2, \cdots, \nu)$ is a binary sequence without consecutive ones. But it is well known ([2], Problem 1(b), p. 14; [3], pp. 166-167) that the total number of binary sequences of length t without consecutive ones is F_{t+2}. Since each of the r sequences $A_1, A_2, \cdots, A_r$ has length m + 1 and each of the remaining $\nu - r$ sequences $A_{r+1}, \cdots, A_\nu$ has length m, it follows that the total number of subsets of A with the desired property is

$$F^r_{m+3} F^{\nu-r}_{m+2}$$

To obtain Church's result, we take $\nu = 2$ and let $n = 2m + r$ where $r = 0$ or $r = 1$, so that $n \equiv r \pmod 2$. Then the total number of k-combinations of the first n integers such that no elements i and i + 2 appear together is

$$F^0_{m+3} F^2_{m+2} = F^2_{m+2} \quad \text{if } r = 0 \text{ (n even)}$$

and

$$F_{m+3} F_{m+2} \quad \text{if } r = 1 \text{ (n odd)} .$$

Additional references dealing with the case $\nu = 2$ may be found in [1].

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EDITORIAL NOTE: The restraint that $0 \le \nu \le n$ can be removed. Set m = 0, so that number of subsets becomes $F^r_{m+3} F^{\nu-r}_{m+2} = F^r_3 F^{\nu-r}_2 = 2^r$ as is well known for the numbers of subsets of $1, 2, 3, \ldots, n$ without constraints. V.E.H.

* * * * *

CIRCULARLY GENERATED ABELIAN GROUPS

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1. INTRODUCTION

A group will be called n-circularly generated if it has a set of $n(\geq 3)$ generators $x_1, x_2, \cdots, x_n$ such that $x_i x_{i+1} = x_{i+2}$ for all i, where the addition of subscripts is modulo n. This notion was suggested to the author by a problem in the American Mathematical Monthly [1], which can be phrased as follows: Show that a 5-circularly generated group is cyclic of order 11. The problem of determining the structure of circularly generated groups in general appears formidable. They are not all abelian, for the familiar quaternionic group [2, p. 8] clearly has this property for $n = 3$. Furthermore, if we don't insist that the generators all be distinct, any dicyclic group is 6-circularly generated with generators S, T, ST, S^{m-1}, $S^{2-m}T$, and ST, in the notation of [2, p. 7]. However, the structure of circularly generated abelian groups can be completely determined, as will be shown below.

It should be observed that an n-circularly generated group on $x_1, x_2, \cdots, x_n$ is clearly generated by x_1 and x_2, so if it is abelian, it must either be cyclic or the direct sum of exactly two cyclic subgroups. Furthermore, any circularly generated abelian group is the homomorphic image of an abelian group for which the circular relations are defining relations, so we will confine our attention to that case.

Henceforth $(G, +)$ will denote an abelian group with generators $x_1, x_2, \cdots, x_n$ and defining relations*

(1) $$x_i + x_{i+1} = x_{i+2}, \quad i = 1, 2, \cdots, n .$$

where addition of subscripts is modulo n.

Supported in part by NSF grant Number GP-4473.

*G is isomorphic to F/N, where F is the free abelian group on n generators $t_1, t_2, \cdots, t_n$ and N is the subgroup generated by all elements of the form $t_i + t_{i+1} - t_{i+2}$, under the correspondence $x_i \Longleftrightarrow t_i + N$. This means simply that all relations in G are consequences of the given relations (1).

The orders of the cyclic summands of G turn out to be various Fibonacci and Lucas numbers. We denote by F_m (respectively, L_m) the m^{th} Fibonacci (Lucas) number, with the usual initial conditions $F_0 = 0$, $L_0 = 2$, $F_1 = L_1 = 1$. Then the results to be proved below may be summarized as follows:

Theorem 1. If $4|n$, then G is the direct sum of two cyclic subgroups, one of order $F_{n/2}$, the other of order $5F_{n/2}$.

Theorem 2. If $2|n$ and $4\nmid n$, then G is the direct sum of two cyclic subgroups, each of order $L_{n/2}$.

Theorem 3. If $2\nmid n$ and $3|n$, then G is the direct sum of two cyclic subgroups, one of order 2, and the other of order $\frac{1}{2}L_n$.

Theorem 4. If $(n, 6) = 1$, then G is cyclic of order L_n.

Note that the direct sum of cyclic groups of orders k and m is itself cyclic of order km if and only if $(k, m) = 1$. It follows that the only cyclic group included among the first three cases is that for $n = 4$, since $F_2 = 1$ (see (10) below). The first eight cases in which G is cyclic are those for which $n = 4, 5, 7, 11, 13, 17, 19, 23$, and the corresponding orders are 5, 11, 29, 199, 521, 3571, 9349, 64079. These numbers are all prime except the last, which is 139 times 451. Thus, the smallest cyclic group G in our list whose order is composite is the one for $n = 23$.

We also observe that every Fibonacci number with even subscript appears among the cyclic summands in Theorem 1. Given any integer $m > 2$, m divides F_k, where k is the period of the Fibonacci sequence modulo m, and k is even [5, Corollary to Theorem 1 and Theorem 4]. Hence a cyclic group of order m is a homomorphic image of at least one of the groups listed above. For $m = 2$, we can take one of the groups of Theorem 3.

Corollary. Every finite cyclic group is n-circularly generated for some n.

2. SOME FIBONACCI AND LUCAS RELATIONS FOR REFERENCE

$$F_{m-2} + F_{m+2} = 3F_m . \tag{2}$$

$$F_{m+3} - F_{m-3} = 4F_m . \tag{3}$$

$$F_{m+3} - F_{m-1} = L_{m+1} . \tag{4}$$

(5) $$F_m + F_{m+2} = L_{m+1} .$$

(6) $$F_{m+3} - F_{m-2} = F_{m+2} + 2F_{m-1} .$$

(7) $$2F_{m+2} - F_{m-3} = 5F_m .$$

(8) $$3F_{m+3} + F_m = 2F_{m+4} .$$

(9) $$\text{If } 3|m, \text{ then } 2|F_m .$$

(10) $$\text{If } 3|m,\ 2\nmid m, \text{ then } 4|L_m .$$

(11) $$2F_mF_{m-1} + F^2_{m+2} = L_{2m+1} .$$

(12) $$2F_{m+2}F_{m+1} - F^2_{m-1} = L_{2m+1} .$$

Relations (2) — (10) are easy, and for the most part well-known, consequences of the definitions. Relations (11) and (12) may be new; their proofs are left as exercises for the reader.

3. A REDUCTION OF THE PROBLEM BY MATRICES

The defining relations for G may be written in matrix form:

$$Ax^t = 0 ,$$

where $x = (x_1, x_2, \cdots, x_n)$ and

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & -1 & 0 & \cdots & 0 \\ & & & \cdots\cdots & & & \\ 0 & \cdots & 0 & 1 & 1 & & -1 \\ -1 & 0 & \cdots & & 0 & 1 & 1 \\ 1 & -1 & 0 & \cdots & & 0 & 1 \end{bmatrix} .$$

The relation matrix A can be reduced via elementary row and column operations (over the integers) to a form from which one can read off the structure of G as a direct sum of cyclic groups [3, 4]. Rather than apply the standard procedure for this, we make some observations about the matrix A. By adding suitable multiples of each of the first $n - 2$ rows to the last two rows, we can reduce A to a matrix of the form

$$\left[\begin{array}{c|cc} \multicolumn{3}{c}{B} \\ \hline 0 & a & b \\ & c & d \end{array}\right] , \tag{13}$$

where B is the $(n - 2)$ by n matrix consisting of the first $n - 2$ rows of A. In this form, it is clear that G is generated by x_{n-1} and x_n subject to the relations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix} = 0 , \tag{14}$$

and that an expression for each of the other x's in terms of these two can be read off from the matrix (13):

$$x_{n-2} = x_n - x_{n-1}, \quad x_{n-3} = x_{n-1} - x_{n-2} = 2x_{n-1} - x_n ,$$

etc. Thus, it suffices to determine the integers a, b, c, d and the structure of an abelian group with relations (14). Observe that row operations involving the first $n - 4$ rows of A do not affect the last two columns.

Lemma 1. After reducing the first k columns of A to zero below the diagonal $(0 \le k \le n - 4$, the last two rows of A have the form:

$$\begin{array}{cccc} 0 \cdots 0 & (-1)^{k+1}F_{k+1} & (-1)^k F_k & 0 \cdots 0 \;\; 1 \;\; 1 \\ \underbrace{0 \cdots 0}_{k} & (-1)^k F_{k+2} & (-1)^{k+1}F_{k+1} & \underbrace{0 \cdots 0}_{n-k-4} \;\; 0 \;\; 1 \end{array} .$$

The proof is by induction on k. Simple induction proofs of this sort will be omitted.

In particular, after $n - 4$ column reductions, the last four rows and columns of (the new) A have the form:

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ (-1)^{n+1} F_{n-3} & (-1)^n F_{n-4} & 1 & 1 \\ (-1)^n F_{n-2} & (-1)^{n+1} F_{n-3} & 0 & 1 \end{bmatrix} . \tag{15}$$

Lemma 2. After $n - 2$ column reductions, A is reduced to the form (13), where $a = d = 1 + (-1)^{n+1}F_{n-1}$, $b = 1 + (-1)^n F_{n-2}$, and $c = (-1)^n F_n$.

Proof. Use the obvious row operations to reduce the first and second columns of (15) to zero below the diagonal.

For each of the cases in Theorems 1-4, we will use elementary row operations to reduce the matrix of (14) to one of the forms

$$\begin{pmatrix} p & 0 \\ kr & r \end{pmatrix}, \begin{pmatrix} kr & r \\ p & 0 \end{pmatrix}, \tag{16}$$

where p, r, k are integers. Then it is clear that G is the direct sum of the cyclic groups generated by x_{n-1} and $x_n + kx_{n-1}$, and that these have orders $|p|$ and $|r|$, respectively. In particular, G is cyclic when $|r| = 1$.

4. THE STRUCTURE OF G FOR EVEN n

Henceforth we will write each relation involving x_{n-1} and x_n by writing only the two coefficients. Thus, we have reduced the problem to the pair of defining relations (with the order reversed from that given above):

R 1 $\qquad (-1)^n F_n , \qquad 1 + (-1)^{n+1} F_{n-1}$

R 2 $\qquad 1 + (-1)^{n+1} F_{n-1} , \qquad 1 + (-1)^n F_{n-2}$.

For each $k > 2$, define the relation Rk to be the sum of the relations R(k - 1) and R(k - 2). Then one verifies by induction the general form

Rk $\qquad F_{k-1} + (-1)^{n-k-1}F_{n-k+1} , \; F_k + (-1)^{n-k}F_{n-k}$.

Clearly, any two consecutive ones of these relations are defining relations for G.

First, suppose that $n = 4q$, and let $m = 2q$. Then we have the defining relations

$$\begin{array}{ll} R(m-2) & F_{m-3} - F_{m+3}\ ,\ F_{m-2} + F_{m+2} \\ R(m-1) & F_{m-2} + F_{m+2}\ ,\ F_{m-1} + F_{m+1}\ . \end{array}$$

Using (2) and (3), we rewrite these as

$$\begin{array}{ll} R(m-2) & -4F_m\ ,\ 3F_m \\ R(m-1) & 3F_m\ ,\ -F_m\ . \end{array}$$

Add 3 times $R(m-1)$ to $R(m-2)$, and we have the relation matrix

$$\begin{pmatrix} 3F_m & -F_m \\ 5F_m & 0 \end{pmatrix}$$

in the form (16), which completes the proof of Theorem 1.

Now suppose $n = 4q + 2$, and again let $m = 2q$. Referring again to the general form for the relation Rk, we have defining relations

$$\begin{array}{ll} Rm & F_{m-1} - F_{m+3}\ ,\ F_m + F_{m+2} \\ R(m+1) & F_m + F_{m+2}\ ,\ F_{m+1} - F_{m+1}\ . \end{array}$$

Using (4) and (5), we have the relation matrix

$$\begin{pmatrix} -L_{m+1} & L_{m+1} \\ L_{m+1} & 0 \end{pmatrix}$$

in the form (16), which completes the proof of Theorem 2.

5. THE STRUCTURE OF G FOR ODD n

The proofs of Theorems 3 and 4 appear to require separate consideration of six cases, depending on the congruence class of n modulo 12.

Case I. Let $n = 12q + 1$ and $m = 6q$. Referring again to Rk in the previous section, we have the defining relations

$$\begin{array}{ll} R(m-1) & F_{m-2} - F_{m+3}, \quad F_{m-1} + F_{m+2} \\ Rm & F_{m-1} + F_{m+2}, \quad F_m - F_{m+1}. \end{array}$$

Use (3) and (6) to rewrite these as

$$\begin{array}{ll} R(m-1) & -2F_{m-1} - F_{m+2}, \quad 5F_{m-1} + F_{m-4} \\ Rm & F_{m-1} + F_{m+2}, \quad -F_{m-1} \end{array}.$$

We ignore the relations Rk for $k > m$ and define $R(m+1)$ by adding 5 times Rm to $R(m-1)$:

$$R(m+1) \qquad 3F_{m-1} + 4F_{m+2}, \quad F_{m-4}.$$

For $k > 1$, define $R(m+k)$ by adding 4 times $R(m+k-1)$ to $R(m+k-2)$. One obtains by induction (using (3)) the general form:

$$R(m+k) \qquad F_{3k+1}F_{m-1} + \tfrac{1}{2}F_{3k+3}F_{m+2}, \quad (-1)^{k+1}F_{m-3k-1}.$$

In particular, for $k = 2q - 2$ and $2q - 1$, we have the defining relations

$$\begin{array}{ll} R(8q-2) & F_{m-5}F_{m-1} + \tfrac{1}{2}F_{m-3}F_{m+2}, \quad -5 \\ R(8q-1) & F_{m-2}F_{m-1} + \tfrac{1}{2}F_mF_{m+2}, \quad 1 \end{array}$$

Add 5 times $R(8q-1)$ to $R(8q-2)$ to get a matrix of the form (16) with $r = 1$. Hence G is cyclic of order.

$$\begin{aligned} &(F_{m-5} + 5F_{m-2})F_{m-1} + \tfrac{1}{2}(F_{m-3} + 5F_m)F_{m+2} \\ &\quad = 2F_mF_{m-1} + F_{m+2}^2 \\ &\quad = L_{2m+1} \\ &\quad = L_n. \end{aligned}$$

(Formulas (7) and (11) were used here.)

Case II. Let $n = 12q + 5$ and $m = 6q + 2$. This leads to the same equations $R(m - 1)$, Rm, and $R(m + k)$ as in Case I. In particular, for $k = 2q - 1$ and $2q$, we have

$$R(8q + 1) \qquad F_{m-4}F_{m-1} + \tfrac{1}{2}F_{m-2}F_{m+2}\,, \quad 3$$

$$R(8q + 2) \qquad F^2_{m-1} + \tfrac{1}{2}F_{m+1}F_{m+2}\,, \quad -1$$

As in Case I, this leads to a cyclic group whose order (using (8) and (11)) is

$$\begin{aligned}(F_{m-4} &+ 3F_{m-1})F_{m-1} + \tfrac{1}{2}(F_{m-2} + 3F_{m+1})F_{m+2} \\ &= 2F_{m-1}F_m + F^2_{m+2} \\ &= L_{2m+1} \\ &= L_n\,.\end{aligned}$$

Case III. Let $n = 12q - 5$ and $m = 6q - 3$. From the general form Rk we have relations

$$Rm \qquad F_{m-1} - F_{m+2}\,, \quad F_{m+2}$$

$$R(m + 1) \qquad F_{m+2}\,, \quad F_{m-1}\,.$$

For $k > 1$, $R(m + k)$ is defined to be $R(m + k - 2)$ minus four times $R(m + k - 1)$. Using (3) and induction on k, we have

$$R(m + k) \qquad \frac{(-1)^k}{2}F_{3k-3}F_{m-1} + (-1)^{k+1}F_{3k-1}F_{m+2}\,, \quad F_{m-3k+2}\,.$$

In particular, for $k = 2q - 2$ and $2q - 1$, we have

$$R(8q - 5) \qquad \tfrac{1}{2}F_{m-6}F_{m-1} - F_{m-4}F_{m+2}\,, \quad 5$$

$$R(8q - 4) \qquad -\tfrac{1}{2}F_{m-3}F_{m-1} + F_{m-1}F_{m-2}\,, \quad 1\,.$$

Again G is cyclic, and the order L_n is computed as in Case I, using (12) instead of (11).

Case IV. Let $n = 12q - 1$ and $m = 6q - 1$. Then relations Rm, R(m + 1), and R(m + k) are as in Case III. For $k = 2q - 1$ and $2q$ we have

$$R(8q-2) \qquad -\tfrac{1}{2}F_{m-5}F_{m-1} + F_{m-3}F_{m+2}, \quad 3$$

$$R(8q-1) \qquad \tfrac{1}{2}F_{m-2}F_{m-1} + F_mF_{m+2}, \quad 1 .$$

Again G is cyclic of order L_n, using (8) and (12) as in the previous cases. This completes the proof of Theorem 4.

Case V. Let $n = 12q + 3$ and $m = 6q + 1$. The relations Rm, R(m + 1), and R(m + k) are the same as in Case III. For $k = 2q$ and $2q + 1$, we have

$$R(8q-1) \qquad \tfrac{1}{2}F_{m-4}F_{m-1} - F_{m-2}F_{m+2}, \quad 2$$

$$R(8q+2) \qquad -\tfrac{1}{2}F^2_{m-1} + F_{m+1}F_{m+2}, \quad 0 .$$

By (9), the first entry in R(8q + 1) is even, hence we have a matrix of the form (16) and G is the direct sum of two cyclic groups, one of order 2, the other (by (12)) of order $\frac{1}{2}L_{2m+1} = \frac{1}{2}L_n$.

Case VI. Let $n = 12q - 3$ and $m = 6q - 2$. Then we have the same relations as in Case I. For $k = 2q - 2$ and $2q - 1$, we have

$$R(8q-4) \qquad F_{m-3}F_{m-1} + \tfrac{1}{2}F_{m-1}F_{m+2}, \quad -2$$

$$R(8q-3) \qquad F_mF_{m-1} + \tfrac{1}{2}F^2_{m+2}, \quad 0 .$$

As in Case V, this leads to the direct sum of a cyclic group of order 2 and one of order $\frac{1}{2}L_n$, which completes the proof of Theorem 3.

6. A FURTHER CONSEQUENCE

It is easy to verify that the second entries in each of the relations appearing in each reduction process above are, except for sign, the remainders in the Euclidean Algorithm, applied to the two entries of relation R1. Thus the smallest non-zero entry appearing is their greatest common divisor.

Corollary. If n is even, then

$$(F_n, F_{n-1} - 1) = \begin{cases} F_{n/2}, & \text{if } 4 \mid n \\ L_{n/2}, & \text{otherwise} . \end{cases}$$

If n is odd, then

$$(F_n, F_{n-1} + 1) = \begin{cases} 2, & \text{if } 3 \mid n \\ 1, & \text{otherwise} . \end{cases}$$

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CONJUGATE GENERALIZED FIBONACCI SEQUENCES

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1. INTRODUCTION

The famous rabbit problem of Leonardo Fibonacci gives the sequence

(1) $$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ \cdots$$

where each element is the number of rabbits present in that time period. In 1634, Albert Girard discovered the law,

(2) $$U_{n+2} = U_{n+1} + U_n$$

for the sequence (1). Any pair of positive integers U_n and U_{n+1} substituted in (2) generates a sequence; if $U_1 = 1$ and $U_2 = 2$ the Fibonacci Sequence is generated and $U_1 = 1$, $U_2 = 3$ generates the Lucas Sequence.

Consider any pair (a_n, a_{n+1}) used in the generating function (2); if $a_n > a_{n+1}$ then a_n must be the first element of the sequence, because no positive integer added to a_n will make the sum smaller, and the same sequence may be generated by the pair (a_{n+1}, a_{n+2}) thus the restriction $a_n \le a_{n+1}$ with no loss of generality. Continuing with the pair (a_n, a_{n+1}) thus restricted the generating function (2). $U_{n-1} = U_{n+1} - U_n$ shows that as n decreases the elements in the sequence become smaller and thus there must be a least positive element. In the case of the Fibonacci Sequence there are two equal least positive elements both one. Let a_1 be the least positive element for some sequence and a_2 the next element for increasing n then the pair (a_1, a_2) characterize the sequence.

If the pair (a_1, a_2) have a common factor, $(a_1, a_2) = k$, then all elements of the sequence have the same common factor and the sequence may be represented by another relatively prime pair $(a_1', a_2') = 1$ where,

$$a_1' = \frac{a_1}{k}, \qquad a_2' = \frac{a_2}{k} \quad .$$

When the elements of the sequence generated by (a_1', a_2') are multiplied by k the original sequence is recovered. A sequence is a primitive sequence if $(a_1, a_2) = 1$. For any pair where $a_2 > 2$, then $2a_1 < a_2$. If $2a_1 \not< a_2$ then there would be some $a_0 > 0$ such that $a_0 + a_1 = a_2$ where $a_0 < a_1$ and a_1 would not be the least positive element. Thus any sequence generated by (2) may be defined by a positive pair of integers $(a_1, a_2) = 1$ and $2a_1 < a_2$. The exception, of course, is the Fibonacci Sequence.

2. DEFINITION OF CONJUGATES

From the pair (a_1, a_2) it has been shown that a_1 is the least positive element in the sequence that (a_1, a_2) defines, but there is also a number a_0 satisfying (2), $a_0 + a_1 = a_2$. For the Fibonacci Sequence $a_0 = 0$ and $a_0 = 2$ for the Lucas Sequence. If negative elements are allowed then there is an a_{-1} satisfying $a_{-1} + a_0 = a_1$ and $a_{-2}, a_{-3}, a_{-4}'''$ can be found. Thus there are values positive or negative for all a_{-n}.

The absolute values of the elements of the sequence formed by a_{-n} form a sequence which is called the conjugate sequence of the original positive sequence. The element a_0 is the zeroth element of both sequences. If the elements of the conjugate sequence equal the elements of the original sequence (if $a_n = |a_{-n}|$) the sequence is called self-conjugate. The Fibonacci and Lucas Sequences are the only self-conjugate Fibonacci-Type sequences. Given a pair (a_1, a_2) defining a sequence, the pair $(a_1^\star, a_2^\star)$ defining its conjugate sequence may be found by solving the equations,

$$a_1^\star = a_2 - 2a, \quad a_2^\star = 2a_2 - 3a \;, \tag{3}$$

These pairs (a_1, a_2) and $(a_1^\star, a_2^\star)$ are called conjugate pairs. The conjugate sequences and pairs are illustrated in the table below.

Table 1

(a_1, a_2)	$(a_1^\star, a_2^\star)$	U_{-4}	U_{-3}	U_{-2}	U_{-1}	U_0	U_1	U_2	U_3	U_4
(1,1)	(1, 1)	-3	2	-1	1	0	1	1	2	3
(1,3)	(1,3)	7	-4	3	-1	2	1	3	4	7
(1,4)	(2,5)	2	-7	5	-2	3	1	4	5	9
(2,5)	(1,4)	9	-5	4	-1	3	2	5	7	12

3. THE CHARACTERISTIC NUMBER D

With any pair (a_1, a_2) there is a number D determined by the equation

$$D = a_2^2 - a_1(a_1 + a_2) \tag{4}$$

A table of D less than 1000 is given in [1]. For a pair (a_1, a_2) determining a D there is associated another conjugate pair (except (1, 1) and (1, 3) that also determine the same D. These pairs are conjugate pairs as defined above. All prime D greater than 5 have the form $P_i = (10k \pm 1)$ and all composite D are in the form,

$$D = 5^{\alpha_0} P_1^{\alpha_1} P_2^{\alpha_2} P_3^{\alpha_3} \cdots P_n^{\alpha_n} \tag{5}$$

all P_i are prime D of the form $(10\,k \pm 1)$.

There may be more than one set of conjugate pairs that give a D. For D less than 1000, the number of conjugate pairs associated with any D may be found from the factorization of D as follows,

$$D = 5^{\alpha_0} P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_n^{\alpha_n} \tag{6}$$

1. One pair $\begin{cases} \alpha_0 \neq 0, & \text{or} \\ \text{any } \alpha_i = 2 \end{cases}$
2. Two pair, $\alpha_0 = 0$, all $d_i < 2$, and at least 2 distinct P_i.

Any two adjacent elements U_n, U_{n+1} from any primitive sequence substituted in the generating function,

$$U_n \cdot U_{n+1} + (U_n + U_{n+1})^2 \tag{7}$$

will give a number from the set of D_3. From the Fibonacci Sequence

$[1, 1] \longrightarrow 5$, $[1, 2] \longrightarrow 11$, $[2, 3] \longrightarrow 31$, $[3, 5] \longrightarrow 79$.

When a D is determined by (7) a conjugate pair that determine the same D by (4) may be found from the adjacent elements used in (7). Let (F_n, F_{n+1}) be the adjacent elements that give D_n from (7) then a conjugate pair that determine the same D_n from (4) is

$$(F_n,\ F_n + F_{n+2}),\ (F_{n+1},\ F_{n+3}) \tag{8}$$

For example the Fibonacci Sequence 1, 1, 2, 3, 5, 8, 13, ⋯ take the adjacent pair [2, 3] in (7); this gives D = 31. The conjugate pair that give D = 31 from (4) is (2, 7), (3, 8) and from (8) where

$$F_n = 2, \quad F_{n+1} = 3, \quad F_{n+2} = 5, \quad F_{n+8} = 8 \ ,$$

the pair is also (2, 7) (3, 8).

If (7) is used to generate D's by using all adjacent elements in all primitive sequences then all D will be generated and each will appear the number of times equal to the number of conjugate pairs associated with it.

REFERENCE

1. Brother U. Alfred, "On the Ordering of Fibonacci Sequences," The Fibonacci Quarterly, 1 (4), 1963, pp. 43-46.

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ERRATA

Please make the following corrections on the article, "A Primer for the Fibonacci Numbers, Part VI," appearing in the December, 1967, Vol. 5, No. 5, issue of the Fibonacci Quarterly, pp. 445-460:

p. 446: In the fifth line from the bottom, replace "indeterminant" with "indeterminate."

p. 452: In the line before relation (3.5), insert "of" before x^n.

p. 455: In the ninth line from the bottom, replace (3.3) by (3.4).

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Please make the following correction in the Vol. Index, Vol. 5, No. 5, December, 1967, issue of the Fibonacci Quarterly: Change S. D. Mohanty to S. G. Mohanty on p. 495 .

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ADVANCED PROBLEMS AND SOLUTIONS

Edited By

RAYMOND E. WHITNEY
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-126 Proposed by L. Carlitz, Duke University.

Let F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively. Sum the series

$$\sum_{m,n=0}^{\infty} F_m F_n F_{m+n} x^m y^n, \qquad \sum_{m,n=0}^{\infty} F_m F_n L_{m+n} x^m y^n .$$

Sum the series

$$\sum_{\substack{m,n=0 \\ m+n \text{ even}}}^{\infty} F_m F_n x^m y^n, \qquad \sum_{\substack{m,n=0 \\ m+n \text{ even}}}^{\infty} L_m L_n x^m y^n .$$

Sum the series

$$S = \sum_{m,n,p=0}^{\infty} F_{n+p} F_{p+m} F_{m+n} x^m y^n z^p .$$

H-127 Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.

The Fibonacci polynomials are defined by

$$f_{n+1}(x) = x \cdot f_n(x) + f_{n-1}(x) \qquad (n \geq 2)$$
$$f_1(x) = 1 \quad \text{and} \quad f_2(x) = x .$$

If $z_r = f_r(x) \cdot f_r(y)$, then show that

(i) z_r satisfies the recurrence relation

$$z_{n+4} - xy \cdot z_{n+3} - (x^2 + y^2 + 2)\, z_{n+2} - xy \cdot z_{n+1} + z_n = 0 \cdot$$

(ii) $$(x + y)^2 \sum_1^n z_r = (z_{n+2} - z_{n-1}) - (xy - 1)(z_{n+1} - z_n) .$$

H-128 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let F_n and L_n denote the Fibonacci and Lucas numbers, respectively. Show that

$$F_n \equiv 2^{2n+3} - 2^{3n+3} \pmod{11} ,$$
$$L_n \equiv 2^{2n} + 2^{3n} \pmod{11} .$$

Generalize.

H-129 Proposed by Stanley Rabinowitz, Far Rockaway, New York.

Define the Fibonacci polynomials by $f_1(x) = 1$,

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_{n+2}(x) = xf_{n+1}(x) + f_n(x), \quad n > 0 .$$

Solve the equation

$$(x^2 + 4)f_n^2(x) = 4k(-1)^{n-1}$$

in terms of radicals, where k is a constant.

SOLUTIONS

GREATEST POWER OF TWO IN N

H-81 Proposed by Vassili Daiev, Sea Cliff, New York.

Find the n^{th} term of the sequence

1, 1, 3, 1, 5, 3, 7, 1, 9, 5, 11, 3, 13, 7, 15, 1, 17, 9, 19, 5, ···

Solution by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pa.

Let u_n denote the n^{th} term of the sequence for $n \geq 1$. Then for $n \geq 1$, each integer n has a unique representation in the form

$$n = 2^{k(n)} \cdot r(n) \quad ,$$

where $k(n)$ is a non-negative integer and $r(n)$ is an odd integer ≥ 1. The given sequence is formed by the rule $u_n = r(n)$.

Also solved by Thomas Dence, L. Carlitz, and C. B. A. Peck.

LEHMER'S FAMOUS PROBLEM GENERALIZED

H-82 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

If $f_0(x) \equiv 0$ and $f_1(x) \equiv 1$, $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$, then show

$$\tan^{-1}\frac{1}{x} = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{x}{f_{2n+1}(x)}\right)$$

Solution by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.

Let $\tan\theta_n = 1/f_n(x)$. Then

$$\tan(\theta_{2n} - \theta_{2n+2}) = \frac{f_{2n+2}(x) - f_{2n}(x)}{1 + f_{2n}(x)f_{2n+2}(x)} = \frac{xf_{2n+1}(x)}{1 + f_{2n}(x)f_{2n+2}(x)}$$

It may be easily established by induction that,

$$f_{n-1}(x)\, f_{n+1}(x) - f_n^2(x) = (-1)^n$$

Hence,

$$\tan(\theta_{2n} - \theta_{2n+2}) = \frac{xf_{2n+1}(x)}{f_{2n+1}^2(x)} = \frac{x}{f_{2n+1}(x)}$$

Or,

$$\tan^{-1}[1/f_{2n}(x)] - \tan^{-1}[1/f_{2n+2}(x)] = \tan^{-1}[x/f_{2n+1}(x)]$$

Hence,

$$\sum_1^m \tan^{-1}\left\{\frac{x}{f_{2m+1}(x)}\right\} = \tan^{-1}\left\{\frac{1}{f_2(x)}\right\} - \tan^{-1}\left\{\frac{x}{f_{2m+2}(x)}\right\}$$

Now $f_2(x) = x$. Also, as $m \to \infty$, $\tan^{-1}(1/f_{2m+2}(x)) \to 0$. Hence,

$$\sum_1^\infty \tan^{-1}\frac{x}{f_{2n+1}(x)} = \tan^{-1}\frac{1}{x}$$

Note: Since $f_n(1) = F_n$, the n^{th} Fibonacci number, we get the interesting result that,

$$\tan^{-1} 1 = \frac{\pi}{4} = \sum_{1}^{\infty} \tan^{-1} \frac{1}{F_{2n+1}}$$

This is Lehmer's famous result.

Also solved by Joseph D. E. Konhauser.

ANOTHER CUTIE

H-83 Proposed by Mrs. William Squire, Morgantown, West Virginia.

Show

$$\sum_{t=1}^{\left[\frac{m+1}{2}\right]} (-1)^{t-1} \binom{m-t}{t-1} 3^{m+1-2t} = F_{2m} ,$$

where $[x]$ is the greatest integer function.

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

We know that the Chebyshev polynomial $S_n(x)$ is given by

(1) $$S_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} x^{n-2j}$$

Also, $S_n(x)$ satisfies the difference equation

$$S_n(x) = xS_{n-1}(x) - S_{n-2}(x) ,$$

with

$$S_0 = 1, \quad S_1 = x .$$

Hence,

$$S_n(x) = \frac{1}{\sqrt{x^2 - 4}}\left[\left\{\frac{x + \sqrt{x^2 - 4}}{2}\right\}^{n+1} - \left\{\frac{x - \sqrt{x^2 - 4}}{2}\right\}^{n+1}\right]$$

Or,

$$S_n(3) = \frac{1}{\sqrt{5}}\left\{\left(\frac{3 + \sqrt{5}}{2}\right)^{n+1} - \left(\frac{3 - \sqrt{5}}{2}\right)^{n+1}\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1 + \sqrt{5}}{2}\right)^{2n+2} - \left(\frac{1 - \sqrt{5}}{2}\right)^{2n+2}\right\} = F_{2n+2}$$

Hence, $F_{2n} = S_{n-1}(3)$. Therefore, from (1) we get

$$F_{2m} = \sum_{j=0}^{[(m-1)/2]} (-1)^j \binom{m - j - 1}{j} 3^{m-1-2j}$$

Changing j to $(t - 1)$ we have,

$$\sum_{t=1}^{[(m+1)/2]} (-1)^{t-1} \binom{m - t}{t - 1} 3^{m+1-2t} = F_{2m}$$

ALPHA AND BETA, AGAIN!

H-85 Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.

Let

$$D_n = f_n x^n - [f_n x^n],$$

where

$$f_{n+1} = f_n + f_{n-1}$$

with

$$f_0 = f_1 = 1, \qquad x = (1 + \sqrt{5})/2,$$

and $[z]$ = greatest integer $\le z$ (so that $z - [z]$ = fractional part of z). Prove (or disprove) the existence of the limits

$$\lim_{n \to \infty} D_{2n} = 0.27 \cdots = A$$

and

$$\lim_{n \to \infty} D_{2n+1} = 0.72 \cdots = B \text{ with } A + B = 1.$$

Generalize to case of

$$\mu_{n+1} = p\mu_n + q\mu_{n-1},$$

where p and q are real and μ_0 and μ_1 are given.

Solution by L. Carlitz, Duke University.

Put

$$x = \frac{1}{2}(1 + \sqrt{5}), \quad y = \frac{1}{2}(1 - \sqrt{5}),$$

so that

$$f_n = \frac{x^{n+1} - y^{n+1}}{x - y}, \quad D_n = \frac{x^{2n+1} - x^n y^{n+1}}{x - y} - \left[\frac{x^{2n+1} - x^n y^{n+1}}{x - y}\right].$$

Then

$$D_{2n} = \frac{x^{4n+1} - y}{x - y} - \left[\frac{x^{4n+1} - y}{x - y}\right],$$

$$\frac{x^{4n+1} - y}{x - y} = F_{4n} + \frac{y^{4n+1} - y}{x - y} = F_{4n} + \frac{y^2 - y^{4n+2}}{y^2 + 1}$$

Since $-1 < y < 0$ it follows that

$$0 < \frac{y^2 - y^{4n+2}}{y^2 - 1} < 1 \qquad (n > 0)\ .$$

Thus

$$\left[\frac{x^{4n+1} - y}{x - y}\right] = F_{4n}\ , \quad D_{2n} = \frac{y^2 - y^{4n+2}}{y^2 + 1}\ .$$

Therefore

$$\lim_{n=\infty} D_{2n} = \frac{y^2}{y^2 + 1} = \frac{1}{x^2 + 1} = \frac{2}{5 + \sqrt{5}} = \frac{5 - \sqrt{5}}{10} = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5} - 1}{2}\right)$$

Similarly

$$D_{2n+1} = \frac{x^{4n+3} + y}{x - y} - \left[\frac{x^{4n+3} + y}{x - y}\right]\ ,$$

$$\frac{x^{4n+3} + y}{x - y} = F_{4n+2} + \frac{y^{4n+3} + y}{x - y} = F_{4n+2} - \frac{y^2 + y^{4n+4}}{y^2 + 1}\ .$$

Since

$$0 < \frac{y^2 + y^{4n+4}}{y^2 + 1} < 1\ ,$$

we have

$$\left[\frac{x^{4n+3} + y}{x - y}\right] = F_{4n+2}\ , \quad D_{2n+1} = 1 - \frac{y^2 + y^{4n+4}}{y^2 + 1}\ ,$$

$$\lim_{n=\infty} D_{2n+1} = 1 - \frac{y^2}{y^2 + 1} = \frac{x^2}{x^2 + 1} = \frac{5 + \sqrt{5}}{10} = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)\ .$$

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RECREATION CORNER
SOLUTION
POPULATION EXPLOSION

Brother Alfred Brousseau
St. Mary's College, Calif.

The problem began with 28 people. Starting with the 28th and working backward taking every third person, it was found that the final two places were the 18th and the 8th. Then for 29, when one takes one step to 26, one is back to the case of 28, so that the answers are 16 and 6. Going to 30, the answers regress back to 14 and 4. This continues until one arrives at 31 with answers 12 and 2. At the next step 2 is replaced by a larger number at the other end of the scale. Starting with 32, one obtains 10 and 31. Again one can continue to 36 with solutions 2 and 23. The next step gives 37 with solutions 36 and 21. The mode of procedure should be evident. The steps are summarized in the following table.

N	P_1	P_2	N	P_1	P_2
28	18	8	618	616	364
32	10	31	800	252	799
37	36	21	926	925	547
48	14	46	1200	377	1198
55	54	32	1389	1387	820
71	22	70	1799	567	1798
82	81	48	2083	2081	1230
106	33	105	2698	852	2697
123	121	71	3124	3122	1845
159	49	157	4047	1276	4045
184	182	107	4685	4684	2769
238	74	236	6070	1914	6068
275	274	162	7027	7026	4154
356	112	355	9104	2872	9103
412	411	243	10540	10539	6231
534	167	532	13656	4307	13654

N	P_1	P_2
15810	15808	9346
20483	6462	20482
23714	23713	14020
30724	9693	30723
35571	35569	21029
46086	14539	46084
53356	53354	31544
69128	21810	69127
80033	80032	47317
103692	32714	103690
120049	120048	70976
155537	49072	15536
180073	180072	106464
233305	73608	233304
270109	270108	159696
349957	110412	349956
405163	405162	239544
524935	165618	524934
607744	607743	359316
787402	248427	787401
911616	911614	538973
1,000,000	734846	362205

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ERRATA

Please make the following corrections to the article "The Bracket Function, Q-Binomial Coefficients, and Some New Stirling Number Formulas," by H. W. Gould, appearing in the December, 1967, Vol. 5, No. 5, issue of the <u>Fibonacci Quarterly</u>, pp. 401-423:

p. 410: In the line after (3), replace "left" by "led."

p. 411: In relation (33), replace $\binom{n}{s}$ by $\begin{bmatrix} n \\ s \end{bmatrix}$.

p. 415: In relation (50), after sigma, replace $G(x, j, q)$ by $F(x, j, q)$.

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RECREATIONAL MATHEMATICS

Joseph S. Madachy *
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This column, hopefully, will serve the need for mathematical relaxation and make the reader look again at the other articles in the Fibonacci Quarterly with a mind more receptive to the fascination of mathematics. Actually, readers of this Journal are already inclined this way since this Journal is devoted to the study of one of the most fascinating series of numbers ever discovered.

Numbers, Fibonacci or otherwise, will not always be touched upon — mathematics, after all, is more than that. I look forward to comments and contributions from readers.

DIGITAL DIVERSIONS

Express the Fibonacci numbers using the ten digits once only and in order and only the common mathematical operations and symbols. Try to avoid expressions included in brackets indicating the nearest whole integer. You should be able to extend the list below. It would be interesting to determine the largest possible Fibonacci number so expressible, or to see in how many different ways a given number can be expressed.

$$F_1 = F_2 = 1 = 0 - 1 + 2 - 3 + 4 - 5 - 6 - 7 + 8 + 9$$
$$F_3 = 2 = 0 + (1)(2) - 3 + 4 - 5 - 6 - 7 + 8 + 9$$
$$F_4 = 3 = 0 - 1 + 2 - 3 - 4 + 5 - 6 - 7 + 8 + 9$$
$$F_5 = 5 = 0 + 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9$$
$$F_6 = 8 = 0 + (1)(2) + 3 + 4 - 5 - 6 - 7 + 8 + 9$$
$$F_7 = 13 = 0 + 1 + 2 + 3 - 4 - 5 + 6 - 7 + 8 + 9$$
$$F_8 = 21 = 0 - 1 + 2 + 3 + 4 - 5 - 6 + 7 + 8 + 9$$
$$F_9 = 34 = 0 + (1)(2) + 3 + 4 - 5 + 6 + 7 + 8 + 9$$

* Editor of The Journal of Recreational Mathematics: co-author, with J. A. H. Hunter, of Mathematical Diversions: author of Mathematics on Vacation: former owner-publisher-editor of the defunct Recreational Mathematics Magazine.

$F_{10} = 55 = 0 + 12 + 34 + 5 - 6 - 7 + 8 + 9$

$F_{11} = 89 = 0 + 1 + 2 + 34 + 56 + 7 - 8 - \sqrt{9}$

$F_{12} = 144 = 0 + 1 + 2 + 3 + 4 + 5! + 6 + 7 - 8 + 9$

A DUDENEY PROBLEM

Henry Ernest Dudeney (1857-1930), one of England's foremost puzzlists once posed the following problem: "It will be found that 32, 547, 891 multiplied by 6 (thus using all the nine digits once, and once only) gives the product 195, 287, 346 (also containing all the nine digits once, and once only). Can you find another number to be multiplied by 6 under the same conditions? Remember that the nine digits must appear once, and once only, in the numbers multiplied and in the product."

Dudeney, in [1], included this problem, with the answer (6)(94, 857, 312) = 569, 143, 872. Martin Gardner, in editing this book, added two solutions supplied by Victor Meally: (6)(89, 745, 321) = 538,471,926 and (6)(98, 745, 231) = 592, 471, 386. With the help of a table constructed in 1963 by Harry L. Nelson of Livermore, California, I found that there are actually 87 solutions to this problem. These are listed in Table 1.

An obvious variation on Dudeney's problem is to ask the same question, but include zero as the tenth digit. There are 174 10-digit solutions derivable from Table 1 by simply appending a zero to one of the factors and the product. Examination of the table discloses many additional 10-digit solutions with the zero not at a terminal position. For example, the first three listed each yield two additional 10-digit solutions:

(6)(201, 578, 943) = 1, 209, 473, 658
(6)(215, 078, 943) = 1, 290, 473, 658
(6)(230, 158, 794) = 1, 380, 952, 764
(6)(231, 508, 794) = 1, 389, 052, 764
(6)(245, 098, 731) = 1, 470, 592, 386
(6)(245, 987, 301) = 1, 475, 923, 806

I leave it to the reader to find the other 10-digit solutions derivable from the table. However, I feel sure there may be other 10-digit solutions to the

Table 1

Solutions to Dudeney's Nine-Digit Problem

6 x 21578943 = 129473658	6 x 42985731 = 257914386	6 x 73195428 = 439172568
6 x 23158794 = 138952764	6 x 43152789 = 258916734	6 x 78195423 = 469172538
6 x 24598731 = 147592386	6 x 43195728 = 259174368	6 x 78219543 = 469317258
6 x 24958731 = 149752386	6 x 43219578 = 259317468	6 x 78549231 = 471295386
6 x 27548913 = 165293478	6 x 43271958 = 259631748	6 x 78942153 = 473652918
6 x 27891543 = 167349258	6 x 45719283 = 274315698	6 x 78943152 = 473658912
6 x 27893154 = 167358924	6 x 45719328 = 274315968	6 x 79854231 = 479125386
6 x 28731594 = 172389564	6 x 45728193 = 274369158	6 x 81954273 = 491725638
6 x 28943157 = 173658942	6 x 45731928 = 274391568	6 x 82719543 = 496317258
6 x 29415873 = 176495238	6 x 45781923 = 274691538	6 x 85473291 = 512839746
6 x 31275489 = 187652934	6 x 45782193 = 274693158	6 x 85491273 = 512947638
6 x 31542789 = 189256734	6 x 45819273 = 274915638	6 x 87249531 = 523497186
6 x 31578942 = 189473652	6 x 45827193 = 274963158	6 x 87294153 = 523764918
6 x 31587294 = 189523764	6 x 47328591 = 283971546	6 x 87315294 = 523891764
6 x 32458971 = 194753826	6 x 47532891 = 285197346	6 x 87495231 = 524971386
6 x 32547891 = 195287346	6 x 48572931 = 291437586	6 x 87941523 = 527649138
6 x 32714589 = 196287534	6 x 48579231 = 291475386	6 x 89145327 = 534871962
6 x 32897541 = 197385246	6 x 48591273 = 291547638	6 x 89532471 = 537194826
6 x 41527893 = 249167358	6 x 48912753 = 293476518	6 x 89532714 = 537196284
6 x 41957283 = 251743698	6 x 49285731 = 295714386	6 x 89745321 = 538471926
6 x 41957328 = 251743968	6 x 52487931 = 314927586	6 x 94152873 = 564917238
6 x 41957823 = 251746938	6 x 52874931 = 317249586	6 x 94857123 = 569142738
6 x 41958273 = 251749638	6 x 52987431 = 317924586	6 x 94857213 = 569143278
6 x 42195783 = 253174698	6 x 71528943 = 429173658	6 x 94857312 = 569143872
6 x 42319578 = 253917468	6 x 71954283 = 431725698	6 x 95248731 = 571492386
6 x 42719583 = 256317498	6 x 71954328 = 431725968	6 x 97328541 = 583971246
6 x 42731958 = 256391748	6 x 72819543 = 436917258	6 x 98541273 = 591247638
6 x 42789153 = 256734918	6 x 72854931 = 437129586	6 x 98724531 = 592347186
6 x 42819573 = 256917438	6 x 72985431 = 437912586	6 x 98745231 = 592471386

problem that are not derivable from the table. At this point, I can only hope that someone will use a computer to find all the 10-digit solutions to this particular problem.

The Nelson table mentioned previously was constructed in answer to a query I had made concerning the solution to the problem: What two or more factors containing the nine (or ten) digits once only yield a product containing the nine (or ten) digits once only? Nelson's computer-calculated table listed all 2,624 solutions to the 9-digit case (zero excluded). (See [4] for a discussion of this table.) The work involved in finding all the 10-digit solutions was not done. Included in Nelson's table are all the solutions to two other variations on Dudeney's problem. Substitute 3 or 9 in place of 6 as a factor. There are 335 solutions to the $(3)(A) = B$ variation and 144 solutions to the $(9)(C) = B$ variation, where A contains eight distinct digits (excluding zero and 3), B contains nine distinct digits (zero excluded,) and C contains eight distinct digits (zero and 9 excluded). Interested readers may obtain one free copy of these $(3)(A) = B$ and $(9)(C) = B$ tables simply by requesting them. Please, only one copy. If you want more, include at least five cents postage for every two copies.

ANOTHER DUDENEY PROBLEM

Dudeney once asked what numbers have cube roots equal to the sum of their digits. Excluding the trivial $1^3 = 1$, Dudeney [2] gave the five solutions: 512, 4913, 5832, 17576, and 19683. That is, $512 = (5 - 1 + 2)^3 = 8^3$; $(4 + 9 + 1 + 3)^3 = 17^3$; and so on.

Some years ago, I asked T. Charles Jones, then a student at Davidson College in Davidson, North Carolina, to run a computer search for solutions to this problem for n^{th} roots to $n = 101$. (The requests I sometimes put to people are not often trivial!) Elsewhere [4] I've shown how one might systematically search for these rather interesting numbers. These numbers — which, by the way, lack a precise name* — can be written as

$$N = abcd\cdots = (a + b + c + d + \cdots)^n = P^n .$$

*In [4] numbers which are representable, in some way, by mathematically manipulating their digits are called narcissistic. Closely related to the above numbers are those which are equal to the sum of the n^{th} powers of their digits; e.g., $153 = 1^3 + 5^3 + 3^3$. Such numbers are called Perfect Digital Invariants (PDI's) by Max Rumney of England, who has studied them extensively [5].

where abcd... represents the digits of N, and $(a + b + c + d + \cdots)$ represents the sum of the digits of N. Table 2 lists the 432 values of P which, when related to the n^{th} power, yield an N, the sum of whose digits is equal to P.

One of the interesting aspects of this problem is that there is at least one representative for every n from $n = 2$ to $n = 101$, with a maximum number (13) of representations at $n = 25$. Trivial representations such as $1^n = 1$ are not listed. The greatest number of times that a given P occurs is five: $P^n = N$ for $P = 90$ and $n = 19, 20, 21, 22$, and 28. The fully printed-out numbers total 19 computer sheets, but readers might be interested in seeing several of the larger examples.

P^n	=	N (Sum of the digits in N is equal to P)							
181^{16}	=	13	26958	06363	75768	00539	94757	97274	10881
187^{16}	=	22	35968	62152	63449	25885	78257	92399	57441
499^{43}	=	10	43094	03484	75692	24451	60376	10004	44524
			27960	69557	10166	43340	61295	76132	73343
			99292	16069	53092	75509	14486	32354	72591
			73992	71499					
999^{75}	=	92770	86733	90001	46643	21616	99937	58761	27716
		93772	92872	78273	34425	52852	00275	13591	27714
		15647	08297	24430	57342	37029	14944	28952	64407
		21199	26192	76548	53218	72362	23108	52440	33783
		01874	09642	00691	32958	96038	80592	97398	10590
		35077	08174	61752	22250	74999			

The largest known number of this type is 1468^{101} which contains 320 digits — whose sum is 1468.

I found, quite by accident, one example of $P^n = N$ where the sum of the digits in N is equal to n:

$$2^{70} = 1,180,591,620,717,411,303,424.$$

Are there any more of this type?

Table 2

$N = P^n$, Where the Sum of the Digits in N equal P

n	P	n	P
2	9	27	305, 307
3	8, 17, 18, 26, 27	28	90, 160, 265, 292, 301, 328
4	7, 22, 25, 28, 36	29	305, 314, 325, 332, 341
5	28, 35, 36, 46	30	396
6	18, 45, 54, 64	31	170, 331, 338, 346, 356, 364, 367, 386, 387, 443
7	18, 27, 31, 34, 43, 53, 58, 68	32	388
8	46, 54, 63	33	170, 352, 359, 378, 406, 422, 423
9	54, 71, 81	34	387, 412, 463
10	82, 85, 94, 97, 106	35	378, 388, 414, 451, 477
11	98, 107, 108, 117	36	388, 424
12	108	37	414, 421, 422, 433, 469, 477, 485, 495
13	20, 40, 86, 103, 104, 106, 107, 126, 134, 135, 146	38	468, 469
14	91, 118, 127, 135, 154	39	449, 523
15	107, 134, 136, 152, 154, 172, 199	40	250, 441, 468, 486, 495, 502
16	133, 142, 163, 169, 181, 187	41	432
17	80, 143, 171, 216	42	280, 487, 523, 531
18	172, 181	43	461, 499, 508, 511, 526, 532, 542, 548, 572
19	80, 90, 155, 157, 171, 173, 181, 189, 207	44	280, 523, 549, 576, 603
20	90, 181, 207	45	360, 503, 523
21	90, 199, 225	46	360, 478, 514, 522, 544, 558, 574, 592
22	90, 169, 193, 217, 225, 234, 256	47	350, 559, 567, 575, 595, 603, 666
23	234, 244, 271	48	370, 513, 631, 667
24	252, 262, 288	49	270, 290, 340, 350, 360, 533, 589, 637, 648, 661, 695
25	140, 211, 221, 236, 256, 257, 261, 277, 295, 296, 298, 299, 337		
26	306, 307, 316, 324		

Table 2

(Continued from P. 65)

n	P
50	685
51	360, 666, 685
52	625, 688, 736, 739
53	648, 683, 703, 746
54	370, 603, 657, 667, 739
55	677, 683
56	684
57	370, 460, 719, 748, 793, 802
58	667, 721, 754
59	370, 440, 693, 845
60	694, 784, 792, 793
61	440, 490, 758, 815, 833
62	855, 865
63	793, 827, 836, 846
64	430, 829, 871
65	818, 856, 891, 928
66	837, 864, 927
67	450, 859, 865, 866, 869, 874 926, 934
68	837
69	540, 936, 962, 963, 1016
70	540, 882, 909
71	917, 991
72	901, 1062
73	853, 882, 928, 1006, 1015
74	936, 1008, 1009, 1018
75	630, 964, 999, 1016, 1053
76	1044, 1075, 1093
77	1061, 1062, 1088
78	964, 1117, 1126, 1134
79	610, 1031, 1043, 1054, 1064, 1091, 1108, 1133
80	1044, 1071, 1134, 1144
81	1062, 1196
82	1048, 1111, 1134, 1231
83	730, 1115, 1151, 1207
84	1188
85	1051, 1103, 1165, 1183, 1277
86	1134, 1225
87	1187, 1216, 1224, 1232, 1278, 1288
88	730, 1084, 1147, 1183, 1186, 1206
89	1151, 1232, 1358
90	1306, 1422
91	720, 1208, 1233, 1253, 1261, 1278
92	720, 1296, 1359
93	810, 820, 1396
94	1285, 1287, 1303, 1327, 1332, 1339, 1341, 1444
95	820, 1323, 1342, 1351, 1385
96	1387
97	1237, 1322, 1324, 1361, 1367, 1397, 1442
98	1359
99	1322, 1403, 1405, 1441
100	1363, 1378, 1408, 1414, 1489
101	1423, 1468.

A FIBONACCI VARIATION

Everyone tries his hand at variations on the Fibonacci theme. Mark Feinberg [3] has given us the Tribonacci and Tetranacci numbers, for example, where the terms of the series are the sums of the previous three or four terms, respectively. I hate to be excluded, so here's mine. The results turned out to be interesting, if not exactly stupendous. Form the ${}_nF$ series in each term is the sum of the NEXT TWO terms, and which starts with ${}_0F = 0$ and ${}_1F = 1$. The series, then, is

$$0,\ 1,\ -1,\ 2,\ -3,\ 5,\ -8,\ 13,\ -21,\ 34,\ -55,\ \text{etc.}$$

Note that if n is zero or even, the ${}_nF = -F_n$; if n is odd, then ${}_nF = F_n$. Can anyone do anything with this series?

SOME FIBONACCI QUERIES

What Fibonacci numbers are integral multiples of the sums of their digits? For example,

$$F_8 = 21,\quad 2 + 1 = 3,\ \text{ and } (3)(7) = 21;\ F_{12} = 144,\ 1 + 4 + 4 = 9,\ \text{ and } (9)(16) = 144;\ F_{18} = 2584,\ 2 + 5 + 8 = 19,\ \text{ and } (19)(136) = 2584.$$

I'm sure there are more. Are there an infinite number of them? Are they a function of n?

Somewhat related to the above is the problem of finding $F_n = N$, such that $N = nk$, where k is a positive integer. For example,

$$F_1 = 1;\ F_5 = 5;\ F_{12} = 144\ (\text{here } k = 12);\ F_{25} = 75025\ (\text{here } k = 3001)$$

Is there a formula relating these?

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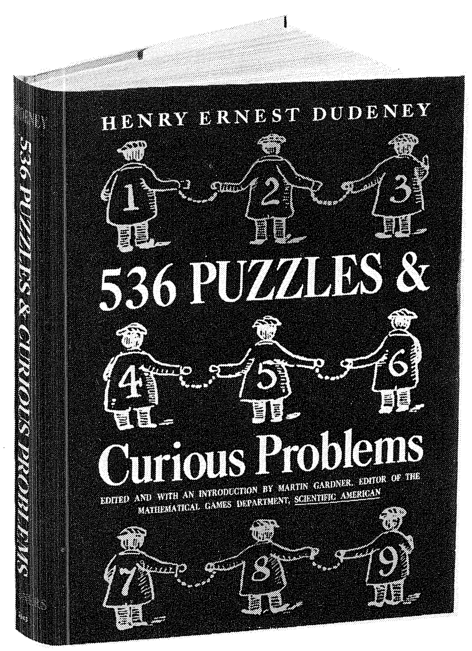

ON THE TRAIL OF THE CALIFORNIA PINE

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The statements in books about pine cones, sunflowers, pineapples and other natural objects manifesting Fibonacci numbers in their structure are interesting, but they are bloodless and academic when compared to the actual experience of seeking out and examining these things. The present account concerns a series of events which began in the summer of 1966 when the author started collecting cones of the California pines with a view to observing and verifying Fibonacci relations.

Being located at Huntington Lake in the Sierra Nevada with a number of pine species in the near vicinity, there was a good opportunity to begin the work of collection. With pine trees of several species all about and with cones in profusion lying on the ground, there seemed to be no problem. However, all these fallen cones were open and here was the first big surprise: it is difficult, if not impossible, in many cases to follow the spirals on open cones. There is a notable element of uncertainty involved which could lead one to using his imagination rather than examining the spirals objectively!

It seemed that the only thing to do was to go about looking for fresh closed cones. This was not too difficult for the lodgepole pine (Pinus Murrayana) but provided a problem with the yellow (Pinus ponderosa) and Jeffrey (Pinus Jeff-eryi) pines. These are large trees and in our area the cones few in number were very high. However, it was possible to collect a fresh cone of each of these species as well as samples of immature closed cones that were found on the ground.

Then followed a hike in the Kaweah Mountains during which many fine specimens of what was taken to be the limber pine (Pinus flexilis) were observed. However, only one fresh cone was collected, though a number of good open specimens were gathered.

On returning to base camp, it was found that the rodents had taken advantage of my absence to upset the boxes in which the fresh cones were stored and of course they had one or more sumptuous meals at my expense. This meant starting all over again with greater security measures against these predators.

A large cardboard box with a substantial board and heavy rock on top of it seemed to provide ample protection.

A special trip was made to the vicinity of Cow Creek (near Mt. Tom) to collect sugar pine (Pinus Lambertiana) cones. There were many immature fallen cones on the ground and these were procured in quantity. Such cones apparently do not come open subsequently, so that with their fine symmetry and large size they furnish an ideal means of bringing out Fibonacci relations.

The white bark pine (Pinus albicaulis) grows at an altitude of from 8,000 to 12,000 ft. Specimens were found on Kaiser Ridge. However, another hazard for the pine cone collector was highlighted at this point. In this area there were not too many trees with cones. But more disappointing, those that had them in quantity had been visited by the Clark nutcracker (or Clark crow) which pecks away at the cones on the trees and gets at the nuts inside. Thus it was not possible to find one good cone. Furthermore, those cones that are on the ground deteriorate so that even here, the one or two open cones collected were very fragile.

At the end of the season in the mountains, therefore, the following species had been obtained either in the form of fresh cones or open cones; lodgepole pine, yellow pine, Jeffrey pine, silver pine (Pinus monticola), what was believed to be limber pine, sugar pine, and white bark pine. On the way down to the valley, there were many digger pines (Pinus Sabiniana) and some fine open specimens were gathered from the ground.

Thus eight species in all had been obtained with the following still awaiting collection:

One-needled pinon (Pinus monophylla)
Pinus edulis (edible pinon pine)
Four-needled pinon (Pinus quadrifolia)
Bristlecone pine (Pinus aristata)
Foxtail pine (Pinus Balfouriana)
Bishop pine (Pinus muricata)
Santa Cruz Island pine (Pinus remorata)
Beach pine (Pinus contorta)
Torrey pine (Pinus Torreyana)
Monterey pine (Pinus radiata)
Knobcone pine (Pinus attenuata)
Coulter pine (Pinus Coulteri)

The work of collecting was continued at St. Mary's College in September, 1966, this institution being located in Contra Costa County about twenty-five miles from San Francisco. The Monterey pine was found on the property. The knobcone pine was located on what is known as Moraga Ridge in the vicinity of a small settlement named Canyon. In the Mt. Diablo area, a fresh digger pine cone was found as well as a green Coulter pine cone. This latter pine is not too easy to identify especially in the Mt. Diablo area as the tree grows there along with digger pines from which it is difficult for the amateur to distinguish it. One certain means of identification is the longer wing on the seed of the Coulter as compared to the digger. The Coulter pine cone is a magnificent large specimen ranging in length from eight to twelve inches.

A special trip was made to Tioga Pass to collect the one-needled pinion pine cone. Not knowing exactly where these trees were to be found, it was necessary to grope around. However, about eight miles beyond the pass on the East side, a prominent group of bushy looking trees was observed on the left side of the road. These were indeed one-needled pinions. After scrambling up steep cliffs to gether specimens, it was found that the Clark nutcracker had again been doing his work. There were quite a few cones on the ground but these dry cones are badly out of shape though not without their artistic aspect. They would render tracing spirals impossible. A number of intact fresh green cones was obtained as well as samples of the needles. These latter are most interesting. As the name implies, the needles come individually and are not in groups of 2, 3, or 5 as with most pines. They are round and sharply pointed!

Bishop pine cones had been found earlier in the spring on the Point Reyes Peninsula. A special trip was made up the coast to Sonoma, Mendocino and Humboldt Counties to look for the beach pine. A friend indicated that trees were located just above Fort Ross. These were found and cones collected. However, on returning home, comparison seemed to indicate that what was in hand was the bishop pine! It was only later on another occasion that the beach pine was found. This tree is similar in some respects (especially in its cone) to the lodgepole pine. A couple of specimens were located along the road going up the coast, but since according to the guide book there were specimens in the vicinity of Albion, this area was explored. In the neighborhood of this settlement there seemed to be none of these trees. However, a road leading inland was followed for a few miles. This led to a plateau area with strange looking soil and here an abundance of these beach pines was found. They appeared to be somewhat stunted in their growth due no doubt to the poor nature of the soil in which they grow. There is also a dwarf variety of this species (Bolanderi) one example of which was apparently found. A forest of these dwarf trees is located according to information just south of Fort Bragg.

A trip to San Diego provided an opportunity to study the Torrey pine. This is a species endemic to California growing naturally only at Del Mar (north of San Diego) and on Santa Rosa Island (one of the Channel Islands). There are only about 3,000 trees of this species in existence. As a result, they are protected to a large extent in Torrey Pines State Park. It was possible to find some trees outside the limits of this park and collect one or two of these very interesting and distinctive cones.

On this same trip an excursion was made to the East with the hope of discovering some pinon pines, but these were not in evidence. As a compensation, however, some interesting Fibonacci specimens (cholla cactus and ocotillo, e. g.), were picked up in the Anza Borrego Desert.

On still another trip to Redding, it was possible to secure two fresh digger pine cones. This was something of an effort as it was necessary to climb to the top of a tree fifteen or twenty feet high, to wrestle with these cones and drop them to the ground. The result was a pair of bruised and badly gummed-up hands.

It was reported that the foxtail pine was located in the Yollo Bolly Mountains and in particular on Mt. Scott (just over 6,000 ft.). Following directions that had been received, this location was found after a trip into the back country amid none too good weather conditions. Various rugged looking trees with cones closed by recent rains were located near the crest and cones and branchlets collected. On returning and submitting these specimens for identification to Dr. Thomas Howell of the Academy of Sciences in San Francisco, it was found that there were no cones of the foxtail pine among them but only cones of the silver pine!

On this same trip some very fine specimens of knobcone pine were obtained West of Redding.

Later in the year, a correspondent in Santa Fe, New Mexico, was contacted and in this way cones of Pinus edulis were obtained.

Thus at the end of this first season there were just four holdouts of the twenty specimens of pine in California, namely; the four-needled pinon, the bristlecone, the foxtail (so it was thought) and the Santa Cruz Island pine. Specimens were studied at the Academy of Sciences in San Francisco where it was noted, for example, that the Santa Cruz Island pine is very close to the bishop pine with which it used to be identified.

The point of collecting all these cones was to use them as part of an exhibit exemplifying Fibonacci relations in nature. The following lessons of experience show how different the book approach to these matters is from actual contact.

In the first place, it was possible to take open cones and close them by soaking them in water. This would take care of a particular exhibit that one would want to make. However, it was found that if such cones were slightly dried and then covered with white glue, one or more coats would be sufficient to keep them closed!

In the second place, most of the fresh cones that were collected dried and in due time opened so that all the effort involved in gathering them was misplaced. There are cones (such as those of the Monterey pine and the bishop pine) which normally remain closed; but if a cone opens on drying, there is not much point in having a fresh cone because sooner or later it will open anyhow.

In examining pine cones, it was found that there are two principal ways of noting the Fibonacci relations. One is to take a particular set of spirals and count their number: this should be a Fibonacci number. The other is to start at some particular bract and follow two spirals proceeding from it until they meet again. Then the number of bracts along each spiral required to go from one intersection to the next should be a Fibonacci number.

It was noted likewise that in many cones there are more than two spirals going through each bract. If there are spirals one, two, and three, for example, any two of these spirals can be related and in each case there will be Fibonacci numbers.

For exhibit purposes the following means of bringing out these relations can be used. In accord with the first system each spiral can be painted a given color, so that the number of colors automatically shows the number of spirals. Following the second system, the spirals issuing from one bract can be colored differently or map pins of different colors, one for each spiral, can be used.

Part two of the pine cone project was completed in the summer and fall of 1967. Once again the mountains were visited, but this time the idea was to find cones and collect them before the birds had done their work. As a result, it was possible to secure some excellent specimens of the cones of the white bark pine. These are things of beauty, roundish in shape and purple in color! Furthermore, they were immediately covered with white glue so that they would hold their appearance and not fall apart or deteriorate. Similarly, fine specimens of silver and yellow pine were obtained in their fresh state and glued.

It must be reported that this gluing of fresh specimens seems to work well for small cones such as the white bark pine cone and the foxtail cone. But the larger cones, such as those of the yellow and Jeffrey pine, opened in spite of the glue. Better results were obtained for these latter cones by taking an older cone, soaking it in water and then gluing. Presumably, the answer may be that the once open cone on being closed has small cracks between the bracts which are filled with glue and thus the bracts are held more strongly together.

The crowning achievement of the summer was a quick trip to the Alta Peak area near Sequoia National Park. It was reported that the foxtail pine could be found there at high altitudes. In this whirlwind trip, the issue was long in doubt: there seemed to be nothing along the slopes but very rugged looking silver pines. As the tree line was approached, however, two somewhat smaller specimens were observed which raised hope. Closer examination showed all the characteristics of the foxtail pine: needles of about an inch length, cones that were definitely different from those of the silver pine and open cones on the ground that corresponded to what had been observed at the Academy of Sciences in San Francisco. There were only two scraggly trees but they represented the end of a long search.

SEQUEL

The final act of the pine cone search came in the fall. In conjunction with a mathematics conference in Squaw Valley at the end of September, it was decided to go to the Bishop area and proceed to Onion Valley west of Independence to collect the limber pine. The reader may recall that the author thought he had this specimen in the summer of 1966 when hiking in the Kaweak area. However, these cones turned out to belong to the foxtail pine, so that all the effort in Yollo Bolly Mountains and the dash to Alta Peak was superflous!

Onion Valley is a very interesting area about nine thousand plus feet, reached by a good mountain road from Independence. According to an article read after the trip, there are eight species of pine within a few miles of this point, the author claiming this to be the greatest density of distinct pine species in the world. Some very interesting trees and cones were found and specimens collected. However, on being brought to Dr. Howell at the Academy of Sciences, it was found that they were excellent specimens of foxtail pine!

Likewise on this trip it was possible to collect some specimens of the bristlecone pine, a unique tree growing at an altitude of over 10,000 ft. in poor dolomite soil. These trees are the oldest living thing in the world, ages of 3,000 years being common and some trees being over 4,000 years old.

In a subsequent trip to the Southern part of the State, a day was taken to do some collecting in the neighborhood of Aguanga (south of San Bernardino) and on Mt. San Gorgonio. The first location is a very small settlement on the road with no pine trees in sight. However, inquiry led to information regarding what the local people call Pine Flat. There, the four-needled pinon was found in quantity and specimens procured.

Not too far beyond this point, what must be literally known as a windfall came to hand. Cresting a gentle pass, trees with very large cones were noted. Investigation showed that they were Coulter pines. However, the cones on the ground were old and broken, while those on the trees were so high it was very difficult to get at them. Continued search revealed a tree with its top broken off: there on the ground was about eight feet of a twenty-five to thirty foot tree with two very large cones on it! Apparently, these cones were sufficiently heavy to snap off the top of the tree in a wind!

About four-thirty in the afternoon after considerable motoring, Poopout Hill parking lot on Mt. San Gorgonio was reached. After about an hour and a half of brisk walking up the trail, the elusive limber pine was found. Specimens were collected for about half an hour and the return trip begin. However, darkness set in at seven o'clock so that there was a full hour of hiking in the dark, though the moon made the going somewhat less difficult!

At the end of this trip, there was a certain feeling of satisfaction: all cones of all California pine species except the Santa Cruz Island pine which is located on this particular Channel Island had been collected.

SPIRAL PATTERNS ON CALIFORNIA PINES

The following designations of spiral patterns do not make any pretense of completeness. Actually bracts can be lined up into sequences in many ways. The following are simply some of the more obvious patterns which have been observed.

As for notation, 8-5, for example, means that starting from a given bract and proceeding along two spirals, 8 bracts will be found on one spiral and 5 on the other when going to the next intersection of the spirals.

Pinus albicaulis (Whitebark pine)	5-3, 8-3, 8-5
Pinus flexilis (Limber pine)	8-5, 5-3, 8-3
Pinus Lambertiana (Sugar pine)	8-5, 13-5, 13-8, 3-5, 3-8, 3-13, 3-21
Pinus monticola (Western white pine, Silver pine)	3-5
Pinus monophylla (One-leaved pinon)	3-5, 3-8
Pinus edulis	5-3
Pinus quadrifolia (Four-leaved pinon)	5-3
Pinus aristata (Bristlecone pine)	8-5, 5-3, 8-3
Pinus Balfouriana (Foxtail pine)	8-5, 5-3, 8-3
Pinus muricata (Bishop pine)	8-13, 5-8
Pinus remorata (Santa Cruz Island pine)	5-8
Pinus contorta (Beach pine)	8-13
Pinus Murrayana (Lodgepole pine, Tamarack pine)	8-5, 13-5, 13-8
Pinus Torreyana (Torrey pine)	8-5, 13-5
Pinus ponderosa (Yellow pine)	13-8, 13-5, 8-5
Pinus Jeffreyi (Jeffrey pine)	13-5, 13-8, 5-8
Pinus radiata (Monterey pine)	13-8, 8-5, 13-5
Pinus attenuata (Knobcone pine)	8-5, 13-5, 3-5, 3-8
Pinus Sabiniana (Digger pine)	13-8
Pinus Coulteri (Coulter pine)	13-8

★ ★ ★ ★ ★

A MAGIC SQUARE INVOLVING FIBONACCI NUMBERS

Herta T. Freitag
Hollins College, Virginia

Is there anything peculiar in the magic square

13	89	97	34
110	21	63	39
68	94	55	16
42	29	18	144 ?

As is well known, the arrangement derives its name from the property that the sum of all numbers contained in either a row, or a column, or a diagonal is a constant, in this case 233, which we will refer to as the "magic number," MN.

It seems that knowledge of such square number arrays, not necessarily four-by-four, were known in China as far back as 2200 B.C., but apparently it was only in the 16th Century that this idea has reached the Christian Occident. From that time onward, this topic has not only held its attraction in a recreational manner, but mathematicians of rank, among them Leonhard Euler, have given it serious attention.

Here is a suggestion for designing a whole set of four-by-four magic squares with the added property that each of the entries is the sum of two Fibonacci numbers (which may, of course, under appropriate conditions, be a Fibonacci number itself) and the magic number is itself a Fibonacci number — even a pre-assigned one, if one so desires.

We will first construct an addition table, denoting by f_{n_i}, $i \in \{1, 2, \cdots, 8\}$, the n_i^{th} Fibonacci number.

If we wish to retain the positions of the numbers in the main diagonal

$$MN = \sum_{i=1}^{8} f_{n_i}$$

	f_{n_5}	f_{n_6}	f_{n_7}	f_{n_8}
f_{n_1}	$f_{n_1} + f_{n_5}$	$f_{n_1} + f_{n_6}$	$f_{n_1} + f_{n_7}$	$f_{n_1} + f_{n_8}$
f_{n_2}	$f_{n_2} + f_{n_5}$	$f_{n_2} + f_{n_6}$	$f_{n_2} + f_{n_7}$	$f_{n_2} + f_{n_8}$
f_{n_3}	$f_{n_3} + f_{n_5}$	$f_{n_3} + f_{n_6}$	$f_{n_3} + f_{n_7}$	$f_{n_3} + f_{n_8}$
f_{n_4}	$f_{n_4} + f_{n_5}$	$f_{n_4} + f_{n_6}$	$f_{n_4} + f_{n_7}$	$f_{n_4} + f_{n_8}$

Array 1

Therefore, we must rearrange the positions of the other numbers. The minor diagonal already does show the desired property. Studying the design of our array, a grouping of the non-diagonal numbers into pairs of positions, which are symmetric with respect to the center of the square, as indicated by the guide lines, suggests itself. We now perform an interchange of the two numbers in each pair. After this transformation, our array becomes:

$f_{n_1} + f_{n_5}$	$f_{n_4} + f_{n_7}$	$f_{n_4} + f_{n_6}$	$f_{n_1} + f_{n_8}$
$f_{n_3} + f_{n_8}$	$f_{n_2} + f_{n_6}$	$f_{n_2} + f_{n_7}$	$f_{n_3} + f_{n_5}$
$f_{n_2} + f_{n_8}$	$f_{n_3} + f_{n_6}$	$f_{n_3} + f_{n_7}$	$f_{n_2} + f_{n_5}$
$f_{n_4} + f_{n_5}$	$f_{n_1} + f_{n_7}$	$f_{n_1} + f_{n_6}$	$f_{n_4} + f_{n_8}$

Array 2

Then MN, as computed from the first row, equals

$$2(f_{n_1} + f_{n_4}) + f_{n_5} + f_{n_6} + f_{n_7} + f_{n_8} .$$

For this number, however, to equal

$$\sum_{i=1}^{8} f_{n_i} ,$$

the condition

$$f_{n_1} + f_{n_4} = f_{n_2} + f_{n_3} ,$$

forces itself upon us. Analogously, we would need to stipulate that

$$f_{n_5} + f_{n_8} = f_{n_6} + f_{n_7} .$$

If we wish to obtain an array which has the properties of a magic square without further restrictions, an interchange of the subset of four numbers which form a square arrangement at the right-hand top of Array 2 with the corresponding one on the left-hand bottom needs to be resorted to. This transformation leads to:

$f_{n_1} + f_{n_5}$	$f_{n_4} + f_{n_7}$	$f_{n_2} + f_{n_8}$	$f_{n_3} + f_{n_6}$
$f_{n_3} + f_{n_8}$	$f_{n_2} + f_{n_6}$	$f_{n_4} + f_{n_5}$	$f_{n_1} + f_{n_7}$
$f_{n_4} + f_{n_6}$	$f_{n_1} + f_{n_8}$	$f_{n_3} + f_{n_7}$	$f_{n_2} + f_{n_5}$
$f_{n_2} + f_{n_7}$	$f_{n_3} + f_{n_5}$	$f_{n_1} + f_{n_6}$	$f_{n_4} + f_{n_8}$

Array 3

Array 3 gives us a prescription for an infinite set of magic squares, such that each entry is the sum of (at most) two Fibonacci numbers.

However, we may wish to have a Fibonacci number for our MN. We need to superimpose the set of conditions: $n_i = n_1 + 2i - 3, \quad i \in \{2, 3, \cdots, 8\}$. Now our magic square reads

$f_a + f_{a+7}$	$f_{a+5} + f_{a+11}$	$f_{a+1} + f_{a+13}$	$f_{a+3} + f_{a+9}$
$f_{a+3} + f_{a+13}$	$f_{a+1} + f_{a+9}$	$f_{a+5} + f_{a+7}$	$f_a + f_{a+11}$
$f_{a+5} + f_{a+9}$	$f_a + f_{a+13}$	$f_{a+3} + f_{a+11}$	$f_{a+1} + f_{a+7}$
$f_{a+1} + f_{a+11}$	$f_{a+3} + f_{a+7}$	$f_a + f_{a+9}$	$f_{a+5} + f_{a+13}$

Array 4

where, for simplicity's sake, we symbolized n_1 by a. Array 4 gives us the means for structuring an infinitude of magic squares. Obviously, we may even pre-assign a magic number, we wish to obtain. Since MN now becomes f_{a+14}, the n^{th} Fibonacci number will be obtained by letting the parameter a equal n - 14.

But we may superimpose one more restriction. Our aim now is to contrive a magic square of the kind just described but with the added property that all elements in the major diagonal are Fibonacci numbers, rather than sums of two Fibonacci numbers. Then — see Array 3 — the restrictions:

$$n_2 = n_5 = n_1 + 1$$
$$n_3 = n_1 + 3$$
$$n_4 = n_1 + 5$$
$$n_6 = n_1 + 2$$
$$n_7 = n_1 + 4$$
$$n_8 = n_1 + 5$$

need to be observed, and the arrangement below (Array 5) will serve our needs. Again, we adopt the simplified symbolism.

f_{a+2}	f_{a+6}	$f_{a+1} + f_{a+6}$	f_{a+4}
$f_{a+3} + f_{a+6}$	f_{a+3}	$f_{a+1} + f_{a+5}$	$f_a + f_{a+4}$
$f_{a+2} + f_{a+5}$	$f_a + f_{a+6}$	f_{a+5}	$2f_{a+1}$
$f_{a+1} + f_{a+4}$	$f_{a+1} + f_{a+3}$	$f_a + f_{a+2}$	f_{a+7}

Array 5

The magic number $MN = f_{a+8}$ is again pre-assignable, and all entries in our major diagonal are Fibonacci numbers. We may test our scheme by wishing to obtain the 13^{th} Fibonacci number, 233, as our magic number. Then $a = 5$, and our magic square becomes the one quoted at the beginning of this paper.

* * * * *

A SEQUENCE OF POWER FORMULAS

Brother Alfred Brousseau
St. Mary's College, Calif.

Starting with the familiar formulas

(1) $$F_{n+1} = F_n + F_{n-1}$$

(2) $$F^2_{n+1} = 2F^2_n + 2F^2_{n-1} - F^2_{n-2}$$

in which a power of a Fibonacci number is expressed as a linear combination of the same power of successive Fibonacci numbers one is led to seek additional formulas of this type.

For the third degree one can start with

$$F^3_{n+1} = F^3_n + 3F^2_n F_{n-1} + 3F_n F^2_{n-1} + F^3_{n-1}$$

$$F^3_{n-2} = F^3_n - 3F^2_n F_{n-1} + 3F_n F^2_{n-1} - F^3_{n-1}$$

$$F^3_{n-3} = -F^3_n + 6F^2_n F_{n-1} - 12F_n F^2_{n-1} + 8F^3_{n-1}$$

which result from cubing familiar linear relations. To arrive at the desired power relation, it is necessary to eliminate the terms that are not simple powers. Multiplying the first relation by a, the second by b and adding the result to the third yields the following relations for this elimination.

$$a - b + 2 = 0$$
$$a + b - 4 = 0$$

from which $b = 3$ and $a = 1$. This gives the desired relation of the third degree:

(3) $$F^3_{n+1} = 3F^3_n + 6F^3_{n-1} - 3F^3_{n-2} - F^3_{n-3}$$

This method can be pursued making use of coefficients without writing out complete expressions. For the fourth degree this gives a table:

F^4_{n+1}	1	4	6	4	1
F^4_{n-2}	1	-4	6	-4	1
F^4_{n-3}	1	-8	24	-32	16
F^4_{n-4}	16	-96	216	-216	81

This table leads to the following equations for eliminating the middle terms.

$$a + b - 2c - 24 = 0$$

$$a + b + 4c + 36 = 0$$

$$a - b - 8c - 54 = 0$$

from which $a = 1$, $b = 15$, $c = 5$, and $d = -1$. This leads to the relation:

$$F^4_{n+1} = 5F^4_n + 15F^4_{n-1} - 15F^4_{n-2} - 5F^4_{n-3} - F^4_{n-4} \tag{4}$$

Fifth and sixth degree relations are:

$$F^5_{n+1} = 8F^5_n + 40F^5_{n-1} - 60F^5_{n-2} - 40F^5_{n-3} + 8F^5_{n-4} + F^5_{n-5} \tag{5}$$

$$F^6_{n+1} = 13F^6_n + 104F^6_{n-1} - 260F^6_{n-2} - 260F^6_{n-3} + 104F^6_{n-4} + 13F^6_{n-5} - F^6_{n-6} \tag{6}$$

Since the algebra at this point was becoming laborious, the coefficients were set up in tabular form for the purpose of discovering a pattern. The heading is the degree; the numbers below are the successive coefficients of the terms on the right-hand side of the relation.

1	2	3	4	5	6
1	2	3	5	8	13
1	2	6	15	40	104
	-1	-3	-15	-60	-260
		-1	-5	-40	-260
			1	8	104
				1	13
					-1

It was observed that one column can be obtained from the previous column by multiplying by a Fibonacci number and dividing successive products by certain Fibonacci numbers in reverse order. Thus to go from the column headed 4 to the column headed 5, multiply each quantity in the 4 column by 8 and divide successive products by 5, 3, 2, 1, 1 respectively. To go from column 5 to column 6, multiply each quantity in column 5 by 13 and divide by 8, 5, 3, 2, 1, 1 respectively. The new elements in each column at the end are all 1's with a plus or minus sign, the order being two minuses, two etc.

With the aid of this empirical result, the table was continued to higher powers.

7	8	9	10	11	12	13
21	34	55	89	144	233	377
273	714	1870	4895	12816	33552	87841
-1092	-4641	-19635	-83215	-352440	-1493064	-6324552
-1820	-12376	-85085	-582505	-3994320	-27372840	-187628376
1092	12376	136136	1514513	16776144	186145312	2063912136
273	4641	85085	1514513	27261234	488605194	8771626578
-21	-714	-19635	-582505	-16776144	-488605194	-14169550626
-1	-34	-1870	-83215	-3994320	-186135312	-8771626578
	1	55	4895	352440	27372840	2063912136
		1	89	12816	1493064	187628376
			-1	-144	-33552	-6324552
				-1	-233	-87841
					1	377
						1

In each instance the coefficients were checked by applying the formula to one particular value of n.

* * * * *

BOOK REVIEW

536 PUZZLIES AND CURIOUS PROBLEMS

by Henry Ernest Dudeney
Edited and with an Introduction by Martin Gardner

Some very pertinent information about the author and this collection of mathematical problems may be gleaned from the introduction by Martin Gardner. Quote:

> "Henry Ernest Dudeney ... was England's greatest maker of puzzles. With respect to mathematical puzzles, especially problems of more than trivial mathematical interest, the quantity and quality of his output surpassed that of any other puzzlist before or since, in or out of England."

He was noteworthy for the originality of his work and his desire to give credit where credit was due when borrowing other people's ideas. He was unusually skillful in geometrical dissections and an expert in magic squares of various types. He was the first to apply the term "digital roots" in recreational mathematics.

The present volume combines two of his books which have been out of print since 1958, namely, "Modern Puzzles" and Puzzles and Curious Problems." The puzzles have been rearranged and reclassified with American words replacing British, American currency substituted for English, etc., to adapt the material to the U. S. public.

A rapid survey of the volume brings about a number of encounters with old familiar favorites, but there is material to spare for just about every one, except possibly the puzzle specialist.

The book is well printed and put together in an attractive manner (see cover photograph p. 68). There are also answers to alleviate the danger of complete psychological frustration.

The publisher is Charles Scribner's Sons (1967) and the book sells for $7.95.

Brother Alfred Brousseau
St. Mary's College, Calif.

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THE FIBONACCI DRAWING BOARD DESIGN OF THE GREAT PYRAMID OF GIZEH

Col. R. S. Beard
Berkeley, Calif.

The comments on the Great Pyramid of Gizeh by Herodotus (484 to 424 B. C.) contained the statement that "The base was a square. The base side was 800 feet. The height was equal."

Apparently some student of the dimensions of this pyramid has interpreted this 'obscure' statement to mean that the square of the vertical height of the pyramid is equal to the area of each of its triangular faces.

Such an imaginative interpretation is not acceptable evidence. It also credits the Egyptians of 3000 B. C. with familiarity with the golden section. However, the facts fit the theory remarkably well.

The elevation of the face triangles of the pyramid is made the unit of measure in the accompanying cross section of the pyramid. Let k symbolize the golden section ratio of $\frac{1}{2}\sqrt{5} - \frac{1}{2}$.

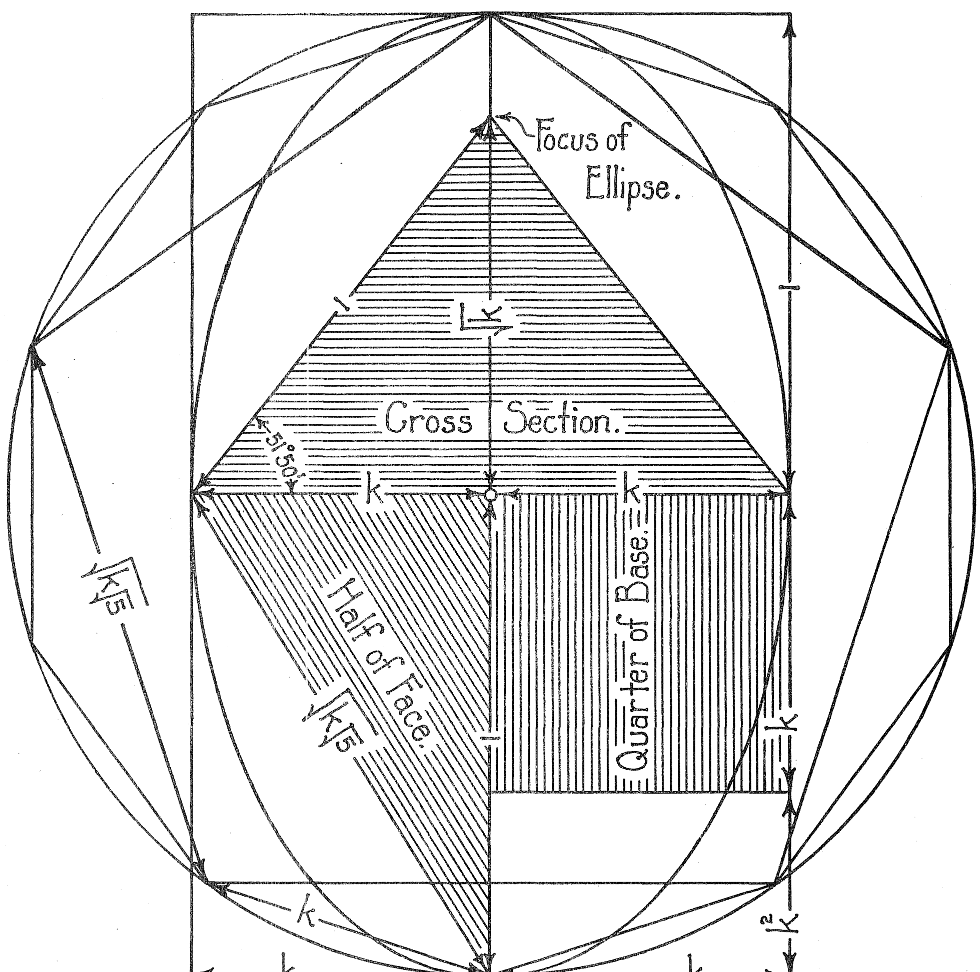
If the square of the vertical height of the pyramid equals the area of one triangular face, each such face is a golden rectangle that has been halved on one diagonal and rejoined on its long sides. The base of the pyramid is then a 2k x 2k square and it has an altitude of $\sqrt{k}$. Each quarter section of the 2 x 2k golden rectangle in the sketch has the area of one triangular face. The inscribed ellipse has one focus at the apex of the pyramid. A circle of radius 1 is centered on the base. The inscribed regular decagon has sides of k length. The sides of the inscribed regular pentagon has sides of the same length as the sloping edges of the pyramid.

Such relationships would certainly have appealed to these Egyptian masters of practical geometry.

The Great Pyramid is now about 750 ft. square at the base. It is 451 ft. high and has a small flat deck on top. Sir William Mathew Flanders Petrie made an exceptionally accurate survey of the pyramid in the early 1880's. On the basis of his painstaking studies, he concluded that the original base of the pyramid was 755.73 ft. square and that its original height was 481.33 ft.

Under the Herodotus design, a base of 755.73 ft. would correspond to the 2k dimension in the drawing. This would make the height of the pyramid

THE GREAT PYRAMID.



Item Intact.–Feet.		Slant Height Ratio.	$k = \frac{1}{2}\sqrt{5} - \frac{1}{2} = 0.618034$*	
Height.—	481.33	$\sqrt{k} = 0.786151$	Square of Height $= (481.33)^2 = \underline{231679.}$	
*Base.—	$(755.73)^2$	$(2k)^2 = (1.236068)^2$	Face Triangle $= \frac{1}{2} \times 611.93 \times 755.73 = \underline{231229.}$	
*Slant Height.—	611.93	$\sqrt{k\sqrt{5}} = 1.175570$	* $\frac{377.87}{611.93} = 0.6175$	$\frac{377.87}{481.33} = 0.7851$

$$\frac{755.73}{2\sqrt{k}}$$

or 480.65 ft.

Surprisingly the dimensions of the pyramid conform equally well to a second and a third theory as to its design.

A widely held second theory makes the height of the pyramid equal to the radius of the circle that has a circumference equal to the perimeter of the base of the pyramid.

$$\frac{4 \times 755.73}{2\pi} = 481.11$$

Sir William Petrie himself was thoroughly convinced that the Egyptians constructed the pyramid with a height-to-width-of-base ratio of seven to eleven.

$$\frac{7}{11} \times 755.73 = 480.92 .$$

Herodotus reports that 100,000 men labored for 30 years to construct this gigantic exhibit of personal egotism. This massive structure has probably settled more than the variations in these computed heights. Nobody will ever know its true original height and early Egyptian knowledge of the golden section remains unconfirmed So roll the dice and choose your own theory.

REFERENCES

1. J. E. Powell, A Lexicon to Herodotus, Cambridge (England) 1938, x + 392.

2. James R. Newman, The World of Mathematics, Vol. 1, p. 10, Simon and Schuster, N. Y., 1956.

3. "The Geometry of the Pentagon and the Golden Section," The Mathematics Teacher, Jan. 1948.

4. Sir William M. F. Petrie, Seventy Years in Archeology.

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BOOK REVIEW

MATHEMATICAL QUICKIES

Charles W. Trigg

Mc-Graw-Hill Book Company, New York, 1967, xi + 210 pp. $7.95

Mathematical "Quickies" were introduced in March, 1950, by Charles W. Trigg, when he was serving as editor of the Problems and Questions Department of the Mathematics Magazine. A Quickie is a problem whose solution appears at first encounter to require laborious methods, but which by proper insight can be disposed of with dispatch. The Quickies caught on at once and have retained their popularity in the Mathematics Magazine ever since.

This book is divided into two parts. In the first part is a superb collection of 270 Quickies, chosen from such diverse fields as arithmetic, algebra, plane and solid geometry, trigonometry, number theory, and recreational mathematics (magic squares, dissections, cryptarithms, etc.). In the second part appear elegant solutions to the Quickies. The problems are all interesting, stimulating, and challenging, and a reader should try to solve them himself before peeking at the ingenious solutions offered in the second part of the book; he may even on occasion come up with a superior solution.

There is room for cleverness in mathematics, and this feature of problem solving is excellently illustrated in this book. The material should appeal to mathematics enthusiasts of all ages and levels of sophistication, and should be invaluable in the arsenals of school teachers and college instructors. It is hoped that Dean Trigg, who is one of the country's outstanding problemists, will dip into his great collection of Quickies and, at some date not too far in the future, give us Some More Mathematical Quickies.

Howard Eves
University of Maine, Orono, Me.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A. P. Hillman
University of New Mexico, Albuquerque, N.M.

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

B-130 Proposed by Sidney Kravitz, Dover, New Jersey

An enterprising entrepreneur in an amusement part challenges the public to play the following game. The player is given five equal circular discs which he must drop from a height of one inch onto a larger circle in such a way that the five smaller discs completely cover the larger one. What is the maximum ratio of the diameter of the larger circle to that of the smaller ones so that the player has the possibility of winning?

B-131 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let $\{H_n\}$ be a generalized Fibonacci sequence, i.e., $H_0 = q$, $H_1 = p$, $H_{n+2} = H_{n+1} + H_n$. Extend, by the recursion formula, the definition to include negative subscripts. Show that if $|H_{-n}| = |H_n|$ for all n, then $\{H_n\}$ is a constant multiple of either the Fibonacci or the Lucas sequence.

B-132 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let u and v be relatively prime integers. We say that u belongs to the exponent d modulo v if d is the smallest positive integer such that $u^d \equiv 1 \pmod{v}$. For $n \geq 3$ show that the exponent to which F_n belongs modulo F_{n+1} is 2 if n is odd and 4 if n is even.

B-133 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let $r = F_{1000}$ and $s = F_{1001}$. Of the two numbers r^s and s^r, which is the larger?

B-134 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Define the sequence $\{a_n\}$ by $a_1 = a_2 = 1$, $a_{2k+1} = a_{2k} + a_{2k-1}$, and $a_{2k} = a_k$ for $k \geq 1$. Show that

$$\sum_{k=1}^{n} a_k = a_{2n+1} - 1, \quad \sum_{k=1}^{n} a_{2k-1} = a_{4n+1} - a_{2n+1} .$$

B-135 Proposed by L. Carlitz, Duke University, Durham, North Carolina

Put

$$F'_n = \sum_{k=0}^{n-1} F_k 2^{n-k-1}, \quad L'_n = \sum_{k=0}^{n-1} L_k 2^{n-k-1} .$$

Show that, for all $n \geq 1$,

$$F'_n = 2^n - F_{n+2}, \quad L'_n = 3 \cdot 2^n - L_{n+2} .$$

SOLUTIONS

GENERALIZATION OF $F_nL_n = F_{2n-1} + F_{2n-2}$

B-112 Proposed by Gerald Edgar, Boulder, Colorado

Let f_n be the generalized Fibonacci sequence (a,b), i.e., $f_1 = a$, $f_2 = b$, and $f_{n+1} = f_n + f_{n-1}$. Let g_n be the associated generalized Lucas sequence defined by $g_n = f_{n-1} + f_{n+1}$. Prove that $f_n g_n = bf_{2n-1} + af_{2n-2}$.

Composite of solutions by David Zeitlin, Minneapolis, Minnesota and Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let r and s be the roots of $x^2 - x - 1 = 0$. Then f_n and g_n are of the form $c_1r^n + c_2s^n$ and hence f_ng_n, f_{2n-1}, and f_{2n-2} are all of the form

$$k_1r^{2n} + k_2(-1)^n + k_3s^{2n}$$

and hence all satisfy the difference equation

$$\text{(E)} \qquad y_{n+3} - 2y_{n+2} - 2y_{n+1} + y_n = 0$$

whose auxiliary polynomial is

$$(x - r^2)(x - rs)(x - s^2) = x^3 - 2x^2 - 2x + 1.$$

Since both sides of the desired formula

$$\text{(F)} \qquad f_n g_n = bf_{2n-1} + af_{2n-2}$$

satisfy (E), formula (F) is established by verifying it for $n = 0, 1,$ and 2 and then using (E) and mathematical induction to prove it for $n \geq 0$.

Also solved by Thomas P. Dence, Douglas Lind, D. V. Jaiswal (India), Stanley Rabinowitz, A. C. Shannon (Australia), M. N. S. Swamy (Canada), and the proposer.

CLUSTER POINTS

B-113 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let (x) denote the fractional part of x, so that if $[x]$ is the greatest integer in x, $(x) = x - [x]$. Let $a = (1 + \sqrt{5})/2$ and let A be the set $\{(a), (a^2), (a^3), \cdots\}$. Find all the cluster points of A.

Solution by the proposer.

If $b = (1 - \sqrt{5})/2$, it is familiar that $L_n = a^n + b^n$, where L_n is the n^{th} Lucas number, which is an integer. Since $-1 < b < 0$, given $\in > 0$, there is an N such that for all $k > N$ we have

$$0 < b^{2k} = L_{2k} - a^{2k} < \in .$$

It follows that $(a^{2k}) \to 1$. Similarly, there is an M such that for all $k > M$ we have

$$0 < -b^{2k+1} = a^{2k+1} - L_{2k+1} < \in ,$$

so $(a^{2k+1}) \to 0$. Clearly these are the only possible cluster points of A.

OUR MAN OF PISA

B-114 Proposed by Gloria C. Padilla, University of New Mexico, Albuquerque, N.M.

Solve the division alphametic

$$\begin{array}{r} \text{PISA} \\ \text{FIB}\overline{\smash{\big)}\text{ONACCI}} \end{array},$$

where each letter is one of the digits 1, 2, ···, 9 and two letters may represent the same digit. (This is suggested by Maxey Brooke's B-80.)

Solution by the proposer.

One solution is the following:

$$\begin{array}{r} 3\,4\,1\,8 \\ 1\,4\,3\overline{\smash{\big)}4\,8\,8\,7\,7\,4} \end{array}$$

IDENTITIES FOR F_{kn} AND L_{kn}

B-115 Proposed by H. H. Ferns, Victoria, B.C., Canada

From the formulas of B-106:

$$2F_{i+j} = F_iL_j + F_jL_i$$
$$2L_{i+j} = 5F_iF_j + L_iL_j$$

one has

$$F_{2n} = F_nL_n$$
$$F_{3n} = (5F_n^3 + 3F_nL_n^2)/4$$
$$L_{2n} = (5F_n^2 + L_n^2)/2$$
$$L_{3n} = (15F_n^2L_n + L_n^3)/4$$

Find and prove the general formulas of these types.

Solution by Stanley Rabinowitz, Far Rockaway, New York.

The formulas look neater when expressed in matrix form. Putting $i = (k - 1)n$ and $j = n$ in the formulas of B-106 gives

$$\text{(R)} \qquad \begin{pmatrix} F_{kn} \\ L_{kn} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} L_n & F_n \\ 5F_n & L_n \end{pmatrix} \begin{pmatrix} F_{(k-1)n} \\ L_{(k-1)n} \end{pmatrix}$$

Repeated application of this formula gives the desired solution:

$$\begin{pmatrix} F_{kn} \\ L_{kn} \end{pmatrix} = \frac{1}{2^k} \begin{pmatrix} L_n & F_n \\ 5F_n & L_n \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

since $F_0 = 0$ and $L_0 = 2$.

Note: From (R) or the formulas of B-106, one can obtain the proposer's formulas:

$$F_{(k+1)n} = \frac{1}{2^k} \sum_{i=0}^{[k/2]} 5^i \binom{k+1}{k-2i} F_n^{2i+1} L_n^{k-2i} ,$$

$$L_{(k+1)n} = \frac{1}{2^k} \sum_{i=0}^{[(k+1)/2]} 5^i \binom{k+1}{k+1-2i} F_n^{2i} L_n^{k+1-2i} .$$

Also solved by David Zeitlin and the proposer.

A GENERATING FUNCTION

B-116 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Find a compact sum for the series

$$\sum_{m,n=0}^{\infty} F_{2m-2n} x^m y^n .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

If $W_{n+2} = aW_{n+1} + bW_n$, then

$$\frac{W_0 + (W_1 - aW_0)t}{1 - at - bt^2} = \sum_{k=0}^{\infty} W_k t^k \qquad (1)$$

Since $W_k = F_{2k\pm p}$ satisfies $W_{k+2} = 3W_{k+1} - W_k$, we have

$$\sum_{m=0}^{\infty} F_{2m-2n} x^m = \frac{F_{-2n} + (F_{2-2n} - 3F_{-2n})x}{1 - 3x + x^2} .$$

Since $F_{-j} = (-1)^{j+1}F_j$, we have the desired sum, S,

$$\begin{aligned} S &= \frac{1}{1 - 3x + x^2}\left((3x - 1)\sum_{n=0}^{\infty} F_{2n}y^n - x\sum_{n=0}^{\infty} F_{2n-2}y^n\right) \\ &= \frac{1}{1 - 3x + x^2}\left(\frac{(3x - 1)y}{1 - 3y + y^2} - \frac{x(-1 + 3y)}{1 - 3y + y^2}\right) \\ &= \frac{x - y}{(1 - 3x + x^2)(1 - 3y + y^2)} \end{aligned}$$

Also solved by Douglas Lind, D. V. Jaiswal (India), M.N.S. Swamy (Canada), and the proposer.

ANOTHER GENERATING FUNCTION

B-117 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Find a compact sum for the series

$$\sum_{m,n=0}^{\infty} F_{2m-2n+1} x^m y^n .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Using (1) in B-116, we have

$$\sum_{m=0}^{\infty} F_{2m-2n+1}x^m = \frac{F_{-2n+1} + (F_{3-2n} - 3F_{-2n+1})x}{1 - 3x + x^2}$$

Since $F_{-j} = (-1)^{j+1} F_j$, we have the desired sum, S,

$$S = \frac{1}{1 - 3x + x^2}\left((1 - 3x)\sum_{n=0}^{\infty} F_{2n-1}y^n + x\sum_{n=0}^{\infty} F_{2n-3}y^n\right)$$

$$= \frac{1}{1 - 3x + x^2}\left(\frac{(1 - 3x)(1 - 2y)}{1 - 3y + y^2} + \frac{x(2 - 5y)}{1 - 3y + y^2}\right)$$

$$= \frac{xy - 2y - x + 1}{(1 - 3x + x^2)(1 - 3y + y^2)} .$$

Also solved by Douglas Lind, D. V. Jaiswal (India), M.N.S. Swamy (Canada), and the proposer.

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FORMULAS FOR DECOMPOSING F_{3n}/F_n, F_{5n}/F_n and L_{5n}/L_n INTO A SUM OR DIFFERENCE OF TWO SQUARES

Dov Jarden
Jerusalem, Israel

(1) $$F_{3n}/F_n = L_n^2 - (-1)^n$$

(1.1) $$F_{6n}/F_{2n} = L_{2n}^2 - 1 = (L_{2n} - 1)(L_{2n} + 1)$$

(1.2) $$F_{3(2n+1)}/F_{2n+1} = L_{2n+1}^2 + 1$$

(2) $$F_{5n}/F_n = (L_{2n} + (-1)^n)^2 - (-1)^n L_n^2$$

(2.1) $$F_{10n}/F_{2n} = (L_{4n} + 1)^2 - L_{2n}^2 = (L_{4n} + 1 - L_{2n})(L_{4n} + 1 + L_{2n})$$

(2.2) $$F_{5(2n+1)}/F_{2n+1} = (L_{4n+2} - 1)^2 + L_{2n+1}^2$$

(3) $$L_{5n}/L_n = (L_{2n} - (-1)^n 3)^2 + (5F_n)^2$$

(3.1) $$L_{10n}/L_{2n} = (L_{4n} - 3)^2 + (5F_n)^2$$

(3.2) $$L_{5(2n+1)}/L_{2n+1} = (L_{4n+2} - 3)^2 - (5F_{2n+1})^2 = (L_{4n+2} - 3 - 5F_{2n+1})(L_{4n+2} - 3 + 5F_{2n+1})$$

The formulas (1), (2), (3) can be easily verified by putting

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad \alpha\beta = -1,$$

and, for (3), also $\alpha - \beta = \sqrt{5}$.

Since for $n > 0$, (3.1) gives a decomposition of L_{10n}/L_{2n} into a sum of two squares, and since any divisor of a sum of two squares is $\equiv 1 \pmod 4$, it follows that any primitive divisor of L_{10n}, $n > 0$, is $\equiv 1 \pmod 4$.

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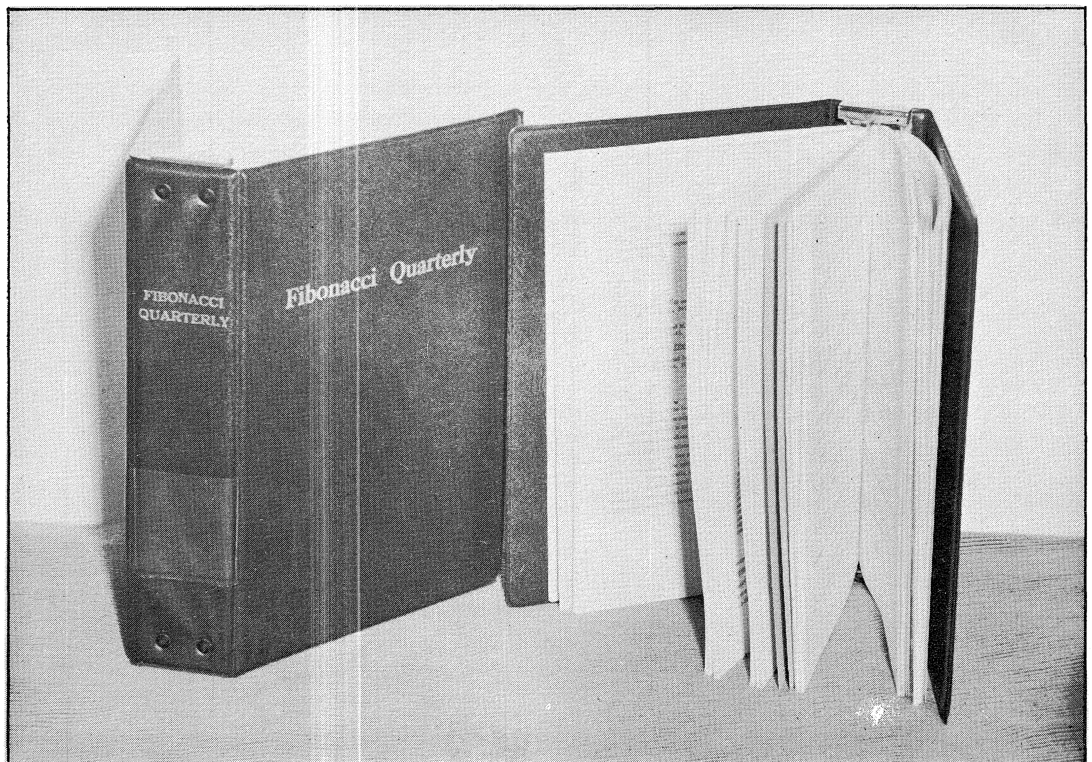

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