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SOME FORMULAE FOR THE FIBONACCI SEQUENCE WITH GENERALIZATIONS

GEORGE H. ANDREWS
Pennsylvania State University, University Park, Pa.

1. INTRODUCTION

In this paper we shall study the following formulae for the Fibonacci numbers.

$$(1.1) \qquad F_n = \sum_{n=-\infty}^{\infty} (-1)^{\alpha} \binom{n-1}{[\frac{1}{2}(n - 1 - 5\alpha)]} \quad ,$$

$$(1.2) \qquad = \sum_{\alpha=-\infty}^{\infty} (-1)^{\alpha} \binom{n}{[\frac{1}{2}(n - 1 - 5\alpha)]} \quad .$$

where $\binom{m}{n}$ is the ordinary binomial coefficient, and $[x]$ is the greatest integer function.

In Section 2, we shall prove these formulae and shall show how directly they imply the following famous congruences [4; p. 150].

$$(1.3) \qquad F_{p-\left(\frac{5}{p}\right)} \equiv 0 \pmod{p} \quad ,$$

$$(1.4) \qquad F_p \equiv \left(\frac{5}{p}\right) \pmod{p},$$

where $\left(\frac{5}{p}\right)$ is the Jacobi-Legendre symbol.

Chapter IV of Dickson's History, Vol. 1 [2; pp. 105-112] is devoted to studying $(u^{p-1} - 1)/p \pmod{p}$. In particular, Einstein made several contributions to this problem among which was the following. If $p \neq 2$,

$$(2^{p-1} - 1)/p \equiv 1 + 1/3 + 1/5 + \cdots + 1/p - 2 \pmod{p} .$$

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We shall prove analogous formulae for

$$F_{p-\left(\frac{5}{p}\right)}\Big/p$$

and

$$\left(F_p - \left(\frac{5}{p}\right)\right)\Big/p$$

in Section 3. Namely, if $p \equiv \pm 2 \pmod 5$,

$$(1.5) \qquad F_{p+1}\Big/p \equiv 2(-1)^{\frac{1}{2}(p-1)} \sum_{\substack{m \equiv 1,5 \pmod{10} \\ |m|<p}} \frac{\left(\frac{m+1}{5}\right)\left(\frac{-1}{m}\right)}{p-m} \pmod p .$$

If $p \equiv \pm 1 \pmod 5$,

$$(1.6) \qquad F_{p-1}\Big/p \equiv 2(-1)^{\frac{1}{2}(p-1)} \sum_{\substack{m \equiv 5,7 \pmod{10} \\ |m|<p}} \frac{\left(\frac{m+1}{5}\right)\left(\frac{-1}{m}\right)}{p-m} \pmod p .$$

For all primes p,

$$(1.7) \qquad \left(F_p - \left(\frac{5}{p}\right)\right)\Big/p \equiv 2(-1)^{\frac{1}{2}(p-1)} \sum_{\substack{m \equiv 1,7 \pmod{10} \\ |m|<p}} \frac{\left(\frac{m+2}{5}\right)\left(\frac{-1}{m}\right)}{p-m} \pmod p .$$

In Section 4, we make the natural generalization of (1.1) and (1.2) by replacing 5 by an arbitrary odd number. This leads us immediately to an n-dimensional analog of the Fibonacci numbers which is closely related to one considered by Raney.

In Section 5, we point out an application of these generalized sequences to the factorization of large numbers, and in Section 6, we discuss related sequences.

2. THE NEW FORMULAE

Let us define

$$F_n(b) = \sum_{\alpha=-\infty}^{\infty} (-1)^{\alpha} \binom{n}{[\frac{1}{2}(n - b - 5\alpha)]} .$$

Then if $\beta = \exp(2\pi i/5)$,

$$
\begin{aligned}
F_n(b) &= (-1)^b \sum_{\alpha=-\infty}^{\infty} (-1)^{5\alpha+\beta} \binom{n}{[\frac{1}{2}(n - b - 5\alpha)]} \\
&= \frac{(-1)^b}{5} \sum_{j=0}^{4} \sum_{\alpha=-\infty}^{\infty} (-1)^{\alpha} B^{j(\alpha-b)} \binom{n}{[\frac{1}{2}(n - \alpha)]} \\
&= \frac{(-1)^{b+n}}{5} \sum_{j=0}^{4} \beta^{-jb} \sum_{\alpha=-\infty}^{\infty} (-1)^{\alpha} \beta^{j(-\alpha+n)} \binom{n}{[\frac{1}{2}\alpha]} \\
&= \frac{(-1)^{b+n}}{5} \sum_{j=0}^{4} \beta^{j(n-b)} \left\{ \sum_{\alpha=-\infty}^{\infty} \beta^{-2j\alpha} \binom{n}{\alpha} \right. \\
&\qquad \left. - \beta^{-j} \sum_{\alpha=-\infty}^{\infty} \beta^{-2j\alpha} \binom{n}{\alpha} \right\} \\
&= \frac{(-1)^{b+n}}{5} \sum_{j=0}^{4} \beta^{j(n-b)} (1 - \beta^{-j})(1 + \beta^{-2j})^n \\
&= \frac{(-1)^b}{5} \sum_{j=1}^{4} \beta^{-jb} (1 - \beta^{-j})(-2 \cos 2\pi j/5)^n \\
&= \frac{(-1)^b}{5} \sum_{j=1}^{2} (\beta^{-jb} + \beta^{jb} - \beta^{-j(b+1)} - \beta^{j(b+1)}) \times \\
&\qquad \times (-2 \cos 2\pi j/5)^n \\
&= \frac{2(-1)^b}{5} \sum_{j=1}^{2} (\cos 2\pi jb/5 - \cos 2\pi j(b + 1)/5) \times \\
&\qquad \times (-2 \cos 2\pi j/5)^n .
\end{aligned}
$$

Now

$$-2\cos 2\pi/5 = -2\cos 8\pi/5 = \tfrac{1}{2}(1 - 5^{\frac{1}{2}}) ,$$

and

$$-2\cos 4\pi/5 = -2\cos 6\pi/5 = \tfrac{1}{2}(1 + 5^{\frac{1}{2}}) .$$

Hence

$$\begin{aligned}
F_n(0) &= \frac{1}{5}(2 + \frac{1}{2}(1 - 5^{\frac{1}{2}}))(\tfrac{1}{2}(1 - 5^{\frac{1}{2}}))^n + \frac{1}{5}(2 + \frac{1}{2}(1 + 5^{\frac{1}{2}}))(\tfrac{1}{2}(1 + 5^{\frac{1}{2}}))^n \\
&= 5^{-\frac{1}{2}}((\tfrac{1}{2}(1 + 5^{\frac{1}{2}}))^{n+1} - (\tfrac{1}{2}(1 - 5^{\frac{1}{2}}))^{n+1}) \\
&= F_{n+1}, \text{ the } (n+1)^{st} \text{ Fibonacci number [4; p. 148]} .
\end{aligned}$$

$$\begin{aligned}
F_n(1) &= -\frac{1}{5}(-\tfrac{1}{2}(1 - 5^{\frac{1}{2}}) + \tfrac{1}{2}(1 + 5^{\frac{1}{2}}))(\tfrac{1}{2}(1 - 5^{\frac{1}{2}}))^n \\
&\qquad - \frac{1}{5}(-\tfrac{1}{2}(1 + 5^{\frac{1}{2}}) + \tfrac{1}{2}(1 - 5^{\frac{1}{2}}))(\tfrac{1}{2}(1 + 5^{\frac{1}{2}}))^n \\
&= 5^{-\frac{1}{2}}((\tfrac{1}{2}(1 + 5^{\frac{1}{2}}))^n - (\tfrac{1}{2}(1 - 5^{\frac{1}{2}}))^n) \\
&= F_n, \text{ the } n^{th} \text{ Fibonacci number [4; p. 148]} .
\end{aligned}$$

Thus we have (1.1) and (1.2).

We now turn our attention to proving (1.3) and (1.4) utilizing (1.1) and (1.2). Our proof rests on the following elementary congruence

$$(2.1) \qquad \binom{p}{a} \equiv \begin{cases} 1 & \text{if } a = 0, p \\ 0 & \text{otherwise} \end{cases} \pmod{p} ,$$

where p is any prime.

If $p = 5m \pm 2$, then for any integer α,

$$[\tfrac{1}{2}(p - 5\alpha)] \neq 0, p ;$$

therefore by (2.1) p divides every term of the sum in (1.1) with $n = p + 1$, and (1.3) is established in this case. Utilizing (1.2) with $n = p = 5m \pm 2$, we may verify that (1.4) holds in this case. If

$$n - 1 = p = 5m \pm 1 ,$$

then by means of (1.1) we verify that

$$F_{p+1} \equiv -1 \pmod{p},$$

and by means of (1.2) with

$$n = p = 5m \pm 1$$

we verify that $F_p \equiv 1 \pmod{p}$. Thus we have completely established (1.4) with $p \neq 5$, and

$$F_{p-1} = F_{p+1} - F_p \equiv -1 + 1 \equiv 0 \pmod{p}$$

establishes completely (1.3) with $p \neq 5$. Finally since $F_5 = 5$ we have (1.3) and (1.4) proved in this exceptional case as well.

3. EINSTEIN FORMULAE FOR F_n.

This section is devoted to proving (1.5), (1.6), and (1.7). We shall utilize the following congruence

$$p^{-1}\binom{p}{a} \equiv -(-1)^a a^{-1} \pmod{p}, \quad 0 < a < p \tag{3.1}$$

In the following sums, we note that the only terms to be considered are those for which initially the lower entry of the binomial coefficient is in the open interval $(0,p)$. We shall thus not trouble to indicate the range of summation until the final line in each case.

From (1.1) with $n - 1 = p = 2m + 1$,

$$F_{2m+2} = \sum_{\alpha=-\infty}^{\infty} (-1)^{\alpha} \binom{p}{[\frac{1}{2}(2m + 1 - 5\alpha)]} = \sum_{\alpha=-\infty}^{\infty} \left\{ \binom{p}{m - 5\alpha} - \binom{p}{m - 2 - 5\alpha} \right\}. \tag{3.2}$$

Hence

$$F_{p+1}/p \equiv \sum \left\{ \frac{-(-1)^{m+\alpha}}{m - 5\alpha} + \frac{(-1)^{m+\alpha}}{m - 5\alpha - 2} \right\} \pmod{p}$$

$$\equiv 2(-1)^{\frac{1}{2}(p-1)} \sum \left\{ \frac{(-1)^{\alpha+1}}{p - 1 - 10\alpha} + \frac{(-1)^{\alpha}}{p - 5 - 10\alpha} \right\} \pmod{p}$$

$$\equiv 2(-1)^{\frac{1}{2}(p-1)} \sum_{\substack{m \equiv 1,5 \pmod{10} \\ |m| < p}} \frac{\left(\frac{m+1}{5}\right)\left(\frac{-1}{m}\right)}{p - m} \pmod{p}\,.$$

From (1.2) with $n = p = 2m + 1$,

$$F_{2m+1} = \sum_{\alpha=-\infty}^{\infty} (-1)^{\alpha} \binom{p}{\frac{1}{2}(2m - 5\alpha)} = \sum_{\alpha=-\infty}^{\infty} \left\{ \binom{p}{m - 5\alpha} - \binom{p}{m - 3 - 5\alpha} \right\}. \tag{3.3}$$

Therefore if p is a prime $\equiv \pm 1 \pmod 5$, we have by (3.2) and (3.3)

$$F_{p-1} = F_{p+1} - F_p = \sum_{\alpha=-\infty}^{\infty} \left\{ \binom{p}{m - 3 - 5\alpha} - \binom{p}{m - 2 - 5\alpha} \right\}. \tag{3.4}$$

Hence from (3.4) with $p \equiv \pm 1 \pmod 5$,

$$F_{p-1}/p \equiv \sum_{\alpha=-\infty}^{\infty} \left\{ \frac{(-1)^{m+\alpha}}{m - 3 - 5\alpha} + \frac{(-1)^{m+\alpha}}{m - 2 - 5\alpha} \right\} \pmod{p}$$

$$\equiv 2(-1)^{\frac{1}{2}(p-1)} \sum \left\{ \frac{(-1)^{\alpha}}{p - 7 - 10\alpha} + \frac{(-1)^{\alpha}}{p - 5 - 10\alpha} \right\} \pmod{p}$$

$$\equiv 2(-1)^{\frac{1}{2}(p-1)} \sum_{\substack{m \equiv 5,7 \pmod{10} \\ |m| < p}} \frac{\left(\frac{m+1}{5}\right)\left(\frac{-1}{m}\right)}{p - m} \pmod{p}\,.$$

Finally from (3.3) with $p = 2m + 1$

$$\left(F_p - \left(\frac{5}{p}\right)\right)/p \equiv -\sum\left\{\frac{(-1)^{m+\alpha}}{m - 5\alpha} + \frac{(-1)^{m+\alpha}}{m - 3 - 5\alpha}\right\} \pmod p$$

$$\equiv -2(-1)^{\frac{1}{2}(p-1)}\sum\left\{\frac{(-1)^{\alpha}}{p - 1 - 10\alpha} + \frac{(-1)^{\alpha}}{p - 7 - 10\alpha}\right\} \pmod p$$

$$\equiv 2(-1)^{\frac{1}{2}(p-1)} \sum_{\substack{m \equiv 1,7 \pmod{10} \\ |m|<p}} \frac{\left(\frac{m+2}{5}\right)\left(\frac{-1}{m}\right)}{p - m} \pmod p .$$

Thus we have established (1.5), (1.6), and (1.7).

Let us now consider a specific example. By (1.1)

$$F_{14} = \binom{13}{6} - \binom{13}{4} - \binom{13}{9} + \binom{13}{1} + \binom{13}{11} = 1716 - 715 - 715 + 13 + 78 = 377$$

By (1.3),

$$F_{14}/13 \equiv 2\left\{\frac{1}{13 - 11} + \frac{1}{13 - 5} - \frac{1}{13 - 1} - \frac{1}{13 + 5} + \frac{1}{13 + 9}\right\}$$
$$\equiv 1 + 1/4 - 1/6 - 1/9 + 1/11 \equiv 1 + 10 - 11 - 3 + 6 \equiv 3 \pmod{13},$$

and indeed,

$$F_{14}/13 = 29 \equiv 3 \pmod{13} .$$

4. GENERALIZATIONS

In this section we discuss the natural generalization of (1.1) and (1.2). We define

$$F_{k,n}(b) = \sum_{\alpha=-\infty}^{\infty} (-1)^{\alpha} \binom{n}{\frac{1}{2}[n - b - (2k + 1)\alpha)]} . \tag{4.1}$$

Exactly as in Section 2, only now setting

$$\beta = \exp(2\pi i/2k+1),$$

we obtain

$$(4.2) \qquad F_{k,n}(b) = \frac{2(-1)^b}{2k+1}\sum_{j=1}^{k}(\cos(2\pi bj/2k+1) - \cos(2\pi(b+1)j/2k+1)) \times$$

$$\times (-2\cos(2\pi j/2k+1))^n,$$

where $k \geq 0$, $n \geq 0$.

From (4.2) we may easily ascertain the linear recurrence in n satisfied by the $F_{k,n}(b)$. Consider the sequence of polynomials defined by

$$f_0(x) = 1, \quad f_1(x) = x - 1, \quad f_k(x) = xf_{k-1}(x) - f_{k-2}(x) .$$

Then the roots of $f_k(x)$ are

$$-2\cos 2\pi j/2k+1, \qquad 1 \leq j \leq k$$

[3; p. 264]. Hence from the elementary theory of finite difference (with

$$E^r a_n = a_{n+r}),$$

we have

$$(4.3) \qquad f_k(E)F_{k,n}(b) = 0 .$$

The n-dimensional Fibonacci sequence studied by Raney [5] has as its auxiliary polynomial $D_n(x)$ [5; p. 347] where in our notation

$$f_n(x) = (-1)^{\frac{1}{2}n(n-1)} x^n D_n(x^{-1}) .$$

Raney remarks that many of the elementary formulae related to the Fibonacci numbers may be generalized to his sequences, and the same is true of $F_{k,n}(b)$. Most of these results may be derived from (4.2); but the proofs are clumsy. It would be nice to relate these sequences to some set of matrices as Raney has done for his sequences; perhaps then easy proofs could be given for analogs of Theorems 7 and 8 of Raney's paper.

5. FACTORIZATION OF LARGE NUMBERS

As is well known the Fibonacci and Lucas numbers are closely related to Lucas's famous test for the primality of the Mersenne numbers $2^p - 1$. We shall derive some similar necessary conditions for the primality of $(k^p - 1)/k - 1$ utilizing some analogs of the Lucas sequence which are related to the generalized Fibonacci sequences discussed in Section 4. For example, when $k = 2$, we shall prove the necessity part of Lucas's theorem on the primality of $2^q - 1$ (with $q \equiv 3 \pmod 4$) [4; p. 224]. When $k = 2$ and $q \equiv 1 \pmod 4$, we shall prove the following result.

Theorem 3. Let r_n be defined by

$$r_1 = 3, \quad r_{n+1} = r_n^2 - 2 .$$

If $q \equiv 1 \pmod 4$ and $M_q = 2^q - 1$ are both primes, then $r_q \equiv 3 \pmod{M_q}$.

When $k = 3$ and $q \equiv 1 \pmod 6$, we have the following theorem.

Theorem 4. Let $s_0 = 1$, $t_0 = -2$, and in general

$$s_{n+1} = s_n^3 - 3s_nt_n - 3; \quad t_{n+1} = t_n^3 + 3s_nt_n + 3 .$$

If $q \equiv 1 \pmod 6$ and $M_q = \frac{1}{2}(3^q - 1)$ are both primes, then

$$s_q \equiv 4 \pmod{M_q},$$
$$t_q \equiv -11 \pmod{M_q}.$$

Our first object in this section will be the derivation of a general theorem which will imply Theorems 3 and 4.

Let $A_j(k)$ denote the set of all ordered j-tuples of the first k positive integers. We define

$$L_{k,n}(j) = \sum (-2 \cos 2\pi m_1/2k+1)^n \cdots (-2 \cos 2\pi m_j/2k+1)^n$$

where the summation is over all

$$(n_1, \cdots, n_j) \in A_j(k) \quad .$$

We shall also need the polynomials

$$w_m(x) = \sum_{j=0}^{m} \binom{2m}{2j} x^{2m-2j}(1-x^2)^j ;$$

these polynomials have the property that

$$\cos 2m\beta = w_m(\cos \beta) .$$

Lemma 1. Let p be an odd prime, $p \equiv n \pmod{2k+1}$, $0 \le n \le 2k$. Then there exists a rational integer $\alpha(k;j;n)$, which depends only on k, j, and n and not on the magnitude of p such that

$$L_{k,(k-1)p+1}(j) \equiv \alpha(k;\ j;\ n) \pmod{p} .$$

Proof. Define n' to be n if n is even and $n+2k+1$ if n is odd; $n\star = \frac{1}{2}n'$. Then in the ring of integers of $Q(-2 \cos 2\pi/2k+1)$

$$(-2 \cos 2\pi j/2k+1)^p = (-2)^p 2^{-p+1} \sum_{i=0}^{\frac{1}{2}(p-1)} \binom{p}{m+i+1} \cos 2\pi(2i+1)j/2k+1$$

$$\equiv -2 \cos 2\pi pj/2k+1 \pmod{(p)}$$

$$\equiv -2 \cos 2\pi mj/2k+1 \pmod{(p)}$$

$$\equiv -2 \cos 2\pi m'j/2k+1 \pmod{(p)} ,$$

where (p) is the principal ideal generated by p in the ring of integers of $Q(-2\cos 2\pi/2k+1)$ and this first equality is from [1; p. 83]. Consequently

$$L_{k,(k-1)p+1}(j) \equiv \sum (-2\cos 2\pi m' n_1/2k+1)^{k-1}(-2\cos 2\pi m_1/2k+1)\cdots \tag{5.1}$$

$$\cdots(-2\cos 2\pi m' n_j/2k+1)^{k-1}(-2\cos 2\pi m_j/2k+1) \pmod{(p)}$$

$$= \sum (-2w_{n*}(\cos 2\pi m_1/2k+1))^{k-1}(-2\cos 2\pi m_1/2k+1)\cdots \tag{5.2}$$

$$\cdots(-2w_{n*}(\cos 2\pi m_j/2k+1))^{k-1}(-2\cos 2\pi m_j/2k+1) \pmod{(p)}.$$

We now define $\alpha(k;j;n)$ to be the expression appearing on the right side of (5.1) (or, what is the same thing, (5.2)). Now (5.2) shows that $\alpha(k;j;n)$ is a symmetric polynomial in $\cos 2\pi m/2k+1$, $1 \le m \le k$; since these are the roots of $f_k(-2x)$ (c.f. Section 4), we see by the symmetric function theorem that $\alpha(k;j;n)$ is a rational number. On the other hand, (5.1) shows that $\alpha(k;j;n)$ is an integer of the field $Q(-2\cos 2\pi/2k+1)$; since the rational integers are integrally closed in $Q(-2\cos 2\pi/2k+1)$, we see that $\alpha(k;j;n)$ must be a rational integer. Hence

$$L_{k,(k-1)p+1}(j) \equiv \alpha(k;j;n) \pmod{(p)}$$

holds in the ring of integers of $Q(-2\cos 2\pi/2k+1)$. Since this congruence involves only rational integers, it must also hold in Z, the ring of rational integers. Thus Lemma 1 is proved.

Corollary 1. If in Lemma 1, $n = 1$ or $2k$, then

$$L_{k,(k-1)p+1}(j) \equiv L_{k,k}(j) \pmod{p}.$$

Proof. In (5.1) with n' either $2k$ or $2k+2$, we have

$$\alpha(k;j;n) = \sum (-2\cos 2\pi m_1/2k+1)^k \cdots (-2\cos 2\pi m_j/2k+1)^k = L_{k,k}(j).$$

The desired results now follow directly from Lemma 1.

We now proceed to our main result.

Theorem 1. Let $k \geq 2$ be an integer. Let

$$\sigma_{k,j} = \sigma_{k,j}(x_1, \cdots, x_n)$$

be the j^{th} elementary symmetric function of $x_1, \cdots, x_k$. Let $g_j(y_1, \cdots, y_k)$ be the polynomial with integral coefficients such that

$$\sigma_{k,j}(x_1^k, \cdots, x_k^k) = g_j(\sigma_{k,1}, \cdots, \sigma_{k,k}) .$$

Let

$$v_{k,0}(j) = L_{k,1}(j)$$

and

$$v_{k,n+1}(j) = g_j(v_{k,n}(1), \cdots, v_{k,n}(k)) .$$

If $k^* = \text{g.c.d}\,(k-1, 2k+1)$, define $m = k^*(2k+1)$, and let $\phi(m) = m'$, $\phi(m') = m''$ where ϕ is Euler's totient function.

If $q > m$ and $M_q = (k^q - 1)/k - 1$ are both primes, then there exist integers $\beta(k;j;i)$, $1 \leq i \leq m''$ depending only on k and j such that

$$v_{k,q}(j) \equiv \beta(k;j;n) \pmod{M_q} ,$$

if $q \equiv a_n \pmod{m''}$, where $a_1, \cdots, a_{\phi(m'')}$ constitute a reduced residue class system $(\mathrm{mod}\ m'')$.

Proof. From the definition of $L_{k,n}(j)$, one easily verifies by induction that $L_{k,k^n}(j) = v_{k,n}(j)$. One also may verify that the residue of M_q (mod 2k + 1), say r, is completely determined by the residue of $q \pmod{m''}$. Therefore if both $q > m$ and M_q are primes,

$$v_{k,q}(j) = L_{k,k^q}(j) = L_{k,(k-1)M_q+1}(j) \equiv \alpha(k;\ j;\ r) \qquad \pmod{M_q} .$$

If we define

$$\beta(k;\ j;\ n) = \alpha(k;\ j;\ r)$$

where $q \equiv a_n \pmod{m''}$, then the theorem follows.

For small values of k we may prove more explicit theorems.

Theorem 2. (Lucas) Let r_n be defined by

$$r_1 = 3, \quad r_{n+1} = r_n^2 - 2 .$$

If $q \equiv 3 \pmod 4$ and $M_q = 2^p - 1$ are both primes, then $r_{q-1} \equiv 0 \pmod{M_q}$.

Proof. In Theorem 1, with $k = 2$, we find that for $n > 0$

$$v_{2,n}(2) = (-2 \cos 2\pi/5)^{2^n}(-2 \cos 4\pi/5)^{2^n} = (-1)^{2^n} = 1.$$

Also

$$x_1^2 + x_2^2 = \sigma_{2,1}^2 - 2\sigma_{2,2} .$$

Hence

$$g_1(y_1, y_2) = y_1^2 - 2y_2 .$$

Thus we see that $r_n = v_{2,n}(1)$.

As in Lemma 1, we have $\pmod{M_q}$

$$\begin{aligned} r_q = L_{2,M_q+1}(1) &\equiv (-2 \cos 4\pi/5)(-2 \cos 2\pi/5) + (-2 \cos 8\pi/5)(-2 \cos 4\pi/5) \\ &= 2(-2 \cos 4\pi/5)(-2 \cos 2\pi/5) = -2 . \end{aligned}$$

Therefore

$$r_{q-1}^2 = r_q + 2 \equiv -2 + 2 \equiv 0 \pmod{M_q}.$$

Thus since M_q was assumed prime,

$$r_{q-1} \equiv 0 \pmod{M_q}.$$

This concludes the proof of Theorem 2.

Proof of Theorem 3. We proceed exactly as in Theorem 2, except that now by Corollary 1

$$r_q \equiv L_{2,M_q+1}(1) \equiv L_{2,2}(1) = 3.$$

Proof of Theorem 4. In Theorem 1, with $k = 3$, we find that for $n \geq 0$

$$v_{3,n}(3) = (-2 \cos 2\pi/7)^{3^n}(-2 \cos 4\pi/7)^{3^n}(-2 \cos 6\pi/7)^{3^n} = (-1)^{3^n} = -1.$$

Now

$$x_1^3 + x_2^3 + x_3^3 = \sigma_{3,1}^3 - 3\sigma_{3,1}\sigma_{3,2} + 3\sigma_{3,3} ,$$

and thus

$$g_1(y_1, y_2, y_3) = y_1^3 - 3y_1y_2 + 3y_3 .$$

Also

$$x_1^3x_2^3 + x_2^3x_3^3 + x_1^3x_3^3 = \sigma_{3,2}^3 - 3\sigma_{3,1}\sigma_{3,2}\sigma_{3,3} + 3\sigma_{3,3}^2 ,$$

and thus

$$g_2(y_1, y_2, y_3) = y_2^2 - 3y_1y_2y_3 + 3y_3^2 .$$

Thus we see that

$$s_n = v_{3,n}(1)$$

and

$$t_n = v_{3,n}(2) .$$

Utilizing Corollary 1, we have $\pmod{M_q}$

$$s_q = L_{3,2M_q+1}(1) \equiv L_{3,3}(1) = 4 \ ;$$

$$t_q = L_{3,2M_q+1}(2) \equiv L_{3,3}(2) = -11 \ .$$

This concludes Theorem 4.

Theorem 5. Under the conditions of Theorem 4, with the single change that $q \equiv 5 \pmod 6$, if both q and M_q are primes, then

$$s_q \equiv 4 \pmod{M_q}.$$

Proof. Since $q \equiv 5 \pmod 6$, $M_q \equiv 2 \pmod 7$. Hence by Lemma 1 we have $\pmod{M_q}$

$$\begin{aligned} s_q = L_{3,2M_q+1}(1) &\equiv \sum_{j=1}^{3} (-2\cos 4\pi j/7)^2(-2\cos 2\pi j/7) \\ &= 4\sum_{j=1}^{3} (2\cos^2 2\pi j/7 - 1)^2(-2\cos 2\pi j/7) \\ &= \sum_{j=1}^{3} ((-2\cos 2\pi j/7)^2 - 4(-2\cos 2\pi j/7)^3 \\ &\qquad + 4(-2\cos 2\pi j/7)) \\ &= L_{3,5}(1) - 4L_{3,3}(1) + 4L_{3,1}(1) \\ &= 16 - 16 + 4 = 4\,. \end{aligned}$$

We now consider some numerical examples of the theorems we have proved. First take $q = 5$, $M_5 = 121$ in Theorem 5. In this case

n	$s_n \pmod{121}$	$t_n \pmod{121}$
0	1	-2
1	4	-11
2	72	-8
3	-6	-59
4	50	-66
5	-18	

Consequently Theorem 5 proves that $121 = \frac{1}{2}(3^5 - 1)$ is not a prime, and indeed $121 = 11^2$.

Next we consider Theorem 4, with $q = 7$, $M_7 = 1093$. In this case

n	s_n (mod 1093)	t_n (mod 1093)
0	1	-2
1	4	-11
2	193	-367
3	-249	-386
4	-510	-96
5	-569	-78
6	-127	-387
7	4	-11

Thus we see that $1093 = \frac{1}{2}(3^7 - 1)$ satisfies the necessity conditions of Theorem 4, and indeed it turns out that 1093 is a prime.

There appears to be a great number of possibilities for further work on the subjects treated in this section. One would hope that Theorem 1 could be strengthened to include sufficiency conditions for the primality of $(k^p - 1)/(k - 1)$. Possibly the arithmetic of the fields $Q(-2\cos(2\pi/2k + 1))$ would yield such results.

6. RELATED SEQUENCES

It is possible to exhibit a large number of sums similar to those given in (1.1), (1.2), or (4.1). To indicate the possibilities we list three such.

$$G_{k,n}(b) = \sum_{\alpha=-\infty}^{\infty} \binom{n}{[\frac{1}{2}(n - b - (2k+1)\alpha)]} ; \tag{6.1}$$

$$J_{k,n}(b) = \sum_{\alpha=-\infty}^{\infty} (-1)^{\alpha} \binom{n}{[\frac{1}{2}(n - b - (2k+1)2\alpha)]} ; \tag{6.2}$$

$$K_{k,n}(b) = \sum_{\alpha=-\infty}^{\infty} (-1)^{\alpha} \binom{n}{[\frac{1}{2}(n - b - (2k+1)(2n+1))]} . \tag{6.3}$$

Following the method of Section 2, we find

$$G_{k,n}(b) = \frac{2}{2k+1} \sum_{j=1}^{k} (\cos 2\pi bj/(2k+1) + \cos 2\pi(b+1)j/2k+1) \times \times (2 \cos 2\pi j/2k + 1)^n \quad ; \tag{6.4}$$

$$J_{k,n}(b) = \frac{1}{2k+1} \sum_{j=1}^{2k} (\cos \pi b(4j + 2k+1)/(4k+2) + \cos \pi (b+1) \times \times (4j + 2k + 1)/4k + 2)(-2 \sin 2\pi j/2k + 1)^n; \tag{6.5}$$

$$K_{k,n}(b) = \frac{-(-1)^k}{2k+1} \sum_{j=1}^{2k} (\sin \pi b(4j + 2k + 1)/(4k+2) + \sin \pi (b + 1) \times \times (4j + 2k + 1)/4k + 2)(-2 \sin 2\pi j/2k + 1)^n. \tag{6.6}$$

As in Section 4 (c. f. (4.3)), we may give linear recurrence formulae for the above expressions as sequences in n.

$$(-1)^k (E - 2) f_k(-E) G_{k,n}(b) = 0 \; ; \tag{6.7}$$

$$E^{-1}((E + 2) f_k^2(E) - 2) J_{k,n}(b) = 0 \; ; \tag{6.8}$$

$$E^{-1}((E + 2) f_k^2(E) - 2) K_{k,n}(b) = 0 \; . \tag{6.9}$$

Equations (6.7) through (6.9) are easily derived from Eqs. (6.4) through (6.6) utilizing the fact that the roots of $(-1)^k(x - 2)f_k(-x)$ are $2 \cos 2\pi j/(2k+1)$, $0 \le j \le k$ [3; p. 264] and the fact that the roots of $x^{-1}((x + 2)f_k^2(x) - 2)$ are $-2 \sin 2\pi j/2k + 1$, $1 \le j \le 2k$ [3; pp. 267-268].

As is clear from their definitions, all these generalized sequences satisfy congruences similar to (1.3) and (1.4). For example if p is an odd prime, $p \neq 2k+1$, then

$$K_{k,p}(0) \equiv 0 \pmod{p} . \tag{6.10}$$

If p is an odd prime, $p \neq 2k+1$, $p \not\equiv \pm 1 \pmod{4k+2}$, then

$$J_{k,p}(0) \equiv 0 \pmod{p} . \tag{6.11}$$

If $p = (2k+1)m + a$ is a prime with $0 < a \leq k$, $m \geq 2$, then

$$G_{k,p+c}(0) \equiv F_{k,p+c}(0) \equiv 0 \pmod{p} , \tag{6.12}$$

where $0 \leq c \leq a - 2$.

If $p = (2k+1)m + a$ is a prime with $k < a \leq 2k$, $m \geq 1$, then

$$G_{k,p+c}(0) \equiv F_{k,p+c}(0) \equiv 0 \pmod{p} , \tag{6.13}$$

where $0 \leq c \leq 2k - 2 - a$. Equations (6.10) through (6.14) are proved exactly the way (1.3) and (1.4) were.

REFERENCES

1. H. T. Davis, The Summation of Series, Principia Press, San Antonio, 1962.
2. L. E. Dickson, History of the Theory of Numbers, Vol. I, Chelsea, New York, 1952.
3. H. Hancock, "Trigonometric Realms of Rationality," Rendiconti del Circolo Matematico di Palermo, 49 (1925), pp. 263-276.
4. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, Oxford, 4th ed., 1960.
5. G. Raney, "Generalization of the Fibonacci Sequence to n-Dimensions," Canadian J. Math., 18 (1966), pp. 332-349.

* * * * *

OVERLAYS OF PASCAL'S TRIANGLE

MONTE B. BOISEN, JR.
University of Nebraska, Lincoln, Nebraska

The purpose of this paper is to demonstrate the versatility of the method presented by V. E. Hoggatt, Jr. It is hoped that the examples presented in this paper will demonstrate to the reader some of the research possibilities opened by this method. (See [1].)

THE METHOD

The basis of the method lies in the concept of generating functions for the columns of a left-adjusted Pascal's triangle. From Figure 1, we see that the generating function for the k^{th} column is

$$\frac{x^k}{(1-x)^{k+1}} .$$

Extensive use will be made of these generating functions and certain variations of them.

row 0	1					
row 1	1	1				
row 2	1	2	1			
row 3	1	3	3	1		
row 4	1	4	6	4	1	
	⋮	⋮	⋮	⋮	⋱	
column	0	1	2	3	4	⋯
generating function	$\frac{1}{1-x}$	$\frac{x}{(1-x)^2}$	$\frac{x^2}{(1-x)^3}$	$\frac{x^3}{(1-x)^4}$	$\frac{x^4}{(1-x)^5}$	⋯

Fig. 1 Left-Adjusted Pascal's Triangle

ARRAY OVERLAYS

Consider the two arrays A and B with integer entries:

$$
\begin{array}{ccccc}
 & & & & \cdot^{\cdot^{\cdot}} \\
 & & & a_{44} & \vdots \cdots \\
 & & a_{33} & a_{34} & \cdots \\
 & a_{22} & a_{23} & a_{24} & \cdots \\
a_{11} & a_{12} & a_{13} & a_{14} & \cdots
\end{array}
$$

$$
\begin{array}{ccccc}
b_{11} & & & & \\
b_{12} & b_{22} & & & \\
b_{13} & b_{23} & b_{33} & & \\
b_{14} & b_{24} & b_{34} & b_{44} & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

An overlay of A on B means that a sequence $C = \{c_1, c_2, \cdots\}$ is produced such that:

$$
\begin{aligned}
c_1 &= a_{11} \cdot b_{11} \\
c_2 &= a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\
c_3 &= a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} + a_{22} \cdot b_{22} \\
&\cdot \\
&\cdot \\
&\cdot \\
c_i &= \sum_{k=0}^{\left[\frac{i}{2}\right]} \sum_{M=k+1}^{i-k} a_{k+1\,M} b_{M\,i-k}\,, \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned}
$$

where $[s]$ as usual represents the greatest integer in s.

FOUR EXAMPLES

Example I. Let us see what type of sequence we can expect if A and B are both left-adjusted Pascal's triangles (i.e., A is a left-adjusted Pascal's triangle placed on its side). The first few terms of such an overlay are

$$1,\ 2,\ 5,\ 12,\ 29,\ \cdots$$

which suggests that there is a recursive relationship that is described by the rule

$$U_{n+2} = 2U_{n+1} + U_n \,.$$

The verification that this recursion indeed holds for the whole sequence can be accomplished by noting that the coefficients of the expansion of $(1 + x)^n$ represent the n^{th} row of Pascal's triangle and that in the overlay the n^{th} row of Pascal's triangle lies on the n^{th} column. Hence we arrive at the conclusion that the generating function for the sequence is

$$\frac{1}{1-x} + (1+x)\frac{x}{(1-x)^2} + (1+x)^2\frac{x^2}{(1-x)^3} + \cdots + (1+x)^{n-1}\frac{x^{n-1}}{(1-x)^n} + \cdots$$
$$= \frac{1}{1-x}\left\{\frac{1}{1 - \frac{x(1+x)}{1-x}}\right\} = \frac{1}{1-2x-x^2} \,.$$

The reader can easily verify that

$$\frac{1}{1-2x-x^2}$$

generates a sequence where the desired recursive relation holds. This example shows that, in spite of the seemingly formidable configuration of the elements of the sequence C, with the column generators one is able to cope with the situation easily.

Example II: This example will concern itself with determining which arrays, when overlayed, will yield the Fibonacci sequence. In order to effect this, we will begin with the generating function for the Fibonacci sequence which is

$$\frac{1}{1-x-x^2} = \frac{1+x}{1-x^2}\left\{\frac{1}{1-\dfrac{x^2(1+x)}{1-x^2}}\right\}$$

$$= \frac{1+x}{1-x^2} + \frac{x^2(1+x)^2}{(1-x^2)^2} + \cdots + \frac{x^{2(n-1)}(1+x)^n}{(1-x^2)^n} + \cdots \;.$$

Remembering Example I, the presence of $(1+x)^{n-1}$ in the n^{th} term suggests that the A array is a left-adjusted Pascal's triangle. Then the B array must have column generators of

$$\frac{1+x}{1-x^2},\ \frac{x^2(1+x)}{(1-x^2)^2},\ \frac{x^4(1+x)}{(1-x^2)^3},\ \cdots,\ \frac{x^{2(n-1)}(1+x)}{(1-x^2)^n}\;.$$

If one notes that x^2 has replaced the x in Figure 1 and that the $1+x$ "fills in" the void left by that replacement, then the array with these column generators is easily seen to be

$$\begin{array}{ccccc}
1 & & & & \\
1 & & & & \\
1 & 1 & & & \\
1 & 1 & & & \\
1 & 2 & 1 & & \\
1 & 2 & 1 & & \\
1 & 3 & 3 & 1 & \\
1 & 3 & 3 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

which is a doubled left-adjusted Pascal's triangle. (Note that, for example, the spot a_{22} is not listed. Consider those spots to contain zero.)

Therefore we conclude that the Fibonacci sequence can be generated by overlaying the left-adjusted Pascal's triangle on the doubled left-adjusted Pascal's triangle.

Example III: In this example the results from Example II will be carried one more step toward a generalization. Instead of considering the Fibonacci sequence, a Fibonacci-like sequence will be considered,

$$U_1 = 1, \quad U_2 = 1, \quad U_3 = 1, \quad U_{n+3} = U_{n+2} + U_{n+1} + U_n ,$$

The generating function for this sequence is easily found, see [2], to be

$$\frac{1 - x^2}{1 - x - x^2 - x^3} = 1 + \frac{x(1+x^2)}{1-x^2} + \frac{x^2(1+x^2)^2}{(1-x^2)^2} + \cdots + \frac{x^n(1+x^2)^n}{(1-x^2)^n} .$$

The presence of $(1 + x^2)^{n-1}$ in the numerator of the n^{th} term suggests that the A array is

$$\begin{array}{ccccc}
 & & & \iddots & \\
 & & & 1 & \cdots \\
 & & & 0 & \cdots \\
 & & 1 & 3 & \cdots \\
 & & 0 & 0 & \cdots \\
 & 1 & 2 & 3 & \cdots \\
 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & \cdots
\end{array}$$

which is simply a left-adjusted Pascal's triangle with a column of zeros placed in between each of its columns. Note that this array is not in the exact form of the array A but the analogous method of overlaying this array is obvious. We are now left with the generating functions

$$1, \ \frac{x}{1 - x^2}, \ \frac{x^2}{(1 - x^2)^2}, \cdots$$

which yield the array

$$
\begin{array}{cccccccc}
1 & & & & & & & \\
0 & 1 & & & & & & \\
0 & 0 & 1 & & & & & \\
0 & 1 & 0 & 1 & & & & \\
0 & 0 & 2 & 0 & 1 & & & \\
0 & 1 & 0 & 3 & 0 & 1 & & \\
0 & 0 & 3 & 0 & 4 & 0 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Therefore, using the method of column generators, we have found the proper arrays which overlay to form the given sequence.

Example IV: Consider the generalized Pascal's triangle whose k^{th} row is determined by the coefficients of the expansion of

$$(1 + x + \cdots + x^{r-1})^k; \quad k = 0, 1, \cdots \text{ and } r \geq 2.$$

Let this triangle be the A-array and let

$$
\begin{array}{cccc}
1 & & & \\
1 & & & \\
1 & 1 & & \\
1 & 2 & & \\
1 & 3 & 1 & \\
1 & 4 & 3 & \ddots \\
\vdots & \vdots & \vdots &
\end{array}
$$

be the B-array. Note that the B-array is formed by "pushing" the columns of Pascal's triangle down so that the first entry of the k^{th} column appears in the $2k^{th}$ row; $k = 0, 1, 2, \cdots$. Hence by our prior experience we know that the generator for the k^{th} column of the B-array is

$$\frac{x^{2k}}{(1 - x)^{k+1}} \quad .$$

By the method used in the previous examples, the generator for the sequence determined by overlaying A on B is

$$\frac{1}{(1 - x)} + (1 + x + \cdots + x^{r-1})\left(\frac{x^2}{(1 - x)^2}\right) + (1 + \cdots + x^{r-1})^2\left(\frac{x^4}{(1 - x)^3}\right) + \cdots$$

$$= \frac{1}{1 - x - x^2 - \cdots - x^{r+1}} \quad .$$

It is easy to verify that

$$\frac{1}{1 - x - x^2 - \cdots - x^{r+1}} = u_1 + u_2x + u_3x^2 + \cdots$$

where

$$u_1 = 1, \quad u_2 = 1, \quad u_3 = 2, \quad \cdots, \quad u_{r+1} = 2^{r+1}, \quad u_{r+2} = 2^r ,$$

$$u_n = \sum_{i=1}^{r+1} u_{n-i}$$

for $n > r + 2$.

It is interesting to note that this sequence of u's is precisely the sequence of the rising diagonal sums in the generalized Pascal's triangle whose k^{th} row is determined by the coefficients of the expansion of $(1 + x + \cdots + x^{r+1})^k$; $k = 0, 1, 2, \cdots$. See [2] for the proof of this fact and [3] for a further discussion of related subjects.

CONCLUSION

The approach used in the preceding examples to find the sequence determined by overlaying an array A on an array B can be described as follows. Let $P_k(x) = a_{1k} + a_{2k}x + \cdots + a_{kk}x^{k-1}$, and let $G_k(x)$ be the generating function for the k^{th} column of the B array; $k = 0, 1, 2, \cdots$. Then

$$\sum_{i=0}^{\infty} P_i(x)G_i(x)$$

is the generating function that determines the desired sequence.

Almost an unlimited number of problems of the type worked in this paper are now open to scrutiny. At the end of this paper there are two such problems stated. The first one is fairly straight forward and the ultimate answer is supplied. The second one seems to be a little tougher and might make a nice project for some ambitious student.

ACKNOWLEDGMENT

I would like to thank Dr. Verner Hoggatt, Jr., in particular for his suggestion of Example IV and more generally for all of his encouragement and for so masterfully growing the tree from which I am privileged to pluck this small fruit.

PROBLEMS

Problem I: Let an unending row of urns be given, the first one labeled 0, the second labeled 1 and so forth. In the urn labeled "k" let there be k distinguishable balls; $k = 0, 1, 2, \cdots$. Suppose a man does a series of events with the n^{th} event, $n = 0, 1, 2, \cdots$, described as follows:

a) He reaches into the urn labeled "n" $n + 1$ times. The first time he takes out 0 balls, the second time 1 ball, the third 2 balls and so forth until the $n + 1$ time he removes all n balls each time replacing the balls he has previously removed.

b) In general he reaches into the urn marked "$n - j$" $n - 2j$ times taking out $j,\ j + 1, \cdots, n - j$ balls respectively (again by replacement), $j \geq 0$.

c) This event ends when he has moved down the line of urns to the one labeled $n - s$ such that $n - s < s$ for the first time.

Since the balls are distinct, associated with each extraction of balls (i.e., each time the man reaches into an urn) there is a number which represents the number of ways the extraction could have occurred. Let S_k be the sum of all these numbers in the k^{th} event. The problem is to find a generating function that determines $\{S_i\}_{i=0}^{\infty}$ as its sequence.

Ans. $$\left(\frac{1}{1 - 3x + 2x^2} - \frac{x}{1 - 2x + x^3} \right)$$

Problem II. Find two non-trivial arrays such that their overlay determines the sequence:

$$U_0 = U_1 = \cdots = U_{n-1} = 1 \quad \text{and} \quad U_k = U_{k-1} + \cdots + U_{k-n}$$

for all $k \geq n$. See [4].

REFERENCES

1. V. E. Hoggatt, Jr., "A New Angle on Pascal's Triangle," the Fibonacci Quarterly, Vol. 6, No. 4, Oct. 1968, p. 221.
2. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Diagonal Sums of Generalized Pascal Triangles," to be published in the Fibonacci Quarterly.
3. V. E. Hoggatt, Jr., "Generalized Fibonacci Numbers and the Polygonal Numbers," J. Recreational Math., Vol. 1, No. 3, July 1968.
4. Problem H-87, Fibonacci Quarterly, Vol. 4, No. 2, April, 1966, page 149, by M. B. Boisen, Jr.

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ON THE DENSITY OF THE k-FREE INTEGERS

R. L. DUNCAN
Lock Haven State College, Lock Haven, Pa. and
Pennsylvania State University, University Park, Pa.

Let T_k denote the set of k-free integers and let $T_k(n)$ be the number of such numbers not exceeding n. Then the Schnirelmann and asymptotic densities of T_k are defined by

$$(1) \qquad d(T_k) = \inf \frac{T_k(n)}{n}$$

and

$$(2) \qquad \delta(T_k) = \lim_{n\to\infty} \frac{T_k(n)}{n} = \frac{1}{\zeta(k)}$$

respectively, where $\zeta(s)$ is the Riemann zeta function. Our purpose is to summarize and extend the known results concerning the relationship between $d(T_k)$ and $\delta(T_k)$.

It has been shown by Rogers [1] that

$$(3) \qquad d(T_2) = \frac{53}{88} < \frac{6}{\pi^2} = \delta(T_2)$$

and it has been shown subsequently [2] that

$$(4) \qquad \delta(T_k) < d(T_{k+1}) \le \delta(T_{k+1}) .$$

The fact that $d(T_k) \le \delta(T_k)$ is an immediate consequence of (1) and (2). More recently, it has been shown by Stark [3] that

$$(5) \qquad d(T_k) < \delta(T_k) .$$

Combining (4) and (5), we have

(6) $$d(T_k) < \delta(T_k) < d(T_{k+1}) ,$$

i.e., the Schnirelmann and asymptotic densities of the k-free integers interlace.

The proofs of (3) and (4) and the second part of (2) are elementary while the proof of (5) is made to depend on what seems to be a much deeper result. Thus it would be very desirable to have a correspondingly simple proof of (5).

It is also easily shown [2] that

$$d(T_k) > 1 - \sum_p p^{-k}$$

from which it follows immediately that

(7) $$d(T_k) > 2 - \zeta(k) .$$

We conclude this survey by showing that $d(T_{k+1})$ is much closer to $\delta(T_{k+1})$ than to $\delta(T_k)$.

To do this we define

(8) $$\Delta(k) = \frac{\delta(T_{k+1}) - d(T_{k+1})}{\delta(T_{k+1}) - \delta(T_k)}$$

since the numerator and denominator in (8) are both positive, the following theorem yields the desired result.

<u>Theorem.</u> $\Delta(k) < 2^{-k}$.

<u>Proof.</u> By (2), (7) and (8) we have

$$\Delta(k-1) < \frac{\frac{1}{\zeta(k)} - 2 + \zeta(k)}{\frac{1}{\zeta(k)} - \frac{1}{\zeta(k-1)}} = \frac{(\zeta(k) - 1)^2}{1 - \frac{\zeta(k)}{\zeta(k-1)}} .$$

But

$$\frac{\zeta(k-1)}{\zeta(k)} = \sum_{n=1}^{\infty} \phi(n)n^{-k} > \zeta(k),$$

where $\phi(n)$ is Euler's function. Hence

$$\Delta(k-1) < \zeta(k)\ (\zeta(k) - 1) \quad .$$

Since $\zeta(3) < 1.203$, the desired result follows from the trivial estimate

$$\zeta(k) < 1 + \frac{1}{2^k} + \frac{1}{3^k} + \int_3^{\infty} \frac{dx}{x^k} \le 1 + \frac{1}{2^k} + \frac{2}{3^k} \quad .$$

It should be observed that this result also furnishes an alternative proof of the second inequality in (6).

REFERENCES

1. Kenneth Rogers, "The Schnirelmann Density of the Square Free Integers," Proc. Amer. Math. Soc. 15(1964), pp. 515-516.
2. R. L. Duncan, "The Schnirelmann Density of the k-Free Integers," Proc. Amer. Math. Soc. 16(1965), pp. 1090-1091.
3. H. M. Stark, "On the Asymptotic Density of the k-Free Integers," Proc. Amer. Math. Soc. 17(1966), pp. 1211-1214.

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SUMMATION OF INFINITE FIBONACCI SERIES

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

In a previous paper, a well-known technique for summing finite or infinite series was employed to arrive at a number of summations of Fibonacci and Lucas infinite series in closed form [1]. This work is rewarding but in reality covers only a limited portion of the possible infinite series that can be constructed. Starting in general with an arbitrary Fibonacci or Lucas infinite series, the probability that it has a closed sum is relatively small. One need only think of the sum of the reciprocals of the Fibonacci numbers themselves which to date has not been determined in a precise manner.

In the face of this situation, what remains to be done? The present article attacks this problem by attempting to accomplish two things: (1) Determining the relations among cognate formulas so that formulas can be grouped into families in which all the members of one family are expressible in terms of one member of the family and other known quantities; (2) Replacing slowly converging sums by those that converge more rapidly.

The combination of these two efforts has this effect. Given families A_1, A_2, $A_3, \cdots$, whose members are expressible in terms of summations $a_1, a_2, a_3, \cdots$, respectively. Then if these quantities a_i can be related to other quantities a_i' which converge more rapidly, the problem of finding the summations in the various families is reduced once and for all to making precise determinations of a very few summations a_i' which can be found in a reasonably small number of steps.

Such is the program. The purpose of the article is to give an illustrative rather than an exhaustive treatment. The investigation, moreover, will be limited to infinite series of the type:

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k_1} F_{n+k_2} F_{n+k_3} \cdots F_{n+k_r}}$$

or of the form:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+k_1} F_{n+k_2} \cdots F_{n+k_r}}$$

with all the Fibonacci numbers in the denominator different and the k_i positive.

NOTATION AND LANGUAGE

To compress notation, the expression $(F_n)_r$ will mean

$$F_n F_{n-1} F_{n-2} \cdots F_{n-r+1} \quad .$$

If there are k Fibonacci numbers in a denominator, we shall speak of this as a "summation of the k^{th} degree."

CONVERGENCE OF THE SUM OF FIBONACCI RECIPROCALS

We shall begin by establishing the fact that the sum of the reciprocals of the Fibonacci numbers:

$$\sum_{n=1}^{\infty} 1/F_n$$

converges. This in turn will be sufficient in itself to enable us to conclude to the convergence of all sums of our two types since their terms are less than or equal to those of this series.

Using the roots of the equation $x^2 - x - 1 = 0$, namely,

$$r = \frac{1+\sqrt{5}}{2} \text{ and } s = \frac{1-\sqrt{5}}{2} \quad ,$$

we have

$$F_n = \frac{r^n - s^n}{\sqrt{5}} \quad . \tag{1}$$

Now $s = -r^{-1}$. Hence when n is odd,

$$1/F_n < \sqrt{5}/r^n$$

and when n is even, it can be shown that

$$1/F_n < \sqrt{5}/r^{n-1} .$$

This follows since the relation for n even, $r^n - r^{-n} > r^{n-1}$ leads to $r^n - r^{n-1} > r^{-n}$, or finally $r - 1 > r^{-2n+1}$ which is certainly true for $n \geq 2$; for $n = 1$, $r - 1 = r^{-1}$.

Thus in either case

$$1/F_n < \sqrt{5}/r^{n-1}, \text{ for } n \geq 2 .$$

Hence

$$\sum_{n=1}^{\infty} 1/F_n < \sum_{n=1}^{\infty} \sqrt{5}/r^{n-1} = \frac{\sqrt{5}}{1 - 1/r} \tag{2}$$

Since the summation of positive terms has an upper bound, it follows that it must converge.

RELATIONS AMONG SECOND-DEGREE SERIES

Essentially, there is only one first degree series of each type in the sense defined in this treatment, so that the first opportunity to relate series comes with the second degree. Here we have a special situation inasmuch as the alternating series can all be evaluated, the final result being:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+k}} = \frac{1}{F_k}\left[kr^{-1} - \sum_{j=1}^{k} F_{j-1}/F_j \right] \tag{3}$$

r being defined as before. The proof is as follows.

$$\sum_{n=1}^{\infty}\left[F_{n-1}/F_n - F_{n+k-1}/F_{n+k}\right] =$$

$$= \lim_{n\to\infty}\left[\sum_{j=1}^{k} F_{j-1}/F_j - \sum_{m=n-k+1}^{n} F_{n+k-1}/F_{m+k}\right] =$$

$$= \sum_{j=1}^{k} F_{j-1}/F_j - k\,r^{-1} .$$

But the initially given summation also equals

$$\sum_{n=1}^{\infty}\left[\frac{F_{n-1}F_{n+k} - F_n F_{n+k-1}}{F_n F_{n+k}}\right]$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n F_k}{F_n F_{n+k}} .$$

Equating the two values and solving gives relation (3).

The non-alternating series of the second degree has closed formulas for the summation

$$\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+k}}$$

when k is even. For the case $k = 2$,

$$\sum_{n=1}^{\infty}\left[\frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+1}F_{n+2}}\right] = 1 .$$

But this likewise equals

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}}$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1 \tag{4}$$

For $k = 4$, the derivation is as follows.

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} - 3\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+4}} =$$

$$= \sum_{n=1}^{\infty} \frac{F_{n+4} - 3F_{n+2}}{F_n F_{n+2} F_{n+4}} = \sum_{n=1}^{\infty} \frac{F_n}{F_n F_{n+2} F_{n+4}}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{F_{n+2} F_{n+4}}$$

$$= \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} - \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} .$$

Solving for the desired summation,

$$3\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+4}} = 2\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} - 5/6$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+4}} = 2/3 - 5/18 = 7/18 . \tag{5}$$

The process can be contained yielding:

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+6}} = 143/960 \tag{6}$$

and an endless series of formulas with a closed value.

For k odd

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} - 2\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+3}} = \sum_{n=1}^{\infty} \frac{1}{F_{n+1} F_{n+3}}$$

$$= \frac{-1}{1 \cdot 2} + \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = -1/2 + 1 = 1/2 .$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+3}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} - 1/4 . \tag{7}$$

It is possible to proceed step-by-step to other formulas in the series. Thus

$$2\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+3}} - 5\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+5}} = \sum_{n=1}^{\infty} \frac{1}{F_{n+3} F_{n+5}}$$

$$= -\frac{1}{1 \cdot 2} - \frac{1}{1 \cdot 3} - \frac{1}{2 \cdot 5} + \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} .$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+5}} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} - 1/10 + 1/5(1/2 + 1/3 + 1/10 - 1)$$

or

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+5}} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} - 17/150 \; . \tag{8}$$

In summary, for second-degree summations of the given types, apart from the results in closed form, the summations

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k}}$$

with k odd are all expressible in the form

$$a + b\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}}$$

where a and b are rational numbers.

AUXILIARY TABLE

In the work with these summations, the formula that is being employed to arrive at Fibonacci numbers which are to be eliminated is:

$$F_k F_{k+n+r} - F_{k+r} F_{k+n} = (-1)^{k-1} F_r F_n \tag{9}$$

or

$$F_n = \frac{(-1)^{k-1}}{F_r}\left[F_k F_{k+n+r} - F_{k+r} F_{k+n}\right] \tag{10}$$

Rather than use this for each instance it is found to be more convenient to make a table which indicates factors a and b in the relation:

$$F_n = a\,F_{n+k} + b\,F_{n+j} \tag{11}$$

The quantities F_{n+k} are at the right; the quantities F_{n+j} are at the top. The tabular values for any given pair are a, b in sequence. Thus, to express F_n in terms of F_{n+6} and F_{n+2}, the quantities a and b are -1/3 and 8/3, respectively, so that

$$F_n = (-F_{n+6} + 8F_{n+2})/3$$

Similarly, to express F_{n+3} in terms of F_{n+8} and F_{n+5}, since the shift in subscripts is relative, we take the table values for F_{n+5} and F_{n+2}. Hence

$$F_{n+3} = (-F_{n+8} + 5F_{n+5})/2$$

Table I
QUANTITIES a,b IN FORMULA (11)

F_{n+k} \ F_{n+j}	F_{n+1}	F_{n+2}	F_{n+3}	F_{n+4}	F_{n+5}	F_{n+6}
F_{n+2}	1,-1					
F_{n+3}	1. -2	-1, 2				
F_{n+4}	1/2(1,-3)	-1, 3	2,-3			
F_{n+5}	1/3(1,-5)	1/2(-1,5)	2,-5	-3,5		
F_{n+6}	1/5(1,-8)	1/3(-1,8)	1/2(2,-8)	-3,8	5,-8	
F_{n+7}	1/8(1,-13)	1/5(01,13)	1/3(2,-13)	1/2(-3,13)	5,-13	-8,13

THIRD-DEGREE SUMMATIONS

For third-degree summations

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+a} F_{n+b}}$$

there is one which has its sum in closed form. The derivation follows.

$$\sum_{n=1}^{\infty}\left[\frac{1}{F_n F_{n+1} F_{n+2}} - \frac{1}{F_{n+1}F_{n+2}F_{n+3}}\right] = \frac{1}{2} .$$

But this also equals

$$\sum_{n=1}^{\infty}\left[\frac{F_{n+3} - F_n}{(F_{n+3})_4}\right] = 2\sum_{n=1}^{\infty}\frac{F_{n+1}}{(F_{n+3})_4}$$

Hence

(12)
$$\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2} F_{n+3}} = \frac{1}{4} .$$

To find

$$\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2} F_{n+4}}$$

in terms of

$$\sum_{n=1}^{\infty}\frac{1}{(F_{n+2})_3} ,$$

we use this result, arranging coefficients so that we obtain F_n in the numerator and then eliminate it from the denominator. Thus

$$2\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2} F_{n+3}} - 3\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2} F_{n+4}}$$

$$\sum_{n=1}^{\infty} \frac{2\,F_{n+4} - 3\,F_{n+3}}{F_n\,F_{n+2}\,F_{n+3}\,F_{n+4}} = \sum_{n=1}^{\infty} \frac{F_n}{F_n\,F_{n+2}\,F_{n+3}\,F_{n+4}} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_3} = \frac{1}{1 \cdot 1 \cdot 2} - \frac{1}{1 \cdot 2 \cdot 3} + \sum_{n=1}^{\infty} \frac{1}{(F_{n+2})_3} \ .$$

Solving for the desired summation,

$$\sum_{n=1}^{\infty} \frac{1}{F_n\,F_{n+2}\,F_{n+4}} = \frac{7}{18} - \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{(F_{n+2})_3} \ . \tag{13}$$

The procedure is similar at each step. There are two formulas (a) and (b) to be combined with appropriate coefficients; a certain F_r is eliminated; a formula (c) is obtained, either

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+2})_3}$$

or one which has previously been expressed in terms of this quantity.

It would occupy altogether too much space to present even a small portion of the derivations. The sequence of steps, however, can be indicated by giving the denominators in the summations (a), (b), and (c) and between (b) and (c), the quantity F_r which was eliminated. The denominator of the desired summation is the same as (b) in this table.

Table II
SCHEMATIC SEQUENCE FOR THIRD-DEGREE SUMMATIONS

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+a} F_{n+b}}$$

Denominator (a)	Denominator (b)	F_r	Denominator (c)
$F_n F_{n+2} F_{n+4}$	$F_n F_{n+2} F_{n+5}$	F_n	$F_{n+2} F_{n+4} F_{n+5}$
$F_n F_{n+2} F_{n+3}$	$F_n F_{n+3} F_{n+4}$	F_n	$F_{n+2} F_{n+3} F_{n+4}$
$F_n F_{n+3} F_{n+4}$	$F_n F_{n+3} F_{n+5}$	F_n	$F_{n+3} F_{n+4} F_{n+5}$
$F_n F_{n+1} F_{n+2}$	$F_n F_{n+1} F_{n+3}$	F_n	$F_{n+1} F_{n+2} F_{n+3}$
$F_n F_{n+1} F_{n+3}$	$F_n F_{n+1} F_{n+4}$	F_n	$F_{n+1} F_{n+3} F_{n+4}$
$F_n F_{n+1} F_{n+4}$	$F_n F_{n+1} F_{n+5}$	F_n	$F_{n+1} F_{n+4} F_{n+5}$
$F_n F_{n+2} F_{n+5}$	$F_n F_{n+2} F_{n+6}$	F_n	$F_{n+2} F_{n+5} F_{n+6}$
$F_n F_{n+3} F_{n+4}$	$F_n F_{n+4} F_{n+5}$	F_n	$F_{n+3} F_{n+4} F_{n+5}$
$F_n F_{n+3} F_{n+4}$	$F_n F_{n+4} F_{n+6}$	F_n	$F_{n+3} F_{n+4} F_{n+6}$

and so on.

The results can be summarized in the form

(14) $$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+s} F_{n+t}} = c + dS$$

where

$$S = \sum_{n=1}^{\infty} \frac{1}{(F_{n+2})_3} .$$

Table III
CONSTANTS c AND d FOR GIVEN s AND t IN FORMULA (14)

s,t	c	d
1,3	-1/4	1
1,4	-7/36	2/3
1,5	-71/450	7/15
1,6	-509/4800	3/10
1,7	-11417/162240	5/26
2,3	1/4	0
2,4	7/18	-1/3
2,5	71/300	-1/5
2,6	509/2880	-1/6
2,7	11417/101400	-7/65
3,4	-5/36	1/3
3,5	-67/300	2/5
3,6	-269/1920	1/4
4,5	19/225	-1/15
4,6	407/2880	-1/6

It should be apparent without formal proof that any summation of the third degree with positive terms can be expressed in the form given by (14). A practical conclusion follows: It is only necessary to find the value of one summation

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+2})_3}$$

which can be done once and for all to any desired number of decimal places. Thereafter for formulas related to this summation their values can be found with a minimum of effort to any desired number of places within the limits established for the one basic formula.

This method of relating a number of formulas to one formula can be continued to higher degrees though the complexities become greater. For example, for seventh-degree expressions:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{i=1}^{7} F_{n+k_i}}$$

we can proceed step-by-step according to the following table. The quantities (a), (b), (c) and F_r have the same meaning as in Table II. The desired summation is indicated by an asterisk.

Table IV
SCHEMATIC SEQUENCE FOR SEVENTH-DEGREE SUMMATIONS
WITH ALTERNATING TERMS

Denominator (a)	Denominator (b)	F_r	Denominator (c)
$(F_{n+6})_7$	$^*(F_{n+5})_6 F_{n+7}$	F_n	$(F_{n+7})_7$
$(F_{n+6})_7$	$^*(F_{n+5})_6 F_{n+8}$	F_n	$(F_{n+6})_6 F_{n+8}$
$(F_{n+6})_7$	$^*(F_{n+5})_6 F_{n+9}$	F_n	$(F_{n+6})_6 F_{n+9}$
and so on.			
$(F_{n+6})_7$	$(F_{n+5})_6 F_{n+7}$	F_{n+5}	$^*(F_{n+4})_5 F_{n+6} F_{n+7}$
$(F_{n+6})_7$	$(F_{n+5})_6 F_{n+8}$	F_{n+5}	$^*(F_{n+4})_5 F_{n+6} F_{n+8}$
$(F_{n+6})_7$	$(F_{n+5})_6 F_{n+9}$	F_{n+5}	$^*(F_{n+4})_5 F_{n+6} F_{n+9}$
and so on.			
$(F_{n+5})_6 F_{n+7}$	$(F_{n+5})_6 F_{n+8}$	F_{n+5}	$^*(F_{n+4})_5 F_{n+7} F_{n+8}$
$(F_{n+5})_6 F_{n+7}$	$(F_{n+5})_6 F_{n+9}$	F_{n+5}	$^*(F_{n+4})_5 F_{n+7} F_{n+9}$
and so on.			
$(F_{n+6})_7$	$(F_{n+5})_6 F_{n+7}$	F_{n+4}	$^*(F_{n+3})_4 (F_{n+7})_3$
$(F_{n+6})_7$	$(F_{n+5})_6 F_{n+8}$	F_{n+4}	$^*(F_{n+3})_4 F_{n+5} F_{n+6} F_{n+8}$
and so on.			

GENERAL CONCLUSION

It is possible to express all summations

$$\sum_{n=1}^{\infty} \frac{1}{\prod_{i=1}^{r} F_{n+k_i}}$$

in the form

$$a + b \sum_{n=1}^{\infty} \frac{1}{(F_{n+r-1})_r} ,$$

where a and b are rational numbers; and all summations

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{i=1}^{r} F_{n+k_i}}$$

in the form

$$c + d \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+r-1})_r} ,$$

where again c and d are rational numbers.

The limitation of this approach is that it is not possible to proceed directly in one step to this final result as a rule. It is necessary to go through a series of formulas and should the desired summation be remote from the final objective, this could be a long operation. Once, however, the various formulas have been linked to the one formula, the problem of calculating these summations becomes relatively simple.

This concludes the discussion of linking formulas of the same degree. We now proceed to a consideration of expressing a summation of lower degree in terms of one of higher degree so as to secure more rapid convergence. But first, formulas will be worked out giving upper bounds for the number of terms required to secure a summation result correct to a given number of decimal places.

APPROXIMATING SUMMATIONS WITH GIVEN ACCURACY

Assuming that we have related the summations of a given degree to one summation, it is only necessary to consider summations of the forms

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+q-1})_q} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+q-1})_q} \quad .$$

Two cases will be taken up according as q is even or odd.

<u>q even</u>

From previous discussion,

$$1/F_n < \sqrt{5}/r^n \quad \text{if } n \text{ is odd}$$

$$1/F_n < \sqrt{5}/r^{n-1} \quad \text{if } n \text{ is even.}$$

For

$$\frac{1}{(F_{n+q-1})_q} \quad ,$$

the result depends on the power of r found on the right-hand side of the inequality. These powers can be calculated by table as follows.

n odd	n even
$2n$	$n - 1$
$2(n + 2)$	$2(n + 1)$
$2(n + 4)$	$2(n + 3)$
$\cdots$	$\cdots$
$2(n + q - 2)$	$2(n + q - 3)$
	$n + q - 1$

The sum in either case is

$$qn + \frac{q(q - 2)}{2} \quad .$$

If we want w terms of the summation

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+q-1})_q}$$

to give a result correct to t decimal places, the $(w + 1)^{st}$ term must be less than 5×10^{-t}. Hence the condition for the desired upper bound is:

$$\frac{1}{(F_{w+q})_q} < \frac{5^{q/2}}{r^{q(w+1)+q(q-2)/2}} < 5 \times 10^{-t}$$

which leads to

$$\frac{t + \frac{(q-2)}{2} \log 5 - \frac{q^2}{2} \log r}{q \log r} < w$$

or

$$w > \frac{4.78514}{q} t + \frac{(q-2)}{2q} (3.34467) - \frac{q}{2} \tag{15}$$

For example, if q is 8 and we want the result correct to 10 decimal places,

$$w > .59814t - 2.74575 \text{ or } w > 3.3565 .$$

Hence four terms would be required. Data from this formula will be found in Table V.

For the summation

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+q-1})_q}$$

to have the result correct to t decimal places,

$$\sum_{k=w+1}^{\infty} \frac{1}{(F_{k+q-1})_q} < \sum_{k=w+1}^{\infty} \frac{5^{q/2}}{r^{qk+q(q-2)/2}}$$

$$= \frac{5^{q/2}}{r^{qw+q^2/2}} \left[1 + r^{-q} + r^{-2q} \cdots\right] < 5 \times 10^{-t}$$

or

$$\frac{5^{q/2}}{r^{qw+q^2/2}} \frac{1}{1 - r^{-q}} < 5 \times 10^{-t}$$

This leads to the inequality

$$w > \frac{t + \frac{(q-2)}{2} \log 5 - \log (1 - r^{-q}) - \frac{q^2 \log r}{2}}{q \log r} .$$

The term

$$-\log (1 - r^{-q}) = r^{-q} + \frac{r^{-2q}}{2} + \frac{r^{-3q}}{3} \cdots$$

$$< r^{-q} + r^{-2q} + r^{-3q} \cdots = \frac{1}{r^q - 1} .$$

This replacement is in the safe direction. Hence

(16) $$w > \frac{4.78514t}{q} + \frac{3.34467(q-2)}{2q} - \frac{q}{2} + \frac{1}{(r^q - 1) q \log r} .$$

Similar considerations applied to the case of q odd give the following results. For the summation with alternating terms:

(17) $$w > \frac{4.78514t}{q} + \frac{3.34467\ (q-2)}{2q} - \frac{(q^2 - 1)}{2q} .$$

For the summation with all terms positive:

$$w > \frac{4.78514t}{q} + \frac{3.34467(q-2)}{2q} + \frac{4.78514}{q(r^q - 1)} - \frac{(q^2-1)}{2q} \quad . \tag{18}$$

Table V

UPPER BOUNDS FOR THE NUMBER OF TERMS REQUIRED FOR RESULTS TO t DECIMAL PLACES FOR THE SUMMATION

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+q-1})_q}$$

q ⟶

t	2	4	6	8	10	12	14	16	18	20
5	11	5	3	1						
10	23	11	7	4	2					
15	35	17	11	7	4	2				
20	47	23	15	10	6	4	2			
25	59	29	19	13	9	6	3	1		
30	71	35	23	16	11	8	5	3	1	
50	119	59	38	28	21	16	12	9	6	4
100	239	119	78	58	45	36	29	24	20	16

Table VI

UPPER BOUNDS FOR THE NUMBER OF TERMS REQUIRED FOR RESULTS TO t DECIMAL PLACES FOR THE SUMMATION

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+q-1})_q}$$

q

t	2	4	6	8	10	12	14	16	18	20
5	13	6	3	1						
10	25	12	7	4	2					
15	37	17	11	7	4	2				
20	49	23	15	10	6	4	2			
25	61	29	19	13	9	6	3	1		
30	73	35	23	16	11	8	5	3	1	
50	121	59	39	28	21	16	12	9	6	4
100	240	119	78	58	45	36	29	24	20	16

Table VII
UPPER BOUNDS FOR THE NUMBER OF TERMS REQUIRED FOR RESULTS TO t DECIMAL PLACES FOR THE SUMMATION

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+q-1})_q}$$

	q									
t	1	3	5	7	9	11	13	15	17	19
5	23	8	4	2						
10	47	16	9	5	3	1				
15	71	24	13	9	5	3	1			
20	95	32	18	12	8	5	3	1		
25	119	40	23	15	11	7	5	2	1	
30	143	48	28	19	13	9	6	4	2	
50	239	79	47	32	24	18	14	10	8	5
100	478	159	95	67	51	40	32	26	22	18

Table VIII
UPPER BOUNDS FOR THE NUMBER OF TERMS REQUIRED FOR RESULTS TO t DECIMAL PLACES FOR THE SUMMATION

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+q-1})_q}$$

t	1	3	5	7	9	11	13	15	17	19
5	31	8	4	2						
10	55	16	9	5	3	1				
15	79	24	14	9	5	3	1			
20	103	32	18	12	8	5	3	1		
25	127	40	23	15	11	7	5	2	1	
30	151	48	28	19	13	9	6	4	2	
50	246	80	47	32	24	18	14	10	8	5
100	486	160	95	67	51	40	32	26	22	18

These tables indicate impressively the gain in efficiency obtained by expressing lower-degree summations in terms of a higher degree summation. The method of achieving this will now be taken up.

LOWER DEGREE SUMMATIONS
IN TERMS OF HIGHER DEGREE SUMMATIONS

The program to be carried out illustrating this process will consist in starting with

$$\sum_{n=1}^{\infty} 1/F_n$$

and establishing a chain of formulas reaching to

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9} .$$

The first step in this chain is found in a result given in the Fibonacci Quarterly [2], namely:

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+2})_3} .$$

The next step is as follows.

(19) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+2})_3} - 3\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+4}} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{n+1} F_{n+2} F_{n+4}}$$

$$= -\frac{1}{1 \cdot 1 \cdot 3} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+3}}$$

$$= -1/3 + 1/4 = -1/12 .$$

It is possible to express

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+4}}$$

in terms of

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_5} .$$

Starting with

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{F_n F_{n+1} F_{n+3}} + \frac{1}{F_{n+1} F_{n+2} F_{n+4}} \right] = \frac{1}{1 \cdot 1 \cdot 3}$$

and noting that

$$F_{n+2} F_{n+4} + F_n F_{n+3} = (-1)^{n-1} + 2F_{n+2} F_{n+3}$$

$$1/3 = \sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_5} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+4}} .$$

Hence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+4}} = \frac{1}{6} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_5} .$$

Substituting into (19),

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+2})_3} = \frac{5}{12} - \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_5} . \tag{20}$$

The next step is as follows.

$$\sum_{n=1}^{\infty}\left[\frac{1}{(F_{n+3})_4 F_{n+5}} - \frac{1}{(F_{n+4})_4 F_{n+6}}\right] = \frac{1}{1\cdot 1\cdot 2\cdot 3\cdot 8}$$

$$= \sum_{n=1}^{\infty}\left[\frac{F_{n+4}F_{n+6} - F_n F_{n+5}}{(F_{n+6})_7}\right]$$

$$= 2\sum_{n=1}^{\infty}\frac{1}{(F_{n+2})_3 F_{n+5} F_{n+6}} + 3\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+1} F_{n+3} F_{n+5} F_{n+6}}$$

$$+ 3\sum_{n=1}^{\infty}\frac{(-1)^n}{(F_{n+6})_7} .$$

We shall not derive the relations for the fifth-degree summations in terms of

$$\sum_{n=1}^{\infty}\frac{1}{(F_{n+4})_5} = S ,$$

but simply state them.

$$\sum_{n=1}^{\infty}\frac{1}{(F_{n+2})_3 F_{n+5} F_{n+6}} = \frac{3S}{20} - \frac{1}{4800}$$

$$\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+1} F_{n+3} F_{n+5} F_{n+6}} = \frac{7S}{40} - \frac{29}{9600} .$$

Substitution leads to the result

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_5} = \frac{97}{2640} + \frac{40}{11} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+6})_7} \quad . \tag{21}$$

The final stage is as follows.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{(F_{n+5})_6 \, F_{n+7}} + \frac{1}{(F_{n+6})_6 \, F_{n+8}} \right] = \frac{1}{1 \cdot 1 \cdot 2 \cdot 3 \cdot 5 \cdot 8 \cdot 21} \; .$$

The numerator of the combined terms inside the brackets is

$$F_{n+6} \, F_{n+8} + F_n \, F_{n+7} = 4 \, F_{n+4} \, F_{n+6} + 5 \, F_{n+4} \, F_{n+5} + 3(-1)^{n-1} \; .$$

Letting

$$A = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+3})_4 \, F_{n+5} \, F_{n+7} \, F_{n+8}}$$

$$B = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+3})_4 \, (F_{n+8})_3}$$

and

$$C = \sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9} \quad ,$$

the result can be written as

$$\frac{1}{1 \cdot 1 \cdot 2 \cdot 3 \cdot 5 \cdot 8 \cdot 21} = 4A + 5B + 3C \; .$$

Similarly,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(F_{n+3})_4 (F_{n+7})_3} + \frac{1}{(F_{n+4})_4 (F_{n+8})_3} = \frac{1}{1\cdot 1\cdot 2\cdot 3\cdot 8\cdot 13\cdot 21}.$$

The numerator of the combined terms within the brackets is

$$F_{n+4}F_{n+8} + F_n F_{n+5} = -F_{n+4}F_{n+6} + 6F_{n+4}F_{n+5} + 3(-1)^{n-1}$$

leading to:

$$\frac{1}{1\cdot 1\cdot 2\cdot 3\cdot 8\cdot 13\cdot 21} = -A + 6B + 3C \ .$$

Solving with the previous relation in A, B, and C gives:

$$A = \frac{53}{1900080} - \frac{3C}{29} \ .$$

It can also be shown that

$$A = \frac{1}{91}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+6})_7} + \frac{73}{2981160} \ .$$

This enables us to arrive at the final conclusion

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+6})_7} = \frac{589}{1900080} - \frac{273}{29}\sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9} \ . \tag{22}$$

Thus a chain has been extended from

$$\sum_{n=1}^{\infty} \frac{1}{F_n} \quad \text{to} \quad \sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9} \ .$$

Connecting the initial and terminal links

$$(23)\qquad \sum_{n=1}^{\infty} \frac{1}{F_n} = \frac{46816051}{13933920} + \frac{16380}{319} \sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9}$$

Another advantage of a summation such as

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9}$$

is that it lends itself readily to calculation. At each stage one factor is added to one end of the denominator and deleted from the other. Table IX shows the calculation of this summation to thirty plus decimal places. The factor applied at each stage to the result on the preceding line is shown at the left.

Table IX

CALCULATION OF $\sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9}$

Term	Factor	Term Multiplied by 10^{33}					
1		4488	97507	72103	71328	01838	684
2	1/55	81	61772	86765	52205	96397	067
3	1/89		91705	31311	97215	79734	799
4	2/144		1273	68490	44405	77496	317
5	3/233		16	39937	64520	24602	957
6	5/377			21749	83614	32687	042
7	8/610			285	24375	26986	060
8	13/987			3	75700	99139	634
9	21/1597				4940	33864	704
10	34/2584				65	00445	588
11	55/4181					85511	721
12	89/6765					1124	988
13	144/10946					14	800
14	233/17711						195
	SUM	4571	52276	20648	18372	59844	456

With the aid of this value

$$\sum_{n=1}^{\infty} \frac{1}{F_n}$$

is found to be to twenty-five decimal places

$$3.35988\ 56662\ 43177\ 55317\ 20113 \quad .$$

CONCLUSION

In this paper two types of infinite Fibonacci series have been considered. Methods have been developed for expressing series of the same degree in terms of one series of that degree. In addition a path has been indicated for proceeding from series of lower degree to those of higher degree so that more rapid convergence may be attained. These two approaches plus the development of closed formulas in a previous article should provide an open door for additional research and calculation along the lines of sums of reciprocals of Fibonacci series of various types.

REFERENCES

1. Brother Alfred Brousseau, "Fibonacci-Lucas Infinite Series Research," the Fibonacci Quarterly, Vol. 7, No. 2, pp. 211-217.
2. The Fibonacci Quarterly, Problem H-10, April 1963, p. 53; Solution to H-10, the Fibonacci Quarterly, Dec. 1963, p. 49.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

Editorial Note: Keep those problem proposals coming, Folks!

H-153 *Proposed by J. Ramanna, Government College, Mercara, India.*

Show that

(i)
$$4\sum_{0}^{n} F_{3k+1}F_{3k+2}\,(2F_{3k+1}^2 + F_{6k+3})(2F_{3k+2}^2 + F_{6k+3}) = F_{3n+3}^6$$

(ii)
$$16\sum_{0}^{n} F_{3k+1}\,F_{3k+2}\,F_{6k+3}\,(2F_{6k+3}^2 - F_{3k}^2\,F_{3k+3}^2) = F_{3n+3}^8\ .$$

Hence generalize (i) and (ii) for F_{3n+3}^{2r}.

H-154 *Proposed by L. Carlitz, Duke University, Durham, North Carolina.*

Show that for $m,\ n,\ p$ integers ≥ 0,

$$\sum_{i,j,k\geq 0} \binom{m+1}{j+k+1}\binom{n+1}{i+k+1}\binom{p+1}{i+j+1}$$
$$= \sum_{a=0}^{m}\sum_{b=0}^{n}\sum_{c=0}^{p}\binom{m-a+b}{b}\binom{n-b+c}{c}\binom{p-c+a}{a},$$

and generalize.

H-155 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.*

The Fibonacci polynomials are defined by

$$f_{n+1}(x) = x f_n(x) + f_{n-1}(x)$$

with $f_1(x) = 1$ and $f_2(x) = x$. Let $z_{r,s} = f_r(x)f_s(y)$. If $z_{r,s}$ satisfies the relation

$$z_{r+4,s+4} + a\, z_{r+3,s+3} + b\, z_{r+2,s+2} + c\, z_{r+1,s+1} + d\, z_{r,s} = 0 ,$$

show that

$$a = c = -xy, \quad b = -(x^2 + y^2 + 2) \quad \text{and} \quad d = 1.$$

H-156 *Proposed by L. Carlitz, Duke University, Durham, North Carolina.*

Prove the identity

$$\sum_{n=0} \frac{q^{n^2} z^n}{(q)_n} \prod_{k=1}^{n} (1 - q^k) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q)_{2k}} z^{-k}$$

$$- \sum_{n=-\infty}^{\infty} q^{n(n+1)} z^n \sum_{k=0}^{\infty} \frac{q^{(k+1)^2}}{(q)_{2k+1}} z^{-k} ,$$

where

$$(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n) .$$

H-157 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.*

A set of polynomials $c_n(x)$, which appears in network theory is defined by

$$c_{n+1}(x) = (x + 2)c_n(x) - c_{n-1}(x) \qquad (n \geq 1)$$

with

$$c_0(x) = 1 \qquad \text{and} \qquad c_1(x) = (x + 2)/2 .$$

(a) Find a polynomial expression for $c_n(x)$.

(b) Show that

$$2c_n(x) = b_n(x) + b_{n-1}(x) = B_n(x) - B_{n-1}(x) ,$$

where $B_n(x)$ and $b_n(x)$ are the Morgan-Voyce polynomials as defined in the Fibonacci Quarterly, Vol. 5, No. 2, p. 167.

(c) Show that $2c_n^2(x) - c_{2n}(x) = 1$.

(d) If

$$Q = \begin{bmatrix} (x+2) & -1 \\ 1 & 0 \end{bmatrix} ,$$

show that

$$\begin{bmatrix} c_n & -c_{n-1} \\ c_{n-1} & -c_{n-2} \end{bmatrix} = \frac{1}{2}(Q^n - Q^{n-2}) \quad \text{for} \quad (n \geq 2) .$$

Hence deduce that $c_{n+1}\, c_{n-1} - c_n^2 = x(x + 4)/4$.

SOLUTIONS
AT LAST

H-98 *Proposed by George Ledin, Jr., San Francisco, California.*

If the sequence of integers is designated as J, the ring identity as I, and the quasi-inverse of J as F, then $(I - J)(I - F) = I$ should be satisfied. For further information see R. G. Buschman, "Quasi Inverses of Sequences," American Mathematical Monthly, Vol. 73, No. 4, III (1966), p. 134.

Find the quasi-inverse sequence of the integers (negative, positive, and zero).

Solution by the proposer.

The sequence $u_{n+2} = au_{n+1} + bu_n$ with initial conditions $u_0 \neq 1$, u_1, has the quasi-inverse

$$v_{n+2} = Av_{n+1} + Bv_n ,$$

where

$$A = a + u_1 / (1 - u_0), \ B = b/(1 - u_0)$$

with initial conditions

$$v_0 = -u_0 / (1 - u_0), \ v_1 = -u_1/(1-u_0)^2 .$$

Since the sequence of integers is defined by the recurrence relation

$$u_{n+2} = 2u_{n+1} - u_n$$

with initial conditions $u_0 = 0$, $u_1 = 1$, its quasi-inverse is then

$$v_{n+2} = 3v_{n+1} - v_n$$

with initial conditions $v_0 = 0$, $v_1 = -1$ which yields

$$0, -1, -3, -8, -21, -55, -144, -377, \cdots, -F_{2n}, \cdots .$$

SUM PRODUCT!

H-120 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.*

The Fibonacci polynomials are defined by

$$f_{n+1}(x) = x \cdot f_n(x) + f_{n-1}(x)$$
$$f_1(x) = 1, \quad f_2(x) = x .$$

If $z_r = f_r(x) \cdot f_r(y)$, then show that

(i) z_r satisfies the recurrence relation,

$$z_{n+4} = xy \cdot z_{n+3} - (x^2 + y^2 + 2)z_{n+2} - xy \cdot z_{n+1} + z_n = 0 .$$

(ii) $$(x + y)^2 \cdot \sum_1^n z_r = (z_{n+2} - z_{n-1}) - (xy - 1)(z_{n+1} - z_n) .$$

Solution by C.B.A. Peck, Ordnance Research Laboratory, State College, Pennsylvania.

(i)
$$\begin{aligned} z_{n+4} &= f_{n+4}(x)f_{n+4}(y) \\ &= (xf_{n+3}(x) + f_{n+2}(x))\,(yf_{n+3}(y) + f_{n+2}(y)) \\ &= xyz_{n+3} + xf_{n+3}(x)\, f_{n+2}(y) + yf_{n+3}(y)\, f_{n+2}(x) \neq z_{n+2} \\ &= xyz_{n+3} + (x^2 + y^2 + 2)\, z_{n+2} - z_{n+2} \\ &\quad + xf_{n+1}(x)\, f_{n+2}(y) + yf_{n+1}(y)\, f_{n+2}(x) \end{aligned}$$

so that

$$\begin{aligned} z_{n+4} - xyz_{n+3} - (x^2 + y^2 + 2)z_{n+2} &= -xyz_{n+1} - xf_{n+1}(x)\, f_n(y) \\ &- yf_{n+1}(y)f_n(x) - z_n + xyz_{n+1} + xf_{n+1}(x)f_n(y) + xyz_{n+1} \\ &+ yf_{n+1}(y)\, f_n(x) = xyz_{n+1} - z_n, \end{aligned}$$ as desired.

(ii) $n = 2$: by expansion,

$$(x + y)^2(1 + xy) = (x^3 + 2x)(y^3 + 2y) - 1 - (xy - 1)(\,(x^2 + 1)(y^2 + 1) - xy).$$

Thus for an inductive proof we need only to show the r.h. and l.h. increments equal. The r.h. one is

$$\begin{aligned} z_{n+2} - z_{n-1} - (xy - 1)(z_{n+1} - z_n) - z_{n+1} + z_{n-2} + (xy - 1)(z_n - z_{n-1}) \\ = z_{n+2} - xyz_{n+1} + 2(xy - 1)z_n - xyz_{n-1} + z_{n-2}, \end{aligned}$$

which by (i) is

$$(x^2 + y^2 + 2)z_n + 2(xy - 1)z_n = (x + y)^2 z_n ,$$

the l.h. one.

Also solved by the proposer, B. King, A. Shannon, L. Carlitz, and C. Bridger.

IN SUMMATION

H-121 *Proposed by H.H. Ferns, University of Victoria, Victoria, B.C., Canada.*

Prove the following identity.

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{F_k}{F_{m-k}}\right)^i F_{mi+\lambda} = \left(\frac{F_m}{F_{m-k}}\right)^n F_{nk+\lambda} - F_\lambda \qquad (m \neq k),$$

where F_n is the n^{th} Fibonacci number, m, λ are any integers or zero and k is an even integer or zero.

Write the form the identity takes if k is an odd integer.

Find an analogous identity involving Lucas numbers.

Solution by the proposer.

The following identities will be required.

$$\text{(1)} \qquad \alpha^k F_m - \alpha^m F_k = (-1)^k F_{m-k}$$

$$\text{(2)} \qquad \beta^k F_m - \beta^m F_k = (-1)^k F_{m-k},$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$ and $F_n = (\alpha^n - \beta^n)/\sqrt{5}$.

The proof of (1) follows. The proof of (2) is similar

$$\begin{aligned}
\alpha^k F_m - \alpha^m F_k &= \alpha^k\left(\frac{\alpha^m - \beta^m}{\sqrt{5}}\right) - \alpha^m\left(\frac{\alpha^k - \beta^k}{\sqrt{5}}\right) \\
&= \frac{\alpha^{m+k} - \alpha^k\beta^m - \alpha^{m+k} + \alpha^m\beta^k}{\sqrt{5}} \\
&= \alpha^k\beta^k\,\frac{\alpha^{m-k} - \beta^{m-k}}{\sqrt{5}} \\
&= (-1)^k F_{m-k} \\
&= F_{m-k},
\end{aligned}$$

since k is even.

Identities (1) and (2) may be written as follows:

$$1 + \left(\frac{F_k}{F_{m-k}}\right)\alpha^m = \left(\frac{F_m}{F_{m-k}}\right)\alpha^k \qquad (m \neq k) \tag{3}$$

$$1 + \left(\frac{F_k}{F_{m-k}}\right)\beta^m = \left(\frac{F_m}{F_{m-k}}\right)\beta^k \qquad (m \neq k) \ . \tag{4}$$

Let

$$\mu = \frac{F_k}{F_{m-k}} \quad \text{and} \quad \nu = \frac{F_m}{F_{m-k}} \ .$$

From (3) and (4) we derive the following:

$$(1 + \mu\alpha^m)^n - (1 + \mu\beta^m)^n = \nu^n(\alpha^{nk} - \beta^{nk}) \tag{5}$$

$$(1 + \mu\alpha^m)^n + (1 + \mu\beta^m)^n = \nu^n(\alpha^{nk} + \beta^{nk}) \ . \tag{6}$$

From (5) we get

$$\sum_{i=1}^{n}\binom{n}{i}\mu^i(\alpha^{mi} - \beta^{mi}) = \nu^n(\alpha^{nk} - \beta^{nk})$$

$$\sum_{i=1}^{n}\binom{n}{i}\mu^i F_{mi} = \nu^n F_{nk} \ . \tag{7}$$

If L_n denotes the n^{th} Lucas number then $L_n = \alpha^n + \beta^n$ and from (6) we obtain

$$\sum_{i=1}^{n}\binom{n}{i}\mu^i L_{mi} = \nu^n L_{nk} - 2 \ . \tag{8}$$

We now add corresponding members of (7) and (8) and simplify the result by applying the identity

$$F_n + L_n = 2F_{n+1} .$$

This gives

$$\sum_{i=1}^{n} \binom{n}{i} \mu^i F_{mi+1} = \nu^n F_{nk+1} - 1 . \tag{9}$$

Adding corresponding members of (7) and (9) and applying the recursion formula

$$F_n + F_{n+1} = F_{n+2}$$

to the result yields

$$\sum_{i=1}^{n} \binom{n}{i} \mu^i F_{mi+2} = \nu^n F_{nk+2} - 1 . \tag{10}$$

Repeating the last operation on (8) and (9) and on each successive pair of identities derived in this manner we get

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{F_k}{F_{m-k}}\right)^i F_{mi+\lambda} = \left(\frac{F_m}{F_{m-k}}\right)^n F_{nk+\lambda} - F_\lambda \qquad (m \neq k)$$

If k is an odd integer this identity takes the form

$$\sum_{i=1}^{n} (-1)^i \binom{n}{i} \left(\frac{F_k}{F_{m-k}}\right)^i F_{mi+\lambda} = \left(\frac{-F_m}{F_{m-k}}\right)^n F_{nk+\lambda} - F_\lambda \qquad (m \neq k) .$$

Beginning with the two identities

$$\alpha^k L_m - \sqrt{5}\,\alpha^m F_k = (-1)^k L_{m-k}$$
$$\beta^k L_m + \sqrt{5}\,\beta^m F_k = (-1)^k L_{m-k}$$

and following the procedure adopted above we arrive at the identity

$$\sum_{i=1}^{n} \binom{n}{i} 5^{\left[\frac{i+1}{2}\right]} \left(\frac{F_k}{L_{m-k}}\right)^i \phi_m^i = \left(\frac{L_m}{L_{m-k}}\right)^n L_{nk+\lambda} - L_\lambda$$

where

$$\phi_m^i = \begin{cases} F_{mi+\lambda} & \text{if } i \text{ is odd} \\ L_{mi+\lambda} & \text{if } i \text{ is even} \end{cases}$$

and k is an even integer or zero. If k is an odd integer this identity takes the form

$$\sum_{i=1}^{n} \binom{n}{i} 5^{\left[\frac{i+1}{2}\right]} \left(\frac{F_k}{L_{m-k}}\right)^i (-\phi_m)^i = \left(\frac{-L_m}{L_{m-k}}\right)^n L_{nk+\lambda} - L_\lambda$$

Examples. If $\lambda = 0$, $m = 1$, $k = 2$ the first identity gives us the well-known formula

$$\sum_{i=1}^{n} \binom{n}{i} F_i = F_{2n} \quad .$$

The same values for these parameters when substituted in the second identity gives the not-so-well-known formula

$$-\binom{n}{1} 5F_1 + \binom{n}{2} 5L_2 - \binom{n}{3} 5^2F_3 + \binom{n}{4} 5^2L_4 - \binom{n}{5} 5^3F_5 + \cdots = (-1)^n L_{2n} - 2.$$

Also solved by L. Carlitz, and A. Shannon.

STIRLING PERFORMANCE

H-123 *Proposed by D. Lind, University of Virginia, Charlottesville, Virginia*

Prove

$$F_n = \sum_{m=0}^{n} \sum_{k=0}^{m} \mathcal{S}_n^{(m)} S_m^{(k)} F_k ,$$

where $S_r^{(s)}$ and $\mathcal{S}_r^{(s)}$ are Stirling numbers of the first and second kinds, respectively, and F_n is the n^{th} Fibonacci number.

Solution by the proposer.

Stirling numbers are defined by

$$x(x-1)\cdots(x-m+1) = \sum_{m=0}^{n} S_n^{(m)} x^m$$

$$x^n = \sum_{m=0}^{n} \mathcal{S}_n^{(m)} x(x-1)\cdots(x-m+1) .$$

Letting $a = (1+\sqrt{5})/2$, $b = (1-\sqrt{5})/2$, we have

$$a^n = \sum_{m=0}^{n} \mathcal{S}_n^{(m)} a(a-1)\cdots(a-m+1)$$

(1)

$$= \sum_{m=0}^{n} \mathcal{S}_n^{(m)} \sum_{k=0}^{m} S_m^{(k)} a^k .$$

Similarly,

$$b^n = \sum_{m=0}^{n}\sum_{k=0}^{m} \mathcal{S}_n^{(m)} S_m^{(k)} b^k .$$

It follows

$$(a^n - b^n)/\sqrt{5} = \sum_{m=0}^{n}\sum_{k=0}^{m} \mathcal{S}_n^{(m)} S_m^{(k)} (a^k - b^k)/\sqrt{5} ,$$

which is the desired result.

[(1) may be found in Jordan's Calculus of Finite Differences, page 183.]

Also solved by David Zeitlin.

BINET?

H-124 *(Corrected). Proposed by J. A. H. Hunter, Toronto, Canada.*

Prove the following identity:

$$F^2_{m+n}L^2_{m+n} - F^2_m L^2_m = F_{2n}F_{2+(2m+n)} ,$$

where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively.

Solution by Paul Smith, University of Victoria, Victoria, B.C., Canada.

A routine computation shows that:

$$F^2_{m+n}L^2_{m+n} - F^2_m L^2_m = \frac{[(\alpha^{m+n} - \beta^{m+n})(\alpha^{m+n} + \beta^{m+n})]^2 - [(\alpha^m - \beta^m)(\alpha^m + \beta^m)]^2}{(\alpha - \beta)^2}$$

$$= \frac{(\alpha^{4(m+n)} + \beta^{4(m+n)} - 1) - (\alpha^{4m} + \beta^{4m} - 1)}{(\alpha - \beta)^2}$$

$$= \frac{(\alpha^{4(m+n)} + \beta^{4(m+n)}) - \alpha^{2n}\beta^{2n}(\alpha^{4m} + \beta^{4n})}{(\alpha - \beta)^2}$$

$$= \frac{(\alpha^{2n} - \beta^{2n})}{(\alpha - \beta)} \cdot \frac{(\alpha^{2(2m+n)} - \beta^{2(2m+n)})}{(\alpha - \beta)}$$

$$= F_{2n}F_{2(2m+n)} .$$

(It is merely necessary to observe that $\alpha\beta = -1$.)

Also solved by C. Bridger, M. Bicknell, A. Shannon, C.B.A. Peck, J. Wessner, F. D. Parker, M. N. S. Swamy, and R. Whitney.

* * * * *

A FOUR-STEP ITERATION ALGORITHM TO GENERATE x in $x^2 + (x+1)^2 = y^2$

EDGAR KARST

University of Arizona, Tuscon, Arizona

Given $x_1 = 3$, $x_2 = 20$, $x_3 = 119$, $x_4 = 696$, and $x_5 = 4059$, we may generate all further x by the simple procedure outlined below:

$20 - 3 = 17 = 4^2 + 1$

$119 - 20 = 99 = 10^2 - 1$

$696 - 119 = 577 = 24^2 + 1$

$4059 - 696 = 3363 = 58^2 - 1$

$6 \cdot 24 = 144,\ 144 - 4 = 140,\ 140^2 + 1 = 19601,\ 19601 + 4059 = 23660 = x_6$

$6 \cdot 58 = 348,\ 348 - 10 = 338,\ 338^2 - 1 = 114243,\ 114243 + 23660 = 137903 = x_7$

$6 \cdot 140 = 840,\ 840 - 24 = 816,\ 816^2 + 1 = 665857,\ 665857 + 137903 = 803760 = x_8$

$6 \cdot 338 = 2028,\ 2028 - 58 = 1970,\ 1970^2 - 1 = 3880899,\ 3880899 + 803760 = 4684659 = x_9$

The author has taken time to check some of the newer lists against print errors. The list in [1, p. 123] should read $y_8 = 1136689$ (instead of 113689). The last column of the list in [2, p. 284] gives the first differences up to $x_{20} - x_{19}$. There are no print errors. The list in [3, p. 104] should read $x_6 = 23660$ (instead of 23360) and $x_{16} = 1070379110496$ (instead of 1070387585472), and correspondingly in the column $x + 1$ there.

REFERENCES

1. Albert H. Beiler, Recreations in the Theory of Numbers, New York, 1964.
2. Otto Emersleben, Uber zweite Binomialkoeffizienten, die Quadratzahlen sind, und Anwendung der Pellschen Gleichung auf Gitterpunktanordnungen. Wissench. Zeitschr. der Ernst-Moritz-Arndt-Universitat Greifswald, XVI (1967), pp. 279-296.
3. T. W. Forget and T. A. Larkin, "Pythagorean Triads of the Form x, x + 1, z Described by Recurrence Sequences," the Fibonacci Quarterly, Vol. 6 (June 1968), pp. 94-104.

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THREE DIOPHANTINE EQUATIONS – PART II*

IRVING ADLER
North Bennington, Vermont

6. THE PELL EQUATIONS

Equation (3) is the special case $d = 2$ of the equation

$$s^2 - dt^2 = 1 , \tag{18}$$

where d is positive and is not a perfect square. Equation (18) is known as the Pell equation. Another way of solving Eq. (3) is provided by the following theorem concerning the Pell equations found in most text books on the theory of numbers. (For a proof of the theorem, see [2].)

Theorem: If (s_1, t_1) is the minimal positive solution of Eq. (18), then every positive solution is given by

$$s_n + t_n\sqrt{d} = (s_1 + t_1\sqrt{d})^n, \quad n > 0. \tag{19}$$

(A solution (s,t) is called positive if $s > 0$, $t > 0$.) The minimal positive solution of Eq. (3) is $(3,2)$. Then, according to this theorem, all positive solutions are given by

$$s_n + t_n\sqrt{2} = (3 + 2\sqrt{2})^n, \quad n = 1, 2, 3, \cdots . \tag{20}$$

Equations (15) and (16) are easily derived from Eq. (20) as follows:

$$\begin{aligned} s_n + t_n\sqrt{2} = (3 + 2\sqrt{2})^n &= (3 + 2\sqrt{2})^{n-1}(3 + 2\sqrt{2}) = (s_{n-1} + t_{n-1}\sqrt{2})(3 + 2\sqrt{2}) \\ &= (3s_{n-1} + 4t_{n-1}) + (2s_{n-1} + 3t_{n-1})\sqrt{2} . \end{aligned}$$

Therefore $s_n = 3s_{n-1} + 4t_{n-1}$, and $t_n = 2s_{n-1} + 3t_{n-1}$.

7. RECURRENCE RELATIONS

If (x_n, z_n) is one of the sequence of non-negative solutions of Eq. (1) with $n \geq 2$, we can derive from Eqs. (7) and (8) a formula that expresses x_n

*Part I appeared in the December 1968 Issue.

as a linear function of x_{n-1} and x_{n-2}, and a formula that expresses z_n as a linear function of z_{n-1} and z_{n-2}. If we replace n by n - 1 in Eqs. (7) and (8), we get

$$x_{n-1} = 3x_{n-2} + 2z_{n-2} + 1 , \tag{21}$$

$$z_{n-1} = 4x_{n-2} + 3z_{n-2} + 2 . \tag{22}$$

From (21) and (22) we get

$$2z_{n-2} = x_{n-1} - 3x_{n-2} - 1 , \tag{23}$$

$$4x_{n-2} = z_{n-1} - 3z_{n-2} - 2 . \tag{24}$$

Then, from Eqs. (7), (22) and (23),

$$x_n = 3x_{n-1} + 2z_{n-1} + 1 .$$

$$x_n = 3x_{n-1} + 2(4x_{n-2} + 3z_{n-2} + 2) + 1 .$$

$$x_n = 3x_{n-1} + 8x_{n-2} + 6z_{n-2} + 5.$$

$$x_n = 3x_{n-1} + 8x_{n-2} + 3(x_{n-1} - 3x_{n-2} - 1) + 5 .$$

$$x_n = 6x_{n-1} - x_{n-2} + 2 . \tag{25}$$

Similarly, from Eqs. (8), (21) and (24),

$$z_n = 4x_{n-1} + 3z_{n-1} + 2 .$$

$$z_n = 4(3x_{n-2} + 2z_{n-2} + 1) + 3z_{n-1} + 2 .$$

$$z_n = 12x_{n-2} + 8z_{n-2} + 3z_{n-1} + 6.$$

$$z_n = 3(z_{n-1} - 3z_{n-2} - 2) + 8z_{n-2} + 3z_{n-1} + 6 .$$

(26) $$z_n = 6z_{n-1} - z_{n-2} \ .$$

EXERCISES

5. Let (u_n, v_n) be the n^{th} solution in positive integers of Eq. (2), $n \geq 2$. Use Eqs. (12) and (13) to derive the recurrence relations

(27) $$u_n = 6u_{n-1} - u_{n-2} + 2 \ ,$$

(28) $$v_n = 6v_{n-1} - v_{n-2} \ .$$

6. Let (s_n, t_n) be the n^{th} solution in positive integers of Eq. (3), $n \geq 2$. Use Eqs. (15) and (16) to derive the recurrence relations

(29) $$s_n = 6s_{n-1} - s_{n-2} \ ,$$

(30) $$t_n = 6t_{n-1} - t_{n-2} \ .$$

8. CLOSED FORMULAS

If a sequence $y_0, y_1, y_2, \cdots, y_7, \cdots$ is defined by specifying the values of the first few terms and determining the values of the rest inductively by means of a linear recurrence relation, then there is a standard technique for finding a formula that expresses y_n in terms of n. For example, it can be shown that if the recurrence relation is the equation

(31) $$y_{n+2} - 6y_{n+1} + y_n = 0 \ ,$$

then

(32) $$y_n = c_1 r_1^n + c_2 r_2^n \ ,$$

where r_1 and r_2 are the roots of the characteristic equation

(33) $$E^2 - 6E + 1 = 0 \ ,$$

and the constants c_1 and c_2 are determined by the values of y_1 and y_2. (See [3] for a proof of this assertion.) The roots of (33) are $3 + 2\sqrt{2}$ and $3 - 2\sqrt{2}$. So in this case

$$y_n = c_1(3 + 2\sqrt{2})^n + c_2(3 - 2\sqrt{2})^n \tag{34}$$

The recurrence relations for z_n, v_n, s_n and t_n all have the form (31) with characteristic equation (33). Hence the closed formulas for z_n, v_n, s_n and t_n all have the form of Eq. (34), and differ only in the values of the constants c_1 and c_2. To determine the constants in the formula

$$z_n = c_1(3 + 2\sqrt{2})^n + c_2(3 - 2\sqrt{2})^n ,$$

we make use of the fact that $z_0 = 1$ and $z_1 = 5$. Then

$$1 = c_1(3 + 2\sqrt{2})^0 + c_2(3 - 2\sqrt{2})^0 ,$$
$$5 = c_1(3 + 2\sqrt{2})^1 + c_2(3 - 2\sqrt{2})^1 .$$

Therefore $c_1 + c_2 = 1$ and $c_1 - c_2 = \frac{1}{2}\sqrt{2}$. Consequently, $c_1 = \frac{1}{4}(2 + \sqrt{2}), c_2 = \frac{1}{4}(2 - \sqrt{2})$, and

$$a = \tfrac{1}{4}\left[(2 + \sqrt{2})\ (3 + 2\sqrt{2})^n + (2 - \sqrt{2})\ (3 - 2\sqrt{2})^n\right] . \tag{35}$$

EXERCISES

7. Determine the values of c_1 and c_2 in each of these closed formulas:

$$s_n = c_1(3 + 2\sqrt{2})^n + c_2(3 - 2\sqrt{2})^n ; \tag{36}$$

$$t_n = c_1(3 + 2\sqrt{2})^n + c_2(3 - 2\sqrt{2})^n ; \tag{37}$$

$$v_n = c_1(3 + 2\sqrt{2})^n + c_2(3 - 2\sqrt{2})^n . \tag{38}$$

It can be shown that if the recurrence relation defining a sequence $\{y_n\}$ is the non-homogeneous equation

(39) $$y_{n+2} - 6y_{n+1} + y_n = 2 ,$$

then

(40) $$y_n = c_1r_1^n + c_2r_2^n - \tfrac{1}{2} ,$$

where r_1 and r_2 are the roots of (33), and c_1 and c_2 are determined by the values of y_0 and y_1. The recurrence relations for x_n and u_n have the form of (40). Hence the closed formulas for x_n and u_n, after evaluation of the constants c_1 and c_2, are

(41) $$x_n = \tfrac{1}{4}\left[(1 + \sqrt{2})(3 + 2\sqrt{2})^n + (1 - \sqrt{2})(3 - 2\sqrt{2})^n - 2\right] ,$$

(42) $$u_n = \tfrac{1}{4}\left[(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n - 2\right] .$$

9. HOW EQUATIONS (1), (2), AND (3) ARE RELATED TO EACH OTHER

The sequence of non-negative integers $\{z_n\}$, $\{v_n\}$ and $\{t_n\}$ which arise in the solution of Eqs. (1), (2) and (3), respectively, all satisfy the same recurrence relation (31). This shows that the solutions of Eqs. (1), (2) and (3) are intimately related to each other. We shall now derive the equations that relate them to each other from the closed formulas for x_n, z_n, s_n, t_n, u_n and v_n. The formulas for z_n, x_n and u_n are Eqs. (35), (41) and (42), respectively. The formulas for s_n, t_n and v_n obtained in Exercise 7 are

(36') $$s_n = \tfrac{1}{2}\left[(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n\right] ,$$

(37') $$t_n = \frac{\sqrt{2}}{4}\left[(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n\right],$$

(38') $$v_n = \frac{\sqrt{2}}{8}\left[(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n\right] .$$

By solving Eqs. (42) and (38') for $(3 + 2\sqrt{2})^n$ and $(3 - 2\sqrt{2})^n$, respectively, we find

(43) $$(3 + 2\sqrt{2})^n = 2u_n + 2\sqrt{2}v_n + 1 ,$$

(44) $$(3 - 2\sqrt{2})^n = 2u_n - 2\sqrt{2}v_n + 1 .$$

Making these substitutions for $(3 + 2\sqrt{2})^n$ and $(3 - 2\sqrt{2})^n$ in Eqs. (41) and (35), we obtain the following equations relating the solutions (x_n, z_n) of Eq. (1) to the solutions (u_n, v_n) of Eq. (2):

(45) $$x_n = u_n + 2v_n ,$$

(46) $$z_n = 2u_n + 2v_n + 1 .$$

If we solve Eqs. (45) and (46) for u_n and v_n, we get these equations:

(47) $$u_n = z_n - x_n - 1 ,$$

(48) $$v_n = \tfrac{1}{2}(2x_n - z_n + 1).$$

EXERCISES

8. Use Eqs. (36'), (37'), (43) and (44) to derive these equations relating the solutions (s_n, t_n) of Eq. (3) to the solutions (u_n, v_n) of Eq. (2):

(49) $$s_n = 2u_n + 1,$$

(50) $$t_n = 2v_n ,$$

(51) $$u_n = \tfrac{1}{2}(s_n - 1),$$

(52) $$v_n = \tfrac{1}{2} t_n .$$

9. Use the results of Exercise 8 and the paragraph that precedes it to derive these equations relating the solutions (s_n, t_n) of Eq. (3) to the solutions (x_n, z_n) of Eq. (1):

(53) $$s_n = 2z_n - 2x_n - 1 ,$$

(54) $$t_n = 2x_n - z_n + 1 ,$$

(55) $$x_n = \tfrac{1}{2}(s_n + 2t_n - 1) ,$$

(56) $$z_n = s_n + t_n .$$

10. Without using the closed formulas (41), (35), (42) and (38') for x_n, z_n, u_n and v_n, respectively, verify that if (x_n, z_n) is a solution of Eq. (1), in non-negative integers, and u_n and v_n are defined by Eqs. (47) and (48), respectively, then u_n and v_n are non-negative integers, and (u_n, v_n) is a solution of Eq. (2). Also verify, conversely, that if (u_n, v_n) is a solution of Eq. (2) in non-negative integers, and x_n and z_n are defined by Eqs. (45) and (46), respectively, then x_n and z_n are non-negative integers, and (x_n, z_n) is a solution of Eq. (1). (See [1], pp. 20-21.)

11. Without using the closed formulas for x_n, z_n, s_n, and t_n, verify that if (x_n, z_n) is a solution of Eq. (1) in non-negative integers, and s_n and t_n are defined by Eqs. (53) and (54), respectively, then s_n and t_n are non-negative integers, and (s_n, t_n) is a solution of Eq. (3). Also verify, conversely, that if (s_n, t_n) is a solution of Eq. (3) in non-negative integers, and x_n and z_n are defined by Eqs. (55) and (56), respectively, then x_n and z_n are non-negative integers, and (x_n, z_n) is a solution of Eq. (1).

If we drop the subscripts in Eqs. (45) through (56), each pair of equations, (45) and (46), (47) and (48), (49) and (50), (51) and (52), (53) and (54), and (55) and (56), defines a linear transformation that converts one of the Eqs. (1), (2) or (3) into one of the other two.

10. FORMULAS FOR GENERATING SIMULTANEOUSLY SUCCESSIVE SOLUTIONS OF EQUATIONS (1), (2), AND (3)

From Eqs. (45) and (50) we get

(57) $$x_n = u_n + t_n .$$

From Eqs. (45), (46), (12) and (13), we get

(58) $$u_{n+1} = x_n + z_n ,$$

(59) $$v_{n+1} = v_n + z_n .$$

Then, starting with $u_0 = 0$, $v_0 = 0$, and applying recursively the sequence of Eqs. (49), (50), (57), (56), (58) and (59), we can generate in succession s_0, t_0, x_0, z_0, u_1, v_1, s_1, t_1, x_1, z_1, u_2, v_2, s_2, t_2, x_2, z_2, and so on. The first ten non-negative solutions to Eqs. (2), (3) and (1), respectively, obtained in this way, are tabulated below.

n	(u_n, v_n)	(s_n, t_n)	(x_n, z_n)
0	(0, 0)	(1, 0)	(0, 1)
1	(1, 1)	(3, 2)	(3, 5)
2	(8, 6)	(17, 12)	(20, 29)
3	(49, 35)	(99, 70)	(119, 169)
4	(288, 204)	(577, 408)	(696, 985)
5	(1681, 1189)	(3363, 2378)	(4059, 5741)
6	(9800, 6930)	(19601, 13860)	(23660, 33461)
7	(57121, 40391)	(114243, 80782)	(137903, 195025)
8	(332928, 235416)	(665857, 470832)	(803760, 1136689)
9	(1940449, 1372105)	(3880899, 2744210)	(4684659, 6625109)

EXERCISE

12. Find (u_{10}, v_{10}), (s_{10}, t_{10}) and (x_{10}, z_{10}).

11. SOLUTIONS WITH EVEN OR ODD INDEX

It is of interest to examine separately the even-numbered solutions (x_{2i}, z_{2i}), (u_{2i}, v_{2i}) and (s_{2i}, t_{2i}) of Eqs. (1), (2) and (3), respectively, and the odd-numbered solutions (x_{2i+1}, z_{2i+1}), (u_{2i+1}, v_{2i+1}) and (s_{2i+1}, t_{2i+1}). These solutions can be expressed in terms of the solutions (x_i, z_i), (u_i, v_i) and (s_i, t_i). For example, we know from Eq. (20) that

$$s_{2i} + t_{2i}\sqrt{2} = (3 + 2\sqrt{2})^{2i} = \left[(3 + 2\sqrt{2})^i\right]^2 = (s_i + t_i\sqrt{2})^2 .$$

That is,

$$s_{2i} + t_{2i}\sqrt{2} = (s_i^2 + 2t_i^2) + 2s_it_i\sqrt{2} .$$

Therefore

(60) $$s_{2i} = s_i^2 + 2t_i^2 = 2s_i^2 - 1 = 1 + 4t_i^2 ,$$

and

(61) $$t_{2i} = 2s_it_i .$$

By using Eqs. (48), (50), (54), (55), (56), (60) and (61), we can show that

(62) $$x_{2i} = 2t_i(t_i + s_i) = 2t_iz_i = 4z_i v_i = 2z_i (2x_i - z_i + 1) ,$$

and

(63) $$z_{2i} = t_i^2 + z_i^2 = (2x_i - z_i + 1)^2 + z_i^2 .$$

By using Eqs. (49), (50), (51), (52), (60) and (61), we can show that

(64) $$u_{2i} = 2t_i^2 = 8v_i^2 ,$$

and

(65) $$v_{2i} = s_it_i = 2v_is_i = 2v_i(2u_i + 1) .$$

By invoking Eqs. (58) and (59), we can show that

(66) $$u_{2i+1} = (v_i + v_{i+1})^2 = (u_{i+1} - u_i)^2 ,$$

and

(67) $$v_{2i+1} = z_i (v_i + v_{i+1}) .$$

The following equations are also easily derived:

(68) $$s_{2i+1} = 2z_i^2 + (v_i + v_{i+1})^2 = 2z_i^2 + (z_i + t_i)^2 ,$$

(69) $$t_{2i+1} = 2z_i + (v_i + v_{i+1}) = 2z_i(z_i + t_i) ,$$

(70) $$x_{2i+1} = (z_i + 2x_i + 1)^2 - z_i^2 ,$$

(71) $$z_{2i+1} = (z_i + 2x_i + 1)^2 + z_i^2 ,$$

12. SUM AND DIFFERENCE RULES

The following rules are either already included among the equations we have derived so far, or are easily derived from them.

(56) $$s_i + t_i = z_i ,$$

(72) $$s_i - t_i = z_{i-1} ,$$

(73) $$u_i + v_i = u_{i+1} - v_{i+1} = \tfrac{1}{2}(z_i - 1) ,$$

(74) $$u_i - v_i = u_{i-1} + v_{i-1} = \tfrac{1}{2}(z_{i-1} - 1) ,$$

(58) $$z_i + x_i = u_{i+1} ,$$

(47) $$z_i - x_i = u_i + 1 ,$$

(75) $$s_{2i} + t_{2i} = t_i^2 + z_i^2 ,$$

(76) $$s_{2i} - t_{2i} = t_i^2 + z_{i-1}^2 ,$$

(77) $$u_{2i} + v_{2i} = 2v_i (v_i + v_{i+1}) ,$$

(78) $$u_{2i} - v_{2i} = 2v_i (t_i - z_i) ,$$

(79) $$z_{2i} + x_{2i} = (z_i + t_i)^2 ,$$

(80) $$z_{2i} - x_{2i} = (z_i - t_i)^2 = s_i^2 ,$$

(81) $$s_{2i+1} + t_{2i+1} = 3(t_i + z_i)^2 + 2t_i z_i + 2 ,$$

(82) $$s_{2i+1} - t_{2i+1} = z_{2i} = t_i^2 + z_i^2 ,$$

(83) $$u_{2i+1} + v_{2i+1} = 2v_{i+1}(v_i + v_{i+1}) ,$$

(84) $$u_{2i+1} - v_{2i+1} = 2v_i(v_i + v_{i+1}) ,$$

(85) $$z_{2i+1} + x_{2i+1} = 2(z_i + 2x_i + 1)^2 ,$$

(86) $$z_{2i+1} - x_{2i+1} = 2z_i^2 ,$$

(87) $$z_{2i+1} - (x_{2i+1} + 1) = (u_{i+1} - u_i)^2 .$$

13. HISTORICAL NOTE

Dickson's History of the Theory of Numbers, Vol. II, contains scattered notes about Eqs. (1) and (2), and denotes a sixty-page chapter to the Pell equation, of which Eq. (3) is a special case. (See [4].) Some of the more interesting facts cited by Dickson are reproduced below.

Concerning Eq. (1).

Fermat showed that if (x, z) is a solution of Eq. (1), then so is $(3x + 2z + 1,\ 4x + 3z + 2)$. (See Eqs. (7) and (8).)

C. Hutton (1842) found that if p_r/q_r is the r^{th} convergent of the continued fraction for the square root of 2, then $p_r p_{r+1}$ and $2q_r q_{r+1}$ are consecutive integers, and the sum of their squares is equal to q^2_{2r+1}.

P. Bachmann (1892) proved that the only integral solutions of $x^2 + y^2 = z^2$, $z > 0$, x and y consecutive, are given by

$$x + y + z\sqrt{2} = (1 + \sqrt{2})(3 + 2\sqrt{2})^k, \quad k = 0, 1, 2, \cdots .$$

R. W. D. Christie (1897) expressed the solutions of Eq. (1) in terms of continuants. The continuant $C(a_1, a_2, \cdots, a_r)$ is the r^{th} order determinant

$$\begin{vmatrix} a_1 & 1 & & & \bigcirc & \\ -1 & a_2 & \cdot & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \bigcirc & & & & -1 & a_r \end{vmatrix}$$

in which the term u_{ij} of the principal diagonal is equal to a_i, $(i = 1, \cdots, r)$, each term $u_{i+1,i}$, $(i = 1, \cdots, r-1)$, immediately under the principal diagonal is equal to -1, and each term $u_{i-1,i}$, $(i = 2, \cdots, r)$, immediately above the principal diagonal, is equal to 1, and every other term is equal to 0. Let Q_r stand for the r^{th} order continuant $C(2, \cdots, 2)$ in which all the diagonal elements are 2, and define $2_0 = 1$. Christie observed that the positive integral solutions of Eq. (1) are given by

$$x_r = Q_0 + Q_1 + \cdots + Q_{2r-1}, \qquad z_r = Q_{2r}, \qquad r = 1, 2, \cdots .$$

This result was proved by T. Muir (1899-1901).

Concerning Eq. (2).

Euler (1732) found solutions to Eq. (2) in the following way: He started with the identity of Plutarch (about 100 AD),

$$\frac{8u(u+1)}{2} + 1 = (2u+1)^2 .$$

By Eq. (2),

$$\frac{u(u+1)}{2} = v^2 .$$

Then $8v^2 + 1 = (2u+1)^2$. Let $s = 2u + 1$, and $t = 2v$. Then s and t satisfy Eq. (3), which Euler solved by using his general method for solving the Pell equation.

Euler proved, too, that u and v satisfy Eq. (2) only when

$$u = \frac{\alpha + \beta - 2}{4}, \quad v = \frac{\alpha - \beta}{4\sqrt{2}} ,$$

where

$$\alpha = (3 + 2\sqrt{2})^n, \qquad \beta = (3 - \sqrt{2})^n, \quad n = 0, 1, 2, \cdots.$$

From this result, he derived the recursion formulas given by Eqs. (27) and (28).

E. Lionnet (1881) stated that 0, 1 and 6 are the only triangular numbers whose squares are triangular numbers. This assertion was proved by Moret-Blanc (1882). In the notation of Section 2, Lionnet's result is that $S(T(n)) = T(m)$ only if $n = 0, 1$ or 3. Since $S(T(0)) = 0 = T(0)$, $S(T(1)) = 1 = T(1)$, and $S(T(3)) = 36 = T(8)$, it follows from Lionnet's result that the equation $S(T(n)) = T(S(n))$ has only the trivial solutions $(0,0)$ and $(1,1)$.

Concerning Eq. (3).

Among those who worked on solving equations of the form $S^2 - dt^2 = 1$ were Diophantus (about 250 AD), and Brahmegupta (born 598 AD).

The general problem of solving all equations of this form was proposed by Fermat in February 1657. Hence an equation of this form should be called a Fermat equation. It came to be known as the Pell equation as a result of an error by Euler, who incorrectly attributed to Pell the method of solution given in Wallis' Opera.

Lagrange gave the first proof that every Pell equation has integral solutions with $t \neq 0$ if d is not a square.

Others who contributed to the voluminous literature on this equation are Legendre, Dirichlet and Gauss.

14. REFERENCES

1. Sierpinski, Waclaw, "Pythagorean Triangles," pp. 16-22, The Scripta Mathematica Studies, Number Nine.
2. LeVeque, William Judson, Topics in Number Theory, pp. 137-143.
3. Jeske, James A., "Linear Recurrence Relations, Part I," The Fibonacci Quarterly, Vol. 1, No. 2, April 1963, pp. 69-74; Part II, Vol. 1, No. 4, December 1963, pp. 35-40.
4. Dickson, Leonard Eugene, History of the Theory of Numbers, Vol. II (Diophantine Analysis), pp. 3, 7, 10, 13, 16, 26, 27, 31, 32, 38, 181, 341-400.

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LINEAR RECURSION RELATIONS – LESSON FOUR
SECOND-ORDER LINEAR RECURSION RELATIONS

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

Given a second-order linear recursion relation

$$(1) \qquad T_{n+1} = a\,T_n + b\,T_{n-1} \;,$$

where a and b are real numbers and the values T_i of the sequence are real as well, there is an auxiliary equation:

$$(2) \qquad x^2 - ax - b = 0 \;,$$

with roots

$$(3) \qquad r = \frac{a + \sqrt{a^2 + 4b}}{2}, \qquad s = \frac{a - \sqrt{a^2 + 4b}}{2} \;.$$

As is usual with quadratic equations, three cases may arise depending on whether

$$(4) \qquad \begin{array}{ll} a^2 + 4b > 0, & \text{roots real and distinct;} \\ a^2 + 4b = 0, & \text{roots real and equal;} \\ a^2 + 4b < 0, & \text{roots complex numbers.} \end{array}$$

CASE 1. $a^2 + 4b > 0$.

In previous lessons we have considered cases of this kind. It has been noted that the roots may be rational or irrational. There seems to be nothing to add for the moment to the discussion of these cases.

CASE 2. $a^2 + 4b = 0$.

The presence of multiple roots in the auxiliary equation clearly requires some modification in the previous development. If

$$x^2 - ax - b = 0$$

$$x^n - ax^{n-1} - bx^{n-2} = 0 .$$

Since the equation has a multiple root $(a/2)$, the derivative of this equation will have this same root. Hence

$$nx^{n-1} - a(n - 1)x^{n-2} - b(n - 2)x^{n-3} = 0 \tag{5}$$

is satisfied by the multiple root also.

Thus the multiple root, r, satisfies the following two relations:

$$\begin{aligned} r^n &= ar^{n-1} + br^{n-2} \\ nr^n &= a(n-1)r^{n-1} + b(n-2)r^{n-2} . \end{aligned} \tag{6}$$

The result is that if we formulate T_n as

$$\begin{aligned} T_n &= An\, r^n + B\, r^n \\ T_{n+1} &= A(n+1)r^{n+1} + Br^{n+1} \end{aligned} \tag{7}$$

it follows that

$$\begin{aligned} T_{n+2} &= a\, T_{n+1} + b\, T_n \\ &= A\,[a(n+1)r^{n+1} + bn\, r^n] + B\,[ar^{n+1} + br^n] \\ &= A(n+2)r^{n+2} + B\, r^{n+2} \end{aligned} \tag{8}$$

so that the form of T is maintained.

EXAMPLE

Find the expression for T_n in terms of the roots of the auxiliary equation corresponding to the linear recursion relation

$$T_{n+1} = 6T_n - 9T_{n-1}$$

if $T_1 = 4$, $T_2 = 7$. Here the auxiliary equation is $x^2 - 6x + 9 = 0$ with a double root of 3. Hence T_n has the form

$$T_n = Anx3^n + Bx3^n .$$

Using the values of T_1 and T_2

$$4 = Ax3 + Bx3$$

$$7 = 2Ax3^2 + Bx3^2$$

with solutions $A = -5/9$, $B = 17/9$. Hence

$$T_n = \frac{-5nx3^n + 17x3^n}{9} = 3^{n-2}[(-5n + 17)] .$$

It may be noted that for any non-zero multiple root r, the determinant of the coefficients in the set of equations for T_1 and T_2 is

$$\begin{vmatrix} r & r \\ 2r^2 & r^2 \end{vmatrix} = -r^3$$

which is not zero, so that these equations will always have a solution.

CASE 3. $a^2 + 4b < 0$.

The case of complex roots is quite similar to that of real and distinct roots as far as determining coefficients from initial value equations is concerned. However, since we have specified that the terms of the sequence and the coefficients in the recursion relation are real, there will have to be a special relation between A and B in the expression for T_n:

$$T_n = A r^n + B s^n .$$

Note that r and s are complex conjugates, so that r^n and s^n are of the form $P + Qi$ and $P - Qi$ respectively, where P and Q are real. If T_n is to be real, then A and B must be complex conjugates as well.

EXAMPLE

Find the expression for T_n in terms of the roots of the auxiliary equation for the linear recursion relation

$$T_{n+1} = 3 T_n - 4 T_{n-1} ,$$

with $T_1 = 5$, $T_2 = 9$. Here the auxiliary equation is:

$$x^2 - 3x + 4 = 0$$

with roots

$$r = \frac{3 + i\sqrt{7}}{2} , \quad s = \frac{3 - i\sqrt{7}}{2} .$$

Then

$$5 = Ar + Bs$$

$$9 = A r^2 + B s^2$$

from which one finds that

$$A = \frac{21 - 11i \sqrt{7}}{28} , \quad B = \frac{21 + 11i \sqrt{7}}{28} .$$

Accordingly,

$$T_n = \left(\frac{21 - 11i\sqrt{7}}{28}\right) r^n + \left(\frac{21 + 11i\sqrt{7}}{28}\right) s^n .$$

AN ANALOGUE

Because of the similarities among second-order linear recursion relations it is possible to find close analogues among them to the Fibonacci and Lucas sequences. Let us consider as an example the second-order linear recursion relation

$$T_{n+1} = 3T_n + T_{n-1} .$$

The auxiliary equation is

$$x^2 - 3x - 1 = 0$$

with roots

$$r = \frac{1 + \sqrt{13}}{2}, \quad s = \frac{1 - \sqrt{13}}{2} .$$

If the initial terms are taken as $T_0 = 0$, $T_1 = 1$, $T_2 = 3$, then

$$1 = Ar + Bs$$
$$3 = Ar^2 + Bs^2 ,$$

with resulting values $A = 1/\sqrt{13}$ and $B = -1/\sqrt{13}$ so that

$$T_n = \frac{r^n - s^n}{\sqrt{13}} = \frac{r^n - s^n}{r - s}$$

has precisely the same form as the expression for F_n with 13 replacing 5 under the square root sign.

If the relation $V_n = T_{n+1} + T_{n-1}$ is used to define the corresponding "Lucas" sequence, the terms of this sequence are:

$$V_0 = 2, \ V_1 = 3, \ V_2 = 11, \ V_3 = 36, \cdots \ .$$

Solving for A and B from

$$3 = Ar + Bs$$

$$11 = Ar^2 + Bs^2$$

gives values of $A = 1$, $B = 1$, so that

$$V_n = r^n + s^n$$

in perfect correspondence to the expression for the Lucas sequence. As a result of this similarity, many relations in the Fibonacci-Lucas complex can be taken over (sometimes with the slight modification of replacing 5 by 13) to this pair of sequences. Thus:

$$\begin{aligned} T_{2n} &= T_n V_n \\ T_{2n+1} &= T_n^2 + T_{n+1}^2 \\ T_{n+1}T_{n-1} - T_n^2 &= (-1)^{n-1} \\ V_{2n} &= V_n^2 + 2(-1)^{n+1} \\ V_n + V_{n+2} &= 13T_{n+1} \\ V_n^2 + V_{n+1}^2 &= 13(T_n^2 + T_{n+1}^2) \ . \end{aligned}$$

PROBLEMS

1. For the sequence $T_1 = 1$, $T_2 = 3$, obeying the linear recursion relation

$$T_{n+1} = 3T_n + T_{n-1}$$

show that every integer divides an infinity of members of the sequence.

2. For the corresponding "Lucas" sequence, prove that if m divides n, where n is odd, then V_m divides V_n.

3. Find the expression for the sequence $T_1 = 2$, $T_2 = 5$ in terms of the roots of the auxiliary equation corresponding to the linear recursion relation

$$T_{n+1} = 4T_n + 4T_{n-1}.$$

4. Prove that the second-order linear recursion relation

$$T_{n+1} = 2T_n - T_{n-1}$$

defines an arithmetic progression.

5. If $T_1 = a$, $T_2 = b$, find the expression for T_n in terms of the roots of the auxiliary equation corresponding to $T_{n+1} = 4T_n - 4T_{n-1}$.

6. If $T_1 = i$, $T_2 = 1$ and $T_{n+1} = -T_{n-1}$, find the general expression for T_n in terms of the roots of the auxiliary equation.

7. $T_1 = 3$, $T_2 = 7$, $T_3 = 17$, $T_4 = 43$, $T_5 = 113, \cdots$ are terms of a second-order linear recursion relation. Find this relation and express T_n in terms of the roots of the auxiliary equation.

8. For the second-order linear recursion relation $T_{n+1} = 5T_n + T_{n-1}$ find the particular sequences analogous to the Fibonacci and Lucas sequences and express their terms as functions of the roots of the auxiliary equation.

9. For $T_1 = 5$, $T_2 = 9$, $T_{n+1} = 3T_n - 5T_{n-1}$, find T_n in terms of the roots of the auxiliary equation.

10. If

$$T_n = \left(\frac{-66 + 13\sqrt{33}}{33}\right)\left(\frac{5 + \sqrt{33}}{2}\right)^n + \left(\frac{-66 - 13\sqrt{33}}{33}\right)\left(\frac{5 - \sqrt{33}}{2}\right)^n$$

determine the recursion relation obeyed by T_n and find T_1 and T_2.

[See page 210 for Solutions to Problems.]

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SOME RESULTS ON FIBONACCI QUATERNIONS

MUTHULAKSHMI R. IYER
Indian Statistical Institute, Calcutta, India

1. INTRODUCTION

Recently the author derived some results about generalized Fibonacci Numbers [3]. In the present paper our object is to derive relations connecting the Fibonacci Quaternions [1] and Lucas Quaternions, to use a similar terminology, with the Fibonacci Numbers [2] and Lucas Numbers [4] as also the inter-relations between them. In Section 3, we give relations connecting Fibonacci and Lucas Numbers; in Section 4, we derive relations of Fibonacci Quaternions to Fibonacci and Lucas Numbers, and in 5, Lucas Quaternions are connected to Fibonacci and Lucas Numbers. Lastly, in Section 6 are listed the relations existing between Fibonacci and Lucas Quaternions.

2. TERMINOLOGY AND NOTATIONS

Following the terminology of A. F. Horadam [1], we define the n^{th} Fibonacci Quaternion as follows:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

where F_n is the n^{th} Fibonacci Number and i, j, k satisfy the relations of the Quaternion viz:

$$i^2 = j^2 = k^2 = -1, \quad ij - ji = k; \quad jk = -kj = i; \quad ki = -ik = j .$$

Now on the same lines we can define the n^{th} Lucas Quaternion T_n say as

$$T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}$$

where L_n is the n^{th} Lucas number. Also, we will denote a quantity of the form

$$F_n - iF_{n-1} + jF_{n-2} - kF_{n-3}$$

by Q_{n^*} and

$$F_n + iF_{n-1} + jF_{n-2} + kF_{n-3}$$

by $Q_{\bar{n}}$. Similar notations hold for T_{n^*} and $T_{\bar{n}}$, that is,

$$L_n - iL_{n-1} + jL_{n-2} - kL_{n-3} = T_{n^*}$$

and

$$L_n + iL_{n-1} + jL_{n-2} + kL_{n-3} = T_{\bar{n}} .$$

Now we proceed to derive the relations one by one. All these results are obtained from the definitions of Fibonacci Numbers and Lucas Numbers, given by

$$F_n = \frac{a^n - b^n}{\sqrt{5}} , \quad L_n = (a^n + b^n)$$

for all n, where a and b are the roots of the equation

$$x^2 - x - 1 = 0 ,$$

obtained from the Fibonacci and Lucas recurrence relations. The roots are connected by

$$a + b = 1, \quad a - b = \sqrt{5} ,$$

and $ab = -1$.

SECTION 3

Consider the following relations:

(1) $$F_{n+r}L_{n+r} = F_{2n+2r}$$

(2) $$F_{n-r}L_{n-r} = F_{2n-2r}$$

Therefore

(3) $$F_{n+r}L_{n+r} + F_{n-r}L_{n-r} = F_{2n}L_{2r}$$

(4) $$F_{n+r}L_{n+r} - F_{n-r}L_{n-r} = F_{2r}L_{2n}$$

(5) $$F_{n+r}L_{n-r} = F_{2n} + (-1)^{n-r}F_{2r}$$

(6) $$F_{n-r}L_{n+r} = F_{2n} - (-1)^{n-r}F_{2r}$$

Therefore

(7) $$F_{n+r}L_{n-r} + F_{n-r}L_{n+r} = 2F_{2n}$$

and

(8) $$F_{n+r}L_{n-r} - F_{n-r}L_{n+r} = 2(-1)^{n-r}F_{2r}$$

(9) $$F_{n+r}L_{n} = F_{2n+r} + (-1)^{n}F_{r}$$

(10) $$F_{n}L_{n+r} = F_{2n+r} - (-1)^{n}F_{r}$$

So

(11) $$F_{n+r}L_{n} + F_{n}L_{n+r} = 2F_{2n+r}$$

(12) $$F_{n+r}L_{n} - F_{n}L_{n+r} = 2(-1)^{n} F_{r}$$

(13) $$F_{n+r}L_{n+s} = F_{2n+r+s} + (-1)^{n+s}F_{r-s}$$

(14) $$F_{n+s}L_{n+r} = F_{2n+r+s} + (-1)^{n+s+1}F_{r-s}$$

(15) $$F_{n+r}L_{n+s} + F_{n+s}L_{n+r} = 2F_{2n+r+s}$$

(16) $$F_{n+r}L_{n+s} - F_{n+s}L_{n+r} = 2(-1)^{n+s}F_{r-s}$$

SECTION 4

In this section, we give the list of relations connecting the Fibonacci Quaternions to Fibonacci and Lucas Numbers. The simplest one is

(17) $$Q_n - iQ_{n+1} - jQ_{n+2} - kQ_{n+3} = L_{n+3}$$

Consider

(18) $$Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n\left[2Q_1 - 3k\right]$$

(19) $$Q_{n-1}^2 + Q_n^2 = 2Q_{2n-1} - 3L_{2n+2}$$

(20) $$Q_{n+1}^2 - Q_{n-1}^2 = Q_nT_n = (2Q_{2n} - 3L_{2n+3}) + 2(-1)^{n+1}(Q_0 - 3k)$$

(21) $$Q_{n-2}Q_{n-1} + Q_nQ_{n+1} = 6F_nQ_{n-1} - 9F_{2n+2} + 2(-1)^{n+1}(Q_{(-1)} - 3k)$$

(22) $$Q_{n-1}Q_{n+3} - Q_{n+1}^2 = (-1)^n\left[2 + 4i + 3j + k\right]$$

(23) $$Q_{n-1}Q_{n+1} - Q_{n-2}Q_{n+2} = (-1)^n\left[2T_0 - k\right] + 4(-1)^{n+1}\left[Q_0 - 2k\right]$$

(24) $$Q_{n-3}Q_{n-2} + Q_nQ_{n+1} = 4Q_{2n-2} - 6L_{2n+1}$$

(25) $$Q_{n-1}^2 + Q_{n+1}^2 = 6F_{n+1}Q_{n-1} - 9F_{2n+3} + 2(-1)^nQ_{(-2)}$$

Also the remarkable relation

(26) $$\frac{Q_{n+r} + (-1)^rQ_{n-r}}{Q_n} = L_r$$

(27) $$Q_{n+1-r}Q_{n+1+r} - Q_{n+1}^2 = (-1)^{n-r}[F_r^2T_0 + F_{2r}(Q_0 - 3r)]$$

Now we turn to relations of the form:

(28) $$Q_{n+r}L_{n+r} = Q_{2n+2r} + (-1)^{n+r}Q_0$$

(29) $$Q_{n-r}L_{n-r} = Q_{2n-2r} + (-1)^{n+r}Q_0$$

(30) $$Q_{n+r}L_{n+r} + Q_{n-r}L_{n-r} = Q_{2n}L_{2r} + 2(-1)^{n+r}Q_0$$

(31) $$Q_{n+r}L_{n+r} - Q_{n-r}L_{n-r} = F_{2r}T_{2n}$$

(32) $$Q_{n+r}L_{n-r} = Q_{2n} + (-1)^{n-r}Q_{2r}$$

(33) $$Q_{n-r}L_{n+r} = Q_{2n} + (-1)^{n-r+1}Q_{2r^\star}$$

(34) $$Q_{n+r}L_{n-r} + Q_{n-r}L_{n+r} = 2Q_{2n} + (-1)^{n-r}L_{2r}Q_0$$

(35) $$Q_{n+r}L_{n-r} - Q_{n-r}L_{n+r} = (-1)^{n-r}F_{2r}T_0$$

(36) $$Q_{n+r}L_n = Q_{2n+r} + (-1)^nQ_r$$

(37) $$Q_nL_{n+r} = Q_{2n+r} - (-1)^nQ_r^\star$$

(38) $$Q_{n+r}L_n + Q_nL_{n+r} = 2Q_{2n+r} + (-1)^nL_rQ_0$$

(39) $$Q_{n+r}L_n - Q_nL_{n+r} = (-1)^nF_rT_0$$

(40) $$Q_{n+r}L_{n+t} = Q_{2n+r+t} + (-1)^{n+t}Q_{r-t}$$

(41) $$Q_{n+t}L_{n+r} = Q_{2n+r+t} + (-1)^{n+r+1}Q_{\overline{r-t}}$$

Therefore:

(42) $$Q_{n+r}L_{n+t} + Q_{n+t}L_{n+r} = 2Q_{2n+r+t} + (-1)^{n+t}L_{r-t}Q_0$$

(43) $$Q_{n+r}L_{n+t} - Q_{n+t}L_{n+r} = (-1)^{n+t}F_{r-t}T_0$$

(44) $$Q_{n+r}F_{n-r} = \frac{1}{5}\left[T_{2n} - (-1)^{n-r}T_{2r}\right]$$

(45) $$Q_{n-r}F_{n+r} = \frac{1}{5}\left[T_{2n} - (-1)^{n-r}T^{\star}_{2r}\right]$$

(46) $$Q_{n+r}F_{n-r} + Q_{n-r}F_{n+r} = \frac{1}{5}\left[2T_{2n} - (-1)^{n-r}L_{2r}T_0\right]$$

(47) $$Q_{n+r}F_{n-r} - Q_{n-r}F_{n+r} = (-1)^{n-r+1}F_{2r}Q_0$$

(48) $$Q_{n+r}F_n = \frac{1}{5}\left[T_{2n+r} - (-1)^n T_r\right]$$

(49) $$Q_nF_{n+r} = \frac{1}{5}\left[T_{2n+r} - (-1)^n T^{\star}_r\right]$$

(50) $$Q_{n+r}F_n + Q_nF_{n+r} = \frac{1}{5}\left[2T_{2n+r} - (-1)^n L_rT_0\right]$$

(51) $$Q_{n+r}F_n - Q_nF_{n+r} = (-1)^{n+1}F_rQ_0$$

(52) $$Q_{n+r}F_{n+t} = \frac{1}{5}\left[T_{2n+r+t} - (-1)^{n+t}T_{r-t}\right]$$

(53) $$Q_{n+t}F_{n+r} = \frac{1}{5}\left[T_{2n+r+t} - (-1)^{n+r+1}T_{\overline{r-t}}\right]$$

(54) $$Q_{n+r}F_{n+t} + Q_{n+t}F_{n+r} = \frac{1}{5}\left[2T_{2n+r+t} - (-1)^{n+t}L_{r-t}T_0\right]$$

(55) $$Q_{n+r}F_{n+t} - Q_{n+t}F_{n+r} = (-1)^{n+t}F_{r-t}Q_0$$

(56) $$Q_{n+r}F_{n-r} + (-1)^r Q_{n-r}F_{n+r} = \frac{1}{5}\left[T_{2n}(1+(-1)^r) - (-1)^{n-r}T_{2r} - (-1)^n T^{\star}_{2r}\right]$$

(57) $$Q_{n+r}L_{n-r} + (-1)^r Q_{n-r}L_{n+r} = Q_{2n}(1+(-1)^r) + (-1)^{n-r}Q_{2r} - (-1)^n Q^{\star}_{2r}$$

(58) $$Q_{n+r}L_{n+t} + (-1)^r Q_{n+t}L_{n+r} = Q_{2n+r+t}(1+(-1)^r) + (-1)^{n+t}Q_{r-t} - (-1)^{n+r+t}Q^{\star}_{r-t}$$

(59) $$Q_{n+r}F_{n+t} + (-1)^r Q_{n+t}F_{n+r} = \frac{1}{5}\left[T_{2n+r+t}(1 + (-1)^r) - (-1)^{n+t}T_{r-t} - (-1)^{n+r+t}T^*_{r-t}\right]$$

SECTION 5

In this section we give the results connecting Lucas Quaternions T_n to Fibonacci and Lucas Numbers. The simplest is:

(60) $$T_n - iT_{n+1} - jT_{n+2} - kT_{n+3} = 15F_{n+3}$$

(61) $$T_{n+r}F_{n+r} = Q_{2n+2r} - (-1)^{n+r}Q_0$$

(62) $$T_{n-r}F_{n-r} = Q_{2n-2r} - (-1)^{n+r}Q_0$$

(63) $$T_{n+r}F_{n+r} + T_{n-r}F_{n-r} = Q_{2n}L_{2r} - 2(-1)^{n+r}Q_0$$

(64) $$T_{n+r}F_{n+r} - T_{n-r}F_{n-r} = F_{2r}T_{2n}$$

(65) $$T_{n+r}F_{n-r} = Q_{2n} - (-1)^{n-r}Q_{2r}$$

(66) $$T_{n-r}F_{n+r} = Q_{2n} + (-1)^{n-r}Q^*_{2r}$$

(67) $$T_{n+r}F_{n-r} + T_{n-r}F_{n+r} = 2Q_{2n} - (-1)^{n-r}L_{2r}Q_0$$

(68) $$T_{n+r}F_{n-r} - T_{n-r}F_{n+r} = (-1)^{n+r+1}F_{2r}T_0$$

(69) $$T_{n+r}F_n = Q_{2n+r} - (-1)^n Q_r$$

(70) $$T_nF_{n+r} = Q_{2n+r} + (-1)^n Q^*_r$$

(71) $$T_{n+r}F_n + T_nF_{n+r} = 2Q_{2n+r} - (-1)^n L_r Q_0$$

(72) $$T_{n+r}F_n - T_nF_{n+r} = (-1)^{n+1}F_rT_0$$

(73) $$T_{n+r}F_{n+t} = Q_{2n+r+t} - (-1)^{n+t}Q_{r-t}$$

(74) $$T_{r+t}F_{n+r} = Q_{2n+r+t} + (-1)^{n+r}Q_{\overline{r-t}}$$

So

(75) $$T_{n+r}F_{n+t} + T_{n+t}F_{n+r} = 2Q_{2n+r+t} - (-1)^{n+t}L_{r-t}Q_0$$

(76) $$T_{n+r}F_{n+t} - T_{n+t}F_{n+r} = (-1)^{n+t+1}F_{r-t}T_0$$

(77) $$T_{n+r}L_{n-r} = T_{2n} + (-1)^{n-r}T_{2r}$$

(78) $$T_{n-r}L_{n+r} = T_{2n} + (-1)^{n-r}T^{\star}_{2r}$$

(79) $$T_{n+r}L_{n-r} + T_{n-r}L_{n+r} = 2T_{2n} + (-1)^{n-r}L_{2r}T_0$$

(80) $$T_{n+r}L_{n-r} - T_{n-r}L_{n+r} = (-1)^{n-r}5F_{2r}Q_0$$

(81) $$T_{n+r}L_n = T_{2n+r} + (-1)^nT_r$$

(82) $$T_nL_{n+r} = T_{2n+r} + (-1)^nT^{\star}_r$$

(83) $$T_{n+r}L_n + T_nL_{n+r} = 2T_{2n+r} + (-1)^nL_rT_0$$

(84) $$T_{n+r}L_n - T_nL_{n+r} = (-1)^n5F_rQ_0$$

(85) $$T_{n+r}L_{n+t} = T_{2n+r+t} + (-1)^{n+t}T_{r-t}$$

(86) $$T_{n+t}L_{n+r} = T_{2n+r+t} + (-1)^{n+r+1}T_{\overline{r-t}}$$

(87) $$T_{n+r}L_{n+t} + T_{n+t}L_{n+r} = 2T_{2n+r+t} + (-1)^{n+t}L_{r-t}T_0$$

(88) $$T_{n+r}L_{n+t} - T_{n+t}L_{n+r} = (-1)^{n+t+1}5F_{r-t}Q_0$$

(89) $$T_{n+r}L_{n-r} + (-1)^rT_{n-r}L_{n+r} = T_{2n}(1 + (-1)^r) + (-1)^{n-r}T_{2r} - (-1)^nT^{\star}_{2r}$$

(90) $$T_{n+r}F_{n-r} + (-1)^r T_{n-r}F_{n+r} = \frac{1}{5}\Big[Q_{2n}(1 + (-1)^r) - (-1)^{n-r}Q_{2r} + (-1)^n Q^{\star}_{2r}\Big]$$

(91) $$T_{n+r}F_{n+t} + (-1)^r T_{n+t}F_{n+r} = \frac{1}{5}\Big[Q_{2n+r+t}(1 + (-1)^r) - (-1)^{n+t}Q_{r-t} + (-1)^{n+r+t}Q^{\star}_{r-t}\Big]$$

(92) $$T_{n+r}L_{n+t} + (-1)^r T_{n+t}L_{n+r} = T_{2n+r+t}(1 + (-1)^r) + (-1)^{n+t}T_{r-t} + (-1)^{n+r+t}T^{\star}_{r-t}$$

SECTION 6

Lastly, in this section we obtain the inter-relations between the Fibonacci and Lucas Quaternions

(93) $$Q_nL_n + T_nF_n = 2Q_{2n}$$

(94) $$Q_nL_n - T_nF_n = 2(-1)^n Q_0$$

(95) $$Q_n + T_n = 2Q_{n+1}$$

(96) $$T_n - Q_n = 2Q_{n-1}$$

(97) $$T_n^2 + Q_n^2 = 6\Big[2F_nQ_n - 3F_{2n+3}\Big] + 4(-1)^n T_0$$

(98) $$T_n^2 - Q_n^2 = 4\Big[2F_nQ_n - 3F_{2n+3} + (-1)^n T_0\Big]$$

(99) $$T_nQ_n + T_{n-1}Q_{n-1} = 2T_{2n-1} - 15F_{2n+2}$$

(100) $$T_nQ_n - T_{n-1}Q_{n-1} = 2Q_{2n-1} - 3L_{2n+2} + 4(-1)^n(Q_0 - 3k)$$

(101) $$T_nQ_{n+1} - T_{n+1}Q_n = 2(-1)^n\Big[2\,Q_1 - 3k\Big]$$

(102) $$T_{n+r}Q_{n+s} - T_{n+s}Q_{n+r} = 2(-1)^{n+s+1}F_{r-s}T_0$$

REFERENCES

1. A. F. Horadam, "Complex Fibonacci Numbers and Fibonacci Quaternions," Amer. Math. Monthly, 70, 1963, pp. 289-291.
2. A. F. Horadam, "A Generalized Fibonacci Sequence," Amer. Math. Monthly, 68, 1961, pp. 455-459.
3. Muthulakshmi R. Iyer, "Identities Involving Generalized Fibonacci Numbers," the Fibonacci Quarterly, Vol. 7, No. 1 (Feb. 1969), pp. 66-72.
4. E. Lucas, Theorie des Numbers, Paris, 1961.

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(Continued from p. 200.)

SOLUTIONS TO PROBLEMS

1. For any modulus m, there are m possible residues $(0, 1, 2, \cdots, m - 1)$. Successive pairs may come in m^2 ways. Two successive residues determine all residues thereafter. Now in an infinite sequence of residues there is bound to be repetition and hence periodicity.

Since m divides T_0, it must by reason of periodicity divide an infinity of members of the sequence.

2. $n = mk$, where m and k are odd. V_n can be written

$$V_n = (r^m)^k + (s^m)^k \quad ,$$

which is divisible by $V_m = r^m + s^m$.

3. $r = 2 + 2i\sqrt{2}$, $s = 2 - 2i\sqrt{2}$.

$$T_n = \left(\frac{2 - 3i\sqrt{2}}{16}\right) r^n + \left(\frac{2 + 3i\sqrt{2}}{16}\right) s^n \quad .$$

4. The auxiliary equation is $(x - 1)^2 = 0$, so that T_n has the form

$$T_n = An \times 1^n + B \times 1^n = An + B \, .$$

5. $$T_n = 2^n\left[\left(\frac{b - 2a}{4}\right) n + \frac{4a - b}{4}\right] \quad .$$

(Continued on p. 224.)

* * * * *

FIBONACCI-LUCAS INFINITE SERIES – RESEARCH TOPIC

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

It is almost an understatement to say that the Fibonacci Quarterly bristles with formulas. A review of this publication, however, reveals that there are very few that involve summations with Fibonacci or Lucas numbers in the denominator. Five problems in all seem to summarize the extent of what has been done along these lines in the Quarterly to February, 1966 (see references 1 to 9 inclusive). The purpose of this paper is to begin the process of filling in this gap by capitalizing on a well-known and favorite method in series summation and to provide an initial set of formulas which may form the groundwork for more extensive developments by other researchers.

The method to be employed may be illustrated by the case of

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad .$$

This can be written in the alternate form

$$\sum_{n=1}^{\infty} [1/n - 1/(n+1)] \, .$$

Let S_n be the sum of the first n terms of either the original series or of the corresponding n parentheses in the remodeled series. It follows that

$$S_n = (1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \cdots + [1/(n-1) - 1/n] \, .$$

Intermediate terms add up to zero in pairs with the result that:

$$S_n = 1 - 1/n \, .$$

Now by definition, the sum of an infinite series is given by the limit of the partial sums, S_n, as n goes to infinity.

Hence

$$\sum_{n=1}^{\infty} 1/n(n+1) = S = \lim_{n \to \infty} S_n = 1 \, .$$

This method with some interesting variations will be employed in working out formulas which will provide in closed form the sums of various Fibonacci-Lucas series.

Case 1. (1) S_n contains two terms. (2) The terms of the revised series go to zero as n goes to infinity.

The example given above would correspond to this type. As an illustration, consider the summation:

$$\sum_{n=1}^{\infty} \left[1/F_{n+1} - 1/F_{n+2}\right] .$$

The sum of the first n parenthesis is:

$$S_n = (1/F_2 - 1/F_3) + (1/F_3 - 1/F_4) + \cdots + \left[1/F_{n+1} - 1/F_{n+2}\right]$$

or

$$S_n = 1 - 1/F_{n+2}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 - 1/F_{n+2}) = 1 .$$

But

$$1/F_{n+1} - 1/F_{n+2} = \frac{F_{n+2} - F_{n+1}}{F_{n+1}\,F_{n+2}} = \frac{F_n}{F_{n+1}\,F_{n+2}} .$$

Accordingly

$$\sum_{n=1}^{\infty} \frac{F_n}{F_{n+1}\,F_{n+2}} = 1 . \tag{1}$$

Case 2. (1) S_n contains more than two terms. (2) The terms of the revised series to to zero as n goes to infinity.

Example.

$$\sum_{n=1}^{\infty} [1/F_n - 1/F_{n+3}]$$

$$\begin{aligned} S_n &= (1/F_1 - 1/F_4) + (1/F_2 - 1/F_5) + (1/F_3 - 1/F_6) \\ &\quad + (1/F_4 - 1/F_7) + \cdots + (1/F_n - 1/F_{n+3}) \\ S_n &= 1/F_1 + 1/F_2 + 1/F_3 - 1/F_{n+1} - 1/F_{n+2} - 1/F_{n+3} . \end{aligned}$$

Hence $$S = \lim_{n\to\infty} S_n = 1 + 1 + 1/2 = 5/2 .$$

But

$$1/F_n - 1/F_{n+3} = (F_{n+3} - F_n)/F_n\,F_{n+3} = 2F_{n+1}/F_n\,F_{n+3} .$$

Hence

$$\sum_{n=1}^{\infty} \frac{F_{n+1}}{F_n\,F_{n+3}} = 5/4 . \tag{2}$$

Case 3. (1) S_n contains two terms. (2) The terms of the revised series approach a limit other than zero.

Example.

$$\sum_{n=1}^{\infty} [F_n/L_{n-1} - F_{n+1}/L_n]$$

$$S_n = (F_1/L_0 - F_2/L_1) + (F_2/L_1 - F_3/L_2) + \cdots + (F_n/L_{n-1} - F_{n+1}/L_n)$$

$$S_n = F_1/L_0 - F_{n+1}/L_n .$$

$$S = \lim_{n\to\infty} (1/2 - F_{n+1}/L_n) = 1/2 - \lim_{n\to\infty} \frac{F_{n+1}}{F_{n-1} + F_{n+1}}$$

$$S = 1/2 - \lim_{n\to\infty} \frac{F_{n+1}/F_{n-1}}{1 + F_{n+1}/F_{n-1}} .$$

If r be the Golden Section ratio

$$\frac{1+\sqrt{5}}{2} ,$$

$$\lim_{n\to\infty} F_{n+1}/F_{n-1} = r^2 .$$

Hence

$$S = 1/2 - r^2/(1 + r^2) = (1 - r^2)/2(1 + r^2).$$

On the other hand,

$$F_n/L_{n-1} - F_{n+1}/L_n = \frac{F_n L_n - F_{n+1}L_{n-1}}{L_{n-1}L_n} = \frac{(-1)^n}{L_{n-1}L_n} .$$

Therefore

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{n-1}L_n} = \frac{r^2 - 1}{2(r^2 + 1)} \approx \frac{\sqrt{5}}{10}. \tag{3}$$

Case 4. S_n contains more than two terms. (2) The terms of the revised series approach a limit not zero.

Example.

$$\sum_{n=1}^{\infty} (F_{n-1}/F_n - F_{n+2}/F_{n+3})$$

$$\begin{aligned} S_n = {} & (F_0/F_1 - F_3/F_4) + (F_1/F_2 - F_4/F_5) + (F_2/F_3 - F_5/F_6) \\ & + (F_3/F_4 - F_6/F_7) \cdots (F_{n-3}/F_{n-2} - F_n/F_{n+1}) \\ & + (F_{n-2}/F_{n-1} - F_{n+1}/F_{n+2}) + (F_{n-1}/F_n - F_{n+2}/F_{n+3}) \end{aligned}$$

$$S_n = F_0/F_1 + F_1/F_2 + F_2/F_3 - F_n/F_{n+1} - F_{n+1}/F_{n+2} - F_{n+2}/F_{n+3}$$

$$S = \lim_{n\to\infty} S_n = 0 + 1 + 1/2 - 3r^{-1} = \frac{3r - 6}{2r} = \frac{9 - 6r}{2} .$$

Now

$$F_{n-1}/F_n - F_{n+2}/F_{n+3} = \frac{F_{n-1}F_{n+3} - F_nF_{n+2}}{F_n F_{n+3}} = 2(-1)^n/F_nF_{n+3} \; .$$

Therefore

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+3}} = \frac{6r - 9}{4} \qquad (4)$$

ANOTHER FAMILY OF SUMMATIONS

Additional formulas can be developed by having sums of two terms in each parenthesis with the signs before the parentheses alternating.

Example 1.

$$S_n = \sum_{k=1}^{n} (-1)^{k-1} \left[1/(F_k L_{k+1}) + 1/(F_{k+1}L_{k+2}) \right] \; .$$

Then

$$S_n = 1/(F_1L_2) + (-1)^{n-1} 1/(F_{n+1}L_{n+2})$$
$$S = \lim_{n\to\infty} S_n = 1/3 \; .$$

On the other hand,

$$1/(F_nL_{n+1}) + 1/(F_{n+1}L_{n+2}) = \frac{F_{n+1}L_{n+2} + F_nL_{n+1}}{F_nF_{n+1}L_{n+1}L_{n+2}} = \frac{L_{2n+2}}{F_nF_{n+1}L_{n+1}L_{n+2}} \; .$$

Accordingly

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}L_{2n+2}}{F_nF_{n+1}L_{n+1}L_{n+2}} = 1/3 \qquad (5)$$

Example 2.

$$\sum_{n=r+2}^{\infty} (-1)^{n-1} \left[1/F^2_{n+r} + 1/F^2_{n-r-1} \right]$$

$$S_n = (1/F^2_{2r+2} + 1/F^2_1) - (1/F^2_{2r+3} + 1/F^2_2) + \cdots + (-1)^{n-1}(1/F^2_{n+2r-1} + 1/F^2_n)$$

$$S_n = \sum_{j=1}^{2r+1} (-1)^{j-1} \Big/ F^2_j + \sum_{j=n+1}^{n+2r-1} (-1)^j \Big/ F^2_j$$

$$S = \lim_{n\to\infty} S_n = \sum_{j=1}^{2r+1} (-1)^{j-1} \Big/ F^2_j \; .$$

But

$$1/F^2_{n+r} + 1/F^2_{n-r-1} = \frac{F^2_{n-r-1} + F^2_{n+r}}{F^2_{n+r}\,F^2_{n-r-1}} = \frac{F_{2r+1}\,F_{2n-1}}{F^2_{n+r}\,F^2_{n-r-1}}$$

Allowing for the fact that F_{2r+1} is a constant factor in all terms, it then follows that:

(6) $$\sum_{n=r+2}^{\infty} \frac{(-1)^{n-1}F_{2n-1}}{F^2_{n+r}\,F^2_{n-r-1}} = \frac{1}{F_{2r+1}} \sum_{j=1}^{2r+1} (-1)^j \Big/ F_j^2$$

SOME ADDITIONAL FORMULAS

Additional formulas together with an indication of the breakdown sums from which they were derived are given below.

(7) $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{F_{2n+2}}{L_n^2\,L^2_{n+2}} = 8/45 \quad .$$

Derived from

$$\sum_{n=1}^{\infty} \left(F_n^2/L_n^2 - F^2_{n+2}/L^2_{n+2}\right)$$

(8) $$\sum_{n=1}^{\infty} \frac{1}{F_n\,F_{n+2}\,F_{n+3}} = 1/4$$

derived from

$$1/2 \sum_{n=1}^{\infty} \left[\frac{1}{F_nF_{n+1}F_{n+2}} - \frac{1}{F_{n+1}F_{n+2}F_{n+3}}\right]$$

(9) $$\sum_{n=1}^{\infty} \frac{F_{n+3}}{F_n\,F_{n+2}\,F_{n+4}\,F_{n+6}} = 17/480$$

derived from

$$\sum_{n=1}^{\infty} \left[\frac{1}{F_n\,F_{n+2}\,F_{n+4}} - \frac{1}{F_{n+2}\,F_{n+4}\,F_{n+6}}\right]$$

(10) $$\sum_{n=1}^{\infty} \frac{F_{4n+3}}{F_{2n}\,F_{2n+1}\,F_{2n+2}\,F_{2n+3}} = 1/2$$

derived from

$$\sum_{n=1}^{\infty} \left[1/(F_{2n}F_{2n+1}) - 1/(F_{2n+2}F_{2n+3})\right]$$

(11) $$\sum_{n=1}^{\infty} L_{n+2}/(F_n F_{n+4}) = 17/6$$

derived from

$$\sum_{n=1}^{\infty} (1/F_n - 1/F_{n+4})$$

(12) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} F_{2n+1}}{F_n^2 F_{n+1}^2} = 1$$

derived from

$$\sum_{n=1}^{\infty} (L_n^2/F_n^2 - L_{n+1}^2/F_{n+1}^2)$$

(13) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} L_{n+1}}{F_n F_{n+1} F_{n+2}} = 1$$

derived from

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[1/(F_n F_{n+1}) + 1/(F_{n+1} F_{n+2})\right]$$

(14) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{3n} L_{3n+3}} = \frac{3 - r}{40(1 + r)}$$

derived from

$$\sum_{n=1}^{\infty} (L_{3n-3}/L_{3n} - L_{3n}/L_{3n+3})$$

(15) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} F_{6n+3}}{F_{3n}^2 F_{3n+3}^2} = 1/8$$

derived from

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[1/F_{3n}^2 + 1/F_{3n+3}^2\right]$$

(16) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+5}} = \frac{150 r^{-1} - 83}{150}$$

derived from

$$\sum_{n=1}^{\infty} (F_{n-1}/F_n - F_{n+4}/F_{n+5})$$

(17) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} F_{6n+3}}{F_{6n} F_{6n+6}} = 1/16$$

(18) $$\sum_{n=1}^{\infty} \frac{F_{2n+5}}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4} F_{n+5}} = 1/15$$

derived from $$\sum_{n=1}^{\infty} \left[\frac{1}{F_n F_{n+1} F_{n+2} F_{n+3}} - \frac{1}{F_{n+2} F_{n+3} F_{n+4} F_{n+5}}\right]$$

(19) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{2n-1} F_{2n+3}} = 1/6$$

derived from
$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{F_{2n-1} F_{2n+1}} + \frac{1}{F_{2n+1} F_{2n+3}}\right]$$

(20) $$\sum_{n=1}^{\infty} \frac{F_{2n}}{F_{n+2}^2 F_{n-2}^2} = 85/108$$

derived from
$$\sum_{n=1}^{\infty} \left[1/F_{n-2}^2 - 1/F_{n+2}^2\right]$$

CONCLUSION

Two main lines of development are open for continuing this research: (1) Building up a collection of formulas; (2) Finding additional methods for arriving at the summation of infinite Fibonacci-Lucas **series.**

Results, whether in the form of isolated formulas (with proof), or other more extensive developments should be reported to the Editor of the Fibonacci Quarterly.

REFERENCES

1. Problem H-10, proposed by R. L. Graham, FQ, April 1963, p. 53.
2. Problem B-9, proposed by R. L. Graham, FQ, April 1963, p. 85.
3. Problem B-19, proposed by L. Carlitz, FQ, Oct. 1963, p. 75.
4. Problem B-23, iii, proposed by S. L. Basin, FQ, Oct. 1963, p. 76.
5. Solution to Problem H-10, solution by L. Carlitz, FQ, Dec. 1963, p. 49.
6. Solution to B-9, solution by Francis D. Parker, FQ, Dec. 1963, p. 76.
7. Solution to B-19, solution to John H. Avila, FQ, Feb. 1964, p. 75.
8. Solution to B-23, iii, solution by J. L. Brown, Jr., FQ, Feb. 1964, p. 79.
9. Problem H-56, proposed by L. Carlitz, FQ, Feb. 1965, p. 45.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contribution are asked to enclose self-addressed stamped postcards.

B-160 *Proposed by Robert H. Anglin, Dan River Mills, Danville, Virginia.*

Show that, if $x = F_nF_{n+3}$, $y = 2F_{n+1}F_{n+2}$, and $z = F_{2n+3}$, then $x^2 + y^2 = z^2$.

B-161 *Proposed by John Ivie, Student at University of California, Berkeley, California*

Given the Pell numbers defined by $P_{n+2} = 2P_{n+1} + P_n$, $P_0 = 0$, $P_1 = 1$, show that for $k > 0$;

(i)
$$P_k = \sum_{r=0}^{[(k-1)/2]} \binom{k}{2r+1} 2^r .$$

(ii)
$$P_{2k} = \sum_{r=1}^{k} \binom{k}{r} 2^r P_r .$$

B-162 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let r be a fixed positive integer and let the sequences $u_1, u_2, \cdots$ satisfy $u_n = u_{n-1} + u_{n-2} + \cdots + u_{n-r}$ for $n > r$ and have initial conditions $u_j = 2^{j-1}$ for $j = 1, 2, \cdots, r$. Show that every representation of U_n as a sum of distinct u_j must be of the form u_n itself or contain explicitly the terms $u_{n-1}, u_{n-2}, \cdots, u_{n-r+1}$ and some representation of u_{n-r}.

B-163 *Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico*

Let n be a positive integer. Clearly

$$(1 + \sqrt{5})^n = a_n + b_n\sqrt{5} ,$$

with a_n and b_n integers. Show that 2^{n-1} is a divisor of a_n and of b_n.

B-164 *Proposed by J. A. H. Hunter, Toronto, Canada.*

A Fibonacci-type sequence is defined by:

$$G_{n+2} = G_{n+1} + G_n ,$$

with $G_1 = a$ and $G_2 = b$. Find the minimum positive values of integers a and b, subject to a being odd, to satisfy:

$$G_{n-1}G_{n+1} - G_n^2 = -11111(-1)^n \quad \text{for } n > 1.$$

B-165 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Define the sequence $\{b(n)\}$ by

$$b(1) = b(2) = 1, \quad b(2k) = b(k), \quad \text{and} \quad b(2k+1) = b(k+1) + b(k)$$

for $k \geq 1$. For $n \geq 1$, show the following:

(a) $$b([2^{n+1} + (-1)^n]/3) = F_{n+1} .$$

(b) $$b([7 \cdot 2^{n-1} + (-1)^n]/3) = L_n .$$

SOLUTIONS

A MULTIPLICATIVE ANALOGUE

B-142 *Proposed by William D. Jackson, SUNY at Buffalo, Amherst, N.Y.*

Define a sequence as follows: $A_1 = 2$, $A_2 = 3$, and $A_n = A_{n-1}A_{n-2}$ for $n > 2$. Find an expression for A_n.

Solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pa.

Let $B_n = \ln A_n$ for $n \geq 1$. Then $B_1 = \ln 2$, $B_2 = \ln 3$ and $B_n = B_{n-1} + B_{n-2}$ for $n > 2$. Clearly

$$B_n = F_{n-2} \cdot \ln 2 + F_{n-1} \cdot \ln 3$$

for $n > 2$, or

$$A_n = 2^{F_{n-2}} \cdot 3^{F_{n-1}}$$

for $n > 2$.

Also solved by Christine Anderson, Richard L. Breisch, Timothy Burns, Herta T. Freitag, J. A. H. Hunter (Canada), John Ivie, Bruce W. King, Leslie M. Klein, Arthur Marshall, C. B. A. Peck, John Wessner, Gregory Wulczyn, Michael Yoder, David Zeitlin, and the proposer.

THE DETERMINANT VANISHES

B-143 *Proposed by Raphael Finkelstein, Tempe, Arizona.*

Show that the following determinant vanishes when a and d are natural numbers:

$$\begin{vmatrix} F_a & F_{a+d} & F_{a+2d} \\ F_{a+3d} & F_{a+4d} & F_{a+5d} \\ F_{a+6d} & F_{a+7d} & F_{a+8d} \end{vmatrix} .$$

What is the value of the determinant one obtains by replacing each Fibonacci number by the corresponding Lucas number?

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico

Let $r = F_{6d}/F_{3d}$ and $s = F_{6d+1} - rF_{3d+1}$. Then

$$rF_{3d} + sF_0 = F_{6d} \quad \text{and} \quad rF_{3d+1} + sF_1 = F_{6d+1}.$$

It follows by induction that

$$F_{n+6d} = rF_{n+3d} + sF_n$$

for all n; in particular, it is true for $n = a$, $n = a + d$, and $n = a + 2d$. Hence the three rows of the matrix are linearly dependent and the determinant is zero.

If each Fibonacci number is replaced by the corresponding Lucas number, the determinant will also be zero by similar reasoning.

Editorial Note: It can be shown that $r = L_{3d}$ and $s = (-1)^{d+1}$.

Also solved by F. D. Parker, C. B. A. Peck, David Zeitlin and the proposer.

LUCAS ALPHAMETIC

B-144 *Proposed by J. A. H. Hunter, Toronto, Canada.*

In this alphametic each distinct letter stands for a particular but different digit, all ten digits being represented here. It must be the Lucas series, but what is the value of the SERIES?

```
    ONE
  THREE
  START
      L
 ------
 SERIES
```

Solution by Charles W. Trigg, San Diego, California

Since they are the initial digits of integers, none of θ, T, S, or L can be zero. Proceeding from the left, clearly $S = 1$, $E = 0$, and T is 8 or 9. In either event, $H + T > 10$, so $T = 8$. Then from the units' column, $L = 3$.

The three integer columns then establish the equalities:

$$\begin{aligned} N + R + 1 &= 10 \\ \theta + R + A + 1 &= I + 10 \\ H + 8 + 1 &= R + 10 . \end{aligned}$$

Whereupon, $N + H = 10$ and $(N,H) = (4,6)$ or $(6,4)$. But $H = R + 1$, so $R = 5$, $H = 6$, $N = 4$.

Then $\theta + A = I + 4$, and $\theta = 9$, $A = 2$, $I = 7$. (θ and A may be interchanged.) Consequently, S E R I E S = 105701.

Also solved by Richard R. Breisch, Timothy Burns, A. Gommel, Edgar Karst, John Milsom, C. B. A. Peck, John Wessner, Michael Yoder, and the proposer.

BINARY N-TUPLES

B-145 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Given an unlimited supply of each of two distinct types of objects, let $f(n)$ be the number of permutations of n of these objects such that no three consecutive objectives are alike. Show that $f(n) = 2F_{n+1}$, where F_n is the n^{th} Fibonacci number.

Solution by Bruce W. King, Adriondack Community College, Glen Falls, N.Y.

Call a permutation of the required type an "admissible n permutation," and let A and B be two of the distinct types of objects. A list of admissible n + 1 permutations can be constructed in the following way:

(a) For each admissible n permutation ending in A, adjoin B on the right; for each distinct admissible n permutation ending in B, adjoin on the right.

(b) For each distinct admissible n - 1 permutation ending in A, adjoin BB on the right; for each distinct admissible n - 1 permutation ending in B, adjoin AA on the right.

Certainly the resulting list contains only admissible n + 1 permutations. Furthermore, there is no possibility of duplication because the permutations described in (b) end with two identical letters, but those described in (a) end with two different letters. Lastly, no n + 1 permutation is unobtainable in this way. For, if there were such a permutation, either the $n - 1$ permutation excluding its last two letters, or the n permutation excluding its last letter would have to be admissible. Consequently, we see that

$$f(n+1) = f(n-1) + f(n) .$$

The rest is an easy proof by induction.

By direct enumeration, $f(3) = 6 = 2F_4$. If $f(n) = 2F_{n+1}$ for integers $n \le N$, then

$$f(N+1) = f(N-1) + f(N) = 2F_N + 2F_{N+1} = 2(F_N + F_{N+1}) = 2F_{N+2} ,$$

and the proof is complete.

Also solved by J. L. Brown, Jr., C. B. A. Peck, Michael Yoder, and the proposer.

ANGLES OF A TRIANGLE

B-146 *Proposed by Walter W. Horner, Pittsburgh, Pennsylvania*

Show that $\pi = \operatorname{Arctan}(1/F_{2n}) + \operatorname{Arctan} F_{2n+1} + \operatorname{Arctan} F_{2n+2}$.

Solution by C. B. A. Peck, Ordnance Research Lab., State College, Pa.

From the solution to H-82 (FQ, 6, 1, 52-54), we get

$$\operatorname{Arctan}(1/F_{2n}) = \operatorname{Arctan}(1/F_{2n+2}) + \operatorname{Arctan}(1/F_{2n+1}).$$

The result now follows from $\operatorname{Arctan} x + \operatorname{Arctan}(1/x) = \pi/2$.

Also solved by Herta T. Freitag, John Ivie, Bruce W. King, John Wessner, Gregory Wulczyn, Michael Yoder, and the proposer.

TWIN PRIMES

B-147 *Proposed by Edgar Karst, University of Arizona, Tuscon, Arizona, in honor of the 66th birthday of Hansraj Gupta on Oct. 9, 1968.*

Let

$$S = (1/3 + 1/5) + (1/5 + 1/7) + \cdots + (1/32717 + 1/32719)$$

be the sum of the sum of the reciprocals of all twin primes below 2^{15}. Indicate which of the following inequalities is true:

(a) $S < \pi^2/6$ (b) $\pi^2/6 < S < \sqrt{e}$ (c) $\sqrt{e} < S$.

Solutions by Paul Sands, Student, University of New Mexico, Albuquerque, New Mexico, and the proposer. (Both used electronic computers.)

	Proposer	Paul Sands
True inequality	(b)	(b)
Number of pairs of primes involved	55	55
S, to six decimal places	1.647986	1.648627

* * * * *

(Continued from p. 210.)

6. $$T_n = -(-i)^n$$

7. $$T_{n+1} = 5T_n - 6T_{n-1}$$
$$T_n = 2^n + 3^{n-1}$$

8. $$r = \frac{5 + \sqrt{29}}{2}, \qquad s = \frac{5 - \sqrt{29}}{2}$$

$$T_n = \frac{r^n - s^n}{\sqrt{29}} \text{ with terms } 1, 5, 26, 135, \cdots$$

$$V_n = r^n + s^n \text{ with terms } 5, 27, 140, \cdots$$

9. $$r = \frac{3 + i\sqrt{11}}{2}, \qquad s = \frac{3 - i\sqrt{11}}{2}$$

$$T_n = \left(\frac{33 - 16i\sqrt{11}}{55}\right)r^n + \left(\frac{33 + 16i\sqrt{11}}{55}\right)s^n$$

10. $$T_{n+1} = 5T_n + 2T_{n-1}; \quad T_1 = 3, \quad T_2 = 7.$$

* * * * *

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