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GENERATION OF STIRLING NUMBERS BY MEANS OF SPECIAL PARTITIONS OF NUMBERS

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1. INTRODUCTION

Stirling numbers of the First and Second Kinds appear as numerical coefficients in expressions relating factorials of variables to powers of the variable and vice versa. Riordan [1] investigates the properties of Stirling numbers in great detail, particularly with respect to recurrence formulas and relationships to other special numbers.

In the series expansions on certain functions of logarithms, Adams [2] develops and tabulates coefficients which run through positive and negative indices. A rearrangement of Adams' table for positive indices together with an appropriate alternation of sign yield Stirling numbers of the First Kind while a different rearrangement for negative indices yields Stirling numbers of the Second Kind.

An excellent summary of the properties of Stirling numbers including recursion and closed form expressions for finding Stirling numbers is presented in a recent Bureau of Standards publication [3]. In this regard, it is interesting to note that members of special partitions of numbers described in the April, 1964, issue of this Journal [4] can also be used to develop Stirling numbers. A discussion of this latter method follows.

2. DESCRIPTION OF COEFFICIENTS

Riordan uses the notation $S(n,k)$ and $s(n,k)$ for Stirling numbers of the Second and First Kinds, respectively, where the integers n and k are positive. Stirling numbers of the First Kind, the sum of whose n and k is odd, are negative. Adams chooses C_k^n, where n is a negative or positive integer and k is zero or a positive integer. Although none of Adams' C's are negative, a negative value for n identifies a C equal to a First Kind Stirling number, neglecting sign. For convenience of manipulation, the obviously subscripted (R for Riordan, A for Adams) indicates n_R, n_A, k_R, k_A replace the n and k's. By direct comparison, it can be seen that

$$\left.\begin{aligned} &(1) \qquad k_R = -n_A \\ &(2) \qquad n_R = k_R + k_A \end{aligned}\right\}\text{(applies for Second Kind only),}$$

$$\left.\begin{aligned} &(3) \qquad n_R = n_A \\ &(4) \qquad n_R = k_R + k_A \end{aligned}\right\}\text{(applies for First Kind only).}$$

The above equations lead to

$$S(n_R, k_R) = C_{n_R - k_R}^{-k_R}, \tag{5}$$

$$C_{k_A}^{n_A} = S(k_A - n_A, -n_A), \tag{6}$$

$$s(n_R, k_R) = (-1)^{n_R + k_R} \cdot C_{n_R - k_R}^{n_R}, \tag{7}$$

$$C_{k_A}^{n_A} = (-1)^{2n_A - k_A} \cdot s(n_A, n_A - k_A). \tag{8}$$

Tabulations of a few Stirling numbers are given below.

Table 1 $S(n_R, k_R)$

$n_R \backslash k_R$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

Table 2 $s(n_R, k_R)$

$n_R \backslash k_R$	1	2	3	4	5
1	1				
2	-1	1			
3	2	-3	1		
4	-6	11	-6	1	
5	24	-50	35	-10	1

In Adams' table, vertical entries for positive n_A are, with appropriate signs, First Kind Stirling numbers, and 45-degree, negative slope, diagonal entries for negative n_A are Second Kind Stirling numbers.

3. GENERATION OF SECOND KIND STIRLING NUMBERS

The negative n_A section of Adams' table suggests a numerical procedure by which Second Kind Stirling numbers can be generated simultaneously with the generation of members of the special partitions described in [4]. For example, in Table 3 consider a few column entries from Adams' table for $n_A = -4$. Differences between the entries are included

Table 3

$k_A \backslash n_A$	-4	Differences
0	1	1
1	10	9
2	65	55
3	350	285

If the differences were known the table entries could be found easily. The differences, however, do not stem from simple recursion formulas. If the manner in which successive sets of Second Kind Stirling numbers are found is investigated, it is seen that the differences are sums of products whose range is controlled by n_A and k_A. As an example from Table 3 ($n_A = -4$, $k_A = 3$) the products can be set up and sums formed vertically and horizontally as is shown in (9).

$$
(9)\qquad
\begin{array}{ccccccccc}
1 & & 2 & & 2\text{x}2 & & 2\text{x}2\text{x}2 & & \\
 & & 3 & & 2\text{x}3 & & 2\text{x}2\text{x}3 & & \\
 & & 4 & & 2\text{x}4 & & 2\text{x}2\text{x}4 & & \\
 & & & & 3\text{x}3 & & 2\text{x}3\text{x}3 & & \\
 & & & & 3\text{x}4 & & 2\text{x}3\text{x}4 & & \\
 & & & & 4\text{x}4 & & 2\text{x}4\text{x}4 & & \text{Vertical Sums} \\
 & & & & & & 3\text{x}3\text{x}3 & & \\
 & & & & & & 3\text{x}3\text{x}4 & & \\
 & & & & & & 3\text{x}4\text{x}4 & & \\
 & & & & & & 4\text{x}4\text{x}4 & & \\
\overline{1} & + & \overline{9} & + & \overline{55} & + & \overline{285} & = & 350 \\
\end{array}
$$

Horizontal Sums

The significant fact demonstrated by (9) is that exclusive of the initial "one," the multiplication signs, and the resultant summations, the array presented by (9) is identically that found in the development of the partition set

$$\mathrm{PV}(\geq 2, \leq 12 | \geq 1, \leq 3 | \geq 2, \leq 4)$$

according to the methods described in [4]. For the purposes of this paper, the PV set designation implies that the set of partitions is arranged in columns, each column consisting of partitions having exactly as many members as the column number. Thus, the set designation

$$\{1, \mathrm{PV}(\geq 2, \leq 12 | \geq 1, \leq 3 | \geq 2, \leq 4)\}$$

includes an initial "one" and the properly arranged partitions.

In general, the set

$$\{1,\ \mathrm{PV}(\geq 2, \leq -n_A k_A | \geq 1, \leq k_A | \geq 2, \leq -n_A)\}$$

when interpreted as in (9) yields Adams'

$$C_{k_A}^{n_A}$$

for negative n_A. Through use of (1) and (2), it is seen that the Second Kind Stirling number $S(n_R, k_R)$ can be found from the set

$$\{1, \mathrm{PV}(\geq 2, \leq k_R(n_R - k_R) | \geq 1, \leq n_R - k_R | \geq 2, \leq k_R)\} .$$

The method suggested above leads directly to

$$C_{k_A}^{n_A}$$

or $S(n_R, k_R)$. An ALGOL language computer program for obtaining the partitions described in [3] was developed as a result of student projects under the author's direction. It is obvious that only a slight modification of this program would be required to generate and store products (as the corresponding partition is formed) needed to obtain C's or S's directly as exemplified by (9).

4. GENERATION OF FIRST KIND STIRLING NUMBERS

Adams lists the following formulas for finding

$$C_0^{n_A},\ C_1^{n_A},\ \text{and}\ C_2^{n_A}\ .$$

The sum forms are applicable for n_A positive, but the product forms apply for n_A positive or negative.

$$C_0^{n_A} = 1\ , \tag{10}$$

$$C_1^{n_A} = 1 + 2 + 3 + \cdots + (n_A - 1) = \frac{n_A(n_A - 1)}{2} \tag{11}$$

$$\begin{aligned} C_2^{n_A} = {} & 1 \times 2 + 1 \times 3 + 1 \times 4 + \cdots + 1 \times (n_A - 1) \\ & + 2 \times 3 + 2 \times 4 + \cdots + 2 \times (n_A - 1) \\ & + 3 \times 4 + \cdots + 3 \times (n_A - 1) \\ & + \cdots\cdots\cdots\cdots \\ & + (n_A - 2)(n_A - 1) \\ = {} & \frac{n_A(n_A - 1)(n_A - 2)(3n_A - 1)}{24} \end{aligned} \tag{12}$$

Although Adams gives no formula for $k_A > 2$, (10), (11), and (12) suggest that tabulations of sums of products might be useful for an extension beyond $k_A =$

2. This is indeed the case as can be demonstrated in an example in which $n_A = 5$. Tabulations corresponding to the known formulas (10), (11), and (12) are listed below. For reasons given later, crossed-out dummy entries are included.

(13)

$$\begin{array}{c|cc|cc}
1 & 1 & 2 & 2 & \cancel{2 \times 2} \\
 & & 3 & 3 & 2 \times 3 \\
 & & 4 & 4 & 2 \times 4 \\
 & & & & \cancel{3 \times 3} \\
 & & & & 3 \times 4 \\
 & & & & \cancel{4 \times 4} \\
\hline
1 = C_0^5 & \multicolumn{2}{c|}{1 + 9 = 10 = C_1^5} & \multicolumn{2}{c}{9 + 26 = 35 = C_2^5}
\end{array}$$

Vertical Sums

Horizontal Sums

Consider the possible extensions beyond (13) for $k_A = 3$ and $k_A = 4$ shown in (14).

(14)

$$\begin{array}{cc|c}
\cancel{2\times2} & \cancel{2\times2\times2} & \cancel{2\times2\times2} \\
2\times3 & \cancel{2\times2\times3} & \cancel{2\times2\times3} \\
2\times4 & \cancel{2\times2\times4} & \cancel{2\times2\times4} \\
\cancel{3\times3} & \cancel{2\times3\times3} & \cancel{2\times3\times3} \\
3\times4 & 2\times3\times4 & 2\times3\times4 \\
\cancel{4\times4} & \cancel{2\times4\times4} & \cancel{2\times4\times}4 \\
 & \cancel{3\times3\times3} & \cancel{3\times3\times3} \\
 & \cancel{3\times3\times4} & \cancel{3\times3\times4} \\
 & \cancel{3\times4\times4} & \cancel{3\times4\times4} \\
 & \cancel{4\times4\times4} & \cancel{4\times4\times4} \\
\hline
\multicolumn{2}{c|}{26 + 24 = 50 = C_3^5} & 24 = C_4^5
\end{array}$$

Vertical Sums

Horizontal Sums

Again, note that the crossed-out entries do not contribute to a sum. The extensions exemplified by (14) yield the correct C_3^5 and C_4^5.

It is seen that exclusive of the initial "ones" (where present), the multiplication signs, the crossed-out lines, and the resultant summations, the tabulations of (13) and (14) are each a partition set of the type described earlier. Further, it is seen that only those entries with repeating members are crossed out. The success of (13) and (14) is not accidental. An investigation of the breakdown of First Kind Stirling numbers reveals that the pattern of (13) and (14) is general.

Exclusion of the crossed-out entries changes a partition set to one with non-repeating members. For identification, the designation changes to P_uV. One way of obtaining P_uV sets would be to generate PV sets and ignore repeating member partitions. This process is, of course, inefficient and can be circumvented as will be shown later.

For the example given, the following implications can be expressed:

$$
\begin{aligned}
&\{1\} \longrightarrow C_0^5 = 1 \\
&\{1,\ P_uV(\geq 2, \leq 4 \,|\, \geq 1, \leq 1 \,|\, \geq 2, \leq 4)\} \longrightarrow C_1^5 = 10 \\
&\{1,\ P_uV(\geq 2, \leq 8 \,|\, \geq 1, \leq 2 \,|\, \geq 2, \leq 4)\} \longrightarrow C_2^5 = 35 \\
&\{0,\ P_uV(\geq 4, \leq 12 \,|\, \geq 2, \leq 3 \,|\, \geq 2, \leq 4)\} \longrightarrow C_3^5 = 50 \\
&\{0,\ P_uV(\geq 6, \leq 12 \,|\, \geq 3, \leq 3 \,|\, \geq 2, \leq 4)\} \longrightarrow C_4^5 = 24\ .
\end{aligned}
\tag{15}
$$

For the general case, the implication is that

$$
\begin{aligned}
&\left\{ \left[\frac{k_A+3}{2k_A+2}\right],\ P_uV\left(\geq 2\left(k_A - 1 + \left[\frac{k_A+3}{2k_A+2}\right]\right), \leq \left(k_A - \left[\frac{k_A}{n_A-1}\right]\right)\cdot(n_A-1) \,\middle|\, \geq k_A \right.\right. \\
&\left.\left. -1 + \left[\frac{k_A+3}{2k_A+2}\right], \leq k_A - \left[\frac{k_A}{n_A-1}\right] - \left[\frac{k_A}{n_A}\right] \,\middle|\, \leq 2, \leq (n_A - 1)\right)\right\} \longrightarrow C_{k_A}^{n_A}\ , \quad n_A > 0\ .
\end{aligned}
\tag{16}
$$

It can be observed from (16) that*

$$
C_{k_A}^{n_A}
$$

does not exist for $k_A \geq n_A$. The corresponding expression for Stirling numbers of the First Kind is found through application of (7) to (16) as

---

*Brackets [ ] except where obviously used for references are used in the customary manner with real numbers to indicate the greatest integer less than or equal to the number bracketed. See Uspensky and Heaslet [5].

$$\begin{aligned} &\Bigg\{\left[\frac{n_R - k_R + 3}{2n_R - 2k_R + 2}\right], P_uV\Bigg(\geq 2\Bigg(n_R - k_R - 1 + \left[\frac{n_R - k_R + 3}{2n_R - 2k_R + 2}\right]\Bigg), \leq \Bigg(n_R - k_R - \\ (17)\qquad &- \left[\frac{n_R - k_R}{n_R - 1}\right]\Bigg)(n_R - 1)\ \Bigg|\ \geq n_R - k_R - 1 + \left[\frac{n_R - k_R + 3}{2n_R - 2k_R + 2}\right], \leq n_R - k_R - \\ &- \left[\frac{n_R - k_R}{n_R - 1}\right] - \left[\frac{n_R - k_R}{n_R}\right]\ \Bigg|\ \geq 2, \leq n_R - 1\Bigg)\Bigg\} \longrightarrow (-1)^{n_R + k_R} s(n_R, k_R) \ . \end{aligned}$$

5. REDUCTION OF P_uV TO SIMPLER PV FORMS

As was indicated earlier, one way of obtaining the P_uV partitions is first to generate PV partitions and then to retain non-repeating member partitions. The repeating member partitions serve only as devices for successive generation of partitions. Equations (13) and (14) illustrate graphically the wastefulness of such a procedure. It is possible to generate simpler PV partitions which easily can be modified to yield the desired P_uV partitions. The method of doing this is described below. While this method applies particularly for the partitions of this paper and is not intended to be general, it has the computational feature of generating exactly as many PV partitions as are needed for conversion to P_uV partitions — no more!

A P_uV partition applicable for this paper can be expressed as

$$(18)\qquad P_uV(\geq 2c, \leq ab \,|\, \geq c, \leq b \,|\, \geq 2, \leq a)$$

whether either $b = c$ alone or $b = c$ and $b = c + 1$, depending on whether the set (1 or 0, P_uV has one or two columns pf partitions. (See (15) for example). Assume that $b = c$. If the PV designation applied for (18), the largest (and last) b-member partition would total ab and would appear as b a's, $(a, a, \cdots, a)$. The u subscript, however, would not permit this partition, the closest approach being

$$(a - b + 1,\ a - b + 2,\ \cdots,\ a) \ .$$

However, $(a - b + 1,\ a - b + 2, \cdots, a)$ can be formed by member addition of

$$(a - b + 1, \ a - b + 1, \cdots, a - b + 1) \text{ and } (0, 1, 2, \cdots, b - 1) .$$

For a given b,

$$(a - b + 1, \ a - b + 1, \cdots, a - b + 1)$$

is an acceptable last partition in a one-partition column PV set and has a greatest member $a - b + 1$ and the sum $ab - b(b - 1)$. The lower limits of the new PV designation remain the same as in (18). Thus, a member-by-member addition of $(0, 1, 2, \cdots, b - 1)$ to the members of

$$PV(\geq 2, \leq ab - b(b - 1) | \geq b, \leq b | \geq 2, \leq 2 - b + 1) \tag{19}$$

produces the desired form of (18) where $b = c$. For the case of two columns of partitions (i.e., $b = c$, $b = c + 1$),

$$PV(\geq 2c, \leq ac - c(c - 1) | \geq c, \leq c | \geq 2, \leq a - c + 1) \tag{20}$$

is augmented by $(0, 1, 2, \cdots, c - 1)$ and

$$PV(\geq 2(c + 1) \leq a(c + 1) - c(c + 1) | \geq c + 1, \leq c + 1 | \geq 2, \leq a - c) \tag{21}$$

is augmented by $(0, 1, 2, \cdots, c)$. An example for $a = 4$, $b = 3$, $c = 2$ follows.

$$
\begin{array}{lll}
\underline{PV(\geq 4, \leq 6 | \geq 2, \leq 2 | \geq 2, \leq 3)} & & \underline{P_uV(\geq 4, \leq 12 | \geq 2, \leq 3 | \geq 2, \leq 4)} \\
2,2 & & 2,3 \qquad 2,3,4 \\
2,3 \ + (0,1) & \longrightarrow & 2,4 \\
3,3 & & 3,4 \\
\underline{PV(\geq 6, \leq 6 | \geq 3, \leq 3 | \geq 2, \leq 2)} & & \\
2,2,2 \ + (0,1,2) & &
\end{array} \tag{22}
$$

Comparison of (22) with (14) shows the reduction in computation.

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1. J. Riordan, An Introduction to Combinatorial Analysis, John Wiley and Sons, Inc., New York, N.Y., 1958, pp. 32-49.

2. E. P. Adams, Smithsonian Mathematical Formulae and Tables of Elliptic Functions, The Smithsonian Institution, Washington, D. C., 1947, pp. 159-160.

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5. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill Book Co., New York, New York, 1939, pp. 94-99.

* * * * *

ERATTA FOR
FACTORIZATION OF 2 X 2 INTEGRAL MATRICES WITH DETERMINANT ±1

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Please make the following corrections to "Factorization of 2x2 Matrices with Determinant ±1," by Gene B. Gale, appearing in the February 1968 issue, Fibonacci Quarterly, pp. 3—22.

Page	Line	Reads	Should Read
5	6	$d < 0$	$d > 0$
5	-8	$c \leq d$	$c < d$
8	-3	$\|ru - \|st\|\|$	$\|\|ru\| - \|st\|\|$
9	5	$ad - bc \geq ad - cd = (a - c)d \geq 0$	$ad - bc > ad - cd = (a - c)d > 0$
9	-1	$\begin{pmatrix} a & r+1 \\ c & w \end{pmatrix}$	$\begin{pmatrix} a & r+1 \\ c & d \end{pmatrix}$
9	4	$cd \geq 0$	$c, d \geq 0$
{11, 12}	{-5, 3}	N	n
12	-6	$ar = (a - 1)(r - 1)$	$ar - (a - 1)(r - 1)$
15	6	$d(rF_k + sF_{k-1})$	$d \mid (rF_k + sF_{k-1})$
16	-4	A_2B	A, B
17	-9	$\left\|\frac{ab - bc}{bd}\right\|$	$\left\|\frac{ad - bc}{bd}\right\|$

Continued on p. 112

ON A CHARACTERIZATION OF THE FIBONACCI SEQUENCE

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For the Fibonacci sequence defined by

$$(1)\qquad \begin{aligned} F_1 &= 1, \quad F_0 = 0 \\ F_n &= F_{n-1} + F_{n-2}, \quad n \geq 2, \end{aligned}$$

it is well known that for all n

$$(2)\qquad F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

We consider the converse problem, i.e., whether or not (1) can be derived from (2).

It is quite easy to prove by induction that if

$$x_{n-1}x_{n+1} - x_n^2 = (-1)^n$$

and

$$x_1 = x_2 = 1,$$

then

$$x_n = x_{n-1} + x_{n-2} .$$

Suppose, however, that x_1 and x_2 are chosen as arbitrary but fixed integers. In this case it will be shown that we cannot conclude (1) from (2), but we do find some interesting results.

Consider the generalized sequence $\{H_n\}$ defined by

*This work was supported by the Undergraduate Research Participation Program of the National Science Foundation through Grant GY-3026.

$$H_n = H_{n-1} + H_{n-2} \tag{3}$$
$$H_0 = H_1 = p, \quad p \text{ and } q \text{ are integers.}$$

Under this definition it can be proved that

$$H_{n-1}H_{n+1} - H_n^2 = (-1)^n(p^2 - qp - q^2), \tag{4}$$

and conversely, given equation (4) then (3) must follow. If $p^2 - pq - q^2 = 1$, then (4) is the same as (2).

Therefore let us consider the integral solutions of an equation of the form

$$y^2 - xy - x^2 \pm 1 = 0$$

First of all it can be shown by induction that the Fibonacci numbers do satisfy this equation. If (2) is to characterize the Fibonacci numbers then we must show that the Fibonacci numbers are the only integral solutions to this equation, and then the sequence $\{H_n\}$, with p, q chosen to satisfy

$$y^2 - xy - x^2 - 1 = 0 \tag{5}$$

would be the sequence $\{F_n\}$. However, given examples such as:

$$y = -1, \quad x = 0; \quad y = -2, \quad x = 3; \quad \text{and} \quad y = -5, \quad x = 8$$

it is seen that (2) and (5) do not characterize the Fibonacci sequence.

The characterizing theorem which can be proved is:

Theorem. If x and y are integers such that $y^2 - xy - x^2 \pm 1 = 0$ and

(1) if x and y are positive, then $x = F_{n-1}$, $y = F_n$ for some n,

(2) if x and y are negative, then $x = -F_{n-1}$, $y = -F_n$ for some n,

(3) if either x or y is negative and the other is positive, then $x = F_{n-1}$, $y = -F_n$ or $x = -F_{n-1}$, $y = +F_n$ for some n.

Proof:

(1) Wasteels proved that if x and y are positive integers such that

$$y^2 - xy - x^2 \pm 1 = 0, \quad (y \geq x)$$

then x and y are consecutive Fibonacci numbers [1].

(2) If x and y are negative, then $-x$ and $-y$ are positive and from the first result we know that $-x = F_{n-1}$, $-y = F_n$ for some n. Therefore,

$$x = -F_{n-1}, \quad y = -F_n$$

for some n.

(3) If either x or y is negative and the other positive then: $y^2 - xy - x^2 \pm 1 = 0$ may be written

$$y^2 = |x||y| - x^2 \pm 1 = 0, \quad |x| \geq |y| \tag{6}$$

Let $|y| > 1$. Then from Eq. (6) we find that $|x| > |y|$ and $|x| < 2|y|$. For if $|x| \geq 2|y|$ then

$$x^2 - |x||y| - y^2 \mp 1 = 0 \tag{7}$$

and

$$|x|(|x| - |y|) - y^2 \mp 1 \geq 2|y||y| - y^2 \mp 1 = y^2 \mp 1 > 0.$$

Thus

$$|x/2| < |y| < |x|$$

Let

$$|x| - |y| = |z|$$

Then

$$0 < |z| < |x/2| < |y|$$

and substituting for $|x|$ in (7)

$$(|z| + |y|)^2 - (|z| + |y|)|y| - y^2 \mp 1 = 0$$

or

$$z^2 + |y|\,|z| - y^2 \pm 1 = 0,$$

so that $|z|$ and $|y|$ satisfy Eq. (6) and $|z|$ is smaller than $|x|$ or $|y|$. If $|z| = 1$ then $|y| = 1$ or 2 so that the theorem is true for $|z|$ and $|y|$ and therefore for $|y|$ and $(|y| + |z|)$ or $|x|$.

If $|z| > 1$ we can repeat the above argument and find z_1 such that

$$|z_1| = |y| - |z|$$

which satisfies Eq. (6) and is less than $|z|$.

If $|z_1| > 1$ we can continue this process until eventually we find a $|z_i|$ such that $|z_i| = 1$. Otherwise we would find an infinite sequence of distinct integers less than x and greater than 1.

If $|z_i| = 1$, then the theorem is true for $|z_i|$ and $|z_{i-1}|$ and also for $|z_{i-1}|$ and $(|z_{i-1}| + |z_i|) = |z_{i-2}|$, and similarly for $|z_1|$ and $(|z_1| + |z_2|) = |z|$ and finally for $|x|$ and $|y|$.

We return to the original problem and consider Eq. (4). If

$$p^2 - pq - q^2 - 1 = 0$$

and p and q are positive, then this identity does indeed characterize the Fibonacci sequence. If, however, p and q are both negative then this identity characterizes the negative of the Fibonacci sequence, and if either p or q is negative while the other is positive then this identity may characterize either the Fibonacci sequence or its negative. There is no way in this case to determine which it will be.

REFERENCE

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* * * * *

THE GENERALIZED FIBONACCI OPERATOR

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I. INTRODUCTION

Some years ago Angus E. Taylor and the author were looking for examples of operators for which spectra could be determined and classified. In the course of this search we chanced upon a bounded linear operator F on the sequence space ℓ_1, defined by the infinite matrix (f_{ij}),

$$f_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } i = 1,\ j > 1 \\ 0 & \text{otherwise} \end{cases}$$

This operator has the interesting property that the norms of its consecutive powers are consecutive Fibonacci numbers, which, as is well known, are defined recursively by

$$f_0 = 0, \quad f_1 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2}, \qquad n \geq 2 .$$

The infinite matrix representations of the n^{th} power of this operator have column vectors such that the first $n+1$ terms of these vectors are, in inverted order, truncated Fibonacci sequences. The spectrum consists of the unit disc together with the point

$$\frac{1+\sqrt{5}}{2} ,$$

the positive zero of the polynomial $P(\lambda) = \lambda^2 - \lambda - 1$, sometimes called the "golden mean" which is well known to be the limit, as n becomes infinite of the positive n^{th} root of the n^{th} term of the Fibonacci sequence. We appropriately enough dubbed this operator the "Fibonacci Operator."

In this paper we define an operator-valued function F of a nonnegative real variable, such that for every nonnegative value of x there is associated with the number x a bounded linear operator $F(x)$ on the sequence space ℓ_1. In addition, there corresponds to each nonnegative value of x:

(1) a sequence $\{f_k(x)\}$

(2) a polynomial $P_x(\lambda)$

(3) an infinite matrix representation $(f_{ij}(x))$ for $F(x)$.

For the case $x = 1$, $F(1)$, $\{f_k(1)\}$, $P_1(\lambda)$ and $(f_{ij}(1))$ are the Fibonacci operator, the Fibonacci sequence, the associated polynomial, and matrix representation, respectively. For all other values of x, $0 \leq x < \infty$, $F(x)$ and the entities referred to in (1), (2), and (3) above have interrelationships similar to those possessed by their counterparts in the case $x = 1$.

II. PRELIMINARY DEFINITIONS AND NOTATION

The operators we shall consider will be bounded linear operators mapping the sequence space ℓ_1 into itself. The space ℓ_1 consists of the set of all absolutely convergent sequences of complex numbers $\xi = \{\xi_i\}$ under the norm defined by

$$\|\xi\| = \sum_{i=1}^{\infty} |\xi_i| .$$

It can be shown (see for example [1]) that every member, A, of the algebra $[\ell_1]$ of bounded linear operators which map ℓ_1 into itself has a matrix representation (a_{ij}), such that the uniform norm of A is given by

$$\|A\| = \sup_j \sum_{i=1}^{\infty} |a_{ij}| .$$

If A is in $[\ell_1]$, then the resolvent set of A, $\rho(A)$, consists of the set of all complex λ for which the operator $(\lambda I - A)^{-1}$, where I is the identity operator, exists as a bounded operator, and the range of $\lambda I - T$ is dense in ℓ_1.

The spectrum of A, $\sigma(A)$, consists of the set of all complex numbers which do not belong to $\rho(A)$. The spectral radius of A, $|\sigma(A)|$, is the radius of the smallest circle, with center at the origin, which contains $\sigma(A)$. We shall have occasion to make use of the following facts: (see 2)

$$(2.1) \qquad |\sigma(A)| = \lim_{n\to\infty} \|A^n\|^{1/n}$$

(2.2) If $|\lambda| \geq |\sigma(A)|$ we can represent $(\lambda I - A)^{-1}$ by its Neumann expansion,

$$(\lambda I - A)^{-1} = \frac{I}{\lambda} + \sum_{n=1}^{\infty} \frac{A^n}{\lambda^{n+1}} \quad .$$

The function F which we wish to consider has for its domain the set of all non-negative real numbers and its range is contained in $[\ell_1]$. If we identify the values of F(x) with their matrix representations under the standard basis, it will be convenient to define F(x) as the sum of two matrices L and C(x).

The infinite matrix $L = (\ell_{ij})$ is defined by

$$\ell_{ij} = \begin{cases} 1 & \text{if } i - j = 1 \\ 0 & \text{otherwise} \end{cases} .$$

When L is used as a left multiplier on a matrix A, we might call it a "lowering matrix." Its effect on A can be crudely described as follows: Each row of A is lowered one step, and the empty first row is replaced by zeros.

The infinite matrix $C(x) = (c_{ij}(x))$ is defined by

$$c_{ij}(x) = \begin{cases} 0 & \text{if } j < [x] + 1 \text{ or } i > 1 \\ j - x & \text{if } j = [x] + 1 \text{ and } i = 1 \\ 1 & \text{if } j > [x] + 1 \text{ and } i = 1 \end{cases} ,$$

where $[x]$ denotes the greatest integer not greater than x. (Note that all entries of C(x) below the first row are zero.) This matrix could be described as "partial column summer." As a left multiplier of a matrix $A = (a_{ij})$, it produces the following effect. In each column of A the elements below the

$[x+1]^{st}$ row are summed, to this is added $(1 - x + [x])$ times the entry in the $[x+1]^{st}$ row and the total is entered as the first row entry of the corresponding column of LA. All other entries in this column of LA are 0.

We are now ready to state our main theorem.

III. PRINCIPAL THEOREM

Theorem 1. Let $F(x)$ be the member of $[\ell_1]$ defined by the infinite matrix $L + C(x)$, $0 \le x < \infty$. With $F(x)$ there are associated

(1) a sequence $\{f_k(x)\}$, defined by

$$f_k(x) = \begin{cases} 0 \text{ if } k = 0 \\ 1 \text{ if } 0 < k \le [x+1] \\ f_{k-1}(x) + ([x+1] - x)f_{k-[x+1]}(x) + (x - [x])f_{k-[x+2]}(x) \\ \qquad\qquad \text{if } k > [x] + 1 \end{cases}$$

and

(2) a polynomial $P_x(\lambda)$,

$$P_x(\lambda) = \left\{ \lambda^{[x+1]} - ([x+1] - x) \right\}(\lambda - 1) - 1$$

such that the following relationships hold.

(a)
$$\|F^n(x)\| = f_{n+[x+2]}(x) - ([x+1] - x)f_{n+1}(x)$$
$$= \sum_{k=0}^{n} f_k^{(x)} + 1$$

(b)
$$\sigma(F) = \left\{ \lambda;\ P(\lambda) = 0 \text{ or } |\lambda| \le 1 \right\}$$

(c)
$$\lim_{n\to\infty} \left\{ f_n(x) - ([x+1] - x)f_{n-[x+1]}(x) \right\}^{1/n} = |\sigma(F(x))| = \sup_{P(\lambda)=0} |\lambda|$$

(d)
$$f_j(x) = f^{(k)}_{(j+1-k)n}(x), \quad j = 1, \cdots, k, \qquad n > [x+1]$$

where

$$F^k(x) = (f_{ij}^{(k)}(x)) .$$

Statement (d) merely says that the first k entries in any column after the $[x+1]^{st}$ of the matrix $F^k(x)$ are the truncated sequence $\{f_k\}_1^k$ in reverse order.

Before proceeding with the proof, we note that in case x is an integer, the sequence $\{f_k(x)\}$ is a sequence of integers similar to the Fibonacci sequence; indeed $\{f_k(1)\}$ is the Fibonacci sequence and $\{f_k(0)\}$ starting with $f_1(0)$ is the geometric progression with first term equal to 1 and common ratio 2. In general, where x is an integer, the sequence $\{f_k(x)\}$ has the following properties:

(i) $$f_0(x) = 0, \quad f_1(x) = f_2(x) = \cdots = f_{x+1}(x) = 1$$
$$f_n(x) = f_{n-1}(x) + f_{n-(1+x)}(x) \text{ if } n > x+1$$

(ii) $$f_{n+x+1}(x) = \sum_{k=0}^{n} f_k(x) + 1$$

(iii) $$\lim_{n\to\infty} \{f_n(x)\}^{1/n} = \sup_{P(\lambda)=0} |\lambda| = |\sigma(F)| ,$$

where

$$P(\lambda) = (\lambda^{x+1} - 1)(\lambda - 1) - 1$$
$$= \lambda^{x+2} - \lambda^{x+1} - \lambda$$

(iv) $$f_{n+x+1}(x) = \|F^n(x)\| .$$

We now turn to the proof of our theorem.

We shall let the matrix representation of $F^n(x)$ be denoted by $(f_{ij}^{(n)}(x))$. However, to simplify the notation in the discussion that follows, we will omit the argument x and represent $F^n(x)$ and $f_{ij}^{(n)}(x)$ merely by F^n and $f_{ij}^{(n)}$. We shall also let

$$\epsilon = 1 - (x - [x])$$

and

$$\ell = \epsilon + x = 1 + [x] .$$

With this notation the matrix representation of $F(x)$ has the appearance:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & \epsilon & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ & \cdots & & & & & \cdots & & & & \end{bmatrix},$$

where ϵ appears in the first row of the ℓ^{th} column.

Since $F^n = (L + C)F^{n-1}$, we see from the description of the effects produced by L and C as left operators that the k^{th} row of F^n is the first row of F^{n-k+1} for $1 \le k \le n$, k an integer. That is

$$(3.1) \qquad f_{kj}^{(n)} = f_{(k-1)j}^{(n-1)} = \cdots = f_{ij}^{(n-k+1)}; \quad 1 \le k \le n.$$

We also see that

$$(3.2) \qquad f_{kj}^{(n)} = \begin{cases} 1 & \text{if } k = n + j \\ 0 & \text{if } k > n \text{ and } k \ne n + j . \end{cases}$$

With the understanding that if $n \le 0$ then $f_{ij}^{(n)} = 0$ and $f_{in}^{(1)} = f_{in} = 0$ we can state the following lemma.

Lemma 1.

$$\text{(a)} \qquad f_{1m}^{(n)} = \epsilon f_{1m}^{(n-\ell)} + \sum_{j=1}^{n-\ell-1} f_{1m}^{(j)} + f_{1(m+n-1)}$$

if m and n are positive integers.

(b) $$f_{1m}^{(n)} = f_{1m}^{(n-1)} + \epsilon f_{1m}^{(n-\ell)} + (1-\epsilon) f_{1m}^{(n-\ell-1)} + (f_{1(m+n-1)} - f_{1(m+n-2)})$$

if m and n are positive integers and $n \geq 2$.

Proof. Part (a) follows easily from the fact that

$$f_{1m}^{(n)} = \sum_{j=1}^{\infty} f_{1j} f_{jm}^{(n-1)}$$

and formulas 3.1 and 3.2. Part (b) is obtained by computing $f_{1m}^{(n-1)}$ from part (a) and subtracting the result from the expression for $f_{1m}^{(n)}$ given in (a).

Lemma 2. If n is an integer and $n \geq 2$ then

$$f_{1m}^{(n)} = g(m) f_{11}^{(n-1)} + f_{1(m+1)}^{(n-1)} ,$$

where

$$g(m) = \begin{cases} 0 & \text{if } m < \ell \\ \epsilon & \text{if } m = \ell \\ 1 & \text{if } m > \ell . \end{cases}$$

Proof. The result follows easily from the fact that

$$f_{1m}^{(n)} = \sum_{j=1}^{\infty} f_{1j}^{(n-1)} \cdot f_{jm} .$$

Lemma 3. If m, n and k are positive integers and $m > k$, then $f_{1m}^{(n)} \geq f_{1k}^{(n)}$. If in addition $k > \ell$, then $f_{1k}^{(n)} = f_{1m}^{(n)}$.

Proof. This result follows from an inductive argument. That the conclusions of the lemma hold for $n = 1$ is evident. From the induction hypotheses that they hold for $n = j$, it quickly follows from Lemma 2 that they hold for $n = j + 1$.

Since from Lemma 3, $f_{1k}^{(n)} = f_{1m}^{(n)}$ for all k and m such that $k > \ell$ and $m > \ell$, we make the following definition.

Definition.

$$f_0 = 0, \quad f_n = f_{1(\ell+1)}^{(n)} \quad \text{if } n \text{ is a positive integer.}$$

From the definitions of ℓ and ϵ and Lemmas 1 and 3, it follows that $\{f_n(x)\}$ is the sequence defined in Part 1 of the conclusion of Theorem 1.

Lemma 4. The norm of F^n is given by

$$F^n = \sum_{k=0}^{n} f_k + 1$$

Proof. Since $0 \le \epsilon \le 1$ and all the entries of the first row of the matrix (f_{ij}) are nonnegative, it follows from part (b) of Lemma 1 that all the elements of the first row of the matrix $(f_{ij}^{(n)})$ are nonnegative. From equation 3.1 we see that the j^{th} component of the m^{th} column vector of $(f_{ij}^{(n)})$ is given by

$$f_{jm}^{(n)} = f_{1m}^{(n-j+1)}, \quad 1 \le j \le n .$$

From this equation and equation 3.2 it follows since all the components are nonnegative that the ℓ_1 norm of the m^{th} column vector of $(f_{ij}^{(n)})$ is given by

$$\sum_{j=1}^{n} f_{1m}^{(j)} + 1 .$$

From Lemma 3 we see that the ℓ_1 norm of the $(\ell + 1)^{st}$ column vector of $(f_{ij}^{(n)})$ is greater than or equal to the ℓ_1 norm of any other column vector of that matrix. The definition of $\|F^n\|$ and that of the sequence $\{f_k\}$ now imply that

$$\|F^n\| = \sum_{k=0}^{n} f_k + 1 .$$

This completes the proof of Lemma 4.

It is a simple matter to use the result part (b) of Lemma 1 to conclude that

$$\sum_{k=0}^{n} f_k + 1 = f_{n+\ell+1} - \epsilon f_{n+1} .$$

This result together with Lemma 4 gives part (a) of part 2 of the conclusion of Theorem 2.

Lemma 5. The formal inverse matrix (g_{ij}) of the matrix representation of $\lambda I - F$ is defined by

$$b_{1j} = \frac{\lambda^{\ell+1}}{P(\lambda)} \begin{cases} \dfrac{\lambda\epsilon + (1-\epsilon)}{\lambda^{\ell-j+3}} & \text{if } j \le \ell \\ \dfrac{1}{\lambda^2} & \text{if } j > \ell \end{cases}$$

where

$$P(\lambda) = \lambda^{\ell+1} - \lambda^{\ell} - \epsilon\lambda - (1-\epsilon) ,$$

and

$$g_{ij} = \begin{cases} \dfrac{1}{\lambda^{i-j+1}} + \dfrac{1}{\lambda^{i-1}} b_{1j} & \text{if } i \ge j \\ \dfrac{1}{\lambda^{i-1}} b_{1j} & \text{if } i < j . \end{cases}$$

Proof. The Neumann expansion for $(\lambda I - F)^{-1}$ converges provided $|\lambda| \geq |\sigma(F)|$. Since

$$|\sigma(F)| = \lim_{n \to \infty} \|F^n\|^{1/n} \leq \|F\| = 2 ,$$

it is clear that the Neumann expansion for $(\lambda I - F)^{-1}$ converges provided $|\lambda| \geq 2$. We, however, are only using the Neumann expansion as a device to obtain the formal matrix inverse of the matrix representation of $(\lambda I - F)$. If we let the matrix for $(\lambda I - F)^{-1}$ be denoted by (g_{ij}), then since

$$(\lambda I - F)^{-1} = \frac{I}{\lambda} + \sum_{n=1}^{\infty} \frac{F^n}{\lambda^{n+1}}$$

it follows that

$$g_{ij} = \frac{\delta_{ij}}{\lambda} + \sum_{n=1}^{\infty} \frac{f_{ij}^{(n)}}{\lambda^{n+1}} .$$

But from 3.1 and 3.2 we see that:

$$f_{ij}^{(n)} = \begin{cases} f_{1j}^{(n-i+1)} & \text{if } 1 \quad i \leq n, \\ \delta_{i(j+n)} & \text{if } i > n . \end{cases}$$

Thus

$$\text{(3.3)} \qquad g_{ij} = \begin{cases} \displaystyle\sum_{n=i}^{\infty} \frac{f_{1j}^{(n-i+1)}}{\lambda^{n+1}} & \text{if } i < j , \\[2ex] \displaystyle\frac{1}{\lambda^{i-j+1}} + \sum_{n=i}^{\infty} \frac{f_{1j}^{(n-i+1)}}{\lambda^{n+1}} & \text{if } i \geq j . \end{cases}$$

If we now consider the matrix (g_{ij}) as the sum of two matrices (a_{ij}) and (b_{ij}) where

$$a_{ij} = \begin{cases} 0 & \text{if } i < j \,, \\ \dfrac{1}{\lambda^{i-j+1}} & \text{if } i \geq j \,. \end{cases} \tag{3.4}$$

we see that

$$b_{ij} = \sum_{n=i}^{\infty} \frac{f_{1j}^{(n-i+1)}}{\lambda^{n+1}} \quad .$$

If $i > 1$ we see that

$$\begin{aligned} b_{ij} &= \sum_{n=i}^{\infty} \frac{f_{1j}^{(n-i+1)}}{\lambda^{n+1}} = \sum_{k=1}^{\infty} \frac{f_{1j}^{(k)}}{\lambda^{k+i}} \\ &= \frac{1}{\lambda^{i-1}} \sum_{k=1}^{\infty} \frac{f_{1j}^{(k)}}{\lambda^{k+1}} = \frac{1}{\lambda^{i-1}} b_{1j} \,. \end{aligned} \tag{3.5}$$

By using part b of Lemma 1, we can solve for values b_{1j} as follows:

$$\begin{aligned} b_{1j} &= \sum_{k=1}^{\infty} \frac{f_{ij}^{(k)}}{\lambda^{k+1}} = \frac{f_{1j}}{\lambda^2} + \sum_{k=2}^{\infty} \frac{f_{1j}^{(k)}}{\lambda^{k+1}} \\ &= \frac{f_{1j}}{\lambda^2} + \sum_{k=2}^{\infty} \frac{\left\{ f_{1j}^{(k-1)} + \epsilon f_{1j}^{k-\ell} + (1-\epsilon) f_{1j}^{(k-\ell-1)} + f_{1(j+k-1)} - f_{1(j+k-2)} \right\}}{\lambda^{k+1}} \end{aligned}$$

or

$$b_{1j} = \frac{f_{1j}}{\lambda^2} + \frac{1}{\lambda} b_{1j} + \frac{\epsilon}{\lambda^{\ell}} b_{1j} + \frac{(1-\epsilon)}{\lambda^{\ell+1}} b_{1j} + \begin{cases} \dfrac{\epsilon}{\lambda^{\ell-j+2}} + \dfrac{1-\epsilon}{\lambda^{\ell-j+3}} & \text{if } j \le \ell \\[2ex] \dfrac{1}{\lambda^2} & \text{if } j > \ell, \end{cases}$$

and therefore

$$b_{1j} = \frac{\lambda^{\ell+1}}{\lambda^{\ell+1} - \lambda - \epsilon\lambda - (1-\epsilon)} \cdot \begin{cases} \dfrac{\lambda\epsilon + (1-\epsilon)}{\lambda^{\ell-j+3}} & \text{if } j \le \ell \\[2ex] \dfrac{1}{\lambda^2} & \text{if } j > \ell \end{cases} \quad . \tag{3.6}$$

Remembering that $g_{ij} = a_{ij} + b_{ij}$ the conclusion of the lemma follows from equations 3.3, 3.4, 3.5, and 3.6.

From Lemma 5 it is easy to see that the matrix (g_{ij}) can be schematically presented as the linear combination of two matrices as follows:

$$(g_{ij}) = \begin{bmatrix} \frac{1}{\lambda} & 0 & 0 & 0 & \cdots & \\ \frac{1}{\lambda^2} & \frac{1}{\lambda} & 0 & 0 & \cdots & \\ \frac{1}{\lambda^3} & \frac{1}{\lambda^2} & \frac{1}{\lambda} & 0 & \cdots & \\ & \cdots & & & \cdots & \\ \frac{1}{\lambda^k} & \frac{1}{\lambda^{k-1}} & \cdots & \frac{1}{\lambda} & 0 & 0 \\ & \cdots & & \cdots & & \end{bmatrix} + \frac{\lambda^{\ell-1}}{P(\lambda)} \begin{bmatrix} h^{(1)}(\epsilon) & h^{(2)}(\epsilon) & \cdots & h^{(\ell)}(\epsilon) & 1 & 1 & \cdots \\ \frac{h^{(1)}(\epsilon)}{\lambda} & \frac{h^{(2)}(\epsilon)}{\lambda} & \cdots & \frac{h^{(\ell)}(\epsilon)}{\lambda} & \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots \\ \frac{h^{(1)}(\epsilon)}{\lambda^2} & \frac{h^{(2)}(\epsilon)}{\lambda^2} & \cdots & \frac{h^{(\ell)}(\epsilon)}{\lambda^2} & \frac{1}{\lambda^2} & \frac{1}{\lambda^2} & \cdots \\ \frac{h^{(1)}(\epsilon)}{\lambda^3} & \frac{h^{(2)}(\epsilon)}{\lambda^3} & \cdots & \frac{h^{(\ell)}(\epsilon)}{\lambda^3} & \frac{1}{\lambda^3} & \frac{1}{\lambda^3} & \cdots \\ \cdots & & & \cdots & & & \end{bmatrix},$$

where

$$h^{(j)}(\epsilon) = \frac{\lambda\epsilon + (1-\epsilon)}{\lambda^{\ell+1-j}}$$

is a factor of each element in each of the first ℓ columns of the second matrix. The first of the above matrices is the matrix representation of

$$I + \sum_{k=1}^{\infty} L^k$$

and the value of its norm is

$$\frac{1}{|\lambda| - 1} = \sum_{k=1}^{\infty} \frac{1}{|\lambda|^k} ,$$

if $|\lambda| > 1$. The value of the norm of the second matrix is

$$\max \begin{cases} \left|h^{(j)}(\epsilon)\right| \dfrac{|\lambda|}{|\lambda| - 1} = \left|h^{(j)}(\epsilon)\right| + \displaystyle\sum_{k=1}^{\infty} \frac{\left|h^{(j)}(\;)\right|}{|\lambda|^k} , \quad j = 1, \cdots, \ell \\ \dfrac{|\lambda|}{|\lambda| - 1} = 1 + \displaystyle\sum_{k=1}^{\infty} \frac{1}{|\lambda|^k} \end{cases}$$

provided $|\lambda| > 1$.

From these facts we can infer that $(\lambda I - F)^{-1}$ is defined and a bounded operator on ℓ_1 into ℓ_1 provided $|\lambda| > 1$ and λ is not a zero of $P(\lambda)$, and that $(\lambda I - F)^{-1}$ is either not defined or is unbounded if λ is a zero of $P(\lambda)$ or $|\lambda| \le 1$. We thus conclude that the resolvent set of F,

$$\rho(F) = \left\{ \lambda \,\middle|\, |\lambda| > 1 \quad \text{and} \quad P(\lambda) \neq 0 \right. ,$$

and therefore the spectrum of F,

$$\sigma(F) = \left\{ \lambda \,\middle|\, |\lambda| \le 1 \;\text{ or }\; P(\lambda) = 0 \right\} .$$

This proves part 2.b of our theorem since if we recall that $\ell = [x+1]$ and $\epsilon = ([x+1] - x)$ we see that the polynomial $P(\lambda)$ is precisely the polynomial $P_x(\lambda)$ defined in the theorem by:

$$P_x(\lambda) = \{\lambda^{[x+1]} - ([x+1] - x)\}(\lambda - 1) - 1 .$$

Lemma 6. For any given value of x, $0 \le x \le \infty$, $P_x(\lambda)$ has precisely one real zero, r_x, with modulus greater than 1 and $1 \le r_x \le 2$.

Proof. As a function of the real variable ξ

$$P_x(\xi) = (\xi^{\ell} - \epsilon)(\xi - 1) - 1 = \xi^{\ell+1} - \xi^{\ell} - \epsilon\xi - (1 - \epsilon)$$

and

$$P'_x(\xi) = (\xi^{\ell} - \epsilon) + (\xi - 1)\ell\xi^{\ell-1} = (\ell + 1)\xi^{\ell} - \ell\xi^{\ell-1} - \epsilon .$$

It is a simple matter to verify that $P'_x(\xi) > 0$ if $\xi > 1$ and $P_x(1) = -1$. From this we infer that $P_x(\xi)$ has precisely one zero greater than 1 and since $P_x(2) > 0$, that this zero lies strictly between 1 and 2 if $x \ne 0$. If $\xi < -1$ and ℓ is odd then $P'_x(\xi) < 0$ and $P_x(-1) > 0$. If $\xi < -1$ and ℓ is even, then $P'_x(\xi) > 0$ and $P_x(-1) > 0$. From these facts it follows that $P_x(\xi)$ has no negative zeros with modulus greater than 1. This completes the proof of the lemma.

Lemma 7. If r_x is the positive real zero of $P_x(\lambda)$, $1 < r_x \le 2$, and if μ is any other zero of $P_x(\lambda)$, then $|\mu| \le r_x$.

Proof. The proof is by contradiction. If we assume $P_x(\mu) = 0$ and $|\mu| > r_x > 1$, then $|\mu|^{\ell} > r_x^{\ell}$ and therefore $|\mu|^{\ell} - \epsilon > r_x^{\ell} - \epsilon > 0$ since $0 \le \epsilon \le 1$. From this last result the following chain of inequalities follows:

$$\frac{1}{|\mu|^{\ell} - \epsilon} < \frac{1}{r_x^{\ell} - \epsilon} ,$$

hence

$$1 + \frac{1}{|\mu|^{\ell} - \epsilon} < 1 + \frac{1}{r_x^{\ell} - \epsilon} = r_x$$

since

$$r_x^{\ell+1} - r_x^{\ell} - \epsilon r_x - (1 - \epsilon) = 0 ,$$

and therefore

$$1 + \frac{1}{|\mu^{\ell} - \epsilon|} < r_x ,$$

or

$$|\mu| < r_x ,$$

since

$$|\mu| = \left|1 + \frac{1}{\mu^{\ell} - \epsilon}\right| \leq 1 + \frac{1}{|\mu^{\ell} - \epsilon|} .$$

But $|\mu| < r_x$ is a contradiction of our assumption that $\mu \geq r_x$.

From Lemmas 6 and 7 and the definition of spectral radius, we immediately deduce the second equality in part 2.c of the conclusion of Theorem 1. That is,

$$|\sigma(F(x))| = \sup_{P_x(\lambda)=0} |\lambda|$$

The first equality of part 2.c of the conclusion of Theorem 1 is an immediate consequence of part 2.a of Theorem 1 and the fact, 2.1, that

$$|\sigma(F(x))| = \lim_{n\to\infty} \|F^n(x)\|^{1/n} .$$

We have now completed the proof of Theorem 1.

IV. A PROPERTY OF $|\sigma(F(x))|$

We conclude this paper with the following theorem.

Theorem 2. The spectral radius of $F(x)$ is a strictly decreasing continuous function of x, $x \geq 0$, and

(a) $$\lim_{x \to \infty} |\sigma(F(x))| = 1$$

(b) $$\lim_{x \to 0} |\sigma(F(x))| = 2 \,.$$

Proof. From Theorem 1 we know that

$$|\sigma(F(x))| = r_x$$

where r_x is the only real root of $P_x(\xi)$, $|\xi| > 1$, and $1 < r_x \leq 2$. Let us assume that n is a positive integer and

$$n - 1 \leq x < y < n \,.$$

It now follows that $r_x > r_y$. The proof is by contradiction.

Assume $r_x \leq r_y$. Then

$$P_x(r_x) = (r_x^n - \epsilon_x)(r_x - 1) - 1 = 0$$

and

$$P_y(r_y) = (r_y^n - \epsilon_y)(r_y - 1) - 1 = 0$$

where

$$\epsilon_x = [x + 1] - x \,.$$

From these equations and the assumption that $r_x \leq r_y$, it follows that

$$r_x = 1 + \frac{1}{r_x^n - \epsilon_x} \leq 1 + \frac{1}{r_y^n - \epsilon_y} = r_y \,,$$

or

$$r_y^n - r_x^n \le \epsilon_x - \epsilon_y = x - y < 0$$

and therefore $r_y < r_x$ which is a contradiction to our assumption that $r_x \le r_y$.

Since we have shown that r_x is strictly decreasing as x increases and is therefore strictly increasing as x decreases for

$$n - 1 < x < n,$$

we see that if

$$n - 1 < y < n,$$

then the following limits exist:

$$\lim_{x \to y+} r_x = \alpha \quad \text{and} \quad \lim_{x \to y-} r_x = \beta .$$

Therefore, since

$$\lim_{x \to y} \epsilon_x = \epsilon_y ,$$

$$\lim_{x \to y+} P_x(r_x) = P_y(\alpha) = 0 \quad \lim_{x \to y-} P_x(r_x) = P_y(\beta) .$$

But since

$$P_y(\xi), \quad |\xi| > 1 ,$$

has only one real root, namely r_y, it follows that $r_y = \alpha = \beta$ and therefore

$$\lim_{x \to y} r_x = r_y$$

or r_x is a continuous function of x on

$$n - 1 < x < n .$$

It is not difficult to see that

$$\lim_{x \to n} r_x = r_n$$

where n is any positive integer. First it is clear that as

$$x \to n^+, \quad P_x(r_x) = (r_x^{n+1} - \epsilon_x)(r_x - 1) - 1 = 0$$

and

$$\epsilon_x = (n + 1) - x ,$$

provided

$$x < n + 1 ,$$

hence

$$\lim_{x \to n+} P_x(r_x) = (\gamma^{n+1} - 1)(\gamma - 1) - 1 = \gamma^{n+2} - \gamma^{n+1} - \gamma = 0 = P_n(\gamma)$$

where

$$\lim_{x \to n+} r_x = \gamma .$$

Similarly as $x \to n^-$,

$$P_x(r_x) = (r_x^n - \epsilon_x)(r_x - 1) - 1 = 0$$

and

$\epsilon_x - n - x$, provided $x > n - 1$, hence

$$\lim_{x \to n^-} P_x(r_x) = (\delta^n - 0)(\delta - 1) - 1 = \delta^{n+1} - \delta^n - 1 = 0 = \frac{P_n(\delta)}{\delta} ,$$

where

$$\lim_{x \to n^-} r_x = \delta .$$

Since both γ and δ must lie between 1 and 2 and $P_n(\xi)$, $|\xi|>1$, has precisely one real root we infer that $\gamma = \delta$ or r_x is continuous at $x = n$ for n an arbitrary positive integer.

It now follows that r_x is a continuous function of x for all $x > 0$ and r_x is a strictly decreasing function of x.

Finally we shall show that

$$\lim_{x \to \infty} r_x = 1 .$$

For assume

$$\lim_{x \to \infty} r_x = r$$

where $r > 1$. In this case

$$\lim_{x \to \infty} \left\{ r_x^{[x+1]} - \epsilon_x \right\} = \lim_{x \to \infty} \frac{1}{r_x - 1} = \frac{1}{r - 1}$$

since for all $x > 0$

$$r_x^{[x+1]} - \epsilon_x = \frac{1}{r_x - 1} .$$

But it is clear that

$$r_x^{[x+1]} - \epsilon_x$$

becomes arbitrarily large as x approaches infinity and therefore cannot have

$$\frac{1}{r - 1}$$

as a limit. This contradicts our assumption that $r > 1$.

That

$$\lim_{x \to 0} r_x = 2$$

follows immediately from the fact that

$$P_0(\lambda) = \lambda^2 - 2\lambda .$$

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A CURIOUS PROPERTY OF A SECOND FRACTION

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In the April, 1968 Fibonacci Quarterly (p. 156), J. Wlodarski discussed some properties of the fraction 878/323 which approximates e. Consider the approximation of π correct to six decimal places given by $355/113 = 3.141592^+$. The sum of the digits of the numerator is 13, and of the denominator, 5. $13/5 = 1 + 8/5$, or one added to the best approximation to the "Golden Ratio" using two one-digit numbers. Also,

$$\frac{355}{113} = \frac{300 + 55}{100 + 13} ,$$

where 55 and 13 are Fibonacci numbers.

Taking 355/226 as an approximation of $\pi/2$ leads to the observation that

$$\frac{355}{226} = \frac{377 - 22}{233 - 7}$$

where 377/233 approximates the golden ratio and 22/7 approximates π, and 377 and 233 are Fibonacci numbers.

A LINEAR ALGEBRA CONSTRUCTED FROM FIBONACCI SEQUENCES
PART I: FUNDAMENTALS AND POLYNOMIAL INTERPRETATIONS

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The purpose of this paper is to demonstrate the construction of a linear algebra with whole Fibonacci sequences as elements. Sequences of complex numbers are considered; hence, this is an algebra over the complex field.

To be of more than curious interest, of course, the algebra must lead somewhere. The vector space leads to geometric interpretation of sequences. The ring leads to polynomial interpretations, and in particular, to binomial expressions. Part II will deal with functions and Taylor series representations.

Only a knowledge of modern algebra at the undergraduate level is required to follow the discussion in Part I. A smattering of topology is required for Part II. Proofs are elementary and are usually based on definitions. In some cases, the reader is asked to fill in the details himself. We begin with:

Definition 1.1. A Fibonacci sequence $U = (u_i)$, $i = 0, 1, \cdots$, is a sequence that has the following properties:

1. u_0, u_1 are arbitrary complex numbers,
2. $u_{n+1} = u_n + u_{n-1}$, $n = 1, 2, \cdots$.

$\mathfrak{F}$ will denote the set of all Fibonacci sequences. Any sequence may be extended to negative subscripts by transposing the recurrence formula; i. e., $u_{n-1} = u_{n+1} - u_n$.

A list of special sequences follows:

$$A = (1, \alpha, \alpha^2, \cdots), \quad \alpha = \frac{1+\sqrt{5}}{2}$$

$$B = (1, \beta, \beta^2, \cdots), \quad \beta = \frac{1-\sqrt{5}}{2}$$

$$F = (0, 1, 1, 2, \cdots)$$
$$I = (1, 0, 1, 1, \cdots)$$
$$L = (2, 1, 3, 4, \cdots)$$
$$O = (0, 0, 0, \cdots)$$

In addition to this, we use the symbols C, R, and Z for the complex, reals, and integers, respectively.

Definition 1.2. For all $U, V \in \mathfrak{F}$, $U = V \Leftrightarrow u_i = v_i$, $i = 0, 1, 2, \cdots$.

Definition 1.3. For $U, V \in \mathfrak{F}$, $U + V = (u_i + v_i)$, $i = 0, 1, \cdots$.

Definition 1.4. For $a \in C$, $U \in \mathfrak{F}$, $aU = (au_i)$, $i = 0, 1, \cdots$.

Theorem 1.1. $\mathfrak{F}$ is a vector space.

Proof. It is a well-known fact that sums and scalar products of Fibonacci sequences yield Fibonacci sequences. The reader is asked to fill in the remainder of the proof from the definition of a vector space. The zero vector is $(0, 0, \cdots)$, and any additive inverse is given by $-U = (-u_0, -u_1, \cdots)$.

Theorem 1.2. The dimension of $\mathfrak{F}$ is 2.

Proof. Consider the vectors I, F, and O, and suppose that $aI + bF = O$. By definitions 2, 3, and 4, the first two terms yield $a = b = 0$. If we insist that a or b be non-zero, then $aI + bF = U \neq O$. We now find that $a = u_0$, $b = u_1$. From $u_0I + u_1F = U$ we find from the n^{th} term that $u_0F_{n-1} + u_1F_n = u_n$, which is a well-known property of all Fibonacci sequences. Hence, an arbitrary vector is uniquely determined by two linearly independent vectors in F, and the theorem is proved.

Theorem 1.3. F is isomorphic to $V_2(C)$, the vector space of all ordered pairs of complex numbers.

Proof. Any vector space is isomorphic to the vector space of n-tuples of its components relative to a fixed basis. Hence, for

$$U \in \mathfrak{F}, \quad U = u_0I + u_1F \Longleftrightarrow U \longleftrightarrow (u_0, u_1) \in V_2(C) .$$

As a consequence of Theorem 1.3, we may agree to identify an arbitrary sequence $U = (u_i)$, $i = 0, 1, \cdots$, with the pair (u_0, u_1), and write $U = (u_0, u_1)$. Property 2 of definition 1.1 has been suppressed, so we turn our attention to the construction of a ring that will bring this property back into evidence.

Definition 1.5. For $U, V \in \mathfrak{F}$, $UV = (u_0 v_0 + u_1v_1, u_0v_1 + u_1v_0 + u_1v_1)$.

Theorem 1.4. $\mathfrak{F}$ is a commutative linear algebra with unity $I = (1, 0)$.

Proof. The reader is asked to fill in the details again.

Associated with each sequence is a complex number, called the characteristic number, that describes many properties of the sequence in the algebra.

Definition 1.6. The characteristic number $C(U)$ of a sequence $U = (u_0, u_1)$ is the complex number $u_0^2 + u_0u_1 - u_1^2 = u_0u_2 - u_1^2$.

Theorem 1.5. $C(U) \neq 0 \Leftrightarrow U$ has a multiplicative inverse $U^{-1} \in \mathfrak{F}$.

Proof. If U has an inverse (x, y), then $(u_0, u_1)(x, y) = (1, 0)$. This is equivalent to the equations

$$\begin{aligned} u_0x + u_1y &= 1 \\ u_0y + u_1x + u_1y &= 0 \;. \end{aligned} \tag{1}$$

Since either $u_0 \neq 0$ or $u_1 \neq 0$, we may reduce equations 1 to

$$\begin{aligned} x(u_0^2 + u_0u_1 - u_1^2) &= u_0 + u_1 \\ y(u_0^2 + u_0u_1 - u_1^2) &= -u_1 \;. \end{aligned} \tag{2}$$

The remainder of the proof is obvious.

Corollary 1.1. If $C(U) \neq 0$, then

$$U^{-1} = \frac{1}{C(U)}(u_2, -u_1) \;.$$

Corollary 1.2. $C(U) = 0 \Leftrightarrow U = a(1, \alpha),\ b(1, \beta)$.

Proof. Solve the equation $u_0^2 + u_0u_1 - u_1^2 = 0$ for u_0/u_1.

The sequence $F = (0, 1)$ plays a major role in the algebra as a shifting operator, and brings property 2 of definition 1.1 back into evidence.

Theorem 1.6. $F^nU = (u_n, u_{n+1}),\ n \in Z$.

Proof. Note that

$$FU = (0, 1)(u_0, u_1) = (u_1, u_0 + u_1),$$

and that

$$F^{-1}U = (-1, 1)(u_0, u_1) = (u_1 - u_0, u_0) \;.$$

The rest of the proof follows easily by mathematical induction.

Theorem 1.7. $C(F^nU) = (-1)^nC(U),\ n \in Z$.

Proof. Note that

$$\begin{aligned} C(FU) &= u_1^2 + u_1u_2 - u_2^2 = u_1^2 + u_1(u_0 + u_1) - (u_0 + u_1)^2 \\ &= -(u_0^2 + u_0u_1 - u_1^2) = -C(U) \;. \end{aligned}$$

The rest of the proof follows easily by induction.

Theorem 1.8. $C(U) \neq 0$, and $n \neq m \Leftrightarrow F^nU$, F^mU are linearly independent in $\mathfrak{F}$.

Proof. We test for linear independence by setting $a(u_n, u_{n+1}) + b(u_m, u_{m+1}) = (0,0)$. This is equivalent to the two equations

$$\begin{aligned} au_n + bu_{n+1} &= 0 \\ au_m + bu_{m+1} &= 0 \end{aligned} \tag{3}$$

Since all $u_i \neq 0$, we may reduce equations 3 to

$$a(u_n u_{m+1} - u_{n+1} u_m) = 0 \quad . \tag{4}$$

$n \neq m$ by hypothesis, so let $m = n + k$, and use the identities $u_m = u_{n+k} = u_n F_{k+1} + u_{n+1} F_k$ and $u_{m+1} = u_{n+1+k} = u_{n+1} F_{k-1} + u_{n+2} F_k$. Equation 4 may now be reduced to

$$a(u_n u_{n+2} - u_{n+1}^2) F_k = aC(F^nU) F_k = 0 \tag{5}$$

Since $C(F^nU) = (-1)^n C(U) \neq 0$, and $F_k \neq 0$ in general, we must conclude that $a = 0$, which in turn implies that $b = 0$.

The converse is proved by assuming that a, b are not both zero. We can, without loss of generality, assume that $a \neq 0$, which implies that $u_n u_{n+2} - u_{n+1}^2 = 0$. Thus $C(F^nU) = 0 \Rightarrow C(U) = 0$.

An alternate form of the product in $\mathfrak{F}$ is now given.

Theorem 1.9. $UV = u_0 V + u_1 FV = v_0 U + v_1 FU$.

Proof. The proof follows immediately from definition 5.

Multiplication in the ring is equivalent to a linear transformation in the vector space, or symbolically, $UV = U(V) = V(U)$, where $U(V)$ means U transforms V. This can be written in matrix form

$$UV = (v_0, v_1) \begin{pmatrix} u_0 & u_1 \\ u_1 & u_0 + u_1 \end{pmatrix} = (u_0, u_1) \begin{pmatrix} v_0 & v_1 \\ v_1 & v_0 + v_1 \end{pmatrix} \tag{6}$$

Any sequence of complex numbers can be decomposed into two sequences of real numbers.

Theorem 1.10. $U = X + iY$, where $U \in \mathfrak{F}(C)$, and $X, Y \in \mathfrak{F}(R)$.

Proof. The reader is asked to supply the details.

The vector space in $\mathfrak{F}$ is obviously a unitary 2-space, and the restriction of $\mathfrak{F}$ to real sequences yields a Euclidean 2-space. Some interesting geometric interpretations follow from this, but lack of space prevents further exposition here.

POLYNOMIALS IN $\mathfrak{F}$ OVER C

The polynomial interpretation of $\mathfrak{F}$ leads to some interesting results. We now establish the conditions for writing polynomials with sequences as "indeterminants" and coefficients in the complex field.

Theorem 1.11. C is embedded in $\mathfrak{F}$.

Proof. Let $\psi: C \to \mathfrak{F}$ be defined by the rule: $\psi(a) = (a, 0) = aI$, $\forall a \in C$. We ask the reader to complete the proof.

Integral powers of Fibonacci sequences make sense as a consequence of our definition of multiplication in $\mathfrak{F}$. The classic conditions for writing polynomials exist, so that $p(X) = a_0 + a_1X + \cdots + a_nX^n$ makes sense, but this is not the whole story. $p(X)$ is a linear combination of the elements $X^i \in \mathfrak{F}$, and can be expressed uniquely as a linear combination of any two linearly independent elements in $\mathfrak{F}$. If it so happens that $C(X) \neq 0$, then by theorem 1.8, X, FX are linearly independent, and there exist k_0, $k_1 \in C$, not both zero, such that $p(X) = k_0X + k_1FX$. But $K = (k_0, k_1) \in \mathfrak{F}$, and by theorem 1.9, $p(X) = KX$. The linear independence of powers of X does not exist in polynomials in $\mathfrak{F}$ over C. This explains why each of the hundreds (possibly thousands) of known summations involving Fibonacci numbers is expressible as a linear combination of at most two Fibonacci numbers. The addition formula for elements of a Fibonacci sequence is a case in point, which can easily be derived in $\mathfrak{F}$. Try it for an exercise.

The sequences $I^n = I = (1, 0)$ and $F^n = (F_{n-1}, F_n)$ may be written down termwise by inspection. L^n follows easily.

Theorem 1.12. $L^{2k} = 5^k(F_{2k-1}, F_{2k})$, and $L^{2k+1} = 5^k(L_{2k}, L_{2k+1})$.

Proof: $L^2 = (2, 1)(2, 1) = 5(1, 1) = 5F^2$, from which $L^{2k} = 5^kF^{2k}$.

$$L^{2k+1} = L^{2k}L = 5^k F^{2k} L \; .$$

Several formulas for the general case U^n will be given.

Definition 1.7. The term of U^n bearing the subscript k will be designated $(U^n)_k$, $k = 0, 1, 2, \cdots$ (the k term is actually the $(k+1)^{st}$ term by ordinal count).

Lemma 1.1. Let $c_i \in C$, $i = 0, 1, \cdots, n$, and let $U \in \mathfrak{F}$. Then

$$\sum_{i=0}^{n} c_i U^i = \left(\sum_{i=0}^{n} c_i (U^i)_0 \, , \sum_{i=0}^{n} c_i (U^i)_1 \right) .$$

Proof. The reader is asked to supply the details.

Theorem 1.13.

$$(U^n)_{k+1} = \sum_{i=0}^{n} \binom{n}{i} u_0^{n-i} u_1^i F_{k+i} \, .$$

Proof.

$$U^n = (u_0 I + u_1 F)^n = \sum_{i=0}^{n} \binom{n}{i} u_0^{n-i} u_1^i F^i \; .$$

Lemma 1.1 and definition 1.7 supply the remainder of the proof.

An alternate form of theorem 1.13 is

Theorem 1.14.

$$(U^n)_{k+1} = \frac{\alpha^k (u_0 + \alpha u_1)^n - \beta^k (u_0 + \beta u_1)^n}{\alpha - \beta}$$

Proof. Substitute the Binet formula,

$$F_j = \frac{\alpha^j - \beta^j}{\alpha - \beta} \qquad ,$$

into theorem 1.13, and reduce it to the form shown.

Example 1.1. Consider the generating function

$$D^n = (I + F^k)^n = \sum_{i=0}^{n} \binom{n}{i} F^{ki} \quad , \tag{7}$$

where

$$(D^n)_{j+1} = \sum_{i=0}^{n} \binom{n}{i} F_{ki+j} \quad . \tag{8}$$

If $k = 1$, $I + F = (1,0) + (0,1) = (1,1) = F^2$, and

$$(D^n)_{j+1} = (F^{2n})_{j+1} = F_{2n+j} = \sum_{i=0}^{n} \binom{n}{i} F_{i+j} \quad . \tag{9}$$

If $k = -1$, $I + F^{-1} = (1,0) + (-1,1) = (0,1) = F$, and

$$(D^n)_{j+1} = (F^n)_{j+1} = F_{n+j} = \sum_{i=1}^{n} \binom{n}{i} F_{-i+j} \quad . \tag{10}$$

But since $F_{-(i-j)} = (-1)^{i-j+1} F_{i-j}$, we have

$$F_{n+j} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i-j+1} F_{i-j} \quad . \tag{11}$$

If $k = 2$, $I + F^2 = (1,0) + (1,1) = (2,1) = L$. From Theorem 1.12, we get

$$5^{n/2} F_{n+j} = \sum_{i=0}^{n} \binom{n}{i} F_{2i+j} \text{, for even n, and } 5^{(n-1)/2} L_{n+j} = \sum_{i=0}^{n} \binom{n}{i} F_{2i+j} \text{, for odd n.} \tag{12}$$

This may be generalized for even k. If the reader will verify that $I + F^{4m} = L_{2m}F^{2m}$, and $I + F^{4m+2} = F_{2m+1}F^{2m}L$, then he may compute $(D^n)_{j+1}$, and complete the problem.

Much of what we know about polynomials may be applied to polynomials in $\mathfrak{F}$ over C. The possibilities of generating term-by-term Fibonacci relations is unbounded.

ADDITIONAL NOTES

1. Let M be the set of all matrices of the form

$$U = \begin{pmatrix} u_0 & u_1 \\ u_1 & u_0 + u_1 \end{pmatrix}, \quad u_0, u_1 \in C,$$

and let the operations be the usual operations of matrix algebra. Then M is isomorphic to F.

2. Let $c[x]$ be the set of polynomials in x over C, and let $s(x) = x^2 - x - 1$. Then $C[x]/s(x)$ is the ring of residue classes of polynomials over C modulo $x^2 - x - 1$. Each residue class has the form $[u_0 + u_1x]$ with operations defined by

$$[u_0 + u_1x] + [v_0 + v_1x] = [u_0 + v_0 + (u_1 + v_1)x]$$

$$[u_0 + u_1x][v_0 + v_1x] = [u_0v_0 + u_1v_1 + (u_0v_1 + u_1v_0 + u_1v_1)x].$$

If we add the redundant operation

$$a[u_0 + u_1x] = [au_0 + au_1x],$$

then $C[x]/s(x)$ is a linear algebra, and furthermore, $C[x]/s(x)$ is isomorphic to $\mathfrak{F}$.

ACKNOWLEDGEMENTS

1. The vector space F is one element of the space of sequences noted by E. D. Cashwell and C. J. Everett, "Fibonacci Spaces," Fibonacci Quarterly, Vol. 4, No. 2, pp. 97-115.

2. The characteristic number of a sequence was used by Brother U. Alfred, "On the Ordering of Fibonacci Sequences," Fibonacci Quarterly, Vol. 1, No. 4, 1963, pp. 43-46. [See Correction, p. 38, Feb. 1964 Quarterly.]

3. A definition for ring multiplication,

$$UV = (u_0v_1 + u_1v_0 - u_1v_1, u_0v_0 + u_1v_1) ,$$

was given by Ken Dill in a paper submitted to the Westinghouse Talent Contest.

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ROMANCE IN MATHEMATICS

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The dome of the famous Taj Mahal, built in 1650 in Agra, India, is ellipsoidal. Now, the ellipse has the geometric property that the angles formed by the focal radii and the normal at a point are congruent. Also, it is a fundamental principle of behavior of sound waves that the angle of incidence equals the angle of reflection. Thus, sound waves issuing from focus A and striking any point on the ellipse will be reflected through focus B.

The builder of the Taj Mahal, Shan Jehan, used these basic principles well in his memorial to his favorite wife who was called Taj Mahal, Crown of the Palace. Honeymooners who visit the shrine are instructed to stand on the two foci which are marked in the tile floor. The husband whispers, "To the memory of an undying love," which can be heard clearly by his wife who is more than fifty feet away but by no one else in the room.

REFERENCE

Kramer, Edna E., "The Mainstream of Mathematics," Premier (paperback), New York, 1961, p. 152.

A LINEAR ALGEBRA CONSTRUCTED FROM FIBONACCI SEQUENCES PART II: FUNCTION SEQUENCES AND TAYLOR SERIES OF FUNCTION SEQUENCES

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In Part I, the algebra $\mathfrak{F}$ was constructed from the set of complex Fibonacci sequences. Finite polynomial and binomial interpretations were considered. We now consider a class of functions defined in $\mathfrak{F}$, which are models of prototype functions in C. These are extended to include Taylor series representations.

We first consider an auxiliary algebra, which is constructed from bits and pieces of easily recognizable structures. As in Part I, many of the proofs are elementary, and the reader is asked to fill in the details himself.

Definition 2.1 Let $G = \{(a, b) : a, b \in C\}$, and define equality and three operations as follows: For $(a_1, a_2), (b_1, b_2) \in G$, $c \in C$,

1. $(a_1, a_2) = (b_1, b_2) \Longleftrightarrow a_1 = b_1, \quad a_2 = b_2$.
2. $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$.
3. $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$.
4. $c(a_1, a_2) = (ca_1, ca_2)$.

Theorem 2.1 G is a commutative linear algebra with unity (1,1).

Proof. The reader is asked to fill in the details.

Definition 2.2 Let $\phi : F \longrightarrow G$ be a function defined by the rule:

$$\phi(u_1, u_1) = (u_1 + \alpha u_1, \; u_0 + \beta u_1) \quad \text{for all } U \in \mathfrak{F}.$$

Theorem 2.2. $\phi : \mathfrak{F} \rightarrow G$ is an isomorphism.

Proof: ϕ is obviously a 1-1 linear transformation from the vector space $\mathfrak{F}$ onto the vector space G. We need only show that ϕ preserves multiplication. For $U, V \in \mathfrak{F}$,

$$
\begin{aligned}
(1) \qquad \phi(UV) &= \phi(u_0v_0 + u_1v_1,\ u_0v_1 + u_1v_0 + u_1v_1) \\
&= (u_0v_0 + u_1v_1 + \alpha(u_0v_1 + u_1v_0 + u_1v_1),\ u_0v_0 + u_1v_1 \\
&\qquad\qquad + \beta(u_0v_1 + u_1v_0 + u_1v_1)) \\
&= (u_0v_0 + \alpha(u_0v_1 + u_1v_0) + (\alpha + 1)u_1v_1\ ,\ u_0v_0 \\
&\qquad\qquad + \beta(u_0v_1 + u_1v_0) + (\beta + 1)u_1v_1) \\
&= ((u_0 + \alpha u_1)(v_0 + \alpha v_1), (u_0 + \beta u_1)(v_0 + \beta v_1)) = \phi(U)\phi(V)\ .
\end{aligned}
$$

The mapping ϕ was motivated by considering the linear factors of the characteristic number; i.e.,

$$C(U) = u_0^2 + u_0u_1 - u_1^2 = (u_0 + \alpha u_1)(u_0 + \beta u_1)\ .$$

Some fundamental vectors are mapped as follows:

1. $\phi(A) = \phi(1, \alpha) = (1 + \alpha^2, 0)$
2. $\phi(B) = \phi(1, \beta) = (0, 1 + \beta^2)$
3. $\phi(I) = \phi(1, 0) = (1, 1)$.

A, B determine the coordinate planes, and I determines a plane of symmetry, which will become significant later. A characteristic number for each

$$X = (x_1, x_2) \in G$$

can be defined as

$$C(X) = x_1 x_2\ .$$

Thus for $U \in \mathfrak{F}$, $C(U) = C(\phi(U))$.

Definition 2.3 Let f be an arbitrary function defined on a domain $D \subseteq C$. Define a corresponding $\hat{f}: D \times D \to G$ by the rule:

$$\hat{f}(X) = \hat{f}(x_1, x_2) = (f(x_1), f(x_2)).$$

whenever no confusion will exist, we will agree to identify $\hat{f}$ with f and write $\hat{f}(X) = f(X)$.

Definition 2.4. If f is defined on $D \subseteq C$, and if $U = (u_1, u_1) \in \mathfrak{F}$ is such that

$$u_0 + \alpha u_1, u_0 + \beta u_1 \in D,$$

define

$$f^\star : \phi^{-1}(D \times D) \longrightarrow F$$

by the rule:

$$f^\star(U) = \phi^{-1}(\hat{f}(X)),$$

where $X = \phi(U)$, or more simply

$$f^\star(U) = \phi \hat{f} \phi^{-1}(U) .$$

The notation used herein for composition of maps is: the order of events reads from left to right, or

$$\phi \hat{f} \phi^{-1}(U) = \phi^{-1}(\hat{f}(\phi(u))) .$$

We may again agree to identify $f^\star$ with $\hat{f}$ whenever no confusion will result, and say $f^\star(U) = \hat{f}(U) = f(U)$.

Theorem 2.3. The formula for $f^\star$ is

$$f^\star(U) = \frac{1}{\alpha - \beta}(\alpha^{-1}f(u_1 + \alpha u_1) - \beta^{-1}f(u_0 + \beta u_1) ,$$
$$f(u_0 + \alpha u_1) - f(u_0 + \beta u_1)).$$

Proof. The proof follows directly from Definition 2.4.

Corollary 2.1. If $f(x) = c$ (a constant), then $f^\star(U) = cI$.

Corollary 2.2 $f^\star(aI) = f(a)I$.

The reader may verify that the functions defined above are well-behaved Fibonacci sequences, and are thus, elements of $\mathfrak{F}$. The reader may further verify the following identities for some elementary functions: For $U, V \in \mathfrak{F}$,

1. $\exp U \exp V = \exp (U + V)$
2. $\exp (-U) = (\exp U)^{-1}$
3. $\sin^2 U + \cos^2 U = I$
4. $\sin U \cos U = \frac{1}{2} \sin 2U$
5. $\sin U (\cos U)^{-1} = \tan U.$

All operations must, of course, be those defined in $\mathfrak{F}$. The brute force approach required by Theorem 2.3 and the subsequent arithmetic in $\mathfrak{F}$ can be tempered by a trick: do the arithmetic in G.

Example 2.1. Show that

$$\sin (U + V) = \sin U \cos V + \cos U \sin V.$$

Since

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

is an identity in C, definition 2.1(1) gives

$$(\sin(x_1 + y_1), \sin (x_2 + y_2)) = (\sin x_1 \cos y_1 + \cos x_1 \sin y_1 , \sin x_2 \cos y_2 + \cos x_2 \sin y_2) \tag{2}$$

as an identity in G. We appeal now to definition 2.3 for the left side of (2) and to definition 2.1(2), (3) for the right side.

$$\widehat{\sin} ((x_1 + y_1), (x_2 + y_2)) = (\sin x_1, \sin x_2)(\cos y_1, \cos y_2) + (\cos x_1, \cos x_2)(\sin y_1, \sin y_2). \tag{3}$$

We now reverse our position and appeal to Definition 2.1 for the left side and Definition 2.3 for the right side of (3).

$$\widehat{\sin} ((x_1, x_2) + (y_1, y_2)) = \widehat{\sin}(x_1, x_2) \widehat{\cos} (y_1, y_2) + \widehat{\cos} (x_1, x_2) \widehat{\sin} (y_1, y_2) , \tag{4}$$

(5) $$\hat{\sin}\,(X + Y) = \hat{\sin}\,X\,\hat{\cos}\,Y + \hat{\cos}\,X\,\hat{\sin}\,Y\,.$$

Definition 2.4, together with Theorem 2.2, yields

(6) $$\sin^\star\,(U + V) = \sin^\star\,U\,\cos^\star\,V + \cos^\star\,U\,\sin^\star\,V\,.$$

the asterisk may be omitted because of our previous agreement.

We have proved in example 2.1 that

$$\sin\,(x + y) = \sin x \cos y + \cos x \sin y \in C \longrightarrow \sin\,(U + V) = \sin U \cos V + \cos V \sin V \in \mathfrak{F}.$$

Notice that, although the work was done in G, no element of G is evident in the final result. This is why G was called an auxiliary algebra in the introduction.

SOME SPECIAL FUNCTIONS

We could continue to define and explore Fibonacci function sequences ad infinitum, but we shall limit the discussion to two very elementary ones. First a theorem must be proved.

Theorem 2.4. If f and f^{-1} both exist on a subset of C, then

$$(f^\star)^{-1} = (f^{-1})^\star$$

on the corresponding subset of $\mathfrak{F}$.

Proof. $f^\star(U)$ is known from Theorem 2.3. Then

(7) $$f^\star(U) \xrightarrow{\phi} \hat{f}(X) = (f(x_1), f(x_2)) \xrightarrow{\hat{f}^{-1}} (f^{-1}(f(x_1)), f^{-1}(f(x_2)))$$
$$= (x_1, x_2) = X \xrightarrow{\phi^{-1}} U\,.$$

From Definition 2.4, we have

(8) $$(f^{-1}(f(x_1)), f^{-1}(f(x_2))) \xrightarrow{\phi^{-1}} (f^{-1})^\star\,(f^\star(U)\,).$$

Since ϕ^{-1} is a mapping,

$$U = (f^{-1})^{\star}(f^{\star}(U)) ,$$

so

$$(f^{-1})^{\star} = (f^{\star})^{-1} .$$

A very fundamental function is now given by:

Definition 2.5. For $U, V \in \mathfrak{F}$, define $U^V = \exp(V \ln U)$.

When written in terms of the components of U, V,

$$U^V = \frac{1}{\alpha - \beta}\left(\alpha^{-1}(u_0 + \alpha u_1)^{v_0+\alpha v_1} - \beta^{-1}(u_0 + \beta u_1)^{v_0+\beta v_1}\right.,$$

$$\left.(u_0 + \alpha u_1)^{v_0+\alpha v_1} - (u_0 + \beta u_1)^{v_0+\beta v_1}\right) . \tag{9}$$

Since $\ln z$ is a many valued function, some trouble may arise from Definition 2.5. The author offers the conjecture that no trouble will arise. Perhaps one of the readers will explore this possibility.

If $V = nI$, Definition 2.5 is specialized to Theorem 1.14. Another elementary but interesting set of relations are the multiple n^{th} roots of a sequence.

Theorem 2.5. There are n^2 distinct n^{th} roots of $U \neq 0 \in \mathfrak{F}$.

Proof. Let

$$r_1^n = |u_0 + \alpha u_1|, \quad r_2^n = |u_0 + \beta u_1| ,$$

and

$$\alpha_i \quad (i = 0, 1, \cdots, n-1)$$

be the complex roots of unity. Then

$$U^{1/n} = \frac{1}{\alpha - \beta}\left(\alpha^{-1} r_1 \omega_i - \beta^{-1} r_2 \omega_j, r_1 \omega_i - r_2 \omega_j\right) . \tag{10}$$

If N is the number of possible solutions, then clearly $N \leq n^2$. We must show $N \nless n^2$. Assume the contrary; i.e., there are at least two identical solutions, which must be termwise equal.

$$\alpha^{-1}r_1\omega_i - \beta^{-1}r_2\omega_j = \alpha^{-1}r_1\omega_k - \beta^{-1}r_2\omega_\ell \tag{11}$$
$$r_1\omega_i - r_2\omega_j = r_1\omega_k - r_2\omega_\ell \ .$$

Both $\omega_i \neq \omega_k$ and $\omega_j \neq \omega_\ell$ must hold or the hypothesis is contradicted immediately. Thus,

$$\alpha r_2 (\omega_j - \omega_\ell) = \beta r_1 (\omega_i - \omega_k) \tag{12}$$
$$r_2 (\omega_j - \omega_\ell) = r_1 (\omega_i - \omega_k) \ .$$

If we substitute from the second equation into the first,

$$\alpha(\omega_i - \omega_k) = \beta(\omega_i - \omega_k) \ , \tag{13}$$

which is clearly impossible unless $\omega_i = \omega_k$. This in turn implies that $\omega_j = \omega_\ell$. Thus, the hypothesis is contradicted, and the theorem is proved.

The reader is invited to find the four square roots of $F^2 = (1, 1)$ (cf. Theorem 1.12).

TAYLOR SERIES REPRESENTATIONS

In order to use the very useful concept of Taylor series representations of complex functions, a definition of convergence in $\mathfrak{J}$ must be formulated. A very short excursion into topology (metric spaces) will furnish the necessary foundation. Let d be the usual metric on C defined by

$$d(z_1, z_2) = |z_2 - z_1|$$

for all $z_1, z_2 \in C$. The next few theorems are so elementary that the proofs are omitted; however, they must be stated. Since the underlying set of G is $C \times C$, we may give

Definition 2.6. Let $\hat{d} : G \times G \longrightarrow R$ be defined by the rule:

$$\hat{d}(X, Y) = \max(d(x_1, y_1),\ d(x_2, y_2)) = \max(|y_1 - x_1|,\ |y_2 - x_2|) \ .$$

Theorem 2.6. $\hat{d}$ is a metric; hence, $(G, \hat{d})$ is a metric space.

An open sphere in G of radius r about the point X is

$$\hat{S}_r(X) = \left\{ Y \in G : \hat{d}(X,Y) \leq r \right\}.$$

If

$$\phi^{-1}(X) = U, \quad \phi^{-1}(Y) = V,$$

then

$$\phi^{-1}(\hat{S}_r(X)) = S_r^\star(U) = \{V \in \mathfrak{F} : \max(|v_0 - u_0 + \alpha(v_1 - u_1)|, \\ |v_0 - u_0 + \beta(v_1 - u_1)|) \leq r\}.$$

If we restrict $\mathfrak{F}$, G to real numbers, then

$$\phi^{-1}(\hat{S}_r(X))$$

is the interior of a golden rectangle with diagonal of length 2r, centered on U, and parallel to the vector I, and with short sides parallel to A, and long sides parallel to B. This fact should delight any true Fibonacciphile, and motivates:

Definition 2.7. Let $d^\star : \mathfrak{F} \times \mathfrak{F} \to R$ be defined by the rule:

$$d^\star(U,V) = \max(|v_0 - u_0 + \alpha(v_1 - u_1)|, |v_0 - u_0 + \beta(v_1 - u_1)|).$$

Theorem 2.7. $d^\star$ is a metric; hence, $(F, d^\star)$ is a metric space.

Theorem 2.8. $\phi : (\mathfrak{F}, d^\star) \to (G, \hat{d})$ is a homeomorphism.

By design the metric spaces $(\mathfrak{F}, d^\star)$, $(G, \hat{d})$ are topologically equivalent. The necessary groundwork has now been laid for the theorem on convergence.

Theorem 2.9. If

$$f(z) = \sum_{i=0}^{\infty} a_i (z - z_0)^i$$

is a Taylor series for $z \in S_r(z_0)$, then

$$f^\star(U) = \sum_{i=0}^{\infty} a_i (U - z_0 I)^i \quad \text{for} \quad U \in S_r^\star(z_0 I) \; .$$

Furthermore:

$$(f^\star(U))_k = \sum_{i=0}^{\infty} a_i ((U - z_0 I)^i)_k \; , \qquad k = 0, 1, \cdots \; .$$

Proof. Let $z_1, z_2 \in S_r(z_0)$. Then for any $\epsilon > 0$, there are N_1, N_2 such that for $n > \max(N_1, N_2)$, we have

$$\sum_{i=0}^{n} a_i \, (z_1 - z_0)^i \in S_\epsilon(f(z_1)) \; , \quad \text{and} \tag{14}$$

$$\sum_{i=0}^{n} a_i \, (z_2 - z_0)^i \in S_\epsilon(f(z_2)) \; . \tag{15}$$

Since these sums are in the coordinate spaces of G, we have

$$\sum_{i=0}^{n} a_i ((z_1 - z_0)^i \, , \, (z_2 - z_0)^i) \in \hat{S}_\epsilon(f(z_1), f(z_2)) \; . \tag{16}$$

But by the definitions of operations in G,

$$\begin{aligned} ((z_1 - z_0)^i, (z_2 - z_0)^i) &= (z_1 - z_0, \, z_2 - z_0)^i \\ &= ((z_1, z_2) - (z_0, z_0))^i \\ &= (Z - Z_0)^i \quad \text{for} \quad i = 0, 1, \cdots \; . \end{aligned} \tag{17}$$

Hence,

$$\sum_{i=0}^{n} a_i (Z - Z_0)^i \in \hat{S}_\epsilon (f(Z)) \; . \tag{18}$$

Let $U = \phi^{-1}(Z)$. Then

$$U \in \phi^{-1}(\hat{S}_r(Z_0))$$

or

$$U \in S_r^\star (z_0 I) \; .$$

By Theorem 2.8,

$$\sum_{i=0}^{n} a_i (U - z_0 I)^k \in S_\epsilon^\star (f^\star(U)) \; . \tag{19}$$

Since $C \times C$ is the underlying set of $\mathfrak{F}$ and G, and since $C \times C$ is always complete as a metric space, the limits exist, which proves the first statement of the theorem.

Now consider a partial sum with remainder in G.

$$f(Z) - \sum_{i=0}^{n} a_i (Z - Z_0)^i = (e_1, e_2) \; . \tag{20}$$

Since this is a finite sum, write the k^{th} term under the mapping ϕ^{-1}.

$$(f(U))_k - \sum_{i=0}^{n} a_i ((U - z_0 I)^i)_k = (e_1, e_2)_k = (E)_k \; . \tag{21}$$

From the first part of the proof, $E \to O$, and by definition $(O)_k = 0$. Hence $(E)_k \to 0$ for each k and the theorem is proved.

Example 2.2. Let

$$(22)\qquad (1 - z)^{-(k+1)} = \sum_{i=0}^{\infty} \binom{k+i}{k} z^i \quad \text{for } z \in S_1(0)$$

be the prototype complex function. Clearly $\frac{1}{2}F \in S_1^{\star}(0)$. By Theorem 2.9 we may write

$$(23)\qquad (I - \tfrac{1}{2}F)^{-(k+1)} = \sum_{i=0}^{\infty} \binom{k+i}{k} (\tfrac{1}{2}F)^i .$$

Reducing the left side of equation 23 yields

$$(24)\qquad \begin{aligned}(I - \tfrac{1}{2}F)^{-(k+1)} &= (((1,0) - (0,\tfrac{1}{2}))^{-1})^{k+1} \\ &= (2(2, -1)^{-1})^{k+1} = 2^{k+1} F^{2k+2} .\end{aligned}$$

Taking the $(j+1)^{st}$ term from each side of equation 23 gives

$$(25)\qquad 2^{k+1} F_{2k+2+j} = \sum_{i=0}^{\infty} \binom{k+i}{k} \frac{F_{i+j}}{2^i}, \qquad \begin{aligned} k &= 0, 1, 2, \cdots, \\ j &= 0, \pm 1, \pm 2, \cdots .\end{aligned}$$

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* * * * *

SYMBOLIC SUBSTITUTIONS INTO FIBONACCI POLYNOMIALS

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1. INTRODUCTION

Symbolic equations give a compact way of representing certain identities. For example, if F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively, then two familiar identities may be written

$$(1 + F)^n = F^{2n}, \quad F^k \equiv F_k ,$$

$$(1 + L)^n = L^{2n}, \quad L^k \equiv L_k ,$$

where the additional qualifiers $F^k \equiv F_k$, $L^k \equiv L_k$ indicate that we drop exponents to subscripts after expanding. Further material on symbolic relations is given in [6, Chapter 15] and [7]. Here we make a similar "symbolic substitution" of certain sequences into the Fibonacci polynomials. We then find the auxiliary polynomials of the recurrence relations which the resulting sequences obey. Finally, we extend these results to the substitution of any recurrent sequence into any sequence of polynomials obeying a recurrence relation with polynomial coefficients.

2. SYMBOLIC SUBSTITUTION OF FIBONACCI NUMBERS INTO FIBONACCI POLYNOMIALS

The Fibonacci numbers F_n are defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n,$$

and the Lucas numbers L_n by

$$L_1 = 1, \; L_2 = 3, \; L_{n+2} = L_{n+1} + L_n .$$

Define the Fibonacci polynomials $f_n(x)$ by

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_{n+2}(x) = xf_{n+1}(x) + f_n(x) .$$

Consider the sequence $\{a_n\}$ given by

$$a_n = f_n(F), \quad F^k \equiv F_k ,$$

that is, a_n is the symbolic substitution of the Fibonacci numbers into the n^{th} Fibonacci polynomial. The first few terms are

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = 1, \quad a_4 = 4, \quad a_5 = 6 .$$

We give four distinct methods of finding the recurrence relation obeyed by the a_n.

The first method applies a technique used by Gould [3]. Write the Fibonacci polynomials as in Figure 1. Our approach to find a_n is to multiply

$$\begin{array}{llll}
1 & & & \\
x & & & \\
x^2 & + 1 & & \\
x^3 & + 2x & & \\
x^4 & + 3x^2 & + 1 & \\
x^5 & + 4x^3 & + 3x & \\
x^6 & + 5x^4 & + 6x^2 & + 1 \\
\cdot & & & \\
\cdot & & & \\
\cdot & & &
\end{array}$$

Figure 1

the coefficient of x^r by F_r and sum the coefficients in the n^{th} row. Now it is known [10] that

$$f_n(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-j-1}{j} x^{n-2j-1} , \tag{1}$$

where $[x]$ represents the greatest integer contained in x. Thus the columns of coefficients in Figure 1 are also those of Pascal's Triangle, so that the generating function $g_k(x)$ for the k^{th} column is

$$g_k(x) = (1 - x)^{-k} .$$

Using Gould's technique, we first find that the generating function for the k^{th} column with the coefficient of x^r multiplied by F_r is

$$[(1 - \alpha x)^{-k} - (1 - \beta x)^{-k}]/(\alpha - \beta)$$

(2)
$$= \left[\sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j} (\alpha^j - \beta^j) x^j\right] \Bigg/ (1 - x - x^2)^k (\alpha - \beta)$$

$$= \left[\sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j} F_j x^j\right] \Bigg/ (1 - x - x^2)^k ,$$

where

$$\alpha = (1 + \sqrt{5})/2, \quad \beta = (1 - \sqrt{5})/2 .$$

We then make all exponents corresponding to coefficients of $f_n(x)$ to be n^{-1} by multiplying the above by x^{2k-1}, which gives the row adjusted generating function for the k^{th} column to be

$$h_k(x) = \frac{x^{2k-1}}{\alpha - \beta} [(1 - \alpha x)^{-k} - (1 - \beta x)^{-k}] .$$

Then

$$G(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{k=1}^{\infty} h_k(x) = \frac{x^{-1}}{\alpha - \beta} \left[\sum_{k=1}^{\infty} \left(\frac{x^2}{1 - \alpha x}\right)^k - \sum_{k=1}^{\infty} \left(\frac{x^2}{1 - \beta x}\right)^k\right]$$

(3)
$$= \frac{x^{-1}}{\alpha - \beta} \left[\frac{\frac{x^2}{1 - \alpha x}}{1 - \frac{x^2}{1 - \alpha x}} - \frac{\frac{x^2}{1 - \beta x}}{1 - \frac{x^2}{1 - \beta x}}\right] = \frac{x^2}{1 - x - 3x^2 + x^3 + x^4} .$$

Result (3) also follows from Problem H-51 [9]. This states that

$$\sum_{k=1}^{\infty} Q_k(x)t^k = \frac{xt}{1 - (2 - x)t + (1 - x - x^2)t^2} ,$$

where

$$Q_k(x) = \sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j} F_j x^j .$$

Thus using (2),

$$G(x) = \sum_{k=1}^{\infty} h_k(x) = x^{-1} \sum_{k=1}^{\infty} Q_k(x) \left(\frac{x^2}{1 - x - x^2} \right)^k$$

$$= \frac{x^2}{1 - x - 3x^2 + x^3 + x^4} .$$

The auxiliary polynomial for the recurrence relation obeyed by the a_n is therefore

(4) $$y^4 - y^3 - 3y^2 + y + 1.$$

The second method uses the generating function for $f_n(t)$. Zeitlin [10] has shown that

$$H(x, t) = \frac{x}{1 - tx - x^2} = \sum_{n=0}^{\infty} f_n(t)x^n .$$

Since

$$a_n = [f_n(\alpha) - f_n(\beta)]/(\alpha - \beta) ,$$

$$
\begin{aligned}
G(x) &= \frac{H(x,\alpha) - H(x,\beta)}{\alpha - \beta} \\
&= \frac{1}{\alpha - \beta}\left[\frac{x}{1 - \alpha x - x^2} - \frac{x}{1 - \beta x - x^2}\right] \\
&= \frac{x^2}{1 - x - 3x^2 + x^3 + x^4}
\end{aligned}
$$

the same as (3).

The third method suggested to the authors by Kathleen Weland, varies the pattern in Figure 1. Write the Fibonacci polynomials as in Figure 2. Then it follows from (1) that the generating function in powers of y for the k^{th} column

$$
\begin{array}{cccccccccc}
 & & & & & & & & & 1 \\
 & & & & & & & x & & \\
 & & & & & x^2 & & & & +\,1 \\
 & & & x^3 & & & & +\,2x & & \\
 & x^4 & & & & +\,3x^2 & & & & +\,1 \\
x^5 & & & +\,4x^3 & & & & +\,3x & & \\
 & & & \vdots & & & & & &
\end{array}
$$

Figure 2

from the right is

$$
\frac{x^k y^{k+1}}{(1 - y^2)^{k+1}} ,
$$

where powers of y for terms on the same row are equal. Then multiplying the k^{th} column by F_k, putting $x = 1$, and summing gives

$$
G(y) = \frac{y}{1 - y^2} \sum_{k=0}^{\infty} F_k \left(\frac{y}{1 - y^2}\right)^k .
$$

Now

$$\sum_{k=0}^{\infty} F_k z^k = \frac{z}{1 - z - z^2}$$

so that

$$G(y) = \frac{y^2}{1 - y - 3y^2 + y^3 + y^4},$$

agreeing with our (3).

Our fourth method uses a matrix approach. It follows by induction that if

$$R(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$R^n(x) = \begin{pmatrix} f_{n+1}(x) & f_n(x) \\ f_n(x) & f_{n-1}(x) \end{pmatrix} \quad (n \geq 0) .$$

Since $f_n(1) = F_n$, we have

$$R^n(1) = Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} .$$

Then the upper right corner of $f_n(Q)$ is a_n. Letting

$$\overline{R}(Q) = \left(\begin{array}{c|c} Q & I \\ \hline I & 0 \end{array}\right),$$

where I is the identity matrix and 0 is the zero matrix, since we may multiply partitioned matrices by blocks, we then have

$$\overline{R}^n(Q) = \left(\begin{array}{c|c} f_{n+1}(Q) & f_n(Q) \\ \hline f_n(Q) & f_{n-1}(Q) \end{array}\right).$$

By the Cayley-Hamilton Theorem, $\overline{R}(Q)$ satisfies its own characteristic polynomial $p(x)$. Since a_n is one of the entries of $\overline{R}^n(Q)$, it obeys a recurrence relation whose auxiliary polynomial is $p(x)$. The desired polynomial is thus

$$p(x) = \det[xI - \overline{R}(Q)] = \det\begin{pmatrix} x-1 & -1 & -1 & 0 \\ -1 & x & 0 & -1 \\ -1 & 0 & x & 0 \\ 0 & -1 & 0 & x \end{pmatrix} \tag{5}$$

$$= x^4 - x^3 - 3x^2 + x + 1,$$

which agrees with (4).

A slight extension of the second method will handle second-order recurrent sequences. A generalization of the matrix method will be described later, and the most general solution to our problem, based on the second method, will be given in the last section.

Let W_n obey

$$W_{n+2} = pW_{n+1} - qW_n, \quad p^2 - 4q \neq 0,$$

and let $a \neq b$ satisfy

$$x^2 - px + q = 0.$$

Then

$$a + b = p, \quad ab = q,$$

and there are constants C and D such that

$$W_n = Ca^n + Db^n$$

for all values of n. We consider the sequence

$$c_n = f_n(W), \quad W^k \equiv W_k .$$

It is easily seen that

$$c_n = Cf_n(a) + Df_n(b) ,$$

implying

$$K(x) = \sum_{n=0}^{\infty} c_n x^n = C\,H(x,a) + D\,H(x,b)$$

$$= \frac{C}{1 - ax - x^2} + \frac{D}{1 - bx - x^2}$$

$$= \frac{(C + D)(1 - x^2) - ab(Ca^{-1} + Db^{-1})x}{1 - px + (q - 2)x^2 + px^3 + x^4}$$

$$= \frac{W_0(1 - x^2) - qxW_{-1}}{1 - px + (q - 2)x^2 + px^3 + x^4} \tag{6}$$

Putting

$$p = 1, \quad q = -1, \quad W_0 = 0, \quad W_1 = 1$$

makes $W_n = F_n$, and $K(x)$ reduces to $G(x)$.

3. A PROPERTY OF 2-BY-2 BLOCK DETERMINANTS

If, in the previous section, we had evaluated

$$\det[xI - \overline{R}(Q)] = \det\left(\begin{array}{c|c} xI - Q & -I \\ \hline -I & xI \end{array}\right)$$

by formally expanding the right side as a usual determinant and taking the determinant of the result, we would have obtained the correct answer; that is,

$$\det\left(\begin{array}{c|c} xI - Q & -I \\ \hline -I & xI \end{array}\right) = \det\,(x^2I - xQ - I^2)\,.$$

We shall encounter such types of 2-by-2 block determinants while generalizing the matrix approach to symbolic substitutions, so it is convenient to state the following

Theorem: Let $A = (a_{ij})$ and $B = (b_{ij})$ $(i, j = 1,\cdots,n)$ by any n-by-n matrices. Then

$$D(k,m) = \det\left(\begin{array}{c|c} A & mI \\ \hline kI & B \end{array}\right) = \det\,(AB - kmI)\ , \tag{7}$$

where k and m are any real constants.

<u>Proof.</u> The result is familiar when $k = 0$ [4, Section 5.4]. Then assume $k \neq 0$, and consider

$$D(k,m) = \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & m & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & m & \cdots & 0 \\ \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & & m \\ k & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & k & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & & \cdot \\ 0 & 0 & \cdots & k & b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

We eliminate the bottom row by multiplying the n^{th} column by b_{nj}/k and subtracting from the $(n + j)^{th}$ column for $j = 1,\cdots, n$, and expanding along the bottom row to yield

$$k(-1)^n\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & m - a_{n1}b_{n1}/k & \cdots & - a_{1n}b_{nn}/k \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & - a_{nn}b_{n1}/k & \cdots & m - a_{nn}b_{nn}/k \\ k & 0 & \cdots & 0 & b_{11} & \cdots & b_{1n} \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & k & b_{n-1,1} & \cdots & b_{n-1,n} \end{pmatrix}\ .$$

Repeating this process of elimination on the resulting bottom row for $n - 1$ more times gives

$$D(k,m) = k^n(-1)^{n^2} \det \begin{pmatrix} m - \sum_{j=1}^{n} a_{1j}b_{j1}/k & \cdots & - \sum_{j=1}^{n} a_{1j}b_{jn}/k \\ \vdots & & \vdots \\ - \sum_{j=1}^{n} a_{nj}b_{j1}/k & \cdots & m - \sum_{j=1}^{n} a_{nj}b_{jn}/k \end{pmatrix}$$

Now $(-1)^{n^2} = (-1)^n$, and for an n-by-n matrix M,

$$(-k)^n \det M = \det(-kM) ,$$

so that

$$D(k,m) = \det(AB - kmI) .$$

A slightly more general form of the above Theorem was located as a problem in [4, Section 5.4].

4. A GENERALIZED MATRIX METHOD FOR SYMBOLIC SUBSTITUTIONS

We shall now extend the matrix technique used in Section 2. Given arbitrary matrices A and B of the same square dimension, let the $(r,s)^{\text{th}}$ entry b_n of B^nA be the n^{th} member of the sequence $\{b_n\}$. We find the auxiliary polynomial for the recurrence obeyed by

$$d_n = f_n(b), \quad b^k \equiv b_k .$$

Clearly the $(r,s)^{\text{th}}$ entry of $f_n(B)A$ is d_n. We also have

$$\overline{R}^n(B) \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right) = \left(\begin{array}{c|c} f_{n+1}(B)A & f_n(B)A \\ \hline f_n(B)A & f_{n-1}(B)A \end{array}\right) .$$

It follows that the sequence $\{d_n\}$ obeys a recurrence relation whose auxiliary polynomial is the characteristic polynomial $p(x)$ of $\overline{R}(B)$. Using (7), the latter becomes

$$
p(x) = \det[xI - \overline{R}(B)] = \det\left(\begin{array}{c|c} xI - B & -I \\ \hline -I & xI \end{array}\right) = \det[(x^2 - 1)I - xB] \,. \tag{8}
$$

The following are some particular cases of this result.

(i) Substitution of Fibonacci numbers. For $B = Q$, as defined above, we obtain (5).

(ii) Substitution of second-order recurrent sequences. Let W_n be as defined in Section 2, and let

$$
A = \begin{pmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{pmatrix}, \quad B = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix} .
$$

Then

$$
B^n A = \begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix},
$$

so letting $r = 1$, $s = 2$, we have $b_n = W_n$. In this case

$$
\begin{aligned}
p(x) &= \det\begin{pmatrix} x^2 - 1 - px & qx \\ -x & x^2 - 1 \end{pmatrix} \\
&= x^4 - px^3 + (q - 2)x^2 + px + 1 ,
\end{aligned}
$$

agreeing with (6).

(iii). Substitution of Fibonacci polynomials. There is nothing to restrict $\{b_n\}$ itself from being a sequence of polynomials. To illustrate this, put $A = I$ and $B = R(t)$, so that if we let b_n be the upper right term of B^nA, $b_n = f_n(t)$. Then the sequence

$$
f_n[f(t)] , \quad f^k(t) \equiv f_k(t) ,
$$

obtained by symbolically substituting the Fibonacci polynomials $f_n(t)$ into the Fibonacci polynomials obeys a recurrence relation whose auxiliary polynomial is

$$\det\left[(x^2 - 1)I - xR(t)\right] = x^4 - tx^3 - 3x^2 + tx + 1.$$

(iv) Substitution of Fibonacci numbers with subscripts in an arithmetic progression. Let the sequence $\{r_n\}$ be generated by

$$r_n = F^s f_n(F^k), \quad F^m \equiv F_m .$$

that is, the sequence is formed by replacing x^n by F_{nk+s} in the Fibonacci polynomials. Now $y_n = F_{nk+s}$ obeys

$$y_{n+2} = L_k y_{n+1} - (-1)^k y_n .$$

Applying (ii), with $p = L_k$ and $q = (-1)^k$, we see that the required auxiliary polynomial is

$$x^4 - L_k x^3 + \left[(-1)^k - 2\right] x^2 + L_k x + 1 .$$

(v) Substitution of powers of the integers. Let $e_n(k) = e_n = n^k$ for fixed $k \geq 0$. We find the auxiliary polynomial of the recurrence obeyed by

$$g_n = f_n(e), \quad e^m \equiv e_m .$$

It is easy to show by induction that

$$B_1^n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix},$$

and in general that

$$B_r^n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \binom{r}{r} & \binom{r}{r-1} & & \cdots & \binom{r}{0} \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ n & 1 & \cdots & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \binom{r}{r}n^r & \binom{r}{r-1}n^{r-1} & \cdots & \binom{r}{0} \end{pmatrix}.$$

Then the lower left term of $f_n(B_k)$ is g_n. The desired polynomial is thus

$$p(x) = \det\left[(x^2 - 1)I - xB_k\right] = \det \begin{pmatrix} x^2 - x - 1 & 0 & \cdots & 0 \\ & x^2 - x - 1 & \cdots & 0 \\ & & \ddots & \\ \star & & & \underbrace{x^2 - x - 1} \end{pmatrix}$$

$$= (x^2 - x - 1)^{k+1} ,$$

where the $\star$ indicates irrevelant terms. Notice that when $k = 0$ the auxiliary polynomial is $x^2 - x - 1$, which agrees with $f_n^{(1)} = F_n$.

(vi) Substitution of powers of Fibonacci numbers. Let $v_k = F_k^m$, m a fixed integer. Consider

$$h_n = f_n(v), \quad v^k \equiv v_k .$$

We require a matrix whose n^{th} power has F_n^m as an entry. Such a matrix is provided by Problem H-26 [8]. Let $B_m = (b_{rs})$, where

$$b_{rs} = \binom{r - 1}{m + 1 - s}$$

for $r, s = 1, \cdots, m + 1$. Then putting $r = m + 1$, $s = 1$, we have that the $(r,s)^{th}$ entry of B_m^n is indeed F_n^m. Thus the $(r,s)^{th}$ entry of $f_n(B_m)$ is h_n, and in this case the auxiliary polynomial is

$$p(x) = \det\left[(x^2 - 1)I - xB_m\right] = x^{m+1} \det\left[\left(\frac{x^2 - 1}{x}\right)I - B_m\right].$$

Put $(x^2 - 1)/x = y$. Now $\det(yI - B_m)$ has been evaluated [1;2;5] to be

$$\det(yI - B_m) = \sum_{r=0}^{m+1} (-1)^{\frac{(m-r)(m-r+1)}{2}} \begin{bmatrix} m+1 \\ r \end{bmatrix} y^r ,$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{F_m F_{m-1} \cdots F_{m-r+1}}{F_1 F_2 \cdots F_r} \quad (r > 0); \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1 ,$$

Now

$$y^2 = x^{-r}(x^2 - 1)^r = \sum_{j=0}^{r} (-1)^j \binom{r}{j} x^{r-2j} ,$$

so that

$$(9) \qquad p(x) = \sum_{r=0}^{m+1} \sum_{j=0}^{r} (-1)^{j+(m-r)(m-r+1)/2} \begin{bmatrix} m+1 \\ r \end{bmatrix} \binom{r}{j} x^{m+r-2j+1}$$

$$= \sum_{s=0}^{2m+2} \left[\sum_{r=0}^{m+1} (-1)^{[(m-r)(m-r+1)+s-m-r-1]/2} \begin{bmatrix} m+1 \\ r \end{bmatrix} \binom{r}{\{s-m-r-1\}/2} \right] x^s ,$$

where in the last expression the summand is zero if $(s - m - r - 1)/2$ is not an integer.

This result may be extended to powers of an arbitrary second-order recurrent sequence $\{W_n\}$, described in Section 2, by using the matrix $C_m = (c_{rs})$, where

$$c_{rs} = \binom{r - 1}{m + 1 - s} p^{r+s-m} q^{m+1-r}$$

for $r, s = 1, \cdots, m + 1$. For a discussion of C_m see [5]. Letting $u_n = (a^n - b^n)/(a - b)$, where a and b are as in Section 2, define

$$\begin{bmatrix} m \\ r \end{bmatrix}_u = \frac{u_m u_{m-1} \cdots u_{m-r+1}}{u_1 u_2 \cdots u_r} \quad (r > 0) \quad \begin{bmatrix} m \\ 0 \end{bmatrix}_u = 1 .$$

Then the counterpart to (9) is

$$p(x) = \sum_{r=0}^{m+1} \sum_{j=0}^{r} (-1)^{m+1-r-j} (-q)^{(m-r+1)(m-r+2)/2} \begin{bmatrix} m+1 \\ r \end{bmatrix}_u \binom{r}{j} x^{m+r-2j+1} . \tag{10}$$

In particular, (10) is the auxiliary polynomial for the recurrence relation obeyed by the symbolic substitution of $\{F^m_{nk+s}\}$ for proper choices of the parameters.

The matrix method developed here is more general than previously indicated. In particular, the full power of (7) has not been exploited. For example, let $\{p_n(x)\}$ be any sequence of polynomials (numbers) obeying

$$p_{n+2}(x) = g(x)p_{n+1}(x) + hp_n(x) , \tag{11}$$

where $g(x)$ is any polynomial in x independent of n, and h is a real constant. Let the sequence $\{b_n\}$ be generated by the matrices A and B as before. We shall find the auxiliary polynomial of the recurrence relation obeyed by

$$s_n = p_n(b), \qquad b^k \equiv b_k .$$

Now the $(r, s)^{th}$ entry of $p_n(B)A$ is s_n. Also, if

$$F(B) = \left(\begin{array}{c|c} g(B) & hI \\ \hline I & 0 \end{array}\right), \quad G(B) = \left(\begin{array}{c|c} p_2(B) & p_1(B) \\ \hline p_1(B) & p_0(B) \end{array}\right) ,$$

then

$$F^n(B)G(B)\left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right) = \left(\begin{array}{c|c} p_{n+2}(B)A & p_{n+1}(B)A \\ \hline p_{n+1}(B)A & p_n(B)\,A \end{array}\right),$$

Since s_n is an entry in the right-hand matrix, it follows that the sequence $\{s_n\}$ obeys a recurrence relation whose auxiliary polynomial is the characteristic polynomial of $F(B)$. Using (7), the latter reduces to

$$\begin{aligned} \det[xI - F(B)] &= \det\left(\begin{array}{c|c} xI - g(B) & -hI \\ \hline -I & xI \end{array}\right) \\ &= \det[(x^2 - h)I - xg(B)] . \end{aligned}$$

Putting $g(x) = x$, $h = 1$, $p_1(x) = 1$, and $p_2(x) = x$ specializes this to (8). As another illustration of this result, we note that $T_n(x)$ and $U_n(x)$, the Chebyshev polynomials of the first and second kind, respectively, obey (11) for $g(x) = 2x$, $h = -1$, along with

$$T_0(x) = 1 = U_0(x), \quad T_1(x) = x, \quad \text{and} \quad U_1(x) = 2x .$$

Then the sequences defined by the symbolic substitutions

$$T_n(F), \quad U_n(F), \quad F^k \equiv F_k ,$$

each obey a recurrence relation whose auxiliary polynomial is

$$\det[(x^2 + 1)I - 2xQ] = x^4 - 2x^3 - 2x^2 - 2x + 1 . \tag{12}$$

5. A GENERAL RESULT

Here we extend the second approach in Section 2 to obtain the most general solution to our problem. Let $\{q_n(x)\}$ be any sequence of polynomials obeying the k^{th} order recurrence relation

$$0 = \sum_{j=0}^{k} a_j(x)q_{n-j}(x), \quad a_0(x)a_k(x) \neq 0 ,$$

where the $a_j(x)$ are polynomials independent of n. Put

$$Q(x,t) = \sum_{j=0}^{k} a_j(x)t^j ,$$

so that

$$M(x,t) = \sum_{n=0}^{\infty} q_n(x)t^n = \frac{P(x,t)}{Q(x,t)} ,$$

where $P(x,t)$ is a polynomial in x and t of degree $<k$ in t. Suppose $\{A_n\}$ is a sequence satisfying an m^{th} order recurrence relation with constant coefficients whose auxiliary polynomial has distinct roots $r_1, r_2, \cdots, r_m$. Then there exist constants $B_1, B_2, \cdots, B_m$ such that

$$A_n = \sum_{i=1}^{m} B_i r_i^n .$$

Define $\{D_n\}$ by

$$D_n = q_n(A), \quad A^k \equiv A_k .$$

Then

$$D_n = \sum_{i=1}^{m} B_i q_n(r_i) ,$$

so that

$$\sum_{n=0}^{\infty} D_n t^n = \sum_{i=1}^{m} B_i M(r_i, t) = \sum_{i=1}^{m} \frac{B_i P(r_i, t)}{Q(r_i, t)} = \frac{r(t)}{Q(r_1, t) \cdots Q(r_m, t)} = \frac{r(t)}{s(t)} ,$$

where the degree of $r(t) < mk$, and the degree of $s(t) = mk$. Therefore $\{D_n\}$ obeys a recurrence relation whose auxiliary polynomial is

$$t^{mk} s(1/t) = \prod_{i=1}^{m} \left[\sum_{j=0}^{k} a_j(r_i) t^{k-j} \right] . \tag{13}$$

Continuing with the illustration of the preceding section, for Chebyshev polynomials of both kinds we have

$$k = 2, \quad a_0(x) = a_2(x) = 1, \quad a_1(x) = -2x ,$$

and if $A_n = F_n$ we see

$$m = 2, \quad r_1 = (1 + \sqrt{5})/2, \quad r_2 = (1 - \sqrt{5})/2 .$$

The desired polynomial is then

$$(t^2 - 2r_1 t + 1)(t^2 - 2r_2 t + 1) = t^4 - 2t^3 - 2t^2 - 2t + 1 ,$$

in agreement with (12).

It happens that (13) is valid even if $r_1, \cdots, r_m$ are not distinct. Then this generalization actually yields the matrix method as a special case. To see this, put

$$k = 2, \quad a_0(x) = 1, \quad a_1(x) = -g(x), \quad a_2(x) = -h,$$

and let b_n be the $(r,s)^{th}$ entry of $B^n A$, where A and B are m-by-m matrices. Then $\{b_n\}$ obeys a recurrence relation whose auxiliary polynomial is

$$\det(xI - B) = (x - r_1) \cdots (x - r_m) .$$

From (13), we have that the sequence

$$\{q_n(b)\},\ b^k \equiv b_k \quad ,$$

obeys a recurrence relation whose auxiliary polynomial is

$$p_1(t) = \prod_{i=1}^{m} [t^2 - g(r_i)t - h].$$

On the other hand, by (8) we find that the matrix method gives the required polynomial as

$$p_2(t) = \det[(t^2 - h) - g(b)t] .$$

To show $p_1(t) = p_2(t)$, we note B is similar to

$$C = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ & r_2 & \cdots & 0 \\ \star & & \ddots & \\ & & & r_m \end{pmatrix} ,$$

so that $g(B)$ is similar to $g(C)$. We also have

$$g(C) = \begin{pmatrix} g(r_1) & 0 & \cdots & 0 \\ & g(r_2) & \cdots & 0 \\ \star & & \ddots & g(r_m) \end{pmatrix} ,$$

where the $\star$ indicates irrevelant entries. Since similar matrices have the same characteristic polynomial,

$$p_2(t) = \det[(t^2 - h) - g(C)t] = \prod_{i=1}^{m} [t^2 - h - tg(r_i)] = p_1(t) .$$

However, the matrix method has the advantage that the roots $r_1, \cdots, r_m$ of the characteristic polynomial of B do not have to be known.

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FIBONACCI SEQUENCES WITH IDENTICAL CHARACTERISTIC VALUES

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A Fibonacci sequence is viewed herein as an integer sequence

$$\{f_n\}_{n=-\infty}^{\infty}$$

which satisfies the recursion

$$f_n = f_{n-1} + f_{n-2} \tag{1}$$

for all n.

Following [1], it is convenient to associate two Fibonacci sequences with each other if one can be transformed into the other by a relabeling of indices. Also, it is apparent that $\{f_n\}$ satisfying (1) implies that $\{-f_n\}$ satisfies (1) and it is convenient to associate a sequence with its negative. These remarks lead to

<u>Definition.</u> Two Fibonacci sequences $\{f_n\}$ and $\{g_n\}$ are equivalent if and only if there exists an integer k such that either

(i) $g_n = f_{n+k}$ for all n;

or

(ii) $g_n = -f_{n+k}$ for all n.

In [1], the discussion pertains to Fibonacci sequences such that there is no common divisor $d > 1$ of every term in the sequence (or equivalently, of any two consecutive terms). In this paper, we will be interested in all integer sequences satisfying (1). However, when there is no common divisor (>1) of the sequence, we will call the sequence primitive.

A well-known identity satisfied by Fibonacci sequences is

(2) $$f_{n+1}f_{n-1} - f_n^2 = \pm D$$

where $D \geq 0$ and the sign alternates with n. We call the integer

$$D = \left| f_{n+1}f_{n-1} - f_n^2 \right|$$

the characteristic of the sequence $\{f_n\}$. The reader may verify that if $\{f_n\}$ is equivalent to $\{g_n\}$ then $\{f_n\}$ and $\{g_n\}$ have the same characteristic.

A table is presented in [1] of all $D < 1000$ for which there exists a primitive sequence. Also, all primitive sequences (up to equivalence) having these characteristics are provided. Such a table leads one to ask the following two questions:

(I) For a given integer $D \geq 0$, how many Fibonacci Sequences are there (up to equivalence) having the characteristic D?

(II) For a given $D \geq 0$, how many primitive Fibonacci sequences are there (up to equivalence) having the characteristic D?

This paper is devoted to providing a complete answer to each of these questions.

For this purpose, we let

$$\alpha = \frac{1+\sqrt{5}}{2}$$

and we consider the field extension $R(\alpha)$ obtained by adjoining α to the rationals. The domain of algebraic integers in $R(\alpha)$ then consists of all numbers of the form $A + B\alpha$, where A and B are rational integers. It is well known (see [2]) that one has unique factorization in this domain of integers. The units in this domain are precisely numbers of the form $\pm\alpha^{\pm n}$ and all primes (up to associates) are

(i) $\sqrt{5} = 2\alpha - 1$

(ii) all rational primes of the form $5k \pm 2$

(iii) numbers of the form $A + B\alpha$ and $A + B\bar{\alpha}$, where $\bar{\alpha}$ is the conjugate of α, i.e.,

$$\bar{\alpha} = \frac{1-\sqrt{5}}{2}$$

and $\left|(A + B\alpha)(A + B\bar{\alpha})\right|$ is a rational prime of the form $5k \pm 1$.

We may assign to each Fibonacci sequence an integer ξ in $R(\alpha)$, namely, the sequence $\{f_n\}$ is assigned the integer $\xi = f_0 + f_1\alpha$. It is easily verified that the assignment of integers in this manner provides a one-to-one correspondence between Fibonacci sequences and integers in $R(\alpha)$. Letting $\xi = A + B\alpha$ be an integer in $R(\alpha)$ we denote by $S(\xi)$ the unique sequence assigned to ξ (i.e., the sequence determined by $f_0 = A$, $f_1 = B$).

The assignment $S(\xi)$ preserves addition in the sense that if $S(\xi_1) = \{f_n\}$ and $S(\xi_2) = \{g_n\}$, then $S(\xi_1 + \xi_2) = \{f_n + g_n\}$. It might also be remarked that the correspondence $S(\xi)$ allows one to define a product of two Fibonacci sequences in a natural way. Namely, for two Fibonacci sequences $S(\xi_1)$ and $S(\xi_2)$, the product sequence is defined as $S(\xi_1\xi_2)$. In this way, one has a ring of Fibonacci sequences which is isomorphic to the ring of integers in $R(\alpha)$.

Two integers ξ_1 and ξ_2 in $R(\alpha)$ are called associates if $\xi_1 = \epsilon\xi_2$ for some unit ϵ (which is one of the integers $\pm\alpha^{\pm n}$). It follows that two sequences $S(\xi_1)$ and $S(\xi_2)$ are equivalent if and only if ξ_1 and ξ_2 are associates.

For a given integer $\xi = A + B\alpha$, we define the (absolute) norm $N(\xi)$ in the usual way as $N(\xi) = \left|\xi\bar{\xi}\right|$, where $\bar{\xi} = A + B\bar{\alpha}$. One can easily verify that the characteristic D of a Fibonacci sequence $S(\xi)$ is simply $N(\xi)$.

As a result of the above remarks, we find that questions (I) and (II) reduce to questions about integers in $R(\alpha)$. Namely, (I) and (II) are equivalent to asking:

(Ia) How many integers in $R(\alpha)$ (up to associates) have a given norm D?

(IIa) How many integers in $R(\alpha)$ (up to associates) with no rational integer divisor $d > 1$ have a given norm D?

To resolve these questions we introduce:

$$P^* = \{\text{set of all positive rational integers } n \text{ such that every prime divisor of } n \text{ is of the form } 5k \pm 1\}$$

and by convention 1 belongs to P^*;

$\omega(n)$ = number of distinct prime divisors of n;

$$d_+(n) = \sum_{\substack{d|n,\, d>0 \\ d\equiv\pm1 \pmod 5}} 1 \;; \qquad d_-(n) = \sum_{\substack{d|n,\, d>0 \\ d\equiv\pm2 \pmod 5}} 1 \;;$$

$$r(n) = d_+(n) - d_-(n),$$

where $r(0) = 1$ by convention. (To illustrate, $\omega(60) = 3$, $d_+(60) = 3$, $d_-(60) = 3$, $r(60) = 0$). The answers to (I) and (II) may now be provided in a compact form as follows:

<u>Theorem 1.</u> For $D \geq 0$, there are exactly $r(D)$ Fibonacci sequences (up to equivalence) having characteristic D.

<u>Theorem 2.</u> There exists a primitive sequence having characteristic $D \geq 0$ if and only if $D = n$ or $D = 5n$, where n belongs to $P^\star$. For such a characteristic D, the number of (inequivalent) primitive sequences is exactly $2^{\omega(n)}$.

<u>Proofs</u>: Letting

$$D = 5^a p_1^{b_1} p_2^{b_2} \cdots p_h^{b_h} q_1^{c_1} \cdots q_k^{c_k}$$

be the prime factorization of D, where p_i is a prime of the form $5m \pm 1$ and q_j a prime of the form $5m \pm 2$ it follows that all integers in $R(\alpha)$ having norm D are

$$A + B\alpha = \epsilon(\sqrt{5})^a \prod_{i=1}^{h} (A_i + B_i\alpha)^{s_i}(A_i + B_i\bar{\alpha})^{t_i} \prod_{j=1}^{k} q_j^{c_j/2}, \tag{3}$$

where ϵ is a unit, $s_i + t_i = b_i$, c_j of necessity is even, and $A_i + B_i\alpha$ is a prime in $R(\alpha)$ having norm p_i. Thus, the number of integers (up to associates) having norm D is the number of ways we can vary each s_i with $0 \leq s_i \leq b_i$. The number of such choices for the s_i is the product $\prod_i (1 + b_i)$. The latter expression (combined with the fact that all c_j must be even) is equivalent to Theorem 1.

This equivalence is a counting exercise which can be ascertained in the following way. The factor 5^a of D has no effect upon the value of $r(D)$. Letting

$$\tilde{D} = \prod_{i=1}^{h} p_i^{b_i} \prod_{j=1}^{k} q_j^{c_j} ,$$

one has $r(D) = r(\tilde{D})$. The divisors of $\tilde{D}$ are the terms in the expansion

$$\prod_{i=1}^{h} (1 + p_i + \cdots + p_i^{b_i}) \prod_{j=1}^{k} (1 + q_j + \cdots + q_j^{c_j}) . \tag{4}$$

By replacing each p_i with the value $+1$ and each q_j with the value -1 in (4), the resulting expansion will yield a term of $+1$ for each divisor of the form $5m \pm 1$ and a term of -1 for each divisor of the form $5m \pm 2$. Thus, the expansion of the modified expression is merely $r(\tilde{D})$. If any c_j is odd the factor $(1 + (-1) + \cdots + (-1)^{c_j})$ is zero which yields $r(\tilde{D}) = 0$. If all c_j are even, then the factor corresponding to q_j is $(1 + (-1) + \cdots + (-1)^{c_j}) = 1$ and the resulting expression for $r(\tilde{D})$ becomes $\prod_i (1 + b_i)$ which is the desired result.

Theorem 2 is obtained by realizing that for (3) to have no rational integer divisor (>1), one must have $a = 0$ or 1, $c_j = 0$ for all j, and the only choices for s_i are 0 and b_i. Thus, there are 2^k choices for s_i, which is theorem 2.

As a final note, it should be pointed out that the proofs of Theorems 1 and 2 proceed in a manner analogous to that which one could take in determining the number of representations of an integer N as the sum of two squares (see Theorem 278 of [2]). In this latter problem one utilizes the ring of gaussian integers whereas in the problems considered above we have relied upon the ring of integers in $R(\alpha)$. It would appear that the above results should extend to other recursions of the form $f_n = af_{n-1} + bf_{n-2}$ provided one has unique factorization in the underlying ring of integers.

For a related paper, see Thoro [3].

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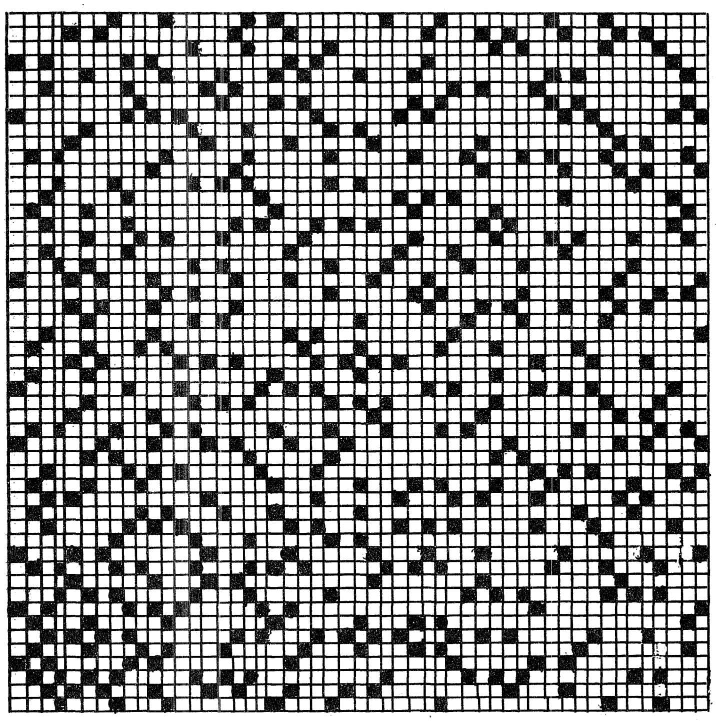
FIRST QUADRANT GRAPH OF GAUSSIAN PRIMES



THE TWIN PRIME PROBLEM
AND GOLDBACH'S CONJECTURE IN THE GAUSSIAN INTEGERS

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1. PRELIMINARIES

The set of Gaussian Integers denoted by G is the set $\{a + bi\}$, a and b real integers and i the "imaginiary unit." It is well known that G with the usual two operations is an integral domain and that the division algorithm holds, if for any α and $\beta \neq 0$ in G, there are γ and δ in G such that $\alpha = \beta\gamma + \delta$, where $|\delta| \leq |\beta|$. Since the division algorithm holds on G, the domain is a unique factorization domain.

A Gaussian prime is a Gaussian integer, ρ, such that:

i) $|\rho| \geq 1$ and

ii) if α divides ρ then $|\alpha| = 1$ or $\alpha = \epsilon\rho$ where ϵ in G and $|\epsilon| = 1$. Here divides means that if α divides β. then there is a Gaussian Integer γ such that $\alpha\gamma = \beta$.

The Gaussian Primes can be separated into the following three classes:

i) if p is a positive real prime of the form $4k + 3$, the $\pm p$ and $\pm ip$ are Gaussian primes.

ii) if p is a positive real prime of the form $4k + 1$, the p can be expressed uniquely as $p = a^2 + b^2$ and the expression generates the 8 Gaussian Primes $\pm a \pm bi$ and $\pm b \pm ai$.

iii) $\pm 1 \pm i$ are Gaussian Primes.

A Gaussian integer, β, is said to be even if $1 + i$ divides β. An easy method of recognizing even Gaussian Integers is the following:

A Gaussian Integer, $\beta = a + bi$, is even if, and only if, 2 divides $a + b$ or in other words, if a and b have the same parity.

Consider the figure which plots the Gaussian Primes in the square with vertices at $\pm 50 \pm 50i$.

2. TWIN PRIMES

A meaningful definition is sought for twin primes in the Gaussian Integers. We have a preference for the following, Definition: Two Gaussian Primes ρ_1 and ρ_2 are called Gaussian twin primes if $\rho_1 - \rho_2 = (1 + i)\epsilon$ where $|\epsilon| = 1$.

Our reason for preferring this definition is that $\pm 1 \pm i$ are the only even primes, and in the real case primes are twins if their differences are ± 2.

Notice in the figure the relative scarcity of primes that are not twins, the smallest odd ones being $17 \pm 12i$ and their associates. It is perhaps coin— cidence but $|17 + 12i| = 20.8+$, which is fairly close to 23, which is the small— est odd real primes, which is not a twin. Notice that 23 and $24+i$ are twin Gaussian Primes and that 47 is not a twin in either system. This serves to point out that there is little, if any, connection between numbers being real twin primes and being Gaussian Twin Primes.

There are two possibilities of definitions for triplets of primes in the Gaussian Integers. The most natural seems to us to be

Definition 2. Three Gaussian Primes, ρ_1, ρ_2, ρ_3 are called Gaussian triplet primes if $\rho_1 - \rho_2 = \rho_2 - \rho_3 = (1 + i)\epsilon$ where $|\epsilon| = 1$.

An example of these triplets would be $20 + 3i$, $21 + 4i$, and $22 + 5i$. The alternate definition would be for the less restrictive condition on the 's: $|\rho_1 - \rho_2| = |\rho_2 - \rho_3| = |1 + i|$. Examples of this less restrictive condition for the triplets would be

(2A) $10 + i$, 11, and $10 - i$

(2B) $19 + 10i$, $20 + 11i$, and $21 + 10i$.

The only real primes that could be considered triplets would be 3, 5, and 7. But it can be noticed from the figure that there are many Gaussian triplet primes.

There are also several possibilities for definitions for Gaussian quadruplet primes. The one we prefer is the more restrictive.

Definition 3: Four Gaussian primes, ρ_1, ρ_2, ρ_3, and ρ_4 are Gaussian quadruplet primes if $\rho_1 - \rho_2 = \rho_2 - \rho_3 = \rho_3 - \rho_4 = (1 + i)\epsilon$ where $|\epsilon| = 1$.

Two examples of these are the primes $31 + 26i$, $32 + 27i$, $33 + 28i$, $34 + 29i$; and $16 + 19i$, $17 + 18i$, $18 + 17i$, and $19 + 16i$.

The less restrictive definition would have

$$|\rho_1 - \rho_2| = |\rho_2 - \rho_3| = |\rho_3 - \rho_4| = |1 + i| .$$

This would not only allow the first definition, but would allow forms like

(3A) $25 + 12i$, $26 + 11i$, $27 + 10i$, and $25 + 9i$, or

(3B) $49 + 34i$, $48 + 35i$, $49 + 36i$, and $48 + 35i$ or

(3C) $24 + 5i$, $25 + 4i$, $26 + 5i$, and $25 + 6i$.

A further loosening might be imposed on the restrictions to allow forms like $43 + 10i$, $44 + 9i$, $45 + 8i$, and $43 + 8i$ by making the condition in the definition that for some j $|\rho_k - \rho_j| = |1 + i|$, for $k \neq j$.

The most restrictive definition for quintuplets would be

Definition 4: The Gaussian Primes ρ_1, ρ_2, ρ_3, ρ_4, and ρ_5 are Gaussian quintuplet primes if $\rho_1 - \rho_2 = \rho_2 - \rho_3 = \rho_3 - \rho_4 = \rho_4 - \rho_5 = (1 + i)\epsilon$ where $|\epsilon| = 1$.

Several less restrictive definitions could be posed that would allow a variety of forms such as the zigzag: $13 + 2i$, $14 + i$, $15 + 2i$, $16 + i$, and $17 + 2i$. We do not wish to list examples of these forms.

We do wish to notice the following:

Theorem 1: There are only finitely many Gaussian quintuplet primes and they are $\pm5\pm2i$, $\pm4\pm i$, ±3, $\pm2\pm i$, $\pm1\pm2i$, $\pm3i$, $\pm1\pm4i$, $\pm2\pm5i$.

Proof: A special division algorithm for $2 + i$ asserts that for any Gaussian Integer γ, there are Gaussian Integers α and β such that $\gamma = \alpha(2 + i) + \beta$ with $|\beta| \leq 1$, hence $\beta = 0$ or ±1 or $\pm i$. (See representation C of [1] for details.)

Now consider

$\rho_1 = \alpha(2 + i) + \beta$ with $|\beta| \leq 1$, and suppose that the ϵ in the theorem is -1, then

i) if $\beta = 0$ then $(2 + i)\alpha = \rho_1$

ii) if $\beta = 1$ then $\rho_2 = (2 + i)(\alpha + 1)$

iii) if $\beta = i$ then $\rho_4 = (2 + i)(\alpha + 2 + i)$

iv) if $\beta = -1$ then $\rho_5 = (2 + i)(\alpha + 2 + i)$

v) if $\beta = -i$ then $\rho_3 = (2 + i)(\alpha + 1)$

So in any case, $2 + i$ is a factor of one of the ρ_j's. Hence at least one of the ρ_j's is composite unless the ρ_j is $(2 + i)\delta$, where $|\delta| = 1$. This only happens when the ρ_j's are in the set specified in the theorems. Similar arguments can be given for $\epsilon = 1$ or $\pm i$.

There are several other definitions that could arise, but we choose to start guessing about the definitions we now have:

Conjecture A: There are infinitely many Gaussian twin primes.

Conjecture B: There are infinitely many Gaussian triplet primes.

Conjecture C: There are infinitely many Gaussian quadruplet primes.

It is clear that if conjecture C is true then the others would be true, and if conjecture A is false, then the others would be false. It seems to us that all three should be either true or false together, but this is only our opinion.

One theorem that can be stated positively about the primes that form a square of twin primes like those of example 3C is the following:

Theorem 2: If $a \pm 1 + bi$, $a + (b \pm 1)i$ are primes with $|a| + |b| > 5$, then a and b are both multiples of 5 and neither is zero.

Proof: Since these four numbers are primes, none is divisible by $2 + i$ nor $2 - i$. The strong division algorithm for $2 + i$ gives $a + bi = (2 + i)\alpha + \delta$ where $|\delta| \leq 1$. But if $\delta = 1$, then $(a - 1) + bi = (2 + i)\alpha$; if $\delta = i$, then $a + (b - 1)i = (2 + i)\alpha$; if $\delta = -1$, then $a + 1 + bi = (2 + i)\alpha$; and if $\delta = -i$, then $a + (b + 1)i = (2 + i)\alpha$ so $\delta = 0$. A similar argument implies that for $a + bi = (2 + 1)\beta + \eta$ then $\eta = 0$. So not only $2 + i$ but also $2 - i$ divides $a + bi$; hence $(2 + i)(2 - i) = 5$ divides $a + bi$, hence 5 divides each component a and b.

Notice that if $b = 0$ then $a + 1$ and $a - 1$ are both primes which is impossible because if $a + 1$ is even, 2 divides it, and if $a + 1$ and $a - 1$ are both odd, one is of the form $4k + 1$, which is not a Gaussian prime. A similar argument settles the case $a = 0$.

Corollary. If ρ_1, ρ_2, ρ_3, and ρ_4 are a set of Gaussian primes as described in theorem 2, then there does not exist a Gaussian prime $\rho \neq \rho_j$ such that $\rho = \rho_j + (1 + i)\epsilon$ for $|\epsilon| = 1$.

Proof. Notice that the eight odd numbers that surround this set have the property that they differ from $a + bi$ by $\pm 2 \pm i$ or $\pm 1 \pm 2i$ hence are divisible by either $2 + i$ or $2 - i$ since $a + bi$ is divisible by both.

This means that forms like 3C that are not near the origin can not have an additional prime attached a checker move away.

3. GOLDBACH'S CONJECTURE

There are several possibilities for generalizing Goldbach's conjecture. One possibility would be

Conjecture D: If α is an even Gaussian Integer, then there are Gaussian Primes ρ_1 and ρ_2 such that $\alpha = \rho_1 + \rho_2$.

This seems to us to be a poor generalization of Goldbach's conjecture. It is more the generalization of the statement, "Every even integer is either the sum or difference of two positive primes."

Since positive is meaningless in the Gaussian Integers, we would like to somehow purge the possibility of allowing differences to creep in. These two possibilities occur: 1) Insist the ρ_1 and ρ_2 lie in that same half plane, or 2) insist that $|\rho_1|$ and $|\rho_2|$ be $\leq |\alpha|$. We however prefer this one.

Conjecture E: If α is an even Gaussian Integer with $|\alpha| > \sqrt{2}$, then there are Gaussian Primes, ρ_1 and ρ_2, such that $\alpha = \rho_1 + \rho_2$ and the angles $\rho_1 0 \alpha$ and $\alpha 0 \rho_2$ are $\leq 45°$.

It is easy to see that conjecture E implies conjecture D and both of the two alternatives mentioned.

Conjecture E has been verified for all even Gaussian Integers in the figure.

Certain conditions stronger than conjecture E might be asserted by reducing 45°. The α may have to be increased in absolute value to avoid certain exceptional cases. For example

Conjecture F: If α is an even Gaussian Integer with $|\alpha| > \sqrt{10}$, then there are primes ρ_1 and ρ_2 with angles $\rho_1 0 \alpha$ and $\alpha 0 \rho_2 \leq 30°$ and $\alpha = \rho_1 + \rho_2$.

This has also been verified for the even integers in the figure. Note that $1 + 3i$, $3 + i$, and 2 and their associates require 45°.

Reducing the angle to 0° doesn't work since 4, 8, 12 $\cdots$ have no representatives as the sum of two Gaussian Primes. There might be some sacred angle θ, which is the dividing point for the truth or falsity of the appropriate conjecture or perhaps if $\theta > 0$ then for all $|\alpha| > N_\theta$ the appropriate conjecture may be true. There might be a universal shaped region that depends on $|\alpha|$ such that the primes ρ_1 and ρ_2 would fall in that region, with this region in some ways minimal.

(Continued on p. 92.)

ON THE LINEAR DIFFERENCE EQUATION WHOSE SOLUTIONS ARE THE PRODUCTS OF SOLUTIONS OF TWO GIVEN LINEAR DIFFERENCE EQUATIONS

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It was shown by Appell [1] that if u_1 and u_2 denote two linearly independent solutions of

$$\left\{D^2 + p(t)D + q(t)\right\}y = 0 ,$$

then u_1^2, u_1u_2, u_2^2 denote three linearly independent solutions of the third-order linear differential equation

$$\left\{D^3 + 3pD^2 + (2p^2 + p' + 4q)D + (4pq + 2q')\right\}y = 0 .$$

Watson [2] shows that if

$$\left\{D^2 + I\right\}v = 0, \quad \left\{D^2 + J\right\}w = 0 ,$$

then $y = vw$ satisfies the fourth-order differential equation

$$D\left\{\frac{y'' + 2(I + J)y' + (I' + J')y}{I - J}\right\} = -(I - J)y, \quad (I \neq J) .$$

Bellman [3] gives a matrix method for obtaining Appell's result and notes that the method can be used to find the linear differential equation of order mn whose solutions are the products of the solutions of a linear differential equation of order m and one of order n.

We now obtain analogous results for linear difference equations.

Let $\left\{A_n\right\}$ and $\left\{B_n\right\}$ denote sequences defined by the second-order linear difference equations

$$A_{n+1} = P_nA_n + Q_nA_{n-1} , \tag{1}$$

$$B_{n+1} = R_nB_n + S_nB_{n-1} , \tag{2}$$

where A_0, A_1, B_0, B_1 are arbitrary and P_n, Q_n, R_n, S_n are given.

If u_1 and u_2, v_1 and v_2 denote pairs of linearly independent solutions of (1) and (2), respectively, then we first obtain the third-order linear difference equation whose solution is given by

$$k_1u_1^2 + k_2u_1u_2 + k_3u_2^2$$

where the k_i's are constants. Squaring (1) and letting $C_n = A_n^2$, we obtain

$$C_{n+1} = P_n^2C_n + Q_n^2C_{n-1} + 2P_nQ_nA_nA_{n-1} \tag{3}$$

or

$$\begin{aligned} C_{n+1} - P_n^2C_n - Q_n^2C_{n-1} &= 2P_nQ_nA_{n-1}(P_{n-1}A_{n-1} + Q_{n-1}A_{n-2}) \\ &= 2P_nP_{n-1}Q_nC_{n-1} + 2P_nQ_nQ_{n-1}A_{n-1}A_{n-2} \ . \end{aligned} \tag{4}$$

By decreasing the index n by 1 in (3), we can eliminate $A_{n-1}A_{n-2}$ to obtain

$$\begin{aligned} P_{n-1}C_{n+1} = P_n(P_nP_{n-1} + Q_n)C_n &+ P_{n-1}Q_n(P_nP_{n-1} + Q_n)C_{n-1} \\ &- P_nQ_nQ_{n-1}^2C_{n-2} \end{aligned} \tag{5}$$

We now obtain the fourth-order equation whose solution is given by $k_1u_1^3 + k_1u_1^3 + k_2u_1^2u_2 + k_3u_1u_2^2 + k_4u_2^3$. Cubing (1) and letting $D_n = A_n^3$, we obtain

$$\begin{aligned} D_{n+1} - P_n^3D_n - Q_n^3D_{n-1} &= 3P_n^2Q_nA_n^2A_{n-1} + 3P_nQ_n^2A_nA_{n-1}^2 \\ &= 3P_n^2Q_nA_n^2A_{n-1} + 3P_nQ_n^2A_{n-1}^2(P_{n-1}A_{n-1} + Q_{n-1}A_{n-2}) \end{aligned} \tag{6}$$

or

$$\begin{aligned} D_{n+1} - P_n^3D_n - Q_n^2(3P_nP_{n-1} + Q_n)D_{n-1} &= 3P_n^2Q_nA_n^2A_{n-1} \\ + 3P_nQ_n^2Q_{n-1}A_{n-1}^2A_{n-2} &= 3P_n^2Q_nA_{n-1}(P_{n-1}A_{n-1} + Q_{n-1}A_{n-2})^2 \\ + 3P_nQ_n^2Q_{n-1}A_{n-1}^2A_{n-2} & \end{aligned} \tag{7}$$

or

$$(8)\qquad D_{n+1} - P_n^3 D_n - Q_n(3P_n P_{n-1} Q_n + 3P_n^2 P_{n-1}^2 + Q_n^2) D_{n-1}$$
$$= 3P_n Q_n Q_{n-1}(2P_n P_{n-1} + Q_n) A_{n-1}^2 A_{n-2} + 3P_n^2 Q_n Q_{n-1}^2 A_{n-2}^2 A_{n-1} .$$

By reducing the index of n by 1 in (6), we can then solve (6) and (8) for $A_{n-1}^2 A_{n-2}$. Then by substituting this expression in (7), we can obtain the desired difference equation.

To find the fourth-order equation satisfied by

$$k_1 u_1 v_1 + k_2 u_1 v_2 + k_3 u_2 v_1 + k_4 u_2 v_2 ,$$

we multiply (1) by (2) and let $E_n = A_n B_n$, to give

$$(9)\qquad E_{n+1} - P_n R_n E_n - Q_n S_n E_{n-1} = P_n S_n A_n B_{n-1} + R_n Q_n B_n A_{n-1}$$
$$= P_n S_n B_{n-1}(P_{n-1} A_{n-1} + Q_{n-1} A_{n-2})$$
$$+ R_n Q_n A_{n-1}(R_{n-1} B_{n-1}' + S_{n-1} B_{n-3})$$

or

$$(10)\qquad E_{n+1} - P_n R_n E_n - (P_n P_{n-1} S_n + R_n R_{n-1} Q_n + Q_n S_n) E_{n-1} = P_n S_n Q_{n-1} B_{n-1} A_{n-2}$$
$$+ R_n Q_n S_{n-1} A_{n-1} B_{n-2} = P_n S_n Q_{n-1} A_{n-2}(R_{n-2} B_{n-2} + S_{n-2} B_{n-3})$$
$$+ R_n Q_n S_{n-1} B_{n-2}(P_{n-2} A_{n-2} + Q_{n-2} A_{n-3})$$

or

$$(11)\qquad E_{n+1} - P_n R_n E_n - (P_n P_{n-1} S_n + R_n R_{n-1} Q_n + Q_n S_n) E_{n-1}$$
$$- (P_n S_n Q_{n-1} R_{n-2} + R_n Q_n S_{n-1} P_{n-2}) E_{n-2}$$
$$= P_n Q_{n-1} S_n S_{n-2} A_{n-2} B_{n-3} + R_n S_{n-1} Q_n Q_{n-2} B_{n-2} A_{n-3} .$$

By now reducing the index n by 2 in (9) and by 1 in (10), we can then eliminate $A_{n-2}B_{n-3}$ and $B_{n-2}A_{n-3}$ from (9), (10), and (11), to obtain the desired difference equation.

If here P_n, Q_n, R_n, S_n are independent of n, the equations simplify and the elimination is rather simple. This special case gives a solution to part (i) of proposed problem H-127 by M. N. S. Swamy (Fibonacci Quarterly, Feb., 1968, p. 51), i.e., "The Fibonacci polynomials are defined by

$$f_{n+1}(x) = xf_n(x), \qquad n \geq 2 ,$$
$$f_1(x) = 1 \quad \text{and} \quad f_2(x) = x .$$

If $z_r = f_r(x)f_r(y)$, then show that (i) z_r satisfies the recurrence relation

$$z_{n+4} - xyz_{n+3} - (x^2 + y^2 + 2)z_{n+2} - xyz_{n+1} + z_n = 0. \text{"}$$

We now extend Bellman's matrix method, with little change, to difference equations.

First we give an analogous lemma for difference equations.

Lemma. Let Y and Z denote, respectively, the solutions of the matrix difference equations

$$EY = A(n)Y , \quad Y(0) = I ,$$
$$EZ = ZB(n) , \quad Z(0) = I ,$$

then the solution of

$$EX = A(n)XB(n) , \quad X(0) = C ,$$

is given by $X = YCZ$. (Here $EY(n) = Y(n + 1)$). An immediate proof follows by substitution.

We now apply this result to finding the third-order linear difference equations whose general solution is $c_1u_1^2 + 2c_2u_1u_2 + c_3u_2^2$ where u_1 and u_2 are linearly independent solutions of

$$\left\{E^2 + p(n)E + q(n)\right\}u = 0 . \tag{12}$$

Without loss of generality, let u_1 and u_2 be determined boundary conditions

$$u_1]_{n=0} = 1, \qquad EU_1]_{n=0} = 0,$$
$$u_2]_{n=0} = 0, \qquad EU_2]_{n=0} = 1.$$

Setting $Eu = v$, (12) is equivalent to

$$\begin{aligned} Eu &= v, \\ Ev &= -pv - qu. \end{aligned}$$

If we now let

$$A(n) = \begin{Vmatrix} 0 & 1 \\ -q(n) & -p(n) \end{Vmatrix},$$

The matrix solution of

$$Eu = A(n)U, \quad U(0) = I,$$

is given by

$$U = \begin{Vmatrix} u_1(n) & u_2(n) \\ Eu_1(n) & Eu_2(n) \end{Vmatrix}$$

and the solution of

$$EV = VA(n)^T, \quad V(0) = I,$$

by $V = U^T$, the transpose of U. From our lemma, the solution of

$$EX = AXA^T, \quad X(0) = C, \tag{13}$$

is given by $X = UCU^T$. Taking C to be the symmetric matrix

$$C = \begin{Vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{Vmatrix},$$

we see that X is given by

$$X = \begin{Vmatrix} x_1 & x_2 \\ x_2 & x_3 \end{Vmatrix}$$

where

$$x_1 = c_1u_1^2 + 2c_2u_1u_2 + c_3u_2^2 ,$$

$$x_2 = c_1u_1Eu_1 + c_2(u_1Eu_2 + u_2Eu_1) + c_3u_2Eu_2 ,$$

$$x_3 = c_1Eu_1^2 + 2c_2(Eu_1)(Eu_2) + c_3Eu_2^2 .$$

Equation (13) can be written as

$$\begin{Vmatrix} Ex_1 & Ex_2 \\ Ex_2 & Ex_3 \end{Vmatrix} = \begin{Vmatrix} 0 & 1 \\ -q & -p \end{Vmatrix} \cdot \begin{Vmatrix} x_1 & x_2 \\ x_2 & x_3 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & -q \\ 1 & -p \end{Vmatrix}$$

and which is also equivalent to the system

$$Ex_1 = x_3 ,$$

$$Ex_2 = qx_2 - px_3 ,$$

$$Ex_3 = q^2x_1 + 2pqx_2 + p^2x_3 .$$

Eliminating x_2 and x_3, we obtain the third-order linear difference equation corresponding to (5). Similarly, eliminating x_1 and x_2, we obtain the equation whose general solution is $c_1Eu_1^2 + 2c_2(Eu_1)(Eu_2) + c_3Eu_2^2$; eliminating x_1 and x_3, we obtain the equation whose general solution is

$$c_1u_1Eu_1 + c_2(u_1Eu_2 + u_2Eu_1) + c_3u_3Eu_3 .$$

In stating our lemma, we ignored any discussion of the dimensionality of Y and Z. It is clear that the result is valid if A(n) and Y are r x r matrices, B(n) and z s x s matrices, and C and X r x s matrices.

Using the same technique as before, but with much more computation, we can obtain the linear difference equation of order rs whose solutions are the products of order r and one of order s.

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(continued from p. 85.)

4. REMARKS

Generalizing these famous conjectures leads to a multitude of conjectures in the Gaussian Integers. Some such as the infinitude of twin primes appears easier to settle and some such as the quadruples of primes seem less attainable than the real case does.

See p. 80 for a First Quadrant Graph of Gaussian Primes.

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LAH NUMBERS FOR FIBONACCI AND LUCAS POLYNOMIALS

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I. INTRODUCTION

In [1] the Fibonacci and Lucas Polynomials are defined as follows:

$$(1) \qquad f_0(x) = 0, \quad f_1(x) = 1, \quad f_{n+2}(x) = xf_{n+1}(x) + f_n(x), \quad n \geq 0,$$

and,

$$(2) \qquad \mathrm{Luc}_0(x) = 2, \quad \mathrm{Luc}_n(x) = f_{n+1}(x) + f_{n-1}(x), \quad n > 0.$$

It is easily seen that

$$(3) \qquad f_n(x) = \sum_{m=0}^{[n/2]} \mathrm{Fi}_n^{n-2m} x^{n-2m-1} = \sum_{s=0}^{n} \mathrm{Fi}_n^s x^{s-1},$$

where n and s have same parity $(n - s - 2k)$, $[n/2]$ is the largest integer contained in $n/2$, i.e.,

$$(4) \qquad [n/2] = \begin{cases} (n/2) & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

and,

$$(5) \qquad \mathrm{Fi}_n^{n-2m} = \binom{n-m-1}{m}, \quad \mathrm{Fi}_n^s = \binom{(n+s-2)/2}{(n-s)/2},$$
$$\mathrm{Fi}_n^s = 0, \quad \text{for } s < 1, \ n < 1, \ n < s, \ n - s = 2k + 1.$$

It follows from (2) that

(6) $$\mathrm{Luc}_n(x) = \sum_{m=0}^{[(n+1)/2]} \mathrm{Lu}_n^{n-2m}\, x^{n-2m} = \sum_{s=0}^{n} \mathrm{Lu}_n^s\, x^s ,$$

where n and s have same parity $(n - s = 2k)$,

and,

(7) $$\mathrm{Lu}_n^{n-2m} = \binom{n-m}{m} + \binom{n-m-1}{m-1}, \quad \mathrm{Lu}_n^s = n\left(\frac{n+s}{2} - 1\right)! \Big/ \left(\frac{n-s}{2}\right)! s!,$$

$\mathrm{Lu}_n^s = 0$, for $n < 0$, $s < 0$, $n < s$, $n - s = 2k + 1$, and $\mathrm{Lu}_0^0 = 2$.

2. FIBONACCI AND LUCAS COEFFICIENTS OF THE SECOND KIND

The numbers Fi_n^s and Lu_n^s will be called Fibonacci and Lucas coefficients of the first kind. According to [2], [3], and [4] we call the numbers fi_n^s and lu_n^s, defined hereafter, Fibonacci and Lucas coefficients of the second kind:

(8) $$x^n = \sum_{m=0}^{[(n+1)/2]} \mathrm{fi}_{n+1}^{n+1-2m}\, f_{n+1-2m}(x) = \sum_{s=0}^{n+1} \mathrm{fi}_{n+1}^s\, f_s(x) ,$$

where $n + 1 - s = 2k$, $\mathrm{fi}_n^s = 0$, for $n - s = 2k + 1$, $n < 1$, $s < 1$, $n < s$,

(9) $$x^n = \sum_{m=0}^{[n/2]} \mathrm{lu}_n^{n-2m}\, \mathrm{Luc}_{n-2m}(x) = \sum_{s=0}^{n} \mathrm{lu}_n^s\, \mathrm{Luc}_s(x) ,$$

where $n - s = 2k$, $\mathrm{lu}_n^s = 0$, for $n - s = 2k + 1$, $n < 0$, $s < 0$, $n < s$.

According to the general theory seen in [2], [3], and [4], the coefficients Fi_n^s, fi_n^s, on the one hand, and the coefficients Lu_n^s, lu_n^s, on the other hand, are quasi-orthogonal, i. e.,

$$\sum_{k=0}^{(n-m)/2} Fi_n^{n-2k}\, fi_{n-2k}^m = \delta_n^m \quad , \tag{10}$$

$$\sum_{k=0}^{(n-m)/2} Lu_n^{n-2k}\, lu_{n-2k}^m = \delta_n^m \quad , \tag{11}$$

where δ_n^m is the Kronecker-delta.

3. NUMERICAL VALUES AND RECURRENCE RELATIONS

Using (1) and (2) we obtain the following table of values for Fi_n^m and Lu_n^m, limited here to $m, n < 11$:

n \ m=	1	2	3	4	5	6	7	8	9	10
1	1									
2	0	1								
3	1	0	1							
4	0	2	0	1						
5	1	0	3	0	1					
6	0	3	0	4	0	1				
7	1	0	6	0	5	0	1			
8	0	4	0	10	0	6	0	1		
9	1	0	10	0	15	0	7	0	1	
10	0	5	0	20	0	21	0	8	0	1

Fi_n^m

It will be observed that the sum of coefficients in one row is equal to the Fibonacci number corresponding to its n, i.e., $f_n(1) = F_n$.

n \ m=	0	1	2	3	4	5	6	7	8	9	10
0	2										
1	0	1									
2	2	0	1								
3	0	3	0	1							
4	2	0	4	0	1						
5	0	5	0	5	0	1					
6	2	0	9	0	6	0	1				
7	0	7	0	14	0	7	0	1			
8	2	0	16	0	20	0	8	0	1		
9	0	9	0	30	0	27	0	9	0	1	
10	2	0	25	0	50	0	35	0	10	0	1

Lu_n^m

It is easily seen that

(12) $$Fi_n^m = Fi_{n-2}^m + Fi_{n-1}^{m-1} ,$$

which is satisfied by (5), as can be easily checked.

(13) $$Lu_n^m = Lu_{n-2}^m + Lu_{n-1}^{m-1} ,$$

for $n > 1$, $m > 1$, but for $m = n = 1$ we have

$$Lu_1^1 = \tfrac{1}{2} Lu_0^0 .$$

It is necessary to introduce the function $N(n)$ which is

(13a) $$N(n) = \begin{cases} 1 & \text{if } n \neq 1 \\ 1/2 & \text{if } n = 1 \end{cases}$$

which allows us to write

(13b) $$Lu_n^m = Lu_{n-2}^m + N(n) Lu_{n-1}^{m-1}$$

for any integer m and n.

According to (9) and (12) of [4] it follows that taking $p = 1$, $k = 2$, the fi_n^m-coefficients satisfy the relation

$$\mathrm{fi}_n^m = \mathrm{fi}_{n-1}^{m-1} - \mathrm{f}_{n-1}^{m+1} \quad , \tag{14}$$

and the lu_n^m-coefficients the relation

$$\mathrm{lu}_n^m = \frac{1}{N(m)}\,\mathrm{lu}_{n-1}^{m-1} - \frac{1}{N(m+2)}\,\mathrm{lu}_{n-1}^{m+1} \quad , \tag{15}$$

but since $m \geq 2$, $N(m + 2) = 1$, thus

$$\mathrm{lu}_n^m = \mathrm{lu}_{n-1}^{m-1} \big/ N(m) - \mathrm{lu}_{n-1}^{m+1} \quad . \tag{15a}$$

The numerical values of the Fibonacci and Lucas coefficients of the second kind can be obtained either from (10) and (11) or from (14) and (15a). Thus, for $n, m \leq 10$.

m \ n=	1	2	3	4	5	6	7	8	9	10
1	1									
2	0	1								
3	-1	0	1							
4	0	-2	0	1						
5	2	0	-3	0	1					
6	0	5	0	-4	0	1				
7	-5	0	9	0	-5	0	1			
8	0	-14	0	14	0	-6	0	1		
9	14	0	-28	0	20	0	-7	0	1	
10	0	42	0	-48	0	27	0	-8	0	1

fi_n^m

m \ n=	0	1	2	3	4	5	6	7	8	9	10
0	1/2										
1	0	1									
2	-1	0	1								
3	0	-3	0	1							
4	3	0	-4	0	1						
5	0	10	0	-5	0	1					
6	-10	0	15	0	-6	0	1				
7	0	-35	0	21	0	-7	0	1			
8	35	0	-56	0	28	0	-8	0	1		
9	0	126	0	-84	0	36	0	-9	0	1	
10	-126	0	210	0	-120	0	45	0	-10	0	1

lu_n^m.

It is easily seen that for n and m having same parity, i.e., $n - m = 2k$, the Fibonacci and Lucas coefficients of the second kind are

$$\mathrm{fi}_n^m = (-1)^{(n-m)/2} \binom{n}{(n-m)/2} m/n , \tag{16}$$

and

$$\mathrm{lu}_n^m = (-1)^m \binom{n}{m} N(m+1) , \tag{17}$$

where $N(m + 1)$, according to (13a) equals 1 if $m \neq 0$, and 1/2 if $m = 0$.

4. LAH NUMBERS

According to [5] and [6] the Lucas-Fibonacci and the Fibonacci-Lucas Lah numbers are defined by the two relations

$$\mathrm{Luc}_n(x) = \sum_{m=0}^{n+1} \mu_n^m f_m(x) , \tag{18}$$

and

$$f_n(x) = \sum_{k=0}^{[(n-1)/2]} \lambda_n^{n-1-2k} \mathrm{Luc}_{n-1-2k}(x) \tag{19}$$

$$= \sum_{m=0}^{n-1} \lambda_n^m \mathrm{Luc}_m(x) ,$$

where n and m are of the same parity, i.e., $n - m = 2p$.

According to the definition of Lucas polynomials given by (2) it follows that

$$\mu_n^m = \begin{cases} 0 & \text{if } m \neq n \pm 1 \\ 1 & \text{if } m = n \pm 1 \end{cases} \tag{20}$$

and

$$\lambda_n^m = (-1)^{(n-m-1)/2} N(m+1) , \tag{21}$$

where n and m are of opposite parity, i.e., $n - m = 2k + 1$, and $N(m)$ is defined by (13a).

According to (8) and (9) of [5], and (3a) and (3b) of [6] we obtain

$$\lambda_n^m = \sum_{s=m}^{n} \mathrm{Fi}_n^s \, \mathrm{lu}_{s-1}^m = (-1)^{(n-m-1)/2} N(m+1) , \tag{22}$$

where n and m have different parity, i.e., $n - m = 2k + 1$, and

$$\sum_{s=m-1}^{n} \mathrm{Lu}_n^s \, \mathrm{fi}_{s+1}^m = \mu_n^m = \begin{cases} 0 & \text{if } m \neq n \pm 1 \\ 1 & \text{if } m = n \pm 1 \end{cases} \tag{23}$$

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LAH NUMBERS FOR R-POLYNOMIALS

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1. INTRODUCTION

According to [1], [2], and [3], given two sequences of polynomials, $P_1(x,n)$ and $P_2(x,n)$, $n = 0, 1, 2, \cdots$.

$$P_k(x,n) = \sum_{m=0}^{n} C_{k,n}^{m} x^m, \qquad k = 1, 2, \tag{1}$$

$$C_{k,n}^{m} = 0, \quad \text{for } n < m, \ m < 0, \ n < 0, \tag{1a}$$

and the inverse expansion

$$x^n = \sum_{m=0}^{n} D_{k,n}^{m} P_k(x,m) \qquad k = 1, 2, \tag{2}$$

$$D_{k,n}^{m} = 0, \quad \text{for } n < m, \ m < 0, \ n < 0, \tag{2a}$$

the coefficients $C_{k,n}^{m}$ and $D_{k,n}^{m}$ are called respectively <u>Generalized Stirling Numbers of First and Second Kind</u> of the polynomials $P_k(x,n)$. Examples of such numbers can be found in [3], [4], and [5].

Let then

$$P_k(x,n) = \sum_{m=0}^{n} L_{k,h,n}^{m} P_h(x,m), \qquad k,h = 1,2, \ k \neq h, \ n = 0,1,2,\cdots \tag{3}$$

$$L_{k,h,n}^{m} = 0, \quad \text{for } n < m, \ m < 0, \ n < 0 \ . \tag{3a}$$

More explicitly,

$$P_k(x,n) = \sum_{s=0}^{n} C_{k,n}^{s} x^s = \sum_{s=0}^{n} C_{k,n}^{s} \sum_{m=0}^{s} D_{h,s}^{m} P_h(x,m) ,$$

$$= \sum_{s=0}^{n} C_{k,n}^{s} \sum_{m=0}^{n} D_{h,s}^{m} P_h(x,m)$$

$$= \sum_{m=0}^{n} P_h(x,m) \sum_{s=m}^{n} C_{k,n}^{s} D_{h,s}^{m} ,$$

so that

(3b) $$L_{k,h,n}^{m} = \sum_{s=m}^{n} C_{k,n}^{s} D_{h,s}^{m} .$$

The coefficients $L_{k,h,n}^{m}$ are called <u>Generalized Lah Numbers</u> for the two sequences of polynomials P_k and P_h, $k \neq h$, $k, h = 1, 2$.

2. QUASI-ORTHOGONALITY

Under the conditions stated, the generalized Stirling numbers of first and second kind for a given sequence of polynomials $P_k(x, n)$ are said to <u>quasi-orthogonal</u> to each other (cf. [3] if

(4) $$\sum_{m=s}^{n} D_{k,n}^{m} C_{k,m}^{s} = \delta_n^s .$$

This result is proved in [3] for both the Q- and R-polynomials, but since the proof does not use the structure of the polynomials it is true for any sequence of polynomials as defined by (1).

Similarly the generalized Lah numbers for two sequences of polynomials P_k and P_h are quasi-orthogonal to the generalized Lah numbers for the sequences of polynomials P_h and P_k, i. e.,

$$\sum_{m=s}^{n} L^m_{k,h,n} L^s_{h,k,m} = \delta^s_n \quad . \tag{5}$$

This result is proved in [2] for the Q-polynomials, but here again the proof does not use the structure of the polynomials, thus is valid for any two sequences of polynomials as defined by (1).

3. RECALL ABOUT R-POLYNOMIALS

In [2] we have studied the generalized Lah numbers for two sequences of Q-polynomials. We shall now study the same for R-polynomials as defined in [3], i. e.,

$$R(x, n) = \sum_{m=0}^{n} C^m_n x^m \tag{6}$$

$$R(x, n + 1) = [K(n + 1) + L(n + 1)x] R(x, n) + \sum_{m=0}^{n} [M(m + 1) + N(m + 2)x] C^m_n x^m \quad , \tag{7}$$

$$R(x,0) = K(0) \tag{8}$$

$$x^n = \sum_{m=0}^{n} D^m_n R(x, m) \quad . \tag{9}$$

In order to simplify the results in [3] it was assumed that $L = 1$. Letting $N(n + 1) + 1 = P(n)$ it was proved that the numbers C^m_n and D^m_n satisfy the recurrence relations

(10) $$C_n^m = \left[K(n) + M(m+1)\right]C_{n-1}^m + P(m)C_{n-1}^{m-1}$$

(11) $$D_n^m = -\left[K(m+1) + M(n)\right]D_{n-1}^m \big/ P(n) + D_{n-1}^{m-1} \big/ P(n) \;.$$

In the following we shall consider two sets of R-polynomials $R_1(x,n)$ and $R_2(x,n)$ and the corresponding generalized Stirling numbers $\left\{C_{1,n}^m,\ C_{2,n}^m\right\}$ and $\left\{D_{1,n}^m,\ D_{2,n}^m\right\}$ which all satisfy the conditions given in sections 1 and 2. The generalized Lah-numbers for the two sequences are $L_{1,2,n}^m$ and $L_{2,1,n}^m$. They satisfy conditions (3a), (3b), and (5). We shall assume that $L_1(n) = L_2(n) = 1$.

4. RECURRENCE RELATIONS

According to relations (6) through (9) we can write

$$\begin{aligned} R_2(x, n+1) &= \sum_{s=0}^{n+1} C_{2,n+1}^m x^m \\ &= \left[K_2(n+1) + x\right]R_2(x,n) + \sum_{s=0}^{n} \left[M_2(s+1) + N_2(s+2)x\right]C_{2,n}^s x^s \;, \end{aligned}$$

and, according to the definition of the generalized Lah-numbers,

$$R_2(x, n+1) = \sum_{m=0}^{n+1} L_{2,1,n+1}^m R_1(x,m) \;,$$

so that

(12) $$\begin{aligned} \sum_{m=0}^{n+1} L_{2,1,n+1}^m R_1(x,m) &= K_2(n+1) \sum_{m=0}^{n} L_{2,1,n}^m R_1(x,m) \\ &\quad + \sum_{m=0}^{n} L_{2,1,n}^m xR_1(x,m) \\ &\quad + \sum_{s=0}^{n} M_2(s+1)C_{2,n}^s \sum_{m=0}^{s} D_{1,s}^m R_1(x,m) \\ &\quad + \sum_{s=0}^{n} N_2(s+2)C_{2,n}^s \sum_{m=0}^{s+1} D_{1,s+1}^m R_1(x,m) \;. \end{aligned}$$

On the other hand we have

$$R_1(x, m+1) = [K_1(m+1) + x]R_1(x, m) + \sum_{p=0}^{m} [M_1(p+1) + N_1(p+2)x]C_{1,m}^{p} x^p$$

thus

$$\text{(13)} \qquad xR_1(x, m) = R_1(x, m+1) - K_1(m+1)R_1(x, m) - \sum_{p=0}^{m} [M_1(p+1) + N_1(p+2)x]C_{1,m}^{p} x^p.$$

Substituting (13) into (12), and reorganizing the last two terms with the help of (1a) and (2a), we obtain

$$\sum_{m=0}^{n+1} L_{2,1,n+1}^{m} R_1(x, m) = K_2(n+1) \sum_{m=0}^{n} L_{2,1,n}^{m} R_1(x, m)$$

$$+ \sum_{m=0}^{n} L_{2,1,n}^{m} \left[R_1(x, m+1) - K_1(m+1)R_1(x, m) - \sum_{p=0}^{m} [M_1(p+1) + N_1(p+2)x]C_{1,m}^{p} x^p \right]$$

$$+ \sum_{m=0}^{n} R_1(x, m) \sum_{s=m}^{n} M_2(s+1)C_{2,n}^{s} D_{1,s}^{m}$$

$$+ \sum_{m=0}^{n+1} R_1(x, m) \sum_{s=m-1}^{n} N_2(s+2)C_{2,n}^{s} D_{1,s+1}^{m},$$

or, interchanging the indices m and s,

$$\sum_{m=0}^{n+1} L_{2,1,n+1}^{m} R_1(x, m) = K_2(n+1) \sum_{m=0}^{n} L_{2,1,n}^{m} R_1(x, m) \tag{14}$$

$$+ \sum_{m=0}^{n} L_{2,1,n}^{m} R_1(x, m+1) - \sum_{m=0}^{n} L_{2,1,n}^{m} K_1(m+1) R_1(x, m)$$

$$- \sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=0}^{s} M_1(p+1) C_{1,s}^{p} \sum_{m=0}^{p} D_{1,p}^{m} R_1(x, m)$$

$$- \sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=0}^{s} N_1(p+2) C_{1,s}^{p} \sum_{m=0}^{p+1} D_{1,p+1}^{m} R_1(x, m)$$

$$+ \sum_{m=0}^{n} R_1(x, m) \sum_{s=m}^{n} M_2(s+1) C_{2,n}^{s} D_{1,s}^{m} + \sum_{m=0}^{n+1} R_1(x, m) \sum_{s=m-1}^{n} N_2(s+2) C_{2,n}^{s} D_{1,s+1}^{m}$$

The fourth and fifth quantities on the right-hand side of (14) can be written as follows:

$$\sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=0}^{s} M_1(p+1) C_{1,s}^{p} \sum_{m=0}^{p} D_{1,p}^{m} R_1(x, m) \tag{15}$$

$$= \sum_{m=0}^{n} R_1(x, m) \sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=m}^{s} M_1(p+1) C_{1,s}^{p} D_{1,p}^{m},$$

$$\sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=0}^{s} N_1(p+2) C_{1,s}^{p} \sum_{m=0}^{p+1} D_{1,p+1}^{m} R_1(x, m) \tag{16}$$

$$= \sum_{m=0}^{n+1} R_1(x, m) \sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=m-1}^{s} N_1(p+2) C_{1,s}^{p} D_{1,p+1}^{m}$$

Substituting (15) and (16) into (14) we obtain by equating the coefficients of $R_1(x, m)$

$$(17)\qquad L^m_{2,1,n+1} = K_2(n+1) - K_1(m+1)\; L^m_{2,1,n} + L^{m-1}_{2,1,n}$$

$$- \sum_{s=m-1}^{n} L^s_{2,1,n} \sum_{p=m}^{s} M_1(p+1)C^p_{1,s}D^m_{1,p} + \sum_{p=m-1}^{s} N_1(p+2)C^p_{1,s}\, D^m_{1,p+1}$$

$$+ \sum_{s=m}^{n} M_2(s+1)C^s_{2,n}\, D^m_{1,s} + \sum_{s=m-1}^{n} N_2(s+2)C^s_{1,n} D^m_{1,s+1} \;,$$

or, changing n into $n-1$,

$$(18)\qquad L^m_{2,1,n} = K_2(n) - K_1(m+1)\; L^m_{2,1,n-1} + L^m_{2,1,n-1}$$

$$- \sum_{s=m-1}^{n-1} L^s_{2,1,n-1} \sum_{p=m}^{s} M_1(p+1)C^p_{1,s}D^m_{1,p} + \sum_{p=m-1}^{s} N_1(p+2)C^p_{1,s}D^m_{1,p+1}$$

$$+ \sum_{s=m}^{n-1} M_2(s+1)C^s_{2,n-1}D^m_{1,s} + \sum_{s=m-1}^{n-1} N_2(s+2)C^s_{2,n-1}\, D^m_{1,s+1}$$

Relation (18) is the recurrence relation for the generalized numbers $L^m_{2,1,n}$. A similar relation for the Lah-numbers $L^m_{1,2,n}$ will be obtained by interchanging the indices 1 and 2 in (18).

5. EXAMPLE

We illustrate by the following example based on examples I and II of section 5 of [3]. Thus:

$$K_1(\alpha) = \alpha + 1,\quad M_1(\alpha) = (\alpha - 1)^2,\quad N_1(\alpha) = 0,\quad K_2(\alpha) = \alpha\,,\quad M_2(\alpha) = \alpha,$$
$$N_2(\alpha) = \alpha\;,$$

where the index 1 corresponds to example I and the index 2 to example II of section 5 of [3]. The numerical values of $C^m_{1,n}$ are those of C^m_n, of $D^m_{1,n}$ those of D^m_n of example I, while $C^m_{2,n}$ and $D^m_{2,n}$ those of C^m_n and D^m_n of example II. Under these conditions we obtain the following for $L^m_{2,1,n}$:

$$(19) \qquad L^{m}_{2,1,n} = (n - m - 2)L^{m}_{2,1,n-1} + L^{m-1}_{2,1,n-1} - \sum_{s=m-1}^{n-1} L^{s}_{2,1,n-1}\left[\sum_{p=m}^{s} p^2 C^{p}_{1,s} D^{m}_{1,p}\right] + \sum_{s=m}^{n-1} (s + 1)C^{s}_{2,n-1}D^{m}_{1,s} + \sum_{s=m-1}^{n-1} (s + 2)C^{s}_{2,n-1}D^{m}_{1,s+1} \quad .$$

m=	0	1	2	3	4
n:					
0	1				
1	-4	3			
2	42	-54	12		
3	-1488	2124	-696	60	
4	99680	-170640	67440	-8880	360

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THE EXISTENCE OF PERFECT 3-SEQUENCES

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For s and n positive integers, a sequence $a_1, a_2, \cdots, a_{sn}$ of length sn is called a perfect s-sequence for the integer n if (a) each of the integers $1, 2, \cdots, n$ occurs exactly s times in the sequence and (b) between any two consecutive occurrences of the integer i there are exactly i entries. Thus 4 1 3 1 2 4 3 2 is a perfect 2-sequence for $n = 4$. The problem of determining all n having a perfect s-sequence is posed in [1] for $s = 2$ and in [4] for $s > 2$.

It is shown in [3] that a perfect 2-sequence exists for an integer n if and only if $n = 3$ or $4 \pmod 4$, and furthermore, an explicit 2-sequence is presented for each such n.

The question of the existence of a perfect s-sequence for any n with $s > 2$ is then raised in [4] and [5]. The problem is partially answered in [5] by providing necessary conditions on n in the case where s is either a multiple of 2 or 3. In the particular case $s = 3$, a necessary condition that there exist a perfect 3-sequence for n is $n \equiv 1, 0,$ or $1 \pmod 9$.

The following examples lead one to believe that for $s = 3$, the above conditions are almost sufficient. Namely, we exhibit perfect 3-sequences for $n = 9, 10, 17, 18$, and 19.

The case $n = 9$:

1 9 1 6 1 8 2 5 7 2 6 9 2 5 8 4 7 6 3 5 4 9 3 8 7 4 3

The case $n = 10$ (with 10 denoted by ϕ):

1 ϕ 1 6 1 7 9 3 5 8 6 3 ϕ 7 5 3 9 6 8 4 5 7 2 ϕ 4 2 9 8 2 4

The case $n = 17$:

17	15	3	16	9	10	3	1	12	1	3	1	13	14	9	6	10
15	17	5	16	12	6	11	9	5	13	10	14	6	7	5	8	15
12	11	17	16	7	4	13	8	2	14	4	2	7	11	2	4	8

The case $n = 18$:

18	16	5	17	11	4	2	9	5	2	4	14	2	15	5	4	11	9
16	18	12	17	13	6	7	8	14	9	11	15	6	10	7	12	8	16
13	6	18	17	7	14	10	8	3	15	12	1	3	1	13	1	3	10

The case $n = 19$:

19	17	13	18	4	11	8	2	16	4	2	9	15	2	4	8	13	11	14
17	19	9	18	12	8	16	5	7	15	11	13	9	5	14	10	7	12	17
5	6	19	18	16	7	15	10	6	3	14	12	1	3	1	6	1	3	10

From the above examples, one has

Conjecture. For $n > 8$, a necessary and sufficient condition that there exist a perfect 3-sequence for n is $n \equiv -1, 0, 1 \pmod 9$.

The necessary condition stated in the above conjecture is proved in [5]. Actually, the results of [5] are a special case of:

Theorem 1. Let $s = pt$ where p is a prime. A necessary condition that a perfect s-sequence exist for an integer n is

$$n \equiv -1, 0, 1, 2, \cdots, \text{ or } p - 2 \pmod{p^2} .$$

Proof. Suppose a perfect s-sequence $a_1, \cdots, a_{sn}$ exists. Then for an integer i occupying positions $c_1, c_2, \cdots, c_s$, we have

$$c_j = c_1 + (j-1)(i+1) \quad (j = 1, \cdots, s) .$$

If $i \not\equiv -1 \pmod p$, the positions c_j range over the residue classes mod p in a manner such that each residue class has an equal number t of occurrences.

On the other hand, for a fixed i such that $i \equiv -1 \pmod p$ the positions c_j are all congruent to each other mod p. Letting r be a residue of p, $0 \leq r \leq p - 1$, we define $N(r)$ as the number of integers $i \equiv -1 \pmod p$ such that the common residue of $c_1, \cdots, c_s$ is r.

We now let $b(n,p)$ denote the number of integers i such that $1 \leq i \leq n$ and $i \equiv -1 \pmod p$. Then, observing that the total number of positions in the sequence $a_1, \cdots, a_{sn}$ congruent to $r \pmod p$ must be nt, it follows (by counting the number of such positions filled by integers i in the range $1 \le i \le n$) that

$$t \cdot b(n,p) + sN(r) = nt .$$

Thus, all $N(r)$ have the common value N expressed by

$$pN = n - b(n,p) = \left[\frac{n+1}{p}\right]$$

where [] is the greatest integer function. Representing n by $n = kp + q$ with $-1 \le q \le p - 2$ it follows that $pN = k$ and $n = p^2N + q$, whence n is out in the assumed range of values.

The fact that theorem 1 is in some sense strong for $s = 3$ does not completely reflect what conditions are required on n for $s > 3$. In particular, if a power (greater than 1) of a prime divides s the conditions on n can be improved over that presented in theorem 1. We shall only treat the case where $p^2 | s$ (with p a prime) although a more general result can be proved for $p^k | s$ with k arbitrary.

Theorem 2. Let $s = p^2t$ where p is a prime. A necessary condition that a perfect s-sequence exist for an integer n is

$$n \equiv -1, 0, 1, 2, \cdots, \text{ or } p - 2 \pmod{p^3} .$$

Proof. Let the integer i (with $1 \le i \le n$) occupy positions $c_1, \cdots, c_s$ in a perfect s-sequence for n. Then

$$c_j = c_1 + (j - 1)(i + 1) \qquad j = 1, \cdots, s .$$

We consider three categories for the integer i as follows:

I.) For the

$$n - \left[\frac{n+1}{p}\right]$$

integers i with $i + 1 \not\equiv 0 \pmod p$ the positions $c_1, \cdots, c_s$ range over the residue classes $(\bmod\ p^2)$ in such a manner that each residue class occurs exactly t times

II.) For the

$$\left[\frac{n+1}{p}\right] - \left[\frac{n+1}{p^2}\right]$$

integers i with $i + 1 \equiv 0 \pmod{p}$ and $i + 1 \not\equiv 0 \pmod{p^2}$ the positions $c_1, \cdots, c_s$ range over the residue classes $c_1, c_1 + p, \cdots, c_1 + (p-1)p \pmod{p^2}$ in a manner whereby each such residue occurs exactly pt times. We let $N(r)$ for $r = 0, 1, \cdots, p-1$ be the number of i in this category with $c_1 \equiv r$ (mod p).

III.) For the

$$\left[\frac{n+1}{p^2}\right]$$

integers with $i + 1 \equiv 0 \pmod{p^2}$ the positions $c_1, \cdots, c_s$ all belong to the same residue class $\pmod{p^2}$.

We let $M(q)$ for $q = 0, 1, \cdots, p^2 - 1$ be the number of i in this category with $c_1 \equiv q \pmod{p^2}$.

Letting q be a residue of p^2 with $q \equiv r \pmod{p}$, the number of positions in the s-sequence for n that are congruent to $q \pmod{p^2}$ is nt. Thus

$$nt = t\left\{n - \left[\frac{n+1}{p}\right]\right\} + ptN(r) + p^2tM(q)$$

or

$$p^2M(q) = \left[\frac{n+1}{p}\right] - pN(r) .$$

The latter implies that $M(q)$ is identical for all residues q of p^2 having the common reduced residue $r \pmod{p}$. Letting $L(r)$ denote this identical value,

$$\left[\frac{n+1}{p^2}\right] = \sum_{q=0}^{p^2-1} M(q) = p \sum_{r=0}^{p-1} L(r) .$$

hence, p divides

$$\left[\frac{n+1}{p^2}\right] .$$

But from theorem 1, $n + 1 = p^2d + e$ where $e = 0, 1, 2, \cdots$, or $p - 1$, hence $d = pd'$ and $n + 1 = p^3d' + e$ which is the desired result.

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Continued from p. 10

18	1	$\frac{1}{dF_{n-1}} > \frac{1}{dF_{n-1}} \ge \frac{1}{bd} = \left\lvert\frac{a}{b} - \frac{c}{d}\right\rvert > \left\lvert\frac{F_n}{F_{n-1}} - \frac{c}{d}\right\rvert > \frac{1}{dF_{n-1}}$	$\frac{1}{dF_{n-1}} \ge \frac{1}{bd} = \left\lvert\frac{a}{b} - \frac{c}{d}\right\rvert \ge\ > \left\lvert\frac{F_n}{F_{n-1}} - \frac{c}{d}\right\rvert \ge \frac{1}{dF_{n-1}}$
18	-7	$\frac{c}{d} - \frac{F_{n+1}}{F_n} \ge \frac{F_{n+1}}{F_n} \ge \frac{F_{n+2}}{F_{n+1}}$	$\frac{c}{d} - \frac{F_{n+1}}{F_n} + \frac{F_{n+1}}{F_n} - \frac{F_{n+2}}{F_{n+1}}$

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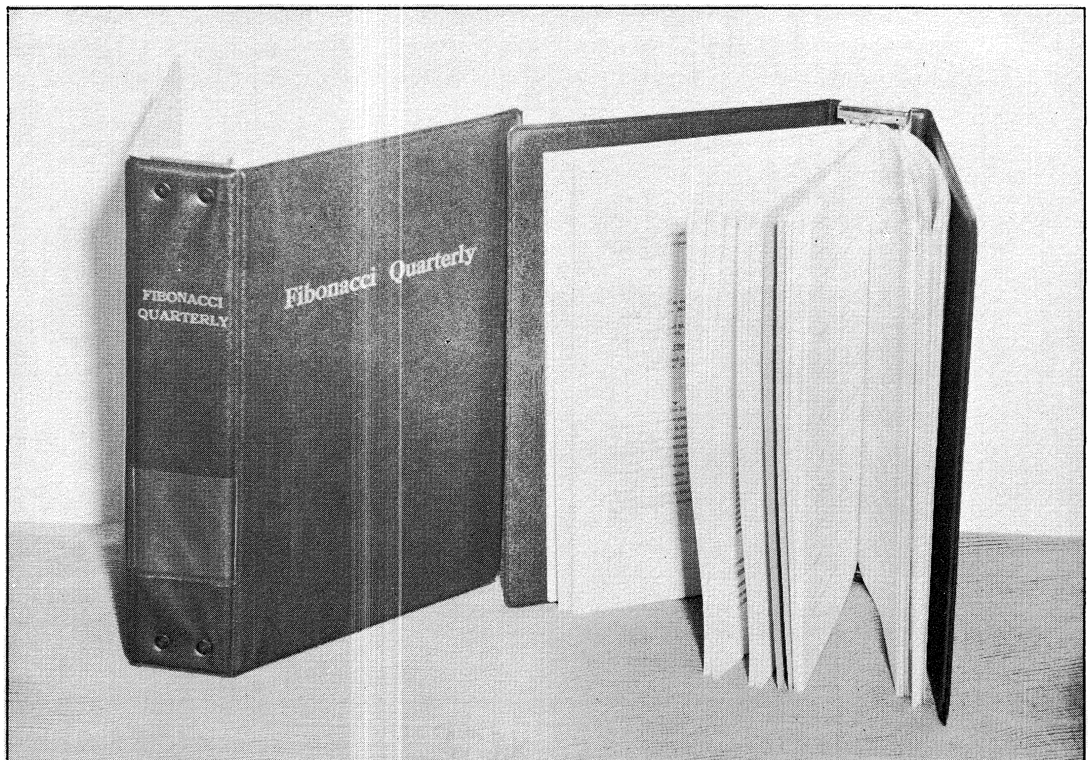

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