THE FIBONACCI QUARTERLY

THE OFFICIAL JOURNAL OF
THE FIBONACCI ASSOCIATION

VOLUME 7 NUMBER 1

CONTENTS

PART I — ADVANCED

PART II — ELEMENTARY

FEBRUARY 1969

THE FIBONACCI QUARTERLY

OFFICIAL ORGAN OF THE FIBONACCI ASSOCIATION

A JOURNAL DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

All subscription correspondence should be addressed to Bro. A. Brousseau, St. Mary's College, Calif. All checks ($6.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Manuscripts intended for publication in the Quarterly should be sent to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, Calif. All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscript sent to the editors.

The Quarterly is entered as third-class mail at the St. Mary's College Post Office, California, as an official publication of the Fibonacci Association.

PRODUCTS OF FIBONACCI AND LUCAS NUMBERS

H. H. FERNS
Victoria, B.C., Canada

Let U_{x_i} denote a Fibonacci or a Lucas number and consider the product

$$U_{x_1} U_{x_2} \cdots U_{x_n} .$$

We are interested in finding a general method by which this product may be "expanded," i.e., expressed as a linear function of Fibonacci or Lucas numbers.

Beginning with the case in which $n = 2$ we find that there are four types of such products. Using Binet's formulas it is easily verified that these may be expressed as follows:

$$F_{x_1} L_{x_2} = F_{x_1+x_2} + (-1)^{x_2} F_{x_1-x_2}$$

$$L_{x_1} F_{x_2} = F_{x_1+x_2} - (-1)^{x_2} F_{x_1-x_2}$$

$$L_{x_1} L_{x_2} = L_{x_1+x_2} + (-1)^{x_2} L_{x_1-x_2}$$

$$F_{x_1} F_{x_2} = \frac{1}{5}\left[L_{x_1+x_2} - (-1)^{x_2} L_{x_1-x_2}\right].$$

From these four identities we make the following observations.

This "multiplication" is not commutative.

The product of a mixed pair (i.e., one factor is a Fibonacci number and the other is a Lucas number) is a linear function of Fibonacci numbers. The product of a Fibonacci and Lucas number is a function of Lucas numbers.

The coefficient of the second term is $(-1)^{x_2}$ or $-(-1)^{x_2}$ according as x_2 comes from the subscript of a Lucas or a Fibonacci number.

The factor $1/5$ occurs when both numbers in the product are Fibonacci.

For convenience we denote -1 by ϵ. Now consider ϵ^{x_i} as playing a dual role. As a coefficient of L_x or F_x it has the value $(-1)^{x_i}$. As an operator applied to these numbers it reduces their subscripts by $2x_i$. With this in mind, we may write

$$F_{x_1} L_{x_2} = (1 + \epsilon^{x_2})F_{x_1+x_2} = F_{x_1+x_2} + (-1)^{x_2}F_{x_1-x_2}$$

$$L_{x_1} F_{x_2} = (1 - \epsilon^{x_2})F_{x_1+x_2} = F_{x_1+x_2} - (-1)^{x_2}F_{x_1-x_2}$$

$$L_{x_1} L_{x_2} = (1 + \epsilon^{x_2})L_{x_1+x_2} = L_{x_1+x_2} + (-1)^{x_2}L_{x_1-x_2}$$

$$F_{x_1} F_{x_2} = (1 - \epsilon^{x_2})L_{x_1+x_2} = \frac{1}{5}\left[L_{x_1+x_2} - (-1)^{x_2}L_{x_1-x_2}\right].$$

We turn now to products containing three factors such as $L_{x_1} L_{x_2} F_{x_3}$.

For the moment we shall understand that $L_{x_1} L_{x_2} F_{x_3}$ means $(L_{x_1} L_{x_2})$ F_{x_3}. Then, making use of the above results, we have

$$\begin{aligned}
(L_{x_1} L_{x_2})F_{x_3} &= \left[L_{x_1+x_2} + (-1)^{x_2}L_{x_1-x_2}\right]F_{x_3} \\
&= L_{x_1+x_2} F_{x_3} + (-1)^{x_2} L_{x_1-x_2} F_{x_3} \\
&= F_{x_1+x_2+x_3} - (-1)^{x_3} F_{x_1+x_2-x_3} + (-1)^{x_2} \times \\
&\qquad \times \left[F_{x_1-x_2+x_3} - (-1)^{x_3} L_{x_1-x_2-x_3}\right] \\
&= F_{x_1+x_2+x_3} - (-1)^{x_3} F_{x_1+x_2-x_3} + (-1)^{x_2} \times \\
&\qquad \times F_{x_1-x_2+x_3} - (-1)^{x_2+x_3} L_{x_1-x_2-x_3} .
\end{aligned}$$

Using ϵ^{x_i} we arrive at the same result.

$$L_{x_1}L_{x_2}F_{x_3} = (1 + \epsilon^{x_2})(1 - \epsilon^{x_3})F_{x_1+x_2+x_3}$$

$$= (1 + \epsilon^{x_2})F_{x_1+x_2+x_3} - (-1)^{x_3}F_{x_1+x_2-x_3}$$

$$= F_{x_1+x_2+x_3} - (-1)^{x_3}F_{x_1+x_2-x_3} + (-1)^{x_2}F_{x_1-x_2+x_3} - (-1)^{x_2+x_3} \times$$

$$\times F_{x_1-x_2-x_3} .$$

Since

$$(1 + \epsilon^{x_2})(1 - \epsilon^{x_3}) = 1 + \epsilon^{x_2} - \epsilon^{x_3} - \epsilon^{x_2+x_3} ,$$

we could proceed as follows:

$$L_{x_1}L_{x_2}F_{x_3} = (1 + \epsilon^{x_2} - \epsilon^{x_3} - \epsilon^{x_2+x_3})F_{x_1+x_2+x_3}$$

$$= F_{x_1+x_2+x_3} + (-1)^{x_2}F_{x_1-x_2+x_3} - (-1)^{x_3}F_{x_1+x_2-x_3} - (-1)^{x_2+x_3} \times$$

$$\times F_{x_1-x_2-x_3} .$$

We leave it as an exercise to show that $L_{x_1}(L_{x_2}F_{x_3})$ when expanded by any of these methods leads to the same result.

There are eight types of products, each consisting of three factors. We list them below.

$$F_{x_1}L_{x_2}L_{x_3} = (1 + \epsilon^{x_2})(1 + \epsilon^{x_3})F_{x_1+x_2+x_3}$$

$$L_{x_1}F_{x_2}L_{x_3} = (1 - \epsilon^{x_2})(1 + \epsilon^{x_3})F_{x_1+x_2+x_3}$$

$$L_{x_1}L_{x_2}F_{x_3} = (1 + \epsilon^{x_2})(1 - \epsilon^{x_3})F_{x_1+x_2+x_3}$$

$$F_{x_1}F_{x_2}F_{x_3} = \frac{1}{5}(1-\epsilon^{x_2})(1-\epsilon^{x_3})F_{x_1+x_2+x_3}$$

$$L_{x_1}F_{x_2}F_{x_3} = \frac{1}{5}(1-\epsilon^{x_2})(1-\epsilon^{x_3})L_{x_1+x_2+x_3}$$

$$F_{x_1}L_{x_2}F_{x_3} = \frac{1}{5}(1+\epsilon^{x_2})(1-\epsilon^{x_3})L_{x_1+x_2+x_3}$$

$$F_{x_1}F_{x_2}L_{x_3} = \frac{1}{5}(1-\epsilon^{x_2})(1+\epsilon^{x_3})L_{x_1+x_2+x_3}$$

$$L_{x_1}L_{x_2}L_{x_3} = (1+\epsilon^{x_2})(1+\epsilon^{x_3})L_{x_1+x_2+x_3} \quad .$$

The preceding results are the bases for the following conjecture.

Let U_{x_i} represent a Fibonacci or a Lucas number. Let p be the number of Fibonacci numbers in a product of both Fibonacci and Lucas numbers. Let

$$\overline{U}_{x_1+x_2+\cdots+x_n}$$

denote a Fibonacci or a Lucas number according as p is odd or even. As a coefficient ϵ^{x_i} has the numerical value $(-1)^{x_i}$ but as an operator applied to

$$\overline{U}_{x_1+x_2+\cdots+x_n} \quad ,$$

it reduces the subscript of the latter by $2x_i$.

Use

$$(1-\epsilon^{x_i}) \qquad \text{or} \qquad (1+\epsilon^{x_i})$$

according as x_i is the subscript of a Fibonacci or a Lucas number in the product. Then

$$\prod_{i=1}^{u} U_{x_i} = \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 \pm \epsilon^{x_2})(1 \pm \epsilon^{x_3}) \cdots (1 \pm \epsilon^{x_n})\overline{U}_{x_1+x_2+\cdots+x_n} .$$

The proof of this conjecture is given at the end of this article. The following example will illustrate

$$F_{15}F_{12}L_{10}F_8 = \frac{1}{5}(1 - \epsilon^{12})(1 + \epsilon^{10})(1 - \epsilon^{8})F_{45}$$

$$= \frac{1}{5}(1 - \epsilon^{12})(1 + \epsilon^{10})(F_{45} - F_{29})$$

$$= \frac{1}{5}(1 - \epsilon^{12})(F_{45} - F_{29} + F_{25} - F_9)$$

$$= \frac{1}{5}(F_{45} - F_{29} + F_{25} - F_9 - F_{21} + F_5 - F_1 + F_{-15})$$

$$= \frac{1}{5}(F_{45} - F_{29} + F_{25} - F_{21} + F_{15} - F_9 + F_5 - F_1) .$$

The above rule also applies if the product consists entirely of Fibonacci or of Lucas numbers each with the same subscript. For example,

$$L_x^5 = (1 + \epsilon^x)^4 L_{5x}$$

$$= (1 + 4\epsilon^x + 6\epsilon^{2x} + 4\epsilon^{3x} + \epsilon^{4x})L_{5x}$$

$$= L_{5x} + 4(-1)^x L_{3x} + 6(-1)^{2x} L_x + 4(-1)^{3x} L_{-x} + (-1)^{4x} L_{-3x}$$

$$= L_{5x} + \left[4(-1)^x + (-1)^x\right] L_{3x} + \left[6(-1)^{2x} + 4(-1)^{2x}\right] L_x$$

$$= L_{5x} + 5(-1)^x L_{3x} + 10\, L_x .$$

More generally, if n is an odd integer we have

$$L_x^n = (1 + \epsilon^x)^{n-1} L_{nx}$$

$$= L_{nx} + \binom{n-1}{1}\epsilon^x L_{(n-2)x} + \binom{n-1}{2}\epsilon^{2x} L_{(n-4)x} + \cdots$$

$$+ \binom{n-1}{n-2}\epsilon^{(n-2)x} L_{-(n-4)x} + \binom{n-1}{n-1}\epsilon^{(n-1)x} L_{-(n-2)x}$$

Since

$$L_{-k} = (-1)^k L_k \quad ,$$

we get

$$L_x^n = L_{nx} + \left[\binom{n-1}{1} + \binom{n-1}{n-1}\right]\epsilon^x L_{(n-2)x} + \left[\binom{n-1}{2} + \binom{n-1}{n-2}\right]\epsilon^{2x} L_{(n-4)x}$$

$$+ \cdots + \left[\binom{n-1}{\frac{n-1}{2}} + \binom{n-1}{\frac{n+1}{2}}\right]\epsilon^{\left(\frac{n-1}{2}\right)} L_x \ .$$

Making use of the identity

$$\binom{n}{m} + \binom{n}{n-m} = \binom{n+1}{m} \quad ,$$

the last equation may be written

$$L_x^n = L_{nx} + \binom{n}{1}\epsilon^x L_{(n-2)x} + \binom{n}{2}\epsilon^{2x} L_{(n-4)x} + \cdots + \binom{n}{\frac{n-1}{2}}\epsilon^{\left(\frac{n-1}{2}\right)x} L_x$$

$$L_x^n = \sum_{i=0}^{\frac{n-1}{2}} (-1)^{xi}\binom{n}{i} L_{(n-2i)x} \qquad n = 1, 3, 5, \cdots \ .$$

Similarly, we get the following:

$$L_x^n = \sum_{i=0}^{\frac{n}{2}-1}\left[(-1)^{xi}\binom{n}{i}L_{(n-2i)x}\right] + 2(-1)^{\frac{n}{2}x}\binom{n-1}{\frac{n}{2}} \qquad (n,\ \text{even})$$

$$F_x^n = \frac{1}{5^{\frac{n-1}{2}}}\sum_{i=0}^{\frac{n-1}{2}}(-1)^{(x+1)i}\binom{n}{i}F_{(n-2i)x} \qquad (n,\ \text{odd})$$

$$F_x^n = \frac{1}{5^{\frac{n}{2}}}\sum_{i=0}^{\frac{n}{2}-1}\left[(-1)^{(x+1)i}\binom{n}{i}L_{(n-2i)x}\right] + 2(-1)^{\frac{n}{2}(x+1)}\binom{n-1}{\frac{n}{2}} \qquad (n,\ \text{even})$$

The proof of the rule which has been used to express products of Fibonacci and Lucas numbers as linear functions of those numbers is a proof by induction.

We have seen that it is true for $n = 2$ and $n = 3$. Assume it is true for all integral values of n up to and including k. Then, if p is even

$$(1) \qquad \prod_{i=1}^{k} U_{x_i} = \frac{1}{5^{\left[\frac{p}{2}\right]}}(1 \pm \epsilon^{x_2})\cdots(1 \pm \epsilon^{x_k})\,L_{x_1+x_2+\cdots+x_k}\ .$$

Multiplying both members of this equation by L_{x+1} we get

$$\begin{aligned}\prod_{i=1}^{k} U_{x_i}L_{x+1} &= \frac{1}{5^{\left[\frac{p}{2}\right]}}(1 \pm \epsilon^{x_2})\cdots(1 \pm \epsilon^{x_k})L_{x_1+x_2+\cdots+x_k}\,L_{x_{k+1}}\\ &= \frac{1}{5^{\left[\frac{p}{2}\right]}}(1 \pm \epsilon^{x_2})\cdots(1 \pm \epsilon^{x_k}) \times\\ &\qquad \times \left(L_{x_1+x_2+\cdots+x_{k+1}} + (-1)^{k+1}L_{x_1+x_2+\cdots+x_k-x_{k+1}}\right)\\ &= \frac{1}{5^{\left[\frac{p}{2}\right]}}(1 + \epsilon^{x_2})\cdots(1 \pm \epsilon^{x_k})(1 + \epsilon^{x_{k+1}})L_{x_1+x_2+\cdots+x_{k+1}}\end{aligned}$$

Next, multiplying both sides of equation (1) by F_{x+1} we get

$$\prod_{i=1}^{k} U_{x_i} F_{x_{k+1}} = \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 + \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k}) L_{x_1+x_2+\cdots+x_k} F_{x_{k+1}}$$

$$= \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 \pm \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k}) \times$$

$$\times \left[F_{x_1+x_2+\cdots+x_{k+1}} - (-1)^{x_{k+1}} F_{x_1+x_2+\cdots+x_k-x_{k+1}} \right]$$

$$= \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 \pm \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k})(1 - \epsilon^{x_{k+1}}) F_{x_1+x_2+\cdots+x_{k+1}} \,.$$

Since both of these results agree with that given by the general rule for $n = k + 1$ the induction is complete for the case in which

$$\overline{U}_{x_1+x_2+\cdots+x_n} = L_{x_1+x_2+\cdots+x_n} \,.$$

We leave the case in which

$$\overline{U}_{x_1+x_2+\cdots+x_n} = F_{x_1+x_2+\cdots+x_n}$$

for the reader to prove.

We now consider the reverse problem; that is, the problem of finding a general method of expressing

$$L_{x_1+x_2+\cdots+x_n} \quad \text{and} \quad F_{x_1+x_2+\cdots+x_n}$$

as a homogeneous function of products, each of the type,

$$F_{x_1} F_{x_2} \cdots F_{x_i} L_{x_{i+1}} L_{x_{i+2}} \cdots L_{x_n} \,.$$

For simplicity let S_i^n denote the sum of all products consisting of i factors which are Fibonacci numbers and $n - i$ which are Lucas numbers. The number of such factors is, of course, $\binom{n}{i}$.

For example,

$$S_2^4 = F_{x_1} F_{x_2} L_{x_3} L_{x_4} + F_{x_1} F_{x_3} L_{x_2} L_{x_4} + F_{x_1} F_{x_4} L_{x_2} L_{x_3} + F_{x_2} F_{x_3} L_{x_1} L_{x_4} + F_{x_2} F_{x_4} L_{x_1} L_{x_3} + F_{x_3} F_{x_4} L_{x_1} L_{x_2} .$$

For later use we note that

$$S_i^n L_{x_{n+1}} + S_{i-1}^n F_{x_{n+1}} = S_i^{n+1} .$$

This follows from the identity

$$\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i} .$$

For the case $n = 2$ we readily prove (using Binet's formulas) that

$$\begin{aligned} F_{x_1+x_2} &= \frac{1}{2} (L_{x_1} F_{x_2} + F_{x_1} L_{x_2}) \\ &= \frac{1}{2} S_1^2 \end{aligned}$$

$$\begin{aligned} L_{x_1+x_2} &= \frac{1}{2} (L_{x_1} L_{x_2} + 5 F_{x_1} F_{x_2}) \\ &= \frac{1}{2} (S_0^2 + 5 S_2^2) . \end{aligned}$$

Using these two identities as a basis, we develop the following for $n = 3$

$$\begin{aligned}
F_{x_1+x_2+x_3} &= F_{(x_1+x_2)+x_3} \\
&= \frac{1}{2}\left[L_{x_1+x_2} F_{x_3} + F_{x_1+x_2} L_{x_3}\right] \\
&= \frac{1}{2}\left[\frac{1}{2}(L_{x_1} L_{x_2} + 5 F_{x_1} F_{x_2})F_{x_3} + \frac{1}{2}(L_{x_1} F_{x_2} + F_{x_1} L_{x_2}) L_{x_3}\right] \\
&= \frac{1}{2^2}\left[L_{x_1} L_{x_2} F_{x_3} + 5 F_{x_1} F_{x_2} F_{x_3} + L_{x_1} F_{x_2} L_{x_3} + F_{x_1} L_{x_2} L_{x_3}\right] \\
&= \frac{1}{2^2}\left[S_1^3 + 5 S_3^3\right]
\end{aligned}$$

$$\begin{aligned}
L_{x_1+x_2+x_3} &= L_{(x_1+x_2)+x_3} \\
&= \frac{1}{2}\left[L_{x_1+x_2} L_{x_3} + 5 F_{x_1+x_2} F_{x_3}\right] \\
&= \frac{1}{2}\left[\frac{1}{2}(L_{x_1} L_{x_2} + 5 F_{x_1} F_{x_2})L_{x_3} + \frac{5}{2}(L_{x_1} F_{x_2} + F_{x_1} L_{x_2}) F_{x_3}\right] \\
&= \frac{1}{2^2}\left[L_{x_1} L_{x_2} L_{x_3} + 5 F_{x_1} F_{x_2} L_{x_3} + 5 F_{x_1} L_{x_2} F_{x_3} + 5 L_{x_1} F_{x_2} F_{x_3}\right] \\
&= \frac{1}{2^2}\left[S_0^3 + 5 S_2^3\right].
\end{aligned}$$

Proceeding in this manner we derive the following identities for $n = 4$ and $n = 5$:

$$F_{x_1+x_2+x_3+x_4} = \frac{1}{2^3}\left[S_1^4 + 5 S_3^4\right]$$

$$F_{x_1+x_2+x_3+x_4+x_5} = \frac{1}{2^4}\left[S_1^5 + 5 S_3^5 + 5^2 S_5^5\right]$$

$$L_{x_1+x_2+x_3+x_4} = \frac{1}{2^3}\left[S_0^4 + 5 S_2^4 + 5^2 S_4^4\right]$$

$$L_{x_1+x_2+x_3+x_4+x_5} = \frac{1}{2^4}\left[S_0^5 + 5S_2^5 + 5^2 S_4^5\right].$$

From the above results we conjecture the validity of the following identities which we will prove later.

$$(2) \quad F_{x_1+x_2+\cdots+x_n} = \frac{1}{2^{n-1}}\left[S_1^n + 5S_3^n + 5^2 S_5^n + \cdots + \begin{cases} 5^{\frac{n-2}{2}} S_{n-1}^n \Big] & (n,\ \text{even}) \\ 5^{\frac{n-1}{2}} S_n^n \Big] & (n,\ \text{odd}) \end{cases}\right.$$

$$(3) \quad L_{x_1+x_2+\cdots+x_n} = \frac{1}{2^{n-1}}\left[S_0^n + 5S_2^n + 5^2 S_4^n + \cdots + \begin{cases} 5^{\frac{n}{2}} S_n^n \Big] & (n,\ \text{even}) \\ 5^{\frac{n-1}{2}} S_{n-1}^n \Big] & (n,\ \text{odd}) . \end{cases}\right.$$

Before proceeding with the proofs of these identities we consider the special case when $x_1 = x_2 = \cdots = x_n = x$. For this case we get the following:

$$F_{nx} = \frac{1}{2^{n-1}}\left[\binom{n}{1} F_x L_x^{n-1} + 5\binom{n}{3} F_x^3 L_x^{n-3} + \cdots + \begin{cases} 5^{\frac{n-2}{2}} \binom{n}{n-1} F_x^{n-1} L_x \Big] & (n,\text{even}) \\ 5^{\frac{n-1}{2}} \binom{n}{n} F_x^n \Big] & (n,\text{odd}) \end{cases}\right.$$

$$L_{nx} = \frac{1}{2^{n-1}}\left[L_x^n + 5\binom{n}{2} F_x^2 L_x^{n-2} + \cdots + \begin{cases} 5^{\frac{n}{2}} \binom{n}{n} F_x^n \Big] & (n,\ \text{even}) \\ 5^{\frac{n-1}{2}} \binom{n}{n-1} F_x^{n-1} L_x \Big] & (n,\ \text{odd}) \end{cases}\right.$$

Note, in particular, if $n = 2$ we get two well-known identities

$$F_{2x} = F_x L_x$$

and

$$L_{2x} = \frac{1}{2}(L_x^2 + 5F_x^2) .$$

We have now to prove the identities (1) and (2). The proof is by induction on n. Both identities are true for $n = 2$. We assume they are valid for all integral values of n up to and including $n = k$.

Then

$$(4)\quad F_{x_1+x_2+\cdots+x_k} = \frac{1}{2^{k-1}}\left[S_1^k + 5S_3^k + 5^2 S_5^k + \cdots + \begin{cases} 5^{\frac{k-2}{2}} S_{k-1}^k\Big] & (k,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_k^k\Big] & (k,\ \text{odd}) \end{cases}\right.$$

$$(5)\quad L_{x_1+x_2+\cdots+x_k} = \frac{1}{2^{k-1}}\left[S_0^k + 5S_2^k + 5^2 S_4^k + \cdots + \begin{cases} 5^{\frac{k}{2}} S_k^k\Big] & (k,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_{k-1}^k\Big] & (k,\ \text{odd}) \end{cases}\right.$$

Now

$$(6)\quad L_{x_1+x_2+\cdots+x_k+x_{k+1}} \equiv L_{(x_1+x_2+\cdots+x_k)+x_{k+1}}$$

$$= \frac{1}{2}\left[L_{x_1+x_2+\cdots+x_k} L_{x_{k+1}} + 5F_{x_1+x_2+\cdots+x_k} F_{x_{k+1}}\right].$$

Applying (4) and (5) to the right member of (6), we get

$$(7)\quad L_{x_1+x_2+\cdots+x_k} L_{x_{k+1}} = \frac{1}{2^{k-1}}\left[S_0^k L_{x_{k+1}} + 5S_2^k L_{x_{k+1}} + \cdots\right.$$

$$\left. + \begin{cases} 5^{\frac{k}{2}} S_k^k L_{x_{k+1}}\Big] & (k,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_{k-1}^k L_{x_{k+1}}\Big] & (k,\ \text{odd}) \end{cases}\right.$$

$$(8)\qquad F_{x_1+x_2+\cdots+x_k} F_{x_{k+1}} = \frac{1}{2^{k-1}}\Bigg[S_1^k F_{x_{k+1}} + 5 S_3^k F_{x_{k+1}} + \cdots + \begin{cases} 5^{\frac{k-2}{2}} S_{k-1}^k F_{x_{k+1}} \Bigg] & (k,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_k^k F_{x_{k+1}} \Bigg] & (k,\ \text{odd})\ . \end{cases}$$

Substituting in (6) from (7) and (8) and regrouping we get the following:

$$L_{x_1+x_2+\cdots+x_{k+1}} = S_0^{k+1} + 5\left(S_2^k L_{x_{k+1}} + S_1^k F_{x_{k+1}}\right) + 5^2\left(S_4^k L_{x_{k+1}} + S_3^k F_{x_{k+1}}\right) + \cdots + \begin{cases} 5^{\frac{k}{2}}\left(S_k^k L_{x_{k+1}} + S_{k-1}^k F_{x_{k+1}}\right) & (k,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_k F_{k+1} & (k,\ \text{odd}) \end{cases}$$

Hence

$$L_{x_1+x_2+\cdots+x_{k+1}} = S_0^{k+1} + 5 S_2^{k+1} + 5^2 S_4^{k+1} + \cdots + \begin{cases} 5^{\frac{k}{2}} S_k^{k+1} & (k+1,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_{k+1}^{k+1} & (k+1,\ \text{odd}) \end{cases}$$

This completes the proof of (3). The proof of (2) is similar.

* * * * *

ERRATA FOR PSEUDO-FIBONACCI NUMBERS

H. H. Ferns
Victoria, B.C., Canada

Please make the following changes in the above-entitled article appearing in Vol. 6, No. 6:

p. 305: in Eq. (3), O_{i+1} should read: O_{i+2}. On p. 306, the 6^{th} line from the bottom: B^{-k+1} should read: B^{k+1}. On page 310, in Eq. (12), $2O_{2n}$ should read: $2\lambda O_{2n}$; in Eq. (13), 30_{2n+1} should read: $3O_{2n+1}$. Equation (17), on p. 312: $(\lambda - 2)O_{2n-1}$ should read: $\lambda(\lambda - 2)O_{2n-1}$. Equation (18s) on p. 313: 40_i^2 should read: $4O_i^2$. In line 3, p. 314, 20_{2n+2} should read $2O_{2n+2}$, and Eq. (20), p. 315: $(\lambda - 2)O_{2n}$ should read $\lambda(\lambda - 2)O_{2n}$.

* * * * *

TSCHEBYSCHEFF AND OTHER FUNCTIONS ASSOCIATED WITH THE SEQUENCE $\{w_n(a,b;\ p,q)\}$

A. F. HORADAM
University of New England, Armidale, Australia,
and University of Leeds, England

1. INTRODUCTION

Previously in this journal [5] and [6], I have defined a generalized sequence $\{w_n(a,b;\ p,q)\}$ and established its fundamental general arithmetical properties, as well as certain special properties of it. In this article, the sequence is related to Tschebyscheff functions and to some combinatorial functions used by Riordan [8]. This is the third of a series of articles developing the theory of $\{w_n(a,b;\ p,q)\}$, as envisaged in [5]. Notation and content of [5] and [6] are assumed when the occasion warrants.

For subsequent reference, we reproduce the Lucas results [7]

$$(1.1)\qquad u_n(p,q) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{p} p^{n-2k} q^k$$

and

$$(1.2)\qquad v_n(p,q) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} p^{n-2k} q^k$$

with reciprocals [3]

$$(1.3)\qquad p_n = \sum_{k=0}^{[n/2]} \left[\binom{n}{k} - \binom{n}{k-1}\right] u_{n-2k}(p,q) q^k$$

and

(1.4) $$p^n = \sum_{k=0}^{[n/2]} \binom{n}{k} v_{n-2k}(p,q)q^k \quad (v_0(p,q) = 1) \quad ,$$

respectively. Consequently, it follows that $(p = -q = 1)$.

(1.5) $$f_n = \sum_{k=0}^{[n/2]} \binom{n-k}{k}$$

from (1.1), and

(1.6) $$l_n = \sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k}$$

from (1.2), with appropriate reciprocals from (1.3) and (1.4).

Making use of (1.1) above together with the first of the forms given in (2.14) [5], we may express w_n as

(1.7) $$w_n(a,b,p,q) = a\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n-k}{k}p^{n-2k}q^k + (b-pa)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n-1-k}{k}p^{n-1-2k}q^k$$

2. TSCHEBYSCHEFF FUNCTIONS

Write

(2.1) $$x = \cos\theta$$

(2.2) $$p = 2x\ , \quad q = 1$$

so that

(2.3) $$d = 2i\sin\theta \quad (i = \sqrt{-1}) \ .$$

Define

(2.4) $$w_n = w_n(a,\ 2x;\ 2x,1) = a \cos n\theta + (2-a)\sin n\theta \cot\theta\ .$$

Using Simpson's formulae (reference Lucas [7]),

(2.5) $$\begin{cases} \sin(n+2)\theta = 2\cos\theta\ \sin(n+1)\theta - \sin n\theta \\ \cos(n+2)\theta = 2\cos\theta\ \cos(n+1)\theta - \cos n\theta \end{cases},$$

we deduce that

(2.6) $$w_{n+2} = pw_{n+1} - w_n\ ,$$

as required by the definition of $w_n(a,b;\ p,q)$ given in [5], in conjunction with (2.1) and (2.2). Notice that (2.1) and (2.2) ensure [5] that

(2.7) $$e = 4(a-1)\cos^2\theta - a^2\ ,$$

whence, for $\{u_n\}$, for which $a = 1$,

(2.8) $$e = -1\ ,$$

while for $\{v_n\}$, for which $a = 2$,

(2.9) $$e = -4\sin^2\theta\ .$$

Immediately from (2.4) we have the Lucas substitutions [7]

(2.10) $$u_n(2x,1) = \frac{\sin(n+1)\theta}{\sin\theta} = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2x)^{n-2k}$$

and

(2.11) $$v_n(2x,1) = 2\cos n\theta = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2x)^{n-2k}$$

with reciprocals

(2.12) $$(2x)^n = \sum_{k=0}^{[n/2]} \left[\binom{n}{k} - \binom{n}{k-1}\right] u_{n-2k}(2x,1)$$

and

(2.13) $$(2x)^n = \sum_{k=0}^{[n/2]} \binom{n}{k} v_{n-2k}(2x,1) ,$$

where we have used (1.1)-(1.4).

But, the expressions in (2.10) and (2.11) exactly describe the Tschebyscheff functions $U_n(x)$ and $2T_n(x) = t_n(x)$ respectively ($T_0 = \frac{1}{2}t_0 = 1$). That is,

(2.14) $$w_n(1, 2x;\, 2x, 1) = u_n(2x, 1) = U_n(x) = 2xU_{n-1}(x) - U_{n-1}(x)$$

and

(2.15) $$w_n(2, 2x;\, 2x, 1) = v_n(2x, 1) = 2T_n(x) = 2(xU_{n-1}(x) - U_{n-2}(x)) .$$

Special cases are

(2.16) $$w_n(1, 1;\, 1, 1) = u_n(1, 1) = U_n(\tfrac{1}{2}) = U_{n-1}(\tfrac{1}{2}) - U_{n-2}(\tfrac{1}{2})$$

and

(2.17) $$w_n(2, 1;\, 1, 1) = v_n(1, 1) = 2T_n(\tfrac{1}{2}) = U_{n-1}(\tfrac{1}{2}) - 2U_{n-2}(\tfrac{1}{2}) .$$

Generally,

$$w_n(a,b;\ 2x,1) = bU_{n-1}(x) - aU_{n-2}(x) \quad . \tag{2.18}$$

By means of the w_n-notation, relationships among Tschebyscheff polynomials may be conveniently expressed. Recalling the known result [8], for instance, that

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \tag{2.19}$$

we may, writing for brevity,

$$w_n = w_n(2, 2x;\ 2x,\ 1) \quad , \tag{2.20}$$

express it in the form

$$\omega_n - 2x\omega_{n-1} - \omega_{n-2} \,. \tag{2.21}$$

Equations (2.4), (2.10) and (2.11) enable us to express every formula in the theory of our second-order recurrences as a corresponding formula involving trigonometrical functions. [Observe that $q = 1$ invalidates any specialized application to the sequences $\{h_n\}$, $\{f_n\}$ and $\{l_n\}$, for all of which $q = -1$.]

Corresponding to the fundamental formula $w_{n+r}w_{n-r} - w_n^2 = eq^{n-r}u_{r-1}^2$ ((4.5) in [5]), for instance, we have

$$\begin{aligned} &a^2\{\cos(n+r)\theta\ \cos(n-r)\theta - \cos^2 n\theta\} \\ &+ (2-a)^2\cot^2\theta\ \{\sin(n+r)\theta\ \sin(n-r)\theta - \sin^2 n\theta\} = e\,\frac{\sin^2 r\theta}{\sin^2\theta} \end{aligned} \tag{2.22}$$

where e is given by (2.7). For $\{u_n\}$ and $\{v_n\}$, we obtain

$$\sin(n+r+1)\theta\ \sin(n-r+1)\theta - \sin^2(n+1)\theta = -\sin^2 r\theta \tag{2.23}$$

and

$$\cos(n+r)\theta \cos(n-r)\theta - \cos^2 n\theta = -\sin^2 r\theta \quad , \tag{2.24}$$

in which e is given by (2.8) and (2.9), respectively. Both results (2.23) and (2.24) are easy to verify. The particular result $w_n^2 + eu_{n-1}^2 = aw_{2n}$ ((4.6) [5]) derived by setting $r = m$ implies the identity

$$\cos 2n\theta - \cos^2 n\theta = -\sin^2 n\theta$$

in (2.24).

Other trigonometrical identities are not hard to detect, but it is interesting to discover just how they are disguised. As further examples, we note that

$$pw_{n+2} - (p^2 - q)w_{n+1} + q^2 w_{n-1} = 0$$

((3.3) [5]), and

$$\frac{w_{n+r} + q^r w_{n-r}}{w_n} = v_r$$

((3.16) [5]) lead to, respectively,

$$\begin{cases} 2\cos\theta \sin(n+3)\theta - (4\cos^2\theta - 1)\sin(n+2)\theta + \sin n\theta = 0 & (2.25) \\ 2\cos\theta \cos(n+2)\theta - (4\cos^2\theta - 1)\cos(n+1)\theta + \cos(n-1)\theta = 0 & (2.26) \end{cases}$$

and

$$\begin{cases} \dfrac{\sin(n+r+1)\theta + \sin(n-r+1)\theta}{\sin(n+1)\theta} = 2\cos r\theta & (2.27) \\[2ex] \dfrac{\cos(n+r)\theta + \cos(n-r)\theta}{\cos n\theta} = 2\cos r\theta & (2.28) \end{cases}$$

where, in each pair of identities, the first refers to $\{u_n\}$ and the second to $\{v_n\}$. A formula also worth investigation is

$$aw_{m+n} + (b - pa)w_{m+n-1} = w_m w_n - qw_{m-1} w_{n-1}$$

((4.1) [5]). Furthermore, the summation formula (3.4) [5] indicates expressions for

$$\sum_{k=0}^{n-1} \cos k\theta$$

and

$$\sum_{k=0}^{n-1} \sin (k + 1)\theta \quad .$$

Similar remarks apply to the formulae for sums of squares and cubes.

Instead of (2.1)-(2.3), we may put

$$y = \cosh \phi = \cos i\phi \tag{2.29}$$

$$p = 2y, \quad q = 1 \tag{2.30}$$

so that

$$d = 2 \sinh \phi = -2i \sin i\phi \tag{2.31}$$

and hence derive a set of parallel results for hyperbolic functions.

Apart from the Carlitz [3] reference quoted earlier, other sources of information regarding the relationships among Tschebyscheff polynomials and Fibonacci-type sequences are, say, Buschman [1] and Gould [4].

3. COMBINATORIAL FUNCTIONS

From (1.1), we have, using the combinatorial function $L_n(x)$ used in Riordan [8],

$$u_n(1,-x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k = L_{n-1}(x) . \tag{3.1}$$

Then, by the second half of the expression (2.14) [5],

$$\begin{aligned} w_n(a,b;\,1,-x) &= b\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-k}{k} x^k + ax\sum_{k=0}^{\left[\frac{n-2}{2}\right]} \binom{n-2-k}{k} x^k \\ &= bL_{n-2}(x) + axL_{n-3}(x) \\ &= b\left\{\frac{(1+g)^n - (1-g)^n}{2^n g^2}\right\} + ax\left\{\frac{(1+g)^{n-1} - (1-g)^{n-1}}{2^{n-1} g^2}\right\} \end{aligned} \tag{3.2}$$

where, for brevity, $g = (1 + 4x)^{\frac{1}{2}}$.

More particularly, notice that

$$w_n(1,1;\,1,-x) = u_n(1,-x) \tag{3.3}$$

affords an alternative expression for the known recurrence relation [8].

$$L_{n-1}(x) = L_{n-2}(x) + xL_{n-3}(x) \qquad [L_0 = 1, L_1 = 1+x] \tag{3.4}$$

while

$$w_{2n}(2,1;\,1,-x) = v_{2n}(1,-x) \tag{3.5}$$

is an alternative expression for the combinatorial function [8]

$$M_n(x) = L_{2n-1}(x) + xL_{2n-3}(x) \qquad (n > 1) . \tag{3.6}$$

Of course,

$$L_{n-1}(1) = f_n \tag{3.7}$$

$$M_n(1) = l_{2n} . \tag{3.8}$$

4. OTHER FUNCTIONS

Besides these combinatorial functions and Tschebyscheff functions (themselves involving trigonometrical and hyperbolic functions), other functions are related to the Fibonacci-type recurrences. In this respect, a recent article by Byrd [2] is worth emphasizing, particularly as, it seems, his work offers possibilities for generalization. In this article, Byrd considers the expansion of analytical functions in a certain set of polynomials which can be associated with Fibonacci numbers. Bessel functions and modified Bessel functions are involved in the process.

Throughout, we have assumed that $p^2 \neq 4q$. The degenerate case $p^2 = 4q$ has been discussed by Carlitz [3], who relates it to the Eulerian polynomial, and, briefly, by the author [5].

REFERENCES

1. R. Buschman, "Fibonacci Numbers, Chebychev Polynomials, Generalizations and Difference Equations," The Fibonacci Quarterly, Vol. 1, No. 4, 1963, pp. 1-7.
2. P. Byrd, "Expansion of Analytic Functions in Polynomials Associated with Fibonacci Numbers," The Fibonacci Quarterly, Vol. 1, No. 1, 1963, p. 16.
3. L. Carlitz, "Generating Functions for Powers of Certain Sequences of Numbers," Duke Math. J., Vol. 29, No. 4, 1962, pp. 521-538.
4. H. Gould, "A New Series Transform with Applications to Bessel, Legendre and Tschebyscheff Polynomials," Duke Math. J., Vol. 31, No. 2, 1964, pp. 325-334.
5. A. Horadam, "Basic Properties of a Certain Generalized Sequence of Numbers," The Fibonacci Quarterly, Vol. 3, No. 3, p. 161.
6. A. Horadam, "Special Properties of the Sequence $\{w_n(a,b;p,q)\}$," The Fibonacci Quarterly, Vol. 5, No. 5, pp. 424-434.
7. E. Lucas, Théorie des Nombres, Paris, 1961, Chapter 18.
8. J. Riordan, An Introduction to Combinatorial Analysis, New York, 1958.

* * * * *

THE BRACKET FUNCTION AND FONTENÉ-WARD GENERALIZED BINOMIAL COEFFICIENTS WITH APPLICATION TO FIBONOMIAL COEFFICIENTS

H. W. GOULD
West Virginia University, Morgantown, West Virginia

INTRODUCTION

In 1915 Georges Fontené (1848-1928) published a one-page note [4] suggesting a generalization of binomial coefficients, replacing the natural numbers by an arbitrary sequence A_n of real or complex numbers. He gave the fundamental recurrence relation for these generalized coefficients and noted that for $A_n = n$ we recover the ordinary binomial coefficients, while for $A_n = q^n - 1$ we obtain the q-binomial coefficients studied by Gauss (as well as Euler, Cauchy, F. H. Jackson, and many others later).

These generalized coefficients of Fontené were later rediscovered by the late Morgan Ward (1901-1963) in a short but remarkable paper [16] in 1936 which developed a symbolic calculus of sequences. He does not mention Fontené. Failing to find other pioneers we shall call the generalized coefficients Fontené-Ward generalized binomial coefficients. We avoid the symbolic method of Ward in our work.

Since 1964, there has been an accelerated interest in Fibonomial coefficients. These correspond to the choice $A_n = F_n$, where F_n is the Fibonacci number defined by

$$F_{n+1} = F_n + F_{n-1}$$

with

$$F_0 = 0, \quad F_1 = 1 .$$

This idea seems to have originated with Dov Jarden [11] in 1949. He actually states the more general definition but only considers the Fibonomial case. Fibonomial coefficients have been quite a popular subject in this Quarterly since 1964 as references [1], [9], [10], [13], and [15] will tell. See also [17].

Because of the restricted nature of the three special cases of Fontené-Ward coefficients cited above, and because so many properties may be obtained in the most general case, we shall develop below a number of very striking general theorems which include a host of special cases among the references at the end of this paper. Despite an intensive study of all available books and journals for twenty years, it is possible that some of our results have been anticipated or extended. Indeed certain notions below are familiar in variant form and we claim only a novel presentation of what seems obvious. However a large body of the results below extend apparently new results of the author [7], [8] and we obtain the following elegant general results: Representation of Fontené-Ward coefficients as a linear combination of greatest integer (bracket function) terms; Representation of the bracket function as a linear combination of Fontené-Ward coefficients; A Lambert series expansion of a new number-theoretic function; A powerful inversion theorem for series of Fontené-Ward coefficients; and some miscellaneous identities including a brief way to study Fontené-Ward multinomial coefficients by avoiding a tedius argument of Kohlbecker [13].

The present paper originated out of discussions with my colleagues, Professors R. P. Agarwal and A. M. Chak, about the feasibility of extending Ward's ideas to broader areas of analysis and number theory. Chak [3] has developed and applied Ward's symbolic calculus of sequences to discuss numerous generalized special functions.

Every result below can be immediately applied to the Fibonacci triangle, or new variants thereof, and the inversion theorem given below is expected to be especially useful to Fibonacci enthusiasts. Such inversion theorems are valuable tools in analysis and have not been previously introduced or applied for Fibonomial coefficients. We may even take our sequence A_n to be the non-Fibonacci numbers and study a non-Fibonomial triangle.

FONTENÉ-WARD COEFFICIENTS: DEFINITION AND PROPERTIES

By the Fontené-Ward generalized binomial coefficient with respect to a sequence A_n we shall mean the following:

$$(1) \qquad \left\{ {n \atop k} \right\}_A = \left\{ {n \atop k} \right\} = \frac{A_n A_{n-1} \cdots A_{n-k+1}}{A_k A_{k-1} \cdots A_1}, \text{ with } \left\{ {n \atop 0} \right\} = \left\{ {n \atop n} \right\} = 1 ,$$

and we also require that

$$\left\{ {n \atop k} \right\} = 0 \qquad \text{whenever} \qquad k < 0 \ \text{ or } \ k > n\,. \tag{2}$$

The sequence A_n is essentially arbitrary but we do require that $A_0 = 0$ and $A_n \neq 0$ for $n \geq 1$. Ward [16] took $A_1 = 1$, and there is no loss of generality in doing that. However we cannot in general simplify very much and we shall retain A_1 as arbitrary. One has only to multiply Ward's sequence by A_1 to obtain our results. When no confusion can occur as to our choice of the basic sequence A_n we shall omit the subscript A in our notation (1). We use braces to set our coefficients apart from ordinary and q-binomial coefficients.

With this definition we can now exhibit the Fontené-Ward Triangle:

$$\begin{array}{c}
1 \\
1 \qquad 1 \\
1 \qquad \frac{A_2}{A_1} \qquad 1 \\
1 \qquad \frac{A_3}{A_1} \qquad \frac{A_3}{A_1} \qquad 1 \\
1 \qquad \frac{A_4}{A_1} \qquad \frac{A_4A_3}{A_1A_2} \qquad \frac{A_4}{A_1} \qquad 1 \\
1 \qquad \frac{A_5}{A_1} \qquad \frac{A_5A_4}{A_1A_2} \qquad \frac{A_5A_4}{A_1A_2} \qquad \frac{A_5}{A_1} \qquad 1 \\
1 \qquad \frac{A_6}{A_1} \qquad \frac{A_6A_5}{A_1A_2} \qquad \frac{A_6A_5A_4}{A_1A_2A_3} \qquad \frac{A_6A_5}{A_1A_2} \qquad \frac{A_6}{A_1} \qquad 1 \\
1 \qquad \frac{A_7}{A_1} \qquad \frac{A_7A_6}{A_1A_2} \qquad \frac{A_7A_6A_5}{A_1A_2A_3} \qquad \frac{A_7A_6A_5}{A_1A_2A_3} \qquad \frac{A_7A_6}{A_1A_2} \qquad \frac{A_7}{A_1} \qquad 1
\end{array}$$

It is evident that the triangle is symmetrical in the sense that

$$\left\{ {n \atop k} \right\} = \left\{ {n \atop n-k} \right\} \qquad 0 \leq k \leq n \tag{3}$$

We can make the definition (1) more symmetrical by introducing generalized factorials. We can define

(4) $$\left\{ n \atop k \right\} = \frac{[n]!}{[k]!\,[n-k]!} ,$$

where

$$[n]! = A_n A_{n-1} \cdots A_2 A_1 \quad \text{with} \quad [0]! = 1 .$$

This is equivalent to the previous definition and allows us to adapt a number of familiar binomial coefficient identities to our study. For example, it is clear that we have

(5) $$\left\{ n \atop k \right\} \left\{ k \atop j \right\} = \left\{ n \atop j \right\} \left\{ n - j \atop k - j \right\} ,$$

which we shall need later.

The basic recurrence relation for the Fontené-Ward coefficients was given by Fontené and is as follows:

(6) $$\left\{ n \atop k \right\} - \left\{ n - 1 \atop k \right\} = \left\{ n - 1 \atop k - 1 \right\} \frac{A_n - A_{n-k}}{A_k} ,$$

In this, change k to $n - k$ and apply (3). We find that

(7) $$\left\{ n \atop k \right\} - \left\{ n - 1 \atop k - 1 \right\} = \left\{ n - 1 \atop k \right\} \frac{A_n - A_k}{A_{n-k}} .$$

In general $A_n - A_k \neq A_{n-k}$. The fraction does equal 1 when we set $A_j = j$, and the fraction equals q^k when we set $A_j = (q^j - 1)/(q - 1)$. Fontené is correct that we get q-binomial coefficients with $A_j = q^j - 1$, but it is better to include the factor $q - 1$ in the denominator so that we can also assert that

$$\lim_{q \to 1} A_j = j$$

making the q-case then agree with ordinary natural numbers.

In the Fibonomial coefficient case, when $A_k = F_k$, write

$$f(n,k) = \frac{F_n - F_k}{F_{n-k}} . \tag{8}$$

It is easily verified that f satisfies the recurrence

$$f(n+1,\ k+1) = f(n,k) + f(n-1, k-1) . \tag{9}$$

By induction it then follows that

$$f(n+r,\ k+r) = F_{r+1} f(n,k) + F_r f(n-1,k-1) . \tag{10}$$

From this one may easily find

$$f(n,k) = F_k \frac{F_{n-k+1} - 1}{F_{n-k}} + F_{k-1} \tag{11}$$

which may also be derived directly from (8) and the relation

$$F_n = F_k F_{n-k+1} + F_{k-1} F_{n-k} . \tag{12}$$

There are then an abundance of ways to modify $f(n,k)$ using known Fibonacci relations, and the particular way we might interpret $f(n,k)$ determines the nature of the Fibonomial relations which will follow from our general theorems.

An important observation is this: $f(n,k)$ is independent of n in the case of ordinary and q-binomial coefficients, but not in the Fibonomial case. This makes the possibility of having certain expansions generalize depend on the way in which we can modify the recurrence.

We return to relation (6) and sum both sides with respect to the upper index. Clearly we obtain the relation

$$\sum_{j=k}^{n} \frac{A_j - A_{j-k}}{A_k} \left\{ \begin{matrix} j-1 \\ k-1 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\} , \tag{13}$$

which is the analogue of the familiar formula

$$\sum_{j=k}^{n} \binom{j-1}{k-1} = \binom{n}{k} .$$

Relation (13) will be very important to us in what follows.

We next define Fontené-Ward multinomial coefficients in the obvious way:

$$(14) \qquad \left\{ {n \atop k_1, k_2, \cdots, k_r} \right\} = \frac{\prod_{i=1}^{n} A_i}{\prod_{i=1}^{k_1} A_i \cdot \prod_{i=1}^{k_2} A_i \cdots \prod_{i=1}^{k_r} A_i} ,$$

subject to $n = k_1 + k_2 + \cdots + k_r$. For $A_i = i$ these pass over to the ordinary multinomial coefficients. What is more, (14) satisfies the following special relation: Set $r = 2$ and write $k_1 = a$, $k_2 = b$ with $a + b = n$. Then

$$(15) \qquad \left\{ {n \atop a, b} \right\} = \left\{ {n \atop a} \right\}$$

in terms of our original definition (1). Moreover, trinomial and higher order coefficients are products of ordinary Fontené-Ward generalized binomial coefficients:

$$(16) \qquad \left\{ {n \atop a, b, c} \right\} = \left\{ {n \atop a} \right\} \left\{ {n-a \atop b} \right\} , \quad a + b + c = n$$

$$(17) \qquad \left\{ {n \atop a, b, c, d} \right\} = \left\{ {n \atop a} \right\} \left\{ {n-a \atop b} \right\} \left\{ {n-a-b \atop c} \right\} , \quad a+b+c+d = n ,$$

$$(18) \qquad \left\{ {n \atop a, b, c, d, e} \right\} = \left\{ {n \atop a} \right\} \left\{ {n-a \atop b} \right\} \left\{ {n-a-b \atop c} \right\} \left\{ {n-a-b-c \atop d} \right\} , a+b+c+d+e = n,$$

and the general result follows at once by induction. This is a well-known device for ordinary multinomial coefficients and the application here is that once one proves that the Fontené-Ward binomial coefficient is an integer for some sequence A_n, then the Fontené-Ward multinomial coefficients, by the above relations, are integers, being just products of integers. This circumvents the tedius argument of Kohlbecker [13] for multinomial Fibonomial coefficients, for example.

Making use of the ideas developed so far and paralleling the steps in a previous paper [8], we are now in a position to state and prove our first major result. We have

Theorem 1. The Fontené-Ward generalized binomial coefficient may be expressed as a linear combination of bracket functions by the formula

$$\left\{ {n \atop k} \right\} = \left[\frac{n}{k}\right] + \sum_{j=k+1}^{n} \left[\frac{n}{j}\right] R_k(j,A) = \sum_{j=k}^{n} \left[\frac{n}{j}\right] R_k(j,A) , \tag{19}$$

where the number-theoretic function R is defined by

$$R_k(j,A) = \sum_{d|j} \frac{A_d - A_{d-k}}{A_k} \left\{ {d-1 \atop k-1} \right\} \mu(j/d) , \tag{20}$$

with $\mu(n)$ being the ordinary Moebius function in number theory.

Proof. Again we use the formula of Meissel

$$\sum_{m \leq x} \left[\frac{x}{m}\right] \mu(m) = 1 , \quad x \geq 1 ,$$

and apply this to formula (13) precisely as was done in [8]. The result follows at once. It is easily seen that $R_k(k,A) = 1$. There will be no confusion of $R_k(j,A)$ with $R_k(j,q)$ in the former paper if we merely make a convention that whenever we have a sequence we denote it by a capital letter and then (20) is meant. Thus $R_k(j,F)$ would mean the Fibonomial case. Thus our first theorem expands the Fibonomial coefficient as a linear combination of bracket functions.

The expansion inverse to this requires a little more care. It was found in [8] by means of a certain inversion theorem for q-binomial coefficients. We must pause and establish the corresponding inversion principle for the most general case. Suppose we set

$$f(n) = \sum_{k=0}^{n} (-1)^{n-k} \left\{ {n \atop k} \right\} g(k) \ . \tag{21}$$

Then we find: $g(0) = f(0)$.

$$f(1) = -g(0) + g(1) \ ,$$

whence

$$g(1) = f(1) + f(0) \ .$$

Then

$$f(2) = g(0) - \frac{A_2}{A_1} g(1) + g(2) \ ,$$

from which we find

$$g(2) = f(2) + \frac{A_2}{A_1} f(1) + \left(\frac{A_2}{A_1} - 1\right) f(0) \ .$$

Similarly it is easily found that

$$\begin{aligned} g(3) &= f(3) + \frac{A_3}{A_1} f(2) + \frac{A_3}{A_1}\left(\frac{A_2}{A_1} - 1\right) f(1) + \left(1 + \frac{A_3}{A_1}\left(\frac{A_2}{A_1} - 2\right)\right) f(0) \\ &= \left\{ {3 \atop 0} \right\} f(3) + \left\{ {3 \atop 1} \right\} f(2) + \left\{ {3 \atop 2} \right\} B_2 f(1) + \left\{ {3 \atop 3} \right\} B_3 f(0) \ , \end{aligned}$$

and it appears that the B_k are independent of n and any number may be found in succession. This is quite correct, for we may readily solve the system of equations necessary to determine such B_k coefficients as will invert (21).

The next step gives

$$B_4 = -1 + \frac{A_4}{A_1}\left(2 - \frac{A_3}{A_2}\left(\frac{A_2}{A_1} - 1\right) + \frac{A_3}{A_1}\left(\frac{A_2}{A_1} - 2\right)\right).$$

Put

$$(22) \qquad g(n) = \sum_{k=0}^{n} \left\{ n \atop k \right\} B_k f(n-k) \quad , \quad \text{with } B_0 = B_1 = 1, \; B_2 = A_2/A_1 - 1, \text{ etc.}$$

It is easily seen by an inductive argument that B_k is independent of f and n. On the one hand, (22) would require us to have

$$(23) \qquad g(n+1) = \sum_{k=0}^{n+1} \left\{ n+1 \atop k \right\} B_k f(n+1-k) = \sum_{j=0}^{n+1} \left\{ n+1 \atop j \right\} B_{n+1-j} f(j) \ .$$

On the other hand we have from (21) that

$$f(n+1) = g(n+1) + \sum_{k=0}^{n} (-1)^{n+1-k} \left\{ n+1 \atop k \right\} g(k) \ ,$$

whence

$$g(n+1) = f(n+1) - \sum_{k=0}^{n} (-1)^{n+1-k} \left\{ n+1 \atop k \right\} g(k) =$$

$$= f(n+1) + \sum_{k=0}^{n} (-1)^{n-k} \left\{ n+1 \atop k \right\} \sum_{j=0}^{k} \left\{ k \atop j \right\} B_j f(k-j) \ ,$$

by (22),

$$= f(n+1) + \sum_{j=0}^{n} f(j) \sum_{k=j}^{n} (-1)^{n-k} \left\{ n+1 \atop k \right\} \left\{ k \atop j \right\} B_{k-j} \ .$$

This expansion must agree with (23) if the induction is to proceed, so we equate coefficients of $f(j)$ to determine a recurrence relation for B_k. At the same time we apply the identity (5) and we have the result that

$$B_{n+1-j} = \sum_{k=j}^{n} (-1)^{n-k} \left\{ {n+1-j \atop k-j} \right\} B_{k-j}, \quad \text{for } 0 \le j \le n. \tag{24}$$

In particular set $j = 0$. We find the remarkably simple recurrence

$$B_{n+1} = \sum_{k=0}^{n} (-1)^{n-k} \left\{ {n+1 \atop k} \right\} B_k, \text{ valid for } n \ge 0. \tag{25}$$

From this it is easily seen that we can summarize our recurrence for B_n in the single formula

$$\sum_{k=0}^{n} (-1)^{n-k} \left\{ {n \atop k} \right\} B_k = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \ge 1. \end{cases} \tag{26}$$

This in turn can be given a handy symbolic expression

$$\{B - 1\}^n = \delta_1^n = \begin{cases} 1, & n = 0 \\ 0, & n \ge 1 \end{cases}, \tag{27}$$

if we just adopt an umbral binomial theorem that

$$\{x + y\}^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x_k y_{n-k}.$$

We shall next evaluate the B coefficients explicitly.

The sequence B_n is determined uniquely by the relation (26), and we can easily solve this by means of determinants. The result of this can be put in the form

$$(28) \qquad B_n = (-1)^n \begin{vmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -\left\{\begin{matrix}2\\1\end{matrix}\right\} & 1 & 0 & \cdots & 0 \\ -1 & \left\{\begin{matrix}3\\1\end{matrix}\right\} & -\left\{\begin{matrix}3\\2\end{matrix}\right\} & 1 & \cdots & 0 \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdots, & \left\{\begin{matrix}n-1\\n-2\end{matrix}\right\} & 1 \\ (-1)^n, & -(-1)^n\left\{\begin{matrix}n\\1\end{matrix}\right\}, & (-1)^n\left\{\begin{matrix}n\\2\end{matrix}\right\}. & \cdots, & \left\{\begin{matrix}n\\n-2\end{matrix}\right\}, & -\left\{\begin{matrix}n\\n-1\end{matrix}\right\} \end{vmatrix}$$

valid for $n \geq 1$.

The n-by-n determinant and the recurrences (25)-(26) allow us to compute as many B's as needed.

It was no accident that we write (26) as (27) and as a Kronecker delta, for not only does (26) allow us to invert (21) to obtain (22), but the converse is also true, (26) allows us to invert (22) back to (21). We have in fact

Theorem 2. For sequences f and g,

$$(29) \qquad f(n) = \sum_{k=0}^{n} (-1)^{n-k} \left\{\begin{matrix}n\\k\end{matrix}\right\} g(k)$$

if and only if

$$(30) \qquad g(n) = \sum_{k=0}^{n} \left\{\begin{matrix}n\\k\end{matrix}\right\} B_{n-k}\, f(k) \quad ,$$

where B_k satisfies recurrence (26), and is given explicitly by (28).

To illustrate the proof we will show that (30) implies (29), assuming (26). We have

$$\sum_{k=0}^{n} (-1)^{n-k} \left\{\begin{matrix}n\\k\end{matrix}\right\} g(k) = \sum_{k=0}^{n} (-1)^{n-k} \left\{\begin{matrix}n\\k\end{matrix}\right\} \sum_{j=0}^{k} \left\{\begin{matrix}k\\j\end{matrix}\right\} B_{k-j}\, f(j) \ ,$$

$$= \sum_{j=0}^{n} \left\{\begin{matrix}n\\j\end{matrix}\right\} f(j) \sum_{k=0}^{n-j} (-1)^{n-j-k} \left\{\begin{matrix}n-j\\k\end{matrix}\right\} B_k = f(n) \quad ,$$

as required, for relation (26)-(27) is equivalent to the Kronecker delta

$$(31) \qquad \sum_{k=0}^{n-j} (-1)^{n-j-k} \left\{ {n-j \atop k} \right\} B_k = \delta_j^n .$$

The reader should have no difficulty in showing that (29) implies (30), relation (31) again being what is needed to cancel out unwanted terms.

These relations are nothing more than extensions of the familiar inversions given in [6], [7], [8].

The application and use of Theorem 2 for Fibonomial expansions needs little elaboration. It allows often to solve for something given implicitly under the summation sign.

As was done in [6] and [8] we need some small variations of Theorem 2. It is easy to see that the theorem can be stated in the equivalent form

$$(32) \qquad f(n) = \sum_{k=0}^{n} (-1)^k \left\{ {n \atop k} \right\} g(k)$$

if and only if

$$(33) \qquad g(n) = \sum_{k=0}^{n} (-1)^k \left\{ {n \atop k} \right\} B_{n-k} f(k) .$$

And we also have

$$(34) \qquad f(n) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} g(k)$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^{n-k} \left\{ n \atop k \right\} B_{n-k} f(k) \ . \tag{35}$$

It is this last form of our theorem that will be used now to find an expansion inverse to Theorem 1. Our steps are the same as in [8].

Theorem 3. The bracket function may be expressed as a linear combination of Fontené-Ward generalized binomial coefficients by the formula

$$\left[\frac{n}{k}\right] = \left\{ n \atop k \right\} + \sum_{j=k+1}^{n} \left\{ n \atop j \right\} Q_k(j, A) = \sum_{j=k}^{n} \left\{ n \atop j \right\} Q_k(j, A) \ , \tag{36}$$

where the coefficients $Q_k(j, A)$ are given by

$$Q_k(j, A) = \sum_{d=k}^{j} (-1)^{j-d} \left\{ j \atop d \right\} \left[\frac{d}{k}\right] B_{j-d} \ , \tag{37}$$

and the B's are given by (26)-(28).

Proof. Assume expansion (36) for unknown Q's. Then by the inversion pair (34)-(35), with $f(n) = [n/k]$ and $g(n) = Q_k(n, A)$, and writing j for k in (34)-(35), the result is immediate.

Hence as a Fibonacci item, this theorem allows one to express the bracket function in terms of Fibonomial coefficients.

The next order of work in [8] was to see if the two expansions, bracket in terms of binomial and conversely, implied a more general inversion theorem; i.e., whether we can now show that our coefficients R and Q are orthogonal in general. Our success in doing this would depend on getting the Lambert series for R and an inverse series for Q. The binomial theorem was used to obtain the latter in [8] and this expansion, the binomial theorem, is more troublesome in our general situation. However we can obtain next the Lambert series for R.

Let us note a general series lemma: For a function $f = f(x,y)$,

$$(38) \qquad \sum_{n=1}^{\infty} \sum_{d|n} f(d,n) = \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} f(d,md)$$

This is merely the limiting case of relation (20) in [7] for example.

Theorem 4. The Lambert series expansion for $R_k(j,A)$ is given by

$$(39) \qquad \sum_{j=k}^{\infty} R_k(j,A) \frac{x^j}{1-x^j} = \sum_{n=k}^{\infty} \frac{A_n - A_{n-k}}{A_k} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} x^n .$$

Proof. First of all the ordinary Moebius inversion theorem applied to relation (20) inverts this to yield

$$(40) \qquad \frac{A_n - A_{n-k}}{A_k} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} = \sum_{d|n} R(d,A) ,$$

which may itself be looked on as a valuable expansion of the Fontené-Ward generalized binomial coefficients in terms of the function $R_k(d,a)$. This is merely the generalization of the combinatorial formula

$$C_k(n) = \binom{n-1}{k-1} = \sum_{d|n} R_k(d) ,$$

found in [7].

Multiply (40) through by x^n and sum both sides on n. We find

$$\sum_{n=1}^{\infty} \frac{A_n - A_{n-k}}{A_k} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} x^n = \sum_{n=1}^{\infty} \sum_{d|n} x^r R_k(d, A)$$

$$= \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} x^{md} R_k(d, A) \quad , \quad \text{by (38)} \quad ,$$

$$= \sum_{d=1}^{\infty} R_k(d, A) \sum_{m=1}^{\infty} x^{md} = \sum_{d=1}^{\infty} R_k(d, A) \frac{x^d}{1 - x^d} \quad .$$

The lower limits of summation in the result can be changed to k instead of 1 since the Fontené-Ward coefficients and R_k are each zero for the first $k - 1$ terms on each side. This proves the theorem.

We have given some detailed steps to illustrate precisely what happens. But let us now try to carry over the binomial theorem. It turns out that we do not need the binomial theorem in a very strong form.

To find the series expansion inverse to (39), we recall the bracket function series (of Hermite) from [8];

$$\sum_{n=k}^{\infty} \left[\frac{n}{k} \right] x^n = \frac{x^k}{(1 - x)(1 - x^k)} \quad , \quad k \geq 1. \tag{41}$$

Substitute the expansion of $[n/k]$ in terms of Fontené-Ward coefficients, and we get

$$\frac{x^k}{1 - x^k} = \sum_{n=k}^{\infty} (1 - x) x^n \sum_{j=k}^{n} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} Q_k(j, A) \tag{42}$$

$$= \sum_{j=k}^{\infty} Q_k(j, A)\,(1 - x) \sum_{n=j}^{\infty} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} x^n \quad .$$

The last inner sum is not conveniently put into closed form by a binomial theorem, but we can transform it as follows:

$$(1 - x)\sum_{n=j}^{\infty}\left\{ {n \atop j} \right\} x^n = x^j + \sum_{n=j+1}^{\infty} x^n \left(\left\{ {n \atop j} \right\} - \left\{ {n-1 \atop j} \right\} \right)$$

and we can now apply the original Fontené recurrence (6) and we recall that $A_0 = 0$ so that x^j can be counted in the sum. The result is the formula

$$(1 - x)\sum_{n=j}^{\infty}\left\{ {n \atop j} \right\} x^n = \sum_{n=j}^{\infty} x^n \left\{ {n-1 \atop j-1} \right\} \frac{A_n - A_{n-j}}{A_j} \quad . \tag{43}$$

This formula is the general counterpart of the familiar formula

$$(1 - x)\sum_{n=j}^{\infty}\binom{n}{j} x^n = \frac{x^j}{(1 - x)^j}$$

used in [7, pp. 241, 252]. The corresponding q-analog in [8, p. 407] was

$$(1 - x)\sum_{n=j}^{\infty}\left[{n \atop j} \right] x^n = x^j \prod_{i=1}^{j} (1 - xq^i) \ .$$

The reader may find it interesting to find the corresponding Fibonomial form.

Finally, we substitute expansion (43) into (42) and we find the formula inverse to (39); i.e., we have proved

Theorem 5. The coefficients $Q_k(j, A)$ satisfy the generating expansion

$$\frac{x^k}{1 - x^k} = \sum_{j=k}^{\infty} Q_k(j, A) \sum_{n=j}^{\infty} x^n \left\{ {n-1 \atop j-1} \right\} \frac{A_n - A_{n-j}}{A_j} \tag{44}$$

We may write the two expansions of Theorems 4 and 5 in the forms

(45) $$\sum_{j=k}^{\infty} R_k(j,A)\,\frac{x^j}{1-x^j} = f(x,k)\ ,$$

and

(46) $$\sum_{j=k}^{\infty} Q_k(j,A)\,f(x,j) = \frac{x^k}{1-x^k}\ ,$$

where $f(x,j)$ is the power series

(47) $$f(x,j) = \sum_{n=j}^{\infty} x^n \left\{ {n-1 \atop j-1} \right\} \frac{A_n - A_{n-j}}{A_j} = (1-x)\sum_{n=j}^{\infty} \left\{ {n \atop j} \right\} x^n\ ,$$

and we may now see easily that substitution of (45) into (46), and conversely, yields our desired orthogonality of R and Q. Thus we evidently have

Theorem 6. The functions R and Q as defined by (20) and (37) satisfy the orthogonality relations

(48) $$\sum_{j=k}^{n} R_k(j,A)Q_j(n,A) = \delta_k^n = \sum_{j=k}^{n} Q_k(j,A)R_j(n,A)\ .$$

Consequently, we also have proved the very general inversion theorem for two sequences that held for the previous cases [7], [8]. That is we have

Theorem 7. For two sequences $f(x,k,A)$, $g(x,k,A)$, then

(49) $$f(x,k,A) = \sum_{k \le j \le x} g(x,j,A)R_k(j,A)$$

if and only if

(50) $$g(x,k,A) = \sum_{k \le j \le x} f(x,j,A)Q_k(j,A) .$$

CONCLUSION

In the present paper we have given a sequence of seven main theorems, generalizing all of the corresponding results previously found for ordinary and q-binomial coefficients to the most general situation for Fontené-Ward generalized binomial coefficients. As a single byproduct we have results universally valid for the popular Fibonomial triangle. The inversion theorems given here are expected to suggest other inversion theorems in the most general setting, which can then be applied to any special case that is covered by the Fontené-Ward Triangle.

REFERENCES

1. Terrance A. Brennan, "Fibonacci Powers and Pascal's Triangle in a Matric," Fibonacci Quarterly, 2 (1964), pp. 93-104, 177-184.
2. Raoul Bricard, "Necrology of Georges Fontené," Nouv. Ann. Math., (5) 1 (1923), pp. 361-363.
3. A. M. Chak, "An Extension of a Class of Polynomials," to appear.
4. G. Fontené, "Généralisation d'une formule connue," Nouv. Ann. Math., (4) 15 (1915), pp. 112.
5. H. W. Gould and L. Carlitz, "Bracket Function Congruences for Binomial Coefficients," Math. Mag., 37 (1964), pp. 91–93.
6. H. W. Gould, "The Operator $(a^x\Delta)^n$ and Stirling Numbers of the First Kind," Amer. Math. Monthly, 71 (1964), pp. 850-858.
7. H. W. Gould, "Binomial Coefficients, the Bracket Function, and Compositions with Relatively Prime Summands," Fibonacci Quarterly, 2 (1964), pp. 241-260.
8. H. W. Gould, "The Bracket Function, q-Binomial Coefficients, and Some New Stirling Number Formulas," Fibonacci Quarterly, 5 (1967), pp. 401-423.
9. V. E. Hoggatt, Jr. and D. A. Lind, "A Power Identity for Second-Order Recurrent Sequences," Fibonacci Quarterly, 4 (1966), pp. 274-288.

[Continued on p. 55.]

CONVERGENCE OF THE COEFFICIENTS IN A RECURRING POWER SERIES

JOSEPH ARKIN
Nanuet, New York

1. INTRODUCTION

In this paper we use the following notation

$$\left(\sum_{w=0}^{\infty} c_w x^w\right)^k = \sum_{w=0}^{\infty} c_w^{(k)} x^w \quad ,$$

(For convenience, we shall write c_w instead of $c_w^{(1)}$.)

We define

$$\sum_{w=0}^{f} b_w x^w = F(x) \neq 0$$

for a finite f,

$$\sum_{w=0}^{t} a_w x^w = \prod_{w=1}^{m} (1 - r_w x)^{d_w} = Q(x)$$

for finite t and m, where the $d_w \neq 0$ and are positive integers. The $r_w \neq 0$ and are distinct and we say $|r_1|$ is the greatest $|r|$ in the $|r_w|$.

2. THEOREM 1

If

$$F(x)/Q(x) = \sum_{w=0}^{\infty} u_w x^w \quad ,$$

then

$$\lim_{n\to\infty} \left| u_n / u_{n-j} \right| \quad \text{(for a finite } j = 0, 1, 2, \cdots) \tag{2.1}$$

converges to $\left|r_1^j\right|$, where the $r_w \neq 0$ in $Q(x)$ are distinct with distinct moduli and $|r_1|$ is the greatest $|r|$ in the $|r_w|$.

<u>Proof</u>. It has been shown by Poincaré [1] that

$$\lim_{n\to\infty} u_n / u_{n-1} \tag{2.2}$$

converges to some root (r) in $Q(x)$. (We must then prove that this root (r) in $Q(x)$ is $|r_1|$.)

Let

$$M(x) = \prod_{w=1}^{m} (1 - r_w x)^{p_w} , \tag{2.3}$$

where the p_w are positive integers or $=0$ and

$$d_1 + p_1 = d_2 + p_2 = \cdots = p_w + d_w = k \qquad (k = 1, 2, 3, \cdots)$$

for a finite $w = 1, 2, 3, \cdots, m$.

Then,

$$M(x)Q(x) = \prod_{w=1}^{m} (1 - r_w x)^k = \phi_k(x) ,$$

so that

$$F(x)M(x)/Q(x)M(x) = F(x)M(x)/\phi_k(x) \tag{2.4}$$

$$= \sum_{w=0}^{\infty} u_w x^w = \sum_{w=0}^{\infty} c(k, w) x^w ,$$

where it is evident

$$u_n = c(k, n) \quad .$$

Now let

$$\phi_k(x) = \sum_{w=0}^{v} c_w^{(k)} x^w \qquad \text{(where } v \text{ is finite)} \quad ,$$

where combining this with (2.4), we write

$$\text{(2.5)} \qquad F(x)M(x)/\phi_{k-1}(x) = \sum_{w=0}^{\infty} c(k-1, w)x^w$$

$$= \left(\sum_{w=0}^{v} c_w x^w\right)\left(\sum_{w=0}^{\infty} c(k, w)x^w\right) \quad ,$$

and combining coefficients leads to

$$\text{(2.5.1)} \qquad c(k-1, n) = \sum_{w=0}^{v} c(k, n-w)c_w = \sum_{w=0}^{v} u_{n-w}\, c_w \quad ,$$

$$k = 2, 3, 4, \cdots \quad .$$

In (2.5.1), we replace k with $k+1$ (where $k = 1, 2, 3, \cdots$) where combining this result with (2.2) leads to

$$\lim_{n\to\infty} \left|c(k+1, n)/c(k+1, n-1)\right| \text{ converges to some root } (r) \text{ in } Q(x).$$

For convenience, we write the convergence as

$$\text{(2.5.2)} \qquad c(k+1, n) = g_{k+1}\, c(k+1, n-1) \quad .$$

Combining (2.5.1) with k replaced by $k + 1$ with (2.5.2), it is easily shown, that for a finite v, we have

$$(2.5.3) \qquad \begin{aligned} c(k,n)/c(k,n-1) &= g_k \\ &= \sum_{w=0}^{v} c(k+1, n-w)c_w \Big/ \sum_{w=0}^{v} c(k+1, n-w-1)c_w \\ &= g_{k+1}\ , \end{aligned}$$

so that

$$(2.5.4) \qquad g_{k+1} = g_k = \cdots = g_1 .$$

Thus to complete the proof of Theorem 1, it remains to show that

$$|g_1| = |r_1| .$$

Then we consider the following (we refer to (2.3))

$$(2.6) \qquad (\phi(x))^{-1} = \prod_{w=1}^{m} (1 - r_w x)^{-1} = \sum_{w=0}^{\infty} e(m, w)x^w \quad \text{(for a finite } m)$$

for the convergence properties of $e(m,n)/e(m,n-1)$, where the $|r_w|$ are distinct and $|r_1|$ is the greatest root.

NOTE. For convenience, we write

$$e(m,n)/e(m,n-j) = r_1^j \qquad \text{(for a finite } j = 0, 1, 2, \cdots) ,$$

in place of

$$\lim_{n \to \infty} |e(m,n)/e(m,n-j)| \text{ converges to } |r_1^j| .$$

For $m = 1$, we have

$$(2.7) \qquad (1 - r_1x)^{-1} = \sum_{w=0}^{\infty} e(1, w)\, x^w ,$$

where

$$e(1, n) = r_1^n ,$$

so that

$$e(1,n)/e(1,n-j) = r_1^j .$$

For $m = 2$, we have

$$(2.8) \qquad [(1 - r_1x)(1 - r_2x)]^{-1} = \sum_{w=0}^{\infty} e(2,w)\, x^w .$$

where

$$e(2,n) = (r_1^{n+1} - r_2^{n+1})/(r_1 - r_2) ,$$

so that

$$e(2,n)/e(2,n-j) = r_1^j .$$

It now remains to consider for finite $m = 3, 4, 5, \cdots$, let

$$(2.9) \qquad \left(1 - \sum_{s=0}^{t-1} a_s x^{t-s}\right)^{-1} = \prod_{s=1}^{t} (1 - r_s x)^{-1} = 1 + \sum_{s=1}^{\infty} U_s x^s ,$$

for a finite $t = 3, 4, 5, \cdots$, where $U_0 = 1$.

Equating the coefficients in this leads to

$$(2.10) \qquad U_n = \sum_{s=1}^{t} a_{t-s} U_{n-s} \qquad (U_0 = 1) ,$$

and

$$U_1 = U_0 a_{t-1} , \quad U_2 = U_1 a_{t-1} + U_0 a_{t-2}, \cdots, U_t = \sum_{s=0}^{t-1} U_s a_s .$$

Also, since in (2.9), we have

$$\prod_{s=1}^{t} (1 - r_s x) = 1 - \sum_{s=0}^{t-1} a_s x^{t-s} ,$$

we may write

$$(2.11) \qquad \prod_{s=1}^{t} (x - r_s) = x^t - \sum_{s=0}^{t-1} a_s x^s = 0 .$$

We now combine (2.10) with (2.11) and write

$$(2.12) \qquad x^t = U_1 x^{t-1} + \sum_{s=2}^{t} \left(U_s - \sum_{r=1}^{s-1} U_r a_{t+r-s} \right) x^{t-s} .$$

Multiplying (2.12) by x and combining the result with

$$U_1 x^t = U_1 \sum_{s=0}^{t-1} a_s x^s$$

in (2.11) leads to

$$(2.13) \qquad x^{t+1} = U_2x^{t-1} + \sum_{r=0}^{t-3}\left(U_{r+3} - \sum_{s=0}^{r} U_{r+2-s}a_{t-s-1}\right)x^{t-r-2}$$

$$+ U_1a_0 .$$

Now, multiplying (2.13) by x and combining the result with

$$U_2x^t = U_2\left(\sum_{s=0}^{t-1} a_sx^s\right)$$

in (2.11), we then have

$$(2.14) \qquad x^{t+2} = U_3x^{t-1} + \sum_{r=0}^{t-3}\left[\left(U_{r+4} - \sum_{s=0}^{r} U_{r+3-s}a_{t-s-1}\right)x^{t-r-2}\right]$$

$$+ a_0U_2 .$$

We continue in the exact way we found (2.13) and (2.14) for $n - 1$ steps to get

$$(2.15) \qquad x^{t+n-1} = U_nx^{t-1} + \sum_{r=0}^{t-3}\left[\left(U_{n+r+1} - \sum_{s=0}^{r} U_{n+r-s}a_{t-s-1}\right)x^{t-r-2}\right]$$

$$+ U_{n-1}a_0 = U_nx^{t-1} + R(x) + U_{n-1}a_0 .$$

We now continue (2.15) with (2.11) to get the following t equations

$$(2.16) \qquad \begin{aligned} r_1^{t+n-1} &= U_nr_1^{t-1} + R(r_1) + U_{n-1}a_0 , \\ &\dots\dots\dots\dots\dots , \\ r_t^{t+n-1} &= U_nr_t^{t-1} + R(r_t) + U_{n-1}a_0 . \end{aligned}$$

Next, we consider the t equations obtained from (2.16). These t equations in the t unknown can be solved by Cramer's rule to obtain

$$U_n D_2 = D_1(n) \quad , \tag{2.17}$$

where $D_1(n)$ and D_2 are the determinants given below:

$$D_1(n) = \begin{vmatrix} r_1^{t+n-1} & r_1^{t-2} & \cdots & r_1 & 1 \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ r_t^{t+n-1} & r_t^{t-2} & \cdots & r_t & 1 \end{vmatrix} \tag{2.18}$$

$$D_2 = \begin{vmatrix} r_1^{t-1} & r_1^{t-2} & \cdots & r_1 & 1 \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ r_t^{t-1} & r_t^{t-2} & \cdots & r_t & 1 \end{vmatrix} \tag{2.19}$$

We now replace n with n - 1 in (2.17) to get

$$U_{n-1} D_2 = D_1(n-1) \quad , \tag{2.20}$$

and dividing (2.17) by (2.20), we get

$$U_n / U_{n-1} = D_1(n) / D_1(n-1) \quad . \tag{2.21}$$

Since the $r_t \neq 0$ and are distinct, then one root (say $|r_1|$ is greater than the other roots, and we write

$$U_n / U_{n-1} = (D_1(n) / r_1^{t+n-2}) / (D_1(n-1)/r_1^{t+n-2}) \tag{2.22}$$

Now in (2.22) we let r_1^{t+n-2} (in the numerator) divide every term of the first column in (2.18) and r_1^{t+n-2} (in the denominator) divide every term in the first column of (2.18) (with n replaced by n - 1). Then if we let $n \to \infty$ it is evident that

(2.23)
$$\lim_{n\to\infty} \left|U_n / U_{n-1}\right| = \left|r_1\right| .$$

Now for a finite t we write

$$\lim_{n\to\infty} \left|U_{n-j} / U_{n-j-1}\right| = \left|r_1\right| \quad (j = 0, 1, 2, \cdots, t-1) ,$$

so that

(2.24)
$$\lim_{n\to\infty} \left|U_n / U_{n-t}\right| = \left|r_1^t\right| .$$

Multiplying the $F(x)$ in (1) with

$$\sum_{s=0}^{\infty} U_s x^s$$

in (2.9), we write

(2.25)
$$\left(\sum_{w=0}^{f} b_w x^w\right)\left(\sum_{s=0}^{\infty} U_s x^s\right) = \sum_{s=0}^{\infty} C_s x^s ,$$

where comparing the coefficients we have

(2.26)
$$C_n = \sum_{s=0}^{f} U_{n-s} b_s .$$

Now, since f is finite, and by the results in (2.23), we write

$$C_n = r_1 \sum_{s=0}^{f} U_{n-s-1} b_s = r_1 C_{n-1} ,$$

where combining this with the $r_t \neq 0$ and are distinct (so that we may add that the r_t have distinct moduli), leads to the completion of the proof for Theorem 1.

From (2.7), (2.8), and (2.17), the following corollary is immediate:

Corollary. If

$$\prod_{s=1}^{t} (1 - r_s x)^{-1} = \sum_{s=0}^{\infty} U_s x^s \quad (U_0 = 1),$$

where the $r_s \neq 0$ and are distinct, then

(2.27) It is always possible to solve for the U_n $(n = 0, 1, 2, \cdots)$ as a function of the r_s.

SECTION 3

Let

$$\left(1 - \sum_{w=1}^{t} a_w x^w\right)^{-k} = \prod_{w=1}^{t} (1 - r_w x)^{-k} = \sum_{w=0}^{\infty} c_w^{(k)} x^w$$

$(c_0^{(k)} = 1$ and $k = 1, 2, 3, \cdots)$ for a finite $t = 2, 3, 4, \cdots$ and the given roots $r_w \neq 0$ and are distinct. We also define

$$S(x) = \sum_{w=1}^{t} \sum_{r=w}^{t} a_r c_{n+w-r} x^{w-1} = 0$$

and

$$b = \sum_{w=2}^{t} a_w x_1^{w-2}$$

where ${}^*x_1 \neq 0$ and is a root in $S(x) = S(x_1) = 0$.

We then have the following:

Theorem 2. If

$$c_0 = 1, \quad c_1 = a_1c_0, \quad c_2 = a_1c_1 + a_2c_0, \cdots$$

$$\cdots, c_t = \sum_{w=0}^{t-1} a_{w+1}\, c_{t-w-1}$$

and

$$p_j = a_1(k + n - j) \quad (j = 1, 2, 3, \cdots, n) ,$$

$$q_{m+1} = b(n - m)(2k + n - m - 1)$$
$$(m = 1, 2, 3, \cdots, n - 1)$$

then

$$nc_n^{(k)} / c_{n-1}^{(k)} = E_n / G_n \quad (k, n = 1, 2, 3, \cdots) , \tag{3.1}$$

where E_n and G_n are the determinants given below.

$$E_n = \begin{vmatrix} p_1 & q_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & p_2 & q_3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & p_3 & q_4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & p_4 & q_5 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & p_{n-1} & q_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & p_n \end{vmatrix} \tag{3.1.1}$$

---

*It should be noted that since the a's are constant for a fixed t, that the root x_1 will be determined as a variable, since it is a function of the c_n and will, of course, change values for different n.

$$
(3.1.2) \qquad G_n = \begin{vmatrix}
p_2 & q_3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & p_3 & q_4 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & p_4 & q_5 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -1 & p_5 & q_6 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & \cdots & -1 & p_{n-1} & q_n \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & p_n
\end{vmatrix}
$$

Proof. Let

$$
(3.2) \qquad 1 = \left(1 - \sum_{w=1}^{t} a_w x^w\right)\left(\sum_{w=0}^{n} c_w x^w\right) \quad \text{(for a finite } n) ,
$$

where the a_w and the c_w are identical to those in (3). Then multiplying and combining the terms in (3.2) leads to $S(x_1) = S(x) = 0$ in (3).

Now, taking each side of (3.2) to the k^{th} power, we write

$$
(3.3) \qquad 1^k = \left(1 - \sum_{w=1}^{t} a_w x^w\right)^k \left(\sum_{w=0}^{n} c_w^{(k)} x^w + J(x)\right) \qquad (k = 2, 3, \cdots),
$$

(where, of course, x_1 is a root in (3.3)).

Using the corresponding values in (3), we write (3.3) as

$$
(3.3.1) \qquad 1 = (1 - a_1 x - bx^2)^k \left(\sum_{w=0}^{n} c_w^{(k)} x^w + J(x)\right)
$$

Differentiation of (3.3.1) leads to

$$
k(a_1 x + 2bx^2)\left(\sum_{w=0}^{n} c_n^{(k)} + J(x)\right) = (1 - a_1 x - bx^2)\left(\sum_{w=1}^{n} nc_n^{(k)} x^n + W(x)\right)
$$

and by comparing coefficients, we conclude that

$$nc_n^{(k)} = a_1(k + n - 1)c_{n-1}^{(k)} + b(2k + n - 2)c_{n-2}^{(k)} \tag{3.4}$$

for

$$k = 2, 3, \cdots, \quad n = 2, 3, \cdots, \quad c_0^{(k)} = 1 \text{ and } c_1^{(k)} = a_1k .$$

When we divide (3.4) by $c_{n-1}^{(k)}$, we get

$$\frac{nc_n^{(k)}}{c_{n-1}^{(k)}} = a_1(k + n - 1) + \frac{b(2k + n - 2)(n - 1)}{\dfrac{(n-1)c_{n-1}^{(k)}}{c_{n-2}^{(k)}}} \qquad (n, k = 2, 3, \cdots) ,$$

which in turn, along with $c_0^{(k)} = 1$ and $c_1^{(k)} = a_1k$, implies (along with the values of p and q in (3)),

$$\frac{nc_n^{(k)}}{c_{n-1}^{(k)}} = p_1 + \frac{q_2}{p_2} + \frac{q_3}{p_3} + \cdots + \frac{q_{n-1}}{p_{n-1}} + \frac{q_n}{p_n} = K(n) . \tag{3.5}$$

We complete the proof of Theorem 2 with Euler's statement [2]

$$K(n) = E_n / G_n ;$$

and we resolve for the case when $k = 1$ with (2.27).

Corollary. In

$$\prod_{w=1}^{t} (1 - r_w x)^{-k} = \left(1 - \sum_{w=1}^{t} a_w x^w\right)^{-k} = 1 + \sum_{w=1}^{\infty} c_w^{(k)} x^w ,$$

it is always possible to solve for

$$nc_n^{(k)} / c_{n-1}^{(k)} = K(n) = E_n / G_n \quad (k \text{ and } n = 2, 3, \cdots) \tag{3.6}$$

when $t = 2, 3, 4,$ or 5, if the $r_w \neq 0$ and are distinct.

Proof. In (2.27), it is seen that the c_n maybe determined. Now, since $t - 1 = 1, 2, 3,$ or 4, then the roots (each root is a function of the c_n) in $S(x)$ (in 3) may always be found, so that we will obtain values for the p and q. We then complete the proof of the corollary by observing that E_n and G_n are both functions of the p and q.

In conclusion: We solve when $t = 1$ and we write

$$(1 - r_x)^{-k} = \sum_{w=0}^{\infty} d_w^{(k)} x^w \qquad (d_0^{(k)} = 1, \quad r \neq 0) \ .$$

Now, differentiating, we have

$$xkr\left(\sum_{w=0}^{\infty} d_w^{(k+1)} x^w\right) = \sum_{w=1}^{\infty} wd_w^{(k)} x^w$$

and comparing the coefficients leads to

$$nd_n^{(k)} = d_{n-1}^{(k+1)} \ r^k$$

so that

$$\prod_{w=1}^{n} wd_w^{(k+n-w)} = r^n \prod_{w=0}^{n-1} (k + n - w - 1) d_w^{(k+n-w)}$$

and we then have

$$d_n^{(k)} = r^n (k + n - 1)!/n!\,(k - 1)! \quad .$$

REFERENCES

1. L. M. Milne-Thomson, The Calculus of Finite Differences, Macmillan and Co., Ltd., London, 1960, p. 526.
2. G. Chrystal, Textbook of Algebra, Vol. II, Dover Publications, Inc., New York, 1961, p. 502.

The author wishes to thank L. Carlitz and V. E. Hoggatt, Jr. for their encouragement.

* * * * *

[Continued from p. 40.]

10. V. E. Hoggatt, Jr., "Fibonacci Numbers and Generalized Binomial Coefficients," Fibonacci Quarterly, 5 (1967), pp. 383-400.
11. Dov Jarden, "The Product of Sequences with a Common Linear Recursion Formula of Order 2," publ. in Recurring Sequences, Jerusalem, 1958, pp. 42-45. Original paper appeared in Hebrew in Riveon Lematematika, 3 (1949), pp. 25-27; 38, being a joint paper with Th. Motzkin.
12. Dov Jarden, "Nullifying Coefficients," Scripta Mathematica, 19(1953), pp. 239-241.
13. Eugene E. Kohlbecker, "On a Generalization of Multinomial Coefficients for Fibonacci Sequences," Fibonacci Quarterly, 4 (1966), pp. 307-312.
14. S. G. Mohanty, "Restricted Compositions," Fibonacci Quarterly, 5(1967), pp. 223-234.
15. Roseanna F. Torretto and J. Allen Fuchs, "Generalized Binomial Coefficients," Fibonacci Quarterly, 2 (1964), pp. 296-302.
16. Morgan Ward, "A Calculus of Sequences," Amer. J. Math., 58 (1936), pp. 255-266.
17. Stephen K. Jerbic, "Fibonomial Coefficients — A Few Summation Properties," Master's Thesis, San Jose State College, San Jose, Calif., 1968.

* * * * *

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-148 Proposed by James E. Desmond, Florida State University, Tallahassee, Florida

Prove or disprove: There exists a positive integer m such that

$$m \text{ times } \left\{ F_{F_{\cdot_{\cdot_{\cdot_{F_{F_n}}}}}} \right.$$

is composite for all integers $n > 5$.

H-149 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

For $s = \sigma + it$ let

$$P(s) = \sum p^{-s} ,$$

where the summation is over the primes. Set

$$\sum_{n=1}^{\infty} a(n)n^{-s} = [1 + P(s)]^{-1} ,$$

$$\sum_{n=1}^{\infty} b(n)n^{-s} = [1 - P(s)]^{-1} .$$

Determine the coefficients $a(n)$ and $b(n)$.

H-150 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada

Show that

$$25\sum_{p=1}^{n-1}\sum_{q=1}^{p}\sum_{r=1}^{q} F_{2r-1}^2 = F_{4n} + (n/3)(5n^2 - 14) ,$$

where F_n is the n^{th} Fibonacci number.

H-151 Proposed by L. Carlitz, Duke University, Durham, N. Carolina

A. Put

$$(1 - ax^2 - bxy - cy^2)^{-1} = \sum_{m,n=0}^{\infty} A_{m,n}x^m y^n .$$

Show that

$$\sum_{n=0}^{\infty} A_{n,n}x^n = \left\{1 - 2bx + (b^2 - 4ac)x^2\right\}^{-\frac{1}{2}} .$$

B. Put

$$(1 - ax - bxy - cy)^{-1} = \sum_{m,n=0}^{\infty} B_{m,n}x^m y^n .$$

Show that

$$\sum_{n=0}^{\infty} B_{n,n}x^n = \left\{ (1 - bx)^2 - 4acx \right\}^{-\frac{1}{2}} .$$

H-152 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let m denote a positive integer and F_n the n^{th} Fibonacci number. Further let $\{c_k\}_{k=1}^{\infty}$ be the sequence defined by

$$\{c_k\}_{k=1} \equiv \{ \underbrace{F_n^m, F_n^m, \cdots, F_n^m}_{2^{m-1} \text{ copies}} \}_{n=1}^{\infty}$$

Prove that $\{c_k\}_{k=1}^{\infty}$ is complete; i.e., show that every positive integer, n, has at least one representation of the form

$$n = \sum_{k=1}^{p} \alpha_k c_k ,$$

where p is a positive integer and

$$\alpha_i = 0 \text{ or } 1 \text{ if } k = 1, 2, \cdots, p - 1$$

$$\alpha_p = 1$$

C.f. V. E. Hoggatt, Jr., and C. King, Problem E1424, American Mathematical Monthly, Vol. 67 (1960), p. 593 and J. L. Brown, Jr., "Note on Complete Sequences of Integers," American Mathematical Monthly, Vol. 67 (1960), pp. 557-560.

SOLUTIONS

POWER PLAY

H-109 Proposed by George Ledin, Jr., San Francisco, Calif.

Solve

$$x^2 + y^2 + 1 = 3xy$$

for all integral solutions and consequently derive the identity

$$F_{6k+7}^2 + F_{6k+5}^2 + 1 = 3F_{6k+7}F_{6k+5} \,.$$

Solution by H. V. Krishna, Manipal Engineering College, Manipal, India

Let the equation in question be expressed as

$$(1) \qquad (x - 3y/2)^2 - 5(y/2)^2 = -1 \quad .$$

The general solution of (1) is therefore given by

$$(2) \qquad \begin{aligned} x - (3y/2) &= \tfrac{1}{2}\left\{(p + \sqrt{5}q)^{2n-1} + (p - \sqrt{5}q)^{2n-1}\right\} \\ (y/2) &= 1/(2\sqrt{5})\left\{(p + \sqrt{5}q)^{2n-1} - (p - \sqrt{5}q)^{2n-1}\right\} \end{aligned}$$

where (p,q) is a particular solution of (1).

Hence (2) reduces to $y = F_{2n-1}$ and $x = (1/2)(L_{2n-1} + 3F_{2n-1})$ for $p = \frac{1}{2}$ and $q = \frac{1}{2}$.

On using $L_{2n-1} + F_{2n-1} = 2F_{2n}$,

$$x = \tfrac{1}{2}\left\{2(F_{2n} + F_{2n-1})\right\} = F_{2n+1} \,,$$

whence the desired identity follows for $n = 3(k + 1)$.

Also Solved by A. Shannon.

TRIG OR TREAT

H-111 Proposed by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

Show that

$$L_n = \prod_{k=1}^{[n/2]} \left\{ 1 + 4 \cos^2 \frac{2k-1}{n} \left(\frac{\pi}{2}\right) \right\} \text{ for } n \ge 1 .$$

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

We know from the solution of Problem H-64 (Fibonacci Quarterly, Vol. 5, Feb. 1967, p. 75), that

$$L_n = \prod_{j=1}^{n} \left\{ 1 - 2i \cos \frac{(2j-1)\pi}{2n} \right\}, \quad i = \sqrt{-1} .$$

If n is odd, then

$$L_{2n+1} = \prod_{j=1}^{2n+1} \left\{ 1 - 2i \cos \frac{(2j-1)\pi}{2(2n+1)} \right\}$$

$$= \prod_{j=1}^{n} \left\{ 1 - 2i \cos \frac{(2j-1)\pi}{2(2n+1)} \right\} \cdot \prod_{k=n+2}^{2n+1} \left\{ 1 - 2i \cos \frac{(2k-1)\pi}{2(2n+1)} \right\}$$

$$\cdot \left\{ 1 - 2 \cos \frac{2(n+1)-1}{4n+2} \pi \right\}$$

$$= \prod_{j=1}^{n} \left\{ 1 - 2i \cos \frac{(2j-1)\pi}{2(2n+1)} \right\} \cdot \prod_{k=n+2}^{2n+1} \left\{ 1 - 2i \cos \frac{(2k-1)\pi}{2(2n+1)} \right\}$$

Letting j = (2n + 2 - k) in the second product, we get

$$L_{2n+1} = \prod_{j=1}^{n}\left[1 - 2i\cos\frac{(2j-1)\pi}{2(2n+1)}\right]\cdot\prod_{j=1}^{n}\left[1 - 2i\cos\left\{\pi - \frac{(2j-1)\pi}{2(2n+1)}\right\}\right]$$

(1)

$$= \prod_{j=1}^{n}\left\{1 + 4\cos^2\frac{2j-1}{2n+1}\cdot\frac{\pi}{2}\right\}.$$

Similarly,

$$\text{(2)} \qquad L_{2n} = \prod_{j=1}^{n}\left\{1 + 4\cos^2\frac{2j-1}{2n}\cdot\frac{\pi}{2}\right\}.$$

Hence from (1) and (2) we have the required result.

Also solved by Charles Wall, Douglas Lind, and David Zeitlin.

VIVA LA DIFFÉRENCE

H-112 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that, for $n \geq 1$,

a) $$L_{n+1}^5 - L_n^5 - L_{n-1}^5 = 5L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)$$

b) $$F_{n+1}^5 - F_n^5 - F_{n-1}^5 = 5F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)$$

c) $$L_{n+1}^7 - L_n^7 - L_{n-1}^7 = 7L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)^2$$

d) $$F_{n+1}^7 - F_n^7 - F_{n-1}^7 = 7F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)^2 .$$

Solution by the proposer.

For parts c) and d), take $x = L_n$, $y = L_{n-1}$ in the identity

$$(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2 + xy + y^2)^2 .$$

Since

$$L_n^2 + L_nL_{n-1} + L_{n-1}^2 = 2L_n^2 - 5(-1)^n ,$$

we get

$$L_{n+1}^7 - L_n^7 - L_{n-1}^7 = 7L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)^2 .$$

Similarly, since

$$F_n^2 + F_nF_{n-1} + F_{n-1}^2 = 2F_n^2 + (-1)^n ,$$

we get

$$F_{n+1}^7 - F_n^7 - F_{n-1}^7 = 7F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)^2 .$$

Parts a) and b) follow in a similar manner, by selecting $x = L_n$, $y = L_{n-1}$; $x = F_n$, $y = F_{n-1}$ in the identity

$$(x + y)^5 - x^5 - y^5 = 5xy(x + y)(x^2 + xy + y^2) .$$

Also solved by Charles Wall.

MINOR EXPANSION

H–117 Proposed by George Ledin, Jr., San Francisco, Calif.

Prove

$$\begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_n \\ F_{n+2} & F_{n+3} & F_n & F_{n+1} \\ F_{n+1} & F_n & F_{n+3} & F_{n+2} \\ F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{vmatrix} = F_{2n+6}\, F_{2n}$$

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa.

The determinant (first evaluated in 1866)

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = (a - b - c + d)\,(a - b + c - d)\,(a + b - c - d)(a+b+c+d) .$$

In this case the product is

$$F_n(F_{n+1} + F_{n-1})F_{n+3}(F_{n+4} + F_{n+2})$$

from the recurrence

$$F_{n+1} = F_n + F_{n-1} .$$

The identities

$$L_n = F_{n+1} + F_{n-1}$$

and

$$F_{2n} = F_n L_n$$

now complete the proof.

Also solved by David Zeitlin, A. Shannon, D. Jaiswal, J. Biggs, F. Parker, S. Lajos, H. Krishna, and Stanley Rabinowitz

GOOD COMBINATION

H-119 Proposed by L. Carlitz, Duke University, Durham, N. Carolina

Put

$$\overline{H}(m,n,p) = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{p} (-1)^{i+j+k} \binom{i+j}{j}\binom{j+k}{k}\binom{k+m-i}{m-i}\binom{m-i+n-j}{n-j} \binom{n-j+p-k}{p-k}\binom{p-k+i}{i} .$$

Show that $\overline{H}(m,n,p) = 0$ unless m,n,p are all even, and that

$$\overline{H}(2m,2n,2p) = \sum_{r=0}^{\min(m,n,p)} (-1)^r \frac{(m+n+p-r)!}{r!\,r!\,(m-r)!\,(n-r)!\,(p-r)!} .$$

(The formula

$$\overline{H}(2m, 2n) = \binom{m+n}{m}^2 ,$$

where

$$\overline{H}(m, n) = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{i+j} \binom{i+j}{j}\binom{m-i+j}{j}\binom{i+n-j}{n-j}\binom{m-i+n-j}{n-j}$$

is proved in the Fibonacci Quarterly, Vol. 4 (1966), pp. 323-325.)

Solution by the proposer.

As a special case of a more general identity (SIAM Review, Vol. 6 (1964) pp. 20-30, formulas (3.1)), we have

$$\sum_{i_1,\cdots,i_6=0}^{\infty} \binom{i_1+i_2}{i_2}\binom{i_2+i_3}{i_3}\binom{i_3+i_4}{i_4}\binom{i_4+i_5}{i_5}\binom{i_5+i_6}{i_6}\binom{i_6+i_1}{i_1} u_1^{i_1} u_2^{i_2} u_3^{i_3} u_4^{i_4} u_5^{i_5} u_6^{i_6}$$

$$= \Big\{ \big[1 - u_1 - u_2 - u_3 - u_4 - u_5 - u_6 + u_1u_4 + u_1u_5 + u_2u_4 + u_2u_5 + u_2u_6 + u_3u_5 + u_3u_6 + u_4u_6 - u_1u_3u_5 - u_2u_4u_6\big]^2 - 4u_1u_2u_3u_4u_5u_6 \Big\}^{-\frac{1}{2}} .$$

In this identity, take

$$u_4 = -u_1, \; u_5 = -u_2, \; u_6 = -u_3 .$$

Changing the notation slightly we get

$$\sum_{m,n,p=o}^{\infty} \overline{H}(m,n,p)\, u^m v^n w^p = \left\{(1 - u^2 - v^2 - w^2)^2 + 4u^2 v^2 w^2 \right\}^{-\frac{1}{2}}$$

$$= \sum_{r=0}^{\infty} (-1)^r \binom{2r}{r} \frac{u^{2r} v^{2r} w^{2r}}{(1 - u^2 - v^2 - w^2)^{2r+1}}$$

$$= \sum_{r=0}^{\infty} (-1)^r \binom{2r}{r} u^{2r} v^{2r} w^{2r} \sum_{n=0}^{\infty} \binom{2r+n}{n} (u^2+v^2+w^2)^n$$

$$= \sum_{r=0}^{\infty} (-1)^r \binom{2r}{r} u^{2r} v^{2r} w^{2r} \times$$

$$\times \sum_{i,j,k=0}^{\infty} \frac{(2r+i+j+k)!}{(2r)!\, i!\, j!\, k!} u^{2i} v^{2j} w^{2k}$$

$$= \sum_{m,n,p=0}^{\infty} u^{2m} v^{2n} w^{2p} \sum_{r=0}^{\min(m,n,p)} (-1)^r \times$$

$$\times \frac{(m+n+p-r)!}{r!\, r!\, (m-r)!\, (n-r)!\, (p-r)!}$$

Comparing coefficients we get

$$\overline{H}(2m, 2n, 2p) = \sum_{r=0}^{\min(m,n,p)} (-1)^r \frac{(m+n+p-r)!}{r!\, r!\, (m-r)!\, (n-r)!\, (p-r)!} \quad .$$

It does not seem possible to sum the series on the right.

* * * * *

IDENTITIES INVOLVING GENERALIZED FIBONACCI NUMBERS

MUTHULAKSHMI R. IYER
Indian Statistical Institute, Calcutta, India

I. INTRODUCTION

K. Subba Rao [4], and more recently V. C. Harris [1] have obtained some identities involving Fibonacci Numbers F_n defined by

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \qquad n \geq 3 .$$

Our object in this paper is to obtain similar results for the generalized Fibonacci Numbers H_n as defined by A. F. Horadam [2],

$$H_1 = p, \quad H_2 = p + q$$

and

$$H_n = H_{n-1} + H_{n-2} \qquad n \geq 3.$$

The numbers p and q are arbitrary. By solving the difference equation for H_n by the usual procedure it is easy to see that

$$H_n = \frac{1}{2\sqrt{5}} \left[la^n - mb^n \right] \qquad [3]$$

where

$$l = 2(p - qb), \quad m = 2(p - qa)$$

and a and b are the roots of the quadratic equation $x^2 - x - 1 = 0$. We call

$$a = \frac{1 + \sqrt{5}}{2} \; ; \qquad b = \frac{1 - \sqrt{5}}{2}$$

so that

$$a + b = 1, \quad ab = -1, \quad a - b = \sqrt{5} .$$

By making use of these results we get

$$1 + m = 2(2p - q), \quad 1 - m = 2q\sqrt{5} ,$$

$$\tfrac{1}{4}\, lm = p^2 - pq - q^2 = e \quad \text{(say)}.$$

It is also easy to see that $H_n = pF_n \neq qF_{n-1}$ where F_n is the n^{th} Fibonacci number given by

$$\frac{a^n - b^n}{\sqrt{5}} .$$

SECTION 2

In this section we obtain certain identities for the generalized Fibonacci numbers. From result (9) of [2] we have the identity

$$H_{r-1}^2 + H_r^2 = (2p - q)\, H_{2r-1} - e\, F_{2r-1} .$$

In this relation putting $r = 2, 3, \cdots, n$ in succession, adding and simplifying, we arrive at the result

$$\sum_{r=1}^{n} H_r^2 = F_n[(p + 2q)H_n + eF_{n-1}] + pq[(-1)^n - 1] . \tag{1}$$

Consider now $H_{2r-1} = pF_{2r-1} + qF_{2r-2}$ so that

$$\sum_{r=1}^{n} H_{2r-1} = p\sum_{r=1}^{n} F_{2r-1} + q\sum_{r=1}^{n} F_{2r-2} .$$

From the formula for F_n this sum reduces to

(2) $$\sum_{r=1}^{n} H_{2r-1} = H_{2n} - H_2 + H_1$$

(3) $$\sum_{r=1}^{n} H_{2r} = H_{2n+1} - H_1$$

On the same lines we get the following identities

(4) $$\sum_{r=1}^{n} H_{3r-2} = \frac{1}{2}\left[H_{3n} - H_2 + H_1\right]$$

(5) $$\sum_{r=1}^{n} H_{3r-1} = \frac{1}{2}\left[H_{3n+1} - H_1\right]$$

(6) $$\sum_{r=1}^{n} H_{3r} = \frac{1}{2}\left[H_{3n+2} - H_2\right]$$

(7) $$\sum_{r=1}^{n} H_{4r-3} = F_{2n-1}H_{2n} - H_2 + H_1$$

(8) $$\sum_{r=1}^{n} H_{4r-2} = F_{2n}H_{2n}$$

(9) $$\sum_{r=1}^{n} H_{4r-1} = F_{2n}H_{2n+1}$$

(10) $$\sum_{r=1}^{n} H_{4r} = F_{2n+1}H_{2n+1} - H_1$$

(11) $$\sum_{r=1}^{n} H_{2r-1}^2 = \frac{1}{5}\,[H_{2n}\,(H_{2n-1} + H_{2n+1}) + 2ne + q(q - 2p)]$$

(12) $$\sum_{r=1}^{n} H_{2r}^2 = \frac{1}{5}\,[H_{2n+1}\,(H_{2n} + H_{2n+2}) - 2ne - p(p + 2q)]$$

Let us now consider product terms as follows:

(13) $$\sum_{r=1}^{n} H_{2r-2}H_{2r-1} = \frac{1}{5}\,[H_{2n-1}^2 + H_{2n}^2 - ne - (p + q)(p + 2q)]$$

(14) $$\sum_{r=1}^{n} H_{2r-1}H_{2r} = \frac{1}{5}\,[H_{2n}^2 + H_{2n+1}^2 + ne - (p^2 + q^2)]$$

(15) $$\sum_{r=1}^{n} H_{2r-1}H_{2r+1} = \frac{1}{5}\,[H_{2n+1}(H_{2n} + H_{2n+2}) + 3ne - p(p + 2q)]$$

(16) $$\sum_{r=1}^{n} H_{2r}H_{2r+2} = \frac{1}{5}\,[H_{2n+2}(H_{2n+1} + H_{2n+3}) - 3ne - (p + q)(3p + q)]$$

Corresponding to the identity

$$F_r^2 - F_{r-k}F_{r+k} = (-1)^{r-k}F_k^2$$

for the generalized Fibonacci numbers we get in the generalized Fibonacci numbers the identity·

$$H_r^2 - H_{r-k}H_{r+k} = (-1)^{r-k}eF_k^2 \tag{17}$$

Now consider the sums

$$\sum_{r=1}^{n} H_{2r-2}H_{2r+2} = \frac{1}{5}\,[H_{2n+1}(H_{2n} + H_{2n+2}) - 7ne - (p^2 + 2pq + 10q^2)] \tag{18}$$

$$\sum_{r=1}^{n} H_{2r-1}H_{2r+3} = \frac{1}{5}\,[H_{2n+2}(H_{2n+1} + H_{2n+3}) + 7ne - (p+q)(3p+q)] \tag{19}$$

Evaluating the quantity $H_kH_{k+1}H_{k+2}$ we get

$$H_kH_{k+1}H_{k+2} = H_{k+1}^3 + (-1)^{k-1}\,eH_{k+1} \tag{20}$$

Therefore

$$H_{2r-1}\,H_{2r}\,H_{2r+1} = H_{2r}^3 + eH_{2r}$$

Hence

$$\sum_{r=1}^{n} H_{2r-1}\,H_{2r}\,H_{2r+1} = \sum_{r=1}^{n} H_{2r}^3 + e\sum_{r=1}^{n} H_{2r}\,.$$

After simplification this becomes,

$$\sum_{r=1}^{n} H_{2r-1}H_{2r}H_{2r+1} = \frac{1}{4}\,[(H_{2n+1}^3 - H_1^3) + e(H_{2n+1} - H_1)] \tag{21}$$

$$\sum_{r=1}^{n} H_r^3 = \frac{1}{2}\,[(H_nH_{n+1}^2 - q^2H_2) + e\{(p-2q)-(-1)^nH_{n-1}\}] \tag{22}$$

Now

$$H_{2r}^3 = (pF_{2r} + qF_{2r-1})^3 .$$

On expanding the right side, taking the sum from $r = 1$ to n and simplifying we get the relation

$$\sum_{r=1}^{n} H_{2r}^3 = \frac{1}{4}\ [(H_{2n+1}^3 - H_1^3) - 3e(H_{2n+1} - H_1)] \tag{23}$$

$$\sum_{r=1}^{n} H_{2r}^2 H_{2r-1} = \frac{1}{4}\ [\ (H_{2n}H_{2n+1}^2 - q^2H_2) + e(H_{2n-1} - H_1)] \tag{24}$$

$$\sum_{r=1}^{n} H_{2r}H_{2r-1}^2 = \frac{1}{4}\ [(H_{2n-1}H_{2n+1}^2 - H_1q^2) + e(H_{2n} - H_2)] \tag{25}$$

$$\sum_{r=1}^{n} H_{2r-1}^2 = \frac{1}{4}\ [(H_{2n}^3 - q^3) + 3e(H_{2n} - q)] \tag{26}$$

From the formula for H_r we can find the sums of the following:

$$\sum_{r=0}^{n} rH_r = nH_{n+2} - H_{n+3} + H_3 \tag{27}$$

$$\sum_{r=0}^{n} (-1)^r rH_r = [(-1)^n \left[(n + 1)H_{n-1} - H_{n-2} \right] + (3q - 2p)] \tag{28}$$

$$\sum_{r=0}^{n} (-1)^r H_{2r} = \frac{1}{5}\ [\ (-1)^{n+1}(H_{2n} + H_{2n+2}) - (p + 2q)] \tag{29}$$

(30) $$\sum_{r=0}^{n} (-1)^r H_{2r+1} = \frac{1}{5}\left[(-1)^n (H_{2n+1} + H_{2n+3}) + (2p - q)\right]$$

(31) $$\sum_{r=0}^{n} rH_{2r} = \left[nH_{2n+1} - H_1\right] - \left[H_{2n} - H_2\right]$$

(32) $$\sum_{r=0}^{n} rH_{2r+1} = nH_{2n+2} - \left[H_{2n+1} - H_1\right]$$

(33) $$\sum_{r=0}^{n} (-1)^r rH_{2r} = \frac{1}{5}\left[(-1)^n((n+1)H_{2n} + nH_{2n+2}) - (H_2 - H_1)\right]$$

(34) $$\sum_{r=0}^{n} (-1)^r rH_{2r+1} = \frac{1}{5}\left[(-1)^n((n+1)H_{2n+1} + nH_{2n+3}) - H_1\right]$$

It is easy to see that the list of identities given by K. Subba Rao can be extended to Fibonacci Quaternions defined by

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} .$$

The author is very grateful to Dr. J. Sethuraman for valuable suggestions.

REFERENCES

1. V. C. Harris, "On Identities Involving Fibonacci Numbers," The Fibonacci Quarterly, Vol. 3, No. 3, 1965, pp. 214-218.
2. A. F. Horadam, "The Generalized Fibonacci Sequence," The Amer. Math. Monthly, Vol. 68, No. 5, 1961, pp. 455-459.
3. A. F. Horadam, "Fibonacci Number Triples," The Amer. Math. Monthly, Vol. 68, No. 8, 1961, pp. 751-753.
4. Subba K. Rao, "Some Properties of Fibonacci Numbers," The Amer. Math. Monthly, Vol. 60, 1953, pp. 680-684.

★ ★ ★ ★ ★

GOLDEN TRIANGLES, RECTANGLES, AND CUBOIDS

MARJORIE BICKNELL and VERNER E. HOGGATT, JR.
A. C. Wilcox High School, Santa Clara, California, and
San Jose State College, San Jose, California

1. INTRODUCTION

One of the most famous of all geometric figures is the Golden Rectangle, which has the ratio of length to width equal to the Golden Section,

$$\phi = (1 + \sqrt{5})/2 \quad .$$

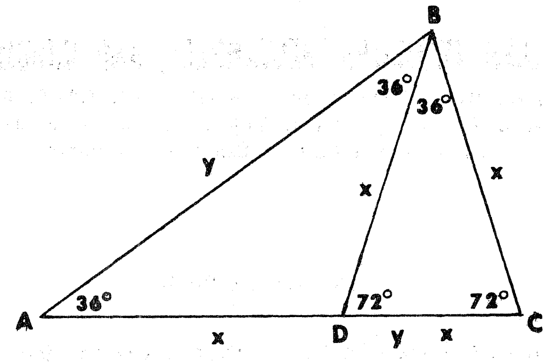
The proportions of the Golden Rectangle appear consistently throughout classical Greek art and architecture. As the German psychologists Fechner and Wundt have shown in a series of psychological experiments, most people do unconsciously favor "golden dimensions" when selecting pictures, cards, mirrors, wrapped parcels, and other rectangular objects. For some reason not fully known by either artists or psychologists, the Golden Rectangle holds great aesthetic appeal. Surprisingly enough, the best integral lengths to use for sides of an approximation to the Golden Rectangle are adjacent members of the Fibonacci series: 1, 1, 2, 3, 5, 8, 13, $\cdots$, and we find 3 x 5 and 5 x 8 filing cards, for instance.

Suppose that, instead of a Golden Rectangle, we study a golden section triangle. If the ratio of a side to the base is

$$\phi = (1 + \sqrt{5})/2 \quad ,$$

then we will call the triangle a Golden Triangle. (See [2], [3].)

Now, consider the isosceles triangle with a vertex angle of 36°. On bisecting the base angle of 72°, two isosceles triangles are formed, and ΔBDC is similar to ΔABC as indicated in the figure:



Since $\Delta ABC \sim \Delta BDC$,

$$\frac{AB}{BD} = \frac{BC}{DC},$$

or,

$$\frac{y}{x} = \frac{x}{y - x},$$

so that

$$y^2 - yx - x^2 = 0 .$$

Dividing through by $x^2 \neq 0$,

$$\frac{y^2}{x^2} - \frac{y}{x} - 1 = 0 .$$

The quadratic equation gives

$$\frac{y}{x} = (1 + \sqrt{5})/2 = \phi$$

as the positive root, so that ΔABC is a Golden Triangle. Notice also, that, using the common altitude from B, the ratio of the area of ΔABC to ΔADB is ϕ.

Since the central angle of a regular decagon is 36°, ΔABC above shows that the ratio of the radius y to the side x of an inscribed decagon is

$$\phi = (1 + \sqrt{5})/2 .$$

Also, in a regular pentagon, the angle at a vertex between two adjacent diagonals is 36°. By reference to the figure above, the ratio of a diagonal to a side of a regular pentagon is also ϕ.

2. A TRIGONOMETRIC PROPERTY OF THE ISOSCELES GOLDEN TRIANGLE

The Golden Triangle with vertex angle 36° can be used for a surprising trigonometric application. Few of the trigonometric functions of an acute angle have values which can be expressed exactly. Usually, a method of approximation is used; most values in trigonometric tables cannot be expressed exactly as terminating decimals, repeating decimals, or even square roots, since they are approximations to transcendental numbers, which are numbers so irrational that they are not the root of any polynomial over the integers.

The smallest integral number of degrees for which the trigonometric functions of the angle can be expressed exactly is three degrees. Then, all multiples of 3° can also be expressed exactly by repeatedly using formulas such as $\sin(A + B)$. Strangely enough, the Golden Triangle can be used to derive the value of $\sin 3°$.

In our Golden Triangle, the ratio of the side to the base was

$$y/x = (1 + \sqrt{5})/2 .$$

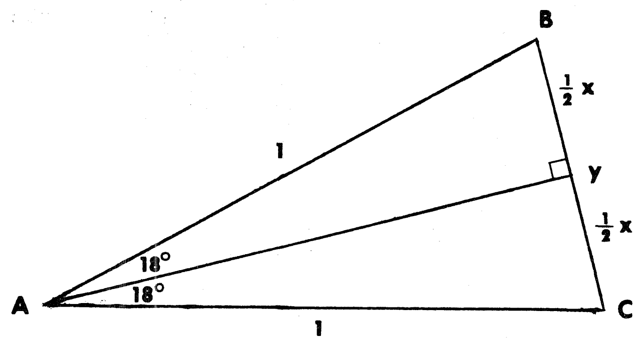
Suppose we let $AB = y = 1$. Then

$$1/x = (1 + \sqrt{5})/2 ,$$

or,

$$x = (\sqrt{5} - 1)/2 .$$

Redrawing the figure and bisecting the 36° angle,



we form right triangle AYC with YC = x/2. Then,

$$\sin 18^\circ = \frac{YC}{AC} = \frac{x}{2} = \frac{\sqrt{5}-1}{4} = \frac{1}{2\phi}\ .$$

Since $\sin^2 A + \cos^2 A = 1$,

$$\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{\sqrt{\sqrt{5}\,\phi}}{2}\ .$$

Since $\sin(A - B) = \sin A \cos B - \sin B \cos A$,

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}\ .$$

Similarly, using $\cos(A - B) = \cos A \cos B + \sin A \sin B$,

$$\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Again using the formula for $\sin(A - B)$,

$$\sin 3^\circ = \sin(18^\circ - 15^\circ) = \left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) - \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)$$

$$= \frac{1}{16}\left[(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - 2(\sqrt{3}-1)\left(\sqrt{5+\sqrt{5}}\right)\right]$$

as given by Ransom in [1].

3. GOLDEN RECTANGLE AND GOLDEN TRIANGLE THEOREMS

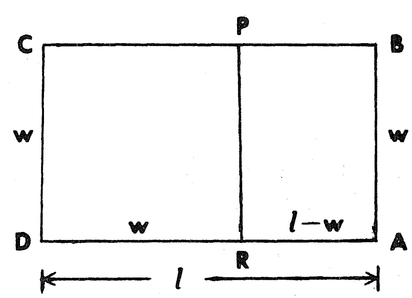
While a common way to describe the Golden Rectangle is to give the ratio of length to width as

$$\phi = (1 + \sqrt{5})/2 ,$$

this ratio is a consequence of the geometric properties of the Golden Rectangle which are discussed in this section.

Theorem. Given that the ratio of length to width of a rectangle is $k > 1$. A square with side equal to the width can be removed to leave a rectangle similar to the original rectangle if and only if $k = (1 + \sqrt{5})/2$.

Proof. Let the square PCDR be removed from rectangle ABCD, leaving rectangle BPRA.



If rectangles ABCD and BPRA have the same ratio of length to width, then

$$k = \frac{w}{l - w} = \frac{l}{w} .$$

Cross-multiplying and dividing by $w^2 \neq 0$ gives a quadratic equation in $\frac{l}{w}$ which has

$$(1 + \sqrt{5})/2$$

as its positive root. If

$$\frac{l}{w} = (1 + \sqrt{5})/2 = \phi,$$

then

$$\frac{w}{l - w} = \frac{1}{\frac{l}{w} - 1} = \frac{1}{\phi - 1} = \phi$$

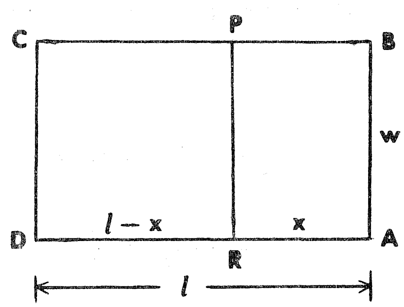
so that both rectangles have the same ratio of length to width.

Theorem. Given that the ratio of length to width of a rectangle is $k > 1$. A rectangle similar to the first can be removed to leave a rectangle such that the ratio of the areas of the original rectangle and the rectangle remaining is k, if and only if

$$k = (1 + \sqrt{5})/2 \quad .$$

Further, the rectangle remaining is a square.

Proof. Remove rectangle BPRA from rectangle ABCD as in the figure:



Then

$$\frac{\text{area ABCD}}{\text{area PCDR}} = \frac{lw}{w(l - x)} \quad .$$

But,

$$\frac{lw}{w(l - x)} = \frac{l}{w} = k ,$$

if and only if

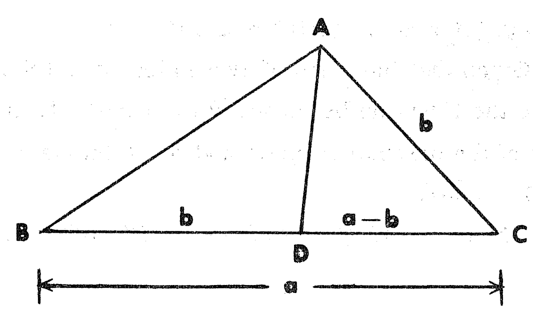
$$\frac{w}{l - x} = 1 ,$$

or $w = l - x$ or PCDR is a square. Thus, our second theorem is a consequence of the first theorem.

Analogous theorems hold for Golden Triangles.

Theorem. Given that the ratio of two sides a and b of a triangle is $a/b = k > 1$. A triangle with side equal to b can be removed to leave a triangle similar to the first if and only if $k = (1 + \sqrt{5})/2$.

Proof. Remove ΔABD from ΔABC.



If $\Delta ADC \sim \Delta BAC$, then

$$\frac{AC}{BC} = \frac{DC}{AC}$$

or

$$\frac{b}{a} = \frac{a - b}{b} .$$

Cross multiply, divide by $b^2 \neq 0$, and solve the quadratic in a/b to give

$$a/b = (1 + \sqrt{5})/2$$

as the only positive root.

If

$$a/b = (1 + \sqrt{5})/2 ,$$

then

$$DC/AC = (a - b)/b = a/b - 1 = (\sqrt{5} - 1)/2$$

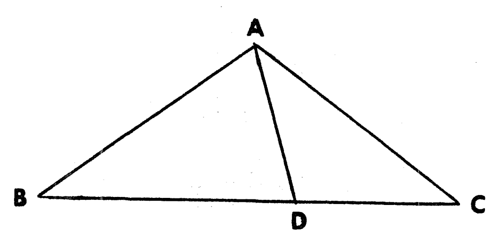
and

$$AC/BC = b/a = 2/(1 + \sqrt{5}) = (\sqrt{5} - 1)/2 = DC/AC .$$

Since $\angle C$ is in both triangles, $\Delta ADC \sim \Delta BAC$.

Theorem. Given that the ratio of two sides of a triangle is $k > 1$. A triangle similar to the first can be removed to leave a triangle such that the ratio of the areas of the original triangle and the triangle remaining is k, if and only if $k = (1 + \sqrt{5})/2$.

Proof. Let $\Delta ADC \sim \Delta BAC$, such that $BC/AC = AC/DC = k$.



If the ratio of areas of the original triangle and the one remaining is k, since there is a common altitude from A,

$$k = \frac{\text{area } \Delta BAC}{\text{area } \Delta BDA} = \frac{(BC)(h/2)}{(BC - DC)(h/2)} = \frac{BC/AC}{BC/AC - DC/AC} = \frac{k}{k - 1/k} .$$

Again cross-multiplying and solving the quadratic in k gives $k = (1 + \sqrt{5})/2$.

If

$$k = (1 + \sqrt{5})/2 ,$$

then

$$BC/AC = AC/DC = (1 + \sqrt{5})/2 ,$$

and the ratio of areas $BC/(BC - DC)$ becomes $(1 + \sqrt{5})/2$ when divided through by AC and then simply substituting the values of BC/AC and DC/AC.

If

$$k = (1 + \sqrt{5})/2 = BC/AC ,$$

and the ratio of areas of ΔBAC and ΔBDA is also k, then

$$k = \frac{BC/AC}{BC/AC - DC/AC} = \frac{k}{k - x} ,$$

which leads to

$$x = k - 1 \quad \text{or} \quad DC/AC = (1 + \sqrt{5})/2 - 1 = 2/(1 + \sqrt{5})$$

so that

$$AC/DC = (1 + \sqrt{5})/2$$

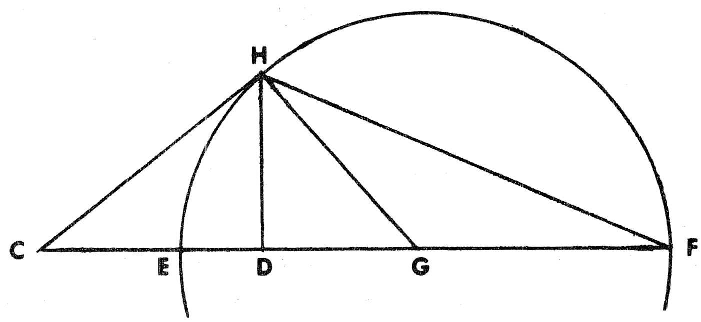
and ΔBAC is similar to ΔADC.

4. THE GENERAL GOLDEN TRIANGLE

Unlike the Golden Rectangle, the Golden Triangle does not have a unique shape. Consider a line segment $\overline{CD}$ of length

$$\phi = (1 + \sqrt{5})/2 \ .$$

Place points E, G, and F on line $\overleftrightarrow{CD}$ such that $CE = 1$, $EG = GF = \phi$ as in the diagram.



Then, $ED = \phi - 1$ and

$$CE/ED = 1(\phi - 1) = \phi,$$
$$CF/DF = (2\phi + 1)(\phi + 1) = \phi^3/\phi^2 = \phi \ ,$$

so that E and F divide segment $\overline{CD}$ internally and externally in the ratio ϕ. Then the circle with center G is the circle of Apollonius for $\overline{CD}$ with ratio ϕ. Incidentally, the circle through C, D, and H is orthogonal to circle with center G and passing through H, and $\overline{HG}$ is tangent to the circle through C, D, and H.

Let H be any point on the circle of Apollonius. Then $CH/HD = \phi$, $CG/HG = \phi$, and $\Delta CHG \sim \Delta HDG$. The area of ΔCHG is

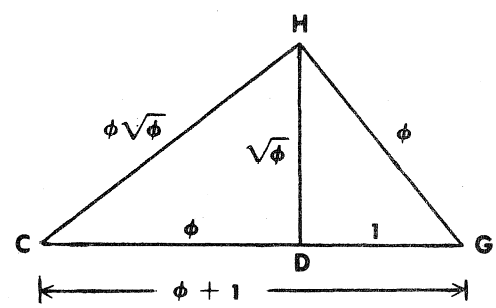
$$h(1 + \phi)/2 = h\phi^2/2 \ ,$$

and when ΔHDG is removed, the area of the remaining ΔCHD is $h\phi/2$, so that the areas have ratio ϕ. Then, ΔCHG is a Golden Triangle, and there are an infinite number of Golden Triangles because H can take an infinite number of positions on circle G.

If we choose H so that $CH = \phi + 1$, then we have the isosceles 36-72-72 Golden Triangle of decagon fame. If we erect a perpendicular at D and let H be the intersection with the circle of Apollonius, then we have a right golden triangle by applying the Pythagorean theorem and its converse. In our right golden triangle ΔCHG, $CH = \phi\sqrt{\phi}$, $HG = \phi$, and $CG = \phi^2$. The two smaller right triangles formed by the altitude to $\overline{CG}$ are each similar to ΔCHG, so that all three triangles are golden. The areas of ΔHDG, ΔCDH, and ΔCHG form the geometric progression,

$$\sqrt{\phi}/2, \quad (\sqrt{\phi}/2)\phi, \quad (\sqrt{\phi}/2)\phi^2 \ .$$

Before going on, notice that the right golden triangle ΔCHG provides an unusual and surprising configuration. While two pairs of sides and all three pairs of angles of ΔCHG and ΔCDH are congruent, yet ΔCHG is not congruent to ΔCDH! Similarly for ΔCDH and ΔHDG. (See Holt [4].)



5. THE GOLDEN CUBOID

H. E. Huntley [5] has described a Golden Cuboid (rectangular parallelopiped) with lengths of edges a, b, and c, such that

$$a : b : c = \phi : 1 : \phi^{-1} \ .$$

The ratios of the areas of the faces are

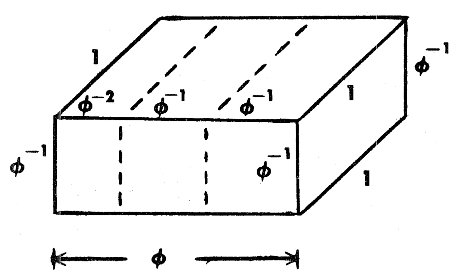
$$\phi : 1 : \phi^{-1} \ ,$$

and four of the six faces of the cuboid are Golden Rectangles.

If two cuboids of dimension

$$\phi^{-1} \times 1 \times \phi^{-1}$$

are removed from the Golden Cuboid, the remaining cuboid is similar to the original and is also a golden cuboid,



If a cuboid similar to the original is removed and has sides b, c, and d, then

$$b : c : d = \phi$$

so that

$$c = d\phi, \quad b = d\phi^2, \quad a = d\phi^3 \; .$$

The volume of the original is $abc = \phi^6 d^3$, and the volume removed is $bcd = \phi^3 d^3$. The remaining volume is $(\phi^6 - \phi^3)d^3$. The ratio of the volume of the original to the volume of the remaining cuboid is

$$\frac{\phi^6 d^3}{(\phi^6 - \phi^3)d^3} = \frac{\phi^3}{\phi^3 - 1} = \frac{2 + \sqrt{5}}{1 + \sqrt{5}} = \frac{3 + \sqrt{5}}{4} = \phi^2/2$$

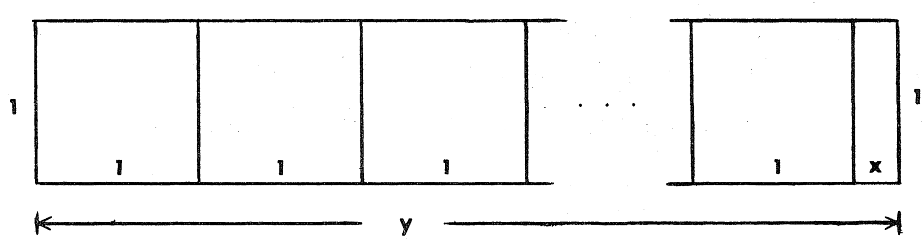
6. LUCAS GOLDEN-TYPE RECTANGLES

Now, in a Golden Rectangle, if one square with side equal to the width is removed, the resulting rectangle is similar to the original. Suppose that we have a rectangle in which when k squares equal to the width are removed, a rectangle similar to the original is formed, as discussed by J. A. Raab [6]. In the figure below, the ratio of length to width in the original rectangle and in the similar one formed after removing k squares is $y:1 = 1:x$ which gives $x = 1/y$. Since each square has side 1,

$$y - x = y - 1/y = k,$$

or,

$$y^2 - ky - 1 = 0 \ .$$



Let us consider only Lucas golden-type rectangles. That is, let $k = L_{2m+1}$, where L_{2m+1} is the $(2m+1)^{st}$ Lucas number defined by

$$L_0 = 2, \ L_1 = 1, \ L_n = L_{n-1} + L_{n-2} \ .$$

A known identity is

$$L_k = \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k = \alpha^k + \beta^k \ ,$$

where α and β are the roots of $x^2 - x - 1 = 0$.

In our problem, if

$$k = L_{2m+1},$$

then

$$y^2 - ky - 1 = 0$$

becomes

$$y^2 - L_{2m+1}\, y - 1 = 0$$

so that

$$y = \alpha^{2m+1}$$

or

$$y = \beta^{2m+1},$$

but

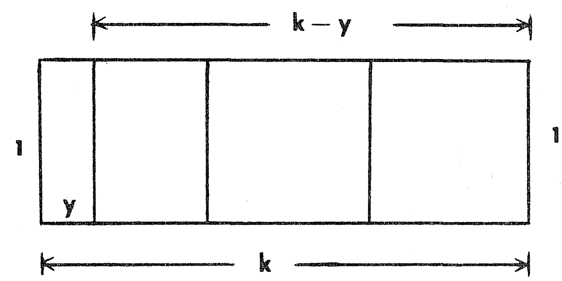
$$y = \alpha^{2m+1}$$

is the only positive root. Then

$$x = 1/\alpha^{2m+1} = -\beta^{2m+1}.$$

On the other hand, suppose we insist that to a given rectangle we add one similar to it such that the result is k squares long. Illustrated for $k = 3$, the equal ratios of length to width in the similar rectangles gives

$$\frac{1}{y} = \frac{k - y}{1} \qquad \text{or } ky - y^2 = 1 \quad \text{or } y^2 - ky + 1 = 0.$$



Now, let $k = L_{2m}$; then $y = \alpha^{2m}$ or $y = \beta^{2m}$. Here, of course, $y = \beta^{2m}$, so that

$$k - y = L_{2m} - \beta^{2m} = \alpha^{2m} .$$

Both of these cases are, of course, in the plane; the reader is invited to extend these ideas into the third dimension.

7. GENERALIZED GOLDEN-TYPE CUBOIDS

Let the dimensions of a cuboid be $a : b : c = k$ and remove a cuboid similar to the first with dimensions $b : c : d = k$. Then

$$c = dk, \qquad b = dk^2, \qquad a = dk^3 .$$

The volume of the original is

$$abc = k^6 d^3 ,$$

the volume removed is

$$bcd = k^3 d^3 ,$$

and the remaining volume is

$$(k^6 - k^3)\, d^3 .$$

The ratio of the original volume to that remaining is

$$\frac{k^6 d^3}{(k^6 - k^3)d^3} = \frac{k^3}{k^3 - 1} .$$

Now, let this ratio equal

$$k^2/L_0 = k^2/2 ,$$

which leads to

$$0 = k^3 - 2k - 1 = (k+1)(k^2 - k - 1)$$

with roots

$$k = -1, \quad (1 \pm \sqrt{5})/2 ,$$

and having

$$k = (1 + \sqrt{5})/2$$

as its only positive root.

Now consider a hypercuboid in a hyperspace of 6 dimensions, with dimensions $a : b : c : d : e : f = k$. Remove a hypercuboid of dimensions

$$b : c : d : e : f : g = k ,$$

and the ratio of the original volume to the volume remaining is

$$\frac{abcdef}{abcdef - bcdefg} = \frac{g^6 k^{21}}{g^6 (k^{21} - k^{15})} = \frac{k^6}{k^6 - 1} ,$$

since

$$f = kg, \quad e = k^2 g, \quad d = k^3 g, \quad e = k^4 g, \quad b = k^5 g, \quad a = k^6 g .$$

Now set this ratio equal to k^2/L_3 or,

$$\frac{k^6}{k^6 - 1} = \frac{k^3}{4}$$

which leads to

$$k^6 - 4k^3 - 1 = 0$$

with roots

$$k = \alpha, \omega\alpha, \omega^2\alpha, \beta, \omega\beta, \omega^2\beta,$$

where ω and ω^2 are cube roots of unity. Then

$$k = \alpha = (1 + \sqrt{5})/2$$

is the only positive real root.

Suppose we have a cuboid in a hyperspace of $4m + 2$ dimensions. Let this have edges

$$a_1, a_2, a_3, \cdots, a_{4m+2} ,$$

and cut off a cuboid similar to it so that

$$k = a_1 = a_2 : a_3 : \cdots : a_{4m+2} = a_2 : a_3 : a_4 : \cdots : a_{4m+2} : a_{4m+3}$$

This implies that the dimensions are related by

$$a_n = k^{4m+3-n} a_{4m+3}$$

for $n = 1, 2, \cdots, 4m + 3$. The volume of the original cuboid is now $a_1a_2a_3 \cdots a_{4m+2}$ while the volume of the cuboid cut off is $a_2a_3 \cdots a_{4m+2}a_{4m+3}$. The remaining cuboid has volume equal to the difference of these, making the ratio of the original volume to that remaining

$$\frac{a_1a_2a_3\cdots a_{4m+2}}{a_2a_3\cdots a_{4m+2}(a_1 - a_{4m+3})} = \frac{a_1}{a_1 - a_{4m+3}} = \frac{k^{4m+2}}{k^{4m+2} - 1}$$

Now let us let this volume ratio equal to

$$k^{2m+1} / L_{2m+1} \quad ,$$

where L_{2m+1} is the $(2m+1)^{st}$ Lucas number, yielding

$$k^{4m+2} - L_{2m+1}\, k^{2m+1} - 1 = 0$$

whose only positive root is

$$\alpha = (1 + \sqrt{5})/2 \quad .$$

The proof is very neat. Since $\alpha\beta = -1$ for α and β the roots of $x^2 - x - 1 = 0$ and since $L_n = \alpha^n + \beta^n$, we can write

$$-1 = \alpha^{2m+1}(\alpha^{2m+1} + \beta^{2m+1}) - \alpha^{4m+2} = \alpha^{2m+1}L_{2m+1} - \alpha^{4m+2} ,$$

and rearrange the terms above to give

$$\begin{aligned} k^{4m+2} - L_{2m+1}k^{2m+1} - 1 &= (k^{4m+2} - \alpha^{4m+2}) - L_{2m+1}(k^{2m+1} - \alpha^{2m+1}) \\ &= (k^{2m+1} - \alpha^{2m+1})(k^{2m+1} + \alpha^{2m+1} - L_{2m+1}) \\ &= (k^{2m+1} - \alpha^{2m+1})(k^{2m+1} - \beta^{2m+1}) = 0 . \end{aligned}$$

Thus, $k = \alpha\omega_j,\ \beta\omega_j$, where ω_j are the $(2m+1)^{st}$ roots of unity, so that

$$k = \alpha = (1 + \sqrt{5})/2$$

is the only positive real root.

Now, let us return to the volumes of the cuboids in the hyperspace of $4m + 2$ dimensions. Let us set $a = a_{4m+3}$. Then, since $k = \alpha$, the volume of the original cuboid is

$$V_1 = a_1 a_2 \cdots a_{4m+2} = a^{4m+2}\alpha^1\alpha^2\alpha^3 \cdots \alpha^{4m+2}$$

and the volume of the cuboid removed is

$$V_2 = a_2 a_3 \cdots a_{4m+2} a_{4m+3} = a^{4m+2}\alpha^1\alpha^2\alpha^3 \cdots \alpha^{4m+1}$$

making the volume of the cuboid remaining

$$V_1 - V_2 = a^{4m+2}\alpha^{T_{4m+1}}(\alpha^{4m+2} - 1)$$

where T_n is the n^{th} triangular number. But,

$$\alpha^{4m+2} - 1 = L_{2m+1}\alpha^{2m+1}$$

so that the remaining cuboid is made up of L_{2m+1} square cuboids with total volume

$$a^{4m+2}\alpha^1\alpha^2\alpha^3 \cdots \alpha^{4m+1}(L_{2m+1}\alpha^{2m+1}) .$$

Thus we have generalized the Golden Cuboid of Huntley [5] and also the golden-type rectangle of Raab [6].

REFERENCES

1. William R. Ransom, Trigonometric Novelties, J. Weston Walch, Portland, Maine, 1959, pp. 22-23.
2. Verner E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin Company, Boston, 1969.
[Continued on p. 98.]

SUMS INVOLVING FIBONACCI NUMBERS

MUTHULAKSHMI R. IYER
Indian Statistical Institute, Calcutta, India

1. INTRODUCTION

In [1] Professor Horadam has defined a certain generalized sequence

$$\{w_n\} \equiv \{w_n(a, b;\ p, q)\} :\ w_0 = a,\quad w_1 = b$$

and

$$w_n = pw_{n-1} - qw_{n-2} \qquad (n \geq 2)$$

for arbitrary integers a and b. The n^{th} term of this sequence satisfies a relation of the form:

$$w_n = A\alpha^n + B\beta^n$$

where

$$A = \frac{b - a\beta}{\alpha - \beta}\ ; \qquad B = \frac{a\alpha - b}{\alpha - \beta}\quad ,$$

α and β being the roots of the equation $x^2 - px + q = 0$. He also mentions the particular cases of $\{w_n\}$ given by

$$\begin{aligned}
w_n(1, p;\ p, q) &= u_n(p, q)\\
w_n(2, p;\ p, q) &= v_n(p, q)\\
w_n(r, r+s;\ 1, -1) &= h_n(r, s)\\
w_n(1, 1;\ 1, -1) &= f_n = u_n(1, -1) = h_n(1,\ 0)\\
w_n(2, 1;\ 1, -1) &= l_n = v_n(1, -1) = h_2(2,\ -1)
\end{aligned}$$

wherein F_n and L_n are the famous Fibonacci and Lucas sequences respectively.

SECTION 2

In this paper our object is to derive some relations connecting the sums of the above sequences up to n terms.

We shall derive a formula for the sum of the most general sequence $\{w_n\}$ and thereby obtain the sums of the other sequences.

Theorem:

$$\sum_{r=0}^{n} w_r = a + \frac{bT_n - aqT_{n-1}}{1 - p + q}$$

where

$$T_n = 1 - \lambda_n ,$$

and

$$\lambda_n = u_n - qu_{n-1} .$$

Consider

$$\sum_{r=0}^{n} w_r = A \sum_{r=0}^{n} \alpha^r + B \sum_{r=0}^{n} \beta^r$$

$$= \frac{b - a\beta}{\alpha - \beta} \frac{\alpha^{n+1} - 1}{\alpha - 1} + \frac{a\alpha - b}{\alpha - \beta} \frac{\beta^{n+1} - 1}{\beta - 1} .$$

This becomes, after simplification by using the facts $(\alpha + \beta) = p$, $\alpha\beta = q$, $\alpha - \beta = d$

$$[(a + b - ap) + aq(u_{n-1} - qu_{n-2}) - b(u_n - qu_{n-1})]/(1 - p + q)$$

Set

$$u_n - qu_{n-1} = \lambda_n .$$

Then, this becomes

$$[(a + b - ap) + aq\lambda_{n-1} - b\lambda_n]/(1 - p + q)$$
$$[a(1 - p + q - q + q\lambda_{n-1}) + b(1 - \lambda_n)]/(1 - p + q)$$
$$a + [-aq(1 - \lambda_{n-1}) + b(1 - \lambda_n)]/(1 - p + q)$$

let now

$$1 - \lambda_n = T_n ,$$

therefore we finally obtain

$$\sum_{r=0}^{n} w_r = a + \frac{bT_n - aqT_{n-1}}{1 - p + q} + \cdots \tag{1}$$

Hence the result.

From this we can obtain immediately the sums of Σu_r, Σv_r, ΣF_r, ΣL_r, etc.

$$\sum_{r=0}^{n} u_r(p, q)$$

is obtained by letting $a = 1$, $b = p$ in (1)

(2)

$$\sum_{r=0}^{n} u_r(p, q) = 1 + \frac{pT_n - qT_{n-1}}{1 - p + q}$$

$$\sum_{r=0}^{n} u_r(p, q) = T_{n+1}/(1 - p + q) \cdots$$

$$\sum_{r=0}^{n} v_n (p,\ q)$$

can be obtained by putting $a = 2, \quad b = p, p, q$ in (1)

(3)
$$\sum_{r=0}^{n} v_r(p,q) = 2 + \frac{p\,T_{n-2q}\,T_{n-1}}{1 - p + q}$$

$$\sum_{r=0}^{n} v_r(p,q) = 1 + \frac{T_{n+1} - q\,T_{n-1}}{1 - p + q} \quad \cdots$$

In particular,

$$\Sigma w_r(1,1;\ 1,-1) = \Sigma F_r = \Sigma u_r(1,-1) = \Sigma h_r(1,0)$$

and

$$\Sigma\, w_r(2,1;\ 1,-1) = \Sigma L_r = \Sigma v_r(1,-1) = \Sigma h_r(2,-1).$$

(i)
$$\sum_{r=0}^{n} u_r(1,\ -1)$$

is derived by putting $a = b = p = 1, \quad q = -1$ in (1).

In this case $\lambda_n = u_n + u_{n-1} = u_{n+1}$. Therefore

$$\sum_{r=0}^{n} u_r(1,-1) = 1 + \frac{(1 - u_{n+1}) + (1 - u_n)}{1 - 1 - 1}$$

$$= 1 - \left[(1 - u_{n+1}) + (1 - u_n)\right]$$

$$\sum_{r=0}^{n} u_r(1,-1) = u_{n+2} - 1 = F_{n+2} - 1 \qquad [3] \cdots (1_i)$$

This can be verified for any n.

(ii) To get $\Sigma v_r(1,-1)$ let $a = 2$, $b = p = 1$, $q = -1$ in (1). Here also $\lambda_n = u_{n+1}$. So

$$\begin{aligned}\sum_{r=0}^{n} v_r(1,-1) &= 2 + \frac{(1 - u_{n+1}) + 2(1 - u_n)}{1 - 1 - 1} \\ &= 2 - [3 - 2u_n - u_{n+1}] \\ &= u_n + u_{n+2} - 1 \\ &= v_{n+2} - 1 \qquad \cdots (1_{ii})\end{aligned}$$

This also can be very easily verified for any n.

(iii) Now to evaluate

$$\sum_{r=0}^{n} h_r(p,q) ,$$

set

$$a = p, \quad b = p + q, \quad p = 1, \quad q = -1$$

in (1). Here again

$$\lambda_n = u_{n+1} = F_{n+1} .$$

Then

$$\sum_{r=0}^{n} h_r(p,q) = p - [(p+q)(1 - F_{n+1}) + p(1 - F_n)]$$

$$= (p+q)F_{n+1} + pF_n - (p+q)$$

$$= (pF_{n+2} + qF_{n+1}) - (p+q)$$

$$\sum_{r=0}^{n} h_r(p,q) = h_{n+2} - (p+q) \text{ by [2]} \qquad \cdots (1_{iii})$$

1, 1_i, (1_{ii}), (1_{iii}) can be proved for all (+ve) integers n by induction. We shall here prove (1) as an illustration. Let us suppose that

$$(1)' \qquad \sum_{r=0}^{k} w_r = a + \frac{bT_k - aqT_{k-1}}{1 - p + q}$$

Next let us add w_{k+1} to both sides, to get

$$\sum_{r=0}^{k+1} w_r = a + \frac{bT_k - aqT_{k-1}}{1 - p + q} + w_{k+1}$$

$$= a + \frac{b(1 - u_k + qu_{k-1}) - aq(1 - u_{k-1} + qu_{k-2})}{1 - p + q}$$

$$+ A\alpha^{k+1} + B\beta^{k+1}$$

$$(4) \qquad \sum_{r=0}^{k+1} w_r = a + \frac{b(1 - u_k + qu_{k-1}) - aq(1 - u_{k-1} + qu_{k-2})}{1 - p + q}$$

$$+ bu_k - aqu_{k-1}$$

$$= a + \frac{1}{1 - p + q}[b(1 - u_{k+1} + qu_k) - aq(1 - u_k + qu_{k-1})]$$

$$\sum_{r=0}^{k+1} w_r = a + \frac{bT_{k+1} - aqT_k}{1 - p + q}$$

Equation (4) is of the same form as (1)' with k replaced by k + 1. Hence, etc.

Similarly other results can be proved for all positive integral values of n.

REFERENCES

1. A. F. Horadam, "Basic Properties of a Certain Generalized Sequence of Numbers," The Fibonacci Quarterly, Vol. 3, 1965, pp. 161-176.
2. A. F. Horadam, "Complex Fibonacci Numbers and Fibonacci Quaternions," Amer. Math. Monthly, Vol. 70, 1963, pp. 289-291.
3. K. Subba Rao, "Some Properties of Fibonacci Numbers," Amer. Math. Monthly, Vol. 60 (1953), pp. 680-684.

* * * * *

[Continued from p. 91.]

3. Tobias Dantzig, Bequest of the Greeks, New York, Charles Scribner's Sons, 1955, pp. 61-62.
4. Marvin H. Holt, "Mystery Puzzler and Phi," The Fibonacci Quarterly, Vol. 3, No. 2, April 1965, pp. 135-138.
5. H. E. Huntley, "The Golden Cuboid," The Fibonacci Quarterly, Vol. 2, No. 3, October 1964, p. 184+.
6. Joseph A. Raab, "A Generalization of the Connection Between the Fibonacci Sequence and Pascal's Triangle," The Fibonacci Quarterly, Vol. 1, No. 3, October 1963, pp. 21-31.

* * * * *

LINEAR RECURSION RELATIONS
LESSON THREE–THE BINET FORMULAS

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

In the previous lesson, the technique of relating the terms of a linear recursion relation to the roots of an auxiliary equation was studied and illustrated. The Fibonacci sequences are characterized by the recursion relation:

(1) $$T_{n+1} = T_n + T_{n-1} ,$$

which is a linear recursion relation of the second order having an auxiliary equation:

(2) $$x^2 = x + 1$$

or

(3) $$x^2 - x - 1 = 0 .$$

The roots of this equation are:

(4) $$r = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad s = \frac{1 - \sqrt{5}}{2}$$

From the theory of the relation of roots to coefficients or by direct calculation it can be ascertained that:

(5) $$r + s = 1 \quad \text{and} \quad rs = -1 .$$

It follows from what has been developed in the previous lesson that the terms of any Fibonacci sequence can be written in the form:

(5) $$T_n = ar^n + bs^n ,$$

where a and b are suitable constants. For example, let

$$T_1 = 2, \quad T_2 = 5 \;.$$

The relations that must be satisfied are:

$$2 = ar + bs$$

$$5 = ar^2 + bs^2 \;.$$

These give solutions:

$$a = \frac{15 + \sqrt{5}}{10} \quad \text{and} \quad b = \frac{15 - \sqrt{5}}{10} \;,$$

so that

$$T_n = \frac{15 + \sqrt{5}}{10} r^n + \frac{15 - \sqrt{5}}{10} s^n \;.$$

Let us apply this technique to what is commonly known as the Fibonacci sequence whose initial terms are $F_1 = 1$ and $F_2 = 1$. Then

$$1 = ar + bs$$

$$1 = ar^2 + bs^2 \;,$$

with solutions

$$a = \frac{1}{\sqrt{5}}$$

and

$$b = \frac{-1}{\sqrt{5}}$$

so that

(6)
$$F_n = \frac{r^n - s^n}{\sqrt{5}} ,$$

the BINET FORMULA for the Fibonacci sequence.

Similarly, for the Lucas sequence with $L_1 = 1$ and $L_2 = 3$,

$$1 = ar + bs$$

$$3 = ar^2 + bs^2 ,$$

one obtains $a = 1$, $b = 1$, so that:

(7)
$$L_n = r^n + s^n ,$$

the BINET FORMULA for the Lucas sequence.

THE GOLDEN SECTION RATIO

With this formulation it is easy to see the connection between the Fibonacci sequences and the Golden Section Ratio. To divide a line segment in what is known as "extreme and mean ratio" or to make a Golden Section of the line segment, one finds a point on the line such that the length of the entire line is to the larger segment as the larger segment is to the smaller segment. To produce an exact parallel with the Fibonacci sequence auxiliary equation, let x be the length of the line and 1 the length of the larger segment. Then:

$$x : 1 = 1 : 1 - x ,$$

which leads to the equation

$$x^2 - x - 1 = 0 .$$

Clearly, we are interested in the positive root

$$r = \frac{1 + \sqrt{5}}{2} .$$

The other root $s = -1/r$ is the negative reciprocal of r, the Golden Section Ratio. (It may be noted that

$$\frac{1}{r} = \frac{\sqrt{5} - 1}{2}$$

is also considered the Golden Section Ratio by some authors. This is a matter of point of view: whether one is taking the ratio of the larger segment to the smaller segment or vice-versa.)

USING THE BINET FORMULAS

The Binet formulas for the Fibonacci and Lucas sequences are certainly not the practical means of calculating the terms of these sequences. Algebraically, however, they provide a powerful tool for creating or verifying Fibonacci-Lucas relations. Let us consider a few examples.

Example 1

If we study the terms of the Fibonacci sequence and the Lucas sequence in the following table:

n	F_n	L_n
1	1	1
2	1	3
3	2	4
4	3	7
5	5	11
6	8	18
7	13	29
8	21	47
9	34	76
10	55	123

it is a matter of observation that:

$$F_4L_4 = 3 \times 7 = 21 = F_8$$

$$F_5L_5 = 5 \times 11 = 55 = F_{10}$$

and in general it appears that:

$$F_n L_n = F_{2n} .$$

Why is this so? Using the Binet formula for F_{2n},

$$F_{2n} = \frac{r^{2n} - s^{2n}}{\sqrt{5}} = \frac{(r^n - s^n)}{\sqrt{5}} (r^n + s^n) = F_n L_n$$

Example 2

$$F_{kn} = \frac{r^{kn} - s^{kn}}{\sqrt{5}} = \frac{(r^k)^n - (s^k)^n}{\sqrt{5}}$$

has a factor

$$\frac{r^k - s^k}{\sqrt{5}} = F_k ,$$

which proves that if k is a divisor of the subscript of a Fibonacci number F_m, then F_k divides F_m.

Example 3

By taking successive values of k, one can intuitively surmise the formula:

$$F_{n+k} F_{n-k} - F_n^2 = (-1)^{n+k+1} F_k^2$$

To prove this relation, use the Binet formula for F. This gives:

$$F_{n+k}F_{n-k} - F_n^2 = \frac{r^{n+k} - s^{n+k}}{\sqrt{5}} \cdot \frac{r^{n-k} - s^{n-k}}{\sqrt{5}} - \frac{(r^n - s^n)^2}{5}$$

$$= r^{2n} + s^{2n} - r^{n+k}s^{n-k} - r^{n-k}s^{n+k} - r^{2n} + 2r^n s^n - s^{2n}$$

$$= - \frac{r^{n-k}s^{n-k}(r^{2k} - 2r^k s^k + s^{2k})}{5} = (-1)^{n+k+1} F_k^2 .$$

PROBLEMS

1. Prove that

$$L_{2n} = L_n^2 + 2(-1)^{n+1} .$$

2. Using the Binet formulas, findthe value of:

$$L_n F_{n-1} - F_n L_{n-1} .$$

3. $F_{3n} = F_n(\qquad)$. Determine the expression for the cofactor of F_n.
4. $F_{5n} = F_n(\qquad)$. Determine the expression for the cofactor of F_n.
5. $L_{3n} = L_n(\qquad)$. Find the expression for the cofactor of L_n.
6. $L_{5n} = L_n(\qquad)$. Find the expression for the cofactor of L_n.
7. For the Fibonacci relation with $T_1 = 3$, $T_2 = 7$, find the expression for T_n in terms of powers of r and s.
8. Using the binomial expansion, find an expression for F_n in terms of powers of 5 and binomial coefficients.
9. Do likewise for L_n.
10. Assuming the relation

$$L_n + L_{n+2} = 5F_{n+1} ,$$

determine an equivalent single Fibonacci number for $F_n^2 + F_{n+1}^2$ using the Binet formula.

[Continued on p. 106.]

ERRATA FOR
A LINEAR ALGEBRA CONSTRUCTED FROM FIBONACCI SEQUENCES

J. W. GOOTHERTS
Lockheed Missiles & Space Company, Sunnyvale, Calif.

Please make the following changes in the above-entitled article, appearing in Vol. 6, No. 5, November 1968:

On page 36, change the eighth line from the end to read:

Definition 1.5. For $U, V \in \mathfrak{F}$, $UV = (u_0v_0 + u_1v_1, u_0v_1 + u_1v_0 + u_1v_1)$.

Equation (3) on p. 38 should read:

$$\begin{aligned} au_n + bu_m &= 0 \\ au_{n+1} + bu_{m+1} &= 0 . \end{aligned} \tag{38}$$

On p. 42, 11 lines from the end, change the "F" to a script $\mathfrak{F}$.

On p. 49, in the equation preceding Eq. (10), change α_i to ω_i.

* * * * *

BOOK REVIEWS

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

FIBONACCI AND LUCAS NUMBERS

Verner E. Hoggatt, Jr.

Houghton Mifflin Company has just released a 92-page booklet in its Enrichment Series entitled "Fibonacci and Lucas Numbers" by our Editor, V. E. Hoggatt, Jr.

If a first impression is valid, this contribution to mathematical literature might be characterized by three words: richness, variety, lucidity. Richness and variety are manifest in relating the Fibonacci and Lucas numbers to many interesting facets of mathematics. The Golden Section Ratio and some unusual geometry receive attention in the early part of the book. Number theory comes into play in the periodic properties of the Fibonacci and Lucas numbers. The prolific Pascal triangle receives its share of attention. The algebra of simple matrices and representation of integers open up many doors to further research and study. Finally, relations with nature round off the treatment and point to the mysterious connection of mathematics with the real world which has fascinated man for untold centuries.

Some examples of lucidity would be the very slick way in which the Binet formulas are introduced; the handling of asymptotic ratios and their relation to the Golden Section in Chapters 5 and 6; the treatment of periodicity of remainders in Chapter 8; the explanation of Fibonacci numbers in nature in Chapter 13.

A helpful feature of the book is an appendix giving solutions of many of the problems in the book.

This book should prove a boon to young and old who wish to enter that magic door which leads to the wonderful world of Fibonacci. All too often we receive pleas for books and materials dealing with this field. There is now a ready answer to these requests for help.

This booklet lists for $1.40, and is also available from the Fibonacci Association.

INVITATION TO NUMBER THEORY

Oystein Ore

As part of its New Mathematical Library, Random House (The L. W. Singer Company) has just released a booklet, "Invitation to Number Theory," by Oystein Ore.

As everyone knows, number theory is a type of mathematics which has fascinated amateur and professional over the centuries. The questions it raises are often quite easy to understand and therefore appealing to the mathematical enthusiast who does not have a great background in mathematics.

The booklet takes up aspects of number theory that are within the range of a good high school student: primes, divisors of numbers, greatest common divisor and least common multiple, the Pythagorean problem, numeration systems, and congruences.

One of the noteworthy features is the way in which the author relates his treatment to the history of mathematics. The following examples bring out this point: figurate numbers, the Euclidean algorithm for finding the greatest common divisor, perfect numbers, amicable numbers, the Pythagorean problem, ancient systems of numeration, and Mersenne numbers.

On the other hand, up-to-date developments are not neglected. There is an interesting discussion of the largest primes discovered by the factorization of Mersenne numbers. In connection with number bases, computers and their mode of arithmetic are introduced.

Finally, the author has introduced interest features throughout the book: magic squares, games with digits, days of the week as related to congruences, tournament schedules.

The book contains problems to be solved and has a section entitled "Solutions to Selected Problems."

The list price is $1.95.

* * * * *

[Continued from p. 104.]

SOLUTIONS TO PROBLEMS

2. $$2(-1)^n$$

3. $$L_{2n} + (-1)^n$$

4. $$L_{4n} + (-1)^n L_{2n} + 1$$

5. $$L_{2n} + (-1)^{n+1}$$

6. $$L_{4n} + (-1)^{n+1} L_{2n} + 1$$

7. $$T_n = \frac{10 + \sqrt{5}}{5} r^n + \frac{10 - \sqrt{5}}{5} s^n$$

8. $$F_n = 2^{-n+1}\left[n + 5\binom{n}{3} + 5^2\binom{n}{5} + 5^3\binom{n}{7} \cdots\right]$$

9 $$L_n = 2^{-n+1}\left[1 + 5\binom{n}{2} + 5^2\binom{n}{4} + 5^3\binom{n}{6} \cdots\right]$$

10. $$F_{2n+1} .$$

* * * * *

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, N. Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United Stated) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-154 Proposed by S. H. L. Kung, Jacksonville University, Jacksonville, Florida

What is special about the following "magic" square?

$$\begin{bmatrix} 11 & 2 & 14 & 19 & 21 \\ 8 & 13 & 3 & 22 & 1 \\ 20 & 17 & 15 & 6 & 9 \\ 7 & 24 & 18 & 10 & 12 \\ 25 & 5 & 23 & 16 & 4 \end{bmatrix}$$

B-155 Composite of Proposals by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada, and Carol Anne Vespe, University of New Mexico, Albuquerque, N. Mex.

Let the n^{th} Pell number be defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. Show that

$$P_{n+a}P_{n+b} - P_{n+a+b}P_n = (-1)^n P_a P_b .$$

B-156 Proposed by Allan Scott, Phoenix, Arizona.

Let F_n be the n^{th} Fibonacci number, $G_n = F_{4n} - 2n$, and H_n be the remainder when G_n is divided by 10.

(a) Show that the sequence $H_2, H_3, H_4, \cdots$ is periodic and find the repeating block.

(b) The last two digits of G_9 and G_{14} give Fibonacci numbers 34 and 89 respectively. Are there any other cases?

B-157 Proposed by Klaus Günther Recke, University of Gottingen, Germany.

Let F_n be the n^{th} Fibonacci number and $\{g_n\}$ any sequence. Show that

$$\sum_{k=1}^{n} (g_{k+2} + g_{k+1} - g_k)F_k = g_{n+2} F_n + g_{n+1}F_{n+1} - g_1 .$$

B-158 Proposed by Klaus Günther Recke, University of Gottingen, Germany.

Show that

$$\sum_{k=1}^{n} (kF_k)^2 = \left[(n^2 + n + 2)F_{n+2}^2 - (n^2 + 3n + 2)F_{n+1}^2 - (n^2 + 3n + 4)F_n^2\right]/2.$$

B-159 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let T_n be the n^{th} triangular number $n(n + 1)/2$ and let $\phi(n)$ be the Euler totient. Show that $\phi(n) \mid \phi(T_n)$ for $n = 1, 2, \cdots$.

SOLUTIONS

NOTE: The name of M. N. S. Swamy was inadvertently omitted from the lists of solvers of B-118, B-119, and B-135.

A PELL ANALOGUE

B-136 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let P_n be the n^{th} Pell number defined by $P_1 = 1$, $P_2 = 2$, and $P_{n+2} = 2P_{n+1} + P_n$. Show that

$$P_{n+1}^2 + P_n^2 = P_{2n+1} .$$

Solution by J. E. Homer, Union Carbide Corporation, Chicago, Ill.

By induction on k it is easily shown that $P_N = P_{k+1}P_{N-k} + P_kP_{N-k-1}$. Letting $N = 2n + 1$ and $k = n$ the desired result follows.

Also solved by Clyde A. Bridger, Timothy Burns, Herta T. Freitag, J. A. H. Hunter (Canada), John Ivie, D. V. Jaiswal (India), Bruce W. King, Douglas Lind, C. B. A. Peck, A. G. Shannon (Australia), M. N. S. Swamy (Canada), Gregory Wulczyn, Michael Yoder, and the proposer.

ANOTHER PELL IDENTITY

B-137 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let P_n be the n^{th} Pell number. Show that $P_{2n+1} + P_{2n} = 2P_{n+1}^2 - 2P_n^2 - (-1)^n$.

Solution by Carol Vespe, University of New Mexico, Albuquerque, N. Mex.

Let $r = 1 + \sqrt{2}$ and $s = 1 - \sqrt{2}$. Both sides of the identity are of the form

$$c_1(r^2)^n + c_2(rs)^n + c_3(s^2)^n$$

with constant c's. Hence both sides satisfy a recurrence relation

$$y_{n+3} = k_2y_{n+2} + k_1y_{n+1} + k_0y_n ,$$

with constant k's. Therefore the identity is proved for all n by the easy verification for $n = 1, 2,$ and 3.

Also solved by Clyde A. Bridger, Herta T. Freitag, J. E. Homer, John Ivie, D. V. Jaiswal (India), Bruce W. King, C. B. A. Peck, A. G. Shannon (Australia), M. N. S. Swamy (Canada), Gregory Wulczyn, Michael Yoder, and the proposer.

B-138 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Show that for any nonnegative integer k and any integer $n > 1$, there is an n-by-n matrix with integral entries whose top row is $F_{k+1}, F_{k+2}, \cdots, F_{k+n}$ and whose determinant is 1.

Solution by J. E. Homer, Union Carbide Corporation, Chicago, Ill.

The g.c.d. of $(F_{k+1}, F_{k+2}, \cdots, F_{k+n})$ is 1. There exists an n-by-n matrix (Problem E1911, American Mathematical Monthly, Aug.-Sept., 1966) with integral entries whose top row is $F_{k+1}, F_{k+2}, \cdots, F_{k+n}$ and whose determinant is the g.c.d. of $(F_{k+1}, F_{k+2}, \cdots, F_{k+n})$.

Solution for $n \geq 4$ by A. C. Shannon, ACER, Hawthorn, Victoria, Australia.

$$\begin{bmatrix} F_{k+1} & F_{k+2} & F_{k+3} & F_{k+4} & \cdots & F_{k+n-2} & F_{k+n-1} & F_{k+n} \\ F_{k+2} & F_{k+3} & 0 & 0 & \cdots & 0 & 0 & F_{k+n-1} \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & F_{k+n-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & F_{k+3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & F_{k+3} & F_{k+2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & F_{k+2} & F_{k+1} \end{bmatrix}$$

Also solved by Michael Yoder and the proposer.

B-139 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show that the sequence $1, 1, 1, 1, 4, 4, 9, 9, 25, 25, \cdots$ defined by $a_{2n-1} = a_{2n} = F_n^2$ is complete even if an a_j with $j \leq 6$ is omitted but that the sequence is not complete if an a_j with $j \geq 7$ is omitted.

Composite of solutions by C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania, and the proposer.

Let $S_n = a_1 + \cdots + a_n$. Then it is easily seen that $S_{2m} = 2F_mF_{m+1}$ and $S_{2m-1} = F_{2m} = F_{m+1}^2 - F_{m-1}^2$.

J. L. Brown's criterion (Amer. Math. Monthly, Vol. 68, pp. 557-560) states that a nondecreasing sequence of positive integers $b_1, b_2, \cdots$ with $b_1 = 1$ is complete if and only if $b_{n+1} \leq 1 + b_1 + \cdots + b_n$ for $n = 1, 2, \cdots$.

Thus it suffices to show that

(A) $$a_{n+1} \le 1 + S_n - a_i \quad \text{for} \quad 1 \le i \le 6 \quad \text{and} \quad n > i$$

and

(B) $$a_{n+1} > 1 + S_n - a_i \quad \text{for} \quad i > 6 \quad \text{and some} \quad n \ge i\,.$$

There is no loss of generality in letting $i = 2k$. Then (B) follows with $n = i = 2k$ since $k \ge 4$, $1 - F_{k-1}^2 \le 1 - 2^2 = -3$, and

$$a_{n+1} = F_{k+1}^2 > 1 + F_{k+1}^2 - F_{k-1}^2 = 1 + S_{2k-1} = 1 + S_{2k} - a_{2k} = 1 + S_n - a_i\,.$$

One easily checks (A) when $n < 6$. With $n = 2m - 1$ and $m \ge 4$, (A) is clear since $S_n - a_i$ contains $a_{n+1} = a_n$ as a term. With $n = 2m$ and $m \ge 3$, (A) holds if

$$a_{n+1} = F_{m+1}^2 \le 1 + S_n - a_6 = S_n - 3 = 2F_mF_{m+1} - 3$$

or if

$$F_{m+1}(2F_m - F_{m+1}) \ge 3$$

or if

$$F_{m+1}(F_m - F_{m-1}) \ge 3\,,$$

which is true for $m \ge 3$.

B-140 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Show that $F_{ab} > F_aF_b$ if a and b are integers greater than 1.

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa.

$ab > a + b - 1$ for this is true for $a, b = 2$ and differentiation with respect to b with a fixed shows that the l.h.s. increases faster than the r.h.s.

in b (and, by symmetry, in a). Then from

$$F_mF_n + F_{m-1}F_{n-1} = F_{m+n-1}$$

(see Fibonacci Quarterly, Vol. 1, No. 1, p. 66),

$$F_{ab} > F_{a+b-1} = F_aF_b + F_{a-1}F_{b-1} > F_aF_b \text{ .}$$

Also solved by J. E. Homer, D. V. Jaiswal (India), A. C. Shannon (Australia), M. N. S. Swamy (Canada), Michael Yoder, and the proposer.

B-141 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Show that no Fibonacci number F_n nor Lucas number L_n is an even perfect number.

Solution by the proposer.

Recall that an even perfect number greater than 6 must leave a remainder of 1 upon division by 9 and must be a multiple of 4. An even perfect number greater than 28 must be a multiple of 16.

If $F_n \equiv 1 \pmod 9$, then $n \equiv 1, 2, 10, 18,$ or $23 \pmod{24}$; if $16|F_n$ then $n \equiv 0 \pmod{12}$. These two sets have no common elements.

If $L_n = 1 \pmod 9$, then $n \equiv 1$ or $11 \pmod{24}$. If $4|L_n$ then $n \equiv 3 \pmod 6$. Again we have an empty intersection.

Problem H-23 asked if there were any triangular Fibonacci numbers beyond 55. If the answer to that question is "no" then the Fibonacci half of the above is immediate.

★ ★ ★ ★ ★

SUSTAINING MEMBERS

*H. L. Alder
V. V. Alderman
G. L. Alexanderson
R. H. Anglin
*Joseph Arkin
Col. R. S. Beard
*Marjorie Bicknell
J. H. Biggs
Frank Boehm
J. L. Bohnert
M. B. Boisen, Jr.
C. A. Bridger
Leonard Bristow
*Bro. A. Brousseau
*J. L. Brown, Jr.
N. S. Cameron
L. Carlitz
P. V. Charland
P. J. Cocuzza
J. L. Crenshaw
D. E. Daykin
J. W. DeCelis
F. DeKoven
J. E. Desmond
A. W. Dickinson
N. A. Draim
D. C. Duncan
M. H. Eastman
C. F. Ellis
Merritt Elmore
H. W. Eves
R. A. Fairbairn
A. J. Faulconbridge
*H. H. Ferns
D. C. Fielder
E. T. Frankel
H. M. Gehman
G. R. Glabe
E. L. Godfrey
*H. W. Gould
Nicholas Grant
G. B. Greene
*J. H. Halton
V. C. Harris
Cletus Hemsteger
*A. P. Hillman
Bruce H. Hoelter
*V. E. Hoggatt, Jr.
*A. F. Horadam
D. F. Howells
*Dov Jarden
*S. K. Jerbic
Kenneth Kloss
Sidney Kravitz
George Ledin, Jr.
Hal Leonard
*C. T. Long
A. F. Lopez
F. W. Ludecke
J. S. Madachy
*Sister M. DeSales McNabb
John Mellish, Jr.
Mrs. Lucille Morton
Stephen Nytch
F. J. Ossiander
Miss Ann Pape
M. M. Risqueno
*D. W. Robinson
*Azriel Rosenfeld
T. J. Ross
F. G. Rothwell
H. J. Schafer
J. A. Schumaker
B. B. Sharpe
L. R. Shenton
G. Singh
A. N. Spitz
H. L. Umansky
*C. R. Wall
*L. A. Walker
V. T. White
P. A. Willis
Charles Ziegenfus

*Charter Members

ACADEMIC OR INSTITUTIONAL MEMBERS

DUKE UNIVERSITY
Durham, No. Carolina

OREGON STATE UNIVERSITY
Corvallis, Oregon

SACRAMENTO STATE COLLEGE
Sacramento, California

SAN JOSE STATE COLLEGE
San Jose, California

ST. MARY'S COLLEGE
St. Mary's College, California

THE CALIFORNIA
MATHEMATICS COUNCIL

UNIVERSITY OF PUGET SOUND
Tacoma, Washington

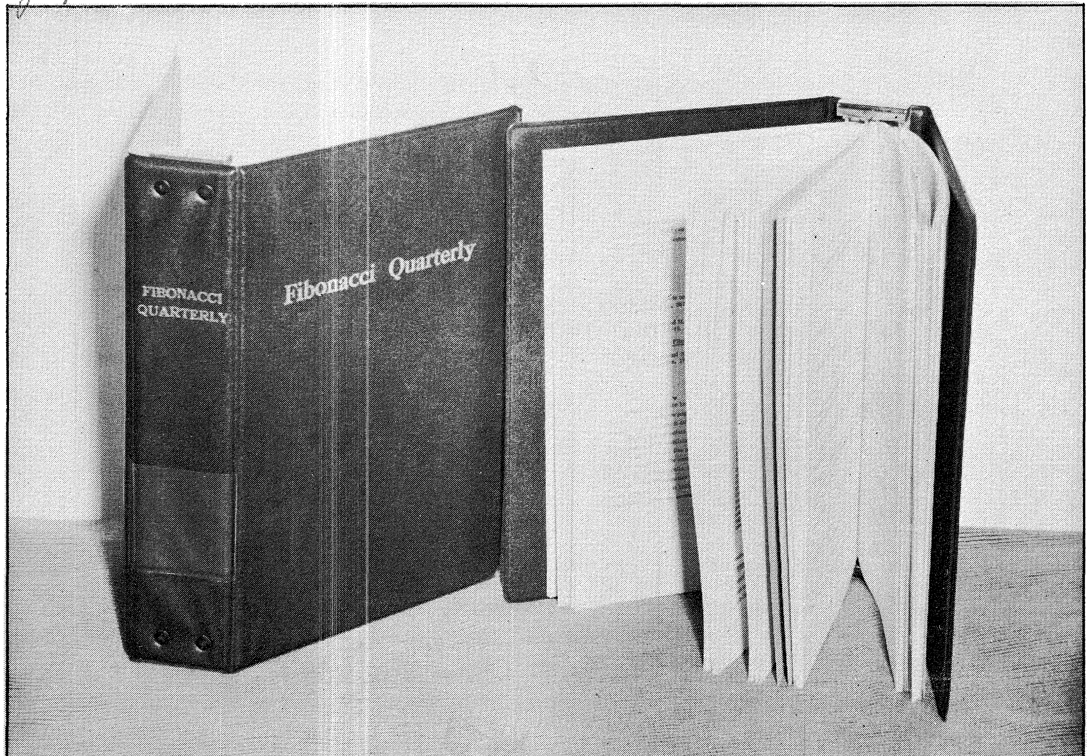
UNIVERSITY OF SANTA CLARA
Santa Clara, California

VALLEJO UNIFIED
SCHOOL DISTRICT

WASHINGTON STATE UNIVERSITY
Pullman, Washington

WESTMINSTER COLLEGE
Fulton, Missouri

July 7



BINDERS NOW AVAILABLE

The Fibonacci Association is making available a binder which can be used to take care of one volume of the publication at a time. This binder is described as follows by the company producing it:

> "....The binder is made of heavy weight virgin vinyl, electronically sealed over rigid board equipped with a clear label holder extending 2 -3/4" high from the bottom of the backbone, round cornered, fitted with a 1 1/2 " multiple mechanism and 4 heavy wires."

The name, FIBONACCI QUARTERLY, is printed in gold on the front of the binder and the spine. The color of the binder is dark green. There is a small pocket on the spine for holding a tab giving year and volume. These latter will be supplied with each order if the volume or volumes to be bound are indicated.

The price per binder is \$3.50 which includes postage (ranging from 50¢ to 80¢ for one binder). The tabs will be sent with the receipt or invoice.

All orders should be sent to: Brother Alfred Brousseau, Managing Editor, St. Mary's College, Calif. 94575