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FIBONACCI REPRESENTATIONS

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1. INTRODUCTION

We define the Fibonacci numbers as usual by means of

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1).$$

We shall be concerned with the problem of determining the number of representations of a given positive integer as a sum of distinct Fibonacci numbers. More precisely we define $R(N)$ as the number of representations

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \tag{1.1}$$

where

$$k_1 > k_2 > \cdots > k_r \geq 2; \tag{1.2}$$

the integer r is allowed to vary. We shall refer to (1.1) as a Fibonacci representation of N provided (1.2) is satisfied.

This definition is equivalent to

$$\prod_{n=2}^{\infty} (1 + y^{F_n}) = \sum_{N=0}^{\infty} R(N)\, y^N \tag{1.3}$$

with $R(0) = 1$. We remark that Hoggatt and Basin [4] have discussed a closely related function defined by

$$\prod_{n=1}^{\infty} (1 + y^{F_n}) = \sum_{N=0}^{\infty} R'(N)\, y^N . \tag{1.4}$$

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Comparing (1.4) with (1.3) it is evident that

$$R'(N) = R(N) + R(N-1) \quad . \tag{1.5}$$

Ferns [3] and Klarner [5] have also discussed the problem of representing an integer as a sum of distinct Fibonacci numbers. We recall that by a theorem of Zeckendorf [1] the representation (1.1) is unique provided the k_j satisfy the inequalities

$$\cdots k_j - k_{j+1} \geq 2 \quad (j = 1, \cdots, r-1); \quad k_r \geq 2 \quad . \tag{1.6}$$

We call such a representation the canonical representation of N.

Rather than work directly with R(N) we shall find it convenient to define the function A(m,n) by means of

$$\prod_{n=1}^{\infty} (1 + x^{F_n} y^{F_{n+1}}) = \sum_{m,n=0}^{\infty} A(m,n) x^m y^n \quad . \tag{1.7}$$

It is easily seen that A(m,n) satisfies the recurrence

$$A(m,n) = A(n-m,n) + A(n-m, m-1) \quad . \tag{1.8}$$

Also, as we shall see,

$$R(N) = A(e(N), N) \quad , \tag{1.9}$$

where

$$e(N) = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_r-1} \quad , \tag{1.10}$$

and the k_s are determined by (1.1); the value of e(N) is independent of the particular Fibonacci representation employed. In particular we may assume that the representation (1.1) is canonical. Indeed most of the theorems of the paper make use of the canonical representation.

In particular it follows from (1.9) that for fixed n there is a unique value of m, namely $e(n)$, such that $A(m,n) \neq 0$.

It is helpful to make a short list of exponent pairs occurring in the right member of (1.7), that is, pairs (m,n) such that $A(m,n) \neq 0$. Using the recurrence (1.8) we get the following:

$$
\begin{array}{l}
1\ 1,\ \ 1\ 2 \mid 2\ 3 \mid 3\ 4,\ \ 3\ 5 \mid 4\ 6,\ \ 4\ 7 \mid 5\ 8 \mid \\
6\ 9,\ \ 6\ 10 \mid 7\ 11 \mid 8\ 12,\ \ 8\ 13 \mid 9\ 14,\ \ 9\ 15 \mid 10\ 16 \mid \\
11\ 17,\ \ 11\ 18 \mid 12\ 19,\ \ 12\ 20 \mid 13\ 21 \mid 14\ 22,\ \ 14\ 23 \mid 15\ 24 \mid \\
\cdots\ 16\ 25,\ \ 16\ 26 \mid 17\ 27,\ \ 17\ 28 \mid 18\ 29 \mid 19\ 30,\ \ 19\ 31 \mid 30\ 32 \mid \\
21\ 33,\ \ 21\ 34 \mid 22\ 35,\ \ 22\ 36 \mid 23\ 37 \mid 24\ 38,\ \ 24\ 39 \mid \\
25\ 40,\ \ 25\ 41 \mid 26\ 42 \mid 27\ 43,\ \ 27\ 44 \mid 28\ 45 \mid
\end{array}
$$

This suggests that for given n, there are just one or two values of m such that $A(m,n) \neq 0$. As we shall see, this is indeed the case.

The first main result of the paper is a reduction formula (Theorem 1) which theoretically enables one to evaluate $R(N)$ for arbitrary N. While explicit formulas are obtained for $r = 1, 2, 3$ in a canonical representation, the general case is very complicated. If, however, we assume that all the k_s have the same parity the situation is much more favorable. Indeed if we assume that

$$N = F_{2k_1} + \cdots + F_{2k_r} \qquad (k_1 > \cdots > k_r \geq 1)$$

and put

$$j_s = k_s - k_{s+1} \qquad (s = 1, \cdots, r-1); \quad j_r = k_r \ ,$$

$$f_r = f(j_1, \cdots, j_r) = R(N), \quad S_r = 1 + f_1 + f_2 + \cdots + f_r \ ,$$

then we have

$$S_0 = 1, \quad S_1 = j_1 + 1, \quad S_r = (j_r + 1)S_{r-1} - S_{r-2} \qquad (r \geq 2)$$

In particular if $j_1 = \cdots = j_r = j$ we have

$$S_r = \sum_{2t \leq r} (-1)^t \binom{r-t}{t} (j+1)^{r-2t}$$

Returning to (1.10) we show also that if $k_r > 2$, then $e(N) = \{\alpha^{-1}N\}$, the integer nearest to $\alpha^{-1}N$, where $\alpha = (1 + \sqrt{5})/2$, while for $k_r = 2$, $e(N) = [\alpha^{-1}N] + 1$.

Additional applications of the method developed in this paper will appear later.

Section 2

As noted above, by the theorem of Zeckendorf, the positive N possesses a unique representation

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r} , \tag{2.1}$$

with

$$k_j - k_{j+1} \geq 2 \quad (j = 1, \cdots, r-1); \quad k_r \geq 2 . \tag{2.2}$$

When (2.2) is satisfied we shall call (2.1) the <u>canonical</u> representation of N. Then the set of integers $(k_1, k_2, \cdots, k_r)$ is uniquely determined by N and conversely.

The following lemma will be required.

<u>Lemma</u>. Let

$$N = F_{k_1} + \cdots + F_{k_r} = F_{j_1} + \cdots + F_{j_s} , \tag{2.3}$$

where

$$k_1 > k_2 > \cdots > k_r \geq 2 ; \quad j_1 > j_2 > \cdots > j_s \geq 2 \tag{2.4}$$

be any two Fibonacci representations of N. Then

(2.5) $$F_{k_1-1} + \cdots + F_{k_r-1} = F_{j_1-1} + \cdots + F_{j_s-1} .$$

Proof. The lemma obviously holds for $N = 1$. We assume that it holds up to and including the value $N - 1$. If $k_1 = j_1$ then (2.3) implies

$$F_{k_2} + \cdots + F_{k_r} = F_{j_2} + \cdots + F_{j_s} < N$$

and (2.5) is an immediate consequence of the inductive hypothesis. We may accordingly assume that $k_1 > j_1$. Since

$$\cdots F_2 + F_3 + \cdots + F_n = F_{n+2} - 2 ,$$

we must have $k_1 = j_1 + 1$. If $k_2 = k_1 - 1$ we can complete the induction as in the previous case. If $k_2 = k_1 - 2$, (2.3) implies

(2.6) $$2F_{k_2} + F_{k_3} + \cdots + F_{k_r} = F_{j_2} + \cdots + F_{j_s} ,$$

with $j_2 \leq k_2$. If $j_2 < k_2$,

$$F_{j_2} + \cdots + F_{j_s} \leq F_2 + F_3 + \cdots + F_{k_2-1} < F_{k_2+1} < 2F_{k_2} ,$$

which contradicts (2.6). If $j_2 = k_2$, (2.6) reduces to

$$F_{k_2} + F_{k_3} + \cdots + F_{kr} = F_{j_3} + \cdots + F_{j_s} < N .$$

Then by the inductive hypothesis

(2.7) $$F_{k_2-1} + F_{k_3-1} + \cdots + F_{k_r-1} = F_{j_3-1} + \cdots + F_{j_s-1} .$$

Since $j_1 = k_1 - 1$, $j_2 = k_2 = k_1 - 2$, we have

$$F_{k_1-1} = F_{j_1} = F_{j_1-1} + F_{j_1-2} = F_{j_1-1} + F_{j_2-1} ,$$

so that (2.7) implies (2.5).

Finally there is the possibility $F_{k_2} < F_{k_1} - 2$. In this case (2.3) reduces to

$$F_{k_1-2} + F_{k_2} + \cdots + F_{k_r} = F_{j_2} + \cdots + F_{j_s} = N' < N ; \tag{2.8}$$

each member of (2.8) is a Fibonacci representation of N'. By the inductive hypothesis

$$\cdots F_{k_1-3} + F_{k_2-1} + \cdots + F_{k_r-1} = F_{j_2-1} + \cdots + F_{j_s-1} . \tag{2.9}$$

Since $j_1 - 1 = k_1 - 2$, (2.9) implies

$$F_{k_1} + F_{k_2-1} + \cdots + F_{k_r-1} = F_{j_1-1} + F_{j_2-1} + \cdots + F_{j_s-1}$$

and the induction is complete.

This evidently completes the proof of the lemma.

We now make the following

<u>Definition.</u> Let

$$N = F_{k_1} + \cdots + F_{k_r} \qquad (k_1 > \cdots > k_r \geq 2) \tag{2.10}$$

be any Fibonacci representation of the positive integer N. Then we define

$$e(N) = F_{k_1-1} + \cdots + F_{k_r-1} . \tag{2.11}$$

It is convenient to define

$$e(0) = 0 . \tag{2.12}$$

In view of the lemma it is immaterial which Fibonacci representation of N we use in defining $e(N)$. In particular we may use the canonical representation (2.1).

Section 3

Returning to (1.7) we put

$$\Phi(x, y) = \prod_{n=1}^{\infty} (1 + x^{F_n} y^{F_{n+1}}) . \tag{3.1}$$

Then

$$\Phi(x,xy) = \prod_{n=1}^{\infty} (1 + x^{F_n+F_{n+1}}) = \prod_{n=2}^{\infty} (1 + y^{F_n}x^{F_{n+1}}) ,$$

$$\cdots$$

so that

$$(1 + xy)\Phi(x,xy) = \Phi(y,x) .$$

Hence

$$(1 + xy) \sum_{m,n=0}^{\infty} A(m,n)x^{m+n}y^n = \sum_{m,n=0}^{\infty} A(m,n)y^m x^n .$$

Comparison of coefficients yields

$$A(m,n) = A(n - m, m) + A(n - m, m - 1) , \tag{3.2}$$

the recurrence stated in the Introduction.

In the next place it is clear from the definition of $e(N)$ that (1.3) reduces to

$$\prod_{n=1}^{\infty} (1 + x^{F_n}y^{F_{n+1}}) = \sum_{N=0}^{\infty} R(N)\, x^{e(N)}y^N , \tag{3.3}$$

where $R(N)$ is defined by

$$\prod_{n=2}^{\infty} (1 + y^{F_n}) = \sum_{N=0}^{\infty} R(N)y^N . \tag{3.4}$$

It follows that

$$R(N) = A(e(N), N) \ . \tag{3.5}$$

In particular we see that, for fixed n, there is a unique value of m, namely $e(n)$, such that $A(m,n) \neq 0$.

If we take $m = e(n)$ in (3.2) we get

$$R(N) = A(N - e(N),\ e(N)) + A(N - e(N), e(N) - 1) \ . \tag{3.6}$$

Now let N have the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r} \tag{3.7}$$

with k_r odd. Then

$$e(N) = F_{k_1-1} + \cdots + F_{k_r-1} \ ,$$

$$N - e(N) = F_{k_1-2} + \cdots + F_{k_r-2} \ .$$

Since $k_r \geq 3$, it follows that

$$N - e(N) = e(e(N)) \ . \tag{3.8}$$

On the other hand, since

$$F_3 + F_5 + \cdots + F_{2t-1} = F_{2t} - 1 \ ,$$

we have, for $k_r = 2t + 1$,

$$e(N) - 1 = F_{k_1-1} + \cdots + F_{k_{r-1}} + (F_3 + F_5 + \cdots + F_{2t-1}) \ ;$$

the right member is evidently a Fibonacci representation, so that

$$\begin{aligned} e(e(N) - 1) &= F_{k_1-2} + \cdots + F_{k_{r-1}-2} + (F_2 + F_4 + \cdots + F_{2t-2}) \\ &= F_{k_1-2} + \cdots + F_{k_{r-1}-2} + F_{k_r-2} - 1 \\ &= N - e(N) - 1 \ . \end{aligned}$$

Thus

$$A(N - e(N), e(N(- 1) = 0$$

and (3.6) becomes

$$R(N) = A(e(e(N)), e(N)) .$$

In view of (3.8) we have

$$R(N) = R(e(N)) \quad (k_r \text{ odd}) . \tag{3.9}$$

Now let k_r in the canonical representation of N be even. We shall show that

$$R(N) = R(e^{2t-1}(N_1)) + (t - 1) R(e^{2t-2}(N_1)) , \tag{3.10}$$

where $k_r = 2t$,

$$N_1 = F_{k_1} + \cdots + F_{k_{r-1}} \tag{3.11}$$

and

$$e^t(N) = e(e^{t-1}(N)), \quad e^0(N) = N . \tag{3.12}$$

To prove (3.10) we take the canonical representation (3.7) with $k_r = 2t$. Then

$$e(N) = F_{k_1-1} + \cdots + F_{k_r-1} , \tag{3.13}$$

which is a Fibonacci representation of e(N) except when t = 1. Excluding this case for the moment, we have as above

$$N - e(N) = e(e(N)) . \tag{3.14}$$

Moreover

$$\begin{aligned} e(N) - 1 &= F_{k_1-1} + \cdots + F_{k_{r-1}-1} + F_{2t-1} - 1 \\ &= F_{k_1-1} + \cdots + F_{k_{r-1}-1} + (F_2 + F_4 + \cdots + F_{2t-2}) \ , \end{aligned}$$

$$\begin{aligned} e(e(N) - 1) &= F_{k_1-2} + \cdots + F_{k_{r-1}-2} + (F_1 + F_3 + \cdots + F_{2t-3}) \\ \cdots \qquad &= F_{k_1-2} + \cdots + F_{k_{r-1}-2} + F_{2t-2} \ , \end{aligned}$$

so that

$$e(e(N) - 1) = e(e(N)) \ . \tag{3.15}$$

Substituting from (3.14) and (3.15) in (3.6) we get

$$R(N) = R(e(N)) + R(e(N) - 1) \qquad (k_r = 2t > 2) \ . \tag{3.16}$$

When $k_r = 2$, (3.13) gives

$$N - e(N) = F_{k_1-2} + \cdots + F_{k_{r-1}-2} = e(e(N_1)) \ ,$$

$$e(N) - 1 = F_{k_1-1} + \cdots + F_{k_{r-1}-1} = e(N_1) \ .$$

Also since

$$e(N) = F_{k_1-1} + \cdots + F_{k_{r-1}-1} + F_2 \ ,$$

we get

$$\begin{aligned} e(e(N)) &= F_{k_1-2} + \cdots + F_{k_{r-1}-2} + F_1 \\ &= N - e(N) + 1 \ . \end{aligned}$$

It therefore follows from (3.5) and (3.6) that

$$R(N) = R(e(N_1)) \qquad (k_R = 2) \ , \tag{3.17}$$

in agreement with (3.10).

Returning to (3.16) we have first

$$R(e(N)) = R(e^2(N)) \qquad (k_r = 2t > 2) , \tag{3.18}$$

by (3.9). Since

$$e(N) - 1 = F_{k_1-1} + \cdots + F_{k_{r-1}-1} + (F_2 + F_4 + \cdots + F_{2t-2}) ,$$

it follows by repeated application of (3.17) and (3.9) that

$$\begin{aligned} \cdots R(e(N) - 1) &= R(F_{k_1-2} + \cdots + F_{k_{r-1}-2} + F_3 + \cdots + F_{2t-3}) \\ &= R(F_{k_1-3} + \cdots + F_{k_{r-1}-3} + F_2 + \cdots + F_{2t-4}) \\ &= R(F_{k_1-2t+2} + \cdots + F_{k_{r-1}-2t+2}) \\ &= R(e^{2t-2}(N_1)) . \end{aligned}$$

Thus (3.16) becomes

$$R(N) = R(e^2(N)) + R(e^{2t-2}(N_1)) \qquad (t > 1) . \tag{3.19}$$

Repeated use of (3.19) gives

$$R(N) = R(e^{2t-2}(N)) +)t - 1)R(e^{2t-2}(N_1)) ;$$

finally, applying (3.17), we get (3.10) .

Combining (3.9) and (3.10) we state the following principal result.

<u>Theorem 1.</u> Let N have the cannonical representation

$$N = F_{k_1} + \cdots + F_{k_r} ,$$

where

$$k_j - k_{j+1} \geq 2 \qquad (j = 1, \cdots, r - 1); \; k_r \geq 2 .$$

Then

(3.20) $$R(N) = R(e^{k_{r-1}}(N_1)) + ([\tfrac{1}{2}k_r] - 1)R(e^{k_{r-2}}(N_1)),$$

where

$$N_1 = F_{k_1} + \cdots + F_{k_{r-1}}.$$

Section 4

Since

$$F_2 + F_4 + \cdots + F_{2t} = F_{2t+1} - 1, \quad F_1 + F_3 + \cdots + F_{2t-1} = F_{2t},$$

it follows that

(4.1) $$e(F_{2t+1} - 1) = F_{2t}, \quad e(F_{2t} - 1) = F_{2t-1} - 1.$$

Also since

$$F_{2t+1} - 2 = F_4 + F_6 + \cdots + F_{2t},$$

$$F_{2t} - 2 = F_2 + F_5 + F_7 + \cdots + F_{2t-1},$$

we get

(4.2) $$e(F_{2t+1} - 2) = F_{2t} - 1, \quad e(F_{2t} - 2) = F_{2t-1} - 1.$$

Now by (3.6), for $k \geq 2$,

$$\begin{aligned} R(F_k) &= A(F_{k-2}, F_{k-1}) + A(F_{k-2}, F_{k-1} - 1) \\ &= R(F_{k-1}) + A(F_{k-2}, F_{k-1} - 1), \end{aligned}$$

$$\begin{aligned} R(F_k - 1) = A(F_k - 1 - e(F_k - 1), e(F_k - 1)) &+ A(F_k - 1 - e(F_k - 1), \\ &\quad e(F_k - 1) - 1). \end{aligned}$$

Then by (4.1),

$$A(F_{2t-2},\ F_{2t-1} - 1) = R(F_{2t-1} - 1),\quad A(F_{2t-1},\ F_{2t} - 1) = 0\ ,$$

so that

$$R(F_{2t}) = R(F_{2t-1}) + R(F_{2t-1} - 1),\quad R(F_{2t-1}) = R(F_{2t-2})\ . \tag{4.3}$$

In the next place by (4.1) and (4.3)

$$\begin{aligned} R(F_{2t} - 1) &= A(F_{2t-2}, F_{2t-1} - 1) + A(F_{2t-2}, F_{2t-1} - 2) \\ &= R(F_{2t-1} - 1)\ , \end{aligned}$$

$$\begin{aligned} R(F_{2t-1} - 1) &= A(F_{2t-3} - 1,\ F_{2t-2}) + A(F_{2t-3} - 1,\ F_{2t-2} - 1) \\ &= R(F_{2t-2} - 1\ ). \end{aligned}$$

Hence we have

$$R(F_k - 1) = R(F_{k-1} - 1) \qquad (k \geq 2)\ , \tag{4.4}$$

which yields

$$R(F_k - 1) = 1 \qquad (k \geq 2)\ . \tag{4.5}$$

Substituting from (4.5) in (4.3), we get

$$R(F_{2t}) = R(F_{2t-1}) + 1,\quad R(F_{2t-1}) = R(F_{2t-2})\ ,$$

which implies

$$R(F_{2t}) = R(F_{2t+1}) = t \qquad (t \geq 1)\ . \tag{4.6}$$

We shall now show that $R(N) = 1$ implies $N = F_k - 1$. Let N have the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r}\ .$$

Then by (3.20)

$$R(e^{k_{r-1}}(N_1)) = 1 \tag{4.7}$$

and $[k_r/2] = 1$, so that $k_r = 2$ or 3. Since

$$e^{k_{r-1}}(N_1) = F_{k_1-k_r+1} + \cdots + F_{k_{r-1}-k_r+1} ,$$

it is necessary that

$$[(k_{r-1} - k_r + 1)/2] = 1$$

and therefore

$$k_{r-1} - k_r = 2 .$$

Similarly

$$k_j - k_{j-1} = 2 \qquad (j = 1, 2, \cdots, r-2) .$$

Hence we have either

$$N = F_{2r} + F_{2r-2} + \cdots + F_2 = F_{2r+1} - 1$$

or

$$N = F_{2r+1} + F_{2r-1} + \cdots + F_3 = F_{2r+2} - 1 .$$

We may sum up the results just obtained in the following theorems.

<u>Theorem 2.</u> We have

$$R(F_k) = [\tfrac{1}{2}k] \qquad (k \geq 2) \tag{4.8}$$

<u>Theorem 3.</u> $R(N) = 1$ if and only if

$$N = F_k - 1, \quad k \geq 1 .$$

If we define $R'(N)$ by means of

$$\prod_{n=1}^{\infty}(1 + y^{F_n}) = \sum_{N=0}^{\infty} R'(N)y^N , \tag{4.9}$$

then

$$R'(N) = R(N) + R(N - 1) \tag{4.10}$$

and it follows immediately that

$$R'(F_k) = [\tfrac{1}{2} k] + 1 \quad (k \geq 2) . \tag{4.11}$$

This result has been proved by Hoggatt and Basin [4].

Further results like (4.5) and (4.8) can be obtained by the same method. For example we can show that

$$R(F_{2t+1} - 2) = 1 + R(F_{2t} - 2) \qquad (t > 1) ,$$

$$R(F_{2t} - 2) = R(F_{2t-1} - 2) \qquad (t > 1) .$$

It follows that

$$R(F_k - 2) = [\tfrac{1}{2}(k - 1)] \quad (k \geq 3) . \tag{4.12}$$

Consequently by (4.11) we have

$$R'(F_k - 1) = [\tfrac{1}{2}(k + 1)] , \tag{4.13}$$

a result proved by Klarner [5, Th. 1].

Section 5

Theorem 1 furnishes a reduction formula by means of which $R(N)$ can be computed by arbitrary N. For example if

(5.1) $$N = F_j + F_k \qquad (j - k \geq 2, \quad k \geq 2)$$

that by (3.20)

$$\begin{aligned} R(N) &= R(e^{k-1}(F_j)) + ([\tfrac{1}{2}k] - 1)R(e^{k-2}(F_j)) \\ &= R(F_{j-k+1}) + ([\tfrac{1}{2}k] - 1)\, R(F_{j-k+2}) . \end{aligned}$$

Applying (4.8) we get

(5.2) $$R(N) = [\tfrac{1}{2}(j - k + 1)] + ([\tfrac{1}{2}k] - 1)[\tfrac{1}{2}(j - k + 2)] .$$

Again if

(5.3) $$N = F_i + F_j + F_k \qquad (i - j \geq 2,\ j - k \geq 2,\ \ k \geq 2),$$

then

$$R(N) = R(F_{i-k+1} + F_{j-k+1}) + ([\tfrac{1}{2}k] - 1)R(F_{i-k+2} + F_{j-k+2}) .$$

Applying (5.2) we get

(5.4) $$\begin{aligned} R(N) = {} & [\tfrac{1}{2}(i - j + 1)] + ([\tfrac{1}{2}(j - k + 1)] - 1)\,[\tfrac{1}{2}(i - j + 2)] \\ & + ([\tfrac{1}{2}k] - 1)\{[\tfrac{1}{2}(i - j + 1)] + ([\tfrac{1}{2}[j - k + 2] - 1)[\tfrac{1}{2}(i - j + 2)]\} . \end{aligned}$$

Unfortunately, for general N the final result is very complicated. However (5.2) and (5.4) contain numerous special cases of interest.

In the first place, taking $k = 2, 3, 4$ in (5.2), we get

(5.5) $$R(F_j + 1) = [\tfrac{1}{2}(j - 1)] \qquad (j \geq 4)$$

(5.6) $$R(F_j + 2) = [\tfrac{1}{2}(j - 2)] \qquad (j \geq 5)$$

(5.7) $$R(F_j + 3) = [\tfrac{1}{2}(j - 3)] + [\tfrac{1}{2}(j - 2)] \quad (j \geq 6) .$$

In the next place for the Lucas number L_k defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{k+1} = L_k + L_{k-1} \qquad (k \geq 1) \ ,$$

since $L_k = F_{k+1} + F_{k-1}$, (5.2) gives

$$R(L_k) = 1 + 2[\tfrac{1}{2}(k-3)] \qquad (k \geq 3) \ .$$

Hence

$$\text{(5.8)} \qquad \begin{aligned} R(L_{2k+1}) &= 2k - 1 && (k \geq 1) \\ R(L_{2k}) &= 2k - 3 && (k > 1) \ . \end{aligned}$$

Since

$$2F_k = F_{k+1} + F_{k-2} \ , \qquad 3F_k = F_{k+2} + F_{k-2} \ ,$$

we get

$$\text{(5.9)} \qquad R(2F_k) = 2 + 2[\tfrac{1}{2}(k-4)] \qquad (k \geq 4) \ ,$$

$$\text{(5.10)} \qquad R(3F_k) = 2 + 3[\tfrac{1}{2}(k-4)] \qquad (k \geq 4) \ .$$

The identity

$$L_{2j} F_k = F_{k+2j} + F_{k-2j}$$

yields

$$\text{(5.11)} \qquad R(L_{2j} F_k) = 2j + (2j+1)([\tfrac{1}{2}k] - j - 1) \qquad (k \geq 2j+2) \ ;$$

for $j = 1$, (5.11) reduces to (5.10).

A few applications of (5.4) may be noted. For $k = 2$ we have

$$\text{(5.12)} \qquad R(F_1 + F_j + 2) = [\tfrac{1}{2}(i-j+1)] + [\tfrac{1}{2}(j-3)][\tfrac{1}{2}(i-j+2)] \ (i-j \geq 2, j \geq 4).$$

while for $k = 3$ we have

$$(5.13)\quad R(F_i + F_j + 2) = [\tfrac{1}{2}(i-j+1)] + [\tfrac{1}{2}(j-4)][\tfrac{1}{2}(i-j+2)] \quad (i \geq j \geq 2,\ j \geq 5)\ .$$

Again, since

$$4F_k = F_{k+2} + F_k + F_{k-2}\ ,$$

it follows that

$$(5.14)\qquad R(4F_k) = 1 + 3[\tfrac{1}{2}(k-4)] \qquad (k \geq 4)\ .$$

Section 6

As remarked above, direct application of Theorem 1 leads to very complicated results for $R(N)$. If, however, all the k_s in the canonical representation of N have the same parity simpler results can be obtained. If all the k_s are odd then by (3.9),

$$(6.1)\qquad R(F_{k_1} + \cdots + F_{k_r}) = R(F_{k_1-1} + \cdots + F_{k_r-1})\ ;$$

we may therefore assume that all the k_s are even.

It will be convenient to introduce the following notation. Put

$$(6.2)\qquad N = F_{2k_1} + \cdots + F_{2k_r}\ ,$$

where

$$(6.3)\qquad k_1 > k_2 > \cdots > k_r \geq 1\ ;$$

also put

$$(6.4)\qquad j_s = k_s - k_{s-1} \quad (s = 1, \cdots, r-1)\ ; \qquad j_r = k_r$$

and

(6.5) $$f_r = f(j_1, \cdots, j_r) = R(N) ,$$

where N is defined by (6.2).

Now by (3.20) and (3.9)

$$R(N) = R(F_{2k_1-2k_r} + \cdots + F_{2k_{r-1}-2k_r}) + (k_r - 1)R(F_{2k_1-2k_r+2} + \cdots + F_{2k_{r-1}-2k_r+2}) .$$

By (6.4) and (6.5) this reduces to

(6.6) $$f(j_1, \cdots, j_r) = f(j_1, \cdots, j_{r-1}) + (j_r - 1)f(j_1, \cdots, j_{r-2}, j_{r-1} + 2) .$$

By (3.19) we have

$$R(F_{2k_1-2k_r+2} + \cdots + F_{2k_{r-1}-2k_r+2}) = R(F_{2k_1-2k_r} + \cdots + F_{2k_{r-1}-2k_r}) + R(F_{2k_1-2k_{r-1}+2} + \cdots + F_{2k_{r-2}-2k_{r-1}+2}) ,$$

so that

$$f(j_1, \cdots, j_{r-2}, j_{r-1} + 2) = f(j_1, \cdots, j_{r-1}) + f(j_1, \cdots, j_{r-3}, j_{r-2} + 2)$$
$$= f(j_1, \cdots, j_{r-1}) + f(j_1, \cdots, j_{r-2}) + \cdots + f(j_1) + 1 .$$

Thus (6.6) reduces to

(6.7) $$f_r = f_{r-1} + (j_r - 1)(f_{r-2} + \cdots + f_1 + 1) .$$

If we define

(6.8) $$S_r = f_r + f_{r-1} + \cdots + f_1 + 1, \quad S_0 = 1 ,$$

then (6.7) becomes

$$f_r - f_{r-1} = (j_r - 1)S_{r-2} \qquad (r \geq 2)$$

and therefore

$$S_r - (j_r + 1)\, S_{r-1} + S_{r-2} = 0 \qquad (r \geq 2)\,. \tag{6.9}$$

We may now state

Theorem 4. With the notation (6.2), (6.3), (6.4), (6.5), $f_r = R(N)$ is determined by means of (6.9) with $S_0 = 1$, $S_1 = j_1 + 1$ and

$$f_r = S_r - S_{r-1}\,.$$

The first few values of S_r are given by

$$S_0 = 1,\quad S_1 = j_1 + 1,\quad S_2 = j_1j_2 + j_1 + j_2,\quad S_3 = j_1j_2j_3 + j_1j_2 + j_1j_3 + j_2j_3 + j_2 - 1.$$

It is evident that $S_r = S(j_1, \cdots, j_r)$ is a polynomial in $j_1, \cdots, j_r$; indeed it is a continuant [1, vol. 2, p. 494].

We have for example

$$S_r = \begin{vmatrix} j_1 + 1 & -1 & 0 & \cdots & 0 \\ -1 & j_2 + 1 & -1 & \cdots & 0 \\ 0 & -1 & j_3 + 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & j_r + 1 \end{vmatrix}$$

and

$$S_r(j_1, j_2, \cdots, j_r) = S(j_r, j_{r-1}, \cdots, j_1)\,.$$

The latter formula implies

$$R(F_{2k_1} + \cdots + F_{2k_r}) = R(F_{2k_1'} + \cdots + F_{2k_r'})\,, \tag{6.10}$$

where

$$k_1' = k_r, \quad k_2' = k_1 - k_r, \quad k_3' = k_1 - k_{r-1}, \cdots, \quad k_r' = k_1 - k_2 .$$

When

$$j_1 = j_2 = \cdots = j_r = j \quad , \tag{6.11}$$

we can obtain a simple explicit formula for S_r. Since in this case

$$S_r - (j+1)S_{r-1} + S_{r-2} = 0 \quad (r \geq 2) , \quad S_0 = 1, \quad S_1 = j+1 \quad ,$$

we find that

$$\sum_{r=0}^{\infty} S_r x^r = (1 - (j+1)x + x^2)^{-1} = \sum_{s=0}^{\infty} x^s (j+1-x)^s$$

$$= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^t \binom{s}{t} (j+1)^{s-t} x^{s+t} \quad ,$$

which gives

$$\cdots \qquad S_r = \sum_{2t \leq r}^{\infty} (-1)^t \binom{r-t}{t} (j+1)^{r-2t} \quad . \tag{6.12}$$

In particular, for $j = 1$, (6.12) reduces to

$$S_r = r + 1 \qquad (j = 1) \quad . \tag{6.13}$$

For certain applications it is of interest to take

$$j_1 = \cdots = j_{r-1} = j, \qquad j_r = k \quad . \tag{6.14}$$

Then $S_1, \cdots, S_{r-1}$ are given by (6.12) while

$$S'_r = (k+1)\, S_{r-1} - S_{r-2} \quad , \tag{6.15}$$

where $S'_r = S(j, \cdots, j, k)$. It follows from (6.15) that

$$f'_r = f(j, \cdots, j, k) = k\, S_{r-1} - S_{r-2} \, . \tag{6.16}$$

In view of the identity

$$L_{2j+1} F_{2k} = F_{2k+2j} + F_{2k+2j-2} + \cdots + F_{2k-2j}$$

we get, using (6.13) and (6.16),

$$R(L_{2j+1} F_{2k}) = (k-j)(2j+1) - 2j \quad (k > j) \, . \tag{6.17}$$

For $k = j$ we have

$$R(L_{2j+1} F_{2j}) = 1 \, . \tag{6.18}$$

Note that

$$L_{2j+1} F_{2j} = F_{4j+1} - 1, \quad L_{2j-1} F_{2j} = F_{4j-1} - 1 \, .$$

When $j = 2$, we have

$$\sum_{r=0}^{\infty} S_r x^{r+1} = \frac{x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n} x^n \quad ,$$

so that

$$S_r = F_{2r+2} \, . \tag{6.19}$$

We now recall the identities

$$F_4 + F_8 + \cdots + F_{4n} = F_{2n+1}^2 - 1 \qquad (n \geq 1),$$

$$F_2 + F_6 + \cdots + F_{4n-2} = F_{2n}^2 \qquad (n \geq 1),$$

$$F_3 + F_7 + \cdots + F_{4n-1} = F_{2n}F_{2n+1} \qquad (n \geq 1),$$

$$F_1 + F_5 + \cdots + F_{4n-3} = F_{2n}F_{2n-1} \qquad (n \geq 1).$$

It follows readily, using (6.16) and (6.19) that

$$R(F_{2n+1}^2 - 1) = F_{2n+1} \qquad (n \geq 0), \tag{6.20}$$

$$R(F_{2n}^2) = F_{2n-1} \qquad (n \geq 1), \tag{6.21}$$

$$R(F_{2n}F_{2n+1}) = F_{2n-1} \qquad (n \geq 1), \tag{6.22}$$

$$R(F_{2n}F_{2n-1}) = F_{2n-1} \qquad (n \geq 1). \tag{6.23}$$

$$R(F_{2n+1}^2 - 2) = F_{2n} \qquad (n \geq 1), \tag{6.24}$$

$$R(F_{2n}^2 - 1) = F_{2n} \qquad (n \geq 1), \tag{6.25}$$

$$R(F_{2n}F_{2n+1} - 1) = F_{2n} \qquad (n \geq 1), \tag{6.26}$$

$$R(F_{2n}F_{2n-1} - 1) = F_{2n-1}. \tag{6.27}$$

$$\cdots$$

Combining (6.20) with (6.24), and so on, we get

$$R'(F_{2n-1}^2 - 1) = F_{2n} \qquad (n \geq 1), \tag{6.28}$$

$$R'(F_{2n}^2) = F_{2n+1} \qquad (n \geq 0), \tag{6.29}$$

$$R'(F_{2n}F_{2n+1}) = F_{2n+1} \qquad (n \geq 0), \tag{6.30}$$

$$R'(F_{2n}F_{2n-1}) = 2F_{2n-1} \qquad (n \geq 1). \tag{6.31}$$

We have also

$$R(F_{2n}^2 - 2) = F_{2n-2} \qquad (n \geq 1)\,, \tag{6.32}$$

$$R(F_{2n+1}^2) = F_{2n-1} \qquad (n \geq 1)\,, \tag{6.33}$$

so that

$$R'(F_{2n}^2 - 1) = L_{2n-1} \qquad (n \geq 1)\,, \tag{6.34}$$

$$R'(F_{2n+1}^2) = L_{2n} \qquad (n \geq 0)\,. \tag{6.35}$$

Several of these results were obtained in [4].

In a similar way one can also prove the following formulas.

$$R(F_{2n}F_{2m}) = R(F_{2n+1}F_{2m}) = (n - m)F_{2m} + F_{2m-1} \qquad (n \geq m)\,, \tag{6.36}$$

$$R(F_{2n}F_{2m+1}) = R(F_{2n+1}F_{2m+1}) = (n - m)F_{2m+1} \qquad (n > m)\,. \tag{6.37}$$

Section 7

We shall now prove

<u>Theorem 5.</u> Let N have the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r}\,. \tag{7.1}$$

Then $e(N + 1) = e(N)$ if and only if $k_r = 2$.

<u>Proof.</u> Take $k_r = 2$. Then

$$N + 1 = F_{k_1} + \cdots + F_{k_{r-1}} + F_3\,,$$

so that

$$e(N + 1) = F_{k_1-1} + \cdots + F_{k_{r-1}-1} + F_2\,.$$

Since

$$e(N) = F_{k_1-1} + \cdots + F_{k_{r-1}-1} + F_1 ,$$

it follows that $e(N + 1) = e(N)$.

Now take $k_r > 2$. Then

$$N + 1 = F_{k_1} + \cdots + F_{k_r} + F_2$$

and

$$e(N + 1) = F_{k_1-1} + \cdots + F_{k_r-1} + 1 .$$

But

$$e(N) = F_{k_1-1} + \cdots + F_{k_r-1} < e(N + 1) .$$

This completes the proof of the theorem.

If N is defined by (7.1) then

$$M = F_{k_1+1} + \cdots + F_{k_r+1}$$

satisfies $e(M) = N$. Moreover, by the last theorem, if $k_r = 2$ then also $e(M - 1) = N$.

Consider

$$N + 1 = F_{k_1+1} + \cdots + F_{k_r+1} + F_2 .$$

Clearly

$$e(M + 1) = F_{k_1} + \cdots + F_{k_r} + 1 = N + 1 .$$

Also, since $F_3 = 2$, we have

$$M - 2 = F_{k_1+1} + \cdots + F_{k_{r-1}} + 1 ,$$

$$e(M - 2) = F_{k_1} + \cdots + F_{k_{r-1}} = N - 1 .$$

It follows that one can have at most two consecutive numbers N, $N + 1$, such that $e(N) = e(N + 1)$. This justifies the assertion about $A(m,n)$ in the introduction.

Section 8

Put

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

Then it is easily verified that

$$\alpha^{-1}F_n = F_{n-1} - \beta^n. \tag{8.1}$$

Hence if N has the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r},$$

it follows that

$$e(N) - \alpha^{-1} N = \beta^{k_1} + \beta^{k_2} + \cdots + \beta^{k_r}. \tag{8.2}$$

Consequently

$$\begin{aligned} \left| e(N) - \alpha^{-1}N \right| &\le \alpha^{-k_1} + \alpha^{-k_2} + \cdots + \alpha^{-k_r} \\ &\le \alpha^{-2} + \alpha^{-4} + \cdots + \alpha^{-2r} \\ &< \frac{\alpha^{-2}}{1 - \alpha^{-2}} + \frac{1}{\alpha^2 - 1} = \frac{1}{\alpha} < 0.62. \end{aligned}$$

If we put

$$\alpha^{-1}N = [\alpha^{-1}N] + \in \qquad (0 < \in < 1),$$

where $[\alpha^{-1}N]$ denotes the greatest integer $\le \alpha^{-1}N$, then

$$-0.62 < e(N) - [\alpha^{-1}N] - \in < 0.62 .$$

This implies

$$[\alpha^{-1}N] \leq e(N) \leq [\alpha^{-1}N] + 1 . \tag{8.3}$$

If $k_r \geq 3$ it follows from (8.2) that

$$\left|e(N) - \alpha^{-1}N\right| \leq \alpha^{-3} + \alpha^{-5} + \cdots + \alpha^{-2r-1}$$

$$< \frac{\alpha^{-3}}{1 - \alpha^{-2}} = \frac{1}{\alpha(\alpha^2 - 1)} = \frac{1}{\alpha^2} < \frac{1}{2}$$

and therefore

$$e(N) = \{\alpha^{-1}N\} \qquad (k_r > 2) , \tag{8.4}$$

where $\{\alpha^{-1}N\}$ denotes the integer nearest to $\alpha^{-1}N$.

Thus the value of $e(N)$ is determined by (8.4) except possibly when $k_r = 2$. Now when $k_r = 2$ we have as above

$$\cdots e(N) - \alpha^{-1}N \geq \alpha^{-2} - \alpha^{-5} - \alpha^{-7} - \cdots - \alpha^{-2r-1} > \alpha^{-2} - \frac{\alpha^{-5}}{1 - \alpha^{-2}}$$

$$= \frac{1}{\alpha^2} - \frac{1}{\alpha^3(\alpha^2 - 1)} = \frac{1}{\alpha^2} - \frac{1}{\alpha^4} = \frac{1}{\alpha^3} > 0 ,$$

so that

$$0 < e(N) - \alpha^{-1}N < 0.62 .$$

It therefore follows that

$$e(N) = [\alpha^{-1}N] + 1 \qquad (k_r = 2) . \tag{8.5}$$

We may now state

Theorem 6. Let N have the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r} .$$

Then if $k_r > 2$,

$$e(N) = \{\alpha^{-1}N\} , \tag{8.6}$$

the integer nearest $\alpha^{-1}N$; if $k_r = 2$,

$$e(N) = [\alpha^{-1}N] + 1 . \tag{8.7}$$

We remark that (8.6) and (8.7) overlap. For example for

$$N = 6 = F_5 + F_2, \quad e(6) = F_4 + F_1 = 4, \quad [6\alpha^{-1}] = [3.72] = 3,$$
$$\{6\alpha^{-1}\} = \{3.72\} = 4.$$

However for

$$N = 25 = F_8 + F_4 + F_2, \quad e(25) = F_7 + F_3 + F_1 = 16, \quad [25\alpha^{-1}] = 15,$$
$$\{25\alpha^{-1}\} = 15 .$$

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A NEW ANGLE ON PASCAL'S TRIANGLE

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1. INTRODUCTION

There has always been such interest in the numbers in Pascal's arithmetic triangle. The sums along the horizontal rows are the powers of two, while the sums along the rising diagonals are the Fibonacci numbers. An early paper by Melvin Hochster [6] generalized the Fibonacci number property by using the left-justified Pascal Triangle and taking other diagonal sums, the first summand being a one on the left edge and subsequent summands are obtained by moving p units up and q units to the right until one is out of the triangle. Unfortunately, he required that $(p,q) = 1$. Harris and Styles [4] produced a generalization of these concepts, and yet a further generalization [5]. We present here a simplifying principle which will make the study of generalizations such as those of Lind [8] easier.

2. COLUMN GENERATORS

Consider the columns of binomial coefficients in the left-justified Pascal Triangle shown in (1). The generating functions for these columns of

$$
(1)\qquad \left\{\begin{array}{ccccc}
1 & & & & \\
1 & 1 & & & \\
1 & 2 & 1 & & \\
1 & 3 & 3 & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \\
\dfrac{1}{(1-x)} & \dfrac{x}{(1-x)^2} & \dfrac{x^2}{(1-x)^3} & \dfrac{x^3}{(1-x)^4} & \text{Column Generator} \\
0^{th} & 1^{st} & 2^{nd} & 3^{rd} & \text{Column}
\end{array}\right.
$$

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coefficients, indicated in (1), are given by the corresponding Maclaurin series. That is,

$$\frac{1}{(1-x)} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \binom{n}{0} x^n$$

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n = \sum_{n=0}^{\infty} \binom{n}{1} x^n$$

$$\frac{x^2}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n = \sum_{n=0}^{\infty} \binom{n}{2} x^n$$

$$\vdots$$

$$\frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n ,$$

where we have used the usual convention that

$$\binom{n}{k} = 0$$

for $n < k$. We should note that the column generators

$$g_k(x) = \frac{x^k}{(1-x)^{k+1}} \quad (k = 0, 1, 2, \cdots)$$

automatically align the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{n}$$

as the coefficients of x^n. Using the above generators, the generating function for the sum of the binomial coefficients across the nth row of Pascal's Triangle is

$$G(x) = \sum_{k=0}^{\infty} g_k(x) = \frac{1}{1-x} \sum_{k=0}^{\infty} \left(\frac{x}{1-x}\right)^k = \frac{1}{(1-x)\left(1 - \frac{x}{1-x}\right)}$$

$$= \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n .$$

This yields the familiar identity

$$\sum_{j=0}^{\infty} \binom{n}{j} = 2^n .$$

If, on the other hand, we multiply each generating function $g_k(x)$ by λ^k and sum again, we find

$$G(x, \lambda) = \sum_{k=0}^{\infty} \lambda^k g_k(x) = \frac{1}{1-x} \sum_{k=0}^{\infty} \left(\frac{\lambda x}{1-x}\right)^k$$

$$= \frac{1}{1-(1+\lambda)x} = \sum_{n=0}^{\infty} (1+\lambda)^n x^n ,$$

Thus by equating coefficients of x^n in each representation we get

$$\sum_{j=0}^{n} \binom{n}{j} \lambda^n = (1+\lambda)^n .$$

If we multiply the generating function $g_k(x)$ by appropriate powers of x, this allows us to vertically shift the separate columns, aligning the numbers along certain diagonals in a horizontal row.

3. THE RISING DIAGONAL SUMS

If we wish to sum the numbers along the rising diagonals, we modify the column generators to be

$$g_k^\star(x) = \frac{x^{2k}}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n-k}{k} x^n .$$

The diagonal sums, derived from (1), are displayed with appropriate column generating functions in (2). We now obtain a generating function

$$(2)\quad \left\{\begin{array}{lllll}
1 & 1 & & & \\
1 & 1 & & & \\
2 & 1 & 1 & & \\
3 & 1 & 2 & & \\
5 & 1 & 3 & 1 & \\
8 & 1 & 4 & 3 & \\
13 & 1 & 5 & 6 & 1 \\
\vdots & \vdots & & \vdots & \\
F_n & \binom{n}{0} & \binom{n-1}{1} & \binom{n-2}{2} & \binom{n-3}{3} \quad \cdots \\
 & \dfrac{1}{1-x} & \dfrac{x^2}{(1-x)^2} & \dfrac{x^4}{(1-x)^3} & \cdots \quad \text{Column Generators}
\end{array}\right.$$

for the sums of the nth row,

$$G(x) = \sum_{k=0}^{\infty} g_k^\star(x) = \frac{1}{1-x} \sum_{k=0}^{\infty} \left(\frac{x^2}{1-x}\right)^k = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n ,$$

a well known result.

4. GENERALIZED FIBONACCI NUMBERS

We now turn to the first generalization of the Fibonacci numbers due to Hochster [6] and Harris and Styles [4]. These numbers are given by

$$u(n;p,q) = \sum_{i=0}^{\left[\frac{n}{p+q}\right]} \binom{n-ip}{iq} \qquad (n \geq 0),$$

where $[x]$ denotes the greatest integer $\leq x$. In particular, $u(n;1,1) = F_{n+1}$ and $u(n;0,1) = 2^n$. To get these sums from the left-adjusted Pascal Triangle we form sums beginning with the $(n+1)^{st}$ one in the leftmost column and add all the coefficients obtained by moving p units up and q units to the right until out of the Triangle. The column generators which yield such summands in a horizontal line are

$$g_k(x) = \frac{x^{k(p+q)}}{(1-x)^{kq+1}} .$$

Thus

$$G(x) = \sum_{k=0}^{\infty} g_k(x) = \frac{1}{1-x} \sum_{k=0}^{\infty} \left\{\frac{x^{p+q}}{(1-x)^q}\right\}^k$$

(3)

$$= \frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u(n;p,q)x^n \quad (p+q \geq 1;\ q \geq 0).$$

This generating function was not given in [4], but is a special case of one given in [9]. We note that in (3) p may be negative. If $p = 1$ and $q = 1$, then (3) becomes

$$\frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n ,$$

while for $q = 2$ and $p = -1$ we have

$$\frac{1 - x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n+1} x^n ,$$

so that there are also Fibonacci numbers in the falling diagonals.

5. A FURTHER GENERALIZATION

In a new paper [5], Harris and Styles consider Pascal's Triangle with each row repeated s times. The column generators for the new array can be easily obtained. The column generator

$$g_k(x) = \frac{1}{(1 - x)^{k+1}}$$

generates the coefficients in the kth column of a left-adjusted Pascal Triangle, and

$$h_k(x) = \frac{1}{(1 - x^s)^{k+1}}$$

has the same coefficients as $g_k(x)$, except each nonzero entry is separated by $s - 1$ consecutive zeros. We can modify the $h_k(x)$ to duplicate each nonzero entry s times by multiplying it by $1 + x + x^2 + \cdots + x^{s-1}$. Thus

$$h_k^{\star}(x) = \frac{1 + x + \cdots + x^{s-1}}{(1 - x^s)^{k+1}} = \frac{1}{(1 - x)(1 - x^s)^k} .$$

To align the coefficients of like powers of x requires

$$g_k^\star(x) = \frac{x^{ks}}{(1-x)(1-x^s)^k} .$$

More generally, if we are interested in summing as before by going along rising diagonals in steps of p units up and q units to the right (see Section 4), then the required column generators will become

$$g_k^\star(x) = \frac{x^{k(p+sq)}}{(1-x)(1-x^s)^{kq}} .$$

The generating function for the numbers

$$u(n;p,q,s) = \sum_{k=0}^{\left[\frac{n}{p+sq}\right]} \binom{\frac{n-pk}{s}}{qk} \quad (n \geq 0)$$

investigated in [5] is thus

$$G(x) = \sum_{k=0}^{\infty} g_k^\star(x) = \frac{1}{1-x} \sum_{k=0}^{\infty} \left(\frac{x^{p+sq}}{(1-x^s)^q} \right)^k$$

(4)

$$= \frac{(1-x^s)^q / (1-x)}{(1-x^s)^q - x^{p+sq}} = \sum_{n=0}^{\infty} u(n;p,q,s)\, x^n .$$

The horizontal sums will be finite if $p + sq \geq 1$, $s > 0$, and $q \geq 0$, so again p may be negative. For example, if $p = -1$, $q = 1$, and $s = 2$, then

$$G(x) = \frac{(1-x^2)/(1-x)}{(1-x^2)-x} = \frac{1+x}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+2}\, x^n ,$$

so there are Fibonacci numbers even in the falling diagonals of the left-adjusted Pascal Triangle with each row repeated s times.

6. THE TRIMMED PASCAL TRIANGLE

Let us return to the numbers $u(n;p,q)$ of [4] (see Section 4). Suppose we define $u^\star(n;p,q)$ as having the same summation pattern (p units up and q units to the right), but in Pascal's Triangle with the first m columns removed. Letting $g_k^\star(x)$ be the generating function for the kth column of this trimmed, left-justified Pascal Triangle, it easily follows that

$$g_k^\star(x) = \frac{x^{k(p+q)}}{(1-x)^{m+1+kq}} .$$

Therefore the generating function for the numbers $u^\star(n;p,q)$ is

$$G^\star(x) = \sum_{k=0}^{\infty} g_k^\star(x) = \frac{1}{(1-x)^{m+1}} \sum_{k=0}^{\infty} \left(\frac{x^{p+q}}{(1-x)^q}\right)^k = \frac{1}{(1-x)^m} \cdot \frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} .$$

We point out that if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n ,$$

then

$$\frac{f(x)}{1-x} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j\right) x^n ,$$

so that multiplying the generating function for the $u(n;p,q)$ by $(1-x)^{-1}$ merely yields the generating function for the partial sums of the $u(n;p,q)$. Repeated application m times yields m-fold partial sums. Thus we note if we take

rising diagonals on Pascal's Triangle with the left column of ones trimmed off, the result will be the sum of the Fibonacci numbers, so that

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1 ,$$

while consideration of row sums gives

$$1 + 2 + \cdots + 2^n = 2^{n+1} - 1$$

(see Figure 1). In general we have

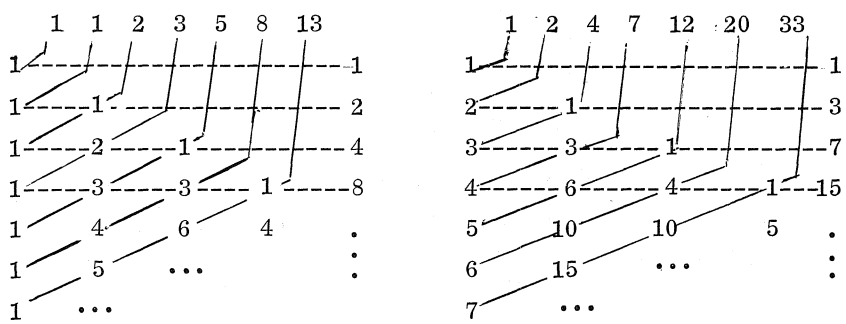

Figure 1

$$\sum_{k=0}^{n} u(k;p,1) = u(n+p+1;\ p,1) - 1 .$$

We also note that the original generating function for the Fibonacci numbers,

$$G(x) = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n ,$$

becomes

$$G^{\star}(x) = \frac{1}{(1-x)^m} \cdot \frac{1}{1 - x - x^2}$$

for Pascal's Triangle trimmed of the first m columns. Thus we have

$$G^{\star}(x) = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{m+k-1}{m-1} F_{n-k} \right\} x^n ,$$

a convolution of the Fibonacci numbers with the $(m-1)^{st}$ column of Pascal's Triangle. If the column of ones is deleted, so that $m = 1$, the generating function for $p = -1$ and $q = 2$ is

$$G(x) = \frac{1}{1-x} \cdot \frac{1-x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n+2}\, x^n ,$$

so Fibonacci numbers are again in the falling diagonals.

Returning to the general case of the generating function for the $u(n;p,q)$ given in (3), we remark that in this particular case we can interpret the sequence generated by

$$\frac{(1-x)^{q-1-m}}{(1-x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u^{\star}(n;p,q)\, x^n \quad (m = 0, 1, \cdots) .$$

7. A SURPRISE CONNECTION

In an important paper concerning unique representations of the positive integers as sums of distinct Fibonacci numbers and the generalization of this representation property, D. E. Daykin ([1], [2], [3]) studied the sequence

$$\{u_n\}_{n=0}^{\infty}$$

defined by

$$\begin{cases} u_n = n & (n = 1, 2, 3, \cdots, r) , \\ u_n = u_{n-1} + u_{n-r} & (n > r) . \end{cases}$$

Now if $r = 1$, we get $u_n = 2^{n-1}$, while if $r=2$, then $u_n = F_{n+1}$. Consider the generating function for the numbers $u(n;\ r-1, 1)$,

$$\frac{1}{1 - x - x^r} = \sum_{n=0}^{\infty} u(n;\ r-1,\ 1)\, x^n .$$

The initial values are $u(n; r-1, 1) = 1$ for $n = 0, 1, \cdots, r-1$, and $u(n; r-1, 1) = n+2-r$ for $n = r,\ r+1,\ \cdots,\ 2r-1$. Thus

$$u_n = u(n + r - 2;\ r - 1,\ 1)$$

for $n \geq 1$. Hence the generating function for the u_n is

$$\sum_{n=1}^{\infty} u(n + r;\ r - 1,\ 1)x^n = \left[\frac{1}{1 - x - x^r} - (1 + x + \cdots + x^{r-2})\right] \Big/ x^{r-1}$$

$$= \frac{(1 - x^r) / (1 - x)}{1 - x - x^r}$$

But this is a special case of (4). Thus the second generalized Fibonacci numbers $u(n; p, q, s)$ of Harris and Styles reduce to the u_n by choosing $s = r$, $q = 1$, and $p = r - 1$.

D. E. Daykin also studied ([1], [2], [3]) the sequence

$$\{v_n\}_{n=1}^{\infty}$$

defined by

$$\begin{cases} v_n = n & (n = 1, 2, \cdots, r), \\ v_n = v_{n-1} + v_{n-r} + 1 & (n > r). \end{cases}$$

It can be easily verified that the numbers $u\ (n + r - 2;\ r - 1, 1)$, summed in Pascal's Triangle with the first column deleted, obey the same recurrence

relation and boundary conditions as the v_n, so that $v_n = u^\star(n + r - 2;\ r - 1, 1)$ for $n \geq 1$. Thus the generating function for the v_n is

$$\sum_{n=0}^{\infty} v_n x^n = \sum_{n=0}^{\infty} u^\star(n + r - 2;\ r - 1, 1)x^n = \frac{1}{(1 - x)(1 - x - x^r)}$$

8. SOME FURTHER RESULTS

Let $f(x)$ be the generating function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n .$$

Suppose we multiply each of the column generators $g_k(x)$ by the corresponding coefficient a_k and sum, yielding

$$G(x) = \sum_{k=0}^{\infty} a_k g_k(x) .$$

In many particular cases the results are quite interesting. For example, let

$$f(x) = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n ,$$

$$g_k(x) = \frac{x^k}{(1 - x)^{k+1}} .$$

Then

$$G(x) = \sum_{k=0}^{\infty} F_{k+1} \frac{x^k}{(1 - x)^{k+1}} = \frac{1}{1 - x} \sum_{k=0}^{\infty} F_{k+1} \left(\frac{x}{1 - x}\right)^k$$

$$= \frac{1}{(1 - x)\left(1 - \frac{x}{1 - x} - \frac{x^2}{(1 - x)^2}\right)} = \frac{1 - x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n+1} x^n .$$

Since in this case

$$g_k(x) = \frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n ,$$

we have that

$$\sum_{k=0}^{n} F_{k+1} \binom{n}{k} = F_{2n+1} .$$

If, on the other hand, we put

$$g_k(x) = \frac{x^{2k}}{(1-x)^{k+1}} ,$$

then we are multiplying F_{k+1} by the corresponding elements of the rising diagonals, and

$$G(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[n/2]} \binom{n-k}{k} F_{k+1} \right) x^n = \frac{1}{1-x} \sum_{k=0}^{\infty} F_{k+1} \left(\frac{x^2}{1-x} \right)^k$$

$$= \frac{1-x}{1-2x+x^3-x^4} .$$

Suppose that

$$f(x) = \frac{1-x}{1-2x-2x^2+x^3} = \sum_{n=0}^{\infty} F_{n+1}^2 x^n .$$

Then

$$G(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]}\binom{n-k}{k}F_{k+1}^2\right)x^n = \frac{1}{1-x}\sum_{k=0}^{\infty}F_{k+1}^2\left(\frac{x}{1-x}\right)^k$$

$$= \frac{1-\dfrac{x}{1-x}}{(1-x)\left(1-2\dfrac{x}{1-x}-2\dfrac{x^2}{(1-x)^2}+\dfrac{x^3}{(1-x)^3}\right)}$$

$$= \frac{(1-2x)(1-x)}{1-5x-5x^2}.$$

There are thus many easily accessible generating functions where the numbers generated are multiplied by the corresponding elements on any of the diagonals whose sums are the $u(n;p,q)$. These methods were discussed in [7].

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PARTITIONS OF N INTO DISTINCT FIBONACCI NUMBERS

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INTRODUCTION

Suppose $\{a_n\}$ is a sequence of natural numbers such that $a_{n+2} = a_{n+1} + a_n$, $n = 1, 2, \cdots$, and let $A(n)$ be the number of sets of numbers $\{i_1, i_2, \cdots\}$ such that $n = a_{i_1} + a_{i_2} + \cdots$. When $a_n = F_n$, F_{n+1} or L_n (where as usual F_n and L_n are the nth Fibonacci and Lucas numbers, respectively) we write $A(n) = R(n)$, $T(n)$, or $S(n)$, respectively. Among other things we proved the following theorems in an earlier paper on this subject [4].

<u>Theorem 1.</u> If $a_n \le K = a_n + k \quad a_{n+1} - a_2$, $n = 3, 4, \cdots$, then

(a) $$A(K) = A(k) + A(a_{n-1} - k - a_2),$$

and

(b) $$A(K) = A(a_{n+1} - k - a_2).$$

Also, if $a_2 \ge 2$ and $1 \le k \le a_2 - 1$, then

(c) $$A(a_{n-1} + k - a_2) = A(a_n - k) = A(a_{n+1} + k - a_2), \quad n = 4, 5, \cdots.$$

<u>Theorem 2:</u>

(a) $T(N) = 1$ if, and only if, $N = F_{n+1} - 1$, $n = 0, 1, \cdots$.

(b) $T(N) = 2$ if, and only if, $N = F_{n+3} + F_n - 1$ or $F_{n+4} - F_n - 1$, $n = 1, 2, \cdots$.

(c) $T(N) = 3$, if and only if, $N = F_{n+5} + F_n - 1$, $F_{n+5} + F_{n+1} - 1$, $F_{n+6} - F_n - 1$, or $F_{n+6} - F_{n+1} - 1$, $n = 1, 2, \cdots$.

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(d) $T(F_{n+k+2} + 2F_{n+2} - 1) = k$, $n = 1, 2, \cdots$, and $k = 4, 5, \cdots$.

For several values of k Hoggatt found solution sets of $T(x) = k$; in each case this solution set could be described as a finite set of sequences having the form $b_n - 1$ where $b_{n+2} = b_{n+1} + b_n$. Thus he was led to conjecture: If $\{b_n\}$ is a sequence of natural numbers such that $b_{n+2} = b_{n+1} + b_n$, then

$$T(b_n - 1) = T(b_{n+1} - 1) = k$$

for all sufficiently large n. Our main purpose in this note is to give proof of Hoggatt's conjecture.

A REPRESENTATION THEOREM

Suppose $\cdots, F_{-1}, F_0, F_1, \cdots$ is the extended sequence of Fibonacci members; that is, $F_0 = 0$, $F_1 = 1$, and $F_{n+1} - F_n - F_{n-1} = 0$, $-\infty \le n \le \infty$. Thus, we have

$$F_{-n} = (-1)^{n+1} F_n, \quad n = 1, 2, \cdots.$$

The following representation theorem should be compared with Zeckendorf's theorem (see for example Brown [1], [2], or Daykin [3]); in particular, is there a sequence essentially different from $\{F_n\}$ which satisfies the conditions of Theorem 3?

Theorem 3. For every pair of non-negative integers A and B there exists a unique set of integers $\{k_1, \cdots, k_i\}$ such that $|k_r - k_s| \ge 2$ whenever $r \ne s$, and

$$A = F_{k_1} + \cdots + F_{k_i} \text{ and } B = F_{k_1+1} + \cdots + F_{k_i+1}.$$

Proof. If a set of integers $\{m_1, \cdots, m_i\}$ has $|m_r - m_s| \ge 2$ whenever $r \ne s$, $F_{m_1} + \cdots + F_{m_i}$ is called a minimal sum. There is a finite algorithm $\mathfrak{A}$ for converting $F_m + F_{m_1} + \cdots + F_{m_i}$ into a minimal sum if $F_{m_1} + \cdots + F_{m_i}$ is a minimal sum $\mathfrak{A}$: First, if $m = m_j$ for some j we can convert $F_{m_1} + \cdots + 2F_{m_j} + \cdots + F_{m_i}$ into a sum involving F's with distinct subscripts since there is a maximal t such that $2F_m + F_{m-2} + \cdots + F_{m-2t}$ is a part of this sum, and this can be replaced with

$$F_{m+1} + F_{m-1} + \cdots + F_{m-2t+1} + F_{m-2t-2} .$$

Second, if $F_m + F_{m_1} + \cdots + F_{m_i}$ is a sum involving F's with distinct subscripts, a minimal sum can be obtained in a finite number of steps by successively replacing $F_v + F_{v-1}$, v maximal, with F_{v+1}. Note that if $\mathfrak{A}$ is applied to $F_{n+m_1} + \cdots + F_{n+m_1}$ and $F_{n+n_1} + \cdots + F_n + n_j$ is the result when $n = 0$, then the same statement holds for $n = 1, 2, \cdots$.

Consider the sequence $\{b_n\}$ defined by

$$b_0 = A, \quad b_1 = B, \quad b_{n+2} = b_{n+1} + b_n, \quad n = 0, 1, \cdots ,$$

then it follows that

$$b_n = F_{n-1}A + F_nB, \quad n = 0, 1, \cdots .$$

Using the algorithm $\mathfrak{A}$ we are going to show by induction on $A + B$ that for every pair of non-negative integers A, B there exists a unique set of integers $\{k_1, \cdots, k_i\}$ such that $|k_r - k_s| \geq 2$ when $r \neq s$, and

$$AF_{n-1} + BF_n = F_{n+k_1} + \cdots + F_{n+k_i}, \quad n = 0, 1, \cdots . \tag{1}$$

If $A + B = 1$, then

$$AF_{n-1} + BF_n$$

is F_{n-1} or F_n, $n = 0, 1, \cdots$. Suppose the statement is true for every pair of non-negative integers A, B with $A + B \leq n$ $(n \geq 1)$. Then if $A + B = n$, there exists a unique set of integers $\{k_1, \cdots, k_i\}$ with $|k_r - k_s| \geq 2$ when $r \neq s$, and

$$AF_{n-1} + BF_n = F_{n+k_1} + \cdots + F_{n+k_i} .$$

Now we can apply $\mathfrak{A}$ to

$$(A + 1)F_{n-1} + BF_n = F_{n-1} + F_{n+k_1} + \cdots + F_{n+k_i}$$

or

$$AF_{n-1} + (B+1)F_n = F_n + F_{n+k_1} + \cdots + F_{n+k_i}$$

to find that there is at least one set of integers which satisfies (1) for every pair of non-negative integers A,B with $A + B = n + 1$. But suppose $AF_{n-1} + BF_n$ can be expressed as a minimal sum in two ways for $n = 0, 1, \cdots$, say

$$AF_{n-1} + BF_n = F_{n+r_1} + \cdots + F_{n+r_i} = F_{n+s_1} + \cdots + F_{n+s_j} .$$

Thus, for every

$$n \geq \max\{r_1, \cdots, r_i, s_1, \cdots, s_j\}$$

the number $AF_{n-1} + BF_n$ has two representations as a sum of non-consecutive Fibonacci numbers (with positive subscripts); this contradicts Zeckendorf's theorem which says that such representations are unique for every natural number.

<u>Corollary</u>: If $\{b_n\}$ is a sequence of natural numbers such that

$$b_{n+2} = b_{n+1} + b_n, \quad n = 0, 1, \cdots ,$$

then there exists a unique set of integers $\{k_1, \cdots, k_i\}$ with $|k_r - k_s| \geq 2$ when $r \neq s$, such that

$$b_n = F_{n+k_1} + \cdots + F_{n+k_i}, \quad n = 0, 1, \cdots . \tag{2}$$

<u>Proof.</u> Put $b_0 = A$, $b_1 = B$ in Theorem 3, then (2) can be proved by induction on n.

HOGGATT'S CONJECTURE

<u>Theorem 4.</u> Suppose $\{b_n\}$ is a sequence of natural numbers such that $b_{n+2} = b_{n+1} + b_n$, then there exists an N such that

$$T(b_n - 1) = T(b_{n+1} - 1), \quad n \geq N; \tag{3}$$

in fact, if

$$(4) \qquad b_n = F_{n+k_1} + \cdots + F_{n+k_i}, \quad k_j \geq k_{j+1} + 2, \quad j = 1, \cdots, i-1.$$

then $N = 2 - k_i$. If $k_i > 2$, the extended sequence found by substituting $n = -1, \cdots, 2 - k_i$ in (4) satisfies (3) for $n \geq 2 - k_i$.

Proof. The Corollary to Theorem 3 guarantees that b_n has the (unique) representation given in (4), so we can assume b_n has this form. If $i = 1$, Theorem 2(a) asserts $T(F_n - 1) = 1$ for $n = 1, 2, \cdots$, so

$$T(F_{n+k_1} - 1) = T(F_{n+k_1+1} - 1)$$

for $n \geq 2 - k_1$ (in fact for $n \geq 1 - k_1$). Now assume $i > 1$. We have

$$F_{n+k_1} \leq b_n - 1 \leq F_{n+k_1+1} - F_3,$$

for $n \geq 3 - k_1 \geq 2 - k_i$, so Theorem 1(a) can be used to write

$$(5) \qquad T(b_n - 1) = T(b_n - F_{n+k_1} - 1) + T(F_{n+k_1+1} - b_n + 1 - F_3).$$

Suppose $1 \leq j \leq i$ is the smallest member such that $k_j > k_{j+1} + 2$, then

$$(6) \qquad F_{n+k_1+1} - b_n + 1 - F_3 = \begin{cases} F_{n+k_{i-1}} - 1, & \text{if } j = i, \\ F_{n+k_{j-2}} + F_{n+k_{j+1}} + \cdots + F_{n+k_j} - 1, & \text{if } j < 1. \end{cases}$$

Now (5) and (6) indicate that Theorem 4 can be proved by means of a double induction on i and $k_1 - k_2 = k \geq 2$; thus, for $i, k \geq 2$ we define proposition $P(i, k)$: If $\{b_n\}$ is a sequence of natural numbers with

$$b_n = F_{n+k_1} + \cdots + F_{n+k_i},$$

such that $k_1 \geq k_2 + 2, \cdots, k_{i-1} \geq k_i + 2$, and $k_1 - k_2 = k$, then

$$T(b_n - 1) = T(b_{n+1} - 1)$$

for all $n \geq 2 - k_1$.

To prove $P(2, 2)$ is true, suppose

$$b_n = F_{n+k_1} + F_{n+k_2}$$

with $k_2 = k_1 - 2$; then using (5) and (6) we have

$$T(F_{n+k_1} + F_{n+k_2} - 1) = T(F_{n+k_2} - 1) + T(F_{n+k_2-1} - 1) , \tag{7}$$

but

$$T(F_{n+k_2} - 1) = T(F_{n+k_2-1} - 1) = 1$$

for all $n \geq 2 - k_2$.

Suppose $P(2, k)$ is true for all $k < K$ $(K > 2)$, and suppose

$$b_n = F_{n+k_1}, \ F_{n+k_2}$$

with $k_1 - k_2 = K$, then using (5) and (6) we have

$$T(F_{n+k_1} + F_{n+k_2} - 1) = T(F_{n+k_2} - 1) + T(F_{n+k_1-2} + F_{n+k_2} - 1) . \tag{8}$$

If

$$K = 3, \ T(F_{n+k_2} - 1) = T(F_{n+k_1-2} + F_{n+k_2} - 1) = T(F_{n+k_1+1} - 1) = 1 ,$$

for all $n \geq 2 - k_2$. If $K > 3$,

$$T(F_{n+k_1-2} + F_{n+k_2} - 1) = T(F_{n+k_1+1} + F_{n+k_2+1} - 1) \quad \text{for all } n \geq 2 - k_2 ,$$

so $P(2,k)$ is true; thus, $P(2,k)$ is true for all $k \geq 2$.

Now we suppose $P(i,k)$ is true for all $i < I$ $(I > 2)$ and all $k > 2$; there is no difficulty in showing that $P(I,2)$ is true and that $P(I,K-1)$ implies $P(I,K)$ for $K > 2$, by using (5) and (6) just as before. This completes the proof.

Corollary:

$$T(F_{n+k} + F_n - 1) = \left[\frac{k+2}{2}\right], \quad k,n = 2,3,\cdots .$$

Proof: Combining (7) and (8) and related results we have

$$T(F_{n+k} + F_n - 1) = \begin{cases} 2, & \text{if } k = 2,3, \\ 1 + T(F_{n+k-2} + F_n - 1), & \text{if } k = 4,5,\cdots . \end{cases} \tag{9}$$

The proof follows by induction on k in (9).

Theorem 5. Suppose $\{b_n\}$ is a sequence of natural numbers such that

$$b_{n+2} = b_{n+1} + b_n ,$$

then $T(b_n)$, $T(b_{n+2})$, $\cdots$, and $R(b_n)$, $R(b_{n+2})$, $\cdots$ form arithmetic progressions for all sufficiently large n.

Proof. The proof that $T(b_n)$, $T(b_{n+2})$, $\cdots$ forms an arithmetic progression follows the proof of Theorem 4, except that we use the fact that a term-by-term sum of two arithmetic progressions is also an arithmetic progression. Theorem 4 and this last result imply $R(b_n)$, $R(b_{n+2})$, $\cdots$ forms an arithmetic progression because $R(N) = T(N) + T(N-1)$, so $R(b_n) + T(b_n - 1)$.

SOLVING $T(x) = j$

In the last section we showed that $T(x) = T(y)$ for every pair $x, y \in S(k_1, \cdots, k_i) = \{F_{n+k_1} + \cdots + F_{n+k_i} - 1;\ n = 2-k_1,\ 3-k_1, \cdots\}$,

where

$$k_1 \geq k_2 + 2, \cdots, k_{i-1} \geq k_i + 2 ;$$

since

$$S(k_1, \cdots, k_i) = S(k_1 + k, \cdots, k_i + k),$$

we will assume $k_i = 0$. The next theorem asserts that every solution x of $T(x) = j$ is contained in one of a finite collection of sets $S(k_1, \cdots, k_i)$ for appropriate sets of numbers $\{k_1, \cdots, k_i\}$.

Theorem 6. (a) Every non-negative integer is contained in exactly one of the sets $S(k_1, \cdots, k_i)$, where $\{k_1, \cdots, k_i\}$ ranges over all sets of integers such that

$$k_1 \geq k_2 + 2, \cdots, k_{i-1} \geq k_i + 2, \quad k_i = 0 .$$

(b) If x, y $S(k_1, \cdots, k_i)$, then

$$T(x) = T(y) \leq \left[\frac{k_1 + 2}{2}\right] .$$

(c) There exists a finite, non-empty collection of sets $S(r_1, \cdots, r_m)$, $S(s_1, \cdots, s_m), \cdots$ such that $T(x) = j$ if, and only if, $x \in S(r_1, \cdots, r_m) \cup S(s_1, s_n) \cup \cdots$.

Proof. (a) This is a reformulation of Zeckendorf's Theorem. (b) The result is true when $i = 1$ or 2 by Theorem 2(a) and the Corollary to Theorem 4, respectively. Now (5) and (6) can be used to prove (b) by induction; the main point of the proof is indicated by the following inequality:

$$\begin{aligned}(10) \qquad T(F_{n+k_1} + \cdots + F_{n+k_i} - 1) &= T(F_{n+k_2} + \cdots + F_{n+k_1} - 1) \\ &\quad + T(F_{n+k_{j-2}} + \cdots + F_{n+k_i} - 1) \\ &\geq \begin{cases} 1 + \left[\frac{k_1}{2}\right] = \left[\frac{k_1 + 2}{2}\right], & \text{if } j = 1, \\ \left[\frac{k_1}{2}\right] + \left[\frac{k_1 - 2j + 2}{2}\right] \geq \left[\frac{k_1 + 2}{2}\right], & \text{if } j > 1. \end{cases}\end{aligned}$$

(c) Every number is in exactly one of the sets $S(k_1, \cdots, k_i)$ by (a) of this Theorem; but $x \in S_j = \{x : T(x) = j\}$ and $x \in S(k_1, \cdots, k_i)$ implies $S(k_1, \cdots, k_i)$ is contained in S_j since $T(x) = T(y) = j$ for every $y \in S(k_1, \cdots, k_i)$ by Theorem 4. There are only finitely many sets $\{k_1, \cdots, k_i\}$ such that

$$k_1 \geq k_2 + 2, \cdots, k_{i-1} \geq k_i + 2, \quad k_i = 0,$$

and

$$\left[\frac{k_1 + 2}{2}\right] \leq j,$$

so S_j is a finite union of sets $S(r_1, \cdots, r_m)$, $S(s_1, \cdots, s_n), \cdots$. The corollary of Theorem 4 implies S_j is non-empty for $j = 1, 2, \cdots$; a different collection of solutions of $T(x) = j$ was given in [4].

Let $t(k_1, \cdots, k_i) = T(x)$, where $x \in S(k_1, \cdots, k_i)$; then if $i = 1$, we have $t(0) = 1$ which is Theorem 2(a). For $i > 1$, if j is the smallest number such that $k_j > k_{j+1} + 2$, then (5) and (6) may be formulated as

$$(11) \qquad t(k_1, \cdots, k_i) = \begin{cases} t(k_2, \cdots, k_i) + 1, & \text{if } j = i \\ t(k_2 + \cdots + k_i) + t(k_j - 1, k_{j+2}, \cdots, k_i) & \\ \quad \text{if } j < i, \ k_j = k_{j+1} + 3, & \\ t(k_2 + \cdots + k_i) + t(k_j - 2, k_{j+1}, \cdots, k_i) & \\ \quad \text{if } j < i, \ k_j \geq k_{j+1} + 4. & \end{cases}$$

Using Theorem 6(b) and (11) we can find all solutions of $T(x) = j$ with a finite amount of checking. This checking would be made easier if we had a non-iterative method for computing $t(k_1, \cdots, k_i)$, but so far we have not been able to find a closed formula for $t(k_1, \cdots, k_i)$.

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MORE ABOUT THE "GOLDEN RATIO" IN THE WORLD OF ATOMS

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In an earlier article (The Fibonacci Quarterly, Issue 4, 1963) the author reported some fundamental asymmetries that appear in the world of atoms.

It has been stated in this article that the numerical values of all these asymmetries approximately are equal to the "golden ratio" ("g. r.").

Two of these asymmetries were found:

1. In the structure of atomic nuclei of protons and neutrons, and
2. In the distribution of nucleons in fission-fragments of the heaviest nuclei appearing in some nuclear reactions.

Recent theoretical studies suggest that an element containing 114 protons and 184 neutrons may be comparitively stable and therefore this hypothetical substance could be produced possibly in some nuclear reactions [1].

One possible reaction involves bombarding element 92 (uranium) with ions (atoms stripped of one or more electrons) of the same element 92, which should yield a hypothetical compound nucleus ${}_{184}[x]^{476}$ that could break up asymmetrically and produce a nucleus with 114 protons:

$$ {}_{92}U^{238} + {}_{92}U^{238} \longrightarrow {}_{184}[x]^{476} \longrightarrow {}_{114}[y]^{298} + {}_{70}Yb^{166} + 12n; $$

12 neutrons (n) would be left over from the reaction [2].

Remark: Both hypothetical (with no names) products of this reaction are designated with the symbols [x] and [y] respectively.

It turns out that the ratio of 114 protons and 184 (298 - 144 = 184) neutrons of the hypothetical element 114 is equal to 0.6195 and differs from the "g. r."-value (if we limit the "g. r."-value to four decimals behind the point) by 0.0015 only.

[Continued on p. 249.]

CONTINUOUS EXTENSIONS OF FIBONACCI IDENTITIES

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1. INTRODUCTION

Some attention has been given to extending the domain of definition of Fibonacci and Lucas numbers from the integers to the real numbers (see, for example, [1]). We give here what seems to be the most natural continuous extension from the point of view of recurrence relations. We then show how several familiar identities have quite natural continuous analogues, providing some support for our contention that these extensions are "the" continuous real extensions of the Fibonacci and Lucas numbers.

2. CONTINUOUS EXTENSIONS

We wish to find real-valued functions $U(x)$ satisfying the difference equation

$$U(x) - c_1U(x-1) - c_2U(x-2) = 0 , \tag{1.1}$$

where c_1 and c_2 are real constants. Let a and b denote the roots of the characteristic polynomial

$$x^2 - c_1x - c_2$$

of (1), where we assume a and b are nonzero real numbers. The quadratic formula gives

$$a = \frac{c_1 + \sqrt{c_1^2 + 4c_2}}{2}, \qquad b = \frac{c_1 - \sqrt{c_1^2 + 4c_2}}{2} .$$

Then

$$a^2 - c_1a - c_2 = 0 ,$$

so, for any real x, multiplying this by a^{x-2} gives

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$$a^x - c_1a^{x-1} - c_2a^{x-2} = 0 ,$$

Similarly,

$$b^x - c_1b^{x-1} - c_2b^{x-2} = 0 .$$

Hence

$$U(x) = k_1a^x + k_2b^x ,$$

where k_1 and k_2 are any real constants, satisfies (1.1). If $a > 0$ and $b > 0$, then $U(x)$ is a continuous real function. However, if $a > 0$ and $b < 0$, as in the Fibonacci case, then b^x assumes imaginary values, so $U(x)$ does not immediately give us the real-valued continuous extension we seek. But since c_1 and c_2 are real, we see

$$V(x) = \mathrm{Re}(U(x))$$

is a real function satisfying (1.1). This $V(x)$ will have the nice properties we are looking for.

Let us make these ideas explicit for the Fibonacci and Lucas case. Here then we let $c_1 = c_2 = 1$, so that

$$a = \tfrac{1}{2}(1 + \sqrt{5}) > 0, \qquad b = \tfrac{1}{2}(1 - \sqrt{5}) < 0 .$$

Letting

$$a^{-1} = \beta = -b > 0 ,$$

we see since $e^{\pi i} = -1$,

$$b^x = (-1)^x\beta^x = e^{\pi i x}\beta^x = \beta^x(\cos \pi x + i \sin \pi x) .$$

To find the continuous Fibonacci extension $F(x)$, we use the initial conditions $F(0) = 0$, $F(1) = 1$ to produce the system

$$0 = k_1 + k_2, \quad 1 = k_1 a + k_2 b ,$$

which has the solution $k_1 = -k_2 = 1/\sqrt{5}$. Then

$$F(x) = \mathrm{Re}[(a^x - b^x)/\sqrt{5}] = \frac{a^x - \beta^x \cos \pi x}{\sqrt{5}} . \tag{2.1}$$

Similar consideration for the Lucas extension $L(x)$ obeying (1.1) with the initial conditions $L(0) = 2$, $L(1) = 1$ give

$$L(x) = a^x + \beta^x \cos \pi x . \tag{2.2}$$

Note that if n is an integer, it follows from the recurrence relation and the chosen initial conditions that

$$F(n) = F_n, \quad L(n) = L_n ,$$

where F_n and L_n denote the usual Fibonacci and Lucas numbers, respectively. Hence $F(x)$ and $L(x)$ are continuous (indeed, infinitely differentiable) real-valued extensions of the Fibonacci and Lucas numbers.

3. CONTINUOUS IDENTITIES

We give in this section the continuous analogues of some familiar Fibonacci and Lucas identities. It follows immediately from (2.1) and (2.2) that

$$F(x + 1) + F(x - 1) = L(x) . \tag{3.1}$$

By multiplying out the left side, and using $\beta = a^{-1}$, one can easily verify that

$$F(x + 1)F(x - 1) - F^2(x) = \cos \pi x . \tag{3.2}$$

This is a particularly neat generalization of the Fibonacci identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n .$$

Similarly, one can show

$$L(x)^2 - L(x+1)L(x-1) = 5\cos \pi x \,, \tag{3.3}$$

which generalizes

$$L_n^2 - L_{n+1}L_{n-1} = 5(-1)^n \,.$$

Equations (2.1) and (2.2) can be solved for a^x to give

$$a^x = \tfrac{1}{2}\{L(x) + \sqrt{5}F(x)\} \,, \tag{3.4}$$

which leads to the deMoivre-type formula

$$\left(\frac{L(x) + \sqrt{5}F(x)}{2}\right)^n = \frac{L(nx) + \sqrt{5}F(nx)}{2} \,. \tag{3.5}$$

Slightly less satisfying is the easily checked formula

$$F(x)L(x) = \tfrac{1}{2}\{F(2x) + (a^{2x} - \beta^{2x})/\sqrt{5}\} \,, \tag{3.6}$$

which reduces to $F_{2n} = F_nL_n$ for n integral. Similarly,

$$F(x+1)^2 + F(x)^2 = \tfrac{1}{2}\{F(2x+1) + (a^{2x+1} + \beta^{2x+1})/\sqrt{5}\} \,, \tag{3.7}$$

which generalizes

$$F_{n+1}^2 + F_n^2 = F_{2n+1} \,,$$

and also

$$F(x+1)^2 - F(x-1)^2 = [F(x+1) - F(x-1)][F(x+1) + F(x-1)] = F(x)L(x). \tag{3.8}$$

We have indicated here how one might continuously extend most Fibonacci and Lucas identities. The functions $F(x)$ and $L(x)$ can be differentiated and

integrated using standard formulas, but the results are not particularly simple. Finally, we note that the above ideas may be carried out to extend general second-order recurring sequences to continuous functions, as indicated in Section 2. However, because of increased complexity, we do not state the more general results here.

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[Continued from p. 244.]

It is well known that the number of protons Z in the lightest stable nuclei is, as a rule, equal to the number of neutrons N. When the atomic number Z increases, the proton-neutron ratio in the nucleus Z/N decreases gradually from 1.0 to about 0.63.

The ratio of Z/N in the heaviest practical stable nucleus ($_{92}U^{238}$) — found in nature — reaches already the value 0.620, but with the still heavier hypothetical element 114 this ratio (114/184 = 0.6195) would yield (if this element could eventually be created) one of the best approximations to the "g. r."-value found in the world of atoms.

It is interesting to note that the ratio of protons of fission-fragments in above nuclear reaction (70/114 = 0.6140) also lies in the range of the "g. r."-value and differs from this value by 0.0040 only.

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*This is an abstract from the statement that the Nobel-Prize-winning chemist G. T. Seaborg made on the occasion of receiving the Willard-Gibbs-medal on 20 May 1966 in Chicago.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-136 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California, and D. A. Lind, University of Virginia, Charlottesville, Virginia.

Let $\{H_n\}$ be defined by $H_1 = p$, $H_2 = q$, $H_{n+2} = H_{n+1} + H_n$ $(n \geq 1)$, where p and q are non-negative integers. Show there are integers N and k such that $F_{n+k} < H_n \leq F_{n+k+1}$ for all $n > N$. Does the conclusion hold if p and q are allowed to be non-negative reals instead of integers?

H-137 Proposed by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pa.

GENERALIZED FORM OF H-70: Consider the set S consisting of the first N positive integers and choose a fixed integer k satisfying $0 < k \leq N$. How many different subsets A of S (including the empty subset) can be formed with the property that $a' - a'' \neq k$ for any two elements a', a'' of A: that is, the integers i and $i + k$ do not both appear in A for any $i = 1, 2, \cdots, N - k$.

H-138 Proposed by George E. Andrews, Pennsylvania State University, University Park, Pa.

If F_n denotes the sequence of polynomials $F_1 = F_2 = 1$, $F_n = F_{n-1} + x^{n-2}F_{n-2}$, prove that $1 + x + x^2 + \cdots + x^{p-1}$ divides F_{p+1} for any prime $p \equiv \pm 2 \pmod 5$.

H-139 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$A_n = \begin{bmatrix} F_n & F_{n+1} & \cdots & F_{n+k-1} \\ F_{n+k-1} & F_n & \cdots & F_{n+k-2} \\ \cdot & \cdot & \cdot & \cdot \\ F_{n+1} & F_{n+2} & \cdots & F_n \end{bmatrix},$$

$$M = \begin{bmatrix} A_n & A_{n+k} & \cdots & A_{n+(m-1)k} \\ A_{n+(m-1)k} & A_n & \cdots & A_{n+(m-2)k} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n+k} & A_{n+2k} & \cdots & A_n \end{bmatrix}.$$

Evaluate det M .

For $m = k = 2$ the problem reduces to H-117 (Fibonacci Quarterly, Vol. 5, No. 2 (1967), p. 162).

H-140 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia

For a positive integer m, let $\alpha = \alpha(m)$ be the least positive integer such that $F_\alpha = 0 \pmod{m}$. Show that the highest power of a prime p dividing $F_1F_2 \cdots F_n$ is

$$\sum_{k=1}^{\infty} \left[\frac{n}{\alpha(p^k)}\right],$$

where $[x]$ denotes the greatest integer contained in x. Using this, show that the Fibonacci binomial coefficients

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{F_m F_{m-1} \cdots F_{m-r+1}}{F_1 F_2 \cdots F_r} \qquad (r > 0)$$

are integers.

H-141 Proposed by H. T. Leonard, Jr., and V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that

(a) $$\frac{F_{3n} + 2^n F_n}{2} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} L_{2(n-(2k+1))} F_{2k+1}$$

(b) $$\frac{L_{2n} - L_n}{2} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k+1} L_{2k+1}$$

(c) $$\frac{L_{2n} + L_n}{2} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} L_{2k}$$

H-142 Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.

With the usual notation for Fibonacci numbers, $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$, show that

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \sum_{k=0}^{n} \binom{\frac{1+\sqrt{5}}{1-\sqrt{5}}k}{k} \binom{n - \frac{1+\sqrt{5}}{1-\sqrt{5}}k}{n-k} = F_{n+1},$$

where

$$\binom{x}{j} = x(x-1)(x-2)\cdots(x-j+1)/j!$$

is the usual binomial coefficient symbol.

SOLUTIONS

ORIGINAL COMPOSITION

H-88 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California (Corrected).

Prove that

$$\sum_{k=0}^{n} F_{4mk} \binom{n}{k} = L_{2m}^{n} F_{2mn}$$

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

Let

$$S = \sum_{k=0}^{n} F_{4mk} \binom{n}{k}$$

$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} (p^{4m})^k - \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} (q^{4m})^k$$

$$= \frac{1}{\sqrt{5}} \left[(1 + p^{4m})^n - (1 + q^{4m})^n \right]$$

where

$$p + q = 1, \quad pq = -1; \quad \text{or} \quad (pq)^{2m} = 1$$

Hence,

$$S = \frac{1}{\sqrt{5}} \left[\left\{ (pq)^{2m} + p^{4m} \right\}^n - \left\{ (pq)^{2m} + q^{4m} \right\}^n \right]$$

$$= \frac{1}{\sqrt{5}} \left[p^{2mn} (p^{2m} + q^{2m})^n - q^{2mn} (p^{2m} + q^{2m})^n \right]$$

$$= (p^{2m} + q^{2m})^n \frac{(p^{2mn} - q^{2mn})}{\sqrt{5}}$$

$$= L_{2m}^{n} F_{2mn}$$

Therefore,

$$\sum_{k=0}^{n} F_{4mk}\binom{n}{k} = L_{2m}^{n}F_{2mn}$$

Also solved by John Wessner, L. Carlitz, and F. D. Parker.

FINE BREEDING

H-96 Proposed by Maxey Brooke, Sweeny, Texas, and V. E. Hoggatt, Jr., San Jose State College, San Jose, California (Corrected).

Suppose a female rabbit produces $F_n(L_n)$ female rabbits at the nth time point and her female offspring follow the same birth sequence, then show that the new arrivals, C_n, (D_n) at the nth time point satisfies

$$C_{n+2} = 2C_{n+1} + C_n\ ;\ C_1 = 1,\ C_2 = 2$$

and

$$D_{n+1} = 3D_n + (-1)^{n+1}$$

Solution by Douglas Lind, University of Virginia.

Hoggatt and Lind ["The Dying Rabbit Problem," to appear, Fibonacci Quarterly] have proved the following result: Let a female rabbit produce B_n female rabbits at the nth time point, her offspring do likewise, and put

$$B(x) = \sum_{n=1}^{\infty} B_n x^n .$$

Then the number R_n of new arrivals has the generating function

$$R(x) = \sum_{n=0}^{\infty} R_n x^n = \frac{1}{1 - B(x)} ,$$

where we use the convention that $R_0 = 1$ (the original female being born at the 0th time point). We apply this result to the cases (i) $B_n = F_n$, and (ii) $B_n = L_n$.

(i) If $B_n = F_n$, then

$$B(x) = \sum_{n=1}^{\infty} F_n x^n = \frac{x}{1 - x - x^2},$$

so

$$R(x) = 1 + \frac{x}{1 - 2x - x^2}.$$

It is clear from the generating function that here the $R_n = C_n$ obey the recurrence relation $C_{n+2} = 2C_{n+1} + C_n$ along with $C_1 = 1$, $C_2 = 2$, thus establishing the desired result.

(ii) The recurrence relation proposed is incorrect, the proper one being shown below. If $B_n = L_n$, then

$$B(x) = \sum_{n=1}^{\infty} L_n x^n = \frac{x + 2x^2}{1 - x - x^2},$$

so that

$$R(x) = \frac{1}{1 - \dfrac{x + 2x^2}{1 - x - x^2}} = 1 + \frac{x + 2x^2}{1 - 2x - 3x^2}.$$

Now

$$\frac{x + 2x^2}{1 - 2x - 3x^2} = -\frac{2}{3} + \frac{\frac{5}{12}}{1 - 3x} + \frac{\frac{1}{4}}{1 + x}.$$

so that

$$D_n = R_n = (5/12)(3^n) + (1/4)(-1)^n \quad (n \geq 1).$$

It follows that $D_1 = 1$, and that $D_{n+1} = 3D_n + (-1)^{n+1}$, the correct relation.

BINOMIAL, ANYONE?

H-97 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show

(a) $$\sum_{k=0}^{n} \binom{n}{k}^2 L_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} L_{n-k}$$

(b) $$\sum_{k=0}^{n} \binom{n}{k}^2 F_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} F_{n=k} \quad .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

If

$$P(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k$$

and

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (x-1)^{n-k} \quad ,$$

then $P(x) = Q(x)$ is a known identity (see elementary problem E799, American

Math. Monthly, 1948, p. 30). If α and β are roots of $x^2 - x - 1 = 0$, then $L_n = \alpha^n + \beta^n$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and thus

(a) $$P(\alpha) + P(\beta) = Q(\alpha) + Q(\beta),$$

since

$$(\alpha - 1)^{n-k} + (\beta - 1)^{n-k} = (\alpha\beta^2)^{n-k} + (\beta\alpha^2)^{n-k}$$
$$= (-1)^{n-k}L_{n-k},$$

(b) $$(P(\alpha) - P(\beta))/\sqrt{5} = (Q(\alpha) - Q(\beta))/\sqrt{5}$$

since

$$(\alpha - 1)^{n-k}(\beta - 1)^{n-k} = (-1)^{n-k}(\beta^{n-k} - \alpha^{n-k})$$
$$= -(-1)^{n-k}(\sqrt{5})F_{n-k}$$

PRODUCTIVE SUMS

H-99 Proposed by Charles R. Wall, Harker Heights, Texas.

Using the notation of H-63 (April 1965 FQJ, p. 116), show that if $\alpha = (1 + \sqrt{5})/2$,

$$\prod_{n=1}^{m} \sqrt{5}F_n \alpha^{-n} = 1 + \sum_{n=1}^{m} (-1)^{n(n-1)/2} F(n,m)\alpha^{-n(m+1)}$$

$$\prod_{n=1}^{m} L_n \alpha^{-n} = 1 + \sum_{n=1}^{m} (-1)^{n(n+1)/2} F(n,m)\alpha^{-n(m+1)},$$

where

$$F(n,m) = \frac{F_m F_{m-1} \cdots F_{m-n+1}}{F_1 F_2 \cdots F_n}.$$

Solution by Douglas Lind, University of Virginia.

We use the familiar identity

(*) $$\prod_{n=0}^{m-1} (1 - q^n x) = \sum_{n=0}^{m} (-1)^n q^{n(n-1)/2} \begin{bmatrix} m \\ n \end{bmatrix} x^n,$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-n+1}}{(1-q)(1-q^2)\cdots(1-q^n)} .$$

If $\beta = (1-\sqrt{5})/2$, then $\sqrt{5}F_n\alpha^{-n} = 1 - (\beta/\alpha)^n$. Putting $q = \beta/\alpha$, then

$$\begin{bmatrix} m \\ n \end{bmatrix} = \alpha^{n^2-mn}F(n,m) ,$$

and putting $x = q$ in (*) gives

$$\prod_{n=1}^{m} \sqrt{5}F_n\bar{\alpha}^n = \prod_{n=1}^{m} (1-q^n) = \sum_{n=0}^{m} (-1)^n F(n,m) q^{n(n+1)/2}\alpha^{n^2-mn}$$

$$= \sum_{n=0}^{m} (-1)^{n(n-1)/2} F(n,m)\alpha^{-n(m+1)}$$

where we have used $\alpha\beta = -1$.

Similarly, $L_n\alpha^{-n} = 1 - (\beta/\alpha)^m$, so putting $q = \beta/\alpha$ and $x = -q$ in (*) gives

$$\prod_{n=1}^{m} L_n\alpha^{-n} = \prod_{n=1}^{m} (1+q^n) = \sum_{n=0}^{m} (-1)^n F(n,m) q^{n(n-1)/2}\alpha^{n^2-mn}(-q)^n$$

$$= \sum_{n=0}^{m} F(n,m) q^{n(n+1)/2}\alpha^{n^2-mn}$$

$$= \sum_{n=0}^{m} (-1)^{n(n+1)/2} F(n,m)\alpha^{-n(m+1)}$$

Also solved by M. N. S. Swamy.

PYTHAGOREANS AND ALL THAT STUFF

H-101 Proposed by Harlan Umansky, Cliffside Park, N. J., and Malcolm Tallman, Brooklyn, N. Y.

Let a, b, c, d be any four consecutive generalized Fibonacci numbers (say $H_1 = p$ and $H_2 = q$ and $H_{n+2} = H_{n+1} + H_n$, $n \geq 1$), then show

$$(cd - ab)^2 = (ad)^2 + (2bc)^2$$

Let $A = L_k L_{k+3}$, $B = 2L_{k+1}L_{k+2}$, and $C = L_{2k+2} + L_{2k+4}$. Then show

$$A^2 + B^2 = C^2 \quad .$$

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

Now

$$\begin{aligned}(cd - ab)^2 &= [c(b + c) - b(c - b)]^2 \\ &= (c^2 + b^2)^2 = (c^2 - b^2)^2 + (2bc)^2 \\ &= (c + b)^2(c - b)^2 + (2bc)^2 = d^2a^2 + (2bc)^2\end{aligned}$$

Hence

$$(1) \qquad (cd - ab)^2 = (ad)^2 + (2bc)^2$$

Since L_k, the Lucas number, is also a generalized Fibonacci sequence with

$$L_1 = p = 1, \quad L_2 = q = 3,$$

we have that for the four consecutive Lucas numbers L_k, L_{k+1}, L_{k+2}, L_{k+3},

$$(2) \qquad (L_{k+2}L_{k+3} - L_kL_{k+1})^2 = (L_kL_{k+3})^2 + (2L_{k+1}L_{k+2})^2 = A^2 + B^2$$

Now

$$
\begin{aligned}
(L_{k+2}L_{k+3} - L_kL_{k+1}) &= L_{k+2}(L_{k+2} + L_{k+1}) - L_{k+1}(L_{k+2} - L_{k+1}) \\
&= L_{k+2}^2 + L_{k+1}^2 = (F_{k+3} + F_{k+1})^2 + (F_{k+2} + F_k)^2 \\
&= (F_{k+2} + 2F_{k+1})^2 + (2F_{k+2} - F_{k+1})^2 \\
&= 5(F_{k+2}^2 + F_{k+1}^2) = 5F_{2k+3} \qquad (3) \\
&= 2F_{2k+3} + (F_{2k+5} - F_{2k+4}) + (F_{2k+2} + F_{2k+1}) + F_{2k+3} \\
&= (F_{2k+3} + F_{2k+1}) + (F_{2k+5} + F_{2k+3}) \\
&= L_{2k+2} + L_{2k+4} = C
\end{aligned}
$$

Thus, from (2) and (3) we have,

$$A^2 + B^2 = C^2 .$$

Also solved by J. A. H. Hunter and A. G. Shannon.

* * * * *

[Continued from p. 285]

RECURRING SEQUENCES — LESSON 1

ANSWERS TO PROBLEMS

1. $a_n = n(n+1)$; $T_{n+3} = 3T_{n+2} - 3T_{n+1} + T_n$
2. $a_n = 3n - 2$; $T_{n+2} = 2T_{n+1} - T_n$
3. $a_n = n^3$; $T_{n+4} = 4T_{n+3} - 6T_{n+2} + 4T_{n+1} - T_n$
4. $T_{6n+k} = 1, 3, 3, 1, 1/3, 1/3$, for $k = 1, 2, 3, 4, 5, 6$, respectively
5. $T_{n+1} = \sqrt{1 + T_n^2}$
6. $T_{n+4} = 4T_{n+3} - 6T_{n+2} + 4T_{n+1} - T_n$
7. $T_{n+1} = aT_n$
8. $T_{n+3} = 3T_{n+2} - 3T_{n+1} + T_n$
9. $T_{2n-1} = a$, $T_{2n} = 1/a$
10. $T_{n+1} = 1/(2 - T_n)$

* * * * *

IS ERATOSTHENES OUT?

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Two thousand years ago the Alexandrian geographer-astronomer Eratosthenes, a friend of Archimedes, devised a procedure for obtaining a list of primes. His procedure is usually identified as "the Sieve" and basically consists of writing a table of consecutive integers starting from 1 and crossing out all multiples of 2, 3, and so on; all those numbers which remain undeleted are the primes sought. This procedure can be extended to larger tables from 1 to N, but when N is large, the sieve is indeed a cumbersome tool. Nevertheless, Eratosthenes' procedure is the only general way of obtaining primes in an orderly fashion today. Extensive tables have been compiled, but no formula that would yield the nth prime for a given n has been found yet; many a mathematician doubt that such a formula exists. When confronted with the question, "What is the nth prime?" all a mathematician can do is look in a table of primes, and if asked, "Is this number a prime?" the mathematician may not be able to reply at all, for although there are tests for primality, they might not be applicable or may prove insufficient, and if the number given is too large, it might not be listed in the tables. The puzzling aspect of the situation is that, although prime numbers are not randomly distributed along the sequence of integers, their distribution has so far defied all attempts at exact description. Despite the countless efforts, number-theorists are not happy with the idea of settling for the "simple-minded" Eratosthenes' Sieve.

This paper presents two elementary glimpses of modified but simple approaches to the Sieve. The first one is a slight improvement on the original procedure of Eratosthenes, although it is basically the same method, cleverly disguised.

Consider the "Semi-Tribonacci" sequence

$$T_k: \ 1,\ 2,\ 4,\ 5,\ 7,\ 8,\ 10,\ 11,\ 13,\ 14,\ 16,\ 17,\ 19,\ 20,\ 22,\ 23,\ \cdots \qquad (k = 1, 2, 3, \cdots)$$

which obeys the recurrence relation

$$T_{k+3} = T_{k+2} + T_{k+1} - T_k\,; \qquad T_1 = 1,\ T_2 = 2\,.$$

Notice that all multiples of 3 are absent, since

$$T_{2k} = 3k - 1$$

and

$$T_{2k+1} = 3k + 1 .$$

The closed-form formula for these Semi-Tribonacci numbers is:

$$T_k = \tfrac{3}{2}k - (\tfrac{3}{4} + \tfrac{1}{4}(-1)^k) \quad (k = 1, 2, 3, \cdots) \tag{1}$$

Now, if we write the above sequence cancelling all T_k such that

$$T_k \equiv T_n \pmod{2T_n + 3} \quad (T_k > T_n) \tag{2}$$

(i.e., cancel all $T_k \equiv 1 \pmod 5$, $T_k \equiv 2 \pmod 7$, $T_k \equiv 4 \pmod{11}$, etc.) we obtain the "Deleted Semi-Tribonacci Sequence:"

$$\overline{T}_k : 1,\ 2,\ 4,\ 5,\ 7,\ 8,\ 10,\ 13,\ 14,\ 17,\ 19,\ 20,\ 22,\ \cdots,\ (k = 1, 2, 3, \cdots)$$

And here we can state the following result:

All numbers

$$P_{k+2} = 2\overline{T}_k + 3$$

are prime numbers, and, in fact, all primes (except 2 and 3) are represented in this way. Thus

$$P_{k+2} = 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ \cdots\ (k = 1, 2, 3, \cdots).$$

The above may seem quite astonishing at first sight. The reader is invited to convince himself that this is, however, true. But, unfortunately, it is only the Sieve covered up. The core of the problem lies in the solution of the following congruences

(3) $$4T_k \equiv (6n - 3 - (-1)^n) \pmod{12n + 6 - 2(-1)^n}$$

which is, to put it mildly, quite a problem by itself.

The second glimpse offers a simpler disguise, but cleverer. Consider the array

(4)
$$\begin{array}{ccccccc}
4 & 7 & 10 & 13 & 16 & 19 & \cdots \\
7 & 12 & 17 & 22 & 27 & 32 & \cdots \\
10 & 17 & 24 & 31 & 38 & 45 & \cdots \\
13 & 22 & 31 & 40 & 49 & 58 & \cdots \\
16 & 27 & 38 & 49 & 60 & 71 & \cdots \\
\cdot & & & & & & \\
\cdot & & & & & & \\
\cdot & \cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots \\
\cdot & \cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots
\end{array}$$

The array is symmetric about its main diagonal, for as it is readily seen, each kth row and k^{th} column are equal, and the numbers are obtained from arithmetic progressions. The differences are: first line, 3, second line, 5, third line, 7, and so on. We are now prepared to formulate the following statement:

If the number N is a member of the above array, then $2N + 1$ is composite; however, if N is not found in this array, then $2N + 1$ is prime. ($2N + 1$ is prime if and only if N is not a member of the above array.)

The proof is very simple. Designate the nth term of the kth row (or k^{th} term of the nth column) by a_{nk}. Then, since

$$a_{n1} = 4 + 3(n - 1), \quad a_{n2} = 7 + 5(n - 1),$$

etc., in general we have

(5) $$a_{nk} = 1 + 3k + (1 + 2k)(n - 1).$$

or more simply

(6) $$a_{nk} = k + (2k + 1)n = a_{kn} = n + (2n + 1)k$$

Now suppose N is found in the array. Then $N = k + (2k + 1)n$ and therefore

$$\begin{aligned} 2N + 1 &= 2(k + (2k + 1)n) + 1 = 2k + 1 + 4kn + 2n = 2k + 1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 2n(2k + 1) \\ &= (2k + 1)(2n + 1) \end{aligned}$$

which means that $2N + 1$ is the product of at least two factors (neither of which is unity) and hence, composite. The converse is proven similarly.

The following example may be useful to compare the powerfulness of the array (4) as opposed to the naive Sieve. Let us suppose that we wanted to find out whether 437 was or was not a prime. Using the rudimentary approach of the Sieve, we would test for divisibility of all primes up to

$$[\sqrt{437}] = 20 ,$$

that is, we would see if 437 is divisible by 3, 5, 7, 11, 13, 17, and 19. Instead of proceeding this way, let us apply the reasoning provided to us by the array's approach.

If 437 is not a prime, we can find an N in the array such that

$$2N + 1 = 437 .$$

This would yield $N = 218$. Is 218 a member of the array? If it is, we should be able to find it as some n^{th} element of some k^{th} row. Thus, we should be able to solve for n the equation

$$k + (2k + 1)n = 218$$

(if we fail, this would mean that 218 is not in the array, and that 437 is prime). First, we find a bound on k by solving the quadratic

$$2(k^2 + k) = 218 ,$$

and this yields

$$k^2 + k - 109 = 0$$

or $k = 10$.

Thus,

$$k = 10, \quad n = 208/21 \quad \text{(no good)}$$
$$k = 9, \quad n = 209/19 = 11$$

and we get

$$a_{9,11} = a_{11,9} = 218.$$

Therefore 218 is contained in the array, and 437 is not prime.

In fact, if we had tried it using the Sieve method we would have found out, sooner or later, that $437 = 19 \cdot 23$. For large numbers, the array test is tedius although shorter than Eratosthenes'.

Nowadays, with the advent of superfast computers, much of the sieve work is done electronically at very high speeds. Still, the job of classifying larger numbers as primes is very difficult and can only be simplified by choosing specific patterns within sequences of identifiable properties. That, for example, is the case of the 3,376-digit number $(2^{11,213} - 1)$ which belongs to the "Mersenne" family of primes and is presently considered the largest known prime number. Other, modern, more effective sieves are inevitably based on the Sieve or its principle.

Despite the fact that mathematics has progressed immeasurably and contemporary mathematicians have the benefit of ultra-sophisticated tools and techniques, Eratosthenes' method has survived the severe test of twenty centuries. Indeed, Erathosthenes is still not out.

* * * * *

FIBONACCI NUMBERS AND THE SLOW LEARNER

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Fibonacci numbers have been used with remarkable success with talented mathematics students from elementary school through graduate level university mathematics. They have been used as both a part of the basic curriculum and as enrichment material.

During the past year I became interested in the possibility of using Fibonacci numbers with a group of twenty-five freshman students in a "low level" basic mathematics class, the required ninth grade general mathematics course designed to fulfill the mathematics requirement for the freshman year. My interest in using Fibonacci numbers was a result of having met considerable frustration in trying to get the class to achieve a basic facility with the fundamental operations using real numbers. After a semester's work with review, explanations, and drill, the class still had difficulty with the same problems.

Before continuing, I should define "low level" to place the balance of my remarks in proper context. During the fall of this past academic year, the students in the class were given a battery of tests including the Differential Aptitude Test, the Gates Reading Survey, and the Lorge Thorndike Intelligence Quotient Test. I compiled a table of the scores on these three tests utilizing the verbal I. Q. score, the numerical ability percentile ranking, and the composite score on the Gates Reading Survey. The following information was compiled as a result of the research; two-thirds of the class had a numerical ability percentile ranking of fifteen percentile or lower. One-half of the class read at the sixth grade level or lower. One-half of the class had I. Q. scores of less than ninety. Only four students were reading at the ninth grade level or higher. Two students ranked above the fiftieth percentile on the numerical ability, and only four students had I. Q. scores of better than one hundred.

Irving Adler in his address at the California Mathematics Council, Northern Section, meeting at Davis during the spring of this year suggested that as teachers we are being a little unrealistic if by repeating the same material we believe we are able to do what competent teachers have failed to do during a student's first eight years. With much the same philosophy I decided to refine

my goals for the course after the first semester. Basically, I hoped to:

(1) effect a change in attitude toward mathematics,

(2) have some of the students get excited over learning something,

(3) achieve basic skills in the four operations working with real numbers.

With the above goals in mind I introduced a five-week unit on Fibonacci numbers via a presentation on the board. Very little introduction was given: I simply announced that we would be working with something new, Fibonacci numbers. The class reaction was a collective, "WHAT ? ? ?". I then proceeded to write 1, 1, 2 on the board and asked the class to follow along as I wrote the next number down; they were to see if they could find out how I was getting the sequence of numbers.

After writing down several more terms of the sequence, the class caught on to the pattern. Within a very short time the entire class was volunteering the next number. We continued until we had the first twenty numbers written down. We then discussed how we could find the numbers of the sequence and ended the session with the simple explanation, "add the first two numbers and you get the third; add the second and third numbers and you get the fourth, ···" Although in the context of this article I will utilize a formal notation to express Fibonacci patterns, no attempt was made in class at this time to express the patterns with a general notation.

I next asked the students to write down on a paper the first fifty terms of the Fibonacci sequence. If they were not able to finish in class, they were welcome to do so at home. To my delight the majority of the class had worked on the first fifty and several had worked on getting the first one hundred Fibonacci numbers.

On the second day I handed out a ditto with the first one hundred Fibonacci numbers written down. After checking the values for their numbers, we discussed the notation F_1, F_2, F_3, ··· which I had used on the ditto. We called the notation the Fibonacci code for telling which Fibonacci number we were discussing. I encouraged each student to use the notation when he worked with a pattern.

During the first week, including the introduction, the class participated in what Brother Alfred terms group research. The class developed the pattern for

$$F_n^2 - F_{n+k}F_{n-k}$$

for $k = 1$ using the group research method. Instead of stating the problem in the preceding form, each person was asked to pick out some Fibonacci number and square it. Then they were to take the product of the two Fibonacci numbers on either side of the number that had been squared. Finally they were to find the difference between the square and the product. The results were tabulated on the board and the class was asked to try a different Fibonacci number. Again the results were written down. The majority of the students quickly saw that we were getting 1 for an answer; however, when I requested that the subtraction must be done in the same order each time a Fibonacci number had been selected, some of the students remarked that you couldn't subtract a larger number from a smaller one. We then had a delightful discussion about directed numbers and ended with the generalization that the answer was 1 if we chose a Fibonacci number with an odd code and -1 otherwise. I discussed $(-1)^{n+1}$ with the more capable students as a way of expressing the pattern.

The class then worked the next day on extending this pattern for different values of k. We started with group research again for $k = 2$ and found the difference of $|1|$. I then had the class work at their desks finding the patterns for other values of k, but not until I had encouraged them to make a conjecture about what they might find. It was a much surprised group of students when the next value of k did not give then $|1|$ for an answer. I was amused at their discovery that different values of k gave what appeared to them to be quite unrelated answers. Although the students became frustrated easily and I found it necessary to spend time helping each one, the problem allowed each student to continue at his own pace. After considerable work, one of the students suddenly saw that the result was a Fibonacci number squared. Some other students were finding this result and sharing their discovery with others around them. It was at this time that I felt I was achieving some of my goals.

The next problem presented was written down in the following way.

$$1 + 1 = 2;\ 1 + 1 + 2 = 4;\ 1 + 1 + 2 + 3 = 7;\ \cdots$$

I asked the students to write down the numbers in Fibonacci code, and the class was asked to find a pattern in the answers. It was necessary to give

some direction by asking if the answer was close to some Fibonacci number. Finally we wrote down the result in the form

$$F_1 + F_2 + 1 = 2 + 1 = 3$$

$$F_1 + F_2 + F_3 + 1 = 4 + 1 = 5$$

I then asked for a verbal generalization from the students and it was decided that if one was added to each of the sums, we obtained a Fibonacci number. I asked them to give the Fibonacci code for the number and we then tabulated our results on the board.

$$F_1 + F_2 = F_4 - 1; \quad F_1 + F_2 + F_3 = F_5 - 1; \cdots F_1 + F_2 + \cdots + F_n = F_{n+2} - 1 .$$

This was another attempt to have the students use notation to express the patterns rather than just verbalizing the result in English. Earlier I had suggested that each student should use a notebook to write down the results of the previous work and I requested that they write down what we had done on the board. Although many of the students felt uncomfortable with the notation and indicated that they did not like using F_n, they understood that the notation said the same as "the sum of the first n Fibonacci numbers can be found by going two more Fibonacci numbers and subtracting 1."

After the students had worked with the pattern for finding the sum of the Fibonacci numbers, I asked them to find

$$F_1 + F_2 + F_3 + \cdots + F_{25}$$

in sixty seconds for a brief quiz. It is interesting that approximately one-third of the class was not able to connect the problem to our previous work; one-fourth of the class found the result correctly; and the remainder of the class used the right idea in trying to find the solution but could not remember which Fibonacci number they should get even though they knew that if they added one to the result they would get a Fibonacci number. However, considering the make-up of the class, I was very encouraged.

We next worked on

$$F_2 + F_4 + F_6 + \cdots + F_{2n} .$$

I placed the problem on the board in the above form and was pleasantly surprised to find a general acceptance of the notation at this point. There was a little concern over the expression $2n$ and we spent some time answering the question of the value of $2n$ for $n = 1$, $n = 2$, $n = 3$, etc. The students indicated they understood; however, as we went on, I had to continually remind the students of this form for the even index in the Fibonacci code.

We then went on to

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} .$$

Again there was concern over $2n - 1$, and we had another chance to discuss an algebraic expression. This was a second opportunity to introduce the concept of the variable without making the process a painful experience.

Interest at this point was running high and I felt that the class was sharing my enthusiasm. Even those who usually were apathetic to any of the material presented during the first semester were becoming involved.

I then presented

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 .$$

At this point there was a little negative reaction that this problem was too hard; some of the students indicated that this problem should be in an algebra class rather than Math I. Since we had discussed scientific notation earlier in the year working with googols and googolplexes, I reviewed what the exponent 2 represented in each term of the series.

The presentation became a little more detailed this time and I found a great number of the students independently making out a table of squares of the Fibonacci numbers to help them find the pattern. I was very enthusiastic over the idea that some of the students were voluntarily doing more than was required. The pattern was finally established but not until we had a chance to discuss what was meant by a factor. In particular

$$F_1^2 + F_2^2 + F_3^2 = 6$$

was an excellent opportunity to both illustrate that they should use the sum of several terms before trying to establish the pattern and to see that the factors would be Fibonacci numbers if they were not the trivial set of 1 and 6. The results of the ensuing discussion were too extensive to record here, but let it suffice to say that we discussed division, multiplication, exponents, factoring, prime numbers, addition, and general notation without appearing to meet any negative reaction.

I was very pleased with our progress but felt that if I were to maintain the existing level of enthusiasm I would have to try to vary the class activity more than I was doing via the presentations on the board and individual work. As such, I decided to have the class work in groups.

Thus, at the end of the second week I told the class that we would start working in groups the following week. I asked seven students, who I felt could act as group leaders, to make three lists of four students each for their possible groups. I indicated I would try to form their groups from these lists. With this done, I reviewed what had been accomplished during the first two weeks.

On Monday I indicated who was to be in each group and handed out two dittos. One summarized our work up till then; and the second was a set of 15 problems which would be done in their group work, Table 1.

Table 1

1. Find the sum of the Lucas numbers.
2. Find the sum of $L_1 + L_2 + L_5 + L_7 + L_9 + \cdots$.
3. Determine the sum of $L_2 + L_4 + L_6 + L_8 + \cdots$.
4. What is the sum of the squares of the Lucas numbers?
5. Form $F_1^2 + F_2^2$; $F_2^2 + F_3^2$; $F_3^2 + F_4^2$; etc.
6. Do the same as in No. 5 for the Lucas numbers.
7. Find the sum of $L_1 + L_5 + L_9 + L_{13} + L_{17} \cdots$.
8. Determine the sum of $L_2 + L_6 + L_{10} + L_{14} \cdots$.
9. Find the sum of $L_3 + L_7 + L_{11} + L_{15} + L_{19} \cdots$.

10. Determi Table 1 (Continued)

10. Determine the sum of $L_4 + L_8 + L_{12} + L_{16} + L_{20} \cdots$.
11. Find a pattern which works for Nos. 7, 8, 9, and 10.
12. Find F_1L_2 and F_2L_1 and their difference; find F_2L_2 and F_3L_2 and their difference; F_3L_4 and F_4L_2 and their difference; etc.
13. Find F_1L_3 and F_3L_1 and their difference; continue as in No. 12.
14. Find F_1L_4 and F_4L_1 and their difference; continue as in No. 13.
15. The process begun in Nos. 12, 13, and 14 can be continued to spacings of three, four, five, etc. Can you find a pattern in the answers?

Each group was to consist of the leader and three students to work with the leader in the group. I placed the seven groups in clusters about the room with the following directions:

(1) If a member of the group had a question, he was to ask the group leader.

(2) The group leader would discuss problems with me so as to explain the problems to the group.

(3) The individuals in the group, excluding the leader, would receive their grade based on their work in the group, their notebooks, and an oral test.

(4) The group leaders would receive their grades based on their understanding of the material discussed, and more importantly on how much knowledge they could impart to each student in their group, i. e., their grades rested on how much their group knew.

Before actually placing them in groups we discussed a second sequence, the Lucas sequence. For homework they were to find the first twenty-five terms of the sequence starting with 1, 3, 4, etc. We also discussed the notation $L_1, L_2, L_3, \cdots$ for the Lucas numbers.

Although the class was homogenous in that it was basically low ability, there was enough diversity in ability so that the leaders were sufficiently advanced in the material to meet their obligations. I believe that much of the success we had in the group work was based on the selection of the groups and the ability of the seven group leaders.

For the next three weeks I worked with the group leaders and the groups themselves encouraging, explaining, and making sure that each group got the

help it needed. I tried to work with each leader and group at least once each day, but I was pleasantly surprised with the way in which the groups maintained their enthusiasm and worked in as mature a fashion as could be desired when I was busy with other groups. Quite often I worked with entire groups discussine one problem. In many cases the group leaders displayed remarkable behavior in directing the research and explaining to the students in their groups a particular pattern. I was particularly impressed with the patience and understanding displayed by the group leaders.

At the end of the first week in groups, I asked each group leader to submit a progress report on his group. Following is an example of the type of response I received.

> "Our group has progressed fairly with one exception,* * *. Now I see how hard it is for a teacher to try to teach her something. She just won't even try to learn, and when I tell her to try, she sais I can't. Sometimes she catches on, but after she gets to a part that is too hard for her (she thinks) she quits and talks or else just plain forgets it. I don't know what I'll do if she won't learn. . . ."

Each student in the group was to keep a record of his work and the patterns discovered were to be listed. I collected these notebooks at the end of each week and was extremely pleased with the results. I wish it were possible to include one of the notebooks in the article. Again let it suffice to say that the notebooks do justice to those collected from students in a freshman algebra course.

During the third week in groups I had each student go before the class using the overhead projector. They were to present the solution to one of the problems from the work done during the five weeks. This problem was given to them when they went up to the overhead projector. They were allowed to take their lists of Fibonacci and Lucas numbers with them to the projector.

The presentations were of fine quality with the students explaining how they were establishing the patterns. I might add that the patterns were not memorized but rediscovered while working in front of the class with the overhead projector. I tried to present a problem geared to the ability of each student and I started the presentations with students who would present their patterns in a good style to serve as examples for the other students. During many of the presentations it was necessary to make suggestions via questions

as to what should be done. I made every attempt to make sure that enough help was given so that the student eventually was able to find the pattern. Grades for the oral presentations were based on how well the presentation was delivered and the amount of help given to the student. During the presentations, the other students were asked to try the pattern at their desks. It was fascinating to see the involvement of the class in some cases as the person giving the presentation would struggle with a pattern.

Generally speaking, the oral presentations were a highlight of the five weeks. I am sure the class approached the presentations with some less enthusiasm, but the cooperation was very satisfying from the majority of the students.

With the conclusion of the unit, grades were given out to each student if such were requested. The student reaction to the work on Fibonacci numbers was very positive. One girl even went so far as to say that the material should be part of the required curriculum for the ninth grade.

The results of the five week unit on Fibonacci numbers were very encouraging. The change in student attitude, one of the three goals, was readily observed. Since the unit was presented later in the year, there was no opportunity to observe whether the changes in attitude observed would have transferred to work presented earlier in the year.

The material presented following the unit involved working with areas, volumes, perimeters, and circumferences of basic geometric figures. Again the material was new to the class and the student reaction was one of acceptance and willingness to work on the problems. Although there was not a great deal of enthusiasm present, the reaction of the class was satisfying, considering we were in the last six weeks of the school year.

I am looking forward to expanding the Fibonacci unit for next year with three classes to include work on phyllotaxis and geometric relationships. Also, I hope to present the unit earlier in the year to explore more fully the transfer of changes of attitude toward mathematics in general.

* * * * *

AMATEUR INTERESTS IN THE FIBONACCI SERIES III
RESIDUES OF u_n WITH RESPECT TO ANY MODULUS

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Dickson [1] reports that "J. L. Lagrange [2] noted that the residues of A_k and B_k with respect to any modulus are periodic." A_k and B_k are described by indicating that "Euler [3] noted that

$$(a+\sqrt{b})^k = A_k + B_k\sqrt{b}$$

implies

$$A_k = \tfrac{1}{2}[(a+\sqrt{b})^k + (a-\sqrt{b})^k], \quad B_k = \frac{1}{2\sqrt{b}}[(a+\sqrt{b})^k - (a-\sqrt{b})^k]\text{''}$$

With this as a hint I tried empirically to determine whether Lagrange's idea would work with the Fibonacci series, u_n. This may not be immediately apparent but simple empirical trials developed a number of significant revelations. Thus, starting with $u_1 = 1$, $u_2 = 1$, $u_3 = 2$, etc., the residues for consecutive u_n, modulus 5 are: 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0. This series then repeats itself endlessly, illustrating Lagrange's periodicity. This is generally true of every modulus tried from 2 to 94. Each modulus has a characteristic period which displays various individual regularities. Thus, the above period, modulus 5, is broken up by zeros into 4 groups of 5 residues each including zero. The following is a resume of the characteristics of all groups and periods determined for all moduli investigated. We define

Group: The residues, starting with the residue from $u_1 = 1$ and continuing to and including the first zero residue obtained after dividing consecutive u_n.

Period: The residues, starting with the residue from $u_1 = 1$ and continuing to and including the first zero residue which follows a residue of 1, obtained after dividing consecutive u_n. From this point, the second period and all succeeding periods will exactly duplicate the first.

Examining the period of modulus 5 given above, from the above definitions, the first group comprises 5 digits, viz., 1, 1, 2, 3, 0. The period comprises 4 groups, containing 20 residues and ending 2, 2, 4, 1, 0.

The characteristics determined in the light of the above are:

1. The sum of all residues in a period (but not, in general, in a group), is divisible by the modulus without remainder. Thus, for modulus 5, the sum of the residues in the period is 40 which is divisible by the modulus.

2. The number of groups in a period is always 1, 2, or 4.

3. If the size of a group is n, then u_n and more generally u_{an} are exactly divisible by the modulus.

4. If P_{n_1} and P_{n_2} are prime factors of the modulus $P_{n_1}P_{n_2}$, the group and period of the modulus are divisible by the group and period respectively of the P_n's. For example, modulus 10 is factored by $P_{n_1} = 2$ and $P_{n_2} = 5$. The group and period of 2 are 3 and the group and period of 5 are 5 and 20 respectively. The group size for modulus 10 is 15 (divisible by 3 and 5); the period, modulus 10, is 60, divisible by 3 and 20. This fact permits ready check of groups and period calculated for composite moduli.

It is evident that the finding listed as 3 above is not particularly helpful in determining the u_n which a given prime modulus will divide, if the group size for that modulus must be determined by actual division of consecutive u_n. Thus, the prime 103 is found to have a group of $n = 104$. To determine that u_{104} is divisible by 103 by dividing 104 consecutive u_n and knowing that u_{104} contains 22 digits, not to mention the large numbers which precede u_{104}, seems to be prohibitively laborious. Fortunately, early in the calculation of groups and periods I found a way to calculate these without any dividing at all! This was determined when it was noted that the residues are additive according to the usual Fibonacci series rule:

$$u_{n+2} = u_n + u_{n+1}$$

until the last residue is equal to or greater than the modulus.

At this time we subtract the modulus from this large residue. If the latter is equal to the modulus, the residue is zero and the group and/or period ends. If it is larger, the difference is set down as the residue in the place of the larger figure. This residue is then added to the previous residue and the

sum is compared with the residue as before. This procedure continues until the group and/or period is determined. As can be seen, all manipulations are additions and subtractions, division is never required.

Example 1. To determine the group and period for modulus 10.

Start with $u_1 = 1$, the residue, $r_1 = 1$. Add this to $u_0 = 0$ and we get the second residue $r_2 = 1$. Add r_1 to

$$r_2 = 1 + 1 = r_3 = 2 .$$

This is still smaller than modulus 10, so we continue.

$r_2 + r_3 = 1 + 2 = r_4 = 3$. $r_3 + r_4 = 2 + 3 = r_5 = 5$. $r_4 + r_5 = 3 + 5 = r_6 = 8$.

Now,

$$r_5 + r_6 = 5 + 8 = 13 .$$

This is larger than modulus 10 so we subtract 10 and get $r_7 = 3$. Now we add

$$r_6 + r_7 = 8 + 3 = 11 .$$

Again, this is larger than the modulus; we subtract 10 and get $r_8 = 1$. Now we add

$r_7 + r_8 = 3 + 1 = r_9 = 4$. $r_8 + r_9 = 1 + 4 = r_{10} = 5$. $r_9 + r_{10} = 4 + 5 = r_{11} = 9$.

Now

$$r_{10} + r_{11} = 5 + 9 = 14 .$$

Subtract 10 and we have $r_{12} = 4$.

$$r_{11} + r_{12} = 9 + 4 = 13$$

from which $r_{13} = 3$.

$$r_{12} + r_{13} = 4 + 3 = r_{14} = 7 .$$

Finally

$$r_{13} + r_{14} = 3 + 7 = 10$$

which is exactly equal to the modulus. When we subtract 10 the result $r_{15} = 0$ and the group ends. Since $r_{14} \neq 1$, the period is not yet complete and is determined by continuing the procedure. Thus, listing consecutive residues starting with r_{14} we get

7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0

ending the second group but the period continues:

9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1, 0

Here the period ends, comprising 4 groups of 15 residues each. Notice that the second period begins exactly the same way as the first: 1, 1, 2, 3, etc. Since all periods are calculated the same way and all periods, regardless of modulus, start with 1, 1, it is obvious that all periods will be exact duplicates of each other and there is no point in continuing operations. Since the group size n, modulus 10, is 15 u_{15} must be divisible by 10. We find $u_{15} = 610$, divisible by 10.

While it is evident that even this procedure is laborious for large prime numbers it is much easier than consecutive divisions of u_n. While short cuts such as this are possible in empirical investigations of the Fibonacci series, it is impossible to avoid labor altogether.

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1. Dickson, History of the Theory of Numbers, 1, 1919, Chapter XVII, p. 393.
2. Cited in Dickson as: "Additions to Euler's Algebra 2, 1774, Sections 78-79, pp. 599-607, Euler, Opra Omnia (1), 1, 619."
3. Cited in Dickson as "Novi Connior. Acad. Petrop., 18, 1773, 185; Corum. Arith., 1, 554."

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RECURRING SEQUENCES—LESSON 1

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The Fibonacci Quarterly has been publishing an abundance of material over the past five years dealing in the main with the Fibonacci sequence and its relatives. Basic to the entire undertaking is the concept of RECURRING SEQUENCE. In view of this fact, a series of some eight lessons has been prepared covering this topic. In line with the word "lesson," examples of principles will be worked out in the articles and a number of "problems" will be included for the purpose of providing "exercise" in the material presented. Answers to these problems will be included on another page so that people may be able to check their work against them.

In this first lesson, the idea of sequence and recursion relation will be considered in a general way. A sequence is an ordered set of quantities. The sequence is finite if the set of quantities terminates; it is infinite if it does not. The prototype of all sequences is the sequence of positive integers: 1, 2, 3, 4, 5, $\cdots$. Other sequences, some quite familiar, are the following:

1, 3, 5, 7, 9, 11, 13, $\cdots$

2, 4, 6, 8, 10, 12, 14, 16, $\cdots$

1, 2, 4, 8, 16, 32, 64, $\cdots$

2, 6, 18, 54, 162, 486, $\cdots$

1, 2, 6, 24, 120, 720, 5040, 40320, $\cdots$

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, $\cdots$

1, 4, 9, 16, 25, 36, 49, 64, $\cdots$.

1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, $\cdots$

For convenience of reference, the terms of sequences can be identified by the following notation: $a_1, a_2, a_3, a_4, a_5, \cdots, a_n, \cdots$. One of the common ways of providing a compact representation of a sequence is to specify a formula for the nth term. For the positive integers, $a_n = n$; for the odd integers 1, 3, 5, 7, $\cdots$, $a_n = 2n - 1$; for the even integers 2, 4, 6, 8, $\cdots$

$a_n = 2n$. The nth terms of the remaining sequences given above are listed herewith.

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ \cdots,\ a_n = 2^{n-1}$$
$$2,\ 6,\ 18,\ 54,\ 162,\ 486,\ \cdots,\ a_n = 2 \cdot 3^{n-1}$$
$$1,\ 2,\ 6,\ 24,\ 120,\ \cdots,\ a_n = n!$$
$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ \cdots,\ a_n = n(n+1)/2$$
$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ \cdots,\ a_n = n^2$$
$$1,\ 1/2,\ 1/3,\ 1/4,\ \cdots,\ a_n = 1/n\ .$$

There is, however, a second way of specifying sequences and that is the recursion approach. The word recursion derives from recur and indicates that something is happening over and over. When in a sequence, there is an operation which enables us to find a subsequent term by using previous terms according to some well-defined method, we have what can be termed a recursion sequence. Again, the prototype is the sequence of positive integers which is completely specified by giving the first term $a_1 = 1$ and stating the recursion relation

$$a_{n+1} = a_n + 1\ .$$

This is the general pattern for a recursion sequence; one or more initial terms must be specified; then an operation (or operations) is set down which enables one to generate any other term of the sequence.

Going once more to some of our previous sequences, the recursion representations are as follows:

$$1,\ 3,\ 5,\ 7,\ \cdots,\ a_1 = 1;\quad a_{n+1} = a_n + 2\ .$$
$$2,\ 4,\ 6,\ 8,\ \cdots,\ a_1 = 2\ ;\quad a_{n+1} = a_n + 2.$$
$$1,\ 2,\ 4,\ 8,\ 16,\ \cdots,\ a_1 = 1;\quad a_{n+1} = 2a_n\ .$$
$$2,\ 6,\ 18,\ 54,\ 162,\ \cdots,\ a_1 = 2;\quad a_{n+1} = 3a_n\ .$$
$$1,\ 2,\ 6,\ 24,\ 120,\ \cdots,\ a_1 = 1;\quad a_{n+1} = (n+1)a_n\ .$$

Is it possible in all instances to give this dual interpretation to a sequence, that is, to specify the nth term on the one hand and to provide a recursion

definition of the sequence on the other? Is it not wise to say in an absolute manner what is possible or impossible in mathematics. But at least it can be stated that sequences which are readily representable by their n^{th} term may be difficult to represent by recursion and on the contrary, sequences which can be easily represented by recursion may not have an obvious n^{th} term. For example, what is the recursion relation for the sequence defined by:

$$a_n = \sqrt[n]{\frac{\log n}{\sqrt[3]{n}}} \qquad ?$$

Or on the other hand, if $a_1 = 2$, $a_2 = 3$, $a_3 = 5$, and

$$a_{n+1} = \frac{7a_n + 5a_{n-1}}{a_{n-2}}$$

what is the expression for the nth term?

However, in most of the usual cases, it is possible to have both the nth term and the recursion formulation of a sequence. Many of the common sequences, for example, have their n^{th} term expressed as a polynomial in n. In such a case, it is possible to find a corresponding recursion relation. In fact, for all polynomials of a given degree, there is just one recursion relation corresponding to them, apart from the initial values that are given. Let us examine this important case.

Our discussion will be based on what are known as finite differences. Given a polynomial in n, such as $f(n) = n^2 + 3n - 1$, we define

$$\Delta f(n) = f(n+1) - f(n)$$

(Read "the first difference of f(n)" for $\Delta f(n)$.) Let us carry out this operation.

$$\Delta f(n) = (n+1)^2 + 3(n+1) - 1 - (n^2 + 3n - 1)$$

$$\Delta f(n) = 2n + 4 .$$

Note that the degree of $\Delta f(n)$ is one less than the degree of the original polynomial. If we take the difference of $\Delta f(n)$ we obtain the second difference of $f(n)$. Thus

$$\Delta^2 f(n) = 2(n+1) + 4 - (2n+4) = 2$$

Finally, the third difference of $f(n)$ is $\Delta^3 f(n) = 2 - 2 = 0$. The situation portrayed here is general. A polynomial of degree m has a first difference of degree $m - 1$, a second difference of degree $m - 2, \cdots$, an mth difference which is constant and an $(m+1)^{st}$ difference which is zero. Basically, this result depends on the lead term of highest degree. We need only consider then what happens to $f(n) = n^m$ when we take a first difference.

$$\Delta f(n) = (n+1)^m - n^m = n^m + mn^{m-1} \cdots - n^m$$

$\Delta f(n) = mn^{m-1} + \cdots$ terms of lower degree. Thus the degree drops by 1.

Suppose we designate the terms of our sequence as T_n. Then

$$\begin{aligned}
\Delta T_n &= T_{n+1} - T_n \\
\Delta^2 T_n &= T_{n+2} - T_{n+1} - (T_{n+1} - T_n) = T_{n+2} - 2T_{n+1} + T_n \\
\Delta^3 T_n &= T_{n+3} - 2T_{n+2} + T_{n+1} - (T_{n+2} - 2T_{n+1} + T_n)
\end{aligned}$$

or

$$\Delta^3 T_n = T_{n+3} - 3T_{n+2} + 3T_{n+1} - T_n .$$

Clearly the coefficients of the Pascal triangle with alternating signs are being generated and it is clear from the operation that this will continue.

We are now ready to transform a sequence with a term expressed as a polynomial in n into a recursion relation. Consider again:

$$T_n = n^2 + 3n - 1 .$$

Take the third difference of both sides. Then

$$\Delta^3 T_n = \Delta^3(n^2 + 3n - 1)$$

But the third difference of a polynomial of the second degree is zero. Hence

$$T_{n+3} - 3T_{n+2} + 3T_{n+1} - T_n = 0$$

or

$$T_{n+3} = 3T_{n+2} - 3T_{n+1} + T_n$$

is the required recursion relation for all sequences whose term can be expressed as a polynomial of the second degree in n.

An interesting particular case is the arithmetic progression whose n^{th} term is

$$T_n = a + (n - 1)d ,$$

where a is the first term and d the common difference. For example, if a is 5 and d is 4,

$$T_n = 5 + 4(n - 1) = 4n - 1 .$$

In any event, an arithmetic progression has a term which can be expressed as a polynomial of the first degree in n. Accordingly the recursion relation for all arithmetic progression is:

$$\Delta^2 T_n = 0$$

or

$$T_{n+2} = 2T_{n+1} - T_n .$$

The recursion relation for the geometric progression with ratio r is evidently

$$T_{n+1} = rT_n .$$

For example, 2, 18, 54, 162, $\cdots$ is specified by $a_1 = 2$, $T_{n+1} = 3T_n$.

This takes care of our listed sequences except the factorial and the reciprocal of n. For the factorial:

$$T_{n+1} = (n + 1)T_n .$$

However, we do not have a pure recursion relation to a subsequent from previous terms of the sequence. We need to eliminate n in the coefficient to bring this about. Now

$$n = T_n / T_{n-1}$$

and

$$n + 1 = T_{n+1} / T_n .$$

Thus

$$T_{n+1} / T_n - T_n / T_{n-1} = 1$$

so that

$$T_{n+1} = T_n (T_n + T_{n-1}) / T_{n-1} .$$

Again for $T_n = 1/n$, we have

$$n = 1/T_n , \quad n + 1 = 1/T_{n+1}, \quad 1/T_{n+1} - 1/T_n = 1$$

so that

$$T_{n+1} = T_n / (1 + T_n) .$$

PROBLEMS

1. Find the nth term and the recursion relation for the sequence: 2, 6, 12, 20, 30, 42, 56, $\cdots$.
2. Find the nth term and the recursion relation for the sequence: 1, 4, 7, 10, 13, 16, $\cdots$.
3. Determine the nth term and the recursion relation for the sequence: 1, 8, 27, 64, 125, 216, 343, $\cdots$.
4. For $T_1 = 1$, $T_2 = 3$ and $T_{n+1} = T_n / T_{n-1}$, find a form of expression for the nth term. (It may be more convenient to do this using a number of formulas.)
5. Find the recursion relation for the sequence with the term $T_n = \sqrt{n}$.
6. What is the recursion relation for a sequence whose term is a cubic polynomial in n?
7. If a is a positive constant, determine the recursion relation for the sequence with the term $T_n = a^n$.
8. Find a recursion relation corresponding to $T_{n+1} = T_n + 2n + 1$ which does not involve n except in the subscripts nor a constant except as a coefficient.
9. Find an expression(s) for the nth term of the sequence the recursion relation $T_n T_{n+1} = 1$, where $T_1 = a$ (a not zero).
10. For the sequence with term $T_n = n/(n + 1)$, find a recursion relation with n occurring only in subscripts.

See page 260 for answers to problems.

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EVEN PERFECT NUMBERS AND SEVEN

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Over the years number theory has given both professional and amateur mathematicians many hours of frustration and enjoyment. The study of perfect numbers is an area of the theory of numbers which dates back to antiquity.

A positive integer n is a perfect number if and only if the sum of its positive integer divisors is $2n$. For example, 28 is a perfect number since the positive integer divisors of 28 are 1, 2, 4, 7, 14, and 28 and

$$1 + 2 + 4 + 7 + 14 + 28 = 56 = 2 \times 28 .$$

The first few perfect numbers are 6; 28; 496; 8,128; 33,550,336; and 8,589,056. Notice that each of these perfect numbers is even. Although no odd perfect number has ever been found, mathematicians have been unable to prove that none exists. It is also unknown whether or not the number of perfect numbers is infinite.

Euclid showed that if n is a positive integer of the form $2^{p-1}(2^p - 1)$ where $2^p - 1$ is a prime then n is a perfect number. Later Euler established that every even perfect number is of the Euclid type. A necessary condition that $2^p - 1$ be a prime is that p be a prime. Thus all even perfect numbers have the form $2^{p-1}(2^p - 1)$ where p is a prime number.

If $p = 3$ then $2^{p-1}(2^p - 1) = 28$ which is a multiple of 7. Since 3 is the only multiple of three which is a prime number, all other prime numbers are of the form $3j + 1$ or $3j + 2$. A careful investigation of the even perfect numbers different from 28 given above yields the following table.

p	$2^{p-1}(2^p - 1)$	p	$2^{p-1}(2^p - 1)$
2	$6 = 7 \cdot 0 + 6$	7	$8128 = 7 \cdot 1161 + 1$
5	$496 = 7 \cdot 70 + 6$	13	$33550336 = 7 \cdot 4792905+1$
17	$8589869056 = 7 \cdot 1227124150 + 6$		

This leads us to conjecture that if $n = 2^{p-1}(2^p - 1)$ is an even perfect number different from 28 then n is of the form $7k + 1$ or $7k + 6$ according as p is

of the form $3j + 1$ or $3j + 2$. Before attempting to prove this conjecture, we shall establish some preliminary results.

Lemma 1. For each positive integer w, $2^{3w} = 7t + 1$ for some positive integer t.

Proof. $2^3 = 8 = 7 \cdot 1 + 1$. Assume that $2^{3w} = 7r + 1$. Then

$$2^{3(x+1)} = 2^{3x+3} = 2^{3x} \cdot 2^3 = (7r + 1)8 = 7(8r+1)$$

and the lemma follows by the principle of mathematical induction.

Lemma 2. For each nonnegative integer z, $2^{3z+1} = 7s + 3$ for some nonnegative integer s.

Proof. If $z = 0$ then $2^{3z+1} = 2 = 7 \cdot 0 + 2$. Assume that $2^{3y+1} = 7m + 2$. Then

$$2^{3(y+1)+1} = 2^{(3y+1)+1} = 2^{3y+1} \cdot 2^3 = (7m + 2)8 = 7(8m+2) + 2$$

and the lemma follows by the principle of mathematical induction.

Theorem. If $n = 2^{p-1}(2^p - 1)$ is an even perfect number different from 28 then n is of the form $7k + 1$ or $7k + 6$ according as p is of the form $3j + 1$ or $3j + 2$.

Proof. $n \neq 28$ implies that $p \neq 3$. Since p is a prime number and $p \neq 3$, p is either of the form $3j + 1$ or $3j + 2$.

Case 1. $p = 3j + 1$. Then $p - 1 = 3j$ and $2^{p-1} = 2^{3j} = 7t + 1$ by Lemma 1. Hence $2^p = 2 \cdot 2^{p-1} = 14t + 2$, from which it follows that $2^p - 1 = 14t + 1$. Thus $n = 2^{p-1}(2^p - 1) = (7t + 1)(14t + 1) = 7(14t^2 + 3t) + 1$.

Case 2. $p = 3j + 2$. Then $p - 1 = 3j + 1$ and $2^{p-1} = 2^{3j+1} = 7s + 2$ by Lemma 2. Hence $2^p = 2 \cdot 2^{p-1} = 14s + 4$, from which it follows that $2^p - 1 = 14s + 3$. Thus $n = 2^{p-1}(2^p - 1) = (7s + 2)(14s + 3) = 7(14s + 7s) + 6$.

Let n be an even perfect number. It can be shown that if $n \neq 6$ then n yields the remainder 1 when divided by 9; if $n \neq 6$ and $n \neq 496$ then n ends with 16, 28, 36, 56, or 76 when n is written in base 10 notation; and if $n \neq 6$ then n has the remainder 1, 2, 3, or 8 when divided by 13.

[Continued on p. 304.]

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A. P. Hillman
University of New Mexico, Albuquerque, N.M.

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-142 Proposed by William D. Jackson, SUNY at Buffalo, Amherst, N. Y.

Define a sequence as follows: $A_1 = 2$, $A_2 = 3$, and $A_n = A_{n-1}A_{n-2}$ for $n > 2$. Find an expression for A_n.

B-143 Proposed by Raphael Finkelstein, Tempe, Arizona.

Show that the following determinant vanishes when a and d are natural numbers:

$$\begin{vmatrix} F_a & F_{a+d} & F_{a+2d} \\ F_{a+3d} & F_{a+4d} & F_{a+5d} \\ F_{a+6d} & F_{a+7d} & F_{a+8d} \end{vmatrix} .$$

What is the value of the determinant one obtains by replacing each Fibonacci number by the corresponding Lucas number?

B-144 Proposed by J. A. H. Hunter, Toronto, Canada

In this alphametic each distinct letter stands for a particular but different digit, all ten digits being represented here. It must be the Lucas series, but what is the value of the SERIES?

ONE
THREE
START
L
SERIES

B-145 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Given an unlimited supply of each of two distinct types of objects, let $f(n)$ be the number of permutations of n of these objects such that no three consecutive objects are alike. Show that $f(n) = 2F_{n+1}$, where F_n is the nth Fibonacci number.

B-146 Proposed by Walter W. Horner, Pittsburgh, Pennsylvania.

Show that $\pi = \text{Arctan}\,(1/F_{2n}) + \text{Arctan}\,F_{2n+1} + \text{Arctan}\,F_{2n+2}$.

B-147 Proposed by Edgar Karst, University of Arizona, Tucson, Arizona, in honor of the 66th birthday of Hansraj Gupta on Oct. 9, 1968.

Let

$$S = (1/3 + 1/5) + (1/5 + 1/7) + \cdots + (1/32717 + 1/32719)$$

be the sum of the sum of the reciprocals of all twin primes below 2^{15}.. Indicate which of the following inequalities is true:

$$\text{(a)}\quad S \le \pi^2/6, \qquad \text{(b)}\quad \pi^2/6 < S < \sqrt{e} \qquad \text{(c)}\quad \sqrt{e} < S.$$

SOLUTIONS

NOTE: The name of A. C. Shannon was inadvertently omitted from the list of solvers of B-109.

LINEAR COMBINATION OF GEOMETRIC SERIES

B-124 Proposed by J.H.Butchart, Northern Arizona University, Flagstaff, Arizona.

Show that

$$\sum_{i=0}^{\infty} (a_i/2^i) = 4 ,$$

where

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \cdots$$

are the Fibonacci numbers.

Solution by R. L. Mercer, University of New Mexico, Albuquerque, N. Mex.

Convergence of the series follows from

$$\lim_{n\to\infty} (a_{n+1}/a_n) = (1+\sqrt{5})/2$$

and the ratio test. Let T be the value of the series. Then

$$T = \sum_{i=0}^{\infty} (a_{i+2} - a_{i+1})/2^i = 4\sum_{i=0}^{\infty} a_{i+2}/2^{i+2} - 2\sum_{i=0}^{\infty} a_{i+1}/2^{i+1}$$

and

$$T = 4(T - a_0 - a_1/2) - 2(T - a_0) .$$

Solving, we find

$$T = 2(a_0 + a_1) = 2a_2 = 4 .$$

Also solved by Dewey C. Duncan, Bruce W. King, J. D. E. Konhauser, F. D. Parker, C. B. A. Peck, A. C. Shannon (Australia), John Wessner, David Zeitlin, and the proposer.

EDITORIAL NOTE:

Since

$$f(x) = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n = \sum_{i=0}^{\infty} a_i x^i$$

Substituting

$$x = \tfrac{1}{2} < \frac{\sqrt{5} - 1}{2} = \left(\frac{1 + \sqrt{5}}{2}\right)^{-1}$$

yields

$$f(\tfrac{1}{2}) = \sum_{i=0}^{\infty} a_i / 2^i = \frac{1}{1 - \frac{1}{2} - \frac{1}{4}} = 4 ,$$

while $f(-1/2) = 4/5$.

V. E. H.

A NON-INTEGRAL SUM

B-125 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Is

$$\sum_{k=3}^{n} \frac{1}{F_k}$$

ever an integer? Explain.

Solution by Dewey C. Duncan, Los Angeles, California.

The summation

$$\sum_{k=3}^{n} \frac{1}{F_k}$$

is never an integer, since

(1) For $n = 3, 4, 5$, the summation yields $1/2$, $5/6$, $31/30$, respectively.

(2) For $n > 5$, the summation yields a sum that is greater than 1 and less than 1.5, since

$$\frac{F_{2k-1}}{F_{2k}} > \frac{F_{2k+1}}{F_{2k+2}}, \qquad \frac{F_{2k}}{F_{2k+1}} < \frac{F_{2k+2}}{F_{2k+3}},$$

and

$$\lim_{n=\infty} \frac{F_n}{F_{n+1}} = \frac{\sqrt{5}-1}{2}$$

From

$$F_{2k-1}F_{2k+1} - F_{2k}^2 = (-1)^{2k}$$

one implies that for all $k \geq 1$,

$$\frac{F_{2k}}{F_{2k+2}} < \frac{F_{2k}}{F_{2k+1}} < \frac{F_{2k-1}}{F_{2k}} < \frac{F_{2k-1}}{F_{2k}}$$

Therefore, since

$$\frac{F_3}{F_4} = \frac{2}{3},$$

we conclude that, for $k > 3$,

$$\frac{F_k}{F_{k+1}} < \frac{2}{3}.$$

Consequently,

$$\sum_{k=3}^{\infty} \frac{1}{F_k} < \frac{1}{2}\left[1 + (2/3) + (2/3)^2 + (2/3)^3 + \cdots\right],$$

whence,

$$\sum_{k=3}^{\infty} \frac{1}{F_k} < 3/2 \quad \text{Q. E. D.}$$

Also solved by R. L. Mercer, C. B. A. Peck, and the proposer.

GOOD ADVICE

B-126 Proposed by J. A. H. Hunter, Toronto, Canada.

Each distinct letter in this alphametic stands, of course, for a particular and different digit. The advice is sound, for our FQ is truly prime. What do you make of it all?

```
 READ
   FQ
 READ
   FQ
 ----
 DEAR
```

Solution by Charles W. Trigg, San Diego, California.

From the units' column R is even. Since $2R + 1 = D$, then $(R,D) = (2, 5)$ or $(4, 9)$.

If $(R,D) = (4,9)$, then (since FQ is prime) $Q = 3$ and $F = 1, 2, 5, 7$, or 8. Furthermore, $2F + A + 2$ is a multiple of ten. Thus $(F,A) = (1,6)$, $(5,8)$ or $(8,2)$. But each of these pairs leads to a value of E which duplicates another digit.

If $(R,D) = (2,5)$ then (since FQ is prime) $Q = 1$, and $F = 3, 4, 6$ or 7. Now $2F + A + 1$ is a multiple of ten, so $(F, A) = (6, 7)$ is the sole solution. Whereupon $E = 10 - 2$ or 8. The unique reconstruction of the addition is

$$\begin{array}{r} 2875 \\ 61 \\ 2875 \\ \underline{61} \\ 5872 \end{array}$$

Additional solution by David Zeitlin, Minneapolis, Minnesota.

$$\begin{array}{r} 0841 \\ 79 \\ 0841 \\ \underline{79} \\ 1840 \end{array}$$

Also solved by H. D. Allen (Canada), A. Gommel, R. L. Mercer, John W. Milsom, C. B. A. Peck, and the proposer.

CONGRUENCES

B-127 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tennessee.

Show that

$$2^n L_n \equiv 2 \pmod 5 ,$$
$$2^n F_n \equiv 2n \pmod 5 .$$

Solution by John Wessner, Melbourne, Florida.

We proceed by induction. Both results are true for $n = 1, 2$. If we assume that the first for $n = k$ and $n = k + 1$, then we have

$$2^k L_k \equiv 2 \pmod 5, \quad 2^{k+1} L_{k+1} \equiv 2 \pmod 5 .$$

Combining these,

$$2^{k+2} L_{k+2} = 2(2^{k+1} L_{k+1} + 2 \cdot 2^k L_k) \equiv 2(2 + 2 \cdot 2) \equiv 2 \pmod 5 .$$

Similarly, in the second case we assume

$$2^k F_k \equiv 2k \pmod 5, \quad 2^{k+2} F_{k+2} \equiv 2(k+1) \pmod 5 .$$

Combining these gives

$$\begin{aligned} 2^{k+2} F_{k+2} &= 2(2^{k+1} F_{k+1} + 2 \cdot 2^k F_k) \\ &\equiv 2\ 2(k+1) + 2 \cdot 2k \equiv 12k + 4 \\ &\equiv 2(k+2) \pmod 5 . \end{aligned}$$

Also solved by Herta T. Freitag, R. L. Mercer, C. B. A. Peck, A. C. Shannon (Australia), Paul Smith (Canada), David Zeitlin, and the proposer.

GENERALIZED SEQUENCES

B-128 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

Let f_n be the generalized Fibonacci sequence with $f_1 = a$, $f_2 = b$, and $f_{n+1} = f_n + f_{n-1}$. Let g_n be the associated generalized Lucas sequence defined by $g_n = f_{n-1} + f_{n+1}$. Also let $S_n = f_1 + f_2 + \cdots + f_n$. It is true that $S_4 = g_4$ and $S_8 = 3g_6$. Generalize these formulas.

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania.

By induction,

$$S_n = f_{n+2} - f_2 , \qquad f_n = F_{n-1} f_2 + F_{n-2} f_1 ,$$

and

$$g_n = L_{n-1} f_2 + L_{n-2} f_1 .$$

Thus

$$S_{4n} = f_{4n-2} - f_2 = (F_{4n-1} - 1) f_2 = F_{4n} f_1$$

and

$$F_{2n}g_{2n+2} = F_{2n}(L_{2n+1}f_2 + L_{2n}f_1) .$$

These are equal, since

$$F_{4n} = F_{2n}L_{2n}$$

and

$$F_{4n-1} - 1 = F_{2n}L_{2n+1} .$$

Thus we have

$$S_{4n} = F_{2n}g_{2n+2} .$$

P. S. $S_n = f_{n+2} - f_2$ occurs in B-20, FQ, Vol. 2, pp. 76-77.

Also solved by Bruce W. King, A. C. Shannon (Australia), David Zeitlin, and the proposer.

MODIFIED GOLDEN RATIO

B-129 Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.

For a given positive integer, k, find

$$\lim_{n\to\infty} (F_{n+k} / L_n) .$$

Solution by Bruce W. King, Burnt Hills — Balston Lake H.S., Burnt Hills, N. Y.

Let $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$. Then $|b/a| < 1$ and it follows that $(b/a)^n \to 0$ as $n \to \infty$. Hence

$$F_{n+k}/L_n = (a^{k+k} - b^{n+k})/\sqrt{5}(a^n + b^n) = (a^k/\sqrt{5})[1 - (b/a)^{n+k}]/[1 + (b/a)^n]$$

approaches $a^k/\sqrt{5}$ as n goes to infinity.

Also solved by R. L. Mercer, C. B. A. Peck, A. C. Shannon (Australia), Paul Smith (Canada), John Wessner, and the proposer.

MULTINOMIAL COEFFICIENTS

B-130 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let coefficients $c_j(n)$ be defined by

$$(1 + x + x^2)^n = c_0(n) + c_1(n)x + c_2(n)x^2 + \cdots + c_{2n}(n)x^{2n}$$

and show that

$$\sum_{j=0}^{2n} c_j(n)^2 = c_{2n}(2n) .$$

Generalize to

$$(1 + x + x^2 + \cdots + x^k)^n ,$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Let

$$\begin{aligned} Q(x) &= Q_{k,n}(x) = (1 + x + x^2 + \cdots + x^k)^n \\ &= q_0(n) + q_1(n)x + q_2(n)x^2 + \cdots + q_{kn}(n)x^{kn} . \end{aligned}$$

Since

$$x^{kn}Q(1/x) = Q(x) ,$$

we have

$$q_j(n) = q_{kn-j}(n)$$

for $j = 0, 1, \cdots, kn$. Equating coefficients of x^{kn} in

$$(1 + x + \cdots + x^k)^{2n} = \left[(1 + x + \cdots + x^k)^n\right]^2 ,$$

we obtain

$$q_{kn}(2n) = \sum_{r=0}^{kn} q_r(n)q_{kn-r}(n) = \sum_{r=0}^{kn} [q_r(n)]^2 .$$

Also solved by R. L. Mercer, R. W. Mercer, A. C. Shannon (Australia), and the proposer.

A FIBONACCI-LUCAS IDENTITY

B-131 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tennessee

Prove that for m odd

$$\frac{L_{n-m} + L_{n+m}}{F_{n-m} + F_{n+m}} = \frac{5F_n}{L_n}$$

and for m even

$$\frac{F_{n-m} + F_{n+m}}{L_{n-m} + L_{n+m}} = \frac{F_n}{L_n} .$$

Solution by John Wessner, Melbourne, Florida.

The following properties of the Fibonacci and Lucas numbers can easily be proved by the use of the Binet formula: (1) For odd values of m,

$$L_{n-m} + L_{n+m} = 5F_nF_m ,$$
$$F_{n-m} + F_{n+m} = L_nF_m .$$

(2) for even values of m,

$$L_{n-m} + L_{n+m} = L_nL_m ,$$

$$F_{n-m} + F_{n+m} = F_nL_m .$$

[Continued on p. 304.]

* * * * *

RECREATIONAL MATHEMATICS

Joseph S. Madachy
4761 Bigger Rd., Kettering, Ohio

Before I go on with new business, readers of this column should make the following corrections in the February 1968 issue of the Fibonacci Quarterly (Vol. 6, No. 1):

Page 64: In 187^{16}, the fifth group of five digits should read 87257 and not 78257.

Page 67: The last few words in the fifth line under "A Fibonacci Variation" should read "... ${}_nF$ series in which each..."

Page 67: Under "Some Fibonacci Queries," for $F_{18} = 2584$, correct the addition to read $2 + 5 + 8 + 4 = 19$.

Some browsing by myself through past issues of the Fibonacci Quarterly disclosed an article by Dewey C. Duncan [2] in which Mr. Duncan anticipated — in a slightly different manner — my Fibonacci variation [4, page 67]. I had formed an ${}_nF$ series in which each term is the sum of the next two terms, starting with ${}_0F = 0$, ${}_1F = 1$:

$$0, \quad 1, \quad -1, \quad 2, \quad -3, \quad 5, \quad -8, \quad 13, \quad -21, \quad 34, \quad -55,$$

etc.

Mr. Duncan introduces Fibonacci number relationships involving zero and negative indices, with

$$F_0 = 0, \quad F_{-1} = 1, \quad F_{-2} = -1, \quad F_{-3} = 2$$

and, generally, $F_{-n} = (-1)^{n+1}F_n$. The Duncan series thus formed is

$$0, \quad 1, \quad -1, \quad 2, \quad -3, \quad 5, \quad -8, \quad 13, \quad -21, \quad 34, \quad -55,$$

etc., which is identical to the ${}_nF$ series given previously.

If n is zero or even, we have $F_{-n} = -F_n$ and ${}_nF = -F_n$ for n odd, we have $F_{-n} = F_n$ and ${}_nF = F_n$.

Such is the beauty of the Fibonacci numbers and their variations!

PRODUCTS WITH DIFFERENT FACTORS CONTAINING THE SAME DIGITS

The following collection was derived from the Nelson table described in [4, pp. 61-63]. The list shows products with two sets of factors containing the same digits, e.g. (6)(4592) = (56)(492). Trivial solutions or those derived from simpler forms, are not listed. For example

$$(23)(794) = (23)(794)$$

$$(6)(500) = (600)(5)$$

and others similar to the above are excluded.

The list contains one set of factors (the 8th set) in which the digits are in the same order, and four sets of factors (the first four) in which the digits are in reverse order.

If the list proves incomplete, I would deeply appreciate new results found by readers.

$(\text{Factors})_1$		$(\text{Factors})_2$	Product
(6)(21)	=	126	126
(3)(51)	=	153	153
(50)(6)	=	(60)(5)	300
(4)(567)	=	(7)(6)(54)	2,268
(6)(3128)	=	(23)(816)	18,768
(4)(72)(86)	=	24, 768	24,768
(6)(4592)	=	(56)(492)	27,552
(7)(3942)	=	(73)(9)(42)	27,594
(9)(3465)	=	(63 x 495)	31,185
(53)(781)	=	(71)(583)	41,393
(9)(7128)	=	(81)(792)	64,152
(4)(56)(729)	=	(9)(24)(756)	163,296
(6)(93)(428)	=	(248)(963)	238,824

(7)(52)(918)	=	(9)(51)(728)	334,152
(92)(8736)	=	(96)(8372)	803,712
(6)(7)(84) (531)	=	(8)(413)(567)	1,873,368
(82)(53671)	=	(562)(7831)	4,401,022
(8) (935721)	=	(9)(831752)	7,485,768
(24)(756)(813)	=	(54)(273168)	14,751,072
(9)(76)(25143)	=	(57)(493)(612)	17,197,812
(4)(86)(53217)	=	(216)(84753)	18,306,648
(34)(96)(5721)	=	(576)(32419)	18,673,344
(4)(657)(8213)	=	21,583,764	21,583,764
(9)(561)(4372)	=	(594)(37162)	22,074,228
(64)(78)(9251)	=	(96)(572)(841)	46,180,992

In the April 1968 issue of the Fibonacci Quarterly [5, p. 166], I had asked you to demonstrate that no consecutive set of Fibonacci numbers could be used to form a magic square. In any $n \times n$ (n must be greater than 2) magic square composed of n^2 positive integers, the magic constant (the sum of the integers in each row, column, and long diagonal) is the sum of all the integers divided by n. Therefore, any integer appearing in a magic square must be smaller than the magic constant.

The demonstration involves showing that the largest integer appearing in an array of consecutive Fibonacci numbers is larger than the magic constant — hence such a magic square is impossible.

The sum of the first p Fibonacci numbers is $F_{p+2} - 1$, where F_{p+2} is the $(p + 2)^{th}$ Fibonacci number. The sum of any q consecutive Fibonacci numbers, where F_p is the first and F_{p+q-1} is the last term is

$$(F_{p+q+1} - 1) - (F_{p+1} - 1) = F_{p+q+1} - F_{p+1} .$$

Let F_p be the first integer in a series of n^2 consecutive Fibonacci numbers. The largest will be F_{p+n^2-1} and the sum of these n^2 terms will be

$$F_{p+n^2+1} - F_{p+1} = S_{array} \tag{1}$$

where S_{array}, then, is the sum of the integers in an $n \times n$ array of n^2 consecutive Fibonacci numbers. From equation (1) we can write

(2) $$S_{array} < F_{p+n^2+1}$$

Three consecutive Fibonacci numbers, starting with F_{p+n^2-1} are:

$$F_{p+n^2-1}, \quad F_{p+n^2}, \quad F_{p+n^2+1}$$

where

$$F_{p+n^2+1} = F_{p+n^2-1} + F_{p+n^2} .$$

Also, in any set of three consecutive Fibonacci numbers (excluding the first three 1, 1, 2), we have

$$F_{p+n^2} - F_{p+n^2-1} < F_{p+n^2-1}$$

or

$$F_{p+n^2} = F_{p+n^2-1} + K ,$$

where

$$K < F_{p+n^2-1} .$$

Then

$$F_{p+n^2+1} = F_{p+n^2-1} + F_{p+n^2-1} + K = 2F_{p+n^2-1} + K .$$

Since $K < F_{p+n^2-1}$ we have

$$2F_{p+n^2-1} + K < 3F_{p+n^2-1}$$

or

(3) $$F_{p+n^2+1} < 3F_{p+n^2-1} .$$

From inequalities (2) and (3) we have

(4) $$S_{array} < 3F_{p+n^2-1} .$$

If we divide (4) by 3 we obtain

$$\frac{S_{array}}{3} < F_{p+n^2-1} .$$

That is, the magic constant for a 3 x 3 array of 9 consecutive Fibonacci numbers will be less than the largest Fibonacci number in the array. It follows that

$$\frac{S_{array}}{n} < F_{p+n^2-1} , \quad (n < 3)$$

where $(S_{array})/n$ is the magic constant for an n x n array, is also true — and so consecutive Fibonacci numbers cannot be used to construct magic squares.

Some general results concerning Fibonacci numbers and magic squares appear in [1]. There Brown proves the general case that no set of distinct Fibonacci numbers can form a magic square.

Also in [3] Freitag shows a magic square constructed with Fibonacci numbers and sums of Fibonacci numbers. One magic square is shown which has terms, each of which is composed of the sum of two Fibonacci numbers.

This last item raised a trick question which I pass on to readers: Can a magic square be constructed in which each term is the sum of two consecutive Fibonacci numbers?

This column for the December 1968 issue will contain an article by Free Jamison and V. E. Hoggatt, Jr., on the dissection of a square into acute isosceles triangles — an extension of a familiar idea. Also, as a result of some work by Charles W. Trigg appearing in the July 1968 issue of the Journal of Recreational Mathematics, I'll present some recreations in instant division.

REFERENCES

1. John L. Brown, Jr., "Reply to Exploring Fibonacci Magic Squares," Fibonacci Quarterly, Vol. 3, No. 2, April 1965, page 146.
2. Dewey C. Duncan, "Chains of Equivalent Fibonacci-Wise Triangles," Fibonacci Quarterly, Vol. 5, No. 1, February 1967, pp. 87-88.
3. Herta T. Freitag, "A Magic Square involving Fibonacci Numbers," Fibonacci Quarterly, Vol. 6, No. 1, February 1968, pp. 77-80.
4. Joseph S. Madachy, "Recreational Mathematics," Fibonacci Quarterly, Vol. 6, No. 1, February 1968, pp. 60-68.
5. Joseph S. Madachy, "Recreational Mathematics," Fibonacci Quarterly, Vol. 6, No. 2, April 1968, pp. 162-166.

* * * * *

[Continued from p. 287.]

It is also relatively easy to demonstrate that a positive integer n is a perfect number if and only if the sum of the reciprocals of the positive integer divisors of n is 2.

If you have some free time why don't you investigate the topic of perfect numbers or, better yet, why don't you suggest it as a possible project for some talented student in one of your high school mathematics classes?

* * * * *

[Continued from p. 298.]

With these the desired results are immediately available.

Also solved by Herta T. Freitag, C. B. A. Peck, A. C. Shannon (Australia), and the proposer.

* * * * *

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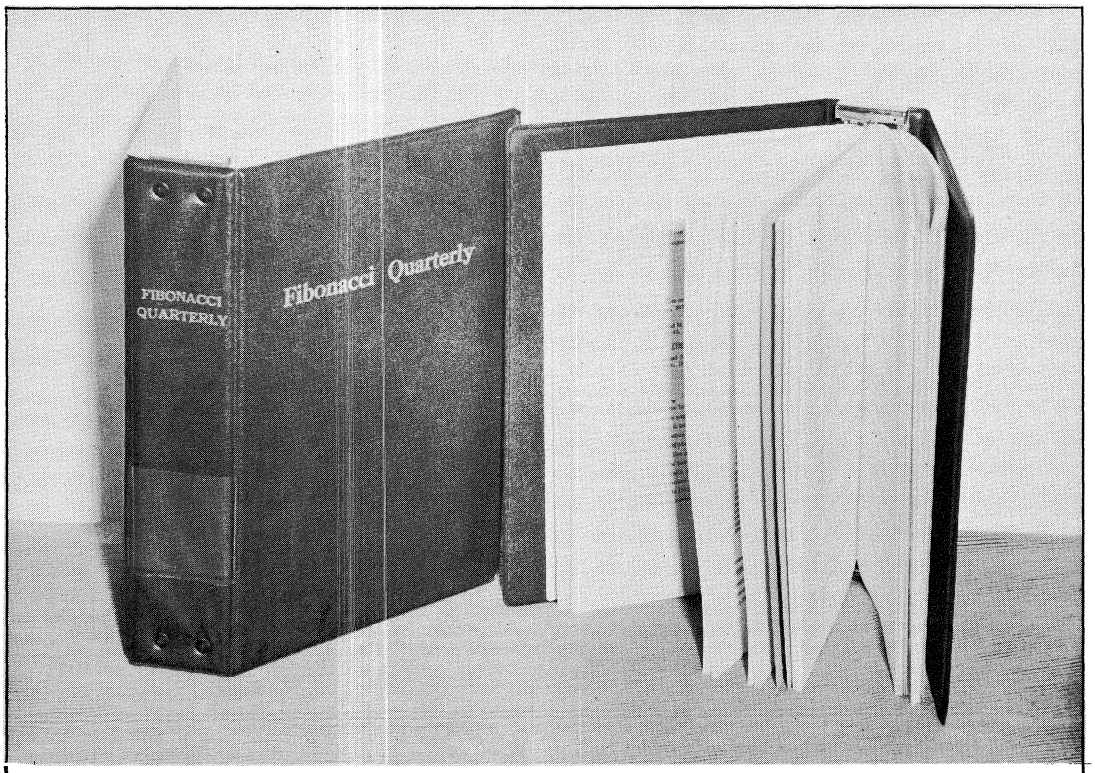

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