# THE FIBONACCI QUARTERLY 

THE OFFICIAL JOURNAL OF<br>THE FIBONACCI ASSOCIATION



VOLUME 6
NUMBER 6
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# THE FIBONACCI QUARTERLY 

OFFICIAL ORGAN OF THE FIBONACCI ASSOCIATION
A JOURNAL DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

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The Quarterly is entered as third-class mail at the St. Mary's College Post Office, California, as an official publication of the Fibonacci Association.

## PSEUDO-FIBONACCI NUMBERS

H. H. FERNS<br>Victoria, B.C., Canada

Consider the two interlocking recursion formulas
(1)

$$
O_{i+1}=O_{i}+P_{i}
$$

(2)

$$
P_{i+1}=O_{i+1}+\lambda O_{i} \quad O_{i}=P_{i}=1
$$

in which $\lambda$ is a positive integer.
For reasons which will shortly become apparent we call $O_{i}$ and $P_{i}$ pseudo-Fibonacci and pseudo-Lucas numbers, respectively.

In fact, eliminating first the $O^{\prime} s$ and then the $P^{\prime} s$, from (1) and (2) we get

$$
\begin{equation*}
O_{i+1}=20_{i+1}+\lambda O_{i} \quad O_{0}=0, \quad O_{1}=1 \tag{3}
\end{equation*}
$$

$$
\begin{equation*}
P_{i+2}=2 P_{i+1}+\lambda P_{i} \quad P_{0}=1, \quad P_{1}=1 \tag{4}
\end{equation*}
$$

Thus the two numbers defined by (1) and (2) satisfy the same recursion formula but with different initial values.

A Binet-type formula for each of $O_{n}$ and $P_{n}$ may be derived from first principles [1]. We leave this as an exercise for the reader.

We shall prove by induction that

$$
\begin{align*}
& O_{n}=\frac{\left(1+{\sqrt{1+\lambda})^{n}-(1-\sqrt{1+\lambda})^{n}}_{2 \sqrt{1+\lambda}}^{P_{n}}=\frac{(1+\sqrt{1+\lambda})^{n}+(1-\sqrt{1+\lambda})^{n}}{2}\right.}{}=\frac{1}{n} \tag{5}
\end{align*}
$$

We introduce the notation $\mathrm{A}=1+\sqrt{1+\lambda}$ and $\mathrm{B}=1-\sqrt{1+\lambda_{0}}$ From thus it follows that (Received 1963--revised Feb. 1968)

\[

\]

## Hence

$$
O_{n}=\frac{A^{n}-B^{n}}{A-B} \quad P_{n}=\frac{A^{n}+B^{n}}{2}
$$

It is immediately apparent from these two forms of $O_{n}$ and $P_{n}$ that $O_{1}=P_{1}=1$. Since $O_{2}=2$, (1) is satisfied for $i=1$. Assume that it is true for $i=2,3, \cdots, k$ Then

$$
\begin{aligned}
O_{k}+P_{k} & =\frac{A^{k}-B^{k}}{A-B}+\frac{A^{k}+B^{k}}{2} \\
& =\frac{2 A^{k}-2 B^{k}+A^{k+1}-B^{k+1}-A^{k} B+A B^{k}}{2(A-B)} \\
& =\frac{A^{k}(2-B)-B^{k}(2-A)+A^{k+1}-B^{k+1}}{2(A+B)} \\
& =\frac{A^{k+1}-B^{k+1}+A^{k+1}-B^{k+1}}{2(A-B)} \\
& =\frac{A^{k+1}-B^{-k+1}}{A-B}=O_{k+1}
\end{aligned}
$$

This completes the proof of (5). A similar proof holds for (6).
If we let $n=-k$ where $k$ is a positive integer we find from (5) and (6) that

$$
O_{-k}=-\frac{O_{k}}{(-\lambda)^{k}} \quad \text { and } \quad P_{-k}=\frac{P_{k}}{(-\lambda)^{k}}
$$

It is left as an exercise to show that $\mathrm{O}_{-k}$ and $\mathrm{P}_{-k}$ satisfy (1) and (2).

In (5) and (6) let $\lambda=4$. We get

$$
\begin{equation*}
O_{n}=2^{n-1} F_{n} \quad \text { and } \quad P_{n}=2^{n-1} L_{n} \quad(\lambda=4) \tag{7}
\end{equation*}
$$

where $F_{n}$ and $L_{n}$ are the $n^{\text {th }}$ Fibonacci and Lucas numbers, respectively.
Thus identities among $O_{n}$ and $P_{n}$ may be transformed by means of the equations in (7) into identities involving Fibonacci and Lucas numbers. Some of the latter will be familiar. The purpose of this article is to find some new or unfamiliar identities among the latter numbers.

We begin with (1) which we write in the form

$$
P_{i}=o_{i+1}-o_{i}
$$

Let $\mathrm{i}=1,2,3, \cdots \mathrm{n}$ in this equation. Adding the resulting equations we get

$$
\begin{equation*}
\sum_{i=1}^{n} P_{i}=\rho_{n+1}-1 \tag{8}
\end{equation*}
$$

Applying the equations in (7) to (8) we have

$$
\sum_{i=1}^{n} 2^{i} L_{i}=2\left(2^{n} F_{n+1}-1\right)
$$

Next, eliminating $\mathbf{O}_{\mathbf{i}+1}$ from (1) and (2) yields

$$
O_{i}-\frac{1}{1+\lambda}\left(P_{i+1}-P_{i}\right)
$$

Following the procedure used above we get
(9)

$$
\sum_{i=1}^{n} o_{i}=\frac{1}{1+\lambda}\left(p_{n+1}-1\right)
$$

$$
\sum_{i=1}^{n} 2^{i} F_{i}=\frac{2}{5}\left(2^{n} L_{n+1}-1\right)
$$

A formula for the sum of the first $n \mathrm{O}^{\prime} \mathrm{s}$ with even numbered subscripts is now derived.

$$
\begin{aligned}
\sum_{i=1}^{n} O_{2 i} & =\sum_{i=1}^{n} \frac{A^{2 i}-B^{2 i}}{A-B} \\
& =\frac{1}{A-B}\left[\sum_{i=1}^{n} A^{2 i}-\sum_{i=1}^{n} B^{2 i}\right] \\
& =\frac{1}{A-B}\left[\frac{A^{2}\left(A^{2 n}-1\right)}{A^{2}-1}-\frac{B^{2}\left(B^{2 n}-1\right)}{B^{2}-1}\right]
\end{aligned}
$$

(10)

$$
\begin{aligned}
& =\frac{1}{A-B}\left[\frac{\left(A^{2} B^{2}-A^{2}\right)\left(A^{2 n}-1\right)-\left(A^{2} B^{2}-B^{2}\right)\left(B^{2 n}-1\right)}{A^{2} B^{2}-A^{2}-B^{2}+1}\right] \\
& =\frac{1}{A-B}\left[\frac{\left.\lambda^{2}\left(A^{2 n}-B^{2 n}\right)-\left(A^{2 n+2}-B^{2 n+2}\right)+A^{2}-B^{2}\right)}{\lambda^{2}-\left(A^{2}+B^{2}\right)+1}\right] \\
& =\frac{1}{A-B}\left[\frac{\lambda^{2}(A-B) O_{2 n}-(A-B) O_{2 n+2}+(A-B) O_{2}}{\lambda^{2}-2 P_{2}+1}\right] \\
& =\frac{\lambda^{2} O_{2 n}-O_{2 n+2}+2}{(\lambda+1)(\lambda-3)} \quad(\lambda \neq 3)
\end{aligned}
$$

Applying recursion formulas (1) and (3) to (10) takes the form
(10a) $\quad \sum_{i=1}^{n} \mathrm{O}_{2 \mathrm{i}}=\frac{\left.(\lambda+1)(\lambda-4) \mathrm{O}_{2 n-1}+\lambda^{2}-\lambda-4\right) \mathrm{P}_{2 n-1}+2}{(\lambda+1)(\lambda+3)} \quad(\lambda \neq 3)$

From (10a) we get

$$
\sum_{i=1}^{n} 2^{2 i} F_{2 i}=\frac{4}{5}\left(2^{2 n} L_{2 n-1}+1\right)
$$

For the special case in which $\lambda=3$ we have

$$
O_{i}=\frac{1}{4}\left[3^{i}-(-1)^{i}\right]
$$

Hence
(10s)

$$
\begin{aligned}
\sum_{i=1}^{n} O_{2 i} & =\frac{1}{4}\left[\sum_{i=1}^{n} 3^{i}-\sum_{i=1}^{n}(-1)^{n}\right] \\
& =\frac{9}{32}\left(9^{n}-1\right)-\frac{n}{4}
\end{aligned}
$$

The following four identities are given without proof. Their derivation follows the same procedure that was used above.

$$
\begin{equation*}
\sum_{i=1}^{n} P_{2 i}=\frac{P_{2 n+2}-2 O_{2 n+2}+2-\lambda}{\lambda-3} \quad(\lambda \neq 3) \tag{11}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{i=1}^{n} P_{2 i}=\frac{(\lambda-4) O_{2 n+1}+\lambda(\lambda-2) O_{2 n-1}-\lambda+2}{\lambda-3} \quad(\lambda \neq 3) \tag{11a}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{i=1}^{n} P_{2 i}=\frac{9}{16}\left(9^{n}-1\right)+\frac{n}{2} \quad(\lambda=3) \tag{11s}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{i=1}^{n} 2^{2 i} L_{2 i}=4\left(2^{2 n} F_{2 n}-1\right) \tag{11'}
\end{equation*}
$$

To find the sum of the first $n O^{\prime} s$ with odd numbered subscripts we use

$$
\sum_{i=1}^{n} O_{2 i-1}=\sum_{i=1}^{2 n} o_{i}-\sum_{i=1}^{n} o_{2 i}
$$

and make use of results already obtained. In this manner we get the following four identities:

$$
\begin{equation*}
\sum_{i=1}^{n} O_{2 i-1}=\frac{(\lambda-1) O_{2 n+1}-2 O_{2 n}+1-\lambda}{(\lambda+1)(\lambda-3)} \quad(\lambda \neq 3) \tag{12}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{i=1}^{n} O_{2 i-1}=\frac{(\lambda-4) P_{2 n-1}+\lambda(\lambda-2) P_{2 n-2}+1-\lambda}{(\lambda+1)(\lambda-3)} \quad(\lambda \neq 3) \tag{12a}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{i=1}^{n} O_{2 i-1}=\frac{3}{32}\left(9^{n}-1\right)+\frac{n}{4} \quad(\lambda=3) \tag{12s}
\end{equation*}
$$

$$
\sum_{i=1}^{n} 2^{2 i-1} F_{2 i-1}=\frac{2}{5}\left[2^{2 n_{L_{2 n-2}}-3}\right]
$$

The next four are derived in a similar manner.

$$
\begin{equation*}
\sum_{i=1}^{n} P_{2 i-1}=\frac{O_{2 n+2}-30_{2 n+1}+1}{\lambda-3} \quad(\lambda \neq 3) \tag{13}
\end{equation*}
$$

(13a)

$$
\sum_{i=1}^{n} P_{2 i-1}=\frac{(\lambda-4) O_{2 n-1}+\lambda(\lambda-2) O_{2 n-2}+1}{\lambda-3}
$$

$$
\sum_{i=1}^{n} P_{2 i-1}=\frac{3}{16}\left(9^{n}-1\right)-\frac{n}{2} \quad(\lambda=3)
$$

$$
\sum_{i=1}^{n} 2^{2 i-1} L_{2 i-1}=2\left(2^{2 n} F_{2 n-2}+1\right)
$$

We now derive the sum of a series with alternating positive and negative signs.

From (10') and (12') we get

$$
\begin{aligned}
\sum_{i=1}^{2 n-1}(-1)^{i+1} 2^{i} F_{i} & =\sum_{i=1}^{n} 2^{2 i-1} F_{2 i-1}-\sum_{i=1}^{n} 2^{2 i} F_{2 i} \\
& =\frac{2^{2 n+1}}{5}\left(L_{2 n-2}-2 L_{2 n-1}\right)-2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\sum_{i=1}^{2 n+1}(-1)^{i+1} 2^{i} F_{i} & =\sum_{i=1}^{n+1} 2^{2 i-1} F_{2 i-1}-\sum_{i=1}^{n} 2^{2 i} F_{2 i} \\
& =\frac{2^{2 n+2}}{5}\left(2 L_{2 n}-L_{2 n-1}\right)-2 .
\end{aligned}
$$

From the last two equations we conclude that
(14)

$$
\sum_{i=1}^{n}(-1)^{i+1} 2^{i} F_{i}=\frac{(-2)^{n+1}}{5}\left(2 L_{n}-L_{n-1}\right)-2
$$

In like manner, beginning with (11') and (13') we get

$$
\sum_{i=1}^{n}(-1)^{i+1} 2^{i} L_{i}=(-2)^{n+1}\left(F_{n-2}-2 F_{n-1}\right)+6
$$

The following identities involve sums of squares. Derivation is given for the first one only.

In several cases the final term is

$$
\pm \frac{\lambda}{1+\lambda}\left[\left(1-(-\lambda)^{n}\right)\right]
$$

For brevity we shall denote this by $\pm R$.
(16)

$$
\sum_{i=1}^{n} O_{i}^{2}=\frac{1}{2(1+\lambda)}\left[\frac{(\lambda-4) O_{2 n}+\lambda(\lambda-2) O_{2 n-1}+2-\lambda}{\lambda-3}+R\right](\lambda \neq 3)
$$

(16s)

$$
\sum_{i=1}^{n} O_{i}^{2}=\frac{1}{128}\left[9^{n+1}-8 n+4(-3)^{n+1}+3\right] \quad(\lambda=3)
$$

(16')

$$
\sum_{i=1}^{n} 2^{2 i} F_{i}^{2}=\frac{2}{5}\left[2^{2 n+1} F_{2 n-i}+\frac{(-4)^{n+1}-6}{5}\right]
$$

$$
\begin{equation*}
\sum_{i=1}^{n} P_{i}^{2}=\frac{1}{2}\left[\frac{(\lambda-4) O_{2 n}+(\lambda-2) O_{2 n-1}}{\lambda-3}-R\right] \quad(\lambda \neq 3) \tag{17}
\end{equation*}
$$

(17s)

$$
\sum_{i=1}^{n} P_{i}^{2}=\frac{1}{32}\left[9^{n+1}+8 n+12(-3)^{n}-12\right] \quad(\lambda=3)
$$

(17)

$$
\sum_{i=1}^{n} 2^{2 i} L_{i}^{2}=2\left[2^{2 n+1} F_{2 n-1}-\frac{(-4)^{n+1}+14}{5}\right]
$$

(18) $\sum_{i=1}^{n}\left[(1+\lambda) \mathrm{O}_{\mathrm{i}}^{2}+\mathrm{P}_{\mathrm{i}}^{2}\right]=\frac{(\lambda-4) \mathrm{O}_{2 n}+\lambda(\lambda-2) \mathrm{O}_{2 n-1}+2-\lambda}{\lambda-3} \quad(\lambda \neq 3)$
(18s)

$$
\sum_{i=1}^{n}\left(40_{i}^{2}+P_{i}^{2}\right)=\frac{1}{16}\left[9^{n+1}+8 n-9\right] \quad \lambda=3
$$

(18')

$$
\sum_{i=1}^{n} 2^{2 i}\left(5 F_{i}^{2}+L_{i}^{2}\right)=8\left[2^{\left.2 n_{F_{2 n-1}}-\eta\right]}\right.
$$

The proof of (16) follows:
$\sum_{i=1}^{n} O_{i}^{2}=\sum_{i=1}^{n}\left[\frac{A^{i}-B^{i}}{A-B}\right]^{2}$

$$
\begin{aligned}
& =\frac{1}{4(1+\lambda)}\left[\sum_{i=1}^{n} A^{2 i}+\sum_{i=1}^{n} B^{2 i}-2 \sum_{i=1}^{n}(A B)^{i}\right] \\
& =\frac{1}{4(1+\lambda)}\left[\frac{A^{2}\left(A^{2 n}-1\right)}{A^{2}-1}+\frac{B^{2}\left(B^{2 n}-1\right)}{B^{2}-1}-2 \frac{A B\left[(A B)^{n}-1\right]}{A B-1}\right] \\
& =\frac{1}{4(1+\lambda)}\left[\frac{\left(B^{2}-1\right) A^{2 n+2}-A^{2} B^{2}+A^{2}+\left(A^{2}-1\right) B^{2 n+2}-A^{2} B^{2}+B^{2}}{A^{2} B^{2}-A^{2}-B^{2}+1}+2 R\right]
\end{aligned}
$$

Since

$$
A^{2}-1=1+\lambda+2 \sqrt{1+\lambda}
$$

and

$$
\mathrm{B}^{2}-1=1+\lambda-2 \sqrt{1+\lambda}
$$

we have

$$
\begin{aligned}
\sum_{i=1}^{n} O_{i}^{2} & =\frac{1}{4(1+\lambda)}\left[\frac{(1+\lambda)\left(A^{2 n+2}+B^{2 n+2}\right)-2 \sqrt{1+\lambda}\left(A^{2 n+2}-B^{2 n+2}\right)+A^{2}+B^{2}-2 A^{2} B^{2}}{A^{2} B^{2}-A^{2}-B^{2}+1}+2 R\right] \\
& =\frac{1}{2(1+\lambda)}\left[\frac{2(1+\lambda) P_{2 n+2}-4(1+\lambda) O_{2 n+2}+2 P_{2}-2 \lambda^{2}}{(\lambda+1)(\lambda-3)}+2 R\right] \quad(\lambda \neq 3) \\
& =\frac{1}{2(1+\lambda)}\left[\frac{P_{2 n+2}-20_{2 n+2}+2-\lambda}{\lambda-3}+R\right] \quad(\lambda \neq 3)
\end{aligned}
$$

From (1) and (3) we get

$$
P_{2 n+2}=20_{2 n+2}=(\lambda-4) O_{2 n}+\lambda(\lambda-2) O_{2 n-1}
$$

## Hence

$$
\sum_{i=1}^{n} O_{i}^{2}=\frac{1}{2(1+\lambda)}\left[\frac{(\lambda-4) O_{2 n}+\lambda(\lambda-2) O_{2 n-1}+2-\lambda}{\lambda-3}+R\right](\lambda \neq 3)
$$

This completes the proof of (16).
We consider next identities involving the sums of products. The proof of the identity

$$
\begin{equation*}
2(1+\lambda) O_{n} O_{m}=P_{n+m}-(-\lambda)^{m_{P}} P_{n-m} \tag{19}
\end{equation*}
$$

follows:

$$
\begin{aligned}
2(1+\lambda) O_{n} O_{m} & =2(1+\lambda)\left[\frac{A^{n}-B^{n}}{2 \sqrt{1+\lambda}}\right] \cdot\left[\frac{A^{m}-B^{m}}{2 \sqrt{1+\lambda}}\right] \\
& =\frac{A^{n+m}+B^{n+m}-A^{n} B^{m}-A^{m} B^{n}}{2} \\
& =\frac{A^{n+m}+B^{n+m}}{2}-\frac{A^{m} B^{m}\left(A^{n-m}-B^{n-m}\right.}{2} \\
& =P_{n+m}-(-\lambda)^{m} P_{n-m}
\end{aligned}
$$

From (19) we may write

$$
\begin{gathered}
2(1+\lambda) \mathrm{O}_{2} \mathrm{O}_{1}=\mathrm{P}_{3}-(-\lambda) \mathrm{P}_{1} \\
2(1+\lambda) \mathrm{O}_{3} \mathrm{O}_{2}=\mathrm{P}_{5}-(-\lambda)^{2} \mathrm{P}_{1} \\
\text { •••••••••••} \\
2(1+\lambda) \mathrm{O}_{2 \mathrm{n}+1} \mathrm{O}_{\mathrm{n}}=\mathrm{P}_{2 \mathrm{n}+1}-(-\lambda)^{\mathrm{n}^{2}} \mathrm{P}_{1} \cdot
\end{gathered}
$$

Adding these n equations gives

$$
2(1+\lambda) \sum_{i=1}^{n} O_{i} O_{i+1}=\sum_{i=1}^{n} P_{2 i+1}-\sum_{i=1}^{n}(-\lambda)^{i} P_{i}
$$

Using (13a) and the fact that $P_{1}=1$ we have
(20) $2(1+\lambda) \sum_{i=1}^{n} \mathrm{O}_{\mathrm{i}} \mathrm{O}_{\mathrm{i}+1}=\frac{(\lambda-4) \mathrm{O}_{2 \mathrm{n}+1}+(\lambda-2) \mathrm{O}_{2 n}+4-\lambda}{\lambda-3}+\mathrm{R} \quad(\lambda \neq 3)$

For the case $\lambda=3$ we get
(20s) $\quad \sum_{i=1}^{n} O_{i} O_{i+1}=\frac{1}{128}\left[3^{2 n+3}+4(-3)^{n+1}-8 n-15\right] \quad(\lambda=3)$

For $\lambda=4$ we have
(20 ${ }^{\circ}$

$$
\sum_{i=1}^{n} 2^{2 i} F_{i} F_{i+1}=\frac{4}{5}\left[2^{2 n_{F}}{ }_{2 n}+\frac{1}{5}\left[1-(-4)^{n}\right]\right]
$$

The proofs of the three following identities are left to the reader.

$$
\begin{aligned}
& 2 P_{n} P_{m}=P_{n+m}+(-\lambda)^{m_{P}} P_{n-m} \\
& 2 O_{n} P_{m}=o_{n+m}+(-\lambda)^{m} O_{n-m} \\
& 2 P_{n} O_{m}=o_{n+m}-(-\lambda)^{m} O_{n-m}
\end{aligned}
$$

Following the same procedure that was used above we arrive at the following identities:

$$
\begin{equation*}
2 \sum_{i=1}^{n} P_{i} P_{i+1}=\frac{(\lambda-4) O_{2 n+1}+\lambda(\lambda-2) O_{2 n}+4-\lambda}{\lambda-3}-R \quad(\lambda \neq 3) \tag{24}
\end{equation*}
$$

(24s)

$$
\sum_{i=1}^{n} P_{i} P_{i+1}=\frac{1}{32}\left[3^{2 n+3}-4(-3)^{n+1}-8 n-39\right] \quad(\lambda=3)
$$

(24)

$$
\sum_{i=1}^{n} 2^{2 i} L_{i} L_{i+1}=4\left[2^{2 n_{F}}{ }_{2 n}-\frac{1}{5}\left[1-(-4)^{n}\right]\right]
$$

$$
\begin{equation*}
2 \sum_{i=1}^{n} P_{i+1} O_{i}=\frac{(\lambda-4) P_{2 n+1}+\lambda(\lambda-2) P_{2 n}-\lambda^{2}+\lambda+4}{(\lambda+1)(\lambda-3)}+R \quad(\lambda \neq 3) \tag{25}
\end{equation*}
$$

(25s)

$$
\begin{equation*}
2 \sum_{i=1}^{n} O_{i+1} L_{i}=\frac{(\lambda-4) P_{2 n+1}+\lambda(\lambda-2) P_{2 n}-\lambda^{2}+\lambda+4}{(\lambda+1)(\lambda-3)}-R \quad(\lambda \neq 3) \tag{26}
\end{equation*}
$$

(26s)

$$
\sum_{i=1}^{n} O_{i+1} L_{i}=\frac{1}{64} \quad\left[3^{2 n+3}+8 n+24(-3)^{n}-51\right] \quad(\lambda=3)
$$

$$
\begin{equation*}
\sum_{i=1}^{n} 2^{2 i} F_{i+1} L_{i}=\frac{4}{5}\left[2^{2 n} L_{2 n}-3+(-4)^{n}\right] \tag{26'}
\end{equation*}
$$

## REFERENCE

1. cf. N. N. Vorob'ev, Fibonacci Numbers, pp. 15-20.
(Continued from p. 369..)
Let the function $h$ be defined by $h(s, t)=(3 s+4 t, 2 s+3 t)$. Using the method employed above, prove that all solutions in positive integers of Eq. (3) are given by

$$
\begin{equation*}
\left(s_{n}, t_{n}\right)=h^{n}(1,0), \quad n=1,2,3, \cdots \tag{17}
\end{equation*}
$$

To be continued in the February issue of this Quarterly.
[Continued from p. 384.]
according to the principles of a highly sophisticated harmonic system based on the canon of proportion of the Fibonacci Series: the system may yet prove to underlie other disparate aspects of Minoan design. ${ }^{1}$

[^0]
# NOTE ON A PAPER OF PAUL F. BYRD, AND A SOLUTION OF PROBLEM P-3 

## H. W. GOULD*

West Virginia University, Morgantown, W. Va.
Paul F. Byrd [1] has shown how to determine the coefficients $c_{k}$ in the expansion
(1)

$$
\mathrm{f}(\mathrm{x})=\sum_{\mathrm{i}=0}^{\infty} \mathrm{c}_{\mathrm{k}} \phi_{\mathrm{k}+1}(\mathrm{x})
$$

where f is an arbitrary power series

$$
\mathrm{f}(\mathrm{x})=\sum_{\mathrm{n}=0}^{\infty} \mathrm{a}_{\mathrm{n}} \mathrm{x}^{\mathrm{n}}
$$

and the polynomials $\phi_{\mathrm{n}}(\mathrm{x})$ are defined by the recurrence
(2)

$$
\phi_{n+2}(x)-2 x \phi_{n+1}(x)-\phi_{n}(x)=0, \quad \phi_{\theta}(x)=0, \quad \phi_{1}(x)=1
$$

or, equivalently, by the generating function

$$
\begin{equation*}
\left(1-2 x t-t^{2}\right)^{-1}=\sum_{n=0}^{\infty} \phi_{n+1}(x) t^{n} \tag{3}
\end{equation*}
$$

It is our object to point out that the expansion theory involved is a special case of a general treatment given by the author in [3]. In that paper the writer has studied generalized Humbert polynomials defined by the generating function

$$
\begin{equation*}
\left(\mathrm{C}-\mathrm{mxt}+\mathrm{yt}^{m}\right)^{p}=\sum_{\mathrm{n}=0}^{\infty} \mathrm{t}^{\mathrm{n}} \mathrm{P}_{\mathrm{n}}(\mathrm{~m}, \mathrm{x}, \mathrm{y}, \mathrm{p}, \mathrm{C}) \tag{4}
\end{equation*}
$$

*Supported by National Sciences Foundation Research Grant GP-482.
(Received Nov. 1965)

The polynomials $P_{n}$ include the polynomial systems of Louville, Legendre, Tchebycheff, Gegenbauer, Pincherle, Humbert, Kinney, Byrd, and several others. In particular, it is clear that
(5)

$$
\phi_{n+1}(x)=P_{n}(2, x,-1,-1,1)
$$

It is shown in [3] that $P_{n}(m, x, y, p, C)$ satisfies

$$
\begin{equation*}
C n P_{n}-m(n-1-p) x P_{n-1}+(n-m-m p) y P_{n-m}=0, \quad n \geq m \geq 1 \tag{6}
\end{equation*}
$$

of which (2) is a corresponding special case.
It is also shown that

$$
\begin{equation*}
P_{n}(m, x, y, p, C)=\sum_{k=0}^{[n / m]}\binom{p}{k}\binom{p-k}{n-m k} C^{p-n+(m-1) k} y^{k}(-m x)^{n-m k} \tag{7}
\end{equation*}
$$

with the corresponding inversion
(8) $\binom{p}{n}(-m x)^{n}=\sum_{k=0}^{[n / m]}(-1)^{k}\left(\frac{p-n+k}{k}\right) \frac{p+m k-n}{p+k-n} C^{n-k-p} y^{k} P_{n-m k}(m, x, y, p, C)$.

For Byrd's special case (5) these reduce to his relations

$$
\begin{equation*}
\phi_{n+1}(x)=\sum_{k=0}^{[n / 2]}\binom{n-k}{k}(2 x)^{n-2 k} \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
(2 x)^{n}=\sum_{k=0}^{[n / 2]}(-1)^{k}\binom{n}{k} \frac{n-2 k+1}{n-k+1} \phi_{n+1-2 k}(x) \tag{10}
\end{equation*}
$$

This, incidentally, solves his problem P-3 [1, page 29] and in a simpler manner than the complicated induction solution given in [4]. Actually, relations (7) and (8), and hence also (9) and (10), are special cases of the general inversion relations $[2,(6.3),(6.4)]$ found by the writer:

$$
F(n)=\sum_{k=0}^{[n / m]}\binom{p-n+m k}{k} f(n-m k)
$$

if and only if

$$
\begin{equation*}
f(n)=\sum_{k=0}^{[n / m]}(-1)^{k}\binom{p-n+k}{k} \frac{p+m k-n}{p+k-n} F(n-m k) \tag{12}
\end{equation*}
$$

Proof of the reci.procal nature of (11) and (12) in turn depends upon general addition theorems for the binomial coefficients, typified by the relation

$$
\begin{equation*}
\sum_{k=0}^{n}(p+q k) A_{k}(a, b) A_{n-k}(c, b)=\frac{p(a+c)+q a n}{a+c} A_{n}(a+c, b) \tag{13}
\end{equation*}
$$

where

$$
A_{k}(a, b)=\frac{a}{a+b k}\binom{a+b k}{k}_{a, b, c},
$$

This relation actually was given in 1793 by Heinrich August Rothe in his Leipzig dissertation, and it is implied by relations in Lagrange's 1770 memoir on solution of equations. The reader may refer to a series of papers by the writer (since 1960) in the Duke Mathematical Journal, and to papers in the 1956 and 1957 volumes of the American Mathematical Monthly. Though not widely known, these general addition theorems enter into something on the order of several hundred papers in the literature. For example, a special case of (13) when $b=4$ was used by Oakley and Wisner to enumerate classes of Flexagons.

We wish to note that relations (11) and (12) were used by the writer [2] to establish certain results about quasi-orthogonal number sets. The relations in [3] may be looked on as a generalization of the Fibonacci polynomials. Finally we note that Byrd's formula (4.4) for the coefficients $c_{k}$ in (1) above are found in the limiting case from the corresponding expansion (6.9)-(6.10) found by the writer [3] for expressing an arbitrary polynomial as a linear combination of generalized Humbert polynomials. The formulas are too complicated to quote here.

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# ON THE EXISTENCE OF AN INFINITUDE OF COMPOSITE PRIMITIVE DIVISORS OF SECOND-ORDER RECURRING SEQUENCES 

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## 1. INTRODUCTION

Let $\alpha \neq 0, \beta=0,|\alpha|>|\beta|$, be any two complex numbers, such that $\alpha+\beta$ and $\alpha \beta$ are two relatively prime integers. Then the numbers

$$
D_{\mathrm{n}}=\frac{\alpha^{\mathrm{n}}-\beta^{\mathrm{n}}}{\alpha-\beta}=\alpha^{\mathrm{n}-1}+\alpha^{\mathrm{n}-2} \beta+\cdots+\beta^{\mathrm{n}-1}, \quad \mathrm{~S}_{\mathrm{n}}=\alpha^{\mathrm{n}}+\beta^{\mathrm{n}}
$$

are integers, since they are expressed as rational integral symmetric functions of the roots $\alpha, \beta$ of an algebraic equation

$$
z^{2}-(\alpha+\beta) z+\alpha \beta=0
$$

with integral coefficients with leading coefficient unity. One may readily verify that $\left\{D_{n}\right\}$ and $\left\{S_{n}\right\}$ are second-order recurring sequences satisfying the common recursion relation

$$
X_{n}=(\alpha+\beta) X_{n-1}-\alpha \beta X_{n-2}
$$

(Since $D_{0}=0, \quad D_{1}=1 ; S_{0}=2, \quad S_{1}=\alpha+\beta$, the recursion relation again shows that the numbers $D_{n}, S_{n}$ are integers.) One may also easily verify that $D_{2 n}=D_{n} S_{n}$.

Adivisor $>1$ of $D_{n}, n>1$, is said to be primitive (or: characteristic) if it is relatively prime to any $D_{i}$ with $1<i<n$. The greatest primitive divisor of $D_{n}$ is denoted by $D_{n}^{\prime}$. A divisor $>1$ of $S_{n}, n>1$, is said to be primitive (or: characteristic) if it is relatively prime to any $S_{i}$ with $0<i<$ $n$. The greatest primitive divisor of $S_{n}$ is denoted by $S_{n}^{f}$. From $D_{2 n}=$ $D_{n} S_{n}$ one may easily deduce that
(1)

$$
\begin{gathered}
\mathrm{D}_{2 n}=\mathrm{S}_{h} \text {. } \\
\text { (Received Nov. } 1966 \text {--revised 1967) } \\
322
\end{gathered}
$$

For any prime $p$ dividing a certain $D_{i}$ with $i>1$, a (p) denotes the smallestpositive subscript $n$, such that $p \mid D_{n}$. Thus $p$ is a primitive divisor of $D_{a(p)}{ }^{\circ}$

By $F_{n}$ we denote the product
(2)

$$
F_{n}=\prod_{\left.d\right|_{n}} D_{d}^{\mu\left(\frac{n}{d}\right)}
$$

where $\mu$ is the Moebius function.
R. D. Carmichael showed in [1] that for any $n \neq 4,6,12$ there is

$$
\begin{equation*}
D_{n}^{\prime}=F_{n} \tag{3}
\end{equation*}
$$

except when $n=a(p) p \lambda, p$ being a primefactor of $D_{n}, \lambda \geq 1$, in which case

$$
\begin{equation*}
D_{n}^{\prime}=\frac{1}{p} F_{n} \tag{4}
\end{equation*}
$$

He showed furthermore that if $n=a(p) p \lambda, \lambda>1$, then $p$ is the greatest divisor of $n$, except when $p=2$, and $a(p)=3$.

Furthermore Carmichael showed, for $\alpha, \beta$ real, the following inequalities

$$
\alpha^{\phi(\mathrm{n})-2^{\omega(\mathrm{n})-1}}<\mathrm{F}_{\mathrm{n}}<\alpha^{\phi(\mathrm{n})+2^{\omega(\mathrm{n})-1}}
$$

where $\phi$ is Euler's totient function, and $\omega(\mathrm{n})$ is the number of distinct prime factors of n .

The main result achieved by Carmichael is the following
Theorem XXIII. If $\alpha$ and $\beta$ are real and $\mathrm{n} \neq 1,2,6$, then $D_{n}$ contains at least one characteristic factor, except when $n=12, \alpha+\beta= \pm 1, \alpha \beta$ $=-1$.

In the present paper the above Carmichael's results are generalized for any two complex numbers $\alpha \neq 0, \beta \neq 0,|\alpha|>|\beta|$, such that $\alpha+\beta$ and $\alpha \beta$ are two relatively prime integers. (However, the exact value of $n$ beginning with which any $D_{n}$ contains at least one characteristic factor, is not calculated
[Dec.
here.) Furthermore, starting from (2), we deduce an asymptotic formula (6) for $F_{n}$ which is stronger than the inequalities given by Carmichael. Finally, the method of proof used here is slightly simpler than the one used by Carmichael. The main results proved here are the existence of an infinitude of composite $D_{n}^{\prime}$ for any $\alpha, \beta$; of composite $D_{2_{n}}$ for $\alpha \beta \neq \square$; and of composite $D_{2 n+1}$ for $(\alpha-\beta)^{2} \neq \pm \square$, or $(\alpha-\beta)^{2}=\square$ and $\alpha \beta \neq-\square$.

## 2. ASYMPTOTIC FORMULA FOR $D_{n}^{\prime}$

By (2)
$\begin{aligned} \log F_{n}=\sum_{d \mid n} \mu\left(\frac{n}{d}\right) \log D_{d}= & \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \log \frac{\alpha^{d}-\beta^{d}}{\alpha-\beta}=\log \alpha \sum_{d \mid n} \mu\left(\frac{n}{d}\right) d \\ & +\sum_{d \mid n} \mu\left(\frac{n}{d}\right) \log \left\{1-\left(\frac{\beta}{\alpha}\right)^{d}\right\}-\log (\alpha-\beta) \sum_{d \mid n} \mu\left(\frac{n}{d}\right) .\end{aligned}$

Noting that

$$
\sum_{\mathrm{d} \mid \mathrm{n}} \mu\left(\frac{\mathrm{n}}{\mathrm{~d}}\right) \mathrm{d}=\phi(\mathrm{n})
$$

and

$$
\sum_{\mathrm{d} \mid \mathrm{n}} \mu\left(\frac{\mathrm{n}}{\mathrm{~d}}\right)=0
$$

for any $n>1$, we get
(5) $\quad \log \mathrm{F}_{\mathrm{n}}=\log \alpha \cdot \phi(\mathrm{n})+\sum_{\mathrm{d} \mid \mathrm{n}} \mu\left(\frac{\mathrm{n}}{\mathrm{d}}\right) \log \left\{1-\left(\frac{\beta}{\alpha}\right)^{\mathrm{d}}\right\}$, for any $\mathrm{n}>1$.

Let us evaluate

$$
\sum_{d \mid n} \mu\left(\frac{n}{d}\right) \log \left\{1-\left(\frac{\beta}{\alpha}\right)^{d}\right\}
$$

Note that for any $0<q<1$ there exists a positive constant $A$, such that, for any complex z , for which $|\mathrm{z}| \leq \mathrm{q}$, there is

$$
|\log (1+z)| \leq A|z|
$$

where by $\log (1+z)$ the principal value of $\log$ is understood. Indeed,

$$
\frac{\log (1+z)}{z}=1-\frac{z}{2}+\frac{z^{2}}{3}-\cdots
$$

is an analytic function in the circle $|z-1|<q<1$, hence it is bounded there. Now, putting $\mathrm{q}=\left|\frac{\beta}{\alpha}\right|$, we have, for any $\mathrm{d}>1,\left|\frac{\beta}{\alpha}\right|^{\mathrm{d}} \leq \mathrm{q}$. Hence

$$
\begin{aligned}
\left|\sum_{d \mid n} \mu\left(\frac{n}{d}\right) \log \left\{1-\left(\frac{\beta}{\alpha}\right)^{\mathrm{d}}\right\}\right| & \leq \sum_{\mathrm{d} \mid \mathrm{n}}\left|\log \left\{1-\left(\frac{\beta}{\alpha}\right)^{\mathrm{d}}\right\}\right|<\sum_{\mathrm{d}=1}^{\infty}\left|\log \left\{1-\left(\frac{\beta}{\alpha}\right)^{\mathrm{d}}\right\}\right| \\
& \left.<A \sum_{d=1}^{\infty}\left|\frac{\beta}{\alpha}\right|^{\mathrm{d}}=A\left|\frac{\beta}{\alpha}\right| \frac{1}{1-\left|\frac{\beta}{\alpha}\right|}=A \right\rvert\, \frac{\beta \mid}{|\alpha|-|\beta|}=B
\end{aligned}
$$

where $B$ is a positive constant.
Hence, by (5) it follows that
(6)

$$
\log \mathrm{F}_{\mathrm{n}}=\log \alpha \cdot \phi(\mathrm{n})+0(1)
$$

Now, by (3), (4), we have the following
Theorem 1. There is

$$
\begin{equation*}
\log D_{n}^{\prime}=\log \alpha \cdot \phi(\mathrm{n})+0(1) \tag{7}
\end{equation*}
$$

[Dec.
exceptwhen $n=a(p) p^{\lambda}, \lambda \geq 1, \quad p$ being a prime factor of $D_{n}$, in which case it is
(8)

$$
\log D_{n}^{\prime}=\log \alpha \cdot \phi(n)-\log p+0(1)
$$

Now, by assumption, $\alpha \beta$ is an integer, and $|\alpha|>|\beta|$, therefore

$$
|\alpha|^{2}>|\alpha| \cdot|\beta|=|\alpha \beta| \geq 1
$$

hence

$$
|\alpha|>1, \quad|\log \alpha| \geq \log |\alpha|>0
$$

By a theorem in [2], p. 114, there exists a positive constant C, such that

$$
\phi(n)>\frac{C \cdot n}{\log \log n} \text { for } n>3
$$

On the other hand $p \mid n$, hence $\log p \leq \log n$. Hence, by Theorem 1,
(9) $\quad \log D_{n}^{\prime}>|\log \alpha| \cdot \phi(n)-\log p-B>\log |\alpha| \frac{C \cdot n}{\log \log n}-\log n-B \underset{n \rightarrow \infty}{\longrightarrow}$,
which means that:

Theorem 2. Beginning with a certain positive $n, D_{n}$ has at least one primitive factor.

Remark. The error term $0(1)$ in (7) cannot be refined, since if $n$ is a prime, then

$$
\sum_{d \mid n} \mu\left(\frac{\mathrm{n}}{\mathrm{~d}}\right) \log \left\{1-\left(\frac{\beta}{\alpha}\right)^{\mathrm{d}}\right\}=-\log \left\{1-\frac{\beta}{\alpha}\right\}+\log \left\{1-\left(\frac{\beta}{\alpha}\right)^{\mathrm{n}}\right\} \underset{\mathrm{n} \rightarrow \infty}{\rightarrow-\log \left\{1-\frac{\beta}{\alpha}\right\} \neq \mathrm{m}}
$$

Theorem 3.

$$
\sum_{n=1}^{\infty} \frac{1}{D_{n}^{\prime}}
$$

converges.
Proof. From (9) it follows that there is a positive constant $D$ such that, for all $\mathrm{n} \geqq 1$,

$$
D_{n}^{\prime} \geq \frac{|\alpha|^{\frac{C \cdot n}{\log \log n}}}{e^{B} \cdot n} \geq D \cdot n^{2}
$$

Hence

$$
\sum_{n=1}^{\infty} \frac{1}{D_{n}^{\prime}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{2}}<\infty
$$

## 3. MAIN RESULTS

Lemma 1. Be $N$ the sequence of natural numbers, $S$ a subsequence of $N$, and A a reduced arithmetic progression. Then, an infinitude of $D_{n}^{\prime}$ is composite for
I)
$\mathrm{n} \in \mathrm{S}$
or
II) $n \in N-S$
according as
I)
any
or
II) no
prime member of $A$ is a factor of a certain $D_{n}^{p}, n \in S$.
Proof. I) Suppose any prime member of $A$ is a factor of a certain $D_{n}^{\prime}$. $n \in S$, and that there is a positive integer $n_{0}$ such that any $D_{n}^{\prime}$, where $n \in$ $S, n>n_{0}$, is a prime. Let $q$ be the greatest prime factor of $D_{n}^{\prime}, n \leq n_{0}$.

Then, by Theor $\in \mathrm{m}$ 3, and noting that

$$
\sum_{p \in A} \frac{1}{p}=\infty
$$

where $p$ denotes a prime number, we have

$$
\infty>\sum_{n \in N} \frac{1}{D_{n}^{\prime}} \geq \sum_{n \in S} \frac{1}{D_{n}^{\prime}} \geq \sum_{\substack{p \in A \\ p>q}} \frac{1}{p}=\infty,
$$

whence $\infty>\infty$, which is absurd. Thus, I) is proved.
II) Suppose no prime member of $A$ is a factor of a certain $D_{n}^{\prime}, n \in S$. Then, noting that any prime $\mathrm{p} \mid 2(\alpha-\beta)^{2} \alpha \beta$ is a factor of a certain $D_{n}^{\prime}$ ([1], p. 45 , Theorem XII), any prime member of A not a factor of $2(\alpha-\beta)^{2} \alpha \beta$ ia a factor of a certain $D_{n}^{\prime}, n \in N-S$, and II) follows as above.

Theorem 4. There is an infinitude of composite $D_{n}^{\prime}$.
Proof. The theorem is an immediate consequence of Lemma 1, noting that any prime $p \mid 2(\alpha-\beta)^{2} \alpha \beta$ is a factor of a certain $D_{n}^{\prime}$.

Lemma 2. If b is an integer, and $\mathrm{b} \neq \square$, then there exists an odd prime $p$, such that $\left(\frac{b}{p}\right)=-1$, where $\left(\frac{b}{p}\right)$ is Legendre's symbol. In particular,
I) If $b= \pm m^{2} p_{1}, \cdots, p_{r}$, where $r \geq 1$ and $p_{1}, \cdots, p_{r}$ are distinct primes, then there exists an integer $u=1(\bmod 4)$, where $\left(u, 4 p_{1}, \cdots, p_{r}\right)=$ 1 , such that, for any prime $p=u\left(\bmod 4 p_{1}, \cdots, p_{r}\right)$, it is

$$
\left(\frac{ \pm \mathrm{b}}{\mathrm{p}}\right)=-1 .
$$

II) If $b=-m^{2}$, then for any prime $p=-1(\bmod 4)$ it is

$$
\left(\frac{\mathrm{b}}{\mathrm{p}}\right)=-1
$$

Proof. [2], p. 75.

Lemma 3. Let $p$ be an odd prime. If $p \mid a x^{2}+b y^{2}$ for some integers $a, b, x, y$, and $p \nmid(x, y)$, then

$$
\left(\frac{-a b}{p}\right)=1
$$

Proof. Since $p \nmid(x, y), p$ cannot divide both $x$ and $y$. Thus, without loss of generality, we may assume that $p \nmid y$. Then there exists an integer $z$, such that $y z=1(\bmod p)$. Hence, from $a x^{2}+b y^{2}=0(\bmod p)$ it follows that

$$
(a x z)^{2}=-a b(\bmod p)
$$

whence

$$
\left(\frac{-a b}{p}\right)=1
$$

Lemmas 2, 3 imply the following:
Lemma 4. I) If $b= \pm m^{2} p_{1}, \cdots, p_{r}$, where $r \geq 1$ and $p_{1}, \cdots, p_{r}$ are distinct primes, then there exists an integer $u=1(\bmod 4)$, where $\left(u, 4 p_{1}\right.$, $\left.\cdots, p_{r}\right)=1$, such that, for any prime $p=u\left(\bmod 4 p_{i}, \cdots, p_{r}\right)$, it is $p \nmid x^{2}$ $+b^{2}$ for any integers $x, y$, such that $p \|(x, y)$.
II) If $b=m^{2}$ and $p \nmid(x, y)$, then $p \nmid x^{2}+b y^{2}$ for any prime $p=-1$ $(\bmod 4)$.

Theorem 5. If $\alpha \beta \neq \square$, then there is an infinitude of composite $D_{2 n}$. Proof. One may readily verify that

$$
\mathrm{D}_{2 \mathrm{n}+1}=\mathrm{D}_{\mathrm{n}+1}^{2}-\alpha \beta \mathrm{D}_{\mathrm{n}}^{2}
$$

On the other hand, $\left(D_{n+1}, D_{n}\right)=1([1]$, p. 38, Corollary). Hence, putting in Lemma 4:

$$
\mathrm{b}=-\alpha \beta, \quad \mathrm{x}=\mathrm{D}_{\mathrm{n}+1}, \quad \mathrm{y}=\mathrm{D}_{\mathrm{n}}
$$

and noting that, according to the assumption, $b=-\alpha \beta \neq-\square$, there exists a reduced arithmetic progression $A$, no prime member of which divides $D_{2 n}+r$.
[Dec.
Hence, no prime member of $A$ is a factor of $D_{2 n+1}$. The theorem follows by Lemma 1, II).

Theorem 6. If
I)

$$
(\alpha-\beta)^{2} \neq \pm \square
$$

or
II) $(\alpha-\beta)^{2}=\square$ and $\alpha \beta \neq-\square$,
then there is an infinitude of composite $D_{2 n+1}$.
Proof. One may readily verify that

$$
\begin{equation*}
\mathrm{S}_{\mathrm{n}}^{2}=(\alpha-\beta)^{2} \mathrm{D}_{\mathrm{n}}^{2}+4(\alpha \beta)^{\mathrm{n}} \tag{9}
\end{equation*}
$$

I) Suppose that $(\alpha-\beta)^{2} \neq \pm$. Then $(\alpha-\beta)^{2}= \pm \mathrm{m}^{2} \mathrm{p}_{1}, \cdots, \mathrm{p}_{\mathrm{r}}$, where $r \geq 1$ and $p_{1}, \cdots, p_{r}$ are distinct primes. Then, by Lemma 2, I), there is an integer $u$, such that

$$
\begin{equation*}
u=1 \quad(\bmod 4) \tag{10}
\end{equation*}
$$

$$
\begin{equation*}
\left(u, 4 p_{1}, \cdots, p_{r}\right)=1 \tag{11}
\end{equation*}
$$

$$
\begin{equation*}
\mathrm{p}=\mathrm{u}\left(\bmod 4 \mathrm{p}_{1}, \cdots, p_{r}\right) \tag{12}
\end{equation*}
$$

implies

$$
\left(\frac{-(\alpha-\beta)^{2}}{p}\right)=-1
$$

for any prime $p$.
Consider the pair of congruences

$$
\left\{\begin{array}{l}
\mathrm{x}=\mathrm{u}\left(\bmod 4 \mathrm{p}_{1}, \cdots, \mathrm{p}_{\mathrm{r}}\right)  \tag{13}\\
\mathrm{x}=1(\bmod 4 \alpha \beta)
\end{array}\right.
$$

From the identity

$$
(\alpha+\beta)^{2}-4 \alpha \beta=(\alpha-\beta)^{2}
$$

and from the assumption

$$
(\alpha+\beta, \alpha \beta)=1,
$$

it follows

$$
1=\left(\alpha \beta, p_{1}, \cdots, p_{r}\right) \leq\left(\alpha \beta, \pm m^{2} p_{1}, \cdots, p_{r}\right)=\left(\alpha \beta,(\alpha-\beta)^{2}\right)=1
$$

Hence

$$
\left(4 \mathrm{p}_{1}, \cdots, \mathrm{p}_{\mathrm{r}}, 4 \alpha \beta\right)=4\left(\mathrm{p}_{1}, \cdots, \mathrm{p}_{\mathrm{r}}, \alpha \beta\right)=4
$$

But, by (10), $4 \mid u-1$, hence (13) has a solution $u^{\prime}$, i. e.,

$$
\begin{equation*}
u^{\prime}=u\left(\bmod 4 p_{1}, \cdots, p_{r}\right), u^{\prime}=1(\bmod 4 \alpha \beta) \tag{14}
\end{equation*}
$$

Let p be a prime satisfying $\mathrm{p}=1(\bmod 4 \alpha \beta)$. If $\alpha \beta$ is odd, then, according to the properties of the Jacobi symbol

$$
\left(\frac{\alpha \beta}{\mathrm{p}}\right)=\left(\frac{\mathrm{p}}{\alpha \beta}\right)=\left(\frac{1}{\alpha \beta}\right)=1 .
$$

If $\alpha \beta$ is even, then $\mathrm{p}=1(\bmod 8)$, and $\alpha \beta=2^{\mathrm{k}}$, where $\mathrm{k} \geq 1$ and $2 \ell \mathrm{t}$. Then

$$
\left(\frac{\alpha \beta}{\mathrm{p}}\right)=\left(\frac{2}{\mathrm{p}}\right)^{\mathrm{k}}\left(\frac{\mathrm{t}}{\mathrm{p}}\right)=\left(\frac{\mathrm{p}}{\mathrm{t}}\right)=\left(\frac{1}{\mathrm{t}}\right)=1 .
$$

in both cases

$$
\left(\frac{\alpha \beta}{p}\right)=1 .
$$

Combining the last result with (11), (12) and (14), we conclude

$$
\begin{equation*}
\left(u^{\prime}, 4 p_{1}, \cdots, p_{r}\right)=1 \tag{15}
\end{equation*}
$$

(16) If $p=u^{\prime}\left(\bmod 4 p_{1}, \cdots, p r\right)$, then

$$
\left(\frac{-(\alpha-\beta)^{2}}{\mathrm{p}}\right)=\left(\frac{-\alpha \beta(\alpha-\beta)^{2}}{\mathrm{p}}\right)=-1
$$

for any prime $p$.
We shall now show that if
(17)

$$
p=u^{\prime} \quad\left(\bmod 4 p_{1}, \cdots, p_{r}\right)
$$

then $p \nmid D_{2 n}$. Indeed, if $p \mid D_{2 n}$, then, by (1), $p \mid S_{n}^{p}$, hence $p \mid S_{n}^{2}$. Hence, by (9),

$$
\mathrm{p} \mid(\alpha-\beta)^{2} \mathrm{D}_{\mathrm{n}}^{2}+4(\alpha \beta)^{\mathrm{n}}
$$

Putting in Lemma 3:

$$
\mathrm{x}=\mathrm{D}_{\mathrm{n}}, \quad \mathrm{y}=2, \quad \mathrm{a}=(\alpha-\beta)^{2}, \quad \mathrm{~b}=(\alpha \beta)^{\mathrm{n}}
$$

we have

$$
\left(\frac{-(\alpha \beta)^{\mathrm{n}}(\alpha-\beta)^{2}}{\mathrm{p}}\right)=1
$$

If $n$ is even, then

$$
1=\left(\frac{-(\alpha \beta)^{\mathrm{n}}(\alpha-\beta)^{2}}{\mathrm{p}}\right)=\left(\frac{-(\alpha-\beta)^{2}}{\mathrm{p}}\right)
$$

If n is odd, then

$$
1=\left(\frac{-(\alpha \beta)^{\mathrm{n}}(\alpha-\beta)^{2}}{\mathrm{p}}\right)=\left(\frac{-\alpha \beta(\alpha-\beta)^{2}}{\mathrm{p}}\right)
$$

Both cases contradict (16). The theorem now follows from (17), (15), and Lemma 1, II).
II) Suppose $(\alpha-\beta)^{2}=m^{2}$, where $m$ is an integer and $\alpha \beta \neq-\square$. Then (9) becomes

$$
\begin{equation*}
\mathrm{S}_{\mathrm{n}}^{2}=\left(\mathrm{mD} \mathrm{D}_{\mathrm{n}}\right)^{2}+4(\alpha \beta)^{\mathrm{n}} \tag{18}
\end{equation*}
$$

This formula implies, by Lemma 3, if
(19)

$$
\mathrm{p} \mid \mathrm{D}_{\mathrm{s}_{\mathrm{n}}}
$$

(and hence $\mathrm{p} \mid \mathrm{S}_{\mathrm{h}}^{2}$ ), then

$$
\left(\frac{-(\alpha \beta)^{\mathrm{n}}}{\mathrm{p}}\right)=1
$$

for any odd prime p. Consider now the three following cases.
Case 1: $\alpha \beta=\mathrm{n}^{2} \cdot 2^{\mathrm{k}}$, where $\mathrm{k} \geq 0$. Then, if $\mathrm{p}=-1(\bmod 8)$, then

$$
\left(\frac{-(\alpha \beta)^{\mathrm{n}}}{\mathrm{p}}\right)=\left(\frac{-1}{\mathrm{p}}\right)\left(\frac{2}{\mathrm{p}}\right)^{\mathrm{k}}=-1
$$

and hence, by (19), $p \nmid D_{2 n}$.
Case 2: $\alpha \beta=n^{2} \cdot 2^{k} \cdot q_{1}, \cdots, q_{r}$, where $k \geq 0, r \geq 1, q_{1}, \cdots, q_{r}$ are distinct odd primes, and $t=q_{1}, \cdots, q_{r}=1(\bmod 4)$.

Consider the pair of congruences
(20)

$$
\left\{\begin{array}{l}
x=-1(\bmod 8) \\
x=1(\bmod t)
\end{array}\right.
$$

Since ( $\mathrm{t}, 8$ ) $=1$, (20) has a solution $u_{0}$ This solution satisfies

$$
\begin{equation*}
(\mathrm{u}, 8 \mathrm{t})=1 \tag{21}
\end{equation*}
$$

If $\mathrm{p}-\mathrm{u}(\bmod 8 \mathrm{t})$ is a prime, then

$$
\begin{equation*}
\left(\frac{-(\alpha \beta)^{\mathrm{n}}}{\mathrm{p}}\right)=\left(\frac{-1}{\mathrm{p}}\right)\left(\frac{2}{\mathrm{p}}\right)^{\mathrm{kn}}\left(\frac{\mathrm{t}}{\mathrm{p}}\right)^{\mathrm{n}}=-\left(\frac{1}{\mathrm{t}}\right)=-1 \tag{22}
\end{equation*}
$$

and hence, by (19), $\mathrm{p} / \mathrm{D}_{2 \mathrm{ln}}$.
Care 3: Everything as in Case 2, except that $t=-1(\bmod 4)$.
Choose a quadratic nonresidue c modulo $q_{1}$, i. e.,

$$
\left(\frac{c}{q_{1}}\right)=-1
$$

Consider the system of congruences
(23)

$$
\left\{\begin{array}{l}
\mathrm{x}=-1(\bmod 8) \\
\mathrm{x}=\mathrm{c} \\
\left.\mathrm{mod} \mathrm{q}_{1}\right) \\
\mathrm{x}=1\left(\bmod \mathrm{q}_{2}\right) \\
\dot{\mathrm{x}}=1 \dot{\left(\bmod \mathrm{q}_{\mathrm{r}}\right)}
\end{array}\right.
$$

If $r \geq 2$, or the system
(24)

$$
\left\{\begin{array}{l}
\mathrm{x}=-1(\bmod 8) \\
\mathrm{x}=\mathrm{c}\left(\bmod \mathrm{q}_{1}\right)
\end{array}\right.
$$

if $r=1$. Since $q_{1}, \cdots, q_{r}$ are distinct odd primes, (23) and (24) have a solution v. v satisfies:
(25)

$$
(v, \quad 8 t)=1
$$

If $\mathrm{p}=\mathrm{v}(\bmod 8 \mathrm{t})$ is a prime, then
(26)
[Cont. on p. 406.]

$$
\begin{aligned}
\left(\frac{-(\alpha \beta)^{n}}{p}\right) & =\left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)^{k n}\left(\frac{t}{p}\right)^{n}=(-1)(+1)\left[-\left(\frac{p}{t}\right)\right]^{n} \\
& =-\left(-\left(\frac{p}{q_{1}}\right)\left(\frac{p}{q_{2}}\right) \cdots\left(\frac{p}{q_{r}}\right)\right)^{n} \\
& =-\left(-\left(\frac{c}{q_{1}}\right)\left(\frac{1}{q_{2}}\right) \cdots\left(\frac{1}{q_{r}}\right)\right)^{n}=1
\end{aligned}
$$

# BASES FOR INTERVALS OF REAL NUMBERS 

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1. INTRODUCTION.

In this paper we discuss the problem of representing uniquely each real number in the interval $(0, c]$, where $c$ is any positive real number, as an infinite series of terms selected from a sequence $\left(b_{n}\right)$ of real numbers. We choose an integer $k \geq 1$ and require that any two terms of ( $b_{n}$ ) whose suffices differ by less than $k$ shall not both be used in the representation of any given real number. The precise definitions and results are given in the next section.

In an earlier paper [2] we discussed an analogous problem of representing the integers in arbitrary infinite intervals.

## 2. STATEMENT OF RESULTS

Throughout this paper $k \geq 1$ is an integer. Also the subscript of the initial term of any sequence is the number 1 ; e. $\mathrm{g}_{0}, \quad\left(\mathrm{c}_{\mathrm{n}}\right)=\left(\mathrm{c}_{1}, \mathrm{c}_{2}, \cdots\right)$.

In order to prove our main result, which is theorem 2 , we need a result which we give in a slightly generalized form as Theorem 1. Let ( $c_{n}$ ) be a sequence of positive real numbers which obey the linear recurrence relation

$$
\begin{equation*}
a_{1} c_{n+k}+a_{2} c_{n+k-1}+\cdots+a_{k} c_{n+1}-c_{n}=0 \tag{2.1}
\end{equation*}
$$

for $n \geq 1$, where $a_{1}, \cdots, a_{k}$ are non-negative real numbers independent of $n$, and $a_{1} \geq 0$. The auxiliary polynomial $g(z)$ of this recurrence relation is given by

$$
g(z)=a_{1} z^{k}+a_{2} z^{k-1}+\cdots+a_{k} z-1
$$

It is clear that $g(z)$ has just one positive real root $\rho$, and that this root is simple.
(Received March 1965)

Theorem 1. If the sequence $\left(c_{n}\right)$ is strictly decreasing, and $\rho$ is smaller than the modulus of any other root of $g(z)$, then $\rho<1$ and $c_{n}=A \rho^{n}$ for $\mathrm{n} \geq 1$, where A is a positive real constant.

We now define a k-series base for the interval of real numbers ( $0, \mathrm{c}]$, where $c$ is any positive real constant. This is analogous to the concept of an $(h, k)$ base for the set of integers as an interval; this concept was given in the earlier paper [2].

Definition. A sequence $\left(b_{n}\right)$ of real numbers is a $k$-series base for ( $0, \mathrm{c}$ ] if each real number $\mathrm{r} \in(0, \mathrm{c}]$ has a unique representation

$$
\begin{equation*}
\mathbf{r}=\mathrm{b}_{\mathrm{i}_{1}}+\mathrm{b}_{\mathrm{i}_{2}}+\cdots, \tag{2.2}
\end{equation*}
$$

where

$$
\mathbf{i}_{\nu+1} \geq i_{\nu}+k
$$

for $\nu \geq 1$, and further, every such series converges to a sum $r \in(0, c]$.
It is clear that the polynomial $f(z)=z^{k}+z-1$ has just one positive real root $\theta$, that $\theta$ is a simple root, and that $\theta<1$. Let $R$ be a real number. We now enunciate our main result.

Theorem 2. Let $\left(b_{n}\right)$ be a sequence of real numbers such that $b_{n} \geq$ $b_{n+1}>0$ for $n \geq 1$. Then $\left(b_{n}\right)$ is a k-series base for $\left(0, \theta^{R}\right]$ if and only if

$$
b_{n}=\theta^{R+n}
$$

for $\mathrm{n} \geq 1$.
It is not true that all k-series bases are decreasing. For instance, when $k=2$, the $\operatorname{series}\left(1,2, \theta, \theta^{2}, \cdots\right)$ is a $k$-series base for $(0,2+\theta]$. However, A. Oppenheim has shown that if the sequence $\left(b_{n}\right)$ is a k-series base for $(0, c]$ for some $c>0$, and if $N$ is an integer such that $b_{n} \geq b_{n+1}$ $>0$ for $n \geq N$ then $b_{n}=A \theta^{n}$ for $n>k$, where $A$ is some positive constant. It is not known if all $k$-series bases (for $k \geq 2$ ) are ultimately decreasing.

It follows from Theorem 2 that the sequence

$$
\begin{equation*}
\left(\theta^{-\mathrm{N}+1}, \theta^{-\mathrm{N}+2}, \cdots, \theta^{-1}, \theta^{0}, \theta^{1}, \cdots\right) \tag{2.3}
\end{equation*}
$$

is a $k$-series base for $\left(0, \theta^{-N}\right.$ ], where $N$ is any positive integer. Hence if $r$ is any positive real number, and $L$ and $M$ are positive integers such that both $\theta^{-L}>r$ and $\theta^{-M}>r$, then the $k$-series representation of $r$ interms of the sequence (2.3) with $N=L$ is the same as with $N=M$. For shortness, therefore, we can refer to this as the ' $\theta$-representation' of $r$. Then, if an initial minus sign is used in representing negative numbers, we can give a unique ' $\theta$-representation' for any real number. A ' $\theta$-representation' of real numbers is akin to decimal representation, but is much more closely related to binary representation since when $k=1$ the ' $\theta$-representation' and the binary representation of the same real number are the same (for when $k=1$, $\theta=\frac{1}{2}$ ).

A further observation is that any sum $T$ of a finite number of terms of the sequence $\left(\theta^{R+1}, \theta^{R+2}, \ldots\right)$, where $R$ is any real number, in the form

$$
\mathrm{T}=\theta^{\mathbf{i}_{1}}+\theta^{\mathbf{i}_{2}}+\cdots+\theta^{\mathbf{i}_{\alpha-1}}+\theta^{\mathbf{i}_{\alpha}-1}
$$

where $i_{\nu+1} \geq i_{\nu}+k$ for $1 \leq \nu<\alpha$, can be written in the form

$$
\mathrm{T}=\sum_{\nu=1}^{\infty} \theta^{\mathrm{i}_{\nu}}
$$

where $\mathrm{i}_{\nu+1} \geq \mathrm{i}_{\nu}+\mathrm{k}$ for $\nu \geq 1$, simply by putting

$$
\begin{equation*}
\theta^{\mathrm{i}} \alpha^{-1}=\sum_{\nu=1}^{\infty} \theta^{\mathrm{i}_{\alpha}+\nu \mathrm{k}} \tag{2.4}
\end{equation*}
$$

(The relation (2.4) follows from the relations

$$
\sum_{\nu=0}^{\infty} \theta^{\nu}<\infty
$$

and

$$
\theta^{\dot{i} \alpha-1+n k}=\theta^{\dot{i} \alpha^{+}+n k}+\theta^{\dot{i} \alpha^{-1}+(n+1) k}
$$

for $\mathrm{n} \geq 0$, both of which are very easily proved.) This fact is analogous to the decimal equation $1=0.9$ or the binary equation $1=0 . \dot{1}$.

## 3. PROOF OF THEOREM 1

We first prove Lemma 1, an equivalent form of which occurred originally in [3] and was also quoted in [4].

Lemma 1. If $\alpha_{1}, \alpha_{2}, \cdots, \alpha_{p}$ are real numbers then there exists an increasing sequence $\left(n_{j}\right)$ of positive integers such that
$\exp \left(\operatorname{in}_{j} \alpha_{1}\right) \rightarrow 1, \quad \exp \left(\operatorname{in}_{j} \alpha_{2}\right) \rightarrow 1, \quad \cdots \exp \left(\operatorname{in}_{j} \alpha_{p}\right) \rightarrow 1$ as $j \rightarrow \infty$.
Proof. For x a real number, let $\bar{x}$ be the number differing from x by a multiple of $2 \pi$ such that $-\pi<\overline{\mathrm{x}} \leq \pi$. We prove the lemma by showing that if we are given any positive real number $\epsilon>0$, and any positive integer $N$, then we find an integer $n \geq N$ such that

$$
\left|\overline{\mathrm{n} \alpha_{\mathrm{s}}}\right|<\epsilon \text { for } \quad 1 \leq \mathrm{s} \leq \mathrm{p}
$$

Let $M$ be the region in p-dimensional space in which each coordinate ranges from $-\pi$ to $\pi$. Let the range of each coordinate be divided into $m$ equal parts, where

$$
\mathrm{m}>\frac{2 \pi}{\epsilon}
$$

is an integer. Then $M$ is divided into $m^{p}$ equal parts. Consider now the $m^{p}+1$ points

$$
\overline{\left(\overline{\mathrm{N} \nu \alpha_{1}}, \overline{\mathrm{~N} \nu \alpha_{2}}, \cdots, \overline{\mathrm{~N} \nu \alpha_{\mathrm{p}}}\right) \text { for } 1 \leq \nu \leq \mathrm{m}^{\mathrm{p}} . . . . . .}
$$

One part of M must contain two of these points; let the corresponding indices be $\nu_{1}$ and $\nu_{2}$. Then clearly

$$
\left\lvert\, \overline{\mathrm{N}}\left(\overline{\left.\nu_{1}-\nu_{2}\right) \alpha_{\mathrm{s}}} \left\lvert\,<\frac{2 \pi}{\mathrm{~m}}<\varepsilon\right.\right.\right.
$$

for $1 \leq s \leq p$, and

$$
\left|\nu_{1}-\nu_{2}\right| \geq 1
$$

We put

$$
\mathrm{n}=\mathrm{N}\left|\nu_{1}-\nu_{2}\right|
$$

this proves Lemma 1.
Since ( $c_{n}$ ) obeys the recurrence relation (2.1), $c_{n}$ can be expressed in the form

$$
\begin{equation*}
c_{n}=\sum_{s=1}^{u}\left(\sum_{t=0}^{v_{s}} n^{t_{B}}\right) \xi_{s}^{n} \quad \text { for } n \geq 1 \tag{3.1}
\end{equation*}
$$

where the numbers $\xi_{S}$ are the distinct roots of $g(z)$, the number $\left(v_{S}+1\right)$ is the multiplicity of the root $\xi_{\mathrm{s}}$ for $1 \leq \mathrm{s} \leq \mathrm{u}$, and the numbers $\mathrm{B}_{\mathrm{st}}$ are suitable complex constants. Let $=\xi_{S^{\circ}}$. We consider two cases.

Case 1. $B_{s t}=0$ when $(s, t) \neq\left(s^{\prime}, 0\right)$. Then by $(3.1)$,

$$
\begin{equation*}
\mathrm{c}_{\mathrm{n}}=\mathrm{B}_{\mathrm{S}^{\prime} 0} \rho^{\mathrm{n}} \quad \text { for } \quad \mathrm{n} \geq 1 \tag{3.2}
\end{equation*}
$$

Since $c_{1}, \quad \rho>0$ it follows by (3.2) that

$$
B_{S^{p} 0}=\frac{c_{1}}{\rho}
$$

a positive constant. Since ( $c_{n}$ ) is a decreasing sequence, $\rho<1$. Hence the theorem is true in this case.

Case 2. $B_{s t} \neq 0$ for at least one pair ( $\left.s, t\right) \neq\left(s^{\prime}, 0\right)$. This implies that $k \geq 2$. We shall deduce a contradiction. By rearranging the terms in (3.1) if necessary, there is a number $p$ where $1 \leq p \leq u$, and a number $q$, where $0 \leq q \leq \min \left(v_{1}, v_{2}, \cdots, v_{p}\right)$ such that
(i) For $1 \leq s \leq p, \quad B_{s q} \neq 0,\left|\xi_{s}\right|=\left|\xi_{1}\right|$ and $B_{s t}=0$ for $q \leq t \leq$ $\mathrm{v}_{\mathrm{S}}$,
(ii) for $p<s \leq u$, if $\left|\xi_{S}\right|=\left|\xi_{1}\right|$ then $B_{s t}=0$ for $q<t \leq v_{s}$, and if $\left|\xi_{\mathrm{s}}\right|^{>}\left|\xi_{1}\right|$ then $\mathrm{B}_{\mathrm{st}}=0$ for $0 \leq \mathrm{t} \leq \mathrm{v}_{\mathrm{s}}$.

Then by (3.1)

$$
\begin{equation*}
c_{n}=\sum_{s=1}^{p} B_{s q} n^{q} \xi_{s}^{n}+R \tag{3.3}
\end{equation*}
$$

where $R$ is the sum of a finite number of non-zero terms of the form $\mathrm{Cn}^{\boldsymbol{\gamma}} \boldsymbol{\xi}_{\boldsymbol{\delta}}{ }^{\mathrm{n}}$, where $C$ is a complex constant and either $\left|\xi_{\delta}\right|=\left|\xi_{1}\right|$ and $\gamma<q$, or $\left|\xi_{\delta}\right|<$ $\left|\xi_{1}\right|$ Our assumption implies that either

$$
\begin{equation*}
\left|\xi_{1}\right|>\rho \quad \text { or } \quad q>0 \tag{3.4}
\end{equation*}
$$

If $\left|\xi_{\delta}\right|<\left|\xi_{1}\right|$ then $n^{\gamma}\left|\xi_{\delta}\right|^{n} /\left|\xi_{1}\right|^{n} \rightarrow 0$ as $\mathrm{n} \rightarrow \infty$. Hence

$$
\begin{equation*}
\mathrm{R} /\left|\xi_{1}\right|^{\mathrm{n}} \mathrm{n}^{\mathrm{q}} \rightarrow 0 \text { as } \mathrm{n} \rightarrow \infty \tag{3.5}
\end{equation*}
$$

For $1 \leq \mathrm{s} \leq \mathrm{p}$, let $\xi_{\mathrm{S}}=\mathrm{r}_{\mathrm{S}} \exp \left(\mathrm{i} \alpha_{\mathrm{S}}\right)$, where $\mathrm{r}_{\mathrm{s}}$ and $\alpha_{\mathrm{s}}$ are the modulus and argument of $\xi_{\mathrm{S}}$ respectively. Then by (3.5) and (3.4) respectively,

$$
\begin{equation*}
R / r_{1}^{n} n^{q} \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty \tag{3.6}
\end{equation*}
$$

and either

$$
\begin{equation*}
\mathrm{r}_{1}>\rho \text { or } \mathrm{q}>0 \tag{3.7}
\end{equation*}
$$

Further, let $w$ be the smallest positive integer such that when $n=w, E_{n} \neq$ 0 , where

$$
E_{n}=\sum_{S=1}^{p} B_{S q} \exp \left(i n \alpha_{S}\right), \text { for } n \geq 1
$$

The number $w$ exists, for otherwise $B_{s q}=0$ for $1 \leq s \leq p$. From (3.3) and (3.6)

$$
\frac{c_{n}}{r_{1}^{n} n^{q}}=\sum_{s=1}^{p} B_{s q} \exp \left(i w \alpha_{s}\right) \cdot \exp \left(i(n-w) \alpha_{s}\right)+0(1) \text { as } n \rightarrow \infty
$$

By Lemma 1 there exists a sequence $\left(n_{j}\right)$ of positive integers such that

$$
\begin{equation*}
\frac{c_{n_{j}}}{r_{1} j_{j} n_{j}^{q}}=E_{w}+0(1) \quad \text { as } \quad j \rightarrow \infty \tag{3.8}
\end{equation*}
$$

It is clear from (3.8) that $E_{w}$ is real and positive; since ( $c_{n}$ ) is a decreasing sequence, we have also that $r_{1}<1$ and hence $\rho<1$.

By (3.7) and (3.8) there exists an integer $m$ such that

$$
\frac{c_{m}}{\rho^{m}}=\frac{c_{m}}{r_{1}^{m} m^{q}} \cdot\left(\frac{r_{1}}{\rho}\right)^{m} \cdot m^{q}>\left(\frac{c_{1}}{\rho}\right) \rho^{1-k}
$$

Hence,
(3.9) $c_{m-k+1}>c_{m-k+2}>\cdots>c_{m}>\left(\frac{c_{1}}{\rho}\right) \rho^{m-k+1}>\left(\frac{c_{1}}{\rho}\right) \rho^{m-k+2}>\cdots>\left(\frac{c_{1}}{\rho}\right)_{\rho}^{m}$.

Therefore,
(3.10) $\quad c_{m-k}=a_{1} c_{m}+a_{2} c_{m-1}+\cdots+a_{k} c_{m-k+1}>\left(\frac{c_{1}}{\rho}\right)\left(a_{1} \rho{ }^{m}+a_{2} p^{m-1}+\right.$

$$
\left.+\cdots+a_{k} m-k+1\right)=\left(\frac{c_{1}}{\rho}\right) \rho^{m-k}
$$

Using (3.9) and (3.10) we find in a similar way that

$$
c_{m-k-1}>\left(\frac{c_{1}}{\rho}\right) \rho^{m-k-1}
$$

and

$$
c_{\mathrm{m}-\mathrm{k}-2}>\left(\frac{\mathrm{c}_{1}}{\rho}\right) \rho^{\mathrm{m}-\mathrm{k}-2}
$$

-••
and so on, until

$$
c_{1}>\left(\frac{\mathrm{c}_{1}}{\rho}\right) \rho
$$

a contradiction. Hence Case 2 does not occur. This proves Theorem 1.

## 4. PROOF OF THEOREM 2

The sequence $\left(b_{n}\right)$ is clearly a k-series base for $\left(0, \theta^{R}\right]$ if and only if

$$
\left(\frac{b_{n}}{\theta^{\mathbf{R}}}\right)
$$

is a k-series base for ( 0,1 ]. Hence without loss of generality we assume that $R=0$, so that we shall be discussing $k$-series bases for $(0,1]$.

## Lemma 2.

$$
\theta^{\mathrm{n}}=\sum_{\nu=0}^{\infty} \theta^{\mathrm{n}+1+\nu \mathrm{k}} \quad \text { for } \quad \mathrm{n} \geq 0
$$

Proof. Since $\theta$ is a root of $f(z)=z^{k}+s-1$ and $0<\theta<1$, we see that

$$
\sum_{\nu=0}^{m} \theta^{\mathrm{n}+1+\nu \mathrm{k}}=\theta^{\mathrm{n}+1} \sum_{\nu=0}^{\infty}\left(\theta^{\mathrm{k}}\right)^{\nu}=\theta^{\mathrm{n}+1}\left(\frac{1}{1-\theta^{\mathrm{k}}}\right)=\frac{\theta^{\mathrm{n}+1}}{\theta}=\theta^{\mathrm{n}} .
$$

for $m \geq 0$. Since $\theta<1$ it follows that

$$
\sum_{\nu=0}^{\infty} \theta^{1+\nu k}=1
$$

and hence

$$
\theta^{\mathrm{n}}=\sum_{\nu=0}^{\infty} \theta^{\mathrm{n}+1+\nu \mathrm{k}} \text { for } \mathrm{n} \geq 0
$$

as required.
Proof of sufficiency. We show that $\left(\theta^{n}\right)$ is a k-series base for $(0,1]$. Let $0<x \leq 1$. First we construct inductively a sequence ( $i_{\nu}$ ) of positive integers such that $\mathrm{i}_{\nu+1} \geq \mathrm{i}_{\nu}+\mathrm{k}$ for $\nu \geq 1$, and
for $m \geq 1$. The integer $i_{1}$ is chosen so that

$$
\theta^{\mathbf{i}_{1}-1} \geq \mathrm{x}>\theta^{\mathbf{i}_{1}}
$$

and since $\theta+\theta^{k}=1$ we see that

$$
\theta^{\mathbf{i}_{1}-1+k}=\theta^{i_{1}-1}-\theta^{\mathbf{i}_{1}} \geq \mathrm{x}-\theta^{\mathrm{i}_{1}}>0 .
$$

Let $t \geq 1$ be an integer and suppose that $i_{1}, i_{2}, \cdots, i_{t}$ are chosen so that (4.1) holds for $\mathrm{m}=\mathrm{t}$, and $\mathrm{i}_{\nu+1} \geq \mathrm{i}_{\nu}+\mathrm{k}$ for $1 \leq \nu<\mathrm{t}_{0}$. Then we choose $\mathrm{i}_{\mathrm{t}+1}$ such that

$$
\begin{equation*}
\theta^{i_{t+1}^{-1}} \geq x-\sum_{\nu=1}^{t} \theta^{i_{\nu}}>\theta^{i_{t+1}} \tag{4.2}
\end{equation*}
$$

Hence

$$
\theta^{i_{t+1}^{-1+k}}=\theta^{i_{t+1}-1}-\theta^{i_{t+1}} \geq x-\sum_{\nu=1}^{t+1} \theta^{i_{\nu}}>0
$$

From (4.2) and the assumption that (4.1) holds for $m=t$ it follows that

$$
\theta^{i_{t} t^{-1+k}} \geq \theta^{i_{t+1}^{-1}}
$$

Hence $i_{t+1} \geq i_{t}+k$. The construction of the sequence $\left(i_{\nu}\right)$ follows by induction.

Since $\theta<1$ it follows from (4.1) that there exists a representation of $x$ in the form

$$
\begin{equation*}
\mathrm{x}=\sum_{\nu=1}^{\infty} \theta^{\mathrm{i}} \nu \tag{4.3}
\end{equation*}
$$

where $i_{1} \geq 1$ and $i_{\nu+1} \geq i_{\nu}+k$ for $\nu \geq 1$.
This representation of $x$ is unique. For otherwise we may assume without loss of generality that

$$
\sum_{\nu=1}^{\infty} \theta^{\mathrm{i}} \nu=\sum_{\nu=1}^{\infty} \theta^{\mathrm{j}_{\nu}}
$$

where $i_{1} \geq 1$ and $i_{\nu+1} \geq i_{\nu}+k$ for $\nu \geq 1, j_{1} \geq 1$ and $j_{\nu+1} \geq j_{\nu}+k$ for $\nu \geq 1$, and $i_{1}<j_{1}$. Then

$$
\theta^{\mathrm{i}_{1}}<\sum_{\nu=1}^{\infty} \theta^{\mathrm{i}_{\nu}}=\sum_{\nu=1}^{\infty} \theta^{\mathrm{j}_{\nu}} \leq \sum_{\nu=0}^{\infty} \theta^{\mathrm{j}_{1}+\nu \mathrm{k}}=\theta^{\mathrm{j}_{1}-1}
$$

by Lemma 2. Hence $i_{1}>j_{1}-1$, which contradicts the assumption that $i_{1}<$ $j_{1}$ 。

Since $\theta>0$, no non-positive numbers can be represented in the form (4.3). By Lemma 2,

$$
\sum_{\nu=0}^{\infty} \theta^{1+\nu \mathrm{k}}=1
$$

and so 1 is the largest number which has a representation in the form (4.3). Hence $\left(\theta^{\mathrm{n}}\right)$ is a k -series base for $(0,1]$. This completes the proof of the sufficiency.

Proof of necessity. We show that if the sequence $\left(b_{n}\right)$ is a k-series base for $(0,1]$, and if $b_{n+1} \geq b_{n}>0$ for $n \geq 1$, then $b_{n}=\theta^{n}$ for $n \geq 1$.

For shortness we write $b_{0}=1$, but as stated earlier, by the sequence $\left(b_{n}\right)$ we mean the sequence $\left(b_{1}, b_{2}, \cdots\right)$. The sequence $\left(b_{n}\right)$ is strictly decreasing, for if $b_{i}=b_{j}$ for $i \neq j$ then clearly some numbers have more than one k-series representation. For $n \geq 1$ we define

$$
\mathrm{B}_{\mathrm{n}}=\left\{\mathrm{r} \mid \mathrm{r}=\sum_{\nu=1}^{\infty} \mathrm{b}_{\mathrm{i}_{\nu}} ; \quad \mathrm{i}_{1}=\mathrm{n}, \quad \mathrm{i}_{\nu+1} \geq \mathrm{i}_{\nu}+\mathrm{k} \text { for } \nu \geq 1\right\}
$$

We denote by $\bar{B}_{n}$ the least upper bound of $B_{n}$. Since $\left(b_{n}\right)$ is a positive strictly decreasing sequence it follows that

$$
\begin{equation*}
\overline{\mathrm{B}}_{\mathrm{n}}=\sum_{\nu=0}^{\infty} \mathrm{b}_{\mathrm{n}+\nu \mathrm{k}} \text { for } \mathrm{n} \geq 1 \tag{4.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\mathrm{B}}_{\mathrm{n}}>\overline{\mathrm{B}}_{\mathrm{n}+1}>0 \text { for } \mathrm{n} \geq 1 \tag{4,5}
\end{equation*}
$$

It follows from (4.5) that there exists a non-negative realnumber $\ell$ such that $\overline{\mathrm{B}}_{\mathrm{n}} \rightarrow \ell$ as $\mathrm{n} \rightarrow \infty$. But, by (4.4)

$$
\sum_{\nu=0}^{\infty} \mathrm{b}_{1+\nu \mathrm{k}}=\lim _{\mathrm{m} \rightarrow \infty}\left(\sum_{\nu=0}^{\mathrm{m}} \mathrm{~b}_{1+\nu \mathrm{k}}+\overline{\mathrm{B}}_{1+(\mathrm{m}+1) \mathrm{k}}\right)=\sum_{\nu=0}^{\infty} \mathrm{b}_{1+\nu \mathrm{k}}+\ell
$$

so that $\ell=0$. Hence

$$
\begin{equation*}
\overline{\mathrm{B}}_{\mathrm{n}} \rightarrow 0 \quad \text { as } \quad \mathrm{n} \rightarrow \infty \tag{4.6}
\end{equation*}
$$

We now prove by induction upon $n$ that

$$
\begin{equation*}
\bar{B}_{n+1}=b_{n} \tag{4.7}
\end{equation*}
$$

for $n \geq 0$. Since $\left(b_{n}\right)$ is a k-series base for ( 0,1 ] it follows from (4.5) that $\overline{\mathrm{B}}_{1}=1$, and so (4.7) is true when $\mathrm{n}=0$. Let $\mathrm{m} \geq 1$ be a positive integer and suppose as an induction hypothesis that (4.7) is true for $0 \leq \mathrm{n}<\mathrm{m}$.

If $b_{m}>\bar{B}_{m+1}$ then there is no k-series representation for $\frac{1}{2}\left(b_{m}+\right.$ $\overline{\mathrm{B}}_{\mathrm{m}+1}$ ). Suppose that $\mathrm{b}_{\mathrm{m}}<\overline{\mathrm{B}}_{\mathrm{m}+1}$. Then we can construct inductively a sequence $\left(j_{\nu}\right)$ of positive integers, where $j_{1}=m$ and $j_{\nu+1} \geq j_{\nu}+\mathrm{k}$ for $\nu \geq 1$, such that for $\nu \geq 1$ there are infinitely many positive integers $n$ satisfying

$$
\overline{\mathrm{B}}_{\mathrm{m}+1}+\frac{1}{\mathrm{n}} \in \mathrm{~B}_{\mathrm{j}} \quad \text { if } \quad \nu=1
$$

or

$$
\begin{equation*}
\bar{B}_{m+1}-b_{j_{1}}-b_{j_{2}}-\cdots-b_{j}+\frac{1}{n} \in B_{j} \quad \text { if } \quad \nu \geq 2 \tag{4.8}
\end{equation*}
$$

By (4.5) and the induction hypothesis,

$$
b_{0}=\bar{B}_{1}>b_{1}=\bar{B}_{2}>\ldots>b_{m-1}=\bar{B}_{m}>\bar{B}_{m+1}
$$

and so there are infinitely many positive integers $n$ such that

$$
\bar{B}_{m+1}+\frac{1}{n} \in B_{m}
$$

Let $\delta^{>1} 1$ be an integer and suppose that the first $\delta-1$ terms of ( $\mathrm{j}_{\nu}$ ) are chosen. Then for infinitely many positive integers $n$,

$$
\left\{\begin{array}{l}
\bar{B}_{m+1}+\frac{1}{n} \in B_{j_{\delta-1}}, \quad \text { if } \delta=2 \\
\bar{B}_{m+1}-b_{j_{1}}-b_{j_{2}}-\cdots-b_{j_{\delta-2}}+\frac{1}{n} \in B_{j_{\delta-1}}, \text { if } \delta \geq 2
\end{array}\right.
$$

Hence

$$
\bar{B}_{m+1}-b_{j_{1}}-b_{j_{2}}-\cdots-b_{j_{\delta-i}}+\frac{1}{n} \in \bigcup_{i=j_{\delta-1}+k}^{\infty} B_{i}
$$

for infinitely many positive integers n. Therefore

$$
\bar{B}_{m+1}-b_{j_{1}}-b_{j_{2}}-\cdots-b_{j_{\delta-i}} \geq 0
$$

However, if $\bar{B}_{m+1}=b_{j_{1}}+b_{j_{2}}+\cdots+b_{j_{\delta_{-1}}}$, then, by replacing $b_{j_{\delta_{-1}}}$ by its $k$-series representation we obtain a $k$-series representation for $\mathcal{B}_{m+1}^{-1}$ different from the k-series representation given in (4.4), and this contradicts the fact that $\left(b_{n}\right)$ is a k-sexies base. Therefore by (4.6) there exists a positive integer $q$ such that

$$
B_{m+1}-b_{j_{1}}-b_{j_{2}}-\cdots-b_{j_{\delta-1}}>\bar{B}_{q}
$$

Hence

$$
\bar{B}_{m+1}-b_{j_{1}}-b_{j_{2}}-\cdots-b_{j_{\delta-1}}+\frac{1}{n} \in \bigcup_{i=j_{\delta_{-1}}+k}^{q} B_{i}
$$

for infinitely many positive integers $n$. Hence there exists $j_{\delta} \geq j_{\delta-1}+k$ such that

$$
\bar{B}_{m+1}-b_{j_{1}}-b_{j_{2}}-\cdots-b_{j_{\delta-1}}+\frac{1}{n} \in B_{j_{\delta}}
$$

for infinitely many positive integers $n$. The construction of the sequence ( $j_{\nu}$ ) follows by induction.

We deduce from (4.8) that for $\nu>1$,

$$
0<\overline{\mathrm{B}}_{\mathrm{m}+1}-\mathrm{b}_{\mathrm{j}_{1}}-\mathrm{b}_{\mathrm{j}_{2}}-\cdots-\mathrm{b}_{\mathrm{j}_{\nu-1}} \leq \mathrm{B}_{\mathrm{j}_{\nu}}
$$

and by (4.6) it follows that

$$
\overline{\mathrm{B}}_{\mathrm{m}+1}=\sum_{\nu=1}^{\infty} \mathrm{b}_{\mathrm{j}_{\nu}}
$$

This k-series representation for $\bar{B}_{m+1}$ is different from that given in (4.4), which contradicts the fact that ( $b_{n}$ ) is a k-series base. Hence $\bar{B}_{m+1}=b_{m}$, and it follows by induction that (4.7) holds for all $n \geq 0$.

By (4.4), for $\mathrm{n} \geq 0$,

$$
\overline{\mathrm{B}}_{\mathrm{n}+1}=\sum_{\nu=0}^{\infty} \mathrm{b}_{\mathrm{n}+1+\nu \mathrm{k}}=\mathrm{b}_{\mathrm{n}+1}+\sum_{\nu=0}^{\infty} \mathrm{b}_{\mathrm{n}+\mathrm{k}+1+\nu \mathrm{k}}=\mathrm{b}_{\mathrm{n}+1}+\overline{\mathrm{B}}_{\mathrm{n}+\mathrm{k}+1}
$$

and therefore, by (4.7)

$$
b_{n}=b_{n+1}+b_{n+k} \quad \text { for } \quad n \geq 0
$$

The number $\theta$ is the positive real root of the auxiliary polynomial $f(z)$ $=z^{k}+z-1$ of this recurrence relation. The modulus of any other root of $f(z)$ is greater than $\theta$. For if $|z|<\theta$, then since $\theta<1$,

$$
|f(z)|=\left|1-z\left(1+z^{k-1}\right)\right| \geq 1-|z|\left(1+|z|^{k-1}\right)>1-\theta\left(1+\theta^{k-1}\right)=0,
$$

whilst if $f(z)=0$ and $|z|=\theta$, then

$$
1-|z|-|1-z|=1-|z|-|f(z)-z+1|=1-|z|-|z|^{k}=0,
$$

so that

$$
|1-z|=1-|z|
$$

and hence $\arg \mathrm{z}=0$ so that $\mathrm{z}=\theta$ 。
By Theorem 1, therefore, for some positive constant $A, b_{n}=A \theta^{n}$ for $\mathrm{n} \geq 1$. However, we have shown that $\left(\theta^{\mathrm{n}}\right)$ is a k-series base for $(0,1]$, and so it follows that $A=1$. This completes the proof of the necessity and of Theorem 2.

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The Editor acknowledges with gratitude the receipt of $\$ 350$ from the Academic Vice President of San Jose State College toward our publication expenses.

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by
RAYMOND E. WHITNEY
Lock Hayen State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-143 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let $\left\{H_{n}\right\}$ be a generalized Fibonacci sequence and, by the recurrence relation, extend the definition to include negative subscripts. Show that

$$
\begin{equation*}
L_{2 j+1} \sum_{k=0}^{n} H_{(2 j+1) k}^{2}=H(2 j+1)(n+1) H(2 j+1) n-H_{0} H_{-(2 j+1)}, \tag{i}
\end{equation*}
$$

(ii)

$$
L_{2 j+1} \sum_{k=0}^{n} H_{(2 j+1) k}=H_{(2 j+1)(n+1)}-H_{-(2 j+1)},
$$

(iii)

$$
\mathrm{L}_{2 \mathrm{j}} \sum_{\mathrm{k}=0}^{\mathrm{n}}(-1)^{\mathrm{k}_{H_{2 j k}^{2}}^{2}=(-1)^{\mathrm{n}} \mathrm{H}_{2 j(n+1)} \mathrm{H}_{2 j n}-\mathrm{H}_{0} \mathrm{H}_{-2 j}, ~, ~, ~}
$$

and derive an expression for
(iv)

$$
\sum_{k=0}^{n}(-1)^{k_{H_{2 j k}}}
$$

H-144 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.
A. Put

$$
[(1-x)(1-y)(1-a x)(1-b y)]^{-1}=\sum_{m, n=0}^{\infty} A_{m, n} x^{m} y^{n}
$$

Show that

$$
\sum_{n=0}^{\infty} A_{n, n} x^{n}=\frac{1-a b x^{2}}{(1-x)(1-a x)(1-b x)(1-a b x)}
$$

B. Put

$$
(1-x)^{-1}(1-y)^{-1}(1-a x y)^{-} \lambda=\sum_{m, n=0}^{\infty} B_{m, n} x^{m} y^{n}
$$

Show that

$$
\sum_{n=0}^{\infty} B_{n, n} x^{n}=(1-x)^{-1}(1-a x)^{-\lambda}
$$

耳-145 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va. If

$$
n=p_{1}^{e_{1}} p_{2}^{e_{2}} \ldots p_{r}^{e_{r}}
$$

is the canonical factorization of $n$, let $\lambda(n)=e_{1}+\ldots+e_{r}$ Show that $\lambda(n)$ $\leq \lambda\left(F_{n}\right)+1$ for all $n$, where $F_{n}$ is the $n{ }^{\text {th }}$ Fibonacci number.

H-146 Proposed by J. A. H. Hunter, Toronto, Canada.
Let $P_{n}$ be the $n^{\text {th }}$ Pell number defined by $P_{1}=1, P_{2}=2$, and $P_{n+2}$ $=2 P_{n+1}+P_{n}$. Prove that the only square Pell numbers are $P_{1}=1$, and $P_{7}$ $=169$.

H-147 Proposed by George Ledin, Jr., University of San Francisco, San Francisco, California.
Find the following limits. $F_{k}$ is the $k^{\text {th }}$ Fibonacci number, $L_{k}$ is the $\mathrm{k}^{\text {th }}$ Lucas number, $\pi=3.14159 \cdots, \alpha=(1+\sqrt{5}) / 2=1.61803 \cdots, \mathrm{k}^{\mathrm{k}}=1$, $2,3, \cdots$ 。

$$
\begin{aligned}
& X_{1}=\lim _{n \rightarrow \infty} \frac{F_{F_{n+1}}}{F_{F_{n}}^{\alpha}} \\
& X_{2}=\lim _{n \rightarrow 0}\left|\frac{\mathrm{~F}{ }_{\mathrm{n}} \mathrm{~m}}{\mathrm{n}^{m}}\right| \\
& \left.\mathrm{X}_{3}=\lim _{\mathrm{n}}\left|\frac{\mathrm{~F}}{\mathrm{~m}}\right| \frac{\mathrm{n}^{\mathrm{m}}}{\mathrm{~F}_{\mathrm{n}}^{\mathrm{m}}} \right\rvert\, \\
& \mathrm{X}_{4}=\lim _{\mathrm{n} \rightarrow 0}\left|\frac{\mathrm{~F}{ }^{m}{ }_{\mathrm{n}}{ }^{m-1} \mathrm{~F}_{\mathrm{n}}}{}\right| \\
& X_{5}=\lim _{n \rightarrow 0}\left|\frac{L_{n}-2}{n}\right|
\end{aligned}
$$

## SOLUTIONS

SUM DAY

H-103 Proposed by David Zeitlin, Minneapolis, Minnesota.

Show that

$$
8 \sum_{k=0}^{n} F_{3 k+1} F_{3 k+2} F_{6 k+3}=F_{3 n+3}^{4}
$$

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa.
We suppose known that

$$
F_{2 X+1}=F_{X}^{2}+F_{X+1}^{2}
$$

and that

$$
F_{x} F_{x+2}=(-1)^{x+1}+F_{x+1}^{2}
$$

Then

$$
\begin{aligned}
\mathrm{F}_{2 \mathrm{x}-3} & =\mathrm{F}_{\mathrm{x}-2}^{2}+\mathrm{F}_{\mathrm{x}-1}^{2}=\left(\mathrm{F}_{\mathrm{x}}-\mathrm{F}_{\mathrm{X}-2}\right)^{2}+\left(\mathrm{F}_{\mathrm{X}-1}-\mathrm{F}_{\mathrm{X}-2}\right)^{2} \\
& =\mathrm{F}_{\mathrm{x}}^{2}+\mathrm{F}_{\mathrm{x}-1}^{2}+\mathrm{F}_{\mathrm{x}-2}^{2}+\mathrm{F}_{\mathrm{x}-3}^{2}-2 \mathrm{~F}_{\mathrm{x}} \mathrm{~F}_{\mathrm{x}-2}-2 \mathrm{~F}_{\mathrm{x}-1} \mathrm{~F}_{\mathrm{x}-3} \\
& =\mathrm{F}_{\mathrm{x}}^{2}+\mathrm{F}_{\mathrm{x}-1}^{2}+\mathrm{F}_{\mathrm{x}-2}^{2}+\mathrm{F}_{\mathrm{x}-3}^{2}-2\left((-1)^{\mathrm{x}-1}+\mathrm{F}_{\mathrm{x}-1}^{2}+(-1)^{\mathrm{x}-2}+\mathrm{F}_{\mathrm{x}-2}^{2}\right) \\
& =\mathrm{F}_{\mathrm{x}}^{2}-\mathrm{F}_{\mathrm{x}-1}^{2}-\mathrm{F}_{\mathrm{x}-2}^{2}+\mathrm{F}_{\mathrm{x}-3}^{2}=\mathrm{F}_{\mathrm{x}}^{2}-\mathrm{F}_{2 \mathrm{x}-3}+\mathrm{F}_{\mathrm{x}-3}^{2},
\end{aligned}
$$

whence

$$
2 \mathrm{~F}_{2 \mathrm{X}-3}=\mathrm{F}_{\mathrm{X}}^{2}+\mathrm{F}_{\mathrm{X}-3}^{2}
$$

Now, the identity to be proved is clearly true for $\mathrm{n}=0$ and we need only show that the right- and left-hand sides increase by the same amount when n is replaced by $n+1$. The right-hand increase is

$$
\mathrm{F}_{3 \mathrm{n}+6}^{4}-\mathrm{F}_{3 \mathrm{n}+3}^{4}=\left(\mathrm{F}_{3 \mathrm{n}+6}-\mathrm{F}_{3 \mathrm{n}+3}\right)\left(\mathrm{F}_{3 \mathrm{n}+6}+\mathrm{F}_{3 \mathrm{n}+3}\right)\left(\mathrm{F}_{3 \mathrm{n}+6}^{2}+\mathrm{F}_{3 \mathrm{n}+3}^{2}\right)
$$

The first factor is

$$
\mathrm{F}_{3 \mathrm{n}+5}+\mathrm{F}_{3 \mathrm{n}+4}-\mathrm{F}_{3 \mathrm{n}+3}=2 \mathrm{~F}_{3 \mathrm{n}+4}
$$

The second is

$$
\mathrm{F}_{3 \mathrm{n}+5}+\mathrm{F}_{3 \mathrm{n}+4}+\mathrm{F}_{3 \mathrm{n}+3}=2 \mathrm{~F}_{3 \mathrm{n}+5}
$$

Thus the total is

$$
4 \mathrm{~F}_{3 n+4} \mathrm{~F}_{3 n+5}\left(\mathrm{~F}_{3 \mathrm{n}+6}^{2}+\mathrm{F}_{3 \mathrm{n}+3}^{2}\right)
$$

The left-hand side increase is

$$
8 \mathrm{~F}_{3 \mathrm{n}+4} \mathrm{~F}_{3 \mathrm{n}+5} \mathrm{~F}_{6 \mathrm{n}+9} .
$$

These increases are equal if

$$
2 \mathrm{~F}_{6 \mathrm{n}+9}=\mathrm{F}_{3 \mathrm{n}+6}^{2}+\mathrm{F}_{3 \mathrm{n}+3}^{2},
$$

which we have already proved.
Also solved by F. D. Parker, Charles R. Wall, and J. Ramanna.

## GENERATOR TROUBLE

H-104 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show

$$
\frac{L_{m} x}{1-5 F_{m} x+(-1)^{m+1} 5 x^{2}}=\sum_{k=0}^{\infty} 5^{k}\left(F_{2 m k}+x L_{(2 k+1) m}\right) x^{2 k}
$$

where $L_{m}$ and $F_{m}$ are the $m^{\text {th }}$ Lucas and Fibonacci numbers, respectively. Solution by David Zeitlin, Minneapolis, Minnesota.

Using (14) and (16) in my paper, "On Summation Formulas for Fibonacci and Lucas Numbers," this Quarterly, Vol. 2, No. 2, 1964, pp. 105-107, we obtain, respectively

$$
\begin{equation*}
\left(1-L_{2 m} y+y^{2}\right) \sum_{k=0}^{\infty} F_{2 m k} y^{k}=F_{2 m y} \tag{1}
\end{equation*}
$$

(2)

$$
\begin{aligned}
\left(1-L_{2 m y}+y^{2}\right) \sum_{k=0}^{\infty} L_{(2 k+1) m} y^{k} & =L_{m}+\left(L_{3 m}-L_{m} L_{2 m}\right) y \\
& = \\
& =L_{m}+(-1)^{m+1} L_{m} y
\end{aligned}
$$

since

$$
L_{m}=(-1)^{m+1}\left(L_{3 m}-L_{m} L_{2 m}\right)
$$

For $y=5 x^{2}$, we obtain

$$
\begin{aligned}
\sum_{k=0}^{\infty} 5^{k}\left(F_{2 m k}\right. & \left.+x L_{(2 k+1) m}\right) x^{2 k} \\
& =\frac{L_{m} x\left(1+5 F_{m} x+5(-1)^{m+1} x^{2}\right)}{1-5 L_{2 m} x^{2}+25 x^{4}} \equiv \frac{L_{m} x}{1-5 F_{m} x+5(-1)^{m+1} x^{2}}
\end{aligned}
$$

since

$$
\mathrm{L}_{2 \mathrm{~m}}=2(-1)^{\mathrm{m}}+5 \mathrm{~F}_{\mathrm{m}}^{2}
$$

and so

$$
1-5 L_{2 m} x^{2}+25 x^{4}=\left(1+5 F_{m} x+5(-1)^{m+1} x^{2}\right)\left(1-5 F_{m}^{\left.x+5(-1)^{m+1} x^{2}\right)}\right.
$$

Also solved by Anthony G. Shannon (Australia).

## OF PRIME INTEREST

H-105 Proposed by Edgar Karst, Norman, Oklahoma, and S. O. Rorem, Davenport, lowa.
Show for all positive integral $n$ and primes $p>2$ that

$$
(\mathrm{n}+1)^{\mathrm{p}}-\mathrm{n}^{\mathrm{p}}=6 \mathrm{~N}+1
$$

where N is a positive integer. Generalize.
Solution by E. W. Bowen, University of New England, Australia.
Let b be a prime, m a positive integer, and $\mu$ the least positive residue of $m$ modulo $b-1$, $i_{0} e_{0}$, for some integer $k, m=k(b-1)+\mu$ where $0<\mu<b-1$.

Clearly $\mathrm{n}^{\mathrm{m}}=0=\mathrm{n}^{\mu}(\bmod b)$ if n is a multiple of b . If n is not a multiple of $b$, we have by Fermat's theorem,

$$
\mathrm{n}^{\mathrm{b}-1} \equiv 1(\bmod \mathrm{~b})
$$

from which we infer

$$
\mathrm{n}^{\mathrm{m}} \equiv \mathrm{n}^{\mathrm{k}(\mathrm{~b}-1)+\mu} \equiv 1_{\mathrm{n}}^{\mathrm{k}} \mathrm{n}^{\mu} \equiv \mathrm{n} \quad(\bmod b)
$$

Thus, for all integers $n$ we have

$$
\mathrm{n}^{\mathrm{m}} \equiv \mathrm{n}^{\mu}(\bmod b)
$$

Using $\Delta$ to denote the difference operator by which

$$
\Delta f(n)=f(n+1)-f(n)
$$

and noting that $\Delta \Delta_{n} \mu=\mu!$, we obtain

$$
\Delta^{\mu} \mathrm{n}^{\mathrm{m}} \equiv \mu \mathrm{~m}!(\bmod \mathrm{b})
$$

and in particular, with $\mu=1$,

$$
\Delta \mathrm{n}^{\mathrm{m}} \equiv 1(\bmod \mathrm{~b}) \text { if } \mathrm{m}=1(\bmod \mathrm{~b}-1)
$$

If $b_{1}, b_{2}, \cdots, b_{S}$ are different primes, we infer immediately that
$\star \Delta n^{m} \equiv 1\left(\bmod b_{1} b_{2} \cdots b_{S}\right)$ if $m \equiv 1\left(\bmod \left(b_{1}-1\right) \cdots\left(b_{S}-1\right)\right)$.

This is a generalization of the required result since, with 2 and 3 as the primes, $b_{i}$, we find that for any odd $m$, and hence for $m=p>2, \Delta n^{m}=1(\bmod$ 6 ), i. e., $(n+1)^{m}-n^{m}=6 N+1$ for some integer $N ; N$ is obviously positive when $n$ is positive and $m>2$.

Examples of other results obtained from $\star$ are:

$$
\begin{aligned}
& \Delta \mathrm{n}^{\mathrm{m}} \equiv 1(\bmod 10) \text { if } \mathrm{m}=4 \mathrm{k}+1 \\
& \Delta \mathrm{n}^{\mathrm{m}} \equiv 1(\bmod 30) \text { if } \mathrm{m}=8 \mathrm{k}+1
\end{aligned}
$$

Summing gives a further generalization of *:

$$
(n+r)^{m}-n^{m} \equiv r \quad\left(\bmod b_{1} \cdots b_{s}\right)
$$

if

$$
m \equiv 1\left(\bmod \left(b_{1}-1\right) \cdots\left(b_{S}-1\right)\right)
$$

Also solved by J. A. H. Hunter, Brother Alfred Brousseau, David Singmaster, Steven Weintraub, and Anthony Shannon.

## BUY MY NOMIAL?

H-106 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.
Show that
(a)

$$
\sum_{k=0}^{n}\binom{n}{k}^{2} L_{2 k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} L_{n-k}
$$

(b)

$$
\sum_{k=0}^{n}\binom{n}{k}^{2} F_{2 k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} F_{n-k}
$$

Solution by David Zeitlin, Minneapolis, Minnesota.
If

$$
P(x) \equiv \sum_{k=0}^{n}\binom{n}{k}^{2} x^{k}
$$

and

$$
Q(x) \equiv \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(x-1)^{n-k},
$$

then $P(x) \equiv Q(x)$ is a known identity (see Elementary Problem E799, American Math. Monthly, 1948, p. 30). If $\alpha$ and $\beta$ are roots of $x^{2}-x-1=0$, then

$$
\mathrm{L}_{\mathrm{n}}=\alpha^{\mathrm{n}}+\beta^{\mathrm{n}}, \quad \mathrm{~F}_{\mathrm{n}}=\left(\alpha^{\mathrm{n}}-\beta^{\mathrm{n}}\right) / \sqrt{5}
$$

and thus
(a)

$$
\mathbf{P}\left(\alpha^{2}\right)+\mathbf{P}\left(\beta^{2}\right)=\mathbf{Q}\left(\alpha^{2}\right)+\mathbf{Q}\left(\beta^{2}\right)
$$

(b)

$$
\frac{\mathrm{P}\left(\alpha^{2}\right)-\mathrm{P}\left(\beta^{2}\right)}{\sqrt{5}}=\frac{\mathrm{Q}\left(\alpha^{2}\right)-\mathrm{Q}\left(\beta^{2}\right)}{\sqrt{5}}
$$

## BE DETERMINANT!

H-107 Proposed by Vladimir Ivanoff, San Carlos, California.
Show that
for all integers $p, q, r$ and $n$.

Solution by C. C. Yalavigi, Government College, Mercara, India.
Let
(1)

$$
D=\left|\begin{array}{lllll}
F_{p_{1}+r n} & \cdots & \cdots & F_{p_{1}+n} & F_{p_{1}} \\
F_{p_{2}+r n} & \cdots & \cdots & F_{p_{2}+n} & F_{p_{2}} \\
\vdots & & & & \\
F_{p_{r+1}+r n} & \cdots & \cdots & F_{p_{r+1}+n} & F F_{p_{r+1}}
\end{array}\right|
$$

On simplifying the first column of this determinant by the use of

$$
F_{i+j}=F_{i+1} F_{j}+F_{i} F_{j-1}
$$

it is easy to show that

$$
D=F_{r n}\left|\begin{array}{lllll}
F_{p_{1}+1} & F_{p_{1}+(r-1) n} & \cdots & F_{p_{1}+n} & F_{p_{1}} \\
F_{p_{2}+1} & F_{p_{2}+(r-1) n} & \cdots & F_{p_{2}+n} & F_{p_{2}} \\
\vdots & & & & \\
F_{p_{r+1}+1} & F_{p_{r+1}+(r-1) n} & \cdots & F_{p_{r+1}+n} & F_{p_{r+1}}
\end{array}\right|
$$

(2)

$$
+F_{r n-1}\left|\begin{array}{lllll}
F_{p_{1}} & F_{p_{1}+(r-1) n} & \cdots & F_{p_{1}+n} & F_{p_{1}} \\
F_{p_{2}} & F_{p_{2}+(r-1) n} & \cdots & F_{p_{2}+n} & F_{p_{2}} \\
\vdots & & & & \\
p_{p_{r+1}} & F_{p_{r+1}+(r-1) n} & \cdots & F_{p_{r+1}+n} & F_{p_{r+1}}
\end{array}\right|
$$

when the subtraction of $F_{n}$ times the first column and $F_{n-1}$ times the last column from the last but one column in the first determinant reduces it to zero and the second determinant also vanishes.

Therefore the desired result follows for $r=2$ 。
Also solved by F. D. Parker, David Zeitlin, Anthony Shannon, C. B. A. Peck, Douglas Lind, William Lombard, Charles R. Wall.

# THREE DIOPHANTINE EQUATIONS -PART I 

IRVING ADLER
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## 1. INTRODUCTION

This article deals with the three Diophantine equations
(1)

$$
x^{2}+(x+1)^{2}=z^{2}
$$

(2)

$$
u^{2}+u=2 v^{2}
$$

$$
\mathrm{s}^{2}+2 \mathrm{t}^{2}=1
$$

These equations have been studied by various methods for hundreds of years, and their solutions in positive integers are well known. (See the historical note at the end of Part II, Feb.) However, as often happens with old problems, people not aware of the long history of these equations keep rediscovering them and their solutions. An article recently submitted to the Fibonacci Quarterly dealt with Eq. (1), and solved it by transforming it into Eq. (3). Elementary Problem B-102 in the December 1966 issue of the Quarterly (page 373) also links Eq. (1) and the solution to Eq. (3). Another article recently submitted to the Quarterly deals with Eq. (2).

The three equations are essentially equivalent because, as we shall see in Section 9, each can be transformed into each of the other two by a linear transformation.

## 2. WHY THE EQUATIONS KEEP COMING UP

The equations come up over and over again because they arise in a natural way from some basic problems of number theory.
A. When the general solution of the equation $x^{2}+y^{2}=z^{2}$ is studied, it is natural to consider the special case in which $x$ and $y$ are consecutive integers. This leads to Eq. (1).
B. When people play with figurate numbers, and, in particular, with the triangular numbers

$$
T(u)=\frac{1}{2} u(u+1)
$$

and the square numbers

$$
\mathrm{S}(\mathrm{v})=\mathrm{v}^{2}
$$

they soon observe that

$$
36=\mathrm{S}(6)=\mathrm{T}(8) .
$$

This observation naturally suggests the problem of finding all the triangular numbers that are also square numbers. This problem leads to Eq. (2).
C. There is no rational number $\mathrm{s} / \mathrm{t}$ equal to the square root of 2 . That is, there are no positive integers $s$ and $t$ such that

$$
\begin{equation*}
S^{2}-2 t^{2}=0 \tag{4}
\end{equation*}
$$

However, it is possible to obtain rational approximations to the square root of 2 with errors smaller than any prescribed amount. The search for rational approximations with a small error naturally leads to consideration of the equation obtained from Eq. (4) by requiring the right-hand member to be 1 instead of 0 . This leads to Eq. (3).

## 3. SOLUTIONS BY TRIAL AND ERROR

One way of finding some positive integers that satisfy Eq. (1) is to substitute first 1 , then 2 , etc., for $x$ in the expression $x^{2}+(x+1)^{2}$ to identify values of x which make the expression a perfect square. Similarly, solutions of Eq. (2) can be found by identifying by trial and error some positive integral values of $u$ that make

$$
\frac{1}{2} u(u+1)
$$

a perfect square. And solutions of Eq. (3) can be found by identifying some positive integral values of $t$ that make $1+2 t^{2}$ a perfect square. Anyone with
patience and a table of squares, or who has access to a computer can discover in this way at least a few of the solutions of each of the three equations.

It will be useful to us to identify not only positive solutions, but nonnegative solutions. The first five non-negative solutions of Eqs. (1), (2), and (3) are shown in the table below:

| Solutions of Equation (1) |  | Solutions of Equation (2) |  | Solutions of Equation (3) |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| x | z | u | V | S | t |
| 0 | 1 | 0 | 0 | 1 | 0 |
| 3 | 5 | 1 | 1 | 3 | 2 |
| 20 | 29 | 8 | 6 | 17 | 12 |
| 119 | 169 | 49 | 35 | 99 | 70 |
| 696 | 985 | 288 | 204 | 577 | 408 |

## 4. CAN WE COMPUTE MORE SOLUTIONS FROM THOSE WE ALREADY HAVE?

Once we have the first few solutions of one of these equations, we may, by inspecting them, find a relationship by which more solutions can be calculated. To facilitate the formulation of such a relationship, let us index the solutions of each equation in order of magnitude, with the non-negative integers $0,1,2, \cdots$, respectively, used as indices. Then, in this notation,

$$
x_{0}=0, \quad z_{0}=1, \quad x_{1}=3, \quad z_{1}=5, \quad x_{2}=20, \quad z_{2}=29
$$

etc. Are there, perhaps, formulas that permit us to calculate $x_{n}$ and $z_{n}$ in terms of $x_{n-1}$ and $z_{n-1}$ ? Let us assume there are such formulas, and let us guess that they are linear. Assume that

$$
\begin{equation*}
x_{n}=a x_{n-1}+b z_{n-1}+c \tag{5}
\end{equation*}
$$

(6)

$$
z_{n}=d x_{n-1}+e z_{n-1}+f
$$

Then we have to use only the first four values of $x$ and $z$ to determine what the values of $a, b, c, d, e$ and $f$ must be. Taking $n$ equal to 1,2 , and 3 in succession, we get the following systems of equations:

Solving these systems of equations, we find that

$$
\mathrm{a}=3, \quad \mathrm{~b}=2, \quad \mathrm{c}=1, \quad \mathrm{~d}=4, \quad \mathrm{e}=3, \quad \text { and } \quad \mathrm{f}=2
$$

Equations (5) and (6) are merely guesses. However, the fact that the values of $a, b, c, d, e$ and $f$ that we calculated on the basis of these guesses turns out to be integers, and small ones, at that, is presumptive evidence in favor of these guesses. Let us continue operating with these guesses. If Eqs. (5) and (6) are true, then they must take this form:

$$
\begin{equation*}
x_{n}=3 x_{n-1}+2 z_{n-1}+1 \tag{7}
\end{equation*}
$$

$$
z_{n}=4 x_{n-1}+3 z_{n-1}+2 .
$$

We can obtain more evidence for or against our guesses by using Eqs. (7) and (8) to calculate $x_{4}$ and $z_{4}$ :

$$
\begin{aligned}
& \mathrm{x}_{4}=3(119)+2(169)+1=696 ; \\
& \mathrm{z}_{4}=4(119)+3(169)+2=985
\end{aligned}
$$

Since these values of $x_{4}$ and $z_{4}$ calculated by means of EqS. (7) and (8) agree with the values of $x_{4}$ and $z_{4}$ in the table, the evidence tends to support the correctness of Eqs. (7) and (8). We now know that Eqs. (7) and (8) are true when $\mathrm{n}=1,2,3$, or 4 . This gives us the confidence to seek a proof that they are true for all positive integral values of $n$. The proof is given in the next section.

## EXERCISES

1. Let $\left(u_{n}, v_{n}\right)$ be the $n^{\text {th }}$ positive integral solution of Eq. (2). If we assume that

$$
u_{n}=a u_{n-1}+b v_{n-1}+c,
$$

and

$$
v_{n}=d u_{n-1}+e v_{n-1}+f,
$$

then what values must $a, b, c, d, e$ and $f$ have in these formulas?
2. Let $\left(S_{n}, t_{n}\right)$ be the $n^{\text {th }}$ solution in positive integers of Eq. (3). If we assume that

$$
s_{n}=a s_{n-1}+b t_{n-1}+c
$$

and

$$
t_{n}=d s_{n-1}+e t_{n-1}+f
$$

then what values must $a, b, c, d, e$ and $f$ have in these formulas?

## 5. PROOF THAT SUCCESSIVE SOLUTIONS ARE LINEARLY RELATED

The preceding section led to the conjecture that successive solutions of Eq. (1) are related by the linear Eqs. (7) and (8). To prove the conjecture, it is necessary to show that
A. If ( $x_{n-1}, z_{n-1}$ ) is a solution of Eq. (1), then $\left(x_{n}, z_{n}\right)$ defined by Eqs. (7) and (8) is also a solution;
B. If we take $x_{0}=0$ and $z_{0}=1$, then every solution of Eq. (1) can be obtained by starting with $\left(x_{0}, z_{0}\right)$ and making repeated use of Eqs. (7) and (8), to generate solutions with greater and greater values of $x$ and $z_{0}^{*}$

Proof of A. Suppose that ( $x_{n-1}, z_{n-1}$ ) is a solution of Eq. (1). Then we want to show that

$$
\left(3 x_{n-1}+2 z_{n-1}+1, \quad 4 x_{n-1}+3 z_{n-1}+2\right)
$$

[^1]is also a solution of Eq. (1). To simplify the notation for the proof, let us drop the subscripts. In this simplified notation, we are assuming that
$$
x^{2}+(x+1)^{2}=z^{2}
$$
and we want to show that
$$
(3 x+2 z+1)^{2}+(3 x+2 z+2)^{2}=(4 x+3 z+2)^{2}
$$
$(3 x+2 z+1)^{2}+(3 x+2 z+2)^{2}$
\[

$$
\begin{aligned}
& =18 x^{2}+24 x z+8 z^{2}+18 x+12 z+5 \\
& =16 x^{2}+24 x z+8 z^{2}+16 x+12 z+4+\left(2 x^{2}+2 x+1\right) \\
& =16 x^{2}+24 x z+9 z^{2}+16 x+12 z+4 \\
& =(4 x+3 z+2)^{2} .
\end{aligned}
$$
\]

in view of the fact that

$$
2 x^{2}+2 x+1=x^{2}+(x+1)^{2}=z^{2}
$$

Proof of B. Equations (7) and (8) determine a function

$$
\mathrm{f}:(\mathrm{x}, \mathrm{z}) \longrightarrow\left(\mathrm{x}^{\prime}, \mathrm{z}^{\prime}\right)
$$

as follows:
(f)

$$
\left\{\begin{array}{l}
x^{\prime}=3 x+2 z+1 \\
z^{\prime}=4 x+3 z+2
\end{array}\right.
$$

If we solve these equations for $x$ and $z$, we obtain the inverse function

$$
\begin{equation*}
\left(x^{\prime}, z^{\prime}\right) \rightarrow(x, z) \tag{g}
\end{equation*}
$$

defined by
(10)

$$
\left\{\begin{array}{l}
x=3 x^{\prime}-2 z^{\prime}+1 \\
z=-4 x^{\prime}+3 z^{\prime}-2
\end{array}\right.
$$

$\mathrm{fg}=\mathrm{i}=$ the identity function. Then

$$
\mathrm{ffgg}=\mathrm{f}(\mathrm{fg}) \mathrm{g}=\mathrm{fig}=\mathrm{fg}=\mathrm{i},
$$

and, in general,

$$
f^{n} g^{n}=i
$$

for every positive integer n. That is,

$$
f^{n} g^{n}(x, z)=(x, z)
$$

We shall show first that if $(x, z)$ is a solution of Eq. (1), with $x>0, z>0$, then

$$
\left(x_{1}, z_{1}\right)=g(x, z)
$$

is a solution of Eq. (1) with $\mathrm{x}_{1} \geq 0$, and $\mathrm{z}_{1}>0$, and $\mathrm{z}_{1}<\mathrm{z}$. If

$$
x^{2}+(x+1)^{2}=z^{2}
$$

then

$$
\begin{aligned}
x_{1}^{2}+\left(x_{1}+1\right)^{2} & =2 x_{1}^{2}+2 x_{1}+1=2(3 x-2 z+1)^{2}+2(3 x-2 z+1)+1 \\
& =18 x^{2}+8 z^{2}-24 x z+18 x-12 z+5 \\
& =16 x^{2}+8 z^{2}-24 x z+16 x-12 z+4+\left(2 x^{2}+2 x+1\right) \\
& =16 x^{2}+9 z^{2}-24 x z+16 x-12 z+4
\end{aligned}
$$

since

$$
2 x^{2}+2 x+1=x^{2}+(x+1)^{2}=z^{2}
$$

then

$$
x_{1}^{2}+\left(x_{1}+1\right)^{2}=(-4 x+3 z-2)^{2}=z_{1}^{2}
$$

Therefore $\left(x_{1}, z_{1}\right)$ is a solution of Eq. (1). Now we aim to show that $x_{1} \geq 0$, $z_{1}>0$, and $z_{1}<z$. The condition $x_{1} \geq 0$ is equivalent to $3 x-2 z+1 \geq 0$. or $2 \mathrm{z} \leq 3 \mathrm{x}+1$. The condition that $\mathrm{z}_{1}>0$ is equivalent to $-4 \mathrm{x}+3 \mathrm{x}-2>0$, or $3 z>4 x+2$. The condition $z_{1}<z$ is equivalent to $-4 x+3 z-2<z$, or $z<2 x+1$. So we shall show that

$$
\mathrm{z}<2 \mathrm{x}+1, \quad 2 \mathrm{z} \leq 3 \mathrm{x}+1
$$

and

$$
\begin{gathered}
3 z>4 x+2 \\
z^{2}=2 x^{2}+2 x+1=4 x^{2}+4 x+1-2 x^{2}-2 x \\
=(2 x+1)^{2}-2 x(x+1)<(2 x+1)^{2},
\end{gathered}
$$

since $x>0$, and hence $2 x(x+1)>0$. Therefore $z<2 x+1$. Since

$$
z^{2}=2 x^{2}+2 x+1
$$

and $x>0$, then

$$
9 z^{2}=18 x^{2}+18 x+9>16 x^{2}+16 x+4=(4 x+2)^{2}
$$

Therefore $3 \mathrm{z}>4 \mathrm{x}+2$ 。

$$
4 z^{2}=8 x^{2}+8 z+4=9 x^{2}+6 x+1-x^{2}+2 x+3
$$

Since $x>0$, we see from the table of solutions of Eq. (1) that $x \geq 3$. Then

$$
x^{2} \geq 3 x=2 x+x \geq 2 x+3
$$

Then

$$
2 x+3-x^{2} \leq 0
$$

Consequently

$$
4 z^{2} \leq 9 x^{2}+6 x+1=(3 x+1)^{2}
$$

and

$$
2 \mathrm{z} \leq 3 \mathrm{x}+1
$$

We have shown that if ( $x, z$ ) is a solution of Eq. (1) for which $x>0$ and $z>0$, then

$$
\left(\mathrm{x}_{1}, \mathrm{z}_{1}\right)=\mathrm{g}(\mathrm{x}, \mathrm{z})
$$

is a solution for which $x_{1} \geq 0, z_{1}>0$, and $z_{1}>z$. If $x_{1}>0$ we can repeat the process to obtain a solution

$$
\left(\mathrm{x}_{2}, \mathrm{z}_{2}\right)=\mathrm{g}\left(\mathrm{x}_{1}, \mathrm{z}_{1}\right)=\mathrm{g}^{2}(\mathrm{x}, \mathrm{z}),
$$

with $x_{2} \geq 0, z_{2}>0$, and $z_{2}<z_{1}$. Continuing in this way as long as $x_{i}>$ $0, \quad i=1,2, \cdots$, we get a descending sequence of positive integers $z>z_{1}>$ $z_{2}>\ldots$. Since this sequence must terminate, there exists a positive integer n for which

$$
\left(\mathrm{x}_{\mathrm{n}}, \mathrm{z}_{\mathrm{n}}\right)=\mathrm{g}^{\mathrm{n}}(\mathrm{x}, \mathrm{z})
$$

is a solution of Eq. (1) with $x_{n}=0$. Then $z_{n}=1$, and

$$
(0,1)=\left(x_{n}, z_{n}\right)=g^{n}(x, z)
$$

Then

$$
\mathrm{f}^{\mathrm{n}}(0,1)=\mathrm{f}^{\mathrm{n}} \mathrm{~g}^{\mathrm{n}}(\mathrm{x}, \mathrm{z})=(\mathrm{x}, \mathrm{z})
$$

This completes the proof of Part B.
If we return now to the notation of Eqs. (7) and (8), we can say that all solutions of Eq. (1) are given by the formula

$$
\begin{equation*}
\left(x_{n}, z_{n}\right)=f^{n}(0,1), \quad n=1,2,3, \cdots \tag{11}
\end{equation*}
$$

where $f$ is defined by (9).

## EXERCISES

3. Exercise 1 leads to the conjecture that successive solutions of Eq. (2) are related by the equations

$$
\begin{equation*}
u_{n}=3 u_{n-1}+4 v_{n-1}+1 \tag{12}
\end{equation*}
$$

$$
\begin{equation*}
v_{n}=2 u_{n-1}+3 v_{n-1}+1 \tag{13}
\end{equation*}
$$

Let the function g be defined by

$$
\mathrm{g}(\mathrm{u}, \mathrm{v})=(3 \mathrm{u}+4 \mathrm{v}+1, \quad 2 \mathrm{u}+3 \mathrm{v}+1)
$$

Using the method employed above, prove that all solutions in positive integers in Eq. (2) are given by

$$
\begin{equation*}
\left(u_{n}, v_{n}\right)=g^{n}(0,0), \quad n=1,2,3, \cdots \tag{14}
\end{equation*}
$$

4. Exercise 2 leads us to the conjecture that successive solutions of Eq. (3) are related by the equations

$$
\begin{equation*}
s_{n}=3 s_{n-1}+4 t_{n-1} \tag{15}
\end{equation*}
$$

$$
\begin{equation*}
t_{n}=2 s_{n-1}+3 t_{n-1} \tag{16}
\end{equation*}
$$

(Continued on p. 317.)

# HARMONIC DESIGN IN MINOAN ARCHITECTURE 

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During 1964-66 a study was undertaken of the remains of Bronze Age construction in the Aegean Basin in an attempt to establish certain underlying principles of Minoan, Mycenaean and other Aegean architectural design. Some 330 structures at 90 sites were examined and measured; evidence for planning and layout procedures, and for modular and proportional design canons were sought. The results of the project, presented (June 1968) in partial fulfillment of the requirements for a doctoral degree at Harvard, are outlined below. ${ }^{1}$

Between $\pm 2000$ BC and $\pm 1500 / 1400 \mathrm{BC}$, Minoan Crete generated the first large-scale, complex townscapes in Europe and a sophisticated architecture comparable to the contemporary architectures of Egypt and the Lewant. During this period a number of large structures, known conventionally as "palaces," were constructed at Knosses, Phaistos and Mallia, and (slightly later) at Kato Zakro, Gournia and Plati (Fig. 1). The largest of these, at Knossos, may be enclosed within a square roughly 150 meters on a side. The palaces are generally similar in groundplan; each consists of a solid mass of construction pierced by a central courtyard and elsewhere by smaller courts and light-wells. The outer trace is not uniplanar but consists of a series of recesses and projections of varying size. The buildings are in most cases two stories in elevation (in certain sections perhaps taller); one palace, at Phaist $s$, spreads over some seven terraces of varying height. The first three palaces named also have extensive paved courtyards bordering their western facades; in all cases building material is stone, frequently in the form of finely squared masonry blocks, particularly on the outer facades.

The remains have come to light only since the turn of the century; ${ }^{2}$ since that time Crete has become one of the most thoroughly explored areas of Greece;

[^2]

Fig. 1 Minoan Palatial Complexes
scores of settlements, large and small, have come to light on this island not much larger than Long Island. Surprises still come forth; one of the six palaces mentioned above (Kato Zakro) was discovered in 1963, and lastSummer (1967) a new town belonging to the Early Bronze Age ( $\pm 2700- \pm 2000 \mathrm{BC}$ ) was uncovered on the South coast.

Despite the archaeological familiarity of Minoan remains, a thorough study of the architecture has yet to be published, which fact is partly responsible for a good deal of misinformation about its nature. Another contributing cause is the great contrast Minoan architecture makes with Greek temple design of a millenium later; the great complexity and seeming irregularity of the former have provided more than one Classical scholar, trained to appreciate the apparent clarity and simplicity of the Greek temple, with nearly insuperable obstacles to understanding。 ${ }^{1}$

There are other factors contributing to the general misunderstanding, the most relevant here being that the palaces (and most notably Knossos) underwent periodic rebuilding and remodelling during the centuries of their use. This has tended to obscure the fact that each large complex was designed initially as a coherent whole. Archaeological research has shown that in some cases (e. g. , Phaistos) ${ }^{2}$ different sections of a palace were constructed at different times: necessarily, construction of such enormous structures would have been phased for varying reasons. The view of Arthur Evans that the palace of Knossos "became" a single structure as a result of the coalescence of separate buildings bordering a central piazza is today not widely held.

Unlike the later Greek temple, the Minoan palace was not designed with bilateral symmetry as its overriding principle; the organizing principles are somewhat more complex and are only now beginning to be understood. That one of the keys to the solution involves the ratios of a Fibonacci Series will
${ }^{\text {I As late as 1957, the author of one of the major textbooks on Greek architec- }}$ ture could write "It appears that the Minoans did not object to disorderly planning as such; they obviously saw no advantage in symmetry and may have been lovers of the picturesque at all costs; in fact their architecture resembles their other arts in showing no sense of form." A. W. Lawrence, Greek Architecture (Penguin, 1957), 34.
${ }^{2}$ The earlier palace at Phaistos was built in at least four separate phases, beginning on the south and working north, E. Fiandra, "I periodi struttivi del primo palazzo di Festos, " Kritika Khronika 15/16 (1961-62) 112 ff. Nevertheless, the entire plan is a unity, as demonstrated by a study of the measurements.
become apparent below. Within the last decade the researches of Prof. Graham of Toronto have provided a number of initial insights into the nature of Minoan metrology. ${ }^{1}$ He postulated the existence of abuilders ${ }^{\imath}$ modules (which he called the "Minoan Foot") used in the layout of the palaces; the value was setat $.3036{ }^{2}{ }^{2}$

That Graham's conclusions were premature was shown by the results of the aforementioned project, in which some 12,000 measurements were made on structures both on Crete and elsewhere in the Aegean and Greece: evidence of four modules was found. Of the 330 structures examined, 217 revealed clear evidence of modular usage (or at least were sufficiently well-preserved to admit of careful measurement):

| MODULE | A | B | C | D | other ${ }^{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| value: | . 2704 | . 3380 | . 4330 | . 3036 |  |
| times found: | 104 | 67 | 17 | 20 | 9 |

An immediate curiosity was that the distinction in usage revealed no consistent geographical or chronological pattern; $i_{0} e_{0}, A$ was not found in area $X$ to the exclusion of $\mathrm{B}, \mathrm{C}$, and D , nor was its use limited to one chronological period within the Bronze Age, etc. Indeed, the impression was gained that a builder was more or less free to choose any of the four in laying out his structure. ${ }^{4}$

A comparison was then made of the scales based on each module (Fig。2); it was noted that there were certain consistent points of contact among the scales of a single unit of measurement. The relationship may be expressed geometrically (Fig. 4); if a rectangle is constructed with the short side equal to .5408 , and the long side $.676(=2 \times .338)$, then the diagonal $=2 \times .434$. The relationship is $10: 16$ or $5: 8$. It is of interest also that the diagonal bisecting
${ }^{1} \mathrm{~J} . \mathrm{W}$. Graham, The Palaces of Crete (Princeton, 1962) ch. XIII, w. refs.
${ }^{2}$ Graham's study involved a sample of measurements of palace sections similar in design at various places; a by-product of a study of window-recesses, the author did not have as his purpose a comprehensive metrological examination of architectural remains of the Bronze Age in Greece.
${ }^{3}$ Six structures yielded evidence of a module of .40 , three of .45 .
${ }^{4}$ As if (to use a New Haven example) Brockett's street grid of 1643 employed rods and each of the Yale colleges was laid out on a different system (meters, feet, fathoms, etc.). Examples may be found in World Weights and Measures, UN Handbook M/21/rev. 1. (1966), for simultaneous usage of several systems in a given country.
[Dec.

| MODULE | A | B | C | D |
| :---: | :---: | :---: | :---: | :---: |
| 1 | . 2704 | . 3380 | . 4330 | . 3036 |
| 2 | . 5408 | . 6760 | . 8660 | . 6072 |
| 3 | . 8112 | 1.0140 | 1.2990 | . 9108 |
| 4 | 1.0816 | 1.3520 | 1.7320 | 1.2144 |
| 5 | 1.3520 | 1.6900 | 2.1650 | 1.5180 |
| 6 | 1.6224 | 2.0280 | 2.5980 | 1.8216 |
| 7 | 1.8928 | 2.3660 | 3.0310 | 2.1252 |
| 8 | 2.1632 | 2.7040 | 3.4640 | 2.4288 |
| 9 | 2.4336 | 3.0420 | 3.8970 | 2.7324 |
| 10 | 2.7040 | 3.3800 | 4.3300 | 3.0360 |
| 11 | 2.9744 | 3.7180 | 4.7630 | 3.3396 |
| 12 | 3.2448 | 4.0560 | 5.1960 | 3.6432 |
| 13 | 3.5152 | 4.3940 | 5.6290 | 3.9468 |
| 14 | 3.7856 | 4.7320 | 6.0620 | 4.2504 |
| 15 | 4.0560 | 5.0700 | 6.4950 | 4.5540 |
| 16 | 4.3264 | 5.4080 | 6.9280 | 4.8576 |
| 17 | 4.5968 | 5.7460 | 7.3610 | 5.1612 |
| 18 | 4.8678 | 6.0840 | 7.7940 | 5.4648 |
| 19 | 5.1376 | 6.4220 | 8.2270 | 5.7684 |
| 20 | 5.4080 | 6.7600 | 8.6600 | 6.0720 |
| 25 | 6.7600 | 8.4500 | 10.8250 | 7.5900 |
| 30 | 8.1120 | 10.1400 | 12.9900 | 9.1080 |
| 35 | 9.4640 | 11.8300 | 15.1550 | 10.6260 |
| 40 | 10.8160 | 13.5200 | 17.3200 | 12.1440 |
| 45 | 12.1680 | 15.2100 | 19.4850 | 13.6620 |
| 50 | 13.5200 | 16.9000 | 21.6500 | 15.1800 |
| 55 | 14.8720 | 18.5900 | 23.8150 | 16.6980 |
| 60 | 16.2240 | 20.2800 | 25.9800 | 18.2160 |
| 65 | 17.5760 | 21.9700 | 28.1450 | 19.7340 |
| 70 | 18.9280 | 23.6600 | 30.3100 | 21.2520 |
| 75 | 20.2800 | 25.3500 | 32.4750 | 22.7700 |
| 80 | 21.6320 | 27.0400 | 34.6400 | 24.2880 |
| 85 | 22.9840 | 28.7300 | 36.8050 | 25.8060 |
| 90 | 24.3360 | 30.4200 | 38.9700 | 27.3240 |
| 95 | 25.6880 | 32.1100 | 41.1350 | 28.8420 |
| 100 | 27.0400 | 33.8000 | 43.3000 | 30.3600 |
| note also: |  |  |  |  |
| 21 | 5.6784 | 7.0980 | 9.0930 | 6.3756 |
| 34 | 9.1936 | 11.4920 | 14.7220 | 10.3224 |
| 55 | 14.8720 | 18.5900 | 23.81 .50 | 16.6980 |
| 89 | 24.0656 | 30.0820 | 38.5370 | 27.0204 |
| 144 | 38.9376 | 48.6720 | 62.3520 | 43.7184 |
| 233 | 63.0032 | 78.7540 | 100.8890 | 70.7388 |
| 377 | 101.9408 | 127.4260 | 143.2410 | 114.4572 |
| 610 | 164.9440 | 206.1800 | 244.1300 | 185.1960 |

Fig. 2 Partial Scale of Values of the Four Modules

| METRIC | .2704 | . 3380 | . 4330 | . 3036 |
| :---: | :---: | :---: | :---: | :---: |
| . 676 | 2.5 | 2 |  |  |
| 1.08 | 4 |  | 2.5 |  |
| 1.352 | 5 | 4 |  | 4.5 |
| 1.52 |  |  | 3.5 | 5 |
| 2.02 | 7.5 | 6 |  |  |
| 2.16 | 8 |  | 5 |  |
| 2.704 | 10 | 8 |  | 9 |
| 3.04 |  | 9 | 7 | 10 |
| 3.38 |  | 10 |  | 11 |
| 4.056 | 15 | 12 |  |  |
| 4.33 | 16 | 13 | 10 |  |
| 5.408 | 20 | 16 |  | 18 |
| 6.07 |  | 18 | 14 | 20 |
| 6.76 | 25 | 20 |  |  |
| 8.112 | 30 | 24 |  |  |
| 9.10 |  | 27 | 21 | 30 |
| 9.464 | 35 | 28 |  |  |
| 9.72 | 36 |  |  | 32 |
| 10.82 | 40 | 32 | 25 | 36 |
| 11.24 |  |  | 26 | 37 |
| 12.16 | 45 | 36 | 28 | 40 |
| 13.52 | 50 | 40 |  | 45 |
| 14.872 | 55 | 44 |  |  |
| 15.21 |  | 45 | 35 | 50 |
| 16.224 | 60 | 48 |  |  |
| 18.20 |  |  | 42 | 60 |
| 20.28 | 75 | 60 |  |  |
| 21.22 |  |  | 49 | 70 |
| 21.64 | 80 |  | 50 |  |
| 27.04 | 100 | 80 |  | 90 |
| 30.36 |  | 90 | 70 | 100 |
| 33.80 | 125 | 100 |  |  |
| 40. 56 | 150 | 120 |  |  |
| 47.32 | 175 | 140 |  |  |
| 54.08 | 200 | 160 | 125 | 180 |
| 60.72 |  | 180 | 140 | 200 |
| 67.60 | 250 | 200 |  |  |
| 75.80 |  | (approximate) | 175 | 250 |

Fig. 3 Similar Metric Dimensions with Disparate Modular Values


Fig. 4 Geometric Interrelationships of the Four Modules
the central axis is . 3025 (.3036 = D). Using . 3380 as base integer, a Fibonacci Series may be generated in which all four modules appear:


The value of the first integer, . 3380, was tentatively taken as the base unit of measurement on which the three variations depend.

This quadripartite system forms the basis of the harmonic system of Minoan architectural design, and brings into focus the complicated system of relative proportions of various subsections of a structure. An excellent example is the western-facade section of the palace at Mallia (Fig. 5).

The section with which we are concerned consists of three subsections further articulated into three wall-planes, two projections and one recess per subsection. The designer gave the wall-planes the proportions shown in Fig. 6, $\mathrm{A}=8, \quad \mathrm{~B}=5+5+5, \quad \mathrm{C}=8$. The Fibonacci integers (base $=2 \times .338=$ .676) are also indicated:

| SECTION: | A | B | C |
| ---: | :---: | :---: | :---: |
| SUBSECTION: | 1.2 .3 | $4,5,6$ | $7,8,9$ |
| (actual) | $3.31 / 3.13 / 3.31$ | $6.03 / 6.06 / 6.09$ | $3.72 / 2.82 / 3.65$ |
| (ideal) | $3.38 / \mathrm{r} / 3.38$ | $6.02 / 6.02 / 6.02$ | $3.72 / \mathrm{r} / 3.72$ |
| INTEGER NO. $:$ | $4 / \mathrm{r} / 4(=.10)$ | $10 / 10 / 10$ | $9 / \mathrm{r} / 9$, | where $\underline{\mathbf{r}}=$ remainder ( $\mathrm{i}_{\mathrm{e}} \mathrm{e}_{\mathrm{o}}, \mathrm{A} 1, \mathrm{~A} 3, \mathrm{C} 7, \mathrm{C} 9$ were staked out from outer edges inward; $\underline{r}$ having a metrological value of null). Note also that both $A$ and C approximate in toto the 11th Fibonacci Integer of this series ( 9.75 vs .9 .75 (A) and 10.19 (C)); the latter is in error by . 44 , or one unit of Module C. (.433), the module generally used in the layout of the palace.

The system of proportions employed by the Minoan architect in the detailed articulation of the perimetral walls extends also to the underlying grid of a palace's groundplan. While space prohibits detailed examination of the procedures in a palace's layout, the following general points may be noted. ${ }^{1}$

The palaces of Knossos, Phaistos and Mallia share the following designcharacteristics. Each plan may be generated by a series of steps involving a

[^3]

Fig. 5 Mallia Palace


Fig. 6 Analysis of Mallia West Facade
grid based on a rectangle with the proportion 5:8 (160B $\times 200 \mathrm{~B}$ ). The basic rectangle was presumably laid out with pegs and ropes of fixed length. The center point was found (by means of diagonals or rope, which would have to be 200 C in length ${ }^{1}$ ), and four quadrants were further indicated. The two eastern quadrants will delimit the central court, the two western the west-central block of the palace. At the center of the overall rectangle (or along an EW axis passing through that point) was constructed a room of (presumably) some ritual significance, the so-called Pillar Crypt.

The entire palace may be generated by subdivisions or additions of fixed modular size to the centralgrid-rectangle; the procedures vary in detail among the palaces. It is noteworthy that the subdivisions of a grid coincide generally with the functional subdivisions of a palace.

A Palace maybe described as a grid of squares of varying size, the sizes determined by a sequence of interrelationships based on proportions such as $3: 5,5: 8,8: 13$, etc., as well as $1: 2$, and $1: 1$. Figure 7 gives an indication of the manner in which the designer generated his plan. Square 2 of this west facade of Knossos is related to 3 as $3: 5$ : square 5 is to square 4 as $2: 3$. It is also notable that the number of long storage magazines in each grid-block is directly related to the modular size of each square; thus, squares of 30 units have three magazines, those of 50 have 5 , etc.

The use of Fibonacci numbers also pervades the design of non-architectural items made by the Minoans. A simple example is shown in Fig. 8, the famous Sarcophagos of Haghia Triadha near Phiastos, whose painted sides are an important source of information on Minoan funerary ritual. The various parts of this limestone coffin reveal the relationship $1,2,3,5$.

A more complex example is the design of a large gaming board found in the Knossos palace. The nature of the game is not yet understood; pieces resembling chess pawns have been found, whose bases in size match the diameters of the circles of the inlaid board. The design may be analyzed as indicated in the diagram; the sequence of six integers with .067 as base is found in various subsections of the board; overall proportion is 8:13 (Fig. 9).

[^4]

Fig. 7 Analysis of Knossos West Facade


Fig. 8 Analysis of Flank of Haghia Triadha Sarcophagos


Fig. 9 Analysis of Knossos Gaming Board: Fibonacci Proportions

This brief consideration of various aspects of Minoan design and its relation to the Fibonacci system ${ }^{1}$ would best be concluded with the following observation. We are not now in a position to understand the full significance of the harmonic system of Minoan design. The principles of the Fibonacci system were certainly understood, as evidenced by the monuments themselves. Whether these principles were a trade secret of a small class of artisans or more widely understood is not known. ${ }^{2}$ Of several scripts used by the Minoans during their history, only one, the last one to be employed, has been deciphered; it is a primitive form of Greek, and it was employed for bureaucratic purposes (listing of commodities, produce, etc.) only; no literature survives, and we certainly have no mathematical treatises. ${ }^{3}$

It is now known that contemporary Egyptian architecture reveals design principles based in part on the Fibonacci system; ${ }^{4}$ it has been known for some time that early in the palatial period Minoan craftsmen were employed in the construction of at least one major Egyptian monument. ${ }^{5}$ It would seem reasonable to assume that such a situation would provide an opportunity for the diffusion to Crete of the principles of this system. If this was the case, ${ }^{6}$ it should be borne in mind that it was only the principles which were diffused, for the Minoan system is grounded in a Minoan metrological system, and the Egyptian is based on a native cubit-measure system.

Whatever the case, the essential point remains: in laying the foundations of architecture in Europe, the Minoan architect designed his structures
${ }^{1}$ Discussed in detail as it applies to some 50 structures in the author's dissertation, "Minoan Palace Planning and its Origins" (Harvard 1968) Chapter III (unpublished).
${ }^{2}$ One of the primary ritual symbols of Minoan Crete, the "double-axe" sign, incised on walls within the palaces, may be an ideogram of the 5:8 triangle of Fig. 4; for a similar situation, cf. A. Badawy, Ancient Egyptian Architectural Design (UCLA 1965) 40-46.
${ }^{3}$ Cf. J. Chadwick, The Decipherment of Linear B (New York 1958); M. Ventris and J. Chadwick, Documents in Mycenaean Greek (Cambridge 1959) 117.
${ }^{4}$ A. Badawy, op. cit., part IV.
${ }^{5}$ W. M. F. Petrie, Illahun, Kahun and Gurob (London 1891) ch. 3.
${ }^{6}$ Ibid., 14. A measuring rod of $.676 \pm$ was found at Kahun $\left(\frac{1}{2}=.338\right)$ which (not being an Egyptian measure) might be connected with the Minoan workmen employed in the construction of the pyramid of Sesostris II. On the other hand, Levantine workmen were also employed there; there is no firmbasis for deciding to whom the rod should be attributed.
[Continued on p. 317.]

# RECREATIONAL MATHEMATICS 

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## A FIBONACCI CONSTANT

I would like to introduce a new constant, if it hasn't been done before. If you evaluate the continued fraction

$$
1+\frac{1}{2+\frac{3}{5+\frac{8}{13+\frac{21}{\text { etc. }}}}}
$$

you obtain $1.3941865502 \cdots$. Readers of this Journal will note immediately that the terms of this continued fraction are successive Fibonacci terms. Perhaps someone will evaluate this constant to many more decimal places, give it a reasonably good name (or Greek letter), and discover some interesting properties of the number.

## INSTANT DIVISION

If you wish to divide 717948 by 4 merely move the initial 7 to the other end, obtaining 179487. This is about as instant as you can be - or is it? Much larger numbers can be divided just as easily:

$$
9,130,434,782,608,695,652,173
$$

can be divided by 7 by transposing the initial 9 to the end, obtaining
1,304,347,826,086,956,521,739 •

An article by Charles W. Trigg [1] described three methods of finding the smallest integer $N$, such that when its initial digit, $d$, is transposed to
the right end of the integer, the result is $N / \mathrm{d}$. Trigg's article restricted d to a single digit and N as the smallest integer satisfying the condition. The idea of instant arithmetic is not new, having appeared in the Fibonacci Quarterly $[2,3]$ and elsewhere $[4,5]$.

I wondered if there were other integers, $N$, such that when any one or more of the initial digits were transposed intact to the right, the result would be $\mathrm{N} / \mathrm{k}$, where k is any integer. In other words, as an example, is there an integer which can be divided by 7 by moving its initial digits, 317 , to the right? The answer is yes. Although not all integers possess the desired property, there are an infinite number of integers that do.

Trigg [1] shows that, for single-digit transposition

$$
F=\frac{d^{2}}{10 d-1}
$$

where $d$ is the initial digit to be transposed to the right and $F$ is the proper fraction which, when written as a decimal for one period, or cycle, represents the integer sought. If $\mathrm{d}=4$, for example, we have

$$
F=16 / 39=.410256410256 \cdots
$$

Therefore, the smallest integer which can be divided by 4 by transposing the initial digit to the right is 410256 .

Now, I will show how to find integers such that the transposition is not restricted to single digits, nor need N be divisible by the transposed digits. Following Trigg's format, let $D$ represent the initial digit or digits to be transposed from left to right, $k$ the divisor of $N$, the integer sought. Then

$$
\mathrm{N}=0 . \mathrm{D} \cdot \mathrm{D} \cdot \mathrm{D} \cdot \mathrm{D}
$$

If D has n digits, we multiply by $10^{\mathrm{n}}$,

$$
10^{\mathrm{n}} \mathrm{~N}=\mathrm{D} \cdots \mathrm{D} \cdots \mathrm{D} \cdots
$$

and

$$
\mathrm{N} / \mathrm{k}=0 . \cdots \mathrm{D} \cdot \mathrm{D} \cdot \cdots
$$

## Therefore

$$
10^{n_{N}}-N / k=D,
$$

or
(1)

$$
\mathrm{N}=\frac{\mathrm{Dk}}{10^{\mathrm{n}} \mathrm{k}-1}
$$

This now allows us to find an $N$ for any integer values of $D$ and $k$. Here are several examples:


Instant division in other bases can be done also. We have, for any base b

$$
N=\frac{D k}{b^{n} k-1}
$$

but, since $b$ in base $b$ is always 10 , we have

$$
\mathrm{N}=\frac{\mathrm{Dk}}{10^{n_{k}-1}}
$$

So the same equation used before works in any base as long as $d, n, k, 10^{n_{k}}$ -1 , and all calculations are in the given base. Some examples:

| Base | $\frac{\mathrm{D}}{n}$ | $\frac{\mathrm{n}}{2}$ | $\frac{\mathrm{k}}{2}$ |  | N |
| :--- | :---: | :---: | :---: | :---: | :--- | :--- |
| Three | 12 | 2 | 2 | $101 / 122$ | $=1202122110201001$ |
| Four | 23 | 2 | 3 | $201 / 233$ | $=23032332220312131331101$ |
| Five | 13 | 2 | 31 | $1003 / 3044$ | $=1310022231202000303$ |

## BIZLEY'S PROBLEM AND INST ANT MULTIPLICATION

In the solution to [5] the Editor notes that M. T. L. Bizley said a more difficult problem would be to determine all rational numbers $q / p$ such that an integer can be found which will increase in the ratio $p: q$ when the digit on the extreme left is moved to the extreme right. Trigg's work in [1] brought me to the general solution to the problem of instant division, and that general solution allowed me to solve Bizley's problem. A solution to Bizley's problem would automatically enable one to multiply instantly by transposing digits from left to right.

In Equation (1) above substitute $q / p$ for $k$, obtaining

$$
\begin{equation*}
\mathrm{N}=\frac{\mathrm{Dq}}{10^{n} \mathrm{q}-\mathrm{p}} \tag{2}
\end{equation*}
$$

A few solutions are given below.

| D | p | q | N |
| :---: | :---: | :---: | :---: |
| 1 | 3 | 2 | 1,176,470,588,235,294 |
| 2 | 3 | 2 | 2,352,941,176,470,588 |
| 3 | 7 | 11 | 3,203,883,495,145,631,067,961,165,048,543,689 |
| 4 | 3 | 2 | 4,705,882,352,941,176 |
| 4 | 2 | 3 | 428,571 |
| 11 | 7 | 3 | $\begin{aligned} & 11,262,798,634,812,286,689,419,795,221,843,003,412,969,- \\ & 283,276,450,511,945,392,491,467,576,791,808,873,720,136,- \\ & 518,771,331,058,020,477,815,699,658,703,071,672,354,948,- \\ & 805,460,750,853,242,320,819 \text { (146 digits) } \end{aligned}$ |

In the examples above, moving $D$ to the right multiplies $N$ by $p / q$.
However, there are certain restrictions on the values of $D, p$, and $q$ in Equation (2), otherwise the results obtained by using the equation are not solutions. For example, if we let $D=6, \stackrel{\mathrm{p}}{\mathrm{D}}=3$, and $\mathrm{q}=2$, we obtain

$$
N=7,058,823,529,411,764
$$

which is not a solution for two reasons: the initial digit, 7 , is not equal to D, nor is the integer produced by transposing the 7 to the right in the ratio $3: 2$ to the calculated N .

Tentatively, I have found that, for proper solutions Dq must be less than $(q / p)\left(10^{n} q-p\right)$. Perhaps readers can provide further insight, or provide definite criteria.

NOTE: In [3, problem 2] it is proven that there is no integer which is doubled when the initial digit is transposed to the right. However, I found several integers which almost meet the condition:

| 124999 | $\cdots$ | 999 | and | 125000 | $\cdots$ | 000 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 249999 | $\cdots$ | 999 | and | 250000 | $\cdots$ | 000 |
| 374999 | $\cdots$ | 999 | and | 375000 | $\cdots$ | 000 |

By including as many $9^{\prime}$ s or $0^{\prime} s$ as necessary, you can get as close to doubling as you desire. It is possible, however, to double by moving two or more digits to the right. Let $\mathrm{D}=10, \mathrm{p}=2$, and $\mathrm{q}=1$ to obtain
$\mathrm{N}=102,040,816,326,530,612,244,897,959,183,673,469,387,755$.

## TRIANGLE DISSECTIONS

Mel Stover first asked [1] if it is possible to cut an obtuse triangle in smaller triangles, all of them acute. It was proven that it canbe done and that no more than seven acute triangles are necessary [2]. Martin Gardner [1] showed that a square can be dissected into no less than eight acute triangles, and then asked if a square could be dissected into less than eleven acute isosceles triangles. In the following paper by V. E. Hoggatt, Jr., and Free Jamison, the answer is given.

DISSECTION OF A SQUARE INTO n ACUTE ISOSCELES TRIANGLES

> VERNER E. HOGGATT, JR. , AND FREE JAMISON

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In answer to Martin Gardner's query [3] as to whether a square can be dissected into less than eleven acute isosceles triangles, the answer is in the affirmative. We will also show that a square can be dissected into $n$ acute isosceles triangles for $\mathrm{n} \geq 10$.
Step 1: The 10-Piece Dissection
Dissect a square into the four triangles shown in Figure 1.
Jamison [4] applies the lemma implied by Figure 2. Thus, since triangle A may be dissected into seven acute isosceles triangles, it follows that a square may be dissected into 10 acute isosceles triangles.


Figure 1


Figure 2

Step 2.
If, in Figure 3 (which is Triangle A of Figure 1), we cut off an isosceles triangle of vertex angle 15 , the remaining triangle is obtuse with $\underline{A}=15^{\circ}$, $\underline{B}=97.5^{\circ}$, and $\underline{C}=67.5^{\circ}$. In [5] it was proven that any obtuse triangle can be dissected into eight acute isosceles triangles. However, if an obtuse triangle is such that $\underline{B}>90^{\circ}, \underline{B}-\underline{A}<90^{\circ}$, and $\underline{B}-\underline{C}<90^{\circ}$, then only seven are needed. Thus, we can also cut a square into eleven acute isosceles triangles.


Figure 3
Step 3.
Let the triangle with angles $15^{\circ}, 97.5^{\circ}$, and $67.5^{\circ}$ (which can be cut into seven already) have an isosceles triangle with vertex angle $15^{\circ}$ removed, leaving a triangle with angles $15^{\circ}, 67.5^{\circ}$, and $97.5^{\circ}$ which can be cutinto seven acute isosceles triangles. Thus we can now cut a square into twelve acute isosceles triangles. But this last step can be repeated as many times as needed to get any $\mathrm{n} \geq 10$ (recall we already have 10, 11, and 12). However, at the point where you had the 10 -piece dissection, you can draw lines joining the midpoints of, say, the equilateral triangle (in Fig. 1) to go from 10 to 13. Then Steps 2 and 3 can go from 13 to 14 to 15 . You can then cut one of the remaining equilateral triangles into four equilateral triangles.

Thus, for any large $n$, we may have mostly equilateral triangles if desired, or, for that matter, one of any shape as in the 10 -piece dissection. REFERENCES

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## LINEAR RECURSION RELATIONS <br> LESSON TWO

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Recursion relations can be set up at will. There is, however, a particular type known as the linear recursion relation which by its simplicity, range of application, and interesting mathematical properties deserves special consideration. In this lesson, the linear recursion relation will be described and the method of expressing its terms by means of the roots of an auxiliary equation analyzed. These basic ideas will be applied and amplified in greater detail in succeeding articles.

The term "linear" in mathematics is used by way of analogy with the equation of a straight line in the plane where the variables $x$ and $y$ do not enter in a degree higher than the first. By extension, there are linear equations in more variables which characterize the plane in three-space, the hyperplane in four-space, etc. By further analogy, one speaks of linear differential equations in which the dependent variable and its derivatives are not found in a degree higher than one. In this context it is natural to call a recursion relation of the form:
(1)

$$
\begin{equation*}
T_{n+1}=a_{1} T_{n}+a_{2} T_{n-1}+a_{3} T_{n-2}+\cdots+a_{r} T_{n-r+1} \tag{1}
\end{equation*}
$$

where the $a_{i}$ are constants, a linear recursion relation. If $a_{r}$ is the last non-zero coefficient, then this would be spoken of as a linear recursion relation of order $r$.

Note that there is no allowance for a constant term. This, however, is no restriction. If, for example,

$$
T_{n+1}=3 T_{n}-2 T_{n-1}+4 T_{n-2}+8
$$

then since

$$
T_{n}=3 T_{n-1}-2 T_{n-2}+4 T_{n-3}+8
$$

it follows by subtraction that

$$
T_{n+1}=4 T_{n}-5 T_{n-1}+6 T_{n-2}-4 T_{n-3}
$$

so that a linear recursion relation of the standard form (1) can be obtained from this variant.

## LINEAR RECURSION RELATION OF THE FIRST ORDER

The linear recursion relation of the first order is
(2)

$$
T_{n+1}=r T_{n}
$$

in which each term is a fixed multiple of the previous term. Evidently, this is the recursion relation of a geometric progression. In terms of the technique that is being developed for relating the terms of the sequence with the roots of an auxiliary equation, we set up the equation corresponding to this recursion relation, namely:

$$
\begin{equation*}
x-x=0 \tag{3}
\end{equation*}
$$

which has the one root $x$. The term of the sequence can be written as a multiple of the $n^{\text {th }}$ power of this root, thus:

$$
\mathrm{T}_{\mathrm{n}}=(\mathrm{a} / \mathrm{r}) \mathrm{r}^{\mathrm{n}}
$$

That this term satisfies the recursion relation (2) follows from (3), since on substituting $r$ for $x$, we have:

$$
\mathrm{r}=\mathrm{r},
$$

and on multiplying both sides by $\mathrm{r}^{\mathrm{n}-1}$,

$$
r^{n}=r \cdot r^{n-1}
$$

Note that the powers of the root have the same recursion relation as the terms (2)! So if

$$
\mathrm{T}_{\mathrm{n}+1}=(\mathrm{a} / \mathrm{r}) \mathrm{r}^{\mathrm{n}+1}
$$

and

$$
\begin{gathered}
r^{n+1}=r \cdot r^{n} \\
T_{n+1}=r(a / r) r^{n}=r T_{n}
\end{gathered}
$$

Perhaps due to the simplicity of this case, the considerations are confusing! But let us pass on to a second-order linear relation where the operations are not so obvious.

## SECOND-ORDER LINEAR RECURSION RELATIONS

In a subsequent article, these relations will be taken up in all possible detail to cover the various situations that may arise. But here we shall start with a simple example to show how the method operates.

Consider then a linear recursion relation

$$
\begin{equation*}
T_{n+1}=5 T_{n}-6 T_{n-1} \tag{4}
\end{equation*}
$$

If all terms are brought to one side and equated to zero, the result is:

$$
\begin{equation*}
T_{n+1}-5 T_{n}+6 T_{n-1}=0 \tag{5}
\end{equation*}
$$

If now the successive terms are replaced bypowers of $x$ one obtains the auxiliary equation

$$
\begin{equation*}
x^{2}-5 x+6=0 \tag{6}
\end{equation*}
$$

whose roots are $r=3, s=2$. Since they satisfy the equation (5), it follows that

$$
\begin{aligned}
& \mathbf{r}^{2}=5 \mathrm{r}-6 \\
& \mathrm{~s}^{2}=5 \mathrm{~s}-6
\end{aligned}
$$

Since we may multiply by any power of $\mathbf{r}$ or s ,

$$
\begin{align*}
& \mathrm{r}^{\mathrm{n}+1}=5 \mathrm{r}^{\mathrm{n}}-6 \mathrm{r}^{\mathrm{n}-1}  \tag{7}\\
& \mathrm{~s}^{\mathrm{n}+1}=5 \mathrm{~s}^{\mathrm{n}}-6 \mathrm{~s}^{\mathrm{n}-1}
\end{align*}
$$

Note that the powers of $r$ and $s$ satisfy the same recursion relation (4) as the terms of the sequence $T_{n}$. Hence if we express these terms as linear combinations of powers of $r$ and $s$, we should obtain expressions that satisfy the recursion relation (4). Set

$$
\begin{align*}
T_{n-1} & =a r^{n-1}+b s^{n-1}  \tag{8}\\
T_{n} & =a r^{n}+b s^{n}
\end{align*}
$$

where a and b are constants. Then

$$
T_{n+1}=5 T_{n}-6 T_{n-1}=a\left(5 r^{n}-6 r^{n-1}\right)+b\left(5 s^{n}-6 s^{n-1}\right)
$$

or

$$
\mathrm{T}_{\mathrm{n}+1}=a \mathrm{r}^{\mathrm{n}+1}+\mathrm{bs} \mathrm{n}^{\mathrm{n}+1}
$$

so that the form of the term persists for all values of $n$ once it is established for two initial values.

What this implies is that given any two starting values $\mathrm{T}_{1}=\mathrm{p}, \mathrm{T}_{2}=\mathrm{q}$ it is possible to find a sequence

$$
\mathrm{T}_{\mathrm{n}}=\mathrm{a} 3^{\mathrm{n}}+\mathrm{b} 2^{\mathrm{n}}
$$

satisfying the recursion relation (4). Consider the particular case $p=2$, $q=7$. Then we should have:

$$
\begin{aligned}
& 2=a \cdot 2+b \cdot 3 \\
& 7=a \cdot 2^{2}+b \cdot 3^{2}
\end{aligned}
$$

Solving for a and b we obtain $\mathrm{a}=-1 / 2, \mathrm{~b}=1$, so that in general,

$$
\mathrm{T}_{\mathrm{n}}=(-1 / 2) 2^{\mathrm{n}}+3^{\mathrm{n}}
$$

If the roots $r$ and $s$ are real and distinct with $r s \neq 0$, it will always be possible to solve the above set of equations for the determinant of the coefficients of the equations:

$$
\begin{aligned}
& \mathrm{p}=\mathrm{ar}+\mathrm{bs} \\
& \mathrm{a}=a r^{2}+b s^{2}
\end{aligned}
$$

is

$$
\left|\begin{array}{ll}
\mathrm{r} & \mathrm{~s} \\
\mathrm{r}^{2} & \mathrm{~s}^{2}
\end{array}\right|=\mathrm{rs}(\mathrm{~s}-\mathrm{r})
$$

which is not zero if $\mathrm{rs} \neq 0$ and $\mathrm{s} \neq \mathrm{r}$.
These considerations can be extended to relations of higher order. For example, suppose we wish to express the terms of a sequence beginning with $3,8,14$ in the form:

$$
\mathrm{T}_{\mathrm{n}}=\mathrm{a} 2^{\mathrm{n}}+\mathrm{b} 3^{\mathrm{n}}+\mathrm{c} 5^{\mathrm{n}}
$$

It is simply necessary to set up a recursion relation with roots 2,3 , and 5 . Thus the auxiliary equation would be

$$
(x-2)(x-3)(x-5)=0
$$

or

$$
x^{3}=10 x^{2}-31 x+30
$$

so that

$$
T_{n+1}=10 T_{n}-31 T_{n-1}+30 T_{n-2}
$$

giving sequence terms as follows:

$$
3,8,14,-18,-374,-2762,-16566, \cdots
$$

To express $T_{n}$ in terms of the powers of the roots use the initial values to form equations as follows.

$$
\begin{aligned}
3 & =2 a+3 b+5 c \\
8 & =4 a+9 b+25 c \\
14 & =8 a+27 b+125 c
\end{aligned}
$$

from which $\mathrm{a}=-5 / 6, \mathrm{~b}=2, \mathrm{c}=-4 / 15$. Thus

$$
\mathrm{T}_{\mathrm{n}}=(-5 / 6) 2^{\mathrm{n}}+2 \cdot 3^{\mathrm{n}}+(-4 / 15) 5^{\mathrm{n}}
$$

Evidently, there are many questions that require further study; the case of equal roots of the auxiliary equation; what happens if the roots are irrational; the situation in which the roots are complex; and various combinations of these cases. Such matters will receive attention in a number of subsequent lessons.

## PROBLEMS

1. Find the recursion relation for the sequence beginning 3,10 with terms in the form

$$
T_{n}=a+2^{n} b
$$

and calculate the first ten terms of the sequence.
2. Given the sequence beginning with 5,12 having a recursion relation

$$
T_{n+1}=8 T_{n}-15 T_{n-1}
$$

express $T_{n}$ as a linear combination of powers of the roots of the auxiliary equation.
3. The sequence

$$
5,13,61,349,2077,12445,74653,447901, \ldots
$$

obeys a linear recursion relation of the second order. Find this relation and express $T_{n}$ as a.linear combination of powers of the roots of the auxiliary equation.
4. A sequence with initial terms $3,7,13$ has an auxiliary equation

$$
x^{3}-6 x^{2}+11 x-6=0
$$

Express the term $T_{n}$ as a linear combination of powers of the roots of this equation.
5. A third-order recursion relation governs the terms of the sequence: $1,6,14,45,131,396,1184,3555,10661,31986,95954,287865,8635.39$.
Determine the coefficients in this recursion relation and express the term $T_{n}$ as a linear combination of powers of the roots of the auxiliary equation.

## LESSON TWO SOLUTIONS

1. $\quad T_{n}=-4+(7 / 2) 2^{n}$

First ten terms: 3, 10, 24, 52, 108, 220, 444, 892, 1788, 3580.
2. $\quad \mathrm{T}_{\mathrm{n}}=(13 / 6) 3^{\mathrm{n}}+(-3 / 10) 5^{\mathrm{n}}$
3.

$$
\begin{aligned}
\mathrm{T}_{\mathrm{n}} & =17 / 5+(4 / 15) 6^{\mathrm{n}} \\
\mathrm{~T}_{\mathrm{n}+\mathrm{i}} & =7 \mathrm{~T}_{\mathrm{n}}-6 T_{\mathrm{n}-1}
\end{aligned}
$$

4. 

$$
\mathrm{T}_{\mathrm{n}}=-2+3 \cdot 2^{\mathrm{n}}+(-1 / 3) 3^{\mathrm{n}}
$$

5. 

$$
\begin{gathered}
T_{n+1}=3 T_{n}+T_{n-1}-3 T_{n-2} \\
T_{n}=1 / 4+(7 / 8)(-1)^{n}+(13 / 24) 3^{n}
\end{gathered}
$$

# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by
A. P. HILLMAN

University of New Mexico, Albuquerque, New Mex.
Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

B-148 Proposed by David Englund, Rockford College, Rockford, Illinois, and Malcolm Tallman, Brooklyn, New York.

Let $\mathrm{F}_{\mathrm{n}}$ and $\mathrm{L}_{\mathrm{n}}$ denote the Fibonacci and Lucas numbers and show that

$$
\left.\left.F_{(2} t_{n}\right)=F_{n} L_{n} L_{2 n} L_{4 n} \cdots L_{(2}^{t-1} n_{n}\right)
$$

B-149 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show that

$$
\mathrm{L}_{\mathrm{n}+1} \mathrm{~L}_{\mathrm{n}+3}+4(-1)^{\mathrm{n}+1}=5 \mathrm{~F}_{\mathrm{n}} \mathrm{~F}_{\mathrm{n}+4}
$$

B-150 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show that

$$
L_{n}^{2}-F_{n}^{2}=4 F_{n-1} F_{n+1}
$$

B-151 Proposed by Hal Leonard, San Jose State College, San Jose, Calif.
Let $m=L_{1}+L_{2}+\cdots+L_{n}$ be the sum of the first $n$ Lucas numbers.
Let

$$
P_{n}(x)=\prod_{n}^{n}\left(1+x^{L_{i}}\right)=a_{0}+a_{1} x+\cdots+a_{m} x^{m}
$$

$$
i=1
$$

Let $q_{n}$ be the number of integers $k$ such that both $0<k<m$ and $a_{k}=0$. Find a recurrence relation for the $q_{n}$.

B-152 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.
Prove that

$$
F_{m+n}=F_{m+1} F_{n+1}-F_{m-1} F_{n-1}
$$

B-153 Proposed by Klaus-Gunther Recke, Gottingen, Germany.
Prove that

$$
\mathrm{F}_{1} \mathrm{~F}_{3}+\mathrm{F}_{2} \mathrm{~F}_{6}+\mathrm{F}_{3} \mathrm{~F}_{9}+\cdots+\mathrm{F}_{\mathrm{n}} \mathrm{~F}_{3 \mathrm{n}}=\mathrm{F}_{\mathrm{n}} \mathrm{~F}_{\mathrm{n}+1} \mathrm{~F}_{2 \mathrm{n}+1}
$$

## SOLUTIONS

GOLDEN RATIO AGAIN?
B-130a Proposed by Sidney Kravitz, Dover, N. Jersey.
An enterprising entrepreneur in an amusementpart challenges the public to play the following game. The player is given five equal circular discs which he must drop from a height of one inch onto a larger circle in such a way that the five smaller discs completely cover the larger one. What is the maximum ratio of the diameter of the larger circle to that of the smaller ones so that the player has the possibility of winning?

## Partial Solution by the Proposer.

With the centers of the smaller circles placed at the vertices of a regular pentagon, the smaller circles cover the larger one with a ratio of diameters equal to the golden ratio $(1+\sqrt{5}) / 2$. There may exist another arrangement of the five circles which results in a smaller ratio.

## EVEN AND ODD SEQUENCES

B-131a Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.
Let $\left\{H_{n}\right\}$ be a generalized Fibonacci sequence, i. e., $H_{0}=q, H_{1}=p$, $H_{n+2}=H_{n+1}+H_{n}$. Extend, by the recursion formula, the definition to include negative subscripts. Show that if $\left|H_{-n}\right|=\left|H_{n}\right|$ for all $n$, then $\left\{H_{n}\right\}$ is a constant multiple of either the Fibonacci or the Lucas sequence.

Solution by David Zeitlin, Minneapolis, Minnesota.

$$
H_{n}=q F_{n+1}+(p-q) F_{n}
$$

and since $F_{-n}=(-1)^{n+1} F_{n}$, we have

$$
\left|H_{-n}\right|=\left|(-1)^{n}\left(q F_{n-1}-(p-q) F_{n}\right)\right|=\left|q F_{n-1}-(p-q) F_{n}\right|
$$

If $\left|H_{1}\right|=\left|H_{-1}\right|$, then (a) $p-q=p$ or (b) $p-q=-p$. If (a) holds, then $q$
$=0$ and $H_{n} \equiv \mathrm{pF}_{\mathrm{n}}$; if (b) holds, then $\mathrm{q}=2 \mathrm{p}$, and

$$
H_{n} \equiv 2 p F_{n+1}-p F_{n}=p L_{n}
$$

Remark. Let $\mathrm{U}_{\mathrm{m}}$ and $\mathrm{V}_{\mathrm{n}}$ be solutions of

$$
\mathrm{W}_{\mathrm{n}+2}=a \mathrm{~W}_{\mathrm{n}+1}+\mathrm{b} \mathrm{~W}_{\mathrm{n}}
$$

where $U_{0}=0, U_{1}=1$ and $V_{0}=2, V_{1}=a$ (if $a=b=1$, then $U_{n} \equiv F_{n}$ and $\mathrm{V}_{\mathrm{n}} \equiv \mathrm{L}_{\mathrm{n}}$ ). If

$$
\left|\mathrm{b}^{\mathrm{n}_{-\mathrm{n}}}\right|=\left|\mathrm{w}_{\mathrm{n}}\right|
$$

for all $n$, then $\left\{W_{n}\right\}$ is a constant multiple of either $\left\{U_{n}\right\}$ or $\left\{V_{n}\right\}$.
Also solved by Herta T. Freitag, John Ivie, D. V. Jaiswal (India), Bruce W. King, C. B. A. Peck, A. C. Shannon (Australia), and the proposer.

## EXPONENT PROBLEM

B-132 Proposed by Charles R.Wall, University of Tennessee, Knoxville, Tenn.
Let $u$ and $\dot{v}$ be relatively prime integers. We say that $u$ belongs to the exponent $d$ modulo $v$ if $d$ is the smallest positive integer such that $u^{d}$ $\equiv 1(\bmod v)$. For $n \geq 3$ show that the exponent to which $F_{n}$ belongs modulo $\mathrm{F}_{\mathrm{n}+1}$ is 2 if n is odd and 4 if n is even.

Solution by the proposer.
From

$$
F_{n+1} F_{n-1}-F_{n}^{2}=(-1)^{n}
$$

we have

$$
F_{n}^{2} \equiv(-1)^{n+1}\left(\bmod F_{n+1}\right)
$$

Now $\mathrm{F}_{\mathrm{n}} \neq 1\left(\bmod \mathrm{~F}_{\mathrm{n}+1}\right)$ as $1 \neq \mathrm{F}_{\mathrm{n}}<\mathrm{F}_{\mathrm{n}+1}$ for $\mathrm{n} \geq 3$. If n is odd then $\mathrm{F}_{\mathrm{n}}^{2} \equiv 1\left(\bmod \mathrm{~F}_{\mathrm{n}+1}\right)$. If n is even then $\mathrm{F}_{\mathrm{n}}^{2} \equiv-1\left(\bmod \mathrm{~F}_{\mathrm{n}+1}\right)$. Now

$$
\mathrm{F}_{\mathrm{n}}^{3} \equiv-\mathrm{F}_{\mathrm{n}} \equiv \mathrm{~F}_{\mathrm{n}-1}\left(\bmod \mathrm{~F}_{\mathrm{n}+1}\right)
$$

and $\mathrm{F}_{\mathrm{n}-1} \neq 1$ as $\mathrm{n} \geqslant 4$ (since n is even). But then

$$
\mathrm{F}_{\mathrm{n}}^{4} \equiv(-1)^{2}=1\left(\bmod \mathrm{~F}_{\mathrm{n}+1}\right)
$$

Also solved by D. V. Jaiswal (India) and A. C. Shannon (Australia).

## AN OLD PROBLEM IN FIBONACCI CLOTHES

B-133 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.
Let $r=F_{1000}$ and $s=F_{1001}$. Of the two numbers $r^{s}$ and $s^{r}$, which is the larger?

Solution by Phil Mana, University of New Mexico, Albuquerque, N. Mexico.

Since $(\ln x) / x$ is monotonically decreasing for $x>e$,

$$
(\ln r) / r>(\ln s) / s
$$

or

$$
\ln r^{1 / r}>\ln s^{1 / s}
$$

Since $\ln x$ is monotonically increasing for $x>0$, this implies that $r^{1 / r}>$ $s^{1 / s}$. Hence $r^{s}>s^{r}$.

Also solved by William D. Jackson, George F. Lowerre, Arthur Marshall, C.B.A. Peck, D. Zeitlin, and the proposer.

## A TELESCOPING SUM

B-134 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.
Define the sequence $\left\{a_{n}\right\}$ by

$$
a_{1}=a_{2}=1, \quad a_{2 k+1}=a_{2 k}+a_{2 k-1},
$$

and

$$
a_{2 k}=a_{k}
$$

for $k \geq 1$. Show that

$$
\sum_{k=1}^{n} a_{k}=a_{2 n+1}-1, \quad \sum_{k=1}^{n} a_{2 k-1}=a_{4 n+1}-a_{2 n+1}
$$

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

$$
\begin{aligned}
\sum_{k=1}^{n} a_{k}=\sum_{k=1}^{n} a_{2 k}=\left(a_{3}-a_{1}\right)+\left(a_{5}-a_{3}\right) & +\cdots+\left(a_{2 n+1}-a_{2 n-1}\right) \\
& =a_{2 n+1}-a_{1}=a_{2 n+1}-1 .
\end{aligned}
$$

Then,

$$
\begin{aligned}
& \sum_{k=1}^{n} a_{2 k-1}=\sum_{k=1}^{n} a_{2 k-1}+\sum_{k=1}^{n} a_{2 k}-\sum_{k=1}^{n} a_{2 k}=\sum_{k=1}^{2 n} a_{k}-\sum_{k=1}^{n} a_{k} \\
&=\left(a_{4 n+1}-1\right)-\left(a_{2 n+1}-1\right)=a_{4 n+1}-a_{2 n+1} .
\end{aligned}
$$

Also solved by L. Carlitz, Herta T. Freitag, John Ivie, D. V. Jaiswal (India), Bruce W. King, George F. Lowerre, C. B. A. Peck, A. C. Shannon (Australia), C. R. Wall, Howard L. Walton, David Zeitlin and the proposer.

GENERALIZED SUMS
B-135 Proposed by L. Carlitz, Duke University, Durham, No. Carolina. Put

$$
F_{n}^{\prime}=\sum_{k=0}^{n-1} F_{k^{2}} 2^{n-k-1}, \quad L_{n}^{\prime}=\sum_{k=0}^{n-1} L_{k} 2^{n-k-1}
$$

Show that, for $\mathrm{n} \geq 1$,

$$
F_{n}^{\prime}=2^{n}-F_{n+2}, \quad L_{n}^{\prime}=3 \cdot 2^{n}-L_{n+2}
$$

Solution by Charles R. Wall, University of Tennessee, Knoxville; Tennessee .

Let $\left\{H_{n}\right\}$ be a generalized Fibonacci sequence, and define

$$
H_{n}^{\prime}=\sum_{k=0}^{n-1} H_{k} 2^{n-k-1}
$$

Then we claim that
(A)

$$
\mathrm{H}_{\mathrm{n}}^{\mathrm{\prime}}=2^{\mathrm{n}^{\mathrm{H}}}{ }_{2}-\mathrm{H}_{\mathrm{n}+2}
$$

for all. $\mathrm{n} \geq 1$.
Identity (A) can be verified for small $n$; assume that (A) holds for $n$.
Then since

$$
2 H_{n+2}-H_{n}=\left(H_{n+3}-H_{n+1}\right)+H_{n+2}-\left(H_{n+2}-H_{n+1}\right)=H_{n+3}
$$

we have

$$
\mathrm{H}_{\mathrm{n}+1}^{\prime}=\sum_{\mathrm{k}=0}^{\mathrm{n}} \mathrm{H}_{\mathrm{k}} 2^{\mathrm{n}-\mathrm{k}}=\mathrm{H}_{\mathrm{n}}+2 \mathrm{H}_{\mathrm{n}}^{\prime}=2^{\mathrm{n}+1} \mathrm{H}_{2}-2 \mathrm{H}_{\mathrm{n}+2}+\mathrm{H}_{\mathrm{n}}=2^{\mathrm{n}+1} \mathrm{H}_{2}-H_{\mathrm{n}+3}
$$

Thus (A) holds for all $\mathrm{n} \geq 1$. To obtain the identities given by Carlitz, we note that $\mathrm{F}_{2}=1, \mathrm{~L}_{2}=3$.
Also solved by Herta T. Freitag, D. V. Jaiswal (India), Bruce W. King, C.B.A. Peck, A. C. Shannon (Australia), David Zeitlin, and the proposer.

## ERRATA

Please make the following correction in the October Elementary Problems and Solutions: In the third equation from the bottom, on p. 292, delete

$$
\frac{F_{2 k}}{F_{2 k+2}}<\frac{F_{2 k}}{F_{2 k+1}}<\frac{F_{2 k \div 1}}{F_{2 k}}<\frac{F_{2 k-1}}{F_{2 k}}
$$

and add, instead,

$$
\frac{F_{2 k}}{F_{2 k+2}}<\frac{F_{2 k+2}}{F_{2 k+3}}<\frac{F_{2 k+1}}{F_{2 k+2}}<\frac{F_{2 k-1}}{F_{2 k}}
$$

[Continued from p. 334.]
Hence, by (13), p | $\mathrm{D}_{2 \mathrm{n}}$
In each case we have found a reduced arithmetic progression no prime member of which is a factor of a certain $D_{2 n}$. Hence, by Lemma 1, II), there is an infinitude of composite $D_{2 n+1}$.

## REFERENCES

1. R. D. Carmichael, "On the Numerical Factors of the Arithmetic Forms $\alpha^{\mathrm{n}} \pm \beta^{\mathrm{n}}, "$ Annals of Mathematics, 15 (1913-1914), pp. 30-70.
2. W. J. LeVeque, Topics in Number Theory, I (1958).

## ANY LUCAS NUMBER $L_{5 p}$, FOR ANY PRIME $p \geq 5$, HAS AT LEAST <br> TWO DISTINCT PRIMITIVE PRIME DIVISORS <br> DOV JARDEN

Hebrew University, Jerusalem, Israel

Proof. It is well known that, for any positive integer $n, L_{5 n} / L_{n}=A_{n} B_{n}$, where

$$
A_{n}=5 F_{n}^{2}-5 F_{n}+1, B_{n}=5 F_{n}^{2}+5 F_{n}+1, A_{n}<B_{n},\left(A_{n}, B_{n}\right)=1
$$

where $F_{\mathrm{n}}$ denotes a Fibonacci number (compare, e.g., Recurring Sequences, Jerusalem, 1966, pp. 16-21. For $n=5$ we have: $A_{n}=101, B_{n}=151$, and the statement is true. In order to prove it for $p>5$, it is sufficient to show that the greatest non-primitive divisor of $L_{5 p}, p>5$, is smaller than $A_{p}$, hence the greatest primitive divisor of $L_{5 p}$ is greater than $B_{p}$, hence both $A_{n}$ and $B_{n}$ have primitive divisors, and since $\left(A_{n}, B_{n}\right)=1$, $A_{n}$ has a primitive prime divisor $a, B_{n}$ has a primitive prime divisor $b$, and a $\neq \mathrm{b}$ 。

Now, the greatest non-primitive divisor of $L_{5 p}$ is $L_{5} L_{p}=11 L_{p}$, and we have to show that $11 L_{p}<A_{p}$ for any prime $p>5$. We shall show that $11 L_{n}<A_{n}$ for any positive integer $n>5$. The proof is based on the following two inequalities:
(1)

$$
\begin{gather*}
L_{n}<3 F_{n} \quad(n>2) \\
33<5\left(F_{n}-1\right) \quad(n>5) \tag{2}
\end{gather*}
$$

Equation (1) is easily verified for $n=3$, 4. If (1) is valid for $n, n+1$, its validity for $n+2$ follows by addition of the corresponding inequalities sidewise. Similarly (2) is shown. Hence

$$
\begin{aligned}
11 \mathrm{~L}_{\mathrm{n}}<11 \cdot 3 \mathrm{~F}_{\mathrm{n}}=33 \mathrm{~F}_{\mathrm{n}}<5\left(\mathrm{~F}_{\mathrm{n}}-1\right) \mathrm{F}_{\mathrm{n}}= & 5 \mathrm{~F}_{\mathrm{n}}^{2}-\mathrm{F}_{\mathrm{n}} \\
& <5 \mathrm{~F}_{\mathrm{n}}^{2}-\mathrm{F}_{\mathrm{n}}+1=A_{\mathrm{n}}
\end{aligned}
$$

This completes the proof.

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[^0]:    ${ }^{1}$ As it does design of structures elsewhere in the Aegean contemporary or later than Minoan palatial construction. There is evidence that the 1:1.6 ratio was employed in design previously in the Early Bronze Age in Greece and western Anatolia (disseration, loc. cit.).

[^1]:    *The proof given here is adapted from that given in [1].

[^2]:    ${ }^{1}$ All measurements here are metric; Bronze Age in Greece: Early: $\pm 2700$ -
    ${ }^{2}$ Beginning with the excavations at Knossos by Sir Arthur Evans in 1899; final publication: A. Evans, The Palace of Minos at Knoss, 4 vols. , 1921-1936; In 1900 the palace at Phaistos began to be uncovered: $\mathrm{L}_{6}$ Pernier and $\mathrm{L}_{0}$ Banti, Il Palazzo Mincico di Festos, 2 vols., 1935 and 1951.

[^3]:    ${ }^{1}$ Factors such as solar orientation of buildings, as well as alignment of certain building axes on prominent landscape features, play a role in design also, as yet not fully understood. Cf. V. Scully, The Earth, the Temple and the Gods (Yale 1962 and 1968) ch. 2; and below, n. 11.

[^4]:    ${ }^{1}$ The Egyptians occasionally used as a module the remen or diagonal of a square laid out in normal cubits (W. M. F. Petrie, Ancient Weights and Measures (London 1926) 41 and passim.

