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TAKE-AWAY GAMES

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I. INTRODUCTION

Several games of "take-away" have become popular. The purpose of this paper is to determine the winning strategy of a general class of take-away games, in which the number of markers which may be removed each turn is a function of the number removed on the preceding turn. By-products of this investigation are a new generalization of Zeckendorf's Theorem [3], and an affirmative answer to a conjecture of Gaskell and Whinihan [2].

Definitions:

(I-1) Let a take-away game be defined as a two-person game in which the players alternately diminish an original stock of markers subject to various restrictions, with the player who removes the last marker being the winner.*

(I-2) A turn or move shall consist of removing a number of these markers.

(I-3) Let the original number of markers in the stock be $N(0)$.

(I-4) After the k^{th} move there will be $N(k)$ markers remaining.

(I-5) The player who takes the first turn shall be called player A. The other player shall be called player B.

(I-6) Let $T(k) = N(k-1) - N(k)$. That is, $T(k)$ is the number of markers removed in the k^{th} move.

(I-7) The winning strategy sought will always be a forced win for Player A.

All games considered in this paper are further restricted by the following rules:

(a) $T(k) \geq 1$ for all $k = 1, 2, \cdots$.

(b) $T(1) < N(0)$ (Thus, $N(0) > 1$.)

(c) For all $k = 2, 3, \cdots$, $T(k) \leq m_k$, where m_k is some function of $T(k-1)$.

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*This definition is essentially taken from Golomb [1].

Rule (a) guarantees that the game will terminate after a finite number of moves since the number of markers in the stock is strictly decreasing, and hence, must reach zero. Rule (b) dispenses with the uninteresting case of immediate victory. Rule (c) is the source of the distinguishing characteristics of the various games which shall be considered.

II. MOTIVATION

Example (II-1)

A simple game occurs when m_k is defined to be constant, m, and we require $T(1) \leq m$. The well known strategy is: If $N(0) \not\equiv 0 \bmod (m+1)$,

remove $N(0) \bmod (m+1)$ markers. On subsequent moves, Player A selects $T(2j+1)$ to be equal to $m+1-T(2j)$.

If $N(0) \equiv 0 \bmod (m+1)$,

Player B can win by applying Player A's strategy above.

A simple way to express this result is to write the integer $N(k)$ in a base $m+1$ number system. Thus,

$$N(k) = a_0 + a_1(m+1) + a_2(m+1)^2 + \cdots + a_j(m+1)^j \quad ,$$

where this representation is unique. Player A's strategy is to remove a_0 markers, provided $a_0 \neq 0$. If $a_0 = 0$, Player A is faced with a losing position.

This result suggests a connection between winning strategies and number systems.

Example (II-2)

Consider the game defined by the rule $m_k = T(k-1)$, the number of markers removed on the preceding move. In other words, $T(k) \leq T(k-1)$. To find a winning strategy, express $N(0)$ as a binary number, e.g., $12 = 1100_B$. Define $/N(0)/$ as follows: If

$$N = (a_n a_{n-1} \cdots a_1 a_0)_B \quad ,$$

in the binary system, then

$$/N/ = a_n + a_{n-1} + \cdots + a_1 + a_0 \quad .$$

If $/N(0)/ = k > 1$,

Player A removes a number corresponding to the last "one" in the binary expansion. (Thus, for $N(0) = 12 = 1100_B$, Player A removes $4 = 100_B$.) Now $/N(1)/ = k - 1 > 0$. Player B now has no move which reduces $/N(1)/$; to do so, he would have to remove twice as many as the rules permit. In addition, any move Player B does make produces an $N(2)$ such that Player A can again remove the last "1" in the expansion of $N(2)$. To see this, note that $N(1)$ can be rewritten as

$$(a_n a_{n-1} \cdots a_r 1 \cdots 11)_B + 1B \,.$$

Now, since $N(k)$ is strictly decreasing, it must reach zero. However, $/0/ = 0$ and $/N/ > 0$ for all positive integers N. Since Player B never decreases $/N(k)/$, Player B cannot produce zero; hence, Player A must win.

If, on the other hand, $/N(0)/ = 1$,

it is clear that Player A cannot win because $N(0) = 100 \cdots 0_B = 11 \cdots 1_B + 1_B$. Any move by Player A permits Player B to remove the last "1" in the expansion, thus applying the strategy formerly used by A above.

Again we see a connection with number systems. A generalization of this method now suggests itself: Find a way to express every positive integer as a unique sum of losing positions. Then a losing position has norm 1. For any other position, the norm reducing strategy described above will work if, given Player A's move,

(i) Player B cannot reduce $/N(k)/$, and

(ii) any move Player B does make permits Player A to reduce $/N(k+1)/$.

III. THE GENERAL GAME

Now consider any game in which m_k is a function of the number of markers removed on the preceding move; i.e., let $m = f(T(k - 1))$. Suppose $f(n) \geq n$ and $f(n) \geq f(n - 1)$ for all positive integers, n. Note that example II-2 satisfied this hypothesis. We want f to be a monotonic nondecreasing function so that if Player B removes more markers he cannot limit Player A to removing fewer markers and thus foil the norm reducing strategy. In addition, we want $f(n) \geq n$ to guarantee the existence of a legal move at all times, and to permit the following definition:

Definition (III-1)

Define a sequence (H_i) by: $H_1 = 1$ and $H_{k+1} = H_k + H_j$ where j is the smallest index such that $f(H_j) \geq H_k$.

Clearly this is well defined because if the above inequality holds for no smaller j, at least we know it holds for $j = k$.

Theorem (III-2)

Every positive integer can be represented as a unique sum of H_i's, such that

$$N = \sum_{i=1}^{n} H_{j_i} \quad \text{and} \quad f(H_{j_i}) < H_{j_{i+1}} \quad \text{for} \quad i = 1, 2, \cdots, n-1 .$$

Proof

The theorem is trivially true when $N = 1$, for $H_1 = 1$.

Assume that the theorem holds for all $N < H_k$; and let $H_k \leq N < H_{k+1}$. By induction,

$$N = H_k + \sum_{i=1}^{n} H_{j_i}$$

where $f(H_{j_i}) < H_{j_{i+1}}$ for $i = 1, 2, \cdots, n-1$. Thus, for the existence of a representation, we need only show that $f(H_{j_n}) < H_k$. Suppose $f(H_{j_n}) \geq H_k$. Then recall that $H_{k+1} = H_k + H_\ell$ where ℓ is the minimal coefficient for which $f(H_\ell) \geq H_k$. Hence $j_n \geq \ell$ and so

$$H_{k+1} = H_k + H_\ell \leq H_k + H_{j_n} \leq N ,$$

contradicting the choice of N. Thus we have existence.

For uniqueness, note that: $f(H_{j_1}) < H_{j_2}$ implies

$$\sum_{i=1}^{2} H_{j_1} < H_{j_2+1}$$

$f(H_{j_2}) < H_{j_3}$ implies

$$\sum_{i=1}^{3} H_{j_i} < H_{j_3+1}$$

$\vdots$

$f(H_{j_{n-1}}) < H_{j_n}$ implies

$$\sum_{i=1}^{n} H_{j_i} < H_{j_n+1}$$

Thus, for $H_{k+1} > N \geq H_k$, the largest term in any sum for N must be H_k. If N has two representations, so does $N - H_k$, but this violates the induction hypothesis. Thus, the representation is unique.

Definition (III-3)

$/N/$ is the number of terms in the "H sum" for N.

Lemma (III-4)

If $/N(k)/ = 1$ and the player cannot move $N(k)$ markers, then any move he does make permits his opponent to reduce $/N(k + 1)/$.

Proof

For simplicity, let us assume that k is odd. Thus, we will prove that Player A can remove an appropriate number of markers so that $/N(k + 2)/ < /N(k + 1)/$.

Rewrite

$$\begin{aligned} N(k) &= H_{j_0} \\ &= H_{j_0-1} + H_{j_1} \\ &\vdots \\ &= H_{j_0-1} + H_{j_1-1} + \cdots + H_{j_n-1} + 1 \end{aligned}$$

for some n where

$$f(H_{j_{i+1}}) \geq H_{j_i - 1} > f(H_{j_{i+1}-1})$$

for $i = 0, 1, \cdots, n - 1$. Note that this is equivalent to

$$H_{j_i} = H_{j_i - 1} + H_{j_{i+1}} .$$

Now Player B removes $T(k + 1)$, with $H_{j_{i+1}} \leq T(k + 1) < H_{j_i}$ for some i between 0 and n, where $H_{j_{n+1}} = 1$. Player A may remove up to

$$\begin{aligned} f(T(k + 1)) &\geq f(H_{j_{i+1}}) \\ &\geq H_{j_i - 1} . \end{aligned}$$

Hence, Player A may elect to remove

$$H_{j_i} - T(k + 1) \leq H_{j_i} - H_{j_{i+1}} = H_{j_i - 1}$$

and

$$N(k + 2) = H_{j_{i-1}-1} + \cdots + H_{j_0 - 1} .$$

Since $f(H_{j_i - 1}) < H_{j_{i-1}-1}$ for $i = 1, 2, \cdots, n$, we have $/N(k + 2)/ = i$.

Let

$$/H_{j_i} - T(k + 1)/ = a > 0 .$$

Now $N(k + 1) = N(k + 2) + H_{j_i} - T(k + 1)$.

Let H_ℓ be the largest term in the H sum of $H_{j_i} - T(k + 1)$. Clearly $H_\ell \leq H_{j_i - 1}$ whence $f(H_\ell) \leq f(H_{j_i - 1}) < H_{j_{i-1}-1}$. Thus

$$/N(k + 1)/ = i + a > i = /N(k + 2)/ ,$$

and this completes the proof of the lemma.

Theorem (III-5)

Let us consider a game defined by f satisfying the properties stated above. Also let (H_i) and the norm be defined as above.

If $/N(0)/ > 1$, Player A can force a win. If $/N(0)/ = 1$, Player B can force a win.

Proof

If $/N(0)/ > 1$,

let $N(0) = H_{j_1} + \cdots + H_{j_n}$ with $f(H_{j_i}) < H_{j_{i+1}}$. Player A removes H_{j_2}. Since Player B can remove at most $f(H_{j_1}) < H_{j_2}$ it is clear that Player B cannot reduce $/N(1)/$ or affect any of the last $n - 2$ terms in the sum, so we may just as well consider $n = 2$. Now we invoke Lemma (III-4), so Player A can reduce $/N(2)/$. Thus, Player A can force a win.

If $/N(0)/ = 1$,

Since Player A cannot remove $N(0)$ markers, Lemma (III-4) tells us that Player B will be able to reduce $/N(1)/$. If $/N(1)/ = 1$, this means that he can remove $N(1)$ and win immediately. If $/N(1)/ > 1$, Player B can apply Player A's strategy from the first part of this proof. Thus, Player B can force a win.

IV. BY-PRODUCTS

In the case when $f(T(k-1)) = 2T(k-1)$, the foregoing results produce the conclusions of Whinihan and Gaskell [2] regarding "Fibonacci Nim." We note that in this case:

$$\begin{aligned} H_1 &= 1 \\ H_2 &= H_1 + H_1 = 2 \\ H_3 &= H_2 + H_1 = 3 \end{aligned}$$

and in general, if

$$H_{n-i} = H_{n-i-1} + H_{n-i-2}$$

for $i = 0$, 1, and 2, then

$$2H_{n-3} \geq H_{n-2} > 2H_{n-4}$$
$$2H_{n-2} \geq H_{n-1} > 2H_{n-3}$$

So

$$2H_{n-1} \geq H_n > 2H_{n-2}$$

by adding the inequalities above. Hence $H_{n+1} = H_n + H_{n-1}$. This process continues by induction so that the sequence (H_i) is indeed the sequence of Fibonacci numbers.

Also in this case, Theorem (III-2) becomes "Zeckendorf's theorem" [3], which states that every positive integer can be uniquely expressed as a Fibonacci sum with no two consecutive subscripts appearing.

Another interesting fact, conjectured by Whinihan and Gaskell [2], is that for the game $m_k = cT(k-1)$, where c is any real number ≥ 1, (H_i) must become a simple recursion sequence for sufficiently large subscripts; i.e., there exist integers k and n_0 such that $H_{n+1} = H_n + H_{n-k}$ for all $n \geq n_0$. Let us now consider how to prove the conjecture, and how to calculate k and n_0 as a function of c.

Lemma (IV-1)

If $cH_{i-1} < H_j \leq cH_i$, then $cH_{i+1} \geq H_{j+1}$.

Proof

Since $cH_{i-1} < H_j \leq cH_i$, we must have $H_{j+1} = H_j + H_i$. Also, $H_{i+1} = H_i + H_k$ where $cH_k \geq H_i$. Now

$$\begin{aligned} cH_{i+1} &= cH_i + cH_k \\ &\geq cH_i + H_i \\ &\geq H_j + H_i = H_{j+1} . \end{aligned}$$

Theorem (IV-2)

There exists an integer k such that $cH_{n-k} < H_n$ for all $n > k$.

Proof

Since $H_{j+1} = H_j + H_i$ where $cH_i \geq H_j$, it follows that

$$\frac{H_{j+1}}{H_j} \geq \left(1 + \frac{1}{c}\right).$$

If we choose k such that

$$\left(1 + \frac{1}{c}\right)^k > c \quad ,$$

then

$$\frac{H_{j+1}}{H_{j-k+1}} \geq \left(1 + \frac{1}{c}\right)^k > c \quad .$$

Thus, $cH_{j-k} < H_j$ for all $j > k$. This completes the proof of the theorem.

Corollary (IV-3)

(H_n) must become a simple recursion sequence for sufficiently large n.

Proof

Lemma (IV-1) says that the difference between successive indices as described before is monotonically nondecreasing. Theorem (IV-2) says that the sequence of differences is bounded. Thus the difference must be constant for all large n. This is equivalent to saying that (H_n) is a recursion sequence for $n \geq n_0$. Q.E.D.

Theorem (IV-4)

If $H_{j+i+1} = H_{j+i} + H_{j+i-k}$ for some j, and for $i = 0, 1, \cdots, k+1$, then this equation holds for every positive integer i.

Proof

By induction, we need only show that $H_{j+k+3} = H_{j+k+2} + H_{j+2}$. By definition, $H_{j+k+2} = H_{j+k+1} + H_{j+1}$ implies $cH_j < H_{j+k+1} \leq cH_{j+1}$. $H_{j+1} = H_j + H_{j-k}$ implies

$$cH_{j-k} < H_{j+1} \leq cH_{j-k+1} \, ,$$

whence

$$c(H_j + H_{j-k}) < H_{j+k+1} + H_{j+1} \leq c(H_{j+1} + H_{j-k+1}) \, ,$$

or

$$cH_{j+1} < H_{j+k+2} \leq cH_{j+2} \quad ,$$

so $H_{j+k+3} = H_{j+k+2} + H_{j+2}$. Q. E. D.

This theorem tells us that k has reached the recursion value when k has been the difference for $k + 2$ successive indices.

V. CONCLUSION

We have discovered some interesting properties of take-away games and their winning strategies. The subject, however, is by no means exhausted.

For example, in Theorem (IV-4) we showed that for every $c \geq 1$ there exists a k such that .... By inspection, I have found:

If c = 1	then	k = 0
= 2		= 1
= 3		= 3
= 4		= 5
= 5		= 7
= 6		= 9
= 7		= 12
= 8		= 14

It is not clear whether or not a simple relation exists between c and k.

In Section IV, we found that $f(x) = cx$ gives rise to a recursion relation for (H_i). Other special cases of f can be studied, to learn about the corresponding sequence (H_i); or one might try to reverse the approach by proceeding from (H_i) to f, as opposed to the approach taken in this paper.

It is also possible to generalize in other ways. For example, if $f(n)$ and $g(n)$ satisfy the hypotheses of Section III, then $(f + g)(n) = f(n) + g(n)$ and $(fg)(n) = f(n)g(n)$ also satisfy the hypotheses. Can the corresponding strategies and sequences be related? Can the procedure be generalized for functions which are not monotonic? These problems are suggested for those interested in pursuing the subject further.

[Continued on page 241.]

ON THE ENUMERATION OF CERTAIN TRIANGULAR ARRAYS

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1. In [2], this quarterly, D. P. Roselle considered the enumeration of certain triangular arrays of integers. He obtained recurrences for these which had a Fibonacci character. In this paper, we obtain explicit formulae for the enumeration of these arrays, with a slight change of notation, and some generalizations. Although difficult to state in its full generality, it will be seen that the method of enumeration can be applied to a rather general class of arrays in a given instance.

By a lattice point in the plane is meant a point with integral coordinates, non-negative unless stated otherwise. By a path (lattice path) is meant a minimal path via lattice points, taking unit horizontal and vertical steps.

It is well known that the number of paths from $(0,0)$ to (p,q) is

$$\binom{p+q}{p},$$

and there are

$$\binom{p+q-1}{p-1}$$

which start with a unit horizontal step.

With $[x]$ the greatest integer $\leq x$, note that

$$\left[\frac{n-1}{m}\right] + 1 = \left[\frac{n+m-1}{m}\right] = \begin{cases} \left[\frac{n}{m}\right] & \text{if } m \mid n, \\ \left[\frac{n}{m}\right] + 1 & \text{if } m \nmid n. \end{cases}$$

2. To fix the idea, we take the simplest case first.

For integral $n \geq 1$, $m \geq 1$, consider the triangular array of integers $a_{ij} = 0$ or 1, $i = 1, 2, \cdots, [(n-1)/m] + 1$ and $j = (i-1)m + 1, \cdots, n$, with the restrictions $1 \geq a_{i,j} \geq a_{i+1,j} \geq 0$ and $1 \geq a_{i,j} \geq a_{i,j+1} \geq 0$, [2, §2]. Let $f(n;m)$ denote the number of these arrays.

For example, with $m = 3$ and $n = 11$, the arrays have the shape

$$
\begin{array}{r}
\text{xxxxxxxxxxx} \\
\text{xxxxxxxx} \\
\text{xxxxx} \\
\text{xx}
\end{array}
$$

$f(11;3) = 88$, and a typical array is

$$
\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & & & & & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & 0 & 0
\end{array}
\quad .
$$

It follows from the restrictions on the a_{ij} that

$$
(2.1) \qquad f(n;m) = f(n-1;m) + f(n-m;m), \qquad n > m,
$$

according as $a_{1,n} = 0$ or 1 with the initial conditions

$$
f(n;m) = n + 1, \qquad 1 \leq n \leq m,
$$

We adjoin the conventional value $f(0;m) = 1$. Compare [2, (1.1) and (1.3)].

We show directly that

$$
(2.2) \qquad f(n;m) = \sum_{k=0}^{\left[\frac{n+m-1}{m}\right]} \binom{n-(m-1)(k-1)}{k} .
$$

It is easy to show that (2.2) satisfies (2.1) and the initial conditions.

As in [2], we note in passing that $f(n;1) = 2^n$; and $f(n;2) = F_{n+2}$, the Fibonacci numbers.

To get (2.2) directly, note first that there is only one array, consisting of all zeros, if $a_{1,1} = 0$. For each $k \geq 1$, we get a new set of arrays in each case where at least $a_{k,(k-1)m+1} = 1$ and all the $a_{k+1,j} = 0$. This adjoins an artificial row of zeros in the case of the last row, but it does not change the count. In view of the restrictions on the a_{ij}, we need only consider the rectangular arrays

$$\begin{matrix} a_{1,(k-1)m+1} & \cdots & a_{1,n} \\ \cdots\cdots & \cdots & \cdot \\ a_{k,(k-1)m+1} & \cdots & a_{k,n} \end{matrix}$$

with

$$a_{i,(k-1)m+1} = 1, \quad i = 1,2,\cdots,k \; .$$

These arrays correspond in a one-one way with the

$$\binom{n - (k - 1)(m - 1)}{k} \tag{2.3}$$

paths from $((k - 1)m, 0)$ to (n, k) which start with a unit horizontal step as follows: for each path, place 1's in the unit squares above and to the **left** (northwest side) of the path and 0's in the unit squares below and to the right (southeast) of the path. For example, see the blocked out section of the preceding example. Sum (2.3) over

$$k = 0, 1, \cdots, [(n - 1)/m] + 1$$

to get (2.2).

The preceding result also enumerates one-line arrays

$$n_1 n_2 \cdots n_s, \qquad s = [(n - 1)/m] + 1 \; , \tag{2.4}$$

where $n_1 = n$ and $0 \le n_{j+1} \le n_j - m$. Compare [2, §4]. This is seen by taking row sums of the a_{ij} in the case that all the $a_{1,j} = 1$. This is precisely the original problem with n replaced by $n - m$. That is, there are

$f(n - m; m)$ such one-line arrays. These may also be thought of as combinations of the first n natural numbers written in descending order; compare [1, p. 222, problem 1].

If, in (2.4), we only require $0 \le n_1 \le n$, we have the obvious additional one-line arrays.

The arrays above had a row depth of one. It is easy to expand the problem to the case of row depth $p \geq 1$. That is, let $f(n; m|p)$ denote the number of arrays of $a_{ij} = 0$ or 1, where $i = 1, 2, \cdots, p[(n + m - 1)/m])$ and for $i = (k - 1)p + s$ $(s = 1, 2, \cdots, p$ and $k = 1, 2, \cdots, [(n + m - 1)/m])$ $j = (k - 1)m + 1, \cdots, n$. We have the same restrictions as before.

For example, with $m = 3$, $n = 11$, and $p = 2$, the arrays have the shape

```
XXXXXXXXXXX
XXXXXXXXXXX
   XXXXXXXX
   XXXXXXXX      ,
      XXXXX
      XXXXX
         XX
         XX
```

and $f(11; 3/2) = 871$.

We shall find in §3 that

$$(2.5) \qquad f(n;m|p) = \sum_{k=0}^{\left[\frac{n+m-1}{m}\right]} \sum_{s=1}^{p} \binom{n - (k - 1)(m - p) + s - 1}{s + (k - 1)p}$$

is a special case of a more general class of arrays. With obvious notational changes, the case $p = m$ of (2.5) is Roselle's $N_k(n,k) = N_n(k)$ [2, (1.11) and (3.10)]. That is, $f(n;k\,k) = N_k(n,k)$. Roselle's (3.10) gives the representation

$$(2.6) \qquad f(n;m|m) = \frac{1}{m} \sum_{j=0}^{m-1} \{(\rho^{-j} + 1)^m - 1\}(\rho^j + 1)^n ,$$

where ρ is a primitive m^{th} root of unity. Now (2.6) and

$$(2.7) \qquad f(n;m|m) = \sum_{k=0}^{\left[\frac{n+m-1}{m}\right]} \sum_{s=1}^{m} \binom{n+s-1}{s+(k-1)m}$$

are the same. To see this first apply the binomial identity

$$(2.8) \qquad \sum_{k=0}^{n} \binom{x+k}{r} = \binom{x+n+1}{r+1} - \binom{x}{r+1}$$

to the inner sum of (2.7) to get

$$(2.9) \qquad f(n;m\ m) = \sum_{k=0}^{\left[\frac{n+m-1}{m}\right]} \left\{ \binom{n+m}{km} - \binom{n}{(k-1)m} \right\} .$$

Next (2.6) can be rewritten as

$$f(n;m|m) = \frac{1}{m}\sum_{j=0}^{m-1} (\rho^j + 1)^{m+n} - \frac{1}{m}\sum_{j=0}^{m-1} (\rho^j + 1)^n .$$

But this is just another way of writing (2.9), c.f. [1, p. 41, problem 7].

Application of (2.8) to (2.5) yields the form

$$f(n;m|p) = \sum_{k=0}^{\left[\frac{n+m-1}{m}\right]} \left\{ \binom{n-(k-1)(m-p)+p}{kp} - \binom{n-(k-1)(m-p)}{(k-1)p} \right\} .$$

which can also be gotten by a direct combinatorial argument.

3. With the same restrictions as before on the a_{ij}, we consider a slightly more general array. In this case, the indentations will still be $m \geq 1$, the first block will have row depth $q \geq 1$, and the successive blocks will have row depth $p + (k - 1)$ respectively, $k = 2, 3, \cdots, [(n - 1)/m] + 1$, $p \geq 1$, $\alpha \geq 0$.

As before, the case $a_{11} = 0$ contributes only one array, all zeros. For each of the cases $a_{s,1} = 1$, $s = 1, 2, \cdots, q$, and $a_{s+1,1} = 0$ there are

$$\binom{n+s-1}{s}$$

arrays — corresponding to the paths from (0,0) to (n,s) with an initial horizontal step. Thus the q by n rectangle contributes

$$1 + \sum_{s=1}^{q} \binom{n+s-1}{s} = \binom{n+q}{n} \tag{3.1}$$

arrays. Note that this rectangle always gives the initial conditions; compare [2, (1.4)]. For the count on the remaining blocks, we consider the case of

$$a_{q+(k-2)p+\binom{k-1}{2}\alpha+s,\,(k-1)m+1} = 1 \quad ,$$

where $k \geq 2$ and $s = 1, 2, \cdots, p + (k - 1)\alpha$ and the next row is all zeros, in each case, these corresponding to the

$$\binom{n + s - 1 - (k - 1)(m - p) + \binom{k-1}{2}\alpha + q - p}{n - (k - 1)m - 1} \tag{3.2}$$

paths from $((k - 1)m, 0)$ to

$$\left(n, q + (k - 2)p + \binom{k-1}{2}\alpha + s\right)$$

with an initial horizontal step.

Thus the total number of arrays is

$$(3.3)\quad 1 + \sum_{s=1}^{q}\binom{n+s-1}{s} + \sum_{k=2}^{\left[\frac{n+m-1}{m}\right]}\ \sum_{s=1}^{p+(k-1)\alpha}\binom{n+s-1(k-1)(m-p)+\binom{k-1}{2}\alpha+q-p}{n-(k-1)m-1}$$

We note that (3.3) can be simplified by replacing the first two terms by the right member of (3.1) and the inner sum by applying (2.8).

We note some special cases of (3.3). First the case $\alpha = 0$ is, with obvious notational changes, Roselle's $N_j(m,k)$ [2, §3]. If, in addition, we take $p = q$, Eq. (3.3) reduces to (2.5), which in turn reduces to (2.2) for $p = 1$.

As we remarked at the beginning, it is now quite clear that the description of a very general case of these types of arrays would be quite complicated. However, it is clear that in any given instance, the method used above is easy to apply.

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[Continued from page 234.]

TAKE-AWAY GAMES

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SOME UNIVERSAL COUNTEREXAMPLES

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In [1], H. H. Ferns discussed minimal and maximal representations of positive integers as sums of distinct Fibonacci numbers. S. G. Mohanty extended those results in [2] by employing a one-parameter family of generalized Fibonacci sequences. This paper provides clarification of the concepts of maximality and minimality as employed by Ferns and Mohanty.

For convenience we will reiterate several definitions and results from [2], with suitably altered notation.

Definition 1: The generalized Fibonacci sequence $\{U_{r,n}\}$ with parameter r is given by

$$U_{r,1} = U_{r,2} = \cdots = U_{r,r} = 1$$
$$U_{r,n} = U_{r,n-1} + U_{r,n-r}$$

for integers n and r such that $n > r > 1$.

For brevity, the parameter r will not be made explicit. Thus $U_{r,n} = U_n$ and $\{U_{r,n}\} = \{U_n\}$. Since we wish to express positive integers as sums of numerically distinct terms of $\{U_n\}$, we make the restriction that the first $r - 1$ terms not be employed in any representation. After Mohanty, we assert without proof that every positive integer N has at least one representation in $\{U_n\}$ subject to that restriction. That is, there exist integers a_i such that $a_i = 0$ or $a_i = 1$ for $i = r, r+1, \cdots, s$; $a_s = 1$; and

$$N = \sum_{i=r}^{s} a_i U_i \ . \tag{1}$$

Definition 2: Given a representation of N of the form indicated above, we define the magnitude of the representation to be the sum of the coefficients a_i.

Definition 3: A representation of N in $\{U_n\}$ is said to be Minimal (or Maximal) if and only if the magnitude of the representation is less than or equal to (or greater than or equal to) the magnitude of every other representation of N in $\{U_n\}$.

This definition agrees with the intuitive notions of minimal and maximal representations in the sense that, for example, a minimal representation employs the fewest possible elements of the sequence $\{U_n\}$. Ferns, working with the special case $r = 2$ (the Fibonacci numbers) defined these ideas in a mathematically simpler but intuitively less satisfying way, which Mohanty generalized essentially as follows:

Definition 4: A representation of the form given by (1) in $\{U_n\}$ is minimal (or maximal) if and only if $a_i a_{i+j} = 0$ (or $a_i + a_{i+j} \geq 1$, respectively) for all $j = 1, 2, \cdots, r - 1$ and $i = r, r + 1, \cdots, s - j$.

It is easy to see that, for $r = 2$, these two definitions are equivalent. For if a representation in $\{F_n\}$ fails Definition 4, then, for some i, $a_i a_{i+1} = 1$ or $a_i + a_{i+1} = 0$ and the relation

$$F_{n+2} = F_{n+1} + F_n$$

can be applied to force conformity to Definition 4 and simultaneously to decrease (or increase) the magnitude of the representation, indicating that the original representation failed Definition 3 also. On the other hand, any representation not in accord with Definition 3 can be made to conform by suitable application of the relation cited above, which applications require the existence of coefficients a_i and a_{i+1} such that Definition 4 fails initially. Hence:

Theorem 1: If $r = 2$, then Definitions 3 and 4 are equivalent.

The main result of this paper is a proof (Theorems 2 and 3) of the converse of Theorem 1. It is clear that every positive integer N has at least one Minimal representation and one Maximal representation in $\{U_n\}$. Further, we have

Lemma 1: Every positive integer has a unique minimal representation in $\{U_n\}$ and a unique maximal representation in $\{U_n\}$.

Proof: This is established in [2], Lemmas 1 and 2.

Therefore, it suffices to display, for each value of r greater than 2, an integer whose minimal (or maximal) representation is not Minimal (or Maximal). Toward that end, we consider the triangular numbers.

Definition 5: The triangular numbers $\{T_n\}$ are given by

$$T_1 = 1; \qquad T_n = n + T_{n-1} .$$

If in the above definition n is allowed to take on successively the values m + 3, m + 4, and m + 5, and the resulting equations are summed, the following useful identity is obtained:

$$T_{m+5} = T_{m+2} + 3m + 12. \tag{2}$$

Lemma 2: If k is an integer such that $1 \leq k \leq r$, then:

$$U_k = 1 \tag{3}$$

$$U_{r+k} = k + 1 \tag{4}$$

$$U_{2r+k} = r + T_{k+1} \tag{5}$$

$$U_{3r+k} = r(k + 2) + T_r + T_1 + T_2 + \cdots + T_{k+1} . \tag{6}$$

Proof: These may be established by infinite induction.

Lemma 3: If $r \geq 6$ and $r = 3m$ for some integer m, then

$$U_{10m+1} + U_{6m+3} + U_{3m+1} = U_{10m} + U_{7m+4} .$$

Proof: Let $r = 3m$ in Equations (4), (5), and (6):

$$U_{3m+k} = k + 1 \tag{4'}$$

$$U_{6m+k} = 3m + T_{k+1} \tag{5'}$$

$$U_{9m+k} = 3m(k + 2) + T_{3m} + T_1 + T_2 + \cdots + T_{k+1} . \tag{6'}$$

Also,

$$(7)\quad U_{10m+1} + U_{6m+3} + U_{3m+1} - U_{10m} - U_{7m+4} = U_{9m+(m+1)} + U_{6m+(3)} + U_{3m+(1)} - U_{9m+(m)} - U_{6m+(m+4)}\,.$$

Since the parenthesized term in each subscript of (7) is less than or equal to $r = 3m > 6$ implies that $m + 4 \leq r$, we can substitute Equations (4'), (5'), (6') in (7) appropriately with k equal to the term in parentheses:

$$\begin{aligned} &U_{10m+1} + U_{6m+3} + U_{3m+1} - U_{10m} - U_{7m+4} \\ &\quad = (3m(m+3) + T_{3m} + T_1 + T_2 + \cdots + T_{m+2}) + (T_4 + 3m) + (2) \\ &\qquad - (3m(m+2) + T_{3m} + T_1 + T_2 + \cdots + T_{m+1}) - (3m + T_{m+5}) \\ &\quad = T_{m+2} + 3m + 12 - T_{m+5}\,. \end{aligned}$$

In view of Equation (2), this establishes the Lemma.

Lemma 4: If $r \geq 6$ and $r = 3m + 1$ for some integer m, then

$$U_{10m+4} + U_{6m+5} + U_{3m+1} = U_{10m+3} + U_{7m+6}\,.$$

Lemma 5: If $r \geq 6$ and $r = 3m + 2$ for some integer m, then

$$U_{10m+8} + U_{6m+7} + U_{3m+4} = U_{10m+7} + U_{7m+9}\,.$$

Proof: Lemmas 4 and 5 are proved in a manner identical with that above, using Equations (2) through (6). Details are omitted.

Theorem 2: Given a sequence U_n satisfying Definition 1 with $r \geq 2$, there exists a positive integer N such that the unique minimal representation of N in U_n is not Minimal.

Proof: For $r \geq 6$, let

$$N = U_{10m+1} + U_{6m+3} + U_{3m+1} \quad \text{if} \quad r = 3m,$$
$$N = U_{10m+4} + U_{6m+5} + U_{3m+1} \quad \text{if} \quad r = 3m + 1,$$

and

$$N = U_{10m+8} + U_{6m+7} + U_{3m+4} \quad \text{if} \quad r = 3m + 2 .$$

The representation given for N is minimal, but in view of Lemmas 3, 4, and 5, is not Minimal. Similarly, let

$$N = 167 = U_{15} + U_{11} + U_8 + U_3 = U_{14} + U_{13} + U_{10} \quad \text{for} \quad r = 3,$$

$$N = 62 = U_{15} + U_{10} + U_5 = U_{14} + U_{13} \quad \text{for} \quad r = 4,$$

and let

$$N = 54 = U_{17} + U_{11} + U_5 = U_{16} + U_{14} \quad \text{for} \quad r = 5.$$

In each of these cases, the first expression for N is minimal but is obviously not Minimal. Thus counterexamples to the minimal-Minimal correlation have been exhibited for all sequences $\{U_n\}$ corresponding to $r > 2$; the proof is complete.

Theorem 3: Given a sequence $\{U_n\}$ satisfying Definition 1 with $r > 2$, there exists an integer N such that the unique maximal representation on N in $\{U_n\}$ is not Maximal.

Proof: For $r \geq 5$, direct substitution using Equations (4) and (5) serves to establish that

$$U_{2r+5} + \sum_{i=0}^{r+2} U_{r+i} = U_{2r+4} + U_{2r+3} + U_{2r+2} + \sum_{i=1}^{r} U_{r+1} .$$

Similarly, we can show that for $r = 4$,

$$U_{13} + U_{10} + U_9 + \sum_{i=3}^{7} U_i = U_{12} + U_{11} + U_{10} + U_8 + U_7 + U_5 + U_4$$

and for $r = 3$,

$$U_{14} + \sum_{i=4}^{11} U_i = U_{13} + U_{12} + U_{11} + U_{10} + U_8 + U_7 + U_6 + U_4 \,.$$

As in the proof of Theorem 2, each of these equations provides two representations for N: the first is maximal by Definition 4, but is of smaller magnitude than the second, and hence not Maximal. This is sufficient to establish the theorem.

Taken together, Theorems 1, 2 and 3 establish that every minimal representation is Minimal and every maximal representation is Maximal in $\{U_n\}$ if and only if $r = 2$, which was the promised result.

Mohanty noted in [2] that $\{U_n\}$ is a special case of the generalized Fibonacci numbers of V. C. Harris and Carolyn C. Styles [3]; specifically,

$$U_n = \sum_{i=0}^{[n/r]} \binom{n - i(r - 1)}{i} \quad ,$$

where $[n/r]$ denotes the greatest integer in n/r. The Tribonacci numbers of Mark Feinberg [4], [5] can be defined as the sums of the rising diagonals of the trinomial triangle generated by $(1 + x + x^2)^n$, and can be generalized in an analogous manner. If the coefficient of x^k in the expansion of $(1+x+x^2)^n$ is denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_3$, then we can define the generalized tribonacci sequence $\{V_{r,n}\}$ by

$$V_{r,n} = \sum_{i=0}^{\infty} \begin{bmatrix} n - i(r - 1) \\ i \end{bmatrix}_3 \quad .$$

As before, we assert without proof that $\{V_n\} = \{V_{r,n}\}$ is complete, even under the restriction that the first $r - 1$ elements of the sequence not to be employed in any integer representations. Further, we extend Definitions 3 and 4 to apply to the new family of sequences, and assert that Theorem 1 can be similarly generalized.

The following theorem is offered without proof.

Theorem 2': If $r \geq 4$, there exists a positive integer N such that the minimal representation of N in $\{V_n\}$ is not Minimal. Specifically,

$$N = V_{4r} + V_{2r+3} + V_{r+1} = V_{4r-1} + V_{3r+2}\,.$$

The left side is in proper form for a minimal but the right side has fewer digits. One can easily find an infinite number of other exceptions for each r. For example, add V_{5r+j} to each side for $j = 1, 2, 3, \cdots$.

One can secure a counterexample for the maximal which is not Maximal by subtracting each of those N's above from

$$\sum_{j=r}^{4r+1} V_j\,.$$

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ERRATA

Please make the following change in the article by London and Finkelstein, "On Fibonacci and Lucas Numbers which are Perfect Powers," Dec. 1969, p. 481:

Equation (14) should read: $Y^2 - 500 = X^3$.

SOME GENERALIZED FIBONACCI IDENTITIES

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1. Let

$$F_{n+1} = F_n + F_{n-1}, \quad F_0 = 0, \; F_1$$

and define

(1.1) $$f_n(x) = \sum_{k=0}^{\infty} F_{n+k}\, x^k/k! \quad (n = 0, 1, 2, \cdots) ,$$

so that

$$f'_n(x) = f_{n+1}(x), \quad f_{n+1}(x) = f_n(x) + f_{n-1}(x) .$$

Note that $f_n(0) = F_n$.

In a recent paper [1] in this Quarterly, Elmore has pointed out that many of the familiar formulas involving the Fibonacci numbers F_n can be extended to the functions $f_n(x)$. For example, the identities

$$F_{m-1}F_{m+1} - F_m^2 = (-1)^m, \quad F_{2m-1} = F_{m-1}^2 + F_m^2$$

become

$$f_{m-1}(x)f_{m+1}(x) - f_m^2(x) = (-1)^m e^x$$

and

$$f_{2m-1}(2x) = f_{m-1}^2(x) + f_m^2(x) ,$$

respectively; the identity

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$$F_{m+n} = F_{m-1}F_n + F_mF_{n-1}$$

becomes

$$f_{m+n}(u + v) = f_{m-1}(u)f_n(v) + f_m(u)f_{n+1}(v) .$$

The formulas

$$f_m(u)f_n(v) = (-1)^r[f_{m+r}(u)f_{n+r}(v) - F_rf_{m+n+r}(u + v)] , \tag{1.2}$$

$$F_mf_m(v - u)e^u = (-1)^r[f_{m+r}(u)f_{n+r}(v) - f_r(u)f_{m+n+r}(v)] , \tag{1.3}$$

seem particularly striking. Elmore remarks that they may be special cases of a more general formula in which no capital F's appear. This is indeed the case, as we shall show below. The formula

$$f_{m+r}(u)f_{n+r}(v) - f_r(x)f_{m+n+r}(y) = (-1)^r e^x f_m(u - x)f_n(v - x) , \tag{1.4}$$

where $x + y = u + v$, reduces to (1.2) when $x = 0$ and reduces to (1.3) when $u = x$, $v = y$.

2. Since it is no more difficult, we consider the following slightly more general situation. Let

$$H_{n+1} = pH_n - qH_{n-1}, \quad H_0 = 0, \quad H_1 = 1 , \tag{2.1}$$

and define

$$h_n(x) = \sum_{k=0}^{\infty} H_{n+k}\, x^k/k! \quad (n = 0,1,2,\cdots) , \tag{2.2}$$

so that

$$h_n'(x) = h_{n+1}(x), \quad h_{n+1}(x) = ph_n(x) - qh_{n-1}(x) .$$

Corresponding to (1.4), we shall show that

$$\text{(2.3)} \quad h_{m+r}(u)h_{n+r}(v) - h_r(x)h_{m+n+r}(y) = q^r e^{px} h_m(u - x)h_n(v - x) \,,$$

provided $x + y = u + v$.

Let α, β denote the roots of $x^2 - px + q = 0$. Then

$$\text{(2.4)} \qquad H_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad ,$$

and (2.2) implies

$$\text{(2.5)} \qquad h_n(x) = \frac{1}{\alpha - \beta}(\alpha^n e^{\alpha x} - \beta^n e^{\beta x}) \;.$$

It follows at once from (2.5) that

$$\text{(2.6)} \qquad \sum_{k=0}^{\infty} h_{n+k}(x)\, y^k/k! = h_n(x + y) \;.$$

Consider

$$\sum_{m,n=0}^{\infty} \{h_{m+r}(u)h_{n+r}(v) = h_r(x)h_{m+n+r}(y)\} \frac{z^m w^n}{m!\,n!}$$

$$\begin{aligned}
&= h_r(u + z)h_r(v + w) - h_r(x)h_r(y + z + w) \\
&= (\alpha - \beta)^{-2}\{(\alpha^r e^{\alpha(u+z)} - \beta^r e^{\beta(u+z)})(\alpha^r e^{\alpha(v+w)} - \beta^r e^{\beta(v+w)} \\
&\qquad\qquad - (\alpha^r e^{\alpha x} - \beta^r e^{\beta x})(\alpha^r e^{\alpha(y+z+w)} - \beta^r e^{\beta(y+z+w)})\} \\
&= (\alpha - \beta)^{-2}\{\alpha^{2r} e^{\alpha(u+v+z+w)} + \beta^{2r} e^{\beta(u+v+z+w)} \\
&\qquad\qquad - q^r(e^{\alpha(u+z)+\beta(v+w)} + e^{\beta(u+z)+\alpha(v+w)})\} \\
&\qquad\qquad - (\alpha - \beta)^{-2}\{\alpha^{2r} e^{\alpha(x+y+z+w)} + \beta^{2r} e^{\beta(x+y+z+w)} \\
&\qquad\qquad - q^r(e^{\alpha x+\beta(y+z+w)} + e^{\beta x+\alpha(y+z+w)})\} \quad .
\end{aligned}$$

If we take $x + y = u + v$, this reduces to

$$(\alpha - \beta)^{-2} q^r \{ e^{\alpha x + \beta(y+z+w)} + e^{\beta x + \alpha(y+z+w)} - e^{\alpha(u+z)+\beta(v+w)} - e^{\beta(u+z)+\alpha(v+w)} \}$$

$$= (\alpha - \beta)^{-2} q^r e^{px} \{ e^{\alpha(-x+y+z+w)} + e^{\beta(-x+y+z+w)} - e^{\alpha(-x+u+z)+\beta(-x+v+w)} - e^{\alpha(-x+v+w)+\beta(-x+u+z)} \}$$

$$= (\alpha - \beta)^{-2} q^r e^{px} (e^{\alpha(-x+v+w)} - e^{\beta(-x+v+w)}) \cdot (e^{\alpha(-x+u+z)} - e^{\beta(-x+u+z)}) .$$

In view of (2.5), we have therefore proved

$$(2.7) \qquad \sum_{m,n=0}^{\infty} \{ h_{m+r}(u) h_{n+r}(v) - h_r(x) h_{m+n+r}(y) \} \frac{z^m w^n}{m!\, n!} = q^r e^{px} h_0(-x + u + z) h_0(-x + v + w) .$$

But by (2.6),

$$h_0(-x + u + z) h_0(-x + v + w) = \sum_{m,n=0}^{\infty} h_m(-x + u) h_n(-x + v) \frac{z^m w^n}{m!\, n!} .$$

Equating coefficients of $z^m w^m$ we immediately get (2.3).

3. Analogous to (2.2), we may define

$$(3.1) \qquad h_n^{\star}(x) = h_n^{\star}(x, \lambda) = \sum_{k=0}^{\infty} H_{n+k} \binom{x}{k} \lambda^k .$$

Then

$$h_n^{\star}(0) = H_n$$

and

$$h^{\star}_{n+1}(x) = h^{\star}_{n}(x) + h^{\star}_{n-1}(x) .$$

Moreover,

$$\underset{x}{\Delta} h^{\star}_{n}(x) = h^{\star}_{n}(x + 1) - h^{\star}_{n}(x) = \sum_{k=1}^{\infty} H_{n+k}\binom{x}{k-1}\lambda^k ,$$

so that

$$\underset{x}{\Delta} h^{\star}_{n}(x) = \lambda h^{\star}_{n+1}(x) .$$

Clearly the series in the right member of (3.1) converges for sufficiently small $|\lambda|$.

It follows at once from (2.4) and (3.1) that

$$h^{\star}_{n}(x) = \frac{1}{\alpha - \beta}[\alpha^n(1 + \lambda\alpha)^x - \beta^n(1 + \lambda\beta)^x]. \tag{3.2}$$

We have also

$$h^{\star}_{n}(x + y) = \sum_{k=0}^{\infty} h^{\star}_{n+k}(x)\binom{y}{k} . \tag{3.3}$$

Now by (3.2),

$$\begin{aligned}
&-qh^{\star}_{m-1}(u)h^{\star}_{n}(v) + h^{\star}_{m}(u)h^{\star}_{n+1}(v) \\
&= (\alpha - \beta)^{-2}\{-q[\alpha^{m-1}(1 + \alpha\lambda)^u - \beta^{m-1}(1 + \beta\lambda)^v][\alpha^n(1 + \alpha\lambda)^u - \beta^n(1 + \beta\lambda)^u] \\
&\qquad + [\alpha^m(1 + \alpha\lambda)^u - \beta^m(1 + \beta\lambda)^v][\alpha^{n+1}(1 + \alpha\lambda)^v - \beta^{n+1}(1 + \beta\lambda)^v]\} \\
&= (\alpha - \beta)^{-2}\{(-q\alpha^{m+n-1} + \alpha^{m+n+1})(1 + \alpha\lambda)^{u+v} \\
&\qquad + (-q\beta^{m+n-1} + \beta^{m+n+1})(1 + \beta\lambda)^{u+v} \\
&\qquad - (q\alpha^{m-1}\beta^n + \alpha^m\beta^{n+1})(1 + \alpha\lambda)^u(1 + \beta\lambda)^v \\
&\qquad - q(\alpha^n\beta^{m-1} + \alpha^{n+1}\beta^m)(1 + \alpha\lambda)^v(1 + \beta\lambda)^u\} .
\end{aligned}$$

Since $\alpha\beta = q$ and

$$\alpha^2 - q = \alpha(\alpha - \beta), \quad \beta^2 - q = -\beta(\alpha - \beta),$$

this reduces to

$$(\alpha - \beta)^{-1} \; \alpha^{m+n}(1 + \alpha\lambda)^{u+v} - \beta^{m+n}(1 + \beta\lambda)^{u+v} .$$

We have therefore,

$$h^{\star}_{m+n}(u + v) = -qh^{*}_{m-1}(u)h^{\star}_{n}(v) + h^{\star}_{m}(u)h^{\star}_{n+1}(v) . \tag{3.4}$$

Similarly, we have

$$h^{\star}_{m-1}(u)h^{\star}_{m+1}(u) - h^{\star 2}_{m}(u) = -q^{m-1}(1 + p\lambda + q\lambda^2)^{u} . \tag{3.5}$$

Finally, corresponding to (2.3), we have

$$\begin{aligned} h^{\star}_{m+r}(u)h^{\star}_{n+r}(v) &- h^{\star}_{r}(x)h^{\star}_{m+n+r}(y) \\ &= q^{r}\, h_{m}(u - x)h_{n}(v - x)(1 + p\lambda + q\lambda^2)^{x} , \end{aligned} \tag{3.6}$$

provided

$$x + y = u + v .$$

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A SIMPLE RECURRENCE RELATION IN FINITE ABELIAN GROUPS

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A finite abelian group G is said to have a simple recurrence relation of length n if there exist distinct nonzero elements $a_1, a_2, \cdots, a_n$ of G such that $a_1 + a_2 = a_3$, $a_2 + a_3 = a_4, \cdots, a_{n-2} + a_{n-1} = a_n$, $a_{n-1} + a_n = a_1$ and $a_n + a_1 = a_2$. It is proved that if $n = 6m$ or $n = 2^{\alpha}3^{\beta}m$ (≥ 3), where $(6, m) = 1$, $\alpha = 0, 2$, or 3 and $\beta = 0, 1$, or 2, then there exists a finite abelian group which has a simple recurrence relation of length n.

Let G be a finite abelian group written additively and $a_1, a_2, \cdots, a_n$ be distinct nonzero elements of G. If

$$a_1 + a_2 = a_3,\ a_2 + a_3 = a_4, \cdots, a_{n-2} + a_{n-1} = a_n,$$
$$a_{n-1} + a_n = a_1 \quad \text{and} \quad a_n + a_1 = a_2,$$

then we say that the ordered set

$$A = \{a_1, a_2, \cdots, a_n\}$$

has a simple recurrence relation (SRR). If G has an ordered subset A such that the cardinal of A is n (≥ 3) and A has a SRR, then we say that G has a SRR of length n. We use the notation $\ell(G) = n$ to mean that G has a SRR of length n.

Suppose

$$A = \{a_1, a_2, \cdots, a_n\}$$

has a SRR; then we have

$$a_3 = a_1 + a_2,\ a_4 = a_1 + 2a_2,\ a_5 = 2a_1 + 3a_2, \cdots .$$

Let

$$U_0 = 0,\ U_1 = 1,\ U_2 = 1,\ U_3 = 2,\ U_4 = 3,\ U_5 = 5, \cdots, U_{i+2} = U_i + U_{i+1}, \cdots,$$

be a Fibonacci sequence ([1], p. 148). Then

(1) $$U_i = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i\right], \quad i \geq 0 .$$

Thus

(2) $$a_{2+i} = U_i a_1 + U_{i+1} a_2 , \quad i \geq 0 .$$

From $a_{n-1} + a_n = a_1$ and $a_n + a_1 = a_2$, we have

(3) $$(U_{n-1} - 1)a_1 + U_n a_2 = 0$$

and

(4) $$(U_{n-2} + 1)a_1 + (u_{n-1} - 1)a_2 = 0 .$$

Let

$$h(n) = (U_{n-2} + 1)U_n - (U_{n-1} - 1)^2, \quad n \geq 2 ,$$
$$d = (U_{n-1} - 1), U_n) ,$$

the g.c.d. of U_{n-1} and U_n, and

$$f(n) = \frac{1}{d} h(n) .$$

Using (1), we can verify

(5) $$U_j U_n - U_{n+1} U_{n-1} = (-1)^{j+1} U_{n-j-1}, \quad j < n .$$

Now

$$\begin{aligned} h(n) &= (U_{n-2} + 1)(U_{n-2} + U_{n-1}) - (U_{n-1} - 1)^2 \\ &= U_{n-2}^2 - U_{n-1}^2 + U_{n-2}U_{n-1} + U_{n-2} + 3U_{n-1} - 1 \\ &= (U_{n-2} + U_{n-1})(U_{n-2} - U_{n-1}) + U_{n-2}U_{n-1} + U_{n-1} + U_{n+1} - 1 \\ &= -U_{n-3}U_n + U_{n-2}U_{n-1} + U_{n-1} + U_{n+1} - 1 \\ &= (-1)^{n-1}U_2 + U_{n-1} + U_{n+1} - 1 \quad \text{(by (5))} \end{aligned}$$

Define

$$\delta_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Then we have

$$h(n) = U_{n-1} + U_{n+1} - 2\delta_n \tag{6}$$

Eliminate a_2 from (3) and (4), and we have $f(n)a_1 = 0$ and thus by permutation, we have

$$f(n)a_i = 0 \quad \text{for every} \quad i = 1, 2, \cdots, n. \tag{7}$$

Before we proceed further, we list some examples below. We use C_m to denote the cyclic group of order m and $C_m \times C_n$ as the cartesian product of C_m and C_n.

(E1) $A = \{(0,1), (1,0), (1,1)\}$ has a SRR in $C_2 \times C_2$;

(E2) $A = \{1, 3, 4, 2\}$ has a SRR in C_5 ;

(E3) $A = \{1, 4, 5, 9, 3\}$ has a SRR in C_{11} ;

(E4) $A = \{(1, 0), (1,1), (0,1), (1,2), (1,3), (0,1)\}$ has a SRR in $C_2 \times C_4$;

(E5) $A = \{1, -5, -4, -9, -13, 7, -6\}$ has a SRR in C_{29} ;

(E6) $A = \{(1,2,1), (1,1,3), (2,0,4), (0,1,2), (2,1,1), (2,2,3), (1,0,4), (0,2,2)\}$ has a SRR in $C_3 \times C_3 \times C_5$;

(E7) $A = \{1, 5, 6, 11, 17, 9, 7, 16, 4\}$ has a SRR in C_{19};

(E8) $A = \{1, 8, 9, 6, 4, 10, 3, 2, 5, 7\}$ has a SRR in C_{11}.

We write $\ell(G) \neq n$ if G does not contain any subset A whose cardinal is n, such that A has a SRR. We note that

(i) because of (7), $\ell(C_4) \neq 3$, $\ell(C_8) \neq 6$;

(ii) since $(7, f(i)) = 1$ for $i = 3, 4, 5, 6$ and $(13, f(i)) = 1$ for $i = 3, 4, \cdots, 12$, therefore both C_7 and C_{13} have no SRR of any length;

(iii) although $f(8) = 15$, $\ell(C_{15}) \neq 8$; if $\{a_1, a_2, \cdots, a_8\}$ has a SRR in C_{15}, then from (4), we have $3a_2 \equiv 9a_1 \pmod{15}$ and thus $a_2 \equiv -2a_1, 3a_1$, or $8a_1 \pmod{15}$.

Case 1: If $a_2 = -2a_1$, then $a_3 = -a_1, \cdots, a_8 = -3a_1 = a_4$, which is impossible.

Case 2: If $a_2 = 3a_1$, then $a_3 = 4a_1, \cdots, a_6 = 3a_1 = a_2$, which is impossible.

Case 3: If $a_2 = 8a_1$, then $a_3 = 9a_1, \cdots, a_7 = 9a_1 = a_3$, which is impossible.

Now we prove

Lemma 1: If

$$(U_n, f(n)) = 1, \quad n \neq 2(2m + 1),$$

then $\ell(C_{f(n)}) = n$.

Proof: Since $(U_n, f(n)) = 1$, therefore

$$d = (U_{n-1} - 1, U_n) = 1,$$

and thus

$$f(n) = h(n) = U_{n-1} + U_{n+1} - 2\delta_n. \tag{8}$$

Also, there exist r and t such that

$$rU_n + tf(n) = 1 \quad . \tag{9}$$

From (3), we have

$$r(U_{n-1} - 1)a_1 + rU_n a_2 = 0 .$$

Substitute $rU_n = 1 - tf(n)$ into the above equation and make use of the result of (7); we have

$$a_2 = r(1 - U_{n-1})a_1 \quad . \tag{10}$$

Thus

$$a_3 = a_1 + a_2 = [r(1 - U_{n-1}) + 1]a_1 ,$$

and in general,

$$a_{2+i} = [rU_{i+1}(1 - U_{n-1}) + U_i]a_1, \quad 0 \le i \le n - 2 \quad . \tag{11}$$

Now we prove that $A = \{a_1, a_2, \cdots, a_n\}$, where a_1 is chosen such that $(a_1, f(n)) = 1$ and a_{2+i}, $0 \le i \le n - 2$ is given by (11), has a SRR.

We need to verify

(I) $a_1, a_2, \cdots, a_n$ considered as elements in $C_{f(n)}$, are distinct and nonzero;

(II) $a_1 + a_2 = a_3$, $a_2 + a_3 = a_4$, $\cdots$, $a_{n-2} + a_{n-1} = a_n$, $a_{n-1} + a_n = a_1$, and $a_n + a_1 = a_2$.

First we prove (II):

For this part, we need only to verify that $a_{n-1} + a_n = a_1$ and $a_n + a_1 = a_2$. In fact,

$$\begin{aligned} a_{n-1} + a_n = [rU_n(1 - U_{n-1}) + U_{n-1}]a_1 &= [(1 - U_{n-1}) + U_{n-1}]a_1 \quad \text{(by (9))} \\ &= a_1 ; \end{aligned}$$

$$a_n + a_1 = [rU_{n-1}(1 - U_{n-1}) + U_{n-2} + 1]a_1$$
$$= U_{n-1}a_2 + (U_{n-2} + 1)a_1 \ .$$

Since

$$f(n) = (U_{n-2} + 1)U_n - (U_{n-1} - 1)^2 ,$$

therefore,

$$rU_n(U_{n-2} + 1)a_1 + r(1 - U_{n-1})(U_{n-1} - 1)a_1 = 0 \ ,$$

from which it follows that

$$(U_{n-2} + 1)a_1 + (U_{n-1} - 1)a_2 = 0 \ .$$

Hence $a_n + a_1 = a_2$.

To prove (I), we shall show that

$$rU_{i+1}(1 - U_{n-1}) + U_i \neq rU_{j+1}(1 - U_{n-1}) + U_j, \quad 0 \le i < j \le n - 2 , \tag{12}$$

$$rU_{i+1}(1 - U_{n-1}) + U_i \neq 1, \quad 0 \le i \le n - 2 ; \tag{13}$$

and

$$rU_{i+1}(1 - U_{n-1}) + U_i \neq 0, \quad 0 \le i \le n - 2 . \tag{14}$$

Suppose for some i,j such that $0 \le i < j \le n - 2$,

$$rU_{i+1}(1 - U_{n-1}) + U_i = rU_{j+1}(1 - U_{n-1}) + U_j \ .$$

then

$$r(U_{j+1} - U_{i+1})(1 - U_{n-1}) + (U_j - U_i) = 0$$

and thus

$$rU_n(U_{j+1} - U_{i+1})(1 - U_{n-1}) + U_n(U_j - U_i) = 0 \ ,$$

from which it follows that

$$(U_{j+1} - U_{i+1})(1 - U_{n-1}) + U_n(U_j - U_i) = 0 ,$$

i.e.,

$$(U_jU_n - U_{j+1}U_{n-1}) - (U_iU_n - U_{i+1}U_{n-1}) + U_{j+1} - U_{i+1} = 0 .$$

Applying (5), we have

$$g(i,j) \equiv (-1)^{j+1}U_{n-j-1} + (-1)^iU_{n-i-1} + U_{j+1} - U_{i+1} = 0, \quad 0 \le i < j \le n - 2 . \tag{15}$$

We can verify that

$$-f(n) \le g(i,j)(\ne 0) \le f(n) .$$

Hence (15) cannot be true.

Similarly, if

$$rU_{i+1}(1 - U_{n-1}) + U_i = 1, \quad 0 \le i \le n - 2 ,$$

then

$$rU_nU_{i+1}(1 - U_{n-1}) + U_n(U_i - 1) = 0 ,$$

which implies that

$$U_{i+1}(1 - U_{n-1}) + U_n(U_i - 1) = 0 ,$$

i.e.,

$$(U_iU_n - U_{i+1}U_{n-1}) + U_{i+1} - U_n = 0$$

or

$$k(i) = (-1)^{i+1}U_{n-i-1} + U_{i+1} - U_n = 0, \quad 0 \le i \le n - 2. \tag{16}$$

We can also verify that

$$-f(n) < k(i)(\neq 0) < f(n).$$

Hence (16) cannot be true.

Finally, if

$$rU_{i+1}(1 - U_{n-1}) + U_i = 0, \quad 0 \le i \le n - 2,$$

then

$$w(i) \equiv (-1)^{i+1}U_{n-i-1} + U_{i+1} = 0, \quad 0 \le i \le n - 2. \tag{17}$$

But for $n \neq 2(2m + 1)$, $w(i) \neq 0$, and $-f(n) < w(i) < f(n)$. Hence (17) cannot be true.

The proof of Lemma 1 is complete.

Lemma 2. Let G_1, G_2 be two finite abelian groups. If

$$\ell(G_1) = m, \quad \ell(G_2) = n, \quad m < n, \quad (m,n) = d,$$

then

$$\ell(G_1 \times G_2) = \frac{1}{d}mn.$$

Proof: Let

$$A = \{a_1, a_2, \cdots, a_m\}$$

be a subset of G_1 such that A has a SRR and

$$B = \{b_1, b_2, \cdots, b_n\}$$

be a subset of G_2 such that B has a SRR. Then we can prove that

$$A \otimes B = \{c_1, c_2, \cdots, c_s\}$$

where

$$s = \frac{1}{d} mn$$

and

$$c_1 = (a_1, b_1), \ c_2 = (a_2, b_2), \ c_3 = c_1 + c_2 = (a_3, b_3), \cdots, c_m = (a_m, b_m),$$
$$c_{m+1} = (a_1, b_{m+1}), \cdots, c_s = (a_m, b_n),$$

has a SRR in $G_1 \times G_2$.

Lemma 3: If $(n, 6) = 1$, then $(U_n, f(n)) = 1$.

Proof: We observe that U_n is even if and only if $n = 3m$. Hence if $(n, 3) = 1$, then U_n is odd.

Now, $(n, 2) = 1$ implies that

$$f(n) = \frac{1}{d} (U_{n-1} + U_{n+1}).$$

It can be proved that if U_n is odd, then $(U_n, U_{n=1} + U_{n+1}) = 1$ ([1], p. 148).

It is clear that $(U_n, h(n)) = 1$ implies that $d = 1$. Hence $f(n) = U_{n-1} + U_{n+1}$, and thus $(U_n, f(n)) = 1$.

From Lemmas 1 and 3, we have

Lemma 4: If $(n, 6) = 1$, then $\ell(C_{f(n)}) = n$.

From (E1), (E2), (E6), (E7), Lemmas 2 and 4, we have

Theorem 1: If

$$n = 6m \quad \text{or} \quad n = 2^{\alpha}3^{\beta}m \ (\geq 3),$$

where $(6, m) = 1$, $\alpha = 0, 2,$ or 3 and $\beta = 0, 1,$ or 3, then there exists a finite abelian group G such that $\ell(G) = n$.

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SOME COUNTEREXAMPLES AND PROBLEMS ON LINEAR RECURRENCE RELATIONS

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In [1, pp. 48-50], several false assertions are made concerning linear recurrence relations (mod m). I will give counterexamples to these and will establish one result on a stronger hypothesis. Theorems 3.6 and 3.7 of [1] are false as stated, and it is an open question what additional hypotheses are required for their validity.

Let

$$u_{n+1} = \sum_{i=0}^{j} a_i u_{n-i} + b \quad . \tag{1}$$

For a given modulus m, let x_n be the least non-negative residue of u_n (mod m). In [1], it is assumed that $a_i \geq 0$, $b \geq 0$, and

$$(a_0, a_1, \cdots, a_j, m) = (x_0, x_1, \cdots, x_j, b, m) = 1 \quad ,$$

although these hypotheses do not appear to be essential. Of course, all quantities are integers. Let $H(m)$ be the period of x_n (mod m). The following <u>false</u> assertions are made in [1; (3.12), 3.6, 3.7 are his numbers]:

x_n is a purely periodic sequence, i. e.,

$$\exists H: \forall n,k \geq 0 \qquad x_{n+kH} \equiv x_n \pmod{m} \quad . \tag{3.12}$$

<u>Theorem 3.6</u> $H(p^{e+1}) = H(p^e)$ or $p \cdot H(p^e)$.

In the supposed proof, c_{ik} is defined by

$$u_{i+kH} = x_i + c_{ik}p^e$$

for $m = p^e$, $H = H(p^e)$. Then $c_{ik} \geq 0$. It is asserted that

(2) $$p \nmid c_{i1} \Rightarrow c_{ik} \equiv k\, c_{i1} \pmod{p},$$

and the proof is completely dependent on this:

Theorem 3.7. If

$$H(p) = H(p^2) = \cdots = H(p^e) \neq H(p^{e+1}),$$

then $H(p^{e+f}) = p^f H(p^e)$.

Example 1. $u_{n+1} = u_n + 2u_{n-1}$, $u_0 = u_1 = 1$. All hypotheses are satisfied for $m = 2^e$. The sequence u_n is given below, together with the x_n sequences (mod 2, 4, 8, and 16).

n	0	1	2	3	4	5	6	7	8	9	10
u_n	1	1	3	5	11	21	43	85	171	341	683
x_n (mod 2)	1	1	1	1	1	1	1	1	1	1	1
x_n (mod 4)	1	1	3	1	3	1	3	1	3	1	3
x_n (mod 8)	1	1	3	5	3	5	3	5	3	5	3
x_n (mod 16)	1	1	3	5	11	5	11	5	11	5	11

We have

$$u_{n+1} = (2^{n+1} + (-1)^n)/3$$

For $e = 1$, x_n is purely periodic with period $H(2) = 1$. For $e > 1$, we have

$$u_0 = u_1 < u_2 < \cdots < u_e < 2^e$$

and

$$u_{e-1} \equiv u_{e-1+2k} \pmod{2^e},$$

and

$$u_e \equiv u_{e+2k} \pmod{2^e}.$$

Clearly $H(p^e) = 2$ for $e > 1$, but x_n is not purely periodic. Further, for (mod 4), we have $c_{12} = 5$, $c_{11} = 1$, $2 \nmid c_{11}$ but $c_{12} \not\equiv 2 \cdot c_{11} \pmod{2}$.

(Of course, $x_n \pmod 4$ is not purely periodic as assumed in the proof of Theorem 3.6, but we can drop the first term by shifting indices.) Equation (2) does not even hold for $p|c_{i1}$ since for $x_n \pmod 2$, we have $c_{02} = 1$, $c_{01} = 0$ but $c_{02} \not\equiv 2 \cdot c_{01} \pmod 2$. Finally, we have $H(2) \neq H(4)$, but $H(8) \neq 4 \cdot H(2)$. So we have shown that equations (3.12) and (2) and Theorem 3.7 are false as stated.

The proper assertion for (3.12) is that x_n is (eventually) periodic, i.e.,

$$\exists n_0,\ \exists H : \forall n \geq n_0,\ \forall k \geq 0 \qquad x_{n+kH} \equiv x_n \pmod m. \tag{3}$$

However, we can obtain pure periodicity under a different assumption.

<u>Theorem</u>. x_n is purely periodic (mod m) if $(a_j, m) = 1$.

<u>Proof</u>. Let n_0 be the least integer ≥ 0 such that (3) holds. From (1) we have

$$a_j x_{n-j} \equiv x_{n+1} - \sum_{i=0}^{j-1} a_i x_{n-i}\ b \pmod m .$$

Since $(a_j, m) = 1$, there is an a_j^{-1} such that $a_j a_j^{-1} \equiv 1 \pmod m$, so we have

$$x_{n-j} \equiv a_j^{-1}\left[x_{n+1} - \sum_{i=0}^{j-1} a_i x_{n-i} - b\right] \pmod m , \tag{4}$$

That is, we can reverse the recurrence relation to get terms of smaller index from terms of larger index. If $n_0 > 0$, set $n = n_0 + j - 1$ and $n = n_0 + kH + j - 1$ in (4) to get

$$x_{n_0-1} \equiv a_j^{-1}\left[x_{n_0+j} - \left(\sum_{i=0}^{j-1} a_i x_{n_0+j-1-i}\right) - b\right] \pmod m . \tag{5}$$

$$(6) \qquad x_{n_0-1+kH} \equiv a_j^{-1}\left[x_{n_0+j+kH} - \left(\sum_{i=0}^{j-1} a_i x_{n_0+j-1-i+kH}\right) - b\right] \pmod{m} .$$

Now (3) shows that the right-hand sides of (5) and (6) are congruent (mod m), so $x_{n_0-1} \equiv x_{n_0-1+kH}$ (mod m). Hence n_0 is not the least integer such that (3) holds, hence $n_0 = 0$, that is x_n is purely periodic (mod m).

In view of this result, one might ask if Theorems 3.6 and 3.7 and Eq. (2) might be valid if $(a_j, m) = 1$.

Example 2.

$$u_{n+1} = u_{n-2} \cdot u_0 = u_1 = 1, \qquad u_2 = 3 .$$

Again, all hypotheses are satisfied for $m = 2^e$ and $a_j = 1$, so $(a_j, m) = 1$. The resulting sequence is $x_n \equiv 1 \pmod 2$ and $x_n = u_n \pmod{2^e}$ $e > 1$. u_n is given by:

n	0	1	2	3	4	5	6	7	8
u_n	1	1	3	1	1	3	1	1	3

Clearly $H(2) = 1$, $H(2^e) = 3$ for $e > 1$, but $H(2^2) \neq 2 \cdot H(2)$ so that Theorems 3.6 and 3.7 both fail. For $p^e = 2$, $c_{02} = 1 \not\equiv 2 \cdot c_{01} = 0 \pmod 2$ and $c_{13} = 0 \not\equiv 3 \cdot c_{11} = 3 \pmod 2$, so (3.12) fails here also.

Further, it is clear that this example can be modified to work for any modulus p^e.

Finally, we remark that we can construct a less artificial example with similar properties from

$$u_{n+1} = u_n + u_{n-1} + 1, \quad u_0 = u_1 = 1 .$$

n	0	1	2	3	4	5	6	7	8	9	10
u_n	1	1	3	5	9	15	25	41	67	109	117
x_n (mod 2)	1	1	1	1	1	1	1	1	1	1	1
x_n (mod 4)	1	1	3	1	1	3	1	1	3	1	1
x_n (mod 8)	1	1	3	5	1	7	1	1	3	5	1

[Continued on page 279.]

ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-169 Proposed by Francis DeKoven, Highland Park, Illinois.

Show $n^2 + 1$ is a prime if and only if $n \neq ab + cd$ with $ad - bc = \pm 1$ for integers a, b, c, d.

H-170 Proposed by H. W. Gould, West Virginia University, Morgantown, West Virginia.

Define the power sequence P to be the sequence of natural numbers which are perfect powers m^r, $r > 1$, arranged in increasing order of magnitude. Define the first term in the sequence as $P_1 = 1$. Then $P = 1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, \cdots$. Find a formula for the n^{th} term, P_n, of the power sequence. Determine the asymptotic behavior of P_n. Define $\psi(n)$ to be the number of terms in the power sequence $\leq p_n$ and relatively prime to p_n. Then the consecutive values of $\psi(n)$ are $1, 1, 3, 2, 5, 5, 4, 2, 9, 5, 8, \cdots$. Find a formula for $\psi(n)$ and determine the behavior of this function ψ. Find suitable generating series for p_n and $\psi(n)$. Finally, find a formula for the n^{th} non-power; i.e., for the n^{th} term in the sequence complementary to P. Note: It may, or may not, be a good idea to include $P_1 = 1$ in the sequence defined above.

H-171 Proposed by Douglas Lind, Stanford University, Stanford, California.

Does there exist a continuous real-valued function f defined on a compact interval I of the real line such that

$$\int_I f(x)^n dx = F_n .$$

What if we require f only be measurable?

SOLUTIONS

SUB MATRICES

H-139 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$A_n = \begin{bmatrix} F_n & F_{n+1} & \cdots & F_{n+k-1} \\ F_{n+k-1} & F_n & \cdots & F_{n+k-1} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n+1} & F_{n+2} & \cdots & F_n \end{bmatrix}$$

$$M = \begin{bmatrix} A_n & A_{n+k} & \cdots & A_{n+(m-1)k} \\ A_{n+(m-1)k} & A_n & \cdots & A_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n+k} & A_{n+2k} & \cdots & A_n \end{bmatrix}$$

Evaluate det M.

For $m = k = 2$, the problem reduces to H-117 (Fibonacci Quarterly, Vol. 5, No. 2 (1967), p. 162).

Solution by the Proposer.

Put $\epsilon = e^{2\pi i/k}$, $\omega = e^{2\pi i/m}$ and define

$$P = (\epsilon^{ij}) \qquad (i,j = 0, 1, \cdots, k-1),$$

$$U = (\omega^{rs}P) \qquad (r,s = 0, 1, \cdots, m-1) .$$

Also put

$$M = (B_{r-s}) \qquad (r,s = 0,1, \cdots, m-1),$$

where $B_0, B_1, \cdots, B_{m-1}$ are arbitrary square matrices of order k and $B_{r+m} = B_r$. Then

$$MU = \left(\sum_t B_{r-t}\omega^{ts}P\right) = \left(\sum_t B_t\omega^{-ts}\omega^{rs}P\right),$$

$$UMU = \left(P\sum_{u,t}\omega^{ru}\omega^{us}B_t\omega^{-ts}\,P\right) \quad ;$$

Since

$$\sum_{u=0}^{m-1} \omega^{u(r+s)} = \begin{cases} m & (m \mid r+s) \\ 0 & (m \nmid r+s) \end{cases} ,$$

it follows that

$$|UMU| = \prod_{s=0}^{m-1} \left| P\left(\sum_t B_t\omega^{-ts}\right)P \right|$$

$$= |P|^{2m} \prod_{s=0}^{m-1} \left| \sum_{t=0}^{m-1} B_t\omega^{-ts} \right| .$$

On the other hand,

$$U^2 = \left(\sum_t \omega^{(r+s)t}P^2\right) ,$$

so that

$$|U^2| = m^m|P|^{2m} .$$

Therefore, since $|P| \neq 0$,

(1) $$|M| = \prod_{s=0}^{m-1} \left| \sum_{t=0}^{m-1} B_t \omega^{ts} \right| .$$

Now take

$$B_t = A_{n+tk} \qquad (t = 0, 1, \cdots, m - 1) .$$

Then

(2) $$\sum_{t=0}^{m-1} B_t \omega^{ts} = \sum_{t=0}^{m-1} A_{n+tk} \omega^{ts} .$$

We shall limit ourselves to the case $k = 2$, so that

$$A_n = \begin{bmatrix} F_n & F_{n+1} \\ F_{n+1} & F_n \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} .$$

Then

$$A_n P = \begin{bmatrix} F_{n+2} & -F_{n-1} \\ F_{n+2} & F_{n-1} \end{bmatrix} ,$$

$$\left(\sum_{t=0}^{m-1} A_{n+tk} \omega^{-ts} \right) P = \begin{bmatrix} \Sigma F_{n+2t+2} \omega^{ts} & - \Sigma F_{n+2t-1} \omega^{ts} \\ \Sigma F_{n+2t+2} \omega^{ts} & \Sigma F_{n+2t-1} \omega^{ts} \end{bmatrix}$$

so that

(3) $$\left| \sum_{t=0}^{m-1} A_{n+2t} \omega^{-ts} \right| = - \left(\sum_{t=0}^{m-1} F_{n+2t+2} \omega^{ts} \right) \left(\sum_{t=0}^{m-1} F_{n+2t-1} \omega^{ts} \right) .$$

Now

$$\sum_{t=0}^{m-1} F_{n+2t}\omega^{ts} = \frac{1}{\alpha - \beta}\left\{\alpha^n \frac{1 - \alpha^{2m}}{1 - \alpha^2\omega^s} - \beta^n \frac{1 - \beta^{2m}}{1 - \beta^2\ {}^s}\right\}$$

$$= \frac{F_n - F_{n+2m} - (F_{n-2} - F_{n+2m-2})\omega^s}{(1 - \alpha^2\omega^s)(1 - \beta^2\omega^s)}$$

and

$$\prod_{s=0}^{m-1} \sum_{t=0}^{m-1} F_{n+2t}\omega^{ts} = \frac{(F_n - F_{n+2m})^m - (F_{n-2} - F_{n+2m-2})^m}{2 - L_{2m}} .$$

It therefore follows from (1), (2), and (3), that

$$|M| = \frac{(-1)^m}{(L_{2m} - 2)^2}\left\{(F_{n+2} - F_{n+2m+2})^m - (F_n - F_{n+2m})^m\right\}$$

$$\cdot \left\{(F_{n-1} - F_{n+2m-2})^m - (F_{n-3} - F_{n+2m-3})^m\right\}.$$

It can be verified that when $m = 2$, the right member reduces to $F_{2n+6}F_{2n}$ in agreement with H-117.

The result for arbitrary k is presumably very complicated.

SUM DIFFERENCE

H-141 Proposed by H. T. Leonard, Jr., and V. E. Hoggatt, Jr., San Jose State College, San Jose, California. (Corrected Version)

Show that

(a) $$\frac{F_{2n} + 2^nF_n}{2} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} L_{2(n-(2k+1))}F_{2k+1}$$

(b) $$\frac{L_{2n} - L_n}{2} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} L_{2k+1}$$

(c) $$\frac{L_{2n} + L_n}{2} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} L_{2k}$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Let $\alpha \neq \beta$ be the roots of $z^2 - z - 1 = 0$ $(\alpha > \beta)$.

(b) and (c). We have

(1) $$(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i = \sum_{k=0}^{[n/2]} \binom{n}{2k} x^{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} x^{2k+1} .$$

Since $L_n = \alpha^n + \beta^n$, $1 + \alpha = \alpha^2$ (also for β), we add (1), for $x = \alpha$, to (1) for $x = \beta$ to obtain

(2) $$L_{2n} = \sum_{k=0}^{[n/2]} \binom{n}{2k} L_{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} L_{2k+1} .$$

Since $\alpha + \beta = 1$, we obtain, by addition of (1), for $x = -\alpha$, to (1), for $x = -\beta$,

(3) $$L_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} L_{2k} - \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} L_{2k+1} .$$

Addition of (2) and (4) gives (c); subtraction of (3) from (2) gives (b).

(a) We have

$$(4) \qquad (y^2 + x)^n = \sum_{i=0}^{n} \binom{n}{i} y^{2(n-i)} x^i$$

$$= \sum_{k=0}^{[n/2]} \binom{n}{2k} y^{2(n-2k)} x^{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} y^{2(n-2k-1)} x^{2k+1} .$$

Since $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, we subtract (4), for $x = \beta$, from (4), for $x = \alpha$, and divide the result by $(\alpha - \beta)$ to obtain

$$(5) \qquad \frac{(y^2 - \alpha)^n - (y^2 + \beta)^n}{\alpha - \beta} = \sum_{k=0}^{[n/2]} \binom{n}{2k} y^{2(n-2k)} F_{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} y^{2(n-2k-1)} F_{2k+1} .$$

Addition of (5), for $y = \alpha$, to (5), for $y = \beta$, simplifies to

$$(6) \qquad F_{3n} = \sum_{k=0}^{[n/2]} \binom{n}{2k} L_{2(n-2k)} F_{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} L_{2(n-2k-1)} F_{2k+1} .$$

Subtraction of (4), for $x = -\beta$, from (4), for $x = -\alpha$, gives

$$(7) \qquad \frac{(y^2 - \alpha)^n - (y^2 - \beta)^n}{\alpha - \beta} = \sum_{k=0}^{[n/2]} \binom{n}{2k} y^{2(n-2k)} F_{2k} - \sum_{k=0}^{[(n-1)/2]}$$

$$- \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} y^{2(n-2k-1)} F_{2k+1} .$$

Addition of (7), for $y = \alpha$, to (7), for $y = \beta$, gives

$$(8) \qquad -2^n F_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} L_{2(n-2k)} F_{2k} - \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} L_{2(n-2k-1)} F_{2k+1} .$$

Subtraction of (8) from (6) gives the desired result.

Also solved by D. Jaiswal (India) and A. C. Shannon (Australia).

ANOTHER SERIES

H-142 Proposed by H. W. Gould, West Virginia University, Morgantown, West Virginia.

With the usual notation for Fibonacci numbers, $F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}$, show that

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \sum_{k=0}^{n} \binom{\frac{1+\sqrt{5}}{1-\sqrt{5}}k}{k}\binom{n - \frac{1+\sqrt{5}}{1-\sqrt{5}}k}{n-k} = F_{n+1} ,$$

where

$$\binom{x}{j} = x(x-1)(x-2)\cdots(x-j+1)/j!$$

is the usual binomial coefficient symbol.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$\beta = \frac{1+\sqrt{5}}{1-\sqrt{5}} = -\frac{3+\sqrt{5}}{2} = -\left(\frac{1+\sqrt{5}}{2}\right)^2$$

$$u_n = \sum_{k=0}^{n} \binom{\beta k}{k}\binom{n-\beta k}{n-k} .$$

Then

$$\sum_{n=0}^{\infty} u_n t^n = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{\beta k}{k}\binom{n-\beta k}{n-k}$$

$$= \sum_{k=0}^{\infty} \binom{\beta k}{k} t^k \sum_{n=0}^{\infty} \binom{n+(1-\beta)k}{n} t^n = \sum_{k=0}^{\infty} \binom{\beta k}{k} t^k (1-t)^{-(1-\beta)k-1} .$$

Now in the formula (see Pólya-Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. 1, p. 126, No. 216)

$$\sum_{n=0}^{\infty} \binom{\beta n}{n} w^n = \frac{x}{(1 - \beta)x + \beta} \quad ,$$

where $1 - x + wx^{\beta} = 0$, take $x = (1 - t)^{-1}$. Then

$$\frac{x - 1}{x^{\beta}} = t(1 - t)^{\beta - 1} = w.$$

It follows that

$$\sum_{n=0}^{\infty} u_n t^n = (1 - t)^{-1} \sum_{k=0}^{\infty} \binom{\beta k}{k} w^k$$

$$= \frac{1}{1 - t} \frac{x}{(1 - \beta)x + \beta} = \frac{1}{(1 - t)(1 - \beta t)} = \frac{1}{1 - (\beta + 1)t + \beta t^2} \quad ,$$

so that

$$\sum_{n=0}^{\infty} \left(\frac{1 - \sqrt{5}}{2}\right)^n u_n t^n = \frac{1}{1 - t - t^2}.$$

Therefore

$$\left(\frac{1 - \quad 5}{2}\right)^n u_n = F_{n+1} \ .$$

Also solved by D. Jaiswal (India).

NEGATIVE ATTITUDE

H-143 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tennessee. (Corrected version)

Let $\{H_n\}$ be a generalized Fibonacci sequence and, by the recurrence relation, extend the definition to include negative subscripts. Show that

(i) $$L_{2j+1} \sum_{k=0}^{n} H^2_{(2j+1)k} = H_{(2j+1)(n+1)} H_{(2j+1)n} - H_0 H_{-(2j+1)} ,$$

(ii) $$L_{2j+1} \sum_{k=0}^{n} H_{(2j+1)k} = H_{(2j+1)(n+1)} - H_{-(2j+1)} - H_0 + H_{(2j+1)n} ,$$

(iii) $$L_{2j} \sum_{k=0}^{n} (-1)^k H^2_{2jk} = (-1)^n H_{2j(n+1)} H_{2jn} + H_0 H_{-2j} ,$$

and derive an expression for

(iv) $$\sum_{k=0}^{n} (-1)^k H_{2jk} .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Our proof uses the fact that if $P(0) = R(0)$ and $\Delta P(n) = \Delta R(n)$, then $P(n) \equiv R(n)$ (where $\Delta P(n) = P(n+1) - P(n)$). We note that $H_{-n} = (-1)^n (H_0 L_n - H_n)$, so that

(A) $$H_0 L_{2j+1} = H_{2j+1} - H_{-(2j+1)}$$

and

(B) $$H_0 L_{2j} = H_{2j} + H_{-2j} .$$

Proof of (i). For $n = 0$, both sides of (i) are equal by (A). Using the Δ operator, it remains to show that

(1) $$L_a H_{an+a} = H_{an+2a} - H_{an} \qquad (a = 2j + 1) .$$

We recall now that

(2) $$H_{m+p} = F_{p-1}H_m + F_p H_{m+1} ,$$

(3) $$F_{m+1}F_{m-1} - F_m^2 = (-1)^m .$$

Thus,

$$H_{an+a} = F_{a-1}H_{an} + F_a H_{an+1} ,$$

and

(4) $$L_a H_{an+a} = L_a F_{a-1} H_{an} + F_{2a} H_{an+1} ,$$

(5) $$H_{an+2a} - H_{an} = (-1 + F_{2a-1})H_{an} + F_{2a}H_{an+1} .$$

By (3),

$$F_{a+1}F_{a-1} - F_a^2 = -1 ,$$

and so

$$L_a F_{a-1} = F_{a+1}F_{a-1} + F_{a-1}^2 = -1 + F_a^2 + F_{a-1}^2 = -1 + F_{2a-1} .$$

Thus, (4) and (5) gives (1) and (i).

Proof of (iii). Both sides of (ii) are equal for $n = 0$ by (B). Using the Δ operator, it remains to show that

(6) $$L_c H_{cn+c} = H_{cn+2c} + H_{cn} \qquad (c = 2j) .$$

Proceeding as in the proof of (i), we obtain (6) by noting that $L_c F_{c-1} = 1 + F_{2c-1}$.

Proof of (ii). Identical to the proof of (i).

Derivation of (iv). Using (5) in my paper, "On Summation Formulas for Fibonacci and Lucas Numbers," this Quarterly, Vol. 2, No. 2, 1964, pp. 105-107, we obtain (for $x = p = -1$, $u_n = H_n$, $a = 2j$, and $d = 0$)

$$\text{(iv)}\quad (2 + L_{2j}) \sum_{k=0}^{n} (-1)^k H_{2jk} = (-1)^n (H_{(2j)(n+1)} + H_{2jn}) + H_0 + H_{-2j} .$$

Also solved by A. Shannon (Australia), C. Wall, and M. Yoder.

[Continued from page 267.]

Here $H(4) = 3H(2)$. But $H(2^{e+2}) = 2^e H(4)$.

This leaves us with the following problems: When do Theorems 3.6 and 3.7 hold? When does (2) hold? For the special case $u_{n+1} = u_n + u_{n-1}$, the theorems hold. A rather incomplete proof is given in [2, Theorem 5]. A complete proof is contained in [3] and will be published soon. It would be nice if these results could be established by the simple approach of [1]. Until then, one must be cautious of any results in [1].

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2. D. D. Wall, "Fibonacci Series Modulo m," American Math. Monthly, 67 (1960), pp. 525-532.
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BOOK REVIEW

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

LEONARD OF PISA

by

Joseph and Frances Gies

Thomas Y. Crowell Company has just published a book entitled Leonard of Pisa with the sub-title, and the New Mathematics of the Middle Ages.

The authors, faced with the well-known paucity of biographical material on their subject, have sought to reconstruct the picture of social life and mathematical practice current in the circumstances in which Leonard of Pisa lived. Their efforts in this direction are quite successful with the result that they have produced a very readable and interesting book.

Since they were dealing with the life of a mathematician, it was necessary to give some idea of his contribution to the progress of this field. This they have done in a manner that is devoid of forbidding technicalities and suitable for the general reading public. In particular, Chapter VI presents a clear summary of what is to be found in Liber Abaci. Chapter VII deals with the Fibonacci sequence which began in an incidental way in Leonard of Pisa's work, but which has achieved considerable development in modern times. It may be noted that the Fibonacci Association is mentioned as part of the continuing history of Leonard of Pisa and in a final note the efforts of Dr. Grimm and Mrs. Marguerite Dunton in producing a reliable English version of Liber Abaci are brought out.

Being a popular work, written by non-mathematicians, certain limitations could be expected. The impression is left, for example, that Leonard of Pisa was almost solely responsible for introducing the Hindu-Arabic system, whereas there were others involved as well in this process (see Boyer, History of Mathematics, John Wiley, 1968, pp. 279ff). Likewise, one could read into the text that there were no notable mathematicians from 1200 to 1500 (pp. 98-99).

A couple of errors might be noted. A. H. Church (p. 82) was not the discoverer of phyllotaxis; he has numerous references to earlier pioneers in
[Continued on page 323.]

MOSAIC UNITS: PATTERNS IN ANCIENT MOSIACS

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Inspecting ancient floor mosaics, I noticed [1] that their geometric patterns tend to fall into the same few size groups, despite the mosaics being in widely separated parts of the classical world.

On measuring all alternative dimensions that it seemed reasonable to measure on each pattern, and doing this for many patterns of the same size group, I obtained a histogram as in Fig. 1.

In every size group I obtained the same basic pattern of histogram; some little peaks followed by a very tall peak, followed by a succession of diminishing waves of small peaks. Examination of these histograms revealed that nearly every pattern has one dimension contributing to the very tall peak. This dimension can be said to be common to every pattern in the size group concerned.

That virtually every pattern of a size group has one dimension of virtually (i.e., within the spread of the very tall peak) the same length, suggests that this dimension was fixed by the mosaicists. Lack of many alternative dimensions would explain why patterns fall into size groups.

Examining equal pattern dimensions on different mosaics, I found that they are not composed of equal numbers of stones. Even on the same mosaic, constant dimensions are often composed of varied numbers of stones. Mosaicists fixing dimensions by measurement, rather than by counting out stones, would explain this.

Measuring (with a class interval of 1 millimeter) 121265 dimensions of patterns that had apparently been originally fixed by mosaicists' measurements, I obtained a frequency distribution as in Figure 2. Measuring more patterns, to a total of more than 310000 pattern dimensions, I found essentially the same distribution; the very tall peaks grew much taller, whilst some extra, but minute, peaks appeared. From the distribution (Fig. 2), it is clear that ancient floor mosaic geometric patterns are remarkably few different absolute sizes. Can we account for it?

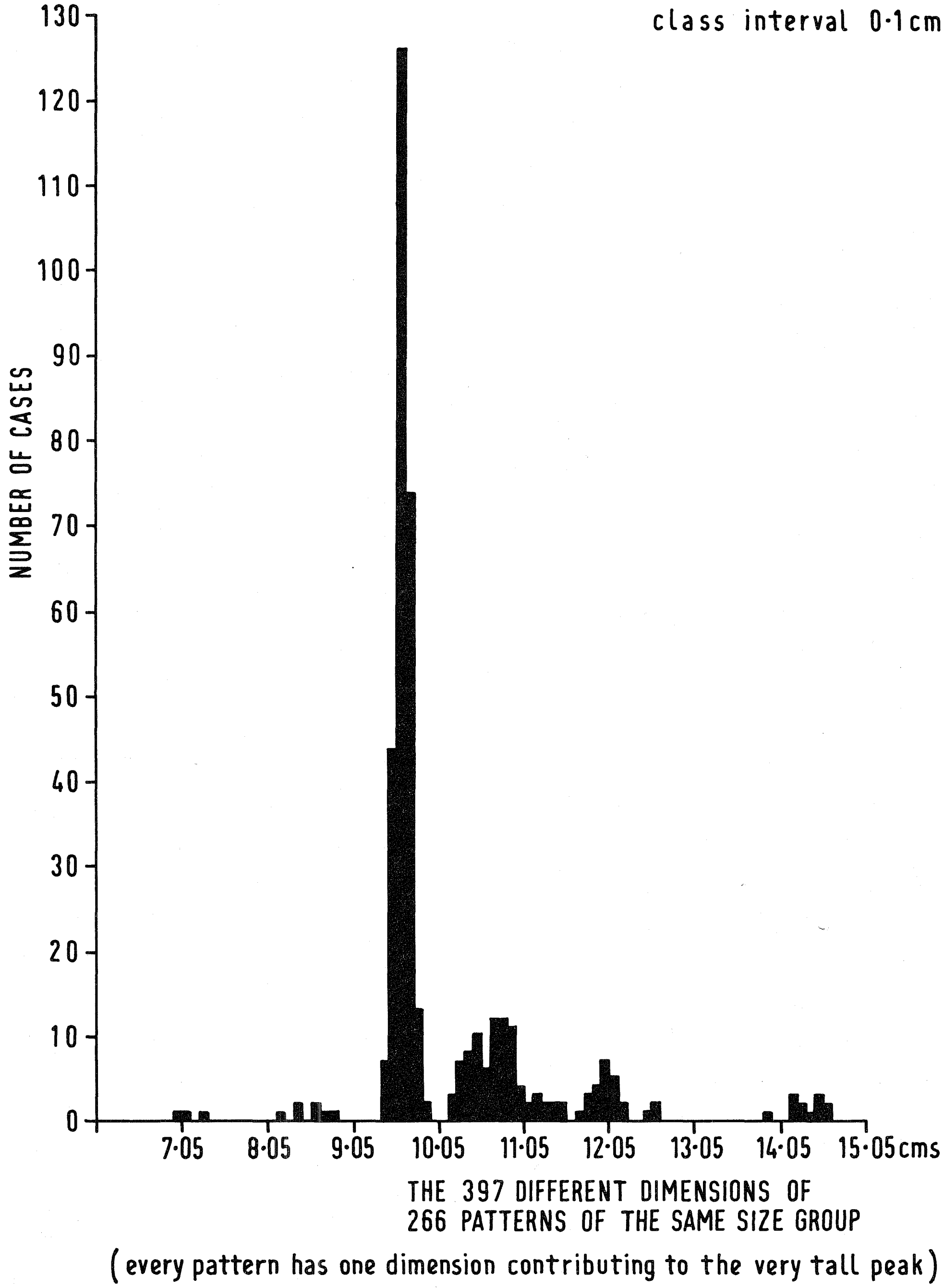
class interval 0·1cm
NUMBER OF CASES
130
120
110
100
90
80
70
60
50
40
30
20
10
0
7·05
8·05
9·05
10·05
11·05
12·05
13·05
14·05
15·05cms
THE 397 DIFFERENT DIMENSIONS OF
266 PATTERNS OF THE SAME SIZE GROUP
(every pattern has one dimension contributing to the very tall peak)

Figure 1

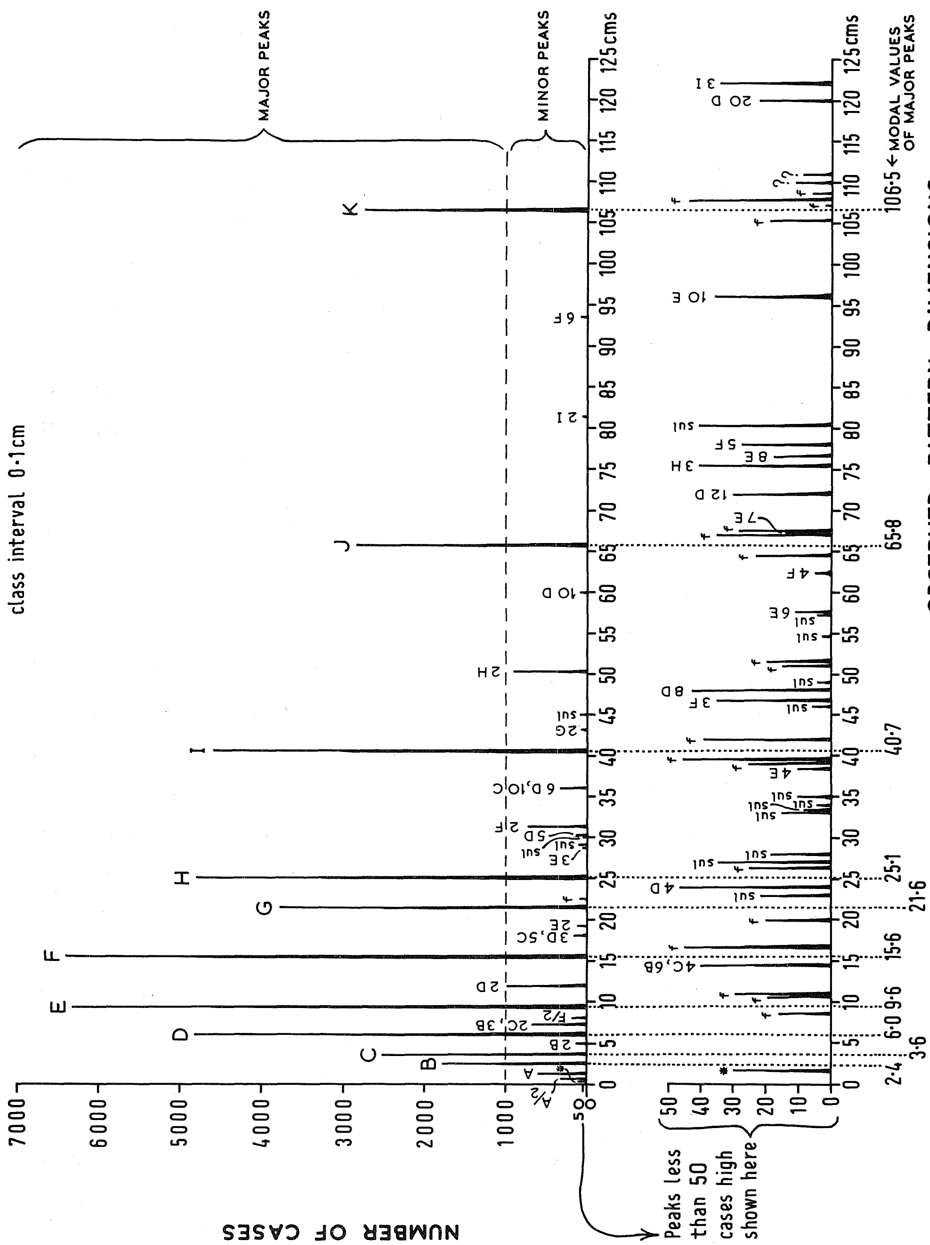
class interval 0·1 cm
MAJOR PEAKS
MINOR PEAKS
NUMBER OF CASES
Peaks less than 50 cases high shown here
OBSERVED PATTERN DIMENSIONS
MODAL VALUES OF MAJOR PEAKS
125 cms
2·4 3·6 6·0 9·6 15·6 21·6 25·1 40·7 65·8 106·5

Figure 2

In the frequency distribution (Fig. 2), there are only 10 peaks higher than 1000 cases. For purposes of analysis, I call these major peaks (labelled B, C, D, ···, Fig. 2). Many of the remaining minor peaks are so small that they have to be shown on an enlarged scale — lower part of Fig. 2.

Peaks occurring at twice or three times the length represented by another peak may owe their existence to just this, i.e., mosaicists having used two or three times a measuring unit, the latter represented by the "basic" peak. Many of the observed peaks exhibit this property. That they arose as being multiples of a more basic unit would be reinforced if peaks which are multiples of another are small peaks, whilst the peak of which they are a multiple is a much higher peak. Many of the minor peaks lie at lengths that are whole multiples of the lengths represented by the major peaks. I regard these as probably having arisen in this way (marked accordingly, Fig. 2).

The values of many ancient standard units of length have come down to us, so it is possible to see whether any observed peaks coincide with known ancient units. Some do, but surprisingly, only a few minor peaks agree with known standard units (marked "s u l," Fig. 2).

We are now left with the major peaks and a few minor ones. Some of the latter are caused by me measuring pattern dimensions which happen to also be the widths of single mosaic stones (marked with an asterisk, Fig. 2). The remaining minor peaks (with the exception of the one at 1.2 cm) have the common property of lying adjacent to one or other of the major peaks (two lie adjacent to one of the tallest minor peaks). Identification of the pattern dimensions that these minor peaks (marked "f," Fig. 2) represent, shows that they account for virtually every case of the few instances where I was unable to decide which of two alternative measurements was the one I should measure, in the sense of trying to measure the distance most likely to have been set down by measurement by the mosaicists. One of these two alternatives must be wrong, in the sense that they cannot both be right. In all but two cases, the alternative measurement lies in the adjacent major peak. That it should coincide with the dimension that the majority of pattern sizes exhibit, is reason to consider this value as the true one. On the other hand, we could reject both alternative measurements. It will not affect the results, for they account for less than 0.85 percent of the observations.

I find unusually wide stones are often associated [2] with distortion of the arrangement of neighboring stones. Construction can be deduced to have proceeded from the unusual stone through the area it distorts. Making maps of such effects leads me to think [2] that mosaics were normally started at their center, and constructed progressively outwards from it.

Assigning imperfections in mosaics values on a numerical scale of increasing imperfection [2] usually yields a map as in Figure 3. Assuming imperfections increase as construction progresses, this again indicates that construction was centrifugal, but also that it was fastest in the four axial directions (A, B, C, D, Fig. 3).

Consequently, the first parts of patterns to be reached in construction would be their parts nearest to the mosaic center (their innermost rim, for patterns centered on the mosaic center) and the first of these parts to be reached will be the part lying on the mosaic axis. Constructional measurements would thus presumably have been made primarily in the mosaic axes and to the inner rims of patterns.

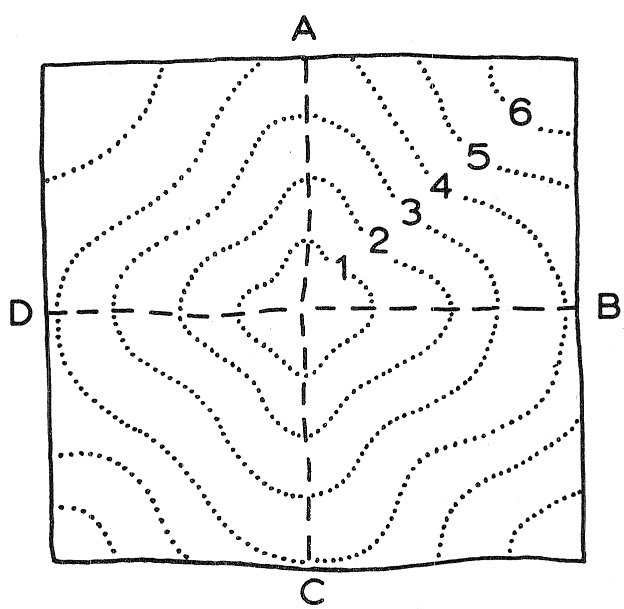

Figure 3

That the ancients usually proceeded like this is supported by the typical shape of the histogram as in Fig. 1. The dimension I deduce as being the one that the mosaicists made (because it lies in the very tall peak (Fig. 1)), is also usually the dimension that I had measured to the inner rim of the pattern. The other alternative dimensions of each pattern, in the vast majority of cases, all lie to the right of this tall peak (Fig. 1). These, representing greater lengths, are those I measured mostly to outer rims of patterns. The rarity of cases where the mosaicists' measurement was apparently not to the inner rim of the pattern, is shown by the scarcity of observations to the left of the very tall peak (Fig. 1).

The crests of the waves of peaks following the very tall peak (Fig. 1) lie at intervals which agree with the lengths represented by the tall peak in each of the smaller size groups of patterns. Consequently, since these crests are caused by including the pattern "thickness" in the measurement, this reveals that pattern thicknesses were often also fixed in terms of the same units as were used to fix the sizes of the smaller patterns.

Resuming analysis of Fig. 2, we are left with the major peaks and a minor peak at 1.2 cm. The modal values of the major peaks are: 2.4, 3.6, 6.0, 9.6, 15.6, 21.6, 25.1, 40.7, 65.8, and 106.5 cm, respectively. Of the 310000 pattern dimensions, 89% lie within ±3 standard deviations ($\sigma \equiv 0.13$ cm for each major peak) of these values. (A further 9% lie at whole multiples of these values. Of the remaining 2%, only approximately 1% can be identified with known standard units of length.)

Presumably we can regard these ten values, responsible for 89% of the observed lengths, as the units that were marked on the rulers which Vitruvius (first century B.C.) tells us [3] that mosaicists "accurately used." I call these values <u>mosaic units</u>.

DETERMINATION OF MORE ACCURATE VALUES FOR MOSAIC UNITS

That it is right to regard mosaic units as a set, is suggested by them lying in a distinct series (ignoring 21.6 cm); each is virtually the sum of the preceding two. On this basis, we might expect, by extrapolation, larger pattern sizes of 172.8, 279.6, and 452.4 cm. I find that the typical pattern sizes greater than 106.5 cm do occur very nearly at these distances, but fall progressively slightly short of these expected values (Fig. 4).

Observed modal value	Hypothetical value k = 1·2 cm	Difference between observed and hypothetical values	Number of cases	Relative reliability of mode
1·2 cm	1·2 cm	0 cm	1313	x 4
2·4	2·4	0	5031	x 7
3·6	3·6	0	6298	x 8
6·0	6·0	0	13970	x 12
9·6	9·6	0	15231	x 12
15·6	15·6	0	16150	x 13
21·6	–	–	10668	x 10
25·1	25·2	0·1	14617	x 12
40·7	40·8	0·1	12785	x 11
65·8	66·0	0·2	5861	x 8
106·5	106·8	0·3	3256	x 6
172·3	172·8	0·5	1553	x 4
278·9	279·6	0·7	426	x 2
451·3	452·4	1·1	485	x 3
730·2	732·0	1·8	144	unity
~1180·3	1184·4	~4·1	>10	–
unknown	–	–	–	–

Figure 4

Unfortunately, very large patterns are rare, for there are few mosaics big enough to exhibit them. I do not yet have sufficient observations to confidently report a value for the observed pattern size corresponding to the expectation of 1184.4 cm, but an approximate observed value is 1180.3 cm.

The two smallest lengths represented by major peaks are 3.6 and 2.4 cm. Extrapolating the series in this direction, we obtain 2.6 - 2.4 = 1.2 cm. This prediction is confirmed by the minor peak at 1.2 cm. Attempting to extrapolate again, we get 2.4 - 1.2 = 1.2 cm, demonstrating that 1.2 cm can be regarded as the basis of the set of mosaic units.

If mosaic units were in fact each the sum of the preceding two, that hypothetical values based on a value of 1.2 cm for the first unit progressively exceed the longer observed lengths by slightly greater amounts (Fig. 4) suggests that the true starting value is slightly less than 1.2 cm. The value of the first mosaic unit (M_1) in the series $M_x = M_{x-1} + M_{x-2}$ which yields values with the best fit to the observations can be determined as follows.

If each unit is the sum of the preceding two, the series can be expressed by the Fibonacci numbers, taking the first value as unity. To give values in a particular system of measure, I introduce a constant k equal to the dimension of the first value in the units of measure desired. A generating relation for mosaic units is therefore:

$$(1) \qquad y = k\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^x - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^x\right]$$

The expression in square brackets yielding Fibonacci numbers by successive substitution of integers 1, 2, 3, $\cdots$ for x; and y, the value of the x^{th} mosaic unit, assumes the units of measure of k.

The observed modal mid-interval value of the first mosaic unit is 1.2 cm. Its true value probably lies somewhere in the range of this modal value: 1.15 $\rightarrow$ 1.25 cm caused by the class interval of 0.1 cm.

From Eq. (1), division of each observed mosaic unit by its pertinent Fibonacci number gives a value for k. The larger the value that this is done for, the more accurate the result. For the first six mosaic units k = 1.2. For units 25.1, 40.7, and 65.8 cm, k begins to be slightly less than 1.2, and for the five units bigger than 65.8 cm, $k = 1.197$ cm for each. It thus appears that k is less than 1.2 cm, probably about 1.197 cm.

Hypothetical values based on 1.197 are shown in Figure 5. Also shown are these values corrected to the nearest whole millimeter, so as to bring them to a form comparable with the observations (class interval 1 mm). In all but one case (Fig. 5), values based on 1.197 cm match the observations.

Trying $k = 1.196$ cm and $k = 1.198$ cm, in both cases the resulting theoretical values for mosaic units progressively diverge from the observed values. Moreover (Figure 6), they diverge in an approximately symmetrical way, indicating that 1.197 cm represents the best value (in cms, to three places of decimals) for k.

A value for k yielding values agreeing with all the observed mosaic units is impossible. Taking $k = 1.197$ cm gives values fitting all observations (ignoring 21.6 cm) except 172.3 cm, for which the theoretical value is 172.36 cm (≡ 172.4 cm). The smallest change in 172.368 cm needed to make it fall into the same class interval as the observed value (172.3 cm) is 0.024 cm. The Fibonacci number for this unit is 144. Thus the necessary change in k is (0.024/144) cm = 0.0001666 cm. This gives a new set of hypothetical values for mosaic units, but whilst fitting the observation 172.3 cm, it begins to diverge from the observations at the 14th and 15th mosaic unit (Fig. 5).

If the first mosaic unit was 1.197 cm long, it explains why some observed mid-interval values appear to be only approximately the sum of the preceding two. For example, we have the observed values 9.6, 15.6, 25.1 cm, but 9.6 + 15.6 = 25.2, not 25.1. However, based on 1.197 cm, we have 9.576 + 15.561 = 25.137 which is exact. Rounding each to the nearest whole millimeter (which is the effect of the class interval of 1 mm), we get

$$9.576\ (\equiv 9.6) + 15.561\ (\equiv 15.6) = 25.137\ (\equiv 25.1)$$

which explains this.

We might expect a similar effect in pattern sizes that are multiples of others. In some cases, this is so; for example, a minor peak occurs (Fig. 2) with modal value 50.3 cm. This could be caused by use of 2x unit 25.1 = 50.2 cm. If the true value of the eighth mosaic unit is 25.137 cm, we get

Observed modal value	Hypothetical value k = 1·197 cm	Previous column correct to nearest whole mm	Difference between previous column and observed value	Hypothetical value k = 1 1968334 cm	Previous column to nearest whole mm, minus observed value
1·2 cm	1·197 cm	1·2 cm	0 cm	1·1968334 cm	0 cm
2·4	2·394	2·4	0	2·3936668	0
3·6	3·591	3·6	0	3·5905002	0
6·0	5·985	6·0	0	5·9841670	0
9·6	9·576	9·6	0	9·5746672	0
15·6	15·561	15·6	0	15·5588342	0
21·6	—	—	—	—	—
25·1	25·137	25·1	0	25·1335014	0
40·7	40·698	40·7	0	40·6923356	0
65·8	65·835	65·8	0	65·8258370	0
106·5	106·533	106·5	0	106·5181726	0
172·3	172·368	172·4	0·1	172·3440096	0
278·9	278·901	278·9	0	278·8621822	0
451·3	451·269	451·3	0	451·2061918	−0·1
730·2	730·170	730·2	0	730·0683740	+0·1
~1180·3	1181·439	1181·4	~1·1	—	—
unknown	1911·609	—	—	—	—

Figure 5

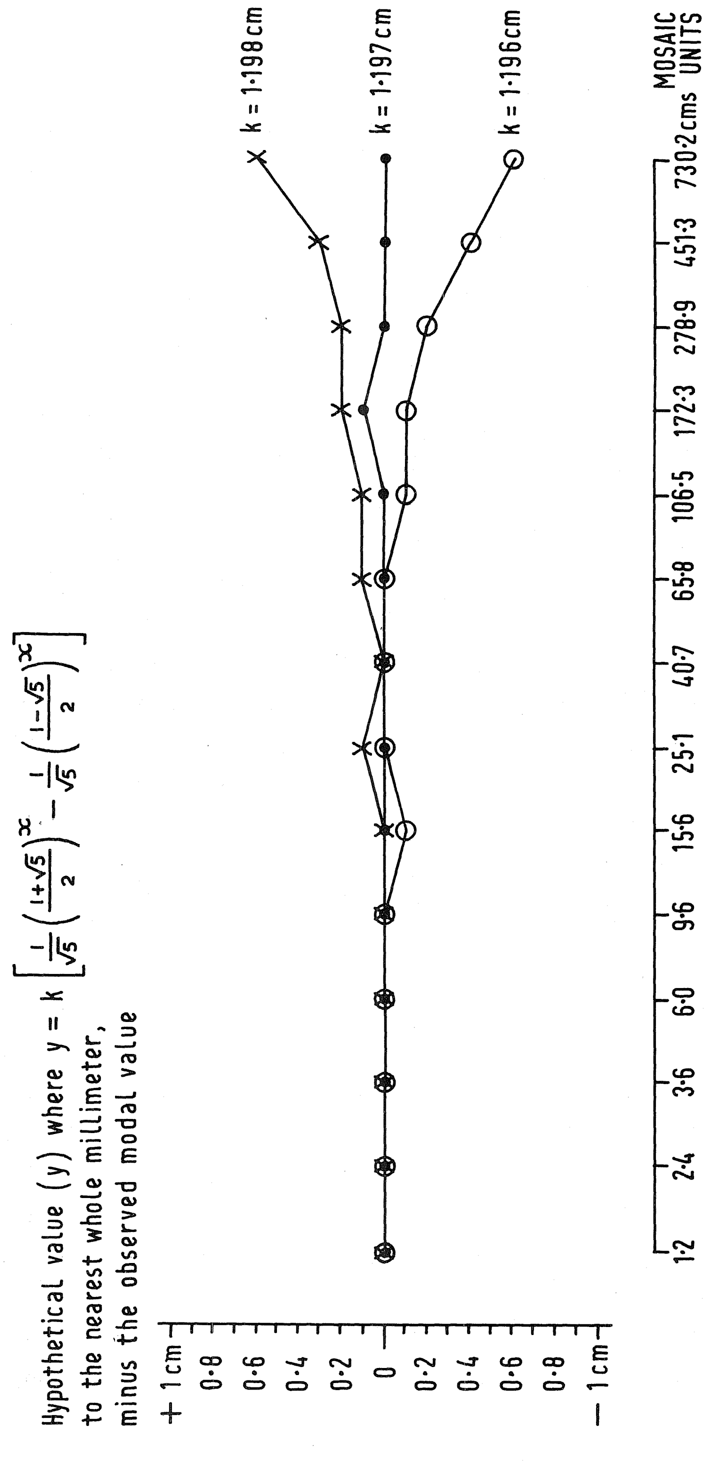
Hypothetical value (y) where $y = k\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^x - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^x\right]$
to the nearest whole millimeter,
minus the observed modal value
+ 1 cm
0·8
0·6
0·4
0·2
0
0·2
0·4
0·6
0·8
— 1 cm
k = 1·198 cm
k = 1·197 cm
k = 1·196 cm
1·2
2·4
3·6
6·0
9·6
15·6
25·1
40·7
65·8
106·5
172·3
278·9
451·3
730·2cms
MOSAIC UNITS

Figure 6

2 x 25.137 = 50.274 ≡ 50.3 cm, matching the observed value. A minor peak occurs at 75.6 cm, but 3 x 25.1 = 75.3, and 3 x 25.137 = 75.4. Here, the observation slightly exceeds both theoretical values.

Small peaks at multiples of unit 9.6 cm occur (Fig. 2). However (Fig. 7) the observed values progressively increase beyond the theoretical values, taking the 5th mosaic unit as 9.576 cm. Although the observed values are fitted (Fig. 7) by values based on the assumption that this unit measures exactly 9.6 cm, this does not necessarily mean that mosaic units were based on 1.2 cm rather than 1.197 cm. If values that are multiples of others were measured out by repeating measurement of the basic unit the desired number of times, greater error would tend to accompany greater multiples. This error would tend mostly to add to the intended length (the observed condition) if rulers were butted end to end to achieve it. Providing rulers are not displaced too much sideways, and as they are unlikely to be compressable, errors will tend to add to the intended value rather than reduce it.

Lack of symmetry of some peaks might be expected if the first unit was 1.197 cm long. For example, in the case of the 6th mosaic unit, the observed peak has a modal value of 15.6 cm. Its theoretical value based on 1.197 cm is 15.561 cm. Although this lies within the ±0.05 cm range of the observed modal mid-interval, it lies very much to the left of the mid-interval (-0.04 cm). We might expect that the observations would form an asymmetrical peak, more values occurring in the left-hand half of the peak. I detect (Fig. 8) no clear tendency for this effect in the present data.

WHY WERE MOSAIC UNITS USED?

The ancient names for some everyday units of length which refer to finger, knuckle, palm, handspan, handlength, etc., suggest that people once actually used their limbs to measure things. Some tradesmen still measure out yards by the tip of their nose to their sideways stretched fist. According to Vitruvius [5], "Besides, the ancients took from the members of the human body the proportional (?) dimensions needed in all constructions, finger, palm, foot, cubit." Some [6] trace this back to Plato in the Theatus, "Man is the measure of all things."

If mosaicists used their fingers, hands, etc., for measuring out their patterns, would this give rise to the observed situation? Whilst it might cause

Observed modal value	Number of cases	Multiple of mosaic unit 9·6 cm	Hypothetical value based on 1·197 cm	Hypothetical value based on 1·2 cm
9·6 cm	15231	1	9·6 cm	9·6 cm
19·2	265	2	19·2	19·2
28·8	97	3	28·7	28·8
38·5	31	4	38·3	38·4
48·0	99	5	47·9	48·0
57·6	36	6	57·5	57·6
67·3	30	7	67·0	67·2
76·8	46	8	76·6	76·8
none	0	9	86·2	86·4
96·0	76	10	95·8	96·0
none	0	>10	—	—

Figure 7

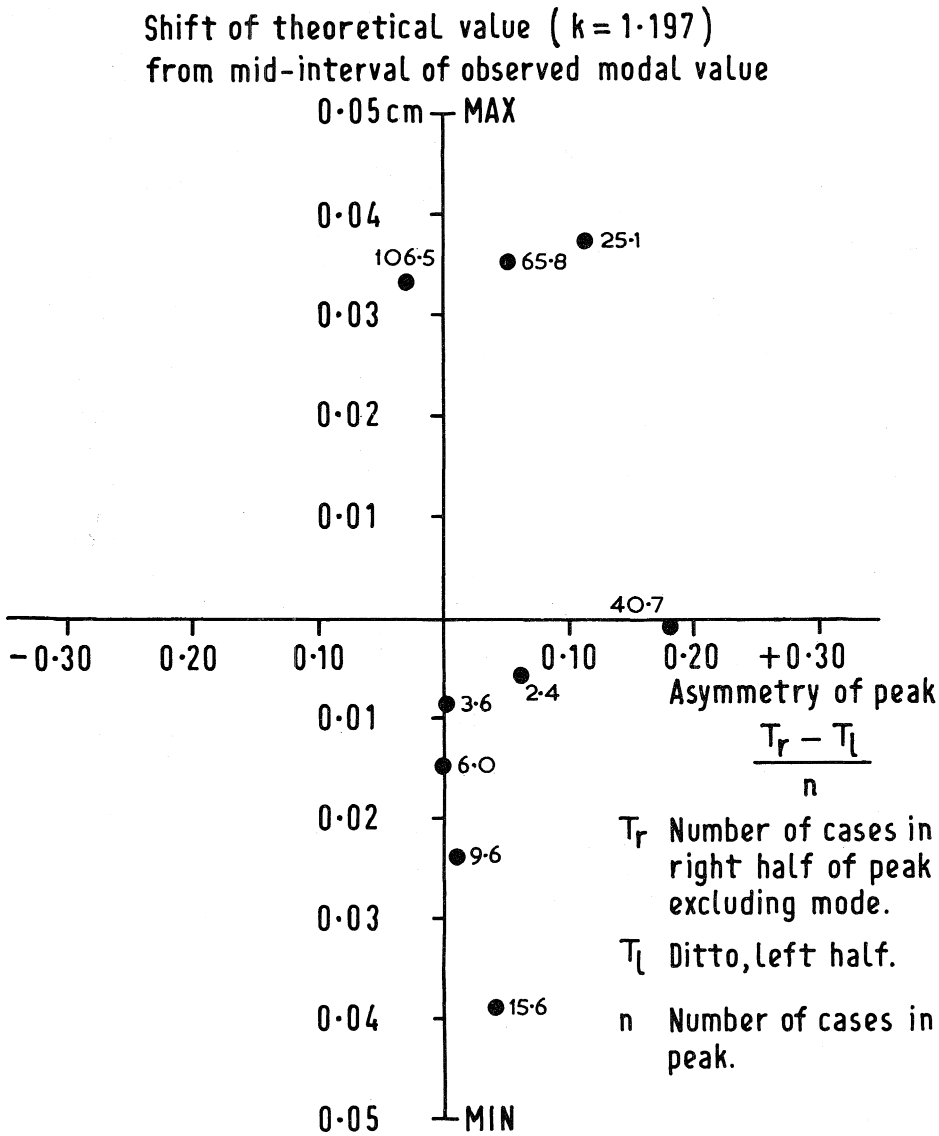
Shift of theoretical value (k = 1·197)
from mid-interval of observed modal value
0·05 cm
MAX
0·04
0·03
0·02
0·01
106·5
65·8
25·1
40·7
−0·30
0·20
0·10
0·10
0·20
+0·30
2·4
3·6
6·0
9·6
15·6
0·01
0·02
0·03
0·04
0·05
MIN
Asymmetry of peak
$\frac{T_r - T_l}{n}$
T_r Number of cases in right half of peak excluding mode.
T_l Ditto, left half.
n Number of cases in peak.

Figure 8

patterns to fall into size groups, the variation between, say, the handlength on different people is far [4] in excess of the range of distances contributing to a typical mosaic pattern size.

However, the variation would be much reduced if, instead of each mosaicist using his own hand length, if each used a ruler calibrated from one single man. Slight support for this exists in that limb dimensions do roughly fit mosaic units. For example, my own approximate dimensions are as follows: First joint of index finger 2.5, first joint of thumb 3.5, thumb length 6.0, length of index finger 9.5, the "spithama" (tip of index finger to tip of thumb spread wide) 15.5, hand span 21.5, foot 25.0, "inner cubit" (tip of index finger to inner bend of bent elbow) 40.0, arm length with fist clenched 66.0 cms, respectively.

Ancient units of length with anthropormorphic names, however, measure distances less coincident with mosaic units than this. For example, a typical Greek standard span is about 23.0 cm. Roman and Greek standard cubits mostly lie between 42.35 cm [7] and 46.000 cm [8], and some Hebrew cubits lie outside this range. Egyptian and Sumerian cubits are mostly longer; there is even a Chino-Sumerian cubit of 74.40 cm!

In discussing why the ancients chose to so consistently make their mosaic patterns one or other of the set of mosaic units, it might be useful to express mosaic units in ancient units of length rather than in a modern system. Comparing 6646 values derived as whole multiples and likely fractions of so ancient units of length possibly pertinent to the mosaic craft, I find very few mosaic units are equal to a whole multiple (or multiple plus likely fraction) of a known standard unit. The only single standard unit which yields more than about two mosaic units appears to be the Greek finger of 1.92 cm [9], and this only fits five of the eleven mosaic units (Fig. 9).

However, expressing the mosaic units (ignoring 21.6 cm) in terms of this Greek finger yields (to the nearest whole number) integers which are the actual Fibonacci numbers up to unit 65.8 cm (Fig. 10). This could be significant, for expressing mosaic units in modern units produces integers (Fig. 10) lacking this property. Neither (Fig. 10) does the Roman digit from most Roman feet fit so well.

Measuring in centimeters does, however, bring out the relationship 10 x unit 3.6 cm = 6 x unit 6.0 cm. This relation could be significant, for sexagesimal relations are common in ancient metrology (some effects of which are

Mosaic unit	Multiple of Greek finger	Length yielded by this multiple
2·4 cm	1 ¼	2·40 cm
9·6	5	9·60
21·6	11 ¼	21·60
25·1	13	24·96
40·7	21 ¼	40·80

Figure 9

MOSAIC UNITS EXPRESSED IN VARIOUS UNITS OF MEASURE

Cms.	nearest integer	Present English inch	integer	Greek finger of 1·92cm	integer	Roman digit of 1·89cm	integer
1·2	1 ●	0·47	0	0·1	0	0·1	0
2·4	2 ●	0·94	1 ●	1·3	1 ●	1·3	1 ●
3·6	4	1·41	1 ●	1·8	2 ●	1·9	2 ●
6·0	6	2·36	2 ●	3·1	3 ●	3·2	3 ●
9·6	10	3·77	4 ●	5·0	5 ●	5·1	5 ●
15·6	16	6·12	6 ●	8·1	8 ●	8·3	8 ●
(21·6)							
25·1	25	9·89	10	13·1	13 ●	13·3	13 ●
40·7	41	16·01	16	21·2	21 ●	21·5	22
65·8	66	25·90	26	34·3	34 ●	34·8	35
106·5	107	41·92	42	55·5	56	56·3	56

Fibonacci number marked ●

Figure 10

still with us, e.g., division of 1° into 60 min. of arc). However, that $1.197 = \sqrt[10]{6}$ can be dismissed as significant, for it is a result of using centimeters. Also the otherwise attractive relation that a right angle triangle of sides 10 and 6 has a hypotenuse of just under 12.

Translating Vitruvius' next remarks, we learn that, "The Ancients grouped these body dimensions into the perfect number called teleon. They decided on the number ten as perfect···. But mathematicians, in disagreement, say the number six is perfect···. Later they realized both six and ten are perfect, and they put them together, making the most perfect number sixteen···." As it happens, 16 x unit 6.0 cm = 10 x unit 9.6 cm.

Dr. George Ledin, Jr. has extracted [10] Fibonacci numbers from mosaic units by dividing the observed values in centimeters by 1.19, obtaining integer 18 from unit 21.6 cm. He has found [10] that the unit 21.6 cm, which is "odd" in the sense that all the others are directly related to Fibonacci numbers, can itself be related to the Fibonacci series, for 18 is a term in the Lucas series. Although multiplying 1.197 cm by 18 gives 21.5, not 21.6 cm as observed, adding unit 5.985 cm to unit 15.561 gives the same result.

Ledin [10] draws attention to the connection: mosaic units → Fibonacci numbers → the so-called "golden section." Firm evidence that the ancients knew, and regarded as special, the "golden" ratio 1:1.618··· is provided by Euclid's Elements Book 6, Definition 3 and Proposition 30. But did the ancients know the Fibonacci series? D'Arcy Thompson has said [11], "...there is no account of it, nor the least allusion to it, in all the history of Greek mathematics...," but also [11], "It is quite inconceivable that the Greeks should have been unacquainted with so simple, so interesting, and so important a series; so clearly connected with, so similar in its properties to, that table of side and diagonal numbers which they knew familiarly."

If the ancients did use mosaic units because of their connection with 1:1.618, it seems to imply that they knew the Fibonacci series. If this could be shown to be their reason, we would apparently have unique evidence of knowledge of the Fibonacci numbers (F_x) before Leonardo of Pisa. It would also mean that the knowledge

$$\lim_{x \to \infty} \frac{F_{x+1}}{F_x} = 1.618 \cdots$$

existed before Kepler, who is apparently [12] regarded as the first to know it. Simply because written record of the series 1, 1, 2, 3, 5, 8, ⋯ has not come down to us from the Greeks of course does not mean that they did not know it. A dramatic example is the recent discovery [13] of the unexpected ancient Greek computing mechanism, complete with dials and gearing, to which no known allusion had reached us either.

The ratio between successive pairs of mosaic units greater than the pair 6.0:9.6 is close to 1:1.616 (ignoring 21.6 cm). The ratio 6.0:9.6 is 1:1.600. The ratios for the smaller units are 1:1.6, 1:1.5, and 1:2. Had the mosaicists invoked the ratio 1:1.618 (without the Fibonacci series) we might expect their units measuring less than the pair 9.6:15.6 cm to exhibit the 1:1.618 ratio. Their smaller units would then measure 5.993, 3.667, 2.666, 1.648 cm, respectively. As such a series can be extrapolated indefinitely, we might expect another unit at 1.002 cm, another at 0.619 cm, and so on. However, the observed frequency distribution of pattern sizes does not support this idea.

Alternatively, if the ancients simply wanted units each the sum of the preceding two, and wanted this in order to invoke the ratio 1:1.618, there is no need for the units to be the lengths I call mosaic units. Consider the general series $U_{n+2} = U_{n+1} + U_n$.

$$U_{n+2} - U_{n+1} - U_n = 0.$$

Put $U_n = At^n$ where A and t are any two constants. Then

$$At^{n+2} - At^{n+1} - At^n = 0.$$

$$t^2 - t - 1 = 0 \qquad (t \neq 0).$$

So

$$t = \frac{1 \pm \sqrt{1 - (-4)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Therefore,

$$U_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$$

where a and b are determined by the initial conditions; the values of the first two terms.

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{a\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + b\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n} .$$

Since

$$0 < \left|\frac{1-\sqrt{5}}{2}\right| < 1 ,$$

then

$$\lim_{n\to\infty} \left(\frac{1-\sqrt{5}}{2}\right)^n = 0 .$$

Thus, by simplification,

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{1+\sqrt{5}}{2} = 1.618\cdots .$$

Thus, very many different sets of units could have been used in mosaic construction, yet all possessing a common property of the ratio between successive units approaching 1:1.618.

As it is, the consistent use of a particular set of absolute lengths (mosaic units) apparently throughout the classical world from about 400 B.C. to 530 A.D. (the limits of my observations) could suggest some particular need. Could it be a practical matter?

Pouring cleaned ancient mosaic stones (given to me by the Italian government and lent by the British Museum for the purpose) into a machine I made to pack stones in adjacent rows in the manner of a mosaic, a strip of mosaic was formed with stones in random sequence. Inspection of successively formed machine made mosaics revealed [14] a tendency for stones in adjacent rows to periodically align transversely across the rows (Fig. 11).

Classical floor mosaicists apparently [2], [15], [16] normally laid their stones without specially selecting them for size, or shaping them to suit. It is possible that they found this alignment phenomenon for themselves, and from then on took advantage of it by making their pattern sizes which coincide with the tendency for alignments, thus causing less erratic rims to their patterns (and for no extra effort).

That the ancients may have noticed alignments is made more probable by the existence of alignments [14] in unpatterned ancient mosaics (Fig. 11). Alternatively, they could simply have noticed that their patterns often turned out most regular when made certain sizes, and from then on intentionally made patterns these sizes, without noticing the alignments as such.

Measurements I made of the distances between alignments in intact unpatterned ancient mosaics largely coincide [14] with the typical intervals between alignments in my machine-made mosaics. These, in turn, agree [14] with the typical pattern sizes — mosaic units.

I find many mosaic patterns lie in sequence such that their dimensions lie in the same sequence as the units would lie if marked in ascending order on a ruler (Fig. 12). It seems likely that mosaicists might notice, even if they did not know beforehand, that the distance between two adjacent calibrations on the ruler also measures one of the units in the same set. From this, it would seem a short step to perceiving that each unit (except 21.6 cm) is the sum of the preceding two. If they regarded the first unit as unity, and ignored unit 21.6 cm, they would have arrived at the Fibonacci series empirically.

The liklihood of the Fibonacci properties of mosaic units being originally intentional is linked with whether it is correct to regard unit 21.6 cm as secondary, thus leaving the units possessing Fibonacci properties. In the sense that 21.6 cm can be expressed in terms of the other units (6.0 + 15.6 cm), 21.6 cm does seem secondary.

On the ruler (Fig. 13A) mosaic unit 21.6 cm is symmetrical with 9.6 cm about unit 15.6 cm, in the same way that 25.1 cm is symmetrical about 15.6 cm with 6.0 cm. The most frequent pattern sizes I have found that are not single

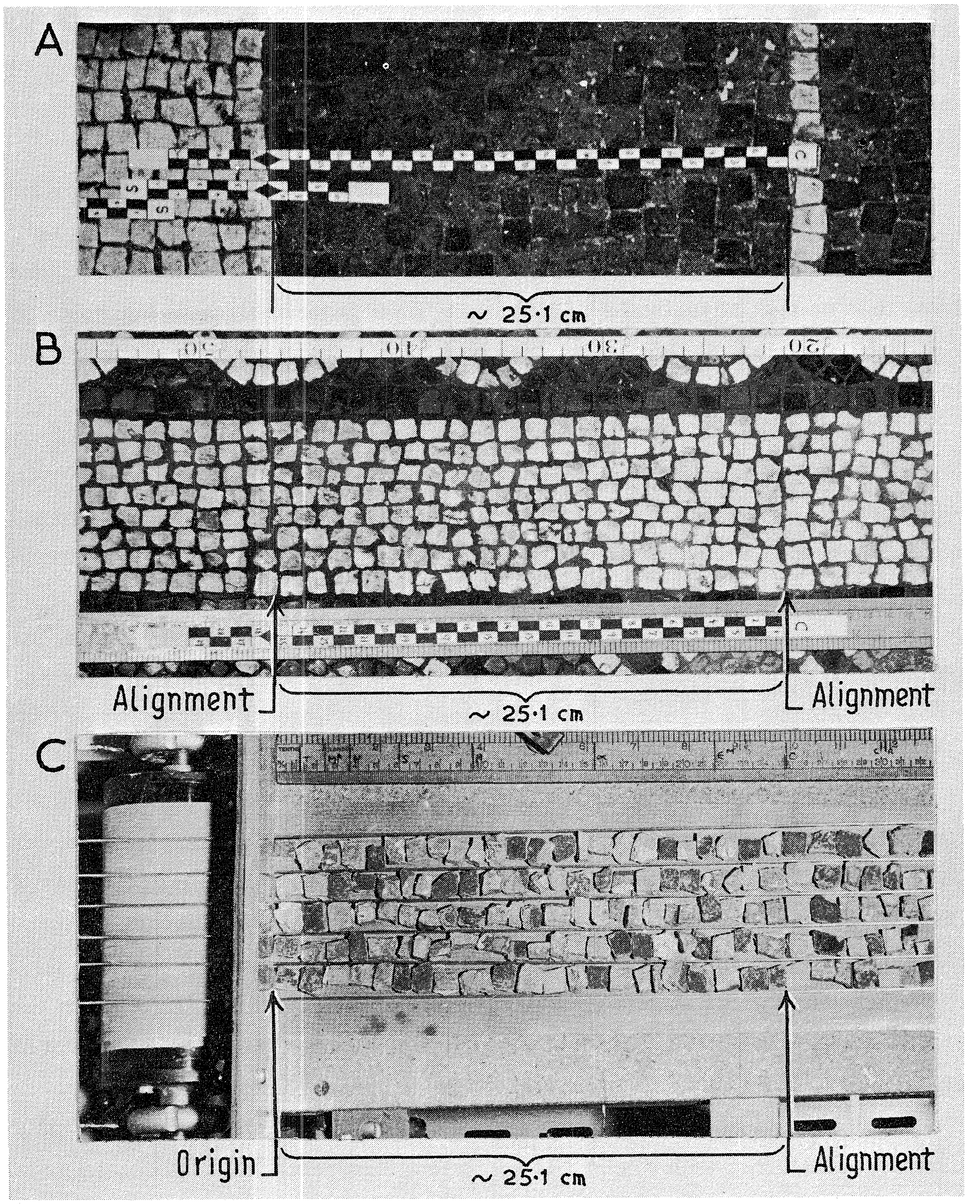

Figure 11

A. Typical ancient mosaic pattern (Markets of Trajan, Rome). Pattern dimension agrees with mosaic unit 25.1 cm.

B. Typical pair of alignments in an intact Roman mosaic (Aldborough, England). Alignment interval agrees with mosaic unit 25.1 cm.

C. Typical alignment which occurred as a packing pehnomenon among loose ancient mosaic stones packed in rows by machine. Distance of alignment from starting place of aggregation agrees with mosaic unit 25.1 cm.

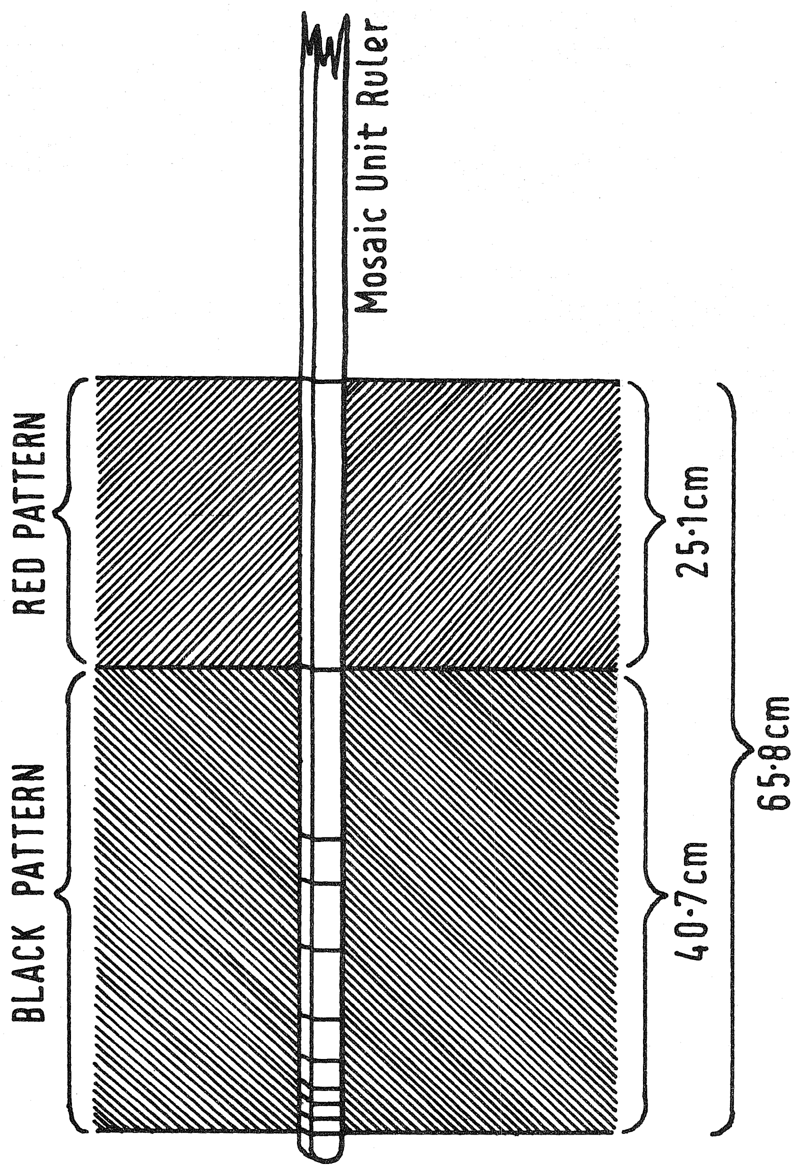
BLACK PATTERN
RED PATTERN
Mosaic Unit Ruler
25·1cm
40·7cm
65·8cm

Figure 12

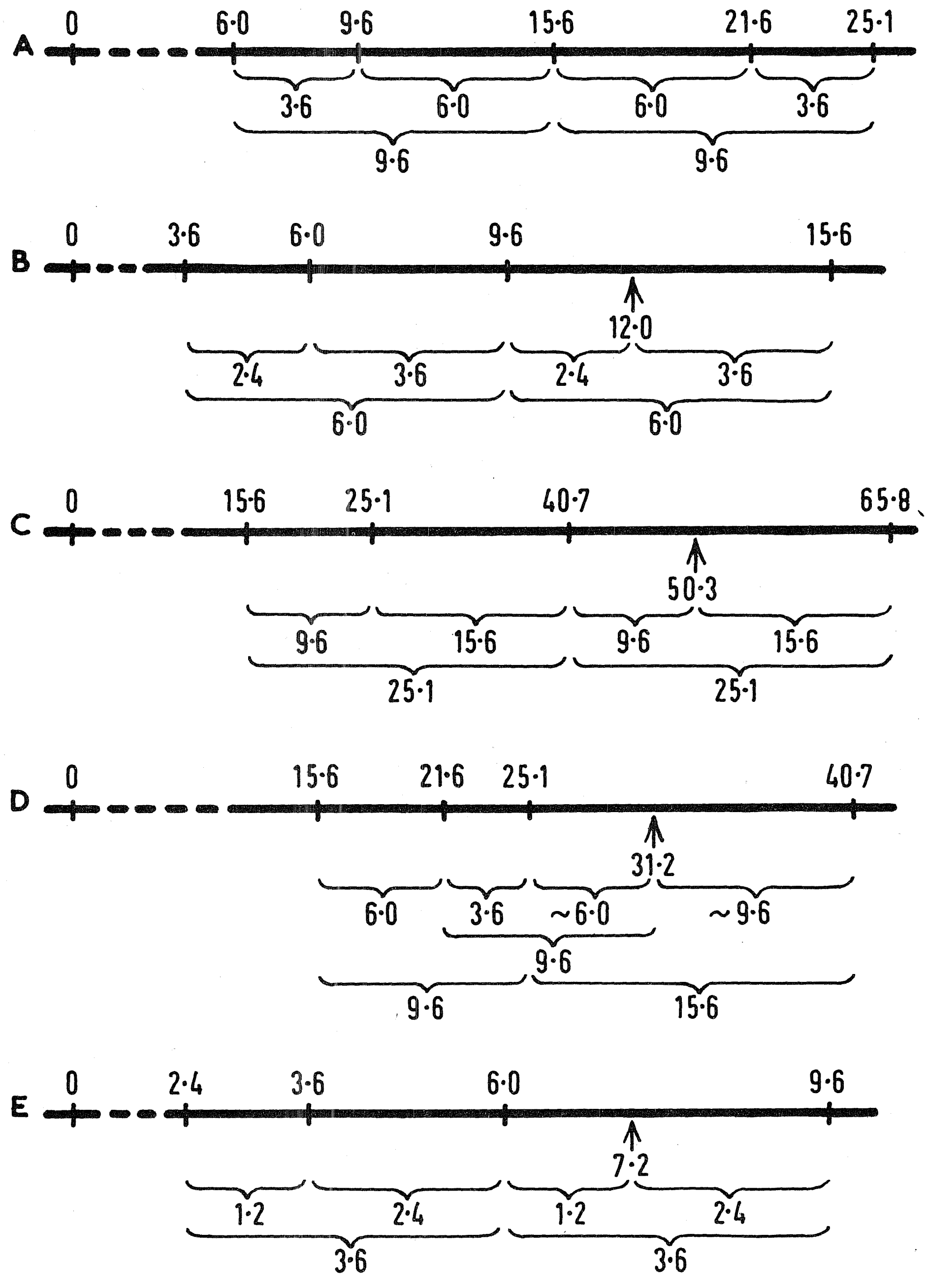
A
0
6·0
9·6
15·6
21·6
25·1
3·6
6·0
6·0
3·6
9·6
9·6
B
0
3·6
6·0
9·6
15·6
12·0
2·4
3·6
2·4
3·6
6·0
6·0
C
0
15·6
25·1
40·7
65·8
50·3
9·6
15·6
9·6
15·6
25·1
25·1
D
0
15·6
21·6
25·1
40·7
31·2
6·0
3·6
~6·0
~9·6
9·6
9·6
15·6
E
0
2·4
3·6
6·0
9·6
7·2
1·2
2·4
1·2
2·4
3·6
3·6

Figure 13

mosaic unit dimensions are 12.0, 50.3, 31.2, and 7.2 cm, respectively. They all possess similar symmetrical properties (Fig. 13, B-E).

A $\sqrt{2}$ relationship is known [17] between the Royal cubit of Herodotus and the Egyptian remen. A square of side 21.6 cm has a diagonal of 30.55 cm (the modern English foot is 30.48 cm). Many ancient standard feet are known, most varying between the 29.3 cm Roman foot [18] to a foot used in Roman Europe and elsewhere of 33.5 cm [7]. A typical value to the nearest whole millimeter is 30.8 cm for the Roman foot and 30.6 cm for the Greek. The latter fits this diagonal of 30.6 cm. That again Greek measure fits better than Roman would fit in with the mosaic craft being passed from the Greeks to the Romans.

However, the value of 21.6 cm occurred, by natural causes, and about as frequently [14] as 6.0, 9.6, 15.6, and 25.1 cm in the form of the interval between alignments in both intact ancient mosaics and in my experimentally produced mosaics. From this point of view it seems as basic as the other values.

That the pattern sizes tend to be the same few lengths for so long a period (1000 years) and over such a geographic extent, seems to me to indicate a practical reason rather than common subscription to some aesthetic. While aesthetic principles were apparently invoked in temple construction, it is less likely that they would be "debased" to the level of crude floor covering. The majority of mosaics in the present study are very humble, rife with imperfections and even errors. Even now, when they have the extra quality of "ancient" to recommend them, many are regarded as not worth bothering about, allowed to fall to pieces, or are openly permitted to be damaged.

In contrast, original mistakes were not normally allowed to remain in the kind of work to which aesthetic principles were applied; witness the remarkable perfection of Greek temples. According to Vitruvius [5], "... the ancients have, in their works, determined that each part be an aliquot (?) part of the total plan... especially in temples, wherein faults as well as beauties will last for all time." The latter is connected with ancient concern over the area contained by a ground plan. For example, Plutarch [19] calls the Parthanon hecatompedon (i.e., the "hundred-footed"). The word <u>templum</u> (a temple) originally meant [20] a space that could be, or is, enclosed. Possibly this concern with the area of a floor came from concern over the area (also called "templum") within which the flight of birds was watched when looking for omens to guide decisions. Fixing the boundary of this area would be crucial to a believer, it must contain any relevant flight that might occur, but must not be too big so as to make proper watching impossible. The notion of an "ideal" also presents itself.

Standardization of building material dimensions [21] is another factor leading to floor areas being defined in whole numbers of units. The latter is used by metrologists in deducing lengths of some ancient standard units. Flinders Petrie found [7] an average of only about 5mm "original" error per meter in ancient measured lengths.

That so many floor mosaics do not fit their floor area suggests lack of relationship between the units used in their construction and the normal standard units fixing the floor area. That "special" units should be used for mosaics would fit the contrast [22] at one time between the cubit used for everyday life and the special cubit reserved for building.

Given a stock of mosaic stones of very high constancy of size (a most unusual condition for ancient mosaic stones) and they are of side length equal to the smallest mosaic unit, one can construct the other mosaic units as follows (Fig. 14A). Set down one stone. Its side gives a length of 1.2 cm. Place another next to it. We have a new dimension of 2.4 cm. Adding the latter stone was equivalent to squaring the original length, for taking the side length of the first stone as unity. $1^2 = 1$, so we added one stone, thereby obtaining the new length of two stones. Square this new dimension (2^2) adding four stones in a square. We obtain a new length of $2 + 1$ stones $= 3.6$ cm. Square this new dimension, and 6.0 emerges. Continue this process, and all the mosaic units (except 21.6 cm) are formed. By simply adding stones until a square is formed on each new dimension, there is no need even to count stones.

However, if, instead of moving around the starting stone, Dr. Michael Whippman of Pennsylvania University has pointed out to me that moving from side to side of the growing diagonal of the overall construction, as in his figure (Fig. 14B), the value 21.6 cm can also be obtained. He suggests that the mosaicists' ruler could have been of this "flat plate" form.

Dr. Wayne Cole of Abbott Laboratories, Illinois, has suggested to me that if the mosaicists used a tool of this form, it need not be bigger than unit 40.7 cm. When the need arises, unit 65.8 cm can be obtained by using first

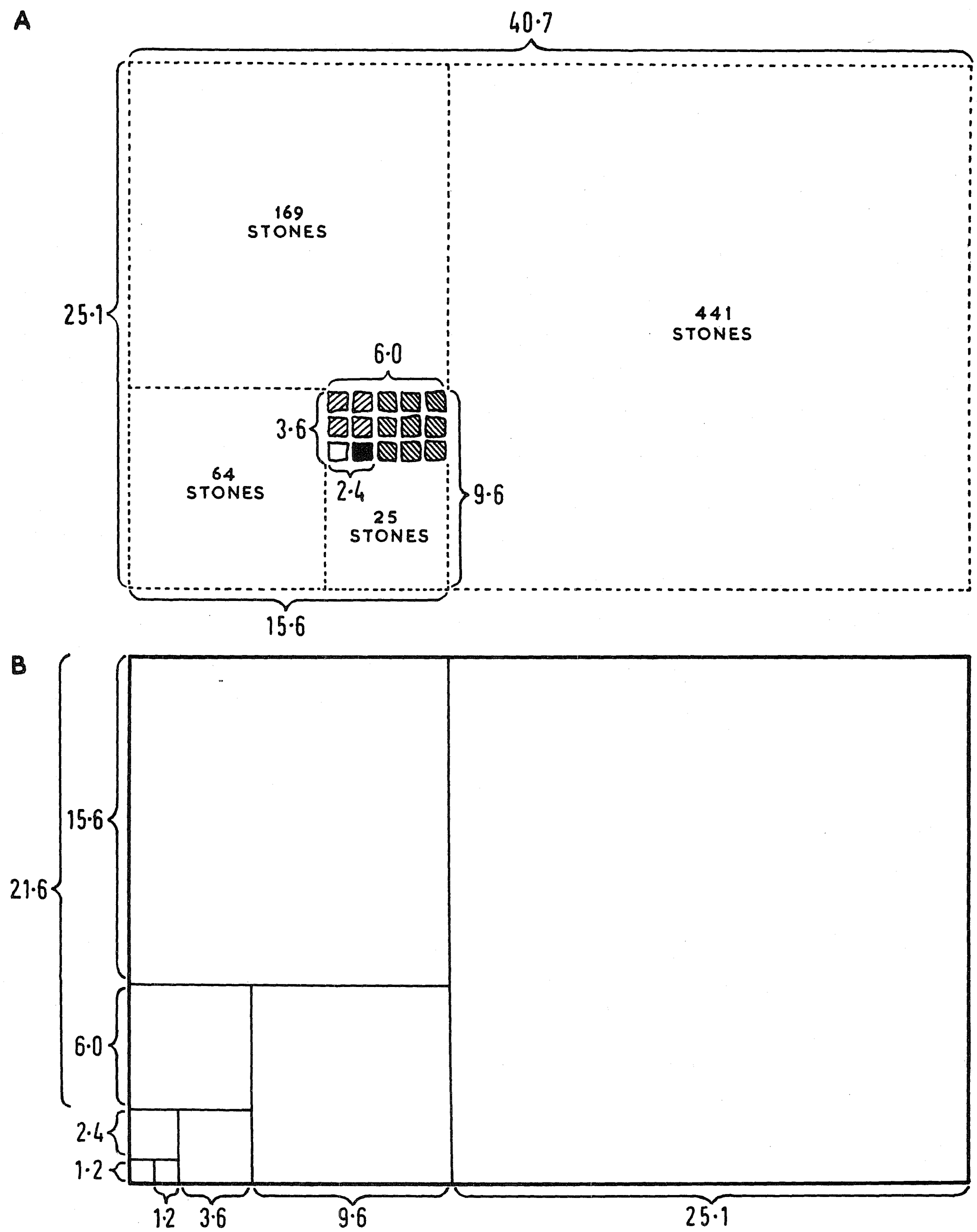
A
40·7
169
STONES
25·1
441
STONES
6·0
3·6
64
STONES
2·4
9·6
25
STONES
15·6
B
15·6
21·6
6·0
2·4
1·2
1·2
3·6
9·6
25·1

Figure 14

one edge of the tool and then the other (40.7 + 25.1 = 65.8 cm). If unit 106.5 cm is required, using the long edge twice and then the other fulfills this (40.7 + 40.7 + 25.1 = 106.5 cm). He also suggests that such a tool could double as a square.

Ancient rulers, often calibrated in inadequately examined units, have come down to us. It is possible that an original mosaicists' ruler may still exist. If one does come to light, it might indicate how mosaic units arose. For example, many ancient rulers are square sectioned sticks with saw cuts for calibrations. On some, the cuts run around all four faces of the ruler for the prime units and only on one or two faces for other values. If the unit 21.6 cm was differentiated in this way, its secondary nature would seem established, emphasizing the Fibonacci properties of the others.

In 1632, A. Bosio [23] gave an engraving showing a Roman tomb on which is depicted two kinds of dividers, a peg and line, a level, a chisel, a punch, a sharp bladed hammer, a square, and a ruler (Fig. 15). The calibrations on rulers shown on other monuments have been found to be accurate [24]. In the engraving, the two smallest divisions on the ruler, each marked by a dot, are equal, and the interval marked "R" is equal to the unmarked interval at the extreme right. The interval marked "dot A dot" is equal to interval "R" plus the two intervals marked with dots. All these intervals can be constructed accordingly, providing we can fix the position for the second calibration from the right, and the size of the interval marked "dot."

Taking the unmarked interval as unity, measurement shows that the interval marked "dot" is 0.618 long. Thus "dot" corresponds to $1/\tau$. "R" corresponds to $1 + (2/\tau)$, where τ is 1:1.618.

If the engraving can be relied on, this is a case where the ancients apparently intentionally used units with 1:1.618 inter-relationships. The proportions

$$\left(1 + \frac{2}{\tau}\right) : 1 : \frac{1}{\tau} \quad ,$$

can be fitted by 21.6:15:6:6.0 cm. Thus the calibrations on Bosio's ruler could be these three mosaic units. I hope to find whether this tomb-face still exists, and whether this ruler does match mosaic units.

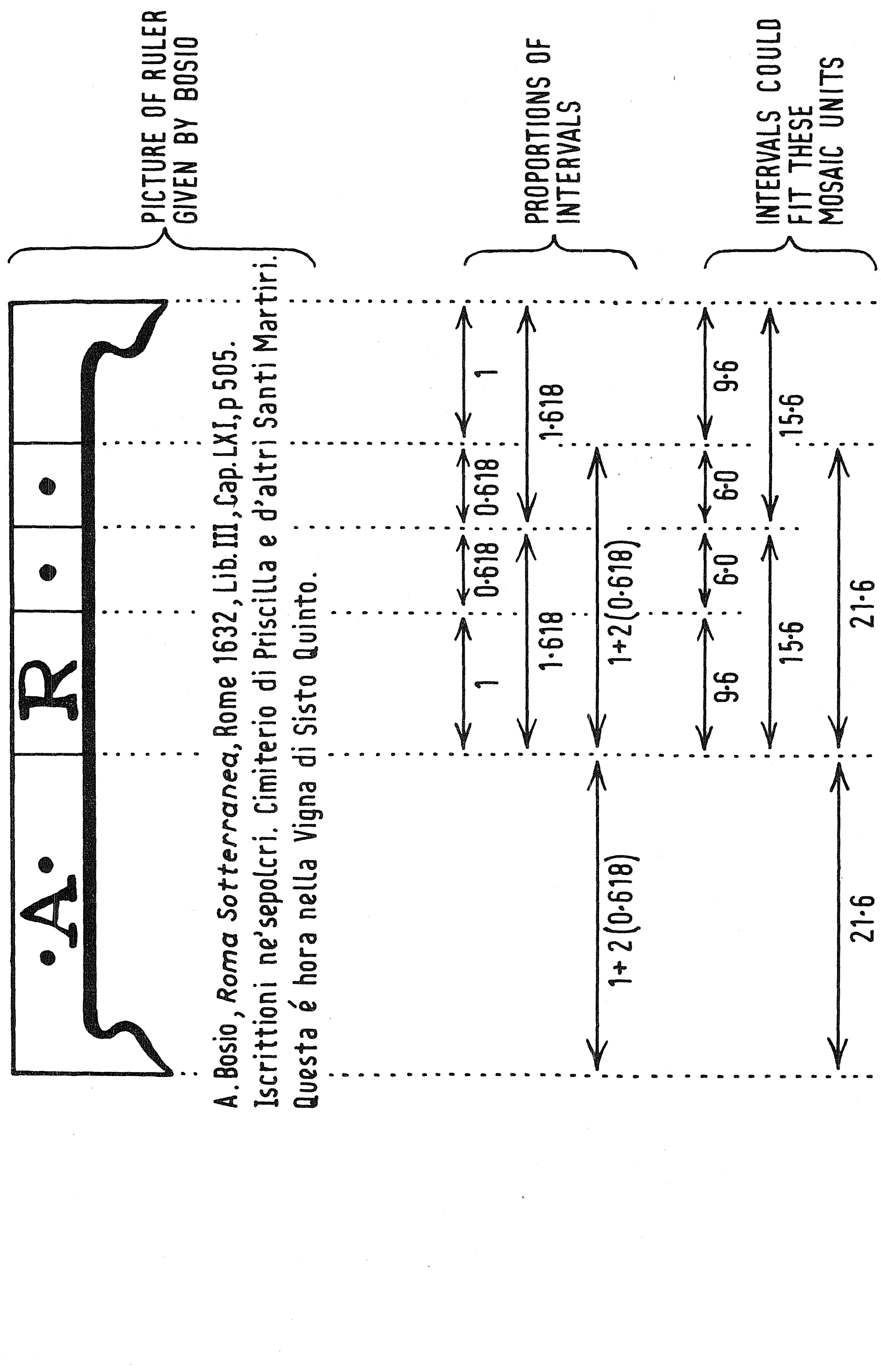
PICTURE OF RULER GIVEN BY BOSIO
·A· R · ·
A. Bosio, Roma Sotterranea, Rome 1632, Lib. III, Cap. LXI, p. 505.
Iscrittioni ne'sepolcri. Cimiterio di Priscilla e d'altri Santi Martiri.
Questa é hora nella Vigna di Sisto Quinto.
PROPORTIONS OF INTERVALS
1
0·618
0·618
1
1·618
1·618
1+2(0·618)
1+2(0·618)
INTERVALS COULD FIT THESE MOSAIC UNITS
9·6
6·0
6·0
9·6
15·6
15·6
21·6
21·6

Figure 15

The origin of the craft of mosaic is unknown to us, so it is hard to know what ideas and knowledge were current during the development of mosaic techniques. From the point of view of patterns sizes, I find the oldest (400 B. C.) known Greek mosaics exhibit the same dimensions as were apparently customarily used throughout the Greek and Roman world thereafter. The floor mosaic patterns at Til Barsib and Arslan Tash (c. 900 B.C.) in Syria, from the information available, appear to be also mosaic unit sizes. There is a distinct lack of primative mosaics in which we might see the mosaic unit phenomenon gradually developing. This could suggest that there are mosaics older than the earliest we at present know, but, if they were of the type, where pebbles are simply placed in earth, they have probably perished.

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[Continued on page 334.]

LINEAR RECURSION RELATIONS — LESSON EIGHT
ASYMPTOTIC RATIOS IN RECURSION RELATIONS

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One of the marvels associated with Fibonacci sequences is the fact that for all such sequences the limit of the ratio T_{n+1}/T_n as n approaches infinity is the Golden Section Ratio

$$\frac{1+\sqrt{5}}{2} \quad .$$

The following table shows the ratio of successive terms for the Fibonacci sequence 2, 5, 7, 12, 19, 31, ⋯ .

n	T_n	T_n/T_{n-1}
1	2	
2	5	
3	7	1.4000000
4	12	1.7142857
5	19	1.5833333
6	31	1.6315789
7	50	1.6129032
8	81	1.6200000
9	131	1.6172839
10	212	1.6183206
11	343	1.6179245
12	555	1.6180758
13	898	1.6180180
14	1453	1.6180400
15	2351	1.6180316
16	3804	1.6180348
17	6155	1.6180336
18	9959	1.6180341
19	16114	1.6180339

But is this indeed so remarkable? There are many other sequences which have limiting ratios and likewise some in which there is no limit. For example, in the Tribonacci Sequence: 1, 2, 4, 7, 13, ⋯ where the last three terms are added together to get the next term, successive ratios are as shown terms are added together to get the next term successive ratios are as shown in the following table.

n	T_n	T_n/T_{n-1}
1	1	
2	2	
3	4	
4	7	1.7500000
5	13	1.8571428
6	24	1.8461538
7	44	1.8333333
8	81	1.8409090
9	149	1.8395061
10	274	1.8389261
11	504	1.8394160
12	927	1.8392857
13	1705	1.8392664

A recursion relation: $T_{n+1} = 3T_n - 4T_{n-1}$ yields a sequence which does not have a limiting ratio. For example, if $F_1 = 5$, $T_2 = 9$, the ratios are as shown in the following table.

n	T_n	T_n/T_{n-1}
1	5	
2	9	
3	7	0.7777777
4	-15	-2.1428571
5	-73	4.8666666
6	-159	2.1780821
7	-185	1.1635220
8	81	-0.4378378
9	983	12.1358024
10	2625	2.6703967
11	3943	1.5020952
12	1329	0.3370530
13	-11785	-8.8675696

Clearly, several questions emerge:

1. Which sequences have a limiting ratio?
2. Which sequences do not have a limiting ratio?
3. On what does the limiting ratio depend?

These questions can be answered conveniently on the basis of expressing T_n in terms of the roots of the auxiliary equation

THE FIBONACCI SEQUENCE

Consider the sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, $\cdots$. Here,

$$F_n = \frac{r^n - s^n}{\sqrt{5}} ,$$

where

$$r = \frac{1 + \sqrt{5}}{2} = 1.61803 \cdots ,$$

and

$$s = \frac{1 - \sqrt{5}}{2} = -0.61803 \cdots .$$

The

$$\lim_{n\to\infty} F_n / F_{n-1} = \frac{r^n - s^n}{r^{n-1} - s^{n-1}} .$$

Dividing the terms of numerator and denominator by the $(n-1)^{st}$ power of r, this ratio takes the form

$$\lim_{n\to\infty} \frac{r - s(s/r)^{n-1}}{1 - (s/r)^{n-1}} .$$

Since the absolute value of s/r is less than 1, the limit of the $(n-1)^{st}$ power of this ratio as n goes to infinity is zero. Thus

$$\lim_{n\to\infty} F_n / F_{n-1} = r .$$

A similar analysis can be made for any Fibonacci sequence. We have found that for such a sequence,

$$T_n = Ar^n + Bs^n .$$

Therefore,

$$\lim_{n\to\infty} T_{n+1}/T_n = \frac{Ar^{n+1} + Bs^{n+1}}{Ar^n + Bs^n}$$

$$= \lim_{n\to\infty} \frac{r + (B/A)s(s/r)^n}{1 + (B/A)(s/r)^n} = r \;.$$

One thing we can learn from this analysis is that the root with larger absolute value, r, dominates the root with smaller absolute value, s.

REAL AND UNEQUAL ROOTS

Clearly, if

$$T_n = Ar^n + Bs^n + Ct^n \cdots ,$$

where the roots are real and unequal and r s t ··· then the limiting ratio of T_{n+1}/T_n will be r.

EQUAL AND REAL ROOTS

If some of the roots are equal, but there is another real root which has the largest absolute value, this latter root will dominate to give the limiting ratio in the sequence. If the equal roots have the largest absolute value, then (consider three equal roots, r).

$$T_n = (An^2 + Bn + C)r^n + Ds^n + Et^n \cdots \;.$$

Therefore $\lim_{n\to\infty} T_{n+1}/R_n$ will equal

$$\lim_{n\to\infty} \frac{\left\{A(n+1)^2 + B(n+1) + C\right\} r^{n+1} + Ds^{n+1} + Et^{n+1} \cdots}{An^2 + Bn + C\; r^n + Ds^n + Et^n \cdots}$$

$$\lim_{n\to\infty} \frac{\left\{(n+1)^2/n^2 + (B/A)(n+1)/n^2 + C/(An^2)\right\} r + (Ds/An^2)(s/r)^n \cdots}{1 + B/(An) + C/(An^2) + (D/An^2)(s/r)^n \cdots} = r$$

Thus the dominant real root again determines the limit of the sequence ratio.

COMPLEX ROOTS

For the type of linear recursion relation we are considering in which the coefficients are real numbers, the complex roots of the auxiliary equation occur in conjugate pairs. Let the portion of T_n dependent on these roots be given by

$$cr^n + c'(r')^n ,$$

where c and c' are complex conjugate coefficients. Now set:

$$c = Ce^{\lambda i} \quad \text{and} \quad c' = Ce^{-\lambda i}$$
$$r = Re^{\phi i} \quad \text{and} \quad r' = Re^{-\phi i}$$

where C and R are the absolute values of the complex quantities c and r, respectively. Then

$$\begin{aligned} cr^n + c'(r')^n &= CR^n e^{(\lambda + n\phi)i} + CR^n e^{-(\lambda+n\phi)i} \\ &= 2CR^n \cos(\lambda + n\phi) . \end{aligned}$$

If there is a real root with greater absolute value than R, this real root will dominate and the sequence ratio will converge. However, if R is greater than any of the real roots, it will dominate them. Only the cosine factor involving n will not converge either directly or in ratio. Thus a sequence in which there is a pair of complex roots whose absolute value is greater than the absolute value of any of the real roots will be a sequence without a limiting ratio.

A COROLLARY

Suppose we are seeking the roots of the cubic

$$x^3 - 7x^2 + 8x - 4 = 0 .$$

From one point of view this might be looked upon as the auxiliary equation of the recursion relation

$$T_{n+1} = 7T_n - 8T_{n-1} + 4T_{n-2} .$$

If we then calculate the terms of a sequence obeying this relation and find that their ratio approaches a limit with increasing n, this limiting ratio would correspond to the largest real root of the cubic. In the present instance, this ratio comes out to be 5.7245767.

PROBLEMS

1. Using the ratio of successive terms of a sequence, determine the largest real root of the equation: $x^3 - 12x^2 + 9x - 7 = 0$.

2. By analyzing the roots of the auxiliary equation, determine the limiting ratio of successive terms in the sequences obeying the recursion relation: $T_{n+1} = 8T_{n-1} + 3T_{n-2}$.

3. By analyzing the roots of the auxiliary equation, determine the limiting ratio of successive terms of sequences having the recursion relation: $T_{n+1} = -3T_n + T_{n-1} + 8T_{n-2} + 4T_{n-3}$.

4. If $R_n = 5(-1)^n$ and $S_n = F_n$, what is the limiting ratio of terms of the sequence $T_n = R_n + S_n$?

5. If

$$R_n = 2^n(n^2 + 3n + 5) \quad \text{and} \quad S_n = 3S_{n-1} + S_{n-2} ,$$

with $S_1 = 1$, $S_2 = 5$, find the limiting ratio of $T_n = R_n + S_n$.

6. By analyzing the auxiliary equation, show that the recursion relation

$$T_{n+1} = 3T_n - 7T_{n-1} + 10T_{n-2}$$

governs sequences which do not have a limiting ratio.

Solutions to problems may be found on page 324.

MAGIC SQUARES CONSISTING OF PRIMES IN A. P.

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Let a be the first prime in A. P. (not necessarily positive), d the common difference, s the last prime in A. P., n the number of primes in A. P., and let the residue r be the smallest positive integer such that $a \equiv r \pmod{d}$; if we keep a constant and increase d, we may speak of a search limit on d, designated by a_d; if we keep d constant and increase a and s, we may speak of a search limit on s, designated by s_n.

The standard magic square of order 3 with elements 1, 3, ⋯, 9 and center element c = 5, may be defined as

8	1	6
3	5	7
4	9	2

and the standard magic square of order 4 with elements 1, 2, ⋯, 16 and center square

8	5
9	12

whose sum is the magic constant C = 34, may be defined as

1	15	14	4
10	8	5	11
7	9	12	6
16	2	3	13

A magic square consisting of primes in A. P. is formed by letting an increasing sequence of 9 or 16 primes in A. P. occupy the locations corresponding to the elements 1, 2, ⋯, 9 or 1, 2, ⋯, 16, respectively, in the standard magic squares. To avoid ambiguity, let $|a| < s$.

The disposition of the prime factors and their powers in d to achieve maximum n was the topic for about 140 years and is reflected in four classical papers by Edward Waring (1734-1798) [15, p. 379], Peter Barlow (1776-1862) [1, p. 67], Moritz Cantor (1829-1920) [3], and Artemas Martin (1835-1918) [8]. In 1910, E. B. Escott (1868-1946) [5, p. 426 and 2, p. 221] found a string of 11 primes in A. P. yielding one case of almost 16 primes in A. P. (except for two composite elements)

-1061	1879	1669	-431
829	409	-13.17	1039
199	619	1249	-11
2089	-23.37	-641	1459

with C = 2056 .

This sequence is treated with loving care in [9, pp. 152-54] and [14]. The concept of magic squares consisting of primes in A. P. may be extended to more or less magic squares of reversible primes [4].

The modern period of this interesting subject starts in 1944 with a paper by Victor Thebault (1882-1960) [13]. In 1958, V. A. Golubev (Kouvshinovo, USSR) [10, p. 348 and 6, p. 120] found a string of 12 primes in A. P. yielding two cases of almost 16 primes in A. P. (except for two composite elements

23143	443563	413533	113233
293413	233353	143263	323443
203323	263383	353473	173293
23.59.349	53173	83203	$17^2.1327$

53173	23.59.349	443563	143263
323443	263383	173293	353473
233353	293413	$17^2.1327$	203323
503623	83203	133233	413533

with C = 993472 and C = 1113592 .

In 1963, V. N. Seredinskij (Moscow, USSR) [11, p. 121 and 12, p. 48] found a string of 14 primes in A.P., yielding four cases of almost 16 primes in A.P. (except for two composite elements).

-19.23.401	665603	605543	4943
365303	245183	65003	425363
185123	305243	485423	125063
725663	-149.773	-55117	545483

-149.773	725663	665603	65003
425363	305243	125063	485423
245183	365303	545483	185123
17.46219	-55117	4943	605543

with C = 1100852 and with C = 1341092,

-55117	17.46219	725663	125063
485423	365303	185123	545483
305243	425363	605543	245183
37.22859	4943	65003	665603

4943	37.22859	17.46219	185123
545483	425363	245183	605543
365303	485423	665603	305243
905843	65003	125063	725663

with C = 1581332 and C = 1821572 .

The magnitude of the last six C could be lowered essentially, when in 1966, the author found a string of 10 primes in A. P. yielding one case of almost 16 primes in A. P. (except for two composite elements) with C = 30824, and in 1967 [7], found a string of 12 primes in A. P. yielding the only known case (so far) of almost 16 primes in A. P. (except for one composite element) with C = 857548.

-9619	22721	20411	-2689
11171	6551	-379	13.17.61
4241	8861	15791	1931
25031	-7309	-4999	23.787

110437	304477	290617	152017
235177	207457	165877	249037
193597	221317	262897	179737
318337	124297	138157	13.61.349

with C = 30824 and with C = 857548 .

Time may be near to find the first entire sequence of 16 primes in A. P.

Even more fascinating is the magic square of order 3. One is tempted to ask: Given any c, is there always a magic square of 9 primes in A. P. belonging to this c? This question may once be answered in the positive. For c = 5 and 7, Golubev found

41	-43	17
-19	5	29
-7	53	-31

and

97	-113	37
-53	7	67
-23	127	-83

Nevertheless, the center element may not be unique, since we have also the magic square

2703607	-3604793	901207
-1802393	7	1802407
-901193	3604807	-2703593

discovered recently by Seredinskij. The author found three further magic squares with low c, namely

2089	-27701	6949
-13841	19	13879
-6911	27739	-20771

127	-83	67
-23	37	97
7	157	-53

1327	-1613	487
-773	67	907
-353	1747	-1193

With the increasing scarcity of primes, one may wonder if there exists a magic square of order 3 and of primes in A. P. whose smallest element is greater than, say, 8 million. In 1968, the author found 10 primes in A. P. starting with $a = 8081737$, and yielding the magic squares

8291947	8081737	8231887
8141797	8201857	8261917
8171827	8321977	8111767

and

8321977	8111767	8261917
8171827	8231887	8291947
8201857	8352007	8141797

Or one could ask for a magic square of order 3 and of primes in A. P. whose greatest element is greater than, say, 40 million. The author found recently 9 primes in A. P. starting with 2657, ending with 49011617, and yielding the magic square

42885497	2657	30633257
12254897	24507137	36759377
18381017	49011617	6128777

All we need for further research is a list of known results of at least 9 primes in A. P. with headings d, r, a, c, s, n, and s (in million). Such a list is published in the Appendix for the first time. Compiled from the newest discoveries around the globe, the author will be pleased to keep them up to date. Two additional tables are available from the author.*

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APPENDIX

MULTIPLES OF d = 210, YIELDING AT LEAST 9 PRIMES IN A. P.

(s_z for 8^{th} term)

d	r	a	c	z	n	s_z
210	47	-373	467	1307	9	0.5
	139	3499	4339	5179	9	
	149	10859	11699	12539	9	
	179	-241	599	1439	9	
	199	-11	829	2089	11	
			1039			
			1249			
420	67	-1613	67	1747	9	0.5
	193	-647	1033	2713	9	
	317	61637	63317	64997	9	
	379	52879	54559	56659	10	
			54979			
	11.37	56267	57947	59627	9	
630	137	279857	282377	284897	9	0.5
840	97	-2423	937	4297	9	0.5
	163	6043	9403	12763	9	
	181	201781	205141	208501	9	
	11.47	103837	107197	110557	9	
	11.71	10861	14221	17581	9	

1050 on next page.

d	r	a	c	z	n	s_z
1050	443	-2707	1493	5693	9	0.5
1260	11.73	2063	7103	12143	9	0.5
1470	859	363949	369829	375709	9	0.5
	11.97	101027	106907	112787	9	
1680	227	216947	223667	230387	9	0.5
	11.71	316621	323341	330061	9	
	1093	31333	38053	44773	9	
	1487	258527	265247	271967	9	
1890	11.37	45767	53327	60887	9	0.5
	487	15607	23167	30727	9	
	31.43	194113	201673	209233	9	
	1543	-4127	3433	10993	9	
2100	13.101	34913	43313	53813	10	0.5
			45413			
	1787	176087	184487	192887	9	
	19.109	102871	111271	119671	9	

[Continued from page 280.]

his works. Also, Fibonacci numbers with prime subscripts need not necessarily be primes (p. 83).

In conclusion, we have in this publication a very readable work that fills a much needed place in the literature. We now have an answer to the many requests for information on Leonard of Pisa which come to the Fibonacci Association.

Specific information regarding the book is as follows:

Publisher: Thomas Y. Crowell Company

Title: <u>Leonard of Pisa and the New Mathematics of the Middle Ages</u>

Authors: Joseph and Frances Gies

Illustrator: Enrico Arno

Number of pages, 128; cover, hard; price $3.95.

(Continued from p. 316.)

SOLUTIONS TO PROBLEMS

1. 11.2556550
2. The roots are 3, and

$$\frac{-3 \pm \sqrt{5}}{2} .$$

Limiting ratio is 3.

3. The roots are -2, -2, r and s. Limiting ratio is -2.
4. The roots of the combined recursion relation will be 1, r, s. Limiting ratio is r.
5. The roots of the combined recursion relation are $+2$, $+2$, $+2$,

$$\frac{3 \pm \sqrt{13}}{2} .$$

The limiting ratio is

$$\frac{3 + \sqrt{13}}{2} = 3.3027756 .$$

6. The roots of the auxiliary equation are 2,

$$\frac{1 \pm \sqrt{19}\, i}{2} .$$

The absolute value of the complex roots is greater than 2. Thus the sequences will not have a limiting ratio.

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-184 Proposed by Bruce W. King, Adirondack Community College, Glens Falls, New York.

Let the sequence $\{T_n\}$ satisfy $T_{n+2} = T_{n+1} + T_n$ with arbitrary initial conditions. Let

$$g(n) = T_n^2 T_{n+3}^2 + 4T_{n+1}^2 T_{n+2}^2 .$$

Show the following:

(i) $$g(n) = (T_{n+1}^2 + T_{n+2}^2)^2 .$$

(ii) If T_n is the Lucas number L_n, $g(n) = 25F_{2n+3}^2$.

(See Fibonacci Quarterly Problems H-101, October, 1968, and B-160, April, 1969.)

B-185 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that

$$L_{5n}/L_n = L_{2n}^2 - (-1)^n L_{2n} - 1 .$$

B-186 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that

$$L_{5n}/L_n = \left[L_{2n} - (-1)^n 5\right]^2 + (-1)^n 25F_n^2 .$$

(For n even, this result has been given by D. Jarden in the Fibonacci Quarterly, Vol. 5 (1967), p. 346.)

B-187 Proposed by Carl Gronemeijer, Saranac Lake, N. York

Find positive integers x and y, with x even, such that

$$(x^2 + y^2)(x^2 + x + y^2)(x^2 + \tfrac{3}{2}x + y^2) = 1{,}608{,}404 .$$

B-188 Proposed by A. G. Shannon, University of Papua and New Guinea, Boroko, Papua.

Two circles are related so that there is a trapezoid ABCD inscribed in one and circumscribed in the other. AB is the diameter of the larger circle which has center O, and AB is parallel to CD. θ is half of angle AOD. Prove that $\sin\theta = (-1 + \sqrt{5})/2$.

B-189 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $a_0 = 1$, $a_1 = 7$, and $a_{n+2} = a_{n+1}a_n$ for $n \geq 0$. Find the last digit (i.e., units digit) of a_{999}.

SOLUTIONS

GENERALIZATIONS OF SECOND-ORDER RECURRENCES

B-166 Suggested by David Zeitlin's solutions to B-148, B-149, and B-150

Let a and b be distinct numbers, $U_n = (a^n - b^n)/(a - b)$, and $V_n = a^n + b^n$. Establish generalizations of the formulas

(a) $$F_{(2^t n)} = F_n L_n L_{2n} \cdots L_{(2^{t-1}n)}$$

(b) $$L_{n+1}L_{n+3} + 4(-1)^{n+1} = 5F_nF_{n+4}$$

of B-148 and B-149 in which one deals with U_n and V_n instead of F_n and L_n.

Solution by C. B. A. Peck, State College, Pennsylvania.

(a) Since $U_nV_n = U_{2n}$, induction on j from 1 yields

$$U_{(2^t n)} = V_{(2^{t-1}n)} \cdots V_{(2^{t-j}n)} U_{(2^{t-j}n)} ,$$

which, for $j = t$, is the desired extension.

(b) $$V_{n+1}V_{n+3} + x = (a^{n+1} + b^{n+1})(a^{n+3} + b^{n+3}) + x$$

$$= a^{2n+4} + (ab)^{n+1}(b^2 + a^2) + b^{2n+4} + x ,$$

while

$$yU_nU_{n+4} = y(a^n - b^n)(a^{n+4} - b^{n+4})/(a - b)^2$$

$$= [a^{2n+4} - (ab)^n(b^4 + a^4) + b^{2n+4}]y/(a - b)^2 .$$

We can take, for instance, $y = (a - b)^2$ and

$$x = -(ab)^n(b^4 + a^4) - (ab)^{n+1}(b^2 + a^2) = -(ab)^n[b^4 + a^4 + ab(b^2 + a^2)]$$

$$= (ab)^n(a^2b^2 - U_5) .$$

Then our generalization is, for instance,

$$V_{n+1}V_{n+3} + (ab)^n[(ab)^2 - U_5] = (a - b)^2U_nU_{n+4} ,$$

which with $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$, simplifies to the Fibonacci case in (b).

Also solved by Wray G. Brady and David Zeitlin.

A LUCAS INEQUALITY

B-167 Proposed by A. G. Shannon, University of Papua and New Guinea, Boroko, Papua.

Let L_n be the n^{th} Lucas number defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$. For which values of n is

$$nL_{n+1} > (n + 1)L_n \quad ?$$

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

The inequality holds for $n = 1$ and $n = 3$. Let it hold for $n = k \geq 3$. Then

$$(k + 1)L_{k+2} = (k + 1)(L_{k+1} + L_k) > (k + 1)L_{k+1} + 2L_k > (k + 1)L_{k+1} + L_k + L_{k-1} = (k + 2)L_{k+1},$$

since $L_k > 0$ and $L_k > L_{k-1}$ for $k \geq 3$. This proves the inequality for $n \geq 3$ by mathematical induction; hence it holds for all positive integers except 2.

Also solved by Herta T. Freitag, Peter A. Lindstrom, C. B. A. Peck, Gerald Satlow, John Wessner, and the Proposer.

AN APPLICATION OF 1/7

B-168 Proposed by S. H. L. Kung, Jacksonville University, Jacksonville, Florida.

Using each of six of the nine positive digits $1, 2, \cdots, 9$ exactly once, form an integer z such that each of z, $2z$, $3z$, $4z$, $5z$, and $6z$ contains the same six digits once and once only.

Solution by Warren Cheves, Littleton, North Carolina.

The solution is $z = 142857$. This was obtained as follows:

Obviously, the first digit of z has to be 1. Otherwise, $6z$ would contain more than 6 digits.

Now consider the last digit of z. It cannot be a 1. It cannot be a 2, 4, 5, 6, or 8, because these numbers when multiplied by 5, 5, 4, 5, and 5, respectively, produce a last digit of 0. This leaves only 3, 7, and 9 as possible candidates for the last digit of z.

Consider

$1 \cdot 7 = 7$	$1 \cdot 3 = 3$	$1 \cdot 9 = 9$
$2 \cdot 7 = 14$	$2 \cdot 3 = 6$	$2 \cdot 9 = 18$
$3 \cdot 7 = 21$	$3 \cdot 3 = 9$	$3 \cdot 9 = 27$
$4 \cdot 7 = 28$	$4 \cdot 3 = 12$	$4 \cdot 9 = 36$
$5 \cdot 7 = 35$	$5 \cdot 3 = 15$	$5 \cdot 9 = 45$
$6 \cdot 7 = 42$	$6 \cdot 3 = 18$	$6 \cdot 9 = 54$

here, the multiples of both 3 and 9 have for their last digits 6 different numbers, none of which is the number 1. Hence, 7 must be the last digit of z. Furthermore, by looking at the last digits of the multiples of 7 (above), we see that the six digits of z must be 1, 2, 4, 5, 7, 8, with 1 being the first and 7 the last.

The order of these six digits was found mainly by trial and error. In other words, multiples of different combinations of the six digits were computed until certain eliminations could be made. (I did find one hint: the "8" could not appear immediately after the "1" or else 6z would contain more than 6 digits.) After my trial and error method, I found that z = 142857 fitted the requirements of B-168.

Also solved by Ed and Martha Clarke, Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, and the Proposer.

A SEQUENCE OF IDENTITIES

B-169 Proposed by C. C. Yalavigi, Government College, Mercara, India.

Prove the following identities:

(a) $$F_n^4 + F_{n-1}^4 + F_{n+1}^4 = 2(F_nF_{n-1} - F_{n+1})^2$$

(b) $$F_n^5 + F_{n-1}^5 - F_{n+1}^5 = 5F_nF_{n-1}F_{n+1}(F_nF_{n-1} - F_{n+1}^2) ,$$

where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Show that these are two cases of an infinite sequence of identities.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

The first identity should read

$$\text{(a)} \qquad F_n^4 + F_{n-1}^4 + F_{n+1}^4 = 2(F_{n+1}^2 - F_nF_{n-1})^2 .$$

This follows from

$$\begin{aligned} F_n^4 + F_{n-1}^4 + F_{n+1}^4 &= F_n^4 + F_{n-1}^4 + (F_n + F_{n-1})^4 \\ &= 2(F_n^4 + 2F_n^3F_{n-1} + 3F_n^2F_{n-1}^2 + 2F_nF_{n-1}^3 + F_{n-1}^4) \\ &= 2(F_n^2 + F_nF_{n-1} + F_{n-1}^2)^2 \\ &= 2(F_{n+1}^2 - F_nF_{n-1}) . \end{aligned}$$

Similarly, to prove (b), we have

$$\begin{aligned} F_{n+1}^5 - F_n^5 - F_{n-1}^5 &= (F_n + F_{n-1})^5 - F_n^5 - F_{n-1}^5 \\ &= 5F_nF_{n-1}(F_n^3 + 2F_n^2F_{n-1} + 2F_nF_{n-1}^2 + F_{n-1}^3) \\ &= 5F_nF_{n-1}(F_n + F_{n-1})(F_n^2 + F_nF_{n-1} + F_{n-1}^2) \\ &= 5F_nF_{n-1}F_{n+1}(F_{n+1}^2 - F_nF_{n-1}) . \end{aligned}$$

To get a general result, we recall that Cauchy (see P. Bachmann, Das Fermatproblem in seiner bisherigen Entwickelung, Berlin, 1919, p. 31) has proven that if p is a prime >3, then

$$\text{(1)} \qquad (x + y)^p - x^p - y^p = pxy(x + y)(x^2 + xy + y^2)f_p(x,y) ,$$

where $f_p(x,y)$ is a polynomial with integral coefficients. For $p \equiv 1 \pmod 3$ there is the stronger result,

$$\text{(2)} \qquad (x + y)^p - x^p - y^p = pxy(x + y)(x^2 + xy + y^2)g_p(x,y) ,$$

where $g_p(x,y)$ is a polynomial with integral coefficients.

Substituting $x = F_n$, $y = F_{n-1}$ in (1) or (2), we get identities of the required kind. In particular, for $p = 7$,

$$F_{n+1}^7 - F_n^7 - F_{n-1}^7 = 7F_nF_{n-1}F_{n+1}(F_{n+1}^2 - F_nF_{n-1}) .$$

For further results of this kind, see "Sums of Powers of Fibonacci and Lucas Numbers," by L. Carlitz and J. A. H. Hunter, Fibonacci Quarterly, December, 1969, p. 467.

Also solved by the Proposer.

A PERIODIC SEQUENCE

B170 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let the binomial coefficient $\binom{m}{r}$ be zero when $m \quad r$ and let

$$S_n = \sum_{j=0}^{\infty} (-1)^j \binom{n-j}{j} .$$

Show that $S_{n+2} - S_{n+1} + S_n = 0$, and hence $S_{n+3} = -S_n$ for $n = 0, 1, 2, \cdots$.

Solution by F. D. Parker, St. Lawrence University, Canton, New York.

If

$$S_n = \sum_{j=0}^{\infty} (-1)^j \binom{n-j}{j} ,$$

then

$$S_{n+2} - S_{n+1} + S_n = \sum_{j=0}^{\infty} (-1)^j \left\{ \binom{n-j+2}{j} - \binom{n-j+1}{j} + \binom{n-j}{j} \right\} .$$

But

$$\binom{n-j+2}{j} - \binom{n-j+1}{j} = \binom{n-j+1}{j-1} ,$$

so that

$$S_{n+2} - S_{n+1} + S_n = \sum_{j=0}^{\infty} (-1)^j \left\{ \binom{n+1-j}{j-1} + \binom{n-j}{j} \right\} .$$

Changing indices, we have

$$\sum_{j=0}^{\infty} (-1)^j \binom{n+1-j}{j-1} = \sum_{j=-1}^{\infty} (-1)^{j+1} \binom{n-j}{j} = \sum_{j=0}^{\infty} (-1)^{j+1} \binom{n-j}{j} ,$$

and therefore

$$S_{n+2} - S_{n+1} + S_n = \sum_{j=0}^{\infty} (-1)^j \left\{ -\binom{n-j}{j} + \binom{n-j}{j} \right\} = 0 .$$

Using this identity, we have

$$0 = S_{n+3} - S_{n+2} + S_{n+1} = S_{n+3} - (S_{n+1} - S_n) + S_{n+1} ,$$

and so $S_{n+3} = -S_n$.

Also solved by A. K. Gupta, C. B. A. Peck, John Wessner, David Zeitlin, and the Proposer.

Zeitlin noted the following:

The Chebyshev polynomial of the second kind, $U_n(x)$, satisfies

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x) ,$$

and is defined by

$$U_n(s) = \sum_{j=0}^{\infty} (-1)^j \binom{n-j}{j} (2x)^{n-2j} .$$

Thus,

$$S_n \equiv U_n(1/2),$$

i.e.,

$$S_{n+2} = S_{n+1} - S_n .$$

AVERAGING EIBONACCI AND PERIODIC SEQUENCES

B-171 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $\binom{m}{r} = 0$ for $m < r$, and let

$$T_n = \sum_{j=0}^{\infty} \binom{n-2j}{2j} .$$

Obtain a fourth-order homogeneous linear recurrence formula for T_n.

Solution by A. K. Gupta, University of Arizona, Tuscon, Arizona.

$$T_{n+3} = \sum_{j=0}^{\infty} \binom{n+3-2j}{2j}$$

$$= \binom{n+3}{0} + \sum_{j=1}^{\infty} \left[\binom{n+2-2j}{2j} + \binom{n+2-2j}{2j-1}\right] ,$$

since

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} ,$$

$$T_{n+3} = T_{n+2} + \sum_{j=1}^{\infty} \binom{n+2-2j}{2j-1}$$

$$= T_{n+2} + \sum_{j=1}^{\infty} \left[\binom{n+3-2j}{2j-1} - \binom{n+2-2j}{2j-2}\right]$$

$$= T_{n+2} - T_n + \sum_{j=1}^{\infty} \binom{n+3-2j}{2j-1}$$

$$= T_{n+2} - T_n + \sum_{j=1}^{\infty} \left[\binom{n+4-2j}{2j} - \binom{n+3-2j}{2j}\right]$$

$$= T_{n+2} - T_n + (T_{n+4} - T_{n+3}) \; .$$

Thus we get

$$T_{n+4} - 2T_{n+3} + T_{n+2} - T_n = 0 \, .$$

Also solved by C. B. A. Peck, John Wessner, David Zeitlin, and the Proposer.

[Continued from page 310.]

20. Servius, Aeneid, IV.
21. For example, titles of standard sizes, Vitruvius De Architectura V.
22. C. R. Lepsius, die Langenmasse der Alten, Berlin (1884).
23. A. Bosio, Roma Sotterranea, Rome (1632).
24. J. Greaves, A Discourse of the Romane foot and denarius, from whence the measures and weights used by the ancients may be deduced, London (1647).

25. Since 1960, this work has benefitted by grants from the worshipful Company of Goldsmiths, University College, London, and the Leverhulme Trust, and particularly from the great encouragement from Prof. Roger Warwick, Guy's Hospital Medical School. I am most grateful to Dr. George Ledin, Jr., for his valuable suggestions, and I thank him and the Fibonacci Association for inviting me to prepare this paper.

UPS AND DOWNS

Dorothy Fifield
Woodside, California

The
Patterns
Perfected
By nature's talent
Repeatedly favor, it seems,
A number sequence named for Senor Fibonacci.
It's one, two, three, five, eight, thirteen, twenty-one,
And up, until you come on down again.
Pine cones, sunflowers and pineapples; what, cacti, too?
No telling what this may lead to.
Accept the challenge.
Explore this
Numbers
Game!

A FIBONACCI RIDDLE

Dorothy Fifield
Woodside, California

I'm
Not dry,
Nor thirsty,
Yet drink a great deal.
But look at me now, I'm flying
Around and down, up and around, plop, plop, up and down.
The longer I spin, the warmer I get,
But I never get dizzy, nor do I tire.
Just when I feel I'm all afire, the trip is over.
Loving hands caress and fold me
Into a neat square
For drying.
What am
I?

(Ans: a towel)

PROGRAM OF THE SEVENTH ANNIVERSARY MEETING OF THE FIBONACCI ASSOCIATION

HARNEY SCIENCE CENTER — UNIVERSITY OF SAN FRANCISCO
Saturday March 14, 1970

INTRODUCTION TOPICS

9:00 Registration

9:20 Welcome, George Ledin, Jr., Institute of Chemical Biology, University of San Francisco

9:30 Matrices and Fibonacci Numbers, John Ivie, Student, University of California at Berkeley

10:45 Fibonacci Geometry, Brother Alfred Brousseau, St. Mary's College

11:00 Recess, Question and Answer Period

11:15 Problems from the Liber Abacci, Richard Grimm, University of Calif. at Davis, and Marguerite Dunton, Sacramento State College

12:00 Lunch

* * *

AFTERNOON SESSIONS ON ADVANCED TOPICS

Hard Analysis vs. Soft Analysis, Ivan Niven, University of Oregon, Eugene, Oregon

Local Distribution of Gaussian Primes, James H. Jordan, Washington State University, Pullman, Washington

Algorithms in Computing, Donald E. Knuth, Computer Sciences Department, Stanford University

Generalized Fibonacci Numbers and Pascal's Pyramid, Verner E. Hoggatt, Jr. San Jose State College

* * * *

Refreshments and a banquet after the meeting for those interested in an opportunity for social contact with other Fibonacci enthusiasts.

SUSTAINING MEMBERS

*H. L. Alder
V. V. Alderman
G. L. Alexanderson
R. H. Anglin
*Joseph Arkin
L. L. Badii
Don Baker
Col. R. S. Beard
Murray Berg
Leon Bernstein
*Marjorie Bicknell
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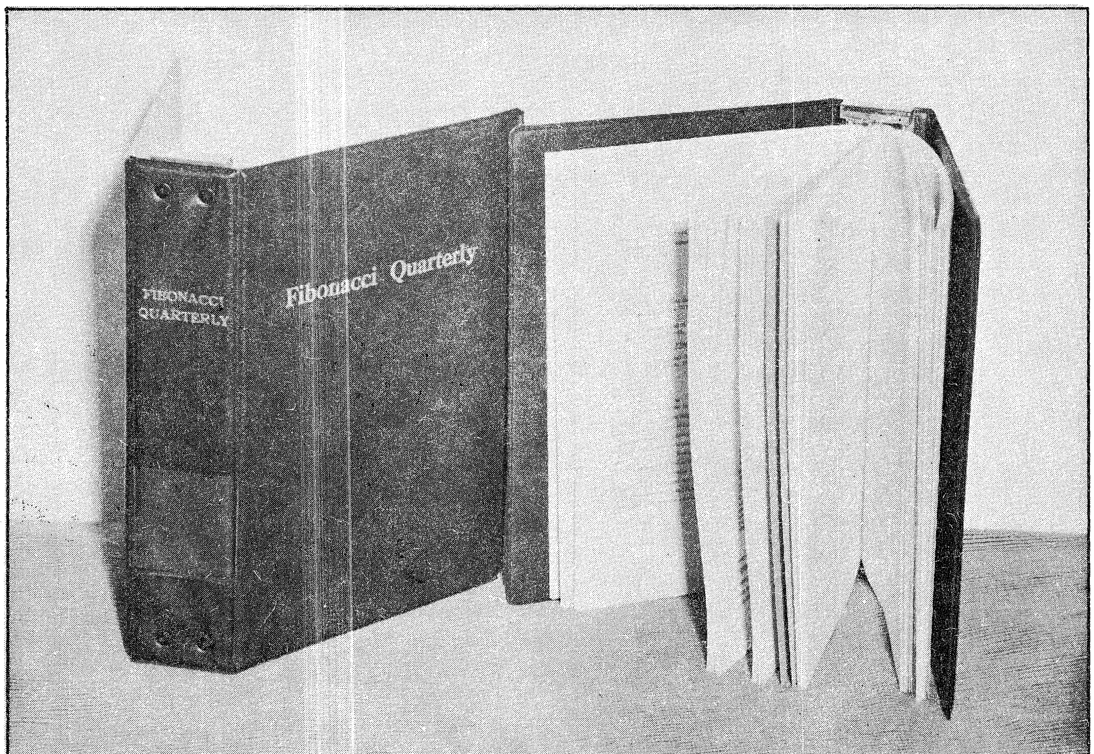
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