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DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

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FIBONACCI REPRESENTATIONS II

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1. Let $R(N)$ denote the number of representations of

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_t}, \tag{1.1}$$

where

$$k_1 > k_2 > \cdots > k_t \geq 2. \tag{1.2}$$

The integer t is allowed to vary. We call (1.1) a Fibonacci representation of N provided (1.2) is satisfied. If in (1.1), we have

$$k_j - k_{j+1} \geq 2 \quad (j = 1, \cdots, t-1); \qquad k_t \geq 2, \tag{1.3}$$

then the representation (1.1) is unique and is called the canonical representation of N.

In a previous paper [1], the writer discussed the function $R(N)$. The paper makes considerable use of the canonical representation and a function $e(N)$ defined by

$$e(N) = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_t-1}. \tag{1.4}$$

It is shown that $e(N)$ is independent of the particular representation. The first main result of [1] is a reduction formula which theoretically enables one to evaluate $R(N)$ for arbitrary N. Unfortunately, the general case is very complicated. However, if all the k_1 in the canonical representation have the same parity, the situation is much more favorable and much simpler results are obtained.

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In the present paper, we consider the function $R(t, N)$ which is defined as the number of representations (1.1) subject to (1.2) where now t is fixed. Again we find a reduction formula which theoretically enables one to evaluate $R(t, N)$ but again leads to very complicated results. However, if all the k_1 in the canonical representation have the same parity, the results simplify considerably. In particular, if

$$N = F_{2k_1} + \cdots + F_{2k_r} \qquad (k_1 > k_2 > \cdots > k_r \geq 1),$$

$$j_s = k_s - k_{s+1} \quad (1 \leq s < r); \quad j_r = k_r,$$

$$f_r(t) = f(t; j_1, \cdots, j_r) = R(t, N),$$

$$F_r(x) = F(x; j_1, \cdots, j_r) = \sum_{t=1}^{\infty} f(t; j_1, \cdots, j_r)\, x^t,$$

$$G_r(x) = F(x; j_1, \cdots, j_{r-1}, j_r + 1),$$

then we have

$$(1.5) \qquad G_r(x) - \frac{x(1 - x^{j_r+1})}{1 - x} G_{r+1}(x) - x^{j_{r-1}+2} G_{r-2}(x) = 0 \qquad (r \geq 2),$$

where

$$G_0(x) = 1, \quad G_1(x) = \frac{x(1 - x^{j_1+1})}{1 - x}.$$

In particular, if $j_1 = \cdots = j_r$, then

$$\sum_{r=0}^{\infty} G_r(x) z^r = \left\{1 - [j + 1]\, xz + x^{j+2} z^2\right\}^{-1},$$

from which an explicit formula for $G_r(x)$ is easily obtained. Also the case

$$j_1 = \cdots = j_{r-1} = j, \qquad j_r = k$$

leads to simple results.

In the final section of the paper some further problems are stated.

2. Put

$$\Phi(a, x, y) = \prod_{n=1}^{\infty} (1 + ax^{F_n} y^{F_{n+1}}) . \tag{2.1}$$

Then

$$\Phi(a, x, xy) = \prod_{n=1}^{\infty} (1 + ay^{F_{n+1}} x^{F_{n+2}}) = \prod_{n=2}^{\infty} (1 + ay^{F_n} x^{F_{n+1}}) ,$$

so that

$$(1 + axy)\Phi(a, x, xy) = \Phi(a, y, x) .$$

Now put

$$\Phi(a, x, y) = \sum_{k,m,n=0}^{\infty} A(k, m, n)\, a^k x^m y^n . \tag{2.2}$$

Comparison of coefficients gives

$$A(k, m, n) = A(k, n - m, m) + A(k - 1, n - m, m - 1) , \tag{2.3}$$

where it is understood that $A(k,m,n) = 0$ when any of the arguments is negative.

In the next place, it is evident from the definition of $e(N)$ and $R(k,N)$ that

$$\prod_{n=1}^{\infty} \left(1 + ax^{F_n} y^{F_{n+1}}\right) = \sum_{N=0}^{\infty} R(k,\ N)\, a^k x^{e(N)} y^N \ . \tag{2.4}$$

Comparing (2.4) with (2.1) and (2.2), we get

$$R(k,\ N) = A(k,\ e(N),\ N) \ . \tag{2.5}$$

In particular, for fixed k, n,

$$A(k,\ m,\ n) = 0 \qquad (m \neq e(n)) \ . \tag{2.6}$$

It should be observed that $A(k, e(n), n)$ may vanish for certain values of k and n. However, since

$$R(n) = \sum_{k=0}^{\infty} R(k,\ n) = \sum_{k=0}^{\infty} A(k,\ e(n),\ n) \ ,$$

it follows that, for fixed n, there is at least one value of k such that

$$A(k,\ e(n),\ n) \neq 0 \ .$$

If we take $m = e(n)$ in (2.3), we get

$$R(t, N) = A(t,\ N - e(N),\ e(N)) + A(t - 1,\ N - e(N),\ e(N) - 1) \ . \tag{2.7}$$

Now let N have the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r} \ , \tag{2.8}$$

with k_r odd. Then

$$\begin{aligned} e(N) &= F_{k_1 - 1} + \cdots + F_{k_r - 1} \ , \\ N - e(N) &= F_{k_1 - 2} + \cdots + F_{k_r - 2} \ . \end{aligned}$$

Since $k_r \geq 3$, it follows that

$$N - e(N) = e(e(N)) . \tag{2.9}$$

On the other hand, exactly as in [1], we find that

$$e(e(N) - 1) = N - e(N) - 1 .$$

It follows that

$$A(t, N - e(N), e(N) - 1) = 0 ,$$

and (2.7) reduces to

$$R(t, N) = A(t, e(e(N)) .$$

We have, therefore,

$$R(t, N) = R(t, e(N)) \quad (k_r \text{ odd}) . \tag{2.10}$$

Now let k_r in the canonical representation of N be even. We shall show that

$$R(t, N) = R(t - 1, e^{k_r - 1}(N_1)) + \sum_{j=2}^{s} R(t - j, e^{k_r - 2}(N_1)) , \tag{2.11}$$

where $k_r = 2s$,

$$N_1 = F_{k_1} + \cdots + F_{k_{r-1}} , \tag{2.12}$$

and

$$e^k(N) = e(e^{k-1}(N)), \qquad e^0(N) = N . \tag{2.13}$$

Assume first that $s > 1$. Then as above

$$N - e(N) = e(e(N)), \tag{2.14}$$

and

$$e(e(N) - 1) = e(e(N)). \tag{2.15}$$

Thus (2.7) becomes

$$R(t, N) = R(t, e(N)) + R(t - 1, e(N) - 1) \qquad (k_r > 2). \tag{2.16}$$

When $k_r = 2$, we have, as in [1],

$$N - e(N) = F_{k_1 - 2} + \cdots + F_{k_{r-1} - 2} = e(e(N_1)),$$
$$e(N) - 1 = F_{k_1 - 1} + \cdots + F_{k_{r-1} - 1} = e(N_1),$$
$$e(e(N)) = N - e(N) - 1.$$

It follows that

$$R(t, N) = R(t - 1, e(N_1)) \qquad (k_r = 2). \tag{2.17}$$

Returning to (2.16), since

$$\begin{aligned} e(N) - 1 &= F_{k_1 - 1} + \cdots + F_{k_{k-1} - 1} + (F_2 + F_4 + \cdots + F_{2t-2}) \\ &= e(N_1) + (F_2 + F_4 + \cdots + F_{2t-2}), \end{aligned}$$

it follows from (2.17) and (2.10) that

$$\begin{aligned} R(t, e(N) - 1) &= R(t - 1, e^2(N_1) + F_3 + \cdots + F_{2t-3}) \\ &= R(t - 1, e^3(N_1) + F_2 + \cdots + F_{2t-4}). \end{aligned}$$

Repeating this process, we get

$$R(t,\ e(N) - 1) = R(t - s,\ e^{2s-2}(N_1))\ ,$$

so that (2.16) becomes

$$R(t, N) = R(t, e^2(N)) + R(t - s,\ e^{2s-2}(N_1)) \qquad (k_r = 2s > 2)\ . \tag{2.18}$$

If $k_r = 4$, Eq. (2.18) reduces, by (2.17) and (2.10), to

$$R(t,\ N) = R(t - 1,\ e^4(N_1)) + R(t - 2,\ e^2(N_1))\ ,$$

since

$$R(t,\ N) = R(t,\ e(N_1)) \qquad (k_r = 2)\ . \tag{2.19}$$

For $k_4 = 2s > 4$, Eq. (2.18) gives

$$\begin{aligned} R(t,\ N) &= R(t,\ e^4(N) + R(t - s + 1,\ e^{2s-2}(N_1)) + R(t - s,\ e^{2s-2}(N_1)) \\ &= R(t,\ e^6(N)) + R(t - s + 2,\ e^{2s-2}(N_1)) + R(t - s + 1,\ e^{2s-2}(N_1)) \\ &\qquad + R(t - s,\ e^{2s-2}(N_1))\ . \end{aligned}$$

Continuing in this way, we ultimately get

$$R(t,\ N) = R(t,\ e^{2s-2}(N)) + \sum_{j=2}^{s} R(t - j,\ e^{2s-2}(N_1))\ . \tag{2.20}$$

By (2.17),

$$R(t,\ e^{2s-2}(N)) = R(t - 1,\ e^{2s-1}(N_1))\ ,$$

so that (2.20) reduces to (2.11).

This proves (2.11) when $k_r > 2$; for $k_r = 2$, it is evident that (2.11) is identical with (2.17).

We may now state

Theorem 1. Let N have the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r} ,$$

where

$$k_j - k_{j+1} \geq 2 \quad (j = 1, \cdots, r - 1); \quad k_r \geq 2 .$$

Then, for $r > 1$, $t > 1$,

$$\text{(2.21)} \qquad R(t, N) = R(t - 1, e^{k_r - 1}(N_1)) + \sum_{j=2}^{s} R(t - j, e^{k_r - 2}(N_1)) ,$$

where $s = [k_r/2]$, $N_1 = F_{k_1} + \cdots + F_{k_{r-1}}$.

3. For $N = F_r$, $r \geq 2$, Eq. (2.7) reduces to

$$\text{(3.1)} \qquad \begin{aligned} R(t, F_r) &= A(t, F_{r-2}, F_{r-1}) + A(t - 1, F_{r-2}, F_{r-1} - 1) \\ &= R(t, F_{r-1}) + A(t - 1, F_{r-2}, F_{r-1} - 1) . \end{aligned}$$

Also,

$$\text{(3.2)} \qquad \begin{aligned} R(t, F_r - 1) &= A(t, F_r - 1 - e(F_r - 1), e(F_r - 1)) \\ &\quad + A(t - 1), F_r - 1 - e(F_r - 1), e(F_r - 1) - 1) . \end{aligned}$$

Since

$$e(F_{2s+1} - 1) = F_{2s}, \quad e(F_{2s} - 1) = F_{2s-1} - 1 ,$$

we have

$$\begin{aligned} A(t - 1, F_{2s-2}, F_{2s-1} - 1) &= R(t - 1, F_{2s-1} - 1), \\ A(t - 1, F_{2s} - 1) &= 0 . \end{aligned}$$

Thus (3.1) becomes

$$\text{(3.2)} \qquad \begin{cases} R(t,\ F_{2s}) = R(t,\ F_{2s-1}) + R(t-1,\ F_{2s-1}-1), \\ R(t,\ F_{2s-1}) = R(t,\ F_{2s}) . \end{cases}$$

In the next place, Eq. (3.2) gives

$$\begin{aligned} R(t,\ F_{2s} - 1) &= A(t,\ F_{2s-2},\ F_{2s-1} - 1) + A(t-1,\ F_{2s-2},\ F_{2s-1} - 2) \\ &= R(t,\ F_{2s-1} - 1)\ , \\ R(t,\ F_{2s+1} - 1) &= A(t,\ F_{2s-1} - 1,\ F_{2s}) + A(t-1,\ F_{2s-1} - 1,\ F_{2s} - 1) \\ &= R(t-1,\ F_{2s} - 1), \end{aligned}$$

that is,

$$\text{(3.3)} \qquad R(t,\ F_r - 1) = R(t - \lambda,\ F_{r-1} - 1) \qquad (r \geq 2)\ ,$$

where

$$\lambda = \begin{cases} 0 & (r \ \text{even}) \\ 1 & (r \ \text{odd})\ . \end{cases}$$

It follows from (3.3) that

$$R(t,\ F_{2s} - 1) = R(t - s + 1, 0), \quad R(t,\ F_{2s+1} - 1) = R(t - s + 1, 1)$$

which gives

$$\text{(3.4)} \qquad \begin{cases} R(t,\ F_{2s} - 1) = \delta_{t,s-1} \\ R(t,\ F_{2s+1} - 1) = \delta_{t,s}\ . \end{cases}$$

Combining (3.2) with (3.4), we get

$$R(t,\ F_{2s}) = R(t,\ F_{2s+1}) = R(t,\ F_{2s-1}) + \delta_{t,s}\ ,$$

so that

$$R(t, F_{2s}) = R(t, F_{2s-2}) + \delta_{t,s} .$$

It follows that

$$R(t, F_{2s}) = \begin{cases} 1 & (1 \le t \le s) \\ 0 & (t > s) \end{cases} .$$

We may now state

<u>Theorem 2.</u> We have, for $s \geq 1$, $t \geq 1$,

$$R(t, F_{2s+1} - 1) = R(t, F_{2s+2} - 1) = \delta_{t,s} ,$$

$$(3.5) \qquad R(t, F_{2s}) = R(t, F_{2s+1}) = \begin{cases} 1 & (1 \le t \le s) \\ 0 & (t > s) \end{cases} .$$

Let $m(N)$ denote the minimum number of summands in a Fibonacci representation of N and let $M(N)$ denote the maximum number of summands. It follows at once from (2.21) that

$$(3.6) \qquad m(N) = r ,$$

where r is the number of summands in the canonical representation of N. Moreover, it is easily proved by induction that

$$(3.7) \qquad R(r, N) = 1.$$

As for $M(N)$, it follows from (2.21) that

$$(3.8) \qquad M(N) \le M(F_{k_1-k_2+2} + \cdots + F_{k_{r-1}-k_r+2}) + [\tfrac{1}{2}k_r] ,$$

where

$$N = F_{k_1} + \cdots + F_{k_r}$$

is the canonical representation. Now, by Theorem 2,

$$M(F_k) = [\tfrac{1}{2}k] .$$

Hence by (3.8),

$$M(F_{k_1} + F_{k_2}) \leq [\tfrac{1}{2}(k_1 - k_2)] + [\tfrac{1}{2}k_2] + 1 .$$

Again, applying (3.8), we get

$$M(F_{k_1} + F_{k_2} + F_{k_3}) \leq [\tfrac{1}{2}(k_1 - k_2)] + [\tfrac{1}{2}(k_2 - k_3)] + [\tfrac{1}{2}k_2] + 2 .$$

It is clear that in general we have

$$M(N) \leq [\tfrac{1}{2}(k_1 - k_2)] + \cdots + [\tfrac{1}{2}(k_{r-1} - k_r)] + [\tfrac{1}{2}k_r] + r - 1 , \tag{3.9}$$

so that

$$M(N) \leq [\tfrac{1}{2}k_1] + r - 1 . \tag{3.10}$$

We note also that (2.21) implies

$$R(M(N), N) = 1 . \tag{3.11}$$

We may state

Theorem 3. Let

$$N = F_{k_1} + \cdots + F_{k_r} \tag{3.12}$$

be the canonical representation of N. Let $m(N)$ denote the minimum number of summands in any Fibonacci representation of N and let $M(N)$ denote the maximum number of summands. Then $m(N) = r$ and $M(N)$ satisfies (3.9). Moreover,

$$R(m(N), N) = R(M(N), N) = 1 . \tag{3.13}$$

It can be shown by examples that (3.9) need not be an equality when $r > 1$.

4. While Theorem 1 theoretically enables one to compute $R(t,N)$ for arbitrary t,N, the results are usually very complicated. Simpler results can be obtained when the k_j in the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r} \tag{4.1}$$

have the same parity. In the first place, if all the k_j are odd, then, by (2.10),

$$R(t, F_{k_1} + \cdots + F_{k_r}) = R(t, F_{k_1-1} + \cdots + F_{k_r-1}) .$$

There is therefore no loss in generality in assuming that all the k_j are even

It will be convenient to use the following notation. Let N have the canonical representation

$$N = F_{2k_1} + \cdots + F_{2k_r} , \tag{4.2}$$

where

$$k_1 > k_2 > \cdots > k_r \geq 1 . \tag{4.3}$$

Then, by (2.21) and (2.10),

$$\begin{aligned} R(t,N) &= R(t - 1, F_{2k_1-2k_r} + \cdots + F_{2k_{r-1}-2k_r}) \\ &\quad + \sum_{j=2}^{k_r} R(t - j, F_{2k_1-2k_r+2} + \cdots + F_{2k_{r-1}-2k_r+2}). \end{aligned} \tag{4.4}$$

Put

$$j_s = k_s - k_{s-1} \quad (s = 1, \cdots, r - 1); \quad j_r = k_r \tag{4.5}$$

and

$$f_r(t) = f(t; j_1, \cdots, j_r) = R(t, N) . \tag{4.6}$$

Then (4.4) becomes

$$(4.7)\qquad f(t;\, j_1, \cdots, j_r) = f(t-1;\, j_1, \cdots, j_{r-1}) + \sum_{u=2}^{j_r} f(t-u;\, j_1, \cdots, j_{r-2}, j_{r-1}+1)\,.$$

By (2.18), we have

$$\begin{aligned} R(t,\ &F_{2k_1-2k_r+2} + \cdots + F_{2k_{r-1}-2k_r+2}) \\ &= R(t,\ F_{2k_1-2k_r} + \cdots + F_{2k_{r-1}-2k_r}) \\ &\quad + R(t-k_{r-1}+k_r-1;\ F_{2k_1-2k_{r-1}+2} + \cdots \\ &\quad + F_{2k_{r-2}-2k_{r-1}+2})\,, \end{aligned}$$

so that

$$\begin{aligned} (4.8)\qquad & f(t;\, j_1, \cdots, j_{r-2}, j_{r-1}+1) \\ &= f(t;\, j_1, \cdots, j_{r-2}, j_{r-1}) + f(t-j_{r-1}-1;\, j_1, \cdots, j_{r-3}, j_{r-2}+1). \end{aligned}$$

If we put

$$(4.9)\qquad F_r(x) = F(x;\, j_1, \cdots, j_r) = \sum_{t=1}^{\infty} f(t;\, j_1, \cdots, j_r)\, x^t\,,$$

it follows from (4.7) that (for $r > 1$),

$$(4.10)\qquad F(x;\, j_1, \cdots, j_r) = xF(x;\, j_1, \cdots, j_{r-1}) + \frac{x(x - x^{j_r})}{1-x} F(x;\, j_1, \cdots, j_{r-2}, j_{r-1}+1).$$

Similarly, by (4.8),

$$(4.11)\quad F(x;\, j_1, \cdots, j_{r-2}, j_{r-1}+1) = F(x;\, j_1, \cdots, j_{r-2}, j_{r-1}) + x^{j_{r-1}+1} F(x; j_1, \cdots, j_{r-3}, j_{r-2}+1),$$

which yields

$$(4.12)\quad \begin{aligned} F(x;\, j_1, \cdots, j_{r-2}, j_{r-1}+1) &= F(x;\, j_1, \cdots, j_{r-2}, j_{r-1}) + x^{j_{r-1}+1} F(x;\, j_1, \cdots, j_{r-3}, j_{r-2}) \\ &\quad + x^{j_{r-1}+j_{r-2}+2} F(x;\, j_1, \cdots, j_{r-3}) + \cdots + x^{j_{r-1}+\cdots+j_2+r-1} F(x;\, j_1). \end{aligned}$$

For brevity, put

$$(4.13)\quad G_r(x) = F(x;\, j_1, \cdots, j_{r-1}, j_r + 1),$$

so that (4.10) becomes

$$(4.14)\quad F_r(x) - xF_{r-1}(x) = \frac{x(x - x^{j_r})}{1 - x} G_{r-1}(x),$$

while (4.11) becomes

$$(4.15)\quad G_{r-1}(x) = F_{r-1}(x) + x^{j_{r-1}+1} G_{r-2}(x).$$

Combining (4.14) with (4.15), we get

$$(4.16)\quad G_r(x) - \frac{x(1 - x^{j_{r+1}})}{1 - x} G_{r-1}(x) + x^{j_{r-1}+2} G_{r-2}(x) = 0.$$

Thus $G_r(x)$ satisfies a recurrence of the second order. Note that

$$\begin{aligned} G_1(x) = F(x;\, j_1 + 1) &= \sum_{t=1}^{\infty} R(t,\, F_{2j_1+2})\, x^t \\ &= \sum_{t=1}^{j_1+1} x^t = \frac{x(1 - x^{j_1+1})}{1 - x}, \end{aligned}$$

$$G_2(x) = F(x;\, j_1,\, j_2 + 1) = \sum_{t=2}^{\infty} R(t,\, F_{2j_1+2j_2+2} + F_{2j_2+2}) \;.$$

Now, by (2.21),

$$R(t,\, F_{2j_1+2j_2+2} + F_{2j_2+2}) = R(t - 1,\, F_{2j_1+1}) + \sum_{u=2}^{j_2+1} R(t - u,\, F_{2j_1+2}) \,,$$

so that

$$G_2(x) = x \sum_{t=1}^{j_1} x^t + \sum_{u=2}^{j_2+1} x^u \sum_{t=1}^{j_1+1} x^t$$

$$= \frac{x^2(1 - x^{j_1})}{1 - x} + \frac{x^2(1 - x^{j_2})}{1 - x}\,\frac{x(1 - x^{j_1+1})}{1 - x} \;.$$

Hence, if we take $G_0(x) = 1$, Eq. (4.16) holds for all $r \geq 2$.

We may state

Theorem 5. With the notation (4.2), (4.6), (4.9), (4.12), $f_r(t) = R(t,N)$ is determined by means of the recurrence (4.16) with

$$G_0(x) = 1, \qquad G_1(x) = \frac{x(1 - x^{j_1+1})}{1 - x}$$

and

$$F_r(x) = G_r(x) - x^{j_r+1}\, G_{r-1}(x) \;.$$

It is easy to show that $G_r(x)$ is equal to the determinant

$$(4.17) \qquad D_r(x) = \begin{vmatrix} x[j_1+1] & -x^{j_1+2} & 0 & \cdots & 0 \\ -1 & x[j_2+1] & -x^{j_2+2} & \cdots & 0 \\ 0 & & x[j_3+1] & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \;, \\ 0 & 0 & 0 & & x[j_r+1] \end{vmatrix} ,$$

where

$$(4.18) \qquad [j] = (1 - x^j)/(1 - x) .$$

Indeed,

$$D_1(x) = x[j_1 + 1] = G_1(x) ,$$

$$D_2(x) = x^2[j_1 + 1][j_2 + 1] - x^{j_1+2} = x^3[j_1 + 1][j_2] + x^2[j_1] = G_2(x) ,$$

and

$$(4.19) \qquad D_r(x) = x[j_r + 1]D_{r-1}(x) - x^{j_{r-1}+2} D_{r-2}(x) .$$

Since the recurrence (4.16) and (4.19) are the same, it follows that $G_r(x) = D_r(x)$.

5. When

$$(5.1) \qquad j_1 = j_2 = \cdots = j_r = j ,$$

we can obtain an explicit formula for $G_r(x)$. The recurrence (4.16) reduces to

$$(5.2) \qquad G_r(x) - x[j - 1]G_{r-1}(x) + x^{j+2}G_{r-2}(x) = 0 \qquad (r \geq 2) .$$

Then

$$\sum_{r=0}^{\infty} G_r(x)z^r = 1 + [j+1]xz + \sum_{r=2}^{\infty} G_r(x)z^r$$

$$= 1 + [j+1]xz + \sum_{r=2}^{\infty} \left\{x[j+1]G_{r-1}(x) - x^{j+2}G_{r-2}(x)\right\} z^r$$

$$= 1 + ([j+1]xz + x^{j+2} z^2) \sum_{r=0}^{\infty} G_r(x)\, z^r \quad ,$$

so that

$$\sum_{r=0}^{\infty} G_r(x)\, z^r = (1 - [j+1]\, xz + x^{j+2} z^2)^{-1}$$

$$= \sum_{s=0}^{\infty} x^s z^s ([j+1] - x^{j+1} z)^s$$

$$= \sum_{s=0}^{\infty} x^s z^s \sum_{t=0}^{s} (-1)^t \binom{s}{t} [j+1]^{s-t} x^{(j+1)t} z^t \ .$$

Hence

$$G_r(x) = \sum_{2t \leq r}^{\infty} (-1)^t \binom{r-t}{t} [j+1]^{r-2t} x^{r+jt} \ . \tag{5.3}$$

Finally, we compute $F_r(x)$ by using

$$F_r(x) = G_r(x) - x^{j+1} G_{r-1}(x) \ . \tag{5.4}$$

When $j = 1$, we have

$$\sum_{r=0}^{\infty} G_r(x)^r = \frac{1}{(1 - xz)(1 - x^2z)} = \frac{1}{1 - x}\left(\frac{1}{1 - xz} - \frac{1}{1 - x^2z}\right) ,$$

which gives

(5.5) $$G_r(x) = x^2[r] = \frac{x^r(1 - x^r)}{1 - x} \qquad (j = 1;\ r \geq 1)$$

(5.6) $$F_r(x) = x^r \qquad (j = 1) .$$

In this case, we evidently have

$$N = F_{2r} + F_{2r-2} + \cdots + F_2 = F_{2r+1} - 1 ,$$

so that (5.6) is in agreement with (3.4).

For certain applications, it is of interest to take

(5.7) $$j_1 = \cdots = j_{r-1} = j; \qquad j_r = k .$$

Then $G_1(x)$, $G_2(x)$, $\cdots$, $G_{r-1}(x)$ are determined by

(5.8) $$G_s(x) = \sum_{2t \leq s} (-1)^t \binom{s-t}{t} [j - 1]^{s-2t} x^{s+jt} \qquad (1 \leq s < r),$$

while

(5.9) $$G'_r(x) = x[k - 1]G_{r-1}(x) - x^{j+2}G_{r-2}(x) ,$$

where

$$G'_r(x) = G_r(x;\ j, \cdots, j, k) .$$

Also,

(5.10) $$F'_r(x) = F_r(x;\ j, \cdots, j, k) = x[k]G_{r-1}(x) - x^{j+2}G_{r-2}(x)\ .$$

We shall now make some applications of these results. Since

$$L_{2j+1}F_{2k} = F_{2k+2j} + F_{2k+2j-2} + \cdots + F_{2k-2j}\ ,$$

it follows from (5.10) that

(5.11) $$\sum_t R(t, L_{2j+1}F_{2k})x^t = x^{2j+1}[2j][k - j] - x^{2j+2}[2j - 1] \qquad (j < k)\ .$$

(Note that formula (6.17) of [1] should read

$$R(L_{2j+1}F_{2k}) = 2j(k - j) - (2j - 1)$$

in agreement with (5.11).) If we rewrite (5.11) as

$$\sum_t R(t,\ L_{2j+1}F_{2k})x^t = x^{2j+1}\{1 + x + \cdots + x^{k-j-1} + (x + \cdots + x^{2j-1})(x + \cdots + x^{k-j-1})\}$$

we can easily evaluate $R(t,\ L_{2j+1}F_{2k})$. In particular, we note that

(5.12) $$R(t,\ L_{2j+1}F_{2k}) > 0 \qquad (j < k)$$

if and only if

$$2j + 1 \le t \le 3j + k - 1\ .$$

Note that, for $k = 3j$,

$$\sum_t R(t,\ L_{2j+1}F_{6j})x^t = x^{2j+1}\{1 + x + \cdots + x^{2j-1} + (x + x^2 + \cdots + x^{2j-1})^2\}\ .$$

This example shows that the function $R(t,N)$ takes on arbitrarily large values.

When $j = k$, we have

$$L_{2k+1}F_{2k} = F_{4k+1} - 1 ,$$

so that, by (3.4),

$$\sum_t R(t, L_{2k+1}F_{2k}) x^t = x^{2k} . \tag{5.13}$$

Next, since

$$L_{2j+1}F_{2k} = F_{2j+2k} + F_{2j+2k-2} + \cdots + F_{2j-2k-2} \qquad (j > k) ,$$

we get

$$\sum_t R(t, L_{2j+1}F_{2k})x^t = x^{2k}[j - k - 1][2k - 1] - x^{2k+1}[2k - 2] \qquad (j > k > 1) . \tag{5.14}$$

Corresponding to (5.15), we now have

$$R(t, L_{2j+1}F_{2k}) > 0 \qquad (j > k > 1) , \tag{5.15}$$

if and only if

$$2k \le t \le j + 3k - 2 .$$

The case $k = 1$ is not included in (5.14), because (5.5) does not hold when $r = 0$. For the excluded case, since

$$L_{2j+1} = F_{2j+2} + F_{2j} ,$$

we get, by Theorem 1,

$$\sum_t R(t, L_{2j+1})x^t = x^2 + (x^2 + x^3)\frac{x - x^j}{1 - x} \qquad (j \geq 1)\,. \tag{5.16}$$

For $t = 1$, Eq. (5.16) reduces to the known result:

$$R(L_{2j+1}) = 2j - 1\,.$$

In [1] a number of formulas of the type

$$R(F^2_{2n+1} - 1) = F_{2n+1} \quad (n \geq 0), \qquad R(F^2_{2n}) = F_{2n} \quad (n \geq 1)$$

were obtained. They depend on the identities

$$F_4 + F_8 + \cdots + F_{4n} = F^2_{4n+1} - 1\,,$$
$$F_2 + F_6 + \cdots + F_{4n+2} = F^2_{2n}\,.$$

We now apply (5.10) to these identities. Then $G_r(x)$ is determined by

$$G_r(x) = \sum_{2t \leq r} (-1)^t \binom{r-t}{t} [3]^{r-2t} x^{r+2t}\,. \tag{5.17}$$

Thus (5.10) yields

$$\sum_t R(t, F^2_{4n+1} - 1)x^t = x(1 + x)G_{n-1}(x) - x^4 G_{n-2}(x)\,, \tag{5.18}$$

$$\sum_t R(t, F^2_{2n})x^t = xG_{n-1}(x) - x^4 G_{n-2}(x)\,, \tag{5.19}$$

with $G_{n-1}(x)$, $G_{n-2}(x)$ given by (5.17).

It may be of interest to note that

$$G_r(1) = \sum_{2t \le r} (-1)^t \binom{r - t}{t} 3^{r-2t} = F_{2r+2} .$$

6. The following problems may be of some interest.
   A. Evaluate $M(N)$ in terms of the canonical representation of N.
   B. Determine whether $R(t,N) \ge 1$ for all t in $m(N) \le t \le M(N)$.
   C. Does $R(t,N)$ have the unimodal property? That is, for given N, does there exist an integer $\mu(N)$ such that

$$R(t, N) \le R(t + 1, N) \qquad (m(N) \le t \le \mu(N)),$$
$$R(t, N) \ge R(t + 1, N) \qquad (\mu(N) \le t < M(N)) ?$$

   D. Is $R(t, N)$ logarithmically concave? That is, does it satisfy

$$R^2(t,N) \ge R(t - 1,N)R(t + 1,N) \qquad (m(N) < t < M(N))?$$

   E. Find the general solution of the equation

$$R(t, N) = 1 .$$

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A LIMITED ARITHMETIC ON SIMPLE CONTINUED FRACTIONS - II

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1. INTRODUCTION

In the first paper [2] in this series, we developed certain properties of the simple continued fraction expansions of integral multiples of quadratic surds with expansions of the form $[a,\dot{b}]$ or $[a,\dot{b},\dot{c}]$ where the notation is that of Hardy and Wright [1, Chapter 10]. For easy reference, we restate the principle results here.

Theorem 1. Let $\zeta = [a,\dot{b}]$, let n be a positive integer, let p_k/q_k denote the k^{th} convergent to ζ and let $t_k = q_{k-1} + q_{k+1}$ for $k \geq 0$ where we take $q_{-1} = 0$. Then $n\zeta = [r,\dot{s}]$ if and only if $n = q_{2m-2}$, $r = p_{2m-2}$, and $s = t_{2m-2}$ for some $m \geq 1$.

Theorem 2. Let ζ, n, p_k/q_k and t_k be as in Theorem 1. Then $n\zeta = [u,\dot{v},\dot{w}]$ if and only if $vn = q_{2m-1}$, $vu = p_{2m-1} - 1$, and $vw = t_{2m-1} - 2$ for some integer $m \geq 1$.

Theorem 3. Let $\zeta = [a,\dot{b},\dot{c}]$, let p_k/q_k be the k^{th} convergent to ζ, let $t_k = q_{k-1} + q_{k+1}$ and $s_k = p_{k-1} + p_{k+1}$ for $k \geq 1$. Then, for every integer $r \geq 1$, we have

$$q_{2r} \cdot \zeta = [p_{2r}, \dot{t}_{2r}, c\dot{t}_{2r}/b],$$
$$q_{2r-1} \cdot \zeta = [p_{2r-1} - 1, \dot{1}, \dot{t}_{2r-1} - 2]$$
$$t_{2r-1} \cdot \zeta = [s_{2r-1}, \dot{q}_{2r-1}, (c^2 + 4\dot{c}/b)q_{2r-1}]$$

and

$$t_{2r} \cdot \zeta = [s_{2r} - 1, \dot{1}, q_{2r} - 2, 1, (bc + \dot{4})q_{2r} - 2].$$

Of course, for $a = b = c = 1$, the preceding theorems give results involving the golden ratio, $(1 + \sqrt{5})/2$, and the Fibonacci and Lucas numbers since, in that case,

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$$\zeta = (1 + \sqrt{5})/2,\ p_k = F_{k+2},\ q_k = F_{k+1},\ t_k = L_{k+1}, \text{ and } s_k = L_{k+2}$$

where F_n and L_n denote respectively the n^{th} Fibonacci and Lucas numbers.

In the present paper, we devote our attention primarily to the study of the simple continued fraction expansions of positive rational multiples of quadratic surds with expansions of the form $[\dot{a}]$. Again, we note that, for $a = 1$, the theorems specialize to results about the golden ratio and Fibonacci and Lucas numbers.

2. PRELIMINARY CONSIDERATIONS

Let the integral sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ be defined as follows:

(1) $$f_0 = 0,\ f_1 = 1,\ f_n = af_{n-1} + f_{n-2},\ n \geq 0\ ,$$

and

(2) $$g_0 = 2,\ g_1 = a,\ g_n = ag_{n-1} + g_{n-2},\ n \geq 0\ ,$$

where a is any positive integer. These difference equations are easily solved to give

(3) $$f_n = \frac{\xi^n - \bar{\xi}^n}{\sqrt{a^2 + 4}}\ ,\quad n \geq 0\ ,$$

and

(4) $$g_n = \xi^n + \bar{\xi}^n\ ,\quad n \geq 0\ ,$$

where

$$\xi = (a + \sqrt{a^2 + 4})/2 \quad\text{and}\quad \bar{\xi} = (a - \sqrt{a^2 + 4})/2$$

are the two irrational roots of the equation

(5) $$x^2 - ax - 1 = 0\ .$$

Incidentally, if β is a quadratic surd, we will always denote the conjugate surd by $\bar{\beta}$. The following formulas, of interest in themselves, generalize results for the Fibonacci and Lucas numbers and are easily proved by induction.

$$(6)\qquad \begin{aligned} f_{2n} &= \sum_{i=0}^{n-1} \binom{n+i}{2i+1} a^{2i+1}, \\ f_{2n+1} &= \sum_{i=0}^{n} \binom{n+i}{2i} a^{2i}, \end{aligned} \qquad n \geq 0,$$

$$(7)\qquad g_n = f_{n-1} + f_{n+1}, \quad n \geq 1,$$

$$(8)\qquad f_{m+n+1} = f_m f_n + f_{m+1} f_{n+1}, \quad m \geq 0, \quad n \geq 0,$$

$$(9)\qquad g_{m+n+1} = f_m g_n + f_{m+1} g_{n+1}, \quad m \geq 0, \quad n \geq 0,$$

$$(10)\qquad f_m f_n - f_{m-1} f_{n+1} = (-1)^{m-1} f_{m-n+1}, \quad 1 \leq m \leq n.$$

Also, we obtain in the usual way from (8) the following lemma.

Lemma 4. For the integral sequence $\{f_n\}_{n \geq 0}$ we have that $f_m | f_n$ if and only if $m | n$, where m and n are positive integers and $m > 2$ if $a = 1$.

3. PRINCIPAL RESULTS

Our first theorem, together with the results of the first paper in this couplet, yields a series of results concerning the simple continued fraction expansion of multiples of $\xi = [\dot{a}]$ by the reciprocals of positive integers. The theorem is also of some interest in its own right.

Theorem 5. Let $\zeta = (a + b\sqrt{c})/d$ with a, b, c, and d integers, c not a perfect square, and c and d positive. Let r be a positive rational number such that $2ar/d$ is an integer. Let $a^2 + d^2 = b^2c$ and let $1 < \zeta \leq r$. Then

$$r\zeta = [a_0, a_1, a_2, \cdots],$$

if and only if

$$\frac{\zeta}{r} = \left[0,\ a_0 - \frac{2ar}{d},\ a_1,\ a_2,\ \cdots\right] .$$

Proof. We note first that $a_0 - 2ar/d$ is positive. This is so since

$$a_0 = \left[\frac{ar + rb\sqrt{c}}{d}\right] > \frac{ra + rb\sqrt{c}}{d} - 1$$

so that

$$\begin{aligned} a_0 - \frac{2ar}{d} &> \frac{-ra + rb\sqrt{c}}{d} - 1 \\ &= \frac{r}{d}\left(\frac{a^2 - b^2c}{-a - b\sqrt{c}}\right) - 1 \\ &= \frac{rd}{a + b\sqrt{c}} - 1 \\ &= \frac{r}{\zeta} - 1 > 0 , \end{aligned}$$

by hypothesis. Now let $\mu = [a_1,\ a_2,\ a_3,\ \cdots]$ so that

$$r\zeta = a_0 + \frac{1}{\mu} .$$

Then

$$\begin{aligned} \left[0,\ a_0 - \frac{2ar}{d},\ a_1,\ a_2,\ \cdots\right] &= \frac{1}{a_0 - \frac{2ar}{d} + \frac{1}{\mu}} \\ &= \frac{1}{r\zeta - \frac{2ar}{d}} \\ &= \frac{1}{r\left(\frac{a + b\sqrt{c}}{d} - \frac{2a}{d}\right)} \\ &= \frac{d}{r(-a + b\sqrt{c})} \\ &= \frac{-d(a + b\sqrt{c})}{r(a^2 - b^2c)} \\ &= \frac{a + b\sqrt{c}}{dr} \\ &= \frac{\zeta}{r} , \end{aligned}$$

and the proof is complete.

Corollary 6. Let a and n be positive integers. Let $\xi = [\dot{a}]$ and let $n > \xi$. Then

$$n\xi = [a_0, a_1, a_2, \cdots]$$

if and only if

$$\frac{\xi}{n} = [0, a_0, -an, a_1, a_2, \cdots] .$$

Proof. Since

$$\xi = [\dot{a}] = \frac{a + \sqrt{a^2 + 4}}{2} ,$$

we may use the preceding theorem with $a = a$, $b = 1$, $c = a^2 + 4$, $d = 2$, and $r = n$. The result then follows immediately since

$$\frac{2ar}{d} = \frac{2an}{2}$$

is an integer and

$$a^2 + d^2 = a^2 + 4 = b^2c ,$$

as required.

Now for

$$\xi = [\dot{a}] = \frac{a + \sqrt{a^2 + 4}}{2} .$$

The convergents p_k/q_k are given by the equations

$$\begin{aligned} &p_0 = a,\ p_1 = a^2 + 1,\ p_n = ap_{n-1} + p_{n-2} , \\ &q_0 = 1,\ q_1 = a,\ q_n = aq_{n-1} + q_{n-2} , \end{aligned} \qquad n \geq 2 , \tag{11}$$

and it is clear that $p_n = f_{n+2}$ and $q_n = f_{n+1}$ for $n \geq 0$. Also, $p'_n = f_n$ and $q'_n = f_{n+1}$ for $n \geq 0$, where p'_n/q'_n is the n^{th} convergent to $1/\xi$. The following results could all be stated in terms of the sequences $\{p_n\}$ and $\{q_n\}$; instead, we use the sequences $\{f_n\}$ and $\{g_n\}$.

Corollary 7. Let r, s, and n be the positive integers with $n > \xi = [\dot{a}]$. Then $\xi/n = [0, r, \dot{s}]$, if and only if, $n = f_{2m-1}$, $r = f_{2m}$, and $s = q_{2m-1}$ for some $m \geq 2$.

Proof. This is an immediate consequence of Theorem 1 with $a = b$, and Corollary 6.

Corollary 8. Let u, v, w, and n be positive integers with $n > \xi = [\dot{a}]$. Then $\xi/n = [0, u, \dot{v}, \dot{w}]$, if and only if, $vn = f_{2m}$, $vu = f_{2m+1} - 1$, and $vw = g_{2m} - 2$ for some integer $m \geq 2$.

Proof. This is an immediate consequence of Corollary 6 and Theorem 2 with $u = v = w = a$.

The next corollary results from Theorem 3 and Corollary 6 by taking $a = b = c$. However, since, in this special case, parts (a) and (b) of Theorem 2 yield results already obtained, we concern ourselves only with parts (c) and (d).

Corollary 9. Let n be a positive integer greater than ξ. Then for $r \geq 1$,

$$\frac{\xi}{g_{2r}} = [0,\ g_{2r-1},\ \dot{f}_{2r},\ (a^2 + \dot{4})f_{2r}]$$

and

$$\frac{\xi}{g_{2r+1}} = [0,\ g_{2r} - 1,\ \dot{1},\ f_{2r+1} - 2,\ 1,\ (a^2 + 4)\dot{f}_{2r+1} - 2].$$

The next theorem shows that the periodic part of the simple continued fraction expansion of n for any positive integer $n > \xi = [\dot{a}]$ is almost symmetric. Of course, by Corollary 6, the same thing is true of ξ/n.

Theorem 10. Let a and n be positive integers with $n > \xi = [\dot{a}]$. Then $n\xi = [a_0, \dot{a}_1, \cdots, \dot{a}_r]$ and the vector $(a_1, a_2, \cdots, a_{r-1})$ is symmetric if $r \geq 2$.

Proof. Since $a_0 = [n\xi]$, we have that

$$0 < n\xi - a_0 < 1$$

and

$$\xi_1 = \frac{1}{n\xi - a_0} > 1 \quad ,$$

where ξ_1 is the first complete quotient in the expansion of n. Moreover,

$$\overline{\xi_1} = \frac{1}{n\overline{\xi} - a_0} = -\frac{1}{\frac{n}{\xi} + a_0}$$

so that

$$-1 < \overline{\xi_1} < 0 \quad ,$$

since $a_0 + n/\xi$ is clearly greater than one. Thus, ξ_1 is a reduced quadratic surd and by the general theory (see, for example [3, Chapter 4]) has a purely periodic simple continued fraction expansion, say

$$\xi_1 = [\dot{a}_1, a_2, \cdots, \dot{a}_r] .$$

Additionally, we also have that $[a_r, a_{r-1}, \cdots, a_1]$ is the expansion of the negative reciprocal of the conjugate of ξ_1. Thus,

$$[\dot{a}_r, a_{r-1}, \cdots, \dot{a}_1] = -\frac{1}{\overline{\xi_1}} = \frac{n}{\xi} + a_0$$

so that

(12) $$\frac{\xi}{n} = [0, a_r - a_0, \dot{a}_{r-1}, a_{r-2}, \cdots, a_1, \dot{a}_r] .$$

But, from above,

$$n\xi = [a_0, \dot{a}_1, \cdots, \dot{a}_r] ,$$

and by Corollary 6,

$$\frac{\xi}{n} = [0, a_0 - an, \dot{a}_1, \dot{a}_2, \cdots, \dot{a}_r] . \tag{13}$$

Thus, comparing (12) and (13), we have that the vector $(a_1, a_2, \cdots, a_{r-1})$ is symmetric.

We now turn our attention to the consideration of more general positive rational multiples of $\xi = [\dot{a}]$.

Theorem 11. Let r be rational with $0 < r < 1$. If the simple continued fraction expansion of $r\xi$ is not purely periodic, then

$$r\xi = [0, a_1, \dot{a}_2, \cdots, \dot{a}_n]$$

and

$$\frac{\xi}{r} = [a_n - a_1, \dot{a}_{n-1}, a_{n-2}, \cdots, a_2, \dot{a}_n]$$

for some $n \geq 2$.

Proof. If $r\xi$ had a purely periodic simple continued fraction expansion, then $r\xi$ would have to be a reduced quadratic surd so that $r\xi > 1$ and $-1 < \overline{r\xi} < 0$. But the first of these inequalities implies that

$$\frac{1}{\xi} < r < 1 , \tag{14}$$

and, since $\overline{\xi} = -1/\xi$, the second implies that

$$\xi > r > 0 , \tag{15}$$

which is already implied by (14). Therefore, since $r\xi$ is not purely periodic, we have

$$0 < r < \frac{1}{\xi} \quad , \tag{16}$$

so that $0 = [r\xi] = a_0$. Now consider

$$\xi_1 = \frac{1}{r\xi} > 1 \quad ,$$

and set $a_1 = [\xi_1] \geq 1$. Again,

$$\frac{1}{\frac{1}{r\xi} - a_1} = \frac{1}{\xi_1 - a_1} = \xi_2 > 1$$

and

$$\overline{\xi}_2 = \frac{1}{\frac{1}{r\overline{\xi}} - a_1} = -\frac{1}{\frac{\xi}{r} + a_1} \quad ,$$

since $\xi\overline{\xi} = -1$. Therefore, $-1 < \overline{\xi}_2 < 0$ and ξ_2 is reduced. Thus, ξ_2 has a purely periodic simple continued fraction expansion,

$$\xi_2 = [\dot{a}_2, a_3, \cdots, \dot{a}_n] \; ,$$

and

$$r\xi = [0, a_1, \dot{a}_2, \cdots, \dot{a}_n] \; ,$$

as claimed. Also,

$$\frac{\xi}{r} + a_1 = -\frac{1}{\overline{\xi}_2} = [\dot{a}_n, a_{n-1}, \cdots, \dot{a}_2] \quad ,$$

so that

$$\frac{\xi}{r} = [a_n - a_1, \dot{a}_{n-1}, a_{r-2}, \cdots, a_2, \dot{a}_n]$$

and the proof is complete.

Theorem 12. Let r be rational with $0 < r < 1$. If the simple continued fraction expansion of $r\xi$ is purely periodic, then

$$r\xi = [\dot{a}_0, a_1, \cdots, \dot{a}_n]$$

and

$$\frac{\xi}{r} = [\dot{a}_n, a_{n-1}, \cdots, \dot{a}_0]$$

for some $n \geq 0$.

Proof. Since the simple continued fraction expansion of $r\xi$ is purely periodic, it is reduced and we have by the preceding proof that

$$\frac{1}{\xi} < r < 1 .$$

Since we also have

$$\frac{\xi}{r} = -\frac{1}{r\bar{\xi}} = [\dot{a}_n, a_{n-1}, \cdots, \dot{a}_0] ,$$

the proof is complete.

In passing, we note that the periodic part of the expansions of $r\xi$ need not exhibit any symmetry or even near symmetry. For example, for

$$\alpha = [\dot{1}] = \frac{1 + \sqrt{5}}{2} ,$$

we have that

$$\frac{2}{9}\alpha = [0,\ 2,\ \dot{1},\ 3,\ 1,\ 1,\ 3,\ \dot{9}]$$

and

$$\frac{3}{4}\alpha = [\dot{1},\ 4,\ 1,\ 2,\ 6,\ \dot{2}]\ .$$

Also, it is easy to find rational numbers r with $0 < r < 1$ such that the surds $r\alpha$ and α/r are not equivalent where we recall that two real numbers μ and ν are said to be equivalent if and only if there exist integers a, b, c, and d with $|ad - bc| = 1$ and such that

$$\mu = \frac{a\nu + b}{c\nu + d}\ .$$

However, as the following theorems show, there exist interesting examples, where near symmetry of the periodic part of the expansions of $r\xi$ and ξ/r and equivalence of $r\xi$ and r/ξ both hold. We will indicate that $r\xi$ and ξ/r are equivalent by the notation

$$r\xi \sim \frac{\xi}{r}\ .$$

<u>Theorem 13.</u> Let a be a positive integer, let $\xi = [\dot{a}]$, and let the sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ be as defined above. Then, for $n \geq 1$,

$$\frac{f_{2n+1}}{f_{2n+2}} \cdot \xi = [\dot{a}_0,\ a_1,\ \cdots,\ \dot{a}_r]$$

and

$$\frac{f_{2n+2}}{f_{2n+1}} \cdot \xi = [\dot{a}_r,\ a_{r-1},\ \cdots,\ \dot{a}_0]$$

where the vector $(a_2, a_3, \cdots, a_r, a_0)$ is symmetric, $a_0 = a_2 = 1$, and $a_1 = f_{4n+3} - 1$.

Proof. We first demonstrate the purely periodic nature of the expansions in question. From the definition, it is clear that f_n is strictly increasing for $n \geq 2$. Also, f_n/f_{n+1} is the n^{th} convergent to $1/\xi$. Therefore,

$$\frac{f_{2n}}{f_{2n+1}} < \frac{1}{\xi} < \frac{f_{2n+1}}{f_{2n+2}} < 1 , \tag{17}$$

and it follows from the proof of Theorem 11 that $\xi f_{2n+1}/f_{2n+2}$ has a purely periodic expansion. Also, from Theorem 12, $\xi f_{2n+2}/f_{2n+1}$ has a purely periodic expansion whose period is the reverse of that for $\xi f_{2n+1}/f_{2n+2}$.

Additionally, from (17), it follows that

$$0 < \frac{f_{2n+1}}{f_{2n+2}} - \frac{1}{\xi} < \frac{1}{2}\left(\frac{f_{2n+1}}{f_{2n+2}} - \frac{f_{2n}}{f_{2n+1}}\right) = \frac{1}{2f_{2n+1}f_{2n+2}}$$

so that

$$1 < \frac{f_{2n+1}}{f_{2n+2}} \cdot \xi < \frac{\xi}{2f_{2n+1}f_{2n+2}} + 1$$

$$< \frac{f_{2n+1}}{2f_{2n+1}f_{2n+2}} + 1 < 2 .$$

Thus, $a_0 = [\xi f_{2n+1}/f_{2n+2}] = 1$. Now

$$\xi_1 = \frac{1}{\dfrac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1} ,$$

and we claim that

$$f_{4n+3} - 1 < \xi_1 < f_{4n+3} ,$$

so that $a_1 = f_{4n+3} - 1$. To see that this is so, we note that, since f_{n+2}/f_{n+1} is the n^{th} convergent to ξ,

$$\frac{f_{2n+2}}{f_{2n+1}} < \frac{f_{4n+4}}{f_{4n+3}} < \xi$$

and

$$1 < \frac{f_{2n+1}f_{4n+4}}{f_{2n+2}f_{4n+3}} < \frac{f_{2n+1}}{f_{2n+2}} \cdot \xi \,.$$

But this gives, using (10),

$$\frac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1 > \frac{f_{2n+1}f_{4n+4} - f_{2n+2}f_{4n+3}}{f_{2n+2}f_{4n+3}}$$

$$= \frac{f_{2n+2}}{f_{2n+2}f_{4n+3}} = \frac{1}{f_{4n+3}}$$

or

$$\xi_1 < f_{4n+3} \tag{18}$$

as desired. Also, we have that

$$\frac{f_{2n+2}}{f_{2n+1}} < \xi < \frac{f_{4n+3}}{f_{4n+2}}$$

so that, again by (10),

$$0 < \frac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1$$

$$< \frac{f_{2n+1}f_{4n+3} - f_{2n+2}f_{4n+2}}{f_{2n+2}f_{4n+2}}$$

$$= \frac{f_{2n+1}}{f_{2n+2}f_{4n+2}}$$

$$= \frac{f_{2n+1}}{f_{2n+1}f_{4n+3} - f_{2n+1}}$$

and

$$\xi_1 > f_{4n+3} - 1 .$$

Thus, $a_1 = [\xi_1] = f_{4n+3} - 1$ as claimed. Finally, to show the symmetry of the vector $(a_2, a_3, \cdots, a_r, a_0)$, it suffices to show that

$$\frac{1}{\dfrac{1}{\dfrac{1}{\dfrac{f_{2n+1}}{f_{2n+2}} \cdot \xi - a_0} - a_1} - a_2} = \frac{f_{2n+2}}{f_{2n+1}} \cdot \xi . \tag{19}$$

Making use of the determined values of a_0 and a_1 and setting $a_2 = 1$, this means that we must show that

$$\frac{1}{\dfrac{1}{\dfrac{1}{\dfrac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1} - (f_{4n+3} - 1)} - 1} = \frac{f_{2n+2}}{f_{2n+1}} \cdot \xi \tag{20}$$

which will also, of course, confirm the fact that $a_2 = 1$. Now (20) is true if and only if

$$\cfrac{1}{\cfrac{1}{\cfrac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1} - (f_{4n+3} - 1)} - 1 = \frac{f_{2n+1}}{\xi f_{2n+2}},$$

which is true if and only if

$$\cfrac{1}{\cfrac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1} - (f_{4n+3} - 1) = \frac{\xi f_{2n+2}}{f_{2n+1} + \xi f_{2n+2}}$$

which, in turn, is true if and only if

$$\frac{f_{2n+2}}{\xi f_{2n+1} - f_{2n+2}} - \frac{\xi f_{2n+2}}{f_{2n+1} + \xi f_{2n+2}} = f_{4n+3} - 1 .$$

To see that this last equation is true we make use of (8), (10), and the fact that $\xi^2 = a\xi + 1$ to obtain

$$\frac{f_{2n+2}}{\xi f_{2n+1} - f_{2n+2}} - \frac{\xi f_{2n+2}}{f_{2n+1} + \xi f_{2n+2}} = \frac{f_{2n+2}f_{2n+1} + \xi f_{2n+2}^2 - \xi^2 f_{2n+1}f_{2n+2} + \xi f_{2n+2}^2}{\xi f_{2n+1}^2 + \xi^2 f_{2n+1}f_{2n+2} - f_{2n+1}f_{2n+2} - \xi f_{2n+2}^2}$$

$$= \frac{2\xi f_{2n+2}^2 - a\xi f_{2n+1}f_{2n+2}}{\xi f_{2n+1}^2 + a\xi f_{2n+1}f_{2n+2} - \xi f_{2n+2}^2}$$

$$= \frac{f_{2n+2}(f_{2n+2} + f_{2n})}{f_{2n+1}f_{2n+3} - f_{2n+2}^2}$$

$$= f_{2n+2}^2 + f_{2n+2}f_{2n} + f_{2n+1}f_{2n+3} - f_{2n+2}^2 - 1$$

$$= f_{4n+3} - 1 .$$

This completes the proof.

Because of the similarity of method, the following theorems are stated without proof. The notation is as before.

Theorem 14. For $n \geq 2$,

$$\frac{f_{2n}}{f_{2n+1}} \cdot \xi = [a_0, a_1, \dot{a}_2, \cdots, \dot{a}_r]$$

and

$$\frac{f_{2n+1}}{f_{2n}} \cdot \xi = [a_3 - 1, \dot{a}_4, a_5, \cdots, a_r, a_2, \dot{a}_3] \quad ,$$

where the vector $(a_3, a_4, \cdots, a_r)$ is symmetric, $a_0 = 0$, $a_1 = 1$, $a_2 = f_{4n+1} - 1$, and $a_3 = f_3 + 1$.

Theorem 15. Let $n \geq 2$ be an integer. Then

$$\frac{f_{n+2}}{g_n} \cdot \xi = [\dot{a}_0, a_1, \cdots, \dot{a}_r] ,$$

$$\frac{g_n}{f_{n+2}} \cdot \xi = [\dot{a}_r, a_{r-1}, \cdots, \dot{a}_0],$$

$$\frac{f_{n+1}}{g_n} \cdot \xi = [\dot{b}_0, b_1, \cdots, \dot{b}_s] ,$$

and

$$\frac{g_n}{f_{n+1}} \cdot \xi = [\dot{b}_s, b_{s-1}, \cdots, \dot{b}_0] .$$

Theorem 16. Let n be a positive integer. Then

$$\frac{g_{2n+1}}{g_{2n+2}} \cdot \xi = [a_0, a_1, \dot{a}_2, \cdots, \dot{a}_r] ,$$

and

$$\frac{g_{2n+2}}{g_{2n+1}} \cdot \xi = [2, \dot{a}_4, a_5, \cdots, a_r, a_2, \dot{a}_3]$$

and the vector $(a_3, a_4, \cdots, a_r)$ is symmetric with $a_0 = 0$, $a_1 = 1$, $a_2 = f_{4n+3} - 1$, and $a_3 = 3$.

Theorem 17. Let n be a positive integer. Then

$$\frac{g_{2n}}{g_{2n+1}} \cdot \xi = [\dot{a}_0, a_1, \cdots, \dot{a}_r]$$

and

$$\frac{g_{2n+1}}{g_{2n}} \cdot \xi = [\dot{a}_r, a_{r-1}, \cdots, a_1, \dot{a}_0] ,$$

where the vector $(a_2, a_3, \cdots, a_r, a_0)$ is symmetric with $a_0 - a_2 = 1$, and $a_1 = f_{4n+1} - 1$.

In view of the preceding results, one would expect an interesting theorem concerning the simple continued fraction expansion of

$$\frac{f_n}{g_n} \cdot \xi \quad \text{and} \quad \frac{g_n}{f_n} \cdot \xi$$

but we were not able to make a general assertion value for all a. To illustrate the difficulty, note that, when $a = 2$ and $\xi = 1 + \sqrt{2}$, we have

$$\frac{f_4}{g_4} \cdot \xi = [0, 1, \dot{5}, 1, 3, 5, 1, \dot{7}] ,$$

$$\frac{f_5}{g_5} \cdot \xi = [0, 1, \dot{5}, 1, 5, 3, 1, 4, 1, \dot{7}] ,$$

and

$$\frac{f_6}{g_6} \cdot \xi = [0, 1, \dot{5}, 1, 4, 1, 3, 5, 1, 4, 1, \dot{7}] .$$

However, for

$$\xi = \alpha = [\dot{1}] = \frac{1 + \sqrt{5}}{2} ,$$

we obtain the following rather elegant result:

Theorem 18. Let $\alpha = (1 + \sqrt{5})/2$ and let F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively. Then, for $n \geq 4$,

$$\frac{F_n}{L_n} \cdot \alpha = [0, 1, \dot{2}, 1, \cdots, 1, 3, 1, \cdots, 1, \dot{4}] \tag{21}$$

and

$$\frac{L_n}{F_n} \cdot \alpha = [3, \dot{1}, \cdots, 1, 3, 1, \cdots, 1, 2, \dot{4}] , \tag{22}$$

where, in (21), there are $n - 4$ ones in the first group and $n - 3$ ones in the second group and just the reverse in (22).

Proof. Set

$$\begin{aligned} x_n &= [\dot{2}, \dot{1}, \cdots, 1, 3, 1, \cdots, 1, \dot{4}] \\ &= [2, 1, \cdots, 1, 3, 1, \cdots, 1, 4, x_n] . \end{aligned}$$

Then it is easy to see by direct computation as on computes convergents, that

$$x_n = \frac{a_n x_n + b_n}{c_n x_n + d_n} , \tag{23}$$

where

$$\begin{aligned}
a_n &= 4(L_{n-1}F_{n-1} + F_{n-2})^2 + (L_{n-2}F_{n-1} + F_{n-3}F_{n-2}) \\
&= 4F_n^2 + F_nF_{n-1} + (-1)^n, \\
b_n &= L_{n-1}F_{n-1} + F_{n-2}^2 = F_n^2, \\
c_n &= 4(L_{n-1}F_{n-3} + F_{n-2}F_{n-4}) + (L_{n-2}F_{n-3} + F_{n-3}F_{n-4}) \\
&= 4F_{n-1}^2 + F_nF_{n-3},
\end{aligned}$$

and

$$d_n = L_{n-1}F_{n-3} + F_{n-2}F_{n-4} = F_{n-1}^2 .$$

Moreover, from (23),

$$x_n = \frac{(a_n - d_n) + \sqrt{(a_n - d_n)^2 + 4b_nc_n}}{2c_n}$$

and

$$\begin{aligned}
y_n &= [0,\ 1,\ x_n] \\
&= \frac{x_n}{x_n + 1} \\
&= \frac{(a_n - d_n) + \sqrt{(a_n - d_n)^2 + 4b_nc_n}}{(a_n - d_n + 2c_n) + \sqrt{(a_n - d_n)^2 + 4b_nc_n}} \\
&= \frac{(a_n - d_n - 2b_n) + \sqrt{(a_n - d_n)^2 + 4b_nc_n}}{2(a_n - b_n + c_n - d_n)} .
\end{aligned}$$

Now

$$
\begin{aligned}
a_n - d_n - 2b_n &= 4F_n^2 + F_nF_{n-1} + (-1)^n - F_{n-1}^2 - 2F_n^2 \\
&= 2F_n^2 + F_nF_{n-1} - F_nF_{n-2} \\
&= F_n(2F_n + F_{n-1} - F_{n-2}) \qquad (25) \\
&= F_n(F_{n+2} - F_{n-2}) \\
&= F_nL_n ,
\end{aligned}
$$

$$
\begin{aligned}
a_n - b_n + c_n - d_n &= (a_n - d_n - 2b_n) + b_n + c_n \\
&= F_nL_n + F_n^2 + 4F_{n-1}^2 + F_nF_{n-3} \qquad (26) \\
&= F_nL_n + 2F_{n-1}L_n \\
&= L_n^2
\end{aligned}
$$

and

$$
\begin{aligned}
(a_n - d_n)^2 + 4b_nc_n &= (4F_n^2 + F_nF_{n-1} + (-1)^n - F_{n-1}^2)^2 \\
&\quad + 4F_n^2(4F_{n-1}^2 + F_nF_{n-3}) \\
&= F_n^2F_{n+3}^2 + 4F_n^2(4F_{n-1}^2 + F_nF_{n-3}) \qquad (27) \\
&= F_n^2(F_{n+3}^2 + 16F_{n-1}^2 + 4F_nF_{n-3}) \\
&= 5F_n^2L_n^2 .
\end{aligned}
$$

Thus, using (25), (26), and (27), in (24), we obtain

$$
y_n = \frac{F_nL_n + F_nL_n\sqrt{5}}{2L_n^2} = \frac{F_n}{L_n} \cdot \frac{1 + \sqrt{5}}{2} ,
$$

as claimed. The other part of the proof is an immediate consequence of Theorem 11.

Finally, we comment on the question of the equivalence of $r\xi$ and ξ/r. If $r = g_m/f_n$ or $r = g_m/g_n$, where m and n are nonnegative integers, it frequently turns out to be the case that $r\xi \sim \xi/r$. However, this is not necessarily the case and hence, a fortiori, it is not necessarily the case for more general r. For example, for $\alpha = (1+\sqrt{5})/2 = [\dot{1}]$,

$$\frac{3}{7} \cdot \alpha = [0, 1, \dot{2}, 3, 1, \dot{4}]$$

and

$$\frac{7}{3} \cdot \alpha = [3, \dot{1}, 3, 2, \dot{4}]$$

where $3 = f_4 = g_2$ and $7 = g_4$; and other examples are easily found. However, if $r = f_m$ and $s = f_n$ for nonnegative integers m and n then we always have

$$\frac{r}{s} \cdot \xi \sim \frac{s}{r} \cdot \xi \quad ,$$

as the following theorem shows.

Theorem 18. If m and n are nonnegative integers, then

$$\frac{f_m}{f_n} \cdot \xi \sim \frac{f_n}{f_m} \cdot \xi \quad .$$

<u>Proof.</u> Without loss of generality, we may assume that $0 < m < n$ and that $(m,n) = 1$. We let

$$a = \frac{f_m f_{2qm+2}}{f_n}, \quad b = c = f_{2qm+1}, \quad d = \frac{f_n f_{2qm}}{f_m},$$

where q is chosen so that

$$2q + 2 \equiv 0 \pmod{n},$$

as may easily be done since $(m,n) = 1$. With this choice for q it follows from Lemma 4 that $f_n | f_{2qm+2}$ and $f_m | f_{2qm}$ so that a, b, c, and d are all integers. Also, by (10),

$$\begin{aligned} ad - bc &= \frac{f_m f_{2qm+2}}{f_n} \cdot \frac{f_n f_{2qm}}{f_m} - f^2_{2qm+1} \\ &= f_{2qm+2} f_{2qm} - f^2_{2qm+1} \\ &= -1 . \end{aligned}$$

Finally, we show that

$$\frac{f_m}{f_n} \cdot \xi = \frac{a\left(\frac{f_n}{f_m} \cdot \xi\right) + b}{c\left(\frac{f_n}{f_m} \cdot \xi\right) + c} \tag{28}$$

for this choice of a, b, c, and d. Making the indicated substitutions, we have that (28) holds if and only if

$$\frac{f_m}{f_n} \cdot \xi = \frac{\frac{f_m f_{2qm+2}}{f_n}\left(\frac{f_n}{f_m} \cdot \xi\right) + f_{2qm+1}}{f_{2qm+1}\left(\frac{f_n}{f_m} \cdot \xi\right) + \frac{f_n f_{2qm}}{f_m}},$$

and this is true if and only if

$$\xi^2 f_{2qm+1} + \xi f_{2qm} = \xi f_{2qm+2} + f_{2qm+1} .$$

But this is clearly true since $a\xi^2 = a\xi + 1$ and $af_{2qm+1} + f_{2qm} = f_{2qm+2}$ and the proof is complete.

Finally, we note that the list of stated theorems is not exhaustive. One could no doubt prove theorems concerning

$$\frac{f_n}{f_{n+2}}\xi, \; \frac{f_n}{f_{n+4}}\xi, \; \frac{f_n}{f_{n+5}}\xi \quad ,$$

and so on. However, we were not able to arrive at general formulations of the expansions of

$$\frac{f_m}{f_n}\cdot\xi, \quad \frac{f_m}{g_n}\cdot\xi, \quad \text{or} \quad \frac{g_m}{g_n}\cdot\xi \quad ,$$

for arbitrary positive integers m and n. The results stated seem to be the most interesting.

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CONVOLUTION TRIANGLES FOR GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

The sequence of integers $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ are called the Fibonacci numbers. The numbers F_1 and F_2 are called the starting pair and $F_{n+2} = F_{n+1} + F_n$ is called the recurrence relation. The long division problem $1/(1 - x - x^2)$ yields

$$\frac{1}{1 - x - x^2} = F_1 + F_2x + F_3x^2 + \cdots + F_{n+1}x^n + \cdots .$$

This expression is called a generating function for the Fibonacci numbers. The generating function yielding

$$\frac{1}{(1 - x - x^2)^{k+1}} = F_1^{(k)} + F_2^{(k)}x + \cdots + F_{n+1}^{(k)}x^n + \cdots$$

is the generating function for the k^{th} convolution of the Fibonacci numbers. For $k = 0$, we get just the Fibonacci numbers. We now show two different ways to get the convolved Fibonacci numbers.

2. CONVOLUTION OF SEQUENCES

If $a_1, a_2, a_3, \cdots, a_n, \cdots$ and $b_1, b_2, b_3, \cdots, b_n, \cdots$ are two sequences, then the convolution of the two sequences is another sequence $c_1, c_2, c_3, \cdots, c_n, \cdots$ whose terms are calculated as shown:

$$\begin{aligned}
c_1 &= a_1b_1 \\
c_2 &= a_1b_2 + a_2b_1 \\
c_3 &= a_1b_3 + a_2b_2 + a_3b_1 \\
&\cdots\cdots\cdots \\
c_n &= a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \cdots + a_kb_{n-k+1} + \cdots + a_nb_1 .
\end{aligned}$$

This last expression may also be written

$$c_n = \sum_{k=1}^{n} a_k b_{n-k+1} \quad .$$

Let us convolve the Fibonacci number sequence with itself. These numbers we call the <u>First Fibonacci Convolution Sequence</u>:

$$\begin{aligned}
F_1^{(1)} &= F_1F_1 & &= 1 \\
F_2^{(1)} &= F_1F_2 + F_2F_1 &= 1 + 1 &= 2 \\
F_3^{(1)} &= F_1F_3 + F_2F_2 + F_3F_1 &= 2 + 1 + 2 &= 5 \\
F_4^{(1)} &= F_1F_4 + F_2F_3 + F_3F_2 + F_4F_1 &= 3 + 2 + 2 + 3 &= 10 \\
F_5^{(1)} &= \sum_{k=1}^{5} F_k F_{5-k+1} & &= 20 \\
F_6^{(1)} &= \sum_{k=1}^{6} F_k F_{6-k+1} & &= 38 \\
F_7^{(1)} &= \sum_{k=1}^{7} F_k F_{7-k+1} & &= 71 \quad .
\end{aligned}$$

Now let us "convolve" the first Fibonacci convolution sequence with the Fibonacci sequence to get the <u>Second Fibonacci Convolution Sequence</u>:

$$\begin{aligned}
F_1^{(2)} &= F_1F_1^{(1)} & &= 1 \\
F_2^{(2)} &= F_2F_1^{(1)} + F_1F_2^{(1)} & &= 3 \\
F_3^{(2)} &= F_3F_1^{(1)} + F_2F_2^{(1)} + F_1F_2^{(1)} & &= 9 \\
F_4^{(2)} &= F_4F_1^{(1)} + F_3F_2^{(1)} + F_2F_3^{(1)} + F_1F_4^{(1)} &= 3\cdot 1 + 2\cdot 2 + 1\cdot 5 + 1\cdot 10 &= 22 \\
F_5^{(2)} &= \sum_{k=1}^{5} F_k^{(1)} F_{5-k+1} & &= 41 \\
F_6^{(2)} &= \sum_{k=1}^{6} F_k^{(1)} F_{6-k+1} & &= 111
\end{aligned}$$

The Fibonacci sequence is obtained from

$$\frac{1}{1 - x - x^2} = F_1 + F_2 x + F_3 x^2 + \cdots + F_{n+1}x^n + \cdots$$

The first Fibonacci convolution sequence is obtained from

$$\frac{1}{(1 - x - x^2)^2} = F_1^{(1)} + F_2^{(1)}x + F_3^{(1)}x^2 + \cdots + F_{n+1}^{(1)}x^n + \cdots \;.$$

The second Fibonacci convolution sequence is obtained from

$$\frac{1}{(1 - x - x^2)^3} = F_1^{(2)} + F_2^{(2)}x + F_3^{(2)}x^2 + \cdots + F_{n+1}^{(2)}x^n + \cdots \;.$$

These could all have been obtained by long division and continued to find as many $F_n^{(k)}$ as desired or one could have found the convoluted sequence by the method of this section. In the next section we shall see yet another way to find the convolved Fibonacci sequences.

3. THE FIBONACCI CONVOLUTION TRIANGLE

Suppose one writes down a column of zeros. To the right and one space down place a one. To get the elements below the one we add the elements one above and the one directly left. This is, of course, the rule of formation for Pascal's arithmetic triangle. Such a rule generates a convolution triangle.

Next suppose instead we add the one above and then diagonally left. Now the row sums are the Fibonacci numbers. We illustrate:

	0						
	0						
	0	1					
	0	1					
	0	1	1				
	0	1	2				
	0	1	3	1			
	0	1	4	3			
	0	1	5	6	1		
	0	1	6	10	4		
	0	1	7	15	10	1	
Column:	0	1	2	3	4	···	n

However, if we add the two elements above and diagonally left, we generate the Fibonacci convolution triangle as follows. Please note these are the same numbers we got in Section 2. The zero-th column are the Fibonacci numbers, F_n; the first column are the first convolution Fibonacci numbers, $F_n^{(1)}$, etc.

	0					
	0					
	0	1				
	0	1				
	0	2				
	0	3	1			
	0	5	2			
	0	8	5			
	0	13	$\boxed{10}$	1		
	0	21	20	3		
	0	34	38	9		
	0	55	71	21	1	
	. . .	. . .	. . .	. . .	. . .	
Column:		0	1	2	3	$\cdots$ n

4. COLUMN GENERATORS OF CONVOLUTION TRIANGLES

It is easily established that the column generating functions for Pascal's triangle are

$$g_k(x) = \frac{x^k}{(1 - x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n ,$$

when the triangle is generated normally as the expansion of $(1 + x)^n$, $n = 0, 1, 2, \cdots$ and as we said to do in the first part of Section 3. The column generators become

$$g_k(x) = \frac{x^{2k}}{(1 - x)^{k+1}}, \qquad k = 0, 1, 2, \cdots$$

if we follow the second set of instructions. These column generators are such that the elements across the rows each are multiplied by the same power of x. We make the column move up or down by changing the power of x in the numer-erator of the column generating function. If we now sum

$$\sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(1-x)^{k+1}} = \left(\frac{1}{1-x}\right)\sum_{k=0}^{\infty}\left(\frac{x^2}{1-x}\right)^k$$

$$= \frac{1}{1-x}\cdot\frac{1}{1-\dfrac{x^2}{1-x}} = \frac{1}{1-x-x^2} .$$

Thus the row sums across the specially positioned (Position 2) Pascal triangle are Fibonacci numbers. These are, of course, the numbers in the zero-th column of the Fibonacci convolution triangle. If we multiply the column generators of Pascal's triangle by a special set of coefficients, we may obtain other columns of the Fibonacci convolution triangle.

Recall that the k^{th} column generator of Pascal's triangle is

$$g_k(x) = \sum_{n=0}^{\infty}\binom{n}{k}x^n = \frac{x^k}{(1-x)^{k+1}} .$$

Replace x by

$$\frac{x^2}{1-x}$$

in the above to obtain

$$g_k\left(\frac{x^2}{1-x}\right) = \sum_{n=0}^{\infty}\binom{n}{k}\left(\frac{x^2}{1-x}\right)^n = \frac{\left(\dfrac{x^2}{(1-x)}\right)^k}{\left(1-\dfrac{x^2}{1-x}\right)^{k+1}}$$

while

$$\left(\frac{1}{1-x}\right) g_k\left(\frac{x^2}{1-x}\right) = \sum_{n=0}^{\infty} \binom{n}{k} g_k = \frac{\left(\frac{1}{1-x}\right) \cdot \left(\frac{x^2}{1-x}\right)^k}{\left(1 - \frac{x^2}{1-x}\right)^{k+1}} = \frac{x^{2k}}{(1-x-x^2)^{k+1}}$$

Thus the row sums are the k^{th} convolution of the Fibonacci numbers since that is the column generator we have obtained. We illustrate:

(First column of Pascal)

Multipliers:	0	1	2	3	4	Row Sums:	
	1						
	1						
	1	1				1	= 1
	1	2				2	= 2
	1	3	1			3·1 + 1·2	= 5
	1	4	3			4·1 + 3·2	= 10
	1	5	6	1		5·1 + 6·2 + 1·3	= 20
	1	6	10	4		6·1 + 10·2 + 4·3	= 38
	1	7	15	10	1	7·1 + 15·2 + 10·3 + 1·4	= 71
	1	8	21	20	5	8·1 + 21·2 + 20·3 + 5·4	=130

First Fibonacci Convolution Sequence

(Second column of Pascal)

Multipliers:	0	0	1	3	6	10	···	Row Sums:
	1							
	1							
	1	1						
	1	2						
	1	3	1					1
	1	4	3					3
	1	5	6	1				9
	1	6	10	4				22
	1	7	15	10	1			51
	1	8	21	20	5			111
	1	9	28	35	15	1		233

Second Fibonacci Convolution Sequence

Thus if we use the numbers in the k^{th} column of Pascal's arithmetic triangle (Position 1) as a set of multipliers with the columns of Pascal's triangle (Position 2), we get row sums which form the k^{th} Fibonacci sequence.

5. EXTENSION TO GENERALIZED FIBONACCI NUMBERS CONVOLUTION TRIANGLES

The Fibonacci numbers are the sums of the rising diagonals of Pascal's triangle which is generated by expanding $(1+x)^n$. The generalized Fibonacci numbers are defined as the sums of the diagonals of generalized Pascal's triangles which are generated by expanding

$$(1 + x + x^2 + \cdots + x^{r-1})^n .$$

The sequences can be shown to satisfy $u_1 = 1$, $u_j = 2^{j-2}$ for $j = 2, 3, \cdots, r$, and

$$u_{n+r} = \sum_{j=1}^{r} u_{n+r-j}, \quad n \geq 1 ,$$

and the generating functions are

$$\frac{1}{1 - x - x^2 - \cdots - x^{r-1}} = \sum_{n=0}^{\infty} u_{n+1} x^n .$$

The simplest instance is the Tribonacci sequence, where $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, and these sums are the rising diagonal sums of the expansions of $(1 + x + x^2)^n$ for $n = 0, 1, 2, 3, \cdots$. The first few terms are 1, 1, 2, 4, 7, 13, 24, 44, 81.

If we return to our Fibonacci convolution triangle at the end of Section 3, we note the row sums are the Tribonacci numbers. The column generators of the Fibonacci convolution are

$$g_k(x) = \frac{x^{3k}}{(1 - x - x^2)^{k+1}} ,$$

where the numbers on each row in the triangle all multiply the same powers of x in the column generators. Adding, we get

$$\sum_{k=0}^{\infty} g_k(x) = \left(\frac{1}{1 - x - x^2}\right)\sum_{k=0}^{\infty}\left(\frac{x^3}{1 - x - x^2}\right)^k = \frac{1}{1 - x - x^2 - x^3}$$

which is the Tribonacci sequence generating function. If we use the special multipliers $\binom{n}{k}$ as before, we get

$$\left(\frac{1}{1 - x - x^2}\right)\sum_{k=0}^{\infty}\binom{n}{k}\left(\frac{x^3}{1 - x - x^2}\right)^k = \sum_{k=0}^{\infty}\binom{n}{k} g_k(x) = \frac{x^{3k}}{(1 - x - x^2 - x^3)^{k+1}}$$

and this is the k^{th} Tribonacci convolution sequence generator and the coefficients appear in the k^{th} column of the Tribonacci convolution triangle. Thus we can obtain all the columns of the Tribonacci convolution triangle from the Fibonacci convolution triangle in the same way we obtained the Fibonacci convolution triangle from Pascal's arithmetic triangle.

We can thus generate a sequence of convolution triangles whose zero-th columns are the rising diagonal sums taken from generalized Pascal triangles induced from expansions of $(1 + x + x^2 + \cdots + x^{r-1})^n$. The column generators for the r^{th} case

$$g_k(x) = \frac{x^{rk}}{(1 - x - x^2 - \cdots - x^{r-1})^{k+1}}$$

can easily be seen to generate the column generators for the $(r+1)^{st}$ case

$$g_k(x) = \frac{x^{(r+1)k}}{(1 - x - x^2 - \cdots - x^r)^{k+1}}$$

using the preceding methods.

Referring back to the Fibonacci convolution triangle of section three, each number in the triangle is the sum of the one number above and the number diagonally left. Because the column generators must obey that law and multiplying by powers of x so that the proper coefficients will be added, we could write a recurrence relation for the column generators of the Pascal convolution triangle as follows:

$$G_k(x) = xG_k(x) + x^2G_{k-1}(x) \quad \text{or} \quad G_k(x) = \frac{x^2}{1-x}G_{k-1}(x) .$$

By similar reasoning, each number of the Fibonacci convolution triangle is the sum of the two terms above it and one diagonally left. Proceeding to column generators, then,

$$G_k(x) = xG_k(x) + x^2G_k(x) + x^3G_{k-1}(x)$$

or

$$G_k(x) = \frac{x^3}{1 - x - x^2} G_{k-1}(x) .$$

6. THE REVERSE PROCESS

One can retrieve the Fibonacci convolution triangle from the Tribonacci convolution triangle quite simply. First recall

$$\sum_{n=0}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}} .$$

Replace x by $-x$; then it becomes

$$\sum_{n=0}^{\infty} \binom{n}{k} (-1)^n x^n = \frac{(-1)^k x^k}{(1+x)^{k+1}} ,$$

or

$$\sum_{n=0} \binom{n}{k} (-1)^{n+k} x^n = \frac{x^k}{(1 + x)^{k+1}} .$$

With these multipliers,

$$\binom{n}{k} (-1)^{n+k} ,$$

we can return from Tribonacci to Fibonacci.

Let the column generators of the Tribonacci case be

$$g_n(x) = \frac{x^{3n}}{(1 - x - x^2 - x^3)^{n+1}} ,$$

and multiplying through by

$$\binom{n}{k} (-1)^{n+k} ,$$

and summing, yields

$$\sum_{n=0}^{\infty} \binom{n}{k} (-1)^{n+k} g_n(x) = \frac{1}{1 - x - x^2 - x^3} \sum_{n=0}^{\infty} \binom{n}{k} (-1)^{n+1} \left(\frac{x^3}{1 - x - x^2 - x^3} \right)^n$$

$$= \frac{x^{3k}/(1 - x - x^2 - x^3)^{k+1}}{\left(1 + \dfrac{x^3}{1 - x - x^2 - x^3} \right)^{k+1}} = \frac{x^{3k}}{(1 - x - x^2)^{k+1}}$$

which are the column generators of the Fibonacci convolution triangle. The same thing applies, in general, to return from the $(r+1)^{st}$ convolution triangle to the r^{th} convolution triangle.

7. SPECIAL PROBLEMS

1. Assuming Pascal's triangle in Position 1 and the column generators are

$$g_k(x) = \frac{x^k}{(1 - x)^{k+1}} ,$$

then show the row sums of Pascal's triangle are the powers of 2. Hint:

$$\frac{1}{1 - 2x} = \sum_{n=0}^{\infty} 2^n x^n .$$

2. Assuming the Fibonacci convolution triangle has its columns positioned so that

$$g_k(x) = \frac{x^k}{(1 - x - x^2)^{k+1}} ,$$

then show the row sums are the Pell numbers $P_1 = 1$, $P_2 = 2$, $P_{n+2} = 2P_{n+1} + P_n$. Hint:

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} P_{n+1}x^n .$$

3. Show that the convolution triangle for the sequence $1, 3, 3^2, \cdots, 3^n, \cdots$ can be obtained from the convolution triangle for the sequence $1, 2, 2^2, 2^3, \cdots, 2^n, \cdots$ using the techniques discussed in this paper.

4. By using the coefficients in

$$\sum_{n=0}^{\infty} \binom{n}{k} (-1)^{n+k} x^n ,$$

as multipliers, show how to get the convolution triangle for the alternate Fibonacci numbers from the convolution triangle for the powers of three. Hint:

$$\frac{1}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n+1} x^n .$$

5. By using the multipliers from

$$\sum_{n=0}^{\infty} 2\left[\binom{n}{k} \cdot 3^n\right] x^n ,$$

on the Fibonacci convolution triangle with column generators

$$g_k(x) = \frac{x^k}{1 - x - x^2} ,$$

obtain the convolution triangle for every third Fibonacci number sequence. Hint:

$$\frac{2}{1 - 4x + x^2} = \sum_{n=0}^{\infty} F_{3n+1} x^n .$$

8. OTHER CONVOLUTION TRIANGLES

Let

$$\left(\frac{(1 - x)^{q-1}}{(1 - x)^q - x^{p+q}}\right)^{k+1} = \sum_{n=0}^{\infty} u^{(k)}(n; p, q)\, x^n ,$$

be the k^{th} convolution of the sequence $u(n; p,q)$, where the sequences $u(n; p,q)$ are the generalized Fibonacci numbers of Harris and Styles [1]. (Also see [2].)

Let

$$g_n(x) = \frac{x^{(p+q)n}}{(1 - x)^{nq+1}}$$

$$\sum_{n=0}^{\infty} g_n(x) = \frac{1}{1 - x} \sum_{n=0}^{\infty} \left(\frac{x^{p+q}}{(1 - x)^q} \right)^n = \frac{(1 - x)^{q-1}}{(1 - x)^q - x^{p+q}} \;.$$

But,

$$\sum_{n=0}^{\infty} \binom{k + n}{k} x^n = \frac{1}{(1 - x)^{k+1}} \;.$$

Thus,

$$\frac{1}{1 - x} \sum_{n=0}^{\infty} \binom{k + n}{k} \left(\frac{x^{p+q}}{(1 - x)^q} \right)^n = \frac{(1 - x)^{(k+1)q-1}}{\left[(1 - x)^q - x^{p+q}\right]^{k+1}}$$

$$\sum_{n=0}^{\infty} \binom{k + n}{k} g_n^{(k)}(x) = \frac{1}{(1 - x)^{k+1}} \sum_{n=0}^{\infty} \binom{k + n}{n} \left(\frac{x^{p+q}}{(1 - x)^q} \right)^n$$

$$= \frac{(1 - x)^{(k+1)q-1-k}}{\left[(1 - x)^q - x^{p+q}\right]^{k+1}}$$

$$= \frac{(1 - x)^{(q-1)(k+1)}}{\left[(1 - x)^q - x^{p+q}\right]^{k+1}} = \sum_{n=0}^{\infty} u^{(k)}(n;\, p, q) x^n$$

and the $g_n^{(k)}(x)$ are the corresponding column generators in the Pascal's triangle with the first k columns trimmed off.

9. REVERSING THE PROCESS, AGAIN

If we consider the convolution triangles whose column generators are

$$g_n(x) = \frac{\left(x^{p+q}\right)^n}{\left((1 - x)^q - x^{p+q}\right)^{n+1}} ,$$

and if we sum these with alternating signs,

$$\sum_{n=0}^{\infty} (-1)^n g_n(x) = \frac{1}{(1 - x)^q - x^{p+q}} \; \frac{1}{1 + \dfrac{x^{p+q}}{(1 - x)^q - x^{p+q}}} = \frac{1}{(1 - x)^q}$$

while

$$\left[\frac{1}{(1 - x)^q - x^{p+q}}\right]^k \sum_{k=0}^{\infty} \binom{n + k}{k} (-1)^n g_n(x) = \frac{1}{(1 - x)^{q(k+1)}} .$$

Thus, we can recover the columns of Pascal's triangle from the above convolution triangle. This may be extended in many ways. Thus, we can obtain the convolution triangles for all the sequences $u(n; p,q)$ by using multipliers from Pascal's triangle on the column generators of Pascal's triangle and taking row sums.

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SOME PROPERTIES OF STIRLING NUMBERS OF THE SECOND KIND

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INTRODUCTION

In attempting to predict the number of demands that will occur during a given period of time, for supplies in military inventory systems, it becomes necessary to formulate suitable probability models for the distribution of demands of individual items of supply. One such model, described in [1], involves two parameters, to be estimated from available data. For example, in the case of predicting demands for items installed on Polaris submarines, the data might consist of items demanded in a series of patrols.

In studying the properties of estimation procedures for parameters of any model, one is led to a consideration of the sampling distributions of the estimates. For the model described in [1], the sampling distributions of some proposed estimates were found to involve Stirling numbers of the second kind, and in the derivation of these distributions from the initial probability assumptions, some properties of these numbers become of interest.

A Stirling number of the second kind, $\mathcal{S}_T^{(m)}$, is the number of ways of partitioning a set of T elements into m non-empty subsets. Thus, if we have the set of elements (1, 2, 3) with T = 3 and m = 2, we have sible partitions

$$\begin{array}{l}(1,2),\ (3)\\(1,3),\ (2)\\(2,3),\ (1)\end{array}$$

with

$$\mathcal{S}_3(2) = 3 .$$

If the order of the partitions is taken into account, that is,

$$(1,2),\ (3) \ ,$$

and

$$(3), \quad (1,2)$$

are considered to be two partitions, the number of ordered partitions is

$$m!\mathcal{S}_T^{(m)} \; .$$

For example, suppose that a given item installed in a Polaris submarine is demanded in each of m partols, with a total quantity demanded of T units, ($T \geq m$). The number of different ways of partitioning T into m demands is $\mathcal{S}_T^{(m)}$; the number of ways in which a particular partition can be assigned to the m patrols is $m!$; thus the number of possible assignments of the total quantity demanded to the m patrols is $m!\mathcal{S}_T^{(m)}$.

PROPERTIES OF STIRLING NUMBERS OF THE SECOND KIND

The generating function of Stirling numbers of the second kind is

$$x^T = \sum_{m=0}^{T} \mathcal{S}_T^{(m)} x(x-1)\cdots(x-m+1) \; .$$

In closed form,

$$\mathcal{S}_T^{(m)} = \frac{1}{m!}\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} k^T \; .$$

Various properties of these numbers are known (e.g., see [2]). Thus,

$$\mathcal{S}_T^{(m)} = 0 \quad \text{for} \quad T < m$$
$$\mathcal{S}_T^{(T)} = 1 \quad \text{for} \quad T = 0, 1, 2, \cdots$$
$$\mathcal{S}_T^{(0)} = 0 \quad \text{for} \quad T = 1, 2, \cdots$$
$$\mathcal{S}_T^{(1)} = 1 \quad \text{for} \quad T = 1, 2, \cdots \; .$$

We have the recurrence relations

$$(2) \qquad \mathscr{S}_{T+1}^{(m)} = m\mathscr{S}_T^{(m)} + \mathscr{S}_T^{(m-1)} \quad \text{for} \quad T \ge m \ge 1$$

$$(3) \qquad \binom{m}{r}\mathscr{S}_T^{(m)} = \sum_{k=m-r}^{T-r}\binom{T}{k}\mathscr{S}_{T-k}^{(r)}\mathscr{S}_k^{(m-r)} \quad \text{for} \quad T \ge m \ge r\,.$$

If $r = 1$, Eq. (3) becomes

$$(4) \qquad m\mathscr{S}_T^{(m)} = \sum_{k=m-1}^{T-1}\binom{T}{k}\mathscr{S}_k^{(m-1)}$$

The following results appear to be less well known.

Lemma 1. For any integers r and k, with $k = 0, 1, \cdots$, and $r = k+1, k+2, \cdots$,

$$(5) \qquad \sum_{j=0}^{r}\binom{r+k}{j}(-1)^j\mathscr{S}_{r-j+k}^{(r-j)} = 0\,.$$

Proof. We prove the lemma by induction on k. In the proof, we use the recurrences

$$\binom{r+k}{j} = \binom{r+k-1}{j} + \binom{r+k-1}{j-1}$$

and

$$\mathscr{S}_{r-j+k}^{(r-j)} = (r-j)\mathscr{S}_{r-j+k-1}^{(r-j)} + \mathscr{S}_{r-j+k-1}^{(r-j-1)}$$

For $k = 0$ and $r = 1, 2, \cdots,$

$$\sum_{j=0}^{r} \binom{r}{j}(-1)^j \mathcal{S}_{r-j}^{(r-j)} = \sum_{j=0}^{r} \binom{r}{j}(-1)^j = (1 - 1)^r = 0 \ .$$

For $k = 1$ and $r = 2, 3, \cdots,$

$$\sum_{j=0}^{r} \binom{r+1}{j}(-1)^j \mathcal{S}_{r-j+1}^{(r-j)} = \sum_{j=0}^{r} \binom{r}{j}(-1)^j \mathcal{S}_{r-j+1}^{(r-j)}$$

$$+ \sum_{j=1}^{r} \binom{r}{j-1}(-1)^j \mathcal{S}_{r-j+1}^{(r-j)}$$

$$= \sum_{j=0}^{r} \binom{r}{j}(-1)^j (r - j)\mathcal{S}_{r-j}^{(r-j)}$$

$$+ \sum_{j=0}^{r-1} \binom{r}{j}(-1)^j \mathcal{S}_{r-j}^{(r-j-1)} - \sum_{j=0}^{r-1} \binom{r}{j}(-1)^j \mathcal{S}_{r-j}^{(r-j-1)}$$

The last two terms cancel each other while the first one becomes

$$r \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j = 0 \ .$$

We suppose the result holds for $k = m - 1$, and $r = m,\ m + 1, \cdots,$ and show that it holds for $k = m$, $r = m + 1, m + 2, \cdots$. We have

$$\sum_{j=0}^{r}\binom{r+m}{j}(-1)^j\mathcal{S}_{r-j+m}^{(r-j)}$$

$$= \sum_{j=0}^{r}\binom{r+m-1}{j}(-1)^j\mathcal{S}_{r-j+m}^{(r-j)}$$

$$+ \sum_{j=1}^{r}\binom{r+m-1}{j-1}(-1)^j\mathcal{S}_{r-j+m}^{(r-j)}$$

$$= \sum_{j=0}^{r}\binom{r+m-1}{j}(-1)^j(r-j)\mathcal{S}_{r-j+m-1}^{(r-j)}$$

$$+ \sum_{j=0}^{r-1}\binom{r+m-1}{j}(-1)^j\mathcal{S}_{r-j+m-1}^{(r-j-1)}$$

$$- \sum_{j=0}^{r-1}\binom{r+m-1}{j}(-1)^j\mathcal{S}_{r-j+m-1}^{(r-j-1)}$$

Again the last two terms cancel each other. The first term becomes

$$r\sum_{j=0}^{r}\binom{r+m-1}{j}(-1)^j\mathcal{S}_{r-j+m-1}^{(r-j)}$$

$$- (r+m-1)\sum_{j=0}^{r-1}\binom{(r-1)+m-1}{j}(-1)^j\mathcal{S}_{(r-1)-j+m-1}^{(r-1-j)}$$

Since $r \geq m + 1$ and $(r - 1) \geq m$, both of these sums are zero, giving the desired result.

<u>Lemma 2.</u> For any integers m and T such that $T \geq 2m$

$$\sum_{k=2(m-1)}^{T-2} \binom{T}{k} (m-1)! \sum_{j=0}^{m-1} \binom{k}{j} (-1)^j \mathscr{S}_{k-j}^{(m-1-j)}$$

(6)

$$= m! \sum_{j=0}^{m} \binom{T}{j} (-1)^j \mathscr{S}_{T-j}^{(m-j)} .$$

<u>Proof.</u>

$$\sum_{k=2(m-1)}^{T-2} \binom{T}{k} (m-1)! \sum_{j=0}^{m-1} \binom{k}{j} (-1)^j \mathscr{S}_{k-j}^{(m-1-j)}$$

$$= (m-1)! \sum_{j=0}^{m-1} \binom{T}{j} (-1)^j \sum_{(k-j)=2(m-1)-j}^{T-j-2} \binom{T-j}{k-j} \mathscr{S}_{k-j}^{(m-j-1)}$$

From Eq. (4),

$$\sum_{(k-j)=m-j-1}^{T-j-1} \binom{T-j}{k-j} \mathscr{S}_{k-j}^{(m-j-1)} = (m-j) \mathscr{S}_{T-j}^{(m-j)} .$$

It follows that

$$\sum_{(k-j)=2(m-1)-j}^{T-j-2} \binom{T-j}{k-j} \mathscr{S}_{k-j}^{(m-j-1)} = \sum_{(k-j)=m-j-1}^{T-j-1} \binom{T-j}{k-j} \mathscr{S}_{k-j}^{(m-j-1)}$$

$$- \binom{T-j}{T-j-1} \mathscr{S}_{T-j-1}^{(m-j-1)}$$

$$- \sum_{(k-j)=m-j-1}^{2m-j-3} \binom{T-j}{k-j} \mathscr{S}_{k-j}^{(m-j-1)}$$

$$= (m-j) \mathscr{S}_{T-1}^{(m-j)} - (T-j) \mathscr{S}_{T-j-k}^{(m-j-1)}$$

$$- \sum_{r=m-j-1}^{2m-j-3} \binom{T-j}{r} \mathscr{S}_{r}^{(m-1-j)} .$$

The right-hand side of Eq. (7) becomes

$$m! \sum_{j=0}^{m} \binom{T}{j} (-1)^j \mathcal{S}_{T-j}^{(m-j)} - (m-1)! \left[\sum_{j=1}^{m-1} \frac{T!\,(-1)^j}{(j-1)!\,(T-j)!} \mathcal{S}_{T-j}^{(m-j)} + \sum_{j=0}^{m-2} \frac{T!\,(-1)^j}{j!\,(T-1-j)!} \mathcal{S}_{T-1-j}^{(m-1-j)} \right]$$

$$- (m-1)! \sum_{j=0}^{m-1} \binom{T}{j} (-1)^j \sum_{r=m-1-j}^{2m-3-j} \binom{T-j}{r} \mathcal{S}_r^{(m-1-j)}$$

The two sums in brackets cancel each other since

$$\sum_{j=1}^{m-1} \frac{T!\,(-1)^j}{(j-1)!\,(T-j)!} \mathcal{S}_{T-j}^{(m-j)} = - \sum_{j=0}^{m-2} \frac{T!\,(-1)^j}{j!\,(T-1-j)!} \mathcal{S}_{T-1-j}^{(m-1-j)} .$$

In the final term, we set $k = (m-1-j)$ so that k ranges from zero to $(m-2)$. Interchanging the order of summation and rewriting the expression, we have

$$(m-1)! \sum_{j=0}^{m-1} \binom{T}{j} (-1)^j \sum_{r=m-1-j}^{2m-3-j} \binom{T-j}{r} \mathcal{S}_r^{(m-1-j)}$$

$$= \sum_{k=0}^{m-2} \binom{T}{m-1+k} \left[\sum_{j=0}^{m-1} \binom{m-1+k}{j} (-1)^j \mathcal{S}_{m-1-j+k}^{(m-1-j)} \right]$$

From Lemma 1, each of the inner sums is zero, so that Eq. (8) becomes

$$m! \sum_{j=0}^{m} \binom{T}{j} (-1)^j \mathcal{S}_{T-j}^{(m-j)}$$

and the lemma follows.

These properties are useful in proving the following theorems.

Theorem 1. Let $t_1, t_2, \cdots, t_m$ be m integers such that

$$t_i \geq 1 \quad i = 1, 2, \cdots, m$$

and

$$\sum_{i=1}^{m} t_i = T \geq m .$$

Then

$$\sum_{t_1} \sum_{t_2} \cdots \sum_{t_m} \frac{T!}{\prod_{i=1}^{m} t_i!} = m! \mathcal{S}_T^{(m)} \tag{9}$$

Proof. We write

$$\begin{aligned} t_1 &= T_1 \\ t_1 + t_2 &= T_2 \\ &\vdots \\ t_1 + t_2 + \cdots + t_m &= T_m = T . \end{aligned}$$

The summation in (9) can be rewritten as

$$\sum_{T_{m-1}}^{T} = m - 1 \binom{T}{T_{m-1}} \left[\cdots \left[\sum_{T_2=2}^{T_3-1} \binom{T_3}{T_2} \left[\sum_{T_1=1}^{T_2-1} \binom{T_2}{T_1} \right] \right] \cdots \right] .$$

But

$$\sum_{T_1=1}^{T_2-1} \binom{T_2}{T_1} = 2^{T_2} - 2 = 2!\,\mathcal{S}_{T_2}^{(2)} \quad .$$

From Eq. (4),

$$2! \sum_{T_2=2}^{T_3-1} \binom{T_3}{T_2} \mathcal{S}_{T_2}^{(2)} = 3!\,\mathcal{S}_{T_3}^{(3)} \, ,$$

and, in general,

$$(r - 1)! \sum_{T_r=(r-1)}^{T_r-1} \binom{T_r}{T_{r-1}} \mathcal{S}_{T_{r-1}}^{(r-1)} = r!\,\mathcal{S}_{T_r}^{(r)} \, ,$$

and the theorem follows.

Theorem 2. Let $t_1, t_2, \cdots, t_m$ be m integers such that

$$t_i \geq 2 \quad i = 1, 2, \cdots, m$$

and

$$\sum_{i=1}^{m} t_i = T \geq 2m \, ,$$

then

$$\sum_{t_1}\sum_{t_2}\cdots\sum_{t_m}\frac{T!}{\prod_{i=1}^{m} t_i!} = m!\sum_{j=0}^{m}\binom{T}{j}(-1)^j\mathcal{S}_{T-j}^{(m-j)} \tag{10}$$

<u>Proof</u>. Again, let

$$\begin{aligned} t_2 &= T_1 \\ t_1 + t_2 &= T_2 \\ &\vdots \\ t_1 + t_2 + \cdots + t_m &= Tm = T\ . \end{aligned}$$

The desired summation can now be written as

$$\sum_{T_{m-1}=2(m-1)}^{T}\binom{T}{T_{m-1}}\left[\cdots\left[\sum_{T_2=4}^{T_3-2}\binom{T_3}{T_2}\left[\sum_{T_1=2}^{T_2-2}\binom{T_2}{T_1}\right]\right]\cdots\right]$$

$$\begin{aligned} \sum_{T_1=2}^{T_2-2}\binom{T_2}{T_1} &= 2^{T_2} - 2 - 2^{T_2} \\ &= 2!\sum_{j=0}^{2}\mathcal{S}_j^{T_2}\ (-1)^j\,\mathcal{S}_{T_2-j}^{(2-j)}\ . \end{aligned}$$

Repeated application of Lemma 2 leads to the desired result.

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A GENERALIZED FIBONACCI SEQUENCE OVER AN ARBITRARY RING

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Let S be a ring with identity I. Consider the sequence $\{M_n\}$ of elements of S, recursively defined by:

$$M_{n+2} = A_1M_{n+1} + A_0M_n \qquad \text{for} \qquad n \geq 0, \tag{1}$$

where M_0, M_1, A_0, and A_1 are arbitrary elements of S.

Special cases of (1) have been considered by Buschman [1], Horadam [2], and Vorobyov [3] where S was taken to be the set of integers. Wyler [4] also worked with such a sequence over a particular commutative ring with identity. In this note, we establish several results for such sequences over S (not necessarily commutative) which are analogues of results derived for similarly defined sequences of integers.

We begin by considering a special case of (1), denoted by $\{F_n\}$ and defined by:

$$F_{n+2} = A_1F_{n+1} + A_0F_n \qquad \text{for} \qquad n \geq 0, \tag{2}$$

where $F_0 = 0$, $F_1 = I$ and A_0, A_1 are arbitrary elements of S.

The fact that S need not be commutative causes difficulty in trying to derive results for the $\{F_n\}$ sequence. However, we note that the terms of this sequence possess an internal symmetry which enables us to make a start at deriving identities.

Theorem 1. If $F_{n+2} = A_1F_{n+1} + A_0F_n$, then

$$F_{n+2} = F_{n+1}A_1 + F_nA_0 . \tag{3}$$

Proof: The proof is straightforward by induction.

Corollary 1:

(i) $$F_{n+1}F_{n-1} - F_n^2 = F_{n-1}A_0F_{n-1} - F_nA_0F_{n-2} ,$$

(ii) $$F_{n-1}F_{n+1} - F_n^2 = F_{n-1}A_0F_{n-1} - F_{n-2}A_0F_n, \qquad n \geq 1 .$$

Proof of (i): From (3), we have

$$\begin{aligned} F_{n+1}F_{n-1} - F_n^2 &= (F_nA_1 + F_{n-1}A_0)F_{n-1} - F_n(A_1F_{n-1} + A_0F_{n-2}) \\ &= F_{n-1}A_0F_{n-1} - F_nA_0F_{n-2} . \end{aligned}$$

The second result can be obtained in a similar manner. We note that the results of Corollary 1 are analogues of Equation (11) of Horadam's paper [2].

The $\{M_n\}$ sequence does not, in general, possess the symmetry of the $\{F_n\}$ sequence and consequently it is even more difficult to work with. There is, however, a relation between the $\{M_n\}$ sequence and the $\{F_n\}$ sequence.

Theorem 2:

$$M_{n+r} = F_rA_0M_{n-1} + F_{r+1}M_n, \qquad n \geq 1, \quad r \geq 0 .$$

Proof: The result is easily established by induction.

Corollary 2:

$$M_n = F_nM_1 + F_{n-1}A_0M_0, \qquad n \geq 1.$$

Proof: Interchange r and n, replace n by $n - 1$ and set $r = 1$ in Theorem 2.

We note that the result of Theorem 2 is identical with Equation (12) of Buschman's paper [1] which was derived for a similarly defined sequence of integers.

For the $\{F_n\}$ sequence, Theorem 2 becomes

(4) $$F_{n+r} = F_rA_0F_{n-1} + F_{r+1}F_n, \qquad n \geq 1 .$$

If we replace n by $n + 1$ and r by n in (4), then we have

(5) $$F_{n+1}^2 + F_nA_0F_n = F_{2n+1} .$$

The commutator of the $\{F_n\}$ sequence is characterized by

Theorem 3:

$$F_nF_{n+r} - F_{n+r}F_n$$
$$= F_nF_rA_oF_{n-1} - F_{n-1}A_0F_rF_n, \qquad n \geq 1, \quad r \geq 1 .$$

Proof: If we replace n by $r + 1$ and r by $n - 1$ in (4), we have

$$F_{n+r} = F_{n-1}A_0F_r + F_nF_{r+1} \tag{6}$$

From (4), (6), and the fact that S satisfies the associative law for multiplication, we have:

$$F_n(F_{r+1}F_n) = (F_nF_{r+1})F_n \quad .$$

$$\therefore F_n(F_rA_0F_{n-1} + F_{r+1}F_n - F_rA_0F_{n-1})$$
$$= (F_{n-1}A_0F_r + F_nF_{r+1} - F_{n-1}A_0F_r)F_n .$$

$$\therefore F_n(F_{n+r} - F_rA_0F_{n-1}) = (F_{n+r} - F_{n-1}A_0F_r)F_n \quad .$$

$$\therefore F_nF_{n+r} - F_{n+r}F_n = F_nF_rA_0F_{n-1} - F_{n-1}A_0F_rF_n .$$

The $\{M_n\}$ sequence appears to be very difficult to work with directly. Investigations indicate that the best that can be done is to concentrate effort on the $\{F_n\}$ sequence and use Theorem 2 and Corollary 2 to derive analogous results for the $\{M_n\}$ sequence.

As a final remark, we note that the sequence obtained from (1) by setting $M_0 = R$, $M_1 = P + Q$, P, Q arbitrary elements of S, and $A_0 = A_1 = I$, yields a nice set of identities which are analogues of those derived by Horadam [2] for a similarly defined sequence of integers.

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[Continued on p. 198.]

A REMARKABLE LATTICE GENERATED BY FIBONACCI NUMBERS

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Functions which can be represented in the s-dimensional unit interval by rapidly convergent Fourier series of unit period in each coordinate can be integrated numerically over this interval with great efficiency by averaging their values over all the points obtained by reducing modulo 1 the coordinates of the multiples of a suitable vector $\overline{g} = \langle g_1/p, \cdots, g_s/p \rangle$, where $g_1, \cdots, g_s$, and p are integers. The crucial property of this vector can be described as follows: For any vector $\overline{h} = \langle h_1, \cdots, h_s \rangle$ put

$$R(\overline{h}) = \max(1, h_1) \cdots \max(1, h_s) ,$$

and denote by $\rho(\overline{g})$ the minimum of $R(\overline{h})$ for all the vectors having integral coordinates not all zero, and satisfying

$$\overline{g} \cdot \overline{h} \equiv 0 \pmod 1 ,$$

where the dot denotes, as usual, the scalar product. Hlawka [5] describes $p\overline{g}$ as a good lattice point modulo p if

$$\rho(\overline{g}) \geq p(8 \log p)^{1-s} \quad \ddagger \tag{1}$$

because upper bounds for the error of integration can be expressed as rapidly decreasing functions of $(\overline{g})$, and he proves the existence of good lattice points modulo any prime for any number of dimensions. The requirement that p should be a prime was introduced only in order to facilitate the proof. Understandably, however, one assumes in any event $(g_1, \cdots, g_s, p) = 1$, so that $\overline{g}$ generates exactly p different multiples modulo 1. Of course, here and in

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†From September, 1969 at the Centre de Recherches Mathématiques, Université de Montreal, Montreal, Canada.

‡As a result of a misprint, the exponent of $8 \log p$ appears to be $-s$ in Hlawka's paper, but his proof applies to lattice points satisfying (1). Thus his results are sharper than those of Korobov ([7], [8]).

what follows, by a multiple modulo 1 of any vector, we understand the result of reducing modulo 1 each coordinate of the multiple of the given vector.

In the case of more than two dimensions no recipe other than trial and error is known for finding good lattice points, and indeed such a recipe seems unlikely to exist. However, in two dimensions, the best lattice points in the sense of maximizing the ratio $(\overline{g})$:p are obtained by putting

$$p = F_n, \quad g_1 = 1, \quad g_2 = F_{n-1},$$

where $\langle F_j \rangle$ are the Fibonacci numbers [9]. One finds, then, $\rho(\overline{g}) = F_{n-2}$, which is of a better order of magnitude than (1).

The case when the integrand has not the required properties of periodicity can be reduced to the periodic case. In the case of two dimensions, denoting the coordinates by x and y, we add to the integrand a suitable polynomial in x with coefficients depending on y, and a polynomial in y with coefficients depending on x. The precise upper bounds for the error ([9], [12]) are too complicated to be discussed here in detail. Let it suffice to say that if the integrand f has partial derivatives up to

$$\frac{\partial^{2r} f}{\partial x^r \partial y^r}$$

of bounded variation in the sense of Hardy and Krause (for a precise definition see, for instance [5] or [9]), and if we add to it suitable polynomials of degree r, this allows us to obtain the value of the integral with an error of the order $F_n^{-(r+1)} \log F_n$ by averaging f over the F_n points defined above. Trial computations carried out by this method [12] gave a very high degree of accuracy. For instance, taking $r = 3$, the value of the integral over the unit square of $\exp(-x^2 - 2y^2)$ (true value 0.446708379 to nine decimals) was obtained with eight correct decimals from $n = 7$ onwards, i.e., using 13 or more points.

The sets of points corresponding to $n = 5$, 6, and 7 are shown in Figs. 1, 2, and 3. It will be seen that they define regular grids, and indeed square grids when n is odd. The importance of these grids lies not so much in the fact that they may be thought to be picturesque, as rather in conclusions of a far-reaching nature which can be drawn from their existence.

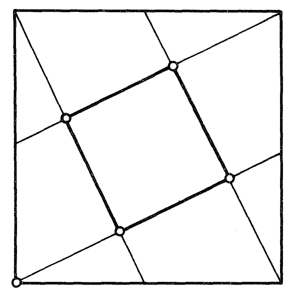

Fig. 1 : n = 5

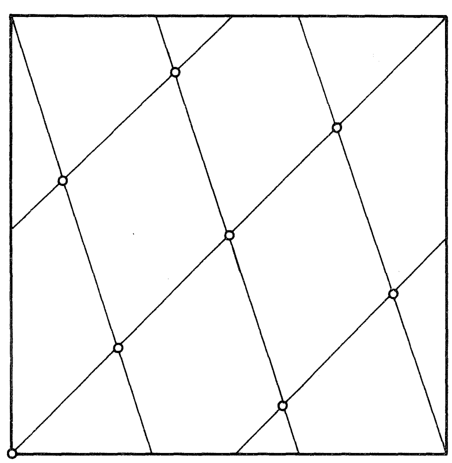

Fig. 2 : n = 6

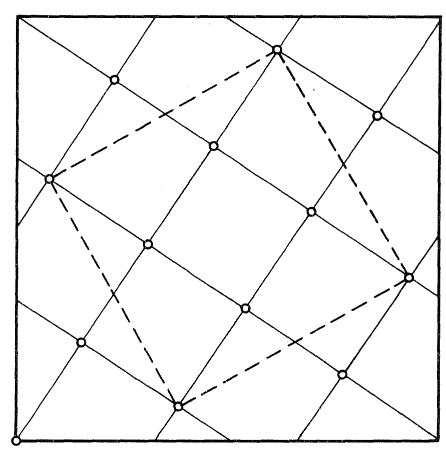

Fig. 3 : n = 7

We begin, however, with a description of the grids themselves. It is easily seen that the sets of points in question form lattices. The lattice generated modulo 1 by the vector

$$\overline{V} = \langle F_n^{-1},\ F_{n-1}F_n^{-1}\rangle$$

will be denoted by L_n. It obviously has a base formed by the vectors $\overline{V}$ and $\overline{e_2} = \langle 0, 1\rangle$. The more detailed nature of L_n depends on the parity of n. In its investigation, we shall repeatedly use the identities

$$(2) \qquad F_{m+1}F_{n+1} + F_mF_n = F_{m+n+1}$$

(see, for instance, I_{26} in [6]), and

$$(3) \qquad F_{-n} = (-1)^{n+1}F_n .$$

When $n = 2\mu + 1$, an alternative basis of L_n is formed by the vectors

$$\overline{V_1} = \langle F_\mu F_{2\mu+1}^{-1},\ -F_{-\mu-1}F_{2\mu+1}^{-1}\rangle \quad \text{and} \quad \overline{V_2} = \langle F_{\mu+1}F_{2\mu+1}^{-1},\ -F_{-\mu}F_{2\mu+1}^{-1}\rangle.$$

Indeed, from (3) and from (2) with $n = -2\mu - 1$ and with $m = \mu - 1$ and $m = \mu$, respectively, we deduce

$$F_\mu F_{2\mu} \equiv -F_{-\mu-1} \pmod{F_{2\mu+1}}$$

and

$$F_{\mu+1}F_{2\mu} \equiv -F_{-\mu} \pmod{F_{2\mu+1}},$$

so that

$$F_\mu \overline{V} \equiv \overline{V_1} \pmod 1 ,$$

and

$$F_{\mu+1}\overline{V} \equiv V_2 \pmod{1}.$$

Thus $L_{2\mu+1}$ contains the lattice generated by $\overline{V_1}$ and $\overline{V_2}$.

To prove that, conversely, $L_{2\mu+1}$ is contained in the lattice generated by $\overline{V_1}$ and $\overline{V_2}$, we note that by (3),

$$-F_{-\mu-1}F_{\mu} - F_{-\mu}F_{\mu+1} = 0,$$

while by (2) and (3),

$$F^2_{-\mu-1} + F^2_{-\mu} = F_{2\mu+1}.$$

Hence

$$-F_{-\mu-1}\overline{V_1} - F_{-\mu}\overline{V_2} = \overline{e_2} .$$

On the other hand, by (3) and by (2) with $m = -\mu$ and with $n = \mu$ and $n = -\mu - 1$, respectively, we find

$$F_{-\mu}F_{\mu} + F_{1-\mu}F_{\mu+1} = F_1 = 1$$

and

$$-F_{-\mu}F_{-\mu-1} = F_{1-\mu}F_{-\mu} = F_{2\mu} ,$$

so that

$$F_{-\mu}\overline{V_1} + F_{1-\mu}\overline{V_2} = \overline{V} .$$

Thus $\overline{V_1}$ and $\overline{V_2}$ generate the same lattice as $\overline{V}$ and $\overline{e_2}$, that is the lattice $L_{2\mu+1}$.

Since $\overline{V_1}$ and $\overline{V_2}$ are orthogonal and of equal length, $L_{2\mu+1}$ forms a grid of squares with sides inclined to the axes of coordinates. It will be seen that this grid is invariant with respect to rotations preserving the unit square.

Since clearly such rotations transform the grid into a parallel one, it suffices to show that a rotation by a right angle about the centre of the square transforms at least one lattice point into a lattice point. Now the point $\overline{V}$ of $L_{2\mu+1}$ is transformed by such a rotation into the point

$$<F_{2\mu-1}F_{2\mu+1}^{-1}, F_{2\mu+1}^{-1}> \equiv F_{2\mu-1}\overline{V} \pmod 1,$$

since, by (2) with $m = 1 - 2\mu$, $n = 2\mu$,

$$F_{2\mu-1}F_{2\mu} \equiv 1 \pmod{F_{2\mu+1}}.$$

Further rotations transform the point in question into

$$<1 - F_{2\mu-1}^{-1}, F_{2\mu-1}F_{2\mu+1}^{-1}> \text{ and } <F_{2\mu}F_{2\mu+1}^{-1}, 1 - F_{2\mu+1}^{-1}>.$$

These points form a square with vertices close to the sides of the unit square, but it does not follow that the sides of this new square are contained in the grid formed by $L_{2\mu+1}$. It is so if, and only if, μ is even, and this can best be seen as follows.

By (I_{25}) in [6] with $n = \mu$, $p = 1$, or by (I_{10}), we have

$$F_{\mu+1}^2 - F_{\mu-1}^2 = F_{2\mu}, \tag{4}$$

and by (I_{19}) in the same book, with $n = \mu - 1$, $k = 2$,

$$F_{\mu-1}^2 = F_{\mu+1}F_{\mu-3} + (-1)^{\mu+1}.$$

Hence

$$F_{\mu+1}(F_{\mu+1} - F_{\mu-3}) = F_{2\mu} - 1$$

when μ is even. It follows that in this case, the abscissa of the point

$$\langle F_{2\mu-1}F_{2\mu+1}^{-1},\ F_{2\mu+1}^{-1}\rangle + (F_{\mu+1} - F_{\mu-3})\overline{V_2}$$

is

$$(F_{2\mu-1} + F_{2\mu} - 1)F_{2\mu+1}^{-1} = 1 - F_{2\mu+1}^{-1} .$$

Since no pair of points $L_{2\mu+1}$ in the unit square can have the same abscissa, this is necessarily the point

$$\langle 1 - F_{2\mu+1}^{-1},\ F_{2\mu-1}F_{2\mu+1}^{-1}\rangle$$

which was mentioned above as another vertex of the square in question. Let it be noted in passing that there are $4(F_{\mu+1} - F_{\mu-3})$ points of $L_{2\mu+1}$ on the perimeter of this square. In Fig. 1, this square is shown by thicker lines.

When μ is odd, the ordinate of $\overline{V_2}$ is negative. Consequently, it is along $\overline{V_1}$ that we should attempt to move from

$$\langle F_{2\mu-1}F_{2\mu+1}^{-1},\ F_{2\mu+1}^{-1}\rangle$$

to

$$\langle 1 - F_{2\mu+1}^{-1},\ F_{2\mu-1}F_{2\mu+1}^{-1}\rangle .$$

But by (2) with $m = \mu$, $n = \mu - 1$,

$$F_{\mu}(F_{\mu+1} + F_{\mu-1}) = F_{2\mu} .$$

Hence if we add $(F_{\mu+1} + F_{\mu-1})\overline{V_1}$ to the starting point, we obtain a point of abscissa

$$(F_{2\mu-1} + F_{2\mu})F_{2\mu+1}^{-1} = 1 ,$$

which shows that, for $\mu > 1$, adding multiples of $\overline{V_1}$ to our starting point cannot produce a point of abscissa $1 - F_{2\mu+1}^{-1}$. Thus indeed the square in question is not formed by the grid; this is illustrated in Fig. 3, where this square is marked in dotted lines.

When n is even, say $n = 2\mu$, the vectors

$$\overline{V_1'} = \langle F_\mu F_{2\mu}^{-1},\ F_{-\mu} F_{2\mu}^{-1} \rangle \quad \text{and} \quad \overline{V_2'} = \langle F_{\mu+1} F_{2\mu}^{-1},\ F_{1-\mu} F_{2\mu}^{-1} \rangle$$

form a basis of $L_{2\mu}$. Indeed, writing $\overline{V}$ as

$$\langle F_{2\mu}^{-1},\ F_{1-2\mu} F_{2\mu}^{-1} \rangle ,$$

we find, by easy applications of (2),

$$(5) \qquad F_\mu \overline{V} \equiv \overline{V_1'} \pmod 1 \quad \text{and} \quad F_{\mu+1}\overline{V} \equiv \overline{V_2'} \pmod 1 .$$

On the other hand, by (2) and (3), we find

$$(6) \qquad F_{-\mu}\overline{V_1'} + F_{1-\mu}\overline{V_2'} = \overline{V} \quad \text{and} \quad -F_{-\mu-1}\overline{V_1'} - F_{-\mu}\overline{V_2'} = \overline{e_2}$$

Now (5) and (6) show that the lattice generated by $\overline{V_1'}$ and $\overline{V_2'}$ is nothing else but $L_{2\mu}$.

However, $\overline{V_1'}$ and $\overline{V_2'}$ are not orthogonal, their scalar product being

$$(F_\mu F_{\mu+1} + F_{-\mu} F_{1-\mu}) F_{2\mu}^{-2} = F_\mu^2 F_{2\mu}^{-2} ,$$

since

$$F_\mu F_{\mu+1} + F_{-\mu} F_{1-\mu} = F_\mu (F_{\mu+1} - F_{\mu-1}) = F_\mu^2 .$$

When $n = 2\mu$, L_n does not form a square grid. The determinant of $L_{2\mu}$ being equal to $F_{2\mu}^{-1}$, this would indeed require a pair of orthogonal vectors of lengths $F_{2\mu}^{-\frac{1}{2}}$ each. The cases of $\mu = 2$ and $\mu = 3$ being trivial, assume $\mu > 3$. In our search for the required vectors, we can dismiss those which have a coordinate equal to, or bigger than, $F_{\mu+1}F_{2\mu}^{-1}$ in absolute value, since by (4), their length exceeds $F_{2\mu}^{-\frac{1}{2}}$. All linear combinations $\alpha V_1' + \beta V_2'$ in which $\beta \neq 0$ are thereby excluded because of their abscissa if $\alpha\beta \geq 0$ and because of their ordinate if $\alpha\beta < 0$. There remain the multiples of $\overline{V_1'}$. But ((78) in [1])

$$F_{2\mu} = F_\mu^2 + 2F_\mu F_{\mu+1} \;; \tag{7}$$

This identity can also be deduced from (4) noting that

$$F_{\mu+1}^2 - F_{\mu-1}^2 = F_\mu^2 + 2F_\mu F_{\mu-1} .$$

It follows from (7) that when $\mu > 3$, we have $F_{2\mu} > 2F_\mu^2$, so that $\overline{V_1'}$ is too short for our purposes, while $2\overline{V_1'}$ has an abscissa exceeding $F_{\mu+1}$, so that it is too long.

The figures representing the lattices with $F_5 = 5$, $F_6 = 8$, and $F_7 = 13$ points show that in each case there is a relatively large number of lattice points on a straight line passing through the origin. In order to evaluate this number in general, we must again distinguish two cases according to the parity of n.

If $n = 2\mu + 1$, one of the vectors $\overline{V_1}$ and $\overline{V_2}$ has both its coordinates positive, one being equal to

$$F_\mu F_{2\mu+1}^{-1}$$

and the other to

$$F_{\mu+1}F_{2\mu+1}^{-1} .$$

The origin being a lattice point, it follows that the line passing through it and parallel to the vector in question contains

$$[F_{2\mu+1}F^{-1}_{\mu+1}] + 1$$

lattice points, where, as usual, $[x]$ denotes the biggest integer not exceeding x. This number is easily determined as follows: By (2) and by (I_{13}) in [6], we have

$$F^2_{\mu+1} + F^2_{\mu} = F_{2\mu+1}$$

and

$$F_{\mu+1}F_{\mu-1} - F^2_{\mu} = (-1)^{\mu} .$$

Hence

$$F_{\mu+1}(F_{\mu+1} + F_{\mu-1}) = F_{2\mu+1} + (-1)^{\mu} ,$$

and consequently the number of lattice points on the line in question is

$$F_{\mu+1} + F_{\mu-1} + \frac{1}{2}(1 + (-1)^{\mu+1}) .$$

When $n = 2\mu$, one of the vectors $\overline{V'_1}$ and $\overline{V'_2}$ has its coordinates equal to $F_{\mu}F^{-1}_{2\mu}$ and $F_{\mu+1}F^{-1}_{2\mu}$ in either order. But by (2),

$$F_{\mu+1}F_{\mu} + F_{\mu}F_{\mu-1} = F_{2\mu} ,$$

and

$$F_{\mu}F_{1-\mu} + F_{\mu+1}F_{2-\mu} = F_2 ,$$

or

$$F_{\mu}F_{\mu-1} - F_{\mu+1}F_{\mu-2} = (-1)^{\mu} .$$

Hence

$$F_{\mu+1}(F_{\mu} + F_{\mu-2}) = F_{2\mu} + (-1)^{\mu+1} ,$$

which shows that the line through the origin parallel to the vector mentioned above contains

$$F_{\mu} + F_{\mu-2} + \frac{1}{2}(1 + (-1)^{\mu})$$

points of L_2 .

Thus, in either case, there is a line, say 1, which contains a number of points of L_n in the unit square which is of the order $\sqrt{F_n}$. The importance of this fact is a consequence of the following considerations.

Let S be any finite set of, say p points of the unit square

$$0 \leq x < 1; \quad 0 \leq y < 1 ,$$

and denote by $\nu(x,y)$ the number of points of S with coordinates smaller than, or equal to, x and y, respectively. The function

$$g(x,y) = p^{-1}\nu(x,y) - xy$$

can be regarded as describing the equidistribution of S over Q^2. If a single number is required to characterize this equidistribution, it can be obtained by taking any of the plausible norms of g. In particular, it has been proposed ([1], [11]) to call

$$D(S) = \sup_{\langle x,y\rangle \in Q^2} |g(x,y)|$$

the extreme discrepancy of S in order to distinguish it from other possible norms of g; the previously used term is simply discrepancy. If f is any function of bounded variation in the sense of Hardy and Krause over the closure of Q^2, then its integral over Q^2 is approximately equal to the average value of f over the points of S, the absolute value of the error not exceeding VD(S), where V is the sum of the two-dimensional variation of f over Q^2 in the sense of Vitali and of the (one-dimensional) variations of f(x, 1) and f(1, y) over [0,1] ([3]; for a slight sharpening of this result, see [9]).

Thus sets of points with low extreme discrepancies provide us with a method of numerical integration over Q^2 even when the integrand cannot be expanded into a uniformly convergent Fourier series. In the case of the set of points determined by the multiples modulo 1 of $\overline{V}$, the extreme discrepancy has been shown to be smaller than $(7/6)F_n^{-1} \log (15\, F_n)$ [9]. However, the integrals we may want evaluated numerically do not necessarily lend themselves to a reduction to integrals over Q^2. It might seem that, if the domain of integration, say D, is contained in Q^2, we could replace the integrand by a function equal to it in D, and to O outside, integrating this new function over the whole of Q^2; this is what would be likely to be done if the Monte Carlo method were applied. The difficulty lies in the fact that in general the new integrand will not be of bounded variation in the sense of Hardy and Krause over Q^2, however regular the initially given function might be, and indeed even if it is a constant, consequently Hlawka's theorem cannot be applied to this situation.

The sets of points which have been described above show that even when the integrand is a constant, say 1, and the domain of integration is, for instance, convex, the integration error can be of the order of $\sqrt{F_n}$ instead of that of $F_n^{-1} \log F_n$. To see this, it suffices to consider two lines, say l_1 and l_2, on opposite sides of 1, parallel to it, and arbitrarily near to each other. Let D_i be the part of Q^2 above l_i $(i = 1,2)$. Then the integrals over D_1 and D_2 will differ arbitrarily little, while the numerical values found for them will differ by the number of points of L_n on 1 divided by F_n, so that for one at least of the two integrals the error of the computation will indeed be of the order of $F_n^{-\frac{1}{2}}$.

When $n = 2\mu + 1$, μ being even, by slightly expanding, or contracting the previously discussed square formed by the grid, it is possible to obtain a similar example of an integration domain leading to errors of the order of $F_n^{-\frac{1}{2}}$ when applying the method in question; of course, many variations on this theme are possible.

All these considerations extend to an arbitrary number of dimensions, and the phenomenon illustrated by the lattice L_n in two dimensions becomes even more accentuated as the number of dimensions increases. The present author, impressed by the pattern of L_n, proved [10] that if a set of p points of the s-dimensional unit interval Q^s is generated by a good lattice point, then there exists an s-1-dimensional linear variety (or hyperplane, to use a rather old-fashioned terminology) which forms with Q^s an intersection containing more than $(4s)^{1-s}p^{1-1/s}$ points of the set. This leads again in an obvious way to an example of convex domains (actually simple polyhedral domains) having the property that by integrating over them, by Hlawka's method, arbitrarily regular functions, which could even reduce to constants, we are always liable to commit errors of the order of $p^{-1/s}$.

This contrasts sharply with the error committed when integrating over the whole of Q^s a function of bounded variation in the sense of Hardy and Krause and using the same set of points; the discrepancy is then $\overline{O}((\log p)^s/p)$ 4 , and the integration error is of the same order of magnitude. With $s > 2$, in the former case the bound obtained for the error (and this is a sharp bound!) is much less favorable than that which is practically claimed by the Monte Carlo method, namely $\overline{O}(p^{-\frac{1}{2}})$.

The irrelevance of some traditional tests applied to so-called random numbers, or to pseudo-random numbers with a view to applications to Monte-Carlo integration was discussed in detail by the present author 11 ; the considerations adduced here show that when the domain of integration is not reduced to a multidimensional unit interval, even discrepancy tests in the appropriate number of dimensions do not guarantee the success of Monte Carlo, although, naturally, nobody can be denied the right of <u>hoping</u> for the best.

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AN EXTENSION OF THE FIBONACCI NUMBERS (PART II)

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In this section we consider the generalized Fibonacci and Tribonacci numbers.

We write the generalized Fibonacci numbers as

$$(1) \qquad (1 - a_1x - a_2x^2)^{-k} = \sum_{v=0}^{\infty} F_v^{(k)} x^v \quad (F_v = F_v^{(1)}) \ ,$$

where

$$F_n = a_1F_{n-1} + a_2F_{n-2}, \quad F_0 = 1, \quad F_1 = a_1, \quad F_2 = a_1^2 + a_2,$$
$$k = 1, 2, 3, \cdots \quad \text{and} \quad n = 0, 1, 2, \cdots \ .$$

The generalized Tribonacci numbers we write as

$$(2) \qquad (1 - a_1x - a_2x^2 - a_3x^3) = \sum_{v=0}^{\infty} T_v^{(k)} x^v \ ,$$

where

$$T_v = T_v^{(1)}, \quad T_0 = 1, \quad T_1 = a_1, \quad T_2 = a_1^2 + a_2, \quad F_3 = a_1^3 + 2a_1a_2 + a_3,$$
$$T_n = a_1T_{n-1} + T_{n-2}a_2 + a_3T_{n-3}, \quad k = 1,2,3,\cdots \quad \text{and} \quad n = 0, 1, 2,\cdots.$$

Note: Throughout this section we consider a_1, a_2, and a_3, as rational integers only.

CONVOLUTED SUM FORMULAS
FOR THE GENERALIZED FIBONACCI AND TRIBONACCI NUMBERS

By elementary means, it is easy to prove, if

$$(1 - y)^{-k} = \sum_{v=0}^{\infty} b_v^{(k)} y^v \tag{3}$$

then

$$\binom{n+k-1}{k-1} = b_n^{(k)},$$

where

$$\binom{n+k-1}{k-1} = (n+k-1)!/n!\,(k-1)!, \quad b_0^{(k)} = 1, \quad k = 1, 2, 3, \cdots \text{ and } n = 0, 1, 2, \cdots.$$

Now, in (1), we replace $a_1x + a_2x^2$ with y so that combining (1) with (3) we may then write

$$\sum_{v=0}^{\infty} b_v^{(k)} y^v = \sum_{v=0}^{\infty} F_v^{(k)} x^v,$$

It is easy to prove with induction that

$$\sum_{j=0} b_{n-j}^{(k)} \binom{n-j}{j} a_1^{n-2j} a_2^j = F_n^{(k)},$$

and combining this result with

$$b^{(k)} = \binom{n+k-1}{k-1}$$

leads to the following generalized Fibonacci convoluted sum formula:

$$F_n^{(k)} = \sum_{j=0} \binom{n+k-1-j}{k-1} \binom{n-j}{j} a_1^{n-2j} a_2^j \tag{4}$$

$$(n = 0,1,2,\cdots, \; k = 1,2,3,\cdots).$$

Now in (2), we replace $a_1x + a_2x^2 + a_3x^3$ with y so that combining (2) with (3), we may then write

$$\sum_{v=0}^{\infty} b_v^{(k)} y^v = \sum_{v=0}^{\infty} T_v^{(k)} x^v \;,$$

and by comparing coefficients, it is easy to prove with induction, that

$$T_n^{(k)} = \sum_{r=0} \sum_{j=0}^{r} \left[b_{n-2r}^{(k)} \binom{n-2r}{2r-j} \binom{2r-j}{j} a_1^{n-4r+j} a_2^{2r-2j} a_3^j + b_{n-2r-1}^{(k)} \binom{n-2r-1}{2r-1+j} \binom{2r+1-j}{j} a_1^{n-4r-2+j} a_2^{2r+1-2j} a_3^j \right]$$

and combining this result with

$$b_n^{(k)} = \binom{n+k-1}{k-1}$$

leads to the following generalized Tribonacci convoluted sum formula:

$$T_n^{(k)} = \sum_{r=0} \sum_{j=0}^{r} \left[\binom{k+n-2r-1}{k-1} \binom{n-2r}{2r-j} \binom{2r-j}{j} a_1^{n-4r+j} a_2^{2r-2j} a_3^j + \binom{k+n-2r-2}{k-1} \binom{n-2r-1}{2r+1-j} \binom{2r+1-j}{j} a_1^{n-4r-2+j} a_2^{2r+1-2j} a_3^j \right] \tag{5}$$

where $n = 0, 1, 2, \cdots$ and $k = 1, 2, 3, \cdots$.

THE GENERALIZED FIBONACCI NUMBER EXPRESSED EXPLICITLY AS A DETERMINANT

We shall now prove the following five statements:

I. $$nF_n^{(k)} = a_1(k + n - 1)F_{n-1}^{(k)} + a_2(2k + n - 2)F_{n-2}^{(k)} ,$$

where

$$F_0^{(k)} = 1, \quad F_1^{(k)} = a_1k, \quad n = 2, 3, \cdots, \quad k = 2, 3, \cdots .$$

II. $$\frac{nF_n^{(k)}}{F_{n-1}^{(k)}} = p_1 + \cfrac{q_2}{p_2 + \cfrac{q_3}{p_3 + \cdots + \cfrac{q_{n-1}}{p_{n-1} + \cfrac{q_n}{p_n}}}}$$

where $p_j = a_1(k + n - j) \quad (j = 1, 2, 3, \cdots, n)$,

$q_{m+1} = a_2(n - m)(2k + n - m - 1), \quad (m = 1, 2, 3, \cdots, n - 1)$,

$(n = 2, 3, \cdots)$

$(k = 2, 3, \cdots)$,

$F_0^{(k)} = 1$,

$F_1^{(k)} = a_1k$.

III. $$(a_1^2 + 4a_2)kF_{n-1}^{(k+1)} = a_1nF_n^{(k)} + a_2(4k + 2n - 2)F_{n-1}^{(k)} ,$$

where $F_0^{(k)} = 1$

$F_1^{(k)} = a_1k$

$n = 1, 2, \cdots$

$k = 1, 2, 3, \cdots$.

IV. $$\sum_{v=1}^{n} F_{n-v}^{(v)} = \frac{((a_1+1+((a_1+1)^2+4a_2)^{\frac{1}{2}}+4a_2)^{\frac{1}{2}})/2)^n - ((a_1+1-((a_1+1)^2+4a_2)^{\frac{1}{2}})/2)^n}{((a_1+1)^2 + 4a_2)^{\frac{1}{2}}}$$

where $n = 1, 2, 3, \cdots$.

V. $F_n^{(k)} = K(p_1, q_2, \cdots, q_n, p_n)/n!$, ($p_n, q_n$ are identical to those in (ii) with $q_1 = 0$).

where $n, k = 1, 2, 3, \cdots$ and $K(p_1, q_2, \cdots, q_n + p_n)$ is the determinant given below in (6).

$$\text{(6)} \qquad K(p_1, q_2, \cdots, q_n, p_n) = \begin{bmatrix} p_1 & q_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & p_2 & q_3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & p_3 & q_4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & p_4 & q_5 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & p_{n-1} & q_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & p_n \end{bmatrix}$$

The table below of the generalized Fibonacci Numbers (in the table, we have replaced a_1, a_2 in (1) with $a_1 = a$ and $a_2 = b$)

(7)

	0	1	2	3	4	$\cdots$
0	0	0	0	0	0	$\cdots$
1	1	a	$a^2 + b$	$a^3 + 2ab$	$a^4 + 3a^2b + b^2$	$\cdots$
2	1	$2a$	$3a^2 + 2b$	$4a^3 + 6ab$	$5a^4 + 12a^2b + 3b^2$	$\cdots$
3	1	$3a$	$6a^2 + 3b$	$10a^3 + 12ab$	$15a^4 + 30a^2b + 6b^2$	$\cdots$
.	.	.	.	.	.	$\cdots$
.	.	.	.	.	.	$\cdots$
k	1	ka	.	.	.	$\cdots$
.	.	.	.	.	.	$\cdots$
.	.	.	.	.	.	$\cdots$

may be constructed as follows:

(8) To get the k^{th} element in the n^{th} column, we add the product of $\underline{a}$ multiplied by the k^{th} element in the $(n-1)^{\text{st}}$ column and the product of $\underline{b}$

multiplied by the k^{th} element in the $(n-2)^{nd}$ column together with the $(k-1)^{st}$ element in the n^{th} column.

We write the k^{th} element in the n^{th} column as $F_n^{(k)}$, so that a restatement of (8) reads

$$F_n^{(k)} = a_1 F_{n-1}^{(k)} + a_2 F_{n-2}^{(k)} + F_n^{(k-1)}, \tag{9}$$

where

$$\begin{aligned}
F_0^{(k)} &= 1 \\
F_1^{(k)} &= a_1 k \\
0 &= F_0^{(0)} = F_1^{(0)} = F_2^{(0)} = \cdots \\
n &= 2, 3, \cdots \\
k &= 1, 2, 3, \cdots .
\end{aligned}$$

PROOF OF I, II, III, AND IV

We use (9) to get

$$\begin{aligned}
\sum_{n=2}^{\infty} F_n^{(k)} x^n &= a_1 \sum_{n=2}^{\infty} F_{n-1}^{(k)} x^n + a_2 \sum_{n=2}^{\infty} F_{n-2}^{(k)} x^n + \sum_{n=2}^{\infty} F_n^{(k-1)} x^n \\
&= a_1 x \sum_{n=1}^{\infty} F_n^{(k)} x^n + a_2 x^2 \sum_{n=0}^{\infty} F_n^{(k)} x^n + \sum_{n=2}^{\infty} F_n^{(k-1)} x^n ,
\end{aligned} \tag{10}$$

for $k = 1, 2, \cdots$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n^{(k)} x^n - F_0^{(k)} - F_1^{(k)} x &= a_1 x \sum_{n=0}^{\infty} F_n^{(k)} x^n - a_1 F_0^{(k)} x + a_2 x^2 \sum_{n=0}^{\infty} F_n^{(k)} x^n \\
&\quad + \sum_{n=0}^{\infty} F_n^{(k-1)} x^n - F_0^{(k-1)} - F_1^{(k-1)} x ,
\end{aligned}$$

and therefore

$$(1 - a_1x - a_2x^2)\sum_{n=0}^{\infty} F_n^{(k)}x^n = F_0^{(k)} - F_0^{(k-1)} + (F_1^{(k)} - a_1F_0^{(k)} - F_1^{(k-1)})x + \sum_{n=0}^{\infty} F_n^{(k-1)}x^n .$$

Now

$$F_0^{(k)} - F_0^{(k-1)} = \begin{cases} 1 - 0 = 1 & \text{if } k = 1 \\ 1 - 1 = 0 & \text{if } k \quad 2 \end{cases}$$

and

$$F_1^{(k)} - a_1F_0^{(k)} - F_1^{(k-1)} = \left\{\begin{array}{ll} a_1 - a_1 - 0 = 0 & \text{if } k = 1 \\ a_1k - a_1 - a_1(k-1) = 0 & \text{if } k \quad 2 \end{array}\right\} = 0 ,$$

for $k = 1, 2, 3, \cdots,$ and

$$\sum_{n=0}^{\infty} F_n^{(0)} = 0 .$$

Therefore

$$\sum_{n=0}^{\infty} F_n^{(k)}x^n = (1 - a_1x - a_2x^2)^{-1}\left(\sum_{n=0}^{\infty} F_n^{(k-1)}x^n\right) \qquad (k = 2, 3, \cdots),$$

and

$$\sum_{n=0}^{\infty} F_n^{(1)}x^n = (1 - a_1x - a_2x^2)^{-1} .$$

From this, we have at once

$$(11) \qquad (1 - a_1x - a_2x^2)^{-k} = \sum_{n=0}^{\infty} F_n^{(k)} x^n \qquad (k = 1,2,3,\cdots) .$$

Differentiation of (11) leads to

$$k(2a_2x + a_1)\left(\sum_{n=0}^{\infty} F_n^{(k+1)}x^n\right) = \sum_{n=1}^{\infty} nF_n^{(k)}x^{n-1} ,$$

and comparing the coefficients we conclude that

$$(12) \quad k(a_1F_{n-1}^{(k+1)} + 2a_2F_{n-2}^{(k+1)}) = nF_n^{(k)} \qquad (k = 1,2,3,\cdots, n = 2,3,\cdots).$$

Combining (12) with (9), we get

$$(13) \qquad nF_n^{(k)} = a_1(k + n - 1)F_{n-1}^{(k)} + a_2(2k + n - 2)F_{n-2}^{(k)}$$

for $k = 2,3,\cdots$, $n = 2,3,\cdots$, $F_0^{(k)} = 1$, and $F_1^{(k)} = a_1k$. This completes the proof for I.

When we divide (13) by $F_{n-1}^{(k)}$, we have

$$\frac{nF_n^{(k)}}{F_{n-1}^{(k)}} = a_1(k + n - 1) + \frac{a_2(2k + n - 2)(n - 1)}{\dfrac{(n - 1)F_{n-1}^{(k)}}{F_{n-2}^{(k)}}} \qquad (n=2,3,\cdots,k=2,3,\cdots)$$

which in turn along with $F_0^{(k)} = 1$ and $F_1^{(k)} = a_1k$, implies II.

The identity

$$a_1^2 + 4a_2 = 4a_2(1 - a_1x - a_2x^2) + (a_1 + 2a_2x)^2$$

may be written as

$$\frac{a_1^2 + 4a_2}{(1 - a_1x - a_2x^2)^k} = \frac{4a_2}{(1 - a_1x - a_2x^2)^{k-1}} + \frac{(a_1 + 2a_2x)^2}{(1 - a_1x - a_2x^2)^k} \tag{14}$$

$$(k = 1,2,3,\cdots) .$$

Differentiation leads to

$$\frac{(a_1^2 + 4a_2)kx}{(1 - a_1x - a_2x^2)^{k+1}} = \frac{4a_2kx}{(1 - a_1x - a_2x^2)^k} + (a_1 + 2a_2x)\left(\sum_{n=1}^{\infty} nF_n^{(k)} x^n\right).$$

Now, by comparing coefficients, we conclude that

$$(a_1^2 + 4a_2)kF_{n-1}^{(k+1)} = a_1nF_n^{(k)} + a_2(4k + 2n - 2)F_{n-1}^{(k)} , \tag{15}$$

when $F_0^{(k)} = 1$, $F_1^{(k)} = a_1k$, $n = 1,2,3,\cdots$, and $k = 1,2,3,\cdots$, which proves III.

We observe that Equations (II) and (III) immediately give an expression for

$$\frac{(a_1^2 + 4a_2)F_{n-1}^{(k+1)}}{F_{n-1}^{(k)}}$$

in the form of a continued fraction, for $n = 2, 3, \cdots$, and $k = 2, 3, \cdots$.

(Proof of IV). In (9), we have

$$F_n^{(k)} = a_1F_{n-1}^{(k)} + a_2F_{n-2}^{(k)} + F_n^{(k-1)} ,$$

so that

$$\sum_{v=1}^{n} F_{n-v}^{(v)} = a_1\sum_{v=1}^{n-1} F_{n-v-1}^{(v)} + a_2\sum_{v=1}^{n-2} F_{n-v-2}^{(v)} + \sum_{v=2}^{n} F_{n-v}^{(v-1)} \qquad (n=2,3,\cdots). \tag{16}$$

We see that

$$\sum_{v=1}^{n-1} F_{n-v-1}^{(v)} = \sum_{v=2}^{n} F_{n-v}^{(v-1)} ,$$

and we write (16) as

$$\text{(17)} \qquad \sum_{v=1}^{n} F_{n-v}^{(v)} = (a_1 + 1)\left(\sum_{v=1}^{n-1} F_{n-v-1}^{(v)}\right) + a_2 \sum_{v=1}^{n-2} F_{n-v-2}^{(v)} .$$

We let

$$u_n = \sum_{v=1}^{n} F_{n-v}^{(v)} ;$$

then

$$u_{n-1} = \sum_{v=1}^{n-1} F_{n-v-1}^{(v)} , \qquad \text{and} \qquad u_{n-2} = \sum_{v=1}^{n-2} F_{n-v-2}^{(v)} ,$$

so that (17) becomes

$$\text{(18)} \qquad u_n = (a_1 + 1)u_{n-1} + a_2 u_{n-2} .$$

Replacing n with $n + 2$ in (18), we have

$$\text{(19)} \qquad u_{n+2} = (a_1 + 1)u_{n+1} + a_2 u_n ,$$

where

$$u_1 = F_0 = 1, \quad F_0^{(2)} + F_1 = 1 + a_1 = u_2, \quad \text{and} \quad n = 1, 2, 3, \cdots .$$

We now solve (19) for u_n by continued fractions (see [1]), and get

$$u_n = ((a_1 + 1 + s)^n - (a_1 + 1 - s)^n)/2^n s = \sum_{v=1}^{n} F_{n-v}^{(v)} ,$$

where

$$s = ((a_1 + 1)^2 + 4a_2)^{\frac{1}{2}}, \quad n = 1, 2, 3, \cdots ,$$

which completes the proof of IV.

PROOF OF V

Combining Euler's expression for a continuant as a determinant (see [2]) with (II) and (6), leads to

$$\frac{nF_n^{(k)}}{F_{n-1}^{(k)}} = \frac{K(p_1, q_2, \cdots, q_n, p_n)}{K(p_2, q_3, \cdots, q_n, p_n)} , \tag{20}$$

for $n,k = 2,3,4,\cdots$.

Note: For convenience we let

$$F_n^{(k)} / F_{n-1}^{(k)} = U_k(n) .$$

Now, using the values of p_j and q_{m+1} in (II), we write

$$nU_k(n) = \frac{K(a_1(k + n - 1), a_2(n - 1)(2k + n - 2), \cdots, a_2(2k), a_1k)}{K(a_1(k + n - 2), a_1(n - 2)(2k + n - 3), \cdots, a_2(2k), a_1k)} , \tag{21}$$

$$(n - 1)U_k(n - 1) = \frac{K(a_1(k + n - 2), a_2(n - 2)(2k + n - 3), \cdots, a_2(2k), a_1k)}{k(a_1(k + n - 3), a_2(n - 3)(2k + n - 4), \cdots, a_2(2k), a_1k)} ,$$

. . . .

$$3U_k(3) = \frac{K(a_1(k + 2), a_2(2)(2k + 1), \cdots, a_2(2k), a_1k)}{\begin{vmatrix} a_1(k + 1) & a_2(2k) \\ -1 & a_1k \end{vmatrix}}$$

$$2U_k(2) = \frac{\begin{vmatrix} a_1(k+1) & a_2(2k) \\ -1 & a_1k \end{vmatrix}}{a_1k} .$$

We now multiply all the equations in (21) from top to bottom to get

$$(22) \qquad n! \prod_{j=2}^{n} U_k(j) = n!\, F_n^{(k)} / F_1^{(k)} = K(p_1, q_2, \cdots, q_n, p_n)/a_1k ,$$

for $n,k = 2, 3, 4, \cdots$.

Now combining (II, with $F_1^{(k)} = a_1k$) with (22) completes the proof of V.

We resolve for $k = 1$ $(n = 0, 1, 2, \cdots)$ by the use of continued fractions (see [1]), and we have

$$F_n = ((a_1 + V)^{n+1} - (a_1 - V)^{n+1})/V2^{n+1} ,$$

where

$$V = (a_1^2 + 4a_2)^{\frac{1}{2}} ,$$

and

$$F_n = a_1F_{n-1} + a_2F_{n-2} \; (F_0 = 1, \; F_1 = a_1) .$$

FORMULAS

For $F_n^{(t)}$ $(t = 2, 3,$ and $4)$ as a function of F_{n-1} and F_n.

Let

$$A = a_1^2 + 4a_2, \; B(k,n) = 4k + 2n - 2 ,$$

where

$$F_0^{(k)} = 1, \quad F_1^{(k)} = a_1 k, \quad n,k = 1,2,3,\cdots,$$

and

$$F_n = a_1 F_{n-1} + a_2 F_{n-2},$$

(where a_1 and a_2 are rational integers); then from (III), we have

$$AkF_{n-1}^{(k+1)} = a_1 n F_n^{(k)} + a_2 B(k,n) F_{n-1}^{(k)} . \tag{23}$$

In (23), we have the following: when $k = 1$, then

$$AF_{n-1}^{(2)} = a_1 n F_n + a_2 B(1,n) F_{n-1}, \tag{24}$$

when $k = 2$, then

$$2AF_{n-1}^{(3)} = a_1 n F_n^{(2)} + a_2 B(2,n) F_{n-1}^{(2)},$$

so that multiplying by 1:A, we get

$$2!\, A^2 F_{n-1}^{(3)} = a_1 n A F_n^{(2)} + a_2 B(2,n) A F_{n-1}^{(2)},$$

and combining this with (24), we write (using the identity $F_n = a_1 F_{n-1} + a_2 F_{n-2}$)

$$\begin{aligned} 2!\, A^2 F_{n-1}^{(3)} &= a_2 B(2,n)(a_1 n F_n + a_2 B(1,n) F_{n-1}) \\ &\quad + a_1 n (a_1 (n+1) F_{n+1} + a_2 B(1, n+1) F_n) \end{aligned} \tag{25}$$

and replacing F_{n+1} (in (25)) with $F_{n+1} = a_1 F_n + a_2 F_{n-1}$ leads to

$$A^2F_{n-1}^{(3)} = \Big[(a_1a_2nB(1,n+1) + a_1a_2nB(2,n) + a_1^3n(n+1))F_n \tag{26}$$
$$+ (a_2^2B(1,n)B(2,n) + a_1^2a_2n(n+1))F_{n-1}\Big],$$

when $k = 3$, then in the exact way we found (26), we prove that

$$3!\,A^3F_{n-1}^{(4)} = M + N, \tag{27}$$

where

$$M = \begin{bmatrix} a_1a_2^2nB(1,n+1)B(3,n) + a_1a_2^2nB(2,n)B(3,n) \\ + a_1^3a_2n(n+1)B(3,n) + a_1a_2^2nB(1,n+1)B(2,n+1) \\ + a_1^3a_2n(n+1)(n+2) + a_1^3a_2n(n+1)B(1,n+2) \\ + a_1^3a_2n(n+1)B(2,n+1) + a_1^5n(n+1)(n+2) \end{bmatrix} F_n,$$

and

$$N = \begin{bmatrix} a_2^3B(1,n)B(2,n)B(3,n) + a_1^2a_2^2n(n+1)B(3,n) \\ + a_1^2a_2^2n(n+1)B(1,n+2) + a_1^2a_2^2n(n+1)B(2,m+1) \\ + a_1^4a_2n(n+1)(n+2) \end{bmatrix} F_{n-1}.$$

REMARKS

The above method may be used to evaluate formulas of the $F_n^{(k)}$ for values of $k = 5$ and higher.

THE GENERALIZED FIBONACCI NUMBER EXPRESSED AS A LIMIT

We now prove that

$$\text{VI} \qquad \lim_{n\to\infty} \left(F_n^{(k+1)}/(n+1)^k F_n\right) = \left(1 + a_1(a_1^2 + 4a_2)^{-\frac{1}{2}}\right)^k / 2^k k! \quad ,$$

when

$$\lim_{n\to\infty} ((4k-2)/n) = 0 \qquad (k, n = 1, 2, 3, \cdots) .$$

Let

$$(28) \qquad A = a_1^2 + 4a_2, \quad V = A^{\frac{1}{2}}, \quad H = \tfrac{1}{2}(a_1 + V) \ ,$$

where

$$F_n = a_1 F_{n-1} + a_2 F_{n-2}, \quad F_0 = 1, \quad F_1 = a_1, \quad \text{and} \quad a_1, a_2$$

are rational integers.

It is easy to prove by use of continued fractions (see [1]) that

$$F_n = ((a_1 + V)^{n+1} - (a_1 - V)^{n+1})/2^{n+1} V \qquad (n = 0, 1, 2, \cdots) ,$$

and then by elementary means we show that

$$(29) \qquad \lim_{n\to\infty} (F_n / F_{n-1}) = \tfrac{1}{2}(a_1 + V) = H .$$

Now, combining (28) with (III), we have

$$(30) \qquad AkF_{n-1}^{(k+1)} = a_1 n F_n^{(k)} + a_2(4k + 2n - 2) F_{n-1}^{(k)} ,$$

where $n, k = 1, 2, 3, \cdots$.

In (30), we have the following: when $k = 1$, then

$$AF_{n-1}^{(2)} = a_1 nF_n + a_2(2n + 2)F_{n-1} ,$$

and dividing this equation by nF_{n-1}, we have

$$\frac{AF_{n-1}^{(2)}}{nF_{n-1}} = \frac{a_1F_n}{F_{n-1}} + a_2\left(\frac{2n + 2}{n}\right) ,$$

where combining this result with (29), we write

$$\text{(31)} \quad A\left(\lim_{n\to\infty} F_{n-1}^{(2)} / nF_{n-1}\right) = \lim_{n\to\infty} (a_1F_n/F_{n-1} + a_2(2n+2)/n) = a_1H + 2a_2 ;$$

when $k = 2$ (in (30)), then

$$\text{(32)} \quad 2AF_{n-1}^{(3)} = a_1 nF_n^{(2)}(F_n / F_n) + a_2(2n + 6)F_{n-1}^{(2)} ,$$

Multiplying both sides of (32) by A/n^2F_{n-1}, we now write

$$\text{(33)} \quad \begin{aligned} 2A^2(F_{n-1}^{(3)} / n^2F_{n-1}) &= a_1(AF_n^{(2)} / F_n)(1/n)(F_n/F_{n-1}) \\ &\quad + a_2\left(\frac{2n + 6}{n}\right)(AF_{n-1}^{(2)} / nF_{n-1}) . \end{aligned}$$

Then combining (33) with (31) leads to

$$\text{(34)} \quad \begin{aligned} 2A^2\left(\lim_{n\to\infty} F_{n-1}^{(3)} / n^2F_{n-1}\right) &= a_1(a_1H + 2a_2)H + 2a_2(a_1H + 2a_2) \\ &= (a_1H + 2a_2)^2 ; \end{aligned}$$

when $k = 3$ (in (30)), then

$$\text{(35)} \quad 3AF_{n-1}^{(4)} = a_1 nF_n^{(3)}(F_n / F_n) + a_2(2n + 10)F_{n-1}^{(3)} ,$$

multiplying both sides of (35) by $2A^2/n^3F_{n-1}$, we now write

$$
(36)\qquad \begin{aligned} 3!\,A^3(F^{(4)}_{n-1}/n^3F_{n-1}) &= a_1(2A^2F^{(3)}_n/n^2F_n)(F_n/F_{n-1}) \\ &\quad + a_2\left(\frac{2n+10}{n}\right)(2A^2F^{(3)}_{n-1}/n^2F_{n-1}) \end{aligned}
$$

where combining (36) with (34) leads to

$$
(37)\qquad \begin{aligned} 3!\,A^3(\lim_{n\to\infty} F^{(4)}_{n-1}/n^3F_{n-1}) &= a_1(a_1H + 2a_2)^2H + 2a_2(a_1H + 2a_2)^2 \\ &= (a_1H + 2a_2)^3 . \end{aligned}
$$

Then, step-by-step, and with induction, we prove that

$$
(38)\qquad k!\,A^k \lim_{n\to\infty} (F_n^{(k+1)}/(n+1)^kF_n) = (a_1H + 2a_2)^k ,
$$

where replacing the A and H in (38) with their respective values in (28), we complete the proof of VI.

REMARK. It may be interesting to note that if $a_1^2 + 4a_2$ is replaced by $a_1^2 + 4a_2 = (a_1k)^2$ in the right side of VI, then of course

$$
\lim_{\substack{n\to\infty \\ k\to\infty}} (2^k k!\, F_n^{(k+1)}/(n+1)^kF_n) = e \qquad (e = 2.71828\cdots) .
$$

AN EXPLICIT FORMULA FOR THE TRIBONACCI NUMBERS

Let

$$
\left(1 - \sum_{r=1}^{t} a_r x^r\right)^{-1} = 1 + \sum_{n=1}^{\infty} c(n,t)x^n ,
$$

where the a_r are rational integers.

In a recent paper (see [3]), it was proved that it is always possible to express the $c(n,t)$ by an explicit formula when $t = 1, 2, 3, 4,$ and 5.

Then, using the methods in [3] we find the following Tribonacci formula ($T_n = c(n,3)$):

$$(39) \qquad T_n = \frac{x_1(x_3^{n+2} - x_2^{n+2}) + x_2(x_1^{n+2} - x_3^{n+2}) + x_3(x_2^{n+2} - x_1^{n+2})}{x_1(x_3^2 - x_2^2) + x_2(x_1^2 - x_3^2) + x_3(x_2^3 - x_1^2)} ,$$

where

$$x_1 = z_1 + 4/9z_1 + 1/3 ,$$
$$x_1 = z_2 + 4/9z_2 + 1/3 ,$$
$$x_3 = z_3 + 4/9z_3 + 1/3 ,$$

with

$$z_1 = (1/3)(3\sqrt{33} + 19)^{1/3} ,$$
$$z_2 = -(z_1/2)(1 - i\sqrt{3}) , \qquad (i = \sqrt{(-1)})$$
$$z_3 = -(z_1/2)(1 + i\sqrt{3}) ,$$

and

$$n = 0, 1, 2, \cdots .$$

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FIBONACCI SEQUENCE MODULO a prime $p \equiv 3 \pmod 4$

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Shah [1] proved: For a prime $q > 7$ the Fibonacci sequence might contain a complete residue system mod q only if $q \equiv 3$ or $7 \pmod{20}$. Here we show the

Theorem. Let p be a prime, $p > 7$, $p \equiv 3 \pmod 4$, then in the Fibonacci sequence, a complete residue system mod p doesn't exist.

It follows from this and Shah's result: The only primes for which the Fibonacci sequence possesses a complete residue system are 2, 3, 5, and 7.

Let p be a prime, $p > 7$, $p \equiv 3 \pmod 4$. In the following all residues and congruences are meant mod p. For the Fibonacci sequences

$$u_{-1} = 0, \quad u_0 = 1, \quad u_1 = 1, \quad u_2 = 2, \cdots$$

is true:

(1) $u_n = u_a u_{n-a} + u_{a-1} u_{n-a-1}, \quad a = 0, \cdots, n; \quad n = 0, 1, \cdots$

(2) $u_{k+b} \equiv \pm u_{k-b}, \quad b = 0, \cdots, k,$

where $g = 2k + 1$ is the minimal index so that $p \mid u_g$ (for $p \equiv 3 \pmod 4$ g is uneven).

(3) $u_{x(g+1)+y} \equiv \pm u_y, \quad y = 0, \cdots, g; \quad x = 0, 1, 2, \cdots$

(You verify these known facts by easy calculations.)

Lemma. The residues

$$u_s u_{s-1}^{-1}, \quad s = 1, \cdots, g,$$

are all different.

Proof. From

$$u_a u_{b-1} \equiv u_b u_{a-1}, \quad 1 \le a \le b \le g,$$

we define (putting $u_a = u_{a-1} + u_{a-2}$ and $u_b = u_{b-1} + u_{b-2}$)

$$u_{a-1} u_{b-2} \equiv u_{b-1} u_{a-2} ,$$

continuing this way, we get

$$u_1 u_{b-a} \equiv u_{b-a+1} u_0 ,$$

this means

$$u_{b-a} \equiv u_{b-a+1} ,$$

hence $u_{b-a-1} \equiv 0$, hence $b = a$.

Corollary 1. $g \le p$.

Corollary 2. The residues

$$u_s u_{s-e}^{-1}, \quad s = e, \cdots, g + e - 1,$$

are all different, e being a given number $1 \le e \le g$.

Proof. From

$$u_a u_{b-e} \equiv u_b u_{a-e} ,$$

we conclude with

$$u_a = u_e u_{a-e} + u_{e-1} u_{a-e-1}$$

and

$$u_b = u_e u_{b-e} + u_{b-e-1}$$

(from (1))

$$u_{a-e-1}u_{b-e} \equiv u_{b-e-1}u_{a-e}$$

and by the Lemma, $a - e = b - e$, $a = b$.

(The Lemma and Corollaries hold, of course, for all primes.)

Proof of the Theorem. From (2) and (3), it is clear that

$$u_n \equiv 0 \quad \text{or} \quad \pm u_c, \quad 1 \leq c \leq k$$

holds for all n. Therefore the question is whether

$$\{0, \pm u_c, 1 \leq c \leq k\}$$

forms a complete residue system or not. This might be the case only if k takes its maximum $(p - 1)/2$. Hence to prove the Theorem, it suffices to prove: Is $g = p$ then there is a congruence

$$(\star) \; u_a \equiv \pm u_b$$

for at least one pair (a,b), $1 \leq a < b \leq (p - 1)/2$.

Putting $e = 5$, Corollary 2 gives: The p residues

$$u_s u_{s-5}^{-1}, \quad s = 5, \cdots, p + 4,$$

are all different. Hence there is a t, $5 \leq t \leq p + 4$, satisfying

$$u_t u_{t-5}^{-1} \equiv 1 .$$

From this,

$$u_t \equiv u_{t-5} \quad \text{for one } t, \quad 5 \leq t \leq p + 4 .$$

We differ 4 cases:

a) $t \geq p$,

b) $p > t > t - 5 \geq (p - 1)/2$,

c) $t > (p-1)/2 > t-5,$

d) $(p-1)/2 \geq t.$

Case a) is impossible:

$$u_{p\pm4} \equiv \pm u_3, \quad u_{p\pm3} \equiv \pm u_2, \quad u_{p\pm2} \equiv \pm u_1, \quad u_{p\pm1} \equiv \pm u_0, \quad u_p \equiv 0$$

(from (2) and (3)). (Check the cases $t = p, \cdots, p+4$ one after the other and take into account $p > 5$.) While Case (d) is a congruence (*) itself, we easily get such a congruence in Case (b) by utilizing (2). In the remaining Case (c), we put

$$t = (p-1)/2 + r, \quad 1 \leq r \leq 4 .$$

We have

$$u_{(p-1)/2+r} \equiv u_{(p-1)/2-(5-r)} .$$

From (2), we conclude

$$u_{(p-1)/2+r} \equiv \pm u_{(p-1)/2-r} ,$$

hence

$$(**) \quad u_{(p-1)/2-(5-r)} \equiv \pm u_{(p-1)/2-r} .$$

$p > 7$ implies $(p-1)/2 > 4$, therefore in (**), both indices are ≥ 1, r and $5-r$ always being different (**) is a congruence (*). This finishes the proof of the Theorem.

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SUMMATION OF POWERS OF ROOTS OF SPECIAL EQUATIONS

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It is well known that the sums of the n^{th} powers of the roots of $x^2 - x - 1 = 0$ give rise to the famous Lucas sequence $1, 3, 4, 7, 11, \cdots, L_n, \cdots$, in which each term is the sum of the preceding two terms. In this case, one can find the roots

$$\alpha = (1 - \sqrt{5})/2,$$

and

$$\beta = (1 + \sqrt{5})/2,$$

and easily calculate $L_n = \alpha^n + \beta^n$ for $n = 1, 2, 3, \cdots$.

But what of the sums of the n^{th} powers of roots of other equations of the form

$$x^n - x^{n-1} - x^{n-2} - \cdots - x - 1 = 0 \;?$$

Soon the roots cannot be found directly but an interesting pattern of sequences of integers emerges.

The problem can be solved using symmetric functions derived in elementary theory of equations, but we prefer matrix theory. We use the following matrix properties. The trace (sum of elements on mail diagonal) of a square matrix of order n is the coefficient of x^{n-1} in its characteristic equation, when the characteristic equation is computed by subtracting x from each main diagonal element and then taking the determinant. If a matrix is raised to the k^{th} power, then its new characteristic equation has as its roots the k^{th} powers of the roots of the original equation. So, raising a matrix to successive powers and summing its main diagonal is a way of calculating sums of powers of the roots of an equation.

To sum the n^{th} powers of the roots of $x^2 - x - 1 = 0$, which is the characteristic equation of

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

simply calculate the trace for successive powers of the matrix. And for $x^3 - x^2 - x - 1 = 0$, we use

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

For

$$x^k - x^{k-1} - \cdots - x - 1 = 0,$$

write the square matrix of order k having each element in the first row equal to one, each element in the k^{th} column except the first equal to zero, and bordering an identity matrix of order $k - 1$, as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

In the table of values occurring that follows, let Σr^n signify the sum of the n^{th} powers of the roots, and

$$\begin{aligned} f(x)_1 &= x - 1 && = 0 \\ f(x)_2 &= x^2 - x - 1 && = 0 \\ \cdots & \quad \cdots \\ f(x)_n &= x^n - x^{n-1} - x^{n-2} - \cdots - x - 1 && = 0 \end{aligned}$$

SUMS OF N^{th} POWERS OF ROOTS

$$f(x)_k = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$$

	1	2	3	4	5		k - 1	k
	$f(x)_1 = 0$	$f(x)_2 = 0$	$f(x)_3 = 0$	$f(x)_4 = 0$	$f(x)_5 = 0$	$\cdots$	$f(x)_{k-1} = 0$	$f(x)_k = 0$
$\sum r^0$	1	2	3	4	5	$\cdots$	k - 1	k
$\sum r^1$	1	1	1	1	1	$\cdots$	1	1
$\sum r^2$	1	3	3	3	3	$\cdots$	3	3
$\sum r^3$	1	4	7	7	7	$\cdots$	7	7
$\sum r^4$	1	7	11	15	15	$\cdots$	15	15
$\sum r^5$	1	11	21	26	31	$\cdots$	31	31
$\sum r^6$	1	18	39	51	57	$\cdots$	63	63
$\sum r^{k-1}$	1	L_{k-1}					$2^{k-1} - 1$	$2^{k-1} - 1$
$\sum r^k$	1	L_k						$2^k - 1$
$\sum r^{k+1}$	1	L_{k+1}						$2^{k+1} - k - 2$

$\sum r^n$ is the sum of the preceding k columnar terms.

Matric theory explains the general form given in the right column of the table. It can be proved by mathematical induction that if the nxn matrix $M = (a_{ij})$ is defined as

$$a_{ij} = \begin{cases} 1, i = 1 \text{ or } i = j+1 \\ 0, i \neq 1 \text{ and } i \neq j+1 \end{cases},$$

then, if $k \leq n$, $M^k = (b_{ij})$ has the following form:

$$b_{ij} = 2^{k-i} \quad \text{for } j = 1, 2, \cdots, n+i-k, \quad i = 1, 2, \cdots, n;$$
$$b_{ij} = 0 \quad \text{if } k < i \text{ and } j \geq i.$$

Thus, the trace of M^k is

$$2^{k-1} + 2^{k-2} + \cdots + 2 + 1 = 2^k - 1, \quad k \leq n.$$

Since M satisfies its own characteristic equation,

$$M^n = M^{n-1} + M^{n-2} + \cdots + M + I$$
$$M^{n+1} = M^n + M^{n-1} + \cdots + M^2 + M,$$

and the trace of M^{n+1} equals the sum of the traces of the matrices on the right, giving trace of

$$M^{n+1} = (2^n - 1) + (2^{n-1} - 1) + \cdots + (2^2 - 1) + (2^1 - 1)$$
$$= 2^{n+1} - n - 2.$$

Since finding the trace of M^k for $k \geq n$ involves summing the n preceding traces already obtained, we can form our table inductively without actually raising the matrices to powers.

For example, to get the series for $n = 5$, the sum of the five roots raised to the zero power is 5. The sum of the five roots to the first power is one, the coefficient of $-x^4$ in

$$x^5 - x^4 - x^3 - x^2 - x - 1 = 0.$$

The sum of the second, third, and fourth powers are given by

$$2^2 - 1, \; 2^3 - 1, \; 2^4 - 1.$$

The sum of the fifth powers is either $2^5 - 1$ or the sum

$$5 + 1 + 3 + 7 + 15.$$

The sum of the sixth powers is the sum of the preceding five entries,

$$1 + 3 + 7 + 15 + 31.$$

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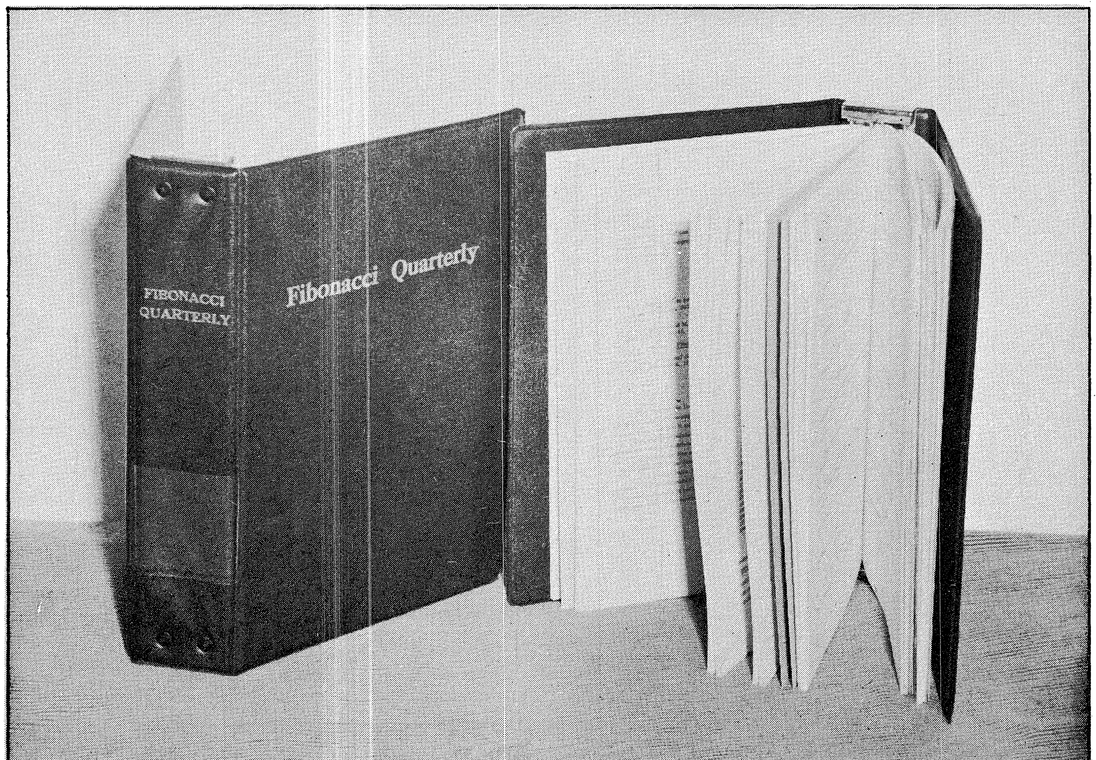
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