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GENERALIZED FIBONACCI SUMMATIONS

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INTRODUCTION

The operator Δ_r is defined [1] by:

$$\Delta_r f(r, a, b \cdots) = f(r, a, b \cdots) - f(r-1, a, b \cdots)$$

and its inverse Σ_r is defined by:

$$\Delta_r \Sigma_r f(r, a, b \cdots) = f(r, a, b \cdots)$$

In this article we will make use of these two operators, which are analogous to the differential and integral operators, to establish several summations involving generalized Fibonacci numbers.

First some elementary properties of Δ_r and Σ_r will be needed. In deriving these and in subsequent work the subscripts to the operators may be omitted if this causes no confusion.

PROPERTIES OF Δ_r AND Σ_r

1. $$\begin{aligned}\Delta(f(r) + g(r)) &= (f(r) + g(r)) - (f(r-1) + g(r-1)) \\ &= (f(r) - f(r-1)) + (g(r) - g(r-1))\end{aligned}$$

$$\Delta(f(r) + g(r)) = \Delta f(r) + \Delta g(r) \tag{0.1}$$

2. $$\begin{aligned}\Delta(f(r) \cdot g(r)) &= f(r) \cdot g(r) - f(r-1) \cdot g(r-1) \\ &= f(r) \cdot (g(r) - g(r-1)) + g(r-1) \cdot (f(r) - f(r-1))\end{aligned}$$

$$(f(r) \cdot g(r)) = f(r)\Delta g(r) + g(r-1)\Delta f(r) \tag{0.2}$$

If $g(r)$ is a constant then $\Delta_r g(r) = 0$ and putting $g(r) = C$ in (0.2) we have:

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(0.3) $$\Delta_r Cf(r) = C\Delta_r f(r) \text{ if } \Delta_r C = 0$$

This covers not only the case when C is a constant but also when it is any function independent of r.

(0.4) $$\Delta_n f(n+p) = (\Delta_r f(r))_{r=n+p}$$

This follows immediately from the definition of Δ_r sinch both left- and right-hand members simplify to $f(n+p) - f(n+p-1)$.

4. Next some properties of Σ_r. Suppose: $\Sigma f(r) = g(r)$. Then from the definitions of Δ and Σ:

$$g(r) - g(r-1) = f(r)$$

Summing these equalities with r taking values from 1 to n

$$g(n) - g(0) = \sum_{r=1}^{n} f(r)$$

i. e.,

(0.5) $$\Sigma f(n) = \sum_{r=1}^{n} f(r) + C \quad ,$$

where $\Delta_n C = 0$ but otherwise C is arbitrary. The connection between the and the summation of f(n) is equivalent to that between indefinite and definite integrals. In particular:

(0.6) $$\sum_{r=1}^{n} f(r) = \Sigma f(n) - (\Sigma f(n))_{n=0}$$

5. From (0.5)

$$\Sigma_n f(n + s) = \sum_{r=1}^{n} f(r + s) + C$$

$$= \sum_{r=1}^{n+s} f(r) + C - \sum_{r=1}^{s} f(r) = \sum_{r=1}^{n+s} f(r) + C'$$

If we ignore the constants:

(0.7) $$\Sigma_n f(n + s) = (\Sigma_r f(r))_{r=n+s}$$

6. In the definition of Σ put $\Delta f(r)$ in place of $f(r)$

$$\Delta(\Sigma\Delta f(r)) = \Delta(f(r))$$

i.e.,

$$\Sigma\Delta f(r) = f(r) + C$$

If we now ignore the constants

(0.8) $$\Sigma\Delta f(r) = f(r)$$

7. In (0.1) replace $f(r)$ by $\Sigma f(r)$ and $g(r)$ by $\Sigma g(r)$

$$\Delta(\Sigma f(r) + \Sigma g(r)) = \Delta\Sigma f(r) + \Delta\Sigma g(r)$$

$$\Sigma\Delta(\Sigma f(r) + \Sigma g(r)) = \Sigma(\Delta\Sigma f(r) + \Delta\Sigma g(r))$$

i.e.,

(0.9) $$\Sigma(f(r) + g(r)) = \Sigma f(r) + \Sigma g(r)$$

8. From (0.2) replace $g(r)$ by $h(r)$ and rearranging:

$$f(r)\Delta h(r) = \Delta(f(r) \cdot h(r)) - h(r-1)\Delta f(r)$$

Let $h(r) = \Sigma g(r)$

$$f(r) \cdot g(r) = \Delta(f(r) \cdot \Sigma g(r)) - \Sigma g(r-1) \cdot \Delta f(r)$$

Thus:

$$\Sigma(f(r) \cdot g(r)) = f(r)\Sigma g(r) - \Sigma(\Sigma g(r-1) \cdot \Delta f(r)) \tag{0.10}$$

This last result, analogous to integration by parts, will be made use of in deriving most of the results which follow.

If $f(r) = C$ where $\Delta_r C = 0$ we can write (0.10) as:

$$\Sigma C g(r) = C \Sigma g(r) \tag{0.11}$$

THE SUMMATIONS

The generalized Fibonacci numbers may be defined by:

$$H_n = H_{n-1} + H_{n-2} \tag{1.1}$$

for all integers n. If $H_0 = 0$ and $H_1 = 1$ we get the Fibonacci sequence which is denoted (F_n).

Two facts about the generalized sequence will be needed. They are:

$$H_{n-1}H_{n+1} - H_n^2 = D(-1)^n \qquad \text{where } D = H_{-1}H_1 - H_0^2 \quad [2] \tag{1.2}$$

and

$$H_{n+r} = F_{r-1}H_n + F_r H_{n+1} \tag{1.3}$$

1. First a very simple (but useful) summation.

$$\Delta H_n = H_n - H_{n-1} = H_{n-2}$$

Thus:

$$\Sigma H_n = H_{n+2} \tag{1.4}$$

2. $\Sigma a^n H_{n+s}$

Note that

$$\Delta a^n = a^n - a^{n-1} = a^{n-1}(a - 1)$$

$$\begin{aligned}\Sigma a^n H_{n+s} &= a^n H_{n+s+2} - \Sigma a^{n-1}(a-1)H_{n+s+1} \\ &= a^n H_{n+s+2} - \frac{a-1}{a^2}\Sigma a^{n+1} H_{n+s+1}\end{aligned}$$

Now using:

$$\Sigma a^{n+1} H_{n+s+1} = \Sigma a^n H_{n+s} + a^{n+1} H_{n+s+1}$$

$$\frac{a^2 + a - 1}{a^2}\Sigma a^n H_{n+s} = a^n H_{n+s+2} - a^{n-1}(a-1)H_{n+s+1}$$

multiplying by a^2

$$(a^2 + a - 1)\Sigma a^n H_{n+s} = a^{n+2} H_{n+s} + a^{n+1} H_{n+s+1}$$

If $a^2 + a - 1 \neq 0$ i.e., $a \neq (-1 \pm \sqrt{5})/2$

$$\Sigma a^n H_{n+s} = \frac{a}{a^2 + a - 1}(a^{n+1} H_{n+s} + a^n H_{n+s+1}) \tag{1.5}$$

3. $\Sigma n^k H_{n+s}$

Before attempting this summation we will find the particular sums when $k = 0, 1, 2$.

$k=0$: this comes straight from (1.4)

$$\Sigma H_{n+s} = H_{n+s+2} \tag{1.6}$$

$k=1$:

$$\Sigma nH_{n+s} = nH_{n+s+2} - \Sigma H_{n+s+1}$$

$$= nH_{n+s+2} - H_{n+s+3} \tag{1.7}$$

$k=2$:

$$\Sigma n^2 H_{n+s} = n^2 H_{n+s+2} - \Sigma(2n-1)H_{n+s+1}$$

$$= n^2 H_{n+s+2} - 2nH_{n+s+3} + 2H_{n+s+4} + H_{n+s+3}$$

$$= (n^2+2)H_{n+s+2} + (3-2n)\,H_{n+s+3} \tag{1.8}$$

Results (1.6), (1.7) and (1.8) suggest that there is a general form:

$$\Sigma n^k H_{n+s} = A_k H_{n+s+2} + B_k H_{n+s+3} \tag{1.9}$$

where A_k, B_k are polynomials in n [3].
To determine the form of these polynomials consider:

$$\Sigma n^k H_{n+s} = n^k H_{n+s+2} - \Sigma(\Delta n^k)H_{n+s+1} \tag{1.10}$$

Now

$$\Delta n^k = n^k - \sum_{r=0}^{k} (-1)^r \binom{k}{r} n^{k-r} = \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} n^{k-r}$$

(1.10) now becomes:

$$\Sigma n^k H_{n+s} = n^k H_{n+s+2} + \left(\sum_{r=1}^{k} (-1)^r \binom{k}{r} n^{k-r} \right) H_{n+s+1}$$

$$= n^k H_{n+s+2} + \sum_{r=1}^{k} (-1)^r \binom{k}{r} (A_{k-r} H_{n+s+3} + B_{k-r} H_{n+s+4})$$

$$= \left(n^k + \sum_{r=1}^{k} (-1)^r \binom{k}{r} B_{k-r} \right) H_{n+s+2}$$

$$+ \left(\sum_{r=1}^{k} (-1)^r \binom{k}{r} (A_{k-r} + B_{k-r}) \right) H_{n+s+3}$$

Compare this with (1.9) and we have:

$$A_k = n^k + \sum_{r=1}^{k} (-1)^r \binom{k}{r} B_{k-r}$$

(1.11)

$$B_k = \sum_{r=1}^{k} (-1)^r \binom{k}{r} (A_{k-r} + B_{k-r})$$

(1.11) and $A_0 = 1$; $B_0 = 0$ give us a way to find A_k, B_k for any non-negative integer k. Using (1.9) we then have the required sum. This is not a very convenient formula to deal with as the values of A_k, B_k given at the end of this article clearly show.

4. $\Sigma H_n H_{n+s}$

This form is chosen rather than one with $n + u$ and $n + v$ as subscripts because we can obtain this sum by putting $n + u$ in place of n and letting $s = v - u$.

Consider:

$$\Delta H_n H_{n+s} = H_n H_{n+s-2} + H_{n+s-1} H_{n-2}$$

(a) put $s = 1$

$$\Delta H_n H_{n+1} = H_n^2 \quad \text{i.e.,} \quad \Sigma H_n^2 = H_n H_{n+1}$$

(b) put $s = 0$

$$\Delta H_n^2 = H_{n-2}H_{n+1} \text{ i.e., } \Sigma H_n H_{n+3} = H_{n+2}^2$$

Combining these last two together

$$\Sigma H_n(AH_n + BH_{n+3}) = AH_nH_{n+1} + BH_{n+2}^2 \tag{1.12}$$

Now

$$AH_n + BH_{n+3} = (A + B)H_n + 2BH_{n+1}$$

so recalling (1.3) we can make (1.12) the required sum if

$$A + B = F_{s-1} \text{ and } 2B = F_s .$$

Let

$$B = \tfrac{1}{2}F_s \quad \text{and} \quad A = F_{s-1} - \tfrac{1}{2}F_s = \tfrac{1}{2}F_{s-3} :$$

(1.12) becomes:

$$\Sigma H_n H_{n+s} = \tfrac{1}{2}(F_{s-3}H_nH_{n+1} + F_sH_{n+2}^2) \tag{1.13}$$

5. $\Sigma H_n H_{n+r} H_{n+s}$

Let

$$h(n) = H_{n-1}H_{n+1} - H_n^2 = D(-1)^n$$

see (1.2)

$$H_{n-1}H_nH_{n+1} - H_n^3 = h(n)H_n$$

Now

$$\Sigma h(n)H_n = D\Sigma(-1)^n H_n = D(-1)^n H_{n-1}$$

from (1.5)

Thus:

$$\Sigma H_{n-1}H_nH_{n+1} - \Sigma H_n^3 = D(-1)^n H_{n-1} \tag{1.14}$$

We can sum H_n^3 by parts:

$$\Sigma H_n^3 = H_n \cdot H_nH_{n+1} - \Sigma H_{n-2}H_{n-1}H_n$$

Rearranging:

$$\Sigma H_{n-1}H_nH_{n+1} + \Sigma H_n^3 = H_n^2H_{n+1} + H_{n-1}H_nH_{n+1} = H_nH_{n+1}^2 \tag{1.15}$$

From (1.14) and (1.15) we have:

$$\Sigma H_{n-1}H_nH_{n+1} = \tfrac{1}{2}(H_nH_{n+1}^2 + D(-1)^n H_{n-1}) \tag{1.16}$$

and:

$$\Sigma H_n^3 = \tfrac{1}{2}(H_nH_{n+1}^2 - D(-1)^n H_{n-1})$$

We now have two particular cases of the summation required. If we had

$$\Sigma H_n^2 H_{n+1}$$

as well as

$$\Sigma H_n^3$$

then by using the method of Section 4, we could generate $\Sigma H_n^2 H_{n+r}$

$$\begin{aligned}\Sigma H_n^2H_{n+1} &= H_{n+1} \cdot H_nH_{n+1} - \Sigma H_{n-1}H_n \cdot H_{n-1} \\ &= H_nH_{n+1}^2 - \Sigma H_n^2H_{n+1} + H_n^2H_{n+1}\end{aligned}$$

Thus:

$$\Sigma H_n^2 H_{n+1} = \tfrac{1}{2} H_n H_{n+1} H_{n+2} \tag{1.17}$$

Combining this with H_n^3 as promised:

$$\Sigma H_n^2 H_{n+r} = \tfrac{1}{2}(F_{r-1}(H_n H_{n+1}^2 - D(-1)^n H_{n-1}) + F_r H_n H_{n+1} H_{n+2}) \tag{1.18}$$

To complete the generalization we require, in addition to the result just derived,

$$\Sigma H_n H_{n+1} H_{n+r}$$

Now:

$$\begin{aligned} H_n H_{n+1} H_{n+r} &= H_n H_{n+1}(F_{r-1} H_n + F_r H_{n+1}) \\ &= F_{r-1} H_n^2 H_{n+1} + F_r H_n H_{n+1}^2 \end{aligned}$$

Using (1.18)

$$\begin{aligned} \Sigma H_n H_{n+1} H_{n+r} &= \tfrac{1}{2} F_{r-1} H_n H_{n+1} H_{n+2} \\ &\quad + \tfrac{1}{2} F_r (H_{n+1}^2 H_{n+2} - D(-1)^n H_n) \end{aligned} \tag{1.19}$$

All that remains now is to combine (1.18) and (1.19) in the same sort of way.

$$\begin{aligned} 2\Sigma H_n H_{n+r} H_{n+s} &= F_{s-1} F_{r-1}(H_n H_{n+1}^2 - D(-1)^n H_{n-1}) + F_{s-1} F_r H_n H_{n+1} H_{n+2} \\ &\quad + F_s F_{r-1} H_n H_{n+1} H_{n+2} + F_s F_r (H_{n+1}^2 H_{n+2} - D(-1)^n H_n) \end{aligned} \tag{1.20}$$

Concentrating for the moment on the last term; this is:

$$\begin{aligned} F_s F_r (H_{n+1}^2 H_{n+2} - D(-1)^n (H_{n+1} - H_{n-1})) &= F_s F_r (H_{n+1}^2 H_{n+2} + D(-1)^n H_{n-1} \\ &\quad + H_{n+1}(H_n H_{n+2} - H_{n+1}^2)) \end{aligned}$$

Substituting this in (1.20) we have:

$$2\Sigma H_n H_{n+r} H_{n+s} = (F_s F_r - F_{s-1} F_{r-1}) D(-1)^n H_{n-1}$$
$$+ (F_{s-1} F_{r-1} + F_s F_r) H_n H_{n+1}^2$$
$$+ (F_s F_r + F_s F_{r-1} + F_{s-1} F_r) H_n H_{n+1} H_{n+2}$$

and this simplifies down to:

$$2\Sigma H_n H_{n+r} H_{n+s} = (F_s F_r - F_{s-1} F_{r-1}) D(-1)^n H_{n-1} + H_{s+r+n+1} H_n H_{n+1}$$

(1.21)

PUTTING IN THE LIMITS

We end by quoting the generalized summations with limits from 1 to n.

(2.1) $$\sum_{r=1}^{n} a^r H_{r+s} = \frac{a}{a^2 + a - 1} (a^{n+1}(H_{n+s} - H_s) + a^n (H_{n+s+1} - H_{s+1}))$$

provided $a^2 + a - 1 \neq 0$.

(2.2) $$\sum_{r=1}^{n} r^k H_{r+s} = A_k(n) H_{n+s+2} + B_k(n) H_{n+s+3} - A_k(0) H_{s+2} - B_k(0) H_{s+3} ,$$

where $A_k(n)$, $B_k(n)$ can be generated from (1.11).

(2.3) $$\sum_{r=1}^{n} H_r H_{r+s} = \tfrac{1}{2}(F_{s-3}(H_n H_{n+1} - H_0 H_1) + F_s (H_{n+2}^2 - H_2^2))$$

(2.4) $$\sum_{r=1}^{n} H_r H_{r+s} H_{r+t} = \tfrac{1}{2}(D(F_s F_t - F_{s-1} F_{t-1})((-1)^n H_{n-1} - H_{-1})$$
$$+ H_{s+t+n+1} H_n H_{n+1} - H_{s+t+1} H_0 H_1)$$

THE POLYNOMIALS A AND B

Let

$$X_k(n) = a_0 + a_1 n + \cdots + a_p n^p + \cdots + a_q n^q .$$

The table below gives the coefficients a_p of the polynomials A_k, B_k .

$X_k(n)$	a_0	a_1	a_2	a_3	a_4	a_5
A_0	1	0	0	0	0	0
B_0	0	0	0	0	0	0
A_1	0	1	0	0	0	0
B_1	-1	0	0	0	0	0
A_2	2	0	1	0	0	0
B_2	3	-2	0	0	0	0
A_3	-12	6	0	1	0	0
B_3	-19	9	-3	0	0	0
A_4	98	-48	12	0	1	0
B_4	129	-76	18	-4	0	0
A_5	-870	490	-120	20	0	1
B_5	-1501	795	-190	30	-5	0

REFERENCES

1. For a different symbolism and slightly different definition see "Finite Difference Equations," Levy and Lessman, Pitman, London, 1959.
2. Solution to H-17, Erbacher and Fuchs, Fibonacci Quarterly, Vol. 2 (1964), No. 1, p. 51.
3. Solution to B-29, Parker, Fibonacci Quarterly, Vol. 2 (1964), No. 2, p. 160.

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PERIODICITY AND DENSITY OF MODIFIED FIBONACCI SEQUENCES

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1. INTRODUCTION

Periodicity of the last digit (or last two digits and so on) in a Fibonacci sequence has been discussed by Geller [1], use being made of a digital computer, and solved theoretically by Jarden [2]. We may regard this as a periodic property of the right-most significant digit(s). There is a similar property for truncated Fibonacci sequences, the truncation being carried out prior to addition and on the right. Although this seems to be a somewhat artificial procedure it is the arithmetic involved on digital computers working in "floating point." The periodic property was noted by chance during a study of error propagation.

We generate a modified Fibonacci sequence from the recurrence

$$u_n = u_{n-1} + u_{n-2} \qquad (n = 2, 3, \cdots) \tag{1}$$

where for the moment u_0 and u_1 are arbitrary, but we retain only a certain number of left-most significant digits. To be more specific we work in an x-digit field $(x = 1, 2, \cdots)$ so that members of the sequence take the form

$$u_n = n_1 n_2 n_3 \cdots n_x , \tag{2}$$

where $n_j = 0, 1, \cdots, 9$ $(j = 1, 2, \cdots, x)$. In the addition of two such numbers

$$n_1 n_2 \cdots, n_x + N_1 N_2 \cdots, N_x$$

the sum is the ordinary arithmetic sum provided there is no overflow on the left; if there is an overflow then the sum is taken to be the first x digits from the left, the last digit on the right being discarded. In other words we are merely describing "floating point" arithmetic in frequent usage (to some base or other) on digital machines. For example, denoting the exponent by the symbol E,

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1-digit field $\quad 4\text{ EO} + 5\text{ EO} = 9\text{ EO}$

$6\text{ EO} + 7\text{ EO} = 1\text{ E1}$

2-digit field $\quad 17\text{ EO} + 82\text{ EO} = 99\text{ EO}$

$99\text{ EO} + 9\text{ EO} = 10\text{ E1}$.

Care is needed when the numbers being added do not belong to the same digit field. Thus

$$6\text{ EO} + 1\text{ E1} = 1\text{ E1}$$
$$74\text{ EO} + 14\text{ E1} = 21\text{ E1}$$

and so on. We confine our attention in this note to arithmetic to base ten and discuss some interesting and challenging properties of Fibonacci sequences in floating point arithmetic which have come to light after extensive work on an IBM 1620 computer.

2. CYCLE DETECTION AND PERIODIC PROPERTIES

<u>One digit field</u>

Take any two one-digit non-negative numbers (not both zero) and set up the modified Fibonacci sequence; then sooner or later the sequence invariably leads into the cyclic six-member set

(3) $\quad 1, 1, 2, 3, 5, 8$.

For examples we have

(a) 3 EO, 6 EO, 9 EO, 1 E1, 1 E1, 2 E1, 3 E1, 5 E1, 8 E1 .

(b) 4 EO, 1 EO, 5 EO, 6 EO, 1 E1, 1 E1, 2 E1, 3 E1, 5 E1, 8 E1.

(c) 1 EO, 0 EO, 1 EO, 1 EO, 2 EO, 3 EO, 5 EO, 8 EO, 1 E1 .

It is convenient to drop the E-field symbol and indicate a change of E-field by a star. Thus (a) - (c) become

(A) $3,\ 6,\ 9,\ 1^\star,\ 1,\ 2,\ 3,\ 5,\ 8\ ;$

(B) $4,\ 1,\ 5,\ 6,\ 1^\star,\ 1,\ 2,\ 3,\ 5,\ 8;$

(C) $1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 1^\star\ ;$

where a change of field applies to all members of the sequence following a starred member. A proof of this cyclic property depends on two facts: first a Fibonacci sequence (modified or not) is determined if any two consecutive members are given, and second in view of the non-decreasing nature of the sequences, $1^\star$ must occur with a non-zero predecessor thus leading into the cycle (if it occurred with a zero predecessor the cycle would already be established).

Two-digit field

For this there is the invariant 34-term cycle

$10,\ 16,\ 26,\ 42,\ 68,\ 11^\star,\ 17,\ 28,\ 45,\ 73,\ 11^\star,\ 18,$
$29,\ 47,\ 76,\ 12^\star,\ 19,\ 31,\ 50,\ 81,\ 13^\star,\ 21,\ 34,\ 55,$
$89,\ 14^\star,\ 22,\ 36,\ 58,\ 94,\ 15^\star,\ 24,\ 39,\ 63\ .$

Reading by columns, a few examples are

37	45	74	02	04	91	18	56	77	99
21	64	00	91	04	19	16	93	34	50
58	$10^\star$	74	93	08	$11^\star$	34	$14^\star$	$11^\star$	$14^\star$
79	16	74	$18^\star$	12	12	50	23	14	19
$13^\star$		$14^\star$	27	20	23	84	37	25	33
20		21	45	32	35	$13^\star$	60	39	52
33		35	72	52	58	21	97	64	85
53		56	$11^\star$	84	93		$15^\star$	$10^\star$	$13^\star$
86		91	18	$13^\star$	$15^\star$		24	16	21
$13^\star$		$14^\star$		21	24				
21		23							
		37							
		60							
		97							
		$15^\star$							
		24							

sequences being terminated as soon as the cycle is joined.

x-digit field

Fields of length up to ten have been partially investigated with the following results:

Digit Field x	Cycle Length $L(x)$
1	6
2	34
3	139
4	67
5	3652
6	7455
7	79287
8	121567
9	1141412
10	4193114

Of course a completely exhaustive search for cycles is more or less out of the question; our search has involved some fifty or more cases with the four-digit field decreasing to less than five for the nine- and ten-digit fields. To say the least, the search in the fields of eight or more digits has been scanty; with this reservation in mind we remark that for the cycles so far found only the four-digit field yields different members in the 67-member cycle; in this case, there appear to be eight different cycles.

In passing we note that a modified Fibonacci sequence in an x-digit field must eventually repeat with cycle length less than 10^{2x}. For the sequence is determined by two consecutive members, and 10^{2x} is the number of different ordered pairs of x-digit numbers on base ten. Interest in the periodicity is heightened by the reduction in the observed cycle length as compared to the possible cycle length.

To identify the cycles the least number u_n and its successor u_{n+1} for the various fields x are as follows:

x	1	2	3	4	4	4	4	4	4	4	4
u_n	1	10	104	1004	1006	1010	1012	1015	1019	1026	1029
u_{n+1}	1	16	168	1625	1627	1634	1637	1642	1649	1660	1665

x	5	6	7	8	9	10
u_n	10002	103670	1616568	16167257	161803186	1618033864
u_{n+1}	16184	167741	2615662	26159171	261803054	2618033786

With these values the complete cycles can be generated without introducing alien members. It will be observed that the ratio u_{n+1}/u_n is near to its expected value $(1 + \sqrt{5})/2 = 1.6180339885$ and increasingly so as the field length increases. In fact for the last six fields the ratio is as follows:

x	5	6	7	8	9	10
u_{n+1}/u_n	1.618076	1.618028	1.6180340	1.61803397	1.618033985	1.6180339887

Cycle Detection

Since members of a cycle beginning with a nine are far less common than for other leading digits, as we shall illustrate in the sequel, cycles are easiest to detect if a search is made for its largest members. Thus if we list the members beginning with nine and their successors, all we have to do is to generate a sequence until a matching pair appears. Cycle lengths are then readily picked up by sorting into order of magnitude the output of largest members at any given stage. The largest members in the various cycles we have found are

Field Length	Largest Member
1	8
2	94
3	958
4	9705, 9765, 9854, 9917
5	99810
6	999916
7	9999866
8	99998612
9	999998685
10	9999999229

3. FRACTION OF CYCLE WITH SPECIFIED LEADING DIGIT

An examination of the two-digit field cycle shows that 11 members have leading digit unity whereas only one member has leading digit nine. Is there an indication here of a general property? With this in mind an analysis of all the cycles available is given in Table 1.

Table 1

Fraction of Members of a Cycle with Stated Leading Digit for Different Fields

x = field length y = leading digit entry = corresponding fraction

x \ y	1	2	3	4	5	6	7	8	9
1	.33333	.16667	.16667	.00000	.16667	.00000	.00000	.16667	.00000
2	.32353	.17647	.11765	.08824	.08824	.05882	.05882	.05882	.02941
3	.30216	.17266	.12950	.09353	.07914	.07194	.05036	.05755	.04317
4	.29851	.17910	.13433	.08955	.07463	.07463	.05970	.04478	.04478
4	.31343	.16418	.13433	.08955	.08955	.05970	.05970	.04478	.04478
4	.29851	.17910	.11940	.10448	.07463	.05970	.05970	.05970	.04478
4	.29851	.17910	.11940	.08955	.08955	.05970	.05970	.05970	.04478
4	.29851	.17910	.11940	.10448	.07463	.07463	.05970	.04478	.04478
4	.29851	.17910	.11940	.10448	.07463	.07463	.05970	.04478	.04478
4	.29851	.17910	.13433	.08955	.07463	.07463	.05970	.04478	.04478
4	.29851	.17910	.11940	.10448	.07463	.07463	.04478	.05970	.04478
5	.30121	.17607	.12486	.09693	.07914	.06709	.05805	.05093	.04573
6	.30101	.17612	.12488	.09698	.07914	.06694	.05795	.05124	.04574
7	.30103	.17608	.12494	.09691	.07918	.06695	.05799	.05116	.04576
8	.30104	.17609	.12494	.09691	.07918	.06694	.05799	.05116	.04575
9	.30103	.17609	.12494	.09691	.07918	.06695	.05799	.05115	.04576

This table of fractional occurrences is of considerable interest. Notice that as the field size increases the fractional values become smoother for a given value of x. Moreover the fractions become closer to $\log_{10}(y+1) - \log_{10} y$ as x increases. In fact we have

y	$\log_{10}(y+1)/y$
1	.301030
2	.176091
3	.124939
4	.096910
5	.079181
6	.066947
7	.057992
8	.051153
9	.045758

For the nine-digit field the fractional values agree with those of the logarithmic difference to six decimal places excepting the two values for $y = 8, 9$, for which there is a discrepancy of one in the last decimal place.

It is interesting to recall that certain distributions of random numbers follow the "abnormal" logarithmic law. For example, it has been observed that there are more physical constants with low order first significant digits than high, and that logarithmic tables show more thumbing for the first few pages than the last. The interested reader in this aspect of the subject may care to refer to a paper by Roger S. Pinkham [3]. Pinkham remarks that the only distribution for first significant digits which is invariant under a scale change is $\log_{10}(y + 1)$. Following up the idea of the effect of a scale change we have taken each field cycle and multiplied the members by $k = 1, 2, \cdots, 9$ and compared the fractional occurrence of members with a given leading digit. A comparison over the k's for a particular field shows remarkable stability. The results of a field of five are given in Table 2. Results for larger fields show about the same stability.

5. CONCLUDING REMARKS

A number of interesting questions suggest themselves as follows:

(a) Is there an analytical tool which could be used to formulate the modified Fibonacci series for a specified field length? Perhaps one of the difficulties here, as pointed out by a referee, is the "one-way" nature of the sequences generated.

Table 2

Density of Members of Cycle According to Leading Digit
For Scaled-Up Field of Five

Scale Factor k Leading Digit y x = 5

k \ y	1	2	3	4	5	6	7	8	9
k=1	.30093	.17606	.12486	.09693	.07913	.06709	.05778	.05148	.04573
2	.30120	.17606	.12486	.09693	.07913	.06709	.05778	.05148	.04545
3	.30093	.17634	.12486	.09693	.07913	.06681	.05805	.05120	.04573
4	.30093	.17606	.12513	.09693	.07913	.06709	.05778	.05120	.04573
5	.30093	.17606	.12486	.09721	.07941	.06681	.05805	.05093	.04573
6	.30093	.17606	.12486	.09721	.07913	.06709	.05778	.05120	.04573
7	.30093	.17606	.12486	.09666	.07941	.06709	.05805	.05120	.04573
8	.30093	.17606	.12486	.09693	.07913	.06681	.05832	.05120	.04573
9	.30120	.17579	.12513	.09666	.07913	.06681	.05805	.05148	.04573

(b) Have all the periods been found for fields of length up to $x = 10$? Are the period lengths the same for a given field length and are there cases similar to $x = 4$ in which there are several periods of the same length?

(c) Is there an asymptotic value for $1(x)$, the cycle length, when x is large?

(d) Is the fact that the density of occurrence of sequence members, with a specified leading digit, follows the so-called logarithmic law, when x is not small, trivial or significant?

6. ACKNOWLEDGEMENTS

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ON A CERTAIN INTEGER ASSOCIATED WITH A GENERALIZED FIBONACCI SEQUENCE

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1. INTRODUCTION

A generalized Fibonacci sequence may be defined by specifying two relatively prime integers and applying the formula

(1)
$$y_n = py_{n-1} + y_{n-2} ,$$

where p is a fixed positive integer ($p = 1$ gives a Fibonacci sequence).

If y_0 is the smallest non-negative term determined by (1), then $y_1 \geq (p + 1)y_0$ with strict inequality for $y_0 > 1$ except in the case $y_0 = y_1 = 1$. In order to avoid trivial exceptions to various statements below, we assume with no real loss of generality that $y_1 > y_0 > 0$ in all that follows.

It has been shown in [1] that the Fibonacci sequences can be ordered using the quantity $y_1^2 - y_0y_1 - y_0^2$. Similarly, the generalized Fibonacci sequences defined in (1) may be ordered using the quantity D defined by

$$D = y_1^2 - py_0y_1 - y_0^2 .$$

It may be of interest to determine for given p the possible values of D and how many generalized Fibonacci sequences can be associated with a given value of D.

We solve completely the cases $p = 1, 2$ which, as will be seen, are essentially simpler than the cases $p \geq 3$. Our proofs make use of the classical theory of binary quadratic forms of positive discriminant

$$d = p^2 + r .$$

A good treatment of this subject is found in [2], which we refer to frequently as a source of the proofs of well-known results.

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Let S_p be the set of positive integers D such that the congruence

$$n^2 \equiv d \mod 4D$$

has solutions for n. We prove the following:

Theorem 1. For $p = 1, 2$, S_p is the set of possible values of the integer $D = y_1^2 - py_0y_1 - y_0^2$ associated with the generalized Fibonacci sequence defined by (1).

Theorem 2. For $p = 1, 2$, let r be the number of distinct odd primes dividing $4D/(d, 4D)$. Then except for the trivial case $p = D = 2$ there are 2^{r+1-p} distinct pairs y_0, y_1 such that $D = y_1^2 - py_0y_1 - y_0^2$ and y_0, y_1 generate a generalized Fibonacci sequence defined by (1), i.e., there are 2^{r+1-p} distinct sequences associated with the given value of D.

The case $p = 1$ of Theorem 1 has been previously proved in [3].

2. REMARKS FOR THE CASE OF GENERAL p

Our problem is to determine all positive integers D which are properly represented (i.e., are represented with x and y relatively prime) by the form

$$Q = x^2 - pxy - y^2$$

with the restriction that

$$x \geq (p + 1)y \geq 0 \tag{2}$$

We denote the quadratic form $ax^2 + bxy + cy^2$ by (a, b, c). We say the ordered pair $(x, y) = (\alpha, \gamma)$ is a proper representation of m by (a, b, c) if α and γ are relatively prime and $a\alpha^2 + b\alpha\gamma + c\gamma^2 = m$.

Lemma 1. Let (α, γ) be a proper representation of the positive integer D by the integral form (a, b, c) of discriminant d. Then there exist unique integers β, δ, n satisfying

$$\alpha\delta - \beta\gamma = 1$$

$$0 \leq n < 2D \tag{3}$$

$$n^2 \equiv d \mod 4D \tag{4}$$

and such that the transformation

$$\begin{aligned} x &= \alpha x' + \beta y' \\ y &= \gamma x' + \delta y' \end{aligned} \tag{5}$$

replaces (a, b, c) by the equivalent form (D, n, k) in which k is determined by

$$n^2 - 4Dk = d$$

Proof. This is a classical result ([2, p. 74, Th. 58]).

Corollary. Q properly represents a positive integer D only if D belongs to the set S_p.

Following [2, p, 74] we call a root n of (4) which satisfies (3) a minimum root. Since n is a root of (4) if and only if $n + 2D$ is also a root, the number of minimum roots is half the total number of roots. By Lemma 1, a proper representation of D by a form (a, b, c) is associated with a unique minimum root of (4).

Lemma 2. Every automorph (5) of the integral form (a, b, c) of discriminant d, where a, b, c have no common divisor 1, has

$$\alpha = \tfrac{1}{2}(u - bv) \qquad \beta = -cv \qquad \gamma = av \qquad \delta = \tfrac{1}{2}(u + bv)\,, \tag{6}$$

where u and v are integral solutions of

$$u^2 - dv^2 = 4\,. \tag{7}$$

Conversely, if u and v are integral solutions of (7), the numbers (6) are integers and define an automorph.

Proof. This is a classical result ([2, p. 112, Th. 87]).

Lemma 3. For given D in S_p, there is associated with a given minimum root n of (4) at most one proper representation of D by $(1, -p, -1)$, which satisfies (2).

Proof. Let (α, γ) be a proper representation of D by (1, -p, -1) satisfying (2) and associated with the minimum root n of (4). For the given D and n, it is clear that any proper representation (α', γ') of D by (1, -p, -1) is the first column of a matrix

$$A\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix},$$

where A is the matrix of some automorph of (1, -p, -1). Thus it is enough to show that (α', γ') does not satisfy (2) unless A is the identity matrix.

Since the smallest positive solution of the equation (7) is obviously (u, v) = $(p^2 + 2, p)$, it follows from Lemma 2 that every automorph of (1, -p, -1) is of the form

$$A = \begin{bmatrix} p^2+1 & p \\ p & 1 \end{bmatrix}^m \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^j \quad \begin{array}{l} j = 1 \text{ or } 2 \\ m = 0, \pm 1, \pm 2, \cdots \end{array}$$

We need only consider non-negative m, because for negative m (α', γ') clearly has components of opposite sign. Obviously (α', γ') does not satisfy (2) for j = 1 and any $m \geq 0$. For j = 2, m = 0, $(\alpha', \gamma') = (\alpha, \gamma)$ satisfies (2) by hypothesis; but this is false for j = 2, m = 1 because

$$(p + 1)(p\alpha + \gamma) \geq (p^2 + 1)\alpha + p\gamma.$$

Then by induction (α', γ') does not satisfy (2) for j = 2 and any $m \geq 1$. This proves the lemma.

3. CASE p = 1 OF THEOREM 1

Lemma 4. S_1 is made up of

1. The integers 1 and 5
2. all primes $\equiv 1$ or 9 mod 10
3. all products of the above integers $\not\equiv 0$ mod 25.

Proof. By definition, S_1 is the set of positive integers D such that the congruence

$$n^2 \equiv 5 \mod 4D \tag{8}$$

has solutions for n. Thus we must have $D \not\equiv 0 \mod 25$ and D odd, since

$$\left(\frac{5}{8}\right) = -1 .$$

So it is enough to show that (8) is soluble for odd prime D if and only if D = 5, or $D \equiv 1$ or 9 mod 10.

By the definition of the Legendre symbol, (8) is soluble for odd prime D if and only if

$$\left(\frac{5}{D}\right) = 1 .$$

But then by quadratic reciprocity and the fact that D is odd

$$\left(\frac{5}{D}\right) = \left(\frac{D}{5}\right) = \begin{cases} 1 & \text{if } D \equiv 1 \text{ or } 4 \mod 5 \\ -1 & \text{if } D \equiv 2 \text{ or } 3 \mod 5 \end{cases}$$

which implies the desired result.

Lemma 5. If D belongs to S_1, then (1, -1, -1) properly represents D. Further, associated with each minimum root of (8) there is at least one proper representation satisfying (2) with p = 1.

Proof. We consider each of the minimum roots of (8). Let (α, γ) be a proper representation of D by (1, -1, -1) associated with a given minimum root n.

We may suppose $\alpha > 0$, $\gamma > 0$. For if $\alpha < 0$, $\gamma < 0$, we consider $(-\alpha, -\gamma)$. If one and only one of α, γ is negative we may suppose it is α. Then we apply the automorph

$$\begin{aligned} x' &= 2x + y \\ y' &= x + y \end{aligned} \tag{9}$$

of (1, -1, -1) successively to (α, γ), getting the sequence

$$(\alpha, \gamma),\ (2\alpha + \gamma,\ \alpha + \gamma),\ \cdots,\ (f_{2m+1}\alpha - f_{2m}\gamma,\ f_{2m}\alpha + f_{2m-1}\gamma),\ \cdots$$

where f_i is the i^{th} member of the Fibonacci sequence $1, 1, 2, 3, 5, \cdots$, If for some m we have

$$f_{2m}\,|\alpha| > f_{2m-1}\,\gamma\ , \tag{10}$$

then

$$(-f_{2m+1}\alpha - f_{2m}\gamma\,,\ -f_{2m}\alpha - f_{2m-1}\gamma)$$

is a proper representation with both members positive, as desired. But (10) must be true for some m because $\gamma = k|\alpha|$ for some rational $k > 0$ and

$$\alpha^2 - \alpha\gamma - \gamma^2 > 0$$

implies

$$k < (1 + \sqrt{5})/2\ ;$$

whereas from the continued fraction expansion of $(1 + \sqrt{5})/2$ we have

$$1 < \frac{3}{2} < \frac{8}{5} < \cdots < \frac{f_{2m}}{f_{2m-1}} < \cdots < \frac{1 + \sqrt{5}}{2}$$

and

$$\lim_{m \to \infty} \frac{f_{2m}}{f_{2m-1}} = \frac{1 + \sqrt{5}}{2}\ .$$

Given a proper representation (α, γ) with both members positive, we apply the inverse of the transformation (9) successively, getting the sequence

$$(\alpha, \gamma),\quad (\alpha - \gamma,\ -\alpha + 2\gamma)\ , \cdots\ ,$$

$$(f_{2m-1}\alpha - f_{2m}\gamma,\ -f_{2m}\alpha + f_{2m+1}\gamma), \cdots$$

Since the successive first members make up a decreasing sequence of positive integers so long as the corresponding second members are positive, we must reach an m such that

$$f_{2m+1}\gamma > f_{2m}\alpha \quad \text{and} \quad f_{2m+3}\gamma < f_{2m+2}\alpha .$$

Then

$$(f_{2m+1}\alpha - f_{2m}\gamma, \; -f_{2m}\alpha + f_{2m+1}\gamma)$$

is a proper representation satisfying (2) with $p = 1$.

All transformations used above of course have determinant 1, so that the minimum root n associated with the originally given proper representation is not changed.

4. CASE $p = 2$ OF THEOREM 1

Lemma 6. S_2 is made up of

1. the integers 1 and 2
2. all primes $\equiv 1$ or $7 \bmod 8$
3. all products of the above integers $\not\equiv 0 \bmod 4$.

Proof. By definition, S_2 is the set of positive integers D such that the congruence

$$n^2 \equiv 8 \mod 4D \tag{11}$$

has solutions for n. Thus we must have $D \not\equiv 0 \mod 4$. Then the result follows from the fact that for odd prime D

$$\left(\frac{2}{D}\right) = \begin{cases} 1 & \text{if } D \equiv 1 \text{ or } 7 \bmod 8 \\ -1 & \text{if } D \equiv 3 \text{ or } 5 \bmod 8 \end{cases}$$

Lemma 7. If D belongs to S_2, then $(1, -2, -1)$ properly represents D. Further, associated with exactly half of the total number of minimum roots of (11) there is at least one proper representation satisfying (2) with $p = 2$.

Proof. We consider each of the minimum roots of (11). Let (α, γ) be a proper representation of D by $(1, -2, -1)$ associated with a given minimum root n.

We argue as in Lemma 5 that we may suppose $\alpha < 0$, $\gamma < 0$. For if $\alpha < 0$, $\gamma < 0$ we consider $(-\alpha, -\gamma)$. If one and only one of α, γ is negative, we may suppose it is α. Then we apply the automorph

$$\begin{aligned} x' &= 5x + 2y \\ y' &= 2x + y \end{aligned} \tag{12}$$

of $(1, -2, -1)$ successively to (α, γ), getting the sequence

$$(\alpha, \gamma),\ (5\alpha + 2\gamma,\ 2\alpha + \gamma), \cdots,$$
$$(g_{2m+1}\alpha + g_{2m}\gamma,\ g_{2m}\alpha + g_{2m-1}\gamma), \cdots$$

where g_i is the i^{th} member of the generalized Fibonacci sequence 1, 2, 5, 12, 29, $\cdots$. If for some m we have

$$g_{2m}|\alpha| > g_{2m-1}\gamma \ , \tag{13}$$

then

$$(-g_{2m+1}\alpha - g_{2m}\gamma,\ -g_{2m}\alpha - g_{2m-1}\gamma)$$

is a proper representation with both members positive. But as in the proof of Lemma 5 a consideration of the continued fraction for $1 + \sqrt{2}$ shows that (13) must be true for some m.

Given a proper representation (α, γ) with both members positive, we apply the inverse of the transformation (12) successively, getting the sequence

$$(\alpha, \gamma),\ (\alpha - 2\gamma,\ -2\alpha + 5\gamma), \cdots, (g_{2m-1}\alpha - g_{2m}\gamma,\ -g_{2m}\alpha + g_{2m+1}\gamma), \cdots$$

Since the successive first members make up a decreasing sequence of positive integers so long as the corresponding second members are positive, we must reach an m such that

$$g_{2m+1}\gamma > g_{2m}\alpha \ \text{ and } \ g_{2m+3}\gamma < g_{2m+2}\alpha \ .$$

Then

$$(\alpha_0, \gamma_0) = (g_{2m-1}\alpha - g_{2m}\gamma, -g_{2m}\alpha + g_{2m+1}\gamma)$$

satisfies

$$\alpha_0 > (5/2)\gamma_0 ,$$

and exactly one of (α_0, γ_0) and

$$(\alpha_1, \gamma_1) = (5\alpha_0 - 12\gamma_0, 2\alpha_0 - 5\gamma_0)$$

satisfies (2) with $p = 2$.

The transformation which takes (α_0, γ_0) to (α_1, γ_1) has determinant -1 and $(\alpha_0, \gamma_0), (\alpha_1, \gamma_1)$ are associated with different minimum roots of (11). Thus the last statement of the lemma is easily verified.

5. PROOF OF THEOREM 2

Lemma 8. Let $(c, m) = 1$. Then

$$x^2 \equiv c \mod m$$

has 2^{r+w} roots if it has any roots, where r is the number of distinct odd primes dividing m and w is given by

$$w = \begin{cases} 0 & \text{if 4 does not divide } m \\ 1 & \text{if 4 but not 8 divides } m \\ 2 & \text{if 8 divides } m . \end{cases}$$

Proof. This is a well-known result ([2, p. 75, Th. 60]).

For $p = 1, 2,$ let r be the number of distinct odd primes dividing $4D/(d, 4D)$. It is easy to verify using Lemma 8 that the congruences (8) and (11) have 2^{r+1} roots. Then Theorem 2 follows from Lemmas 3, 5, and 7.

6. CONCLUDING REMARKS

We comment briefly on the reasons for confining detailed discussion above to the cases $p = 1, 2$.

Let $h(d)$ be the number of distinct non-equivalent reduced forms of discriminant d. We can make little progress if $h(d) > 1$, because for such d the problem of determining all positive integers properly represented by $(1, -p, -1)$ even without the restriction (2) is unsolved. We remark that $h(d) = 1$ for $p = 1, 2, 3, 5, 7$, but $h(d) = 2$ for $p = 4, 6$.

However, it is not enough simply to confine ourselves to the study of those p for which $h(d) = 1$. We have seen that for $p = 1, 2$ the converse of Lemma 1 Corollary is true and for any properly representable D a proper representable D a proper representation satisfying (2) can be found. However, for $p \geq 3$ there exist integers D which are properly represented by $(1, -p, -1)$ but which have no proper representation satisfying (2), and it is not simple to describe the subset of S_p composed of such integers.

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ON Q-FIBONACCI POLYNOMIALS

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INTRODUCTION

Throughout this paper we shall use the following notation:

$$\sum_{s_1=a_1}^{b_1} \sum_{s_2=a_2}^{b_2} \cdots \sum_{s_n=a_n}^{b_n} = \left(\Sigma^{(n)},\ s_1\Big|_{a_1}^{b_1},\ \ s_2\Big|_{a_2}^{b_2},\cdots s_n\Big|_{a_n}^{b_n}\right) .$$

Let $F_0, F_1, F_2, \cdots, F_n, \cdots$ be the sequence of Fibonacci numbers, i.e., 0, 1, 1, 2, 3, 5, 8, $\cdots$. According to [1] we define $n, m, k \geq 0$.

$$(1) \qquad Q(x;1,-F,n) = \eta(x,k,n) = \prod_{m=1}^{n} (1 - xF_{k+m}) = \sum_{s=0}^{n} A(k,n,s)x^s ,$$

$$(2) \qquad \eta(x,k,0) = 1 ,$$

$$(3) \qquad x^n = \sum_{m=0}^{n} B(k,n,m)\eta(x,k,m)$$

$$(4) \qquad 1 = B(k,0,0)\eta(x,k,0) ,$$

$$(5) \qquad A(k,n,s), \quad B(k,n,m) = 0 \quad \text{for} \quad n < m, \ n < 0, \ m < 0 .$$

The A and B numbers are quasi-orthogonal. (For a set of comprehensive definitions of orthogonality and quasi-orthogonality cf. [3].) Thus

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$$\sum_{s=m}^{n} B(k,n,s)A(k,s,m) = \delta_n^m \quad , \tag{6}$$

where δ_n^m is the Kronecker Delta.

Still according to [1] the A and B numbers satisfy the difference equations

$$A(k,n,m) = A(k,n-1,m) - F_{n+k}A(k,n-1,m-1) \tag{7}$$

$$B(k,n,m) = (F_{m+1+k})^{-1}B(k,n-1,m) - (F_{m+k})^{-1}B(k,n-1,m-1) \quad , \tag{8}$$

where the error in Eqs. (10) and (12) of [1] has been corrected.

2. BASIC RELATIONS

According to the preceding definitions we can write

$$\eta(x,k,n) = \prod_{m=1}^{n}(1 - xF_{k+m}) = \prod_{m=1}^{p}(1 - xF_{k+m})\prod_{m=p+1}^{n}(1 - xF_{k+m})$$

$$= \eta(x,k,p)\prod_{m=p+1}^{n}(1 - xF_{k+m}) \ .$$

In the last product we take $m - p = s$, $m = s + p$, so that for $m = p + 1$, $s = 1$, and for $m = n$, $s = n - p$, thus

$$\prod_{m=p+1}^{n}(1 - xF_{k+m}) = \prod_{s=1}^{n-p}(1 - xF_{k+p+s}) = \eta(x,k+p,n-p),$$

i.e.,

$$\eta(x,k,n) = \eta(x,k,p)\eta(x,k+p,\ n-p) \quad , \tag{9}$$

or,

$$(10) \qquad \eta(x, k, n + p) = \eta(x, k, p)\eta(x, k + p, n) \ .$$

By substitution into (10) of the polynomial form for the η's we obtain

$$(11) \qquad \sum_{m=0}^{n} A(k, n + p, m)x^m = \left[\sum_{s=0}^{p} A(k, p, s)x^s\right]\left[\sum_{t=0}^{n} A(k + p, n, t)x^t\right] ,$$

so that equating the coefficients of same powers of x we have with $s + t = m$,

$$(12) \qquad A(k, n + p, m) = \sum_{s=0}^{m} A(k, p, s)A(k + p, n, m - s)$$

which is a convolution formula for the A numbers. Also

$$x^n = \sum_{m=0}^{n} B(k, n, m)\eta(x, k, m) \ , \quad x^p = \sum_{s=0}^{p} B(k + p, p, s)\eta(x, k + p, s) \ ,$$

hence,

$$\begin{aligned} x^{n+p} &= \sum_{t=0}^{n+p} B(k, n + p, t)\eta(x, k, t) \\ &= \left[\sum_{m=0}^{n} B(k, n, m)\eta(x, k, m)\right]\left[\sum_{s=0}^{p} B(k + p, p, s)\eta(x, k + p, s)\right] \\ &= \left(\sum\nolimits^{(2)}, \ m\Big|_{0}^{n}, \ s\Big|_{0}^{p}\right) B(k, n, m)B(k + p, p, s)\eta(x, k, m)\eta(x, k + p, s) \ . \end{aligned}$$

By comparing the coefficients of $\eta(x, k, t)$ and using (10) with $m + s = t$ we obtain

$$B(k, n + p, t) = \sum_{m=0}^{t} B(k, n, m) B(k + p, p, t - m) , \tag{13}$$

which is a convolution formula for the B numbers.

3. LAH TYPE RELATIONS

According to [2] we have for $k \neq h$

$$\sum_{s=m}^{n} A(k, n, s) B(h, s, m) = L(k, h, n, m) \tag{14}$$

$$\sum_{s=m}^{n} A(h, n, s) B(k, s, m) = L(h, k, n, m) \tag{15}$$

$$\eta(x, j, n) = \sum_{m=0}^{n} \eta(x, i, m) L(j, i, n, m) , \tag{16}$$

where $k, h = i, j$, with $i \neq j$. Again according to [2] there is a quasi-orthogonality relation between the Lah numbers:

$$\sum_{s=m}^{n} L(i, j, n, s) L(j, i, s, m) = \delta_n^m . \tag{17}$$

Still according to [2] the recurrence relation for Lah numbers is

$$\text{(18)}\qquad L(i,j,n,m) = \left[1 - (F_{j+n}\big/F_{i+m+1})\right]L(i,j,n-1,m) + (F_{j+n}\big/F_{i+m})L(i,j,n-1,m-1)\ .$$

4. GENERALIZATION TO THREE VARIABLES

Although we could generalize to p variables we prefer to limit ourselves to $p = 3$ for the sake of simplicity. Let

$$\text{(19)}\qquad \begin{aligned}\eta(x,y,z;\,k,h,j;\,n) &= \prod_{m=1}^{n}(3 - xF_{k+m} - yF_{h+m} - zF_{j+m})\\ &= \left(\sum^{(3)}, r\Big|_0^n,\ s\Big|_0^n,\ t\Big|_0^n\right)A(k,h,j;n,n,n;r,s,t)\cdot\\ &\quad\cdot x^r y^s z^t\,, \qquad r+s+t \le n\,.\end{aligned}$$

$$\text{(20)}\qquad \eta(x,y,z;\,k,h,j;\,0) = 1\,.$$

To find an inversion formula for (19) we use (3), i.e.,

$$x^r = \sum_{m=0}^{r} B(k,r,m)\eta(x,k,m)$$

$$y^s = \sum_{p=0}^{s} B(h,s,p)\eta(y,h,p)$$

$$z^t = \sum_{q=0}^{t} B(j,t,q)\eta(z,j,q)\ ,$$

so that

$$x^r y^s z^t = \left(\sum^{(3)}, m\Big|_0^r, \; p\Big|_0^s, \; q\Big|_0^t \right) B(k,r,m)B(h,s,p)B(j,t,q) \cdot$$

(21)
$$\cdot \; \eta(x,k,m)\eta(y,h,p)\eta(z,j,q)$$

$$= \left(\sum^{(3)}, m\Big|_0^r, \; p\Big|_0^s, \; q\Big|_0^t \right) B(k,h,j;\; r,s,t;\; m,p,q) \cdot$$

$$\cdot \; \eta(x,k,m)\eta(y,h,p)\eta(z,j,q) \;,$$

where

(22)
$$B(k,h,j;\; r,s,t;\; m,p,q) = B(k,r,m)B(h,s,p)B(j,t,q).$$

5. QUASI-ORTHOGONALITY RELATIONS

If in the second form of (21) we substitute according to (1) we obtain

$$x^r y^s z^t = \left(\sum^{(3)}, \; m\Big|_0^r, \; p\Big|_0^s, \; q\Big|_0^t \right) B(k,h,j;\; r,s,t;\; m,p,q) \sum_{a=0}^{m} A(k,m,a)x^a \cdot$$

$$\cdot \sum_{b=0}^{p} A(h,p,b)y^b \sum_{c=0}^{q} A(j,q,c)z^c \;,$$

$$= \left(\sum^{(6)}, m\Big|_0^r, \; p\Big|_0^s, \; q\Big|_0^t, \; a\Big|_0^m, \; b\Big|_0^p, \; c\Big|_0^q \right) B(k,h,j;\; r,s,t;\; m,p,q)$$
$$A(k,m,a)A(h,p,b)A(j,q,c)x^a y^b z^c \;.$$

Since the A and B numbers are zero under the conditions stated in the introduction we can extend the limits m, p, q of the summation to n, change the order of summations, and obtain after taking out the zero coefficients

(23)
$$\left(\sum^{(3)}, \; m\Big|_a^r, \; p\Big|_b^s, \; q\Big|_c^t \right) B(k,h,j;\; r,s,t;\; m,p,q) A(k,m,a)A(h,p,b) \cdot$$
$$\cdot \; A(j,q,a) = \delta_a^r \delta_b^s \delta_c^t \;.$$

This relation is actually nothing but the product of three relations of the form given by (6).

6. RECURRENCE RELATIONS

By writing

$$\eta(x, y, z; k, h, j; n + 1) = (3 - xF_{k+n+1} - yF_{h+n+1} - zF_{j+n+1})\eta(x, y, z; k, h, j, n)$$

and substituting according to (19) and equating the coefficients of the same monomials we obtain

$$\begin{aligned} A(k, h, j; n + 1, n + 1, n + 1; r, s, t) &= 3A(k, h, j; n, n, n; r, s, t) \\ &- F_{k+n+1}A(k, h, j; n, n, n; r - 1, s, t) - F_{h+n+1}A(k, h, j; n, n, n; r, s - 1, t) \\ &- F_{j+n+1}A(k, h, j; n, n, n; r, s, t - 1) \quad , \end{aligned} \tag{25}$$

which is a recurrence realtion satisfied by the A numbers.

To find a recurrence relation satisfied by the B numbers we use (8) and obtain

$$\begin{aligned} B(k, r, m) &= (F_{m+1+k})^{-1}B(k, r - 1, m) - (F_{m+k})^{-1}B(k, r - 1, m - 1) \\ B(h, s, p) &= (F_{p+1+h})^{-1}B(h, s - 1, p) - (F_{p+h})^{-1}B(h, s - 1, p - 1) \\ B(j, t, q) &= (F_{q+1+j})^{-1}B(j, t - 1, q) - (F_{q+j})^{-1}B(j, t - 1, q - 1) \ , \end{aligned}$$

and by multiplying these three relations by each other and using (22) we have the following recurrence relation for the B numbers:

$$\begin{aligned} B(k, h, j; r, s, t; m, p, q) &= (F_{m+1+k}F_{p+1+h}F_{q+1+j})^{-1} \cdot \\ &\quad B(k, h, j; r - 1, s - 1, t - 1; m, p, q) \\ &-(F_{m+1+k}F_{p+1+h}F_{q+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m, p, q - 1) \\ &-(F_{m+1+k}F_{p+h}F_{q+1+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m, p - 1, q) \\ &-(F_{m+k}F_{p+1+h}F_{q+1+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m - 1, p, q) \\ &+(F_{m+1+k}F_{p+h}F_{q+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m, p - 1, q - 1) \\ &+(F_{m+k}F_{p+1+h}F_{q+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m - 1, p, q - 1) \\ &+(F_{m+k}F_{p+h}F_{q+1+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1, m - 1, p - 1, q) \\ &-(F_{m+k}F_{p+h}F_{q+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m - 1, p - 1, q - 1) \ . \end{aligned}$$

7. CONCLUDING REMARKS

(i) Equations (7), (8), (12), (13), (18), (25), and (26) indicate that the coefficients A and B involved are particular solutions of corresponding partial difference equations which may be of interest.

(ii) Although in this paper we have assumed that the numbers F_k are Fibonacci numbers the same relations would hold for any sequence that is defined for k being a positive integer or zero.

(iii) We have not attempted to define Lah numbers corresponding to the A and B numbers in the case of several variables although this seems possible.

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ON THE GENERALIZED LANGFORD PROBLEM

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For n a positive integer, the sequence $a_1, \cdots, a_{2n}$ is said to be a perfect sequence for n if (a) each integer i in the range $1 \le i \le n$ appears exactly twice in the sequence, and (b) the double occurrence of i in the sequence is separated by exactly i entries. Thus 4 1 3 1 2 4 3 2 is a perfect sequence for $n = 4$. The problem of determining all integers n having a perfect sequence is posed in [1] and resolved in [2] and [3]. In particular, n has an associated perfect sequence if and only if $n \equiv 3$ or $4 \pmod 4$.

In [4], the problem is generalized by introducing the notion of a perfect s-sequence for an integer n. Namely, a perfect s-sequence for n (with s, $n > 0$) is a sequence of length sn such that (a) each of the integers $1, 2, \cdots,$ n occurs exactly s times in the sequence and (b) between any two consecutive occurrences of the integer i there are exactly i entries. The problem of determining all s and n for which there are perfect s-sequences is then posed in [4]. (The existence of a perfect s-sequence for any n with $s > 2$ is yet in doubt.) It is shown in [4] that no perfect 3-sequences exist for $n = 2$, 3, 4, 5, and 6.

The following theorems expand upon the above results pertaining to the non-existence of perfect s-sequences for various classes of n and s.

Theorem 1. Let $s = 2t$. Then there is no generalized s-sequence for $n \equiv 1$ or $2 \pmod 4$.

Proof. Let p_i denote the position of the first occurrence of the integer i $(1 \le i \le n)$ in the sequence. The integer i then occurs in positions p_i, $p_i + (i + 1), \cdots,$ $p_i + (s - 1)(i + 1)$. The sn integers $p_i + j(i + 1)$ (with $i = 1,$ $\cdots, n;$ $j = 0, 1, \cdots, s - 1$) are however the integers $1, \cdots, sn$ in some order. Thus

$$\sum_{i=1}^{n} \sum_{j=0}^{s-1} \{p_i + j(i + 1)\} = \sum_{k=1}^{sn} k \quad .$$

Letting

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$$P = \sum_{i=1}^{n} p_i \quad ,$$

the latter equality yields

$$sP + \frac{(s-1)s}{2}\left\{\frac{(n+1)(n+2)}{2} - 1\right\} = \frac{sn(sn+1)}{2}$$

or

$$P = \frac{n\{(s+1)n - (3s-5)\}}{4}$$

Inasmuch as P is an integer, the numerator $N = n\{(s+1)n - (3s-5)\}$ must be divisible by 4. But for $n \equiv 1 \pmod 4$,

$$N \equiv (s+1) - (3s-5) \equiv -4t + 6 \equiv 2 \pmod 4 \; ,$$

where $s = 2t$, which is impossible. Similarly, for $n \equiv 2 \pmod 4$,

$$N \equiv 2\{2(s+1) - (3s-5)\} \equiv -4t + 14 \equiv 2 \pmod 4$$

which is also impossible.

We now extend the results in [4] by proving there is no 3-sequence for $n \equiv 2, 3, 4, 5, 6$, or $7 \pmod 9$. Actually we show somewhat more in the next theorem.

Theorem 2. Let $s = 6r + 3$ (with $r \geq 0$). Then there is no perfect s-sequence for any $n \equiv 2, 3, 4, 5, 6$, or $7 \pmod 9$.

Proof. Let q_i denote the position that integer i occurs for the $(3r + 2)^{th}$ time (i.e., q_i is the position of the "middle" occurrence of i). Then i occurs in positions $q_i + j(i+1)$ for $j = -2(2r+1), -3r, \cdots, 3r, (3r+1)$. The sn integers $q_i + j(i+1)$ (with $i = 1, \cdots, n$; $j = -(3r+1), \cdots, 3r+1$) are then the integers $1, 2, 3, \cdots, sn$ in some order. Thus

$$\sum_{i=1}^{n} \sum_{j=-(3r+1)}^{3r+1} \{q_i + j(i+1)\}^2 = \sum_{k=1}^{sn} k^2 .$$

Letting

$$Q = \sum_{i=1}^{n} q_i^2 ,$$

and noting that the linear terms on the left-hand side of the last equation cancel, we have

$$sQ + 2\left\{\frac{(3r+1)(3r+2)s}{6}\right\}\left\{\frac{(n+1)(n+2)(2n+3)}{6} - 1\right\}$$

$$= \frac{sn(sn+1)(2sn+1)}{6}$$

Cancelling out s and collecting terms yields $Q = M/18$, where the numerator M is given by

$$M = (198r^2 + 198r + 50)n^3 - (81r^2 + 27r - 9)n^2 - (117r^2 + 117r + 23)n .$$

Inasmuch as Q is an integer, the numerator M must be divisible by 9. But

$$M \equiv 50n^3 - 23n \equiv 5(n^3 - n) \pmod 9.$$

It is easily verified from the latter that for the values of n under consideration, namely, $n \equiv 2, 3, 4, 5, 6,$ or $7 \pmod 9$ we have $M \equiv 3$ or $6 \pmod 9$. Thus M is not divisible by 9 which provides a contradiction.

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FIBONACCIAN ILLUSTRATION OF L'HOSPITAL'S RULE

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In [1] there is the statement: using the convention $F_0/F_0 = 1$." [$F_n = F_{n+1} + F_{n-2}$, $F_0 = 0, F_1 = 1$].

In this note it will be shown how the equation $F_0/F_0 = 1$ follows naturally from L'Hospital's Rule applied to the continuous function

$$F_x \equiv \frac{1}{\sqrt{5}}\,(\phi^x - \phi^{-x}\cos\pi x) \qquad [\phi = 2^{-1}(1+\sqrt{5})]\ .$$

F_x obviously reduces to the Fibonacci numbers F_n when n = 0, ±1, ±2, ±3, $\cdots$. Then

$$\frac{F_0}{F_0} = \left.\frac{\frac{1}{\sqrt{5}}(\phi^x - \phi^{-x}\cos\pi x)}{\frac{1}{\sqrt{5}}(\phi^x - \phi^{-x}\cos\pi x)}\right]_{x=0} = \left.\frac{\frac{d}{dx}(\phi^x - \phi^{-x}\cos\pi x)}{\frac{d}{dx}(\phi^x - \phi^{-x}\cos\pi x)}\right]_{x=0}$$

$$= \left.\frac{(\log\phi)\phi^x - (\log\phi^{-1})\phi^{-x}\cos\pi x + \phi^{-x}\pi\sin\pi x}{(\log\phi)\phi^x - (\log\phi^{-1})\phi^{-x}\cos\pi x + \phi^{-x}\pi\sin\pi x}\right]_{x=0}$$

$$= \frac{\log\phi - \log\phi^{-1}}{\log\phi - \log\phi^{-1}} = 1\ .$$

(Continued on p. 150.)

FIBONACCI SEQUENCE MODULO m

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Wall [1] has discussed the period $k(m)$ of Fibonacci sequence modulo m. Here we discuss a somewhat related question of the existence of a complete residue system mod m in the Fibonacci sequence.

We say that a positive integer m is <u>defective</u> if a complete residue system mod m does not exist in the Fibonacci sequence.

It is clear that not more than $k(m)$ distinct residues mod m can exist in the Fibonacci sequence, so that we have:

<u>Theorem 1.</u> If $k(m) < m$, then m is defective.

<u>Theorem 2.</u> If m is defective, so is every multiple of m.

<u>Proof.</u> Suppose tm is not defective. Then for every r, $0 \leq r \leq m - 1$, there exists a Fibonacci number u_n (which, of course, depends on r) for which $u_n \equiv r \pmod{tm}$. But then $u_n \equiv r \pmod{m}$, so that m is not defective.

Remark: The converse is not true; i.e., if m is a composite defective number, it does not follow that some proper divisor of m is defective. For example, 12 is defective, but none of 2, 3, 4 and 6 is.

<u>Theorem 3.</u> For $r \geq 3$ and m odd, $2^r m$ is defective.

<u>Proof.</u> The Fibonacci sequence (mod 8) is

$$1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, 2, 3, 5, \cdots .$$

The sequence is periodic and $k(8) = 12$. It is seen that the residues 4 and 6 do not occur. This proves that 8 is defective. Since for $r \geq 3$, $2^r m$ is a multiple of 8, the theorem follows from Theorem 2.

<u>Theorem 4.</u> If a prime $p \equiv \pm 1 \pmod{10}$, then p is defective.

<u>Proof.</u> For $p \equiv \pm 1 \pmod{10}$, $k(p)\ (p - 1)$ ([1]), and hence $k(p) \leq p - 1 < p$. Therefore by Theorem 1, p is defective.

<u>Theorem 5.</u> If a prime $p \equiv 13$ or $17 \pmod{20}$, then p is defective.

<u>Proof.</u> Let u_n denote the n^{th} Fibonacci number. Since [1] for $p \equiv \pm 3 \pmod{10}$, $k(p) \mid 2(p + 1)$, it is clear that all the distinct residues of p that

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occur in the Fibonacci sequence are to be found in the set $\{u_1, u_2, u_3, \cdots, u_{2(p+1)}\}$. We shall prove that for each t, $1 \leq t \leq 2(p+1)$,

$$u_t \equiv 0 \quad \text{or} \quad u_t \equiv \pm u_r \pmod{p}, \tag{5.1}$$

for some r, where $1 \leq r \leq (p-1)/2$.

Granting for the moment that (5.1) has been proved, it follows that all the distinct residues of p occurring in the Fibonacci sequence are to be found in the set

$$\{0, \pm u_1, \pm u_2, \pm u_3, \cdots, \pm u_m\}, \tag{5.2}$$

where $m = (p-1)/2$; or, since $u_1 = u_2 = 1$, the set (5.2) may be replaced by

$$\{0, \pm 1, \pm u_3, \pm u_4, \cdots, \pm u_m\}. \tag{5.3}$$

But this set contains at most $2(m-1) + 1 = p - 2$ distinct elements. Thus the number of distinct residues of p occurring in the Fibonacci sequence is not more than $p - 2$. Therefore p is defective.

Proof of (5.1). It is easily proved inductively that for $0 \leq r \leq p - 1$,

$$u_{p-r} \equiv (-1)^{r+1} u_{r+1} \pmod{p} \tag{5.4}$$

and that for $1 \leq r \leq p + 1$

$$u_{p+1+r} \equiv -u_r \pmod{p}. \tag{5.5}$$

We note that since $p \equiv \pm 3 \pmod{10}$, $p \mid u_{p+1}$, $u_p \equiv -1 \pmod{p}$ [2, Theorem 180]. (5.4) and (5.5) are valid for all such primes. Replacing r by $(p-1)/2 - s$ in (5.4), we get for $0 \leq s \leq (p-1)/2$.

$$u_{h+s} \equiv (-1)^{s+1} u_{h-s} \pmod{p}, \text{ where } h = (p+1)/2. \tag{5.6}$$

In particular, we note that $p \mid u_m$ for $m = (p+1)/2$, $p+1$, $3(p+1)/2$ and $2(p+1)$.

(5.5) and (5.6) clearly imply (5.1). This completes the proof. Combining Theorems 4 and 5, we have

Theorem 6. If a prime $p \equiv 1, 9, 11, 13, 17$ or $19 \pmod{20}$, then p is defective.

Remarks: This implies that if p is a non-defective odd prime, then $p = 5$ or $p \equiv 3$ or $7 \pmod{20}$. While it is easily seen that 2, 3, 5 and 7 are non-defective, the author has not been able to find any other non-defective primes.

From Theorems 2 and 6, we have

Theorem 7. If $n > 1$ is non-defective, then n must be of the form $n - 2^t m$, m odd, where $t = 0, 1,$ or 2 and all prime divisors of m (if any) are either 5 or $\equiv 3$ or $7 \pmod{20}$. Finally, we prove

Theorem 8. If a prime $p \equiv 3$ or $7 \pmod{20}$, then a necessary and sufficient condition for p to be non-defective is that the set

$$\{0,\ \pm 1,\ \pm 3,\ \pm 4,\ \cdots,\ \pm u_h\}\quad,$$

where $h = (p+1)/2$, is a complete residue system mod p.

Proof. The formulae (5.5) and (5.6) still remain valid. However, for primes $p \equiv 3 \pmod 4$, we cannot prove that $p \mid u_h$ (in fact, $p \nmid u_h$). So that all distinct residues of p occurring in the Fibonacci sequence must be found in the set

$$\{0,\ \pm 1,\ \pm u_3,\ \pm u_4,\ \cdots,\ \pm u_h\}\quad.$$

Since this set contains only p numbers, it can possess all the p distinct residues of p if and only if it is a complete residue system mod p.

The author wishes to express his gratitude to Professor A. M. Vaidya for suggesting the problem and for his encouragement and help in the preparation of this note.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

H-131 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Consider the left-adjusted Pascal triangle. Denote the left-most column of ones as the zeroth column. If we take sums along the rising diagonals, we get Fibonacci numbers. Multiply each column by its column number and again take sums, C_n, along these same diagonals. Show $C_1 = 0$ and

$$C_{n+1} = \sum_{j=0}^{n} F_{n-j} F_j$$

H-132 Proposed by J.L. Brown, Jr., Ordnance Research Lab., State College, Pa.

Let $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n > 0$. Define the Fibonacci sequence. Show that the Fibonacci sequence is not a basis of order k for any positive integer k; that is, show that not every positive integer can be represented as a sum of k Fibonacci numbers, where repetitions are allowed and k is a fixed positive integer.

H-133 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Characterize the sequences

i. $$F_n = u_n + \sum_{j=1}^{n-2} u_j$$

ii.
$$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4}\sum_{j=1}^{i} u_j$$

iii.
$$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4}\sum_{j=1}^{i} u_j + \sum_{m=1}^{n-6}\sum_{i=1}^{m}\sum_{j=1}^{i} u_j$$

by finding starting values and recurrence relations. Generalize.

H-134 Proposed by L. Carlitz, Duke University

Evaluate the circulants

$$\begin{vmatrix} F_n & F_{n+k} & \cdots & F_{n+(m-1)k} \\ F_{n+(m-1)k} & F_n & \cdots & F_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n+k} & F_{n+2k} & \cdots & F_n \end{vmatrix}, \quad \begin{vmatrix} L_n & L_{n+k} & \cdots & L_{n+(m-1)k} \\ L_{n+(m-1)k} & L_n & \cdots & L_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ L_{n+k} & L_{n+2k} & \cdots & L_n \end{vmatrix}$$

H-135 Proposed by James E. Desmond, Florida State University, Tallahassee, Fla.

PART I:

Show that

$$j + 1 = \sum_{d=0}^{[j/2]} \binom{j-d}{d} 2^{j-2d}(-1)^d ,$$

where $j \geq 0$ and $[j/2]$ is the greatest integer not exceeding $j/2$.

PART 2:

Show that

$$F_{(j+1)n} = F_n \sum_{d=0}^{[j/2]} \binom{j-d}{d} L_n^{j-2d}(-1)^{(n+1)d}$$

where $j \geq 0$ and $[j/2]$ is the greatest integer not exceeding $j/2$.

SOLUTIONS

RECURSIVE BREEDING

H-89 Proposed by Maxey Brooke, Sweeny, Texas

Fibonacci started out with a pair of rabbits, a male and a female. A female will begin bearing after two months and will bear monthly thereafter. The first litter a female bears is twin males, thereafter she alternately bears female and male.

Find a recurrence relation for the number of males and females born at the end of the n^{th} month and the total rabbit population at that time.

Solution by F. D. Parker, St. Lawrence University

The number of females at the end of n months, $F(n)$, is equal to the number of females at the end of the previous plus the number of females who are at least three months old. Thus we have

$$F(n) = F(n-1) + F(n-3) .$$

The number of males at the end of n months, $M(n)$, will be the sum of the males at the end of the previous month, the number of females at least three months old, and twice the number of females who are exactly two months old. Thus

$$M(n) = M(n-1) + F(n-3) + 2(F(n) - F(n-2)) .$$

The total rabbit population is the same as it would be if each pair of offspring were of mixed sex, that is,

$$M(n) + F(n) = 2f(n) ,$$

where $f(n)$ is the n^{th} Fibonacci number.

DIVIDED WE FALL!

H-92 Proposed by Brother Alfred Brousseau, St. Mary's College, California

Prove or disprove: Apart from F_1, F_2, F_3, F_4, no Fibonacci number, F_i $(i > 0)$ is a divisor of a Lucas number.

Solution by L. Carlitz, Duke University

Put

$$L_n = \alpha^n + \beta^n, \quad F_n = (\alpha^n - \beta^n)/(\alpha - \beta),$$

where

$$\alpha = \tfrac{1}{2}(1 + \sqrt{5}), \quad \beta = \tfrac{1}{2}(1 - \sqrt{5}) .$$

Also put $n = mk + r$, $0 \leq r < k$. Since

$$\alpha^n + \beta^n = \alpha^r(\alpha^{mk} - \beta^{mk}) + \beta^{mk}(\alpha^r + \beta^r) ,$$

it follows from $F_k | L_n$ that $F_k | \beta^{mk} L_r$. Since β is a unit of $Q(\sqrt{5})$ it follows that $F_k | L_r$. Now from $L_r = F_{r-1} + F_{r+1}$ we get $L_r < F_{r+2}$ for $r > 2$. Hence we need only consider $F_{r+1} | L_r$. However this implies $F_{r+1} | F_{r-1}$ which is impossible for $r \geq 2$. Therefore $F_k \mid L_n$ is impossible for $k > 4$.

Also solved by James Desmond.

OOPS!!

H-93 Proposed by Douglas Lind, Univ. of Virginia, Charlottesville, Virginia. (corrected).

Show that

$$F_n = \overline{\prod_{k=1}^{n-1}} (3 + 2 \cos 2k\pi/n)$$

$$L_n = \prod_{k=0}^{\overline{n-2}} (3 + \cos (2k + 1)\pi/n)$$

where $\bar{n}$ is the greatest integer contained in $n/2$.

Solution by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.

We know from Problem H-64 (FQ. Vol. 3, April 1965, p. 116) that,

$$F_n = \prod_{j=1}^{n-1}\left(1 - 2i \cos \frac{j\pi}{n}\right),$$

where $i = \sqrt{-1}$.

If n is odd,

$$\begin{aligned} F_{2n+1} &= \prod_{j=1}^{2n}\left(1 - 2i\cos\frac{j\pi}{2n+1}\right) \\ &= \prod_{1}^{n}\left(1 - 2i\cos\frac{j\pi}{2n+1}\right)\prod_{n+1}^{2n}\left(1 - 2i\cos\frac{j\pi}{2n+1}\right) \\ &= \prod_{j=1}^{n}\left(1 - 2i\cos\frac{j\pi}{2n+1}\right)\prod_{k=n+1}^{2n}\left[1 + 2i\cos\pi\left(1 - \frac{k}{2n+1}\right)\right] \end{aligned}$$

Letting $j = (2n + 1 - k)$ in the second product we get

$$\begin{aligned} F_{2n+1} &= \prod_{1}^{n}\left(1 - 2i\cos\frac{j\pi}{2n+1}\right)\prod_{1}^{n}\left(1 + 2i\cos\frac{j\pi}{2n+1}\right) \\ &= \prod_{1}^{n}\left(1 + 4\cos^2\frac{\pi j}{2n+1}\right) = \prod_{1}^{n}\left(3 + 2\cos\frac{2j\pi}{2n+1}\right) \quad \cdots \text{(A)} \end{aligned}$$

Similarly when n is even,

$$F_{2n} = \prod_{j=1}^{2n-1}\left(1 - 2i \cos \frac{j}{2n}\right)$$

$$= \prod_{1}^{n-1}\left(1 - 2i\cos\frac{j\pi}{2n}\right)\prod_{1}^{n-1}\left(1 + 2i\cos\frac{j\pi}{2n}\right)\cdot\left(1 + 2i\cos\frac{\pi}{2}\right)$$

$$= \prod_{1}^{n-1}\left(1 + 4\cos^2\frac{j\pi}{2n}\right)$$

$$= \prod_{1}^{n-1}\left(3 + 2\cos\frac{2j\pi}{2n}\right) \qquad \cdots \text{(B)}$$

From (A) and (B) we see that

$$F_n = \prod_{k=1}^{\overline{n-1}}\left(3 + 2\cos\frac{2k\pi}{n}\right) \qquad \cdots \text{(C)}$$

Hence,

$$F_{2n} = \prod_{k=1}^{\overline{2n-1}}\left(3 + 2\cos\frac{k\pi}{n}\right)$$

$$= \prod_{i=2,4,\cdots 2(\overline{n-1})}\left(3 + 2\cos\frac{i\pi}{n}\right)\prod_{j=1,3,\cdots,2,(\overline{n-2})+1}\left(3 + \cos\frac{j\pi}{n}\right)$$

Letting $i = 2k$ and $j = (2k + 1)$ we have

$$F_{2n} = \prod_{k=1}^{n-1}\left(3 + 2\cos\frac{2k\pi}{n}\right)\prod_{k=0}^{n-2}\left(3 + 2\cos\frac{(2k+1)\pi}{n}\right)$$

$$= F_n \prod_{k=0}^{n-2}\left\{3 + 2\cos\frac{(2k+1)\pi}{n}\right\}$$

Since $F_{2n} = F_nL_n$, we have

$$L_n = \prod_{k=0}^{n-2}\left[3 + 2\cos\frac{(2k+1)\pi}{n}\right] \qquad \cdots \text{(D)}$$

Also solved by L. Carlitz.

ANOTHER IDENTITY

H-95 Proposed by J. A. H. Hunter, Toronto, Canada.

Show

$$F^3_{n+k} + (-1)^k F^3_{n-k} = L_k F^2_k F_{3n} + (-1)^k F^3_n \quad .$$

Solution by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.

$$\begin{aligned} F_{n-k} &= F_nF_{-(k+1)} + F_{-k}F_{n+1} \\ &= (-1)^kF_nF_{k+1} + (-1)^{k-1}F_kF_{n+1} \quad , \end{aligned}$$

since

$$F_{-n} = (-1)^{n-1}F_n \quad .$$

Hence,

$$(-1)^kF_{n-k} = F_nF_{k+1} - F_kF_{n+1} \quad .$$

Also,

$$F_{n+k} = F_nF_{k-1} + F_kF_{n+1} .$$

Hence we have,

$$F^3_{n+k} + (-1)^{3k} F^3_{n-k} = (F_nF_{k-1} + F_kF_{n+1})^3 + (F_nF_{k+1} - F_kF_{n+1})^3$$

Or,

$$\begin{aligned}
I = F^3_{n+k} + (-1)^kF^3_{n-k} &= F^3_n(F^3_{k+1} + F^3_{k-1}) \\
&\quad + 3F_nF_kF_{n+1}F_{k-1}(F_nF_{k-1} + F_kF_{n+1}) \\
&\quad - 3F_nF_kF_{n+1}F_{k+1}(F_nF_{k+1} - F_kF_{n+1}) \\
&= F^3_n(F_{k+1} + F_{k-1})(F^2_{k+1} - F^2_{k-1} - F_{k+1}F_{k-1}) \\
&\qquad - 3F^2_nF_kF_{n+1}(F^2_{k+1} - F^2_{k-1}) + 3F_nF^2_kF^2_{n+1}(F_{k+1} + F_{k-1}) \\
&= F^3_n L_k (F_{k+1} - F_{k-1})^2 + F_{k+1} F_{k-1} \\
&\qquad - 3F^2_nF_kF_{n+1}(F_{k+1} + F_{k-1})(F_{k+1} - F_{k-1}) \\
&\qquad + 3F_nF^2_kF^2_{n+1}L_k \\
&= L_kF^3_n(F^2_k + F_{k+1}F_{k-1}) - 3F^2_nF^2_kF_{n+1}L_k + 3F_nF^2_kF^2_{n+1}L_k
\end{aligned}$$

Using the identity,

$$F^2_k - F_{k+1}F_{k-1} = (-1)^k$$

we obtain

$$\begin{aligned}
I &= L_kF^3_n\left(2F^2_k + (-1)^k\right) + 3L_kF^2_kF_nF_{n+1}(F_{n+1} - F_n) \\
&= L_kF^2_k(2F^3_n + 3F_nF_{n+1}F_{n-1}) + (-1)^kF^3_nL_k
\end{aligned} \tag{1}$$

Now,

$$\begin{aligned}
F_{3n} &= F_n F_{2n-1} + F_{2n} F_{n+1} \\
&= F_n(F_{n-1}^2 + F_n^2) + (F_n F_{n-1} + F_n F_{n+1}) F_{n+1} \\
&= F_n^3 + F_n F_{n+1}^2 + F_{n-1} F_n (F_{n+1} + F_{n-1}) \\
&= F_n^3 + F_n(F_n^2 + 2F_n F_{n-1} + F_{n-1}^2) + F_{n-1} F_n (F_{n+1} + F_{n-1}) \\
&= 2F_n^3 + 2F_{n-1} F_n (F_{n-1} + F_n) + F_{n-1} F_n F_{n+1} \\
&= 2F_n^3 + 3F_{n-1} F_n F_{n+1}
\end{aligned}$$

Substituting this in (1) we get

$$I = L_k F_k^2 F_{3n} + (-1)^k F_n^3 L_k$$

Therefore,

$$F_{n+k}^3 + (-1)^k F_{n-k}^3 = L_k \left[F_k^2 F_{3n} + (-1)^k F_n^3 \right]$$

Also solved by Charles R. Wall.

LATE ACKNOWLEDGEMENTS

Clyde Bridger: H-79, H-80.

C. B. A. Peck: H-32, H-44, H-45, H-67.

* * * * *

(Continued from p. 138.)

All known Fibonacci equations using F_n are theoretically generalizable to equations using F_x. For some examples, see [2]. See [3] also.

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* * * * *

MATHEMATICAL MODELS FOR THE STUDY OF THE PROPAGATION OF NOVEL SOCIAL BEHAVIOR

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Suppose we wish to develop a mathematical model for the spread of novel, social behavior, such as rumors, newly coined words, new hobbies or habits, new ideas, etc. Let us illustrate the development of a highly simplified model of this sort, where we are concerned only with behavior which spreads on a person-to-person basis. We shall assume that all individuals who are capable of being potential transmitters of the new behavior adopt it after only one single exposure to it. We shall further assume that all potential transmitters contact exactly m <u>different</u> persons per unit time. Finally, we shall assume a population sufficiently large so that no <u>convergence effects</u> occur during the <u>initial</u> period of growth. By this we mean a population of potential converts whose size, in relation to the actual number of increasing converts, in great enough for <u>practical</u> purposes to warrant the assumption that those who are spreading the novel social behavior will meet for quite some time only individuals who have not as yet been subject to contact with it. This last assumption can be expressed by stating that the rate of repetitious contacts with those who already display the novel behavior in question, is zero.

Under these several constraints it can be shown that the increment of growth, G_i, at any time $t = i$ will be given by

$$G_i = m(m+1)^{i-1}, \quad i \geq 1 \tag{1}$$

and the <u>cumulative</u> or <u>total growth</u>, $N(t)$, in the number of persons who exhibit the novel social behavior at time, t, will be given by

$$N(t) = (m+1)^t, \tag{2}$$

where equation (2) holds only for discrete time instants, that is, where $t = 1, 2, \cdots$.

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We now assume that every person possesses a circle of acquaintances and that, for each person in the population, there are exactly D persons in his circle of acquaintances. We further assume that each person succeeds in contacting all of these D persons only after k units of time have elapsed. In short, $D = mk$. When $t \geq k + 1$ each person continues to exhibit the novel, social behavior but he no longer transmits it to anyone else. G_0 is defined as one. When k units of time have elapsed, the population of converts to the new behavior is $N(k)$. When $t = k + 1$, G_0 will cease to transmit the new behavior but he will still exhibit it. We therefore have

$$N(k + 1) = [N(k) - G_0]\, m + N(k) \tag{3}$$

$$= N(k)Y - G_0 m \ , \tag{4}$$

where $Y = (m + 1)$.

At time instant, $t = k + 2$, the number of people who cease to be transmitters will be G_1, and $N(k + 2)$ will be given by the following recursion relationship.

$$N(k + 2) = [N(k + 1) - G_1]\, m + N(k + 1) \tag{5}$$

$$= N(k + 1)Y - G_1 m \tag{6}$$

Substituting equation (4) into equation (6) we obtain

$$N(k + 2) = [N(k)Y - G_0 m]\, Y - G_1 m \tag{7}$$

which in turn becomes

$$N(k + 2) = N(k)Y^2 - m(G_0 Y + G_1) \tag{8}$$

If we proceed to develop the recursion relationships exhibited in equations (3) through (8), we obtain the following model for $1 \leq i \leq 6$.

$$
\begin{aligned}
N(k+1) &= N(k)Y - G_0 m \\
N(k+2) &= N(k)Y^2 - mY^0(G_0 Y + G_1) \\
N(k+3) &= N(k)Y^3 - mY(G_0 Y + G_1) - G_2 m \\
N(k+4) &= N(k)Y^4 - mY^2(G_0 Y + G_1) - mY^0(G_2 Y + G_3) \\
N(k+5) &= N(k)Y^5 - my^3(G_0 Y + G_1) - mY(G_2 Y + G_3) - G_4 m \\
N(k+6) &= N(k)Y^6 - mY^4(G_0 Y + G_1) - mY^2(G_2 Y + G_3) - mY^0(G_4 Y + G_5) \\
&\ \vdots
\end{aligned} \tag{9}
$$

From the preceding it can be readily seen that if we wish to determine the value of $N(k + i)$ and if i is even, then

$$
\begin{aligned}
N(k+i) = N(k)Y^i &- mY^{i-2}(G_0 Y + G_1) - mY^{i-4}(G_2 Y + G_3) - \\
&- mY^{i-6}(G_4 Y + G_5) - \cdots - mY^{i-i}(G_{i-2} Y + G_{i-1}),
\end{aligned} \tag{10a}
$$

while if i is odd, then

$$
\begin{aligned}
N(k+i) = N(k)Y^i &- mY^{i-2}(G_0 Y + G_1) - mY^{i-4}(G_2 Y + G_3) - \cdots \\
&- mY^{i-(i-1)}(G_{i-3} Y + G_{i-2}) - G_{i-1} m
\end{aligned} \tag{10b}
$$

Both equations (10a) and (10b) can be summarized formally as follows.

$$
N(k+i) = N(k)Y^i - m\sum_{n=0}^{i-1} G_n Y^{i-1-n}, \quad 1 \le i \le k \tag{11}
$$

If we substitute $(m + 1)$ for Y into equations (10a) or (10b) and the appropriate value of G_i as given by equation (1), then $N(k + i)$ can be computed. The computed value will reflect the propagation or cumulative growth of the novel social behavior, under all the assumptions and conditions which have been mentioned above.

We now define

$$A \equiv -m\sum_{n=0}^{i-1} G_n Y^{i-1-n} = -mY^{i-1} - m\sum_{n=1}^{i-1} G_n Y^{i-1-n} \tag{12}$$

But by equation (1) we have

$$G_n = m(m+1)^{n-1} = mY^{n-1}, \quad n \geq 1 \tag{13}$$

Hence

$$A = -mY^{i-1} - m^2\sum_{n=1}^{i-1} Y^{i-2} \tag{14}$$

$$= -mY^{i-1} - m^2(i-1)Y^{i-2} \tag{15}$$

If we now substitute the value for A, as given by equation (15), for the second expression on the right-hand side of equation (11), we obtain

$$N(k+i) = N(k)Y^i - mY^{i-1} - m^2(i-1)Y^{i-2}, \quad 1 \leq i \leq k \tag{16}$$

There are two justifications for the constraint that $1 \leq i \leq k$. First is the fact that the growth of the novel behavior will be initially exponential, if the potential population of converts is very much larger than the actual and increasing population of converts for a relatively modest time period occurring at the beginning of the growth phenomenon in question. The actual length of the growth interval assumed is, of course, $2k$ units of time. The second reason for assuming the constraint that $1 \leq i \leq k$ is that the substitution of $i = 0$ in either equations (10a), (10b) or (16), or their analogues, would make no sense. The correction for the fact that transmitters of the novel social behavior possess only a limited circle of acquaintances, D, holds only for those situations in which converted individuals have begun to exhaust their circles of acquaintanceship and, in mathematical terms, this means that $i \neq 0$.

Substituting $(m + 1) = Y$ in equation (16) will yield

$$N(k + i) = (m + 1)^{k+i} - m(m + 1)^{i-1} - (i - 1)m^2(m + 1)^{i-2} \tag{17}$$

$$= (m + 1)^{i-2} (m + 1)^{k+2} - m^2(i - 1) - m(m + 1) \tag{18}$$

$$= (m + 1)^{i-2} (m + 1)^{k+2} - m(im + 1) \tag{19}$$

The equivalence of either equation (10a) with equation (16) or equation (10b) with equation (16), can be seen from the relations given by equations (12) through (15).

The argument of the preceding exposition suggests to some extent how the mathematical model may be of use to the sociologist for a variety of phenomena which are of interest to him.

Models for behavioral diffusion theory have been developed over the last two decades. They may be highly sophisticated or relatively simple, mathematically speaking. Sophisticated examples of models for diffusion theory, intended for some specifically designed experiments, may be found in the work of Rapoport [1]. An early and systematic development of a *predominantly* algebraic treatment of diffusion theory, intended for experimental designs of an aggregative type, was worked out by Winthrop [2]. The formulation of some early *ad hoc* models intended for empirical use, was undertaken by Dodd [3]. The relationship of Dodd's S-Theory to those formulations of diffusion theory for which the present writer has been responsible, has been worked out jointly by Dodd and Winthrop [4]. The model presented in this paper is an example of the strictly algebraic type of model. Models of this kind make it somewhat easier to present the exposition of diffusion theory.

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★ ★ ★ ★ ★

CURIOUS PROPERTY OF ONE FRACTION

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It is well known that an integral fraction, with no more than three digits above the line and three below, gives the best possible approximation of the famous mathematical constant "e".

This fraction is 878/323. In decimal form (2,71826 ···) it yields the correct value for "e" to four decimal places.

If the denominator of this fraction is subtracted from the numerator the difference is 555.

Now, the iterated cross sum of the numerator is 5 and the same cross sum of the denominator is 8. The ratio 5/8 gives the best possible approximation to the "Golden Ratio" with no more than one digit in the numerator and one in the denominator.

★ ★ ★ ★ ★

A THEOREM ON POWER SUMS

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Allison [1, p. 272] showed that the identity

$$(1)\qquad \left\{\sum_{x=1}^{n} x^r\right\}^p = \left\{\sum_{x=1}^{n} x^s\right\}^q \qquad (n = 1, 2, 3, \cdots)$$

holds if and only if $r = 1$, $p = 2$, $s = 3$, and $q = 1$. In this paper we consider the more general problem of finding polynomials

$$f(x) = \sum_{i=0}^{r} a_i x^i \qquad \text{and} \qquad g(x) = \sum_{i=0}^{s} b_i x^i$$

over the real field which satisfy

$$(2)\qquad \{f(1) + \cdots + f(n)\}^p = \{g(1) + \cdots + g(n)\}^q \quad (n = 1, 2, 3, \cdots),$$

where r, p, s and q are positive integers.

First we note that

$$\sum_{x=1}^{n} f(x) = \sum_{i=0}^{r} a_i S_i ,$$

where

$$S_k = \sum_{x=1}^{n} x^k, \qquad k = 0, 1, 2, \cdots .$$

Thus the left member of (2) becomes

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$$\left\{a_r \frac{n^{r+1}}{r+1} + \cdots\right\}^p ,$$

since S_r is a polynomial in n having degree $r + 1$ and leading coefficient

$$\frac{1}{r+1} .$$

Similarly the right member of (2) becomes

$$\left\{b_s \frac{n^{s+1}}{s+1} + \cdots\right\}^q ,$$

so (2) can be written

$$\left\{a_r \frac{n^{r+1}}{r+1} + \cdots\right\}^p = \left\{b_s \frac{n^{s+1}}{s+1} + \cdots\right\}^q . \tag{3}$$

For (3) to hold we must have

$$(r + 1)p = (s + 1)q \tag{4}$$

and

$$\left(\frac{a_r}{r+1}\right)^p = \left(\frac{b_s}{s+1}\right)^q . \tag{5}$$

Case 1. Suppose $p = q$. From (2) we find $f(n) = g(n)$, $n = 1, 2, 3, \cdots$, so $f(x) = g(x)$.

Case 2. Suppose $p \neq q$. We may assume without loss of generality that $p > q$ and $(p, q) = 1$. We will also assume that $a_r = b_s = 1$. Following Allison [op. cit.] we see that for (3) to hold we must have $r = 1$, $p = 2$, $s = 3$, and $q = 1$. Specifically,

$$(S_1 + a_0S_0)^2 = S_3 + b_2S_2 + b_1S_1 + b_0S_0 . \tag{6}$$

Using well-known formulas for S_k, $k = 0, 1, 2, 3$, we write (6) as

$$\left\{\frac{n(n+1)}{2} + a_0 n\right\}^2 = \left\{\frac{n(n+1)}{2}\right\}^2 + b_2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + b_1 \frac{n(n+1)}{2} + b_0 n. \tag{7}$$

Rewriting (7) in powers of n, we find

$$\begin{aligned} \frac{n^4}{4} + \left(\frac{1}{2} + a_0\right)n^3 + \left(\frac{1}{2} + a_1\right)^2 n^2 &= \frac{n^4}{4} + \left(\frac{1}{2} + \frac{b_2}{3}\right)n^3 \\ &+ \left(\frac{1}{4} + \frac{b_2}{2} + \frac{b_1}{2}\right)n^2 + \left(\frac{b_2}{6} + \frac{b_1}{6} + b_0\right)n . \end{aligned} \tag{8}$$

Equating coefficients in (8) yields

$$\begin{aligned} a_0 &= \frac{b_2}{3} \\ \left(\frac{1}{2} + a_0\right)^2 &= \frac{1}{4} + \frac{b_2}{2} + \frac{b_1}{2} \\ 0 &= \frac{b_2}{6} + \frac{b_1}{2} + b_0 . \end{aligned} \tag{9}$$

Let a_0 be arbitrary and regard (9) as the linear system

$$\sum_{j=0}^{2} a_{ij}\, b_j = c_i \qquad (i = 0, 1, 2). \tag{10}$$

Since the determinant $|a_{ij}| \neq 0$, we can solve for b_0, b_1, b_2 in terms of a_0. Easy calculations show

$$b_0 = -a^2 , \quad b_1 = 2a^2 - a, \quad b_2 = 3a , \tag{11}$$

where a_0 is replaced by a for simplicity. Thus

$$f(x) = x + a, \quad g(x) = x^3 + 3ax^2 + (2a^2 - a)x - a^2 \tag{12}$$

When $a = 0$, (12) yields the result of Allison.

If we do not require $a_r = b_s = 1$, it is interesting to note that for arbitrary p, q one can always find non-monic polynomials $f(x)$, $g(x)$ to satisfy (2). Specifically $f(x)$ and $g(x)$ are chosen to satisfy

$$\sum_{x=1}^{n} g(x) = n^q, \quad \sum_{x=1}^{n} g(x) = n^p . \tag{13}$$

If (13) holds, obviously (2) does.

In general the construction of a function $f_t(x)$ satisfying

$$\Sigma f_t(x) = n^t \qquad (t = 1, 2, 3, \cdots) \tag{14}$$

is recursive. First note that $f_1(x) = 1$. We find $f_{t+1}(x)$ as follows. Recall that

$$\sum_{x=1}^{n} x^t = \frac{n^{t+1}}{t+1} + s_t n^t + \cdots + s_1 n .$$

Thus

$$(t+1)\sum_{x=1}^{n}\left\{x^t - s_t f_t(x) - \cdots - s_1 f_1(x)\right\} = n^{t+1} , \tag{15}$$

so

$$f_{t+1}(x) = (t+1)\left\{x^t - \sum_{k=1}^{t} s_k f_k(x)\right\} . \tag{16}$$

We summarize these results in the following.

Theorem. The solutions of (2) are as follows. If $p = q$, $f(x)$ is arbitrary and $g(x) = f(x)$. If $p \neq q$, the only monic solutions occur when $p = 2$ and $q = 1$, in which case $f(x)$ and $g(x)$ are defined by (12), where a is an arbitrary real constant. Non-monic solutions for that case can be found using (13).

As an example of these results suppose that $p = 3$ and $q = 4$. By (14) and (17) we have

$$\left\{\sum_{x=1}^{n}(4x^3 - 6x^2 + 4x - 1)\right\}^3 = \left\{\sum_{x=1}^{n}(3x^2 - 3x + 1)\right\}^4 , \quad (n = 1, 2, 3, \cdots) .$$

REFERENCE

1. Allison, "A Note on Sums of Powers of Integers," American Mathematical Monthly, Vol. 68, 1961, p. 272.

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A NUMBER PROBLEM

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There are infinite many numbers with the property: if units digit of a positive integer, M, is 6 and this is taken from its place and put on the left of the remaining digits of M, then a new integer, N, will be formed, such that $N = 6M$. The smallest M for which this is possible is a number with 58 digits (1016949 $\cdots$ 677966).

Solution: Using formula

$$\frac{6x}{1 - 4x - x^2} = 3\sum_{n=0}^{\infty} F_{3n}\, x^n ,$$

with $x = 0,1$ we have $1,01016949 \cdots 677966$, where the period number (behind the first zero) is M.*

*1016949152542372881355932203389830508474576271186440677966.

(Continued on p. 175.)

RECREATIONAL MATHEMATICS

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DIGITAL DIVERSIONS

In the February 1968 issue of The Fibonacci Quarterly, I had asked readers to work at expressing Fibonacci numbers using the ten digits once only, in order, and using only the common mathematical operations and symbols. V. E. Hoggatt, Jr., the General Editor of this Journal, came up with a set of equations which, though not exactly what I had in mind, are of special interest because of their versatility. All ten digits are used and logarithms are required.

We start with

$$\log_2 2^n = n$$

or

$$\log_{\underbrace{\sqrt{\sqrt{\cdots\sqrt{2}}}}_{(n \text{ radicals})}} 2 = 2^n$$

then

$$\log_2 \Bigl(\log_{\underbrace{\sqrt{\sqrt{\cdots\sqrt{2}}}}_{(n \text{ radicals})}} 2\Bigr) = n$$

This leads to

$$0 + \log_{(5-1)/2}\left[\log_{\sqrt{6/3}}(8-4)/(9-7)\right] = 1$$

or

$$0 + \log_{(5-1)/2}\left[\log_{\underbrace{\sqrt{\sqrt{\cdots\sqrt{6/3}}}}_{(n \text{ radicals})}}(8-4)/(9-7)\right] = n \; .$$

The study of all this eventually leads to the following:

$$\log_2 \left(\log_{\underbrace{\sqrt{\sqrt{\cdots\sqrt{m}}}}_{(\text{ n radicals})}} m\right) = n$$

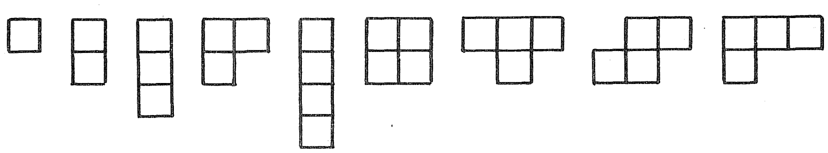
which further leads to the desired ten-digits-in-order form for any Fibonacci number, F_n :

$$\log_{(0+1+2+3+4)/5} \left(\log_{\underbrace{\sqrt{\sqrt{\cdots\sqrt{-6+7+8}}}}_{(F_n \text{ radicals})}} 9\right) = F_n .$$

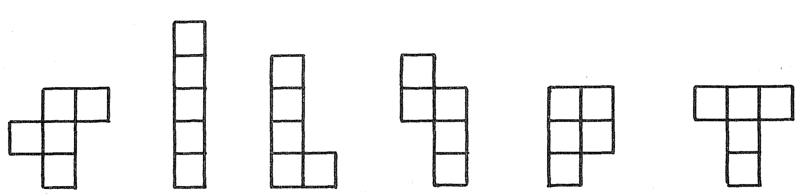
How about something more along these lines?

A PENTOMINO TILING PROBLEM

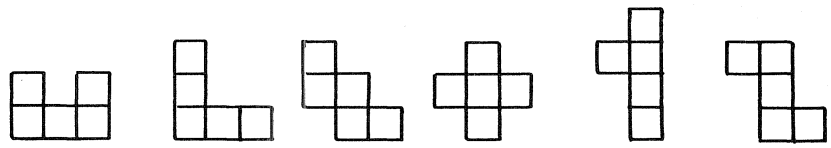
Ever since Solomon W. Golomb's article [1] appeared, much time has been devoted to the study of polyominoes and their properties. Polyominoes are configurations made up of squares connected edge-to-edge. The figures below show the first nine members of the polyomino family:



The first is a monomino, the second is a domino. The third and fourth figures are the two trominoes. The remaining figures are the five tetrominoes. Continued construction shows there are twelve pentominoes — those made with five squares. The pentominoes have proven so popular that they have had names assigned to them corresponding to their resemblance to certain letters of the alphabet. They are shown below.

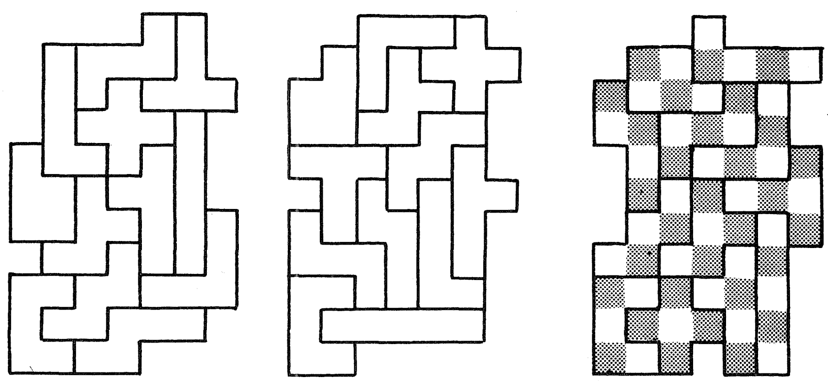


F I L N P T



U V W X Y Z

Many polyomino problems have been posed, but here's a pentomino problem from Maurice J. Pova of Lancanshire, England: Find irregular patterns of the twelve pentominoes which form tessellation patterns; i.e., which cover a plane. There are 2339 distinct 6 x 10 rectangles which can be made from the pentominoes, but we are looking for irregular patterns. Three examples found by Povah are shown below. You should be able to find others.



The third figure has a bonus feature: the checkerboard pattern is maintained throughout the tessellation. The black and white squares fall on the same parts of each pentomino as it repeats in the plane.

ARE FIBONACCI NUMBERS "NORMAL"?

A "normal" number is one which contains the statistically expected number of each of the digits and combinations of digits. A random 100-digit number, if normal, ought to contain approximately 10 zeroes, 10 ones, 10 twos, and so on. For larger numbers, one could check for the expected occurrences of the pairs 10, 11, 12, 13, ⋯, 97, 98, 99. There is even a "poker hand" test for large enough numbers, in which groups of five digits are examined to see if the statistically expected number of "busts," "one pair," "full house," and other poker hands are present. Such a statistical study has been made of the digits of π[2].

I wondered if the Fibonacci numbers are normal. There are at least two ways of attacking the problem. The first method consists of examining each Fibonacci number and counting the number of distinct digits. By so doing I found some typical results.

F_n	Number of digits in F_n	Number of each of the following digits 0	1	2	3	4	5	6	7	8	9
F_{100}	21	1	3	3	1	3	3	1	2	2	2
F_{500}	105	9	8	19	8	7	11	11	11	11	10
F_{1000}	209	20	13	21	18	21	23	26	21	20	26

F_{100} is reasonably normal; F_{500} has more twos than expected; F_{1000} has a slightly low count of ones.

The second method consists of noting the cumulative sums of the digits. I did this up to F_{100} counting all the digits in all those 100 Fibonacci numbers. The results are tabulated below.

Number of each of the following digits to F_{100}

0	1	2	3	4	5	6	7	8	9
110	136	107	102	111	95	95	117	92	106

The total number of digits in the first 100 Fibonacci numbers is 1071. The distribution of the digits to F_{100} appears to be reasonably normal, except for the somewhat large number of ones.

Further work on this matter might lead to interesting speculation — depending on the results. The work of counting digits is tedious, but a computer could be programmed to calculate the Fibonacci numbers, count their digits, and print cumulative totals as well. Other statistical tests could be applied with the aid of a computer.

OBSERVATION

Has anyone noticed this before? While trying to see if the Fibonacci numbers could be used to make magic squares, I discovered that no set of consecutive Fibonacci numbers could be so used. Can you demonstrate this?

REFERENCES

1. Solomon W. Golomb, "Checkerboards and Polyominoes," Amer. Math. Monthly, Vol. 61, No. 10 (December 1954), pp. 675-682.

2. R. K. Pathria, "A Statistical Study of Randomness Among the First 10,000 Digits of π," Mathematics of Computation, Vol. 16, No. 78 (April 1962), pp. 188-197.

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(continued from p. 191.)

$$\sum_{n=0}^{\infty} F_{n+k}^7 x^n = \frac{P_k(x)}{1 - 21x - 273x^2 + 1092x^3 + 1820x^4 - 1092x^5 - 273x^6 + 21x^7 + x^8}, \quad k=0,1,2,3,4,5,6,7$$

$P_0(x) = x(1 - 20x - 166x^2 + 318x^3 + 166x^4 - 20x^5 - x^6)$

$P_1(x) = 1 - 20x - 166x^2 + 318x^3 + 166x^4 - 20x^5 - x^6$

$P_2(x) = 1 + 107x - 774x^2 - 1654x^3 + 1072x^4 + 272x^5 - 21x^6 - x^7$

$P_3(x) = 128 - 501x - 2746x^2 - 748x^3 + 1364x^4 + 252x^5 - 22x^6 - x^7$

$P_4(x) = 2187 + 32{,}198x - 140{,}524x^2 - 231{,}596x^3 + 140{,}028x^4 + 34{,}922x^5 - 2687x^6 - 128x^7$

$P_5(x) = 78{,}125 + 456{,}527x - 2{,}619{,}800x^2 - 3{,}840{,}312x^3 + 2{,}423{,}126x^4 + 594{,}364x^6 - 46{,}055x^6 - 2187x^7$

$P_6(x) = 2{,}097{,}152 + 18{,}708{,}325x - 89{,}152{,}812x^2 - 139{,}764{,}374x^3 + 85{,}906{,}864x^4 + 21{,}332{,}070x^5 - 1{,}642{,}812x^6 - 78{,}125x^7$

$P_7(x) = 62{,}748{,}417 + 483{,}369{,}684x - 2{,}429{,}854{,}358x^2 - 3{,}730{,}909{,}776x^3 + + 2{,}311{,}422{,}054x^4 + 570{,}879{,}684x^5 - 44{,}118{,}317x^6 - 2{,}097{,}152x^7.$

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FURTHER PROPERTIES OF MORGAN-VOYCE POLYNOMIALS

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1. INTRODUCTION

A set of polynomials $B_n(x)$ and $b_n(x)$ were first defined by Morgan-Voyce [1] as,

$$(1)\qquad b_n(x) = x B_{n-1}(x) + b_{n-1}(x) \qquad (n \geq 1)$$

$$(2)\qquad B_n(x) = (x + 1)B_{n-1}(x) + b_{n-1}(x) \qquad (n \geq 1)$$

with

$$(3)\qquad b_0(x) = B_0(x) = 1.$$

In an earlier article [2], a number of properties of these polynomials $B_n(x)$ and $b_n(x)$ were derived and these were used in a later article to establish some interesting Fibonacci identities [3]. We shall now consider some further properties of these polynomials and establish their relations with the Fibonacci polynomials $f_n(x)$.

2. GENERATING MATRIX

The matrix Q defined by,

$$(4)\qquad Q = \begin{bmatrix} (x+2) & -1 \\ 1 & 0 \end{bmatrix}$$

may be called as the generating matrix, since we may establish by induction that,

$$(5)\qquad Q^n = \begin{bmatrix} B_n & -B_{n-1} \\ B_{n-1} & -B_{n-2} \end{bmatrix}$$

(Received February 1967)

Hence,

$$\begin{bmatrix} b_n & -b_{n-1} \\ b_{n-1} & -b_{n-2} \end{bmatrix} = \begin{bmatrix} (B_n - B_{n-1}) & -(B_{n-1} - B_{n-2}) \\ (B_{n-1} - B_{n-2}) & -(B_{n-2} - B_{n-3}) \end{bmatrix} = Q^n - Q^{n-1}$$

$$= Q^{n-1}(Q - I) \tag{6}$$

Since the determinant of $Q = 1$, we have

$$B_{n+1}B_{n-1} - B_n^2 = -1 \tag{7}$$

and

$$\begin{vmatrix} b_n & -b_{n-1} \\ b_{n-1} & -b_{n-2} \end{vmatrix} = |Q - I| = \begin{vmatrix} x+1 & -1 \\ 1 & -1 \end{vmatrix} = x$$

or

$$b_{n+1}b_{n-1} - b_n^2 = x . \tag{8}$$

3. B_n AND b_n AS TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Letting $\cos\theta = (x+2)/2$ in the identity

$$\sin(n+1)\theta + \sin(n-1)\theta = 2\sin(n\theta)\cos\theta$$

we have

$$\frac{\sin(n+1)\theta}{\sin\theta} + \frac{\sin(n-1)\theta}{\sin\theta} = (x+2)\frac{\sin n\theta}{\sin\theta} \qquad (-4 \le x \le 0) ,$$

with

$$\frac{\sin(n+1)\theta}{\sin\theta} = 1 \qquad \text{for } n = 0$$

$$= (x+2) \qquad \text{for } n = 1 .$$

Thus,

$$\frac{\sin (n+1)\theta}{\sin \theta}$$

satisfies the difference equation for B_n. Hence,

$$B_n(x) = \frac{\sin (n+1)\theta}{\sin \theta} \qquad (-4 \le x \le 0) \tag{9}$$

Similarly, if $\cosh \phi = (x+2)/2$, then

$$B_n(x) = \frac{\sin h\,(n+1)\phi}{\sin h\phi} \qquad (x \ge 0) \tag{10}$$

Since $b_n = B_n - B_{n-1}$, we have

$$b_n(x) = \frac{\cos\,(2n+1)\,\theta/2}{\cos\,\theta/2} \qquad (-4 \le x \le 0) \tag{11a}$$

and

$$b_n(x) = \frac{\cosh\,(2n+1)\,\phi/2}{\cosh\,\phi/2} \qquad (x \ge 0) \tag{11b}$$

4. DIFFERENTIAL EQUATIONS FOR $B_n(x)$ AND $b_n(x)$

It has been shown earlier [2] that

$$B_n(x) = \sum_{k=0}^{n} \binom{n+k-1}{n-k} x^k = \sum_{k=0}^{n} c_n^k x^k \tag{12}$$

and

$$b_n(x) = \sum_{k=0}^{n} \binom{n+k}{n-k} x^k = \sum_{k=0}^{n} d_n^k x^k \quad . \tag{13}$$

Hence

$$\frac{c_n^{k+1}}{c_n^k} = \frac{\binom{n+k+2}{n-k-1}}{\binom{n+k+1}{n-k}} = \frac{(n-k)(n+k+2)}{(2k+3)(2k+2)}$$

Thus, the coefficients of x^k and x^{k+1} of $B_n(x)$ are related by

$$k(k-1)c_n^k + 4(k+1)k\, c_n^{k+1} + 3k\, c_n^k + 6(k+1)c_n^{k+1} - n(n+2)c_n^k = 0 \cdots \tag{14}$$

But the coefficient of x^k in the expansion of

$$x^2B_n'' + 4\,x\,B_n' + 3x\,B_n' + 6\,B_n' - n(n+2)B_n$$

is the same as the left-hand side expression of (14). Hence, $B_n(x)$ satisfies the differential equation

$$x(x+4)y'' + 3(x+2)y' - n(n+2)y = 0 \tag{15}$$

Similarly, starting with (13) we can show that $b_n(x)$ satisfies the differential equation

$$x(x+4)y'' + 2(x+1)y' - n(n+1)y = 0 \tag{16}$$

Using (15) and (16) we shall now derive some identities for $B_n(x)$ and $b_n(x)$. We have from (15)

$$x(x+4)(B_n'' - B_{n-1}'') + 3(x+2)(B_n' - B_{n-1}') - n(n+2)B_n + (n+1)(n-1)B_{n-1} =$$

or,

$$x(x+4)b_n'' + 3(x+2)b_n' - n(n+1)b_n - nB_n - (n+1)B_{n-1} = 0\ .$$

Using (16) this may be reduced to

$$(x + 4)b_n'(x) = nB_n(x) + (n + 1)B_{n-1}(x) \quad . \tag{17}$$

Hence,

$$(x + 4)(b_{n+1}' - b_n') = (n + 1)B_{n+1} + (n + 2)B_n - nB_n - (n + 1)B_{n-1} \tag{18}$$

Differentiating (1) we get,

$$b_{n+1}' - b_n' = x B_n' + B_n \tag{19}$$

Substituting (19) in (18) and simplifying we have

$$x(x + 4)B_n'(x) = nB_{n+1}(x) - (n + 2)B_{n-1}(x) \tag{20}$$

From (20) we may derive that

$$x(x + 4)b_n'(x) = nb_{n+1}(x) + b_n(x) - (n + 1)b_{n-1}(x) \quad . \tag{21}$$

5. INTEGRAL PROPERTIES

It has been shown earlier [2] that,

$$\int b_n(x)dx = \frac{B_{n+1}(x) - B_{n-1}(x)}{(n + 1)} + c \tag{22}$$

c being an arbitrary constant. We also know that,

$$\begin{aligned} B_n(0) &= (n + 1); \quad B_n(-4) = (-1)^n(n + 1) \\ b_n(0) &= 1 \quad ; \quad b_n(-4) = (-1)^n(2n + 1) \end{aligned} \tag{23}$$

Hence, from (22) and (23) we have the two special integrals,

$$\int_{-4}^{0} B_{2n}(x)\, dx = 4/(2n + 1) \tag{24a}$$

and

(24b)
$$\int_{-4}^{0} B_{2n+1}(x)\,dx = 0$$

Since

$$B_n^2(x) = \sum_0^n B_{2m}$$

we have

(25)
$$\int_{-4}^{0} B_n^2(x)\,dx = \sum_0^n 4/(2m+1)$$

Similarly, the following integrals may be established:

$$\int_{-4}^{0} b_n^2(x)\,dx = -\int_{-4}^{0} b_{2n+1}(x)\,dx = 4/(2n+1)$$

$$\int_{-4}^{0} B_n(x)B_{n+1}(x)\,dx = 0$$

$$\int_{-4}^{0} b_n(x)\,B_n(x)\,dx = -\int_{-4}^{0} b_{n+1}(x)B_n(x)dx = -4\sum_0^n 1/(2m+1)$$

$$\int_{-4}^{0} b_n(x)b_{n+1}(x)\,dx = -4 - 8\sum_1^n 1(2m+1)$$

$$\int_{-4}^{0} B_{n+1}(x)\, B_{n-1}(x)\, dx = 4 \sum_{1}^{n} 1/(2m+1)$$

$$\int_{-4}^{0} b_{n+1}(x)\, b_{n-1}(x)\, dx = 8 \sum_{1}^{n-1} 1/(2m+1) + 4/(2n+1) - 8$$

$$\int_{-4}^{0} b_n^2(x)\, dx = 8 \sum_{1}^{n-1} 1/(2m+1) + 4/(2n+1) .$$

6. ZEROS OF $B_n(x)$ AND $b_n(x)$

From (9) we see that the zeros of $B_n(x)$ are given by $\sin (n+1)\theta = 0$. Hence,

$$\theta = (r\pi)/(n+1), \quad r = 1, 2, \cdots, n .$$

Therefore,

$$(x+2) = 2 \cos \frac{r}{n+1} \pi$$

or,

$$x = -4 \sin^2 \left\{ \frac{r}{n+1} \cdot \frac{\pi}{2} \right\}, \quad r = 1, 2, \cdots, n .$$

Similarly, the zeros of $b_n(x)$ are given by

$$-4 \sin^2 \left\{ \frac{2r-1}{2r+1} \cdot \frac{\pi}{2} \right\}, \quad r = 1, 2, \cdots, n.$$

Thus the zeros of $B_n(x)$ and $b_n(x)$ are real, negative and distinct.

7. $B_n(x)$, $b_n(x)$ AND $f_n(x)$

The Fibonacci polynomials $f_n(x)$ are defined by

$$f_{n+1}(x) = x f_n(x) + f_{n-1}(x) \qquad (n \geq 2) \tag{26}$$

with

$$f_1(x) = 1 \quad \text{and} \quad f_2(x) = x .$$

It is also known [4] that

$$f_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1} , \tag{27}$$

where $[n/2]$ is the greatest integer in $(n/2)$. Hence

$$f_{2n+1}(x) = \sum_{j=0}^{n} \binom{2n-j}{j} x^{2n-2j} = \sum_{r=0}^{n} \binom{n+r}{n-r} (x^2)^r$$

$$= b_n(x^2) ,$$

from (13). Hence,

$$b_n(x^2) = f_{2n+1}(x) . \tag{28}$$

Now

$$f_{2n+3}(x) - f_{2n+1}(x) = xf_{2n+2}(x)$$

or

$$b_{n+1}(x^2) - b_n(x^2) = xf_{2n+2}(x)$$

Hence from (1) we have

$$x^2 B_n(x^2) = x f_{2n+2}(x)$$

or

$$B_n(x^2) = \frac{1}{x} f_{2n+2}(x) \tag{29}$$

Thus, $B_n(x)$, $b_n(x)$ and $f_n(x)$ are interrelated.

(See also H-73 Oct. 1967 pp 255-56)

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1. A. M. Morgan-Voyce, "Ladder Network Analysis Using Fibonacci Numbers," IRE. Transactions on Circuit Theory, Vol. CT-6, Sept. 1959, pp. 321-322.

2. M. N. S. Swamy, "Properties of the Polynomials Defined by Morgan-Voyce," Fibonacci Quarterly, Vol. 4, Feb. 1966, pp. 73-81.

3. M. N. S. Swamy, "More Fibonacci Identities," Fibonacci Quarterly, Vol. 4, Dec. 1966, pp. 369-372.

4. M. N. S. Swamy, Problem B-74, Fibonacci Quarterly, Vol. 3, Oct. 1965, p. 236.

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(Continued from p. 161.)

(Compare this problem with H-65 and above solution formula with the formula

$$\frac{2x}{1 - 4x - x^2} = \sum_{n=0}^{\infty} F_{3n} x^n$$

in the Fibonacci Quarterly, Vol. 2, No. 3, p. 208.)

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SCOTT'S FIBONACCI SCRAPBOOK

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The following generating functions are submitted to continue the list in "A Primer for the Fibonacci Numbers, Part VI," by V. E. Hoggatt, Jr., and D. A. Lind, Fibonacci Quarterly, Vol. 5, No. 5, 1967, pp. 445-460. From time to time, as space permits, more generating functions and special results will be placed in this column in order that they may be properly recorded. Thanks to Kathleen Weland for verifying these.

$$\sum_{n=0}^{\infty} L_{n+k}^3 x^n = \frac{P_k(x)}{1 - 3x - 6x^2 + 3x^3 + x^4} \qquad k = 0, 1, 2, 3$$

$P_0(x) = 8 - 23x - 24x^2 + x^2$

$P_1(x) = 1 + 24x - 23x^2 - 8x^3$

$P_2(x) = 27 - 17x - 11x^2 - x^3$

$P_3(x) = 64 + 151x - 82x^2 - 27x^2$

$$\sum_{n=0}^{\infty} F_{n+k}^4 x^n = \frac{P_k(x)}{1 - 5x - 15x^2 + 15x^3 + 5x^4 - x^5} \qquad k = 0, 1, 2, 3, 4$$

$P_0(x) = x - 4x^2 - 4x + x^4$

$P_1(x) = 1 - 4x - 4x^2 + x^3$

$P_2(x) = 1 + 11x - 14x^2 - 5x^3 + x^4$

$P_3(x) = 16 + x - 20x^2 - 4x^2 + x^4$

$P_4(x) = 81 - 220x - 244x^2 - 79x^3 + 16x^4$

$$\sum_{n=0}^{\infty} F_{n+k}^5 x^n = \frac{P_k(x)}{1 - 8x - 40x^2 + 60x^3 + 40x^4 - 8x^5 - x^6}$$

$k = 0, 1, 2, 3, 4, 5$

$P_0(x) = x - 7x^2 - 16x^3 + 7x^4 + x^5$

$P_1(x) = 1 - 7x - 16x^2 + 7x^3 + x^4$

$P_2(x) = 1 + 24x - 53x^2 - 39x^3 + 8x^4 + x^5$

(Continued on p. 191)

LINEAR DIOPHANTINE EQUATIONS WITH NON-NEGATIVE PARAMETERS AND SOLUTIONS

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1. INTRODUCTION

Solving equations, where we are required to find only the integral solutions, has some historical interest. These equations are known as Diophantine equations, after Diophantus of Alexandria, the first to treat these problems in an algebraic manner.

There are innumerable problems that result in first degree equations with two unknowns, where it is required to find integral solutions. Such an equation is called a linear Diophantine equation and is written as

$$ax + by = n . \tag{1}$$

It is usually stipulated that the parameters, a, b and n, are also integers. However, if these parameters are rational numbers, (1) can be easily transformed so that each parameter becomes integral.

Equation (1) is indeterminant in that there is an unlimited number of solutions, and if we did not require integral solutions, the problem of finding a solution would be simple. If, however, we restrict the solutions to be integral, the problem of finding these solutions is no longer simple, and in fact there may be no solution. Yet, if a solution does exist, the total number of solutions is still unlimited.

The problem warrants more attention by the added restriction that the solutions be non-negative pairs. If this restriction is imposed upon the parameters as well, then if a solution exists, the number of solutions is finite. The problem of finding these solutions and determining the number of such solutions has occupied much attention throughout the history of number theory [1, Chap. II].

The purpose of this paper is to give an explicit formula for the general solution of (1) and to establish the relationship that exists between the parameters when no solution exists.

(Received December 1966)

2. PRELIMINARY REMARKS

Before developing the relationships above some remarks pertaining to historical developments, topical concepts, and the existence of solutions are in order.

Euler proved that Eq. (1) is solvable in integers when $(a,b) = 1$, i.e., they are relatively prime [1, p. 47], and Gauss proved that the equivalent of (1) is solvable in integers if and only if $(a,b)|n$ [1, p. 54]. In view of these results and the general conditions imposed on (1), i.e., the solutions and parameters are to be non-negative integers, there is no loss of generality by assuming $(a,b) = 1$.

If (x_1, y_1) is a solution of (1) in integers and $(a,b) = 1$, then all other solutions will be given by

$$\begin{aligned} x &= x_1 + bj \\ y &= y_1 - aj \end{aligned} \tag{2}$$

where j is an integer [2, p. 29].

It is for this first solution that we seek an explicit formula. This can be accomplished easily with the Fermat-Euler Theorem applied to the congruence $ax \equiv n \pmod b$. Such a result has advantages over other methods of solution, such as, algorithms involving a succession of recursive steps. The Fermat-Euler Theorem involves the concept of Euler's function, denoted $\phi(b)$, which is equal to the number of natural numbers less than b that are coprime with b. An explicit formula for this value is given by

$$\phi(b) = b \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right), \tag{3}$$

where $p_1, p_2, \cdots, p_r$ are the different prime factors of the natural number b [2, p. 24]. The statement of the Fermat-Euler Theorem then becomes [3, p. 63]

$$a^{\phi(b)} \equiv 1 \pmod b \tag{4}$$

Assuming that a, b and n are non-negative and $(a,b) = 1$, then if either a or b equals 0 or 1, then the determination of the solution of (1) becomes a simple case. Hence, in what follows we assume that both a and b are greater than 1. This implies that $a \neq b$, since if $a = b$, then $(a,b) = (a,a) = a$, but $(a,b) = 1$, hence $a = b = 1$, a contradiction.

As a final remark we might consider the graphical representation of this problem. Under the imposed restrictions, the graph of equation (1) is confined to the first quadrant. We note that (1) with non-negative parameters represents the family of all line segments whose endpoints are the rational points of the x and y axes. Thus, the line segment determined by the endpoints

$$\left(\frac{p}{q}, 0\right) \quad \text{and} \quad \left(0, \frac{r}{s}\right)$$

has the equation

$$rqx + spy = pr.$$

This is a form of (1) where $rq = a$ and $sp = b$, and $pr = n$. Now we are ready to examine the general solution.

3. THE SOLUTION

The explicit formula involves the concept of the principal remainder modulo m, for which we may use the following notation:

DFN. (PRINCIPAL REMAINDER):

$$(5) \qquad [\![y \ (\mathrm{mod}\ m)]\!] = x \quad \text{iff} \quad x \equiv y \ (\mathrm{mod}\ m) \ \text{and} \ 0 \leq x \leq m - 1$$

The following lemma, that is easily verified, though not essential to the derivation of the explicit formula, makes the solution of a specific example feasible.

Lemma.

$$\left[\!\left[\prod_{i=1}^{n} y_i \pmod m \right]\!\right] \equiv \prod_{i=1}^{n} [\![y_i \pmod m]\!] \pmod m . \tag{6}$$

As a special consequence of this lemma we may note that,

$$[\![y^n \pmod m]\!] \equiv [\![y \pmod m]\!]^n \pmod m . \tag{7}$$

Since the number of solutions of (1) will be finite, the method of solution will be to find the minimum positive integral value of x (or y), and then to find the corresponding value of y (or x) which will necessarily be maximum and then to subtract multiples of a (or b) to obtain the set of all possible non-negative solutions of (1). The following formula for the minimum value of x is essentially due to Bouniakowski, and independently, Cauchy [1, pp. 55-56].

Theorem. If the equation, $ax + by = n$, has non-negative parameters, a, b, and n, and $a \geq 1$ and $b \geq 1$, and $(a, b) = 1$, then when non-negative integral solutions exist, the minimum non-negative integral value of x, which satisfies the equation such that y is also a non-negative integer, is given by

$$X_{min} = [\![na^{\phi(b)-1} \pmod b]\!] . \tag{8}$$

Proof. The remarks made in Sec. 2 claim that there is no essential loss of generality by assuming the above conditions. It is important to note that we must assume a value of n such that non-negative solutions do exist. There does exist a finite number of values of n, for a given a and b, such that the equation will not have the non-negative solutions that we are seeking. This is proved in the next section. Formula (8) is independent of this consideration, therefore, we could obtain an erroneous value of X_{min} if used without circumspection, however, we would soon be aware of the error when the attempt to solve for the corresponding value of y was made.

The equation $ax + by = n$ is equivalent to the congruence

$$ax \equiv n \pmod b . \tag{9}$$

Now by the Fermat-Euler Theorem if we multiply both sides of (9) by $a^{\phi(b)-1}$, we obtain

$$x \equiv n \cdot a^{\phi(b)-1} \pmod{b} \quad , \tag{10}$$

whereby, the least value of x is the principal remainder,

$$X_{min} = [\![n \cdot a^{\phi(b)-1} \pmod{b}]\!] \qquad \text{Q. E. D.} \tag{11}$$

Using the same principals we may derive a formula for X_{max}.

$$X_{max} = \left[\frac{n}{a}\right] - \left[\!\!\left[-[\![n \pmod{a}]\!] \cdot a^{\phi(b)-1} \pmod{b} \right]\!\!\right] \quad , \tag{12}$$

where $[\;]$ denotes the greatest integer function.

Proof. Equation (1) is equivalent to the congruence,

$$by \equiv n \pmod{a} \quad . \tag{13}$$

Now if $m = [\![n \pmod{a}]\!]$, then $b \cdot y \equiv m \pmod{a}$. Therefore,

$$by \in \{m, \; m + a, \; m + 2a, \cdots, m + ka\} \quad . \tag{14}$$

where $k = [n/a]$. We note that $m + ka = n$, hence, this is the maximum value that $b \cdot y$ can achieve. By substituting these values of $b \cdot y$ into (1) we obtain the corresponding values of x, that is,

$$x \in \{k, \; k - 1, \; k - 2, \cdots, k - k\} \quad , \tag{15}$$

in that order. Therefore, there is an integral solution of (1) when

$$y = \frac{ja + m}{b}$$

is an integer, where $0 \le j \le k$. This situation is equivalent to the congruence

$$ja + m \equiv 0 \pmod{b} \ . \tag{16}$$

The corresponding value of x is

$$x = k - j \ . \tag{17}$$

Now since $(a, b) = 1$, we can solve (16) for j by using (4), the Fermat-Euler Theorem. This gives

$$j \equiv -ma^{\phi(b)-1} \pmod{b} \ . \tag{18}$$

Now that j has been found, we can find x from (17). Since k represents the maximum value that x can be, we will have the maximum integral value of x that satisfies (1) by subtracting the least value of j from k as represented in (17). That least value of j is the principal remainder, $[\![j \pmod{b}]\!]$. By substituting for k, j, and m in (17), we arrive at (12). Q. E. D.

Corollary. If a non-negative integral solution of (1) exists and $(a, b) = 1$, then there are at least

$$\left[\frac{n}{ab}\right]$$

and at most

$$\left[\frac{n}{ab}\right] + 1$$

solutions.

Proof. We note that (13) has just one solution such that $0 \leq y < a$ [3, p. 51]. Therefore, from (14) we also note that there are at least $[k/b]$ and at most $[k/b] + 1$ solutions. Also, since $k = [n/a]$, the corollary is proved.

4. THE NON-EXISTENCE OF SOLUTIONS

Even when a and b are relatively prime, in deference to Sec. 2, there will be cases when (1) has no solutions, due to the restriction that they be non-negative. Naturally, we would wish to know just what cases have no solutions, therefore, it is necessary to state the following.

Theorem. The equation $ax + by = n$, where a, b, and n are non-negative integers and $ab \neq 0$ and $(a,b) = 1$, will not have integral solutions ≥ 0 when $n = ab - (ja + kb)$, where $j,k = 1, 2, 3, \cdots$.

Proof. Assume (1) has a non-negative integral solution and that $n = ab - (ja + kb)$ for some j and k. Then

$$ax + by = ab - ja - kb \tag{19}$$

$$a(x + j) + b(y + k) = ab \tag{20}$$

Let $X = x + j$ and $Y = y + k$; then,

$$aX + bY = ab. \tag{21}$$

It is important to note that X, Y, a, and b are all greater than or equal to one. From (21) we obtain

$$b = X + \frac{bY}{a} \text{ or } b - X = \frac{bY}{a} . \tag{22}$$

Now $b \equiv X$ is an integer, therefore a divides Y, since a and b are relatively prime. Suppose $Y/a = r$, then $b - X = br$, but this is impossible since all the quantities are positive integers, therefore, there is a contradiction. Q. E. D.

It was proved by E. Lucas [1, p. 68] that there are $\frac{1}{2}(a - 1)(b - 1)$ such values of n which afford no solutions. He also gave a method to determine if a given case was solvable, but it involved long computations.

5. AN EXAMPLE

Find all the non-negative integral solutions of $20x + 14y = 410$. $(20, 14) = 2$, therefore, by dividing through by 2 we have the parameters 10, 7, and 205. Now since $205 \geq 10 \cdot 7$ and $(10,7) = 1$ we can solve for X_{min}.

$$\begin{aligned} X_{min} &= [\![n \cdot a^{\phi(b)-1} \bmod b)]\!] \\ &= [\![205 \cdot 10^{\phi(7)-1} (\bmod 7)]\!] \\ &= [\![205 \cdot 10^{5} (\bmod 7)]\!] \end{aligned}$$

By using the Lemma, this simplifies to

$$\begin{aligned} X_{min} &= [\![2 \cdot 3^{5} \ (\bmod 7)]\!] \\ &= [\![2 \cdot 3 \cdot 9 \cdot 9 \ (\bmod 7)]\!] \\ &= [\![2 \cdot 3 \cdot 2 \cdot 2 \ (\bmod 7)]\!] \\ &= [\![3 \ (\bmod 7)]\!] \\ &= 3 \ . \end{aligned}$$

By substituting X_{min} into the original equation we obtain Y_{max}.

$$\begin{aligned} 10 \cdot 3 + 7y &= 205 \\ y &= 25 \end{aligned}$$

Now subtract multiples of a, (10). In this case $y \in \{25, 15, 5\}$. The corresponding values of x are found by adding multiples of b, (7). In this case $x \in \{3, 10, 17\}$. The three pairs of non-negative integral solutions of the original equation are (3, 25), (10, 15) and (17, 5).

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1. L. Dickson, History of the Theory of Numbers, Vol. 2, Chelsea, New York, 1952.
2. T. Nagell, Introduction to Number Theory, Wiley and Sons, New York, 1951.
3. G. Hardy and E. Wright, An Introduction to the Theory of Numbers, Oxford University Press, London, 1954.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-136 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let P_n be the n^{th} Pell number defined by $P_1 = 1$, $P_2 = 2$, and $P_{n+2} = 2P_{n+1} + P_n$. Show that $P^2_{n+1} + P^2_n = P_{2n+1}$.

B-137 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let P_n be the n^{th} Pell number. Show that $P_{2n+1} + P_{2n} = 2P^2_{n+1} - 2P^2_n - (-1)^n$.

B-138 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Show that for any non-negative integer k and any integer $n > 1$ there is an n by n matrix with integral entries whose top row is $F_{k+1}, F_{k+2}, \cdots, F_{k+n}$ and whose determinant is 1.

B-139 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show that the sequence 1, 1, 1, 1, 4, 4, 9, 9, 25, 25, $\cdots$ defined by $a_{2n-1} = a_{2n} = F^2_n$ is complete even if an a_j with $j \le 6$ is omitted but that the sequence is not complete if an a_j with $j \ge 7$ is omitted.

B-140 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Show that $F_{ab} > F_aF_b$ if a and b are integers greater than 1.

B-141 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Show that no Fibonacci number F_n nor Lucas number L_n is an even perfect number.

SOLUTIONS

ERRATUM

A line was omitted in the printing of the solution of B-95 in Vol. 5, No. 2, p. 204. The submitted solution follows:

Solution by Charles W. Trigg, San Diego, California.

For $n \geq 3$, F_k is divisible by 2^n if k is of the form $2^{n-2} \cdot 3(1+2m)$, $m = 0, 1, 2, \cdots$. If k is of the form $3(1+2m)$, F_k is divisible by 2 but by no higher power of 2. Hence, the highest power of 2 that exactly divides $F_1F_2F_3 \cdots F_{100}$ is

$$[(100 - 3)/6 + 1] + 3[(100+6)/12] + 4[112/24] + 5[124/48] + 6[148/96] + 7[196/192]$$

or 80. As usual, $[x]$ indicates the largest integer in x.

A PARTIAL SUM INEQUALITY

B-118 Proposed by J. L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania.

Let $F_1 = 1 = F_2$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. Show that for all $n \geq 1$ that

$$\sum_{k=1}^{n} (F_k/2^k) < 2 .$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

From the standard generating function

$$\sum_{k=1}^{\infty} F_k x^k = \frac{x}{1 - x - x^2}$$

which converges for $|x| < 2/(1+\sqrt{5})$, we find for $x = \frac{1}{2}$ that

$$\sum_{k=1}^{\infty} (F_k / 2^k) = 2 ,$$

from which the result follows.

Also solved by E. M. Clark, Lawrence D. Gould, John Ivie, Clifford Juhlke, Bruce W. King, Geoffrey Lee, Robert L. Mercer, F. D. Parker, C. B. A. Peck, J. Ramanna, A. C. Shannon, John Wessner, David Zeitlin, and the proposer.

A FIBONACCI TRAPEZOID

B-119 Proposed by Jim Woolum, Clayton Valley High School, Concord, Calif.

What is the area of an equilateral trapezoid whose bases are F_{n-1} and F_{n+1} and whose lateral side is F_n ?

Solution by F. D. Parker, St. Lawrence University, Canton, New York.

The difference between the two bases is F_n, so that the base angles of the trapezoid are $\pi/3$, and the altitude is $F_n\sqrt{3}/2$. Thus the area is given by $\sqrt{3}F_n(F_{n+1} + F_{n-1})/4$. This result can be simplified to $\sqrt{3}F_{2n}/4$.

Also solved by Herta T. Freitag, Lawrence D. Gould, J. A. H. Hunter, John Ivie, Bruce W. King, Geoffrey Lee, Douglas Lind, John W. Milsom, C. B. A. Peck, A. C. Shannon, John Wessner, and the proposer.

A TRIANGULAR NUMBERS RELATION

B-120 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Find a simple function g such that g(n) is an integer when n is an integer and $g(m + n) - g(m) - g(n) = mn$.

Solution by J. A. H. Hunter, Toronto, Canada.

Taking suitable integral values for m and n we find that:

$$g(2) = 2g(1) + 1 \qquad g(3) = 3g(1) + 3$$
$$g(4) = 4g(1) + 6 \qquad g(5) = 5g(1) + 10$$

The sequence 1, 3, 6, 10 suggests triangular numbers. Hence, taking $g(1) = 1$, we have: $g(n) = n(n + 1)/2$. But g(1) may be any positive integer, so we take the generak function:

$$g(m) = m(m + 2k + 1)/2, \qquad k \text{ an integer.}$$

Also solved by L. Carlitz, E. M. Clarke, Douglas Lind, C. B. A. Peck, J. Ramanna, David Zeitlin, and the proposer.

A CONGRUENCE MODULO F_d

B-121 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let n, q, d and r be integers with $n \geq 0$, $d > 0$, $n = qd + r$, and $0 \leq r < d$. Prove that

$$F_n \equiv F_{d+1})^q F_r \pmod{F_d}.$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

Vinson ["The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence," Fibonacci Quarterly, 1(1963), No. 2, 37-45] has shown that

$$F_n = F_{qd+r} = \sum_{j=0}^{q} \binom{q}{j} F_d^j F_{d-1}^{q-j} F_{r+j} \quad (q \geq 0) .$$

Hence

$$F_n \equiv F_{d-1}^q F_r \equiv F_{d+1}^q F_r \qquad (\text{mod } F_d)$$

since

$$F_{d+1} \equiv F_{d-1} \quad (\text{mod } F_d) .$$

Also solved by David L. Estrin and the proposer.

ANALOG OF A MULTIPLE ANGLE FORMULA

B-122 Proposed by A. J. Montleaf, University of New Mexico, Albuquerque, N.M.

Show that

$$\sin[(2k+1)\theta]/\sin\theta = 2\cos[2k\theta] + 2\cos[2(k-1)\theta] + 2\cos[2(k-2)\theta] + \cdots + \\ + 2\cos[2\theta] + 1$$

and obtain the analogous formula for $F_{(2k+1)m}/F_m$ in terms of Lucas numbers.

Solution by Paul A. Anderson, University of Minnesota, Minneapolis, Minn.

The stated formula is well known (see, for example, Taylor, Advanced Calculus, p. 729). The analogous formula for $F_{(2k+1)m}/F_m$ is

$$\frac{F_{(2k+1)m}}{F_m} = (-1)^{km} + (-1)^{(k+1)m} L_{2m} + (-1)^{(k+2)m} L_{4m} + \cdots + (-1)^{2km} L_{2km} .$$

The proof is by induction on k. For $k = 1$,

$$\frac{F_{3m}}{F_m} = \frac{a^{3m} - b^{3m}}{a^m - b^m} = a^{2m} + (ab)^m + b^{2m} = L_{2m} + (-1)^m,$$

where

$$a = \frac{1 + \sqrt{5}}{2}, \; b = \frac{1 - \sqrt{5}}{2} \quad .$$

If

$$\frac{F_{(2k+1)m}}{F_m} = (-1)^{km} + \sum_{i=1}^{k} (-1)^{(k+i)m} L_{2mi} \;,$$

then

$$L_{(2k+2)m} + (-1)^m \frac{F_{(2k+1)m}}{F_m} = a^{(2k+2)m} + b^{(2k+2)m} + (-1)^m \frac{a^{(2k+1)m} - b^{(2k+1)m}}{a^m - b^m} =$$

$$\frac{a^{(2k+3)m} - b^{(2k+3)m} + a^m b^{(2k+2)m} - a^{(2k+2)m} b^m + (-1)^m (a^{(2k+1)m} - b^{(2k+1)m})}{a^m - b^m} =$$

$$\frac{a^{(2k+3)m} - b^{(2k+3)m}}{a^m - b^m} = \frac{F_{(2k+3)m}}{F_m} \; .$$

Also solved by A. C. Shannon and the proposer.

SQUARE SUM OF SUCCESSIVE SQUARES

B-123 (From B-102, Proposed by G. L. Alexanderson, University of Santa Clara, Santa Clara, California)

Show that all the positive integral solutions of $x^2 + (x \pm 1)^2 = z^2$ are given by

$$x_n = (P_{n+1})^2 - (P_n)^2 \; ; \; z_n = (P_{n+1})^2 + (P_n)^2; \quad n = 1, 2, \cdots ;$$

where P_n is the Pell number defined by $P_1 = 1$, $P_2 = 2$, and $P_{n+2} = 2P_{n+1} + P_n$.

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Letting $w = 2x \pm 1$ changes $x^2 + (x \pm 1)^2 = z^2$ into $w^2 - 2z^2 = -1$. Let Z be the ring of the integers and let $Z\sqrt{2}$ be the ring consisting of the real numbers $\alpha = z + b\sqrt{2}$ with a and b in Z. Let V consist of the positive real numbers $\alpha = a + b\sqrt{2}$ of $Z[\sqrt{2}]$ such that $a^2 - 2b^2 = -1$. Then V can be shown to be a group under multiplication. Since V has no number between 1 and $1 + \sqrt{2}$, it follows that V is the cyclic group generated by $1 + \sqrt{2}$. The odd powers $(1 + \sqrt{2})^{2n-1}$ lead to $a^2 - 2b^2 = -1$. Therefore the positive integral solutions of $w^2 - 2z^2 = -1$ are obtained by equating "rational" and "irrational" parts of $w_n + z_n\sqrt{2} = (1 + \sqrt{2})^{2n-1}$, i.e.,

$$w_n = [(1 + \sqrt{2})^{2n-1} + (1 - \sqrt{2})^{2n-1}]/2, \quad z_n = [(1 + \sqrt{2})^{2n-1} - (1 - \sqrt{2})^{2n-1}]/2\sqrt{2}.$$

The desired formulas then may be found using the analogue $P_n = [(1 + \sqrt{2})^n - (1 - \sqrt{2})^n]/2\sqrt{2}$ of one of the Binet formulas.

Also solved by A. C. Shannon and the proposer.

* * * * *

(Continued from p. 176)

$$P_3(x) = 32 - 13x - 99x^2 - 32x^3 + 9x^4 + x^5$$
$$P_4(x) = 243 + 1181x - 1952x^2 - 1271x^3 + 257x^4 + 32x^5$$
$$P_5(x) = 3125 + 7768x - 15851x^2 - 9752x^3 + 1944x^4 + 243x^5$$

$$\sum_{n=0}^{\infty} F^6_{n+k}x^n = \frac{P_k(x)}{1 - 13x - 104x^2 + 260x^3 + 260x^4 - 104x^5 - 13x^6 + x^7}$$

$$k = 0, 1, 2, 3, 4, 5, 6$$

$$P_0(x) = x(1 - 12x - 53x^2 + 53x^3 + 12x^4 - x^5)$$
$$P_1(x) = 1 - 12x - 53x^2 + 53x^3 + 12x^4 - x^5$$
$$P_2(x) = 1 + 51x - 207x^2 - 248x^3 + 103x^4 + 13x^5 - x^6$$
$$P_3(x) = 64 - 103x - 508x^2 - 157x^3 + 117x^4 + 12x^5 - x^6$$
$$P_4(x) = 729 + 6148x - 16,797x^2 - 16,523x^3 + 6,668x^4 + 831x^5 - 64x^6$$
$$P_5(x) = 15,625 + 59,019x - 206,063x^2 - 182,872x^3 + 76,644x^4 + 9413x^5 - 729x^6$$
$$P_6(x) = 262,144 + 1,418,937x - 4,245,372x^2 - 3,985,856x^3 + 1,634,413x^4 + 202,396x^5 - 15,625x^6$$

(Continued on p. 166.)

PASCAL'S TRIANGLE AND SOME FAMOUS NUMBER SEQUENCES

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The Fibonacci sequence has a well-known relationship to certain diagonals of Pascal's Triangle.

Another interesting relationship exists between the double numbers of Pascal's Triangle and each of two sequences well known in atomic and nuclear physics.

One of these two sequences represents the numbers of electrons (2, 8, 18, 32, 50,···), and another — the numbers of nucleons (2, 8, 20,··· and 28, 50, 82, 126,···) in the occupied shell structures of atoms and their nuclei respectively.

The details are shown in the following figure.

2. (Fibonacci numbers)

2·1 2·1 2·2 2·3 2·5 2·8

big "magic" nucleonic numbers (sums of two numbers on the 2 and 4 lines)

28 50 82 126

every adjacent pair of numbers sums up to the number of electrons in the occupied shell of the atom, e. g., 2+6=8, 6+12 = 18, etc.*

little "magic" nucleonic numbers

2									
2	2								
2	4	2							
<u>2</u>	6	6	2						
2	<u>8</u>	12	8	2					
2	10	<u>20</u>	20	10	2				
2	12	30	40	30	12	2			
2	14	42	70	70	42	14	2		
2	16	56	112	140	112	56	16	2	
2	18	72	168	252	252	168	72	18	2

Pascal's Triangle (with double numbers)

*Remark. The definition does not exclude: 0 + 2 = 2.

* * * * *

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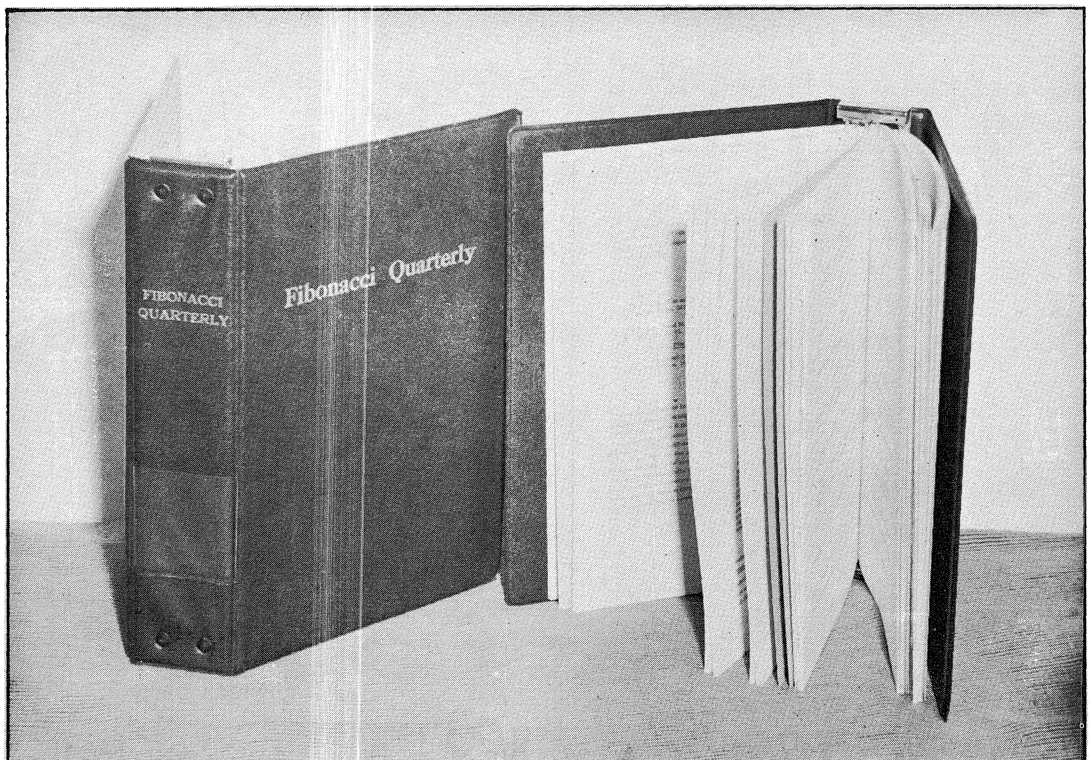
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