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THE LINEAR DIOPHANTINE EQUATION IN n VARIABLES AND ITS APPLICATION TO GENERALIZED FIBONACCI NUMBERS

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1. SUMMARY OF RESULTS

The solution of the Linear Diophantine Equation in n unknowns, viz.

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = c$$

with

$$n \geq 2; \quad c_1, c_2, \cdots, c_n, \; c$$

integers is a problem which may occupy more space in the future development of linear programming. For $n = 2$ this is achieved by known methods — either by developing c_2/c_1 in a continued fraction by Euclid's algorithm or by solving the linear congruence $c_1x_1 \equiv c(c_2)$. For $n > 2$ refuge is usually taken to solving separately the equation $c_1x_1 + c_2x_2 = c$ and the homogeneous linear equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ and adding the general solution of the latter to a special solution of the former. This is usually a most cumbersome method which becomes especially unhappy under the restriction that none of the unknowns $x_i (i = 3, \cdots, n)$ vanishes, since in the opposite case the rank of the Diophantine equation is lowered. The first part of the present paper, therefore, suggests a method of solving the linear Diophantine equation in $n > 2$ unknowns with the restriction $x_i \neq 0$ $(i = 1, \cdots, n)$ based on a modified algorithm of Jacobi-Perron; it is proved that if the equation is consistent, this method always leads to a solution; numerical examples illustrate the theory.

In the second part of this paper these results are being used to state explicitly the solution of a linear Diophantine equation whose coefficients are generalized Fibonacci numbers. The periodicity of the ratios of generalized Fibonacci numbers of the third degree is proved using rational ratios only.

Concluding, an explicit formula is stated for the limiting ratio of two subsequent generalized Fibonacci numbers of any degree by means of two simple infinite series. For this purpose the author repeatedly utilizes results of his previous papers on a modified algorithm of Jacobi-Perron.

2. THE STANDARD EQUATION

A Linear Diophantine Equation in n unknowns

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 1, \quad n > 2 \tag{1.1}$$

will be called a Standard Equation of Degree n (abbreviated S. E. n) if the following restrictions on its coefficients hold:

(1.2)
- a) c_i a natural number for every $i = 1, \cdots, n$;
- b) $1 < c_1 < c_2 < \cdots < c_n$;
- c) $(c_1, c_2, \cdots, c_n) = 1$;
- d) $c_i \nmid c_{i+j}$; $i, j \geq 1, \quad i + j \leq n$;
- e) $(c_{k_1}, c_{k_2}, \cdots, c_{k_{n-1}}) = d > 1; \quad k_i, k_j = 1, \cdots, n$; $k_i \neq k_j$; $(i, j = 1, \cdots, n - 1)$.

A linear Diophantine equation in m unknowns with integral coefficients

$$a_1y_1 + a_2y_2 + \cdots + a_my_m = A, \quad (m > 1;\ a_i \neq 0;\ i = 1, \cdots, m) \tag{1.3}$$

will be called trivial, if

$$a_i = 1 \quad \text{for at least one } i \ ; \tag{1.4}$$

otherwise it will be called nontrivial. This notation is justified; for let be $|a_i| = 1$ in (1.3). Then all the solutions of (1.3) are given by

$$\begin{aligned} &y_1, y_2, \cdots, y_{i-1}, y_{i+1}, \cdots, y_m \quad \text{any integers}, \quad 1 < i < m \ ; \\ &y_i = a_i(A - a_1y_1 - a_2y_2 - \cdots - a_{i-1}y_{i-1} - a_{i+1}y_{i+1} - \cdots - a_my_m) \ ; \end{aligned} \tag{1.5}$$

and similar for $i = 1$, $i = m$.

Let equation (1.3) be nontrivial; it will be called reduced, if

$$(a_1, a_2, \cdots, a_m, A) = 1 \tag{1.6}$$

nonreduced, if

$$(a_1, a_2, \cdots, a_m, A) = d > 1 \, . \tag{1.7}$$

With the meaning of (1.7), (1.3) can always w. l. o. g. be reduced by cancelling d from the coefficients $a_1, \cdots, a_m, A$.

As is well known, (1.3) is solvable if

$$(a_1, a_2, \cdots, a_m) \mid A \, , \tag{1.8}$$

otherwise unsolvable.

Theorem 1.1. Every reduced nontrivial solvable equation (1.3) can be transformed into an S. E. n.

Proof. We obtain from the conditions of Theorem 1.1.

$$(a_1, a_2, \cdots, a_m, A) = 1; \qquad |a_i| > 1, \quad (i = 1, \cdots, m) \, . \tag{1.9}$$

Substituting in (1.3)

$$y_i = Az_i \, , \quad (i = 1, \cdots, m) \tag{1.10}$$

we obtain

$$a_1z_1 + a_2z_2 + \cdots + a_mz_m = 1 \, . \tag{1.11}$$

Since (1.3) is solvable, we have $(a_1, a_2, \cdots, a_m) \mid A$, which, together with (1.9), yields

$$(a_1, a_2, \cdots, a_m) = 1 \, . \tag{1.12}$$

Let denote

$$z_{k_i} = u_{k_i} \quad \text{if} \quad b_{k_i} = a_{k_i} > 0\ , \tag{1.13}$$

$$z_{k_i} = -u_{k_i} \quad \text{if} \quad b_{k_i} = -a_{k_i} > 0 \quad (k_i = 1, \cdots, m)\ . \tag{1.14}$$

In virtue of (1.13), (1.14), equation (1.11) takes the form

$$b_1u_1 + b_2u_2 + b_mu_m = 1; \quad (b_1, b_2, \cdots, b_m) = 1\ . \tag{1.15}$$

We can now presume, without loss of generality,

$$1 < b_1 \le b_2 \le b_3 \le \cdots \le b_m\ . \tag{1.16}$$

Let b_i be the first coefficient in (1.16) such that

$$b_i \mid b_{k_s}, k_s > i, s = 1, \cdots, m-n; \; m-n \quad m-i; \; i+1 \le k_s \le m\ . \tag{1.17}$$

Putting

$$\begin{gathered} b_{k_s} = t_s b_i\ , \qquad (s = 1, \cdots, m-n) \\ u_i + t_1u_{k_1} + t_2u_{k_2} + \cdots + t_{m-n}u_{k_{m-n}} = v_i\ , \end{gathered} \tag{1.18}$$

we obtain from (1.15), (1.18)

$$\begin{gathered} b_1u_1 + b_2u_2 + \cdots + b_{i-1}u_{i-1} + b_iv_i + b_{r_1}u_{r_1} + \cdots + b_{r_{n-i}}u_{r_{n-i}} = 1, \\ b_i \nmid b_{r_1}, b_{r_2}, \cdots, b_{r_{n-i}}; \; i+1 \le r_q \le m, \quad (q = 1, \cdots, n-i)\ . \end{gathered} \tag{1.19}$$

We shall prove

$$(b_1, b_2, \cdots, b_{i-1}, b_i, b_{r_1}, b_{r_2}, \cdots, b_{r_{n-i}}) = 1\ . \tag{1.20}$$

Suppose,

$$(b_1, b_2, \cdots, b_{i-1}, b_i, b_{r_1}, b_{r_2}, \cdots, b_{r_{n-i}}) = d > 1;$$

we would then obtain, in view of (1.17),

$$(b_1, b_2, \cdots, b_i, b_{i+1}, \cdots, b_m) =$$
$$(b_1, b_2, \cdots, b_{i-1}, b_i, b_{k_1}, \cdots, b_{k_{m-n}}, b_{r_1}, \cdots, b_{r_{n-i}}) \geq$$
$$(b_1, b_2, \cdots, b_{i-1}, b_i, b_{r_1}, b_{r_2}, \cdots, b_{r_{n-i}}) = d > 1 ,$$

contrary to (1.15).

If there exists a b_{r_q} such that $b_{r_q} | b_{r_p}$, $(p > q)$ this process is repeated as before; otherwise we obtain from (1.19) denoting

$$\text{(1.21)} \qquad \begin{aligned} &b_j = h_j, \quad (j = 1, \cdots, i); \; u_j = v_j, \quad (j = 1, \cdots, i - 1) ; \\ &b_{r_j} = h_{i+j} ; \; u_{r_j} = v_{i+j} , \quad (j = 1, \cdots, n - i) , \end{aligned}$$

$$\text{(1.22)} \qquad \begin{aligned} &h_1v_1 + h_2v_2 + \cdots + h_iv_i + h_{i+1}v_{i+1} + \cdots + h_nv_n = 1 , \\ &1 < h_1 < h_2 < \cdots < h_n ; \quad (h_1, \cdots, h_n) = 1, \; h_i \nmid h_j; \; j > i. \end{aligned}$$

It should be noted that, in virtue of (1.18), the values of $u_1, u_{k_1}, u_{k_2}, \cdots, u_{k_{m-n}}$ are obtained from those of v_i in (1.22) as follows

$$\text{(1.23)} \qquad u_{k_1}, \cdots, u_{k_{m-n}} \text{ any integers; } u_i = v_i - t_1u_{k_1} - \cdots - t_{m-n}u_{k_{m-n}}.$$

If the h_i $(i = 1, \cdots, n)$ of (1.22) do not fulfill conditions e) of (1.2), we choose n different primes p_i such that

$$\text{(1.24)} \qquad p_i \nmid h_1h_2 \cdots h_n , \quad (i = 1, \cdots, n) ; \; p_1 > p_2 > \cdots > p_n ,$$

and denote

$$\text{(1.25)} \qquad p_1p_2 \cdots p_n = P; \; v_i = p_i^{-1}Px_i ; \; c_i = p_i^{-1}Ph_i , \quad (i = 1, \cdots, n) .$$

With (1.25) equation (1.22) takes the form (1.1). Since

$$c_1 = h_1p_1^{-1}P = h_1p_2p_3 \cdots p_m > h_1 ,$$

we obtain

$$c_1 > 1 \tag{1.26}$$

We further obtain, for $i \geq 1$, and in virtue of (1.24)

$$\begin{aligned} c_i &= h_i p_i^{-1}P < h_{i+1}p_i^{-1}P < h_{i+1}p_{i+1}^{-1}P = c_{i+1}\ , \\ c_i &< c_{i+1} \qquad (i = 1, 2, \cdots, n-1)\ . \end{aligned} \tag{1.27}$$

But

$$(p_1^{-1}P,\ \cdots,\ p_n^{-1}P) = 1, \quad \text{and} \quad (h_1, h_2,\ \cdots,\ h_n) = 1\ ,$$

and since $p_i \nmid h_1 h_2 \cdots h_n$, we obtain, on ground of a known theorem

$$(h_1 p_1^{-1}P,\ h_2 p_2^{-1},\ \cdots,\ h_n p_n^{-1}P) = 1\ ,$$

so that

$$(c_1, c_2,\ \cdots,\ c_n) = 1\ . \tag{1.28}$$

We shall now prove that the numbers c_i $(i = 1, \cdots, n)$ from (1.25) fulfill the conditions e) of (1.2). We shall prove it for one $(n-1)$ tuple of the c_i; the general proof for any $(n-1)$ tuple is analogous. We obtain

$$\begin{aligned} &(c_1, c_2,\ \cdots,\ c_{n-1}) = (h_1 p_1^{-1}P,\ h_2 p_2^{-1}P,\ \cdots,\ h_{n-1}p_{n-1}^{-1}P) = \\ &(h_1 p_2 p_3 \cdots p_n,\ h_2 p_1 p_3 \cdots p_n,\ \cdots,\ h_{n-1}p_1 \cdots p_{n-2}p_n) \geq p_n > 1\ . \end{aligned}$$

By this method we obtain, indeed, generally

$$(c_{k_1},\ c_{k_2},\ \cdots,\ x_{k_{n-1}}) = p_{k_n} > 1, \quad k_i \neq k_j \ \text{ for } i \neq j\ . \tag{1.29}$$

Thus Theorem 1.1 is completely proved.

A Linear Diophantine Equation in n unknowns which satisfies conditions a), b), c), d) of (1.1) will be called a Deleted Standard Equation of Degree n (abbreviated S'. E. n). Let

$$h_1v_1 + h_2v_2 + \cdots + h_nv_n = 1$$

be an S'. E. n. We have proved that every nontrivial reduced solvable Diophantine equation can be transformed into an S'. E. n, whereby $n > 2$.

An n-tuple of integers $(x_1, x_2, \cdots, x_n)$ for which

$$h_1x_1 + h_2x_2 + \cdots + h_nx_n = 1 \quad , \tag{1.30}$$

is a solution vector of S'. E. n; it will be called a standard solution vector, if $x_i \neq 0$ for all $i = 1, \cdots, n$. As already pointed out in the Summary of Results, we are aiming at finding a standard solution vector of S'. E. n. Since in the S'. E. n condition e) of (1.2) it is not fulfilled, there must be at least one (n - 1)-tuple of numbers among the $h_1, \cdots, h_n$ which are relatively prime. We shall presume, without loss of generality,

$$(h_1, h_2, \cdots, h_{n-1}) = 1 \tag{1.31}$$

and let $(x_1, x_2, \cdots, x_{n-1})$ be a standard solution vector of

$$h_1v_1 + h_2v_2 + \cdots + h_{n-1}v_{n-1} = 1 \quad .$$

Then $(x_1, x_2, \cdots, x_{n-1}, 0)$ is a solution vector of the S'. E. n, but it is not a standard solution vector; such one would be given by the n-tuple,

$$(x_1, x_2, \cdots, x_{n-1} - th_n, th_{n-1}),$$
$$t \text{ any integer}, \quad x_{n-1} \neq th_n .$$

Thus the problem for an S'. E. n which is not an S. E. n is reduced to find a standard solution vector of an S'. E. n - 1; this can be either an S . E. n - 1, or only an S'. E. n - 1.

Theorem 1.2. An S. E. n has only standard solution vectors.

Proof. Let $(x_1, x_2, \cdots, x_k, 0, 0, \cdots, 0)$ be a solution vector of an S. E. n, and let $x_i \neq 0$, $(i = 1, \cdots, k)$. It is easy to verify that $k \geq 2$, and let be $k \leq n - 1$. The arrangement of the components of the solution vector can be assumed without loss of generality. Then

$$c_1x_1 + c_2x_2 + \cdots + c_kx_k = 1 ; \tag{1.32}$$

but since

$$(c_1, c_2, \cdots, c_k, c_{k+1}, \cdots, c_{n-1}) = p_n ,$$

we obtain

$$(c_1, c_2, \cdots, c_k) \geq p_n > 1 ,$$

which is inconsistent with (1.32). This proves Theorem 1.2.

Let again

$$h_1v_1 + h_2v_2 + \cdots + h_nv_n = 1$$

be an S'. E. n and

$$h_1v_1 + h_2v_2 + \cdots + h_nv_n = 0 \tag{1.33}$$

its homogeneous part. We shall denote

$$D(h_1, \cdots, h_n) = \begin{vmatrix} th_1 & v_{1,1} & v_{1,2} & \cdots & v_{1,n-2} & h_1 \\ th_2 & v_{2,1} & v_{2,2} & \cdots & v_{2,n-2} & h_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ th_n & v_{n,1} & v_{n,2} & \cdots & v_{n,n-2} & h_n \end{vmatrix} \tag{1.34}$$

t, $v_{i,j}$ any integers,
$(i = 1, \cdots, n;\ j = 1, \cdots, n-2)$

(1.35) $H_{k,n}$ is the algebraic cofactor of the element $a_{k,n}$.

For any $v_{i,j}$ the following identity holds

$$D(h_1, \cdots, h_n) = h_1H_{1,n} + h_2H_{2,n} + \cdots + h_nH_{n,n} = 0. \tag{1.36}$$

Theorem 1.3. Let $(x_1, x_2, \cdots, x_n)$ be a solution vector of an S'. E. n and $(H_{1,n}, H_{2,n}, \cdots, H_{n,n})$ be any solution vector of its homogeneous part; then infinitely many solution vectors of S'. E. n are given by

$$(1.37) \qquad (x_1 + H_{1,n}, x_2 + H_{2,n}, \cdots, x_n + H_{n,n}) \; .$$

Proof. This follows immediately from (1.30), (1.36) adding these two equations.

2. A MODIFIED ALGORITHM OF JACOBI-PERRON

Pursuing ideas of Jacobi [2] and Perron [3], the author [1, a) - q)] has modified the algorithm named after the two great mathematicians (see especially [1, m), n), p)]; one of these [1, p)] will be used in the second part of this paper. In order to find a standard solution vector of an S'. E. n, the author suggests a new modification of the Jacobi-Perron algorithm as outlined below.

We shall denote, as usually, by V_{n-1} the set of all ordered $(n - 1)$-tuples of real numbers $(a_1, a_2, \cdots, a_{n-1})$, $(n = 2, 3, \cdots)$ and call V_{n-1} the real number vector space of dimension $n - 1$ and the $(n-1)$-tuples its vectors. Let

$$(2.1) \qquad a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$$

be a given vector in V_{n-1}, and let

$$(2.2) \qquad b^{(v)} = (b_1^{(v)}, b_2^{(v)}, \cdots, b_{n-1}^{(v)})$$

be a sequence of vectors in V_{n-1}, which are either arbitrarily given or derived from $a^{(0)}$ by a certain transformation of V_{n-1}. We shall now introduce the following transformation

$$(2.3) \qquad Ta^{(v)} = a^{(v+1)} = \frac{1}{a_1^{(v)} - b_1^{(v)}} (a_2^{(v)} - b_2^{(v)}, \cdots, a_{n-1}^{(v)} - b_{n-1}^{(v)}, 1)$$
$$a_1^{(v)} \neq b_1^{(v)}, \quad v = 0, 1, \cdots$$

If we define the real numbers $A_i^{(v)}$ by the recursion formulas

$$A_i^{(i)} = 1;\ A_i^{(v)} = 0;\ (i,\ v = 0, 1, \cdots, n-1;\ i \neq v)\ ,$$

$$A_i^{(v+n)} = A_i^{(v)} + \sum_{j=1}^{n-1} b_j^{(v)} A_i^{(v+j)},\quad (i = 0, \cdots, n-1;\ v = 0, 1, \cdots)$$

then, as has been proved by the author and previously stated by Perron, the following formulas hold

$$(2.5)\qquad D_v = \begin{vmatrix} A_0^{(v)} & A_0^{(v+1)} & \cdots & A_0^{(v+n-1)} \\ A_1^{(v)} & A_1^{(v+1)} & \cdots & A_1^{(v+n-1)} \\ \cdot & \cdot & \cdots & \cdot \\ A_{n-1}^{(v)} & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} = (-1)^{v(n-1)},\quad (v = 0, 1, \cdots)$$

$$(2.6)\qquad a_i^{(0)} = \frac{A_i^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_i^{(v+j)}}{A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)}}\ ,\quad (i = 1, \cdots, n-1;\ v = 0, 1, \cdots)$$

(2.5) is the determinant of the transformation matrix of $Ta^{(v)}$; a further important formula proved by the author in [1, p] is

$$(2.6a)\qquad \begin{vmatrix} 1 & A_0^{(v+1)} & \cdots & A_0^{(v+n-1)} \\ a_1^{(0)} & A_1^{(v+1)} & \cdots & A_1^{(v+n-1)} \\ a_2^{(0)} & A_2^{(v+1)} & \cdots & A_2^{(v+n-1)} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-1}^{(0)} & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} = \frac{(-1)^{v(n-1)}}{A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)}}$$

$v = 0, 1, \cdots$

In the previous papers of the author the vectors $b^{(v)}$ were not arbitrarily chosen, but derived from the vectors $a^{(v)}$ by a special formation law. The nature of this formation law plays a decisive role in the theory of the modified algorithms of Jacobi-Perron. Both Jacobi and my admired teacher Perron used only the formation law:

$$b_i^{(v)} = [a_i^{(v)}], \quad (i = 1, \cdots, n-1; \; v = 0, 1, \cdots) \tag{2.7}$$

where $[x]$ denotes, as customary, the greatest integer not exceeding x. In this paper the modification of Jacobi-Perron's algorithm rests with the following different formation law of the $b_i^{(v)}$

$$\begin{aligned} b_1^{(v)} &= a_1^{(v)} \quad \text{if } a_1^{(v)} \neq [a_1^{(v)}]; \\ b_1^{(v)} &= a_1^{(v)} - 1 \quad \text{if } a_1^{(v)} = [a_1^{(v)}]; \\ b_k^{(v)} &= [a_k^{(v)}] \qquad (k = 2, \cdots, n-1; \; v = 0, 1, \cdots). \end{aligned} \tag{2.8}$$

It may happen that for some v $a_i^{(v)} = [a_i^{(v)}]$ for every i. In this case the algorithm with the formation law (2.8) must be regarded as finished, and $b_i^{(v)} = a_i^{(v)}$, $(i = 1, \cdots, n-1)$. The algorithm of the vectors $a^{(v)}$ as given by (2.3) is called periodic if there exist two integers p, q $(p \geq 0, \; q \geq 1)$ such that the transformation T yields

$$T^{v+q} = T^v, \quad (v = p, p+1, \cdots) \tag{2.9}$$

In case of periodicity the vectors $a^{(v)}$ $(v = 0, p, \cdots, p-1)$ are said to form the preperiod, and the vectors $a^{(v)}$ $(v = p, p+1, \cdots, p+q-1)$ are said to form the period of the algorithm; $\min p = s$ and $\min q = t$ are called respectively the lengths of the preperiod and period; $s + t$ is called the length of the algorithm which is purely periodic if $s = 0$.

3. A STANDARD SOLUTION VECTOR OF S. E. n

Let

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 1 \tag{3.1}$$

be an S. E. n; let the given vector $a^{(0)}$ in V_{n-1} have the form

$$a^{(0)} = (a_1^{(0)}, \cdots, a_{n-1}^{(0)}); \quad a_i^{(0)} = c_{i+1}/c_1 \qquad (i = 1, \cdots, n-1)\,. \tag{3.2}$$

The main result of this chapter is stated in

Theorem 3.1. Let the vectors $a^{(v)}$ be transforms of the vector $a^{(0)}$ from (3.2), obtained from (2.3) by means of the formation law (2.8); then there exists a natural number t such that the components of the vector $a^{(t)}$ are integers, viz.

$$a^{(t)} = (a_1^{(t)}, \ldots, a_{n-1}^{(t)}), \qquad a_i^{(t)} \text{ integers} \quad (i = 1, \cdots, n-1)\,. \tag{3.3}$$

Proof. We obtain from (2.8), since $c_1 \nmid c_2$ and, therefore, $[a_1^{(0)}] \neq a_1^{(0)}$,

$$b_i^{(0)} = [c_{i+1}/c_1]\,, \qquad (i = 1, \cdots, n-1)\,. \tag{3.4}$$

From (3.4) we obtain

$$\begin{aligned} c_{i+1} &= b_i^{(0)}c_1 + c_i^{(1)}\,, \qquad (c_i^{(1)} \text{ an integer})\,, \\ 0 &< c_i^{(1)} < c_n^{(1)}\,; \quad c_n^{(1)} = c_1; \quad (i = 1, \cdots, n-1) \end{aligned} \tag{3.5}$$

From (3.2), (3.4) and (3.5) we obtain

$$a_i^{(0)} - b_i^{(0)} = \frac{c_{i+1}}{c_1} - \frac{c_{i+1} - c_i^{(1)}}{c_1}\,,$$

$$a_1^{(0)} - b_1^{(0)} = \frac{c_1^{(1)}}{c_1}\,; \quad a_{i+1}^{(0)} - b_{k+1}^{(0)} = \frac{c_{k+1}^{(1)}}{c_1}\,, \quad (k = 1, \cdots, n-2) \tag{3.6}$$

and from (3.6), in view of (2.3)

$$a_i^{(1)} = c_{i+1}^{(1)} \Big/ c_1^{(1)}, \quad (i = 1, \cdots, n-1), \tag{3.7}$$

so that

$$\begin{aligned} b_i^{(1)} &= [c_{i+1}^{(1)} / c_1^{(1)}], && (i = 2, \cdots, n-1); \\ b_1^{(1)} &= [c_2^{(1)} / c_1^{(1)}], && \text{if } c_1^{(1)} \nmid c_2^{(1)}, \\ b_1^{(1)} &= (c_2^{(1)} / c_1^{(1)}) - 1, && \text{if } c_1^{(1)} \mid c_2^{(1)}. \end{aligned} \tag{3.8}$$

If $c_1^{(1)} = 1$, Theorem 3.1 is true with $t = 1$; let us, therefore, presume that $c_1^{(1)} > 1$. Of the two possible cases, viz. I) $c_1^{(1)} \mid c_2^{(1)}$ and II) $c_1^{(1)} \nmid c_2^{(1)}$, we shall first investigate case II). Here we obtain

$$\begin{aligned} & c_{i+1}^{(1)} = b_i^{(1)} c_1^{(1)} + c_i^{(2)}, \qquad (c_i^{(2)} \text{ an integer}), \\ & 0 \le c_i^{(2)} < c_n^{(2)}; \quad c_n^{(2)} = c_1^{(1)}, \quad (i = 2, \cdots, n-1); \\ & 0 < c_1^{(2)} < c_n^{(2)} \end{aligned} \tag{3.9}$$

We obtain, comparing (3.5) and (3.9)

$$0 < c_1^{(2)} < c_1^{(1)} < c_1. \tag{3.10}$$

Before investigating case I), we shall prove the following

Lemma 3.1.1. Let the vector $a^{(v)}$ in the modified algorithm of Jacobi-Perron with the formation law (2.8) and the given vector (3.2) have the form

$$a^{(v)} = \left(\frac{c_2^{(v)}}{c_1^{(v)}}, \frac{c_3^{(v)}}{c_1^{(v)}}, \cdots, \frac{c_n^{(v)}}{c_1^{(v)}} \right) \quad (v = 0, 1, \cdots) \tag{3.11}$$

then

$$(c_1^{(v)}, c_2^{(v)}, \cdots, c_n^{(v)}) = 1 . \tag{3.12}$$

Proof. The lemma is correct for $v = 0$, in virtue of (3.1) and (3.2). Let it be true for $v = k$, viz.

$$a^{(k)} = \frac{1}{c_1^{(k)}}(c_2^{(k)}, c_3^{(k)}, \cdots, c_n^{(k)}), \ (c_1^{(k)}, c_2^{(k)}, \cdots, c_n^{(k)}) = 1 . \tag{3.13}$$

From (3.13) we obtain

$$\begin{gathered} c_{i+1}^{(k)} = b_i^{(k)} c_1^{(k)} + c_i^{(k+1)} ; \ c_i^{(k+1)} \text{ integers}, \quad (i = 1, \cdots, n-1). \\ 0 < c_i^{(k+1)} < c_1^{(k)} . \end{gathered} \tag{3.14}$$

Let us denote

$$c_1^{(k)} = c_n^{(k+1)} , \tag{3.15}$$

$$(c_1^{(k+1)}, c_2^{(k+1)}, \cdots, c_n^{(k+1)}) = d . \tag{3.16}$$

If $d = 1$, Lemma 3.1.1 is proved; let us, therefore, presume

$$d > 1 . \tag{3.17}$$

We then obtain from (3.14), (3.15), (3.16)

$$d \mid c_n^{(k+1)}; \ c_n^{(k+1)} = c_1^{(k)} ; \ d \mid c_{i+1}^{(k)} , \qquad (i = 1, \cdots, n-1) , \tag{3.18}$$

so that

$$(c_1^{(k)}, c_2^{(k)}, \cdots, c_n^{(k)}) \geq d > 1 ; \tag{3.19}$$

but (3.19) contradicts (3.13), and the assumption that $d > 1$ is false which proves the lemma. We shall return to case I) and presume

$$c_1^{(1)} \mid c_{i+1}^{(1)}, \quad (i = 1, 2, \cdots, m). \tag{3.20}$$

In view of Lemma 3.1.1, the restriction holds

$$m \le n - 2, \tag{3.21}$$

since, permitting $m = n - 1$, we would obtain

$$(c_1^{(1)}, \cdots, c_n^{(1)}) = c_1^{(1)} > 1,$$

contrary to Lemma 3.1.1. It then follows from (3.20), in view of (2.8)

$$\begin{aligned} &c_2^{(1)} = (b_1^{(1)} + 1)c_1^{(1)}; \; c_{i+1}^{(1)} = b_i^{(1)} c_1^{(1)}, \quad (i = \;\;, \cdots, m); \\ &c_{m+2}^{(1)} = b_{m+1}^{(1)} c_1^{(1)} + c_{m+1}^{(2)}; \; 1 \le c_{m+1}^{(2)} \le c_1^{(1)}; \\ &c_{m+2+j}^{(1)} = b_{m+1+j}^{(1)} c_1^{(1)} + c_{m+1+j}^{(2)}; \\ &0 \le c_{m+1+j}^{(2)} < c_1^{(1)}, \qquad (j = 1, \cdots, n - m - 2). \end{aligned} \tag{3.22}$$

From (3.7), (3.22), we obtain, denoting

$$c_1^{(1)} = c_n^{(2)} \tag{3.23}$$

$$\begin{aligned} &a_1^{(1)} - b_1^{(1)} = 1; \; a_i^{(1)} - b_i^{(1)} = 0, \qquad (i = 2, \cdots, m); \\ &a_{m+1}^{(1)} - b_{m+1}^{(1)} = c_{m+1}^{(2)} \big/ c_n^{(2)}; \\ &a_{m+1+j}^{(1)} - b_{m+1+j}^{(1)} = c_{m+1+j}^{(2)} \big/ c_n^{(2)}, \quad (j = 1, \cdots, n - m - 2). \end{aligned} \tag{3.24}$$

From (3.24) we obtain, in view of (2.3),

$$\begin{aligned} &a_i^{(2)} = 0, \quad (i = 1, \cdots, m - 1); \; a_m^{(2)} = c_{m+1}^{(2)} \big/ c_n^{(2)}; \\ &a_{m+j}^{(2)} = c_{m+1+j}^{(2)} \big/ c_n^{(2)}, \qquad (j = 1, \cdots, n - m - 2); \; a_{n-1}^{(2)} = 1. \end{aligned} \tag{3.25}$$

The reader should well note that all the $a_i^{(2)}$ $(i = 1, \cdots, n-1)$ have the same denominator $c_n^{(2)}$; for if $a_i^{(2)} = 0$ we put $a_i^{(2)} = 0/c_n^{(2)}$; if $a_{n-1}^{(2)} = 1$, we put

$$a_{n-1}^{(2)} = c_n^{(2)} \Big/ c_n^{(2)}$$

Combining (3.5) and (3.22), we obtain

$$1 < c_{m+1}^{(2)} < c_1^{(1)} < c_1 \,. \tag{3.26}$$

From (3.25) we obtain, in view of (2.8) and recalling that

$$c_{m+1+j}^{(2)} < c_1^{(1)} = c_n^{(2)}, \quad (j = 1, \cdots, n-m-2) \,,$$

$$b_1^{(2)} = -1; \; b_{i+1}^{(2)} = 0 \,; \; (i = 1, \cdots, n-3) \; b_{n-1}^{(2)} = 1 \,, \tag{3.26a}$$

and from (3.25), (3.26a)

$$\begin{aligned}
&a_i^{(2)} - b_i^{(2)} = 1; \; a_{1+i}^{(2)} - b_{1+i}^{(2)} = 0 \,, \quad (i = 1, \cdots, m-2) \,; \\
&a_m^{(2)} - b_m^{(2)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \\
&a_{m+j}^{(2)} - b_{m+j}^{(2)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)} \,, \quad (j = 1, \cdots, n-m-2) \,; \\
&a_{n-1}^{(2)} - b_{n-1}^{(2)} = 0 \,.
\end{aligned} \tag{3.27}$$

From (3.27), we obtain, in view of (2.3),

$$\begin{aligned}
&a_i^{(3)} = 0 \,, \quad (i = 1, \cdots, m-2); \; a_{m-1}^{(3)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \,; \\
&a_{m-1+j}^{(3)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)} \,, \quad (j = 1, \cdots, n-m-2) \,; \\
&a_{n-2}^{(3)} = 0 \,; \; a_{n-1}^{(3)} = 1
\end{aligned} \tag{3.28}$$

We shall now prove the formula

$$
(3.29)\qquad
\begin{aligned}
&a_i^{(k+1)} = 0, \quad (i = 1, \cdots, m-k); \quad a_{m-k+1}^{(k+1)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \;; \\
&a_{m-k+1+j}^{(k+1)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)} \,, \quad (j = 1, \cdots, n-m-2) \;; \\
&a_{n-k-1+s}^{(k+1)} = 0, \quad (s = 1, \cdots, k-1); \quad a_{n-1}^{(k+1)} = 1 \;; \\
&k = 2, \cdots, m-1 \;.
\end{aligned}
$$

Proof by induction. Formula (3.29) is valid for $k = 2$, in virtue of (3.28). Let it be true for $k = v$, viz.

$$
(3.30)\qquad
\begin{aligned}
&a_i^{(v+1} = 0, \quad (i = 1, \cdots, m-v); \quad a_{m-v+1}^{(k+1)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \\
&a_{m-v+1+j}^{(k+1)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)} \,, \quad (j = 1, \cdots, n-m-2) \;; \\
&a_{n-v-1+s}^{(v+1)} = 0 \,, \quad (s = 1, \cdots, v-1) \;; \quad a_{n-1}^{(v+1)} = 1 \;.
\end{aligned}
$$

From (3.30) we obtain, in virtue of (2.8) and (3.22),

$$
(3.31)\qquad b_1^{(v+1)} = -1; \quad b_{1+i}^{(v+1)} = 0, \quad (i = 1, \cdots, n-3); \quad b_{n-1}^{(v+1)} = 1 \;,
$$

and from (3.30) and (3.31),

$$
(3.32)\qquad
\begin{aligned}
&a_1^{(v+1)} - b_1^{(v+1)} = 1 \;; \quad a_{1+i}^{(v+1)} - b_{1+i}^{(v+1)} = 0, \quad (i = 1, \cdots, m-v-1); \\
&a_{m-v+1}^{(v+1)} - b_{m-v+1}^{(v+1)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \;; \\
&a_{m-v+1+j}^{(v+1)} - b_{m-v+1+j}^{(v+1)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)} \,, \quad (j = 1, \cdots, n-m-2) \;; \\
&a_{n-v-1+s}^{(v+1)} - b_{n-v-1+s}^{(v+1)} = 0 \,, \quad (s = 1, \cdots, v-1) \\
&a_{n-1}^{(v+1)} - b_{n-1}^{(v+1)} = 0 \;.
\end{aligned}
$$

From (3.32) we obtain, in view of (2.3),

$$
\begin{aligned}
&a_i^{(v+2)} = 0, \quad (i = 1,\cdots, m - v - 1);\ a_{m-v}^{(v+2)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \ ; \\
(3.33) \qquad &a_{m-v+j}^{(v+2)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)}, \quad (j = 1,\cdots, n - m - 2) \ ; \\
&a_{n-v-2+s}^{(v+2)} = 0, \quad (s = 1,\cdots, v) \ ; \ a_{n-1}^{(v+2)} = 1 \ .
\end{aligned}
$$

But (3.33) is formula (3.29) for $k = v + 1$; thus formula (3.29) is completely proved. We now obtain from (3.29), for $k = m - 1$,

$$
\begin{aligned}
&a_1^{(m)} = 0 \ ; \ a_2^{(m)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \ ; \\
(3.34) \qquad &a_{2+j}^{(m)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)}, \quad (j = 1,\cdots, n - m - 2) \ ; \\
&a_{n-m+s}^{(m)} = 0, \quad (s = 1,\cdots, m - 2) \ ; \ a_{n-1}^{(m)} = 1 \ ,
\end{aligned}
$$

and from (3.34), in virtue of (2.8) and (3.22)

$$
(3.35) \qquad b_1^{(m)} = -1; \ b_{1+i}^{(m)} = 0, \quad (i = 1,\cdots, n - 3); \ b_{n-1}^{(m)} = 1 \ .
$$

From (3.34), (3.35) we obtain

$$
\begin{aligned}
&a_1^{(m)} - b_1^{(m)} = 1; \ a_2^{(m)} - b_2^{(m)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \\
(3.36) \qquad &a_{2+j}^{(m)} - b_{2+j}^{(m)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)}, \quad (j = 1,\cdots, n - m - 2) \ ; \\
&a_{n-m+s}^{(m)} - b_{n-m+s}^{(m)} = 0 \ , \quad (s = 1,\cdots, m - 1) \ ,
\end{aligned}
$$

and from (3.36), in view of (2.3)

$$
\begin{aligned}
&a_1^{(m+1)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \ ; \ a_{1+j}^{(m+1)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)}, \ (j = 1,\cdots, n-m-2); \\
(3.37) \qquad & \\
&a_{n-m-1+s}^{(m+1)} = 0 \ , \quad (s = 1,\cdots, m - 1) \ ; \ a_{n-1}^{(m+1)} = 1 \ .
\end{aligned}
$$

From (3.37) we obtain, in virtue of (2.8) and (3.22),

$$b_i^{(m+1)} = 0, \quad (i = 1, \cdots, n-2) \; ; \; b_{n-1}^{(m+1)} = 1 \quad , \tag{3.38}$$

and from (3.37), (3.38)

$$\begin{aligned} &a_i^{(m+1)} - b_i^{(m+1)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)} \quad ; \\ &a_{i+j}^{(m+1)} - b_{i+j}^{(m+1)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)} , \quad (j = 1, \cdots, n-m-2) \\ &a_{n-m-1+s}^{(m+1)} - b_{n-m-1+s}^{(m+1)} = 0 \quad (s = 1, \cdots, m) \end{aligned} \tag{3.39}$$

From (3.39) we obtain, in virtue of (2.3)

$$\begin{aligned} &a_j^{(m+2)} = c_{m+1+j}^{(2)} \Big/ c_{m+1}^{(2)} , \quad (j = 1, \cdots, n-m-2) \; ; \\ &a_{n-m-2+s}^{(m+2)} = 0, \quad (s = 1, \cdots, m) \; ; \; a_{n-1}^{(m+2)} = c_n^{(2)} \Big/ c_{m+1}^{(2)} \quad , \end{aligned}$$

or

$$\begin{aligned} &a_j^{(m+2)} = c_{j+1}^{(m+2)} \Big/ c_1^{(m+2)} , \quad (j = 1, \cdots, n-m-2) \; ; \\ &a_{n-m-2+s}^{(m+2)} = 0, \quad (s = 1, \cdots, m) \; ; \; a_{n-1}^{(m+2)} = c_n^{(m+2)} \Big/ c_1^{(m+2)} \; ; \\ &c_{m+i}^{(2)} = c_i^{(m+2)} , \quad (i = 1, \cdots, n-m-1) \; ; \; c_n^{(2)} = c_n^{(m+2)} \; . \end{aligned} \tag{3.40}$$

From (3.7), (3.9), we obtain

$$\begin{aligned} &a_1^{(1)} - b_1^{(1)} = c_1^{(2)} \Big/ c_1^{(1)} ; \; a_{1+j}^{(1)} - b_{1+j}^{(1)} = c_{1+j}^{(2)} \Big/ c_1^{(1)} , \\ &\qquad (j = 1, \cdots, n-2) \end{aligned} \tag{3.41}$$

and from (3.41), in view of (2.3),

$$(3.42) \qquad a_j^{(2)} = c_{i+j}^{(2)} / c_i^{(2)}, \quad (j = 1, \cdots, n-1); \quad c_i^{(1)} = c_n^{(2)}$$

We have thus obtained two chains of inequalities

$$0 < c_i^{(2)} < c_i^{(1)} < c_1; \quad 0 < c_i^{(m+2)} < c_i^{(1)} < c_1 .$$

If $c_i^{(2)}$ or $c_i^{(m+2)} = 1$, Theorem 3.1 is proved. Otherwise we deduce from (3.40) or (3.42), which show that the vectors $a^{(2)}$ and $a^{(m+2)}$ have the same structure of their components, how the algorithm is to be continued. In any case we obtain a chain of inequalities

$$(3.43) \qquad \begin{aligned} &0 < c_1^{(m_k)} < c_1^{(m_{k-1})} < \cdots < c_1^{(m_2)} < c_1^{(1)} < c_1 , \\ &m_2 = 2 \text{ if } c_1^{(1)} \nmid c_2^{(1)}; \quad m_2 = m + 2 \text{ if } c_1^{(1)} \mid c_2^{(1)}, \cdots \end{aligned}$$

and since the $c_i^{(m_i)}$ are natural numbers, we must necessarily arrive at

$$(3.44) \qquad c_i^{(t)} = 1, \quad t = m_k \geq 1 .$$

This proves Theorem 3.1.

We are now able to state explicitly the standard solution vector of the S. E. n (3.1) and prove, to this end,

<u>Theorem 3.2.</u> A solution vector of the S. E. n is given by the formula

$$(3.45) \qquad \begin{aligned} &X = (x_1, x_2, \cdots, x_n); \quad x_i = (-1)^{(t+1)(n-1)} B_{i,n} , \\ &(i = 1, \cdots, n) \end{aligned}$$

where the $B_{i,n}$ are the cofactors of the elements of the n^{th} row in the determinant

$$D_{t+1} = \begin{vmatrix} A_0^{(t+1)} & A_0^{(t+2)} & \cdots & A_0^{(t+n)} \\ A_1^{(t+1)} & A_1^{(t+2)} & \cdots & A_1^{(t+n)} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n-1}^{(t+1)} & A_{n-1}^{(t+2)} & \cdots & A_{n-1}^{(t+n)} \end{vmatrix} \tag{3.46}$$

In D_{t+1} t has the meaning of (3.44) and the

$$A_i^{(v)} \quad (i = 0, 1, \cdots, n-1;\ v = t+1, t+2, \cdots, t+n)$$

have the meaning of (2.4) and are obtainable from the modified Jacobi-Perron algorithm of the given vector $a^{(0)}$ from (3.2) by means of the formation law (3.8).

<u>Proof.</u> We shall recall that, in virtue of the formation law (3.8) all the numbers $b_i^{(v)}$ and, therefore, the numbers

$$A_i^{(v)} \quad (i = 0, 1, \cdots, n-1;\ v = 0, 1, \cdots)$$

are integers. For $c_1^{(t)} = 1$ we obtain

$$\begin{aligned} a^{(t)} &= (c_2^{(t)}, c_3^{(t)}, \cdots, c_n^{(t)}) = (a_1^{(t)}, \cdots, a_{n-1}^{(t)}), \\ b_i^{(t)} &= a_i^{(t)} = c_{i+1}^{(t)}, \qquad (i = 1, \cdots, n-1). \end{aligned} \tag{3.47}$$

Recalling formulas (2.4), (2.6), and (3.2), we obtain

$$a_i^{(0)} = \frac{A_i^{(t)} + \sum_{j=1}^{n-1} a_j^{(t)} A_i^{(t+j)}}{A_0^{(t)} + \sum_{j=1}^{n-1} a_j^{(t)} A_0^{(t+j)}} =$$

$$\frac{A_i^{(t)} + \sum_{j=1}^{n-1} b_j^{(t)} A_i^{(t+j)}}{A_0^{(t)} + \sum_{j=1}^{n-1} b_j^{(t)} A_0^{(t+j)}} = \frac{A_i^{(t+n)}}{A_0^{(t+n)}} ,$$

so that

$$c_{i+1}\Big/c_1 = A_i^{(t+n)}\Big/A_0^{(t+n)}, \quad (i = 1,\cdots,n-1)\,. \tag{3.48}$$

From (3.48) we obtain

$$c_{i+1} = c_1A_i^{(t+n)}\Big/A_0^{(t+n)}\ ,$$

and, since $(c_1, c_2, \cdots, c_n) = 1$,

$$\left(c_1, c_1A_1^{(t+n)}\Big/A_0^{(t+n)},\ c_1A_2^{(t+n)}\Big/A_0^{(t+n)}, \cdots, c_1A_{n-1}^{(t+n)}\Big/A_0^{(t+n)}\right) = 1 \tag{3.49}$$

and from (3.49), in virtue of a known theorem,

$$\left(c_1A_0^{(t+n)}, c_1A_1^{(t+n)}, c_1A_2^{(t+n)}, \cdots, c_1A_{n-1}^{(t+n)}\right) = A_0^{(t+n)},$$

or

$$c_1\left(A_0^{(t+n)},\ A_1^{(t+n)},\ A_2^{(t+n)}, \cdots, A_{n-1}^{(t+n)}\right) = A_0^{(t+n)} \tag{3.50}$$

From (2.5) we obtain

$$D_{t+1} = \begin{vmatrix} A_0^{(t+1)} & A_0^{(t+2)} & \cdots & A_0^{(t+n)} \\ A_1^{(t+1)} & A_1^{(t+2)} & \cdots & A_1^{(t+n)} \\ \cdot & \cdot & \cdots & \cdot \\ A_{n-1}^{(t+1)} & A_{n-1}^{(t+2)} & \cdots & A_{n-1}^{(t+n)} \end{vmatrix} = (-1)^{(t+1)(n-1)}$$

so that

$$\left(A_0^{(t+n)},\ A_1^{(t+n)},\ A_2^{(t+n)},\ \cdots,\ A_{n-1}^{(t+n)}\right) = 1\,. \tag{3.51}$$

From (3.50), (3.51), we obtain

$$c_1 = A_0^{(t+n)} , \tag{3.52}$$

and from (3.48), (3.52),

$$c_{i+1} = A_i^{(t+n)} , \quad (i = 0, 1, \cdots, n-1) . \tag{3.53}$$

(3.53) is a most decisive result; we obtain, in virtue of it,

$$D_{t+1} = \begin{vmatrix} A_0^{(t+1)} & A_0^{(t+2)} & \cdots & A_0^{(t+n-1)} & c_1 \\ A_1^{(t+1)} & A_1^{(t+2)} & \cdots & A_1^{(t+n-1)} & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{n-1}^{(t+1)} & A_{n-1}^{(t+2)} & \cdots & A_{n-1}^{(t+n-1)} & c_n \end{vmatrix} = (-1)^{(t+1)(n-1)} , \tag{3.54}$$

and from (3.54), denoting the cofactors of the c_i in D_{t+1} by $B_{i,n}$ $(i = 1, \cdots, n)$

$$\sum_{i=1}^{n} B_{i,n} c_i = (-1)^{(t+1)(n-1)} ,$$

or, multiplying both sides of this equation by $(-1)^{(t+1)(n-1)}$,

$$\sum_{i=1}^{n} ((-1)^{(t+1)(n-1)} B_{i,n}) c_i = 1 , \tag{3.55}$$

which proves Theorem 3.2.

4. NUMERICAL EXAMPLES FOR SOLUTION OF S'. E. n and S. E. n

In this chapter we shall illustrate our theory with three numerical examples.

Let the S'. E. 4 have the form

$$53x + 117y + 209z + 300u = 1 \quad . \tag{4.1}$$

The given vector $a^{(0)}$ has the components

$$a_1^{(0)} = 117/53\ ; \quad a_2^{(0)} = 209/53; \quad a_3^{(0)} = 300/53\ . \tag{4.2}$$

Carrying out the modified Jacobi-Perron algorithm (2.8) for the vector (4.2), we obtain the sequence of vectors

$$\begin{aligned} b_1^{(0)} &= (2,\ 3,\ 5)\ ; \\ b^{(1)} &= (4,\ 3,\ 4)\ ; \\ b^{(2)} &= (0,\ 1,\ 1)\ ; \\ b^{(3)} &= (1,\ 2,\ 3)\ ; \\ b^{(4)} &= (1,\ 0,\ 2)\ . \end{aligned} \tag{4.3}$$

We find that $a^{(4)} = b^{(4)}$, so that

$$t = 4;\ t + 1 = 5\ . \tag{4.4}$$

From (4.3) we calculate easily, in virtue of (2.4)

$$\begin{aligned} &A_0^{(5)} = 4; \quad A_0^{(6)} = 5; \quad A_0^{(7)} = 24; \quad A_0^{(8)} = 53\ . \\ &A_1^{(5)} = 9; \quad A_1^{(6)} = 11; \quad A_1^{(7)} = 53; \quad A_1^{(8)} = 117\ . \\ &A_2^{(5)} = 16; \quad A_2^{(6)} = 20; \quad A_2^{(7)} = 95; \quad A_2^{(8)} = 209\ . \\ &A_3^{(5)} = 23; \quad A_3^{(6)} = 28; \quad A_3^{(7)} = 136; \quad A_3^{(8)} = 300\ . \end{aligned} \tag{4.5}$$

Since here $(t + 1)(n + 1) = 5.3 = 15$, the determinant (3.54) is of the following form

$$(4.6) \qquad \begin{vmatrix} 4 & 5 & 24 & 53 \\ 9 & 11 & 53 & 117 \\ 16 & 20 & 95 & 209 \\ 23 & 28 & 136 & 300 \end{vmatrix} = -1$$

from which we obtain, developing D_5 in elements of the last column

$$53 \cdot 3 + 117 \cdot 3 + 209 \cdot (-1) + 300 \cdot (-1) = 1.$$

A solution vector of (4.1) is, therefore, given by

$$(4.7) \qquad X = (3, 3, -1, -1) .$$

Since X is a standard solution vector, there is not need to transform (4.1) into an S. E. 4.

Let the S'. E. 4 have the form

$$(4.8) \qquad 37x + 89y + 131z + 401u = 1.$$

Proceeding as before, we obtain for the D_{t+1} of (3.54)

$$(4.9) \qquad \begin{vmatrix} 1 & 2 & 7 & 37 \\ 2 & 5 & 17 & 89 \\ 3 & 7 & 25 & 131 \\ 10 & 22 & 76 & 401 \end{vmatrix} = 1 ,$$

which gives the solution vector for (4.8)

$$(4.10) \qquad X = (-6, -2, 0, +1)$$

Since this vector has a zero among its components, we have to transform the S'. E. 4 of (4.8) into an S. E. 4. Here we choose

$$P = 2 \cdot 3 \cdot 5 \cdot 7; \; x = 30x'; \; y = 42y'; \; z = 70z'; \; u = 105u' . \tag{4.11}$$

Now the S'. E. 4 takes the form of an S. E. 4, viz.

$$1110x' + 3738y' + 9170z' + 42105u' = 1 . \tag{4.12}$$

Carrying out the algorithm (2.8) of the given vector

$$a^{(0)} = (3738/1110, \; 9170/1110, 42105/1110) \tag{4.13}$$

we obtain the vectors $b^{(v)}$

$$\begin{aligned} &b^{(0)} = (3, 8, 37); \; b^{(1)} = (0, 2, 2); \; b^{(2)} = (0, 1, 1); \\ &b^{(3)} = (0, 0, 1); \; b^{(4)} = (29, 17, 54); \; b^{(5)} = (1, 1, 2); \\ &b^{(6)} = (1, 0, 2) . \end{aligned} \tag{4.14}$$

Here

$$t = 6, \quad t + 1 = 7, \quad (t+1)(n-1) = 21, \; D_7 = -1 \quad ;$$

after calculating the $A_i^{(v)}$, the determinant D_7 from (3.54) becomes

$$\begin{vmatrix} 3 & 272 & 552 & 1110 \\ 10 & 916 & 1859 & 3738 \\ 25 & 2247 & 4560 & 9170 \\ 114 & 10318 & 20930 & 42105 \end{vmatrix} = -1 , \tag{4.15}$$

which gives the standard solution vector of (4.12)

$$X' = (198, \; -23, \; -10, \; -1) , \tag{4.16}$$

and, in view of (4.11) the standard solution vector of (4.8)

$$X = (5940, \; -966, \; -700, \; -105) . \tag{4.17}$$

Let the S'E. 5 be

$$73x + 199y + 471z + 800u + 2001v = 1\,. \tag{4.18}$$

Proceeding as before, we obtain for the determinant (3.54)

$$\begin{vmatrix} 4 & 21 & 21 & 22 & 73 \\ 11 & 57 & 57 & 60 & 199 \\ 26 & 136 & 135 & 142 & 471 \\ 44 & 230 & 230 & 241 & 800 \\ 110 & 576 & 576 & 603 & 2001 \end{vmatrix} = 1 \tag{4.19}$$

which gives the vector solution

$$X = (0, \; -2, \; 0, \; 3, \; -1)\;. \tag{4.20}$$

Since this vector has zero components, we have to transform the S'. E. 5 (4.18) into an S. E. 5. Here we choose

$$P = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11;\; x = 210x';\; y = 330y';\; z = 462z';\; y = 770u';\; v = 1155v'\;. \tag{4.21}$$

The S. E. 5 takes the form

$$15330x' + 65670y' + 217602z' + 616000u' + 2311155v' = 1\,. \tag{4.22}$$

Carrying out the algorithm of the given vector

$$a^{(0)} = \left(\frac{65670}{15330}, \frac{217602}{15330}, \frac{616000}{15330}, \frac{2311155}{15330}\right)\,. \tag{4.23}$$

we obtain the vectors $b^{(v)}$

$$\begin{aligned} b^{(0)} &= (4, 14, 40, 150);\; & b^{(1)} &= (0, 0, 2, 3);\; & b^{(2)} &= (0, 0, 0, 1)\,; \\ b^{(3)} &= (1, 0, 0, 1); & b^{(4)} &= (14, 8, 1, 18);\; & b^{(5)} &= (1, 0, 0, 1)\,; \\ b^{(6)} &= (1, 0, 2, 6); & b^{(7)} &= (1, 1, 0, 2);\; & b^{(8)} &= (0, 2, 0, 9)\,. \end{aligned} \tag{4.24}$$

Here

$$t = 8, \quad t + 1 = 9, \quad (t + 1)(n - 1) = 36;$$

after calculating the $A_i^{(v)}$, the determinant D_9 from (3.54) takes the form

$$(4.25) \qquad \begin{vmatrix} 95 & 99 & 790 & 1681 & 15330 \\ 407 & 424 & 3384 & 7201 & 65670 \\ 1349 & 1405 & 11213 & 23861 & 217602 \\ 3818 & 3978 & 31744 & 67547 & 616000 \\ 14323 & 14925 & 119100 & 253428 & 2311155 \end{vmatrix} = 1 ,$$

which gives the standard solution vectors of (4.22) and (4.18)

$$(4.26) \qquad X' = (1053, \ 26, \ -2, \ 13, \ -11) ,$$

$$(4.27) \qquad X = (221130, \ 8580, \ -924, \ 10010, \ -12705) .$$

5. THE CONJUGATE STANDARD EQUATIONS

DEFINITION. The Diophantine equations

$$\begin{aligned} & c_1x_1 + c_2x_2 + \cdots + c_nx_n = c_1^{(v)} , && (v = 1,\ldots,t - 1) ; \\ & c_j \quad \text{from} \quad (1.2) , && (j = 1,\ldots,n) ; \\ & c_1^{(v)} \quad \text{from} \quad (3.11); \ t \text{ from Theorem 3.1.} && \end{aligned}$$

will be called Conjugate Standard Equations.

In this chapter we shall find a solution vector for a conjugate standard equation and prove, to this end,

Theorem 5.1. A solution vector of the conjugate standard equation (5.1) is given by the vector whose j^{th} component is

$$(5.2) \qquad x_j = (-1)^{(v+1)(n-1)} B_{j,n}^{(v+1)} , \quad (v = 1,\cdots, t - 1)$$

where the $B_{j,n}^{(v+1)}$ are the cofactors of the elements in the n^{th} row of the determinant

$$(5.3)\qquad \begin{vmatrix} A_0^{(v+1)} & A_0^{(v+2)} & \cdots & A_0^{(v+n-1)} & c_1 \\ A_1^{(v+1)} & A_1^{(v+2)} & \cdots & A_1^{(v+n-1)} & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{n-1}^{(v+1)} & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} & c_n \end{vmatrix}$$

If $(x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})$ is a solution vector of the standard equation

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 1 ,$$

then (5.2) is different from

$$(\cdots, x_j, \cdots) = (\cdots, x_j^{(0)}c_1^{(0)}, \cdots)(j = 1, 2, \cdots, n) .$$

Proof. As can be easily verified from the proof of Theorem 3.1, the relation holds

$$(5.4)\qquad a_{n-1}^{(v)} = c_1^{(v-1)}\big/c_1^{(v)} , \qquad (v = 1, 2, \cdots); \quad c_1^{(0)} = c_1 .$$

We shall first prove the formula

$$(5.5)\qquad A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)}A_0^{(v+j)} = a_{n-1}^{(1)}a_{n-1}^{(2)} \cdots a_{n-1}^{(v)}, \ (v=1,2,\cdots) .$$

We obtain, for $v = 1$, in view of (2.4),

$$A_0^{(1)} + \sum_{j=1}^{n-1} a_j^{(1)}A_0^{(1+j)} = a_{n-1}^{(1)}A_0^{(n)} = a_{n-1}^{(1)} ,$$

so that formula (5.5) is correct for $v = 1$. Let it be correct for $v = k$, viz.

$$A_0^{(k)} + \sum_{j=1}^{n-1} a_j^{(k)} A_0^{(k+j)} = a_{n-1}^{(1)} a_{n-1}^{(2)} \cdots a_{n-1}^{(k)}, \quad (k = 1, 2, \cdots). \tag{5.6}$$

From (2.3) we obtain

$$\begin{aligned} a_j^{(k)} &= \left(a_{j-1}^{(k+1)} \Big/ a_{n-1}^{(k+1)}\right) + b_j^{(k)}, \quad (j = 2, \cdots, n-1; k = 1, 2, \cdots) \\ a_1^{(k)} &= \left(1 \Big/ a_{n-1}^{(k+1)}\right) + b_1^{(k)} \end{aligned} \tag{5.7}$$

Rearranging the left side of the (5.6) by substituting there for $a_j^{(k)}$ the values from (5.7), we obtain

$$A_0^{(k)} + a_1^{(k)} A_0^{(k+1)} + \sum_{j=2}^{n-1} \left(\frac{a_{j-1}^{(k+1)} A_0^{(k+j)}}{a_{n-1}^{(k+1)}} + b_j^{(k)} A_0^{(k+j)} \right)$$
$$= a_{n-1}^{(1)} a_{n-1}^{(2)} \cdots a_{n-1}^{(k)};$$

The left side of this equation has the form

$$A_0^{(k)} + \frac{A_0^{(k+1)}}{a_{n-1}^{(k+1)}} + b_1 A_0^{(k+1)} + \sum_{j=2}^{n-1} \left(\frac{a_{j-1}^{(k+1)} A_0^{(k+j)}}{a_{n-1}^{(k+1)}} \right) + \sum_{j=2}^{n-1} b_j^{(k)} A_0^{(k+j)}$$

$$= \frac{A_0^{(k+1)} + \Sigma_{j=2}^{n-1} a_{j-1}^{(k+1)} A_0^{(k+j)}}{a_{n-1}^{(k+1)}} + \left(A_0^{(k)} + \sum_{j=1}^{n-1} b_j^{(k)} A_0^{(k+j)} \right)$$

$$= \left(A_0^{(k+1)} + \sum_{j=2}^{n-1} a_{j-1}^{(k+1)} A_0^{(k+1)} \right) \Big/ a_{n-1}^{(k+1)} + A_0^{(k+n)} = \left(A_0^{(k+1)} + \sum_{j=1}^{n-1} a_j^{(k+1)} A_0^{(k+1+j)} \right.$$
$$\left. + a_{n-1}^{k+1} A_0^{(k+n)} \right) \Big/ a_{n-1}^{(k+1)} = \left(A_0^{(k+1)} + \sum_{j=1}^{n-1} a_j^{(k+1)} A_0^{(k+1+j)} \right) \Big/ a_{n-1}^{(k+1)}.$$

We thus obtain

$$\left(A_0^{(k+1)} + \sum_{j=1}^{n-1} a_j^{(k+1)} A_0^{(k+1+j)}\right) \Big/ a_{n-1}^{(k+1)} = a_{n-1}^{(1)} a_{n-1}^{(2)} \cdots a_{n-1}^{(k)} \; ,$$

or

$$A_0^{(k+1)} + \sum_{j=1}^{n-1} a_j^{(k+1)} A_0^{(k+1+j)} = a_{n-1}^{(1)} a_{n-1}^{(2)} \cdots a_{n-1}^{(k+1)} \; . \tag{5.8}$$

But (5.8) is (5.5) for $v = k + 1$, which proves (5.5). From (5.4), (5.5), we now obtain

$$A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)} = \frac{c_1}{c_1^{(1)}} \cdot \frac{c_1^{(1)}}{c_2^{(1)}} \cdots \frac{c_1^{(v-1)}}{c_1^{(v)}} \; ,$$

$$A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)} = c_1 \Big/ c_1^{(v)} \; , \quad (v = 1, 2, \cdots) \; . \tag{5.9}$$

The reader should note that (5.9) holds for $v = 0$, too. We shall now return to formula (2.6. a), viz.

$$\begin{vmatrix} 1 & A_0^{(v+1)} & \cdots & A_0^{(v+n-1)} \\ a_1^{(0)} & A_1^{(v+1)} & \cdots & A_1^{(v+n-1)} \\ a_2^{(0)} & A_2^{(v+1)} & \cdots & A_2^{(v+n-1)} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-1}^{(0)} & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} = \frac{(-1)^{v(n-1)}}{A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)}}$$

Substituting here the values of $a_j^{(0)}$ from (3.2) and for

$$A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)}$$

from (5.9), we obtain

$$\begin{vmatrix} 1 & A_0^{(v+1)} & \cdots & A_0^{(v+n-1)} \\ c_2/c_1 & A_1^{(v+1)} & \cdots & A_1^{(v+n-1)} \\ c_3/c_1 & A_2^{(v+1)} & \cdots & A_2^{(v+n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ c_n/c_1 & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} = \frac{(-1)^{v(n-1)}}{c_1/c_1^{(v)}} .$$

or, multiplying both sides by c_1 and interchanging the first and the last row of the determinant,

$$\begin{vmatrix} A_0^{(v+1)} & A_0^{(v+2)} & \cdots & A_0^{(v+n-1)} & c_1 \\ A_1^{(v+1)} & A_1^{(v+2)} & \cdots & A_1^{(v+n-1)} & c_2 \\ A_2^{(v+1)} & A_2^{(v+2)} & \cdots & A_2^{(v+n-1)} & c_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n-1}^{(v+1)} & A_{n-1}^{(v+2)} & \cdots & A_{n-1}^{(v+n-1)} & c_n \end{vmatrix} = (-1)^{(v+1)(n-1)} c_1^{(v)} . \tag{5.10}$$

From (5.10) we obtain

$$c_1 B_{1,n}^{(v+1)} + c_2 B_{2,n}^{(v+1)} + \cdots + c_n B_{n,n}^{(v+1)} = (-1)^{(v+1)(n-1)} c_1^{(v)} ,$$

or, multiplying both sides by $(-1)^{(v+1)(n-1)}$

$$\sum_{j=1}^{n} c_j(-1)^{(v+1)(n-1)} B_{j,n}^{(v+1)} = c_1^{(v)} \quad . \tag{5.11}$$

(5.11) proves the first statement of Theorem (5.1). To prove the second statement, we have to show that $c_1^{(v)}$ cannot be a divisor of all the

$$x_j = (-1)^{(v+1)(n-1)} B_{j,n}^{(v+1)} , \qquad (j = 1, \cdots, n)$$

To prove this, we recall formula (2.5), viz.

$$D_{v+1} = (-1)^{(v+1)(n-1)} \quad , \tag{5.12}$$

so that

$$A_0^{(v+n)} B_{1,n}^{(v+1)} + A_1^{(v+n)} B_{2,n} + \cdots + A_{n-1}^{(v+n)} B_{n,n}^{(v+1)} = (-1)^{(v+1)(n-1)},$$

or

$$A_0^{(v+n)} x_1 + A_1^{(v+n)} x_2 + \cdots + A_{n-1}^{(v+n)} x_n = 1 \quad . \tag{5.13}$$

From (5.13) we obtain

$$(x_1, x_2, \cdots, x_n) = 1 \quad , \tag{5.14}$$

and since $c_1^{(v)} > 1$ for $v < t$, the second statement of Theorem 5.1 is proved. It should be stressed that the case

$$c_1^{(v_1)} = c_1^{(v_2)} = \cdots = c_1^{(v_k)}$$

is possible $(1 < k < t)$. In this case we shall consider the conjugate equations $c_1x_1 + c_2x_2 + \cdots + c_nx_n = c_1^{(v_j)}$, $(j = 1, \cdots, k)$ as different ones, since each of them will provide a different solution of (5.1) for the same $c_1^{(v)}$.

We shall solve some conjugate standard equations of (4.12), viz.

$$1110x' + 3738y' + 9170z' + 4210u' = 1 .$$

We calculate easily

$$c_1^{(1)} = 408;\ c_1^{(2)} = 290;\ c_1^{(3)} = 219;\ c_1^{(4)} = 4;\ c_1^{(5)} = 2;\ t = 6. \tag{5.15}$$

Calculating the $A_i^{(v)}$ on basis of (4.14) we obtain a solution of

$$1110x' + 3738y' + 9170z' + 42105u' = 219 , \quad (v = 3)$$

$$X' = (-31, \ -2, \ 0, \ 1) \quad .$$

Similarly we obtain a solution of

$$1110x' + 3738y' + 9170z' + 42105u' = -4 \quad (v = 4)$$

$$X' = (-15, \ 2, \ 1, \ 0)$$

It should be well noted that the solution vectors of the conjugate standard equations are not necessarily standard solution vectors.

6. GENERALIZED FIBONACCI NUMBERS

The generalized Fibonacci numbers are defined by the initial values and the recursion formula as follows

$$\begin{aligned} &F_1^{(n)} = F_2^{(n)} = \cdots = F_{n-1}^{(n)} = 0, \quad F_n^{(n)} = 1 ; \\ &F_{k+n}^{(n)} = \sum_{j=0}^{n-1} F_{k+j}^{(n)} ; \quad k + 1, n = 2, 3, \cdots . \end{aligned} \tag{6.1}$$

The numbers $F_i^{(n)}$ $(i = 1, 2, \cdots)$ will be called generalized Fibonacci numbers of degree n and order i. They are calculated by the generating function

(6.2) $$x^{n-1} \Big/ (1 - x - x^2 - \cdots - x^n) = \sum_{i=1}^{\infty} F_i^{(n)} x^{i-1} .$$

Let denote

(6.3) $$f(x) = x^n + x^{n-1} + \cdots + x - 1 .$$

$f(x)$ from (6.3) is called the generating polynomial. This can be transformed into

(6.4) $$f(x) = (x^{n+1} - 2x + 1)/(x - 1), \quad x \neq 1 .$$

The equation

(6.5) $$(x - 1)f(x) = x^{n+1} - 2x + 1 = 0, \quad x \neq 1 ,$$

has 2 real roots and $(n - 2)/2$ pairs of conjugate complex roots for $n = 2m$ $(m = 1, 2, \cdots)$ and one real root and $(n - 1)/2$ pairs of conjugate complex roots for $n = 2m + 1$ $(m = 1, 2, \cdots)$. This is easily proved by analyzing the derivative of $f(x)$. The roots of $f(x)$ are, of course, irrationals. From (6.2) we obtain

(6.6) $$F_v^{(n)} = F_v^{(n)}(x_1, x_2, \cdots, x_n) , \quad (v = 1, 2, \cdots)$$

where $F_v^{(n)}(x_1, x_2, \cdots, x_n)$ is a symmetric function of the n roots of $f(x)$. It will be a main result of the next chapter to find an explicit formula for the ratio

(6.7) $$\lim_{v \to \infty} F_{v+1}^{(n)} \Big/ F_v^{(n)} .$$

In the case of the original Fibonacci numbers, viz. $n = 2$, this is a well-known fact. As can be easily verified from (6.2), the $F_v^{(2)}$ have the form

(6.8) $$F_{m+1}^{(2)} = \left(\left(\frac{\sqrt{5} + 1}{2} \right)^m \Big/ \sqrt{5} \right) + (-1)^{m-1} \left(\left(\frac{\sqrt{5} - 1}{2} \right)^m \Big/ \sqrt{5} \right) , \quad (m = 0, 1, \cdots) .$$

From (6.8) we obtain easily

$$\lim_{m \to \infty} F^{(2)}_{m+1} \Big/ F^{(2)}_{m} = (\sqrt{5} + 1)/2 \ . \tag{6.9}$$

Of course, for generalized Fibonacci numbers, a limiting formula analogous to (6.9) can be given by infinite series, as will be solved in the next chapter. We shall use the notation

$$D^{(n)}_v = \begin{vmatrix} F^{(n)}_v & F^{(n)}_{v+1} & \cdots & F^{(n)}_{v+n-1} \\ F^{(n)}_{v+1} & F^{(n)}_{v+2} & \cdots & F^{(n)}_{v+n} \\ \cdots & \cdots & \cdots & \cdots \\ F^{(n)}_{v+n-1} & F^{(n)}_{v+n} & \cdots & F^{(n)}_{v+2n-2} \end{vmatrix} , \quad (v = 1, 2, \cdots) . \tag{6.10}$$

We shall prove the formula

$$D^{(n)}_v = (-1)^{(n(n-1)/2)+(v-1)(n-1)} \ . \tag{6.11}$$

Proof by induction. We obtain from (6.1)

$$D^{(n)}_1 = \begin{vmatrix} F^{(n)}_1 & F^{(n)}_2 & \cdots & & & F^{(n)}_n \\ F^{(n)}_2 & F^{(n)}_3 & \cdots & & F^{(n)}_n & F^{(n)}_{n+1} \\ F^{(n)}_3 & F^{(n)}_4 & \cdots & F^{(n)}_n & F^{(n)}_{n+1} & F^{(n)}_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ F^{(n)}_n & F^{(n)}_{n+1} & \cdots & & & F^{(n)}_{2n-1} \end{vmatrix} =$$

$$= \begin{vmatrix} 0 & 0 & \cdots & & & 1 \\ 0 & 0 & & & 1 & F^{(n)}_{n+1} \\ 0 & 0 & \cdots & 1 & F^{(n)}_{n+1} & F^{(n)}_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & F^{(n)}_{n+1} & \cdots & & & F^{(n)}_{2n-1} \end{vmatrix} ,$$

$$D_1^{(n)} = (-1)^{n(n-1)/2}, \quad (n = 2, 3, \cdots) . \tag{6.12}$$

We further obtain from (6.1)

$$D_v^{(n)} = \begin{vmatrix} F_v^{(n)} & F_{v+1}^{(n)} & \cdots & F_{v+n-2}^{(n)} & F_{v+n-1}^{(n)} \\ F_{v+1}^{(n)} & F_{v+2}^{(n)} & \cdots & F_{v+n-1}^{(n)} & F_{v+n}^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{v+n-1}^{(n)} & F_{v+n}^{(n)} & \cdots & F_{v+2n-3}^{(n)} & F_{v+2n-2}^{(n)} \end{vmatrix} =$$

$$\begin{vmatrix} F_v^{(n)} & F_{v+1}^{(n)} & \cdots & F_{v+n-2}^{(n)} & (F_{v-1}^{(n)} + \sum_{j=1}^{n-1} F_{v-1+j}^{(n)}) \\ F_{v+1}^{(n)} & F_{v+2}^{(n)} & \cdots & F_{v+n-1}^{(n)} & (F_v^{(n)} + \sum_{j=1}^{n-1} F_{v+j}^{(n)}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{v+n-1}^{(n)} & F_{v+n}^{(n)} & \cdots & F_{v+2n-3}^{(n)} & (F_{v+n-2}^{(n)} + \sum_{j=1}^{n-1} F_{v+n-2+j}^{(n)} \end{vmatrix} =$$

$$\begin{vmatrix} F_v^{(n)} & F_{v+1}^{(n)} & \cdots & F_{v+n-2}^{(n)} & F_{v-1}^{(n)} \\ F_{v+1}^{(n)} & F_{v+2}^{(n)} & \cdots & F_{v+n-1}^{(n)} & F_v^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{v+n-1}^{(n)} & F_{v+n}^{(n)} & \cdots & F_{v+2n-3}^{(n)} & F_{v+n-2}^{(n)} \end{vmatrix} =$$

$$(-1)^{n-1} \begin{vmatrix} F_{v-1}^{(n)} & F_v^{(n)} & F_{v+1}^{(n)} & \cdots & F_{v+n-2}^{(n)} \\ F_v^{(n)} & F_{v+1}^{(n)} & F_{v+2}^{(n)} & \cdots & F_{v+n-1}^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{v+n-2}^{(n)} & F_{v+n-1}^{(n)} & F_{v+n}^{(n)} & \cdots & F_{v+2n-3} \end{vmatrix} .$$

We have thus proved the formula

$$D_v^{(n)} = (-1)^{n-1} D_{v-1}^{(n)} . \tag{6.13}$$

From (6.13) we obtain

$$D_v^{(n)} = (-1)^{n-1} D_{v-1}^{(n)} = (-1)^{n-1}(-1)^{n-1} D_{v-2} = \cdots = (-1)^{(v-1)(n-1)} D_1^{(n)}$$

which, together with (6.12), proves (6.11). We have simultaneously proved

Theorem 6.1. A vector solution of the S'. E. n

$$F_{v+n-1}^{(n)} x_1 + F_{v+n}^{(n)} x_2 + \cdots + F_{v+2n-2}^{(n)} x_n = 1 \tag{6.14}$$

is given by the formula

$$x_i = (-1)^{(n(n-1)/2)+(v-1)(n-1)} B_{i,n}, \quad (i = 1, \cdots, n), \tag{6.15}$$

where the $B_{i,n}$ are the cofactors of the elements in the n^{th} row of the determinant (6.10).

We shall now turn to the periodicity of the algorithm for ratios of cubic Fibonacci numbers and prove

Theorem 6.2. The Jacobi-Perron algorithm of the two irrationals

$$a_1^{(0)} = \lim_{v \to \infty} (F_{v+3}^{(3)} / F_{v+2}^{(3)}) ; \; a_2^{(0)} = \lim_{v \to \infty} (F_{v+4}^{(3)} / F_{v+2}^{(3)}) \tag{6.16}$$

is periodic; the preperiod has the length $S = 2$ and the form

$$\begin{matrix} 1 & 3 & , \\ 0 & 1 & . \end{matrix} \tag{6.17}$$

The period has the length $T = 6$ and the form

$$\begin{matrix} 0 & 2 & , \\ 0 & 2 & , \\ 0 & 2 & , \\ 0 & 2 & , \\ 0 & 1 & , \\ 0 & 4 & , \\ 0 & 1 & . \end{matrix} \tag{6.18}$$

Proof. We shall first prove the following inequalities

$$F^{(3)}_{v+3} < F^{(3)}_{v+4} < 2F^{(3)}_{v+3}, \quad (v = 3, 4, \cdots) ; \tag{6.19}$$

$$3F^{(3)}_{v+2} < F^{(3)}_{v+4} < 4F^{(3)}_{v+2}, \quad v \text{ as above} . \tag{6.20}$$

From

$$F^{(3)}_{v+4} = F^{(3)}_{v+3} + F^{(3)}_{v+2} + F^{(3)}_{v+1} ; \; F^{(3)}_{v+1}, F^{(3)}_{v+2} > 0 \quad \text{for} \quad v \geq 2 ,$$

we obtain

$$F^{(3)}_{v+4} > F^{(3)}_{v+3} .$$

We further obtain

$$F^{(3)}_{v+4} = 2F^{(3)}_{v+3} - (F^{(3)}_{v+3} - F^{(3)}_{v+2} - F^{(3)}_{v+1}) ,$$

but

$$F^{(3)}_{v+3} - F^{(3)}_{v+2} - F^{(3)}_{v+1} = F^{(3)}_{v} > 0, \quad \text{for } v = 3, 4, \cdots$$

therefore

$$F^{(3)}_{v+4} < 2F^{(3)}_{v+3} ,$$

which proves (6.19). We further obtain

$$\begin{aligned} F^{(3)}_{v+4} &= F^{(3)}_{v+3} + F^{(3)}_{v+2} + F^{(3)}_{v+1} \\ &= (F^{(3)}_{v+2} + F^{(3)}_{v+1} + F^{(3)}_{v}) + F^{(3)}_{v+2} + F^{(3)}_{v+1} \\ &= 2F^{(3)}_{v+2} + 2F^{(3)}_{v+1} + F^{(3)}_{v} \\ &= 2F^{(3)}_{v+2} + (F^{(3)}_{v+1} + F^{(3)}_{v} + F^{(3)}_{v-1}) + F^{(3)}_{v+1} - F^{(3)}_{v-1} \\ &= 3F^{(3)}_{v+2} + F^{(3)}_{v+1} - F^{(3)}_{v-1} ; \end{aligned}$$

but

$$F_{v+1}^{(3)} - F_{v-1}^{(3)} = F_v^{(3)} + F_{v-2}^{(3)} > 0 \text{ for } v \geq 3 ,$$

therefore

$$F_{v+4}^{(3)} > 3F_{v+2}^{(3)} .$$

Since

$$F_{v+2}^{(3)} = F_{v+1}^{(3)} + F_v^{(3)} + F_{v-1}^{(3)} = 2F_v^{(3)} + 2F_{v-1}^{(3)} + F_{v-2}^{(3)} > F_v^{(3)} + F_{v-2}^{(3)}$$

for $v \geq 3$, we obtain

$$F_{v+1}^{(3)} - F_{v-1}^{(3)} = F_v^{(3)} + F_{v-2}^{(3)} < F_{v+2}^{(3)} ,$$

and, therefore, from the previous result

$$F_{v+4}^{(3)} < 4F_{v+2}^{(3)} ,$$

which proves (20).

We shall now carry out the algorithm of Jacobi-Perron for the numbers

$$(6.21) \qquad a_1^{(0)} = F_{v+3}^{(3)} \Big/ F_{v+2}^{(3)} \;;\; a_2^{(0)} = F_{v+4}^{(3)} \Big/ F_{v+2}^{(3)} , \quad v \geq 12 .$$

Though the proof is carried out for the rationals

$$F_{v+3}^{(3)} \Big/ F_{v+2}^{(3)} \qquad \text{and} \qquad F_{v+4}^{(3)} \Big/ F_{v+2}^{(3)} ,$$

and not for their limiting values, the reader will understand, after having read Chapter 7, that this is permissible.

We obtain from (6.19), substituting $v - 1$ for v, and in virtue of $v \geq 12$,

$$F_{v+2}^{(3)} < F_{v+3}^{(3)} < 2F_{v+2}^{(3)} \; ; \; 1 < F_{v+3}^{(3)} \Big/ F_{v+2}^{(3)} < 2 \; ,$$

so that

$$b_1^{(0)} = [a_1^{(0)}] = 1 \; . \tag{6.22}$$

From (6.20), we obtain

$$3 < F_{v+4}^{(3)} \Big/ F_{v+2}^{(3)} < 4 \; ,$$

so that

$$b_2^{(0)} = [a_2^{(0)}] = 3 \; . \tag{6.23}$$

From (6.21), (6.22), (6.23), we obtain

$$\begin{aligned} a_2^{(1)} &= 1 \Big/ (a_1^{(0)} - b_1^{(0)}) = 1 \Big/ \left((F_{v+3}^{(3)} \Big/ F_{v+2}^{(3)}) - 1 \right) \\ &= F_{v+2}^{(3)} \Big/ (F_{v+3}^{(3)} - F_{v+2}^{(3)}) = F_{v+2}^{(3)} \Big/ (F_{v+1}^{(3)} + F_v^{(3)}) \; ; \end{aligned}$$

$$\begin{aligned} a_1^{(1)} &= (a_2^{(0)} - b_2^{(0)}) \Big/ (a_1^{(0)} - b_1^{(0)}) \\ &= \left(\frac{F_{v+4}^{(3)}}{F_{v+2}^{(3)}} - 3 \right) \frac{F_{v+2}^{(3)}}{F_{v+1}^{(3)} + F_v^{(3)}} = (F_{v+4}^{(3)} - 3F_{v+2}^{(3)} \Big/ (F_{v+1}^{(3)} + F_v^{(3)}) \; ; \end{aligned}$$

but, as has been proved before,

$$F_{v+4}^{(3)} - 3F_{v+2}^{(3)} = F_v^{(3)} + F_{v-2}^{(3)} \; ;$$

we thus obtain

$$a_1^{(1)} = \frac{F_v^{(3)} + F_{v-2}^{(3)}}{F_{v+1}^{(3)} + F_v^{(3)}} \; ; \quad a_2^{(1)} = \frac{F_{v+2}^{(3)}}{F_{v+1}^{(3)} + F_v^{(3)}} \; . \tag{6.24}$$

Since

$$0 < F_v^{(3)} + F_{v-2}^{(3)} < F_{v+1}^{(3)} + F_v^{(3)} \,,$$

we obtain

$$0 < (F_v^{(3)} + F_{v-2}^{(3)})/(F_{v+1}^{(3)} + F_v^{(3)}) < 1 \,;$$

since further

$$F_{v+2}^{(3)} / (F_{v+1}^{(3)} + F_v^{(3)}) = (F_{v+1}^{(3)} + F_v^{(3)} + F_{v-1}^{(3)}) / (F_{v+1}^{(3)} + F_v^{(3)}) =$$

$$1 + (F_{v-1}^{(3)} / (F_{v+1}^{(3)} + F_v^{(3)})), \text{ and since } F_{v-1}^{(3)} < F_{v+1}^{(3)} + F_v^{(3)} \,,$$

we obtain

$$b_1^{(1)} = 0; \; b_2^{(1)} = 1 \,. \tag{6.25}$$

From (6.24), (6.25), we obtain

$$1/(a_1^{(1)} - b_1^{(1)}) = (F_{v+1}^{(3)} + F_v^{(3)}) / (F_v^{(3)} + F_{v-2}^{(3)}) \;;$$

$$a_2^{(1)} - b_2^{(1)} = (F_{v+2}^{(3)} - F_{v+1}^{(3)} - F_v^{(3)}) / (F_{v+1}^{(3)} + F_v^{(3)}) =$$

$$F_{v-1}^{(3)} / (F_{v+1}^{(3)} + F_v^{(3)}) \;;$$

we thus obtain, in virtue of (2.3)

$$a_1^{(2)} = \frac{F_{v-1}^{(3)}}{F_v^{(3)} + F_{v-2}^{(3)}} \,; \; a_2^{(2)} = \frac{F_{v+1}^{(3)} + F_v^{(3)}}{F_v^{(3)} + F_{v-2}^{(3)}} \,. \tag{6.26}$$

From (6.26) we obtain, since

$$0 < F_{v-1}^{(3)} < F_v^{(3)} + F_{v-2}^{(3)} \,, \; 0 < F_{v-1}^{(3)} / (F_v^{(3)} + F_{v-2}^{(3)}) < 1 \,,$$

and further, since

$$\begin{aligned}(F^{(3)}_{v+1} + F^{(3)}_{v}) / (F^{(3)}_{v} + F^{(3)}_{v-2}) &= (2F^{(3)}_{v} + F^{(3)}_{v-1} + F^{(3)}_{v-2}) / (F^{(3)}_{v} + F^{(3)}_{v-2}) = \\ &= (2F^{(3)}_{v} + 2F^{(3)}_{v-2} + F^{(3)}_{v-3} + F^{(3)}_{v-4}) / (F^{(3)}_{v} + F^{(3)}_{v-2}) = \\ &= 2 + ((F^{(3)}_{v-3} + F^{(3)}_{v-4}) / (F^{(3)}_{v} + F^{(3)}_{v-2})) < 3 ,\end{aligned}$$

so that

$$b^{(2)}_1 = 0; \quad b^{(2)}_2 = 2 . \tag{6.27}$$

From (6.26), (6.27), we obtain, on basis of the previous results

$$1 / (a^{(2)}_1 - b^{(2)}_1) = (F^{(3)}_{v} + F^{(3)}_{v-2}) / F^{(3)}_{v-1} ;$$

$$a^{(2)}_2 - b^{(2)}_2 = ((F^{(3)}_{v+1} + F^{(3)}_{v}) / (F^{(3)}_{v} + F^{(3)}_{v-2})) - 2 = (F^{(3)}_{v-3} + F^{(3)}_{v-4}) / (F^{(3)}_{v} + F^{(3)}_{v-2});$$

we thus obtain, in virtue of (2.3),

$$a^{(3)}_1 = \frac{F^{(3)}_{v-3} + F^{(3)}_{v-4}}{F^{(3)}_{v-1}} ; \quad a^{(3)}_2 = \frac{F^{(3)}_{v} + F^{(3)}_{v-2}}{F^{(3)}_{v-1}} . \tag{6.28}$$

Since

$$F^{(3)}_{v-3} + F^{(3)}_{v-4} < F^{(3)}_{v-3} + F^{(3)}_{v-4} + F^{(3)}_{v-2} = F^{(3)}_{v-1} ,$$

we obtain

$$b^{(3)}_1 = [a^{(3)}_1] = 0 .$$

We further obtain

$$\begin{aligned}F^{(3)}_{v} + F^{(3)}_{v-2} &= F^{(3)}_{v-1} + 2F^{(3)}_{v-2} + F^{(3)}_{v-3} \\ &= F^{(3)}_{v-1} + (F^{(3)}_{v-2} + F^{(3)}_{v-3} + F^{(3)}_{v-4}) + F^{(3)}_{v-2} - F^{(3)}_{v-4} \\ &= 2F^{(3)}_{v-1} + F^{(3)}_{v-2} - F^{(3)}_{v-4} ;\end{aligned}$$

Therefore

$$2F^{(3)}_{v-1} < F^{(3)}_{v} + F^{(3)}_{v-2} < 3F^{(3)}_{v-1} \; ; \; 2 < \frac{F^{(3)}_{v} + F^{(3)}_{v-2}}{F^{(3)}_{v-1}} < 3 \; ; \tag{6.29}$$

$$b^{(3)}_1 = 0 \; ; \qquad b^{(3)}_2 = 2 \; .$$

From (6.28), (6.29), we obtain

$$1 \Big/ (a^{(3)}_1 - b^{(3)}_1) = F^{(3)}_{v-1} / (F^{(3)}_{v-3} + F^{(3)}_{v-4}) \; ;$$

$$\begin{aligned} a^{(3)}_2 - b^{(3)}_2 &= \left((F^{(3)}_{v} + F^{(3)}_{v-2}) \Big/ F^{(3)}_{v-1} \right) - 2 \\ &= (F^{(3)}_{v-2} - F^{(3)}_{v-4}) \Big/ F^{(3)}_{v-1} = (F^{(3)}_{v-3} + F^{(3)}_{v-5} \Big/ F^{(3)}_{v-1} \, , \end{aligned}$$

so that, in virtue of (2.3),

$$a^{(4)}_1 = \frac{F^{(3)}_{v-3} + F^{(3)}_{v-5}}{F^{(3)}_{v-3} + F^{(3)}_{v-4}} \; ; \quad a^{(4)}_2 = \frac{F^{(3)}_{v-1}}{F^{(3)}_{v-3} + F^{(3)}_{v-4}} \; . \tag{6.30}$$

From (6.30) we obtain

$$b^{(4)}_1 = [a^{(4)}_1] = 0 \; ,$$

and further

$$F^{(3)}_{v-1} = F^{(3)}_{v-2} + F^{(3)}_{v-3} + F^{(3)}_{v-4} = 2(F^{(3)}_{v-3} + F^{(3)}_{v-4}) + F^{(3)}_{v-5} \, ,$$

so that

$$F^{(3)}_{v-1} \Big/ (F^{(3)}_{v-3} + F^{(3)}_{v-4}) = 2 + (F^{(3)}_{v-5} \Big/ (F^{(3)}_{v-3} + F^{(3)}_{v-4})) \, ,$$

or

$$2 < \left(F_{v-1}^{(3)} \Big/ \left(F_{v-3}^{(3)} + F_{v-4}^{(3)}\right)\right) < 3 ,$$

which finally yields

$$b_1^{(4)} = 0; \quad b_2^{(4)} = 2 . \tag{6.31}$$

From (6.30), (6.31), we obtain

$$1 \Big/ \left(a_1^{(4)} - b_1^{(4)}\right) = \left(F_{v-3}^{(3)} + F_{v-4}^{(3)}\right) \Big/ \left(F_{v-3}^{(3)} + F_{v-5}^{(3)}\right) ;$$

$$a_2^{(4)} - b_2^{(4)} = F_{v-5}^{(3)} \Big/ \left(F_{v-3}^{(3)} + F_{v-4}^{(3)}\right) ,$$

so that, in virtue of (2.3),

$$a_1^{(5)} = \frac{F_{v-5}^{(3)}}{F_{v-3}^{(3)} + F_{v-5}^{(3)}} , \qquad a_2^{(5)} = \frac{F_{v-3}^{(3)} + F_{v-4}^{(3)}}{F_{v-3}^{(3)} + F_{v-5}^{(3)}} . \tag{6.32}$$

From (6.32) we obtain

$$\left[a_1^{(5)} \right] = b_1^{(5)} = 0 ,$$

and further,

$$\left(F_{v-3}^{(3)} + F_{v-4}^{(3)}\right) \Big/ \left(F_{v-3}^{(3)} + F_{v-5}^{(3)}\right)$$

$$= \left(F_{v-3}^{(3)} + F_{v-5}^{(3)} + F_{v-6}^{(3)} + F_{v-7}^{(3)}\right) \Big/ \left(F_{v-3}^{(3)} + F_{v-5}^{(3)}\right) = 1 + \frac{F_{v-6}^{(3)} + F_{v-7}^{(3)}}{F_{v-3}^{(3)} + F_{v-5}^{(3)}} ,$$

so that

$$1 < \left(\left(F_{v-3}^{(3)} + F_{v-4}^{(3)}\right) \Big/ \left(F_{v-3}^{(3)} + F_{v-5}^{(3)}\right)\right) < 2 ,$$

which yields

$$b_1^{(5)} = 0; \quad b_2^{(5)} = 1 . \tag{6.33}$$

From (6.23), (6.33), we obtain easily

$$a_1^{(6)} = \frac{F_{v-6}^{(3)} + F_{v-7}^{(3)}}{F_{v-5}^{(3)}}; \quad a_2^{(6)} = \frac{F_{v-3}^{(3)} + F_{v-5}^{(3)}}{F_{v-5}^{(3)}} . \tag{6.34}$$

From (6.34) we obtain

$$b_1^{(6)} = [a_1^{(6)}] = 0 ,$$

and further

$$\begin{aligned} F_{v-3}^{(3)} + F_{v-5}^{(3)} &= F_{v-4}^{(3)} + 2F_{v-5}^{(3)} + F_{v-6}^{(3)} = 3F_{v-5}^{(3)} + 2F_{v-6}^{(3)} + F_{v-7}^{(3)} \\ &= 3F_{v-5}^{(3)} + (F_{v-6}^{(3)} + F_{v-7}^{(3)} + F_{v-8}^{(3)}) + F_{v-6}^{(3)} - F_{v-8}^{(3)} \\ &= 4F_{v-5}^{(3)} + F_{v-7}^{(3)} + F_{v-9}^{(3)} < 4F_{v-5}^{(3)} < 4F_{v-6}^{(3)} + 5F_{v-5}^{(3)} ; \end{aligned}$$

therefore,

$$4 < \left((F_{v-3}^{(3)} + F_{v-5}^{(3)}) \big/ F_{v-5}^{(3)} \right) < 5 ,$$

so that

$$b_1^{(6)} = 0; \quad b_2^{(6)} = 4 . \tag{6.35}$$

From (6.34), (6.35), we obtain

$$\begin{aligned} 1 / (a_1^{(6)} - b_1^{(6)}) &= (F_{v-5}^{(3)} \big/ (F_{v-6}^{(3)} + F_{v-7}^{(3)}) , \\ a_2^{(6)} - b_2^{(6)} &= (F_{v-7}^{(3)} + F_{v-9}^{(3)}) \big/ F_{v-5}^{(3)} , \end{aligned}$$

so that, in virtue of (2.3)

$$a_1^{(7)} = \frac{F_{v-7}^{(3)} + F_{v-9}^{(3)}}{F_{v-6}^{(3)} + F_{v-7}^{(3)}} ; \quad a_2^{(7)} = \frac{F_{v-5}^{(3)}}{F_{v-6}^{(3)} + F_{v-7}^{(3)}} . \tag{6.36}$$

From (6.36) we obtain

$$b_1^{(7)} = [a_1^{(7)}] = 0 \;,$$

and further

$$F_{v-5}^{(3)} \Big/ (F_{v-6}^{(3)} + F_{v-7}^{(3)}) = (F_{v-6}^{(3)} + F_{v-7}^{(3)} + F_{v-8}^{(3)}) \Big/ (F_{v-6}^{(3)} + F_{v-7}^{(3)})$$
$$= 1 + (F_{v-8}^{(3)} \Big/ (F_{v-6}^{(3)} + F_{v-7}^{(3)})) \;,$$

so that

$$b_1^{(7)} = 0 \;; \quad b_2^{(7)} = 1 \;. \tag{6.37}$$

From (6.36, (6.37), we obtain

$$1 \Big/ (a_1^{(7)} - b_1^{(7)}) = (F_{v-6}^{(3)} + F_{v-7}^{(3)}) \Big/ (F_{v-7}^{(3)} + F_{v-9}^{(3)}) \;,$$
$$a_2^{(7)} - b_2^{(7)} = F_{v-8}^{(3)} \Big/ (F_{v-6}^{(3)} + F_{v-7}^{(3)}) \;,$$

so that, in virtue of (2.3),

$$a_1^{(8)} = \frac{F_{v-8}^{(3)}}{F_{v-7}^{(3)} + F_{v-9}^{(3)}} \;; \quad a_2^{(8)} = \frac{F_{v-6}^{(3)} + F_{v-7}^{(3)}}{F_{v-7}^{(3)} + F_{v-9}^{(3)}} \;. \tag{6.38}$$

Substituting in (6.38) for v the value

$$v = u + 7 \;, \tag{6.39}$$

we obtain

$$a_1^{(8)} = \frac{F_{u-1}^{(3)}}{F_u^{(3)} + F_{u-2}^{(3)}} \;; \quad a_2^{(8)} = \frac{F_{u+1}^{(3)} + F_u^{(3)}}{F_u^{(3)} + F_{u-2}^{(3)}} \;. \tag{6.40}$$

Comparing (6.26) with (6.40), we see that

$$(6.41) \qquad a_1^{(8)} = a_1^{(2)} ; \quad a_2^{(8)} = a_2^{(2)} \quad \text{for} \quad u = v \to +\infty ,$$

which proves the first statement of Theorem 6.2. The forms of the preperiod (6.17) and the period (6.18) is verified by the formulas (6.22) and (6.23, 25, $\cdots$, 35, 37).

Applying Theorem (5.1) to the Jacobi-Perron algorithm of the numbers

$$F_{v+3}^{(3)} \big/ F_{v+2}^{(3)} , \qquad F_{v+4}^{(3)} \big/ F_{v+2}^{(3)}$$

(this Theorem holds for any algorithm (2.3), as long as the formation law of the $b_i^{(v)}$ generates integers) and singling out the denominators

$$\begin{aligned} c_1^{(2)} &= F_v^{(3)} + F_{v-2}^{(3)} , \\ c_1^{(3)} &= F_{v-1}^{(3)} , \\ c_1^{(4)} &= F_{v-3}^{(3)} + F_{v-4}^{(3)} , \end{aligned}$$

we obtain, on ground of (6.41) and the vector equations $a^{(9)} = a^{(3)}$, $a^{(10)} = a^{(4)}$,

$$(6.42) \qquad \begin{aligned} c_1^{(2+6k)} &= F_{v-7k}^{(3)} + F_{v-2-7k}^{(3)} , \\ c_1^{(3+6k)} &= F_{v-1-7k} , \\ c_1^{(4+6k)} &= F_{v-3-7k}^{(3)} + F_{v-4-7k}^{(3)} . \end{aligned}$$

From (6.42), we obtain, in virtue of (5.3), where $n = 3$,

$$(6.43) \qquad \begin{vmatrix} A_0^{(3+6k)} & A_0^{(4+6k)} & F_{v+2}^{(3)} \\ A_1^{(3+6k)} & A_1^{(4+6k)} & F_{v+3}^{(3)} \\ A_2^{(3+6k)} & A_2^{(4+6k)} & F_{v+4}^{(3)} \end{vmatrix} = F_{v-7k}^{(3)} + F_{v-2-7k}^{(3)}, \quad v \geq 7k + 3.$$

Substituting in (6.43) $v = u + 7k$, we obtain that a solution vector of the S'.E.3

$$(6.44) \qquad xF_{u+2+7k}^{(3)} + yF_{u+3+7k}^{(3)} + zF_{u+4+7k}^{(3)} = F_u^{(3)} + F_{u-2}^{(3)} ,$$

$$k = 0, 1, \cdots ; \quad u = 3, 4, \cdots$$

is given by

$$
(6.45)\qquad
\begin{aligned}
x &= A_1^{(3+6k)}\,A_2^{(4+6k)} - A_1^{(4+6k)}\,A_2^{(3+6k)} \quad ,\\
y &= A_2^{(3+6k)}\,A_0^{(4+6k)} - A_2^{(4+6k)}\,A_0^{(3+6k)} \quad ,\\
z &= A_0^{(3+6k)}\,A_1^{(4+6k)} - A_0^{(4+6k)}\,A_1^{(3+6k)} \quad .
\end{aligned}
$$

Substituting in (6.44) $u = 5$, we obtain that (6.45) is a solution vector of

$$
(6.46)\qquad xF_{7(k+1)}^{(3)} + yF_{7(k+1)+1}^{(3)} + zF_{7(k+2)+2}^{(3)} = 3 \quad .
$$

We further obtain from (6.42), in virtue of (5.3),

$$
(6.47)\qquad
\begin{vmatrix}
A_0^{(4+6k)} & A_0^{(5+6k)} & F_{v+2}^{(3)} \\
A_1^{(4+6k)} & A_1^{(5+6k)} & F_{v+3}^{(3)} \\
A_2^{(4+6k)} & A_2^{(5+6k)} & F_{v+4}^{(3)}
\end{vmatrix}
= F_{v-1-7k}^{(3)}
$$

Substituting in (6.47) $v = u + 7k$, we obtain that a solution vector of the S'.E.3

$$
(6.48)\qquad
\begin{gathered}
xF_{u+2+7k}^{(3)} + yF_{u+3+7k}^{(3)} + zF_{u+4+7k}^{(3)} = F_{u-1}^{(3)} \quad ,\\
k = 0, 1, \cdots ;\ u = 4,\ 5, \cdots \quad ,
\end{gathered}
$$

is given by

$$
(6.49)\qquad
\begin{aligned}
x &= A_1^{(4+6k)}\,A_2^{(5+6k)} - A_1^{(5+6k)}\,A_2^{(4+6k)} \quad ;\\
y &= A_2^{(4+6k)}\,A_0^{(5+6k)} - A_2^{(5+6k)}\,A_0^{(4+6k)} \quad ;\\
z &= A_0^{(4+6k)}\,A_1^{(5+6k)} - A_0^{(5+6k)}\,A_1^{(4+6k)} \quad .
\end{aligned}
$$

We obtain from (6.48), for $u = 6$, that the equation

$$
(6.50)\qquad xF_{7(k+1)+1}^{(3)} + yF_{7(k+1)+2}^{(3)} + zF_{7(k+1)+3}^{(3)} = 2
$$

has the vector solution (6.49).

We further obtain from (6.42), in virtue of (5.3)

(6.51)
$$\begin{vmatrix} A_0^{(5+6k)} & A_0^{(6+6k)} & F_{v+2}^{(3)} \\ A_1^{(5+6k)} & A_1^{(6+6k)} & F_{v+3}^{(3)} \\ A_2^{(5+6k)} & A_2^{(6+6k)} & F_{v+4}^{(3)} \end{vmatrix} = F_{v-3-7k}^{(3)} + F_{v-4-7k}^{(3)} .$$

Substituting in (6.51) $v = u + 7k$, we obtain that a solution vector of

(6.52)
$$xF_{u+2+7k}^{(3)} + yF_{u+3+7k}^{(3)} + zF_{u+4+7k}^{(3)} = F_{u-3}^{(3)} + F_{u-4}^{(3)} ;$$
$$k = 0, 1, \cdots ; \quad u = 6, 7, \cdots .$$

is given by

(6.53)
$$x = A_1^{(5+6k)}A_2^{(6+6k)} - A_1^{(6+6k)}A_2^{(5+6k)}; \; y = A_2^{(5+6k)}A_0^{(6+6k)} - A_2^{(6+6k)}A_0^{(5+6k)}$$
$$z = A_0^{(5+6k)}A_1^{(6+6k)} - A_0^{(6+6k)}A_1^{(5+6k)} .$$

We obtain from (6.52), for $u = 9$, that a solution vector of

(6.54)
$$xF_{7(k+1)+4}^{(3)} + yF_{7(k+1)+5}^{(3)} + zF_{7(k+1)+6}^{(3)} = 6$$

is given by (6.53).

We shall give a few numeric examples for this theory. If we put $k = 1$ in (6.50), we obtain

$$xF_{15}^{(3)} + yF_{16}^{(3)} + zF_{17}^{(3)} = 2 .$$

From (6.49), we calculate easily

$$x = -20; \quad y = -2; \quad z = 7$$

so that

(6.55)
$$7F_{17}^{(3)} - 2F_{16}^{(3)} - 20F_{15}^{(3)} = 2 .$$

We calculate easily

$$F_{15}^{(3)} = 927;\ F_{16}^{(3)} = 1705;\ F_{17}^{(3)} = 3136,$$

which verifies (6.55).

If we put $k = 1$ in (6.54), we obtain

$$xF_{18}^{(3)} + yF_{19}^{(3)} + zF_{20}^{(3)} = 6 .$$

From (6.53), we calculate easily

$$x = -38; \quad y = -29; \quad z = 27 ,$$

so that

(6.56) $$27F_{20}^{(3)} - 29F_{19}^{(3)} - 38F_{18}^{(3)} = 6 .$$

We calculate easily

$$F_{18}^{(3)} = 5768; \quad F_{19}^{(3)} = 10609; \quad F_{20}^{(3)} = 19513$$

which verifies (6.56).

7. THE GENERATING POLYNOMIAL OF GENERALIZED FIBONACCI NUMBERS

The main purpose of this chapter will be the statement of an explicit formula for the limiting value of the ratio

$$F_{v-1}^{(n)} \Big/ F_v^{(n)}$$

of two successive generalized Fibonacci numbers of degree $n \geq 2$. To this end, we shall investigate the generating polynomial $f(x)$ from (6.3) recalling a few results of the author stated in a previous paper [1. p)]. We obtain from (6.3)

$$f(0) = -1; \quad f(1) = n - 1 > 0 ;$$

$$f'(x) = \sum_{k=0}^{n-1} (n - k)x^{n-1-k} > 0 \quad \text{for } x > 0.$$

Therefore $f(x)$ has one and only one real root w in the open interval $(0, 1)$, so that

$$w^n + w^{n-1} + \cdots + w - 1 = 0; \quad 0 < w < 1 . \tag{7.1}$$

We shall now carry out the modified Jacobi-Perron algorithm of the numbers

$$a_s^{(0)} = \sum_{i=0}^{s} w^{s-i} , \quad (s = 1, \cdots, n - 1) , \tag{7.2}$$

which are the components of the given vector $a^{(0)}$. These have, therefore, the form of (7.2), viz.

$$a_1^{(0)} = w + 1; \; a_2^{(0)} = w^2 + w + 1; \; \cdots ; \; a_{n-1}^{(0)} = w^{n-1} + w^{n-2} + \cdots + 1.$$

Then the numbers $a_s^{(v)}$ are functions of w, viz.

$$a_s^{(v)} = a_s^{(v)}(w) , \quad (s = 1, \cdots, n - 1; \; v = 0, 1, \cdots) . \tag{7.3}$$

For the formation law of the rationals $b_s^{(v)}$ we use the formation law

$$b_s^{(v)} = a_s^{(v)}(0) , \quad (s = 1, \cdots, n - 1; \; v = 0, 1, \cdots) . \tag{7.4}$$

The author has proved in [1, p)] that under these assumptions the modified Jacobi-Perron algorithm of the given vector (6.2) is purely periodic; the length of the period is $T = 1$, and it has the form

$$b_s^{(v)} = 1, \quad (s = 1, \cdots, n - 1; \; v = 0, 1, \cdots) . \tag{7.5}$$

As has been proved by the author in 1, p) , the formula holds

$$(7.6) \qquad w = \lim_{v \to \infty} \left(A_0^{(v-1)} \Big/ A_0^{(v)}\right) ,$$

where the $A_0^{(v)}$ have the meaning of (2.4). From (2.4) and (7.5), we obtain

$$\begin{aligned} A_0^{(0)} &= 1 , \\ A_0^{(1)} &= 0 = F_1^{(n)} , \\ A_0^{(2)} &= 0 = F_2^{(n)} , \\ &\cdots\cdots\cdots \\ A_0^{(n-1)} &= 0 = F_{n-1}^{(n)} . \end{aligned}$$

Since

$$A_0^{(n)} = A_0^{(0)} + \sum_{j=1}^{n-1} b_j^{(0)} A_0^{(j)} = 1 + \sum_{j=1}^{n-1} A_0^{(j)} = 1 ,$$

we have

$$A_0^{(n)} = F_n^{(n)} = 1 .$$

We have thus obtained

$$(7.7) \qquad A_0^{(i)} = F_i^{(n)} , \quad (v = 1, 2, \cdots) .$$

We shall now prove that (7.7) holds for any $i \geq 1$, viz.

$$(7.8) \qquad A_0^{(v)} = F_v^{(n)} , \quad (v = 1, 2, \cdots) .$$

<u>Proof by induction</u>. In virtue of (7.7) formula (7.8) is correct for $v = 1, 2, \cdots, n$. Let (7.8) be correct for

$$(7.9) \qquad v = k, k+1, \cdots, k+(n-1) , \quad k \geq 1$$

We shall now prove that (7.8) is correct for $v = k + n$. We obtain from (2.4) and (7.5), (7.9)

$$A_0^{(k+n)} = A_0^{(k)} + \sum_{j=1}^{n-1} b_j^{(k)} A_0^{(k+j)}$$

$$= A_0^{(k)} + \sum_{j=1}^{n-1} A_0^{(k+j)}$$

$$= F_k^{(n)} + \sum_{j=1}^{n-1} F_{k+j}^{(n)} = F_{k+n}^{(n)} \quad ,$$

which proves formula (7.8).

Combining (7.6) and (7.8), we obtain the formula

$$w = \lim_{v\to\infty} \left(F_{v-1}^{(n)} \Big/ F_v^{(n)}\right) . \tag{7.10}$$

Theoretically (7.10) is a very significant formula and answers the questions posed in (6.7). But practically it is of no great value, since neither w nor $F_v^{(n)}$ can be calculated easily because of lack of an explicit formula for either of them. This problem will be solved in the forthcoming passages.

The polynomial $x^{n+1} - 2x + 1$, $x \neq 1$, has the same roots as the generating polynomial $f(x) = x^n + x^{n-1} + \cdots + x - 1$. Particularly, it has one, and only one, real root in the open interval $(0,1)$, viz. w from (7.1). In a previous paper [1. p)] the author has proved the following

<u>Theorem.</u> Let be

$$F(w) = w^{n+1} - 2w + 1 = 0 , \quad 0 < w < 1 . \tag{7.11}$$

If we carry out the modified algorithm of Jacobi-Perron for the given vector $a^{(0)}$ with the components

$$a_s^{(0)} = w^s, \quad (s = 1, \cdots, n-1); \; a_n^{(0)} = w^n - 2, \tag{7.12}$$

then the algorithm becomes purely periodic; the length of the period is $T = n + 1$, and it has the form

$$\begin{array}{c} \\ n+1 \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. \end{array} \begin{array}{ccccc} & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{n}} & \\ 0 & 0 & \cdots & 0 & -2 \\ 0 & 0 & \cdots & 0 & 2 \\ 0 & 0 & \cdots & 0 & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 2 \end{array} \tag{7.13}$$

If, for $v > v_0$,

$$\frac{\left|A_0^{(v)}\right| + \sum_{j=1}^{n-1} \left|a_j^{(0)}\right| \left|A_0^{(v+j)}\right|}{\left|a_n^{(0)}\right| \left|A_0^{(v+n)}\right|} \le m < 1, \tag{7.14}$$

then

$$w = \lim \left(A_0^{(v-1)} \Big/ A_0^{(v)}\right). \tag{7.15}$$

We thus have only to prove that (7.14) holds for the modified algorithm of Jacobi-Perron of (7.12). We obtain from (2.14) and (7.13)

$$A_0^{(0)} = 1; \; A_0^{(v)} = 0, \; (v = 1, \cdots, n); \; A_0^{(n+1)} = 1;$$

$$A_0^{(n+2)} = A_0^{(1)} + \sum_{j=1}^{n} b_j^{(1)} A_0^{(1+j)} = b_n^{(1)} A_0^{(n+1)} = 2; \tag{7.16}$$

$$A_0^{(n+3)} = A_0^{(2)} + \sum_{j=1}^{n} b_j^{(2)} A_0^{(2+j)} = b_n^{(2)} A_0^{(n+2)} = 2^2.$$

We shall now prove

$$A_0^{(n+1+v)} = 2^v, \quad (v = 0, 1, \cdots, n) . \tag{7.17}$$

Proof by induction. (7.17) is correct for $v = 0, 1, 2$, in virtue of (7.16). Let it be correct for $v = k$, viz.

$$A_0^{(n+1+k)} = 2^k, \quad (k = 0, 1, \cdots, n-1) . \tag{7.18}$$

From (7.18) we obtain

$$\begin{aligned} A_0^{(n+1+k+1)} &= A_0^{(k+1)} + \sum_{j=1}^{n} b_j^{(k+1)} A_0^{(k+1+j)} = b_n^{(k+1)} A_0^{(n+1+k)} \\ &= 2 \cdot 2^k = 2^{k+1} , \end{aligned}$$

which proves (7.17). We further obtain from (7.16), (7.17)

$$\begin{aligned} A_0^{(n+1+n+1)} &= A_0^{(n+1)} + \sum_{j=1}^{n} b_j^{(n+1)} A_0^{(n+1+j)} \\ &= 2 + b_n^{(n+1)} A_0^{(n+1+n)} = 2 + b_n^{(0)} A_0^{(n+1+j)} \\ &= 2 + (-2) \cdot 2^n = 2 - 2^{n+1}; \left| A_0^{(n+1+n+1)} \right| \geq \frac{2n+1}{n+1} \cdot 2^n, \quad n \geq 3, \end{aligned}$$

$$\left| A_0^{(n+1+n+1)} \right| \geqq \frac{2n+1}{n+1} \cdot \left| A_0^{(n+1+n)} \right| . \tag{7.19}$$

We now deduce from (7.17), (7.19),

$$\left| A_0^{(n+1+v)} \right| \geq \frac{2n+1}{n+1} \left| A_0^{(n+v)} \right| \text{ for } v = 0, 1, \cdots, n+1 \tag{7.20}$$

and shall prove generally

$$\left| A_0^{(n+1+v)} \right| > \frac{2n+1}{n+1} \left| A_0^{(n+v)} \right| , \quad (v = 0, 1, \cdots) . \tag{7.21}$$

Proof by induction. Let be

$$\left| A_0^{(n+1+v)} \right| \geq \frac{2n+1}{n+1} \left| A_0^{(n+v)} \right| , \quad \text{for } v = k, k+1, \cdots, k+n-1. \tag{7.22}$$

(7.22) is correct for $k = 0, 1, 2$, in virtue of (7.20). We now obtain, in virtue of (2.4), (7.13),

$$\begin{aligned} A_0^{(n+1+k+n)} &= A_0^{(k+n)} + \sum_{j=1}^{n} b_j^{(n+k)} A_0^{(k+n+j)} \\ &= A_0^{(k+n)} + b_n^{(k+n)} A_0^{(k+n+n)} \\ &= A_0^{(k+n)} \pm 2A_0^{(k+n+n)} \quad , \end{aligned}$$

$$\left| A_0^{(n+1+k+n)} \right| \geq 2 \left| A_0^{(k+n+n)} \right| - \left| A_0^{(k+n)} \right| . \tag{7.23}$$

But from (7.22) we obtain

$$\begin{aligned} \left| A_0^{(n+k)} \right| &\leq \frac{n+1}{2n+1} \left| A_0^{(n+k+1)} \right| \leq \left(\frac{n+1}{2n+1} \right)^2 \left| A_0^{(n+k+2)} \right| \\ &\cdots \leq \left(\frac{n+1}{2n+1} \right)^n \left| A_0^{(k+n+n)} \right| , \end{aligned}$$

$$\left| A_0^{(k+n)} \right| \leq \left(\frac{n+1}{2n+1} \right)^n \left| A_0^{(k+n+n)} \right| . \tag{7.24}$$

From (7.23), (7.24) we obtain

$$\left| A_0^{(n+1+k+n)} \right| \geq \left(2 - \left(\frac{n+1}{2n+1} \right)^n \right) \left| A_0^{(k+n+n)} \right| . \tag{7.25}$$

We shall now prove

$$2 - \left(\frac{n+1}{2n+1} \right)^n > \frac{2n+1}{n+1} , \quad \text{for } n = 3, 4, \cdots \tag{7.26}$$

We have to prove

$$2 - \left(\frac{n+1}{2n+1}\right)^n > 2 - \frac{1}{n+1} , \quad \text{or} \quad n + 1 < \left(\frac{2n+1}{n+1}\right)^n \quad \text{or}$$

$$n + 1 < \left(1 + \frac{n}{n+1}\right)^n , \qquad n = 3, 4, \cdots .$$

But, for $n \geq 3$,

$$1 + \binom{n}{1} \cdot \frac{n}{n+1} + \binom{n}{2}\left(\frac{n}{n+1}\right)^2 < \left(1 + \frac{n}{n+1}\right)^n .$$

We shall prove

$$n + 1 \leq 1 + \binom{n}{1} \cdot \frac{n}{n+1} + \binom{n}{2}\left(\frac{n}{n+1}\right)^2 ,$$

or

$$n \leq \frac{n^2}{n+1} + \frac{n^3(n-1)}{2(n+1)^2} ,$$

or

$$1 \leq \frac{n}{n+1} + \frac{n^2(n-1)}{2(n+1)^2} ,$$

or

$$\frac{1}{n+1} \leq \frac{n^2(n-1)}{2(n+1)^2} ; \quad 2(n+1) \leq n^2(n-1) .$$

But, for $n \geq 3$,

$$n^2(n-1) \geq 2n^2 \geq 6n = 2n + 4n \geq 2n + 12 > 2n + 2$$
$$= 2(n+1) .$$

Thus (7.26) is proved.

From (7.25), (7.26), we obtain

$$\left| A_0^{(n+1+k+n)} \right| > \frac{2n+1}{n+1} \left| A_0^{(k+n+n)} \right| ,$$

which proves (7.21).

From (7.21) we obtain

$$(7.27)\qquad \left|A_0^{(k+v)}\right| > \left(\frac{2n+1}{n+1}\right)^k \left|A_0^{(v)}\right| , \quad (k+v \geq n+1) .$$

We shall now prove formula (7.14). We obtain, since

$$\left|a_j^{(0)}\right| = w^j < 1 , \qquad (j = 1, \cdots, n-1) ;$$
$$\left|a_n^{(0)}\right| = 2 - w^n , \qquad n \geq 3 ,$$

$$\frac{\left|A_0^{(v)}\right| + \sum_{j=1}^{n-1} \left|a_j^{(0)}\right| \left|A_0^{(v+j)}\right|}{\left|a_n^{(0)}\right| \left|A_0^{(v+n)}\right|} < \frac{\sum_{j=0}^{n-1} \left|A_0^{(v+j)}\right|}{(2-w^n)\left|A_0^{(v+n)}\right|} .$$

But from (7.22) we obtain

$$\left|A_0^{(v+j)}\right| < \left(\frac{n+1}{2n+1}\right)^{n-j} \left|A_0^{(v+n)}\right| ,$$

therefore

$$\frac{\left|A_0^{(v)}\right| + \sum_{j=1}^{n-1} \left|a_j^{(0)}\right| \left|A_0^{(v+j)}\right|}{\left|a_n^{(0)}\right| \left|A_0^{(v+n)}\right|} < \frac{\sum_{j=0}^{n-1} \left(\frac{n+1}{2n+1}\right)^{n-j}}{2-w^n}$$

$$= \frac{\frac{n+1}{2n+1}\left(1 - \left(\frac{n+1}{2n+1}\right)^n\right)}{\left(1 - \frac{n+1}{2n+1}\right)(2-w^n)} = \frac{(n+1)\left(1 - \left(\frac{n+1}{2n+1}\right)^n\right)}{(2-w^n)n} ,$$

so that

$$(7.28)\qquad \frac{\left|A_0^{(v)}\right| + \sum_{j=1}^{n-1} \left|a_j^{(0)}\right| \left|A_0^{(v+j)}\right|}{\left|a_n^{(0)}\right| \left|A_0^{(v+n)}\right|} < \frac{(n+1)}{(2-w^n)n} \left(1 - \left(\frac{n+1}{2n+1}\right)^n\right)$$

We shall now prove

(7.29) $$(n+1)/n < 2 - w^n, \quad n \geqq 3 .$$

We obtain from

$$F(x) = x^{n+1} - 2x + 1 ,$$

$$F(0) = 1, \quad F(1) = 0; \quad F'(x) = (n+1)x^n - 2 ;$$

therefore

$$F'(x) < 0 \quad \text{for} \quad 0 < x^n < 2/(n+1) ,$$
$$F'(x) > 0 \quad \text{for} \quad x^n > 2/(n+1) .$$

Since w is the only root in the open interval (0.1), we obtain

(7.30) $$w^n < \frac{2}{n+1} .$$

From (7.30) we obtain

$$2 - \frac{2}{n+1} < 2 - w^n .$$

It is easy to prove the following formula

$$\frac{n+1}{n} < 2 - \frac{2}{n+1} .$$

With (7.31) and the previous result (7.29) is proved. From (7.28), (7.29), we obtain

$$\frac{\left|A_0^{(v)}\right| + \sum_{j=1}^{n-1} \left|a_j^{(0)}\right| \left|A_0^{(v+j)}\right|}{\left|a_n^{(0)}\right| \left|A_0^{(v+n)}\right|} < 1 - \left(\frac{n+1}{2n+1}\right)^n . \tag{7.32}$$

But (7.32) verifies (7.14) with

$$m = 1 - \left(\frac{n+1}{2n+1}\right)^n < 1 . \tag{7.33}$$

We shall use a formula for the $A_0^{(v)}$ of an algorithm with the period (7.13) proved by the author in [1. p)], viz.

$$\begin{gathered} A_0^{((s+1)(n+1)+k)} = b^k \sum_{i=0}^{s} \binom{i(n+1) + s + k - i}{i(n+1) + k} z^i \\ b = 2; \quad z = -2^{n+1}; \quad (s = 0, 1, \cdots; k = 0, 1, \cdots, n) \end{gathered} \tag{7.34}$$

Writing in formula (7.15) $v = (s+1)(n+1) + 1$, we obtain

$$w = \lim_{s \to \infty} \left(A_0^{((s+1)(n+1))} \Big/ A_0^{((s+1)(n+1)+1)}\right) ,$$

and, using (7.34),

$$w = \lim_{s \to \infty} \frac{\sum_{i=0}^{s} (-1)^i \binom{(n+1)i + s - i}{(n+1)i} 2^{(n+1)i}}{2\Sigma_{i=0}^{s} (-1)^i \binom{(n+1)i + s + 1 - i}{(n+1)i + 1} 2^{(n+1)i}} . \tag{7.35}$$

Comparing (7.10) and (7.35), we obtain the wanted relation

$$\lim_{s \to \infty} \frac{F_{s-1}^{(n)}}{F_s^{(n)}} = \frac{1}{2} \lim_{s \to \infty} \frac{\sum_{i=0}^{s} (-1)^i \binom{(n+1)i + s - i}{(n+1)i} 2^{(n+1)i}}{\sum_{i=0}^{s} (-1)^i \binom{(n+1)i + s + 1 - i}{(n+1)i + 1} 2^{(n+1)i}} . \tag{7.36}$$

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SPECIAL INTEGER SEQUENCES CONTROLLED BY THREE PARAMETERS

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1. INTRODUCTION

The positive integers h, n, and k are used as parameters to postulate a set of rules for generating a family of sequences of positive integers. It is shown that some of the sequences are directly related to sums of the k^{th} powers of roots of selected n^{th} degree polynomials in which the coefficient of the $(n-h)^{th}$ power is zero. The remaining sequences are the Lucas-like sequences described in a previous paper [1] plus a transition sequence.

2. FIRST-TYPE SEQUENCE

For a given n, the k^{th} member of a sequence is u_{kn}. For each h, n has the values specified by $n \geq h + 1$. There are, in general, four types of behavior within a sequence. A general sequence is formularized in (1) with boundaries between types of behavior indicated by xxxxx, ooooo, or _____.

For the special case $h = 1$, there are no values above the xxxxx divider. By interpreting a summation as zero when its upper limit is zero, it is seen that the first term (i.e., the $k = 1$ term) for $h = 1$ appears between the xxxxx and ooooo dividers and is zero. For $h \geq 2$ there are always some terms for each type of behavior, and the first term of a sequence is always one. Some examples are given in Table 1.

Table 1

k	h=1, n=2	h=1, n=6	h=3, n=7	h=5, n=8
1	0	0	1	1
	0000000	0000000		
2	2	2	3	3
	_____		xxxxxxx	
3	0	3	4	7
			0000000	
4	2	6	11	15
				xxxxxxx
5	0	10	21	26
				0000000
6	2	17	42	57
		_____		
7	0	21	78	113
			_____	
8	2	38	139	223
				_____

$$\left.\begin{array}{l} u_{1n} = 2^1 - 1 \\ \cdots\cdots \\ u_{kn} = 2^k - 1, \text{ (general term)} \\ \cdots\cdots \\ u_{h-1,n} = 2^{h-1} - 1, \end{array}\right\} \quad (1 \le k \le h - 1)$$

xxxxxxxxxxx

$$u_{hn} = \sum_{b=1}^{h-1} u_{bn} , \qquad (k = h)$$

ooooooooooo

(1)
$$\left.\begin{array}{l} u_{h+1,n} = \left(\displaystyle\sum_{b=1}^{h} u_{bn}\right) - u_{1n} + h + 1 , \\ \cdots\cdots\cdots\cdots \\ u_{kn} = \left(\displaystyle\sum_{b=1}^{k-1} u_{bn}\right) - u_{k-h,n} + k \quad \text{(general term)} \\ \cdots\cdots\cdots\cdots \\ u_{nn} = \left(\displaystyle\sum_{b=1}^{n-1} u_{bn}\right) - u_{n-h,n} + n \end{array}\right\} \quad (h + 1 \le k \le n)$$

____

$$\left.\begin{array}{l} u_{n+1,n} = \left(\displaystyle\sum_{b=1}^{n} u_{bn}\right) - u_{n+1-h,n} \\ \cdots\cdots\cdots\cdots \\ u_{kn} = \left(\displaystyle\sum_{b=k-n}^{k-1} u_{bn}\right) - u_{k-h,n} \quad \text{(general term)} \\ \cdots\cdots\cdots\cdots \end{array}\right\} \quad k \ge n + 1$$

It is interesting to note that there are $h - 1$ terms prior to a xxxxx divider and n terms prior to a _____ divider. Inspection of (1) shows that for $h \ge 2$ the first $h - 1$ terms follow the pattern $1, 3, 7, 15, 31, \cdots, 2^k - 1, \cdots$. For values of $k > h$, it is seen from (1) that u_{kn} is found from a sum which includes u_{kn} 's in an order which would be consecutive except for an always excluded $u_{k-h,n}$ term. Behavior of the first-type sequences is included in tables in the Appendix for $h = 1(1)5$, $n = 1(1)11$, and $k = 1(1)11$.

3. A USE OF THE FIRST-TYPE SEQUENCE

For selected h and n, the k^{th} term of a first-type sequence is the same as $S_k^{(n)}$, the sum of the k^{th} powers of the roots of

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n , \tag{2}$$

if the choices $a_0 = 1$, $a_h = 0$, and all other a's = -1 are made. Verification over a limited range can be made by direct comparison of Table 1 of [1] and the corresponding table of the Appendix. The interpretation is, of course, that $S_k^{(n)} = u_{kn}$ for a given h.

4. SECOND-TYPE SEQUENCE

The first-type sequence applied for $n \geq h + 1$ and the u_{kn} 's were identically the $S_k^{(n)}$'s in that range. If for $2 \leq n \leq h$ the $S_k^{(n)}$'s are calculated and interpreted as u_{kn} 's, the u_{kn} 's so determined are members of a second-type sequence. The tables of the Appendix include second-type sequences.

For $n \leq h - 1$, (2) does not have an a_hx^{n-h} term, and does not have the missing term resulting from $a_n = 0$. Since the Lucas-like sequences of [1] are found from (2) with no missing terms, the second-type sequences are the Lucas-like sequences for $n \leq h - 1$.

For $n = h - 1$ and $n = h$, the second-type sequences are the same since setting $a_h = 0$ in each case produces equations (2) differing only by a root factor $(x - 0)$ which contributes nothing to the sum of powers of roots. The sequence for $n = h > 2$ accordingly is equal to the Lucas-like sequence obtained for $n = h - 1$. Alternatively, it is seen that the sequence for $n = h > 2$ is related to the second-type sequences. This is demonstrated in (3) which is applicable for $n = h > 2$ only.

$$\left.\begin{aligned} u_{1n} &= 2^1 - 1 \\ &\cdots\cdots \\ u_{kn} &= 2^k - 1 \quad \text{(general term)} \\ &\cdots\cdots \\ u_{h-1,n} &= 2^{h-1} - 1 \end{aligned}\right\} \quad (1 \le k \le h - 1)$$

xxxxxxxxxxxxx

(3)

$$u_{hn} = \sum_{b=1}^{h-1} u_{bn} \qquad (k = h)$$

ooooooooooooo

____

$$\left.\begin{aligned} u_{h+1,n} &= \left(\sum_{b=1}^{n} u_{bn}\right) - u_{1n} \\ &\cdots\cdots\cdots\cdots\cdots \\ u_{kn} &= \left(\sum_{b=k-n}^{k-1} u_{bn}\right) - u_{k-n,n} \quad \text{(general term)} \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad (k \ge h + 1)$$

Comparison of (3) with (1) indicates that (3) is essentially (1) with the 000000000 and _________ boundaries coalesced. Thus, it is seen that a second-type sequence for $n = h > 2$ is a transition between Lucas-like sequences and a first-type sequence.

5. APPENDIX

Table 2 h = 1

k / n	1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0	0
2	0	2	2	2	2	2	2	2	2	2	2
3	0	0	3	3	3	3	3	3	3	3	3
4	0	2	2	6	6	6	6	6	6	6	6
5	0	0	5	5	10	10	10	10	10	10	10
6	0	2	5	11	11	17	17	17	17	17	17
7	0	0	7	14	21	21	28	28	28	28	28
8	0	2	10	22	30	38	38	46	46	46	46
9	0	0	12	30	48	57	66	66	75	75	75
10	0	2	17	47	72	92	102	112	112	122	122
11	0	0	22	66	110	143	165	176	187	187	198

First-Type Sequences

Second-Type Sequence

Table 3 h = 2

k / n	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1
3	1	1	4	4	4	4	4	4	4	4	4
4	1	1	5	9	9	9	9	9	9	9	9
5	1	1	6	11	16	16	16	16	16	16	16
6	1	1	10	16	22	28	28	28	28	28	28
7	1	1	15	29	36	43	50	50	50	50	50
8	1	1	21	39	67	73	81	89	89	89	89
9	1	1	31	66	114	130	139	148	157	157	157
10	1	1	46	111	188	226	246	256	266	276	276
11	1	1	67	179	313	386	430	452	463	474	485

First-Type Sequences

Second-Type Sequences

Table 4 h = 3

k / n	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2	1	3	3	3	3	3	3	3	3	3	3
3	1	4	4	4	4	4	4	4	4	4	4
4	1	7	7	11	11	11	11	11	11	11	11
5	1	11	11	16	21	21	21	21	21	21	21
6	1	18	18	30	36	42	42	42	42	42	42
7	1	29	29	50	64	71	78	78	78	78	78
8	1	47	47	91	115	131	139	147	147	147	147
9	1	76	76	157	211	238	256	265	274	274	274
10	1	123	123	278	383	443	473	493	503	513	513
11	1	199	199	485	694	815	881	914	936	947	958

Second-Type Sequences (n = 2–3) First-Type Sequences (n = 4–11)

Table 5 h = 4

k/n	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2	1	3	3	3	3	3	3	3	3	3	3
3	1	4	7	7	7	7	7	7	7	7	7
4	1	7	11	11	11	11	11	11	11	11	11
5	1	11	21	21	26	26	26	26	26	26	26
6	1	18	39	39	45	51	51	51	51	51	51
7	1	29	71	71	85	92	99	99	99	99	99
8	1	47	131	131	163	179	187	195	195	195	195
9	1	76	241	241	304	340	358	367	376	376	376
10	1	123	442	442	578	648	688	708	718	728	728
11	1	199	814	814	1090	1244	1321	1365	1387	1398	1409

Second-Type Sequences (n = 2–4) First-Type Sequences (n = 5–11)

Table 6 $h = 5$

k/n	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2	1	3	3	3	3	3	3	3	3	3	3
3	1	4	7	7	7	7	7	7	7	7	7
4	1	7	11	15	15	15	15	15	15	15	15
5	1	11	21	26	26	26	26	26	26	26	26
6	1	18	39	51	51	57	57	57	57	57	57
7	1	29	71	99	99	106	113	113	113	113	113
8	1	47	131	191	191	207	215	223	223	223	223
9	1	76	241	367	367	403	421	430	439	439	439
10	1	123	443	708	708	788	828	848	858	868	868
11	1	199	815	1365	1365	1530	1618	1662	1684	1695	1706

Second-Type Sequences First-Type Sequences

6. REFERENCE

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ERRATA
SCOTT'S FIBONACCI SCRAPBOOK

In the equations on p. 176, please arrange all the exponents in ascending order. Also on p. 176, please change the sign in the line beginning with $P_4(x)$ to a plus instead of minus. On p. 191 (continuation of Scott's article), please make the line beginning with $P_5(x)$ read as follows:

$$P_5(x) = 3125 + 7768x - 15851x^2 - 9463x^3 + 1976x^4 + 243x^5$$

On page 166, please make the following corrections: In $P_4(x)$, change the next-to last number to $2689x^6$. In $P_5(x)$, change the last number on the first line to read: $594,362x^5$. In $P_6(x)$, change the last number on the first line to read: $85,906,862x^4$, and the following number to $21,282,070x^5$. In $P_7(x)$, please change the last number of the first line to read: $3,730,909,778x^3$, and the following number to $2,311,372,054x^4$.

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BERNOULLI NUMBERS*

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1. INTRODUCTION

The purpose of this paper is to discuss some of the properties of the Bernoulli and related numbers and to indicate the relationship of these numbers to cyclotomic fields. We shall use the notation of Nörlund [25].

The Bernoulli numbers may be defined by means of

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (|x| < 2\pi) \quad . \tag{1.1}$$

This is equivalent to

$$\sum_{r=0}^{n} \binom{n}{r} B_r = B_n \quad (n > 1) \ . \tag{1.2}$$

together with $B_0 = 1$.

It is convenient to write (1.2) in the following symbolic form:

$$(B + 1)^n = B^n \qquad (n > 1) \tag{1.3}$$

where it is understood that after expansion of the left member we replace B^k by B_k.

We next define the Bernoulli polynomial $B_n(a)$ by means of

$$\frac{xe^{ax}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(a) \frac{x^n}{n!} \quad . \tag{1.4}$$

It follows that

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(1.5) $$B_n(a) = \sum_{r=0}^{n} \binom{n}{r} B_r a^{n-r}$$

or symbolically

(1.6) $$B_n(a) = (B + a)^n .$$

Moreover, we have from (1.4)

(1.7) $$B_n(0) = B_n ,$$

(1.8) $$B_n(a + 1) - B_n(a) = na^{n-1} ,$$

(1.9) $$B'_n(a) = nB_{n-1}(a) .$$

The polynomial $B_n(a)$ is uniquely determined by means of (1.7) and (1.8). Additional properties of interest are

(1.10) $$B_n(1 - a) = (-1)^n B_n(a)$$

and the multiplication theorem.

(1.11) $$B_n(ka) = k^{n-1} \sum_{s=0}^{k-1} B_n\left(a + \frac{s}{k}\right)$$

valid for all integral $k \geq 1$. Nielsen [24] has observed that if a polynomial $f_n(a)$ satisfies

$$f_n(ka) = k^{n-1} \sum_{s=0}^{k-1} f_n\left(a + \frac{s}{k}\right)$$

for some $k > 1$ then we have

$$f_n(a) = C_n \cdot B_n(a) ,$$

where C_n is independent of a.

It is not difficult to show that

$$B_{2n+1} = 0 \quad (n > 0) \tag{1.12}$$

and that

$$(-1)^{n-1}B_{2n} > 0 \quad (n > 0) . \tag{1.13}$$

The Euler numbers E_n may be defined by means of

$$\frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} , \tag{1.14}$$

which is equivalent to

$$(E + 1)^n + (E - 1)^n = \begin{cases} 2 & (n = 0) \\ 0 & (n > 0) \end{cases} . \tag{1.15}$$

It follows that

$$E_{2n+1} = 0 \quad (n \geq 0) \tag{1.16}$$

while

$$(-1)^n E_{2n} > 0 \quad (n \geq 1) ; \tag{1.17}$$

the E_{2n} are odd integers.

The Euler polynomial $E_n(a)$ is defined by means of

$$\frac{2e^{ax}}{e^x + 1} = \sum_{n=0}^{\infty} E_n(a) \frac{x^n}{n!} \tag{1.18}$$

It follows that

$$E_n = 2^n E_n(1/2) \tag{1.19}$$

Clearly

$$E_n(a+1) + E_n(a) = 2a^n \tag{1.20}$$

Corresponding to (1.10) and (1.11) we have

$$E_n(1-a) = (-1)^n E_n(a) , \tag{1.21}$$

$$E_n(kx) = k^n \sum_{s=0}^{k-1} (-1)^s E_n\left(a + \frac{s}{k}\right) \quad (k \text{ odd}) , \tag{1.22}$$

$$E_n(kx) = \frac{-2k^n}{n+1} \sum_{s=0}^{k-1} (-1)^s E_{n+1}\left(a + \frac{s}{k}\right) \quad (k \text{ even}) . \tag{1.23}$$

2. THE STAUDT-CLAUSEN THEOREM

The B_n are rational numbers, as is evident from the definition. The denominator of B_{2n} is determined by the following remarkable theorem.

<u>Theorem 1.</u> We have, for $n \geq 1$,

$$B_{2n} = G_{2n} - \sum_{p-1 \mid 2n} \frac{1}{p} , \tag{2.1}$$

where G_{2n} is an integer and the summation on the right is over all primes p (including 2) such that $p - 1$ divides $2n$.

For example, we have

$$B_2 = \frac{1}{6} = 1 - \frac{1}{2} - \frac{1}{3} , \quad B_4 = \frac{-1}{30} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} ,$$

$$B_6 = \frac{1}{42} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{7} .$$

We shall sketch a proof of Theorem 1. It follows from (1.1) that

$$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{s=0}^{k} (-1)^s \binom{k}{s} s^n . \tag{2.2}$$

Now it is familiar that

$$\frac{1}{k!} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} s^n$$

is an integer (Stirling number of the second kind). Thus (2.2) becomes

$$B_n = \sum_{k=0}^{n} \frac{k!}{k+1} c(n,k) ,$$

where $c(n,k)$ is an integer. In the next place if $a \geq 2$, $b \geq 2$, $ab > 4$, we can easily verify that $(ab-1)!/ab$ is integral. Hence in the right member of (2.2) it is only necessary to consider $k = 4$ and k equal to a prime p. Since

$$\sum_{s=0}^{p-1} (-1)^s \binom{p-1}{s} s^n \equiv \sum_{s=0}^{p-1} s^n$$

$$\equiv \begin{cases} -1 \pmod{p} & (p-1 \mid n,\ n > 0) \\ 0 \pmod{p} & (p-1 \nmid n) , \end{cases}$$

(2.2) reduces to

$$B_{2n} = G'_{2n} - \sum_{p-1 \mid 2n} \frac{1}{p} + \frac{1}{4} \sum_{s=0}^{3} (-1)^s \binom{3}{s} s^{2n} , \tag{2.3}$$

where G'_{2n} is an integer. But

$$\sum_{s=0}^{3}(-1)^s \binom{3}{s} s^{2n} \equiv -3 - 3^{2n} \equiv 0 \pmod{4}$$

so that (2.3) reduces to (2.1).

Hurwitz [12] has proved the following elegant analog of the Staudt-Clausen theorem. Let $\zeta(u)$ be the lemniscate function defined by means of

$$\zeta'^2(u) = 4\zeta^3(u) - 4\zeta(u) . \tag{2.4}$$

We may put

$$\zeta(u) = \frac{1}{u^2} + \sum_{1} \frac{2^{4n}E_n}{4n} \frac{u^{4n-2}}{(4n-2)!} \tag{2.5}$$

(The E_n in (2.5) should not be confused with the Euler number defined by (1.14).) Corresponding to (2.1) we have

$$E_n = G_n + \frac{1}{2} + \frac{(2a)^{4n/(p-1)}}{p} , \tag{2.6}$$

where G_n is an integer and the sum on the right is over all primes $p \equiv 1 \pmod{4}$ such that $p - 1$ divides $4n$; moreover, a is uniquely determined by means of

$$p = a^2 + b^2, \quad a \equiv b + 1 \pmod{4} .$$

Hurwitz's proof makes use of the complex multiplication of the function $\zeta(u)$. However the present writer [7] has proved the following generalized Staudt-Clausen theorem in an elementary manner.

Put

$$f(x) = \sum_{n=1}^{\infty} a_n x^n / n! \quad (a_1 = 1) , \tag{2.7}$$

where the a_n are arbitrary rational integers and assume that the inverse function is of the type

(2.8) $$\lambda(x) = \sum_{n=1}^{\infty} c_n x^n / n \qquad (c_1 = 1),$$

where the c_n are integers. Note that the denominator in (2.8) is n, not $n!$. Now put

(2.9) $$\frac{x}{f(x)} = \sum_0^{\infty} \beta_n x^n / n! \ .$$

Then we have

(2.10) $$\beta_n = G_n - \sum_{p-1|n} \frac{1}{p} c_p^{n/(p-1)} ,$$

where G_n is integral and the summation is over all primes p such that $p - 1$ divides n.

When $f(x) = e^x - 1$, $\lambda(x) = \log(1 + x)$, (2.10) reduces to (2.1).

3. KUMMER'S CONGRUENCES

Kummer obtained certain congruences for both the Bernoulli and Euler numbers that are of considerable importance in applications. We state first the result for Euler numbers.

Theorem 2. Let $r \geq 1$, $n \geq r$ and let p denote an arbitrary odd prime. Then

(3.1) $$\sum_{s=0}^{r} (-1)^s \binom{r}{s} E_{n+s(p-1)} \equiv 0 \pmod{p^r} .$$

A more general result is contained in Theorem 3. Let $r \geq 1$, $e \geq 1$, $n \geq re$ and put $w = p^{e-1}(p-1)$, where p is an odd prime. Then

$$(3.2) \qquad \sum_{s=0}^{r} (-1)^s \binom{r}{s} E_{n+sw} \equiv 0 \pmod{p^{re}} .$$

For the Bernoulli numbers we have Theorem 4. Let $r \geq 1$, $e \geq 1$, $n > re$ and put $w = p^{e-1}(p-1)$, where p is a prime such that $p - 1 \nmid n$. Then

$$(3.3) \qquad \sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{B_{n+sw}}{n+sw} \equiv 0 \pmod{p^{re}} .$$

For proof of these theorems see Nielsen [24, Ch. 14] or Bachmann [26]. Note that $p = 2$ is excluded in Theorems 2 and 3. Frobenius [9] has proved a result for the case $p = 2$. There is a fallacious proof in Bachmann's book.

Vandiver [19] obtained a result like (3.3) without the denominator $n + sw$ but under more restrictive hypotheses. He proved that

$$(3.4) \qquad \sum_{s=0}^{r} (-1)^s \binom{r}{s} B_{(a+s)(p-1)} \equiv 0 \pmod{p^{r-1}} ,$$

where

$$a > 0, \quad r > 0, \quad a + r \leq p - 1 .$$

For more general results in this direction see [3].

The quotient B_n/n occurring in (3.3) is integral (mod p) provided $p - 1 \nmid n$. More precisely we state

Theorem 5. If p is prime and $p - 1 \nmid 2n$, $p^r | n$ then the numerator of B_{2n} is divisible by p^r.

The case $p - 1 | 2n$ is covered by the following supplementary theorem.

Theorem 6. Let $p^r (p - 1) | n$. Then p^r divides the numerator of

$$B_{2n} + \frac{1}{p} - 1 .$$

For proof of Theorem 6, see [3].

4. RECURRENCES

In addition to the fundamental recurrence (1.2), the B_n satisfy many more recurrences. Many are derived in Nielsen's book. The following two occur in a paper by D. H. Lehmer [13].

$$\sum_{r=0}^{n} \binom{6n+3}{6r} B_{6r} = 2n + 1 , \tag{4.1}$$

$$\sum_{r=0}^{n} \binom{6n+5}{6r+2} B_{6r+2} = \frac{1}{3}(6n + 5) . \tag{4.2}$$

In all the known recurrences the number of terms is of order An, where A is a positive constant. Thus it is of interest to ask whether B_n can satisfy a relation of the form

$$\sum_{r=0}^{k} A_r(n)B_{n-r} = A(n) ,$$

where the $A_j(n)$ and $A(n)$ satisfy certain restrictions and k is independent of n.

We may state

Theorem 7. The equation

$$\sum_{r=0}^{k} A_r(n)B_{n-r} = A(n) \quad (n > No) \tag{4.3}$$

where $A_0(n)$ is a polynomial in n with integral coefficients, $A_1(n), \cdots, A_k(n)$, $A(n)$ are arbitrary integral-valued functions of n and k is independent of n, is impossible.

Theorem 8. The equation

$$\sum_{r=0}^{k} A_r(n)E_{n-r} = A(n) \qquad (n > No), \tag{4.4}$$

where $A_0(n)$, $A_1(n)$, $\cdots$, $A_k(n)$, $A(n)$ are polynomials in n with integral coefficients and k is independent of n, is impossible.

Theorem 7 is proved by means of the Staudt-Clausen Theorem; Theorem 8 by means of Kummer's Congruences. For these and more general results, see [5], [6].

5. IRREGULAR PRIMES

A prime p is said to be regular if it does not divide the numerator of any of the numbers

$$B_2, B_4, \cdots, B_{p-3}. \tag{5.1}$$

The prime p is irregular if it does divide the numerator of at least one of the numbers (5.1). The motivation for these definitions will appear presently.

The first few irregular primes are

37, 59, 67, 101, 103, 131, 149, 157, 233, 257, 263, 271, 283, 293.

It might appear that the irregular primes are relatively rare. Actually, it is not known whether infinitely many regular primes exist. In the opposite direction we have

Theorem 9. The number of irregular primes is infinite.

This theorem is due to Jensen; for the proof see [23, p. 82]. A simpler proof is given in [2]. Jensen proved a slightly stronger result, namely that there exist infinitely many irregular primes congruent to 5 (mod 6). This result has very recently been improved by Montgomery [14].

Theorem 10. Let T be a fixed integer >2. Then there exist infinitely many irregular primes that are not congruent to 1 (mod T).

Paralleling the above definition, we may say that a prime p is irregular relative to the Euler numbers provided it divides at least one of the Euler numbers

$$E_2, E_4, \cdots, E_{p-3}. \tag{5.2}$$

Theorem 11. There exist inifinitely many primes that are irregular relative to the Euler numbers.

For proof see [2]. Here again nothing is known about the number of regular primes relative to the Euler numbers. Also it is not known how the two kinds of regular primes are related.

6. CONNECTION WITH CLASS NUMBERS AND FERMAT'S LAST THEOREM

Let p denote a fixed odd prime and put $\zeta = e^{2\pi i/p}$. Let $h = h(\zeta)$ denote the class number of the cyclotomic field $Q(\zeta)$. It is customary to put

$$h = AB; \tag{6.1}$$

A is called the first factor of the class number and B is called the second factor. The number B appears as the quotient of two determinants involving logarithms of units; it is equal to the class number of the real field $Q(\zeta + \zeta^{-1})$.

It is of considerable interest to know when h is divisible by p. We have the following criterion.

Theorem 12. The class number of $Q(\zeta)$ is divisible by p if and only if p is irregular.

It can be proved that if p divides B then necessarily p divides A. This yields

Theorem 13. $p \mid h \Leftrightarrow p \mid A$.

Vandiver [18] has proved

Theorem 14. Let $n \geq 1$. Then A satisfies

$$A \equiv 2^{-1/2(p-3)} p \prod_s B_{sp^n+1} \pmod{p^n}, \tag{6.2}$$

where the product is over $s = 1, 3, 5, \cdots, p-2$.

When $n = 1$, (6.2) reduces to

$$A \equiv 2^{-1/2(p-3)} p \prod_s B_{sp+1} \pmod{p} \quad .$$

Now by Theorem 4 with $r = 1$ we have

$$\frac{B_{sp+1}}{sp+1} = \frac{B_{s+1}}{s+1} \pmod{p} \quad (1 \leq s < p-2) ;$$

for $s = p - 2$ we have by the Staudt-Clausen Theorem

$$pB_{p(p-2)+1} = pB_{(p-1)^2} \equiv -1 \pmod{p} \quad .$$

Thus (6.2) reduces to

$$A \equiv \frac{-4}{(1/2(p-3))!} \prod_{s=1}^{1/2(p-3)} B_{2s} \pmod{p}. \tag{6.3}$$

Kummer has proved the following result concerning Fermat's last theorem.

Theorem 15. If p is regular the equation

$$\alpha^p + \beta^p + \nu p = 0 \quad (\alpha, \beta, \nu \in Q(\zeta)) \tag{6.4}$$

has only the trivial solution $\alpha = \beta = \nu = 0$.

Nicol, Selfridge and Vandiver [16] have proved that Fermat's last theorem holds for prime exponents less than 4002.

The equation (in rational integers)

$$x^p + y^p + z^p = 0 \qquad (p \nmid xyz) \tag{6.5}$$

is known as the first case of Fermat's last theorem.

It has been proved that if (6.5) is satisfied then

$$2^p \equiv 2 \pmod{p^2} \tag{6.6}$$

and

$$3^p \equiv 3 \pmod{p^2} \tag{6.7}$$

Indeed considerably more is known in this direction.

It has also been proved that if (6.5) holds then

$$B_{p-3} \equiv B_{p-5} \equiv B_{p-7} \equiv B_{p-9} \equiv 0 \pmod{p} . \tag{6.8}$$

Finally we state some criteria involving the Euler numbers. Vandiver [20] has proved that if (6.5) is satisfied then

$$E_{p-3} \equiv 0 \pmod{p}. \tag{6.9}$$

M. Gut [10] has proved that if

$$x^{2p} + y^{2p} = z^{2p} \quad (p \nmid xyz) \tag{6.10}$$

is satisfied, then

$$E_{p-3} \equiv E_{p-5} \equiv E_{p-7} \equiv E_{p-9} \equiv E_{p-11} \equiv 0 \pmod{p} . \tag{6.11}$$

7. CONCLUDING REMARKS

The references that follow include mainly papers that have been referred to above. Vandiver in his expository paper [22] remarks that some 1500 papers on Bernoulli numbers have been published!

For Fermat's last theorem, the reader is referred to Vandiver's expository paper [21] as well as Dickson [8], Hilbert [11] and Vandiver-Wahlin [23].

For the Euler numbers and related matters see Salie [17].

We conclude with some remarks about real quadratic fields. Let p be a prime $\equiv 1 \pmod 4$ and let $E = 1/2(t + u\sqrt{p}) > 1$ denote the fundamental unit of $Q(\sqrt{p})$. Ankeny, Artin and Chowla [1] have conjectured that $u \not\equiv 0 \pmod{p}$; Mordell [15] has proved the following results:

(1) If p is regular then $u \not\equiv 0 \pmod{p}$.

(2) If $p \equiv 5 \pmod 8$ then $u \equiv 0 \pmod p$ if and only if $B_{(p-1)/2} \equiv 0 \pmod p$. Chowla had proved (2) for all $p \equiv 1 \pmod 4$.

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The first use of the Q-matrix to generate the Fibonacci Numbers appears in an abstract of a paper by Professor J. L. Brenner by the title "Lucas' Matrix." This abstract appeared in the March, 1951 American Mathematical Monthly on pages 221 and 222. The basic exploitation of the Q-matrix appeared in 1960 intthe San Jose State College Master's thesis of Charles H. King with the title 'Some Further Properties of the Fibonacci Numbers. Further utilization of the Q-matrix appears in the Fibonacci Primer sequence parts I-V.

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THE QUADRATIC FIELD $Q(\sqrt{5})$ AND A CERTAIN DIOPHANTINE EQUATION

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1. INTRODUCTION

We establish here a characterization of the Fibonacci and Lucas numbers while determining the units of the quadratic field extension $Q(\sqrt{5})$ of the rational field Q. Using an appropriate norm on $Q(\sqrt{5})$, we also find all solutions to the Diophantine equation $x^2 - 5y^2 = \pm 4$ and solve a certain binomial coefficient equation. Except for the definitions of basic algebraic structures, the treatment is self-contained, and so should also serve as a brief introduction to algebraic number theory. We hope the reader sees the beauty of one branch of mathematics interacting profitably with another, wherein both gain.

For the definitions of group, ring, and field, we refer the reader to [1]. Let u be an element of the field of complexnumbers C. We say u is an algebraic number if there is a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (a_i \in Q, \ a_n \neq 0) \tag{1}$$

with coefficients in Q not all zero which is satisfied by u, i.e., such that

$$p(u) = a_n u^n + a_{n-1} u^{n-1} + \cdots + a_1 u + a_0 = 0 .$$

Thus $\sqrt{2}$ and $i = \sqrt{-1}$ are algebraic numbers, while π is not. Among all the polynomials satisfied by u, there is one of least positive degree, say of the form p(x) in (1). Since $p(u) = 0$ implies $a_n^{-1}p(u) = 0$, we may choose p(x) with leading coefficient 1, i.e., so that p(x) is monic. The monic polynomial of least positive degree satisfied by u is called the minimal polynomial of u. For example, the minimal polynomial of $\frac{1}{2}\sqrt{2}$ is $x^2 - \frac{1}{2}$. The reason we insist that the leading coefficient of p(x) be 1 is that with this provision the minimal polynomial is unique (see [1, Chap. 14]).

An algebraic number is said to be an algebraic integer if its minimal polynomial has integral coefficients. For example, any rational r is an algebraic number (it satisfies $x - r$), but among the rationals only the integers are algebraic integers (the reader should prove this). For this reason the ordinary integers are sometimes referred to as rational integers. An algebraic number $u \neq 0$ is called a unit if both u and u^{-1} are algebraic integers. As an example, -1 and i are units. A unit should be distinguished from the unit (multiplicative identity) element 1 of the field, although the unit element is also a unit.

3. THE QUADRATIC FIELD $Q(\sqrt{5})$

Denote by $Q(\sqrt{5})$ the smallest field contained in the field of real numbers R which contains both Q and $\sqrt{5}$. We first expose the form of the elements in $Q(\sqrt{5})$.

Theorem 1. $Q(\sqrt{5}) = \{r + s\sqrt{5} \mid r, s \in Q\}$.

Proof. Denote the right side in Theorem 1 by S. Then since the elements of S are formed using the field operations from those in Q and $\sqrt{5}$, we have $S \subset Q(\sqrt{5})$. But we claim S is already a field. Clearly it inherits the necessary additive and associative properties from R, and the product of any two elements in S is easily shown to be again in S. Hence we must only show the existence of inverses in S. If $r + s\sqrt{5} \neq 0$, then

$$\frac{1}{r + s\sqrt{5}} = \frac{r - s\sqrt{5}}{r^2 - 5s^2} = \frac{r}{r^2 - 5s^2} - \left(\frac{s}{r^2 - 5s^2}\right)\sqrt{5} \in S .$$

Since $Q(\sqrt{5})$ is the smallest subfield of R containing Q and $\sqrt{5}$, we have $Q(\sqrt{5}) \subset S$. Thus $S = Q(\sqrt{5})$.

Because of the irrationality of $\sqrt{5}$, we note that two elements in $Q(\sqrt{5})$ are equal if and only if they are equal componentwise, i.e., $a + b\sqrt{5} = c + d\sqrt{5}$ for $a, b, c, d \in Q$ if and only if $a = c$ and $b = d$. $Q(\sqrt{5})$ is called a quadratic field because it is formed by adjoining $\sqrt{5}$ to Q, and the minimal polynomial of $\sqrt{5}$ is a quadratic.

We next describe the set $Q_i(\sqrt{5})$ of algebraic integers in R which also occur in $Q(\sqrt{5})$.

Theorem 2. The set $Q_i(\sqrt{5})$ of algebraic integers in $Q(\sqrt{5})$ consists of precisely the numbers $\frac{1}{2}(a + b\sqrt{5})$, where a and b are integers such that $a \equiv b \pmod{2}$.

Proof. Using Theorem 1, any number u in $Q(\sqrt{5})$ may be expressed as $u = (a + b\sqrt{5})/c$, where the integers a, b, and c have no common factor except ± 1. We may assume $b \neq 0$ to exclude the trivial case when u is rational. Then the monic polynomial of lowest degree satisfied by u is

$$p(x) = \left(x - \frac{a + b\sqrt{5}}{c}\right)\left(x - \frac{a - b\sqrt{5}}{c}\right) = x^2 - \left(\frac{2a}{c}\right)x + \frac{a^2 - 5b^2}{c^2}\,. \tag{2}$$

If u is to be an algebraic integer, then the coefficients $2a/c$ and $(a^2 - 5b^2)/c^2$ must be integers. Thus $4a^2/c^2$, $(4a^2 - 20b^2)/c^2$, and hence $20b^2/c^2$ must all be integers, so that $c \mid 2a$ and $c^2 \mid 20b^2$, where $n \mid m$ means n divides m. Now any prime factor $p \neq 2$ of c must divide both a and b by the above, contrary to our assumption that a, b, c have no common factor except ± 1. Similarly $4 \mid c$ is impossible, so the only choices left are $c = 1$ and $c = 2$.

If $c = 1$, $p(x)$ has integral coefficients and u is an algebraic integer. In this case u has the form $\frac{1}{2}(2a + 2b\sqrt{5})$, and $2a \equiv 2b \equiv 0 \pmod 2$, so the conclusion of the theorem is true. If $c = 2$, then $(a^2 - 5b^2)/c^2 = (a^2 - 5b^2)/4$ is an integer if and only if a and b are either both odd or both even, or equivalently $a \equiv b \pmod 2$. Hence the theorem also holds here, completing the proof.

We remark the $Q_i(\sqrt{5})$ actually forms a ring because it is closed under multiplication. The reader is urged to verify the details.

We next investigate the question of units in $Q(\sqrt{5})$. First note that by definition if u_1 and u_2 are units, then u_1, u_1^{-1}, u_2, u_2^{-1}, $-u_1$ are all in $Q_i(\sqrt{5})$. Using Theorem 2, it is straightforward to verify that then u_1u_2, $(u_1u_2)^{-1}$, $u_1u_2^{-1}$, $(u_1u_2^{-1})^{-1}$, $(-u_1)^{-1}$ are also in $Q_i(\sqrt{5})$. Hence u_1u_2, $u_1u_2^{-1}$, and $-u_1$ are units in $Q(\sqrt{5})$. In particular, if u is a unit, so is u^{-1}.

The Gaussian integers J are the set of complex numbers with integral real and imaginary parts. A useful function from J to the nonnegative integers is the norm defined by $|a + bi| = a^2 + b^2$. This norm is handy because $|xy| = |x||y|$ for $x, y \in J$, so it preserves the multiplicative structure of J. We now introduce an analogous function on $Q_i(\sqrt{5})$. If $u = \frac{1}{2}(a + b\sqrt{5}) \in Q_i(\sqrt{5})$, define the norm of u by

$$N(u) = \tfrac{1}{2}(a + b\sqrt{5})\tfrac{1}{2}(a - b\sqrt{5}) = \tfrac{1}{4}(a^2 - 5b^2)\,.$$

The reader should verify that $N(u)$ is always an integer (possibly negative), and that $N(u_1u_2) = N(u_1)N(u_2)$ for all $u_1, u_2 \in Q_i(\sqrt{5})$. We use this norm to obtain a characterization of units.

Theorem 3. An element $u \in Q_i(\sqrt{5})$ is a unit if and only if $N(u) = \pm 1$.

Proof. If u is a unit, then $u, u^{-1} \in Q_i(\sqrt{5})$, so that $1 = N(1) = N(uu^{-1}) = N(u)N(u^{-1})$. Since $N(u)$ and $N(u^{-1})$ are integers, $N(u) = \pm 1$. Conversely, if $u = \frac{1}{2}(a + b\sqrt{5}) \in Q_i(\sqrt{5})$ such that $N(u) = \pm 1$, then

$$\tfrac{1}{2}(a + b\sqrt{5})\,\tfrac{1}{2}(a - b\sqrt{5}) = \pm 1 ,$$

so that

$$u^{-1} = \pm\tfrac{1}{2}(a - b\sqrt{5}) \in Q_i(\sqrt{5})$$

by Theorem 2. Thus u is a unit.

Using the norm function on $Q_i(\sqrt{5})$ and recalling that a unit in $Q(\sqrt{5})$ must already be in $Q_i(\sqrt{5})$, we can obtain a complete accounting of the units in $Q(\sqrt{5})$. Let $\alpha = (1 + \sqrt{5})/2 \in Q_i(\sqrt{5})$. Then $N(\alpha) = -1$, so by Theorem 3 α is a unit in $Q(\sqrt{5})$. By the above remarks we therefore know that $\pm\alpha$, $\pm\alpha^2$, $\pm\alpha^3, \cdots, \pm 1$, $\pm\alpha^{-1}$, $\pm\alpha^{-2}, \cdots$ are units in $Q(\sqrt{5})$. Thus in contrast with the Gaussian integers J, where the only units are ± 1, $\pm i$, in $Q(\sqrt{5})$ there are units of either sign as large or as small as we please.

Theorem 4. The numbers

$$\pm\alpha^n, \quad \pm\alpha^{-n} \qquad (n = 0, 1, 2, \cdots) \tag{3}$$

are the only units in $Q(\sqrt{5})$.

Proof. We first prove there is no unit between 1 and α. Suppose that there is a unit $u \in Q_i(\sqrt{5})$ such that $1 < u < \alpha$. By Theorem 2, $u = \frac{1}{2}(x + y\sqrt{5})$, where x and y are integers. Then by Theorem 3

$$\pm 1 = N(u) = \frac{x^2 - 5y^2}{4} = \left(\frac{x + y\sqrt{5}}{2}\right)\left(\frac{x - y\sqrt{5}}{2}\right),$$

so that using $1 < u$ we find

$$-\tfrac{1}{2}(x + y\sqrt{5}) < -1 \le \tfrac{1}{2}(x + y\sqrt{5})\tfrac{1}{2}(x - y\sqrt{5}) \le 1 < \tfrac{1}{2}(x + y\sqrt{5}) .$$

Dividing by $u \neq 0$ yields

$$-1 < \tfrac{1}{2}(x - y\sqrt{5}) < 1 . \tag{4}$$

Adding (4) to $1 < u < \alpha$ gives

$$0 < x < 1 + \alpha,$$

showing that $x = 1$ or 2. But in either case there is no integer y such that $1 < u < \alpha$ holds. This contradiction shows there is no unit between 1 and α.

Now to finish the proof. Suppose $u \neq 0$ is a unit, where we may assume u is positive since $-u$ is also a unit. Then either $u = \alpha^n$, or there is an integer n such that $\alpha^n < u < \alpha^{n+1}$. Now α^{-n} is a unit, implying $\alpha^{-n}u$ also is. But then $1 < \alpha^{-n}u < \alpha$, which was shown impossible in the first part of the proof. Hence the only units in $Q(\sqrt{5})$ are given in (3).

We now use Theorem 4 to give a characterization of Fibonacci and Lucas numbers. But we first need,

Theorem 5. Define the Fibonacci numbers F_n by $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_{n+1} + F_n$, and the Lucas numbers L_n by $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$. Then

$$\alpha^n = \tfrac{1}{2}(L_n + F_n\sqrt{5}) .$$

Proof. We establish this by induction. It is certainly true for $n = 0, 1$. If it is valid for $n = k$, $k + 1$, simply adding the corresponding equations together with the fact that $\alpha^{k+2} = \alpha^{k+1} + \alpha^k$ shows it holds for $n = k + 2$, completing the induction step and the proof.

Theorem 6. The algebraic number $\tfrac{1}{2}(a + b\sqrt{5}) \in Q(\sqrt{5})$ is a unit if and only if $a = L_n$ and $b = F_n$ for some integer n.

Proof. This is a combination of Theorems 4 and 5.

Thus we have characterized the Fibonacci and Lucas numbers in terms of the units in $Q(\sqrt{5})$. We note in passing that since α^n is a unit of $Q(\sqrt{5})$, Theorem 2 implies $F_n \equiv L_n \pmod 2$.

An application of these properties of $Q(\sqrt{5})$ to prove the converse of a familiar property of the Fibonacci numbers has been given by Carlitz [2]. This type of development is capable of generalization to $Q(\sqrt{d})$, where d may be assumed to be a squarefree integer. One striking fact is that the analogue of unique factorization of elements into powers of irreducible (prime) elements holds for only a finite number of d ($d = 5$ is one of them). For further information about this, we refer the reader to [3; Chap. 15] for a number theoretic approach, and to [1; Chap. 14] for an algebraic one.

4. THE SOLUTION OF $x^2 - 5y^2 = \pm 4$

We show here how the solutions of the Diophantine equation $x^2 - 5y^2 = \pm 4$ may be easily obtained as a byproduct of the preceding algebraic material. Note that $N(\alpha) = -1$, so that $N(\alpha^n) = (-1)^n$. Then if $u \in Q_i(\sqrt{5})$, $N(u) = 1$ if and only if $u = \alpha^{2n}$, and $N(u) = -1$. if and only if $u = \alpha^{2n+1}$ for some integer n. This observation leads to the

Theorem 7. (i) All rational integral solutions of $x^2 - 5y^2 = 4$ are given by $x = L_{2n}$, $y = F_{2n}$, and (ii) all of $x^2 - 5y^2 = -4$ by $x = L_{2n+1}$, $y = F_{2n+1}$ $(n = 0, \pm 1, \pm 2, \cdots)$.

Proof. (i) Since $N(\alpha^{2n}) = 1$, Theorem 5 shows that the purported solutions actually satisfy $x^2 - 5y^2 = 4$. Conversely, if $x^2 - 5y^2 = 4$, then $x \equiv y \pmod 2$ and $N[\frac{1}{2}(x + y\sqrt{5})] = 1$. By the preceeding remarks, $\frac{1}{2}(x + y\sqrt{5}) = \alpha^{2n}$ for some n, so that by Theorem 5 $x = L_{2n}$, $y = F_{2n}$, showing that these are all the solutions.

(ii) As in (i), $N(\alpha^{2n+1}) = -1$ and Theorem 5 show that $x = L_{2n+1}$, $y = F_{2n+1}$ are actually solutions. On the other hand, if $x^2 - 5y^2 = -4$, then $x \equiv y \pmod 2$ and $N[\frac{1}{2}(x + y\sqrt{5})] = -1$. Then $\frac{1}{2}(x + y\sqrt{5}) = \alpha^{2n+1}$ for some n, so by Theorem 5 $x = L_{2n+1}$, $y = F_{2n+1}$, completing the proof.

We remark that Theorem 7 was proved by Long and Jordan [4] by using the classical theory of the Pell equation, from which the result follows easily. Theorem 7 also provides a characterization of Fibonacci and Lucas numbers analogous to Theorem 6, but in terms of a Diophantine equation.

5. THE SOLUTION OF A CERTAIN BINOMIAL COEFFICIENT EQUATION

We shall use the preceding results to solve completely the seemingly unrelated binomial coefficient equation,

$$\binom{n}{k} = \binom{n-1}{k+1}. \tag{5}$$

For example, the three solutions of (5) with smallest n are

$$\binom{2}{0} = \binom{1}{1} = 1, \binom{15}{5} = \binom{14}{6} = 3003, \binom{104}{39} = \binom{103}{40}. \tag{6}$$

First note that by cancelling common factors, (5) is equivalent to

$$n(k+1) = (n-k)(n-k-1) ,$$

or

$$k^2 + (1 - 3n)k + n^2 - 2n = 0 .$$

This quadratic in k has a solution in integers if and only if its discriminant $5n^2 + 2n + 1$ is a perfect square, say

$$5n^2 + 2n + 1 = t^2 .$$

Then

$$25n^2 + 10n + 1 = 5t^2 - 5 + 1,$$

so that

$$(5n+1)^2 - 5t^2 = -4 , \tag{7}$$

which is the form of the Diophantine equation which we solved in the previous section. Then by (ii) of Theorem 7, (7) has an integral solution if and only if

$x = L_{2r+1}$, $y = F_{2r+1}$, and $x \equiv 1 \pmod 5$, the last condition being imposed so that n is an integer. Now it is easy to verify that $L_{2r+1} \equiv 1 \pmod 5$ if and only if r is even, say $r = 2s$, so all solutions of (7) are given by

$$n = \frac{L_{4s+1} - 1}{5}, \quad t = F_{4s+1} .$$

Using the Binet form for Fibonacci and Lucas numbers, we have

$$n = \frac{L_{4s+1} - 1}{5} = F_{2s}F_{2s+1} .$$

Also,

$$k = \frac{3n - 1 - t}{2} = \tfrac{1}{2}(3F_{2s}F_{2s+1} - 1 - F_{4s+1}) = F_{2s-2}F_{2s+1} .$$

Hence all solutions of our original equation (5) are given by

$$n = F_{2s}F_{2s+1}, \quad k = F_{2s-2}F_{2s+1}, \quad s = 1, 2, 3, \cdots,$$

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EDITORIAL COMMENT

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PYTHAGOREAN TRIADS OF THE FORM X, X+1, Z DESCRIBED BY RECURRENCE SEQUENCES

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The term Pythagorean Triples or Triads is applied to those integers which describe all right triangles with integral sides. The sub-class which is the subject of this paper, is restricted to those of sides x, $x + 1$, $\sqrt{2x^2 + 2x + 1}$. It is obvious that the smallest such triangle has sides 3, 4, 5. The problem is to find a general method of sequential progress through the family of all such triangles. In the course of this development, and consequent to a solution of Pell's equation, it is shown that these triangles bear a curious relationship to a series which, with the exception of a single coefficient, is identical with the Fibonacci series.

It can be shown that in a right triangle $x^2 + y^2 = z^2$, primitive solutions are given by integers a, b such that $x = a^2 - b^2$, $y = 2ab$ and $z = a^2 + b^2$ where $a > b$, and (a, b) are relatively prime. This paper will be concerned with triangles in which $y = x \pm 1$, or $x^2 + (x \pm 1)^2 = z^2$, the primitive solutions of which also take this form.

A. If x is odd and

$$x = a^2 - b^2 \quad \text{and} \quad x + 1 = 2ab,$$

then

$$-1 = a^2 - 2ab - b^2$$

$$-1 = a^2 - 2ab - b^2 + b^2 - b^2$$

$$-1 = a^2 - 2ab + b^2 - 2b^2$$

$$-1 = (a - b)^2 - 2b^2$$

B. If x is even and

$x = 2ab$ and $x + 1 = a^2 - b^2$ (Note: In A, x was odd and in B, x is even in order to account for all possibilities.)

then

$$+1 = a^2 - 2ab - b^2$$

$$+1 = (a - b)^2 - 2b^2$$

Let $p = a - b$ and $q = b$, then by A and B above

(1) $$\pm 1 = p^2 - 2q^2 .$$

Equation (1) is an example of Pell's equation. By inspection, the smallest integral solution greater than zero of this equation is $p = 1$, $q = 1$.

Equation (1) can be factored into

$$(p - q\sqrt{2})(p + q\sqrt{2}) = \pm 1$$

which, when raised to the n^{th} power, becomes

$$(p - q\sqrt{2})^n (p + q\sqrt{2})^n = \pm 1$$

Specifically

$$(1 - \sqrt{2})^n (1 + \sqrt{2})^n = \pm 1$$

since $p = 1$, $q = 1$ is a solution of equation (1).

Now let

(2) $$p_n + q_n\sqrt{2} = (1 + \sqrt{2})^n$$

then

(3) $$p_n - q_n\sqrt{2} = (1 - \sqrt{2})^n$$

Then, by solving these simultaneous equations,

$$p_n = 1/2\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right] \tag{4}$$

$$q_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right] \tag{5}$$

Since $p = 1$, $q = 1$ is the smallest solution of equation (1), then the general solution is given by (2) or (3) above and, therefore, by (4) and (5). (This can be found in most texts on Number Theory.)

Adding equations (4), (5)

$$p_n = 1/2\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right] \tag{4a}$$

$$q_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right] \tag{5a}$$

$$p_n + q_n = \frac{1}{2\sqrt{2}}\left[\sqrt{2}(1+\sqrt{2})^n + \sqrt{2}(1-\sqrt{2})^n + (1+\sqrt{2})^n - (1-\sqrt{2})^n\right]$$

$$= \frac{1}{2\sqrt{2}}\left[(\sqrt{2}+1)(1+\sqrt{2})^n - (1-\sqrt{2})(1-\sqrt{2})^n\right]$$

$$= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}\right]$$

$$p_n + q_n = q_{n+1} \tag{6}$$

Since $p_n = a - b$ and $q_n = b$, then

$$a = p_n + q_n$$

or

$$a = q_{n+i}$$

and, of course,

$$b = q_n$$

Equation (2) can be rewritten

$$\begin{aligned} p_{n+1} + q_{n+1}\sqrt{2} &= (1 + \sqrt{2})^{n+1} \\ &= (1 + \sqrt{2})^n (1 + \sqrt{2}) \\ &= (p_n + q_n\sqrt{2})\ (1 + \sqrt{2}) \\ &= p_n + p_n\sqrt{2} + q_n\sqrt{2} + 2q_n \\ &= (p_n + 2q_n) + \sqrt{2}(p_n + q_n) \end{aligned}$$

But

$$p_n + q_n = q_{n+1}$$

(7) $$\therefore\ p_{n+1} = p_n + 2q_n$$

Rewriting equations (7), (6) and subtracting,

(7.a) $$p_{n-1} = p_{n-2} + 2q_{n-2}$$

(6.a) $$q_{n-1} = p_{n-2} + q_{n-2}$$

(8) $$p_{n-1} = q_{n-1} + q_{n-2}$$

Now rewriting equation (6)

(6.b) $$q_n = p_{n-1} + q_{n-1}$$

Substitute equation (8)

(9) $$\begin{aligned} q_n &= q_{n-1} + q_{n-2} + q_{n-1} \\ q_n &= 2q_{n-1} + q_{n-2} \end{aligned}$$

In both A and B above, the term 2ab was used, once for x and once for $x + 1$. If p and q satisfy $p^2 - 2q^2 = -1$, then $x + 1 = 2ab$. If p and q satisfy $p^2 - 2q^2 = +1$, then $x = 2ab$. Equations (2) and (3) state that the only way for the negative portion of equation (1) to be satisfied is for $(1 - \sqrt{2})^n$ to be negative. If $(1 - \sqrt{2})^n$ is negative, then $x + 1 = 2ab$; if $(1 - \sqrt{2})^n$ is positive, then $x = 2ab$. Since $(1 - \sqrt{2})$ is a negative term $(\sqrt{2} > 1)$, $(1 - \sqrt{2})^n$ is positive when n is even and negative when n is odd. Now the formula for one side of the triangle becomes

$$2q_n q_{n+1} = \begin{cases} x & \text{for even values of } n \\ x + 1 & \text{for odd values of } n \end{cases} \tag{10}$$

We have now developed a recurrence relationship for the q terms in relation to previous q terms (equation 9).

Except for the coefficient 2 of q_{n-1}, this is the Fibonacci Series. Note that in this same manner the expression $p_n = 2p_{n-1} + p_{n-2}$ can also be proved.

Until now nothing has been formulated concerning the hypotenuse or z term of the Pythagorean Triple. Since squaring and taking the root of very large numbers is difficult, it would be advantageous to have a recursive formula for the z terms. We propose to prove that

$$z_n = q_{2n+1} \tag{11}$$

is such a formula. Then any Pythagorean Triad of the form x, $x + 1$, z can be found recursively by using equations (9), (10), and (11). Further, by use of equation (6), any two consecutive q terms can be found and the sequence proceeds from there. See Appendix A. Proof for equation (11) follows.

From A and B above, two conditions are possible, either $x = a^2 - b^2$ and $x + 1 = 2ab$ or $x = 2ab$ and $x + 1 = a^2 - b^2$. In either case,

$$x^2 + (x + 1)^2 = (a^2 - b^2)^2 + (2ab)^2 .$$

As stated before,

$$2ab = 2q_n q_{n+1}$$

for the n^{th} triad. Also,

$$a^2 - b^2 = q_{n+1}^2 - q_n^2$$

since

$$a = q_{n+1} \text{ and } b = q_n$$

Then,

$$x^2 + (x + 1)^2 = \left(q_{n+1}^2 - q_n^2\right)^2 + \left(2\,q_n\,q_{n+1}\right)^2$$

$$= q_{n+1}^4 - 2q_n^2 q_{n+1}^2 + q_n^4 + 4q_n^2 q_{n+1}^2$$

$$= q_{n+1}^4 + 2q_n^2\,q_{n+1}^2 + q_n^4$$

$$= \left(q_{n+1}^2 + q_n^2\right)^2$$

$$\sqrt{x^2 + (x + 1)^2} = z_n = q_{n+1}^2 + q_n^2$$

To prove equation (11) all that remains is to prove that

$$q_{2n+1} = q_{n+1}^2 + q_n^2$$

To do this we will prove by induction on k that

$$q_{2n+1} = q_{k+2}\,q_{2n-k} + q_{k+1}\,q_{2n-(k+1)} \cdot$$

If $k = 0$

$$q_{2n+1} = 2q_{2n} + q_{2n-1}$$

$$q_{2n} = 2q_{2n-1} + q_{2n-2}$$

$$q_{2n+1} = 2\left[2q_{2n-1} + q_{2n-2}\right] + q_{2n-1}$$

If $k = 1$

$$q_{2n+1} = 5q_{2n-1} + 2q_{2n-2}$$

Notice now that q_{2n+1} is represented in terms of

$$(q_3 = 5,\ q_{2n-1}),\ (q_2 = 2, \text{ and } q_{2n-2}).$$

Assume that the k^{th} relationship is of the form

$$q_{2n+1} = q_{k+2}\,q_{2n-k} + q_{k+1}\,q_{2n-(k+1)}$$

Certainly the first relationship is true as we have just shown. Assume the k^{th} relationship is true. Then,

$$q_{2n+1} = q_{k+2}\,q_{2n-k} + q_{k+1}\,q_{2n-(k+1)}$$

From equation (9) we know

$$q_{2n-k} = 2q_{2n-k-1} + q_{2n-k-2}$$

Then

$$q_{2n+1} = q_{k+2}\left[2q_{2n-k-1} + q_{2n-k-2}\right] + q_{k+1}q_{2n-k-1}$$

$$q_{2n+1} = 2q_{k+2}\,q_{2n-k-1} + q_{k+2}\,q_{2n-k-2} + q_{k+1}\,q_{2n-k-1}$$

$$q_{2n+1} = q_{2n-k-1}\left[2q_{k+2} + q_{k+1}\right] + q_{k+2}\,q_{2n-k-2}$$

Since

$$2q_{k+2} + q_{k+1} = q_{k+3}\ ,$$

$$q_{2n+1} = q_{k+3}\,q_{2n-k-1} + q_{k+2}q_{2n-k-2}$$

This is the $(k + 1)^{st}$ relationship and this proves the general equation inductively. Specifically, when $k = n - 1$,

$$q_{2n+1} = q_{(n-1)+2}\, q_{2n-(n-1)} + q_{(n-1)+1}\, q_{2n-[(n-1)+1]}$$

$$q_{2n+1} = q_{n+1}\, q_{n+1} + q_n\, q_n$$

$$q_{2n+1} = q_{n+1}^2 + q_n^2$$

Then this completes the proof for equation (11).

APPENDIX A

n		q_n	$2q_nq_{n+1}$	= {	x
1		1	4	x_1 =	3
2		2	20	x_2 =	20
3	z_1 =	5	120	x_3 =	119
4		12	696	x_4 =	696
5	z_2 =	29	4060	x_5 =	4059
6		70	23360	x_6 =	23360
7	z_3 =	169	137904	x_7 =	137903
8		408	803760	x_8 =	803760
9	z_4 =	985	4684660	x_9 =	4684659
10		2378	27304196	x_{10} =	27304196
11	z_5 =	5741	159140520	x_{11} =	159140519
12		13860	927538920	x_{12} =	927538920
13	z_6 =	33461	5406093004	x_{13} =	5406093003
14		80782	31509019100	x_{14} =	3150919100
15	z_7 =	195025	183648021600	x_{15} =	183648021599
16		470832	1070387585472	x_{16} =	1070387585472
17	z_8 =	1136689	6238626641380	x_{17} =	6238626641379
18		2744210	36361380737780	x_{18} =	36361380737780
19	z_9 =	6625109	211929657785304	x_{19} =	211929657785303
20		15994428	1235216565974040	x_{20} =	1235216565974040
21	z_{10} =	38613965			
22		93222358			
23	z_{11} =	225058681			

APPENDIX A (Continued)

n			q_n	$2q_n\, q_{n+1}$	= {	x
24			54339720			
25	$\underline{z_{12}}$	=	1311738121			
26			3166815962			
27	$\underline{z_{13}}$	=	7645370045			
28			18457556052			
29	$\underline{z_{14}}$	=	44560482149			
30			107578520350			
31	$\underline{z_{15}}$	=	259717522849			
32			527013566048			
33	$\underline{z_{16}}$	=	1513744654945			
34			4074502875938			
35	$\underline{z_{17}}$	=	9662750406821			
36			23400003689580			
37	$\underline{z_{18}}$	=	56462757785981			
38			136325519261542			
39	$\underline{z_{19}}$	=	329113796309065			
40			794553111879672			
41	$\underline{z_{20}}$	=	1918220020068409			

APPENDIX B

x	x + 1	z
3	4	5
20	21	29
119	120	169
696	697	985
4059	4060	5741
23360	23361	33461
137903	137904	195025
803760	803761	1136689
4684659	4684660	6625109
27304196	27304197	38613965
159140519	159140520	225058681
927538920	927538921	1311738121
5406093003	5406093004	7645370045
31509019100	31509019101	44560482149
183648021599	183648021600	259717522849
1070387585472	1070387585473	1513744654945
6238626641379	6238626641380	9662750406821
36361380737780	36361380737781	56462757785981
211929657785303	211929657785304	329113796309065
1235216565974040	1235216565974041	1918220020068409

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GENERALIZED RABBITS FOR GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

The original Fibonacci number sequence arose from an academic rabbit production problem (see [1] and [5], pp. 2-3). In this paper we generalize the birth sequence pattern and determine the sequences of new arrivals and total population. We shall obtain the Fibonacci sequence in several different ways.

2. GENERAL BIRTH SEQUENCE

Consider a new-born pair of rabbits which produce a sequence of litters. Let the number of rabbit pairs in the n^{th} litter, which is delivered at the n^{th} time point, be B_n. Assume that each offspring pair also breeds in the same manner. Clearly $B_0 = 0$, and the B_n are nonnegative integers for $n \geq 1$.

The array (1) will aid us in our formalization. Let

$$B(x) = \sum_{n=0}^{\infty} B_n x^n \quad (B_0 = 0),$$

$$(1) \qquad \left\{ \begin{aligned} R_0 &= 1 \\ R_1 &= B_1R_0 \\ R_2 &= B_2R_0 + B_1R_1 \\ R_3 &= B_3R_0 + B_2R_1 + B_1R_2 \\ &\vdots \\ R_n &= B_nR_0 + \cdots + B_1R_{n-1} \end{aligned} \right.$$

$$R(x) = \sum_{n=0}^{\infty} R_n x^n \quad (R_0 = 1), \qquad \text{and} \qquad T(x) = \sum_{n=0}^{\infty} T_n x^n \quad (T_0 = 1)$$

be the generating functions for the birth sequence, new arrival sequence, and total population sequence, respectively. Remembering that $B_0 = 0$, it is clear that

$$(2) \qquad R_n = \sum_{j=0}^{n-1} B_{n-j}R_j = \sum_{j=0}^{n} B_{n-j}R_j \qquad (n \geq 1) .$$

Noticing that (2) gives the incorrect result $R_0 = 0$ instead of the correct $R_0 = 1$, we have

$$R(x) - 1 = -1 + \sum_{j=0}^{\infty} R_j x^j = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} B_{n-j}R_j\right) x^n .$$

$$= R(x)B(x) ,$$

so that

(3) $$R(x) = \frac{1}{1 - B(x)}$$

and

$$B(x) = \frac{R(x) - 1}{R(x)} .$$

Now

$$T_n = \sum_{j=0}^{n} R_j ,$$

so by summing the array (1) along the diagonals we can also write

$$T_n = 1 + \sum_{j=0}^{n-1} B_{n-j}T_j = 1 + \sum_{j=0}^{n} B_{n-j}T_j ,$$

since $B_0 = 0$. Thus

$$T(x) - \frac{1}{1 - x} = T(x)\, B(x) ,$$

so that

(4) $$T(x) = \frac{1}{(1 - x)(1 - B(x))} = \frac{R(x)}{1 - x} ,$$

$$B(x) = 1 - \frac{1}{(1 - x)T(x)} .$$

3. SOME INTERESTING SPECIAL CASES

The original Fibonacci rabbit problem has the birth sequence generating function as

$$B(x) = \frac{x^2}{1 - x} = \sum_{n=0}^{\infty} B_n x^n .$$

Here the new-born rabbit pairs mature for one period and then each pair gives birth to a new rabbit pair at each time point thereafter in their private times. Using (3),

$$R(x) = \frac{1}{1 - \dfrac{x^2}{1-x}} = \frac{1-x}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n-1}x^n ,$$

while equation (4) yields

$$T(x) = \frac{1}{(1-x)\left(1 - \dfrac{x^2}{1-x}\right)} = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1}\, x^n .$$

Thus both the new arrival sequence and the total population sequence are Fibonacci sequences (see the chart in [1], p. 57).

We may also get Fibonacci sequences in other ways. Let

$$B(x) = \frac{x}{1-x^2} .$$

Then

$$R(x) = \frac{1}{1 - \dfrac{x}{1-x^2}} = \frac{1-x^2}{1-x-x^2} = 1 + \sum_{n=0}^{\infty} F_n x^n ,$$

and

$$T(x) = \frac{1+x}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+2}\, x^n .$$

In this birth sequence a rabbit pair produces and rests in alternate time periods.

If, on the other hand,

$$B(x) = x + x^2 ,$$

then

$$R(x) = \frac{1}{1-(x+x^2)} = \sum_{n=0}^{\infty} F_{n+1}\, x^n$$

and

$$T(x) = \frac{1}{(1-x)(1-x-x^2)} = \sum_{n=0}^{\infty} (F_{n+3} - 1)\, x^n .$$

Suppose instead

$$B(x) = \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n .$$

Then

$$R(x) = \frac{(1-x)^2}{1-3x+x^2} = 1 + \sum_{n=0}^{\infty} F_{2n} x^n ,$$

and

$$T(x) = \frac{1-x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n+1} x^n .$$

Suppose we let the pair produce with a birth sequence which is the Fibonacci sequence. Then

$$B(x) = \frac{x}{1-x-x^2} ,$$

$$R(x) = \frac{1-x-x^2}{1-2x-x^2} = 1 + \sum_{n=0}^{\infty} C_n x^n ,$$

where $C_0 = 0$, $C_1 = 1$, and $C_{n+2} = 2C_{n+1} + C_n$ $(n \geq 0)$. We note that if

$$f_0(x) = 0, \quad f_1(x) = 1, \quad \text{and} \quad f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$$

define the sequence of Fibonacci polynomials $\{f_n(x)\}$, then $C_n = f_n(2)$.

There is a typographical error in Weland [6]. When

$$B(x) = \sum_{n=2}^{\infty} \left(1 + \sum_{j=1}^{n} 6C_{j-1} \right) x^n ,$$

then

$$T(x) = \sum_{n=0}^{\infty} F_{n+1}^3 x^n ,$$

where the C_n are the same as in the example immediately above.

4. SOME FURTHER FIBONACCI RESULTS

Since $F_{k-1} + F_{k+1} = L_k$, and every k^{th} Fibonacci number obeys the recurrence relations

$$y_{n+2} = L_k y_{n+1} - (-1)^k y_n ,$$

we can now give the following results. If

$$B(x) = \frac{F_{k+1}x - (-1)^k x^2}{1 - F_{k-1}x} = xF_{k+1} + x^2 F_k^2 \sum_{j=0}^{\infty} F_{k-1}^j x^j ,$$

then

$$R(x) = \frac{1 - F_{k-1}x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn+1} x^n .$$

If

$$B(x) = \frac{F_{k-1}x - (-1)^k x^2}{1 - F_{k+1}x} = xF_{k-1} + x^2 F_k^2 \sum_{j=0}^{\infty} F_{k+1}^j x^j ,$$

then

$$R(x) = \frac{1 - F_{k+1}x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn-1} x^n .$$

If

$$B(x) = \frac{(F_{k+1} - 1)x + (F_{k-1} - (-1)^k)x^2}{(1 - x)(1 - F_{k-1}x)} ,$$

then

$$T(x) = \sum_{n=0}^{\infty} F_{kn+1} x^n .$$

If

$$B(x) = \frac{(F_{k-1} - 1)x + (F_{k+1} - (-1)^k)x^2}{(1 - x)(1 - F_{k+1}x)} ,$$

then

$$T(x) = \sum_{n=0}^{\infty} F_{kn-1} x^n .$$

We conclude this section with two final examples. Suppose the birth sequence is given by $B_0 = B_1 = 0$, $B_n = 2n - 1$ $(n \geq 2)$. Then we find

$$R(x) = \sum_{n=0}^{\infty} F_{n-1} F_{n+2} x^n$$

and

$$T(x) = \sum_{n=0}^{\infty} F_{n+1}^2 x^n .$$

We must not leave out the Lucas birth sequence. If

$$B(x) = \frac{x(2-x)}{1-x-x^2} = \sum_{n=0}^{\infty} L_n x^{n+1} ,$$

then

$$R(x) = \frac{1-x-x^2}{1-3x} = \sum_{n=0}^{\infty} R_n x^n$$

with

$$R_0 = 1, \quad R_1 = 2, \quad R_n = 5 \cdot 3^{n-2} \qquad n \geq 2$$

5. BIRTH SEQUENCES YIELDING GENERALIZED FIBONACCI NUMBERS

The generalized Fibonacci numbers $u(n; p, q)$ of Harris and Styles [2] have the generating function [4]

$$\frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u(n; p, q) x^n .$$

If

$$B(x) = \frac{x^{p+q}}{(1-x)^q} ,$$

then

$$T(x) = \frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} \qquad (q \geq 1) .$$

We note that here the birth sequence $\{B_n\}$ starts with $p + q - 1$ zeros (maturing periods) and then proceeds down the $(q - 1)^{st}$ column of the left-justified Pascal's Triangle [4]. We note further that if

$$B(x) = x + \frac{x^{p+q}}{(1 - x)^{q-1}} ,$$

$$R(x) = \frac{(1 - x)^{q-1}}{(1 - x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u(n;\, p,\, q)\, x^n \quad (q \geq 2) .$$

In this case $B_0 = 0$, $B_1 = 1$, $B_j = 0$ $(j = 2, \cdots, p + q - 1)$, and the sequence then proceeds down the $(q - 2)^{nd}$ column of the left-justified Pascal's Triangle. It was this interesting problem that inspired further research resulting in this paper.

6. A SECOND GENERALIZATION

Harris and Styles [3] gave a further generalization of the Fibonacci numbers by introducing the numbers

$$u(n;\, p,\, q,\, s) = \sum_{i=0}^{\left[\frac{n}{p+sq}\right]} \binom{\left[\frac{n - ip}{s}\right]}{iq} ,$$

where $[x]$ represents the greatest integer contained in x. It is shown in [4] that the generating function for these numbers is

$$\frac{(1 - x^s)^q \big/ (1 - x)}{(1 - x^s)^q - x^{p+sq}} = \sum_{n=0}^{\infty} u(n;\, p,\, q,\, s)\, x^n .$$

If

$$B(x) = \frac{x^{p+sq}}{(1 - x^s)^q} ,$$

then

$$T(x) = \frac{(1-x^s)^q \big/ (1-x)}{(1-x^s)^q - x^{p+sq}} \qquad (p + sq \geq 1) \qquad q \geq 1\,.$$

Therefore the birth sequence yielding $u(n;\ p,\ q,\ s)$ as the total population sequence begins with $p + sq - 1$ zeros (maturing periods) and then has the entries of the $(q-1)^{st}$ column of the left-justified Pascal's Triangle alternated with $s - 1$ zeros. The pair thus alternately produces and then rests for $s - 1$ periods after maturing for $p + sq - 1$ periods.

Note: Lucile Morton has now completed her San Jose State College Master's Thesis, "The Generalized Fibonacci Rabbit Problem," and the results will be written up in a paper to appear soon in the Fibonacci Quarterly.

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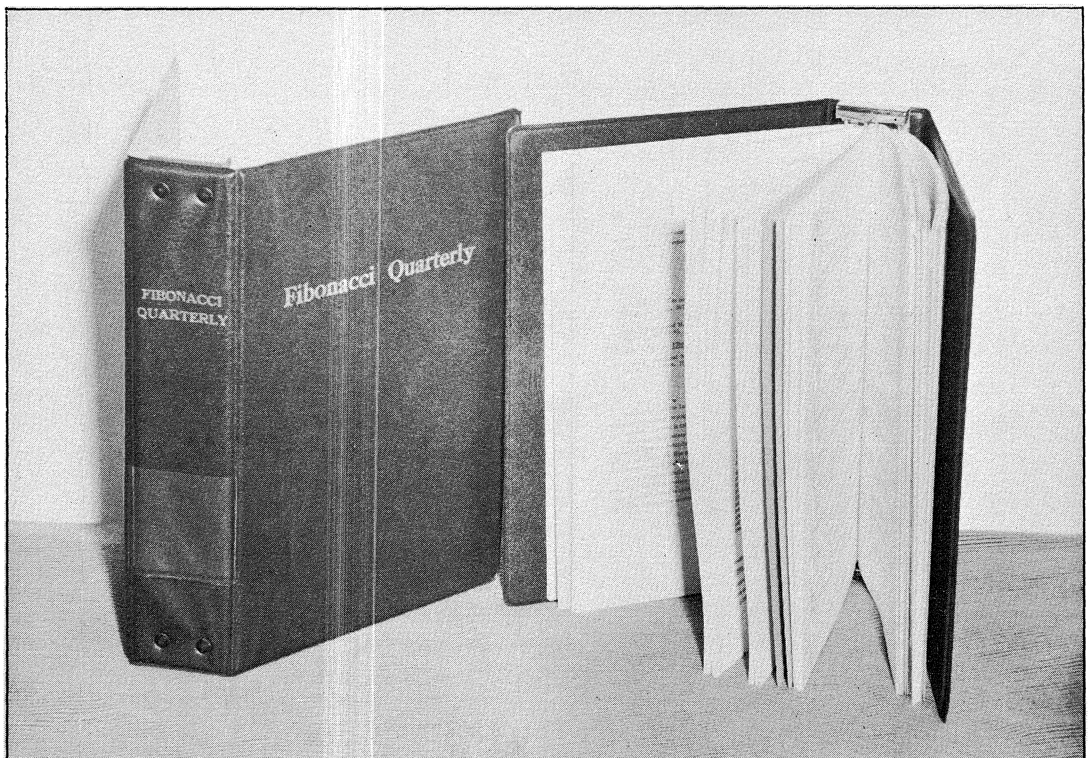

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