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DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

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ON THE NUMBER OF DIVISIONS NEEDED IN FINDING THE GREATEST COMMON DIVISOR

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Let $n(a,b)$ and $N(a,b)$ be the number of divisions needed in finding the greatest common divisor of positive integers a,b using the Euclidean algorithm and the least absolute value algorithm, respectively. In addition to showing some properties of periodicity of $n(a,b)$ and $N(a,b)$, the paper gives a proof of the following theorems:

Theorem 1. If $n(a,b) = k > 1$, then $a + b \geq f_{k+3}$ and the pair (a,b) with smallest sum such that $n(a,b) = k$ is the pair (f_{k+1}, f_{k+2}), where $f_1 = 1$, $f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$, $n = 1, 2, 3, \cdots$.

Theorem 2. If $N(a,b) = k > 1$, then $a + b \geq x_{k+1}$ and the pair (a,b) with smallest sum such that $N(a,b) = k$ is the pair $(x_k, x_k + x_{k-1})$, where $x_1 = 1$, $x_2 = 2$ and $x_k = 2x_{k-1} + x_{k-2}$, $k = 3,4,\cdots$. These results may be compared with other results found in [1], [2].

Since $n(a,b) = n(b,a)$, we can assume $a \leq b$. To prove the first theorem, let $n(a,b) = k$ and assume the k steps in finding (a,b) are

$$\begin{aligned} b &= q_1 a + r_1 \\ a &= q_2 r_1 + r_2 \\ &\cdots \\ r_{k-3} &= q_{k-1} r_{k-2} + r_{k-1} \\ r_{k-2} &= q_k r_{k-1} \end{aligned}$$

If $k = 1$, then $r_1 = 0$ so $b = q_1 a$ and the smallest pair (a,b) is $(1,1)$ so

$$a = f_1, \quad b = f_2, \quad a + b = f_3 = 2 .$$

Note this case is not included in the theorem. In case $k > 1$, it is evident that the smallest values of a,b will be obtained for $r_{k-1} = 1$ and all the q's $= 1$ except q_k, which cannot be 1 but is 2. Thus the pairs $(r_{k-1}, r_{k-2}), \cdots$, (a,b) are $(1,2)$, $\cdots$, (f_{k+1}, f_{k+2}). Since

$$a + b = f_{k+1} + f_{k+2} = f_{k+3} ,$$

the theorem is proved.

We have

Corollary 1. If $a + b < f_{k+3}$, then $n(a,b) < k$ for $k > 1$.

For $b = a + i$, i a fixed positive integer so that $b < 2a$, the quantities satisfy

$$(1) \qquad n(a + mi, \quad a + [m + 1]i) = n(a, a + i), \quad m = 0, 1, 2, \cdots .$$

This follows from the remark that if $n(a,b) = k$, then

$$n(a + b, 2a + b) = k + 1, \quad k = 1, 2, 3, \cdots .$$

This is evident since the first division would be $(2a + b) = 1(a + b) + a$ and $n(a, a + b) = n(a,b) = k$. Equation (1) is a consequence since each n is one more than $n(i, a + mi) = n(i, a)$. The periodicity is evident in the table of values of $n(a,b)$ for $a \leq b < 2a$.

a = 1	1
2	1 2
3	1 2 3
4	1 2 2 3
5	1 2 3 4 3
6	1 2 2 2 3 3
7	1 2 3 3 4 4 3
8	1 2 2 4 2 5 3 3
9	1 2 3 2 3 4 3 4 3
10	1 2 2 3 3 2 4 4 3 3
11	1 2 3 4 4 3 4 5 5 4 3
12	1 2 2 2 2 4 2 5 3 3 3 3
13	1 2 3 3 3 5 3 4 6 4 4 4 3
14	1 2 2 4 3 4 3 2 4 5 4 5 3 3
15	1 2 3 2 4 2 3 3 4 4 3 5 3 4 3

Fig. 1 $n(a,b)$ for $b = a, a + 1, \cdots, 2a - 1$

To prove Theorem 2, assume the steps in finding (a,b) with $n(a,b) = k$ are

$$\begin{aligned} b &= q_1 a \pm r_1 \\ a &= q_2 r_1 \pm r_2 \\ &\cdots \\ r_{k-3} &= q_{k-1} r_{k-2} \pm r_{k-1} \\ r_{k-2} &= q_k\, r_{k-1} \end{aligned} ,$$

where

$$0 \le r_1 \le \tfrac{1}{2} a, \quad 0 < r_2 \le \tfrac{1}{2} r_1, \cdots, 0 < r_{k-1} \le \tfrac{1}{2} r_{k-2} .$$

Because of the restriction on the remainders, we must have $q_2, q_3, \cdots, q_k$ equal to or greater than 2. But since

$$2r_i + r_{i+1} \le 3r_i - r_{i+1}, \quad i = 1, \cdots, k - 1,$$

in each case, we obtain the smallest sum $a + b$ with $q_2 = \cdots = q_k = 2$ and with $q_1 = 1$. For $k = 1$, we have $1 = 1 \cdot 1$ so $a = b = 1$. Set $x_i = r_{k-1}$. For $k > 1$,

$$a = x_k = 2x_{k-1} + x_{k-2} \quad \text{and} \quad b = x_{k+1} = x_k + x_{k-1} .$$

Then

$$a + b = 2x_k + x_{k-1} = x_{k+1} .$$

This completes the proof of the theorem.

Corollary 2. If $a + b < x_{k+1}$, then $N(a,b) < k$ for $k > 1$.

Figure 2 exhibits the periodicity (for i fixed):

(2) $$N(a, a + i) = N(a + mi,\ a + [m + 1]i), \quad 1 \le i \le a/2 ,$$

and the symmetry:

$$N(a, a+i) = N(a, 2a-i), \quad 1 \le i \le a-1. \tag{3}$$

a = 1	1
2	2
3	2 2
4	2 2 2
5	2 3 3 2
6	2 2 2 2 2
7	2 3 3 3 3 2
8	2 2 3 2 3 2 2
9	2 3 2 3 3 2 3 2
10	2 2 3 3 2 3 3 2 2
11	2 3 3 3 3 3 3 3 3 2
12	2 2 2 2 4 2 4 2 2 2 2
13	2 3 3 3 4 3 3 4 3 3 3 2
14	2 2 3 3 3 3 2 3 3 3 3 2 2
15	2 3 2 3 2 3 3 3 3 2 3 2 3 2
16	2 2 3 2 3 2 4 2 4 2 3 2 3 2 2
17	2 3 3 3 4 3 4 3 3 4 3 4 3 3 2 2
18	2 2 2 3 4 2 4 2 2 2 4 2 4 3 2 2 2
19	2 3 3 3 3 3 4 4 3 3 4 4 3 3 3 3 3 2
20	2 2 3 2 2 3 3 3 4 2 4 3 3 3 2 2 3 2 2
21	2 3 2 3 3 3 2 4 3 3 3 3 4 2 3 3 3 2 3 2
22	2 2 3 3 4 2 3 3 4 3 2 3 4 3 3 2 4 3 3 2 2
23	2 3 3 3 4 3 4 3 4 4 3 3 4 4 3 4 3 4 3 3 3 2

Fig. 2 $N(a,b)$ for $b = a+1, \cdots, 2a-1$

I wish to acknowledge the assistance of Professor V. C. Harris in shortening the proofs.

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DIAGONAL SUMS OF GENERALIZED PASCAL TRIANGLES

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1. INTRODUCTION

A sequence of generalized Fibonacci numbers $u(n; p, q)$ which can be interpreted as sums along diagonals in Pascal's triangle appear in papers by Harris and Styles [1], [2]. In this paper, Pascal's binomial coefficient triangle is generalized to trinomial and other polynomial coefficient arrays, and a method is given for finding the sum of terms along any rising diagonals in any such array, given by $(1 + x + \cdots + x^{r-1})^n$, $n = 0, 1, 2, 3, \cdots$, $r \geq 2$.

2. THE TRINOMIAL TRIANGLE

If we write only the coefficients appearing in expansions of the trinomial, $(1 + x + x^2)^n$, we are led to the following array:

1												
1	1	1										
1	2	3	3	1								
1	3	6	7	6	3	1						
1	4	10	16	19	16	10	4	1				
1	5	15	30	45	51	45	30	15	5	1		
1	6	21	50	90	126	141	126	90	50	21	6	1

$\cdots$

Call the top row the zero$^{\text{th}}$ row and the left column the zero$^{\text{th}}$ column. Then, let

$$G_0 = \frac{1}{1 - x}, \qquad G_1 = \frac{x}{(1 - x)^2}, \qquad G_2 = \frac{x}{(1 - x)^3}$$

be the column generators as the columns are positioned above. The general recurrence for the column generators is

(1) $$G_{n+2} = \frac{x}{1-x}(G_{n+1} + G_n) \; .$$

Let

$$G = \sum_{n=0}^{\infty} G_n = \sum_{n=0}^{\infty} u(n;\, 0,1)x^n \; .$$

The sum G in the general case will have for the coefficient of x^n the number $u(n;\, p,q)$, which, as applied to the trinomial triangle, will be the sum of the term in the left column and the n^{th} row and the terms obtained by taking steps p units up and q units to the right. That is, $u(n;\, p,q)$ is a member of a sequence of sums whose terms lie on particular diagonals of the trinomial triangle. To find G, for $p = 0$ and $q = 1$, we use the method of Polya [3] and the recurrence relation (1). Let S_n be the sum of the first n terms of G.

$$G_2 = \frac{x}{1-x}(G_1 + G_0)$$

$$G_3 = \frac{x}{1-x}(G_2 + G_1)$$

$$\cdots$$

$$G_{n+1} = \frac{x}{1-x}(G_n + G_{n-1})$$

$$G_{n+2} = \frac{x}{1-x}(G_{n+1} + G_n) \; .$$

Summing vertically,

$$S_n + G_{n+2} + G_{n+1} - G_0 - G_1 = \frac{x}{1-x}(S_n + G_{n+1} - G_0 + S_n)$$

$$S_n\left(1 - \frac{2x}{1-x}\right) = G_0\left(1 - \frac{x}{1-x}\right) + G_1 + G_{n+1}\left(\frac{x}{1-x} - 1\right) - G_{n+2} \; .$$

It can be shown that $\lim_{n\to\infty} G_n = 0$ for $|x| < 1/r$, $r > 2$, so that

$$G = \lim_{n\to\infty} S_n = \frac{1-x}{1-3x}\left(\frac{1-2x}{(1-x)^2} + \frac{x}{(1-x)^2}\right) = \frac{1}{1-3x},$$

which was to be expected, since

$$\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + 81x^4 + 245x^5 + \cdots,$$

where each coefficient is the sum of an appropriate row in the triangle. In fact, each coefficient of x^n is $u(n; 0,1) = 3^n$.

Now, let us consider $u(n; p,1)$. Here

$$G_0^\star = 1/(1-x) \quad \text{and} \quad G_1^\star = x^{p+1}/(1-x)^2$$

with recurrence

$$G_{n+2}^\star = \frac{x}{1-x}(x^p G_{n+1}^\star + x^{2p} G_n^\star) . \tag{2}$$

(Notice that multiplication by x^p and x^{2p} allows for moving up p rows in the triangle.) Following Polya's method of summing vertically as before,

$$S_n^\star\left(1 - \frac{x^{p+1} + x^{2p+1}}{1-x}\right) = \frac{1}{1-x}\left(1 - \frac{x^{p+1}}{1-x}\right) + \frac{x^{p+1}}{(1-x)^2}$$
$$+ \left(\frac{x^{p+1}}{1-x} - 1\right) G_{n+1}^\star - G_{n+2}^\star .$$

Since, again, $\lim_{n\to\infty} G_n^\star = 0$, for $|x| < 1/r$, $r > 2$, so that

$$G^\star = \lim_{n\to\infty} S_n^\star = \frac{1}{1 - x - x^{p+1} - x^{2p+1}} .$$

Now, if $p = 1$, we get the generating function for the Tribonacci numbers, $G = 1/(1 - x - x^2 - x^3)$. The Tribonacci numbers T_n (see [4]) are 1, 1, 2, 4, 7, 13, 24, $\cdots$, where each term after the third is the sum of the preceding three terms. That is, $u(n;\ 1,1) = T_{n+1}$. For a particular verification, the reader is invited in each case to perform the indicated division.

Now, if we let $q = 2$, then we must deal with every other term of the column generator recurrence relation. To solve $u(n;\ 0,2)$, $G_0 = 1/(1-x)$, $G_2 = x/(1-x)^3$, and the recurrence (1) originally considered, leads to

$$G_{2n+4} = \left(\frac{x^2}{(1-x)^2} + \frac{2x}{1-x}\right)G_{2n+2} - \frac{x^2}{(1-x)^2}\cdot G_{2n} \tag{3}$$

Following the same method as before, we have, for

$$S_n = \sum_{i=0}^{n} G_{2i} ,$$

$$S_n\left(1 - \frac{2x - x^2}{(1-x)^2} + \frac{x^2}{(1-x)^2}\right) = G_0\left(1 - \frac{2x - x^2}{(1-x)^2}\right) + G_2 + R_n ,$$

where R_n is a term involving G_{2n+4} and G_{2n+2}. Again, since

$$\lim_{n\to\infty} G_n = 0, \quad \lim_{n\to\infty} R_n = 0, \quad |x| < 1/r, \quad r > 2,$$

$$\lim_{n\to\infty} S_n = G = \frac{1 - 2x}{1 - 4x + 3x^2} = \sum_{n=0}^{\infty} u(n;\ 0,2)x^n .$$

This gives us a generating function for sums of alternate terms of rows in the trinomial triangle.

Let $p = 1$ and $q = 2$ and return to $G_0 = 1/(1-x)$, $G_2 = x^2/(1-x)^2$, and, from recurrence (3),

$$G^\star_{2n+4} = x\left(\frac{x^2}{(1-x)^2} + \frac{2x}{1-x}\right)G^\star_{2n+2} - \frac{x^2\cdot x^2}{(1-x)^2}G^\star_{2n} ,$$

where we must multiply by x and x^2 to account for moving up one row through the trinomial array. Going to the Polya method again to find $u(n;\ 1,2)$ we have, for

$$S_n^\star = \sum_{i=0}^{n} G_{2i}^\star ,$$

$$S_n^\star\left(1 - \frac{(2x - x^2)x}{(1 - x)^2} + \frac{x^2 \cdot x^2}{(1 - x)^2}\right) = G_0^\star\left(1 - \frac{x(2x - x^2)}{(1 - x)^2}\right) + G_2^\star + R_n ,$$

where R_n involves only terms $G_{2n+2}^\star$ and $G_{2n+4}^\star$, so that $\lim_{n\to\infty} R_n = 0$, $|x| < 1/r$, $r > 2$.

$$S_n^\star\left(\frac{1 - 2x - x^2 + x^3 + x^4}{(1 - x)^2}\right) = (1 - 2x + x^2 - 2x^2 + x^3 + x^2)/(1 - x)^3 + R_n$$

$$G^\star = \lim_{n\to\infty} S_n^\star = \frac{1 - 2x + x^3}{(1 - x)(1 - 2x - x^2 + x^3 + x^4)} ,$$

which simplifies to

$$\frac{1 - x - x^2}{1 - 2x - x^2 + x^3 + x^4} = \sum_{n=0}^{\infty} u(n;\ 1,2)x^n .$$

Returning now to the more general case, we find the generating function for the numbers $u(n;\ p,2)$. Using the recurrence relation (3), but allowing for moving up p rows in the triangle, and then summing vertically as before yields

$$S_n\left(1 - \frac{(2x - x^2)x^p}{(1 - x)^2} + \frac{x^2 \cdot x^{2p}}{(1 - x)^2}\right) = \frac{1}{1 - x}\left(1 - \frac{x^p(2x - x^2)}{(1 - x)^2}\right)$$

$$+ \frac{x^{p+1}}{(1 - x)^3} + R_n ,$$

where again $\lim_{n\to\infty} R_n = 0$ for $|x| < 1/r$. Simplifying the above, and letting $\lim_{n\to\infty} S_n = G$,

$$G = \frac{(1 - 2x + x^2 + x^{p+2} - x^{p+1})/(1 - x)}{(1 - x)^2 - 2x^{p+1} + x^{p+2} + x^{2p+2}}$$

$$= \frac{1 - x - x^{p+1}}{(1 - x)^2 - 2x^{p+1} + x^{p+2} + x^{2p+2}} = \sum_{n=0}^{\infty} u(n;\ p,2)\, x^n .$$

This agrees with the previous cases for $p = 1,\ q = 2$ and for $p = 0,\ p = 2$.

In seeking the numbers $u(n;\ 0,3)$, we need the recurrence relation

$$G_{3(n+2)} = \frac{3x^2 - 2x^3}{(1 - x)^3} G_{3(n+1)} + \frac{x^3}{(1 - x)^3} G_{3n} ,$$

which, following the previous method, gives

$$S_n\left(1 - \frac{3x^2 - 2x^3}{(1 - x)^3} - \frac{x^3}{(1 - x)^3}\right) = \frac{1}{1 - x}\left(1 - \frac{3x^2 - 2x^3}{(1 - x)^3}\right)$$

$$+ \frac{2x^2 - x^3}{(1 - x)^4} + R_n ,$$

and

$$\sum_{n=0}^{\infty} u(n;\ 0,3)x^n = \frac{1 - 2x}{1 - 3x} = 1 + \sum_{n=0}^{\infty} 3^n x^{n+1} .$$

In fact,

$$\sum_{n=0}^{\infty} u(n;\ p,3)x^n = \frac{1 - 2x + x^2 - x^{p+2}}{(1 - x)^3 - 3x^{p+2} + 2x^{p+3} - x^{2p+3}} .$$

3. QUADRINOMIALS, PENTANOMIALS, AND HEXANOMIALS

If we consider the array of coefficients which arise in the expansion of the quadrinomial $(1 + x + x^2 + x^3)^n$,

$$
\begin{array}{lllllllllllll}
1 \\
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 3 & 2 & 1 \\
1 & 3 & 6 & 10 & 12 & 12 & 10 & 6 & 3 & 1 \\
1 & 4 & 10 & 20 & 31 & 40 & 44 & 40 & 31 & 20 & 10 & 4 & 1 \\
 & & & & & & \cdots
\end{array}
$$

and use the methods of the preceding section, the expressions given below can be derived without undue difficulty. For the quadrinomial coefficients, the generating functions are given by

$$G_{n+3} = \frac{x}{1 - x}(G_{n+2} + G_{n+1} + G_n)$$

where

$$G_0 = 1/(1 - x), \quad G_1 = x/(1 - x)^2, \quad G_2 = x/(1 - x)^3, \quad G_3 = x/(1 - x)^4,$$
$$G_4 = (3x^2 - 3x^3 + x^4)/(1 - x)^5 \quad .$$

It is easy to find that $u(n;\ 0,1) = 4^n$ and $u(n;\ 1,1) = Q_{n+1}$, where Q_n is the quadrinacci number given by $1, 1, 2, 4, 8, 15, 29, \cdots$, where each term after the fourth is the sum of the preceding four terms (see [4]). The generating function for Q_n is $1/(1 - x - x^2 - x^3 - x^4)$, and

$$\sum_{n=0}^{\infty} u(n;\ p,1)x^n = \frac{1}{1 - x - x^{p+1} - x^{2p+1} - x^{3p+1}} \quad .$$

From the recursion

$$G_{2(n+3)} = \frac{2x - x^2}{(1 - x)^2} G_{2(n+2)} + \frac{x^2}{(1 - x)^2} G_{2(n+1)} + \frac{x^2}{(1 - x)^2} G_{2n} \ ,$$

one finds

$$\sum_{n=0}^{\infty} u(n;\ 0,2)x^n = \frac{1 - 2x}{1 - 4x} = 1 + \sum_{n=1}^{\infty} 2^{2n-1} x^n ;$$

$$\sum_{n=0}^{\infty} u(n;\ 1,2)\ x^n = \frac{1 - x - x^2}{1 - 2x - x^2 + x^3 - x^4 - x^5} ;$$

$$\sum_{n=0}^{\infty} u(n;\ p,2)\ x^n = \frac{1 - x - x^{p+1}}{(1 - x)^2 - 2x^{p+1} + x^{p+2} - x^{2p+2} - x^{3p+2}} .$$

Also, from

$$G_{3(n+3)} = \frac{3x - 3x^2 + x^3}{(1 - x)^3} G_{3(n+2)} - \frac{3x^2 - x^3}{(1 - x)^3} G_{3(n+1)} + \frac{x^3}{(1 - x)^3} G_{3n} ,$$

one finds

$$\sum_{n=0}^{\infty} u(n;\ 0,3)x^n = \frac{1 - 3x}{1 - 5x + 4x^2}$$

If one continues in a similar way, the analogous results for the pentanomial becomes $u(n;\ 0,1) = 5^n$;

$$\sum_{n=0}^{\infty} u(n;\ p,1)\ x^n = \frac{1}{1 - x - x^{p+1} - x^{2p+1} - x^{3p+1} - x^{4p+1}}$$

where $u(n;\ 1,1) = 1, 1, 2, 4, 8, 16, 31, 61, \cdots$, and each term after the fifth is the sum of the preceding five terms:

$$\sum_{n=0}^{\infty} u(n;\ 0,2)x^n = \frac{1 - 3x}{1 - 6x + 5x^2} ;$$

$$\sum_{n=0}^{\infty} u(n;\ p,2)x^n = \frac{1 - x - x^{p+1} - x^{2p+1}}{(1 - x)^2 - 2x^{p+1} + x^{p+2} - 2x^{2p+1} + x^{2p+2} + x^{3p+2} + x^{4p+2}} .$$

For the hexanomial, we can derive $u(n; 0, 1) = 6^n$;

$$\sum_{n=0}^{\infty} u(n; p, 1)\, x^n = \frac{1}{1 - x - x^{p+1} - x^{2p+1} - x^{3p+1} - x^{4p+1} - x^{5p+1}} ;$$

$$\sum_{n=0}^{\infty} u(n; 0,2)x^n = \frac{1 - 3x}{1 - 6x} ;$$

$$\sum_{n=0}^{\infty} u(n; p, 2)x^n = \frac{1 - x - x^{p+1} - x^{2p+1}}{(1-x)^2 - 2x^{p+1} + x^{p+2} - 2x^{2p+1} + x^{2p+2} - x^{3p+2} - x^{4p+2} - x^{5p+2}}$$

In general, for a k-nomial (k terms) coefficient array, one discovers that $u(n; 0, 1) = k^n$ and $u(n; 0, k) = k^{n-1}$, $n \geq 1$. Now we can readily generalize our results.

4. GENERALIZATION OF TRINOMIAL CASE

In the quadratic equation $y^2 - ay + b = 0$, let $a = b = x/(1 - x)$. Then, if r_1 and r_2 are the roots of the above quadratic, let

$$r_1^k + r_2^k = P_k\left(\frac{x}{1 - x}, \frac{x}{1 - x}\right),$$

given by $P_0 = 2$, $P_1 = x/(1 - x)$,

$$P_2 = \left(\frac{x}{1 - x}\right)^2 + \frac{2x}{1 - x} ,$$

and satisfying

$$P_{k+2} = \frac{x}{1 - x}(P_{k+1} + P_k) .$$

Now, the recurrence relation for the column generators for the trinomial case is (let $q = k$)

$$G_{(n+2)k} = P_k G_{(n+1)k} + (-1)^{k+1}\left(\frac{x}{1 - x}\right)^k G_{nk} ,$$

leading to

$$G^{\star}_{(n+2)k} = x^p P_k G^{\star}_{(n+1)k} + \frac{(-1)^{k+1} x^{2p} x^k}{(1 - x)^k} G^{\star}_{nk}$$

where $G^{\star}_{nk} = x^{np} G_{nk}$ to allow for moving p steps up through the triangle. Then, summing vertically gives

$$S_n \left(1 - P_k x^p + \frac{(-1)^k x^{2p+k}}{(1 - x)^k} \right) = G^{\star}_0 (1 - P_k x^p) + G^{\star}_k + R_n ,$$

where $\lim\limits_{n \to \infty} R_n = 0$, $|x| < 1/r$, $r > 2$.

Hence,

$$G \left(\frac{(1 - x)^k - x^p P_k (1 - x)^k + (-1)^k x^{2p+k}}{(1 - x)^k} \right) = \frac{1}{1 - x} (1 - P_k x^p) + x^p G_k$$

for the column generators defined in Eq. (1).

Applying the formula given by Bicknell and Draim [5],

$$P_k = \sum_{i=0}^{[k/2]} \frac{k(k - i - 1)!}{(k - 2i)!\, i!} \cdot \left(\frac{x}{1 - x} \right)^{k-i} ,$$

$[x]$ the greatest integer function, gives an explicit formula for G. Since G is the generating function for the numbers $u(n; p, k)$, we have resolved our problem for the trinomial triangle. Harris and Styles [1] have solved the binomial case by summing diagonals of Pascal's triangle. Feinberg in [6] has given series convergents for $u(n; p, 1)$ for the trinomial and quadrinomial cases. We now move on to the solution of the general case for the array of coefficients formed from polynomials of n terms.

5. SYMMETRIC FUNCTIONS AND COLUMN GENERATORS: THE GENERAL CASE

Let

$$P(x) = x^n - p_1x^{n-1} + p_2x^{n-2} - \cdots + (-1)^j p_j x^{n-j} + \cdots + (-1)^n p_n ,$$

where p_j is the j^{th} symmetric function of the roots of $P(x) = 0$. (For a discussion of symmetric functions, see [7] and [8].) Now let $p_j(k)$ be the j^{th} symmetric function of the k^{th} powers of the roots of $P(x)$. Then

$$p_1(m + n) - p_1(m + n - 1)p_1 + p_1(m + n - 2)p_2 - \cdots + (-1)^n p_1(m)p_n \equiv 0 ,$$

since each p_1 represents sums of the products of solutions which are geometric progression solutions to the original difference equation whose auxiliaty polynomial is listed above. Thus we need n starting values for each such sequence.

If

$$G_{n+2} = \frac{x}{1 - x}(G_{n+1} + G_n) ,$$

then

$$G_{(n+2)q} = p_1(q)G_{(n+1)q} + (-1)^{q+1}\left(\frac{x}{1 - x}\right)^q G_{nq} ,$$

where

$$p_1(0) = 2, \quad p_1(1) = x/(1 - x);$$

and

$$p_1(m + 2) = \frac{x}{1 - x}(p_1(m + 1) + p_1(m)) ,$$

with auxiliary polynomial

$$y^2 - \frac{x}{1-x}y - \frac{x}{1-x} .$$

This is the resolution of our trinomial case, expressed in a modified form.

The column generators for the quadrinomial case will be related by

$$G_{n+3} = \frac{x}{1-x}(G_{n+2} + G_{n+1} + G_n)$$

where

$$G_{(n+3)q} = p_1(q)G_{(n+2)q} - p_2(q)G_{(n+1)q} + p_3(q)G_{nq} .$$

Here, the auxiliary polynomial is

$$y^3 - p_1(1)y^2 + p_2(1)y - p_3(1)$$

where

$$p_1(1) = p_3(1) = -p_2(1) = \frac{x}{1-x} .$$

Now,

$$p_2(k) = (p_1^2(k) - p_1(2k))/2$$

$$p_3(k) = \left(\frac{x}{1-x}\right)^k .$$

Next,

$$p_1(0) = 3, \quad p_1(1) = x/(1-x), \quad p_1(2) = \left(\frac{x}{1-x}\right)^2 + \frac{2x}{1-x} ,$$

and

$$p_1(m+3) = \frac{x}{1-x}(p_1(m+2) + p_1(m+1) + p_1(m)) .$$

Notice that, since our values for $p_1(q)$, $p_2(q)$, and $p_j(q)$ are defined sequentially and since moving up p rows can be adjusted by multiplying by x^p, we can solve the quadrinomial case. To derive $u(n; p,q)$, we can use $(G^\star_{iq} = x^{ip}G_q)$

$$G^\star_{(n+3)q} = x^p p_1(q) G_{(n+2)q} - x^{2p} p_2(q) G_{(n+1)q} + x^{3p} p_3(q) G^\star_{nq} ,$$

leading to

$$S_n(1 - x^p p_1(q) + x^{2p} p_2(q) - x^{3p} p_3(q)) = G_0(1 - x^p p_1(q) + x^{2p} p_2(q))$$
$$+ x^{2p} G_q (1 - x^p p_1(q)) + x^{4p} G_{2q} + R_n ,$$

where $\lim_{n\to\infty} R_n = 0$, $|x| < 1/r$, $r > 2$.

Using formulas given by Bicknell and Draim [9],

$$p_1(q) = \sum_{k=0}^{[q/3]} \sum_{n=0}^{\left[\frac{q-3k}{2}\right]} \frac{q(q - m - 2k - 1)!}{(q - 2n - 3k)!\,m!\,k!} \cdot \left(\frac{x}{1-x}\right)^{q-m-2k} ,$$

$$p_2(q) = \sum_{n=0}^{q/3} \sum_{n=0}^{\left[\frac{q-3k}{2}\right]} \frac{q(q - m - 2k - 1)!}{(q - 2n - 3k)!\,m!\,k!} \cdot \left(\frac{x}{1-x}\right)^{q-k} (-1)^{q-m-3k} ,$$

$$p_3(q) = \left(\frac{x}{1-x}\right)^q ,$$ [x] the greatest integer function,

we actually could write an explicit formula for G, the generating function for the numbers $u(n; p,q)$ for the quadrinomial case.

For the pentanomial case, we would go to

$$G_{n+4} = \frac{x}{1-x} (G_{n+3} + G_{n+2} + G_{n+1} + G_n) ,$$

with auxiliary polynomial

$$y^4 - p_1(1)y^3 + p_2(1)y^2 - p_3(1)y + p_4(1) = 0 ,$$

where

$$p_1(1) = p_3(1) = -p_2(1) = -p_4(1) = x/(1 - x) .$$

Then we need

$$p_1(0) = 4,\quad p_1(1) = x/(1 - x),\quad p_1(2) = (2x - x^2)/(1 - x)^2,$$
$$p_1(3) = (3x - 3x^2 + x^3)/(1 - x)^3,$$

and

$$p_1(n + 4) = \frac{x}{1 - x}\,(p_1(n + 3) + p_1(n + 2) + p_1(n + 1) + p_1(n));$$

$$p_2(k) = (p_1^2(k) - p_1(2k))/2 ,$$

$$p_3(k) = (p_1^3(k) - 3p_1(2k)p_1(k) + 2p_1(3k))/6,\ \text{(see [7])}$$

$$p_4(k) = (-1)^k\left(\frac{x}{1 - x}\right)^k .$$

The relationship

$$G^\star_{(n+4)q} = x^p p_1(q)G^\star_{(n+3)q} - x^{2p}p_2(q)G_{(n+2)q} + x^{3p}p_3(q)G_{(n+1)q} - x^{4p}p_4(q)G_{nq} ,$$

$G^\star_{iq} = x^{ip}G_{iq}$, combined with our earlier techniques provides a general solution for $u(n; p,q)$ for the pentanomial case, although it would be a messy computation. However, if one notes some of the relationships between the $p_1(k)$ for the polynomials

$$y^{n-1} - \frac{x}{1 - x}\,(y^{n-2} + y^{n-3} + \cdots + y + 1) = 0$$

for different values of n, much of the labor is taken out of the computation. The expressions $p_1(k)$ are identical for the polynomial with n terms and the polynomial with $(n - 1)$ terms for $k = 1, 2, 3, \cdots, n - 2$; $p_1(0) = n$ for all cases; and

$$P_1(m + n - 1) = \frac{x}{1 - x}\left(\sum_{i=2}^{n} p_1(m + m - i)\right)$$

In fact,

$$p_1(k) = \frac{1}{(1 - x)^k} - 1$$

for $k = 1, 2, \cdots, n - 1$ for the polynomial with n terms. Thus, $p_1(k)$ can be derived sequentially for any value of k for the polynomial with n terms given by

$$y^{n-1} - \frac{x}{1 - x}(y^{n-2} + \cdots + y + 1) = 0 .$$

We can sequentially generate all sums of powers of the roots of any polynomial because we can get the proper starting values sequentially as well as find higher powers sequentially.

Now, it is well known that, given all the sums of the powers of the roots, $p_1(0)$, $p_1(1)$, $p_1(2)$, $\cdots$, $p_1(n)$, for a given fixed polynomial, one can determine the other symmetric functions of the roots in terms of the $p_1(k)$. (See [7], [8].) Waring's formula gives

$$p_m(k) = (-1)^r \cdot \frac{(p_1(k))^{r_1} \cdot (p_1(2k))^{r_2} \cdot (p_1(3k))^{r_3} \cdots (p_1(mk))^{r_m}}{(r_1!\, r_2!\, r_3! \cdots r_m!)(1^{r_1} \cdot 2^{r_2} \cdot 3^{r_3} \cdots m^{r_m})}$$

$$r_1 + r_2 + r_3 + \cdots + r_m = r$$

$$r_1 + 2r_2 + 3r_3 + \cdots + mr_m = m .$$

Also, the generating functions for the coefficients arising in the expansion of the n-nomial $(1 + x + x^2 + \cdots + x^{n-1})^k$ can be derived sequentially by $G_1 = x^i/(1-x)^{i+1}$, $i = 0,1,2,\cdots, n-1$, $G_n = ((1-x)^{n-1} - 1)/(1-x)^{n+1}$, and $G_{n+1} = x/(1-x) \cdot (G_n + G_{n-1} + \cdots + G_2 + G_1 + G_0)$. Thus, for the polynomial with n terms, by taking $G^\star_{iq} = x^{ip}G_{iq}$, letting

$$G^\star_{(m+n-1)q} = \sum_{i=1}^{n-1} (-1)^{i+1} p_i(q) G^\star_{(m+n-1-i)} ,$$

and using the methods of this paper, the generating function for the numbers $u(n; p,q)$ could be derived.

In [11] it was promised a proof that, for $p = 1$,

$$\sum_{n=0}^{\infty} u(n;p,1)x^n = \frac{1}{1 - x - x^{p+1} - x^{2p+1} - \cdots - x^{(r-1)p+1}}$$

for the general r-nomial triangle induced by the expansion

$$(1 + x + x^2 + \cdots + x^{r-1})^n \qquad n = 0,1,2,3,\cdots .$$

This follows from the definition. Let the r-nomial triangle be left justified and take sums by starting on the left edge and jumping up p and over 1 entry repeatedly until out of the triangle. Thus,

$$u(n;p,1) = \sum_{k=0}^{\left[\frac{n(r-1)}{p+1}\right]} \left\{ {n - kp \atop k} \right\}_r ,$$

where

$$(1 + x + x^2 + \cdots + x^{r-1})^n = \sum_{j=0}^{n(r-1)} \left\{ {n \atop j} \right\}_r x^j .$$

The r-nomial coefficient $\left\{ {n \atop j} \right\}_r$ is the entry in the n^{th} row and j^{th} column of the generalized Pascal triangle. Thus

$$\frac{1}{1 - x(1 + x^p + x^{2p} + \cdots + x^{p(r-1)})} = \sum_{n=0}^{\infty} \left[x(1 + x^p + x^{2p} + \cdots + x^{p(r-1)})\right]^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\left[\frac{n(r-1)}{p+1}\right]} \left\{ {n - kp \atop k} \right\}_r \right) x^n .$$

(Continued on p. 393.)

TABLE OF GENERATING FUNCTIONS FOR POLYNOMIAL TRIANGLE DIAGONAL SUMS

u(n; p,q)	Binomial [1]	Trinomial	Quadrinomial
u(n; 0,1)	$\frac{1}{1 - 2x}$	$\frac{1}{1 - 3x}$	$\frac{1}{1 - 4x}$
u(n; 1,1)	$\frac{1}{1 - x - x^2}$	$\frac{1}{1 - x - x^2 - x^3}$	$\frac{1}{1 - x - x^2 - x^3 - x^4}$
u(n; p,1)	$\frac{1}{1 - x - x^{p+1}}$	$\frac{1}{1 - x - x^{p+1} - x^{2p+1}}$	$\frac{1}{1 - x - x^{p+1} - x^{2p+1} - x^{3p+1}}$
u(n; 0,2)	$\frac{1 - x}{1 - 2x}$	$\frac{1 - 2x}{1 - 4x + 3x^2}$	$\frac{1 - 2x}{1 - 4x}$
u(n; 1,2)	$\frac{1 - x}{1 - 2x + x^2 - x^3}$	$\frac{1 - x - x^2}{1 - 2x - x^2 + x^3 + x^4}$	$\frac{1 - x - x^2}{1 - 2x - x^2 + x^3 - x^4 - x^5}$
u(n; p,2)	$\frac{1 - x}{(1 - x)^2 - x^{p+2}}$	$\frac{1 - x - x^{p+1}}{(1 - x)^2 - 2x^{p+1} + x^{p+2} = x^{2p+2}}$	$\frac{1 - x - x^{p+1}}{(1 - x)^2 - 2x^{p+1} + x^{p+2} - x^{2p+2} - x^{3p+2}}$
u(n; 0,3)	$\frac{1 - 2x + x^2}{1 - 3x + 3x^2 - 2x^3}$	$\frac{1 - 2x}{1 - 3x}$	$\frac{1 - 3x}{1 - 5x + 4x^2}$
u(n; p,3)	$\frac{(1 - x)^2}{(1 - x)^3 - x^{p+3}}$	$\frac{(1 - x)^2 - x^{p+2}}{(1 - x)^3 - 3x^{p+2} + 2x^{p+3} - x^{2p+3}}$	
u(n; 0,4)	$\frac{1 - 3x + 3x^2 - x^3}{1 - 4x + 6x^2 - 4x^3}$	$\frac{1 - 3x + 2x^2 - x^3}{1 - 4x + 4x^2 - 4x^3 + 3x^4}$	$\frac{1 - 3x}{1 - 4x}$
u(n; p,q)	$\frac{(1 - x)^{q-1}}{(1 - x)^q - x^{p+q}}$		

TABLE OF GENERATING FUNCTIONS FOR POLYNOMIAL TRIANGLE DIAGONAL SUMS

u(n; p,q)	Pentanomial	Hexanomial
u(n; 0,1)	$\frac{1}{1-5x}$	$\frac{1}{1-6x}$
u(n; 1,1)	$\frac{1}{1-x-x^2-x^3-x^4-x^5}$	$\frac{1}{1-x-x^2-x^3-x^4-x^5-x^6}$
u(n; p,1)	$\frac{1}{1-x-x^{p+1}-x^{2p+1}-x^{3p+1}-x^{4p+1}}$	$\frac{1}{1-x-x^{p+1}-x^{2p+1}-x^{3p+1}-x^{4p+1}-x^{5p+1}}$
u(n; 0,2)	$\frac{1-3x}{1-6x+5x^2}$	$\frac{1-3x}{1-6x}$
u(n; 1,2)	$\frac{1-x-x^2-x^3}{1-2x-x^2-x^3+x^4+x^5+x^6}$	$\frac{1-x-x^2-x^3}{1-2x-x^2-x^3+x^4-x^5-x^6-x^7}$
u(n; p,2)	$\frac{1-x-x^{p+1}-x^{2p+1}}{(1-x)^2-2x^{p+1}+x^{p+2}-2x^{2p+1}+x^{2p+2}+x^{3p+2}+x^{4p+2}}$	$\frac{1-x-x^{p+1}-x^{2p+1}}{(1-x)^2-2x^{p+1}+x^{p+2}-2x^{2p+1}+x^{2p+2}-x^{3p+2}-x^{4p+2}-x^{5p+2}}$
u(n; 0,3)	$\frac{1-4x+2x^2}{1-6x+6x^2-5x^3}$	$\frac{1-4x}{1-6x}$

r-nomial

u(n;p,1) $$\frac{1}{1-x-x^{p+1}-x^{2p+1}-\ldots-x^{(r-1)p+1}}$$

GENERATING FUNCTIONS

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1. INTRODUCTION

With an arbitrary sequence of (complex) numbers $\{a_n\} = \{a_0, a_1, a_2, \cdots\}$ we associate the (formal) power series

$$a(x) = \sum_{n=0}^{\infty} a_n x^n . \tag{1.1}$$

The definition is purely formal; convergence of the series need not be assumed. The series (1.1) is usually called an ordinary generating function.

Let $\{b_n\} = \{b_0, b_1, b_2, \cdots\}$ be another sequence and

$$b(x) = \sum_{n=0}^{\infty} b_n x^n$$

the corresponding generating function. We define the sum of $\{a_n\}$ and $\{b_n\}$ by means of

$$\{a_n\} + \{b_n\} = \{c_n\}, \quad c_n = a_n + b_n \quad (n = 0,1,2,\cdots) ;$$

then clearly

$$c(x) = \sum_{n=0}^{\infty} c_n x^n = a(x) + b(x) .$$

Similarly, if we define the product

$$\{a_n\}\{b_n\} = \{p_n\},$$

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by means of

$$(1.2) \qquad p_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, 2, \cdots),$$

then it is easily seen that

$$p(x) = \sum_{n=0}^{\infty} p_n x^n = a(x)\, b(x) .$$

The product defined by (1.2) is called the Cauchy product of $\{a_n\}$ and $\{b_n\}$.

In contrast with (1.1) we may define the exponential generating function

$$(1.3) \qquad A(x) = \sum_{n=0}^{\infty} a_n x^n/n!$$

which again is a formal definition. The product is now defined by means of

$$(1.4) \qquad p_n = \sum_{k=0}^{\infty} \binom{n}{k} a_k b_{n-k} ;$$

this is known as the Hurwitz product and is of particular interest in certain number-theoretic questions (see for example [15, p. 147]).

One can develop an algebra of sequences using either the Cauchy or Hurwitz product. In either case multiplication is associative and commutative and distributive with respect to addition. Moreover the product of two sequences is equal to the zero sequence

$$\{z_n\} = \{0, 0, 0, \cdots\}$$

if and only if at least one factor is equal to $\{z_n\}$; thus the set of all sequences constitute a domain of integrity.

In the present paper, however, we shall be primarily interested in showing how generating functions can be employed to sum or transform finite series of various kinds. We shall also illustrate the use of generating functions in solving several enumerative problems. For a fuller treatment the reader is referred to [18].

In the definitions above we have considered only the case of one dimensional sequences. This can of course be generalized in an obvious way, namely with the double sequence $\{a_{m,n}\}$ we associate the series

$$a(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} x^m y^n . \tag{1.5}$$

Also factorials may be inserted as in (1.3). Indeed, there is now a certain amount of choice; for example both

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} \frac{x^m y^n}{m!\,n!}, \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} x^m y^n/n! \tag{1.6}$$

are useful. As we shall see in Section 10, other possibilities also occur.

More generally, we may consider

$$a(x_1,\cdots,x_k) = \sum_{n_1,\cdots,n_k=0}^{\infty} a_{n_1,\cdots,n_k} x_1^{n_1} \cdots x_k^{n_k} \tag{1.7}$$

and its various modifications as in (1.6). Of particular interest in the theory of numbers is the Dirichlet series

$$\sum_{n=1}^{\infty} a_n /n^s ; \tag{1.8}$$

the product is now defined by

$$p_n = \sum_{rs=n} a_r b_s . \tag{1.9}$$

We may think of (1.8) as a generalization of (1.7). For let $q_1, q_2, \cdots, q_k$ denote the first k primes and let $a_n = 0$ unless

$$n = q_1^{f_1} q_2^{f_2} \cdots q_k^{f_k} .$$

If we put

$$a_n = a_{f_1, \cdots, f_k}$$

it follows that

$$\sum_{n=1}^{\infty} a_n / n^s = a(q_1^{-s}, \cdots, q_k^{-s}) , \tag{1.10}$$

where the right member is defined by (1.7).

2. As a first simple illustration of the generating function technique, we take the binomial expansion

$$(1 + x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k , \tag{2.1}$$

where, to begin with, we assume m is a nonnegative integer. Combining (2.1) with

$$(1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j$$

we immediately get

$$\sum_{s=0}^{k} \binom{m}{s} \binom{n}{k - s} = \binom{m + n}{k} \quad (k = 0, 1, 2, \cdots) . \tag{2.2}$$

It is to be understood that the binomial coefficient $\binom{n}{k} = 0$ if $k > n$ or $k < 0$.

Each side of (2.2) is a polynomial in m and n. Since (2.2) holds for all nonnegative values of m,n it follows that it holds when m,n are arbitrary complex numbers.

It is convenient to introduce the following notation:

$$(a)_n = a(a + 1)\cdots(a + n - 1), \quad (a)_0 = 1 .$$

It is easily verified that

$$\binom{a}{k} = (-1)^k \frac{(-a)_k}{k!} ,$$

and that (2.2) becomes

$$\sum_{s=0}^{\infty} \frac{(-k)_s (a)_s}{s!\,(b)_s} = \frac{(b - a)_k}{(b)_k} . \tag{2.3}$$

In (2.3) a and b are arbitrary except that b is not a negative integer.

The formula

$$\sum_{k=0}^{m} (-1)^{k-n} \binom{m}{n}\binom{k}{n} = \begin{cases} 1 & (m = n) \\ 0 & (m \neq n) \end{cases} \tag{2.4}$$

is very useful. The proof is quite simple. We may evidently assume $m \geq n$. Since

$$\binom{m}{k}\binom{k}{n} = \binom{m}{n}\binom{m - n}{k - n} ,$$

it is clear that the left member of (2.4) is equal to

$$\binom{m}{n} \sum_{k=n}^{m} (-1)^{k-n} \binom{m - n}{k - n} = \binom{m}{n} (-1)^{m-n}$$

and (2.4) follows at once.

As an immediate application of (2.4) we have the following theorem: If

$$b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k \quad (n = 0,1,2,\cdots) \tag{2.5}$$

then

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k \quad (n = 0,1,2,\cdots) \tag{2.6}$$

and conversely.

It is of interest to express the equivalence of (2.5) and (2.6) in terms of generating functions. As above, put

$$A(x) = \sum_{n=0}^{\infty} a_n x^n/n!\,, \quad B(x) = \sum_{n=0}^{\infty} b_n x^n/n! \;.$$

Then (2.5) becomes

$$B(x) = e^x A(-x) \tag{2.7}$$

while (2.6) becomes

$$A(x) = e^x B(-x)\,. \tag{2.8}$$

It is easy to extend the above to multiple sequences. If

$$a_{m,n} = \sum_{j=0}^{m} \sum_{k=0}^{n} (-1)^{j+k} \binom{m}{j} \binom{n}{k} b_{j,k} \tag{2.9}$$

then

$$b_{m,n} = \sum_{j=0}^{m} \sum_{k=0}^{n} (-1)^{j+k} \binom{m}{j} \binom{n}{k} a_{j,k} \tag{2.10}$$

and conversely. Moreover if

$$A(x,y) = \sum_{m,n=0}^{\infty} a_{m,n} \frac{x^m y^n}{m!\,n!}, \quad B(x,y) = \sum_{m,n=0}^{\infty} b_{m,n} \frac{x^m y^n}{m!\,n!}$$

then

$$A(x,y) = e^{x+y} B(-x,-y) \tag{2.11}$$

and

$$B(x,y) = e^{x+y} A(-x,-y) . \tag{2.12}$$

3. As a second illustration we shall prove the formula

$$\sum_{k=0}^{m} \binom{x}{k}\binom{n}{m-k}\binom{y+n-k}{n} = \sum_{k=0}^{m} \binom{y-x+n}{k}\binom{x}{m-k}\binom{y+n-k}{n-k}. \tag{3.1}$$

This result is a slight generalization of a formula due to Greenwood and Gleason [10] and Gould [9].

Put

$$A_{m,n} = \sum_{k=0}^{m} \binom{x}{k}\binom{n}{m-k}\binom{y+n-k}{n}, \quad B_{m,n} = \sum_{k=0}^{m} \binom{y-x+n}{k}\binom{x}{m-k}\binom{y+n-k}{n-k}.$$

Then

$$\begin{aligned}\sum_{m=0}^{\infty} A_{m,n} t^m &= \sum_{k=0}^{\infty} \binom{x}{k}\binom{y+n-k}{n} \sum_{m=k}^{n+k} \binom{n}{m-k} t^m \\ &= \sum_{k=0}^{\infty} \binom{x}{k}\binom{y+n-k}{n} t^k (1+t)^n ,\end{aligned}$$

$$\sum_{m,n=0}^{\infty} A_{m,n} t^m u^n = \sum_{k=0}^{\infty} \binom{x}{k} t^k \sum_{n=0}^{\infty} \binom{y+n-k}{n} (1+t)^n u^n$$

$$= \sum_{k=0}^{\infty} \binom{x}{k} t^k (1-u-tu)^{-y+k-1}$$

$$= (1-u-tu)^{-y-1} \sum_{k=0}^{\infty} \binom{x}{k} t^k (1-u-tu)^k$$

$$= (1-u-tu)^{-y-1} \left[1 + t(1-u-tu)\right]^x ,$$

so that

$$\sum_{m,n=0}^{\infty} A_{m,n} t^m u^n = \frac{(1+t)^x (1-tu)^x}{(1-u-tu)^{y+1}} . \tag{3.2}$$

On the other hand,

$$\sum_{m=0}^{\infty} B_{m,n} t^m = \sum_{k=0}^{n} \binom{y-x+n}{k} \binom{y+n-k}{n-k} \sum_{m=k}^{\infty} \binom{x}{m-k} t^m \tag{3.3}$$

$$= \sum_{k=0}^{n} \binom{y-x+n}{k} \binom{y+n-k}{n-k} t^k (1+t)^x ,$$

$$\frac{(1-tu)^x}{(1-u-tu)^{y+1}} = (1-tu)^{x-y-1} \left(1 - \frac{u}{1-tu}\right)^{-y-1}$$

$$= \sum_{r=0}^{\infty} \binom{y+r}{r} u^r (1-tu)^{x-y-r-1}$$

$$= \sum_{r=0}^{\infty} \binom{y+r}{r} u^r \sum_{k=0}^{\infty} \binom{y-x+r+k}{k} t^k u^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{y-x+n}{k} \binom{y+n-k}{n-k} t^k u^n ,$$

so that by (3.3),

$$\sum_{m,n=0}^{\infty} B_{m,n}\, t^m u^n = \frac{(1+t)^x(1-tu)^x}{(1-u-tu)^{y+1}} \tag{3.4}$$

Comparing (3.4) with (3.2), (3.1) follows at once.

We remark that if we put

$${}_3F_2\begin{bmatrix} a,b,c \\ d,e \end{bmatrix} = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{k!\,(d)_k (e)_k}$$

then (3.1) becomes

$$\binom{n}{m}\binom{y+n}{n}\,{}_3F_2\begin{bmatrix} -x,-y,-m \\ -y-n,n-m+1 \end{bmatrix} = \binom{x}{m}\binom{y+n}{n}\,{}_3F_2\begin{bmatrix} x-y-n,-n,-m \\ -y-n,x-m+1 \end{bmatrix},$$

which is a special case of a known transformation formula [1, p. 98, ex. 7].

4. A set of polynomials $A_n(x)$ that satisfy

$$A_n'(x) = n\,A_{n-1}(x) \quad (n = 0,1,2,\cdots)\,, \tag{4.1}$$

where the prime denotes differentiation, is called an Appell set. It is easily proved that such a set may be defined by

$$\sum_{n=0}^{\infty} A_n(x)\, z^n/n! = e^{xz} \sum_{n=0}^{\infty} a_n\, z^n/n!\,, \tag{4.2}$$

where the a_n are independent of x. Also it is evident from (4.1) that

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k\, x^{n-k}\,. \tag{4.3}$$

This formula is sometimes written in the suggestive form

$$A_n(x) = (x+a)^n\,,$$

where it is understood that after expansion of the right member, a^k is replaced by a_k.

It also follows at once from (4.2) that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k A_{n-k}(x) = a_n . \tag{4.4}$$

We may view (4.3) and (4.4) as an instance of the equivalence of (2.5) and (2.6).

If $a_0 \neq 0$, we may define the sequence $\{b_n\}$ by means of

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} = \begin{cases} 1 & (n = 0) \\ 0 & (n > 0) \end{cases} , \tag{4.5}$$

or equivalently $A(z)B(z) = 1$, where

$$B(z) = \sum_{n=0}^{\infty} b_n z^n/n! \quad .$$

It then follows from (4.2) and (4.5) that

$$x^n = \sum_{k=0}^{n} \binom{n}{k} b_k A_{n-k}(x) \quad . \tag{4.6}$$

As an illustration we take the Bernoulli polynomial $B_n(x)$ defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) z^n/n! \quad ; \tag{4.7}$$

the Bernoulli number $B_n = B_n(0)$ is defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n z^n/n! \quad . \tag{4.8}$$

Since

$$\frac{e^z - 1}{z} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} ,$$

it follows that

$$x^n = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} B_{n-k}(x) . \tag{4.9}$$

By means of (4.7) we can easily obtain the following basic properties of $B_n(x)$.

$$B_n(x+1) - B_n(x) = nx^n , \tag{4.10}$$

$$B_n(1-x) = (-1)^n B_n(x) , \tag{4.11}$$

$$\sum_{s=0}^{k-1} B_n\left(x + \frac{s}{k}\right) = k^{1-n} B_n(kx) \qquad (k = 1, 2, 3, \cdots) . \tag{4.12}$$

Closely related to $B_n(x)$ is the Euler polynomial $E_n(x)$ defined by

$$\frac{2e^{xz}}{2^z + 1} = \sum_{n=0}^{\infty} E_n(x)\, z^n/n! . \tag{4.13}$$

Corresponding to (4.10), (4.11), (4.12) we have

$$E_n(x+1) + E_n(x) = 2x^n , \tag{4.14}$$

$$E_n(1-x) = (-1)^n E_n(x) , \tag{4.15}$$

$$\sum_{s=0}^{k-1} (-1)^s E_n\left(x + \frac{s}{k}\right) = k^{-n} E_n(kx) \qquad (k \text{ odd}) , \tag{4.16}$$

$$(4.17)\quad \sum_{s=0}^{k-1} (-1)^s B_{n+1}\left(x + \frac{s}{k}\right) = -\frac{n+1}{2k^n} E_n(kx) \qquad (k \text{ even}) .$$

For further developments the reader is referred to [14, Ch. 2].

5. Another important Appell set is furnished by the Hermite polynomials which may be defined by

$$(5.1)\qquad e^{2xz-z^2} = \sum_{n=0}^{\infty} H_n(x)\, z^n/n! \ .$$

Differentiating with respect to x we get

$$(5.2)\qquad H_n'(x) = 2n\, H_{n-1}(x)$$

so that the definition (4.1) is modified slightly. If we differentiate (5.1) with respect to z we get

$$\sum_{n=0}^{\infty} H_{n+1}(x) z^n/n! = 2(x - z)e^{2xz-z^2} ,$$

so that

$$(5.3)\qquad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \qquad (n \geq 1) .$$

Also, multiplying (5.1) by e^{z^2}, we get

$$(5.4)\qquad (2x)^n = \sum_{2k \leq n} \frac{n!}{k!\,(n-2k)!} H_{n-2k}(x) \ .$$

In the next place it follows from (5.1) that

$$\sum_{m,n=0}^{\infty} H_m(x) H_n(x) \frac{u^m v^n}{m!\,n!} = e^{2x(u+v)-u^2-v^2} = e^{2x(u+v)-(u+v)^2} e^{2uv}$$

$$= e^{2uv} \sum_{n=0}^{\infty} H_n(x)(u+v)^n/n! = \sum_{k=0}^{\infty} \frac{(2uv)^k}{k!} \sum_{m,n=0}^{\infty} H_{m+n}(x) \frac{u^m v^n}{m!\,n!} .$$

Equating coefficients we get

$$(5.5)\qquad H_m(x)H_n(x) = \sum_{k=0}^{\min(m,n)} 2^k k! \binom{m}{k}\binom{n}{k} H_{m+n-2k}(x) .$$

Similarl y we have the inverse formula

$$(5.6)\qquad H_{m+n}(x) = \sum_{k=0}^{\min(m,n)} (-1)^k 2^k k! \binom{m}{k}\binom{n}{k} H_{m-k}(x)H_{n-k}(x) .$$

The formulas (5.5), (5.6) are due to Nielsen [13]; (5.5) was rediscovered by Feldheim [8]. The above proof is due to Watson [20].

Another interesting formula is

$$(5.7)\qquad \sum_{n=0}^{\infty} H_n(x)H_n(y)\, z^n/n! = (1 - 4z^2)^{-\frac{1}{2}} \exp\left\{\frac{4xyz - 4(x^2 + y^2)z^2}{1 - 4z^2}\right\}$$

We note first that

$$\begin{aligned} \sum_{n,k=0}^{\infty} H_{n+k}(x) \frac{z^n t^k}{n!\, k!} &= \sum_{n=0}^{\infty} H_n(x) \frac{(z + t)^n}{n!} \\ &= e^{2x(z+t)-(z+t)^2} \\ &= e^{2xz-z^2} e^{2(x-z)t-t^2} \\ &= e^{2xz-z^2} \sum_{k=0}^{\infty} H_k(x - z) \frac{t^k}{k!} . \end{aligned}$$

Equating coefficients, we get

$$(5.8)\qquad \sum_{n=0}^{\infty} H_{n+k}(x) z^n/n! = e^{2xz-z^2} H_k(x - z) ,$$

which reduces to (5.1) when $k = 0$.

Since, by (5.1),

$$H_n(x) = \sum_{2k \le n} (-1)^k \frac{n!}{k!\,(n-2k)!} (2x)^{n-2k} , \tag{5.9}$$

we have

$$\sum_{n=0}^{\infty} H_n(x)\, H_n(y) z^n/n!$$

$$= \sum_{n=0}^{\infty} \sum_{2k \le n} (-1)^k \frac{(2x)^{n-2k}}{k!\,(n-2k)!} H_n(y)\, z^n$$

$$= \sum_{n,k=0}^{\infty} (-1)^k \frac{(2x)^n z^{n+2k}}{k!\, n!} H_{n+2k}(y)$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{n!} \sum_{n=0}^{\infty} H_{n+2k}(y) \frac{(2xz)^n}{n!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{k!} e^{4xyz-4x^2z^2} H_{2k}(y - 2xz)$$

$$= e^{4xyz-4x^2z^2} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{k!} \sum_{s=0}^{k} (-1)^s \frac{(2k)!}{s!\,(2k-2s)!} (2y - 4xz)^{2k-2s}$$

$$= e^{4xyz-4x^2z^2} \sum_{k,s=0}^{\infty} (-1)^k \frac{(2k+2s)!}{s!\,(2k)!\,(k+s)!} z^{2k+2s} (2y - 4xz)^{2k} .$$

Since

$$(2k)! = 2^{2k} k!\, (\tfrac{1}{2})_k ,$$

we get

$$e^{4xyz-4x^2z^2} \sum_{k,s=0}^{\infty} (-1)^k \frac{(\frac{1}{2})_{k+s}}{s!\,k!\,(\frac{1}{2})_k} 2^{2s} z^{2k+2s} (2y - 4xz)^{2k}$$

$$= e^{4xyz-4x^2z^2} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}(2y - 4xz)^{2k}}{k!} \sum_{s=0}^{\infty} \frac{(k+\frac{1}{2})_s}{s!} (2z)^{2s}$$

$$= e^{4xyz-4x^2z^2} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}(2y - 4xz)^{2k}}{k!} (1 - 4z^2)^{-k-\frac{1}{2}}$$

$$= (1 - 4x^2)^{-\frac{1}{2}} \exp\left\{4xyz - 4x^2z^2 - \frac{z^2(2y - 4xz)^2}{1 - 4z^2}\right\} .$$

$$= (1 - 4x^2)^{-\frac{1}{2}} \exp\left\{\frac{4xyz - 4(x^2 + y^2)z^2}{1 - 4z^2}\right\} .$$

This completes the proof of (5.7). The proof is taken from Rainville [16, p. 197].

6. The formula of Saalschutz [1, p. 9],

$$\sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{k!\,(c)_k (d)_k} = \frac{(c - a)_n (c - b)_n}{(c)_n (c - a - b)_n} , \tag{6.1}$$

where

$$c + d = -n + a + b + 1, \tag{6.2}$$

is very useful in many instances.

If we replace c by $c - n$, (6.1) becomes

$$\sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{k!\,(c-n)_k (d)_k} = \frac{(d - a)_n (d - b)_n}{(d)_n (d - a - b)_n} , \tag{6.3}$$

where now

$$c + d = a + b + 1 . \tag{6.4}$$

Now by (6.3)

$$\sum_{n=0}^{\infty} \frac{(d-a)_n (d-b)_n}{x!\,(d)_n} x^n = \sum_{n=0}^{\infty} \frac{(d-a-b)_n}{n!} x^n \sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{k!\,(c-n)_k (d)_k}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (b)_k}{k!\,(d)_k} x^k \sum_{n=0}^{\infty} \frac{(d-a-b)_n}{n!} x^n$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (b)_k}{k!\,(d)_k} x^k (1-x)^{a+b-d} .$$

Thus (6.3) is equivalent to

$$(6.5) \qquad F(a,b;\, d;\, x) = (1-x)^{d-a-b} F(d-a,\, d-b;\, d;\, x) ,$$

where $F(a,b;\, d;\, x)$ denotes the hypergeometric function.

It is customary to prove (6.5) by making use of the differential equation of the second order satisfied by $F(a,b;c;x)$. We shall, however, give an inductive proof of (6.1) which we now write in the form

$$(6.6) \qquad \sum_{k=0}^{n} \frac{(-n)_k (a+n)_k (b)_k}{k!\,(c)_k (d)_k} = \frac{(c-b)_n (d-b)_n}{(c)_n (d)_n} ,$$

where

$$(6.7) \qquad c + d = a + b + 1 .$$

Let

$$S_n(a,b,c,d) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+n)_k (b)_k}{(c)_k (d)_k} ,$$

where a,b,c,d satisfy (6.7). Then

$$S_{n+1}(a,b,c,d) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+n+1)_k (b)_k}{(c)_k (d)_k} - \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+n+1)_{k+1} (b)_{k+1}}{(c)_{k+1} (d)_{k+1}}$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+n+1)_k (b)_k}{(c)_{k+1} (d)_{k+1}} \left\{(c+k)(d+k) - (a+n+k+1)(b+k)\right\}.$$

Now put

$$(c+k)(d+k) - (a+n+k+1)(b+k) = A(d+k) + B(c+k),$$

where A, B are independent of k. Then

$$\text{(6.8)} \qquad \begin{cases} (d-c)A = (c-b)(a-c+n+1), \\ (c-d)B = (d-b)(a-d+n+1). \end{cases}$$

It follows that

$$S_{n+1}(a,b,c,d) = \frac{A}{c} S_n(a+1,b,c+1,d) + \frac{B}{d} S_n(a+1,b,c,d+1).$$

Assuming that (6.6) holds, we therefore get

$$S_{n+1}(a,b,c,d) = \frac{A}{c} \frac{(c-b+1)_n (d-b)_n}{(c+1)_n (d)_n} + \frac{B}{d} \frac{(c-b)_n (d-b+1)_n}{(c)_n (d+1)_n}$$

$$= \frac{(c-b+1)_{n-1} (d-b+1)_{n-1}}{(c)_{n+1} (d)_{n+1}} \left\{A(d-b)(c-b+n)(d+n) + B(c-b)(d-b+n)(c+n)\right\}$$

By (6.8),

$$(d-c)\left\{A(d-b)(c-b+n)(d+n) + B(c-b)(d-b+n)(c+n)\right\}$$
$$= (c-b)(d-b)\left\{(c-b+n)(d+n)(a-c+n+1) - (d-b+n)(c+n)(a-d+n+1)\right\}$$
$$= (c-b)(d-b)\left\{(c-b+n)(d+n)(d-b+n) - (d-b+n)(c+n)(c-b+n)\right\}$$
$$= (c-b)(d-b)(c-b+n)(d-b+n)(d-c).$$

Therefore

$$S_{n+1}(a,b,c,d) = \frac{(c-b)_{n+1}(d-b)_{n+1}}{(c)_{n+1}(d)_{n+1}} ,$$

which completes the induction.

As an application we take (6.6) in the form

$$\text{(6.9)} \qquad \sum_{j=0}^{k} \frac{(-k)_j(a+k)_j(-a+b+c+1)_j}{j!\,(b+1)_j(c+1)_j} = \frac{(a-b)_k(a-c)_k}{(b+1)_k(c+1)_k} ,$$

where now a, b, c are arbitrary. Then

$$\sum_{k=0}^{\infty} \frac{(a)_k(a-b)_k(a-c)_k}{k!\,(b+1)_k(c+1)_k} x^k$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} \frac{(-k)_j(a)_{j+k}(-a+b+c+1)_j}{j!\,(b+1)_j(c+1)_j}$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{(a)_{2j}(-a+b+c+1)_j}{j!\,(b+1)_j(c+1)_j} x^j \sum_{k=0}^{\infty} \frac{(a+2j)_k}{k!} x^k ,$$

so that we have

$$\text{(6.10)} \quad \sum_{k=0}^{\infty} \frac{(a)_k(a-b)_k(a-c)_k}{k!\,(b+1)_k(c+1)_k} x^k = \sum_{j=0}^{\infty} (-1)^j \frac{(a)_{2j}(-a+b+c+1)_j}{j!\,(b+1)_j(c+1)_j} x^j(1-x)^{-a-2}$$

If we take $a = -2n$, $x = 1$; (6.10) reduces to

$$\text{(6.11)} \quad \sum_{k=0}^{2n} \frac{(-2n)_k(-2n-b)_k(-2n-c)_k}{k!\,(b+1)_k(c+1)_k} = (-1)^n \frac{(2n)!\,(b+c+2n+1)_n}{n!\,(b+1)_n(c+1)_n} .$$

In particular, for $b = c = 0$, (6.11) becomes Dixon's theorem:

$$\text{(6.12)} \qquad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1) \frac{(3n)!}{(n!)^3} .$$

Note also that (6.10) implies, for $a = -n$, $b = c = 0$,

$$\sum_{k=0}^{n} \binom{n}{k}^3 x^k = \sum_{2j \le n} \frac{(n+j)!}{(j!)^3 (n-j)!} x^j (1+x)^{n-2j} , \tag{6.13}$$

and in particular

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{2j \le n} \frac{(n+j)!}{(j!)^3 (n-j)!} 2^{n-2j} , \tag{6.14}$$

a result due to MacMahon. For other proofs of these formulas see [17, pp. 41, 42].

7. We now turn to some problems involving multiple generating functions. To begin with, we take

$$\begin{aligned}
(1 - 2x - 2y + x^2 - 2xy + y^2)^{-\frac{1}{2}} &= \left[(1-x-y)^2 - 4xy\right]^{-\frac{1}{2}} \\
&= (1-x-y)^{-1}\left[1 - \frac{4xy}{(1-x-y)^2}\right]^{-\frac{1}{2}} \\
&= \sum_{r=0}^{\infty} \binom{2r}{r} \frac{(xy)^r}{(1-x-y)^{2r+1}} \\
&= \sum_{r=0}^{\infty} \binom{2r}{r} (xy)^r \sum_{s,t=0}^{\infty} \frac{(2r+s+t)!}{(2r)!\, s!\, t!} x^s y^t \\
&= \sum_{m,n=0}^{\infty} x^m y^n \sum_{r=0}^{\min(m,n)} \frac{(m+n)!}{r!\, r!\, (m-r)!\, (n-r)!} .
\end{aligned}$$

Since

$$\sum_{r=0}^{\min(m,n)} \frac{(m+n)!}{r!\, r!\, (m-r)!\, (n-r)!} = \binom{m+n}{m} \sum_{r=0}^{\min(m,n)} \binom{m}{r}\binom{n}{r} = \binom{m+n}{m}^2 ,$$

we have

$$(7.1) \qquad (1 - 2x - 2y + x^2 - 2xy + y^2)^{-\frac{1}{2}} = \sum_{m,n=0}^{\infty} \binom{m+n}{m}^2 x^m y^n .$$

This is in fact a disguised form of the generating function for Legendre polynomials:

$$(7.2) \qquad (1 - 2xz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) z^n .$$

However to save space, we shall not elaborate this point.

One can extend (7.1) in various ways. For example, we can construct the generating function for the Jacobi polynomial

$$(7.3) \qquad P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k} .$$

It is known that

$$(7.4) \qquad \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) z^n = 2^{\alpha+\beta} R^{-1}(1 - z + r)^{-\alpha}(1 - z + R)^{-\beta} ,$$

where

$$R = (1 - 2xz + z^2)^{\frac{1}{2}} .$$

For a proof of (7.4) see, for example, [16, p. 140].

If we put

$$u = \tfrac{1}{2}(x - 1)z, \quad v = \tfrac{1}{2}(x + 1)z ,$$

we have

$$(7.5) \qquad R = [(1 - u - v)^2 - 4uv]^{\frac{1}{2}}$$

and (7.4) becomes

$$(7.5)\qquad \sum_{j,k=0}^{\infty}\binom{\alpha+j+k}{j}\binom{\beta+j+k}{k}u^j v^k = 2^{\alpha+\beta}R^{-1}(1-u+v+R)^{-\alpha}(1+u-v+R)^{-\beta}$$

with R defined by (7.5).

We shall now give a simple proof of (7.6). Consider the expression

$$(1-x)^{-\alpha-1}(1-y)^{-\beta-1}\sum_{j,k=0}^{\infty}\frac{(\alpha+1)_{j+k}(\beta+1)_{j+k}}{j!\,k!\,(\alpha+1)_k(\beta+1)_j}\frac{(-1)^{j+k}x^j y^k}{(1-x)^{j+k}(1-y)^{j+k}}$$

$$= \sum_{j,k=0}^{\infty}(-1)^{j+k}\frac{(\alpha+1)_{j+k}(\beta+1)_{j+k}}{j!\,k!\,(\alpha+1)_k(\beta+1)_j}x^j y^k\sum_{r,s=0}^{\infty}\frac{(\alpha+j+k+1)_r(\beta+j+k+1)_s}{r!\,s!}x^r y^s$$

$$= \sum_{m,n=0}^{\infty}(\alpha+1)_m(\beta+1)_n x^m y^n\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{(-m)_j(-n)_k}{j!\,k!}\frac{(\alpha+m+1)_k(\beta+n+1)_j}{(\alpha+1)_k(\beta+1)_j}\ .$$

The inner sum is equal to

$$\sum_{j=0}^{m}\frac{(-m)_j}{j!}\frac{(\beta+n+1)_j}{(\beta+1)_j}\sum_{k=0}^{k}\frac{(-n)_k(\alpha+m+1)_k}{k!\,(\alpha+1)_k} = \frac{(-n)_m}{(\beta+1)_m}\frac{(-m)_n}{(\alpha+1)_n}$$

by (2.3), which vanishes unless m = n. It follows that

$$(7.7)\qquad (1-x)^{-\alpha-1}(1-y)^{-\beta-1}\sum_{j,k=0}^{\infty}\binom{\alpha+j+k}{j}\binom{\beta+j+k}{k}\frac{(-1)^{j+k}x^j y^k}{(1-x)^{j+k}(j-y)^{j+k}} = \frac{1}{1-xy}\,.$$

Now put

$$u = -\frac{x}{(1-x)(1-y)},\quad v = \frac{y}{(1-x)(1-y)}\ .$$

Then

$$1-x = \frac{2}{1-u+v+R},\quad 1-y = \frac{2}{1+u-v+R},\quad \frac{1-xy}{(1-x)(1-y)} = R$$

and (7.7) reduces to (7.6).

8. We shall now extend (7.1) in another direction, namely a larger number of variables. Consider first

$$[(1 - x - y - z)^2 - 4xyz]^{-\frac{1}{2}} = \sum_{r=0}^{\infty} \binom{2r}{r} \frac{(xyz)^r}{(1 - x - y - z)^{2r+1}} .$$

Since

$$(1 - x - y - z)^{-2r-1} = \sum_{k=0}^{\infty} \binom{2r + k}{k} (x + y + z)^k$$

$$= \sum_{s,t,u=0}^{\infty} \frac{(2r+s+t+u)!}{(2r)!\, s!\, t!\, u!} x^s y^t z^u ,$$

we get

$$[(1 - x - y - z)^2 - 4xyz]^{-\frac{1}{2}} = \sum_{r=0}^{\infty} \binom{2r}{r} (xyz)^r \sum_{s,t,u=0}^{\infty} \frac{(2r + s + t + u)!}{(2r)!\, s!\, t!\, u!} x^s y^t z^u$$

$$= \sum_{m,n,p=1}^{\infty} x^m y^y z^p \sum_{r=0}^{\min(m,n,p)} \frac{(m+n+p-r)!}{r!\, r!\, (m-r)!\, (n-r)!\, (p-r)!}$$

Now by (6.1)

$$\sum_{r=0}^{\min(m,n,p)} \frac{(m + n + p - r)!}{r!\, r!\, (m - r)!\, (n - r)!\, (p - r)!} = \frac{(m + n + p)!}{m!\, n!\, p!} \sum_{r=0}^{-m} \frac{(-m)_r (-n)_r (-p)_r}{r!\, r!\, (-m - n - p)_r}$$

$$= \frac{(m + n + p)!}{m!\, n!\, p!} \frac{(n + 1)_m (p + 1)_m}{m!\, (n + p + 1)_m}$$

$$= \frac{(m + n)!\, (m + p)!\, (n + p)!}{m!\, m!\, n!\, n!\, p!\, p!}$$

$$= \binom{m + n}{m} \binom{n + p}{n} \binom{p + m}{p} .$$

Finally therefore we have

$$(8.1)\quad [(1-x-y-z)^2-4xyz]^{-\frac{1}{2}} = \sum_{m,n,p=0}^{\infty} \binom{m+n}{m}\binom{n+p}{n}\binom{p+m}{p} x^m y^n z^p .$$

To carry this further a different approach seems necessary. In the expansion

$$(1-v)^{-1-i} = \sum_{j=0}^{\infty} \binom{i+j}{j} v^j$$

replace v by $v/(1-w)$ and multiply by $(1-w)^{-1}$. Then

$$\frac{(1-w)^i}{(1-v-w)^{i+1}} = \sum_{j=0}^{\infty} \binom{i+j}{j} v^j (1-w)^{-j-1}$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{i+j}{j}\binom{j+k}{k} v^j w^k .$$

Next replacing w by $w/(1-x)$, we get

$$\frac{(1-w-x)^i}{[(1-v)(1-x)-w]^{i+1}} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty} \binom{i+j}{j}\binom{j+k}{k}\binom{k+r}{r} v^j w^k x^r .$$

Now replace x by $x/(1-y)$. This yields

$$(8.2)\quad \frac{[(1-w)(1-y)-x]^i}{[(1-v)(1-x-y)-(1-y)w]^{i+1}} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}$$

$$\cdot \binom{i+j}{j}\binom{j+k}{k}\binom{k+r}{r}\binom{r+s}{s} v^j w^k x^r y^s .$$

Now multiply both sides of (8.2) by $u^i y^{-i}$ and sum over i. It follows that

$$(8.3)\qquad \{(1-v)(1-x-y)-(1-y)w-[(1-w)(1-y)-x]uy^{-1}\}^{-1}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{i+j}{j}\binom{j+k}{k}\binom{k+r}{r}\binom{r+s}{s}u^i v^j w^k x^r y^{s-i} .$$

We are concerned with that part of the multiple sum that is independent of y. The left member of (8.3) is equal to

$$\{[(1-v)(1-x)-w+u(1-w)]-(1-v-w)y-(1-w-x)xy^{-1}\}^{-1}$$

$$= \sum_{r=0}^{\infty}\frac{[(1-v-w)y+(1-w-x)uy^{-1}]^r}{[(1-v)(1-x)-w+u(1-w)]^{r+1}} .$$

Expanding the numerator by the binomial theorem, it is clear that the terms independent of y contribute

$$\sum_{r=0}^{\infty}\binom{2r}{r}\frac{(1-v-w)^r(1-w-x)^r u^r}{[(1-v)(1-x)-w+u\,(1-w)]^{2r+1}}$$

$$= \{[(1-v)(1-x)-w+u\,(1-w)]^2-4u(1-v-w)(1-w-x)\}^{-\frac{1}{2}}$$

$$= \{(1-u-v-w-x+uw+vx)^2-4uvwx\}^{-\frac{1}{2}}$$

We have therefore proved

$$(8.4)\qquad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\binom{i+j}{j}\binom{j+k}{k}\binom{k+r}{r}\binom{r+i}{i}u^i v^j w^k x^r$$

$$= \{(1-u-v-w-x+uw+vx)^2-4uvwx\}^{-\frac{1}{2}} .$$

We now specialize (8.4) by taking $u = w$, $v = x$. Since

$$(1-2u-2w+u^2+w^2)^2-4u^2w^2 = (1-u-v)^2(1-2u-2v+u^2-2uv+v^2)$$
$$= (1-u-v)^2[(1-u-v)^2-4uv],$$

Eq. (8.4) becomes

$$\sum_{m,n=0} H(m,n)u^m v^n = (1-u-v)^{-1}[(1-u-v)^2 - 4uv]^{-\frac{1}{2}} \tag{8.5}$$

where

$$H(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{i+j}{j}\binom{m-i+j}{j}\binom{i+n-j}{n-j}\binom{m-i+n-j}{n-j}. \tag{8.6}$$

If we multiply (8.5) by $1-u-v$ and apply (7.1), we get

$$H(m,n) - H(m-1,n) - H(m,n-1) = \binom{m+n}{m}^2 , \tag{8.7}$$

an identity due to Paul Brock [2], [3]. We remark also that (8.5) implies

$$H(m,n) = \sum_{r=0}^{m}\sum_{s=0}^{n}\binom{r+s}{s}^2\binom{m-r+n-s}{m-r} . \tag{8.8}$$

Also, since

$$\begin{aligned}(1-u-v)^{-1}[(1-u-v)^2 - 4uv]^{-\frac{1}{2}} &= \sum_{r=0}^{\infty}\binom{2r}{r}(uv)^r(1-u-v)^{-2r-2} \\ &= \sum_{r=0}^{\infty}\binom{2r}{r}(uv)^r\sum_{s,t=0}^{\infty}\frac{(2r+s+t+1)!}{(2r+1)!\,s!\,t!}u^s v^t \\ &= \sum_{m,n=0}^{\infty}u^m v^n\sum_{r=0}^{\min(m,n)}\binom{2r}{r}\frac{(m+n+1)!}{(2r+1)!(m-r)!(n-r)!}\end{aligned}$$

it follows that

$$H(m,n) = \binom{m+n}{m}\sum_{r=0}^{\min(m,n)}\frac{m+n+1}{2r+1}\binom{m}{r}\binom{n}{r}. \tag{8.9}$$

For the generalized version of (8.4), see [4], [6], [18, Ch. 4].

9. We shall now briefly discuss some enumerative problems. The problem of permutations with a given number of inversions was called to the writer's attention by H. W. Gould. Let $\{a_1, a_2, \cdots, a_n\}$ denote a permutation of $\{1,2,\cdots,n\}$. The pair a_i, a_j is called an <u>inversion</u> provided that $i < j$ but $a_i > a_j$. Thus $\{1,2,\cdots,n\}$ has no inversions, while $\{n, n-1, \cdots, 1\}$ has $n(n-1)/2$ inversions. Let $B(n,r)$ denote the number of permutations of $\{1, 2, \cdots, n\}$ with r inversions. Clearly, $0 \leq r \leq n(n-1)/2$.

From the definition, it follows that

$$B(n+1,r) = \sum_{\substack{s=0 \\ s \leq n}}^{r} B(n, r-s) . \tag{9.1}$$

This recurrence is obtained when the element $n+1$ is adjoined to any permutation of $\{1,2,\cdots,n\}$. Now put

$$\beta_n(x) = \sum_{r=0}^{n(n-1)/2} B(n,r)x^r .$$

Then by (9.1),

$$\beta_{n+1}(x) = \sum_{r=0}^{n(n+1)/2} x^r \sum_{\substack{s=0 \\ s \leq n}}^{r} B(n, r-s)$$

$$= \sum_{s=0}^{n} x^s \sum_{r=0}^{n(n-1)/2} B(n,r)x^r ,$$

so that

$$\beta_{n+1}(x) = (1 + x + \cdots + x^n)\beta_n(x) . \tag{9.2}$$

Since $\beta_1(x) = 1$, (9.2) yields

$$\beta_n(x) = \frac{(1-x)(1-x^2)\cdots(1-x^n)}{(1-x)^n} . \tag{9.3}$$

Thus, for example,

$$B(n,0) = 1, \quad B(n,1) = n - 1, \quad B(n,2) = \tfrac{1}{2}(n + 1)(n - 2) \qquad (n > 1),$$

$$B(n,3) = \tfrac{1}{6}\, n(n^2 - 7) \qquad (n > 2)\ ,$$

$$B(n,4) = \tfrac{1}{24}\, n(n + 1)(n^2 - n - 14) \qquad (n > 3)\ .$$

From (9.3), we get the generating function

$$\sum_{n=0}^{\infty} \beta_n(x) z^n/(x)_n = \frac{1 - x}{1 - x - z}\ , \tag{9.4}$$

where

$$(x)_n = (1 - x)(1 - x^2)\cdots(1 - x^n)\ , \qquad (x)_0 = 1\ .$$

This is the first occurrence in the present paper of a generating function with denominator $(x)_n$; see the remark in Section 11 below.

If we make use of Euler's formula

$$\begin{aligned} \prod_{n=1}^{\infty} (1 - x^n) &= \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{1}{2}k(3k+1)} \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots\ , \end{aligned} \tag{9.5}$$

we obtain an explicit formula for $B(n,r)$ when $r \leq n$. For example, we have

$$B(n,4) = \binom{n+3}{4} - \binom{n+2}{3} - \binom{n+1}{2} \qquad (n \geq 4)\ ,$$

$$B(n,5) = \binom{n+4}{5} - \binom{n+3}{4} - \binom{n+2}{3} + 1 \qquad (n \geq 5)\ ,$$

$$B(n,6) = \binom{n+5}{6} - \binom{n+4}{5} - \binom{n+3}{4} + n \qquad (n \geq 6)\ .$$

If we rewrite (9.3) in the form

$$\beta_n(x) = (1 + x)(1 + x + x^2) \cdots (1 + x + \cdots + x^{n-1}) ,$$

we obtain the following combinatorial theorem: $B(n, r)$ is equal to the number of (integral) solutions $x_1, x_2, \cdots, x_n$ of the equation

$$x_1 + x_2 + \cdots + x_n = r \tag{9.6}$$

subject to the conditions

$$0 \le x_k < k \qquad (k = 1, 2, \cdots, n) .$$

We remark also that (9.3) implies

$$\sum_{r=0}^{n(n-1)/2} B(n,r) = n! \quad ,$$

$$\sum_{r=0}^{n(n-1)/2} (-1)^r B(n,r) = 0 \qquad (n > 1) ,$$

$$\sum_{r=0}^{n(n-1)/2} r\, B(n,r) = n! \sum_{k=1}^{n} \frac{1}{k} \binom{k}{2} = \frac{1}{4} n(n-1) \cdot n! \quad .$$

For references, see [12, pp. 94-97].

10. As a second enumerative problem, we consider permutations with a given number of rises. If $\{a_1, a_2, \cdots, a_n\}$ is a permutation of $\{1, 2, \cdots, n\}$, a_j, a_{j+1} is a rise provided $a_j < a_{j+1}$. By convention there is always a rise preceding a_1. For example, the permutation $\{3, 4, 1, 2\}$ has 3 rises.

Let $A_{n,k}$ denote the number of permutations of $\{1, 2, \cdots, n\}$ with k rises. Then we have the recurrence

$$A_{n+1,k} = (n - k + 2)A_{n,k-1} + kA_{n,k} \quad . \tag{10.1}$$

The proof is simple. Let $\{a_1, \cdots, a_n\}$ be a permutation of $\{1, 2, \cdots, n\}$. If $a_i < a_{i+1}$ and we place $n + 1$ between a_i and a_{i+1} the number of rises is unchanged. If, however, $a_i > a_{i+1}$, the number of rises is increased by 1; this is also true when $n + 1$ is placed to the right of a_n.

It is also clear from the definition that

$$A_{n,1} = A_{n,n} = 1 \qquad (n = 1,2,3,\cdots); \tag{10.2}$$

the permutations in question are $\{n, n-1, \cdots, 1\}$ and $\{1,2,\cdots,n\}$, respectively. By means of (10.1) and (10.2), we can easily compute the first few values of $A_{n,k}$.

1				
1	1			
1	4	1		
1	11	11	1	
1	26	66	26	1

If, in a given permutation $\{a_1,a_2,\cdots,a_k\}$, we replace a_k by $n - a_k + 1$ $(k = 1,2,\cdots,n)$, it follows that

$$A_{n,k} = A_{n,n-k+1}\ . \tag{10.3}$$

Also it is evident that

$$\sum_{k=1}^{n} A_{n,k} = n!\ . \tag{10.4}$$

Put

$$A_0(x) = 1, \quad A_n(x) = \sum_{k=1}^{n} A_{n,k}\, x^{k-1} \qquad (n = 1,2,3,\cdots).$$

Then it can be shown that

$$\frac{1 - x}{e^z - x} = \sum_{n=0}^{\infty} (x - 1)^{-n} A_n(x)\, z^n/n! \tag{10.5}$$

We shall not give the proof of (10.5). It is indeed easier to define $A_n(x)$ by means of (10.5) and show that the other properties follow from this definition.

For references, see [5], [18, Ch. 8].

The symmetry property (10.3) is not obvious from (10.5). This suggests the following change in notation. Put

$$A(r,s) = A_{r+s+1,\, r+1} . \tag{10.6}$$

Then by (10.3),

$$A(r,s) = A(s,r) . \tag{10.7}$$

Also (10.5) implies, after a little manipulation,

$$F(x,y) = \frac{e^x - e^y}{xe^y - ye^x} = \sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s+1)!} . \tag{10.8}$$

Another symmetrical generating function is

$$(1 + xF(x,y))(1 + xF(x,y)) = \sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s)!} . \tag{10.9}$$

The denominator in the right members of (10.8) and (10.9) should be noticed.

11. We conclude with a few remarks about q-series; an instance has appeared in (9.4). Simple examples are

$$\prod_{n=0}^{\infty} (1 - x^n s)^{-1} = \sum_{n=0}^{\infty} z^n/(x)_n , \tag{11.1}$$

$$\prod_{n=0}^{\infty} (1 + x^n z) = \sum_{n=0}^{\infty} x^{\frac{1}{2}n(n-1)} z^n/(x)_n , \tag{11.2}$$

where as above

$$(x)_0 = 1, \quad (x)_n = (1 - x)(1 - x^2) \cdots (1 - x^n) .$$

A more general result that includes both (11.1) and (11.2) is

$$\prod_{n=0}^{\infty} \frac{1 - ax^n z}{1 - x^n z} = \sum_{n=0}^{\infty} \frac{(a)_n}{(x)_n} z^n , \tag{11.3}$$

where

$$(a)_0 = 1, \quad (a)_n = (1 - a)(1 - ax) \cdots (1 - ax^{n-1}) .$$

To prove (11.3), put

$$F(z) = \prod_{n=0}^{\infty} \frac{1 - ax^n z}{1 - x^n z} = \sum_{n=0}^{\infty} A_n z^n ,$$

where A_n is independent of z. Then

$$F(xz) = \frac{1 - z}{1 - az} F(z) ,$$

so that

$$(1 - az) \sum_{n=0}^{\infty} A_n x^n z^n = (1 - z) \sum_{n=0}^{\infty} A_n z^n .$$

This gives

$$(1 - x^n)A_n = (1 - ax^{n-1})A_{n-1} ,$$

and (11.3) follows at once.

In particular, for $a = x^k$, (11.3) becomes

(11.4) $$\prod_{n=0}^{k-1}(1 - x^n z)^{-1} = \sum_{n=0}^{\infty} \frac{(x^k)_n}{(x)_n} z^n = \sum_{n=0}^{\infty} \begin{bmatrix} k+n-1 \\ n \end{bmatrix} z^n ,$$

where

$$\begin{bmatrix} k \\ n \end{bmatrix} = \frac{(x)_k}{(x)_n (x)_{k-n}} \qquad (0 < n \le k) .$$

If we take $a = x^{-k}$ and replace z by $x^k z$ we get

(11.5) $$\prod_{n=0}^{k-1}(1 - x^n z) = \sum_{n=0}^{k} (-1)^n \begin{bmatrix} k \\ n \end{bmatrix} x^{\frac{1}{2}n(n-1)} z^n .$$

Note that when $x = 1$, $\begin{bmatrix} k \\ n \end{bmatrix}$ reduces to $\binom{k}{n}$.

It also follows from (11.3) that

(11.6) $$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k (b)_{n-k} a^{n-k} = (ab)_n ,$$

for arbitrary a,b. Specializing a,b or using (11.5), we get

(11.7) $$\sum_{=0}^{k} \begin{bmatrix} m \\ k-s \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} x^{s^2-ks+ms} = \begin{bmatrix} m+n \\ k \end{bmatrix} .$$

which evidently generalizes (2.2).

The function

$$e(z) = \prod_{n=0}^{\infty} (1 - x^n z)^{-1}$$

can be thought of as an analog of the exponential function. This suggests the definition (compare (4.2)),

$$e(tz) \sum_{n=0}^{\infty} a_n z^n/(x)_n = \sum_{n=0}^{\infty} A_n(t) z^n/(x)_n , \tag{11.8}$$

where a_n is a function of x that is independent of t and z. Using (11.1), we get

$$A_n(t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_k t^{n-k} . \tag{11.9}$$

If we define the operator Δ by means of

$$\Delta f(t) = f(t) - f(xt) ,$$

it follows at once from (11.9) that

$$\Delta A_n(t) = (1 - x^n)A_{n-1}(t) . \tag{11.10}$$

Conversely if a set of polynomials in t satisfy (11.10), then there exists a sequence $\{a_n\}$ independent of t such that (11.8) holds.

The special case $a_n = 1$ is of particular interest. Put

$$e(tz)e(z) = \sum_{n=0}^{\infty} H_n(t)z^n/(x)_n ,$$

so that

$$H_n(t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} t^k .$$

For properties of these and related polynomials, see [7], [11], [19]. The $H_n(t)$ are in some respects analogous to the Hermite polynomials. We cite the bilinear generating function

$$(11.11)\quad \sum_{n=0}^{\infty} H_n(u)H_n(v)z^n/(x)_n = \frac{e(z)e(uz)e(vz)e(uvz)}{e(uvz^2)}$$

$$= \prod_{n=0}^{\infty} \frac{1 - x^n uvz^2}{(1 - x^n z)(1 - x^n uz)(1 - x^n vz)(1 - x^n uvz)}$$

which may be compared with (5.7). For proof of (11.11), see [7].

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EXPLICIT DETERMINATION OF THE PERRON MATRICES IN PERIODIC ALGORITHMS OF THE PERRON-JACOBI TYPE WITH APPLICATION TO GENERALIZED FIBONACCI NUMBERS WITH TIME IMPULSES

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0. By the Perron matrices P_k in an n-dimensional algorithm of the Jacobi-Perron type [1] we understand the analogue to the 2-dimensional matrices

$$\begin{bmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{bmatrix}$$

built up from two consecutive "convergents"

$$\frac{p_{k-1}}{q_{k-1}}, \quad \frac{p_k}{q_k} \quad .$$

of an ordinary continued fraction.

As explained in detail in Chapter I of a previous joint paper of ours [2] these $n \times n$ matrices P are defined recurrently by

$$P_k = P_{k-1}A \qquad (k = 0, 1, \cdots) ,$$

with the initial condition

$$P_{-1} = I \qquad \text{(n-rowed unit matrix)} ,$$

where the matrices

$$A_k = \begin{bmatrix} 0 & \cdots & 0 & a_{0k} \\ 1 & & & a_{1k} \\ & \ddots & & \\ & & 1 & a_{n-1,k} \end{bmatrix} \qquad (k = 0, 1, \cdots) ,$$

are built up from the "partial quotients"

$$a_{0k} = 1, \quad a_{1k}, \quad \cdots \quad a_{n-1},$$

in the algorithm, which in the special case $n = 2$ of ordinary continued fractions reduce essentially to only one a_{1k} in each step.

From this recurrent definition it follows that the Perron matrices P_{-1} are built up from an infinite sequence of n-termed columns $\mathfrak{M}_{k-1}$ in the form

$$P_{k-1} = (\mathfrak{M}_{k-n}, \cdots, \mathfrak{M}_{k-1}) ,$$

satisfying the recurrency formulae

$$(0.1) \qquad a_{0k}\mathfrak{M}_{k-n} + \cdots + a_{n-1,k}\mathfrak{M}_{k-1} \qquad (k \geq 0) ,$$

with the initial condition that

$$\mathfrak{M}_{-n} = W_0, \cdots, \mathfrak{M}_{-1} = W_{n-1}$$

are the columns of the n-rowed unit matrix I.

In the present paper the entries of the Perron matrices P_{k-1} shall be denoted by $p^{(\nu)}_{k-(n-\nu')}$, where the super- and subscripts $\nu = 0, \cdots, n-1$ and $\nu' = 0, \cdots, n-1$ indicate the lines and columns, respectively:

$$P_{k-1} = \left(p^{(\nu)}_{k-(n-\nu')}\right)_{\substack{\nu \text{ lines} \\ \nu' \text{ columns}}} \qquad \begin{pmatrix} \nu = 0, \cdots, n-1 \\ \nu' = 0, \cdots, n-1 \end{pmatrix}$$

Thus the recurrency formulae (0.1) with the initial conditions (0.2) become

$$(0.3) \qquad p^{(\nu)}_k = \sum_{\nu'=0}^{n-1} a_{\nu'} \, p^{(\nu)}_{k-(n-\nu')} \qquad \begin{pmatrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{pmatrix}$$

with

$$(0.4) \qquad p^{(\nu)}_{-(n-\nu')} = e^{(\nu)}_{\nu'} = \begin{Bmatrix} 1 \text{ for } \nu = \nu' \\ 0 \text{ for } \nu \neq \nu' \end{Bmatrix}$$

(entries of the unit matrix I) .

In Perron's original paper [2] these $p^{(\nu)}$ would be the $A^{(k+n)}$.

We shall consider only purely periodical algorithms. Let ℓ be the length of the period. Then in the recurrency formulae (0.3) there are only ℓ different n-termed coefficient sets $a_{\nu' k}$ $(\nu' = 0, \cdots, n-1)$, which recur periodically. In our first, purely algebraic part these ℓ sets will be considered as algebraically independent indeterminates and denoted by $a_{\nu'}^{(\lambda)}$ $(\lambda = 0, \cdots, \ell - 1)$. For the sake of algebraic generality and formal symmetry we include in this stipulation also the coefficients $a_0^{(\lambda)}$ which in the actual algorithm are throughout equal to 1.

For purely periodical algorithms, the infinite sequence of recurrency formulae (0.3) reduces to a finite system

$$p_{k\ell+\lambda}^{(\nu)} = \sum_{\nu'=0}^{n-1} a_{\nu'}^{(\lambda)} p_{(k\ell+\lambda)-(n-\nu')} \quad \begin{pmatrix} k \quad 0 \\ \lambda = 0, \cdots, \ell - 1 \\ \nu = 0, \cdots, n-1 \end{pmatrix} \tag{0.5}$$

of ℓ linear recurrencies with the n linearly independent initial conditions (0.4).

We shall chiefly be concerned with the special case of period length $\ell = 1$, where there remains only one single linear recurrency

$$p_k^{(\nu)} = \sum_{\nu'=0}^{n-1} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} \quad \begin{pmatrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{pmatrix} \tag{0.6}$$

with the n linearly independent initial conditions (0.4). In this case we shall obtain the following simple explicit expressions for the entries $p_k^{(\nu)}$ of the Perron matrices P_k (last column):

$$\left\{ \begin{aligned} p_k^{(\nu)} = & \sum_{\substack{L(k_0, \cdots, k_{n-1}) = k + (n-\nu) \\ k_0, \cdots, k_{n-1} \geq 0}} \binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots, k_{n-1}} \\ & \frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} \begin{pmatrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{pmatrix} \end{aligned} \right. , \tag{0.7}$$

with summation restricted by the linear form

(0.8) $$L(k_0, \cdots, k_{n-1}) = nk_0 + (n-1)k_1 + \cdots + 1k_{n-1}$$

in the summation variables $k_0, \cdots, k_{n-1}$, and with the polynomial coefficients

(0.9) $$\binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots \ k_{n-1}} = \frac{(k_0 + \cdots + k_{n-1})!}{k_0! \cdots k_{n-1}!} \quad .$$

The procedure by which we reach our aim (0.7) is the very old method of Euler, viz., to translate the recurrency formula (0.6) for the sequences $p_k^{(\nu)}$ into algebraic expressions for the generating functions

$$p^{(\nu)}(x) = \sum_{k \geq 0} p_k^{(\nu)} x^k \qquad (\nu = 0, \cdots, n-1) \ ,$$

and to determine the power series coefficients $p_k^{(\nu)}$ from those algebraic expressions.

In the general case of arbitrary period length ℓ we shall show that the same object can be achieved in principle. The explicit formulae, however, would be so complicated that one can hardly expect to write them down <u>in extenso</u>, but for simpler special cases. As an example, we shall carry through <u>in extenso</u> the very special case $\ell = 2$ with $n = 2$, i.e., the case of purely periodic ordinary continued fractions with period length 2.

There is, however, a special case of a more general type in which we can obtain as definite a result as (0.7). Amongst the numerous periodic algorithms, discovered by the first author in previous papers*, a particular period structure prevails, viz., of length $\ell = n$ and with the following specialization of the coefficients in (0.5):

(0.10) $$a_{\nu'}^{(\lambda)} = t^{d_n(\lambda, \nu')} a_{\nu'} \qquad (\lambda, \nu' = 0, \cdots, n-1) \ ,$$

where

*See the complete list of references in [3].

(0.11)
$$d_n(\lambda, \nu') = \begin{matrix} 0 \text{ for } \lambda + \nu' < n \\ 1 \text{ for } \lambda + \nu' \geq n \end{matrix}$$

is the so-called "number to be carried over" in the addition of the n-adic digits λ, ν'. In this important case we shall derive from (0.7) the following generalization:

(0.12)
$$p_k^{(\nu)} = t^{-\left[\frac{k}{n}\right]-1} \sum_{L(k_0,\cdots,k_{n-1})=k+(n-\nu)} \binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots, k_{n-1}} \frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} \times$$
$$t^{k_0+\cdots+k_{n-1}} a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} \quad \left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix}\right) .$$

We shall come back to another significance of this case in our second chapter.

CHAPTER I: ALGEBRAIC FOUNDATIONS

1. We begin with considering the special case of period length $\ell = 1$. To the recurrency formula (0.6), viz.,

(1.1)
$$p^{(\nu)} = \sum_{\nu'=0}^{n-1} a_{\nu'} p^{(\nu)}_{-(n-\nu')} \qquad (k \geq 0)$$

with the initial conditions (0.4), viz.,

(1.2)
$$p^{(\nu)}_{-(n-\nu')} = e^{(\nu)}_{\nu'} \qquad (\nu, \nu' = 0, \cdots, n-1) \quad ,$$

we let correspond the characteristic polynomial

$$F = F(x) = 1 - \sum_{\nu'=0}^{n-1} a_{\nu'} x^{n-\nu'} \quad ,$$

and the n generating functions

$$p^{(\nu)} = p^{(\nu)}(x) = \sum_{k \geq 0} p_k^{(\nu)} x^k .$$

Now

$$\begin{aligned}
a_{\nu'} x^{n-\nu'} p^{(\nu)} &= \sum_{k \geq 0} a_{\nu'} p_k^{(\nu)} x^{k+(n-\nu')} \\
&= \sum_{k \geq n-\nu'} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} x^k \\
&= \sum_{k \geq 0} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} x^k \\
&\quad - \sum_{0 \leq k \leq (n-\nu')-1} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} x^k \\
&= \sum_{k \geq 0} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} x^k \\
&\quad - \begin{cases} a_{\nu'} x^{\nu-\nu'} & \text{for } \nu' \leq \nu \\ 0 & \text{for } \nu' > \nu \end{cases} ,
\end{aligned}$$

the latter because the summation condition $0 \leq k \leq (n - \nu') - 1$ is equivalent to $-(n - \nu') < k - (n - \nu') \leq -1$, so that the initial conditions (1.2) are applicable.

Summation over ν' then yields

$$\begin{aligned}
FP^{(\nu)} - P^{(\nu)} &= -\sum_{k \geq 0} \left(\sum_{\nu'=0}^{n-1} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} \right) x^k + \\
&\quad + \sum_{\nu'=0}^{\nu} a_{\nu'} x^{\nu-\nu'} .
\end{aligned}$$

Here the negative terms on the left and right are equal to each other on account of the recurrency formula (1.1). This gives the algebraic expressions

$$p^{(\nu)} = \frac{A^{(\nu)}}{F} \quad \text{with} \quad A^{(\nu)}$$

(1.3)
$$= A^{(\nu)}(x) = \sum_{\nu'=0}^{\nu} a_{\nu'} x^{\nu-\nu'} \qquad (\nu = 0,\cdots,n-1) ,$$

for the generating functions $P^{(\nu)}$.

2. In order to obtain explicit expressions for the recurrent sequences $p_k^{(\nu)}$, we have to develop the rational functions (1.3) into power series in x. The power series for $1/F$ is obtained easily from the geometrical series:

(2.1)
$$\begin{aligned}\frac{1}{F} &= \sum_{k\geq 0}\left(\sum_{\nu'=0}^{n-1} a_{\nu'} x^{n-\nu'}\right)^k \\ &= \sum_{k_0,\cdots,k_{n-1}\geq 0}\binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}}\times \\ &\quad \times a_0^{k_0}\cdots a_{n-1}^{k_{n-1}} x^{nk_0+(n-1)k_1+\cdots+1k_{n-1}} \\ &= \sum_{k\geq 0}\left(\sum_{L(\mathfrak{M})=k}\binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}} a_0^{k_0}\cdots a_{n-1}^{k_{n-1}}\right)x^k ,\end{aligned}$$

with the linear form

(2.2)
$$L(\mathfrak{M}) = nk_0 + (n-1)k_1 + \cdots + 1k_{n-1}$$

in the summation variable vector

$$\mathfrak{M} = (k_0, \cdots, k_{n-1}) .$$

In what follows the summation variables $k_0,\cdots,k_{n-1}$ are throughout silently supposed to be 0. The solutions $\mathfrak{M}$ of $L(\mathfrak{M}) = k$ correspond to the partitions of k into summands from $1,\cdots,n$; their number $p_{n(k)}$ is well known.

In order to obtain from (2.1) the power series for the rational functions $p^{(\nu)}$ in (1.3), we have to multiply by the single terms $a_{\nu'}x^{\nu-\nu'}$ of the polynomials $A^{(\nu)}$ in the numerator and then sum up over ν'. Multiplication by one of these terms and subsequent transformation of the summation yields in the first place

$$
\begin{aligned}
\frac{a_{\nu'}x^{\nu-\nu'}}{F} &= \sum_{k\geq 0}\left(\sum_{L(\mathfrak{M})=k}\binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}}\right. \\
&\qquad \left.\times\, a_0^{k_0}\cdots a_{\nu'}^{k_{\nu'}+1}\cdots a_{n-1}^{k_{n-1}}\right)x^{k+(\nu-\nu')} \\
&= \sum_{k\geq 0}\left(\sum_{L(\mathfrak{M})=k-(\nu-\nu')}\binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}}\right. \\
&\qquad \left. a_0^{k_0}\cdots a_{\,'}^{k_{\,'}+1}\cdots a_{n-1}^{k_{n-1}}\right)x^k .
\end{aligned}
$$

In order to simplify the subsequent summation over ν' we have here formally admitted terms with $L(k_0,\cdots,k_{n-1}) < 0$, which actually vanish because the summation condition is empty. Summation over ν' then yields the development

$$
\begin{aligned}
p^{(\nu)} &= \sum_{k\geq 0}\left(\sum_{\nu'=0}^{\nu}\ \sum_{L(\mathfrak{M})=k-(\nu-\nu')}\binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}}\right. \\
&\qquad \left.\times\, a_0^{k_0}\cdots a_{\nu'}^{k_{\nu'}+1}\cdots a_{n-1}^{k_{n-1}}\right)x^k
\end{aligned}
$$

for the generating functions, and thus the explicit expressions

$$
\begin{aligned}
p^{(\nu)} &= \sum_{\nu'=0}^{\nu}\ \sum_{L(\mathfrak{M})=k-(\nu-\nu')}\binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}} \\
&\qquad \times\, a_0^{k_0}\cdots a_{\nu'}^{k_{\nu'}+1}\cdots a_{n-1}^{k_{n-1}} ,
\end{aligned}
\tag{2.3}
$$

for the recurrent sequences in question.

3. As a last step, the sum (2.3) of polynomials in $a_0, \cdots, a_{n-1}$ can be put into canonical form, i.e., represented as a single polynomial in $a_0, \cdots, a_{n-1}$. This is achieved by a further transformation of summation which, in its turn, allows to reverse the order of the two summations.

The transformation, leading to this, is

$$k_{\nu'} \to k_{\nu'} - 1 \quad . \tag{3.1}$$

It is true that by it the silent summation condition $k_{\nu'} \geq 0$ is transformed into $k_{\nu'} \geq 1$. However here, too, after the transformation, the summation may again be extended formally over all $k_{\nu'} \geq 0$, because the polynomial coefficients with a negative term in the "denominator" vanish, if only the sum of all terms in the "numerator" is non-negative. The truth of this assertion is easily seen by expressing the factorials in the definition (0.9) of the polynomial coefficients as values of the Gamma-function and observing that this function has no zeros at all, and has poles only at $0, -1, -2, \cdots$. That the "numerator" here is non-negative, is seen as follows. Under the transformation (3.1), according to the definition (0.8), one has

$$L(\mathfrak{M}) \to L(\mathfrak{M}) - (n - \nu')$$

and hence

$$p_\kappa^{(\nu)} = \sum_{\nu'=0}^{\nu} \sum_{L(\mathfrak{M})=k+(n-\nu)} \binom{k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1}}{k_0, \quad \cdots, \quad k_{\nu'} - 1, \quad \cdots, \quad k_{n-1}} \times a_0^{k_0} \cdots a_{\nu'}^{k_{\nu'}} \cdots a_{n-1}^{k_{n-1}} \quad . \tag{3.2}$$

Here the sum of all terms in the "numerator" is surely non-negative, because $L(\mathfrak{M}) = k + (n - \nu) \geq k + 1 \geq 1$ and hence not all $k_0, \cdots, k_{n-1}$ vanish.

Since by this transformation the inner summation condition in (3.2) has become independent of the outer summation variable ν', the order of the two summations may now be reversed:

$$p_k^{(\nu)} = \sum_{L(\mathfrak{M})=k+(n-\nu)} \left(\sum_{\nu'=0}^{} \binom{k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{\nu'} - 1\ ,\ \cdots,\ k_{n-1}} \right) \times \tag{3.3}$$
$$\times\, a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} .$$

Thus the polynomial (2.3) has already been put into canonical form. But, moreover, it is even possible to consummate the inner sum in (3.3). For, by definition

$$\binom{k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{\nu'} - 1\ ,\ \cdots,\ k_{n-1}} = \frac{(k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1})!}{k_0! \cdots (k_{\nu'} - 1)! \cdots k_{n-1}!}$$
$$= \binom{k_0 + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{n-1}} \frac{k_{\nu'}}{k_0 + \cdots + k_{n-1}} \qquad (\text{also for } k_{\nu'} = 0) .$$

and hence

$$\sum_{\nu'=0}^{\nu} \binom{k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{\nu'} - 1\ ,\ \cdots\ ,\ k_{n-1}} \tag{3.4}$$
$$= \binom{k_0 + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{n-1}} \frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} .$$

Thus (3.3) yields our first chief result

$$p_k^{(\nu)} = \sum_{L(\mathfrak{M})=k+(n-\nu)} \binom{k_0 + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{n-1}} \times \tag{3.5}$$
$$\times \frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} , \left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix} \right) .$$

as announced in (0.7).

We remark that (3.5), conveniently interpreted, holds even for $k \geq -n$, i.e., including the initial values corresponding to $k = -(n - \nu')$ ($\nu' = 0, \cdots , n - 1$). For in these cases the summation condition $L(\mathfrak{M}) = \nu' - \nu$ has no non-negative solutions if $\nu' < \nu$, only one such solution, viz., $k_0, \cdots, k_{n-1} = 0$, if $\nu' = \nu$, and only such solutions with $k_0, \cdots, k_\nu = 0$ if $\nu' > \nu$. Hence for $\nu' < \nu$ the sum is 0 by the usual convention for empty sums, for $\nu' > \nu$ it is also zero with regard to the factor

$$\frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} ,$$

and for $\nu' = \nu$ it is 1 if this factor of the indeterminate form $0/0$ is understood as 1.

It is furthermore perhaps not useless to remark that for the first initial condition (1.2), i.e., for $\nu = 0$ this result can also be written in the simpler form

$$(3.6) \qquad p_k^{(0)} = \sum_{L(\mathfrak{M})=k} \binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots, k_{n-1}} a_0^{k_0+1} \cdots a_1^{k_1} \cdots a_{n-1}^{k_{n-1}} \qquad (k \geq 0)$$

as is already clear from the intermediate result (2.3).

4. Since operating with polynomial coefficients, and in particular with their fundamental recurrency property

$$(4.1) \qquad \sum_{\nu'=0}^{n-1} \binom{k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1}}{k_0, \cdots, k_{\nu'} - 1, \cdots, k_{n-1}} = \binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots, k_{n-1}}$$

(special case $\nu = n - 1$ of (3.4)), is not so familiar and handy as in the special case $n = 2$ of binomial coefficients, we attach here the following simple reduction of the former to the latter.

From the definition (0.9) one has

$$(4.2) \qquad \begin{aligned} \binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots, k_{n-1}} &= \binom{k_0 + \cdots + k_\nu}{k_0, \cdots, k_\nu} \times \\ &\times \binom{(k_0 + \cdots + k_\nu) + k_{\nu+1} + \cdots + k_{n-1}}{k_0 + \cdots + k_\nu, k_{\nu+1}, \cdots, k_{n-1}} , \end{aligned}$$

for any $\nu = 1, \cdots, n-2$. For $\nu = 1$, the first factor on the right is the binomial coefficient

$$\binom{k_0 + k_1}{k_0}$$

Iterating this case of (4.2) in the second factor on the right, and putting

$$(4.3)\qquad \begin{aligned} k'_0 &= k_0 \\ k'_1 &= k_0 + k_1 \\ &\vdots \\ k'_{n-1} &= k_0 + k_1 + \cdots + k_{n-1}\,, \end{aligned}$$

one obtains the reduction

$$(4.4)\qquad \binom{k_0 + \cdots + k_{n-1}}{k_0\,, \cdots\,, k_{n-1}} = \binom{k'_1}{k'_0}\binom{k'_2}{k'_1}\cdots\binom{k'_{n-1}}{k'_{n-2}}.$$

Application of this reduction to our final result (3.5) yields the equivalent expression

$$(4.5)\qquad \begin{aligned} p_k^{(\nu)} = \sum_{S(\mathfrak{M})=k+(n-\nu)} &\binom{k'_1}{k'_0}\binom{k'_2}{k'_1}\cdots\binom{k'_{n-1}}{k'_{n-2}} \times \\ &\times \frac{k'_\nu}{k'_{n-1}} a_0^{k'_0} a_1^{k'_1-k'_0} \cdots a_{n-1}^{k'_{n-1}-k'_{n-2}} \quad \binom{k \geq 0}{\nu = 0, \cdots, n-1} \end{aligned}$$

where

$$(4.6)\qquad S(\mathfrak{M}) = k'_0 + \cdots + k'_{n-1}$$

is the simpler linear form obtained by the transformation (4.3) from $L(\mathfrak{M})$ in (2.2). The silent summation condition $k_0, \cdots, k_{n-1} \geq 0$ is transformed in $0 \leq k'_0 \leq \cdots \leq k'_{n-1}$.

Special cases of the formulae (4.5), with $n = 2$ and $n = 3$, have recently been developed by Arkin-Hoggatt [4].

5. We now turn to the general case of an arbitrary period length ℓ. To the ℓ recurrency formula (0.5), viz.,

$$p^{(\nu)}_{k\ell+\lambda} = \sum_{\nu'=0}^{n-1} a^{(\lambda)}_{\nu'} p^{(\nu)}_{(k\ell+\lambda)-(n-\nu')} \qquad \left(\lambda = 0, \overset{k \geq 0}{\cdots}, n - 1\right) \tag{5.1}$$

with the initial conditions (0.4), viz.,

$$p^{(\nu)}_{-(n-\nu')} = e^{(\nu)}_{\nu'} \qquad (\nu, \nu' = 0, \cdots, n - 1) \tag{5.2}$$

we let correspond the ℓ polynomials

$$F^{(\lambda)} = F^{(\lambda)}(x) = 1 - \sum_{\nu'=0}^{n-1} a^{(\lambda)}_{\nu'} x^{n-\nu'}$$

and the n generating functions

$$p^{(\nu)} = p^{(\nu)}(x) = \sum_{k \geq 0} p^{(\nu)} x^k .$$

We split these polynomials and functions into components, corresponding to the residue classes mod ℓ of the x-exponents:

$$\begin{aligned} F^{(\lambda)} &= \sum_{\lambda'=0}^{\ell-1} F^{(\lambda)}_{\lambda'} \qquad \text{with} \qquad F^{(\lambda)}_{\lambda'} = F^{(\lambda)}_{\lambda'}(x) \\ &= e^{(0)}_{\lambda'} - \sum_{\substack{\nu'=0 \\ n-\nu' \equiv \lambda' \bmod \ell}}^{n-1} a^{(\lambda)}_{\nu'} x^{n-\nu'} , \end{aligned} \tag{5.3}$$

$$p^{(\nu)} = \sum_{\lambda''=0}^{\ell-1} p_{\lambda''}^{(\nu)} \quad \text{with} \quad p_{\lambda''}^{(\nu)} = p_{\lambda''}^{(\nu)}(x) =$$

(5.4)

$$= \sum_{k\geq 0} p_{k\ell+\lambda''}^{(\nu)} x^{k\ell+\lambda''} .$$

In order to translate the recurrency formulae (5.1) with the initial conditions (5.2) into algebraic expressions for the generating functions, we multiply, for each fixed λ and ν, the terms $a_{\nu'}^{(\lambda)} x^{n-\nu'}$ of a component $F_{\lambda'}^{(\lambda)}$ by that component $p_{\lambda''}^{(\nu)}$ for which

(5.5) $$\lambda' + \lambda'' \equiv \lambda \bmod \ell .$$

Subsequently we sum up, first over the ν' with

(5.6) $$n - \nu' \equiv \lambda' \bmod \ell ,$$

and then over the ℓ pairs λ', λ'' with (5.5). According to the congruences (5.5) and (5.6), we put

(5.7) $$(n - \nu') + \lambda'' = \lambda + h\ell ,$$

with an integer $h \geq 0$. The whole procedure will be quite analogous to that in Section 1 for the special case $\ell = 1$. In the first place, one has

$$a_{\nu'}^{(\lambda)} x^{n-\nu'} p_{\lambda''}^{(\nu)} = \sum_{k\geq 0} a_{\nu'}^{(\lambda)} p_{k\ell+\lambda''}^{(\nu)} x^{(k\ell+\lambda'')+(n-\nu')}$$

$$= \sum_{k\geq 0} a_{\nu'}^{(\lambda)} p_{k\ell+\lambda''}^{(\nu)} x^{(k+h)\ell+\lambda} \qquad \text{(by (5.7))}$$

$$= \sum_{k\geq h} a_{\nu'}^{(\lambda)} p_{(k-h)\ell+\lambda''}^{(\nu)} x^{k\ell+\lambda}$$

$$= \sum_{k\geq h} a_{\nu'}^{(\lambda)} p_{(k-h)\ell+\lambda''}^{(\nu)} x^{k\ell+\lambda} \qquad \text{(by (5.7))}$$

$$= \sum_{k \geq 0} a_{\nu'}^{(\lambda)} p_{(k\ell+\lambda)-(n-\nu')}^{(\nu)} x^{k\ell+\lambda}$$

$$- \sum_{0 \leq k \leq h-1} a_{\nu'}^{(\lambda)} p_{(k\ell+\lambda)-(n-\nu')}^{(\nu)} x^{k\ell+\lambda}$$

$$= \sum_{k \geq 0} a_{\nu'}^{(\lambda)} p_{(k\ell+\lambda)-(n-\nu')}^{(\nu)} x^{k\ell+\lambda}$$

$$- \begin{Bmatrix} a_{\nu'}^{(\lambda)} x^{\nu-\nu'} & \text{for } \nu' \equiv \nu - \lambda \bmod \ell \text{ and } \nu' \leq \nu \\ 0 \text{ otherwise} & \end{Bmatrix} .$$

The latter one sees as follows. The summation condition $0 \leq k \leq h - 1$ implies, again by (5.7), the inequality chain

$$-n \leq -(n - \nu') \leq \lambda - (n - \nu') \leq (k\ell + \lambda) - (n - \nu') \leq ((h - 1)\ell + \lambda'')$$
$$- (n - \nu') = -(\ell - \lambda'') \leq -1 ,$$

so that the initial conditions (5.2) are applicable. They say that almost all terms of the sum in question vanish, save only one with

$$(k\ell + \lambda) - (n - \nu') = -(n - \nu), \qquad \text{or else,} \qquad k\ell + \lambda = \nu - \nu' .$$

Such a term can occur only if $\nu' \equiv \nu - \lambda \bmod \ell$ and $\nu' \leq \nu$. If these conditions are satisfied, it actually occurs, because then the equation $k\ell + \lambda = \nu - \nu'$ has a solution $k \geq 0$ with

$$k\ell = (\nu - \nu') - \lambda < (n - \nu') - \lambda = h\ell - \lambda'' \leq h\ell ,$$

and hence $k \leq h - 1$.

Summation over the $\nu' = 0, \cdots, n - 1$ with $n - \nu' \equiv \lambda' \bmod \ell$, according to (5.3) now yields

$$(F_{\lambda'}^{(\lambda)} - e_{\lambda'}^{(0)})p_{\lambda''}^{(\nu)} = -\sum_{k\geq 0}\left(\sum_{\substack{\nu'=0 \\ n-\nu'\equiv\lambda'\bmod \ell}}^{n-1} a_{\nu'}^{(\lambda)} p_{(\ell+\lambda)-(n-\nu')}^{(\nu)}\right) x^{k\ell+\lambda} +$$

$$+ \sum_{\substack{\nu'=0 \\ n-\nu'\equiv\lambda' \bmod \ell \\ \nu-\nu'\equiv\lambda \bmod \ell}}^{\nu} a_{\nu'}^{(\lambda)} x^{\nu-\nu'} ,$$

and summation over the pairs λ', λ'' with $\lambda' + \lambda'' \equiv \lambda \bmod \ell$ further yields

$$\sum_{\lambda'+\lambda''\equiv \bmod \ell} F_{\lambda'}^{(\lambda)} P_{\lambda''}^{(\nu)} - P_{\lambda}^{(\nu)} = -\sum_{k\geq 0}\left[\sum_{\nu'=0}^{n-1} a_{\nu'}^{(\lambda)} p_{(\ell+\lambda)-(n-\nu')}^{(\nu)}\right] \times$$

$$\times\ x^{k\ell+\lambda} + \sum_{\substack{\nu'=0 \\ \nu-\nu'\equiv\lambda \bmod \ell}}^{n-1} a_{\nu'}^{(\lambda)} x^{\nu-\nu'} .$$

Here the negative terms on the left and right are equal to each other on account of the recurrency formulae (5.1). Thus the following system of ℓ linear equations for the ℓ components $P_{\lambda''}^{(\nu)}$ of the generating function $P^{(\nu)}$ results:

$$\sum_{\lambda'+\lambda''\equiv\lambda \bmod \ell} F_{\lambda'}^{(\lambda)} P_{\lambda''}^{(\nu)} = A^{(\lambda,\nu)} \quad \text{with} \quad A^{(\lambda,\nu)}$$

$$(5.8) \qquad = A^{(\lambda,\nu)}(x) = \sum_{\substack{\nu'=0 \\ \nu-\nu'\equiv\lambda \bmod \ell}}^{n-1} a_{\nu'}^{(\lambda)} x^{\nu-\nu'} .$$

The matrix of its coefficients is built up from the components $F_{\lambda'}^{(\lambda)}$, of the characteristic polynomials $F^{(\lambda)}$. Lines and columns of this matrix are specified by λ and $\lambda'' \equiv \lambda - \lambda' \bmod \ell$ (not by λ and λ'). Written out fully, it is the matrix

$$\left(F^{(\lambda)}_{\lambda-\lambda''}\right)_{\substack{\lambda \text{ lines} \\ \lambda'' \text{ columns}}} = \begin{pmatrix} F_0^{(0)} & F_{\ell-1}^{(0)} & \cdots & F_1^{(0)} \\ F_1^{(1)} & F_0^{(1)} & \cdots & F_2^{(1)} \\ \vdots & & & \\ F_{\ell-1}^{(\ell-1)} & F_{\ell-2}^{(\ell-1)} & \cdots & F_0^{(\ell-1)} \end{pmatrix} .$$

Here $\lambda - \lambda''$ on the left is to be understood as reduced to its least non-negative residue mod ℓ.

Now let

$$D = \left| F^{(\lambda)}_{\lambda-\lambda''} \right|_{\substack{\lambda \text{ lines} \\ \lambda'' \text{ columns}}}$$

denote the determinant of this matrix and $(D^{(\lambda)}_{\lambda-\lambda''})$ its transposed adjoined matrix. Then the linear system (5.8) has the solution

$$(5.9) \qquad P^{(\nu)}_{\lambda''} = \frac{\sum_{\lambda=0}^{\ell-1} D^{(\lambda)}_{\lambda-\lambda''} A^{(\lambda,\nu)}}{D} , \qquad (\lambda'' = 0, \cdots, \ell - 1) .$$

From this one obtains the following algebraic expressions for the generating functions themselves:

$$(5.10) \qquad P^{(\nu)} = \frac{\sum_{\lambda=0}^{\ell-1} \left(\sum_{\lambda''=0}^{\ell-1} D^{(\lambda)}_{\lambda-\lambda''} \right) A^{(\lambda,\nu)}}{D} , \qquad (\nu = 0, \cdots, n - 1) .$$

In order to obtain explicit expressions for the recurrent sequences $p^{(\nu)}$, one has to develop these rational functions of x into power series in x. This seems however extremely difficult. One would first have to find a sufficiently smooth expression for the determinant D and its minors $D^{(\lambda)}_{\lambda-\lambda''}$.

In the following two sections we illustrate this on the next-simplest case $\ell = 2$ and carry it through to the end under the special assumption $n = 2$. After what has been delineated in the preceding sections, we can be brief in doing this.

6. In the special case $\ell = 2$ we have to consider two alternating recurrency formulae

$$p_{2k}^{(\nu)} = \sum_{\nu'=0}^{n-1} a_{\nu'} p_{2k-(n-\nu')}^{(\nu)} ,$$

$$p_{2k+1}^{(\nu)} = \sum_{\nu'=0}^{n-1} b_{\nu'} p_{(2k+1)-(n-\nu')}^{(\nu)} ,$$

for each of the n linearly independent initial conditions

$$p_{-(n-\nu')}^{(\nu)} = e_{\nu'}^{(\nu)} \qquad (\nu, \nu' = 0, \cdots, n-1) .$$

For the sake of easier readability, we here have distinguished the two coefficient sequences, hitherto denoted by $a_{\nu'}^{(\lambda)}$, instead by the upper indices $\lambda = 0,1$ by writing them with two different letters a, b. In the same manner we denote the polynomial pairs $F^{(\lambda)}$ and $A^{(\lambda,\nu)}$ $(\lambda = 0,1)$ now by F, G and $A^{(\nu)}$, $B^{(\nu)}$, respectively.

From the pair of characteristic polynomials

$$F = F(x) = 1 - \sum_{\nu'=0}^{n-1} a_{\nu'} x^{n-\nu'} = F_0 + F_1 ,$$

$$G = G(x) = 1 - \sum_{\nu'=0}^{n-1} b_{\nu'} x^{n-\nu'} = G_0 + G_1 ,$$

each decomposed in its even and odd components, algebraic expressions for the generating functions

$$P^{(\nu)} = P^{(\nu)}(x) = \sum_{k \geq 0} p_{\kappa}^{(\nu)} x^k = P_0^{(\nu)} + P_1^{(\nu)} ,$$

likewise decomposed, are found as follows.

The linear equation pair (5.8) for the component pair $P_0^{(\nu)}$, $P_1^{(\nu)}$ has the matrix

$$\begin{pmatrix} F_0 & F_1 \\ G_1 & G_0 \end{pmatrix}$$

with the determinant

$$D = F_0G_0 - F_1G_1 \quad ,$$

and with the transposed adjoined matrix

$$\begin{pmatrix} G_0 & -F_1 \\ -G_1 & F_0 \end{pmatrix} \quad .$$

The terms on the right are

$$A^{(\nu)} = \sum_{\substack{\nu'=0 \\ \nu-\nu' \equiv 0 \bmod 2}}^{\nu} a_{\nu'} x^{\nu-\nu'}$$

$$B^{(\nu)} = \sum_{\substack{\nu'=0 \\ \nu-\nu' \equiv 1 \bmod 2}}^{\nu} b_{\nu'} x^{\nu-\nu'} \quad .$$

Hence the solution (5.9) for the components is

$$P_0^{(\nu)} = \frac{G_0A^{(\nu)} - F_1B^{(\nu)}}{F_0G_0 - F_1G_1} , \qquad P_1^{(\nu)} = \frac{-G_1A^{(\nu)} + F_0B^{(\nu)}}{F_0G_0 - F_1G_1} ,$$

and the generating functions (5.10) themselves are

$$P^{(\nu)} = \frac{(G_0 - G_1)A^{(\nu)} + (F_0 - F_1)B^{(\nu)}}{F_0G_0 - F_1G_1} \quad . \tag{6.1}$$

It is worth remarking that this can be written in such a way that only the characteristic polynomials F, G themselves, not their components, figure in it. For, the component pairs are given by

$$F_0(x) = \frac{F(x) + F(-x)}{2}, \qquad F_1(x) = \frac{F(x) - F(-x)}{2},$$

$$G_0(x) = \frac{G(x) + G(-x)}{2}, \qquad G_1(x) = \frac{G(x) - G(-x)}{2},$$

Thus the determinant becomes

$$D(x) = \frac{F(x)G(-x) + F(-x)G(x)}{2},$$

and the generating functions become

$$P^{(\nu)}(x) = \frac{G(-x)A^{(\nu)}(x) + F(-x)B^{(\nu)}(x)}{D(x)} . \tag{6.2}$$

7. Under the special assumption $n = 2$, one has

$$\begin{aligned} F &= 1 - a_1x - z_0x^2 = (1 - a_0x^2) - a_1x , \\ G &= 1 - b_1x - b_0x^2 = (1 - b_0x^2) - b_1x , \end{aligned}$$

$$\begin{array}{rl|rl} A^{(0)} & = a_0 & A^{(1)} & = a_1 \\ B^{(0)} & = 0 & B^{(1)} & = b_0x , \end{array}$$

$$\begin{aligned} D &= (1 - a_0x^2)(1 - b_0x^2) - a_1b_1x^2 \\ &= 1 - (a_0 + b_0 + a_1b_1)x^2 + a_1b_1x^4 , \end{aligned}$$

$$P^{(0)} = \frac{a_0 + a_0b_1x - a_0b_0x^2}{D}, \qquad P^{(1)} = \frac{a_1 + (b_0 + a_1b_1)x - a_0b_0x^3}{D}.$$

The power series development of $1/D$ is

$$\frac{1}{D} = \sum_{k\geq 0} \left((a_0 + b_0 + a_1b_1)x^2 - a_0b_0x^4 \right)^k$$

$$= \sum_{k_0,k_1\geq 0} \binom{k_0 + k_1}{k_1} (a_0 + b_0 + a_1b_1)^{k_0}(-a_0b_0)^{k_1}x^{2k_0+4k_1}$$

$$= \sum_{k\geq 0} \sum_{k_0+2k_1=k} (-1)^{k_1}\binom{k_0 + k_1}{k_1}(a_0 + b_0 + a_1b_1)^{k_0}(a_0b_0)^{k_1} \quad x^{2k} .$$

From this, one obtains easily the following power series developments for the even and odd components of the two generating functions:

$$(7.1)\quad \begin{cases} P_0^{(0)} = \dfrac{a_0 - a_0b_0x^2}{D} = \displaystyle\sum_{k\geq 0}\left[\sum_{k_0+2k_1=k} (-1)^{k_1}\binom{k_0+k_1}{k_1}(a_0+b_0+a_1b_1)^{k_0}a_0^{k_1+1}b_0^{k_1}\right]x^{2k} \\ \qquad - \displaystyle\sum_{k\geq 0}\left[\sum_{\substack{k_0+2k_1=k\\(k_0\geq 1)}} (-1)^{k_1}\binom{k_0+k_1-1}{k_1}(a_0+b_0+a_1b_1)^{k_0-1}a_0^{k_1+1}b_0^{k_1+1}\right]x^{2k} \\ P_1^{(0)} = \dfrac{a_0b_1}{D} = \displaystyle\sum_{k\geq 0}\left[\sum_{k_0+2k_1=k} (-1)^{k_1}\binom{k_0+k_1}{k_1}b_1(a_0+b_0+a_1b_1)^{k_0}a_0^{k_1+1}b_0^{k_1}\right]x^{2k+1} \\ P_0^{(1)} = \dfrac{a_1}{D} = \displaystyle\sum_{k\geq 0}\left[\sum_{k_0+2k_1=k} (-1)^{k_1}\binom{k_0+k_1}{k_1}a_1(a_0+b_0+a_1b_1)^{k_0}a_0^{k_1}b_0^{k_1}\right]x^{2k} \\ P_1^{(1)} = \dfrac{(b_0+a_1b_1)x - a_0b_0x^3}{D} = \displaystyle\sum_{k\geq 0}\left[\sum_{k_0+2k_1=k} (-1)^{k_1}\times\right. \\ \qquad \left.\times\binom{k_0+k_1}{k_1}(b_0+a_1b_1)(a_0+b_0+a_1b_1)^{k_0}a_0^{k_1}b_0^{k_1}\right]x^{2k+1} \\ \qquad - \displaystyle\sum_{k\geq 0}\left[\sum_{\substack{k_0+2k_1=k\\(k_1\geq 1)}} (-1)^{k_1}\binom{k_0+k_1-1}{k_1}\times\right. \\ \qquad \left.\times(a_0+b_0+a_1b_1)^{k_0-1}a_0^{k_1+1}b_0^{k_1+1}\right]x^{2k+1} . \end{cases}$$

The sums in square brackets — or in the first and fourth cases, more exactly, their differences — are the looked for explicit expressions for the recurrent sequences

$$p_{2k}^{(0)},\ p_{2k+1}^{(0)} \quad\text{and}\quad p_{2k}^{(1)},\ p_{2k+1}^{(1)} .$$

8. We finally come to consider the important special case, where $\ell = n$, i.e., the period length coincides with the dimension of the algorithm, and where n^2 indeterminate recurrency coefficients $a_{\nu'}^{(\lambda)}$ are specialized to combinations of only $n+1$ indeterminates $a_{\nu'}$ and t as specified in (0.10), (0.11), viz.,

$$(8.1)\qquad a_{\nu'}^{(\lambda)} = t^{d_n(\lambda,\nu')} a_{\nu'} \quad \text{with} \quad d_n(\lambda,\nu') = \begin{Bmatrix} 0 & \text{for} & \lambda + \nu' < n \\ 1 & \text{for} & \lambda + \nu' \geq n \end{Bmatrix} .$$

In this case the recurrency formulae (0.5) specialize to

$$(8.2)\qquad p_{kn+\lambda}^{(\nu)} = \sum_{\nu'=0}^{n-1} t^{d_n(\lambda,\nu')} a_{\nu'} p_{(kn+\lambda)-(n-\nu')}^{(\nu)} \qquad \begin{pmatrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{pmatrix} ,$$

with the n linearly independent initial conditions (0.4), viz.,

$$(8.3)\qquad p_{-(n-\nu')}^{(\nu)} = e_{\nu'}^{(\nu)} \qquad (\nu, \nu' = 0, \cdots, n-1) .$$

These recurrency formulae can be reduced to those of the special case $t = 1$, but with new coefficients. For this purpose consider the modified sequences

$$(8.4)\qquad \bar{p}_{kn+\lambda}^{(\nu)} = t^{k+1} p_{kn+\lambda}^{(\nu)} .$$

They satisfy again the initial conditions (8.3). Now the $p^{(\nu)}$-subscripts on the right of (8.2) reduce as follows to the canonical form on the left:

$$(kn + \lambda) - (n - \nu') = (k - 1)n + (\lambda + \nu') = (k - 1 + d_n(\lambda, \nu'))n + \lambda'$$

with $0 \leq \lambda' \leq n - 1$.

Hence,

$$p^{-(\nu)}_{(kn+\lambda)-(n-\nu')} = t^{k+d_n(\lambda,\nu')} p^{(\nu)}_{(kn+\lambda)-(n-\nu')} \quad . \tag{8.5}$$

From (8.4), (8.5), we obtain the following transformation of the recurrency formulae (8.2):

$$\begin{aligned} p^{-(\nu)}_{kn+\lambda} &= t^{k+1} p^{(\nu)}_{kn+\lambda} \\ &= \sum_{\nu'=0}^{n-1} t^{k+1+d_n(\lambda,\nu')} a_{\nu'} p^{(\nu)}_{(\ n+\lambda)-(n-\nu')} \\ &= \sum_{\nu'=0}^{n-1} t a_{\nu'} p^{-(\nu)}_{(\ n+\lambda)-(n-\nu')} \quad . \end{aligned}$$

Thus the modified sequences $p^{-(\nu)}_{kn+\lambda}$ satisfy the linear recurrency (0.6) with the modified coefficients $ta_{\nu'}$, and, as already said, with the same initial conditions (0.4). According to (0.7), they are therefore given explicitly by

$$\begin{aligned} p^{-(\nu)}_{kn+\lambda} &= \sum_{L(\mathfrak{M})=(kn+\lambda)+(n-\nu)} \binom{k_0 + \cdots + k_n}{k_0\ ,\ \cdots\ ,\ k_n} \times \\ &\times \frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} t^{k_0+\cdots+k_{n-1}} a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} \\ &\qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix}\right) , \end{aligned} \tag{8.6}$$

Going back to the original sequences $p^{(\nu)}_{nk+\lambda}$ by (8.4) and replacing the no longer necessary detailed subscripts $nk+\lambda$ by simply k, we obtain our second chief result,

$$p_k^{(\nu)} = t^{-\left[\frac{k}{n}\right]-1} \sum_{L(\mathfrak{M})=k+(n-\nu)} \binom{k_0 + \cdots + k_n}{k_0, \cdots, k_n} \times$$

(8.7)
$$\times \frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} t^{k_0+\cdots+k_{n-1}} a_0^{k_0} \cdots a_{n-1}^{k_{n-1}}$$
$$\left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix}\right)$$

as announced in (0.12). The remark after (3.5), concerning validity even for $k \geq -n$, i.e., including the initial values holds obviously for (8.7) as well.

Application of the reduction (4.3), (4.4) of polynomial to binomial coefficients to this result yields, in analogy to (4.5), the equivalent expression

$$p_k^{(\nu)} = t^{-\left[\frac{k}{n}\right]-1} \sum_{S(\mathfrak{M})= +(n-\nu)} \binom{k'_1}{k'_0}\binom{k'_2}{k'_1} \cdots \binom{k'_{n-1}}{k'_{n-2}} \times$$

(8.8)
$$\times \frac{k'_\nu}{k'_{n-1}} t^{k'_{n-1}} a_0^{k'_0} a_1^{k'_1-k'_0} \cdots a_{n-1}^{k'_{n-1}-k'_{n-2}}$$
$$\left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix}\right)$$

CHAPTER II. GENERALIZED FIBONACCI NUMBERS WITH TIME IMPULSES

9. It is known from the history of mathematics [5] that the original Fibonacci numbers F_k, named after their discoverer, and defined by the recurrency formula

(9.1) $$F_{k+2} = F_k + F_{k+1} \qquad (k \geq 1)$$

with the initial values

(9.2) $$F_1 = 1, \quad F_2 = 1,$$

describe the mathematical structure of a biological process in nature, viz., of the way rabbits would multiply if no outside factors would interfere with this idealized fertility. From a purely speculative viewpoint this recurrency definition could be replaced by a variety of other structures. So, for instance, the initial values could be replaced by others, as was done by E. Lucas. Thus (9.2) by (in new notation) becomes

(9.2') $$L_1 = 1, \quad L_2 = 3.$$

Or the dimension 2 of the recurrency could be increased to any $n \geq 2$, as was done by the first author [3] who substituted (9.1), (9.2) by

$$\overset{n}{F}_{k+n} = \overset{n}{F}_{k} + \cdots + \overset{n}{F}_{k+(n-1)} \qquad (k \geq 1), \tag{9.3}$$

$$\overset{n}{F}_{1}, \cdots, \overset{n}{F}_{n-1} = 0, \quad \overset{n}{F}_{n} = 1 . \tag{9.4}$$

This generalization to higher dimension could be carried further by considering recurrencies with constant weights $a_0, \cdots, a_{n-1}$ given to the preceding terms, viz.,

$$\overset{n}{F}_{k+n} = a_0\overset{n}{F}_{k} + \cdots + a_{n-1}\overset{n}{F}_{k+n-1} \qquad (k \geq 1) \tag{9.5}$$

with arbitrary initial values

$$\overset{n}{F}_{1}, \cdots, \overset{n}{F}_{n} .$$

Formula (9.5) is actually the recurrency law (0.6) of our introductory section.

The question which is the natural generalization of the original Fibonacci numbers is idle. The answer to it depends on the viewpoint one takes and is a matter of mathematical taste and preferences. Raney [6], for instance, has proposed a generalization widely different in viewpoint and preferences from those mentioned above.

From a purely biological, or even mechanical, viewpoint one would rather expect that a process in nature, depending on n preceding positions, would not go on with such an idealized uniform law of passing to the next position as are those mentioned above, but rather with additional impulses, acting on this law, which are themselves functions of time. It is already a daring presumption that such impulses, imposed by nature, would be recurring regularly. But the purely mathematical applications which will be given in a subsequent paper are some justification for the subsequent new, and in the view of the authors, more "natural" generalization.

For this proposed generalization of the Fibonacci numbers we modify the recurrency law (9.5), i.e., (0.6) by time impulses in the shape of a constant time factor $t \neq 0$, attached to some of the weights $a_0, \cdots, a_{n-1}$ according to the more general recurrency law (0.5) of our introductory section. As initial values we admit throughout the n linearly independent standard sets (0.4).

From them any set of n initial values may be linearly combined, and the corresponding recurrent sequence will then be obtained from those corresponding to (0.4) by the same linear combination.

10. Before we apply the general results (8.7), (8.8) of our first chapter to special cases of the generalized Fibonacci numbers with time impulses, let us make some preliminary remarks.

1.) The restriction of summation

$$L(\mathfrak{M}) = nk_0 + (n-1)k_1 + \cdots + 1k_{n-1} = k + (n - \nu)$$

in the sums (8.7) with multinomial coefficients

$$\binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots, k_{n-1}}$$

can be removed by eliminating the last summation variable k_{n-1} (the only one with coefficient 1) on the strength of that restriction, viz., by putting

$$k_{n-1} = k + (n - \nu) - (nk_0 + \cdots + 2k_{n-2}) \tag{10.1}$$

wherever k_{n-1} occurs in the terms of the sum. It is convenient to combine this elimination with the reduction (4.2) of the multinomial coefficients of order n to such of order n - 1 and binomial coefficients. Thus the formulae (8.7) become

$$\begin{aligned} p_\kappa^{(\nu)} = \sum_{k_0, \cdots, k_{n-2}} \binom{S_0(\mathfrak{M})}{k_0, \cdots, k_{n-2}} \binom{k + (n-\nu) - L_0(\mathfrak{M})}{S_0(\mathfrak{M})} \times \\ \times \frac{k_0 + \cdots + k_\nu}{k + (n - \nu) - L_0(\mathfrak{M})} \times \\ \times t^{k - \left[\frac{k}{n}\right] + (n-\nu) - 1 - L_0(\mathfrak{M})} \times \\ \times a_0^{k_0} a_1^{k_1} \cdots a_{n-2}^{k_{n-2}} a_{n-1}^{k + (n-\nu) - L_0(\mathfrak{M}) - S_0(\mathfrak{M})}. \end{aligned} \tag{10.2}$$

$$\left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-2 \end{matrix}\right)$$

$$(10.2')\left\{\begin{aligned} p^{(n-1)} &= \sum_{k_0,\cdots,k_{n-2}} \binom{S_0(\mathfrak{M})}{k_0,\cdots,k_{n-2}} \binom{k+1-L_0(\mathfrak{M})}{S_0(\mathfrak{M})} \times \\ &\times t^{k-\left[\frac{k}{n}\right]-L_0(\mathfrak{M})} \times \\ &\times a_0^{k_0} a_1^{k_1} \cdots a_{n-2}^{k_{n-2}} a_{n-1}^{k+1-L_0(\mathfrak{M})-S_0(\mathfrak{M})} \\ &\qquad \begin{pmatrix} k \geq 0 \\ \nu = n-1 \end{pmatrix}, \end{aligned}\right.$$

with the reduced linear forms

$$L_0(\mathfrak{M}) = (n-1)k_0 + \cdots + 1k_{n-2}, \quad S_0(\mathfrak{M}) = k_0 + \cdots + k_{n-2} .$$

For confirmation of (10.2), (10.2'), notice that with the help of these two linear forms the substitution (10.1) takes the form

$$k_{n-1} = k + (n - \nu) - L_0(\mathfrak{M}) - S_0(\mathfrak{M}) .$$

Notice further that the silent summation condition $k_{n-1} \geq 0$ is transformed into the upper limitation of summation

$$L_0(\mathfrak{M}) + S_0(\mathfrak{M}) \leq k + (n - \nu) .$$

This limitation may be passed over silently by the following conventions. For $L_0(\mathfrak{M}) < k + (n - \nu)$ no convention is necessary, because in this case the binomial coefficient vanishes if $S_0(\mathfrak{M}) > k + (n - \nu) - L_0(\mathfrak{M})$; in particular for $L_0(\mathfrak{M}) = k + (n - \nu)$, however, we convene to consider the denominator of the subsequent fraction cancelled against the same factor of the factorial in the numerator of the binomial coefficient, as will actually be done later. For $L_0(\mathfrak{M}) > k + (n - \nu)$, we convene to consider the binomial coefficient as being 0; this is not in accordance with the usual extension of Pascal's triangle to negative "numerators"-k by means of the fundamental recurrency property, fixing arbitrarily,

$$\binom{-k}{0} = 1 \quad ,$$

since this extension gives them non-zero values as long as the "denominator" is non-negative.

Observe, by the way, that for $\nu = n - 2$ one has $k_0 + \cdots + k_\nu = S_0(\mathfrak{M})$. Hence in this case the binomial coefficient can be combined with the subsequent fraction to

$$\binom{k + (n - \nu) - L_0(\mathfrak{M}) - 1}{S_0(\mathfrak{M}) - 1} .$$

In (10.2), (10.2'), the restriction of summation $L(\mathfrak{M}) = k + (n - \nu)$ has disappeared. This is deceptive, however, in cases where the recurrency coefficient a_{n-1} is specialized to 0. For, in such cases only the terms in which a_{n-1} has exponent $k_{n-1} = 0$ remain in the sum. Thus the restriction reappears, so to say, by the backdoor, in a slightly modified form, viz., without the term $1k_{n-1}$. This is a change to the worse, even to the worst, into the bargain since now there is no longer a term with coefficient 1 which would allow a further elimination.

2.) Things stand better with the sums (8.8), in which the polynomial coefficients have been reduced to products

$$\binom{k'_1}{k'_0} \cdots \binom{k'_{n-1}}{k'_{n-2}}$$

of binomial coefficients. Here, in the restriction of summation

$$S(\mathfrak{M}) = k'_0 + \cdots + k'_{n-1} = k + (n - \nu) \quad ,$$

<u>each</u> of the n summation variables $k'_0, \cdots, k'_{n-1}$ has coefficient 1, so that there are n different ways of removing the restriction by elimination. However, in cases where a recurrency coefficient $a_{\nu'}$ with $\nu' \geq 1$ is specialized to 0, only the terms with $k'_{\nu'} = k'_{\nu'-1}$ remain in the sum, so that the coefficient of $k'_{\nu'}$ becomes higher than 1, and thus elimination of $k'_{\nu'}$ is barred. For this reason the restriction can be removed only in cases where either at least one consecutive pair $a_{\nu'}, a_{\nu'+1}$ with $0 \leq \nu' \leq n - 2$ or a_{n-1} alone is not specialized to 0.

We shall chiefly be concerned with the latter case $a_{n-1} \neq 0$, in which reduction of the sums (8.7) to unrestricted summation has already been achieved in (10.2), (10.2'). For treating the cases where some of the preceding a_{ν}, are specialized to 0, it will, however, be more convenient to start from the corresponding reduction of the sums (8.8), viz.,

$$
(10.3)\quad \left\{
\begin{aligned}
p_{\kappa}^{(\nu)} = {} & \sum_{k'_0,\cdots,k'_{n-2}} \binom{k'_1}{k'_0} \cdots \binom{k'_{n-2}}{k'_{n-3}} \binom{k+(n-\nu)-S_0(\mathfrak{M}')}{k'_{n-2}} \times \\
& \times \frac{k'_{\nu}}{k+(n-\nu)-S_0(\mathfrak{M}')} \times \\
& \times t^{k-\left[\frac{k}{n}\right]+(n-\nu)-1-S_0(\mathfrak{M}')} \times \\
& \times a_0^{k'_0} a_1^{k'_1-k'_0} \cdots a_{n-2}^{k'_{n-2}-k'_{n-3}} \times \\
& \times a_{n-1}^{k+(n-\nu)-S_0(\mathfrak{M}')-k'_{n-2}} \\
& \qquad\qquad \left(\begin{array}{c} k \geq 0 \\ \nu = 0,\cdots,n-2 \end{array}\right)
\end{aligned}
\right.
$$

$$
(10.3')\quad \left\{
\begin{aligned}
p_{\kappa}^{(n-1)} = {} & \sum_{k'_0\cdots k'_{n-2}} \binom{k'_1}{k'_0} \cdots \binom{k'_{n-2}}{k'_{n-1}} \binom{k+1-S_0(\mathfrak{M}')}{k'_{n-2}} \times \\
& \times t^{k-\left[\frac{k}{n}\right]-S_0(\mathfrak{M}')} \times \\
& \times a_0^{k'_0} a_1^{k'_1-k'_0} \cdots a_{n-2}^{k'_{n-2}-k'_{n-3}} \times \\
& \times a_{n-1}^{k+1-S_0(\mathfrak{M}')-k'_{n-2}}, \\
& \qquad\qquad \left(\begin{array}{c} k \geq 0 \\ \nu = n-1 \end{array}\right)
\end{aligned}
\right.
$$

with the reduced linear form

$$S_0(\mathfrak{M}') = k'_0 + \cdots + k'_{n-2}$$

The remark made after (10.2), (10.2') about the silent summation condition $k'_{n-1} \geq 0$ holds, mutatis mutandis, also for the silent summation condition $k'_{n-1} \geq k'_{n-2}$ in (10.3), (10.3'), the latter corresponding to the former under the transformation (4.3). We uphold the conventions made in that remark.

We must enlarge, however, on the subsequent observation about the possibility of combining the binomial coefficient in (10.2) with the subsequent fraction for $\nu = n - 2$, because this observation generalizes here to all $\nu = 0, \cdots, n - 2$ and thus allows to get rid of these fractions altogether. This is seen by the following chain of reductions:

$$\binom{k + (n - \nu) - S_0(\mathfrak{M}')}{k'_{n-2}} \frac{k'_\nu}{k + (n - \nu) - S_0(\mathfrak{M}')} = \frac{k'_\nu}{k'_{n-2}} \binom{k + (n - \nu) - S_0(\mathfrak{M}') = 1}{k'_{n-2} - 1} .$$

$$\binom{k'_{n-2}}{k'_{n-3}} \frac{k'_\nu}{k'_{n-2}} = \frac{k'_\nu}{k'_{n-3}} \binom{k'_{n-2} - 1}{k'_{n-3} - 1}$$

$$\vdots$$

$$\binom{k'_{\nu+1}}{k'_\nu} \frac{k'_{\nu+1}}{k'_\nu} = \binom{k'_{\nu+1} - 1}{k'_\nu - 1} .$$

which, of course, has to be considered only for $k'_\nu \geq 1$ and hence all subsequent $k'_{\nu+1}, \cdots, k'_{n-2} \geq 1$, too. This chain of reduction yields

$$\binom{k'_{\nu+1}}{k'_\nu} \cdots \binom{k'_{n-2}}{k'_{n-3}} \binom{k + (n - \nu) - S_0(\mathfrak{M}')}{k'_{n-2} - 1} \frac{k'_\nu}{k + (n - \nu) - S_0(\mathfrak{M}')}$$

$$= \binom{k'_{\nu+1} - 1}{k'_\nu - 1} \cdots \binom{k'_{n-2} - 1}{k'_{n-3} - 1} \binom{k + (n - \nu) - 1 - S_0(\mathfrak{M}')}{k'_{n-2} - 1}$$

By the transformation

$$k'_\nu - 1 \to k'_\nu, \cdots, k'_{n+2} - 1 \to k'_{n-2} ,$$

after which the summation range is again $k'_{\nu+1}, \cdots, k'_{n-2} \geq 0$, then

$$S(\mathfrak{M}') \qquad S(\mathfrak{M}') + (n - \nu) - 1 \ ,$$

and thus (10.3) becomes

$$(10.4)\quad \left\{\begin{aligned} p_k^{(\nu)} = \sum_{k_0',\cdots,k_{n-2}'} &\binom{k_1'}{k_0'}\cdots\binom{k_\nu' + 1}{k_{\nu-1}'}\cdots\binom{k_{n-2}'}{k_{n-3}'}\times \\ &\times\binom{k + e_{n-1}^{(\nu)} - S_0(\mathfrak{M}')}{k_{n-2}'} t^{k-\left[\frac{k}{n}\right]-S_0(\mathfrak{M}')}\times \\ &\times a_0^{k_0'+e_0^{(\nu)}} a_1^{k_1'-k_0'}\cdots a_\nu^{k_{\nu+1}'-k_{\nu-1}'}\cdots\times \\ &\times a_{n-2}^{k_{n-2}'-k_{n-3}'} a_{n-1}^{k+e_{n-1}^{(\nu)}-S_0(\mathfrak{M}')-k_{n-2}'} \\ &\qquad\qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0,\cdots,n-1 \end{matrix}\right) \end{aligned}\right.$$

where the modified middle terms

$$\binom{k_\nu' + 1}{k_{\nu-1}'}$$

and

$$a_\nu^{k_\nu'+1-k_{\nu-1}'}$$

are only meant for $\nu = 1,\cdots,n-2$, and where $e_0^{(\nu)}, e_{n-1}^{(\nu)}$ are the coefficients in the first and last column of the unit matrix, introduced in (0.4); by inserting e_{n-1} at the two places, the case $\nu = n - 1$, split off in (10.2'), (10.3'), could now be re-included. Formulae (10.4) could be expressed more concisely introducing also the other $e_{\nu'}^{(\nu)}$ $(\nu' = 1,\cdots,n-1)$ and using the product sign:

$$(10.5)\quad \begin{cases} p_k^{(\nu)} = \sum\limits_{k'_0,\cdots,k'_{n-2}} \left(\prod\limits_{\nu'=0}^{n-2} \binom{k'_{\nu'} + e^{(\nu)}_{\nu'}}{k'_{\nu'-1}} a_{\nu'}^{k'_{\nu'}+e^{(\nu)}_{\nu'}-k'_{\nu'-1}} \right) \times \\ \times\, t^{k-\left[\frac{k}{n}\right]-S_0(\mathfrak{M}')} \binom{k + e^{(\nu)}_{n-1} - S_0(\mathfrak{M}')}{k_{n-2}} \times \\ \times\, a_{n-1}^{k+e^{(\nu)}_{n-1}-S_0(\mathfrak{M}')-k'_{n-2}} \\ \qquad\qquad\qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0,\cdots,n-1 \end{matrix} \right) \end{cases}$$

where one has to understand formally $k'_{-1} = 0$. For our intention of passing to special cases, though, formulae (10.4) allow a better survey.

Notice that for each $\nu = 0,\cdots,n-2$ the silent summation condition for k'_ν in the formula (10.4) or (10.5) for $p_k^{(\nu)}$ has to be modified into $k'_\nu + 1 \geq k'_{\nu-1}$.

Since the original formulae (3.5), (8.7), (10.2) and (10.2') with the polynomial coefficients will not be referred to again, we shall hence forward simplify the notation by omitting the dashes on $k_1,\cdots,k_{n-2}$.

3.) As to specialization of the recurrency coefficients $a_0,a_1,\cdots,a_{n-1}$, we may suppose without loss of generality $a_0 \neq 0$, by considering only recurrencies of the exact order n. In the Jacobi-Perron algorithm there is always even $a_0 = 1$; see (0.1) and what was explained before and afterwards.

4.) For $a_0 = 1$ and $t = 1$ the two recurrent sequences $p_k^{(0)}$ and $p_k^{(n-1)}$ with the first and last set of our standard initial values (0.4) are essentially equal to each other, i. e., they differ only by a translation of the sequence variable k:

$$(10.6)\qquad p_k^{(n-1)} = p_{k+1}^{(0)} \qquad (k \geq -n) \quad .$$

For, $p_k^{(0)}$ has the initial values, $1,0,\cdots,0$. Hence by the recurrency formula $p_0^{(0)} = a_0 = 1$. Therefore $p_{k+1}^{(0)}$ has the initial values $0,\cdots,0,1$. Since for $t = 1$ the recurrency formulae for $p_k^{(0)}$ and $p_k^{(n-1)}$ are the same, (10.6) follows.

11. We now apply our general results to special cases of the generalized Fibonacci numbers. We base these applications as far as possible on our appropriately adapted main result (10.4) for cases with recurrency coefficient $a_{n-1} \neq 0$. Only in the cases with $a_{n-1} = 0$, treated at the end, we have to go back to the original result (8.8).

1.) <u>The uniform case: $a_0, a_1, \cdots, a_{n-1} = 1$; $t = 1$.</u>

In this case we found it convenient, in order to avoid confusion, to put the recurrency order n on top of the sequence letter, as already done in (9.3-5). Here (10.4) becomes simply

$$\overset{n}{p}{}_k^{(\nu)} = \sum_{k_0,\cdots,k_{n-2}} \binom{k_1}{k_0} \cdots \binom{k_\nu + 1}{k_{\nu-1}} \cdots \binom{k_{n-2}}{k_{n-3}} \times \tag{11.1}$$
$$\times \binom{k + e_{n-1}^{(\nu)} - S_0(\mathfrak{M})}{k_{n-2}} \qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix}\right),$$

with

$$S_0(\mathfrak{M}) = k_0 + \cdots + k_{n-2} \quad .$$

The first and last of these sequences, essentially equal to each other according to (10.6), are essentially equal to the sequence of generalized Fibonacci numbers considered by the first author in his previous paper [3], and mentioned above in (9.3). For, adaptation to the initial values (9.4) of those latter yields

$$\overset{n}{F}_k = \overset{n}{p}{}_{k-n}^{(0)} = \overset{n}{p}{}_{k-(n+1)}^{(n-1)} \qquad (k \geq 1) . \tag{11.2}$$

In particular, for $n = 2$ there remains only one summation variable $k_0 = s$, and (11.1) becomes

$$\overset{2}{p}{}_k^{(\nu)} = \sum_s \binom{k + \nu - s}{s} \qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0, 1 \end{matrix}\right) , \tag{11.3}$$

These two sequences are essentially equal to the sequence (9.1) of the original Fibonacci numbers. For, adaptation to the initial values (9.2) of those latter yields

$$(11.4) \qquad F_k = \overset{2}{p}{}^{(0)}_{k-1} = \overset{2}{p}{}^{(1)}_{k-2}\binom{k-1-s}{s} \qquad (k \geq 1) .$$

Notice that, unfortunately, the initial values (9.4) of the generalized Fibonacci numbers

$$\overset{n}{F}_k$$

are <u>not</u> in accordance with the traditional initial values (9.2) of the original Fibonacci numbers F_k, corresponding to the special case $n = 2$. By (11.2), (11.4) the connection is

$$(11.5) \qquad \overset{2}{F}_{k+1} = F_k ,$$

i.e., a translation by 1. The traditional initial values (9.2) are in accordance with the representation

$$F_k = \frac{\epsilon^k - \epsilon'^k}{\epsilon - \epsilon'} , \qquad (k \geq 0)$$

where

$$\epsilon = \frac{1 + \sqrt{5}}{2} ,$$

whose analogue for the Lucas numbers is

$$L_k = \epsilon^k + \epsilon'^k \qquad (k \geq 0) .$$

The Lucas numbers, according to their initial values (9.2'), are obtained by the linear combination

$$(11.6)\qquad L_k = \overset{2}{p}{}^{(0)}_{k-3} + 3\overset{2}{p}{}^{(1)}_{k-3} = \sum_s \binom{k-3-s}{s} + 3\binom{k-2-s}{s} \qquad (k \geq 3).$$

The representations (11.4) and (11.6) of the historical Fibonacci and Lucas numbers are well known [5].

In all following cases we presuppose

$$a_0 = 1, \qquad t \text{ arbitrary },$$

the latter with the only natural restriction $t \neq 0$.

2.) <u>The multiple uniform case: all $a_1, \cdots, a_{n-1} = a \neq 0$.</u>

In this case we have to attach to the expression (11.1) the powers of t and a according to (10.4). In order to determine the exponent of a in the simplest possible manner, observe that the sum of the exponents of $a_0, a_1, \cdots, a_{n-1}$ in (10.4) (or (10.5)) reduces to $k + 1 - S_0(\mathfrak{M})$. But since here only $a_1, \cdots, a_{n-1} = a$ whereas $a_0 = 1$, the exponent $k_0 + e_0^{(\nu)}$ has to be subtracted. Thus

$$(11.7)\quad \begin{cases} p_k^{(\nu)} = \displaystyle\sum_{k_0,\cdots,k_{n-1}} \binom{k_1}{k_0} \cdots \binom{k_\nu + 1}{k_{\nu-1}} \cdots \binom{k_{n-2}}{k_{n-3}} \binom{k + e_{n-1}^{(\nu)} - S_0(\mathfrak{M})}{k_{n-2}} \times \\ \qquad \times\, t^{k - \left[\frac{k}{n}\right] - S_0(\mathfrak{M})}\, a^{k+1-e_0^{(\nu)} - S_0(\mathfrak{M}) - k_0} \\ \qquad\qquad\qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix} \right) \end{cases}$$

with

$$S_0(\mathfrak{M}) = k_0 + \cdots + k_{n-2} .$$

We illustrate this by the two lowest cases:

<u>$n = 2$</u>

$$(11.8)\qquad p_k^{(\nu)} = \sum_{k'} \binom{k + \nu - k'}{k'} t^{k - \left[\frac{k}{2}\right] - k'} a^{k - 2k' + \nu} \qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0, 1 \end{matrix} \right) ;$$

<u>n = 3</u>

$$(11.9)\quad \begin{cases} p_k^{(0)} = \sum_{k_0,k_1} \binom{k_1}{k_0}\binom{k-(k_0+k_1)}{k_1} t^{k-\left[\frac{k}{3}\right]-(k_0+k_1)} a^{k-(2k_0+k_1)} \\ p_k^{(1)} = \sum_{k_0,k_1} \binom{k_1+1}{k_0}\binom{k-(k_0+k_1)}{k_1} t^{k-\left[\frac{k}{3}\right]-(k_0+k_1)} a^{k+1-(2k_0+k_1)} \\ p_k^{(2)} = \sum_{k_0,k_1} \binom{k_1}{k_0}\binom{k+1-(k_0+k_1)}{k_1} t^{k-\left[\frac{k}{3}\right]-(k_0+k_1)} a^{k+1-(2k_0+k_1)} \end{cases}$$

$$(k \geq 0)\ .$$

It would be worthwhile to confirm (11.8) from (7.1) by specializing there $a_0 = 1$, $b_0 = 1$, $a_1 = a$, $b_1 = ta$.

3.) <u>Reduced multiple uniform cases: some $a_{\nu'} = 0$, the other $a_{\nu'} = a \neq 0$ $(\nu' = 1, \cdots, n-1)$.</u>

a) <u>Cases with $a_{n-1} = a \neq 0$.</u>

As we saw in Section 10, in these cases, the general reduction (10.4) to unrestricted summation is effective. The results are obtained from (10.4) by simply adding the summation conditions

$$k_{\nu'} = k_{\nu'-1} \quad \text{for all } \nu' \neq \nu \text{ with } a_{\nu'} = 0\ ,$$
$$k_\nu = k_{\nu-1} - 1 \quad \text{if } a_\nu = 0\ .$$

They effect that the correspondent binomial coefficients

$$\binom{k_{\nu'}}{k_{\nu'} - 1} \quad \text{or} \quad \binom{k_\nu + 1}{k_{\nu-1}}$$

drop out becoming 1, and that the linear form $S_0(\mathfrak{M})$ is changed to no longer homogeneous linear functions $S_\nu(\mathfrak{M})$ of the remaining summation variables.

We illustrate this in the two cases where all but one or all of the coefficients $a_1, \cdots, a_{n-2}$ are specialized to 0.

(i) $\underline{a_1, \cdots, a_{r-1},\ a_{r+1}, \cdots, a_{n-2} = 0;\ a_r = a \neq 0}$
$\underline{(1 \le r \le n-2)}$

$$
(11.10)\quad \left\{\begin{array}{ll}
p_k^{(\nu)} = \displaystyle\sum_{k',k''} \binom{k''}{k'}\binom{k - S_\nu(k',k'')}{k''} t^{k - \left[\frac{k}{n}\right] - S_\nu(k',k'')} a^{k - S'_\nu(k',k'')}, & \left(\begin{array}{c} k \ge 0 \\ \nu = 0, \cdots, r-1 \end{array}\right) \\
p_k^{(\nu)} = \displaystyle\sum_{k',k''} \binom{k''+1}{k'}\binom{k - S_\nu(k',k'')}{k''} t^{k - \left[\frac{k}{n}\right] - S_\nu(k',k'')} a^{k+1-S'_\nu(k',k'')}, & \left(\begin{array}{c} k \ge 0 \\ \nu = r, \cdots, n-2 \end{array}\right) \\
p_k^{(n-1)} = \displaystyle\sum_{k',k''} \binom{k''}{k'}\binom{k+1 - S'_\nu(k',k'')}{k''} t^{k - \left[\frac{k}{n}\right] - S'_\nu(k',k'')} a^{k+1-S'_\nu(k',k'')} & \left(\begin{array}{c} k = 0 \\ \nu = n-1 \end{array}\right)
\end{array}\right.
$$

with the linear functions

$$
S_\nu(k',k'') = rk' + (n-1-r)k'' + \left\{\begin{array}{ll} \nu & \text{for } \nu = 0, \cdots, r-1 \\ \nu - r & \text{for } \nu = r, \cdots, n-2 \\ 0 & \text{for } \nu = n-1 \end{array}\right\}
$$

and

$$
S'_\nu(k',k'') = S_\nu(k',k'') + k' = (r+1)k' + (n-1-r)k'' + \left\{\begin{array}{ll} \nu & \text{for } \nu = 0, \cdots, r-1 \\ \nu - r & \text{for } \nu = r, \cdots, n-2 \\ 0 & \text{for } \nu = n-1 \end{array}\right\} .
$$

(ii) $\underline{a_1, \cdots, a_{n-2} = 0}$

$$(11.11)\qquad p^{(\nu)} = \sum_{k'} \binom{k + e_{n-1}^{(\nu)} - S_\nu(k')}{k'} t^{k-\left[\frac{k}{n}\right]-S_\nu(k')} a^{k+e_{n-1}^{(\nu)}-S'_\nu(k')} \qquad \begin{pmatrix} k = 0 \\ \nu = n-1 \end{pmatrix},$$

with the linear functions

$$S_\nu(k') = \begin{Bmatrix} (n-1)k' + \nu & \text{for } \nu = 0,\cdots,n-2 \\ (n-1)k' & \text{for } \nu = n-1 \end{Bmatrix}$$

and

$$S'_\nu(k') = S_\nu(k') + k' = \begin{Bmatrix} nk' + \nu & \text{for } \nu = 0,\cdots,n-2 \\ nk' & \text{for } \nu = n-1 \end{Bmatrix}.$$

We illustrate (11.10) and (11.11) by the lowest case:

$$\underline{n = 3}$$

In (11.10) for $n = 3$ the only possibility is $r = 1$. But then $a_0 = 1$; $a_1, a_2 = a \neq 0$, and no coefficient is specialized to 0. Hence formulae (11.10) must coincide with (11.9), which is confirmed at once.

Formulae (11.11) for $n = 3$ specialize to

$$(11.12)\quad \left\{ \begin{aligned} p_\kappa^{(0)} &= \sum_{k'} \binom{k - 2k'}{k'} t^{k-\left[\frac{k}{3}\right]-2k'} a^{k-3k'} \\ p_\kappa^{(1)} &= \sum_{k'} \binom{k - 2k' - 1}{k'} t^{k-\left[\frac{k}{3}\right]-2k'-1} a^{k-3k'-1} \qquad (k \geq 0)\ . \\ p_\kappa^{(2)} &= \sum_{k'} \binom{k + 1 - 2k'}{k'} t^{k-\left[\frac{k}{3}\right]-2k'} a^{k+1-3k'} \end{aligned} \right.$$

The term with $k = 0$, $k' = 0$ in the second formula is an example for the necessity of our deviating convention after (10.2), (10.2') about the binomial coefficients with negative "numerator." From the recurrency

$$p_0^{(1)} = 1p_{-3}^{(1)} + 0p_{-2}^{(1)} + 1p_{-1}^{(1)}$$

with

$$p_{-3}^{(1)} = 0, \quad p_{-2}^{(1)} = 1, \quad p_{-1}^{(1)} = 0 \ ,$$

it is obvious that $p_0^{(1)} = 0$. But (11.12) would yield a non-zero value $p_0^{(1)}$, with negative exponents of t and a into the bargain, if the binomial coefficient $\binom{-1}{0}$ of the first term of the sum would be given the usual value 1.

b) Cases with $a_{n-1} = 0$

As we saw in Section 10, in these cases, the general reduction (10.4) to unrestricted summation is ineffective, and we can achieve our aim in the same way only if there is at least one consecutive pair of recurrency coefficients $a_{\nu'}$, $a_{\nu'+1}$ with $0 \leq \nu' \leq n - 2$, which are not specialized to 0.

We shall consider here again only cases where all but one of the coefficients $a_1, \cdots, a_{n-2}$ are specialized to 0; in the case where all of them are 0, the recurrency

$$p_k^{(\nu)} = p_{k-n}^{(\nu)}$$

is trivial.

Let $a_r = a \neq 0$ be the only coefficient remaining intact. For $r = 1$ the pair $a_0 = 1$, $a_r = a$ satisfies the above condition, for $r = 2, \cdots, n-2$ however it is not satisfied. In both cases, we have to go back to our general result (8.8).

(i) $a_1 = a \neq 0; \ a_2, \cdots, a_{n-1} = 0$

Here, in (8.8) are to be added the summation conditions

$$K_2 = \cdots = K_{n-2} = K \quad ,$$

so that now

$$S(\mathfrak{M}) = S(K_0, K) = K_0 + (n - 1)K \ .$$

Thus (8.8) becomes

$$
\begin{aligned}
p_k^{(0)} &= \sum_{S(K_0,K)=k+n} \binom{K}{K_0} \frac{K_0}{K}\, t^{K-\left[\frac{k}{n}\right]-1} a^{K-K_0} \\
&= \sum_{S(K_0,K)=k+n} \binom{K-1}{K_0-1} t^{K-\left[\frac{k}{n}\right]-1} a^{K-K_0} \\
&= \sum_{S(K_0,K)=k} \binom{K}{K_0} t^{K-\left[\frac{k}{n}\right]} a^{K-K_0} \qquad \begin{pmatrix} k \geq 0 \\ \nu = 0 \end{pmatrix} \\
p_k^{(\nu)} &= \sum_{S(K_0,K)=k+(n-\nu)} \binom{K}{K_0} t^{K-\left[\frac{k}{n}\right]-1} a^{K-K_0} \\
&\qquad \begin{pmatrix} k \geq 0 \\ \nu = 1,\cdots,n-1 \end{pmatrix}.
\end{aligned}
\tag{11.13}
$$

Since in the summation condition K_0 has coefficient 1, it can be eliminated, putting

$$
K_0 = \begin{Bmatrix} k - (n-1)K & \text{for } \nu = 0 \\ k + (n-\nu) - (n-1)K & \text{for } \nu = 1,\cdots,n-1 \end{Bmatrix} .
$$

Making this substitution, we can however no longer silently pass over the summation conditions $0 \leq K_0 \leq K$. Thus we obtain

$$
\left\{
\begin{aligned}
p_k^{(0)} &= \sum_{(n-1)K\leq k\leq nK} \binom{K}{k-(n-1)K} t^{K-\left[\frac{k}{n}\right]} a^{nK-k} \\
&\qquad \begin{pmatrix} k \geq 0 \\ \nu = 0 \end{pmatrix} \\
p_k^{(\nu)} &= \sum_{(n-1)K\leq k+(n-\nu)\leq nK} \binom{K}{k+(n-\nu)-(n-1)K} \times \\
&\quad \times t^{K-\left[\frac{k}{n}\right]-1} a^{nK-k-(n-\nu)} \\
&\qquad \begin{pmatrix} k \geq 0 \\ \nu = 1,\cdots,n-1 \end{pmatrix} .
\end{aligned}
\right.
\tag{11.14}
$$

We illustrate this by the lowest case:

$\underline{n = 3}$

$$
(11.15)\quad
\begin{cases}
p_k^{(0)} = \displaystyle\sum_{2K \le k \le 3k} \binom{K}{k - 2K} t^{K - \left[\frac{k}{3}\right]} a^{3K-k} \\[2ex]
p_k^{(1)} = \displaystyle\sum_{2K \le k+2 \le 3K} \binom{K}{k + 2 - 2K} t^{K - \left[\frac{k}{3}\right] - 1} a^{3K-k-2} \\[2ex]
p_k^{(2)} = \displaystyle\sum_{2K \le k+1 \le 3K} \binom{K}{k + 1 - 2K} t^{K - \left[\frac{k}{3}\right] - 1} a^{3K-k-1} \qquad (k \ge 0)
\end{cases}
$$

Formulae (11.9), (11.12), (11.15) together cover all possible cases of generalized Fibonacci numbers of order $n = 3$ with time impulses.

(ii) $a_1, \cdots, a_{n-1},\ a_{r+1}, \cdots, a_{n-1} = 0;\ a_r = a \ne 0$
$\underline{(2 \le r \le n - 2)}$

Here, in (8.8) are to be added the summation conditions

$$K_0 = \cdots = K_{r-1} = K, \qquad K_r = \cdots = K_{n-1} = K' ,$$

so that now

$$S(\mathfrak{M}) = S(K, K') = rK + (n - r)K' .$$

Thus (8.8) becomes

$$
(11.16)\quad
\begin{cases}
p_k^{(\nu)} = \displaystyle\sum_{S(K,K')=k+(n-\nu)} \binom{K'}{K} \frac{K}{K'}\, t^{K' - \left[\frac{k}{n}\right] - 1} a^{K'-K} \\[2ex]
\quad = \displaystyle\sum_{S(K,K')=k+(n-\nu)} \binom{K' - 1}{K - 1} t^{K' - \left[\frac{k}{n}\right] - 1} a^{K'-K} \\[2ex]
\quad = \displaystyle\sum_{S(K,K')=k-\nu} \binom{K'}{K} t^{K' - \left[\frac{k}{n}\right]} a^{K'-K} \qquad \left(\begin{matrix} k \ge 0 \\ \nu = 0, \cdots, r - 1 \end{matrix}\right) \\[2ex]
p_k^{(\nu)} = \displaystyle\sum_{S(K,K')=k+(n-\nu)} \binom{K'}{K} t^{K' - \left[\frac{k}{n}\right] - 1} a^{K'-K} \qquad \left(\begin{matrix} k \ge 0 \\ \nu = r, \cdots, n - 1 \end{matrix}\right)
\end{cases}
$$

Since here in the summation condition, both variables K, K' have coefficients $r,\ n - r > 1$, neither of them can be eliminated, so that by (11.16), other than (11.13), has to be considered as the final result.

There is, however, one very special case in which a different possibility of achieving unrestricted summation presents itself, viz., if both coefficients $r,\ n - r$ are equal, or else:

$$\underline{n = 2r}$$

In this case the summation restriction is

$$\frac{n}{2}(K + K') = \begin{Bmatrix} k - \nu & \text{for } \nu = 0,\cdots,n/2 - 1 \\ k + (n - \nu) & \text{for } \nu = n/2,\cdots,n - 1 \end{Bmatrix} .$$

Hence the sequences $p^{(\nu)}$ contain non-zero terms only for $k \equiv \nu \bmod n/2$, respectively. Putting accordingly

$$k = \begin{Bmatrix} \frac{n}{2}h + \nu & \text{for } \nu = 0,\cdots,\frac{n}{2} - 1 \\ \frac{n}{2}h + \left(\nu - \frac{n}{2}\right) & \text{for } \nu = \frac{n}{2},\cdots,n - 1 \end{Bmatrix} \quad (h \geq 0) ,$$

the restriction becomes

$$K + K' = \begin{Bmatrix} h & \text{for } \nu = 0,\cdots,\frac{n}{2} - 1 \\ h + 1 & \text{for } \nu = \frac{n}{2},\cdots,n - 1 \end{Bmatrix} .$$

Here K', say, can be eliminated by the substitution

$$K' = \begin{Bmatrix} h - K & \text{for } \nu = 0,\cdots,\frac{n}{2} - 1 \\ h + 1 - K & \text{for } \nu = \frac{n}{2},\cdots,n - 1 \end{Bmatrix} .$$

Thus in this very special case the non-zero terms of the sequences $p_k^{(\nu)}$ are the unrestricted sums

(11.17)

$$p^{(\nu)}_{\frac{n}{2}h+\nu} = \sum_K \binom{h-K}{K} t^{h-\left[\frac{h}{2}\right]-K} a^{h-2K} \qquad \left(\begin{matrix} h \geq 0 \\ \nu = 0,\cdots,\frac{n}{2}-1 \end{matrix}\right)$$

$$p^{(\nu)}_{\frac{n}{2}h+\left(\nu-\frac{n}{2}\right)} = \sum_K \binom{h+1-K}{K} t^{h-\left[\frac{h}{2}\right]-K} a^{h+1-2K} \qquad \left(\begin{matrix} h \geq 0 \\ \nu = \frac{n}{2},\cdots,n-1 \end{matrix}\right)$$

We illustrate this by the lowest case:

$$\underline{n = 4}$$

(11.18)

$$p^{(\nu)}_{2h+\nu} = \sum_K \binom{h-K}{K} t^{h-\left[\frac{h}{2}\right]-K} a^{h-2K} \qquad \left(\begin{matrix} h \geq 0 \\ \nu = 0,1 \end{matrix}\right)$$

$$p^{(\nu)}_{2h+(\nu-2)} = \sum_K \binom{h+1-K}{K} t^{h-\left[\frac{h}{2}\right]-K} a^{h+1-2K} \qquad \left(\begin{matrix} h \geq 0 \\ \nu = 2,3 \end{matrix}\right).$$

However formulae (11.17), (11.18) are immediate consequences of the general result (11.8) for $n = 2$, because considering only the non-zero terms, the corresponding recurrency formulae reduce to those for the generalized Fibonacci numbers of order $n = 2$ with time impulse. This shows the underlying true reason why reduction to unrestricted summation is possible in this very special case (and in similar cases with any proper division of n instead of 2 as well), in spite of what has been said in Section 10.

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NUMBERS GENERATED BY THE FUNCTION exp $(1-e^x)$

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A sequence of numbers $\{C_n,\ n = 0,1,2,\cdots\}$ is defined from its generating function $\exp(1 - e^x)$. A series representation for C_n (which is analogous to Dobinski's formula), a relationship with the Stirling numbers of the second kind, a recurrence relation between the C_n and a difference equation satisfied by C_n are obtained. The relationships between the Bell numbers and $\{C_n\}$ are also investigated. Finally, three determinantal representations for C_n are given. The 'Aitken Array' for C_n, $1 \le n \le 21$ is given in the appendix.

1. INTRODUCTION AND SUMMARY

While studying the moment properties of a discrete random variable associated with the Stirling numbers of the second kind, σ_n^j, we encountered an interesting sequence of numbers. More explicitly, let X be a discrete random variable with probability distribution

$$(1.1)\qquad P\{X = j\} = \sigma_n^j / B_n, \qquad j = 1,2,\cdots,n$$

where

$$\sum_{j=1}^{n} \sigma_n^j = B_n, \quad n = 1,2,\cdots$$

are called the Bell numbers. The k^{th} moment of the random variable X is given by

$$(1.2)\qquad D(X^k) = \sum_{j=1}^{n} j^k \sigma_n^j / B_n = B_n^{(k)} / B_n \quad \text{(say)};$$

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and the first six values of $B_n^{(k)}$ are given by

$$
\begin{aligned}
B_n^{(0)} &= B_n \\
B_n^{(1)} &= B_{n+1} - B_n \\
B_n^{(2)} &= B_{n+2} - 2B_{n+1} \\
B_n^{(3)} &= B_{n+3} - 3B_{n+2} + 0B_{n+1} + B_n \\
B_n^{(4)} &= B_{n+4} - 4B_{n+3} + 0B_{n+2} + 4B_{n+1} + B_n \\
B_n^{(5)} &= B_{n+5} - 5B_{n+4} + 0B_{n+3} + 10B_{n+2} + 5B_{n+1} - 2B_n \ .
\end{aligned}
\tag{1.3}
$$

This led us to look for an expression for $B_n^{(k)}$ in terms of the Bell numbers $B_{n+k}, B_{n+k-1}, \cdots, \cdots, B_n$ of the form

$$
B_n^{(k)} = \sum_{i=0}^{k} \binom{k}{i} C_i \, B_{n+k-1} \tag{1.4}
$$

The first few C_i, $i = 1, 2, \cdots$ are given by $C_0 = 1$, $C_1 = -1$, $C_2 = 0$, $C_3 = 1$, $C_4 = 1$, $C_5 = -2$, $C_6 = -9$, $C_7 = -9$ and $C_8 = 50$. In this paper we will study some properties of the sequence $\{C_n\}$. In the next section, we give an ad hoc definition of $\{C_n\}$ in terms of the generating function $\exp(1 - e^x)$ and prove some properties. We also derive a relationship between Stirling numbers of the second kind and the C_n. In Section 3, we will derive some relationships between the Bell numbers and the C_n. In Section 4, we will obtain some determinantal representations for the C_n. The proofs are closely related to the proofs (due to several authors) in the case of Bell numbers as summarized by Finlayson in his thesis [1].

2. THE NUMBERS GENERATED BY THE FUNCTION $\exp(1 - e^x)$

Definition: The sequence $\{C_n, \ n = 0, 1, 2, \cdots\}$ is defined by its exponential generating function,

$$
\sum_{k=0}^{\infty} C_k \frac{x^k}{k!} = \exp(1 - e^x) \ . \tag{2.1}
$$

From the power series expansion of $\exp(1 - e^x)$ we will give an infinite series representation for C_k.

Proposition 1:

$$C_k = e \sum_{r=0}^{\infty} (-1)^r \frac{r^k}{r!}, \qquad k = 0, 1, 2, \cdots . \tag{2.2}$$

Proof: From the definition we note that C_k is the coefficient of $x^k/k!$ in the Maclaurin series expansion of $\exp(1 - e^x)$.

$$\begin{aligned} \exp(1 - e^x) &= e \sum_{r=0}^{\infty} (-1)^r e^{xr}/r! \\ &= e \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{k=0}^{\infty} \frac{x^k r^k}{k!} \\ &= e \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{r=0}^{\infty} (-1)^r \frac{r^k}{k!} \qquad , \end{aligned}$$

which shows that

$$C_k = e \sum_{r=0}^{\infty} (-1)^r \frac{r^k}{r!}, \qquad k = 0, 1, 2, \cdots .$$

We will use this series representation to obtain the relationship between the Stirling numbers of the second kind σ_k^j and C_k. We define $\sigma_0^0 = 1$ and $\sigma_k^0 = 0$, $k = 1, 2, \cdots$.

Proposition 2:

$$C_k = \sum_{j=1}^{k} (-1)^j \sigma_k^j \quad . \tag{2.3}$$

Proof. In terms of the j^{th} differences of powers of zero, $\Delta^j(0^k)$, we have, according to Jordan [3],

$$r^k = \sum_{j=0}^{k} \binom{r}{j} \frac{\Delta^j(0^k)}{j!}$$

$$C_k = r \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} r^k$$

$$= e \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{j=0}^{k} \binom{r}{j} \Delta^j(0^k)$$

$$= e \sum_{r=0}^{\infty} (-1)^r \sum_{j=0}^{k} \frac{\Delta^j(0^k)}{j!\,(r-j)!}$$

$$= e \sum_{j=0}^{k} \frac{\Delta^j(0^k)}{j!} (-1)^j \sum_{r=j}^{\infty} \frac{(-1)^r(-1)^{-j}}{(r-j)!}$$

$$= \sum_{j=0}^{k} (-1)^j \frac{\Delta^j(0^k)}{j!}$$

which proves the result since $\Delta^j(0^k) = j!\sigma_k^j$.

Customarily, Stirling numbers of the first kind are defined as numbers with alternate signs, whereas Stirling numbers of the second kind are defined as numbers with positive signs. The relation (2.3) for the C_n, and the corresponding relation for the Bell numbers B_n, given by

$$B_n = \sum_{j=0}^{n} \sigma_n^j ,$$

suggest that the Stirling numbers of the second kind may also be defined with alternate signs.

Using proposition 1, we now obtain a recursive relation between the C-numbers.

Proposition 3.

$$(2.4)\qquad C_{k+1} = -\sum_{j=0}^{k}\binom{k}{j}C_j \qquad k = 0,1,\cdots;\ C_0 = 1\ .$$

Proof:

$$C_{k+1} = e\sum_{r=1}^{\infty}(-1)^r\frac{r^{k+1}}{r!}$$

$$= e\sum_{s=0}^{\infty}(-1)^{s+1}\frac{(s+1)^k}{s!}$$

$$= e\sum_{s=0}^{\infty}\frac{(-1)^{s+1}}{s!}\sum_{j=0}^{k}\binom{k}{j}s^j$$

$$= -\sum_{j=0}^{k}\binom{k}{j}e\sum_{s=0}^{\infty}\frac{(-1)^s s^j}{j!} = -\sum_{j=0}^{k}\binom{k}{j}C_j\ .$$

In the next proposition we will show that C_n satisfies an n^{th} order difference equation. As before, let Δ denote the difference operator and let $E \equiv 1 + \Delta$, so that $E^jC_0 = C_j$, $j = 1,2,\cdots$.

Proposition 4:

$$(2.5)\qquad \Delta^n C_1 = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}C_{j+1} = -C_n, \qquad n = 1,2,\cdots\ .$$

Proof. The first equality will be established by the binomial expansion of $(E - 1)^n$, and the second equality follows from proposition 1. For completeness, the proof is sketched on the following page.

$$\Delta^n C_1 = (E - 1)^n E C_0 = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} E^j E C_0$$

$$= e\sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j}\sum_{r=0}^{\infty} (-1)^r \frac{r^{j+1}}{r!}\ , \quad \text{from (2.2)}$$

$$= e\sum_{r=1}^{\infty} \frac{(-1)^r r}{r!}\sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} r^j$$

$$= -e\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(r-1)!}(r - 1)^n = -C_n\ .$$

The difference equation $\Delta^n C_1 = -C_n$ can be used on computing $C_1, C_2, \cdots, C_n$ for small values of n. This computation can be arranged in a triangular array

$$\begin{array}{llllll} C_1 & \Delta C_1 & \Delta^2 C_1 & \Delta^3 C_1 & \Delta^4 C_1 & \cdots \\ C_2 & \Delta C_2 & \Delta^2 C_2 & \Delta^3 C_2 & \cdots & \\ C_3 & \Delta C_3 & \Delta^2 C_3 & \cdots & & \\ C_4 & \Delta C_4 & \cdots & & & \\ C_5 & \cdots & & & & \end{array} \tag{2.6}$$

The first column gives us the value of C_n, $n = 1,2,3,\cdots$, the second column gives us the first differences, and the j^{th} column gives us the j^{th} differences of C_n, $n = 1,2,3,\cdots$. This table can be filled up as follows: Let us assume that we know $C_1 = -1$. Equation (2.5) for $n = 1$, with $\Delta C_1 = -C_1$ enables us to find $C_2 = C_1 + \Delta C_1 = 0$. Now using (2.5) again for $n = 2$, we find $\Delta^2 C_1 = -C_2 = 0$. Since $\Delta^2 C_1 + \Delta C_1 = \Delta C_2$ we find $\Delta C_2 = 1$ and since $\Delta^2 C_2 + C_2 = C_3$, we find $C_3 = 1$. Now using (2.5) again for $n = 3$, with $\Delta^3 C_1 = -C_3$, we find $\Delta^3 C_1 = -1$, and so on. A part of the difference array is as follows:

$$\begin{array}{rrrrrr} -1 & 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & -1 & -2 & 1 & \\ 1 & 0 & -3 & -1 & & \\ 1 & -3 & -4 & & & \\ -2 & -7 & & & & \\ -9 & & & & & \end{array} \tag{2.7}$$

The corresponding table for the Bell numbers B_n and their differences, based on $\Delta^n B_1 = B_n$ is given in Table 1 of Finlayson [1]. He used the same method of construction, which is at times referred to as the Aitken array by Moser and Wyman [4]. In the appendix we give the Aitken array for the C_n for $1 \leq n \leq 21$.

3. RELATIONSHIPS BETWEEN THE BELL NUMBERS B_n, AND THE C_n

It is well known (Riordan [5]) that the exponential generating function of the Bell numbers B_n is given by

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \exp(e^x - 1)\,. \tag{3.1}$$

Since the generating functions of

$$b_n = \frac{B_n}{n!} \quad \text{and} \quad c_n = \frac{C_n}{n!}$$

are reciprocals of each other, following Riordan [5] we could have defined the sequence $\{c_n\}$ as the inverse sequence of $\{b_n\}$. From this property we can easily derive the following

Proposition 5:

$$\sum_{k=0}^{n} \binom{n}{k} B_k C_{n-k} = 0, \quad n = 1, 2, \cdots, \quad \text{with} \quad B_0 = C_0 = 1\,. \tag{3.2}$$

A less obvious relationship between B_n and C_n is given by the following:

Proposition 6:

$$\sum_{j=0}^{n} \binom{n}{j} C_j B_{n+1-j} = 1, \qquad n = 0, 1, 2, \cdots \tag{3.3}$$

Proof: Differentiating (3.1) with respect to x, we obtain

$$\sum_{k=1}^{\infty} B_k \frac{x^{k-1}}{(k-1)!} = e^x \exp(e^x - 1) .$$

Multiplying this by the exponential generating function of C_n we obtain

$$\left(\sum_{j=0}^{\infty} C_j \frac{x^j}{j!}\right)\left(\sum_{k=1}^{\infty} B_k \frac{x^{k-1}}{(k-1)!}\right) = e^x$$

which implies that

$$\sum_{n=0}^{\infty} B_1^{(n)} \frac{x^n}{n!} = e^x$$

where

$$B_1^{(n)} = \sum_{j=0}^{n} \binom{n}{j} C_j B_{n+1-j} ,$$

as defined in the introduction.

Now it follows that $B_1^{(n)} = 1$, $n = 0, 1, 2, \cdots$, since

$$e^x = \sum_{n=0}^{\infty} 1 \frac{x^n}{n!}$$

is the exponential generating function of the sequence with unity in every place. A 'dual' to proposition 6 can be stated as

Proposition 7:

$$\sum_{j=0}^{n} \binom{n}{j} B_j C_{n+1-j} = -1, \qquad n = 0,1,2,\cdots \tag{3.4}$$

Proof. This follows along the same lines as that of Proposition 6, where we now differentiate the exponential generating function of the C_n.

4. DETERMINANTAL REPRESENTATIONS OF C_n

We noted in Section 3 that the sequences

$$\{b_n\} = \left\{\frac{B_n}{n!}\right\} \quad \text{and} \quad \{c_n\} = \left\{\frac{C_n}{n!}\right\}$$

are inverse sequences as defined on page 25 of Riordan [5]. On page 45, Riordan gives as a problem the representation of n^{th} number of the sequence $\{a'_n\}$ as a determinant of the elements of the inverse sequence $\{a_n\}$. This says

$$a'_n = (-1)^n a_0^{-n-1} \begin{vmatrix} a_1 & a_0 & 0 & \cdots & \\ a_2 & a_1 & a_0 & \cdots & \\ a_3 & a_2 & a_1 & \cdots & \\ \vdots & \vdots & \vdots & a_0 & \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix} = (-1)^n a_0^{-n-1} \delta_n \quad \text{(say)} .$$

The following recursive relation for δ_n can be shown,

$$\delta_n = \sum_{k=0}^{n-1} (-1)^k a_0^k a_{k+1} \; {}_{n-k-1}, \quad \delta_0 = 1 .$$

Applying this result for the Bell numbers B_n, and C_n we will have

Proposition 8:

a)

$$\frac{C_n}{n!} = (-1)^n \begin{vmatrix} \frac{B_1}{1!} & B_0 & & & \\ \frac{B_2}{2!} & \frac{B_1}{1!} & B_0 & & \\ \frac{B_3}{3!} & \frac{B_2}{2!} & \frac{B_1}{1!} & & \\ \cdots & \cdots & \cdots & \cdots & B_0 \\ \frac{B_n}{n!} & \frac{B_{n-1}}{(n-1)!} & \frac{B_{n-2}}{(n-2)!} & & \frac{B_1}{1!} \end{vmatrix} = (-1)^n \xi_n \quad \text{(say)} \tag{4.1}$$

b)

$$(-1)^n \frac{C_n}{n!} = \sum_{k=0}^{n-1} (-1)^k \frac{B_{k+1}}{(k+1)!} \xi_{n-k-1} \quad . \tag{4.2}$$

In Proposition 3, we have shown that

$$C_{n+1} = -\sum_{j=0}^{n} \binom{n}{j} C_j, \qquad n = 0,1,2,\cdots$$

with $C_0 = 1$. From this nonsingular system of equations, using Cramer's rule, we can derive the following:

Proposition 9:

$$C_{n+1} = (-1)^n \begin{vmatrix} 1 & 1 & & & & & \\ 1 & 1 & 1 & & & 0 & \\ 1 & 2 & 1 & 1 & & & \\ 1 & 3 & 3 & 1 & 1 & & \\ \vdots & & & & & & 1 \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdot & \cdot & \cdots & \binom{n}{n} \end{vmatrix} \tag{4.3}$$

The corresponding determinantal representation for the Bell numbers which seems to be due to Ginsburg [2], is also quoted by Finlayson [1]. Ginsburg [2] derived another determinantal expression for the Bell numbers (also quoted by Finlayson [1]) and the corresponding representation for the C-numbers is given by the following:

Proposition 10:

$$C_{n+1} = (-1)^{n+1} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \\ \frac{1}{1!} & 1 & 2 & 0 & 0 & \\ \frac{1}{2!} & \frac{1}{1!} & 1 & 3 & 0 & \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 1 & 4\cdots & \\ \vdots & \vdots & \vdots & & & n \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & & \cdots & 1 \end{vmatrix}$$

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* * * * *

APPENDIX

AITKEN ARRAY FOR THE C NUMBERS

n	C_n	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7
1	-1	1	-0	-1	-1	2	9	9
2	0	1	-1	-2	1	11	18	-41
3	1	0	-3	-1	12	29	-23	-358
4	1	-3	-4	11	41	6	-381	-1355
5	-2	-7	7	52	47	-375	-1736	-1265
6	-9	0	59	99	-328	-2111	-3001	21590
7	-9	59	158	-229	-2439	-5112	18589	177063
8	50	217	-71	-2668	-7551	13477	195652	671803
9	267	146	-2739	-10219	5926	209129	867455	-318740
10	413	-2593	-12958	-4293	215055	1076584	548715	-24897365
11	-2180	-15551	-17251	210762	1291639	1625299	-24348650	-194276517
12	-17731	-32802	193511	1502401	2916938	-22723351	-218625167	-691883220
13	-50533	160709	1695912	4419339	-19806413	-241348518	-910508387	2126876237
14	110176	1856621	6115251	-15387074	-261154931	-1151856905	1216367850	52384151835
15	1966797	7971872	-9271823	-276542005	-1413011836	64510945	53600519685	
16	9938669	-1299951	-285813828	-1689553841	-1348500891	53665030630		
17	8638718	-287113779	-1975367569	-3038054732	52316529739			
18	-278475061	-2262481448	-5013422401	49278475007				
19	-2540956509	-7275903849	44265052506					
20	-9816860358	36989148757						
21	27172288399							

n	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}	Δ_{13}	Δ_{14}	Δ_{15}
1	-50	-267	-413	2180	17731	50533	-110176	-1966797
2	-317	-680	1767	19911	68264	-59643	-2076973	-11905466
3	-997	1087	21578	88175	8621	-2136616	-13982439	-30482853
4	90	22765	109853	96796	-2127995	-16119055	-44465292	220776103
5	22855	132618	206549	-2031199	-18247050	-60584347	176310811	3561302972
6	155473	339267	-1824550	-20278249	-78831397	115726464	3737613783	25168346191
7	494740	-1485283	-22102799	-99109646	36895067	3853340247	28905959974	
8	-990543	-23588082	-121212445	-62214579	3890235314	32759300221		
9	-24578625	-144800527	-183427024	3828020735	36649535535			
10	-169379152	-328227551	3644593711	40477556270				
11	-497606703	3316366160	44122149981					
12	2818759457	47438516141						
13	50257275598							

n	Δ_{16}	Δ_{17}	Δ_{18}	Δ_{19}	Δ_{20}
1	-9938669	-8638718	278475061	2540956509	9816860358
2	-18577387	269836343	2819431570	12357816867	
3	251258956	3089267913	15177248437		
4	3340526869	18266516350			
5	21607043219				

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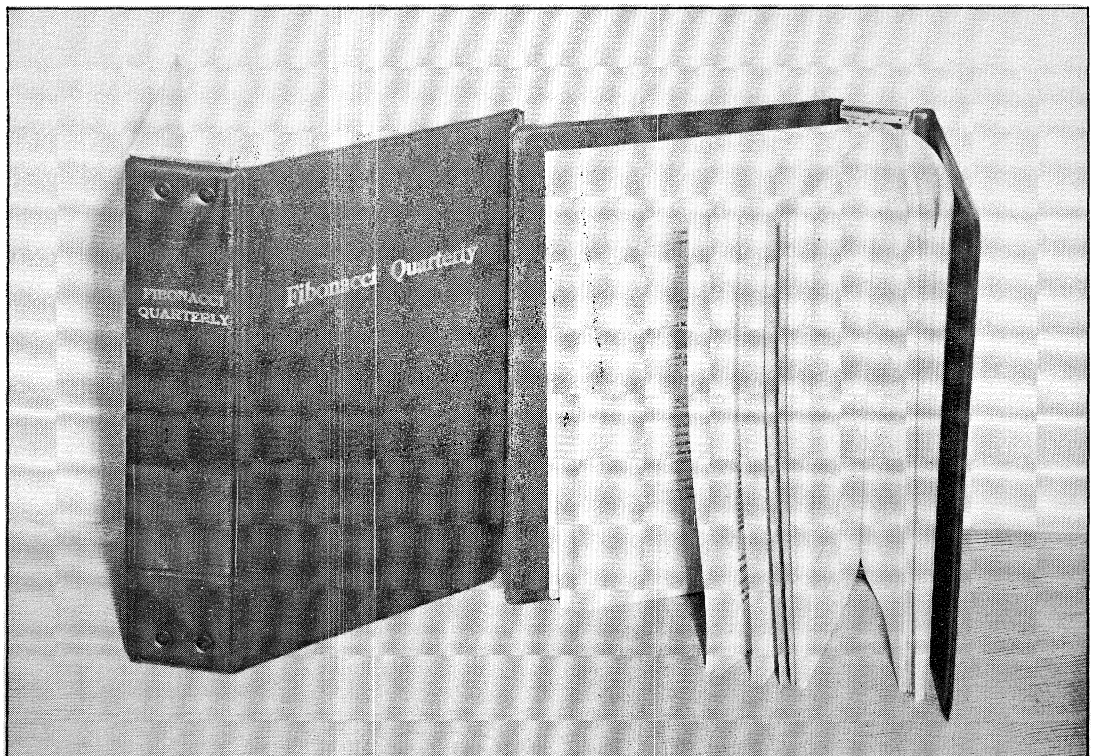

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> "....The binder is made of heavy weight virgin vinyl, electronically sealed over rigid board equipped with a clear label holder extending 2 -3/4" high from the bottom of the backbone, round cornered, fitted with a 1 1/2 " multiple mechanism and 4 heavy wires."

The name, FIBONACCI QUARTERLY, is printed in gold on the front of the binder and the spine. The color of the binder is dark green. There is a small pocket on the spine for holding a tab giving year and volume. These latter will be supplied with each order if the volume or volumes to be bound are indicated.

The price per binder is \$3.50 which includes postage (ranging from 50¢ to 80¢ for one binder). The tabs will be sent with the receipt or invoice.

All orders should be sent to: Brother Alfred Brousseau, Managing Editor, St. Mary's College, Calif. 94575