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A RECURSION RELATION FOR POPULATIONS OF DIATOMS

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Diatoms are a type of one-celled algae whose unusual reproduction cycle gives rise to an interesting problem in number theory. Sparing the morphological details, the cycle can be described as follows. Each diatom when it reproduces (by cell-division) gives rise to one just like itself, and one a size smaller. This process continues to produce smaller and smaller members of the population until a size is reached where cell-division is no longer physiologically possible. These smallest members then grow until they become as large as the first size, and then begin reproducing normally.

The problem is to determine U_n, the population on the n^{th} generation as a function of both the number of sizes possible, and the growing period.

Let $(m + 1)$ be the number of sizes possible including the growing size, and let r be the number of generations it takes for the smallest size to become mature.

We will show that

$$(1) \qquad U_n(m,r) = 1 + \sum_{j=1}^{m} \sum_{i=0}^{\infty} \binom{n - ir}{im + j} ;$$

and that U_n satisfies the following $m+r)^{th}$ order linear recurrence relation:

$$(2) \qquad \sum_{k=0}^{\infty} (-1)^k \binom{m}{k} U_{(n-k)} = U_{n-(m+r)} .$$

Diagram 1 illustrates the derivation of equation (1). The n^{th} horizontal row represents the population on the n^{th} generation. In the first group of columns, each entry is the sum of the two entries north and northwest of it. This is because for $1 \le k < m + 1$, the k^{th} and $(k - 1)^{th}$ sizes each give rise to a k^{th} size, and because in the $(m + 1)^{th}$ column we have an individual either growing, or mature; in either case contributing one to the same column in the

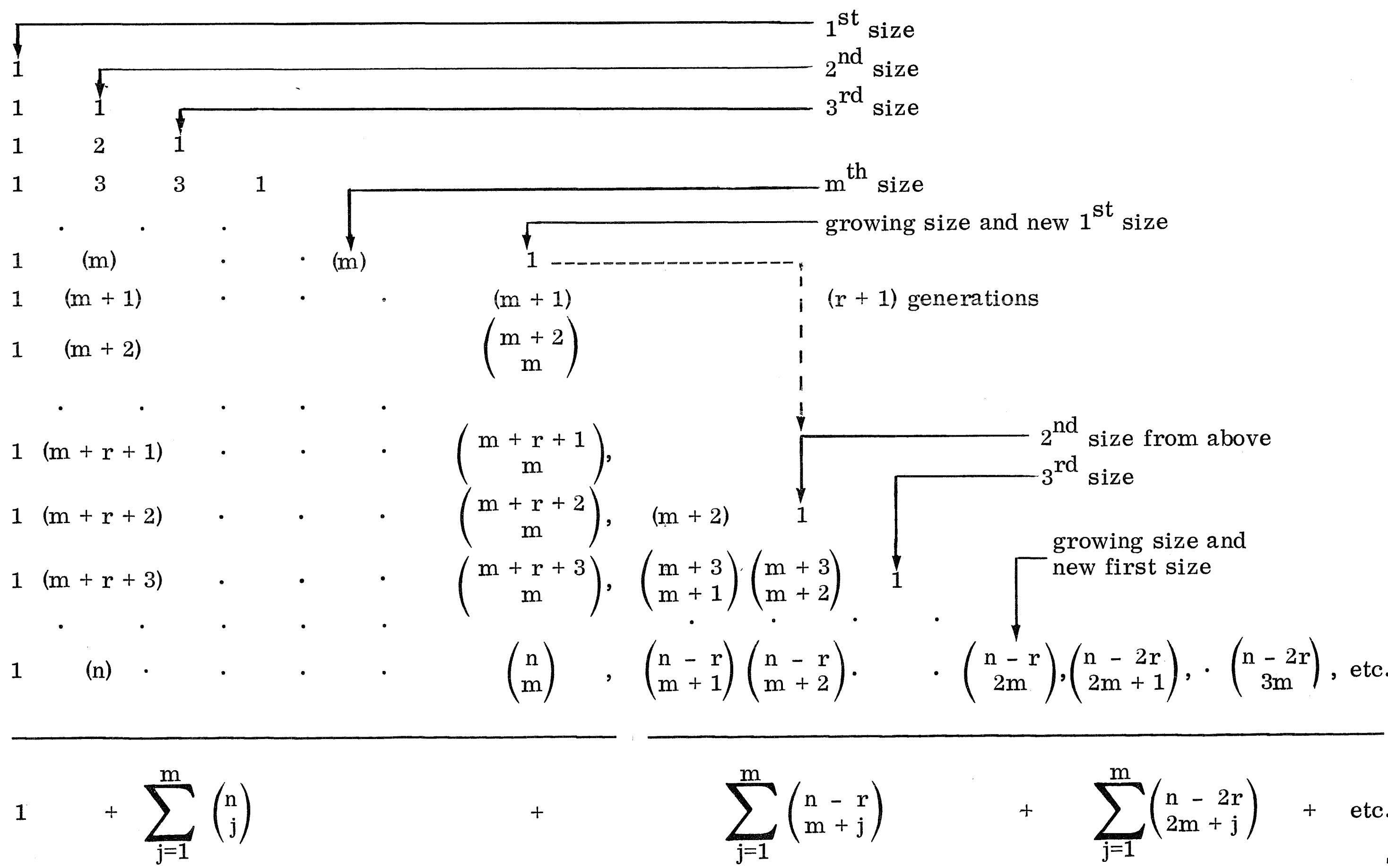
1st size
2nd size
3rd size
mth size
growing size and new 1st size
(r + 1) generations
2nd size from above
3rd size
growing size and
new first size
etc.

Diagram 1

next generation. Clearly binomial coefficients are an efficient representation since

$$(3) \qquad \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k} .$$

In the succeeding groups of columns (only the second group is shown) the same procedure is followed except for the first column of the group. This column represents the second size members which arise from the new first sizes in the last column of the previous group of columns and from the second sizes in the past generation. Thus each element in the first row here is gotten by adding the element north of it to the element $(r + 1)$ places above it and in the last column of the previous group.

Continuing in like manner, we see that in the $(i + 1)^{th}$ group of columns on the n^{th} generation, the top index of the binomial coefficients is $n - ir$, and the bottom index runs from $i(n) + 1$ to $(i + 1)(m)$. This gives equation (1) since all the terms in (1) are zero as soon as the bottom index becomes larger than the top.

We now derive the recurrence relation (2) using the expression in (1) for $U_n(m,r)$. The indices in the double sums on the right will always be from $j = 1$ to m, and $i = 0$ to ∞.

From (1) and (3) we have:

$$(4) \qquad U_n = 1 + \sum_j \sum_i \binom{n - ir}{im + j} = 1 + \sum_j \sum_i \left[\binom{n-1-ir}{im + j} + \binom{n - i - ir}{im + j - 1} \right] ;$$

therefore

$$(5) \qquad U_n - U_{n-1} = \sum_j \sum_i \binom{n - 1 - ir}{im + j - 1} .$$

Now by induction on t we show:

$$(6) \qquad \sum_{k=0}^{\infty} (-1)^k \binom{t}{k} U_{n-k} = \sum_j \sum_i \binom{n - t - ir}{im + j - t} ;$$

$$(7) \qquad = \sum_j \sum_i \left[\binom{n - t - 1 - ir}{im + j - t} + \binom{n - t - 1 - ir}{im + j - t - 1} \right] .$$

Equation (5) shows that (6) holds for $t = 1$, now assume it holds for t; then replacing n with $n - 1$ in (6) and subtracting from (7), we have:

$$\sum_{k=0}^{\infty} (-1)^k \binom{t}{k} U_{n-k} - \sum_{k=0}^{\infty} (-1)^k \binom{t}{k} U_{n-k-1} = \sum_j \sum_i \binom{n - (t+1) - ir}{im + j - (t+1)} .$$

$$\sum_j \sum_i \binom{n - (t+1) - ir}{im + j - (t+1)} = U_n + \sum_{k=1}^{\infty} \left[(-1)^k \binom{t}{k} U_{n-k} + (-1)^k \binom{t}{k-1} U_{n-k} \right]$$

$$= U_n + \sum_{k=1}^{\infty} (-1)^k \binom{t+1}{k} U_{n-k} = \sum_{k=0}^{\infty} (-1)^k \binom{t+1}{k} U_{n-k} ,$$

hence (6) holds for $(t + 1)$ and therefore for all $t \geq 1$.

Now letting $t = m$ in (6), we have:

$$\sum_{k=0}^{\infty} (-1)^k \binom{m}{k} U_{n-k} = \sum_j \sum_i \binom{n - m - ir}{im + j - m}$$

$$= \sum_{j=1}^{m} \sum_{i=0}^{\infty} \binom{n - (m+r) - (i-1)r}{(i-1)m + j}$$

$$= \sum_{j=1}^{m} \binom{n - (m+r) + r}{j - m} + \sum_{j=1}^{m} \sum_{i=1}^{\infty} \binom{n - (m+r) - (i-1)r}{(i-1)m + j}$$

$$= 1 + \sum_{j=1}^{m} \sum_{i=0}^{\infty} \binom{n - (m+r) - ir}{im + j}$$

$$= U_{n-(m+r)} .$$

which establishes (2).

Note that from the diagram we get the following $(m + r)$ initial conditions on $U_n(m,r)$:

$$U_n = \sum_{k=0}^{m} \binom{n}{k} \qquad 1 \le n \le (m + r) ,$$

and also that $U_n(1,1) = F_{n+2}$ where F_n is the n^{th} Fibonacci number. Indeed, the diatom problem is a generalization of the famous Fibonacci rabbit problem.

PART II — GENERATING FUNCTIONS

We may find the generating function for $U_n(m,r)$ by using the recursion above, however it is simpler to calculate the generating functions for each individual size and then add them.

We use the following notation, with m and r fixed.

$a(i,n)$ = total size i in n^{th} generation, $1 \le i \le m$.
$b(j,n)$ = total of growing size which are j generations old in the n^{th} generation, $0 \le j \le r - 1$.

Then we have,

$$(3) \quad \begin{aligned} a(i,n) &= a(i-1,n-1) + a(i,n-1), & a(i,0) &= 0, \quad 2 \le i \le m; \\ a(1,n) &= b(r-1,n-1) + a(1,n-1), & a(1,0) &= 1 . \\ b(j,n) &= b(j-1,n-1) &, \; b(j,0) &= 0, \quad 1 \le j \le r-1; \\ b(0,n) &= a(m,n-1) &, \; b(0,0) &= 0 . \end{aligned}$$

Now we let

$$(4) \qquad A(i,x) = \sum_{n=0}^{\infty} a(i,n)x^n \qquad B(j,x) = \sum_{n=0}^{\infty} b(j,n)x^n ,$$

which, combined with (3), gives

$$
\begin{aligned}
A(i,x) &= xA(i-1,x) + xA(i,x), \qquad 2 \le i \le m \\
A(1,x) &= xA(1,x) + xB(r-1,x) + 1\,. \\
B(j,x) &= xB(j-1,x) \qquad\qquad , \qquad 1 \le j \le r-1 \\
B(0,x) &= xA(m,x)\,.
\end{aligned} \tag{5}
$$

Solving (5), we get

$$
\begin{aligned}
A(m,x) &= \left(\frac{x}{1-x}\right)^{m-1} A(1,x) = \left(\frac{x}{1-x}\right)^{m-1}\left(\frac{x}{1-x}B(r-1,x) + \frac{1}{1-x}\right), \\
&= \left(\frac{x}{1-x}\right)^{m} x^{r-1}B(0,x) + \frac{x^{m-1}}{(1-x)^m} \\
&= \frac{x^{m+r}}{(1-x)^m}A(m,x) + \frac{x^{m-1}}{(1-x)^m}
\end{aligned}
$$

$$
\begin{aligned}
A(m,x) &= \frac{x^{m-1}}{(1-x)^m - x^{m+r}} \\
A(i\,,\,x) &= A(m,x)\left(\frac{1-x}{x}\right)^{m-i} \\
B(j\,,\,x) &= A(m,x)\,x^{j+1}
\end{aligned} \tag{6}
$$

Now we define $P(m,\,r;x)$ as

$$
P(m,r;x) = \sum_{n=0}^{\infty} U_n(m,r)x^n = \sum_{n=0}^{\infty}\left(\sum_{i=1}^{m} a(i,n) + \sum_{j=0}^{r-1} b(j,n)\right)x^n
$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} A(i,x) + \sum_{i=0}^{r-1} B(j,x) \\
&= A(m,x)\left(\sum_{i=1}^{m} \frac{1-x}{x}^{\,m-i} + \sum_{j=1}^{r} x^j\right) \\
&= \frac{(1-x)^m - x^m}{((1-x)^m - x^{m+r})(2x-1)} + \frac{x^m(1-x)^r}{((1-x)^m - x^{m+r})(1-x)}
\end{aligned} \tag{7}
$$

It is of interest to know whether or not the polynomial

$$p(x) = (1 - x)^m - x^{m+r}$$

has any repeated roots. To show that there are none, we set

$$p'(x) = -m(1 - x)^{m-1} - (m + r)x^{m+r-1}$$

and $p(x)$ equal to zero simultaneously, and note that this implies

$$x = \frac{m + r}{r} .$$

This cannot be a root of $p(x)$ since the only possible rational roots of $p(x)$ are ± 1.

Let $\alpha_1, \alpha_2, \cdots, \alpha_{m+r}$ be the roots of $p(x)$ where $|\alpha_i| \leq |\alpha_{i+1}|$. Then

$$P(m,r;x) = g(x) \Big/ \prod_{i=1}^{m+r} (x - \alpha_i) ,$$

where $g(x)$ is a polynomial. This expression has the partial fraction expansion,

$$P(m,r;x) = \sum_{i=1}^{m+r} \frac{\beta_i}{(1 - x/\alpha_2)} ,$$

hence

$$P(m,r;x) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{m+r} \beta_i \left(\frac{1}{\alpha_i} \right)^n \right] x^n ,$$

therefore

$$U_n(m,r) = \sum_{i=1}^{m+r} \beta_i \left(\frac{1}{\alpha_i}\right)^n . \tag{8}$$

From (8) we see that

$$U_n(m,r) = 0\left(\frac{1}{|\alpha_1|^n}\right) .$$

We know that for $r = 0$, $U_n(m,0) = 2^n$ since then we have ordinary cell division. Also, for $r > 0$, $U_n(m,0)$ grows slower than 2^n because of the time lag. Indeed, we now show that for $r > 0$, α_1 is real and $1/2 < \alpha_1 < 1$.

Since

$$(1 - \alpha_i) = \alpha_i^{\frac{m+r}{r}} ,$$

we see that if α_i is real, it must satisfy both $1 - \alpha_i = y$ and

$$\alpha_i^{\frac{m+r}{r}} = y .$$

In Fig. 1 we see that for any m and $r > 0$, there is always a positive real solution, α, to these simultaneous equations where $1/2 < \alpha < 1$. Also, when $m + r$ is even, there is a large (< -1) negative solution. We now show that α is actually the smallest possible in absolute value. We note that for all i,

$$|1 - \alpha_i| = |\alpha_i|^{\frac{m+r}{r}} .$$

Hence $1 - \alpha_i$ must lie on the intersection of the circle about the origin with radius

[Continued on p. 463.]

* * * * *

DIVISIBILITY PROPERTIES OF FIBONACCCI POLYNOMIALS

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1. INTRODUCTION

A famous unsolved problem in number theory asks the question, "Are there infinitely many prime numbers in the Fibonacci sequence?" It is well known that if $\{U_n\}$ is the sequence defined by:

$$U_n = U_{n-1} + U_{n-2}; \quad U_0 = 0, \quad U_1 = 1,$$

then U_n is prime only if n is prime. The converse, however, is not true since, for example, $U_{19} = 113 \cdot 37$. Whether there are infinitely many primes p such that U_p is prime, or indeed whether there are infinitely many exceptions, has been an elusive problem for over a century.

In this paper we parametrize the sequence by using the recursion:

$$U_n(x) = xU_{n-1}(x) + U_{n-2}(x); \quad U_0(x) = 0, \quad U_1(x) = 1 .$$

(Note that $U_n(1) = U_n$.) The resulting sequence: $0, 1, x, x^2 + 1, x^3 + 2x, x^4 + 3x^2 + 1$, etc., satisfies all of the important divisibility relations of the original sequence with the following welcome exception:

Theorem 1. $U_n(x)$ is irreducible if and only if n is prime, which we will prove here.

The following notation will be used throughout the paper.

$$\omega = \frac{x + \sqrt{x^2 + 4}}{2}, \qquad \bar{\omega} = \frac{x - \sqrt{x^2 + 4}}{2}$$

$$V_n(x) = xV_{n-1}(x) + V_{n-2}(x); \quad V_0(x) = 2, \quad V_1(x) = x .$$

2. SOME PROPERTIES OF THE SEQUENCE

The following are just a few of the results concerning the sequence which may be readily proved.

$$U_n(x) = \frac{\omega^n - \overline{\omega}^n}{\omega - \overline{\omega}} \quad . \tag{1}$$

$$\omega\overline{\omega} = -1 \quad . \tag{2}$$

$$V_n(x) = \omega^n + \overline{\omega}^n \, . \tag{3}$$

$$U_n(x) \mid U_{nm}(x) \tag{4}$$

If

$$U_n(x) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \Delta(n,m)x^{n-2m-1} \, ,$$

then,

$$\Delta(n,m) = \frac{4^m}{2^{n-1}} \sum_{j=m}^{\left[\frac{n-1}{2}\right]} \binom{n}{2j+1}\binom{j}{m} = \binom{n-m-1}{m} \, . \tag{5}$$

(6)

$$\text{i)} \quad U_{2n+1}(x) \equiv (-1)^n(2n+1) \pmod{(x^2+4)} \, ;$$

$$\text{ii)} \quad U_{2n}(x) \equiv (-1)^{n-1}\, nx \pmod{(x^2+4)} \, .$$

(7)

$$\text{i)} \quad U_{a+b}(x) = U_a(x)V_b(x) - (-1)^b U_{a-b}(x)$$

$$\text{ii)} \quad U_{a+b}(x) = U_b(x)V_a(x) + (-1)^b U_{a-b}(x) \quad .$$

$$(U_a(x), U_b(x)) = U_{(a,b)}(x) \quad . \tag{8}$$

If p is a prime,

$$U_p(x) \equiv (x^2+4)^{\frac{p-1}{2}} \pmod{p} \quad . \tag{9}$$

Equations (1), (2), and (3) are well known, and (4) follows immediately from (1). Equation (5) follows from (1) by expanding and comparing coefficients, while (6) and (7) may be proved by routine calculation using (1), (2), and (3). To prove (8), let

$$I = \{n : f(x) \mid U_n(x)\} \quad ,$$

where

$$f(x) = (U_a(x),\ U_b(x)) \ .$$

If $r \in I$, then by (4) $mr \in I$ for any integer m. If $r \in I$ and $t \in I$, then by (7), $r - t \in I$. Hence I is an ideal containing a and b, and therefore $(a,b) \in I$, which shows that

$$(U_a(x),\ U_b(x)) \mid U_{(a,b)}(x) \quad ,$$

and by (4) we have

$$U_{(a,b)}(x) \mid (U_a(x),\ U_b(x)) \ .$$

The proof of the identity in (9) goes as follows.

By (5) we have,

$$\Delta(p,m) \equiv \binom{\frac{p-1}{2}}{m} 4^m \pmod{p} \quad ,$$

hence

$$U_p(x) \equiv \sum_{m=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{m} x^{2\left(\frac{p-1}{2}-m\right)} 4^m \equiv (x^2 + 4)^{\frac{p-1}{2}} \pmod{p} \ .$$

3. PROOF OF THEOREM 1

That $U_n(x)$ is irreducible only if n is prime, follows immediately from (8). We now prove that $U_p(x)$ is always irreducible.

Suppose that for some odd prime, p, $U_p(x)$ is reducible. Then we may write

$$U_p(x) = \prod_{i=1}^{m} f_i(x) ,$$

where the $f_i(x)$ are all monic irreducibles.

<u>Case 1.</u> $m \geq 3$. Since $U_p(x)$ contains only even powers of x, $U_p(x) = U_p(-x)$. Hence for each i there exists a j such that $f_i(x) = \pm f_j(-x)$, and for that same j, $f_i(-x) = \pm f_j(x)$. Therefore,

$$f_i(x)f_j(x) = (\pm f_j(-x))(\pm f_i(-x)) = f_i(-x)f_j(-x) .$$

Hence if $i \neq j$, $U_p(x)$ is divisible by an even polynomial. On the other hand, if $i = j$, $f_i(x)$ is even since $f_i(0) \neq 0$. In either instance, we have some factorization $h(x)g(x) = U_p(x)$, where $h(x)$ and $g(x)$ have degree ≥ 2 and both are even functions of x. Now by the division algorithm, we may write

$$h(x) = \ell_1(x)(x^2 + 4) + h ,$$

and

$$g(x) = \ell_2(x)(x^2 + 4) + g ,$$

where h and g are integers. Now by (6), we see that

$$h(x)g(x) \equiv \pm p (\operatorname{mod} x^2 + 4) ,$$

hence $h = \pm p$ and $g = \pm 1$ without loss of generality. On the other hand, by (9), we have

$$g(x) \equiv (x^2 + 4)^k \ (\operatorname{mod} p) \quad \text{when} \quad p \equiv 3 \bmod 4 ,$$

and

$$g(x) \equiv (x + \alpha)^{k_1}(x - \alpha)^{k_2} \pmod p \quad \text{when} \quad p \equiv 1 \bmod 4,$$

where $\alpha = 2\sqrt{-1} \bmod p$. In the second case, we note that $k_1 = k_2$ since $g(x) = g(-x) \pmod p$. Hence, in either instance, we may write

$$g(x) = \ell_3(x)p + (x^2 + 4)^k ,$$

where $\ell_3(x)$ is even since $g(x)$ and $(x^2 + 4)$ are. Therefore $\ell_3(x) \equiv c \pmod{x^2 + 4}$ for some integer c, and we have

$$\pm 1 \equiv g(x) \equiv cp \pmod{x^2 + 4} ,$$

a contradiction. Hence if $U_p(x)$ is reducible, it must have only two factors.

Case 2. $m = 2$. Let $U_p(x) = f(x)g(x)$ where $f(x)$ and $g(x)$ are irreducible and monic. Now either $f(-x) = f(x)$ or $f(-x) = g(x)$. (Note: since $\operatorname{sgn} f(0) - \operatorname{sgn} g(0) \neq 0$, $f(-x) \neq -f(x)$ or $-g(x)$). If $f(-x) = f(x)$, the argument in Case 1 is applicable, since $f(x)$ and $g(x)$ are even. Hence we may assume $f(-x) = g(x)$. Now if $p \equiv 3 \pmod 4$, we get an immediate contradiction. Since

$$\deg f(x) = \deg g(x) = \frac{p-1}{2} ,$$

which is odd, we have that the leading coefficients of $f(-x)$ and $g(x)$ have opposite signs. Therefore $p \equiv 1 \pmod 4$. Now if we let

$$f(x) = \sum_{n=0}^{\frac{p-1}{2}} a_n x^{\frac{p-1}{2}-n} \quad \text{and} \quad g(x) = \sum_{n=0}^{\frac{p-1}{2}} (-1)^n a_n x^{\frac{p-1}{2}-n},$$

then we have

$$f(x)g(x) = x^{p-1} + (2a_2 - a_1^2)x^{p-3} + (2a_4 - 2a_3a_1 + a_2^2)x^{p-5} + \cdots .$$

Now from (5) we have that $(2a_2 - a_1^2) = p - 2$ which means a_1 must be odd and consequently a_2 is even since $2a_2 \equiv 0 \pmod 4$. But also from (5), we have that

$$(2a_4 - 2a_3a_1 + a_2^2) = \frac{(p - 3)(p - 4)}{2} ,$$

which is odd; this is a contradiction since a_2 is even. Therefore $U_p(x)$ is irreducible.

4. FURTHER CONSIDERATIONS

The generating function for $\{U_n(x)\}$ is quite easy to derive, but not very illuminating for number theoretic purposes. We include it here for the sake of completeness.

Let

$$f(x,y) = \sum_{n=0}^{\infty} U_n(x)y^n ,$$

then

$$f(x,y) = \frac{y}{1 - xy - y^2} ,$$

by using the recursion relation and the fact that $U_n(x) = (-1)^{n-1}U_{-n}(x)$.

The main theorem of this paper brings to mind the sequence of cyclotomic polynomials which are also irreducible for prime numbers. We conclude this paper by showing the following inherent connection between the two sequences.

<u>Theorem.</u> The $n - 1$ roots of $U_n(x)$ are given by

$$U_n\left(2i \cos \frac{k\pi}{n}\right) = 0 ,$$

for $k = 1, 2, \cdots, n - 1$.

Proof. Let $x = 2i\cos\theta$, $0 \le \theta \le \pi$, then from (1),

$$U_n(2i\cos\theta) = \frac{(i\cos\theta + \sin\theta)^n - (i\cos\theta - \sin\theta)^n}{2\sin\theta}$$

$$= \frac{(-i)^n(e^{-i\theta n} - e^{i\theta n})}{2\sin\theta}$$

$$U_n(2i\cos\theta) = \frac{(i)^{n-1}\sin n\theta}{\sin\theta}$$

which is zero for

$$\theta = \frac{k\pi}{n}, \qquad k = 1, 2, \cdots, n-1.$$

* * * * *

[Continued from page 456.]

$$|\alpha_i|^{\frac{m+r}{r}},$$

and the circle about (1,0) with radius $|\alpha_i|$. Now, for $\alpha_i = \alpha$, the two circles must be tangent externally (tangent, because $1 - \alpha$ is real; and externally, since $0 < 1 - \alpha < 1$). Now if there exists an i such that $|\alpha_i| < \alpha$, then the radii of both circles would be smaller, and hence they couldn't intersect. This shows that $\alpha = \alpha_i$.

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ON THE COMPLETENESS OF THE LUCAS SEQUENCE

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It is well known* that the Lucas sequence

$$L_0,\ L_1,\ L_2,\ \cdots\ =\ 2,\ 1,\ 3,\ \cdots$$

is complete. It is easy to see that if $0 \le m < n$, the integer $L_{n+1} - 1$ can't be represented as a sum of distinct L_i with $i \ne m, n$. Thus $\{L_j\}$ is not complete after the removal of two arbitrary terms L_m, L_n. We will also show that the sequence is complete after the removal of any one term L_n with $n \ge 2$.

Let N be a positive integer. It is well known that N is a (maximal) sum of L_i's, that is,

$$\text{(1)} \qquad N = L_{i_1} + L_{i_2} + \cdots + L_{i_\beta} \quad \text{with} \begin{cases} i_1 \ge 0 \text{ and} \\ i_{\nu+1} - i_\nu \ge 2 \text{ for } 1 \le \nu < \beta . \end{cases}$$

We suppose L_n is one of the terms in the representation (1), for otherwise we have nothing to show, say $n = i_\alpha \le i_\beta$. Then

$$\text{(2)} \qquad M = L_{i_1} + L_{i_2} + \cdots + L_{i_\alpha} \le L_n + L_{n-2} + \cdots + L_k + L_0$$
$$= \begin{cases} L_{n+1} + 1 \text{ and } k = 2 \text{ if } n \text{ is even,} \\ L_{n+1} - 1 \text{ and } k = 3 \text{ if } n \text{ is odd .} \end{cases}$$

If $M = L_{n+1} + 1$, we replace the sum (2) for M by $L_1 + L_{n+1}$ in (1). If $M = L_{n+1}$ we replace the sum (2) for M by L_{n+1} in (1). Observe that L_{n+1} does not appear in (1). If $M \le L_{n+1} - 1$, we can re-represent it as a sum of distinct terms L_i with $0 \le i \le n - 1$, and so we are through in this final case.

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*V. E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin Co., Boston, 1969.

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REMARK ON A PAPER BY R. L. DUNCAN CONCERNING THE UNIFORM DISTRIBUTION MOD 1 OF THE SEQUENCE OF THE LOGARITHMS OF THE FIBONACCI NUMBERS

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In the following we present a short proof of a theorem shown by R. L. Duncan [1]:

Theorem 1. If $\mu_1, \mu_2, \cdots$ is the sequence of the Fibonacci numbers, then the sequence $\log\mu_1, \log\mu_2, \cdots$ is uniformly distributed mod 1.

Moreover, we show the following proposition.

Theorem 2. The sequence of the integral parts $[\log \mu_1]$, $[\log \mu_2], \cdots$ of the logarithms of the Fibonacci numbers is uniformly distributed mod m for every positive integer $m \geq 2$.

Proof of Theorem 1. It is well known that

$$\frac{\mu_{n+1}}{\mu_n} \rightarrow \frac{1 + \sqrt{5}}{2} ,$$

or

$$\text{(1)} \qquad \log \mu_{n+1} - \log \mu_n \rightarrow \log \frac{1 + \sqrt{5}}{2} , \quad \text{as } n \rightarrow \infty .$$

In [2] (see th. 12.2.1), it is shown that if $\omega \neq 0$ is real and algebraic, then θ^{ω} is not an algebraic number. Therefore,

$$\frac{1 + \sqrt{5}}{2}$$

being an algebraic number, we conclude that

$$\log \frac{1 + \sqrt{5}}{2}$$

is transcendental. (One can also argue as follows: let be given that $\theta > 0$ is algebraic. Now suppose that $\log \theta = u/v$ where u and v are integers. Then we would have $\theta^v = e^u$. But this is impossible since θ^v is algebraic and e^u is transcendental (orally communicated by A. M. Mark).

According to a theorem due to J. G. van der Corput we have that a sequence of real numbers $\lambda_1, \lambda_2, \cdots$ is uniformly distributed mod 1 if

$$\lambda_{n+1} - \lambda_n \longrightarrow \theta \quad \text{(an irrational number) as } n \to \infty .$$

(see [3]). By the property (1) we see that the sequence $\log \mu_1, \log \mu_2, \cdots$ is uniformly distributed mod 1.

Proof of Theorem 2. First, we use the fact that the sequence

$$\frac{\log \mu_n}{m} \quad (m, \text{ an integer} \neq 0), \quad n = 1, 2, \cdots ,$$

is uniformly distributed mod 1 which follows by the same argument used in the proof of Theorem 1: we have namely

$$\frac{\log \mu_{n+1}}{m} - \frac{\log \mu_n}{m} \to \frac{\log \frac{1+\sqrt{5}}{2}}{m} \quad \text{(non-algebraic) as } n \to \infty .$$

Then according to a theorem of G. L. van den Eynden [4], quoted in [5] the sequence

$$[\log \mu_1],\ [\log \mu_2],\ \cdots$$

is uniformly distributed modulo m for every integer $m \geq 2$, that is, if $A(N, j, m)$ is the number of elements of the set

$$\{[\log \mu_n]\} \quad (n = 1, 2, \cdots, N) ,$$

satisfying

$$[\log \mu_n] \equiv j \pmod{m}, \quad (0 \leq j \leq m - 1) ,$$

then

[Continued on page 473.]

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SUMS OF POWERS OF FIBONACCI AND LUCAS NUMBERS

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1. Hunter has stated as a problem in this Quarterly [2] the identity

$$F_{n-1}^4 + F_n^4 + F_{n+1}^4 = 2[2F_n^2 + (-1)^n]^2 . \tag{1}$$

This can be proved rapidly in the following way. In the identity

$$x^4 + y^4 + (x + y)^4 = 2(x^2 + xy + y^2)^2 , \tag{2}$$

take $x = F_{n-1}$, $y = F_n$. Then

$$F_{n-1}^4 + F_n^4 + F_{n+1}^4 = 2(F_{n-1}^2 + F_{n-1}F_n + F_n^2)^2 .$$

Since

$$F_{n-1}^2 + F_{n-1}F_n + F_n^2 = F_{n-1}F_{n+1} + F_n^2 = 2F_n^2 + (-1)^n ,$$

we immediately get (1).

Similarly if we take $x = L_{n-1}$, $y = L_n$ in (2), then since

$$L_{n-1}^2 + L_{n-1}L_n + L_n^2 = L_{n-1}L_{n+1} + L_n^2 = 2L_n^2 - 5(-1)^n ,$$

we get the companion formula

$$L_{n-1}^4 + L_n^4 + L_{n+1}^4 = 2[2L_n^2 - 5(-1)^n]^2 . \tag{3}$$

In the same way the identities

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$$(x + y)^5 - x^5 - y^5 = 5xy(x + y)(x^2 + xy + y^2),$$

$$(x + y)^7 - x^7 - y^7 = 7xy(x + y)(x^2 + xy + y^2)^2 ,$$

lead to the following:

$$F_{n+1}^5 - F_n^5 - F_{n-1}^5 = 5F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n) , \tag{4}$$

$$L_{n+1}^5 - L_n^5 - L_{n-1}^5 = 5L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n), \tag{5}$$

$$F_{n+1}^7 - F_n^7 - F_{n-1}^7 = 7F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)^2 , \tag{6}$$

$$L_{n+1}^7 - L_n^7 - L_{n-1}^7 = 7L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)^2 . \tag{7}$$

Cauchy has proved (see [1, p. 31]) that if p is a prime 3 then

$$(x + y)^p - x^p - y^p = pxy(x + y)(x^2 + xy + y^2)f_p(x,y) , \tag{8}$$

where $f_p(x,y)$ is a polynomial with integral coefficients. For $p \equiv 1 \pmod 6$ there is the stronger result:

$$(x + y)^p - x^p - y^p = pxy(x + y)(x^2 + xy + y^2)^2 g_p(x,y) , \tag{9}$$

where $g_p(x,y)$ is a polynomial with integral coefficients. Substituting $x = F_{n-1}$, $y = F_n$, we get

$$F_{n+1}^p - F_n^p - F_{n-1}^p = pF_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)F_{n,p} ,$$

$$L_{n+1}^p - L_n^p - L_{n-1}^p = pL_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)L_{n,p} ,$$

where $F_{n,p}$ and $L_{n,p}$ are integers. If $p \equiv 1 \pmod 6$ we get

$$F_{n+1}^p - F_n^p - F_{n-1}^p = pF_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)^2 F'_{n,p} ,$$

$$L_{n+1}^p - L_n^p - L_{n-1}^p = pL_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)^2 L'_{n,p} ,$$

where $F'_{n,p}$ and $L'_{n,p}$ are integers.

2. To get more explicit results, we proceed as follows. Consider the identity

$$(10)\qquad \frac{x}{1 - xw} + \frac{y}{1 - yw} + \frac{z}{1 - zw} = \frac{(x + y + z) - 2(xy + xa + yz)w + 3xyzw^2}{1 - (x + y + z)w + (xy + xz + yz)w^2 - xyzw^3} .$$

We take $z = -x - y$. Then (10) becomes

$$(11)\qquad -\frac{x}{1 - xw} - \frac{y}{1 - yw} + \frac{x + y}{1 + (x + y)w} = \frac{-2Uw + 3Vw^2}{1 - Uw^2 + Vw^3} ,$$

where

$$(12)\qquad U = x^2 + xy + y^2, \quad V = xy(x + y) .$$

We have

$$\begin{aligned}(1 - Uw^2 + Vw^3)^{-1} &= \sum_{r=0}^{\infty} w^{2r}(U - Vw)^r \\ &= \sum_{r=0}^{\infty} w^{2r} \sum_{s=0}^{r} (-1)^s \binom{r}{s} U^{r-s}V^s w^s \\ &= \sum_{k=0}^{\infty} (-1)^k w^k \sum_{r} \binom{r}{k - 2r} U^{3r-k}V^{k-2r} .\end{aligned}$$

Since the left member of (11) is equal to

$$\sum_{k=0}^{\infty} [(-1)^k(x + y)^{k+1} - x^{k+1} - y^{k+1}]\, w^k ,$$

it follows that

$$(-1)^k(x+y)^{k+1} - x^{k+1} - y^{k+1}$$

$$= (-1)^k\, 2 \sum_r \binom{r}{k-2r-1} U^{3r-k+2}\, V^{k-2r-1}$$

$$+ (-1)^k 3 \sum_r \binom{r}{k-2r-2} U^{3r-k+2} V^{k-2r-1} .$$

Since

$$2\binom{r}{k-2r-1} + 3\binom{r}{k-2r-2} = \frac{k+1}{k-2r-1}\binom{r}{k-2r-2} ,$$

we have

$$(x+y)^{k+1} - (-1)^k(x^{k+1} + y^{k+1})$$

$$= \sum_r \frac{k+1}{k-2r-1}\binom{r}{k-2r-2} U^{3r-k+2} V^{k-2r-1}. \tag{13}$$

When k is odd, it is to be understood that for $r = (k-1)/2$, the coefficient on the right is 2.

Replacing k by $2k$ in (13), we get

$$(x+y)^{2k+1} - x^{2k+1} - y^{2k+1}$$

$$= \sum_r \frac{2k+1}{2k-2r-1}\binom{r}{2k-2r-2} U^{3r-2k+2} V^{2k-2r-1};$$

the range of r is determined by

$$r < k, \quad 2k-2 \le 3r . \tag{15}$$

In particular (14) implies

$$(x + y)^{6k+1} - x^{6k+1} - y^{6k+1}$$

(16)
$$= \sum_{r=0}^{k-1} \frac{6k + 1}{2r + 1} \binom{3k - r - 1}{2r} U^{3k-3r-1} V^{2r+1} .$$

For example, we have

$$(x + y)^7 - x^7 - y^7 = 7 U^2V$$
$$(x + y)^{13} - x^{13} - y^{13} = 13 U^2V(U^3 + 2V^2)$$
$$(x + y)^{19} - x^{19} - y^{19} = 19U^2V(U^6 + 7U^3V^2 + 3V^4) .$$

We also have from (14)

$$(x + y)^{6k+5} - x^{6k+5} - y^{6k+5}$$

(17)
$$= \sum_{r=0}^{k} \frac{6k + 5}{2r + 1} \binom{3k - r + 1}{2r} U^{3k-3r+1} V^{2r+1} .$$

For example,

$$(x + y)^5 - x^5 - y^5 = 5UV$$
$$(x + y)^{11} - x^{11} - y^{11} = 11 UV(U^3 + V^2)$$
$$(x + y)^{17} - x^{17} - y^{17} = 17(UV(U^6 + 5U^3V^2 + V^4).$$

When $6k + 1$ is prime, the coefficients on the right of (16) are divisible by $6k + 1$; moreover the right member has the polynomial factor U^2. When $6k + 5$ is prime, the coefficients on the right of (17) are divisible by $6k + 5$; moreover the right member has the polynomial factor U. Thus (16) and (17) furnish explicit formulas for the factors $f_p(x,y)$ and $g_p(x,y)$ occurring in (8) and (9).

In addition we have the identity

$$(x + y)^{6k+3} - x^{6k+3} - y^{6k+3}$$

(18)
$$= \sum_{r=0}^{k} \frac{6k + 3}{2r + 1} \binom{3k - r}{2r} U^{3k-3r} V^{2r+1} .$$

For example

$$(x + y)^9 - x^9 - y^9 = 9U^3V + 3V^3$$

$$(x + y)^{15} - x^{15} - y^{15} = 15U^6V + 50U^3V^3 + 3V^5 .$$

For even exponents we get

$$(x + y)^{2k} + x^{2k} + y^{2k}$$

(19)
$$= 2U^k + \sum_{0<3r<k} \frac{k}{r}\binom{k - r - 1}{sr - 1} U^{k-3r}V^{2r} .$$

In particular, (19) yields

$$(x + y)^{6k} + x^{6k} + y^{6k}$$

(20)
$$= 2U^{3k} + \sum_{r=1}^{k} \frac{3k}{r}\binom{3k - r - 1}{2r - 1} U^{3k-3r}V^{2r} .$$

The first few coefficients in the right member of (19) are given by the following table.

k \ r	0	1	2	3
1	2			
2	2			
3	2	3		
4	2	8		
5	2	15		
6	2	24	3	
7	2	35	14	
8	2	48	40	
9	2	63	90	3
10	2	80	175	20

REFERENCES

1. P. Bachmann, Das Fermatproblem in seiner bisherigen Entwickling, Berlin and Lupzig, 1919.
2. Problem H-79, this Quarterly, Vol. 4 (1966), p. 57.

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$$\lim_{n \to \infty} \frac{1}{N} A(N,j,m) = \frac{1}{m} \qquad \text{for} \qquad j = 0, 1, \cdots, n-1.$$

(see [5]).

REFERENCES

1. R. L. Duncan, "An Application of Uniform Distributions to the Fibonacci Numbers," The Fibonacci Quarterly, Vol. 5, No. 2, 1967, pp. 137-140.
2. A. O. Gel'fond and Yu. V. Linnik, Elementary Methods in the Analytic Theory of Numbers, 1968.
3. J. G. van der Corput, "Diophantische Ungleichungen I: Zur Gleichverteilung modulo Eins," Acta Mathematica, 55-56, 1930-31 (378).
4. C. L. van den Eynden, "The Uniform Distribution of Sequences," Dissertation, University of Oregon, 1962.
5. L. Kuipers and S. Uchiyama, "Notes on the Uniform Distribution of Sequences of Integers," Proc. Japan Ac., Vol. 4, No. 7, 1968 (609).

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NOTE ON THE INITIAL DIGIT PROBLEM

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The initial digit problem is concerned with the distribution of the first digits which occur in the set of all positive integers. There are many possible interpretations of the heuristic question, "What is the probability that an integer chosen at random has initial digit equal to a?" [1]. If $A = \{a_n\}$ is the set of all positive integers with initial digit a, then the asymptotic density [2] of A would provide a suitable answer to this question if it exists. However, it is easily shown that the asymptotic density doesn't exist.

The purpose of this note is to show that the logarithmic density [2] of A exists and is equal to $\log(1 + 1/a)$, where $\log x$ is the common logarithm. This result is in agreement with previous solutions of the initial digit problem [1]. It is also of interest to note that the logarithmic density exists and is equal to the asymptotic density whenever the latter exists [2].

The logarithmic density $\delta(A)$ is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{\ln n} \sum_{a_\nu \le n} \frac{1}{a_\nu} ,$$

and the lower and upper logarithmic densities $\underline{\delta}(A)$ and $\overline{\delta}(A)$ are obtained by replacing lim by $\underline{\lim}$ and $\overline{\lim}$ respectively. Now it is obvious that

$$\begin{aligned}\underline{\delta}(A) &= \underline{\lim}_{k\to\infty} \frac{1}{\ln(a\,10^k - 1)} \sum_{a_\nu \le a10^k - 1} \frac{1}{a_\nu} \\ &= \underline{\lim}_{k\to\infty} \frac{1}{k \ln 10} \sum_{\nu=1}^{k-1} [H((a+1)10^\nu - 1) - H(a\,10^\nu - 1)] \\ &= \underline{\lim}_{k\to\infty} \frac{1}{k \ln 10} \sum_{\nu=1}^{k-1} [H((a+1)10^\nu) - H(a\,10^\nu)] ,\end{aligned}$$

where

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} .$$

Using the well-known asymptotic formula [3] $H(n) = \ln n + \gamma + 0(1/n)$, we get

$$\begin{aligned}\underline{\delta}(A) &= \lim_{k\to\infty} (k - 1)(\ln(a + 1) - \ln a)/k \ln 10 \\ &= \ln(1 + 1/a)/\ln 10 = \log(1 + 1/a) .\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{\delta}(A) &= \lim_{k\to\infty} \frac{1}{\ln((a + 1)10^k - 1} \sum_{a_\nu \leq (a+1)10^k - 1} \frac{1}{a_\nu} \\ &= \lim_{k\to\infty} \frac{1}{k \ln 10} \sum_{\nu=1}^{k} [H((a + 1)\, 10^\nu) - H(a\, 10^\nu)] \\ &= \log(1 + 1/a) = \underline{\delta}(A) ,\end{aligned}$$

and the desired result follows.

REFERENCES

1. B. J. Flehinger, "On the Probability that a Random Integer has Initial Digit A," Amer. Math. Monthly, 73 (1966), 1056-1061.
2. H. Halberstam and K. F. Roth, Sequences, Vol. 1, Oxford, 1966.
3. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1960.

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ON FIBONACCI AND LUCAS NUMBERS WHICH ARE PERFECT POWERS

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The Fibonacci numbers, defined for all rational integers n by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n ,$$

have for several centuries engaged the attention of mathematicians, and while many of their properties may be established by very simple methods, there are many unsolved problems connected with them. One such problem is to determine which Fibonacci numbers are perfect powers. The case of the Fibonacci squares was solved by J. H. E. Cohn in [3] and also in [4]. (See [5] for some applications of Cohn's method to other Diophantine problems.) Cohn showed that the only squares in the sequence F_n are given by

$$F_{-1} = F_1 = F_2 = 1, \quad F_0 = 0 \text{ and } F_{12} = 144 .$$

Having solved the problem of the Fibonacci squares, one is led to inquire as to which numbers F_n can be perfect cubes, fifth powers, etc. A proof that

$$F_1 = F_2 = 1, \quad F_6 = 8 \text{ and } F_{12} = 144$$

are the only perfect powers in the sequence F_n for positive n was given by Buchanan [1], but, unfortunately, Buchanan's proof was incomplete and was later retracted by him [2]. Thus the problem of determining all the perfect powers in the sequence F_n remains unsolved. In the present paper we first present a general criterion for solving this problem. We then apply our result to the case of the Fibonacci cubes and give the complete solution for this case. Finally, we give a similar criterion for determining which Lucas numbers are perfect powers, and determine all Lucas numbers which are perfect cubes.

To determine which numbers F_n are perfect k^{th} powers, we may assume, by Cohn's result, that $k = p$, where p is an odd prime, and also that n is positive, since $F_0 = 0$ and $F_{-n} = (-1)^{n+1}F_n$. Let L_n be the n^{th} term in the Lucas sequence, defined by

$$L_1 = 1, \quad L_2 = 3, \quad L_{n+2} = L_{n+1} + L_n ,$$

and let

$$a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2} .$$

By induction, it is easily verified that

$$F_n = \frac{a^n - b^n}{\sqrt{5}}, \quad L_n = a^n + b^n ,$$

and since $ab = -1$, we have finally that

(1) $$L_n^2 - 5F_n^2 = 4(-1)^n .$$

Let us first assume that n is even and that $F_n = z^p$, $L_n = y$, where p is an odd prime. Then (1) becomes

(2) $$y^2 - 5z^{2p} = 4 .$$

Now it is clear that the solution of (2) reduces to the solution of

(3) $$y^2 - 5x^p = 4 ,$$

subject to

(4) $$x = z^2 .$$

In (3) we set

$$x = \frac{X}{5}, \quad y = \frac{Y}{5^{(p-1)/2}} ,$$

which yields

$$Y^2 - 4 \cdot 5^{p-1} = X^p \;,$$

subject to

(5) $\frac{X}{5} = z^2, \quad X > 0, \quad Y > 0, \quad X \equiv 0 \pmod 5, \quad Y \equiv 0 \pmod{5^{(p-1)/2}}\,.$

Similarly, if n is odd, the problem reduces to solving

$$Y^2 + 4 \cdot 5^{p-1} = X^p \;,$$

subject to (5), and we have proved

Theorem 1. The problem of determining which numbers F_n, $n > 0$, are perfect p^{th} powers, where p is an odd prime, reduces to the solution of the equations

$$Y^2 + 4 \cdot 5^{p-1}(-1)^{n-1} = X^p \,,$$

subject to the conditions

$$\frac{X}{5} = z^2, \quad X > 0, \quad Y > 0, \quad X \equiv 0 \pmod 5, \quad Y \equiv 0 \pmod{5^{(p-1)/2}}\,.$$

Let us apply Theorem 1 to the case $p = 3$. Here the problem reduces to solving

(6) $$Y^2 - 100 = X^3 \;,$$

and

(7) $$Y^2 + 100 = X^3 \;,$$

subject to the conditions

(8) $\frac{X}{5} = z^2, \quad X > 0, \quad Y > 0, \quad X \equiv Y \equiv 0 \pmod 5\,.$

Now Hemer proved [7], [8], that the only solutions of (6) with $Y > 0$ are

$$[X,Y] = [-4,6],\quad [0,10],\quad [5,15],\quad [20,90],\ [24,118] \text{ and } [2660,137190].$$

Of these solutions, the only ones satisfying (8) are [5,15] and [20,90]. This yields $[x,y] = [1,3]$ and $[4,18]$ as the only solutions of (3) (with $p = 3$) which satisfy (4), and from these solutions we derive

$$L_2 = 3,\quad F_2 = 1 \text{ and } L_6 = 18,\quad F_6 = 8\ .$$

Thus the only cubes in the sequence F_n with n positive and even are $F_2 = 1$ and $F_6 = 8$.

In two previous papers [6], [10], we showed that the only integer solutions of (7) with $Y > 0$ are $[X,Y] = [5,5]$, $[10,30]$ and $[34,198]$. Of these solutions only $[5,5]$ satisfies (8), and from this solution we derive $F_1 = L_1 = 1$. Thus the only cube in the sequence F_n with n positive and odd is $F_1 = 1$, and we have

<u>Theorem 2.</u> The only cubes in the Fibonacci sequence F_n are

$$F_{-6} = -8,\ F_{-2} = -1,\ F_0 = 0,\ F_{-1} = F_1 = F_2 = 1 \text{ and } F_6 = 8\ .$$

Next, we give a criterion for determining which Lucas numbers are perfect p^{th} powers, where p is an odd prime. We note that the case of the Lucas squares was solved by Cohn [3], who showed that the only Lucas squares are $L_1 = 1$ and $L_3 = 4$.

In (1) let $F_n = z$, $L_n = y^p$ and $n > 0$, and assume first that n is even. Then we get

$$y^{2p} - 5z^2 = 4\ . \tag{9}$$

It is clear that (9) reduces to solving

$$x^p - 5z^2 = 4\ , \tag{10}$$

subject to $x = y^2$.

Equation (10) may be written

$$5x^p - (5z)^2 = 20 \quad ,$$

and, setting $v = 5z$, it reduces to

$$(11) \qquad 5x^p - v^2 = 20 \; ,$$

subject to

$$x = y^2, \quad v \equiv 0 \pmod 5 \; .$$

Finally, setting

$$x = \frac{X}{5}, \quad v = \frac{Y}{5^{(p-1)/2}} \quad ,$$

(11) reduces to

$$(12) \qquad Y^2 + 4 \cdot 5^p = X^p \quad ,$$

subject to the conditions

$$(13) \qquad \frac{X}{5} = y^2, \quad X > 0, \quad Y > 0, \quad X \equiv 0 \pmod 5, \quad Y \equiv 0 \pmod{5^{(p+1)/2}}.$$

Similarly, if n is odd, the problem reduces to

$$Y^2 - 4 \cdot 5^p = X^p \quad ,$$

subject to the conditions (13), and we have

Theorem 3. The problem of determining all the perfect p^{th} powers in the sequence L_n, where p is an odd prime, reduces to solving the two equations

$$Y^2 + 4 \cdot 5^p (-1)^n = X^p \quad ,$$

subject to the conditions

$$\frac{X}{5} = y^2, \quad X > 0, \quad Y > 0, \quad X \equiv 0 \pmod 5), \quad Y \equiv 0 \pmod{5^{(p+1)/2}}) .$$

Finally, we apply Theorem 3 to the case $p = 3$. Here the problem reduces to solving

$$Y^2 - 300 = X^3 , \tag{14}$$

and

$$Y^2 + 500 = X^3 , \tag{15}$$

subject to

$$\frac{X}{5} = y^2, \quad X > 0, \quad Y > 0, \quad X \equiv 0 \pmod 5), \quad Y \equiv 0 \pmod{25}) . \tag{16}$$

In a previous paper [9], we showed that (15) is insoluble and that the only solution of (14) with $Y > 0$ is $[X,Y] = [5,25]$. This solution clearly fulfills (16) and also implies that $L_1 = F_1 = 1$. Thus we have proved

Theorem 4. The only cube in the Lucas sequence L_n, $n > 0$, is $L_1 = 1$.

In conclusion, we wish to point out that Siegel [11] has shown that the problem of determining all the complex quadratic fields of class number 1 can be reduced to the problem of finding all the cubes in the sequences F_n and L_n. Thus we have completed yet another proof of Gauss' famous conjecture on complex quadratic fields of class number 1.

REFERENCES

1. F. Buchanan, "N^{th} Powers in the Fibonacci Series," Am. Math. Monthly, 71 (1964), 647-649.
2. F. Buchanan, Retraction of "N^{th} Powers in the Fibonacci Series," Am. Math. Monthly, 71 (1964), 1112.

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THE DYING RABBIT PROBLEM

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1. INTRODUCTION

Fibonacci numbers originally arose in the answer to the following problem posed by Leonardo de Pisa in 1202. Suppose there is one pair of rabbits in an enclosure at the 0^{th} month, and that this pair breeds another pair in each of the succeeding months. Also suppose that pairs of rabbits breed in the second month following birth, and thereafter produce one pair monthly. What is the number of pairs of rabbits at the end of the n^{th} month? It is not difficult to establish by induction that the answer is F_{n+2}, where F_n is the n^{th} Fibonacci number. In [1] Brother Alfred asked for a solution to this problem if, like Socrates, our rabbits are motral, say each pair dies one year after birth. His answer [2], however, contained an error. The mistake was noted by Cohn [3], who also supplied the correct solution. In this paper we generalize the dying rabbit problem to arbitrary breeding patterns and death times.

2. SOLUTION TO THE GENERALIZED DYING RABBIT PROBLEM

Suppose that there is one pair of rabbits at the 0^{th} time point, that this pair produces B_1 pairs at the first time point, B_2 pairs at the second time point, and so forth, and that each offspring pair breeds in the same manner. We shall let $B_0 = 0$, and put

$$B(x) = \sum_{n=0}^{\infty} B_n x^n ,$$

so that $B(x)$ is the birth polynomial associated with the birth sequence

$$\{B_n\}_{n=0}^{\infty} .$$

The degree of $B(x)$, $\deg B(x)$, may be finite or infinite. Now suppose a pair of rabbits dies at the m^{th} time point after birth (after possible breeding), and let $D(x) = x^m$ be the associated death polynomial. If our rabbits are immortal, put $D(x) = 0$. Clearly $\deg D(x) > 0$ implies $\deg D(x) \geq \deg B(x)$, unless the rabbits have strange mating habits. Let T_n be the total number of live pairs of rabbits at the n^{th} time point, and put

$$T(x) = \sum_{n=0}^{\infty} T_n x^n ,$$

where $T_0 = 1$. Our problem is then to determine $T(x)$, where $B(x)$ and $D(x)$ are known.

Let R_n be the number of pairs of rabbits born at the n^{th} time point assuming <u>no</u> deaths. With the convention that the original pair was born at the 0^{th} time point, and recalling that $B_0 = 0$, we have

$$\begin{aligned} R_0 &= 1, \\ R_1 &= B_0R_1 + B_1R_0 , \\ R_2 &= B_0R_2 + B_1R_1 + B_2R_0 . \end{aligned}$$

and in general that

$$R_n = \sum_{j=0}^{n} B_j R_{n-j} \qquad (n \geq 1) . \tag{1}$$

Note that for $n = 0$ this expression yields the incorrect $R_0 = 0$. Then if

$$R(x) = \sum_{n=0}^{\infty} R_n x^n ,$$

equation (1) is equivalent to

$$R(x) = R(x)B(x) + 1 ,$$

so that

$$R(x) = \frac{1}{1 - B(x)} .$$

The total number $T_n^\star$ of pairs at the n^{th} time point assuming no deaths is given by

$$T_n^\star = \sum_{j=0}^{n} R_j \ ,$$

and we find

$$\begin{aligned}\frac{1}{(1-x)[1-B(x)]} &= \frac{R(x)}{1-x} = \left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{n=0}^{\infty} R_n x^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} R_j\right)x^n = \sum_{n=0}^{\infty} T_n^\star x^n = T^\star(x) \ .\end{aligned} \tag{2}$$

Hoggatt [4] used slightly different methods to show both (1) and (2).

We must now allow for deaths. Since each pair dies m time points after birth, the number of deaths D_n at the n^{th} time point equals the number of births R_{n-m} at the $(n-m)^{th}$ time point. Therefore

$$\sum_{n=0}^{\infty} D_n x^n = D(x) \sum_{n=0}^{\infty} R_n x^n = \frac{D(x)}{1-B(x)} \ .$$

Letting the total number of dead pairs of rabbits at the n^{th} time point be

$$C_n = \sum_{j=0}^{n} D_j \ ,$$

we have

$$\begin{aligned}\frac{D(x)}{(1-x)[1-B(x)]} &= \left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{n=0}^{\infty} D_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} D_j\right)x^n \\ &= \sum_{n=0}^{\infty} C_n x^n = C(x) \ .\end{aligned}$$

Now the total number of live pairs of rabbits T_n at the n^{th} time point is $T_n^* - C_n$, so that

(3) $$T(x) = T^*(x) - C(x) = \frac{1 - D(x)}{(1 - x)[1 - B(x)]}$$

3. SOME PARTICULAR CASES

To solve Brother Alfred's problem, we put $B(x) = x^2 + x^3 + \cdots + x^{12}$ and $D(x) = x^{12}$ in (3) to give

$$T(x) = \frac{1 - x^{12}}{(1 - x)(1 - x^2 - x^3 - \cdots - x^{12})} = \frac{1 - x^{12}}{1 - x - x^2 + x^{13}} .$$

It follows that the sequence $\{T_n\}$ obeys

$$T_{n+13} = T_{n+12} + T_{n+11} - T_n \quad (n \geq 0) ,$$

together with the initial conditions $T_n = F_{n+1}$ for $n = 0, 1, \cdots, 11$, and $T_{12} = F_{13} - 1$, which agrees with the answer given by Cohn [3].

As another example of (3), suppose each pair produce a pair at each of the two time points following birth, and then die at the m^{th} time point after birth $(m \geq 2)$. In this case, $B(x) = x + x^2$ and $D(x) = x^m$. From (3), we see

$$T(x) = \frac{1 - x^m}{(1 - x)(1 - x - x^2)} .$$

Making use of the generating function

$$\frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1}x^n ,$$

we get

$$T(x) = \frac{1 + x + \cdots + x^{m-1}}{1 - x - x^2} = \sum_{j=0}^{m-1} \frac{x^j}{1 - x - x^2}$$

$$(4) \quad = \sum_{j=0}^{m-1}\left(\sum_{n=0}^{\infty} F_{n+1}x^{n+j}\right) = \sum_{n=0}^{m-1}\left(\sum_{k=0}^{n} F_{k+1}\right)x^n + \sum_{n=m}^{\infty}\left(\sum_{k=0}^{m-1} F_{n-k+1}\right)x^n$$

$$= \sum_{n=0}^{m-1} (F_{n+3} - 1)x^n + \sum_{n=m}^{\infty} (F_{n+3} - F_{n-m+3})x^n .$$

For $m = 4r$ it is known [5] that

$$F_{n+3} - F_{n-4r+3} = F_{2r}L_{n-2r+3} ,$$

where L_n is the n^{th} Lucas number, while for $m = 4r + 2$,

$$F_{n+3} - F_{n-4r+1} = L_{2r+1}F_{n-2r+2} ,$$

which may be used to further simplify (4). In particular, for $m = 2$,

$$T(x) = 1 + 2x + \sum_{n=0}^{\infty} F_{n+2}x^n = \sum_{n=0}^{\infty} F_{n+2}x^n ,$$

while for $m = 4$ we have

$$T(x) = 1 + 2x + 4x^2 + 7x^3 + \sum_{n=4}^{\infty} L_{n+1}x^n$$

$$= -x + \sum_{n=0}^{\infty} L_{n+1}x^n .$$

Thus for proper choices of $B(x)$ and $D(x)$ we are able to get both Fibonacci and Lucas numbers as the total population numbers.

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REFERENCES

1. Brother U. Alfred, "Exploring Fibonacci Numbers," Fibonacci Quarterly, 1 (1963), No. 1, 57-63.
2. Brother U. Alfred, "Dying Rabbit Problem Revived," Fibonacci Quarterly, 1 (1963), No. 4, 53-56.
3. John H. E. Cohn, "Letter to the Editor," Fibonacci Quarterly, 2 (1964), 108.
4. V. E. Hoggatt, Jr., "Generalized Rabbits for Generalized Fibonacci Numbers," Fibonacci Quarterly, 6 (1968), No. 3, 105-112.
5. I. Dale Ruggles, "Some Fibonacci Results Using Fibonacci-Type Sequences," Fibonacci Quarterly, 1 (1963), No. 2, 75-80.

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[Continued from page 481.]

3. J. H. E. Cohn, "Square Fibonacci Numbers, Etc.," Fibonacci Quarterly, 2 (1964), 109-113.
4. J. H. E. Cohn, "On Square Fibonacci Numbers," Jour. London Math. Soc., 39 (1964), 537-540.
5. J. H. E. Cohn, "Lucas and Fibonacci Numbers and Some Diophantine Equations," Proc. Glasgow Math. Assoc., 7 (1965), 24-28.
6. Finkelstein, R. P., "On the Units in a Quartic Field with Applications to Mordell's Equation," Doctoral Dissertation, Arizona State University, 1968.
7. O. Hemer, "On the Diophantine Equation $y^2 - k = x^3$," Doctoral Dissertation, Uppsala, 1952.
8. O. Hemer, "On Some Diophantine Equations of the Type $y^2 - f^2 = x^3$," Math. Scand., 4 (1956), 95-107.
9. H. London and R. Finkelstein, "On Mordell's Equation $y^2 - k = x^3$," to appear.
10. H. London and R. Finkelstein, "On Mordell's Equation $y^2 - k = x^3$: II. The Case $k < 0$," to be submitted.
11. C. Siegel, "Zum Beweise des Starkschen Sätzes," Inventiones Math., 5 (1968), 180-191.

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ORTHOGONAL EXPANSION
DERIVED FROM THE EXTREME VALUE DISTRIBUTION

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1. INTRODUCTION

The cumulative distribution function, $F(x)$, of the extreme value distribution is given by

$$F(x) = e^{-e^{-x}}, \quad \text{for} \quad -\infty < x < \infty, \tag{1}$$

and the density function, $f(x) = F'(x)$, is obtained as

$$f(x) = e^{-(x+e^{-x})}, \quad \text{for} \quad -\infty < x < \infty. \tag{2}$$

The extreme value distribution has found a number of applications. Cramer [2] derives (2) as an asymptotic density of the first value from the top for certain transformed variates in a random sample of n observations drawn from Laplace's and normal distributions. The distribution function (1) was first used by Gompertz [3] in connection with actuarial life tables and later on has been used extensively in the study of growth.

The purpose of this paper is (i) to find an explicit expression for the moment generating function of the standardized extreme value distribution and (ii) to derive an orthogonal expansion (Type A series) from the extreme value density in a manner similar to the way in which Gram [4] and Charlier [1] derived an orthogonal expansion from the normal density by making use of the Hermite polynomials which are orthogonal with respect to the normal density. The orthogonal expansion requires the calculation of first eight standardized moments of (2) which in turn involve the evaluation of the Riemann zeta function. This difficulty is overcome by using the tabular values of the Riemann zeta function given by Steiljes [6].

2. MOMENT GENERATING FUNCTION

The moment generating function, $M_x(u)$, of the density function $f(x)$ is

$$M_x(u) = \int_{-\infty}^{\infty} e^{ux} e^{-(x+e^{-x})} dx \quad ,$$

which, on substituting $s = e^{-x}$, becomes

$$(3) \qquad M_x(u) = \int_0^{\infty} s^{-u} e^{-s} ds$$

$$= \Gamma(1 - u)$$

$$= \sum_{k=0}^{\infty} \Gamma^{(k)}(1)(-u)^k/k!$$

where $\Gamma^{(k)}(1)$ is the k^{th} derivative of the gamma function, $\Gamma(p)$, at $p = 1$. This proves the following.

Lemma 1. The moment generating function of the extreme value density $f(x)$ is given by (3).

According to Jordan [5], the n^{th} derivative of $\Gamma(p)$ at $p = 1$ is

$$(4) \qquad \Gamma^{(n)}(1) = (-1)^n \sum \frac{n!}{d_1!\, d_2! \cdots d_n!} C^{d_1}(S_2/2)^{d_2} \cdots (S_n/n)^{d_n} \quad ,$$

where the summation is over non-negative integers $d_1, d_2, \cdots, d_n$ such that $d_1 + 2d_2 + 3d_3 + \cdots + nd_n = n$; S_k is the Riemann zeta function defined by

$$S_k = \sum_{n=1}^{\infty} n^{-k} \quad ,$$

and C is Euler's constant which, correct to nine decimal places, is 0.577215665^{-}.

If μ_1' and μ_2 denote the mean and variance of $f(x)$, then (3) and (4) give us

$$\mu_1' = C \quad \text{and} \quad \mu_2 = S_2 \ .$$

Defining $z = (x - C)/\sqrt{S_2}$, we get the standardized extreme value density function

$$g(z) = \sqrt{S_2}\, e^{-[C + \sqrt{S_2}\, z + e^{-(C + \sqrt{S_2}\, z)}]}, \quad \text{for } -\infty < z < \infty \quad . \tag{5}$$

The moment generating function, $M_z(u)$, of $g(z)$ is obtained as

$$\begin{aligned} M_z(u) &= E(e^{uz}) \\ &= e^{-Cu/\sqrt{S_2}}\, M_x(u/\sqrt{S_2}) \ , \end{aligned}$$

which, by Lemma 1, becomes

$$\begin{aligned} M_z(u) &= \left[\sum_{h=0}^{\infty} (C/\sqrt{S_2})^h (-u)^h/h!\right]\left[\sum_{k=0}^{\infty} \Gamma^{(k)}(1)(-u/\sqrt{S_2})^k/k!\right] \\ &= \sum_{r=0}^{\infty} \alpha_r u^r/r! \qquad , \end{aligned} \tag{6}$$

where α_r is the r^{th} standardized moment of $g(z)$ and

$$\alpha_r = \sum_{j=0}^{r} (-1)^r (1/S_2)^{r/2} \binom{r}{j} C^{r-j}\Gamma^{(j)}(1) \ . \tag{7}$$

This completes the proof of the following:

Theorem 1. The moment generating function of the standardized extreme value distribution $g(z)$ is given by (6).

The first eight of the expressions in (7), using (4), are

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= 1 \\ \alpha_3 &= 2S_3/\sqrt{S_2^3} \end{aligned}$$

$$\alpha_4 = (3S_2^2 + 6S_4)/S_2^2$$

$$\alpha_5 = (20S_2S_3 + 24S_5)/\sqrt{S_2^5}$$

$$\alpha_6 = (15S_2^3 + 40S_3^2 + 90S_2S_4 + 120S_6)/S_2^3$$

$$\alpha_7 = (210S_2^2S_3 + 420S_3S_4 + 504S_2S_5 + 720S_7)/\sqrt{S_2^7}$$

$$\alpha_8 = (105S_2^4 + 1120S_2S_3^2 + 1260S_2^2S_4 + 1260S_4^2 + 2688S_3S_5 + 3360S_2S_6 + 5040S_8)/S_2^4 \quad .$$

The values of S_k for $k = 2, 3, \cdots, 70$ have been computed by Stieltjes [6] up to 32 decimal places. Using his tabular values, we have

$$S_2 = 1.644934067 \qquad S_6 = 1.017343062$$
$$S_3 = 1.202056903 \qquad S_7 = 1.008349277$$
$$S_4 = 1.082323234 \qquad S_8 = 1.004077356$$
$$S_5 = 1.036927755^{+}$$

The substitution of S's give the numerical values of α's as

$$\alpha_1 = 0.000000000 \qquad \alpha_5 = 18.566615980$$
$$\alpha_2 = 1.000000000 \qquad \alpha_6 = 91.414247335^{-}$$
$$\alpha_3 = 1.139547099 \qquad \alpha_7 = 493.149891500$$
$$\alpha_4 = 5.400000000 \qquad \alpha_8 = 3091.022943246$$

3. ORTHOGONAL POLYNOMIALS

If α_r denotes the r^{th} standardized moment of $g(z)$, then, according to Szego [7], the orthogonal polynomials $q_n(z)$ associated with the density function $g(z)$ are given by

$$q_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix} 1 & 0 & 1 & \alpha_3 & \cdots & \alpha_n \\ 0 & 1 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n+1} \\ & & \vdots & & & \\ \alpha_{n-1} & \alpha_n & \alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{2n-1} \\ 1 & z & z^2 & z^3 & \cdots & z^n \end{vmatrix} \tag{8}$$

where the leading coefficient of $q_n(z)$ is one and

$$(9) \qquad D_n = \begin{vmatrix} 1 & 0 & 1 & \alpha_3 & \cdots & \alpha_n \\ 0 & 1 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n+1} \\ & & \vdots & & & \\ \alpha_{n-1} & \alpha_n & \alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{2n-1} \\ \alpha_n & \alpha_{n+1} & \alpha_{n+2} & \alpha_{n+3} & \cdots & \alpha_{2n} \end{vmatrix}$$

The polynomials $q_n(z)$ have the orthogonality property that

$$(10) \qquad \int_{-\infty}^{\infty} q_m(z)q_n(z)g(z)dz = \begin{cases} D_n/D_{n-1} & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

Substituting for α's in (8), the polynomials $q_n(z)$, correct to six decimal places, for $n = 0, 1, 2, 3,$ and 4, are obtained as

$$\begin{aligned} q_0(z) &= 1 \\ q_1(z) &= z \\ q_2(z) &= z^2 - 1.139547z - 1 \\ q_3(z) &= z^3 - 3.634938z^2 - 1.257817z + 2.495391 \\ q_4(z) &= z^4 - 7.557958z^3 + 6.560849z^2 + 14.769958z - 3.348201 \ . \end{aligned}$$

4. DERIVATION OF ORTHOGONAL EXPANSION

Suppose that a density function, $h(z)$, can be represented formally by an infinite series of the form

$$(11) \qquad h(z) = g(z) \sum_{n=0}^{\infty} a_n q_n(z) \ ,$$

where the $q_n(z)$ are orthogonal polynomials associated with the density function $g(z)$.

Multiplying both sides of (11) by $q_n(z)$ and integrating from $-\infty$ to ∞, we have, in virtue of the orthogonality relationship (10),

$$a_n = \frac{D_{n-1}}{D_n} \int_{-\infty}^{\infty} h(z) q_n(z) dz \quad . \tag{12}$$

The reader familiar with harmonic analysis will recognize the resemblance between this procedure and the evaluation of constants in a Fourier series.

The first five values of a's, given by (12), are computed as

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0.0500572(\beta_3 - 1.139547) \\
a_4 &= 0.0045512(\beta_4 - 7.557958\beta_3 + 3.212648)
\end{aligned}$$

where β_r is the r^{th} standardized moment of $h(z)$.

Substituting for the a's in (11), we have

Theorem 2. The orthogonal expansion (Type A series) derived from the standardized extreme value density $g(z)$ is

$$\begin{aligned}
h(z) = g(z)[1 &+ 0.0500572(\beta_3 - 1.139547)q_3(z) + 0.0045512(\beta_4 \\
&- 7.557958\beta_3 + 3.212648)q_4(z) + \cdots]
\end{aligned}$$

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REFERENCES

1. C. V. L. Charlier, Researches into the Theory of Probability, K. Fysiogr. Sällsk. Handl. B 16 (1906).
2. H. Cramér, Mathematical Methods of Statistics, Princeton University Press, Princeton, New Jersey (1945).

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REPRESENTATION OF NATURAL NUMBERS AS SUMS OF GENERALIZED FIBONACCI NUMBERS – II

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The well-known observation of Zeckendorf is that every positive integer N has a unique representation

$$N = u_{i_1} + u_{i_2} + \cdots + u_{i_d} ,$$

where

(1) $$i_1 \geqq 1 \text{ and } i_{\nu+1} - i_\nu \geqq 2 \text{ for } i \leqq \nu < d ,$$

and $\{u_n\}$ is the Fibonacci sequence

$$\cdots, 0, 0, 1, 2, 3, 5, 8, 13, \cdots$$

defined by

(2) $$\begin{cases} u_n = 0 & \text{for } n \leqq 0 , \\ u_1 = 1, \; u_2 = 2, & \text{and} \\ u_{n+1} = u_n + u_{n-1} & \text{for } n \geqq 2 . \end{cases}$$

Existence of such a representation follows from (2), and its uniqueness follows easily from the identity

(3) $$u_{n+1} = 1 + u_n + u_{n-2} + u_{n-4} + \cdots \quad \text{for } n \geqq 0 .$$

The object of this note is to discuss very general methods for uniquely representing integers, of which Zeckendorf's theorem is a special case. I feel that my results give a fairly complete description of the representations; they certainly extend the treatment of an earlier paper of the same name [5].

Here are some remarks on the notation which will be followed throughout this paper. We reserve the brackets $\{\cdots\}$, $(\cdots)$ and $[\cdots]$ for sequences, vectors and matrices, respectively. By V we denote the set of all vectors $(i_1, i_2, \cdots, i_d)$ of various dimensions $d \geq 1$, whose components i_ν are integers with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_d$. Often we will write I instead of $(i_1, i_2, \cdots, i_d)$ and M instead of $[m_{\mu,\nu}]$. Also $\{a_n\}$, $n = 1, 2, 3, \cdots$ will denote any sequence of integers satisfying axiom 1.

Axiom 1. The sequence is strictly increasing and its first term is 1.

For convenience, we write $a(I)$ or $a(i_1, i_2, \cdots, i_d)$ for the number

$$a(I) = a(i_1, i_2, \cdots, i_d) = a_{i_1} + a_{i_2} + \cdots + a_{i_d} .$$

It will be noted that all small letter symbols stand for non-negative integers.

In [5] I discussed pairs $\{a_n\}$, $\{k_n\}$ which represent the integers according to

Definition 1. $\{a_n\}$, $\{k_n\}$ represent the integers if, for each positive integer N there is one and only one vector $I = (i_1, i_2, \cdots, i_d)$ in V such that $N = a(I)$ and

$$i_{\nu+1} - i_\nu \geq k_\nu \quad \text{for } 1 \leq \nu < d . \tag{4}$$

Let us write h and k for k_1 and k_2, respectively. Then it turns out ([5], theorems C and D) that $\{a_n\}$, $\{k_n\}$ represent the integers if and only if

$$0 \leq k - 1 \leq h \leq k = k_\nu \quad \text{for} \quad \nu \geq 2 , \tag{5}$$

and $\{a_n\}$ is the $(h,k)^{\text{th}}$ Fibonacci sequence $\{v_n\}$ defined by

$$\begin{cases} v_n = n & \text{for } 1 \leq n \leq k , \\ v_n = v_{n-1} + v_{n-h} & \text{for } k < n < h + k, \\ v_n = k - h + v_{n-1} + v_{n-k} & \text{for } n \geq h + k . \end{cases} \tag{6}$$

The Fibonacci sequence $\{u_n\}$ has been defined by authors in various ways, such as

$$\cdots, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 3, 5, 8, 13, \cdots,$$
$$\cdots, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 8, 13, \cdots,$$

and

$$\cdots, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, \cdots .$$

One can sometimes simplify an argument by changing from one definition to another. We chose to define $\{u_n\}$ by (2) in order to use (3). Sometimes it is more convenient to define (v_n) by

$$\begin{aligned} v_n &= 0 \quad \text{for } n < k^\star , \\ v_n &= 1 \quad \text{for } k^\star \le n \le 1, \quad \text{and} \\ v_n &= k - h + v_{n-1} + v_{n-k} \quad \text{for } n \ge 2, \end{aligned} \tag{6.1}$$

where $k^\star = 1$ if $h = k - 1$ but $k^\star = -k + 2$ if $h = k$.

In the sequel, when we define a sequence, we will only consider the argument on hand at the time.

Next observe that the $(2,2)^{\text{th}}$ Fibonacci sequence is the ordinary Fibonacci sequence $\{u_n\}$, $n \ge 1$, and if $k_\nu = 2$ for all ν then condition (4) becomes condition (1). Thus in [5] I generalized Zeckendorf's theorem by replacing the constant 2 in (1) by a sequence $\{k_n\}$. Later, I replaced $\{k_n\}$ by an infinite matrix $M = [m_{\mu,\nu}]$, where $\mu, \nu \ge 1$, of non-negative integers $m_{\mu,\nu}$ as described in definition 2.

Definition 2. $\{a_n\}$, M represent the integers if, for each positive integer N there is one and only one vector $I \in V$ such that $N = a(I)$ and

$$i_\mu - i_\nu \ge m_{\mu-\nu,\nu} \quad \text{for } 1 \le \nu < \mu \le d. \tag{7}$$

I described all such pairs $\{a_n\}$, M to a splinter group of the 1962 International Congress of Mathematicians in Stockholm (see the programme). However in an effort to simplify my proofs, I made one further generalization as follows.

Definition 3. $\{a_n\}$, W represent the integers, where $W \subseteq V$, if for each positive integer N there is one and only one vector $I \in W$ such that $N = a(I)$.

There is very little one can say about $\{a_n\}$, W as this definition stands, so with my eye on condition (7), I make W satisfy axiom 2.

Axiom 2. If

$$(i_1, i_2, \cdots, i_d) \in V; \quad (j_1, j_2, \cdots, j_e) \in W; \quad 1 \le d \le e$$

and

$$i_{\nu+1} - i_\nu \ge j_{\nu+1} - j_\nu$$

for $1 \le \nu < d$ then

$$(i_1, i_2, \cdots, i_d) \in W .$$

This axiom merely says that if a vector is in W and we "cut its tail off" or "stretch" it, or do both things, it will still be in W. Important trivial consequences of axiom 2 are the laws

$$(8) \quad \begin{cases} (i_1, i_2, \cdots, i_d) \in W \Longleftrightarrow (i_1 + 1, i_2 + 1, \cdots, i_d + 1) \in W \text{ and } i_1 \ge 1, \\ \text{and } (i_1, i_2, \cdots, i_{d-1}, i_d) \in W \Longrightarrow (i_1, i_2, \cdots, i_{d-1}, i_d + 1) \in W . \end{cases}$$

If $M = \left[m_{\mu,\nu}\right]$ is any matrix, and W is the set of all vectors $I = (i_1, i_2, \cdots, i_d)$ satisfying (7), then clearly axiom 2 holds for W. Hence definition 3 with axiom 2 is more general than definition 2, which in turn is more general than definition 1.

I will now state the fundamental theorem of all this work.

Theorem 1. Suppose $\{a_n\}$, W represent the integers, $W \subseteq V$, and axioms 1 and 2 hold. Then for $t = 1, 2, 3, \cdots$ all the integers N such that $a_t \le N < a_{t+1}$, and only these integers, each have a representation $N = a(I)$ with $I = (i_1, i_2, \cdots, i_d)$ in W and $i_d = t$.

It follows from the theorem that any part of a representation is a representation. In other words, if

$$(i_1, i_2, \cdots, i_d) \in W; \quad i \le e \le d$$

and

$$1 \leq \nu_1 < \nu_2 < \cdots < \nu_e \leq d$$

then

$$(i_{\nu_1}, i_{\nu_2}, \cdots, i_{\nu_e}) \in W \ .$$

Also the theorem shows that the representations of the successive integers $1, 2, 3, \cdots$ change "continuously," in the same way as their representations in the binary scale do. All possible representations using only $a_1, a_2, \cdots, a_t$ are exhausted before a_{t+1} is used. To determine the representation of a given integer N you find the suffix t such that $a_t \leq N < a_{t+1}$, then the suffix s such that $a_s \leq N - a_t < a_{s+1}$, and so on.

Now suppose W satisfies axiom 2. Then clearly $(1) \in W$, there is a least integer p such that $(1,p) \in W$, and there is a largest integer q such that the vector $(1,1,\cdots,1)$ of dimension q is in W. One of the numbers p and q is 1 and the other is greater than 1. My proofs, of theorem 1 and the results below for representations under definition 2, all split into the two cases $p = 1$ and $q = 1$. I always establish a chain of lemmas, each of which involves a number of complicated statements, and has a proof depending on nested induction arguments. One can gain some idea of the lengths of the proofs by inspecting [5]. For this reason, I do not intend to publish any proofs in this paper. I have tried repeatedly, but unsuccessfully, to find analytic proofs. I think that such proofs would be elegant, and would at the same time settle my monotonicity conjecture below.

An important result contained in the lemmas is the following:

If N is an integer $N \geq 1$, and the representations of N and $N + 1$ are respectively

$$N = a(i_1, i_2, \cdots, i_d)$$

and

$$N + 1 = (j_1, j_2, \cdots, j_e)$$

then

$$1 \le a(j_1 + 1,\ j_2 + 1, \cdots, j_e + 1) - a(i_1 + 1, i_2 + 1, \cdots, i_d + 1) \le q + 1 .$$

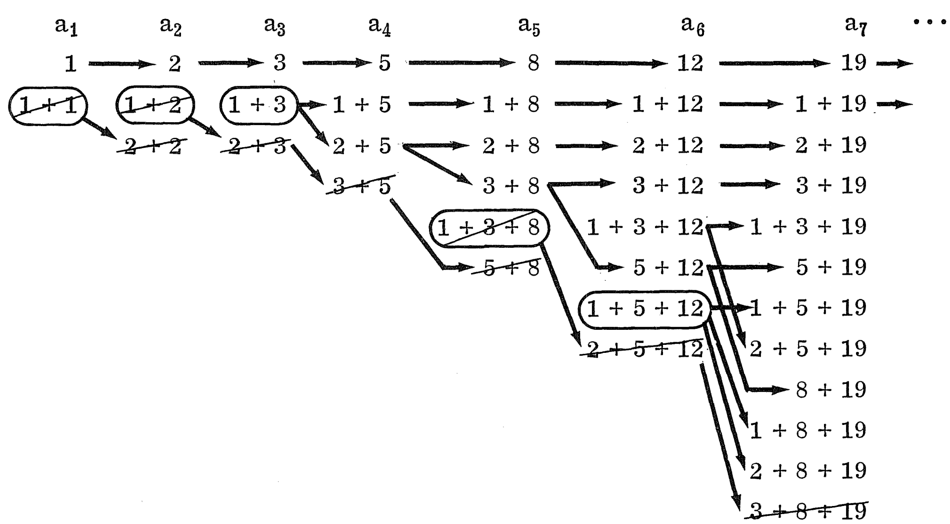
Notice the revelence of (8) to this result. Moreover the result enables us to give bounds for the rate of growth of $\{a_n\}$, and these bounds are necessary in the proofs. Taking $N = a_t - 1$ so that $N + 1 = a_t = a(t)$, we find that

$$1 \le a_{t+1} - a(i_1 + 1, i_2 + 1, \cdots, i_d + 1) \le q + 1.$$

We can in fact say more than the above, and I will illustrate the account by starting to construct a pair $\{a_n\}$, W inductively. We must have $a_1 = 1$, and the vector (1) in W. We are free to have (1, 1) in W or not. Suppose we choose not to have it in. Then we can choose to have (1, 2) in W or not. Suppose not. Then we are free to have (1, 3) in W or not. Suppose we have it in. Then our construction could proceed as shown in Table 1.

Table 1

Construction of $\{a_n\}$, W when p = 3



In the table, a representation is circled if at the appropriate stage of the construction, we had freedom to admit or reject it. A representation is crossed out iff it is not admitted. A representation at the head of an arrow must be admitted, or not as the case may be, by virtue of (8) or axiom 2, because the representation at the tail of the arrow was admitted or not. Notice that we had no freedom over the values of a_5 or a_8. Also the representation 1+3+12 must be admitted even though it is not controlled by (8) and earlier representations. If 1+3+12 is rejected, then $a_7 = 16$ and we have $17 = a(1,7) = a(4,6)$ contradicting the uniqueness of the representations. In general, for $p > 1$, when we have freedom over the value of a_t, (i.e., we can accept or reject some representation $N = a(i_1, i_2, \cdots, i_d)$ with $i_d = t - 1$), if we choose the lower value for a_t we will have freedom of choice over a_{t+1}. On the other hand, if we choose the higher value for a_t we will have no freedom over a_{t+1}, a_{t+2}, $\cdots, a_{t+p-2}$, and possibly over more terms, and sometimes over all further terms.

A typical construction with $q > 1$ is shown in Table 2.

Table 2

Construction of $\{a_n\}$, W when $q = 3$

a_1	a_2	a_3	a_4
1	4	15	55
1+1	1+4	1+15	1+55
1+1+1	1+1+4	1+1+15	1+1+55
~~1+1+1+1~~	1+1+1+4	$\vdots$	$\vdots$
	4+4		
	1+4+4	15+15	
	1+1+4+4	$\vdots$	
	1+1+1+4+4		
	4+4+4	15+15+15	
	1+4+4+4	$\vdots$	
	1+1+4+4+4		
	~~1+1+1+4+4+4~~	4+4+15+15+15	
	~~4+4+4+4~~	1+4+4+15+15+15	
		~~1+1+4+4+15+15+15~~	
		~~1+1+1+4+4+15+15+15~~	
		~~4+4+4+15+15+15~~	

Whichever way the pair $\{a_n\}$, W arise there will be a sequence $\{m_n\}$ of integers, $0 \le m_1 \le m_2 \le m_3 \le \cdots$, which may be finite or infinite, such that if we put

$$a_n = 0 \quad \text{for} \quad n \le 0, \tag{3}$$

then we have the identity

$$a_{n+1} = 1 + a_n + a_{n-m_1} + a_{n-m_2} + \cdots \quad \text{for} \quad n \ge 0 \ . \tag{10}$$

This identity corresponds to (3). Moreover, if our use of the freedom of choice discussed above has a cyclic pattern, then $\{m_n\}$ is eventually periodic. It will then follow by subtracting equations (10) in pairs that high up terms in $\{a_n\}$ satisfy a finite recurrence relation. For example, continuing the construction of Table 1, let us use our freedom in column $3,4,5,\cdots$ according to the pattern: admit, no choice, reject, admit, no choice, reject,$\cdots$. Then $\{m_n\} = 2,5,8,11,14,\cdots$, an arithmetical progression with common difference 3, and $\{a_n\}$ is given by

$$\begin{cases} a_n = 0 \ \text{for} \ n \le 0, \ a_1 = 1, a_2 = 2, \ a_3 = 3, \ \text{and} \\ a_{n+1} = a_n + a_{n-2} + a_{n-2} - a_{n-3} \quad \text{for} \quad n \ge 3 \ . \end{cases} \tag{11}$$

The first 8 terms of $\{a_n\}$ are 1, 2, 3, 5, 8, 12, 19, 30 and the next 7 appear in (11.1) below.

We can use the above facts to obtain bounds for any sequence $\{a_n\}$ as follows. We define a sequence $\{b_n\}$ which has the same construction as $\{a_n\}$ to some particular stage, then from that stage on, whenever freedom arises we choose the largest (smallest) possible value for b_t. The sequence $\{b_n\}$ so constructed will satisfy a finite recurrence relation which we can use to evaluate the terms of $\{b_n\}$, and hence obtain upper (lower) bounds for $\{a_n\}$. As an example, let us find bounds for the sequence $\{a_n\}$ started in Table 1. If we admit as many representations as possible in the remainder of the construction, we find that $\{m_n\} = 2,5,7,10,12,15,\cdots$ (first differences $m_{n+1} - m_n$ are $3,2,3,2,3,2,\cdots$) and that $\{a_n\}$ may be defined as

$$(12)\qquad \begin{cases} a_1 = 1, \ a_2 = 2, \ a_3 = 3, \ a_4 = 5, \ a_5 = 8, \text{ and} \\ a_{n+1} = a_n + a_{n-2} + a_{n-4} \quad \text{for} \quad n \geq 5, \end{cases}$$

This sequence is the most highly divergent one which starts like Table 1.

Again starting from Table 1, we this time reject as many representations as possible. Then $\{m_n\}$ is the finite sequence $2, 5$ and

$$(13)\qquad \begin{cases} a_n = 0 & \text{for } n < 0 \\ a_{n+1} = 1 + a_n + a_{n-2} + a_{n-5} & \text{for } n \geq 0. \end{cases}$$

This sequence is the most slowly divergent one which starts like Table 1. The first 8 terms of any sequence starting like Table 1 are $1,2,3,5,8,12,19,30$. I will now show some of the terms which follow these for the bounds (12) and (13), and for the example (11).

	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	$\cdots$
(12.1)	47	74	116	182	286	449	705	1107	$\cdots$
(11.1)	46	72	113	175	273	427	664	1035	$\cdots$
(13.1)	46	71	110	169	260	401	617	949	$\cdots$

Let us now consider the matrix M of definition 2. If there are three fixed integers r, s, t such that

$$m_{r,s} \leq m_{r-t,s+t} + m_{t,s} \qquad \text{with} \qquad 1 \leq t < r \quad \text{and} \quad 1 \leq s,$$

then we will say that the element $m_{r,s}$ of M is redundant. We do so because if in some representation $N = a(I)$ we have

$$i_{r+s} - i_{s+t} \geq m_{r-t,s+t} \text{ and } i_{s+t} - i_s \geq m_{t,s}$$

then automatically

$$i_{r+s} - i_s \geq m_{r,s}\,.$$

There is in fact no loss of generality in assuming that redundant elements take the largest possible value (which does not alter the representations under definition 2). In other words (applying an extension of the above argument) we assume that

$$m_{r,s} \geq m_{r-t,s+t} + m_{t,s} \quad \text{for all } 1 \leq t < r \text{ and } 1 \leq s .$$

We extend the definition of redundancy to rows, by saying that a row of M is redundant if every element of the row is redundant. If any one element of a row is not redundant then we say that the row is non-redundant.

Next let us assume that $\{a_n\}$, M represent the integers. Then it turns out that the matrix M has only two kinds of row, namely "straight" rows like

$$(\alpha, \alpha, \alpha, \alpha, \cdots), \quad 0 \leq \alpha$$

and "bent" rows of the form

$$(\beta, \alpha, \alpha, \alpha, \cdots) \text{ where } 0 \leq \beta = \alpha - 1.$$

If either type of row is non-redundant then every element α in it is non-redundant. If a bent row is non-redundant then every succeeding row is redundant (the bent row is the last non-redundant row of M). Moreover, if a bent row is non-redundant, and its element β is non-redundant, then it is the first non-redundant row of M, and if in addition $\beta > 0$ then it is the very first row of M. It follows from these facts that if M has infinitely many non-redundant rows, then all its rows are straight.

If the row

$$(m_{r,1}, m_{r,2}, m_{r,3}, \cdots)$$

of M is non-redundant, then

$$m_r^\star \leq m_{r,1} \leq 1 + m_r^\star ,$$

where

$$m_r^* = \underset{1 \le t < r}{\text{maximum}} \left| m_{r-t,1+t} + m_{t,1} \right| .$$

This condition merely says that either $m_{r,1}$ is redundant or it lays the weakest possible new condition on the representations. Now we already know that

$$m_{r,1} \le m_{r,2} \le m_{r,1} + 1 .$$

Hence it follows that (even if $m_{r,1}$ is redundant), either $m_{r,2}$ imposes the same condition as $m_{r,1}$, or $m_{r,2}$ imposes the weakest condition on the representations, which is stronger than that imposed by $m_{r,1}$.

Satisfying the above rules in all possible ways produces all possible matrices M for which there is a sequence $\{a_n\}$ such that $\{a_n\}$, M represent the integers. For example, the first corner of any matrix which starts with $m_{11} = 2$ looks like one of the matrices in Table 3.

Table 3

$$\begin{bmatrix} 2 & 2 & 2 & \cdots \\ 4 & 4 & 4 & \cdots \\ 6 & 6 & 6 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 4 & 4 & 4 & \cdots \\ 6 & 7 & 7 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 4 & 4 & 4 & \cdots \\ 7 & 7 & 7 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 4 & 4 & 4 & \cdots \\ 7 & 8 & 8 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 & \cdots \\ 4 & 5 & 5 & \cdots \\ 6 & 7 & 7 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & \cdots \\ 7 & 7 & 7 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & \cdots \\ 7 & 8 & 8 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & \cdots \\ 8 & 8 & 8 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & \cdots \\ 8 & 9 & 9 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 3 & \cdots \\ 5 & 6 & 6 & \cdots \\ 8 & 9 & 9 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

Once the matrix $M = [m_{\mu,\nu}]$ is given, the sequence $\{a_n\}$ is determined by (9) and (10), provided $\{m_\mu\}$ is derived from M as follows. If M has no bent row then $\{m_\mu\}$ is infinite and

$$m_\mu = m_{\mu,1} \quad \text{for} \quad 1 \le \mu .$$

On the other hand if M has a first bent row, and this row is the ρ^{th} row, then $\{m_\mu\}$ is finite with ρ terms given by

$$m_\mu = m_{\mu,1} \quad \text{for } 1 \le \mu \le \rho .$$

We get a simplification of (9) and (10) in the case when M has no bent row, but it has only a finite number of non-redundant rows. In this case, if the last non-redundant row is the ρ^{th}, then $\{m_\mu\}$ is periodic with period $m_\rho = m_{\rho,1}$. Hence not only (9) and (10) hold, but we also find by subtraction that

$$(14) \qquad a_{n+1} = a_n + a_{n-m_1} + a_{n-m_2} + \cdots + a_{n-m_{\rho-1}} + a_{n-m_\rho+1} \quad \text{for } n \ge m_\rho .$$

It is easy to see how relations (9), (10), and (14) generalize the definition (6) of the $(h,k)^{th}$ Fibonacci sequence.

When we know that all rows of M after the ρ^{th} row are redundant, we usually remove them from M. Then M has order $\rho \times \infty$ instead of $\infty \times \infty$. However, the fact that M has order $\rho \times \infty$ does not necessarily imply that the ρ^{th} row is non-redundant.

The bounding sequences (12) and (13) which we found earlier are in fact the sequences $\{a_n\}$ for the matrices

$$M = \begin{bmatrix} 2 & 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & 5 & \cdots \end{bmatrix} \qquad \text{and} \qquad M = \begin{bmatrix} 2 & 2 & 2 & 2 & \cdots \\ 5 & 6 & 6 & 6 & \cdots \end{bmatrix},$$

respectively. In these cases, our constructive process of "admitting (rejecting) as many representations as possible" is equivalent to saying, "let all rows of the matrix after the 2nd be redundant." The sequence (11) corresponds to the $\infty \times \infty$ matrix

$$M = \begin{bmatrix} 2 & 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & 5 & \cdots \\ 8 & 8 & 8 & 8 & \cdots \\ 11 & 11 & 11 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

With this matrix, a vector $(i_1, i_2, \cdots, i_d)$ satisfies condition (7) iff (i) we have $i_{\nu+1} - i_\nu \geq 2$ for $1 \leq \nu < d$, and (ii) if $1 \leq \eta < \theta < d$ and

$$i_{\eta+1} - i_\eta = i_{\theta+1} - i_\theta = 2$$

then there is an integer λ such that $\eta < \lambda < \theta$ and $i_{\lambda+1} - i_\lambda \geq 4$.

It has long been known that the Fibonacci sequence $\{u_n\}$ can be obtained from Pascal's triangle. The triangle is set out on the lattice points of the first quadrant of the (x, y)-plane. Then one draws a family of equispaced parallel lines on the triangle, choosing the slope and spacing of the lines, so that the sum of all the numbers of the triangle, whose lattice points lie on the n^{th} line of the family, is the n^{th} term u_n of the sequence. In 1959, I observed that the $(h, k)^{th}$ Fibonacci sequence $\{v_n\}$ could be obtained in the same way, provided that when $h = k - 1$ the first row $(1, 1, 1, \cdots)$ of the triangle is removed from the triangle ([6] theorem 8).

Harris and Styles defined sequences by means of Pascal's triangle in [9], and discussed the properties of their sequences. Suppose Pascal's triangle lies on the lattice points of the first quadrant of the (x, y)-plane. Then for $p \geq 0$, $q > 0$ they let $u(n, p, q)$ be the sum of the n^{th} term in the first row $(1, 1, 1, \cdots)$ of the triangle and those terms of the triangle which can be reached from it by taking steps $(x, y) \to (x - p - q,\ y + q)$. When $q = 1$, we have

$$u(n, p, 1) = v_{n-p+1} \quad \text{for} \quad n = 0, \pm 1, \pm 2, \cdots$$

where $\{v_n\}$ is the $(p+1, p+1)^{th}$ Fibonacci sequence (6.1).

Now suppose that M, $\{a_n\}$ represent the integers, and that all rows of M after the ρ^{th} are redundant. Then the terms of $\{a_n\}$ can be obtained from a $\rho + 1$ dimensional Pascal's triangle. The n^{th} term of $\{a_n\}$ is the sum of all the numbers of the generalized triangle which lie on the n^{th} number of a ρ-dimensional family of equispaced parallel hyperplanes. I will give the details for $\rho = 2$ and the second row non-redundant. The reader will immediately see the result for general ρ. With slight modifications, the method can be applied to a wide class of sequences satisfying finite recurrence relations.

When $\rho = 2$ and the second row is non-redundant, the matrix M is of the form

$$M = \begin{bmatrix} \alpha & \alpha & \alpha & \alpha & \cdots \\ \beta & \gamma & \gamma & \gamma & \cdots \end{bmatrix}$$

where

$$0 \le 2\alpha \le \beta \le 2\alpha + 1 \le \gamma \quad \text{and} \quad \beta \le \gamma \le \beta + 1 .$$

The second row of M could be either straight or bent. Also the sequence $\{a_n\}$ is given by

$$\text{(15)} \qquad \begin{cases} a_n = 0 & \text{for} \quad n < \alpha^\star , \\ a_n = 1 & \text{for} \quad \alpha^\star \le n \le 1 , \\ a_{n+1} = \gamma - \beta + a_n + a_{n-\alpha} + a_{n-\gamma+1} & \text{for} \quad n \ge 1, \end{cases}$$

where $\alpha^\star = 1$ if $\beta = \gamma - 1$ but $\alpha^\star = -\alpha + 1$ if $\beta = \gamma$.

Notice that when $\alpha = 2$ and $\beta = \gamma = 5$ then we get back to the sequence (12) again.

We now define our 3-dimensional Pascal's triangle. In other words, we define an integer-valued function $\pi(x,y,z)$ on the 3-dimensional lattice by the relations

$$\pi(x,y,z) = \begin{cases} 0 \text{ if } x < 0 \text{ or } y < 0 \text{ or } z < 0, \\ 1 \text{ if } x = y = z = 0, \\ \pi(x-1,y,z) + \pi(x,y-1,z) + \pi(x,y,z-1) \text{ otherwise} \end{cases}$$

It is easy to see that Pascal's triangle appears on each of the three planes $x = 0$, $y = 0$ and $z = 0$. My result is that the n^{th} term of $\{a_n\}$ of (15) is the sum of all the values of $\pi(x,y,z)$ (whose lattice points lie) on the plane

$$x + (\alpha + 1)y + \gamma z = n + \alpha - 1 + (\gamma - \beta)\alpha$$

provided that if $\gamma = \beta - 1$ we remove the x-axis (i.e., if $\gamma = \beta - 1$, we replace π by $\pi^\star$ where $\pi^\star(x,y,z) = 0$ if $y = z = 0$ but $\pi^\star = \pi$ otherwise). The proof is by induction.

Next let r be a fixed integer $r \ge 1$. Let $(i_1, i_2, \cdots, i_d) \in W$ iff

$$i_{\nu+r} - i_\nu \geq 1$$

for $1 \leq \nu \leq d - r$, and put $b_n = (r+1)^{n-1}$ for $n \geq 1$. Then $\{b_n\}$, W represent the integers in the familiar scale of powers of $r + 1$, and the order of the terms in $\{b_n\}$ is immaterial. Suppose, on the other hand, that

$$(i_1, i_2, \cdots, i_d) \in W$$

iff

$$i_{\nu+1} - i_\nu \geq 2$$

for

$$1 \leq \nu < d,$$

and $\{b_n\}$, W represent the integers. Then as I showed in [1] axiom 1 must hold, and in fact $\{b_n\} = \{u_n\}$.

I would now like to state my monotonicity conjecture, which extends a conjecture that I made in [5].

Conjecture. Suppose $\{b_n\}$, W represent the integers and axiom 2 holds. Then either axiom 1 holds for $\{b_n\}$ or $\{b_n\}$ is $s^0, s^1, s^2, \cdots$ in some order and s is an integer $s \geq 1$.

Another result which gives weight to the conjecture is

Theorem 2. Let $r \geq 1$ be a fixed integer. Let M be the matrix whose only non-redundant row is its r^{th} row, and this r^{th} row is $(0, 1, 1, \cdots)$. If $\{b_n\}$, M represent the integers then axiom 1 holds for $\{b_n\}$. Moreover, $b_1 = 1$ and $b_{n+1} = (r+1)b_n + 1$ for $n \geq 1$.

The first example of a pair $\{a_n\}$, W which is not equivalent to a pair $\{a_n\}$, M was given by my student A. J. W. Hilton in 1963. He took a fixed integer $r \geq 4$, and let W be the set of all vectors I of V such that

$$1 \leq i_1 < i_2 < \cdots < i_d$$

and, for $1 < \nu < d$, if $i_{\nu+1} - i_\nu = 1$ then $i_\nu - i_{\nu-1} \geq r$. Then he put

$$w_n = 1 \quad \text{for} \quad n \le 1$$
$$w_2 = 2$$
$$w_n = w_{n-1} + w_{n-2} + w_{n-p} \quad \text{for} \quad n \ge 3,$$

With these definitions, we have

Theorem 3 (Hilton). $\{w_n\}$, W represent the integers but are not equivalent to any system $\{b_n\}$, M.

I have tried to find an elegant classification for all sets $\{a_n\}$, W. However, I have so far been unable to improve on the constructive method which I have described for obtaining all sets $\{a_n\}$, W.

In this paper, I have been concerned with unique representations. It would be interesting to know what happens if the uniqueness condition was dropped, and perhaps only sufficiently large numbers N had to have a representation (N > constant). Results in this direction have been found by Brown, Ferns, Hoggatt, King, and others [1], [2], [3], [7], and [8], respectively. I feel that the results I have given in this paper are complete in the same sense as N. G. de Bruijn's discussion is complete for representations

$$N = s_1 + s_2 + s_3 + \cdots \quad ,$$

where each s_i belongs to a finite or infinite set S_i of non-negative integers containing 0. In a paper [4] which is now classical, he showed that all such systems are what he called "degenerate British number systems."

Some results have been obtained concerning representing all integers in some interval by Harris, Hilton, Hoggatt, Mohanty, Styles, myself and others [6], [9]. However, the problems concerning representations for all the integers $0, \pm 1, \pm 2, \cdots$ are much more difficult and only a few special theorems are known.

REFERENCES

1. J. L. Brown, Jr., "Note on Complete Sequences of Integers," Amer. Math. Monthly, Vol. 68 (1961), pp. 557-560.
2. J. L. Brown, Jr., "Generalized Bases for the Integers," Amer. Math. Monthly, Vol. 71 (1964), pp. 973-980.

3. J. L. Brown, Jr., "A New Characterization of the Fibonacci Numbers," The Fibonacci Quarterly, Vol. 3 (1965), pp. 1-8.
4. N. G. de Bruijn, "On Number Systems," Nieuw. Archief voor Wiskunde, Vol. 3 (1956), pp. 15-17.
5. D. E. Daykin, "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers," Jour. London Math., Vol. 35 (1960), pp. 143-161.
6. D. E. Daykin and A. J. W. Hilton, "Bases for Infinite Intervals of Integers," The Fibonacci Quarterly, Vol. 5, No. 4 (1967), pp. 329-346.
7. H. H. Ferns, "On the Representation of Integers as Sums of Distinct Fibonacci Numbers," The Fibonacci Quarterly, Vol. 3 (1965), pp. 21-30.
8. R. L. Graham, "A Property of Fibonacci Numbers," The Fibonacci Quarterly, Vol. 2 (1964), pp. 1-10.
9. V. C. Harris and C. C. Styles, "A Generalization of Fibonacci Numbers," The Fibonacci Quarterly, Vol. 2 (1964), pp. 277-289.

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3. G. Gompertz, 'On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies," Phil. Trans. Roy. Soc., London, 115 (1825), 513-585.
4. J. P. Gram, Om Räkkeudviklinger Bestemte ved Hjälp af de mindste Kvadraters Methode, Kjöpenhavn (1879).
5. C. Jordan, Calculus of Finite Differences, Chelsea Pub. Co., New York, 1950.
6. T. J. Steiljes, "Tables des valeurs des Sommes $S_k = \sum_{n=1}^{\infty} n^{-k}$," Acta Mathematica, 10 (1887), 299-302.
7. G. Szegö, "Orthogonal Polynomials," Amer. Math. Soc. Colloquium Publications, 23 (1959).

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ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

H-162 Proposed by David A. Klarner, University of Alberta, Edmonton Alberta, Canada.

Suppose $a_{ij} \geq 1$ for $i, j = 1, 2, \cdots$, show there exists an $x \geq 1$ such that

$$(-1)^n \begin{vmatrix} a_{11} - x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - x^2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x^n \end{vmatrix} \leq 0$$

for all n.

H-163 Proposed by H. H. Ferns, Victoria, B. C., Canada.

Prove the following identities:

1. $$\sum_{k=1}^{n} 2^{2k-2} L_k F_{k+3} = 2^{2n} F_{n+1}^2 - 1$$

2. $$5 \sum_{k=1}^{n} 2^{2k-2} F_k L_{k+3} = 2^{2n} L_{n+1}^2 - 1 ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively.

H-164 Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.

Generalize H-127 and find a recurrence relation for the product $C_n = A_n(x)B_n(y)$, where A_n and B_n satisfy the general second-order recurrence equations:

$$A_{n+1}(x) = R(x)A_n(x) + S(x)A_{n-1}(x) \tag{1}$$

$$B_{n+1}(y) = P(y)B_n(y) + Q(y)B_{n-1}(y) , \tag{2}$$

$$n \geq 1 \quad \text{and} \quad A_0, A_1, B_0, B_1 \text{ arbitrary} .$$

H-165 Proposed by H. H. Ferns, Victoria, B. C., Canada.

Prove the identity

$$\sum_{i=1}^{n} \binom{n}{i} \frac{F_{ki}}{F_{k-2}^{i}} = \left(\frac{F_k}{F_{k-2}}\right)^n F_{2n} \qquad (k \neq 2) ,$$

where F_i denotes the i^{th} Fibonacci number.

SOLUTIONS

A BASIS OF FACT?

H-132 Proposed by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania.

Let

$$F_1 = 1, \; F_2 = 1, \; F_{n+2} = F_{n+1} + F_n \quad \text{for} \quad n > 0$$

define the Fibonacci sequence. Show that the Fibonacci sequence is not a basis of order k for any positive integer k; that is, show that not every positive integer can be represented as a sum of k Fibonacci numbers, where repetitions are allowed and k is a fixed positive integer.

Solution by the Proposer.

Assume $\{F_n\}_1^\infty$ is a basis of order k, where k is some fixed positive integer. Then, in particular, for given $n > 0$, any positive integer $r \le F_n$ would have a representation in the form

$$r = \sum_{i=1}^{k} F_{n_i} , \tag{1}$$

where $n_1 \le n_2 \le \cdots \le n_k$ and $n_k \le n$. But the maximum number of distinct integers which could be formed by the right-hand side of (1) is clearly $\le n^k$. Thus each of the F_n integers $1, 2, 3, \cdots, F_n$ would have to be expressed in a form capable of representing at most n^k distinct integers. Since, by choosing n large enough, we can make $F_n > n^k$, a contradiction is obtained for the value of k under consideration. [The inequality $F_n > n^k$ follows from the fact that F_n is approximately $a^n/\sqrt{5}$ for large n, where $a = (1 + \sqrt{5})/2$].

SUM SHINE

H-133 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Characterize the sequences

i. $$F_n = u_n + \sum_{j=1}^{n-2} u_j$$

ii. $$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j$$

iii. $$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-6} \sum_{j=1}^{m} \sum_{j=1}^{i} u_j ,$$

by finding starting values and recurrence relations. Generalize.

Solution by D. V. Jaiswal, Holkar Science College, Indore, India.

We shall first prove the iii part.

$$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-6} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j$$

$$\therefore \quad F_{n-1} = u_{n-1} + \sum_{j=1}^{n-3} u_j + \sum_{i=1}^{n-5} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-7} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j$$

$$F_{n-2} = u_{n-2} + \sum_{j=1}^{n-4} u_j + \sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-8} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j \quad .$$

Since $F_n - F_{n-1} - F_{n-2} = 0$, we have

$$0 = (u_n - u_{n-1} - u_{n-2}) + \left(u_{n-2} - \sum_{j=1}^{n-4} u_j\right)$$

$$+ \left(\sum_{j=1}^{n-4} u_j - \sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j\right) + \left(\sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j - \sum_{m=1}^{n-8} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j\right).$$

Cancelling out the terms, we get

$$u_n = u_{n-1} + \sum_{m=1}^{n-8} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j \quad .$$

(ii) Proceeding as above, we shall get

$$u_n = u_{n-1} + \sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j \quad .$$

(i) Proceeding as above we shall get

$$u_n = u_{n-1} + \sum_{j=1}^{n-4} u_j \quad .$$

Generalization. If

$$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j + \cdots +$$

$$+ \sum_{s=1}^{n-2r} \sum_{q=1}^{s} \sum_{p=1}^{q} \cdots \sum_{i=1} \sum_{j=1}^{i} u_j \qquad ,$$

(r summations)

then proceeding as above, we shall get

$$u_n = u_{n-1} + \sum_{s=1}^{n-2r-2} \sum_{q=1}^{s} \sum_{p=1}^{q} \cdots \sum_{i=1} \sum_{j=1}^{i} u_j$$

Editorial Note: Professor Hoggatt obtained the solutions:

i) $u_n = u(n;\ 2,2)$

ii) $u_n = u(n;\ 3,3)$

iii) $u_n = u(n;\ 4,4)$ where $u(n;\ p,q)$ represents the generalized Fibonacci number.

See V. C. Harris and C. C. Styles, "A Generalization of Fibonacci Numbers," The Fibonacci Quarterly, Vol. 2, No. 4, pp. 277-289.

CIRCLE TO THE RIGHT

H-134 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Evaluate the circulants

$$\begin{vmatrix} F_n & F_{n+k} & \cdots & F_{n+(m-1)k} \\ F_{n+(m-1)k} & F_n & \cdots & F_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n+k} & F_{n+2k} & \cdots & F_n \end{vmatrix}, \quad \begin{vmatrix} L_n & L_{n+k} & \cdots & L_{n+(m-1)k} \\ L_{n+(m-1)k} & L_n & \cdots & L_{n+(m-2)k} \\ \cdots & & \cdots & \cdots \\ L_{n+1} & L_{n+2k} & \cdots & L_n \end{vmatrix}$$

Solution by the Proposer.

We recall that

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ a_{m-1} & a_0 & \cdots & a_{m-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_0 \end{vmatrix} = \prod_{r=0}^{m-1} \sum_{s=0}^{m-1} a_s \omega^{rs} \qquad (\omega = e^{2\pi i/m}).$$

Hence if we put

$$\Delta_m(F) = \begin{vmatrix} F_n & F_{n+k} & \cdots & F_{n+(m-1)k} \\ F_{n+(m-1)k} & F_n & \cdots & F_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n+2k} & F_{n+2k} & \cdots & F_n \end{vmatrix},$$

$$\Delta_m(L) = \begin{vmatrix} L_n & L_{n+k} & \cdots & L_{n+(m-1)k} \\ L_{n+(m-1)k} & L_n & \cdots & L_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ L_{n+k} & L_{n+2k} & \cdots & L_n \end{vmatrix},$$

we have

$$\Delta_m(F) = \prod_{r=0}^{m-1} \sum_{s=0}^{m-1} F_{n+sk}\omega^{rs}, \quad \Delta_m(L) = \prod_{r=0}^{m-1} \sum_{s=0}^{m-1} L_{n+sk}\omega^{rs}.$$

Put

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}, \quad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n.$$

Then

$$\sum_{s=0}^{m-1} F_{n+sk}\omega^{rs} = \frac{1}{\alpha - \beta} \sum_{s=0}^{m-1} (\alpha^{n+sk} - \beta^{n+sk})\omega^{rs}$$

$$= \frac{1}{\alpha - \beta} \left\{ \alpha^n \frac{1 - \alpha^{mk}}{1 - \omega^r\alpha^k} - \beta^n \frac{1 - \beta^{mk}}{1 - \omega^r\beta^k} \right\}$$

$$= \frac{\alpha^n(1 - \alpha^{mk})(1 - \omega^r\beta^k) - \beta^n(1 - \beta^{mk})(1 - \omega^r\alpha^k)}{(\alpha - \beta)(1 - \omega^r\alpha)(1 - \omega^r\beta)}$$

$$= \frac{\alpha^n - \beta^n - (\alpha^{n+mk} - \beta^{n+mk}) - \omega^r(\alpha^n\beta^k - \alpha^k\beta^n - \alpha^{n+mk}\beta^k + \beta^{n+mk}\alpha^k)}{(\alpha - \beta)(1 - \omega^r\alpha)(1 - \omega^r\beta)}$$

$$= \frac{F_n - F_{n+mk} - (-1)^k\omega^r(F_{n-k} - F_{n+(m-1)k})}{(1 - \omega^r\alpha^k)(1 - \omega^r\beta^k)}$$

so that

$$\prod_{r=0}^{m-1}\sum_{s=0}^{m-1} F_{n+sk}\omega^{rs} = \prod_{r=0}^{m-1} \frac{F_n - F_{n+mk} - (-1)^k\omega^r(F_{n-k} - F_{n+(m-1)k})}{(1 - \omega^r\alpha^k)(1 - \omega^r\beta^k)}$$

$$= \frac{(F_n - F_{n+mk})^m - (-1)^{mk}(F_{n-k} - F_{n+(m-1)k})^m}{(1 - \alpha^{mk})(1 - \beta^{mk})}$$

Therefore

$$(\star) \qquad \Delta_m(F) = \frac{(F_n - F_{n+mk})^m - (-1)^{mk}(F_{n-k} - F_{n+(m-1)k})^m}{1 + (-1)^{mk} - L_{mk}} .$$

Similarly,

$$\sum_{s=0}^{m-1} L_{n+sk}\omega^{rs} = \sum_{s=0}^{m-1} (\alpha^{n+sk} + \beta^{n+sk})\omega^{rs}$$

$$= \alpha^n \frac{1 - \alpha^{mk}}{1 - \omega^r\alpha^k} + \beta^n \frac{1 - \beta^{mk}}{1 - \omega^r\beta^k}$$

$$= \frac{\alpha^n + \beta^n - \alpha^{n+mk} - \beta^{n+mk} - \omega^r(\alpha^n\beta^k + \beta^n\alpha^k - \alpha^{n+mk}\beta^k - \beta^{n+m}\alpha)}{(1 - \omega^r\alpha)(1 - \omega^r\beta)}$$

$$= \frac{L_n - L_{n+mk} - (-1)^k\omega^r(L_{n-k} - L_{n+(m-1)k})}{(1 - \omega^r\alpha^k)(1 - \omega^r\beta^k)} .$$

It follows that

$$(\star\star) \qquad \Delta_m(L) = \frac{(L_n - L_{n+mk})^m - (-1)^{mk}(L_{n-k} - L_{n+(m-1)k})^m}{1 + (-1)^{mk} - L_{mk}} .$$

Also solved by D. Jaiswal (India).

THE GREATEST INTEGER!

H-135 Proposed by James E. Desmond, Florida State University, Tallahassee, Florida.

PART I

Show that

$$j + 1 = \sum_{d=0}^{\left[\frac{j}{2}\right]} \binom{j - d}{d} 2^{j-2d}(-1)^d \quad ,$$

where $j \geq 0$ and $[j/2]$ is the greatest integer not exceeding $j/2$.

PART II

Show that

$$F_{(j+1)n} = F_n \sum_{d=0}^{\left[\frac{j}{2}\right]} \binom{j - d}{d} L_n^{j-2d} (-1)^{(n+1)d} \quad ,$$

where $j \geq 0$ and $[j/2]$ is the greatest integer not exceeding $j/2$.

Solution by the Proposer.

PART I

We have (see "A Generalization of the Connection between the Fibonacci Sequence and Pascal's Triangle," by Joseph A. Raab, this quarterly, Vol. 1, No. 3, October 1963, pp. 25-26) that

$$\sum_{d=0}^{\left[\frac{j}{2}\right]} \binom{j - d}{d} 2^{j-2d}(-1)^d = x_j$$

and $x_{j+2} = 2x_{j+1} - x_j$ for all $j \geq 0$. Let S be the set of all integers $(j + 1) > 0$ for which the theorem is true, $1 = x_0$ and $2 = x_1$, so 1 and 2 are in S. Suppose q and q + 1 are in S, so that $q = x_{q-1}$ and $q + 1 = x_q$. Then

$$x_{q+1} = 2x_q - x_{q-1} = 2(q+1) - q = q+2 .$$

Thus $q + 2$ is in the set S and the proof is complete by mathematical induction.

PART II

The same reference as given in Part I yields the result that

$$\sum_{d=0}^{\left[\frac{j}{2}\right]} \binom{j-d}{d} L_n^{j-2d}(-1)^{(n+1)d} = x_j$$

and

$$x_{j+2} = L_n x_{j+1} + (-1)^{n+1} x_j$$

for all $j \geq 0$. Let S be the set of all integers $(j+1) > 0$ for which the theorem is true. $F_n = F_n x_0$ and $F_{2n} = F_n L_n = F_n x_1$, so 1 and 2 are in S. Suppose q and $q+1$ are in S, so that $F_{qn} = F_n x_{q-1}$ and $F_{(q+1)n} = F_n x_q$. Then

$$F_n x_{q+1} = F_n L_n x_q + F_n(-1)^{n+1} x_{q-1} = L_n F_{(q+1n} + (-1)^{n+1} F_{qn} = F_{(q+2)n}$$

by a known identity (see "Some Fibonacci Results Using Fibonacci-Type Sequences," by I. Dale Ruggles, this quarterly, Vol. 1, No. 2, April, 1963, p. 77). Thus $q+2$ is in the set S and the proof is complete by mathematical induction.

Also solved by B. King, L. Carlitz, D. Jaiswal (India), and D. Zeitlin.

SQUEEZE PLAY

H-136 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California, and D. A. Lind, University of Virginia, Charlottesville, Va.

Let $\{H_n\}$ be defined by $H_1 = p$, $H_2 = q$, $H_{n+2} = H_{n+1} + H_n$ $(n \geq 1)$ where p and q are non-negative integers. Sh there are integers N and k such that $F_{n+k} < H_n \le F_{n+k+1}$ for all $n > N$. Does the conclusion hold if p and q are allowed to be non-negative reals instead of integers?

Solution by Gerald A. Edgar, Student, University of California, Santa Barbara, California.

In order for the result to be true, we must have $p > 0$ or $q > 0$. Let

$$a = (1 + \sqrt{5})/2, \quad b = (1 - \sqrt{5})/2 .$$

Define $f(n) = [an + 1/2]$, for n a positive integer, where $[x]$ is the greatest integer in x (thus f(n) is the nearest integer to an). We now prove that $f(f(n)) = f(n) + n$. The definition of f gives

$$\text{(1)} \qquad an + \tfrac{1}{2} \ge f(n) > an - \tfrac{1}{2}$$

$$\text{(2)} \qquad af(n) + \tfrac{1}{2} \ge f(f(n)) > af(n) - \tfrac{1}{2} .$$

But (1) is the same as

$$\frac{f(n)}{a} + \frac{1}{2a} > n \ge \frac{f(n)}{a} - \frac{1}{2a} ,$$

or, since $(1/a) = a - 1$,

$$(a - 1)f(n) + (a - 1)/2 > n \ge (a - 1)f(n) - (a - 1)/2$$

or

$$\text{(3)} \qquad af(n) + \frac{a}{2} - \frac{1}{2} > n + f(n) \ge af(n) - \frac{a}{2} + \frac{1}{2} .$$

Equations (2) and (3) give

$$\frac{a}{2} > f(n) + n - f(f(n)) \ge -\frac{a}{2} .$$

But $a/2 < 1$, and $f(n) + n - f(f(n))$ is an integer, so it must be zero, and we have

(4) $$f(f(n)) = f(n) + n .$$

Because of its recurrence, H_n must have the form $H_n = ca^n + db^n$ for some constants c and d. Now $|b| < 1$, so

$$\lim_{n\to\infty} (aH_n - H_{n+1}) = \lim_{n\to\infty} (ca^{n+1} + dab^n - ca^{n+1} - db^{n+1})$$
$$= \lim_{n\to\infty} \sqrt{5}\,db^n = 0$$

Thus there is an integer N such that $|aH_n - H_{n+1}| < \frac{1}{2}$ for all $n \geq N$. In particular, $|aH_N - H_{N+1}| < \frac{1}{2}$, so, since H_{N+1} is an integer,

$$H_{N+1} = [aH_N + \tfrac{1}{2}] = f(H_N) .$$

It is now an easy induction to show that

(5) $$H_{N+m} = f^m(H_N)$$

for $m = 0, 1, 2, \cdots$, where f^m is the m^{th} iterate of f defined by

$$f^0(x) = x$$
$$f^{n+1}(x) = f(f^n(x)) .$$

(Note that in particular, $f(F_n) = F_{n+1}$ for $n = 2, 3, \cdots$ for the Fibonacci numbers.) Since H_N is a positive integer, there is an integer k such that

(6) $$F_{N+k} < H_N \leq F_{N+k+1} .$$

We may then obtain by induction (using the fact that f is strictly increasing on the positive integers)

$$F_{n+k} < H_n \leq F_{n+k+1} ,$$

for all $n \geq N$.

The result does not hold for non-negative reals in general; take

$$p = a/\sqrt{5}, \quad q = a^2/\sqrt{5};$$

then $H_n > F_n$ when n is even and $H_n < F_n$ when n is odd.

Also solved by J. Desmond, A. Shannon, and M. Yoder.

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FIBONACCI AND THE ATOM

H. E. HUNTLEY
(Sometimes) Professor of Physics, University of Ghana

Occasions of the appearance in the natural world of the Fibonacci series and of the golden section of Greek mathematics will be known to readers of this journal. In biology references, for example, the series crops up in connection with the genealogy of the drone bee, with the Nautilus sea-shell, with the florets of compositae blossoms, and in Phyllotaxis. Its appearance in the inorganic world, however, is less frequently recorded. One example is the multiple reflection of a light ray by two sheets of glass (Vol. 1, No. 1, p. 56). Another, set out below, concerns the ideally simplified atoms of a quantity of hydrogen gas.

Suppose that the single electron in one of the atoms is initially in the ground level of energy and that it gains and loses, successively, either one or two quanta of energy, so that the electron in its history occupies either the ground level (state 0) or the first energy level (state 1) or the second energy level (state 2). In this idealized case, the number of different possible histories of an atomic electron is a Fibonacci number (diagram, p. 000).

Let us make the following assumptions:

1. When the gas gains radiant energy, all state 1 atoms rise to state 2; half state zero atoms rise to state 1 and half to state 2.

2. When the gas loses energy by radiation, all the atoms in state 1 fall to state zero; half those in state 2 fall to state 1, and half to state zero.

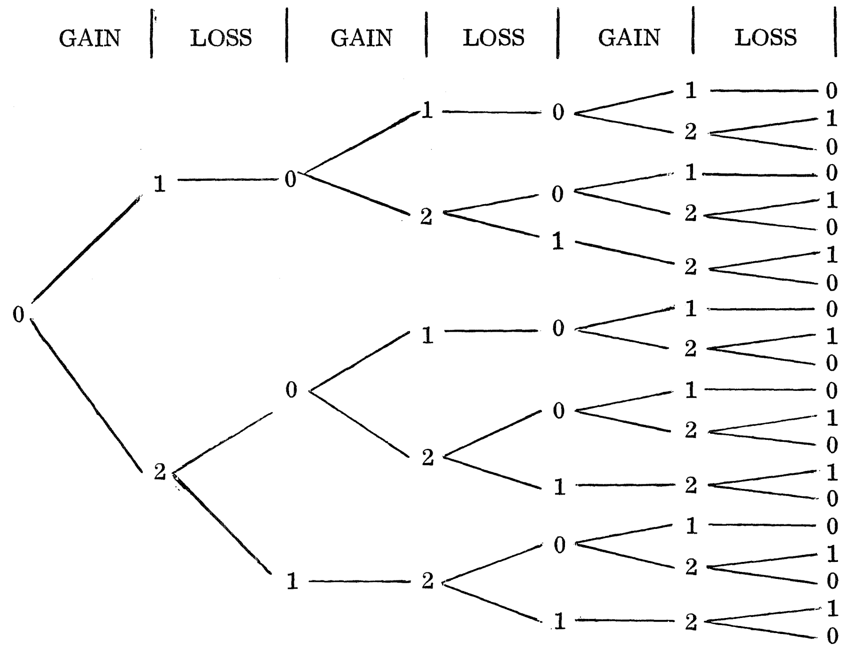
The Table shows the successive fractions of the total number of atoms found in each state. These fractions are formed exclusively of Fibonacci numbers.

A point of interest is that the fraction of atoms in the intermediate energy level (state 1) remains constant at 38.2%. If u_n is the n^{th} term of the Fibonacci series, this fraction is u_n/u_{n-1} as n tends to infinity.

$$u_n/u_{n-1} = 1 - u_{n+1}/u_{n+2} \to \varphi'^2, \quad \text{i.e.,} \quad 38.2\%.$$

The symbols φ and φ' stand for the limits of u_{n+1}/u_n and u_n/u_{n+1}, respectively as n tends to infinity. They are the roots of the equation: $x^2 - x - 1 = 0$.

POSSIBLE HISTORIES OF AN ATOMIC ELECTRON



The number of possibilities of different histories of an electron are:

		2	3	5	8	13	21 ...
Level 0	$\frac{1}{1}$	0	$\frac{2}{3}$	0	$\frac{5}{8}$	0	$\frac{13}{21} \rightarrow 0$ or $-\varphi'$
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{5}{13}$	$\frac{8}{21} \rightarrow \varphi'^2$
2		$\frac{1}{2}$	0	$\frac{3}{5}$	0	$\frac{8}{13}$	$0 \rightarrow -\varphi'$ or 0

The above fractions, showing the changing proportions of atoms in each state are formed of Fibonacci numbers.

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FIBONACCI STATISTICS IN CONIFERS

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

The Editor of the Fibonacci Quarterly has received an urgent phone call from a Houghton-Mifflin representative: "Is the picture of the pine cone in your manuscript spiralling correctly?" The thought was that possibly the negative had been turned over and so what should be steep spirals going to the left would become steep spirals going to the right. The Editor relayed the question to the Managing Editor who hurried to the basement, picked up a pine cone and found on the first try that the direction of the spirals agreed with the picture.

Another life situation. After giving a talk on Fibonacci numbers in nature or exhibiting specimens which show the spirals and Fibonacci numbers, the query naturally arises: "How constant are these numbers in nature?"

With such questions in mind, an investigation was begun in the summer of 1969. Very quickly it was discovered that spirals on cones go in both directions. For example, if we consider two particular sets of spirals, one steep and the other more gradual where the count from one intersection to the next along the spirals is eight on one spiral and five on the other, then on some cones the steep spiral goes to the right and the more gradual spiral goes to the left, while on others, it is just the reverse.

This led to the following general approach. Wherever possible cones would be studied for individual trees; approximately four hundred cones would be examined for each species. The information and results for the various species are set forth in the remainder of this article.

LODGEPOLE PINE (Pinus Murrayana), also known as Tamarack Pine

The cones on this tree are small and abundant. They were collected in the neighborhood of Huntington Lake in the middle Sierra. Because they are open and difficult to count in this state they were soaked in water to close them after which it was relatively easy to follow the spirals.

In this report and those that follow the notation 8R means that the count along the gradual spiral from one intersection of the two spirals to the next was 8, while that along the steep spiral was 5. Thus in the 8R case the gradual spiral goes to the right. It should be noted that this method of counting simply reflects the fact that there are eight steep spirals and five gradual spirals on the cone (i.e., of the spirals we are considering). NS means non-standard: it was not possible to find the 8-5 pattern on cones listed under this heading. This does not mean that in all cases there was no Fibonacci pattern: sometimes there was double a Fibonacci number, for example. But we are not interested in these deviants as such, but simply in their relative abundance.

TREE	8R	8L	NS	%8R	%8L	%NS
1	95	68	4	56.9	40.7	2.4
2	84	70	1	54.2	45.2	0.6
Various	285	282	3			
TOTAL	464	420	8	52.0	47.1	0.9

JEFFREY PINE (Pinus Jeffreyi)

These large cones were collected in the vicinity of Huntington Lake. The count was made after they were closed by soaking.

TREE	8R	8L	NS	%8R	%8L	%NS
1	42	40	1	50.6	48.2	1.2
2	21	22	1	47.7	50.0	2.3
3	38	38	13	42.7	42.7	14.6
Various	90	93	3			
TOTAL	191	193	18	47.5	48.0	4.5

SUGAR PINE (Pinus Lambertiana)

The cones were studied on the spot in the area west of Kaiser Peak in the middle Sierra region. In many cases, due to the fact that they were not closed it was not possible to determine whether they had the pattern or not. Thus these cones do not provide positive information on the presence or absence of the given pattern.

TREE	8R	8L	NS	%8R	%8L	%NS
1	25	28		47.2	52.8	
2	60	29	1	66.7	32.2	1.1
3	57	53		51.8	48.2	
Various	68	80	1			
TOTAL	210	190	2	52.2	47.3	0.5

It should be noted that this is a very regular cone and that only in the few cases noted was there positive evidence of the lack of the usual pattern.

SILVER PINE (Pinus monticola)

The cones were collected on Kaiser Ridge not far from Huntington Lake. They were soaked so as to make it possible to follow the spirals conveniently. The count was five along the gradual spiral and three along the steep spiral from one intersection to the next.

TREE	8R	8L	NS	%8R	%8L	%NS
1	15	16		48.4	51.6	
2	26	33		44.1	55.9	
3	48	56		46.2	53.8	
4	64	65	5	47.8	48.5	3.7
Various	65	56				
TOTAL	218	226	5	48.6	50.3	1.1

YELLOW PINE (Pinus ponderosa)

The cones were collected in the middle Sierra between Auberry and Pine Ridge. They were soaked before the cones were examined.

TREE	8R	8L	NS	%8R	%8L	%NS
1	44	52		45.8	54.2	S1
2	59	46	2	55.1	43.0	1.9
3	74	37		66.7	33.3	
4	35	58		37.6	62.4	
Various	3	19				
TOTAL	215	212	2	50.1	49.4	0.5

ONE-NEEDLED PINYON (Pinus monophylla)

There is a fine stand of these trees about eight miles from Tioga Pass on the east side of the Sierra. In one notable case it was possible to study 140 fresh cones on the tree. Cones picked up from the ground were soaked before the count was made. The best way to study this cone is when it is fresh and green. Often after closing with water the old cones tend to retain some of their irregularities.

(See table on next page.)

TREE	5R	5L	NS	%5R	% 5L	% NS
1	6	8		42.9	57.1	
2	2	5		28.6	71.4	
3	67	72	1	47.9	51.4	0.7
4	18	18		50.0	50.0	
Various	107	97	4			
TOTAL	200	200	5	49.4	49.4	1.2

FOXTAIL PINE (Pinus Balfouriana)

Some forty miles south of Bishop is the town of Independence. Thirteen miles west of this township at over 9,000 ft. is a spot known as Onion Valley. It was there that specimens of foxtail pine cones were collected. They were soaked before the count was made and hence this species provides evidence of exceptions to the regular pattern.

	5R	5L	NS	%5R	%5L	%NS
TOTAL	212	212	36	46.1	46.1	7.8

LIMBER PINE (Pinus flexilis)

These cones were examined on the spot at Onion Valley. Cones in which the pattern could not be discerned were simply not considered and hence these statistics give no evidence regarding exceptions.

	5R	5L	%5R	%5L
TOTAL	226	182	55.4	44.6

BRISTLECONE PINE (Pinus aristata)

About fifteen miles below Bishop just above Big Pine there is a turnoff leading to the Ancient Bristlecone Pine Area. Since this is a protected area under the control of the Forest Service, it was necessary to study the cones on the spot. Those on which the pattern could not be discerned were eliminated from consideration and hence the following statistics give no evidence regarding possible exceptions.

(See table on next page.)

TREE	8R	8L	%8R	%8L
1	27	23	54.0	46.0
2	3	11	21.4	78.6
3	24	18	57.1	42.9
4	5	5	50.0	50.0
5	13	23	36.1	63.9
6	9	7	56.3	43.7
Various	93	88		
TOTAL	174	175	49.9	50.0

DIGGER PINE (Pinus Sabiniana)

About 70 of these cones were found near Auberry, 30 on Mt. Diablo, and approximately 225 on Mt. Hamilton. These are very large cones and it would have been quite difficult to collect them, soak them, and thus arrive at positive evidence regarding exceptions. Hence they were counted on the spot, the uncountable specimens not being given consideration.

TREE	8R	8L	%8R	%8L
1	6	10	37.5	62.5
2	7	15	31.8	68.2
3	3	11	21.4	78.6
4	12	19	38.7	61.3
Various	135	121		
TOTAL	163	176	48.1	51.5

COUNTER PINE (Pinus Coulteri)

About fifty of these cones were examined on Mt. Diablo and the rest on Mt. Hamilton and its vicinity. Again, these cones are very large and it would have been quite a problem to collect them, soak them, and thus get positive evidence regarding deviations from the usual pattern.

TREE	8R	8L	%8R	%8L
1	9	6	60.0	40.0
Various	159	152		
TOTAL	168	158	51.5	48.5

KNOBCONE PINE (Pinus attenuata)

About 140 of these cones were examined at St. Mary's College (Contra Costa County) and the rest were found west of Redding. Most of the cones were countable but those that were open were collected and soaked before the count was made. Thus the fact that there are no exceptions to the pattern is significant in this case.

TREE	8R	8L	%8R	%8L
1	3	19	13.6	86.4
2	4	5	44.4	55.6
3	22	10	68.8	31.2
4	14	8	63.6	36.4
5	6	47	11.3	88.7
6	9	1	90.0	10.0
7	13	8	61.9	38.1
8	8	4	66.7	33.3
9	4	6	40.0	60.0
10	7	10	41.2	58.8
11	10	33	23.3	76.7
12	9	10	47.4	52.6
Various	66	89		
TOTAL	175	250	41.2	58.8

MONTEREY PINE (Pinus radiata)

With the exception of about 20 cones examined at St. Mary's College, the rest were collected on Grizzly Peak (near Berkeley). Where necessary, the cones were soaked so that these statistics provide information regarding exceptions to the usually observed pattern.

TREE	8R	8L	NS	%8R	%8L	%NS
1	5	5		50.0	50.0	
2	13	8		61.9	38.1	
3	20	23	5	41.7	47.9	10.4
Various	148	132	2			
TOTAL	186	168	7	51.5	46.5	1.9

BISHOP PINE

Cones were studied on trees north of Port Ross. The fact that there are no exceptions to the pattern is significant.

TREE	8R	8L	%8R	%8L
1	21	9	70.0	30.0
2	22	16	57.9	42.1
3	2	15	11.8	88.2
4	15	7	68.2	31.8
5	50	32	61.0	39.0
Various	3	1		
TOTAL	113	80	58.5	41.4

BEACH PINE (Pinus contorta)

The trees studied were found on Albion Ridge some ten miles south of Fort Bragg. The fact that there are no exceptions to the standard patterns is significant.

TREE	8R	8L	%8R	%8L
1	28	21	57.1	42.9
2	26	27	49.1	50.9
3	13	23	36.1	63.9
4	23	23	50.0	50.0
Various	68	22		
TOTAL	158	116	57.7	42.3

DOUGLAS FIR (Pseudotsuga Menziesii according to Munz)

Cones were collected from trees near St. Helena in Napa County.

TREE	5R	5L	NS	%5R	%5L	%NS
1	54	45		54.5	45.5	
2	38	42	1	46.9	51.9	1.2
3	31	60		34.1	65.9	
Various	47	40				
TOTAL	170	187	1	47.5	52.2	0.3

CONCLUSIONS AND OBSERVATIONS

On the basis of this very qualitative investigation it is possible to arrive at a few conclusions and observations.

1. Overall for most species of pines there appears to be a tendency to have about as many right as left spiral cones. (Right and left are purely relative to the definition of this investigation.) Overall for the pine cones (apart from Douglas fir) considered in this study, of the 8R-8L groups, 50.9% were 8R and 49.1% 8L; of the 5R-5L groups, 51.1% were 5R and 48.9% were 5L. (The exceptional cases are not included in these percentages.)

2. For any tree with ten or more cones, there were always both right and left spiral cones.

3. If the probability of right and left spiral cones is approximately 50%, then there were certain trees which seemed to deviate from this figure to a marked degree. Apparently they had a tendency to produce more of one type than the other.

4. For lodgepole, Jeffrey, silver, yellow, one-needled pinyon, foxtail, knobcone, Monterey, bishop and beach pines, there were 74 exceptional cones out of 4290 which is 1.7%. If the foxtail pine is eliminated, the percent drops to 1.0. Note also that knobcone, bishop and beach pines showed no exceptions for the samples considered.

5. Occasionally one finds a tree with a high percent of cones which deviate from the usual pattern.

* * * * *

LINEAR RECURSION RELATIONS — LESSON SIX
COMBINING LINEAR RECURSION RELATIONS

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Suppose we have two sequences $P_i(1, 5, 25, 125, 625, 3125, \cdots)$ with a recursion relation:

$$(1) \qquad P_{n+1} = 5P_n \;,$$

and $Q_i(3, 10, 13, 23, 36, 59, \cdots)$, A a Fibonacci sequence with recursion relation:

$$(2) \qquad Q_{n+1} = Q_n + Q_{n-1} \;.$$

Let

$$(3) \qquad T_n = P_n + Q_n \;.$$

What is the recursion relation of T_n and how can it be conveniently obtained from the recursion relations of P_n and Q_n?

Proceeding in a straightforward manner, we could first eliminate P_n as follows:

$$T_{n+1} = P_{n+1} + Q_{n+1}$$

$$5T_n = 5P_n + 5Q_n \;.$$

Subtracting and using relation (1),

$$T_{n+1} - 5T_n = Q_{n+1} - 5Q_n \;.$$

We can proceed likewise for Q. ,Thus

$$\begin{aligned} T_{n+1} - 5T_n &= Q_{n+1} - 5Q_n \\ T_n - 5T_{n-1} &= Q_n - 5Q_{n-1} \\ T_{n-1} - 5T_{n-2} &= Q_{n-1} - 5Q_{n-2} \quad . \end{aligned}$$

Now subtract the sum of the last two equations from the first and use relation (2). The result is:

$$T_{n+1} - 6T_n + 4T_{n-1} + 5T_{n-2} = 0 \quad ,$$

a recursion relation involving only T_i.

A much simpler approach is by means of an operator E, such that

(3) $$(E)T_n = T_{n+1}.$$

The effect of E is to increase the subscript by 1. A relation

$$P_{n+1} - 5P_n = 0 \quad ,$$

can be written

$$(E - 5)P_n = 0,$$

and a relation

$$Q_{n+1} - Q_n - Q_{n-1} = 0 \quad ,$$

can be written

$$(E^2 - E - 1)Q_{n-1} = 0 \quad .$$

It is not difficult to convince oneself that these operators obey the usual algebraic laws. As a result, if

$$T_n = P_n + Q_n \quad ,$$

$(E - 5)(E^2 - E - 1)T_n = (E - 5)(E^2 - E - 1)P_n + (E - 5)(E^2 - E - 1)Q_n$. But $(E - 5)P_n = 0$ and $(E^2 - E - 1)Q_n = 0$, so that

$$(E - 5)(E^2 - E - 1)T_n = 0$$

or

$$(E^3 - 6E^2 + 4E + 5)T_n = 0 \ ,$$

which is valent to the recursion relation

$$T_{n+3} = 6T_{n+2} - 4T_{n+1} - 5T_n \ .$$

In general, if we have linear operators such that:

$$f(E)P_n = 0 \qquad \text{and} \qquad g(E)Q_n = 0 \qquad \text{and} \qquad T_n = AP_n + BQ_n,$$

where A and B are constants, then

$$f(E)g(E)T_n = Af(E)g(E)P_n + Bf(E)g(E)Q_n = 0 \ ,$$

since $f(E)P_n = 0$, and $g(E)Q_n = 0$. Thus when T_n is the sum of terms of two sequences with different recursion relations, the recursion relation for T_n is found by multiplying T_n by the two recursion operators for the two sequences.

Example. What is the recursion relation for $T_n = 2 \times 5^n + n^2 - n + 4$? The recursion relation for 2×5^n is $(E - 5)P_n = 0$, and that for $n^2 - 4 + 4$ is $(E^3 - 3E^2 + 3E - 1)Q_n = 0$. Thus the recursion relation for the given sequence is

$$(E - 5)(E^3 - 3E^2 + 3E - 1)T_n = 0,$$

which is equivalent to:

$$T_{n+4} = 8T_{n+3} - 18T_{n+2} + 16T_{n+1} - 5T_n.$$

Example. Find the recursion relation corresponding to T_n if

$$P_{n+1} = P_n + P_{n-1} + P_{n-2} \quad \text{and} \quad Q_n = 3n^2 - 4n + 5 \quad \text{and} \quad T_n = P_n + Q_n .$$

The operator expressions for these recursion relations are:

$$(E^3 - E^2 - E - 1)P_{n-2} = 0 \quad \text{and} \quad (E^3 - 3E^2 + 3E - 1)Q_{n-2} = 0 .$$

Thus the recursion relation for T_n is:

$$(E^3 - E^2 - E - 1)(E^3 - 3E^2 + 3E - 1)T_n = 0 ,$$

which is equivalent to:

$$T_{n+6} = 4T_{n+5} - 5T_{n+4} + 2T_{n+3} - T_{n+2} + 2T_{n+1} - T_n .$$

It may be noted that two apprently different recursion relations may conceal the fact that they embody partly identical recursion relations. For example, if

$$\begin{aligned} P_n &= 4P_{n-1} - 3P_{n-2} - 2P_{n-3} + P_{n-4} \\ Q_n &= 3Q_{n-1} - 2Q_{n-2} - Q_{n-3} + Q_{n-4} , \end{aligned}$$

and we proceed directly to find the recursion operator and corresponding recursion relation for $T_n = P_n + Q_n$, we arrive at a recursion relation of order eight. However, in factored form, we have:

$$(E^2 - E - 1)(E^2 - 3E + 1)P_{n-4} = 0 ,$$

and

$$(E^2 - E - 1)(E^2 - 2E + 1)Q_{n-4} = 0 .$$

The recursion relation for T_n in simpler form would thus be:

$$(E^2 - E - 1)(E^2 - 3E + 1)(E^2 - 2E + 1)T_n = 0,$$

which is only of order six.

If the terms of the two sequences are given explicitly, a slightly different but equivalent procedure using the auxiliary equation is possible. Thus if

$$P_n = 5n + 2 + 2 \times 3^n + F_n$$
$$Q_n = n^2 - 3n + 5 - 6 \times 2^n + L_n \ ,$$

the roots of the auxiliary equation for P_n are 1, 1, 3, r, and s, while those of the auxiliary equation for Q_n are 1, 1, 1, 2, r, s. Hence the roots for the auxiliary equation of T_n would be 1, 1, 1, 2, 3, r, s, where r and s are the roots of the equation $x^2 - x - 1 = 0$. Thus the auxiliary equation for T_n would be:

$$(x - 1)^3(x - 2)(x^2 - x - 1) = 0$$

which leads equivalently to the recursion relation

$$T_{n+7} = 9T_{n+6} - 31T_{n+5} + 50T_{n+4} - 33T_{n+3} - 5T_{n+2} + 17T_{n+1} - 6T_n \,.$$

PROBLEMS

1. If P_n is the geometric progression 3, 15, 75, 375, 1875, $\cdots$ and

$$Q_n = 5F_n + 2(-1)^n \ ,$$

what is the recursion relation for $T_n = P_n + Q_n$?

2. Given recursion relations

$$P_{n+1} = 4P_n - P_{n-1} - 6P_{n-2} \quad \text{and} \quad Q_{n+1} = 6Q_n - 10Q_{n-1} + Q_{n-2} + 6Q_{n-3} \,,$$

with $T_{n+1} = P_{n+1} + Q_{n+1}$, determine the recursion relation of lowest order satisfied by T_{n+1}.

3. Determine the recursion relation for $T_n = P_n + Q_n$ where P_n is the arithmetic progression 3, 7, 11, 15, 19, $\cdots$ and Q_n is the geometric progression 2, 6, 18, 54, $\cdots$.

4. Determine the recursion relation for $T_n = 2^n + F_n^2$ given that the recursion relation for F_n^2 is

$$F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \ .$$

5. Determine the recursion relation for $T_n = 5L_n^2 + (-1)^{n-1} + 4F_n$.

(See page 544 for solutions to problems.)

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A SHORTER PROOF

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In his article (April 1967) on 1967 as the sum of squares, Brother Brousseau proves that 1967 is not the sum of three squares. This fact can be proved more briefly as follows:

If $1967 = a^2 + b^2 + c^2$, where a, b and c are positive integers, then, as Brother Brousseau has shown, a, b and c are all odd. Then $a = 2x + 1$, $b = 2y + 1$, and $c = 2z + 1$, where x, y and z are integers.

Consequently,

$$\begin{aligned} 1967 &= (2x + 1)^2 + (2y + 1)^2 + (2z + 1)^2 \\ &= 4x^2 + 4x + 4y^2 + 4y + 4z^2 + 4z + 3. \end{aligned}$$

Then

$$1964 = 4x^2 + 4x + 4y^2 + 4y + 4z^2 + 4z \ .$$

Dividing by 4, we get

$$491 = x^2 + x + y^2 + y + z^2 + z \ ,$$

[Continued on page 551.]

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A FIBONACCI MATRIX AND THE PERMANENT FUNCTION

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The permanent of an n-square matrix $[a_{ij}]$ is defined to be

$$\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{ij_i} ,$$

where

$$\sigma = (j_1, j_2, \cdots, j_n)^{\star}$$

is a member of the symmetric group S_n of permutations on n distinct objects. For example, the permanent of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is

$$a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31} .$$

This is similar to the definition of the determinant of $[a_{ij}]$, which is

$$\sum_{\sigma \in S_n} \epsilon_\sigma \prod_{i=1}^{n} a_{ij_i} ,$$

where ϵ_σ is 1 or -1 depending upon whether σ is an even or an odd permutation.

There are other similarities between the permanent and the determinant functions, among them:

(a) interchanging two rows, or two columns, of a matrix changes the sign of the determinant — but it does not change the permanent at all. Thus, the permanent of a matrix remains invariant under arbitrary permutations of its rows and columns; and

*In this notation, $(j_1, j_2, \cdots, j_n)$ is an abbreviation for the permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$.

(b) there is a Laplace expansion for the permanent of a matrix as well as for the determinant. In particular, there is a row or column expansion for the permanent. For example, if we use "per $[a_{ij}]$" for the permanent of the matrix $[a_{ij}]$, then expansion along the first column yields that

$$\operatorname{per}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\operatorname{per}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{21}\operatorname{per}\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{31}\operatorname{per}\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}.$$

For further information on properties of the permanent, the reader should see [1, p. 578] and [3, pp. 25-26].

Unfortunately, one of the most useful properties of the determinant — its invariance under the addition of a multiple of a row (or column) to another row (or column) — is false for the permanent function. As a result, evaluating the permanent of a matrix is, generally, a much more difficult problem than evaluating the corresponding determinant.

Let P_n be the n-square matrix whose entries are all 0, except that each entry along the first diagonal above the main diagonal is equal to 1, and the entry in the n^{th} row and first column also is 1. (P_n is a "permutation matrix.") For example,

$$P_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The reader can verify that

$$P_5^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$P_5^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

We now define the matrix $Q(n,r)$ to be

$$\sum_{j=1}^{r} P_n^j .$$

For example,

$$Q(5,2) = P_5 + P_5^2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

It is not difficult to see that $\operatorname{per} Q(n,1) = \operatorname{per} P_n = 1$, $\operatorname{per} Q(n,2) = 2$, and that $\operatorname{per} Q(n,n) = n!$. It has been shown [2] that

$$\text{(1)} \qquad \operatorname{per} Q(n,3) = 2 + \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n .$$

The strategy used in the derivation of (1) was to use techniques for the solution of a linear difference equation on a certain recurrence involving $\operatorname{per} Q(n,3)$. There are, also, expressions available for $\operatorname{per} Q(n,4)$, $\operatorname{per} Q(n,n-1)$ and $\operatorname{per} Q(n,n-2)$. (See [3], [3, pp. 22-28] and [3, pp. 31-35], respectively.) However, $\operatorname{per} Q(n,r)$ has not been determined for $5 \le r \le n-3$. The objectives of this paper are to use a "Fibonacci matrix" to derive (1), and to derive an explicit expression for $\operatorname{per} Q(n,3)$ other than that provided by (1). (By a "Fibonacci matrix" we mean a matrix M_n for which $M_n = \operatorname{per} M_{n-1} + \operatorname{per} M_{n-2}$.)

Let F_n be the matrix $[f_{ij}]$, where $f_{ij} = 1$ if $|i-j| \le 1$ and $f_{ij} = 0$ otherwise. Then, by starting with an expansion along the first column, we find that F_n is a Fibonacci matrix.* Since $\operatorname{per} F_2 = 2$ and $\operatorname{per} F_3 = 3$, $\operatorname{per} F_n$ yields the $(n+1)^{\text{th}}$ term of the Fibonacci sequence 1, 1, 2, 3, 5, ⋯. It is well known that the n^{th} Fibonacci number is given by

*There are other Fibonacci matrices. See problem E1553 in the 1962 volume of the <u>American Mathematical Monthly</u>, for example.

$$\frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} .$$

This, it follows that

$$\text{per } F_n = \frac{(1 + \sqrt{5})^{n+1} - (1 - \sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} . \tag{2}$$

It is not quite as well known that the n^{th} Fibonacci number is also given by

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n - k - 1}{k} ,$$

where

$$\left[\frac{n - 1}{2}\right]$$

is the greatest integer in

$$\frac{n - 1}{2} .$$

(See [4, pp. 13–14] for a proof.) From this it follows that

$$\text{per } F_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n - k}{k} . \tag{3}$$

Now let $U_n(i, j)$ be the n-square matrix all of whose entries are 0 except the entry in row i and column j, which is 1. If we let $R_n = F_n + U_n(n, 1)$, by expansion along the first row of R we find that

$$\text{per } R_n = \text{per } F_{n-1} + \text{per}\,[F_{n-1} - U_{n-1}(2,1) + U_{n-1}(n-1,1)]\,.$$

But, by expanding along the first column,

$$\text{per}\,[F_{n-1} - U_{n-1}(2,1) + U_{n-1}(n-1,1)] = \text{per } F_{n-2} + 1\,. \tag{4}$$

Thus,

$$\text{per } R_n = \text{per } F_{n-1} + \text{per } F_{n-2} + 1 = 1 + \text{per } F_n\,.$$

If we now let $S_n = R_n + U_n(1,n)$, by expansion along the first row of S_n we find that

$$\begin{aligned} \text{per } S_n = \text{per } F_{n-1} + \text{per}\,&[F_{n-1} - U_{n-1}(2,1) + U_{n-1}(n-1,1)] \\ &+ \text{per}\,[Q(n-1,2) - U_{n-1}(n-2,1) \\ &- U_{n-1}(n-1,2) + P_{n-1}^{n-1}]. \end{aligned} \tag{5}$$

If we substitute from (4) and use Z for the matrix of the third term of the right member of (5), we have

$$\begin{aligned} \text{per } S_n &= \text{per } F_{n-1} + \text{per } F_{n-2} + 1 + \text{per } Z \\ &= \text{per } F_n + 1 + \text{per } Z. \end{aligned}$$

Now expand Z along its first column to obtain $\text{per } Z = 1 + \text{per } F_{n-2}$. Then

$$\text{per } S_n = 2 + \text{per } F_n + \text{per } F_{n-2}\,.$$

Since $\text{per } S_n = \text{per } Q(n,3)$ (because S_n can be obtained from $Q(n,3)$ by a permutation of columns), it follows that

$$\text{per } Q(n,3) = 2 + \text{per } F_n + \text{per } F_{n-2}\,.$$

By using (2), we obtain an expression for $\text{per } Q(n,3)$ which reduces to that given by Minc in [1]. By using (3), we obtain:

$$\operatorname{per} Q(n,3) = 2 + \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} + \sum_{k=0}^{\left[\frac{n-2}{2}\right]} \binom{n-k-2}{k} .$$

REFERENCES

1. Marvin Marcus and Henryk Minc, "Permanents," The American Mathematical Monthly, Vol. 72 (1965), No. 6, pp. 477-591.
2. Henryk Minc, "Permanents of (0,1)-Circulants," Canadian Mathematical Bulletin, Vol. 7 (1964), pp. 253-263.
3. Herbert J. Ryser, Combinatorial Mathematics, MAA Carus Monograph No. 14, J. Wiley and Sons, New York, 1963.
4. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell Publishing Company, New York, 1961.

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SOLUTIONS TO PROBLEMS

1. $$T_{n+1} = 5T_n + 2T_{n-1} - 9T_{n-2} - 5T_{n-3} .$$

2. $$T_{n+1} = 5T_n - 4T_{n-1} - 9T_{n-2} + 7T_{n-3} + 6T_{n-4} .$$

3. $$T_{n+1} = 5T_n - 7T_{n-1} + 3T_{n-2} .$$

4. $$T_{n+4} = 4T_{n+3} - 2T_{n+2} - 5T_{n+1} + 2T_n .$$

5. $$T_{n+6} = 2T_{n+5} + 4T_{n+4} - 4T_{n+3} - 6T_{n+2} + T_n .$$

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-172 Proposed by Gloria C. Padilla, Albuquerque High School, Albuquerque, New Mexico.

Let $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$ for $n = 0, 1, \cdots$. Show that

$$F_{n+2}^3 = F_n^3 + F_{n+1}^3 + 3F_nF_{n+1}F_{n+2} .$$

B-173 Proposed by Gloria C. Padilla, Albuquerque High School, Albuquerque, New Mexico.

Show that

$$F_{3n} = F_{n+2}^3 - F_{n-1}^3 - 3F_nF_{n+1}F_{n+2} .$$

B-174 Proposed by Mel Most, Ridgefield Park, New Jersey.

Let a be a non-negative integer. Show that in the sequence

$$2F_{a+1},\quad 2^2F_{a+2},\quad 2^3F_{a+3},\ \cdots$$

all differences between successive terms must end in the same digit.

B-175 Composed from the Solution by David Zeitlin to B-155.

Let r and q be constants and let $U_0 = 0$, $U_1 = 1$, $U_{n+2} = rU_{n+1} - qU_n$. Show that

$$U_{n+a}U_{n+b} - U_{n+a+b}U_n = q^nU_aU_b .$$

B-176 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. M.

Let $\begin{bmatrix} n \\ r \end{bmatrix}$ denote the Fibonomial Coefficient

$$F_nF_{n-1}\cdots F_{n-r+1}/F_1F_2\cdots F_r .$$

Show that

$$F_n^3 = \begin{bmatrix} n+2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} n+1 \\ 3 \end{bmatrix} - \begin{bmatrix} n \\ 3 \end{bmatrix} .$$

B-177 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. M.

Using the notation of B-176, show that

$$F_n^4 = \begin{bmatrix} n+3 \\ 4 \end{bmatrix} - a\begin{bmatrix} n+2 \\ 4 \end{bmatrix} - a\begin{bmatrix} n+1 \\ 4 \end{bmatrix} + \begin{bmatrix} n \\ 4 \end{bmatrix} ,$$

for some integer a and find a.

SOLUTIONS

A VERY MAGIC SQUARE

B-154 Proposed by S. H. L. Kung, Jacksonville University, Jacksonville, Fla.

What is special about the following "magic" square?

$$\begin{bmatrix} 11 & 2 & 14 & 19 & 21 \\ 8 & 13 & 3 & 22 & 1 \\ 20 & 17 & 15 & 6 & 9 \\ 7 & 24 & 18 & 10 & 12 \\ 25 & 5 & 23 & 16 & 4 \end{bmatrix}$$

Solution by the Proposer.

(a) The sum of all the numbers contained in a row, or in a column, or in a diagonal is a prime.

(b) The sum of the squares of all numbers contained in a row, column, or diagonal is also a prime.

Solvers Guy A. Guillottee (Quebec, Canada) and Michael Yoder listed observation (a) above.

A PELL NUMBERS IDENTITY

B-155 Composite of proposals by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada, and Carol Anne Vespe, University of New Mexico, Albuquerque, New Mexico.

Let the n^{th} Pell number be defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. Show that

$$P_{n+a}P_{n+b} - P_{n+a+b}P_n = (-1)^n P_a P_b \ .$$

Solution by Wray G. Brady, University of Bridgeport, Bridgeport, Conn.

One finds that

$$P_n = \frac{r^n - s^n}{2\sqrt{2}} \ ,$$

where $r = 1 + \sqrt{2}$, $s = 1 - \sqrt{2}$, and $rs = -1$. Then

$$\begin{aligned} 8(P_{n+a}P_{n+b} - P_{n+a+b}P_n) &= r^{2n+ab} - r^{n+b}s^{n+a} - r^{n+a}s^{n+b} + s^{2n+a+b} \\ &\quad - (r^{2n+a+b} - r^{n+a+b}s^n - r^n s^{n+a+b} + s^{2n+a+b}) \\ &= (-1)^n(r^{a+b} + s^{a+b} - r^a s^b - r^b s^a) \\ &= (-1)^n(r^a - s^a)(r^b - s^b) = 8P_aP_b(-1)^n \ , \end{aligned}$$

and the desired result follows.

EDITORIAL NOTE: Let $f_a(x) = 0$, $f_1(x) = 1$ and $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$; then $f_n(2) = P_n$.

Also solved by Herta T. Freitag, Guy A. Guillottee (Quebec, Canada), Serge Hamelin (Quebec, Canada), Bruce W. King, C. B. A. Peck, A. G. Shannon (Boroko, T.P.N.G.), Gregory Wulczyn, Michael Yoder, David Zeitlin, and the Proposers.

PERIODIC REMAINDERS

B-156 Proposed by Allan Scott, Phoenix, Arizona.

Let F_n be the n^{th} Fibonacci number, $G_n = F_{4n} - 2n$, and H_n be the remainder when G_n is divided by 10.

(a) Show that the sequence $H_2, H_3, H_4, \cdots$ is periodic and find the repeating block.

(b) The last two digits of G_9 and G_{14} give Fibonacci numbers 34 and 89, respectively. Are there any other cases?

Solution by Herta T. Freitag, Hollins, Virginia.

(a) Since $F_1 \equiv 1 \pmod{10}, \cdots, F_{59} \equiv 1 \pmod{10}$, $F_{60} \equiv 0$ we have $F_1 \equiv F_{1+4 \cdot 15k} \pmod{10}$, where k is any positive. Also $G_n = F_{4n} - 2n \equiv H_n \pmod{10}$ and $2n \equiv 2, 4, 6, 8, 0 \pmod{10}$ for $n \equiv 1, 2, 3, 4, 0 \pmod 5$. Thus the repeating block of 15 terms $H_1, H_2, \cdots, H_{15}$ is 1, 7, 8, 9, 5, 6, 7, 3, 4, 5, 1, 2, 3, 9, 0 and $H_i = H_{i+15k}$ for integers i and k.

(b) By studying the corresponding pattern modulo 100 we detect another periodicity cycle such that all Fibonacci numbers smaller than 100 must occur within the last two digits of G_n provided $1 \le n \le 150$. More explicitly, we indicate the G corresponding to a given F in the following table:

Subscript on F	1 or 2	3	4	5	6
Subscript on G	1 or 26	87	118 or 143	55 or 80	48

Subscript on F	7	8	9	10	11
Subscript on G	103 or 128	121 or 146	9	5 or 130	14 or 139

Also solved by Serge Hamelin (Quebec, Canada), C. B. A. Peck, and the Proposer.

A TELESCOPING SUM

B-157 Proposed by Klaus Gunther Recke, University of Gottingen, Germany.

Let F_n be the n^{th} Fibonacci number and $\{g_n\}$ any sequence. Show that

$$\sum_{k=1}^{n} (g_{k+2} + g_{k+1} - g_k)F_k = g_{n+2}F_n + g_{n+1}F_{n+1} - g_1 .$$

Solution by John E. Homer, Jr., Union Carbide Corporation, Chicago, Illinois.

The sum is equivalent to

$$\sum_{k=3}^{n} g_k(F_{k-2} + F_{k-1} - F_k) + g_{n+2}F_n + g_{n+1}(F_n + F_{n-1}) + g_2F_1 - g_2F_2 - g_1F_1$$

$$= g_{n+2}F_n + g_{n+1}F_{n+1} - g_1,$$

since

$$F_{k-2} + F_{k-1} - F_k = 0 .$$

Also solved by Wray G. Brady, Herta T. Freitag, Serge Hamelin (Quebec, Canada), Bruce W. King, Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T.P.N.G.), Michael Yoder, David Zeitlin, and the Proposer.

ANOTHER TELESCOPING SUM

B-158 Proposed by Klaus Günther Recke, University of Gottingen, Germany.

Show that

$$\sum_{k=1}^{n} (kF_k)^2 = [(n^2 + n + 2)F_{n+2}^2 - (n^2 + 3n + 2)F_{n+1}^2 - (n^2 + 3n + 4)F_n^2]/2 .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Let H_n satisfy $H_{n+2} = H_{n+1} + H_n$. Noting that

$$H_{n+3}^2 = 2H_{n+2}^2 + 2H_{n+1}^2 - H_n^2 ,$$

it is easy to show, using mathematical induction, that

$$2\sum_{k=1}^{n} k^2H_k^2 = (n^2 + n + 2)H_{n+2}^2 - (n^2 + 3n + 2)H_{n+1}^2 - (n^2 + 3n + 4)H_n^2 + C,$$

where

$$C = 6H_1^2 + 2H_2^2 - 8H_1H_2 .$$

We note that $C = 0$ when $H_k = F_k$ or $H_k = L_k$.

Also solved by Herta T. Freitag, Serge Hamelin (Quebec, Canada), John E. Homer, Jr., Bruce W. King, Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T.P.N.G.), Michael Yoder, and the Proposer.

THE EULER TOTIENT

B-159 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let T_n be the n^{th} triangular number $n(n+1)/2$ and let $\varphi(n)$ be the Euler totient. Show that

$$\varphi(n) \mid \varphi(T_n)$$

for $n = 1, 2, \cdots$.

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, N. M.

We assume it is known that $\varphi(ab) = \varphi(a)\varphi(b)$ if $(a,b) = 1$; $\varphi(n)$ is even if $n > 2$; and $\varphi(2^k)$ if $k \geq 1$. Let $n = 2^k s$, where $2 \nmid s$. The proof is in three cases.

<u>Case 1.</u> $k = 0$. Then

$$\varphi(T_n) = \varphi\left(n \cdot \frac{n+1}{2}\right) = \varphi(n)\varphi\left(\frac{n+1}{2}\right),$$

since

$$\left(n, \frac{n+1}{2}\right) = 1 .$$

Case 2. $k = 1$. Note that

$$\varphi(n) = \varphi(2)\varphi(s) = \varphi(s) = \varphi\left(\frac{n}{2}\right) ,$$

so

$$\varphi(T_n) = \varphi\left(\frac{n}{2}\right)\varphi(n + 1) = \varphi(n)\varphi(n + 1) .$$

Case 3. $k > 1$. Now

$$\varphi(n) = \varphi(2^k)\varphi(s) = 2^{k-1}\varphi(s) ,$$

and

$$\varphi\left(\frac{n}{2}\right) = 2^{k-2}\varphi(s) .$$

Also we obviously have $n + 1 > 2$; so let $\varphi(n + 1) = 2m$, where m is an integer. Then

$$\varphi(T_n) = \varphi\left(\frac{n}{2}\right)\varphi(n + 1) = 2^{k-2}\varphi(s)2m = m \cdot 2^{k-1}\varphi(s) = m\varphi(n) .$$

Also solved by Herta T. Freitag, Guy A. Guillottee (Canada), Serge Hamelin (Canada), Douglas Lind (England), C. B. A. Peck, Gregory Wulczyn, and the Proposer.

* * * * *

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or

$$491 = x(x + 1) + y(y + 1) + z(z + 1) .$$

This is impossible, since $x(x + 1)$, $y(y + 1)$, and $z(z + 1)$ are all even.

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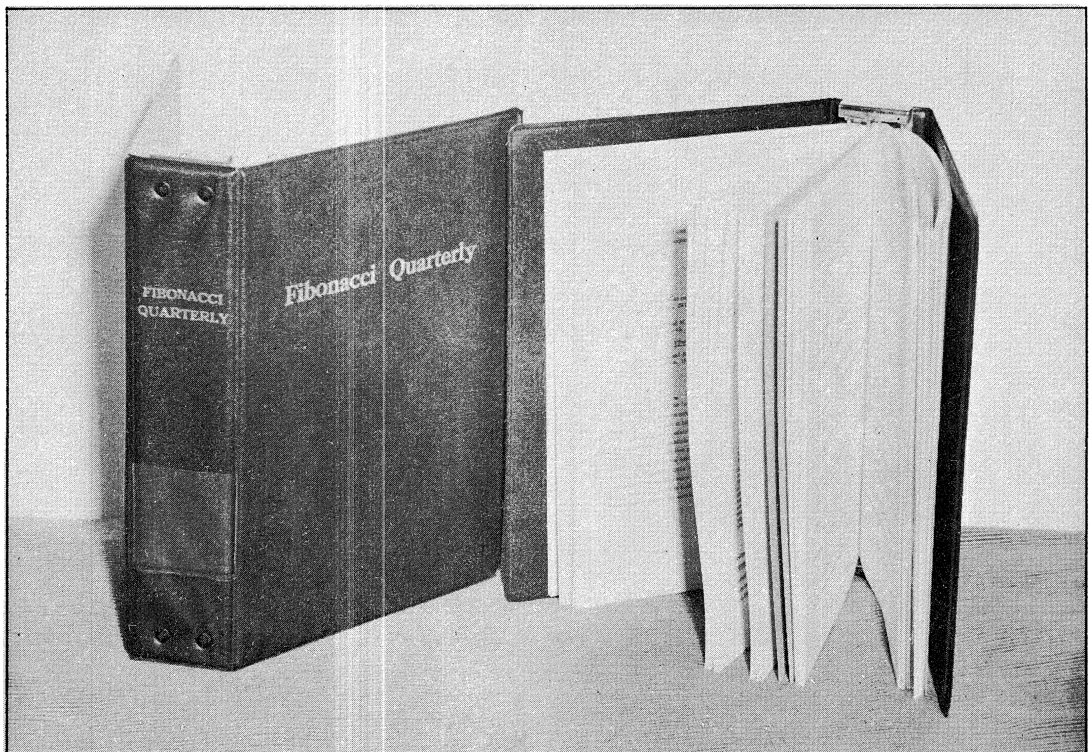

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