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PART I — ADVANCED
A Theorem Concerning Odd
Perfect Numbers
An Addition Algorithm for Greatest Common Divisor
On Determinants Whose Elements
Are Products of Recursive Sequences.

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# A THEOREM CONCERNING ODD PERFECT NUMBERS 

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Although the question of the existence of odd perfect numbers is still open, many necessary conditions for an odd integer to be perfect have been established. The oldest of these is due to Euler (see p. 19 in [1]), who proved that if $n$ is an odd perfect number then $n=p^{a_{k}}{ }^{2}$ where $p$ is a prime, $\mathrm{k}>1$, $(\mathrm{p}, \mathrm{k})=1$, and $\mathrm{p} \equiv \mathrm{a} \equiv 1(\bmod 4)$. In 1953, Touchard [6] proved that if n is odd and perfect, then either $\mathrm{n}=12 \mathrm{t}+1$ or $\mathrm{n}=36 \mathrm{t}+9$ 。 More recently the first author [5] has established upper and lower bounds for

$$
\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}
$$

where n is an odd perfect number. In fact, these bounds are improved ones over those established in [3] and [4]. For convenience, we give in Table 1 the results of [5] correct to five decimal places.

Table 1

|  | Lower Bound | Upper Bound | Range |
| :---: | :---: | :---: | :---: |
| (A) | .64412 | .67841 | .03429 |
| (B) | .65696 | .69315 | .03619 |
| (C) | .59595 | .67377 | .07782 |
| (D) | .59993 | .66172 | .06179 |

Our objective in the present paper is to improve (some of) the results of [5]. Our bounds for

$$
\sum_{n \ln } \frac{1}{p}
$$

$\mathrm{p} \mid \mathrm{n}$
are given in Theorem 1 while the five decimal place approximations appear in Table 2. In what follows, $n$ denotes an odd perfect number, and $p$ denotes a prime. The notation

$$
\sum_{p=5}^{11} \frac{1}{p}
$$

for example, will be used to represent the sum

$$
\frac{1}{5}+\frac{1}{7}+\frac{1}{11}
$$

Table 2

|  | Lower Bound | Upper Bound | Range |
| :---: | :---: | :---: | :---: |
| (A) | .64738 | .67804 | .03066 |
| (B) | .66745 | .69315 | .02570 |
| (C) | .59606 | .67377 | .07771 |
| (D) | .60383 | .65731 | .05348 |

Theorem 1. If n is an odd perfect number, then
(A) if $n=12 t+1$ and $5 \mid n$,

$$
\sum_{p=5}^{19} \frac{1}{p}+\frac{\log \left\{2 \prod_{p=5}^{19}(p-1) / p\right\}}{23 \log (23 / 22)}<\sum_{p \mid n} \frac{1}{p}<\frac{1}{5}+\log (50 / 31):
$$

(B) if $\mathrm{n}=12 \mathrm{t}+1$ and $5 / \mathrm{n}$,

$$
\sum_{p=7}^{59} \frac{1}{p}+\frac{\log \left\{2 \prod_{p=7}^{59}(p-1) / p\right\}}{61 \log (61 / 60)}<\sum_{p \mid n} \frac{1}{p}<\log 2
$$

(C) if $\mathrm{n}=36 \mathrm{t}+9$ and $5 \mid \mathrm{n}$,

$$
\frac{1}{3}+\frac{1}{5}+\frac{1}{17}+\frac{\log (256 / 255)}{257 \log (257 / 256)}<\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}<\frac{1}{3}+\frac{1}{5}+\frac{1}{13}+\log (65 / 61)
$$

(D) if $\mathrm{n}=36 \mathrm{t}+9$ and $5 / \mathrm{n}$,
$\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{\log (80 / 77)}{13 \log (13 / 12)}<\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}<\frac{1}{3}+\frac{1}{13}+\frac{1}{17}+\log (37349 / 30941)$.

The upper bounds in (B) and (C) are due to the first author [5]. The rest of the theorem is new.

## 2. THE UPPER BOUNDS

In this section, we shall establish the upper bounds for

$$
\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}
$$

given in (A) and (D) of Theorem 1. Our argument parallels that in [5].
According to Euler's theorem, we can write

$$
\mathrm{n}=\mathrm{p}_{0}^{\mathrm{a}_{0}} \mathrm{p}_{1}^{\mathrm{a}_{1}} \mathrm{p}_{2}^{\mathrm{a}_{2}} \ldots \mathrm{p}_{\mathrm{k}}^{\mathrm{a}_{\mathrm{k}}}
$$

where $\mathrm{p}_{0} \equiv \mathrm{a}_{0} \equiv 1(\bmod 4)$ and $\mathrm{a}_{\mathrm{j}} \equiv 0(\bmod 2)$ for $1 \leqq \mathrm{j} \leqq \mathrm{k}$ 。 We assume that $\mathrm{p}_{1}<\mathrm{p}_{2}<\ldots<\mathrm{p}_{\mathrm{k}}$. Since n is perfect, we have immediately
(2.0) $2=\prod_{j=0}^{k} 1-1 / p_{j}^{a} j^{+1} \prod_{j=0}^{k}\left(1-1 / p_{j}\right)^{-1}$,
so that

$$
\begin{align*}
\log 2=\sum_{j=0}^{k} \frac{1}{p_{j}} & +\sum_{j=1}^{k} \sum_{i=1}^{\infty} 1 /(i+1) p_{j}^{i+1}-1 / i p_{j}^{\left(a_{j}+1\right) i}  \tag{2.1}\\
& +1 / 2 p_{0}^{2}-1 / p_{0}^{a_{0}+1}+\sum_{i=2}^{\infty} 1 /(i+1) p_{0}^{i+1}-1 / i p_{0}^{\left(a_{0}+1\right) i}
\end{align*}
$$

Remark 1. Since $a_{j} \geqq 2$ for $1 \leqq j \leqq k$ each term is positive in the second summation of (2.1).

Remark 2. Since $\mathrm{a}_{0} \geqq 1$ and $\mathrm{i} \geqq 2$, each term is positive in the last summation of (2.1).

Remark 3. Since $\mathrm{a}_{0} \geqq 1$, we have

$$
1 / 2 p_{0}^{2}-1 / p_{0}^{a_{0}+1} \geqq-1 / 2 p_{0}^{2}
$$

Remark 4. Since $a_{0}$ is odd $\left(p_{0}+1\right) \mid \sigma\left(p_{0}^{a_{0}}\right)$, and since $n=\sigma(n) / 2$, it follows that $\left(p_{0}+1\right) / 2 \mid \mathrm{n}$ and a fortiori that n is divisible by a prime $\mathrm{p}_{\mathrm{S}} \leqq$ $\left(p_{0}+1\right) / 2$.

Remark 5. If $\mathrm{p}_{\mathrm{s}}$ is the prime mentioned in Remark 4 and $\mathrm{p}_{0}>5$, then

$$
\mathrm{W}=1 / 2 \mathrm{p}_{\mathrm{s}}^{2}-1 / \mathrm{p}_{\mathrm{s}}^{\mathrm{a}_{\mathrm{s}}+1}+1 / 2 \mathrm{p}_{0}^{2}=1 / \mathrm{p}_{0}^{\mathrm{a}_{0}+1}>0 .
$$

For since $3 \leqq p_{S} \leqq\left(p_{0}+1\right) / 2, a_{S} \geqq 2, a_{0} \geqq 1$, we have

$$
\mathrm{W} \geqq 1 / 2 \mathrm{p}_{\mathrm{S}}^{2}-1 / 3 \mathrm{p}_{\mathrm{S}}^{2}-1 / 2 \mathrm{p}_{0}^{2} \geqq 2 / 3\left(\mathrm{p}_{0}+1\right)^{2}-1 / 2 \mathrm{p}_{0}^{2}>0
$$

We consider first the case $n=12 t+1$ and $5 \mid n$. Since $3 / \mathrm{n}$, we see from Remark 4 that $p_{0} \neq 5$. Therefore, $p_{1}=5$.

If $\left(p_{0}+1\right) / 2 \neq 5^{m}$, then we can assume that the $p_{S}$ of Remark 4 is not 5. Since $a_{1} \geqq 2$, it follows from (2.1) and Remarks 1, 2, 5 that

$$
\begin{aligned}
\log 2 & >\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}+\sum_{\mathrm{i}=1}^{\infty} 1 /(\mathrm{i}+1) 5^{\mathrm{i}+1}-1 / \mathrm{i} 5^{3 \mathrm{i}} \\
& =\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}-\frac{1}{5}-\log (1-1 / 5)+\log \left(1-1 / 5^{3}\right) .
\end{aligned}
$$

Therefore,

$$
\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}<\frac{1}{5}+\log (50 / 31)
$$

Since the smallest prime such that $(p+1) / 2=5^{m}$ is $p=1249=2 \cdot 5^{4}$ -1 , we see that if $\left(p_{0}+1\right) / 2=5^{m}$, then $p_{0} \geqq 1249$, so that $-1 / 2 p_{0}^{2} \geqq$ $-1 / 2(1249)^{2}$. Also, in this case, it follows from Remark 4 that $a_{1} \geqq 4$. From (2.1) and Remarks 1, 2, 3, we have

$$
\log 2>\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}-\frac{1}{5}-\log (1-1 / 5)+\log \left(1-1 / 5^{5}\right)-1 / 2(1249)^{2}
$$

Therefore,

$$
\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}<\frac{1}{5}+1 / 2(1249)^{2}+\log (1250 / 781)<\frac{1}{5}+\log (50 / 31)
$$

This completes the discussion of the upper bound for this case. We remark that the upper bound established in [5] for (A) exceeds ours by $1 / 2738$.

Turning to the case $n=36 t+9$ and $5 / \mathrm{f}$ n we have $\mathrm{p}_{1}=3$. We consider four mutually exclusive and exhaustive possibilities.

First, suppose that $\mathrm{a}_{1}=2$ and $\mathrm{p}_{0}=17$. Since $\sigma\left(3^{2}\right)=13$ and $\mathrm{n}=$ $\sigma(\mathrm{n}) / 2$, we see that $13 \mid \mathrm{n}$. Let $13=\mathrm{p}_{\mathrm{r}}$. If $\mathrm{a}_{\mathrm{r}}=2$ then since $\sigma\left(13^{2}\right)=183$ and since $p_{0}+1=18$, it would follow from Remark 4 that $3^{3} \mid n$. Since this is impossible, we conclude that $a_{r} \geqq 4$. Since $a_{0} \geqq 1$, it follows from (2.1) and Remark 1 that

$$
\begin{aligned}
\log 2>\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}} & -\frac{1}{3}-\log (1-1 / 3)+\log \left(1-1 / 3^{3}\right)-\frac{1}{13}-\log (1-1 / 13) \\
& +\log \left(1-1 / 13^{5}\right)-\frac{1}{17}-\log (1-1 / 17)+\log \left(1-17^{2}\right)
\end{aligned}
$$

Therefore,

$$
\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}<\frac{1}{3}+\frac{1}{13}+\frac{1}{17}+\log (37349 / 30941)
$$

Second, suppose that $\mathrm{a}_{1}=2$ and $\mathrm{p}_{0}=13$. Then the $\mathrm{p}_{\mathrm{S}}$ of Remark 4 is 7, and it follows from (2.1) and Remark 1 that
$\log 2>\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}-\frac{1}{3}-\log (1-1 / 3)+\log \left(1-1 / 3^{3}\right)-\frac{1}{7}-\log (1-1 / 7)$

$$
+\log \left(1-1 / 7^{3}\right)-\frac{1}{13}-\log (1-1 / 13)+\log \left(1-1 / 13^{2}\right)
$$

Therefore,
$\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}<\frac{1}{3}+\frac{1}{7}+\frac{1}{13}+\log (21 / 19)<\frac{1}{3} \quad \frac{1}{13} \quad \frac{1}{17}+\log (37349 / 30941)$.

Next, suppose that $a_{1}=2$ and $p_{0}>17$. As before, we have $13 \mid n$, while $p_{0} \geqq 37$. For if $p_{0}=29$, it would follow from Remark 4 that $5 \mid n$ which is impossible. From (2.1) and Remarks 1, 2, 3, we have

$$
\begin{aligned}
\log 2>\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}} & -\frac{1}{3}-\log (1-1 / 3)+\log \left(1-1 / 3^{3}\right)-\frac{1}{13} \\
& -\log (1-1 / 13)+\log \left(1-1 / 13^{3}\right)-1 / 2(37)^{2}
\end{aligned}
$$

Therefore,

$$
\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}<\frac{1}{3}+\frac{1}{13}+\frac{1}{2738}+\log (78 / 61)<\frac{1}{3}+\frac{1}{13}+\frac{1}{17}+\log (37349 / 30941)
$$

Finally, suppose that $a_{1} \geqq 4$. Since $p_{0} \geqq 13$, we have $-1 / 2 p_{0}^{2} \geqq$ $-1 / 338$. From (2.1) and Remarks 1, 2, 3, it follows that

$$
\log 2>\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}-\frac{1}{3}-\log (1-1 / 3)+\log \left(1-1 / 3^{5}\right)-\frac{1}{338} .
$$

Therefore,

$$
\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}<\frac{1}{3}+\frac{1}{338}+\log (162 / 121)<\frac{1}{3}+\frac{1}{13}+\frac{1}{17}+\log (37349 / 30941)
$$

This completes the discussion of the upper bound for this case.

## 3. THE LOWER BOUNDS

In this section, we change our notation and write simply

$$
\mathrm{n}=\mathrm{p}_{1}^{\mathrm{a}_{1}} \mathrm{p}_{2}^{\mathrm{a}_{2}} \ldots \mathrm{p}_{\mathrm{k}}^{\mathrm{a}_{\mathrm{k}}}
$$

where $p_{1}<p_{2}<\ldots<p_{k^{*}}$ We first establish two lemmas.
Lemma 1. If

$$
\mathrm{n}=\mathrm{p}_{1}^{\mathrm{a}} 1 \mathrm{p}_{2}^{\mathrm{a}_{2}} \cdots \mathrm{p}_{\mathrm{k}}^{\mathrm{a}_{\mathrm{k}}}
$$

is an odd perfect number and $q$ is a prime such that $p_{N}<q \leqq p_{N+1}$, then

$$
\log 2<\log \left\{\prod_{j=1}^{N} p_{j} /\left(p_{j}-1\right)\right\}+q \log \frac{q}{q-1}\left(\sum_{p \mid n} \frac{1}{p}-\sum_{j=1}^{N} \frac{1}{p_{j}}\right)
$$

Proof. From (2.0), it follows that

$$
2<\prod_{\mathrm{p} \mid \mathrm{n}}^{\Pi(1-1 / \mathrm{p})^{-1}}
$$

Taking logarithms, we have

$$
\begin{aligned}
& \log 2<\log \prod_{j=1}^{N} p_{j} /\left(p_{j}-1\right)+\sum_{j=N+1}^{k} \sum_{i=1}^{\infty} 1 / i p_{j}^{i} \\
& \leqq \log p_{j=1}^{N} /\left(p_{j}-1+\sum_{j=N+1}^{k} \sum_{i=1}^{\infty} 1 /\left(i p_{j} q^{i-1}\right)\right. \\
& =\log \prod_{j=1}^{N} p_{j} /\left(p_{j}-1\right)+\sum_{j=N+1}^{k} \frac{1}{p_{j}} \sum_{i=1}^{\infty} q / i q^{i} \\
& =\log \prod_{j=1}^{N} p_{j} /\left(p_{j}-1\right)+q \log \frac{q}{q-1}\left(\sum_{p \mid n} \frac{1}{p}-\sum_{j=1}^{N} \frac{1}{p_{j}}\right)
\end{aligned}
$$

The necessary modifications in both the statement and proof of this lemma in case $q \leqq p_{1}$ or $p_{k}<q$ are obvious and are therefore omitted. Lemma 2. The function $f(x)=x \log x /(x-1)$ is monotonic decreasing on the interval $[2, \infty)$.

Proof. We easily verify that

$$
\mathrm{f}^{\mathrm{l}}(\mathrm{x})=\log \left(1+\frac{1}{\mathrm{x}-1}\right)-\frac{1}{\mathrm{x}-1}
$$

Since $\log (1+z)<z$ if $0<z \leqq 1$, we see immediately that $f^{\prime}(x)<0$ if $x \geqq 2$.

We are now prepared to prove the lower bounds for

$$
\sum \frac{1}{\mathrm{p}}
$$

$$
\mathrm{p} \mid \mathrm{n}
$$

stated in Theorem 1. We shall defer the proof of (C) until last since it differs in spirit from the others.

From Lemma 1, we have

$$
\begin{equation*}
\sum_{p \mid n} \frac{1}{p}>\sum_{j=1}^{N} \frac{1}{p_{j}}+\frac{\log \left\{2 \prod_{j=1}^{N}\left(p_{j}-1\right) / p_{j}\right\}}{q \log \{q /(q-1)\}} \tag{3.0}
\end{equation*}
$$

while from Lemma 2 it follows easily that if $s$ is a prime such that $s<q$ then

$$
\begin{equation*}
\frac{1}{s}+\frac{\log \{(s-1) / s\}}{q \log \{q /(q-1)\}}<0 \tag{3.1}
\end{equation*}
$$

If $n=12 t+1$ and $5 \ln$, then $p_{1}=5$. If $r$ is the greatest prime less than $q$ then it follows from $(3.0)$ and (3.1) that

$$
\begin{equation*}
\sum_{p \mid n} \frac{1}{p}>\sum_{p=5}^{r} \frac{1}{p}+\frac{\log \left\{2 \prod_{p=5}^{r}(p-1) / p\right\}}{q \log \{q /(q-1)\}} \tag{3.2}
\end{equation*}
$$

An hour's work on a desk calculator shows that the right-hand member of (3.2) is maximal for $q=23, r=19$. This completes the proof for this case. We remark that the lower bound for (A) established in [5] is (3.2) with $\mathrm{q}=11, \mathrm{r}=7$ 。

If $n=12 t+1$ and $5 /\left(n\right.$, then $p_{1} \geqq 7$. With $r$ defined as before, it follows from (3.0) and (3.1) that

$$
\begin{equation*}
\sum_{p \not n} \frac{1}{p}>\sum_{p=7}^{r} \frac{1}{p}+\frac{\log \left\{2 \prod_{p=7}^{r}(p-1) / p\right\}}{q \log \{q /(q-1)\}} \tag{3.3}
\end{equation*}
$$

Some rather tedious calculations verify that the right-hand member of (3.3) is maximal for $q=61, r=59$. The lower bound for (B) established in [5] is (3.3) with $q=11, r=7$.

If $\mathrm{n}=36 \mathrm{t}+9$ and $5 / \mathrm{n}$, then $\mathrm{p}_{1}=3$ and $\mathrm{p}_{2} \geqq 7$. With r defined as before, we have from (3.0) and (3.1),

$$
\begin{equation*}
\sum_{p \mid n} \frac{1}{p}>\sum_{p=3}^{r} \frac{1}{p}+\frac{\log \left\{2{\underset{p}{\prod}}_{r_{*}^{*}}^{r^{*}}(p-1) / p\right\}}{q \log \{q /(q-1)\}} \tag{3.4}
\end{equation*}
$$

where the asterisk indicates that the prime 5 is to be omitted from consideration. A few minutes of calculation verifies that the right-hand member of (3.4) is maximal for $q=13, r=11$. The lower bound for (D) established in [5] is (3.4) with $q=7, r=3$.

Now suppose that $n=36 t+9$ and $5 \mid n$. Then $7 / / n$ by a result of Kuhnel [2]. We consider three mutually exclusive and exhaustive possibilities.

If either 11 or 13 divides $n$, then

$$
\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}>\frac{1}{3}+\frac{1}{5}+\frac{1}{13}>\frac{1}{3}+\frac{1}{5}+\frac{1}{17}+\frac{\log (256 / 255)}{257 \log (257 / 256)}
$$

If neither 11 nor 13 divides $n$ but $17 \mid n$, then $p_{3}=17$ and either (i) $\mathrm{p}_{4}<251$, or (ii) $\mathrm{p}_{4}>251$. In case (i), we have

$$
\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}>\frac{1}{3}+\frac{1}{5}+\frac{1}{17}+\frac{1}{251}>\frac{1}{3}+\frac{1}{5}+\frac{1}{17}+\frac{\log (256 / 255)}{257 \log (257 / 256)}
$$

In case (ii), if we take $q=257$ in (3.0), we have

$$
\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}>\frac{1}{3}+\frac{1}{5}+\frac{1}{17}+\frac{\log (256 / 255)}{257 \log (257 / 256)}
$$

[Continued on p. 374.]

# AN ADDITION ALGORITHM FOR GREATEST COMMON DIVISOR 

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\begin{abstract}
An elementary algorithm is presented for finding the greatest common divisor of two numbers. It is trivial to programme and fast, even for large numbers. Only addition is used, and the only storage space needed is enough to hold the two numbers.

About three years ago, I discovered an algorithm which K. Y. Choong, C. R. Rathbone and I used to obtain the first 20,000 partial quotients of the continued fraction of $\pi$. I here show how an adaptation of the algorithm may be used to find the greatest common divisor (g.c.d.) of any two positive integers. The complete process when the numbers are 1168 and 2847 is:


I will now describe the general process. Let $P, Q$ be the two given positive integers, and suppose that k is the number of digits in the larger of $P$ and $Q$. From a finite sequence of numbers, each of not more than $k$ digits, according to the rules:
(a) The first two numbers are $10^{k}-P$ and $Q$. Of these $10^{k}-P$ is an $N$ number and $Q$ is a $Z$ number.
(b) At each subsequent stage, the next number is the last N number plus the last $Z$ number. Carries beyond k digits are ignored. If there is no such carry, this next number is an N number; otherwise, it is a $Z$ number.
(c) Stop when the next number would be zero.

Than the last $Z$ number is the g.c.d. of $P$ and $Q$.
We start with $10^{\mathrm{k}}-\mathrm{P}$ and it might be argued that this requires a subtraction. However, we can obtain $10^{k}-P$ by applying the transformation $0 \rightarrow 9,1 \rightarrow 8, \cdots, 9 \rightarrow 0$ to the digits of $P$, and then adding 1 to $P$. Hence, I am justified in saying that the algorithm only uses addition.

It will be noticed that, as we move down the sequence, the $N$ numbers begin with more and more nines, while the $Z$ numbers begin with more and more zeros. Hence the designations N and Z . It is not necessary to evaluate the next number in order to determine whether it is an $N$ or a $Z$. Suppose the last N and Z numbers, respectively, are


Then we look for the largest integer $j$ such that the sum $n_{j}+z_{j}$ of the $j^{\text {th }}$ digits is not 9. If there is no such j then the $\mathrm{g} . \mathrm{c} . \mathrm{d}$. is 1 . If there is such a $j$, and $n_{j}+z_{j}<9$, we will get no carry and so, by the definition in rule (b), the next number will be an N number. Moreover, the addition is worked out step-by-step from the right, and so we can over-write the digits of the last N number, step-by-step, with the digits of the new N number. Similarly, if $n_{j}+n_{k}>9$, the next number is written over the last $Z$ number. Hence, the space for the storage of the $2 k$ digits of $P$ and $Q$ is sufficient for the complete calculation. Once we have $n_{k}=p$ and $z_{k}=0$ we can ignore these digits, and similarly for $n_{k-1}, z_{k-1}$, etc. Thus the amount of work required in the additions steadily diminishes, and this is indicated by the dotted line in the example. The simplest flow diagram is also shown.

I will now prove that the algorithm does produce the g.c.d. At each stage, we have $\mathrm{N}=10^{\mathrm{k}}-\mathrm{p}$ with $0<\mathrm{p}<10^{\mathrm{k}}$ and $\mathrm{Z}=\mathrm{q}$ with $0<\mathrm{q}<10^{\mathrm{k}}$ and we want the g.c.d. ( $p, q$ ). By rule (a), this is certainly the case initially.

Case 1. Suppose $N+Z$ would give no carry; that is, $10^{k}-p+q \leq 10^{k}$ - 1 or $q<p$. Then the next number will be $10^{k}-(p-q)$. It will be an $N$ number bigger than $N$ but less than $10^{k}$. Since g.c.d. $(p-q, q)=$ g.c.d. ( $p, q$ ) the next stage will be of the correct form.

Case 2. Suppose $N+Z$ would give a carry; that is, $10^{k}-p+q \geq 10^{k}$ or $q \geq$ p. This carry is ignored, so the next number will be $\left(10^{k}-p\right)+(q)$ $-10^{\mathrm{k}}=\mathrm{q}-\mathrm{p}$, and $0 \leq \mathrm{q}-\mathrm{p}<\mathrm{q}$. Again the next stage will be of the correct form. Since the size of $Z$ foes down at each stage, we will reach a stage where the next $Z$ would be zero; then $p=q$ so that the g.c.d. $(p, q)=q$ in Z.

I have described all this in terms of arithmetic to the base 10. Clearly, the algorithm works with any base, and in particular in binary. Sometimes it is convenient to work to the base 10 but with several consecutive digits of the numbers in a computer word. With alittle more programming effort, one can speed the process up as follows.

Let $r$ and $s$ be the largest integers such that $n_{r} \neq 9$ and $z_{s} \neq 0$ respectively, and suppose that $\mathrm{r}>\mathrm{s}$. Then, instead of replacing N by $\mathrm{N}+$ $Z$, it saves work to replace $N$ by $N+10^{r-s-1} Z$. For this operation, we again only need addition with the appropriate shift. In most cases, we are in fact able to improve this to replacing N by $\mathrm{N}+10^{\mathrm{r}-\mathrm{S}} \mathrm{Z}$, it is not difficult to distinguish the exceptional cases. As one would expect, the corresponding situation obtains if $r<s$.


Flow Diagram for g. c. d.

# ON DETERMINANTS WHOSE ELEMENTS ARE PRODUCTS OF RECURSIVE SEQUENCES 

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## 1. INTRODUCTION

Let $W_{0}, W_{1}, p \neq 0$, and $q \neq 0$ be arbitrary real numbers, and define
(1.1) $\quad W_{n+2}=p W_{n+1}-q W_{n}, \quad p^{2}-4 q \neq 0, \quad(n=0,1, \cdots)$,

$$
\begin{equation*}
\mathrm{U}_{\mathrm{n}}=\left(\mathrm{A}^{\mathrm{n}}-\mathrm{B}^{\mathrm{n}}\right) /(\mathrm{A}-\mathrm{B}) \quad(\mathrm{n}=0,1, \cdots) \tag{1.2}
\end{equation*}
$$

$$
\begin{equation*}
V_{n}=A^{n}+B^{n}, \quad V_{-n}=V_{n} / q^{n} \tag{1.3}
\end{equation*}
$$

$$
(\mathrm{n}=0,1, \cdots)
$$

where $A \neq B$ are roots of $y^{2}-p y+q=0$. Carlitz $[1, p .132$ (6)], using a well-known result for linear transformations of a quadratic form, has given a closed form for the class of determinants

$$
\begin{equation*}
\left|W_{n+r+s}^{k}\right| \quad(r, s=0,1, \cdots, k) \tag{1.4}
\end{equation*}
$$

As a first generalization of (1.4), we will show that for $m=1,2, \cdots$, and $n_{0}=0,1, \cdots$,

$$
\begin{align*}
& \left|W_{m(n+r+s)+n_{0}}^{k}\right| \quad(r, s=0,1, \cdots, k)  \tag{1.5}\\
& =(-1)^{(\mathrm{k}+1)(\mathrm{k} / 2)} \cdot \mathrm{q}^{\left(\mathrm{mn}+\mathrm{n}_{0}\right)(\mathrm{k}+1)(\mathrm{k} / 2)+(\mathrm{mk} / 3)\left(\mathrm{k}^{2}-1\right)} \cdot \prod_{\mathrm{j}=0}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{j}} \\
& \text { k } \\
& \text { - }\left(\mathrm{W}_{1}^{2}-\mathrm{pW}_{0} \mathrm{~W}_{1}+\mathrm{qW}_{0}^{2}\right)^{(\mathrm{k}+1) \mathrm{k} / 2} \cdot \Pi \mathrm{U}_{\mathrm{mi}}^{2(\mathrm{k}+1-\mathrm{i})} . \\
& i=1
\end{align*}
$$

For $m=1$ and $n_{0}=0$, Eq. (1.5) gives the main result (1.4) of [1]. As in [1] , our proof of (1.5) will require the following known result for quadratic forms (e. g., see [2, pp. 127-128]):

Lemma 1. Let a quadratic form

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i j} x_{i} x_{j} \quad\left(\alpha_{i j}=\alpha_{j i}\right)
$$

be transformed by a linear transformation

$$
x_{i}=\sum_{k=1}^{n} \beta_{i k} Y_{k} \quad(i=1,2, \cdots, n)
$$

to

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} c_{i j} Y_{i} Y_{j} \quad\left(c_{i j}=c_{j i}\right)
$$

Then

$$
\begin{equation*}
\left|c_{i j}\right|=\left|\alpha_{i j}\right| \cdot\left|\beta_{i j}\right| \quad(i, j=1,2, \cdots, n) \tag{1.6}
\end{equation*}
$$

## 2. STATEMENT OF THEOREIM 1

We note that (1.5) is a special case of Theorem 1.
Theorem 1. Let $W_{n}, n=0,1, \cdots$, satisfy (1.1), where $A \neq B \neq 0$ are the roots of $y^{2}-p y+q=0$. Let $m, k=1,2, \cdots$, and define

$$
P_{n}=\prod_{i=1}^{k} W_{m n+n_{i}} \quad(n=0,1, \cdots)
$$ where $n_{i}, i=1,2, \cdots, k$, are arbitrary integers or zero. Let $N_{k}=n_{1}$ $+n_{2}+\cdots+n_{k}$. Then, with $u+1$ as the row index and $v+1$ as the column index, we have



$$
\cdot\left(W_{1}^{2}-\mathrm{pW}_{0} \mathrm{~W}_{1}+\mathrm{qW}_{0}^{2}\right)^{(\mathrm{k}+1) \mathrm{k} / 2} \prod_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{U}_{\mathrm{mi}}^{2(\mathrm{k}+1-\mathrm{i})}
$$

with $\mathrm{C}_{0}=\mathrm{A}^{\mathrm{N}_{\mathrm{k}}}$,
(2.3)

$$
C_{r}=\sum_{j=1}^{\binom{k}{r}} A^{N_{k}-S(j, r)}{ }_{B^{S}(j, r)} \quad(r=1,2, \cdots, k)
$$

$$
S(j, r)=n_{1}^{(j)}+n_{2}^{(j)}+n_{3}^{(j)}+\cdots+n_{r}^{(j)} \quad\left(j=1,2, \cdots,\binom{k}{r}\right)
$$

where, for each $j, S(j, r)$, as the sum of $r$ integers, $n_{i}^{(j)}, i=1,2, \ldots$, $r$, represents one of the $\binom{k}{r}$ combinations obtained by choosing $r$ numbers from the $k$ numbers, $n_{1}, n_{2}, n_{3}, \cdots, n_{k}$.

Remarks. If $n_{i} \equiv n_{0}, i=1,2, \cdots, k$, then $N_{k}=k n_{0}, S(j, r)=r n_{0}$, and

$$
\mathrm{C}_{\mathrm{r}}=\binom{\mathrm{k}}{\mathrm{r}} \mathrm{~A}^{(\mathrm{k}-\mathrm{r}) \mathrm{n}_{0}} \mathrm{~B}^{\mathrm{rn}}
$$

Since $A B=q$, we have

$$
\prod_{r=0}^{k} C_{r}=q^{n_{0}(k+1) k / 2} \cdot \prod_{j=0}^{k}\binom{k}{j},
$$

and thus (2.2) gives (1.5) as a special case.

$$
\text { For the case } n_{1}=n_{2}=\cdots=n_{k-1}=d \text { and } n_{k} \neq d_{\bullet} \text { it is readily seen }
$$ that

$$
C_{r}=\binom{k-1}{r-1} A^{(k-r) d_{B}(r-1) d+n_{k}}+\binom{k-1}{r} A^{(k-r-1) d+n_{k_{B}} r d}
$$

As a footnote to Theorem 1, we have

$$
\text { Lemma 2. For } \mathrm{r}<\mathrm{k}-\mathrm{r}, \mathrm{r}=0,1, \cdots \text {, we have }
$$

(2.5) $\quad C_{r} C_{k-r}=\binom{k}{r} q^{N_{k}}+\binom{k}{r} \sum_{j=2} \sum_{i=1} q^{S(i, r)-S(j, r)+N_{k}} \cdot V_{2 S(j, r)-2 S(i, r)}$.

Thus,

where
(2.8) $\quad C_{k / 2}=\sum_{j=1}^{\binom{k-1}{k / 2}} q^{S(j, k / 2)} \cdot V_{N_{k}}-2 S(j, k / 2) \quad(k=2,4,6, \cdots)$.

Proof of Lemma 2. Since $A B=q$, we obtain from (2.3),

$$
C_{r}=\sum_{j=1}^{\binom{k}{r}} q^{S(j, r)} \cdot A^{N_{k}-2 S(j, r)}
$$

Noting that a choice of $r$ numbers from $k$ numbers leaves a complement choice of $k-r$ numbers, we have from (2.3)

$$
\begin{align*}
& C_{k-r}=\sum_{j=1}^{\binom{k}{r}} A^{N_{k}-S(j, k-r)} B_{B}^{S(j, k-r)}=\sum_{j=1}^{\binom{k}{r}} A^{S(j, r)} B_{k} N_{k}-S(j, r)  \tag{2.9}\\
&=\binom{k}{r} \\
& i=1
\end{align*} q^{S(i, r)} \cdot B^{N_{k}-2 S(i, r)} .
$$

In forming the product $\mathrm{C}_{\mathrm{r}} \mathrm{C}_{\mathrm{k}-\mathrm{r}}$, we note that $\binom{\mathrm{k}}{\mathrm{r}}$ product pairs have equal $i$ and $j$ indices and the same value $q N_{k}$. For the cross products with $i \neq$ $j$, we combine those pairs having the same values of $i$ and $j$, noting that

$$
\begin{aligned}
& =q^{S(i, r)-S(j, r)+N_{k}} V_{2 S(j, r)-2 S(i, r)} \quad .
\end{aligned}
$$

Set $k=2 r$ in (2.3). Since a choice of $r$ numbers from a set of $2 r$ numbers leaves another set of $r$ numbers, we may again pair off related terms of the sum in (2.3). Since

$$
A^{N_{2} r^{-S(j, r)} S_{B}^{S(j, r)}}+A_{B}^{S(j, r)}{ }_{B}^{N_{2 r^{-S(j, r)}}}=q^{S(j, r)} V_{N} r_{2}-2 S(j, r)
$$

and

$$
\binom{2 \mathrm{r}}{\mathrm{r}}=2\binom{2 \mathrm{r}-1}{\mathrm{r}}
$$

we obtain (2.8) from (2.3) with $r=k / 2$.

## 3. PROOF OF THEOREM 1

Since $A \neq B$, the general solution to (1.1) is

$$
\mathrm{W}_{\mathrm{n}}=\mathrm{aA}^{\mathrm{n}}+\mathrm{bB}^{\mathrm{n}}, \quad \mathrm{n}=0,1, \cdots
$$

where $a$ and $b$ are arbitrary constants whose values satisfy $W_{0}=a+b$ and $W_{1}=a A+b B$. We readily find that $(B-A) a=W_{0} B-W_{1}$ and $(B-A) b$ $=W_{1}-A W_{0}$. Since $A+B=p$ and $A B=q$, we have that

$$
\begin{equation*}
(A-B)^{2} a b=-\left(W_{1}^{2}-p W_{0} W_{1}+q W_{0}^{2}\right) . \tag{3.1}
\end{equation*}
$$

We observe that
(3.2) $\quad P_{n}=\prod_{i=1}^{k} W_{m n+n_{i}}=\sum_{j=0}^{k} K_{j}\left(B^{m(k-j)} A^{m j}\right)^{n} \quad(n=0,1, \cdots)$,
where $K_{j}, j=0,1, \cdots, k$, denote arbitrary constants independent of $n$. The quadratic form

$$
\begin{align*}
Q & =\sum_{r, s=0}^{k} P_{n+r+s} Y_{r} Y_{S}=\sum_{r, s=0}^{k} Y_{r} Y_{S} \cdot \sum_{j=0}^{k} K_{j}\left(B^{m(k-j)} A^{m j}\right)^{n+r+s} \\
= & \sum_{j=0}^{k} K_{j}\left(B^{m(k-j)} A^{m j}\right)^{n} \sum_{r, s=0}^{k} A^{m j(r+s)} B^{m(k-j)(r+s)} Y_{r} Y_{S}  \tag{3.3}\\
& \sum_{j} \\
= & \sum_{j=0}^{k} K_{j}\left(B^{m(k-j)} A^{m j}\right)^{n} x_{j}^{2},
\end{align*}
$$

$$
\begin{equation*}
x_{j}=\sum_{r=0}^{k}\left(A^{m j_{B}} m(k-j)\right)^{r} Y_{r} \quad(j=0,1, \cdots, k) \tag{3.4}
\end{equation*}
$$

Thus, by means of the linear transformation (3.4), we have reduced $Q$ to a diagonal form. If $M$ denotes the determinant of the linear transformation (3.4), it follows from Lemma 1 (see (1.6)), that
(3.5) $\left|P_{n+r+s}\right|=M^{2} \cdot \underset{j=0}{k} K_{j}\left(B^{m(k-j)} A^{m j}\right)^{n}=M^{2} \cdot q^{m n(k+1) k / 2} \cdot \prod_{j=0}^{k} K_{j}$,
where

$$
\begin{equation*}
M=\left|\left(A^{m j_{B}}{ }^{m(k-j)}\right)^{r}\right| \quad(j, r=0,1, \cdots, k) \tag{3.6}
\end{equation*}
$$

is a Vandermonde determinant.
We find now that

$$
\begin{aligned}
& \mathrm{k}-1 \mathrm{k} \\
& M=\underset{0 \leq j<r \leq k}{I I}\left(A^{m r} B^{m(k-r)}-A^{m j_{B} m(k-j)}\right)=\underset{j=0}{\prod=j+1} \prod^{m} A^{m j} B^{m(k-r)}(A-B) U_{m(r-j)} \\
& \mathrm{k}-1 \mathrm{k}-\mathrm{j} \\
& =(A-B)^{k(k+1) / 2} \cdot \prod_{j=0 \mathrm{~s}=1}^{\prod} A^{m j} B^{m(k-j-s)} U_{m s} \\
& j=0 \mathrm{~s}=1 \\
& =(A-B)^{k(k+1) / 2} \cdot \prod_{j=0}^{k-1} A^{m j(k-j)} B^{m(k-j)(k-j-1) / 2} \cdot \prod_{i=0}^{k-1} \prod_{s=1}^{\Pi} U_{m s} \\
& =(A-B)^{\mathrm{k}(\mathrm{k}+1) / 2} \cdot q^{\mathrm{mk}\left(\mathrm{k}^{2}-1\right) / 6} \cdot \prod_{i=1}^{\mathrm{k}} \mathrm{U}_{\mathrm{mi}}^{\mathrm{k}+1-\mathrm{i}} \cdot
\end{aligned}
$$

(3.7)

We proceed now to evaluate

$$
{\underset{j=0}{\mathrm{k}} \mathrm{~K}_{\mathrm{j}}, ~}_{\text {in }}
$$

of (3.5). From (3.2) we have

$$
\begin{equation*}
\prod_{i=1}^{k} W_{m n+n_{i}}=B^{m k n} \cdot \sum_{j=0}^{k} K_{j}\left((A / B)^{m n}\right)^{j} \tag{3.8}
\end{equation*}
$$

which is a polynomial in the variable $(A / B)^{m n}$. Since $W_{n}=a A^{n}+b B^{n}$, we have

$$
\mathrm{W}_{\mathrm{mn}+\mathrm{n}_{\mathrm{i}}}=\mathrm{B}^{\mathrm{mn}}\left(a A^{n_{i}}(\mathrm{~A} / \mathrm{B})^{\mathrm{mn}}+\mathrm{bB}^{\mathrm{n}_{\mathrm{i}}}\right)
$$

and thus


Recalling the definition of the elementary symmetric functions of the roots of a polynomial, we conclude, after comparing (3.8) and (3.9), that (see (2.3))


Using (3.1), we obtain from (3.10)

$$
\prod_{\mathrm{r}=0}^{\mathrm{k}} \mathrm{~K}_{\mathrm{r}}=(\mathrm{ab})^{\mathrm{k}(\mathrm{k}+1) / 2}{ }_{\mathrm{r}=0}^{\mathrm{k}} \mathrm{C}_{\mathrm{r}}
$$

$$
\begin{equation*}
=(-1)^{\mathrm{k}(\mathrm{k}+1) / 2}(\mathrm{~A}-\mathrm{B})^{-\mathrm{k}(\mathrm{k}+1)}\left(\mathrm{W}_{1}^{2}-\mathrm{pW} W_{0} W_{1}+q W_{0}^{2}\right)^{\mathrm{k}(\mathrm{k}+1) / 2} \prod_{\mathrm{r}=0}^{\mathrm{k}} \mathrm{C}_{\mathrm{r}} \tag{3.11}
\end{equation*}
$$

Thus, (3.5), with the use of (3.7) and (3.11), gives the desired result, (2.2).

## 4. THE CASE $p^{2}-4 q=0$

In [1], Carlitz gave an alternate proof of (1.4) for the case $p^{2}-4 q=0$. Although (1.4) was proved for the case $p^{2}-4 q \neq 0$, the two results are shown to be the same for the case $p^{2}-4 q=0$.

In the derivation of $(2.2)$, we assumed that $p^{2}-4 q \neq 0$. It can be shown (by a repetition of the argument in [1]) that (2.2) is also valid for the case $p^{2}-4 q=0$, where now $U_{n}=n(p / 2)^{n-1}$, and $W_{n}=(a+b n)(p / 2)^{n}$, with $\mathrm{a}=\mathrm{W}_{0}$ and $\mathrm{pb}=2 \mathrm{~W}_{1}=\mathrm{pW}_{0}$. Since $\mathrm{A}=\mathrm{B}=\mathrm{p} / 2$, we obtain from (2.3) that

$$
\mathrm{C}_{\mathrm{r}}=\binom{\mathrm{k}}{\mathrm{r}}(\mathrm{p} / 2)^{\mathrm{N}_{\mathrm{k}}}
$$

Moreover, we have

and

$$
\prod_{\mathrm{r}=0}^{\mathrm{k}}\binom{\mathrm{k}^{\prime}}{\mathrm{r}}(\mathrm{r}!)^{2}=(\mathrm{k}!)^{\mathrm{k}+1}
$$

Thus, from Theorem 1, we obtain the simplified result
Theorem 2
$\prod_{i=1}^{k}\left(a+b n_{i}+b m(n+r+s)\right)(p / 2)^{m(n+r+s)+n_{i}} \mid \quad(r, s=0,1, \cdots, k)$
$(4.1)=(-1)^{(\mathrm{k}+1) \mathrm{k} / 2} \cdot(\mathrm{p} / 2)^{(\mathrm{k}+1)\left(\mathrm{k}(\mathrm{mn}+1)+(2 / 3) \mathrm{mk}(\mathrm{k}-1)+(\mathrm{k} / 3)(\mathrm{mk}+2 \mathrm{~m}-3)+\mathrm{N}_{\mathrm{k}}\right)}$

- $(\mathrm{bm})^{\mathrm{k}(\mathrm{k}+1)} \cdot\left(\mathrm{k}_{\mathrm{g}}\right)^{\mathrm{k}+1}$ 。

Remarks. If $m=1$ and $n_{i} \equiv 0, i=1,2, \cdots, k$, then $N_{k}=0$, and thus (4.1) contains, as a special case, the second (and the last) principal result, (7), of [1].

Additional simplifications of (4.1) are readily obtained.

## REFERENCES

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2. W. L. Ferrar, Algebra, Oxford University Press, London, 1941.
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## AN APPLICATION OF THE LUCAS TRIANGLE

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1. INTRODUCTION

Consider the integer triangle whose entries are given by

$$
\begin{aligned}
& A_{j, 0}=1, \quad A_{j, j}=2, \quad j=1,2,3, \cdots ; \\
& A_{n+1, j}=A_{n, j}+A_{n, j-1} \quad(0<j<n, n \geq 1)
\end{aligned}
$$

The first few lines of the triangle are listed left-justified below:

| 1 | 2 |  |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
|  | 1 | 3 | 2 |  |  |  |  |
|  | 1 | 4 | 5 | 2 |  |  |  |
|  | 1 | 5 | 9 | 7 | 2 |  |  |
|  | 1 | 6 | 14 | 16 | 9 | 2 |  |
|  | 1 | 7 | 20 | 30 | 25 | 11 | 2 |

One notes that the recurrence relation is the same as the one for Pascal's triangle. Apart from no $\mathrm{A}_{0,0}$ term the array is really the sum of two Pascal triangles. The rising diagonal sums are the Lucas numbers, $L_{1}=1, L_{2}=3, L_{n+2}=L_{n+1}+L_{n^{\circ}}$ The $A_{0,0}=2$ would also add $L_{0}=2$ to the rising diagonal sum sequence. The triangular array is now the Lucas triangle of Mark Feinberg [1]. It is also closely related to a convolution triangle [3].

Consider the new array obtained in a simple way from our first array A by shifting the $j^{\text {th }}$ column down $j$ places $(j=1,2,3, \cdots)$. The column on the left is the $0^{\text {th }}$ column.

B: | 1 |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | 2 |  |  |  |
|  | 1 | 3 |  |  |  |
|  | 1 | 4 | 2 |  |  |
|  | 1 | 5 | 5 |  |  |
|  | 1 | 6 | 9 | 2 |  |
|  | 1 | 7 | 14 | 7 |  |
|  | 1 | 8 | 20 | 16 | 2 |
| 1 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |  |

The relationship is

$$
B_{i, j}=A_{i-j, j} \quad 0 \leq j \leq[i / 2]
$$

where $[\mathrm{x}$ ] is the greatest integer not exceeding x . The recurrence relation for $B_{i, j}$ is

$$
\begin{array}{ll}
B_{i, o}=1 & \text { for all } i \\
B_{i, j}=B_{i-1, j}+B_{i-2, j-1}, & 1 \leq j \leq[i / 2]
\end{array}
$$

along with other useful relations true for all j :

$$
\begin{aligned}
\mathrm{B}_{2 \mathrm{j}, \mathrm{j}} & =2 \\
\mathrm{~B}_{2 \mathrm{j}+1, \mathrm{j}} & =2 \mathrm{j}+1 \\
\mathrm{~B}_{2 \mathrm{j}+1, \mathrm{j}+1} & =0
\end{aligned}
$$

## 2. ANOTHER ARRAY

Harlan Umansky [2] laid out the following display of formulas for powers of Lucas numbers.
[Oct.

$$
\begin{aligned}
& L_{n}^{1}=L_{n} \\
& L_{n}^{2}=L_{2 n}+2(-1)^{n} \\
& L_{n}^{3}=L_{3 n}+3(-1)^{n} L_{n} \\
& \text { C: } \quad L_{n}^{4}=L_{4 n}+4(-1)^{n} L_{n}^{2}-2 \\
& L_{n}^{5}=L_{5 n}+5(-1)^{n_{n}} L_{n}^{3}-5 L_{n} \\
& \mathrm{~L}_{\mathrm{n}}^{6}=\mathrm{L}_{6 \mathrm{n}}+6(-1)^{\mathrm{n}} \mathrm{~L}_{\mathrm{n}}^{4}-9 \mathrm{~L}_{\mathrm{n}}^{2}+2(-1)^{\mathrm{n}} \\
& L_{n}^{7}=L_{7 n}+7(-1)^{n_{n}} L_{n}^{5}-14 L_{n}^{3}+7(-1)^{n_{n}} L_{n} \\
& L_{n}^{8}=L_{8 n}+8(-1)^{n_{L}}{ }_{n}^{6}-20 L_{n}^{4}+16(-1)^{n^{n}}{ }_{n}^{2}-2
\end{aligned}
$$

The display given in [2] contains 7 missing pairs of parentheses. The above displayed form was suggested by Edgar Karst who, along with Brother Alfred Brousseau, noted the typing errors in [2]. Surely, we note that exclusive of signs, the coefficients in display C are precisely those of Array B. We shall prove the theorem:

Theorem 1.

$$
L_{n}^{m}=L_{m n}+\sum_{j=1}^{[m / 2]} c_{m, j}(-1)^{n j+j-1} L_{n}^{m-2 j}
$$

where

$$
\begin{aligned}
& C_{k, 0}=1 \\
& C_{m, j}=C_{m-1, j}+C_{m-2, j-1}, \quad 1 \leq j \leq[m / 2] \text { for } m \geq 2
\end{aligned}
$$

Proof. The proof shall proceed by induction. For all $n$, the theorem is true for $m=1$, the sum being empty. Assume, for $n \geq 1$,

$$
L_{n}^{k}=L_{n k}+\sum_{j=1}^{[k / 2]} C_{k, j}(-1)^{n j+j-1} L_{n}^{k-2 j}
$$

for $\mathrm{k}=1,2,3, \cdots, \mathrm{~m}$ along with

$$
\mathrm{C}_{\mathrm{k}, 0}=1, \quad \mathrm{C}_{2 \mathrm{k}, \mathrm{k}}=2, \quad \mathrm{C}_{2 \mathrm{k}+1, \mathrm{k}}=2 \mathrm{k}+1, \quad \text { and } \quad \mathrm{C}_{2 \mathrm{k}+1, \mathrm{k}+1}=0
$$

Therefore,

$$
L_{n}^{m}=L_{m n}+\sum_{j=1}^{[m / 2]} C_{m, j}(-1)^{n j+j-1} L_{n}^{m-2 j}
$$

and

$$
L_{n}^{m+1}=L_{n} L_{m n}+\sum_{j=1}^{[m / 2]} C_{m, j}(-1)^{n j+j-1} L_{n}^{m+1-2 j}
$$

But,

$$
L_{n} L_{m n}=L_{(m+1) n}+(-1)^{n_{1}} L_{(m-1) n}
$$

Thus,

$$
L_{n}^{m+1}=L_{(m+1) n}+(-1)^{n} L_{(m-1) n}+\sum_{j=1}^{[m / 2]} c_{m, j}(-1)^{n j+j-1} L_{n}^{m+1-2 j}
$$

Returning to the inductive assumption for $\mathrm{k}=\mathrm{m}-1$ yields

$$
\begin{aligned}
(-1)^{n} L_{(m-1) n} & =(-1)^{n} L_{n}^{m-1}+(-1)^{n+1} \sum_{j=1}^{[(m-1) / 2]} C_{m-1, j}^{(-1)^{n j+j-1}} L_{n}^{m-1-2 j} \\
& =(-1)^{n} L_{n}^{m-1}+\sum_{j=1}^{[(m-1) / 2]} C_{m-1, j}(-1)^{n(j+1)+(j+1)-1} L_{n}^{m-1-2 j}
\end{aligned}
$$

Now let $p=j+1$; then since $[(m-1) / 2]+1=[(m+1) / 2]$,

$$
(-1)^{n} L_{(m-1) n}=(-1)^{n} L_{n}^{m-1}+\sum_{p=2}^{[(m+1) / 2]} C_{m-1, p-1}(-1)^{n p+p-1} L_{n}^{m+1-2 p}
$$

Therefore,

$$
\begin{aligned}
L_{n}^{m+1}= & L_{(m+1) n}+\left\{(-1)^{n} L_{n}^{m-1}+\sum_{p=2}^{[(m+1) / 2]} C_{m-1, p-1}^{\left.(-1)^{n p+p-1} L_{n}^{m+1-2 p}\right\}}\right\} \\
& +\sum_{p=1}^{[m / 2]} C_{m, p}(-1)^{n p+p-1} L_{n}^{m+1-2 p} \\
= & L_{(m+1) n}+\sum_{p=1}^{[(m+1) / 2]}\left(C_{m, p}+C_{m-1, p-1}\right)(-1)^{n p+p-1} L_{n}^{m+1-2 p}
\end{aligned}
$$

We examine the possible extra term added to the second summation. If m is 2 k , then $[\mathrm{m} / 2]=[(\mathrm{m}+1) / 2]=\mathrm{k}$ and $\mathrm{C}_{2 \mathrm{k}, \mathrm{k}}=2$ and $\mathrm{C}_{2 \mathrm{k}-1, \mathrm{k}-1}=$ $2 \mathrm{k}-1$; thus, $\mathrm{C}_{2 \mathrm{k}+1, \mathrm{k}}=2 \mathrm{k}+1$. If $\mathrm{m}=2 \mathrm{k}+1$, then $[\mathrm{m} / 2]+1=$ $[(\mathrm{m}+1) / 2]=\mathrm{k}+1$ and the term $\mathrm{C}_{2 \mathrm{k}+1, \mathrm{k}+1}=0$ and $\mathrm{C}_{2 \mathrm{k}, \mathrm{k}}=2$; thus $\mathrm{C}_{2 \mathrm{k}+2, \mathrm{k}+1}=2$. Thus, if one defines
$\mathrm{C}_{\mathrm{k}-1,0}=1, \quad \mathrm{C}_{2 \mathrm{k}, \mathrm{k}}=2, \quad \mathrm{C}_{2 \mathrm{k}+1, \mathrm{k}}=2 \mathrm{k}+1, \quad \mathrm{C}_{2 \mathrm{k}+1, \mathrm{k}+1}=0$
for $k \geq 1$, and

$$
C_{m+1, p}=C_{m, p}+C_{m-1, p-1}, \quad 1 \leq p \leq\left[\frac{m+1}{2}\right], \quad m \geq 1
$$

then

$$
L_{n}^{m+1}=L_{(m+1) n}+\sum_{p=1}^{[(m+1) / 2]} C_{m+1, p}(-1)^{n p+p-1} L_{n}^{m+1-2 p}
$$

[Continued on p. 427.]

# ONE-ONE CORRESPONDENCES between the SET N OF POSItive INTEGERS AND THE SETS $N^{n}$ AND $\cup_{n \in N} N^{n}$ <br> EUGENE A. MAIER 

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1. Let $N$ be the set of positive integers and let $N^{n}$ be the set of all n-tuples of positive integers. It is well known that there exist one-one correspondences between $N^{n}$ and $N$ for all $N$, and between $\underset{n \in N^{n}}{\cup} N^{n}$ and $N$. In this paper, we give examples of such functions.
2. Theorem 1. Define $f_{n}: N^{n} \rightarrow N$ by

$$
\begin{equation*}
\mathrm{f}_{\mathrm{n}}\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \cdots, \mathrm{x}_{\mathrm{n}}\right)=\binom{\mathrm{s}_{\mathrm{n}}}{\mathrm{n}}-\sum_{\mathrm{k}=1}^{\mathrm{n}-1}\binom{\mathrm{~s}_{\mathrm{k}}-1}{\mathrm{k}}, \tag{1}
\end{equation*}
$$

where

$$
s_{k}=\sum_{i=1}^{k} x_{i}
$$

for $\mathrm{k} \leq \mathrm{n}$ and the combinatorial symbol $\binom{\mathrm{m}}{\mathrm{k}}$ is defined to be 0 if $\mathrm{m}<\mathrm{k}$. Then $f_{n}$ is a one-one correspondence.

Proof. We begin by defining a relation $<$ on $N^{n}$ as follows:
Definition. $\left(x_{1}^{\prime}, x_{2}^{\prime}, \cdots, x_{n}^{\prime}\right)<\left(x_{1}, x_{2}, \cdots, x_{n}\right)$ if and only if $s_{n}^{\prime}<$ $\mathrm{s}_{\mathrm{n}}$, or $\mathrm{s}_{\mathrm{n}}^{\prime}=\mathrm{s}_{\mathrm{n}}$ and there exists $\mathrm{k} \leq \mathrm{n}$ such that $\mathrm{x}_{\mathrm{k}}^{\prime}<\mathrm{x}_{\mathrm{k}}$ and $\mathrm{x}_{\mathrm{i}}^{\prime}=\mathrm{x}_{\mathrm{i}}$ for $\mathrm{k}<\mathrm{i} \leq \mathrm{n}$.

It is readily established that $<$ well-orders $N^{n}$. For $\alpha \in N^{n}$, let $\mathrm{M}_{\alpha}=\left\{\beta \in \mathrm{N}^{\mathrm{n}} \mid \beta \leq \alpha\right\}$ and let $\mathrm{f}_{\mathrm{n}}(\alpha)=\#\left(\mathrm{M}_{\alpha}\right)$ where $\#\left(\mathrm{M}_{\alpha}\right)$ is the number of elements in $M_{\alpha^{*}}$. Since $M_{\alpha}$ is a finite set, it follows that $f_{n}$ is a oneone mapping from $N^{n}$ onto $N$. We prove by induction on $n$ that $f_{n}(\alpha)$ is given by (1).

If $\mathrm{n}=1$, we have $\mathrm{f}_{1}\left(\mathrm{x}_{1}\right)=\#\left\{\beta \in \mathrm{~N} \mid \beta \leq \mathrm{x}_{1}\right\}=\mathrm{x}_{1}$ which is the value (1) gives for $f_{1}\left(x_{1}\right)$. Assume (1) is valid for $n$. Observe that

$$
\left(x_{1}^{\prime}, x_{2}^{\prime}, \cdots, x_{n+1}^{\prime}\right)=\left(x_{1}, x_{2}, \cdots, x_{n+1}\right)
$$

[Oct.
if and only if
(i)

$$
s_{n+1}^{\prime}<s_{n+1},
$$

or
(ii)

$$
s_{n+1}^{\prime}=s_{n+1} \text { and } x_{n+1}^{\prime}<x_{n+1}
$$

or
(iii) $s_{n+1}^{\prime}=s_{n+1}, x_{n+1}^{\prime}=x_{n+1}$ and $\left(x_{1}^{\prime}, \cdots, x_{n}^{\prime}\right) \leq\left(x_{1}, \cdots, x_{n}\right)$.

Thus if

$$
\alpha=\left(x_{1}, x_{2}, \cdots, x_{n+1}\right)
$$

$\mathrm{M}_{\alpha}$ may be expressed as the union of three disjoint sets $\mathrm{A}, \mathrm{B}$ and C which consist of those elements of $\mathrm{N}^{\mathrm{n}+1}$ satisfying, respectively, conditions (i), (ii), and (iii). Thus,

$$
\mathrm{f}_{\mathrm{n}+1}(\alpha)=\#\left(\mathrm{M}_{\alpha}\right)=\#(\mathrm{~A})+\#(\mathrm{~B})+\#(\mathrm{C})
$$

We now compute $\#(A)+\#(B)+\#(C)$. We will have occasion to use the combinatorial identity,
(2)

$$
\sum_{j=t+1}^{t+r}\binom{j-1}{t}=\binom{t+r}{t+1}
$$

(which may be established by induction on $r$ ) and the fact that the number of n -tuples of positive integers which satisfy the equation $\mathrm{x}_{1}+\cdots+\mathrm{x}_{\mathrm{n}}=\mathrm{t}$ is

$$
\binom{\mathrm{t}-1}{\mathrm{n}-1}
$$

(Think of placing $t$ objects in a row and placing dividers into $n-1$ of the $t-1$ spaces between the objects. Then $x_{1}$ is the number of objects before the first divider, $x_{2}$ is the number between the first and second dividers, etc.)

Note that $\beta=\left(\mathrm{y}_{1}, \mathrm{y}_{2}, \cdots, \mathrm{yn}_{\mathrm{n}}, 1\right)$ is an element of A if and only if $\mathrm{y}_{1}+\mathrm{y}_{2}+\cdots+\mathrm{y}_{\mathrm{n}+1}=\mathrm{j}$ where $\mathrm{n}+1 \leq \mathrm{j}<\mathrm{s}_{\mathrm{n}+1^{\circ}}$. Thus,

$$
\#(A)=\sum_{j=n+1}^{s_{n+1}^{-1}}\binom{j-1}{n}
$$

and hence, using (2),

$$
\#(A)=\binom{s_{n+1}-1}{n+1} .
$$

Now $\beta \in \mathrm{B}$ if and only if $1 \leq \mathrm{y}_{\mathrm{n}+1} \leq \mathrm{x}_{\mathrm{n}+1}-1$ and

$$
y_{1}+\cdots+y_{n+1}=x_{1}+\cdots+x_{n+1}=s_{n+1}
$$

Thus $\beta \in B$ if and only if $y_{1}+\cdots+y_{n}=j$ where $s_{n}+1 \leq j \leq s_{n+1}-1$. Hence,

$$
\#(B)=\sum_{j=s_{n}+1}^{s_{n+1}-1}\binom{j-1}{n-1}
$$

Using (2), we have

$$
\#(B)=\sum_{j=n}^{s_{n+1}-1}\binom{j-1}{n-1}-\sum_{j=n}^{s_{n}}\binom{j-1}{n-1}=\binom{s_{n+1}-1}{n}-\binom{s_{n}}{n}
$$

Finally, $\beta \in \mathrm{C}$ if and only if

$$
\begin{gathered}
\mathrm{y}_{\mathrm{n}+1}=\mathrm{x}_{\mathrm{n}+1} \\
\mathrm{y}_{1}+\cdots+\mathrm{y}_{\mathrm{n}}=\mathrm{s}_{\mathrm{n}},
\end{gathered}
$$

and

$$
\left(y_{1}, \cdots, y_{n}\right)=\left(x_{1}, \cdots, x_{n}\right) .
$$

The least such $\beta$ is the $(\mathrm{n}+1)$-tuple

$$
\left(s_{n}-n+1,1,1, \cdots, 1, x_{n+1}\right)
$$

Thus $\beta \in \mathrm{C}$ if and only if

$$
\left(s_{n}-n+1,1, \cdots, 1\right)=\left(y_{1}, \cdots, y_{n}\right) \leftrightharpoons\left(x_{1}, \cdots, x_{n}\right)
$$

Hence,

$$
\#(\mathrm{C})=\mathrm{f}_{\mathrm{n}}\left(\mathrm{x}_{1}, \cdots, \mathrm{x}_{\mathrm{n}}\right)-\mathrm{f}_{\mathrm{n}}\left(\mathrm{~s}_{\mathrm{n}}-\mathrm{n}+1,1, \cdots, 1\right)+1
$$

Therefore, using the induction hypothesis and (2), we have

$$
\begin{aligned}
\#(C) & =\left[\binom{s_{n}}{n}-\sum_{k-1}^{n-1}\binom{s_{k}-1}{k}\right]-\left[\binom{s_{n}}{n}-\sum_{k=1}^{n-1}\binom{s_{n}-n+k-1}{k}\right]+1 \\
& =-\sum_{k=1}^{n-1}\binom{s_{k}-1}{k}+\sum_{k=s_{n}-n}^{s_{n-1}}\binom{k-1}{s_{n}-n-1} \\
& =-\sum_{k=1}^{n}\binom{s_{k}-1}{k}+\binom{s_{n}-1}{n}+\binom{s_{n}-1}{s_{n}-n_{n}} .
\end{aligned}
$$

Thus, since

$$
\binom{s_{n+1}-1}{n+1}+\binom{s_{n+1}-1}{n}=\binom{s_{n+1}}{n+1}
$$

and

$$
\binom{s_{n}-1}{n}+\binom{s_{n}-1}{s_{n}-n}=\binom{s_{n}}{n}
$$

we have

$$
f_{n+1}\left(x_{1}, \cdots, x_{n+1}\right)=\#(A)+\#(B)+\#(C)=\binom{s_{n+1}}{n+1}-\sum_{k=1}^{n}\binom{s_{k}-1}{k}
$$

and the theorem is established.
3. Theorem 2. Define $g: ~ \bigcup_{n \in N} N^{n} \rightarrow N$ by

$$
g\left(x_{1}, \ldots, x_{n}\right)=2^{s_{n}-1}-1+\sum_{k=1}^{n}\binom{s_{n}-1}{k-1}-\sum_{k=1}^{n-1}\binom{s_{k}-1}{k}
$$

where

$$
s_{k}=\sum_{i=1}^{\mathrm{k}} \mathrm{x}_{\mathrm{i}}
$$

for $\mathrm{k} \leq \mathrm{n}$ and $\binom{\mathrm{m}}{\mathrm{k}}$ is defined to be 0 if $\mathrm{m}<\mathrm{k}$. Then g is a one-one correspondence.

Proof. Define a relation $\triangleleft$ on $\bigcup_{n \in N} N^{n}$ as follows:
Definition. $\left(x_{1}^{1}, \cdots, x_{m}^{\prime}\right) \triangleleft\left(x_{1}, \cdots, x_{n}\right)$ if and only if
(i)

$$
\mathrm{s}_{\mathrm{m}}^{\prime}<\mathrm{s}_{\mathrm{n}}
$$

or
(ii)

$$
\mathrm{s}_{\mathrm{m}}^{\mathrm{p}}=\mathrm{s}_{\mathrm{n}} \text { and } \mathrm{m}<\mathrm{n}
$$

or
(iii)

$$
s_{m}^{\prime}=s_{n}, m=n \text { and }\left(x_{1}^{\prime}, \cdots, x_{n}^{\prime}\right)<\left(x_{1}, \cdots, x_{n}\right)
$$

The relation $\triangleleft$ well-orders ${ }_{n \in N} N^{n}$. For $\alpha \in \cup N^{n}$, let

$$
\mathbf{s}_{\alpha}=\left\{\beta \in \mathrm{N}^{\mathrm{n}} \mid \beta \unlhd \alpha\right\}
$$

and let $\mathrm{g}(\alpha)=\#\left(\mathrm{~S}_{\alpha}\right)$. Then g is a one-one mapping from $\bigcup_{n \in N} N^{n}$ onto $N$. We may express $S_{\alpha}$ as the union of three disjoint sets $X, Y$, and $Z$ which consist of those elements of $\bigcup_{n \in N} N^{n}$ satisfying, respectively, conditions (i), (ii), and (iii) in the definition of $\triangleleft$.

Now $\beta=\left(\mathrm{y}_{1}, \cdots, \mathrm{y}_{\mathrm{m}}\right) \in \mathrm{X}$ if and only if $\mathrm{y}_{1}+\cdots+\mathrm{y}_{\mathrm{m}}=\mathrm{j}$ where $1 \leq j \leq s_{n}-1$. The number of elements in $\bigcup_{n \in N} N^{n}$ satisfying this equation for fixed j is

$$
\sum_{m \in N}\binom{j-1}{m-1}=\sum_{m=1}^{j}\binom{j-1}{m-1}=2^{j-1}
$$

Thus

$$
\#(X)=\sum_{j=1}^{s_{n}-1} 2^{j-1}=2^{s_{n}-1}-1
$$

We have $\beta \quad \mathrm{Y}$ if and only if $\mathrm{y}_{1}+\cdots+\mathrm{y}_{\mathrm{m}}=\mathrm{s}_{\mathrm{n}}$ where $\mathrm{m}<\mathrm{n}$. Thus

$$
\#(\mathrm{Y})=\sum_{m=1}^{n-1}\binom{s_{n}-1}{m-1} .
$$

Finally, $\beta \in \mathrm{Z}$ if and only if $\beta \in \mathrm{N}^{\mathrm{n}}$ and $\beta_{0}=\beta=\left(\mathrm{x}_{1}, \cdots, \mathrm{x}_{\mathrm{n}}\right)$ where $\beta_{0}$ is the n-tuple $\left(\mathrm{s}_{\mathrm{n}}-\mathrm{n}+1,1,1, \cdots, 1\right)$. Thus, using the result of Theorem 1 and (2), we have

$$
\begin{aligned}
\#(Z) & =f_{n}\left(x_{1}, \cdots, x_{n}\right)-f_{n}\left(s_{n}-n+1,1, \cdots, 1\right)+1= \\
& =\binom{s_{n}}{n}-\sum_{k=1}^{n-1}\binom{s_{k}-1}{k}-\left[\binom{s_{n}}{n}-\sum_{k=1}^{n-1}\binom{s_{n}-n+k-1}{k}\right]+1 \\
& =-\sum_{k=1}^{n-1}\binom{s_{k}-1}{k}+\sum_{k=s_{n}-n}^{s_{n}-1}\binom{k-1}{s_{n}-n-1} \\
& =-\sum_{k=1}^{n-1}\binom{s_{k}-1}{k}+\binom{s_{n}-1}{s_{n}-n}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
g\left(x_{1}, \cdots, x_{n}\right) & =\#(X)+\#(Y)+\#(\mathrm{Z})= \\
& =2^{s_{n}-1}-1+\sum_{k=1}^{n-1}\binom{s_{n}-1}{k-1}-\sum_{k=1}^{n-1}\binom{s_{k}-1}{k}+\binom{s_{n}-1}{s_{n}-n} \\
& =2^{s_{n}-1}-1+\sum_{k=1}^{n}\binom{s_{n}-1}{k-1}-\sum_{k=1}^{n-1}\binom{s_{k}-1}{k}
\end{aligned}
$$

## SOME RESULTS IN TRIGONOMETRY

## BROTHER L. RAPHAEL, F.S.C.

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Graphs of the six circular functions in the first quadrant yield some particularly elegant results involving the Golden Section.

Let $\varphi^{2}+\varphi=1$, so that $\varphi=(\sqrt{5}-1) / 2=0.61803$ and notice that:
$\arccos \varphi=\arcsin \sqrt{1-\varphi^{2}}=\arcsin \sqrt{\varphi}=0.90459$
$\arcsin \varphi=\arccos \sqrt{1-\varphi^{2}}=\arccos \sqrt{\varphi}=0.66621$
Further, if $\tan x=\cos x$, then $\sin x=\cos ^{2} x$ and $\sin ^{2} x+\sin x=1$, that is, $x=\arcsin \varphi$ in which case $\tan \arcsin \varphi=\cos \arcsin \varphi=\cos \arccos \sqrt{\varphi}=\sqrt{\varphi}$ [Continued on p. 392.]

# THE "DIFFERENCE SERIES" OF MADACHY 

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In a recent issue, Madachy [1] has raised two conjectures concerning difference series which resultat the various levels in the Sieve of Eratosthenes; both conjectures are valid. In this modified sieve, the sieving prime is discarded along with its multiples. The difference series associated with a prime is the series of differences which occur between the members of the remaining list after sievings through that particular prime.

Conjecture 1. If $d(d>2)$ is the number of terms in one period of the difference series, then the series is symmetrical about the ( $d / 2$ )th term. Conjecture 2. The $(\mathrm{d} / 2)^{\text {th }}$ term $(\mathrm{d}>2)$ will always be 4 .
The validity of Conjecture 1 can be argued as follows. Consider the numbers which remain after we have sieved by $2,3,5, \cdots, p_{n}$ and let $\mathrm{P}=2 \cdot 3 \cdot 5 \cdots \mathrm{p}_{\mathrm{n}}$. Then, as Madachy shows, there are $\mathrm{P}+1$ numbers in the original set of consecutive integers from which one period of the difference series is formed; we note that both endpoints must be included in order to generate the differences. Starting from $P-1$ of these integers, sieving by 2 can be done by sieving each second number and beginning at either end, since 2 divides $P$. The numbers 0 and $P$ play like roles, as do the numbers 1 and $P$ - 1. Similarly, since 3 divides $P$ we can again count from either end of the original set and sieve each third number. Likewise we do this for each number through $\mathrm{p}_{\mathrm{n}}$. To illustrate, consider $\mathrm{n}=3$ so that P-1 = 29. We underline to indicate sieving.


Thus the difference series begins $6,4,2,4, \cdots$. The last element of a period of the difference series occurs from the pair $P-1, P+1$, and is always 2. Since we can apply the process forward or backward over the first period (deleting this last difference), the differences are symmetrically arranged about the "middle difference," i. e., about the (d/2) th term.

Conjecture 2 can be settled by considering how this "middle difference" must be formed. Since $d>2$, the middle difference must span the number $\mathrm{P} / 2$ in the original set of consecutive integers, which is an odd number; actually $P / 2=3.5 \cdots p_{n}$. Since $(P / 2)+1$ and $(P / 2)-1$ are even, this "middle difference" must be at least 4. From the results of Conjecture 1, symmetry then shows that if it is not 4 , it must be at least 8 , In this case, both of the odd numbers $(P / 2)+2$ and ( $\mathrm{P} / 2$ ) - 2 must have been sieved out at some stage; $i_{.}$e., by some number $3,5, \cdots, p_{n}$. However, if we consider a decomposition of $(\mathrm{P} / 2) \pm 2$; we see that this is not possible, for

$$
(\mathrm{P} / 2) \pm 2=3 \cdot 5 \cdots \mathrm{p}_{\mathrm{n}} \pm 2,
$$

and the remainder is $\pm 2$ when it is divided by any of these primes, hence it could not have been sieved out.

It is of interest to note that no sieving with any prime $p$ greater than 3, although the actual numbers which are sieved out are not regularly arranged, a ratio of exactly $1 / \mathrm{p}$ of them disappear. In terms of the difference series this means that within one period of that series formed after sieving by $2,3,5, \cdots, p$, exactly $1 / p$ of the pairs of members of the previous difference series are combined. An example for $P=30$ illustrates this.

$$
\begin{array}{ccccccccc}
\frac{4}{4} & 4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 \\
6 & 4 & 2 & 4 & 2 & 4 & \frac{4}{6} & 2 & 4 \\
6 & 4 & 2 & \cdots
\end{array}
$$

From the 10 differences (of 5 periods) of the previous series we form 2 new differences, a ratio of $2 / 10=1 / 5$ for the sieving prime 5 .

A very difficult problem is to try to determine in general exactly which pairs of differences are to be combined to form the next difference series. In the above example, pairs numbered $1,8,11,18, \ldots$ are combined. For the next sequences we combine pairs numbered:
(7) $1,13,20,24,31,35,42,54 ; 1+56, \cdots \cdot$
(11) $1,27,32,42,47,58,73,77,93,103, \cdots$
(13) $1,35,44,51,62,77,84,99,110,115, \cdots \circ$
(17) $1,56,62,75,94,100,119,132,139, \cdots \cdot$

No general formula for the $n^{\text {th }}$ difference series seems to exist.

## REFERENCE

1. Joseph S. Madachy, 'Recreational Mathematics - 'Difference Series' Resulting from Sieving Primes," Fibonacci Quarterly, 7 (1969), pp. 315318.

[Continued from p. 346.]
If n is not divisible by 11,13 , or 17 , then $\mathrm{p}_{2}<19 \leqq \mathrm{p}_{3}$. Taking $q=19$ in (3.0), we have

$$
\sum_{\mathrm{p} \mid \mathrm{n}} \frac{1}{\mathrm{p}}>\frac{1}{3}+\frac{1}{5}+\frac{\log (16 / 15)}{19 \log (19 / 18)}>\frac{1}{3}+\frac{1}{5}+\frac{1}{17}+\frac{\log (256 / 255)}{257 \log (257 / 256)}
$$

This completes the proof of the lower bound for (C) and also that of the new parts of Theorem 1.

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# INFINITELY MANY GENERALIZATIONS OF ABEL'S PARTIAL SUMMATION IDENTITY 

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It is well known that if $\Sigma_{k=1}^{m} B_{k}(x)$ is bounded independently of $m$ and $x$ (say for all $x$ in an interval I) and $A_{k}$ tends to zero monotonically as $k \rightarrow \infty$, then $\Sigma_{k=1}^{\infty} A_{k} B_{k}(x)$ is uniformly convergent on I. This follows from a finite identity first used systematically by Abel, namely,

$$
\begin{equation*}
\sum_{k=m}^{n} A_{k} B_{k}=s_{n} A_{n}-s_{m-1} A_{m-1}+\sum_{k=m-1}^{n-1} s_{k}\left(A_{k}-A_{k+1}\right) \tag{1}
\end{equation*}
$$

where

$$
s_{k}=\sum_{i=1}^{k} B_{i}
$$

The purpose of this paper is to show that an infinite sequence of finite identities involving summations (of which (1) is the simplest example) can be deduced from the so-called "P. Hall commutator collecting process" which is fundamental in the theory of finitely generated nilpotent groups.

Let $G$ be the free group on two generators $a$ and $b,\left\{G_{n}\right\}$ itslower central series $\left(G_{1}=G, G_{n+1}=\left[G_{n}, G\right]\right)$, and $\left\{\phi_{n}\right\}$ the corresponding natural homomorphisms, so $\phi_{n}: G \rightarrow G / G_{n}$. P. Hall's commutator collecting process yields for every $g \in G$ an integer $r=r(n)$ such that

$$
\begin{equation*}
\phi_{\mathrm{n}}(\mathrm{~g})=\mathrm{c}_{1}^{\mathrm{e}_{1}} \mathrm{c}_{2}^{\mathrm{e}_{2}} \ldots \mathrm{c}_{\mathrm{r}}^{\mathrm{e}^{r_{r}} G_{\mathrm{n}}} \tag{2}
\end{equation*}
$$

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where the $c_{i} \in G$ are the so-called basic commutators $\left(c_{1}=a, c_{2}=b\right.$, $c_{3}=[b, a], c_{4}=[b, a, a], c_{5}=[b, a, b], c_{6}=[b, a, a, a], c_{7}=[b, a, a, b]$, $\left.c_{8}=[b, a, b, b], \cdots\right)$ and the $e_{i}$ are integers uniquely determined by $g$ and n. A detailed explanation of these concepts can be found in Chapter 5 of Magnus, Karrass, and Solitar, Combinatorial Group Theory, Interscience Publishers, John Wiley and Sons, Inc., New York, 1966.

Now let $*$ denote the operator on $G$ which turns words backwards; e. g., $\left(a^{3} b^{2} a b\right)^{*}=b a b^{2} a^{3}$. If $\phi_{n}(g)$ is given by (2), define $\phi_{n}^{*}(g)$ by

$$
\begin{equation*}
\phi_{\mathrm{n}}^{*}(\mathrm{~g})=\mathrm{c}_{1}^{\mathrm{e}_{1}} \mathrm{c}_{2}^{*}{ }^{\mathrm{e}_{2}} \cdots \mathrm{c}_{\mathrm{r}}^{*}{ }^{\mathrm{e}} \mathrm{r}_{\mathrm{G}_{\mathrm{n}}} . \tag{3}
\end{equation*}
$$

Since $\mathrm{g}^{*}$ can be formed by making the substitutions $\mathrm{a} \rightarrow \mathrm{a}^{-1}$ and $\mathrm{b} \rightarrow \mathrm{b}^{-1}$ in $\mathrm{g}^{-1}$, it follows that

$$
\begin{equation*}
\phi_{\mathrm{n}}^{*}(\mathrm{~g})=\phi_{\mathrm{n}}\left(\mathrm{~g}^{*}\right) \tag{4}
\end{equation*}
$$

Similarly, let ' denote the operator on $G$ which interchanges $a$ and $b$; e. g., ( $\left.a^{3} b^{2} a b\right)^{\prime}=b^{3} a^{2} b a$. Then,

$$
\begin{equation*}
\phi_{\mathrm{n}}^{\prime}(\mathrm{g})=\phi_{\mathrm{n}}\left(\mathrm{~g}^{\prime}\right) \tag{5}
\end{equation*}
$$

Equations (4) and (5) provide infinitely many generalizations of (1).
To obtain specific identities from (4) and (5) write $g$ in the form

$$
\begin{equation*}
g=b^{x_{0} x_{1} x_{2}}{ }^{2} \cdot b^{x_{2 m}} \tag{6}
\end{equation*}
$$

where the $x_{i}$ are integers. Then $g^{*}$ is obtained from (6) by replacing $x_{i}$ with $x_{2 m-i}$, and $g^{\prime}$ is similarly obtained by replacing $m$ with $m+1$ and $\mathrm{x}_{\mathrm{i}}$ with $\mathrm{y}_{\mathrm{i}}$, where $\mathrm{y}_{0}=\mathrm{y}_{2 \mathrm{~m}+2}=0$ and $\mathrm{y}_{\mathrm{i}}=\mathrm{x}_{\mathrm{i}-1}$ for $1 \leq \mathrm{i} \leq 2 \mathrm{~m}+1$. Tables I, II, and III of the appendix show how to calculate $\phi_{5}(\mathrm{~g})$ from g , where g has the form (6), and $\phi_{5}^{*}(\mathrm{~g}), \phi_{5}(\mathrm{~g})$ from $\phi_{5}(\mathrm{~g})$, where $\phi_{5}(\mathrm{~g})$ has the form (2).

Example 1. By equating the exponents of $c_{3}$ in (5), we obtain
(7)

$$
\left(\sum_{i=0}^{m-1} x_{2 i+1}\right)\left(\sum_{i=0}^{m} x_{2 i}\right)-\sum_{i=0}^{m-1}\left(\sum_{j=i}^{m-1} x_{2 j+1}\right) x_{2 i}
$$

$$
=\sum_{i=0}^{m+1} y_{2 i}\left(\sum_{j=i}^{m+1} y_{2 j+i}\right)=\sum_{i=1}^{m}\left(\sum_{j=i}^{m} x_{2 j}\right) x_{2 i-1} .
$$

By letting

$$
u(t)=\sum_{i=0}^{t} x_{2 i}
$$

and

$$
\mathrm{v}(\mathrm{t})=\sum_{\mathrm{i}=0}^{\mathrm{t}} \mathrm{x}_{2 \mathrm{i}+1}
$$

this may be expressed in the more familiar form

$$
u(m-1) v(m-1)-\sum_{i=0}^{m-1} v(i-1)(u(i)-u(i-1))=
$$

(8)

$$
=\sum_{i=1}^{m} u(i-1)(v(i-1)-v(i-2))
$$

which is the discrete analogue of the familiar $u v-\int v d u=\int u d v$. Equation (1) is easily verified from (8).

Example 2. By equating the exponents of $c_{5}$ in (4), we obtain

$$
\begin{aligned}
& \sum_{i=1}^{m} x_{2 i}\left(\sum_{j=0}^{i-1} x_{2 j}\right)\left(\sum_{j=0}^{i-1} x_{2 j+1}\right)+\sum_{i=0}^{m}\binom{x_{2 i}}{2}\left(\sum_{j=0}^{i-1} x_{2 j+1}\right) \\
&= \sum_{i=0}^{m-1} x_{2 i}\left(\sum_{j=i+1}^{m} x_{2 j}\right)\left(\sum_{j=i}^{m-1} x_{2 j+1}\right)+\sum_{i=0}^{m}\binom{x_{2 i}}{2}\left(\sum_{j=i}^{m-1} x_{2 j+1}\right) \\
&+\left[\frac{1}{2}\left(\sum_{j=0}^{m-1} x_{2 j+1}\right)\left(\sum_{j=0}^{m} x_{2 j}\right)-\sum_{i=0}^{m} x_{2 i}\left(\sum_{j=i}^{m-1} x_{2 j+i}\right)\right]\left(\sum_{j=0}^{m} x_{2 j}-1\right) .
\end{aligned}
$$

(9)

It follows by equating the coefficients of $x_{1}$ in ${ }^{\prime}(9)$ that

$$
\begin{equation*}
\sum_{i=1}^{m} x_{2 i}\left(\sum_{j=0}^{i-1} x_{2 j}\right)+\sum_{i=0}^{m}\binom{x_{2 i}}{2}=\frac{1}{2}\left(\sum_{j=0}^{m} x_{2 j}\right)\left(\sum_{j=0}^{m} x_{2 j}-1\right) \tag{10}
\end{equation*}
$$

Equation (10) is also directly obvious, and can be considered a discrete analogue of

$$
\int_{u d u}=\frac{1}{2} u^{2} .
$$

It is clear that these identities provide new tests for the convergence of infinite series, but the author has neither been able to use them to decide the convergence of any series whose convergence is presently unknown, nor to show that these identities always have integral analogues.

## APPENDIX

For a $g$ given in the form (6), Table 1 gives the exponents $e_{i}$ of (2) for $r(5)=8$, and Tables 2 and 3 give the exponents $f_{i}$ and $h_{i}$ of $\phi_{5}^{*}(g)$ and $\phi_{5}^{\prime}(\mathrm{g})$, respectively. If p is a complicated expression, ( $)_{q}$ shall denote the binomial coefficient $\binom{p}{q}$. The author has extended these tables (by hand) to $r(6)=14$. The formula for $e_{14}$ is an unwieldy sum of five terms, one of which is

$$
\sum_{i=0}^{m-1}(p)_{2}\left(\sum_{j=i+1}^{m} x_{2 j}\right),
$$

where

$$
p=x_{2 i} \sum_{j=i}^{m-1} x_{2 j+1}
$$

TABLE 1

$$
\begin{aligned}
& e_{1}=\sum_{i=0}^{m-1} x_{2 i+1} \quad e_{2}=\sum_{i=0}^{m} x_{2 i} \\
& e_{3}=\sum_{i=0}^{m-1} x_{2 i}\left(\sum_{j=i}^{m-1} x_{2 j+1}\right) \quad e_{4}=\sum_{i=0}^{m=1} x_{2 i}\left(\sum_{j=i}^{m-1} x_{2 j+1}\right) 2 \\
& e_{5}=\sum_{i=0}^{m-1} x_{2 i}\left(\sum_{j=i}^{m-1} x_{2 j+1}\right)\left(\sum_{j=i+1}^{m} x_{2 j}\right)+\sum_{i=0}^{m-1}\binom{x_{2 i}}{2}\left(\sum_{j=i}^{m-1} x_{2 j+1}\right) \\
& e_{6}=\sum_{i=0}^{m-1} x_{2 i}\left(\sum_{j=i}^{m-1} x_{2 j+1}\right)_{3} \\
& e_{7}=\sum_{i=0}^{m-1}\binom{x_{2 i}}{2}\left(\sum_{j=i}^{m-1} x_{2 j+1}\right)_{2}+\sum_{i=0}^{m-1} x_{2 i}\left(\sum_{j=i}^{m-1} x_{2 j+1}\right)\left(\sum_{j=i+1}^{m} x_{2 j}\right) \\
& e_{8}=\sum_{i=0}^{m-1}\binom{x_{2 i}}{3}\left(\sum_{j=i}^{m-1} x_{2 j+1}\right)+\sum_{i=0}^{m-1}\binom{x_{2 i}}{2}\left(\sum_{j=i}^{m-1} x_{2 j+1}\right)\left(\sum_{j=i+1}^{m} x_{2 j}\right) \\
& +\sum_{i=0}^{m-1} x_{2 i}\left(\sum_{j=i}^{m-1} x_{2 j+i}\right)\left(\sum_{j=i+1}^{m} x_{2 j}\right),
\end{aligned}
$$

[Continued on p. 405.]

# THE SMALLEST NUMBER WITH DIVISORS A PRODUCT OF DISTINCT PRIMES 

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## 1. INTRODUCTION

Let $P_{i}$ denote the $i^{\text {th }}$ prime. This paper contains a proof that there is a number k such that for $\mathrm{k}>\mathrm{K}$, the number

$$
P_{1} P_{k+s^{-1}} P_{P_{2}}^{P_{k-1+s^{-1}}} \cdots P_{r}^{P_{k-r+1+s^{-1}}} \cdots P_{k}^{P_{1+s}}
$$

is the smallest number having

$$
P_{k+s} P_{k-1+s} \cdots P_{1+s}
$$

divisors, where $\mathrm{s} \geq 0,1 \leq \mathrm{r} \leq \mathrm{k}-1$.

## 2. LEMMAS

The following Lemma is repeatedly used in the proof of Lemma 2.
Lemma 1. There exist positive constants

$$
C=\frac{1}{9 \log 2}
$$

and $d$ such that $\mathrm{cr} \log \mathrm{r}<\mathrm{P}_{\mathrm{r}}<\mathrm{dr} \log \mathrm{r}$. See [2p. 186].
Lemma 2. Let $P_{i}$ denote the $i^{\text {th }}$ prime. There exists a number $K$ large enough such that for $k>K$, we have

$$
\mathrm{P}_{\mathrm{r}}^{\mathrm{P}_{\mathrm{k}-\mathrm{r}+1}>\mathrm{P}_{\mathrm{k}}, ~}
$$

for $r=1,2, \cdots, k-1$.
Proof. For $r=1$, we do have

$$
\begin{gather*}
2^{\mathrm{P}_{\mathrm{k}}}>{ }^{\mathrm{P}_{\mathrm{k}}}  \tag{1}\\
380
\end{gather*}
$$

for all k .
For $\mathrm{r}=2$, by Lemma 1, we have

$$
3^{P_{k-1}}>3^{c(k-1) \log (k-1)}
$$

There exists a $k_{2}$ such that for $k>k_{2}$, we have

$$
\begin{equation*}
3^{\mathrm{P}_{\mathrm{k}-1}}>3^{\mathrm{c}(\mathrm{k}-1) \log (\mathrm{k}-1)}>\mathrm{dk} \log \mathrm{k} \tag{2}
\end{equation*}
$$

By Lemma 1 and Eq. (2), there is a constant $k_{2}$ such that for $k>k_{2}$, we have
(3)

$$
3^{P_{k-1}}>P_{k}
$$

Similarly, for

$$
3 \leq \mathrm{r} \leq \frac{\mathrm{k}+1}{2}
$$

there is a constant $k_{r}$ such that for $k>k_{r}$, we have

$$
\begin{equation*}
\mathrm{P}_{\mathrm{r}}^{\mathrm{P}_{\mathrm{k}-\mathrm{r}+1}}>\mathrm{P}_{\mathrm{k}} \tag{4}
\end{equation*}
$$

For $\mathrm{r}=\mathrm{k}-1$, by Lemma 1, we have

$$
\begin{aligned}
\mathrm{P}_{\mathrm{k}-1}^{2} & >(\mathrm{c}(\mathrm{k}-1) \log (\mathrm{k}-1))^{2} \\
& =\mathrm{c}^{2}(\mathrm{k}-1)^{2} \log ^{2}(\mathrm{k}-1)
\end{aligned}
$$

Hence, there is a constant $\mathrm{k}_{\mathrm{k}-1}$ such that for $\mathrm{k}>\mathrm{k}_{\mathrm{k}-1}$, we have

$$
\begin{equation*}
\mathrm{P}_{\mathrm{k}-1}^{2}>\mathrm{c}^{2}(\mathrm{k}-1)^{2} \log ^{2}(\mathrm{k}-1)>d \mathrm{k} \log \mathrm{k}>\mathrm{P}_{\mathrm{k}} \tag{5}
\end{equation*}
$$

Similarly, for

$$
\frac{\mathrm{k}+1}{2}<\mathrm{r} \leq \mathrm{k}-2
$$

there is a constant $k_{r}$ such that

$$
\begin{aligned}
\mathrm{P}_{\mathrm{r}}^{\mathrm{P}_{\mathrm{k}-\mathrm{r}+1}} & >{\text { (cr } \log \mathrm{r})^{\mathrm{P}_{\mathrm{k}-\mathrm{r}+1}}} \\
& >\mathrm{dk} \log \mathrm{k} \\
& >\mathrm{P}_{\mathrm{k}}
\end{aligned}
$$

for $k>k_{r}$.
Let K be the maximum of $\mathrm{k}_{1}, \mathrm{k}_{2}, \cdots, \mathrm{k}_{\mathrm{k}-1^{-}}$. Then for $\mathrm{k}>\mathrm{K}$, we have

$$
P_{r}^{P_{k-r+1}}>P_{k} \quad(r=1,2, \cdots, k-1) \quad \text { Q.E.P. }
$$

Immediately following from Lemma 2, we have
Lemma 3. There is a constant $K$ such that for $k>K$, we have

$$
\mathrm{P}_{\mathrm{r}}^{\mathrm{P}_{\mathrm{k}-\mathrm{r}+1+\mathrm{S}}}>\mathrm{P}_{\mathrm{k}}>\mathrm{P}_{\mathrm{i}}
$$

for $r=1,2, \cdots, k-1$, and $s \geq 0$, where $r<i \leq k-1$.
Since we know $P_{1}^{A B-1}>P_{1}^{A-1} P_{2}^{B-1}$ if $A>1, B>1$, and $P_{1}^{A}>P_{2}$ ([1, Lemma 1]), together with Lemma 3, we conclude the following theorem. Theorem. There is a constant $K$ such that for $k>K$,

$$
P_{1} P_{k+s^{-1}} P_{P_{2}} P_{k-1+s^{-1}} \ldots P_{r}^{P_{k-r+1+s^{-1}}} \ldots P_{k}^{P_{1+s^{-1}}}
$$

is the smallest number such that it has $P_{k+s} P_{k-1+s} \cdots P_{k-r+1+s} \cdots P_{1+s}$ divisors.

## REFERENCES

1. M. E. Grost, "The Smallest Number with Given Number of Divisors," American Mathematical Monthly, Vol. 75, No. 7, 1968.
2. Ivan Niven and Herbert S. Zuckerman, An Introduction to the Theory of Numbers, John Wiley and Sons, Inc. , Second Edition, 1966.

# ADVANCED PROBLEMS AND SOLUTIONS 

## Edited by

RAYMOND E. WHITNEY

## Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania, 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

## H-172 Proposed by David Englund, Rockford College, Rockford, Illinois.

Prove or disprove the "identity,"

$$
F_{k n}=F_{n} \sum_{t=1}^{\left[\frac{k+1}{2}\right]}(-1)^{(n+1)(t+1)}\binom{k-t}{t-1} L_{n}^{k-2 t+1}
$$

where $F_{n}$ and $L_{n}$ denote the $n^{\text {th }}$ Fibonacci and Lucas numbers, respectively, and [x] denotes the greatest integer function.

H-173 Proposed by George Ledin, Jr., Institute of Chemical Biology, University of San Francisco, San Francisco, California.

Solve the Diophantine equation,

$$
x^{2}+y^{2}+1=3 x y
$$

H-174 Proposed by Daniel W. Burns, Chicago, Illinois.
Let $k$ be any non-zero integer and $\left\{\mathrm{S}_{\mathrm{n}}\right\}_{\mathrm{n}=1}^{\infty}$ be the sequence defined by $S_{n}=n k$ 。

Define the Burn's Function, $B(k)$, as follows: $B(k)$ is the minimal value of n for which each of the ten digits, $0,1, \cdots, 9$, have occurred
in at least one $S_{m}$ where $1 \leq m \leq n$. For example, $B(1)=10, B(2)=$ 45. Does $B(k)$ exist for all $k$ ? If so, find an effective formula or algorithm for calculating it.

## SOLUTIONS

OLDIES BUT GOODIES
The following problems are still lacking solutions:

| $\mathrm{H}-22$ | $\mathrm{H}-46$ | $\mathrm{H}-74$ | $\mathrm{H}-86$ | $\mathrm{H}-94$ | $\mathrm{H}-104$ | $\mathrm{H}-108$ | $\mathrm{H}-115$ | $\mathrm{H}-125$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{H}-23$ | $\mathrm{H}-60$ | $\mathrm{H}-76$ | $\mathrm{H}-87$ | $\mathrm{H}-100$ | $\mathrm{H}-105$ | $\mathrm{H}-110$ | $\mathrm{H}-116$ | $\mathrm{H}-127$ |
| $\mathrm{H}-40$ | $\mathrm{H}-61$ | $\mathrm{H}-77$ | $\mathrm{H}-90$ | $\mathrm{H}-102$ | $\mathrm{H}-106$ | $\mathrm{H}-113$ | $\mathrm{H}-118$ | $\mathrm{H}-130$ |
| $\mathrm{H}-43$ | $\mathrm{H}-73$ | $\mathrm{H}-84$ | $\mathrm{H}-91$ | $\mathrm{H}-103$ | $\mathrm{H}-107$ | $\mathrm{H}-114$ | $\mathrm{H}-122$ |  |

GENERATING FUNCTIONS
H-144 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
A. Put

$$
[(1-x)(1-y)(1-a x)(1-b y)]^{-1}=\sum_{m, n=0}^{\infty} A_{m, n} x^{m} y^{n}
$$

Show that

$$
\sum_{n=0}^{\infty} A_{n, n} x^{n}=\frac{1-a b x^{2}}{(1-x)(1-a x)(1-b x)(1-a b x)}
$$

B. Put

$$
(1-x)^{-1}(1-y)^{-1}(1-a x y)^{-\lambda}=\sum_{m, n=0}^{\infty} B_{m, n} x^{m} y^{n}
$$

Show that

$$
\sum_{n=0}^{\infty} B_{n, n} x^{n}=(1-x)^{-1}(1-a x)^{-\lambda}
$$

Solution by the Proposer.
Solution, A. We have

$$
A_{m, n}=\sum_{i=0}^{m} \sum_{j=0}^{n} a^{i} b^{j}=\frac{\left(1-a^{m+1}\right)\left(1-b^{n+1}\right)}{(1-a)(1-b)},
$$

so that

$$
\begin{aligned}
\sum_{n=0}^{\infty} A_{n, n} x^{n} & =\sum_{n=0}^{\infty} \frac{\left(1-a^{n+1}\right)\left(1-b^{n+1}\right)}{(1-a)(1-b)} x^{n} \\
& =\frac{1}{(1-a)(1-b)}\left\{\frac{1}{1-x}-\frac{a}{1-a x}-\frac{b}{1-b x}+\frac{a b}{1-a b x}\right\} \\
& =\frac{1}{1-b}\left\{\frac{1}{(1-x)(1-a x)}-\frac{b}{(1-b x)(1-a b x)}\right\} \\
& =\frac{1-a b x^{2}}{(1-x)(1-a x)(1-b x)(1-a b x)}
\end{aligned}
$$

Solution, B. We have

$$
\begin{aligned}
&(1-x)^{-1}(1-y)^{-1}(1-a x y)^{-\lambda} \\
&=\sum_{r, s, t=0}^{\infty} \frac{(\lambda)}{t!} a^{t} x^{r+t} y^{s+t}
\end{aligned}
$$

where

$$
(\lambda)_{t}=(\lambda-1)(\lambda-2) \cdots(\lambda-t+1) \quad(t \geq 1) \quad \text { and } \quad(\lambda)_{0}=1,
$$

so that

$$
B_{m, n}=\sum_{t=0}^{\min (m, n)} \frac{(\lambda)_{t}}{t!} a^{t}
$$

Hence

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_{n, n} x^{n} & =\sum_{n=0}^{\infty} x^{n} \sum_{t=0}^{\infty} \frac{(\lambda)}{t!} a^{t} \\
& =\sum_{t=0}^{\infty} \frac{(\lambda)}{t!}(a x)^{t} \sum_{n=0}^{\infty} x^{n} \\
& =(1-x)^{-1}(1-a x)^{-\lambda} .
\end{aligned}
$$

Also solved by M. Yoder and D. Jaiswal.

## FACTOR ANALYSIS

H-145 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia. If

$$
\mathrm{n}=\mathrm{p}_{1}^{\mathrm{e}_{1}} \mathrm{p}_{2}^{\mathrm{e}_{2}} \cdots \mathrm{p}_{\mathrm{r}}^{\mathrm{e}_{\mathrm{r}}}
$$

is the canonical factorization of $n$, let $\lambda(n)=e_{1}+\cdots+e_{r}$. Show that $\lambda(n)$ $\leq \lambda\left(F_{n}\right)+1$ for all $n$, where $F_{n}$ is the $n^{\text {th }}$ Fibonacci number.

Solution by the Proposer.
Clearly, $\lambda(m n)=\lambda(m)+\lambda(n)$, and if $m \mid n$ then $\lambda(m) \leq \lambda(n)$. Also, $1=\lambda(p) \leq \lambda\left(F_{p}\right)$ for any prime $p$. We show by induction that $\lambda\left(p^{k}\right) \leq$ $\lambda\left(F_{p} k\right)$ for all $k$, except when $p=k=2$, when $\lambda(4)=\lambda\left(F_{4}\right)+1$. The cases when $p^{k} \leq 12$ are checked directly. Assume the result is true for $\mathrm{p}^{\mathrm{k}-1}$. Then since $\mathrm{p}^{\mathrm{k}}>12$, by Carmichael's theorem ("On the Numerical

Factors of the Arithmetical Forms $\alpha^{\mathrm{n}} \pm \beta^{\mathrm{n}}$," Annals of Math. (2 ${ }^{\text {nd }}$ Ser.), 15, pp. 30-70, Theorem XXIII) there is a prime dividing $F_{p^{k}}$ not dividing $F_{p^{k-1}}$. Then since $\mathrm{F}_{\mathrm{p}^{\mathrm{k}-1}} \mid \mathrm{F}_{\mathrm{p}^{\mathrm{k}}}$, we have

$$
\lambda\left(\mathrm{F}_{\mathrm{p}^{\mathrm{k}}}\right) \geq 1+\lambda\left(\mathrm{F}_{\mathrm{p}^{\mathrm{k}-1}}\right) \geq \mathrm{k}
$$

completing the induction. Hence $\lambda\left(p^{k}\right) \leq \lambda\left(F_{p^{k}}\right)$ except when $p=k=2$. In the factorization

$$
\mathrm{n}=\mathrm{p}_{1}^{\mathrm{e}} \ldots \mathrm{p}_{\mathrm{r}}^{\mathrm{e}_{\mathrm{r}}}
$$

we can assume $p_{1}=2$, and $e_{1}=0$ if necessary. Then

$$
{ }_{p_{i}} e_{i}, \cdots, F{ }_{p_{r}} e_{r}
$$

are pairwise relatively prime since $p_{1}^{e_{1}}, \cdots, p_{r}^{e_{r}}$ are, and since $F p_{i} e_{i}$
divides $F_{n}$ for each $i$, so their product

$$
\underset{p_{1}}{\mathrm{e}_{1}} \cdots \mathrm{~F}_{\mathrm{p}_{\mathrm{r}}} \mathrm{e}_{\mathrm{r}} \mid \mathrm{F}_{\mathrm{n}}
$$

Hence,

$$
\begin{aligned}
\lambda\left(\mathrm{F}_{\mathrm{n}}\right) \geq \lambda\left(\mathrm{F}_{\mathrm{p}_{1}} \mathrm{e}_{1} \cdots \mathrm{p}_{\mathrm{r}}^{\mathrm{e}} \mathrm{e}_{\mathrm{r}}\right) & =\lambda\left(\mathrm{F}_{\mathrm{p}_{1}}\right)+\cdots+\lambda\left(\mathrm{F}_{\mathrm{p}_{\mathrm{r}}}\right) \geq \\
& \geq\left(\mathrm{e}_{1}-1\right)+\mathrm{e}_{2}+\cdots+\mathrm{e}_{\mathrm{r}}=\lambda(\mathrm{n})-1,
\end{aligned}
$$

which completes the proof.
Also solved by M. Yoder.

## CONVERGING FRACTIONS

H-147 Proposed by George Ledin, Jr., University of San Francisco, San Francisco, California.
Find the following limits. $\mathrm{F}_{\mathrm{k}}$ is the $\mathrm{k}^{\text {th }}$ Fibonacci number, $\mathrm{L}_{\mathrm{k}}$ is the $\mathrm{k}^{\text {th }}$ Lucas number, $\pi=3.14159 \cdots, \alpha=(1+\sqrt{5}) / 2=1.61803 \ldots$, $\mathrm{m}=1,2,3, \cdots$.

$$
\begin{aligned}
& \mathrm{X}_{1}=\lim _{\mathrm{n} \rightarrow \infty} \frac{\mathrm{~F}_{\mathrm{F}+1}}{\mathrm{~F}_{\mathrm{F}_{\mathrm{n}}}^{\alpha}} \\
& X_{2}=\lim _{\mathrm{n} \rightarrow 0}\left|\frac{\mathrm{~F} \mathrm{n}_{\mathrm{m}}}{\mathrm{n}^{\mathrm{m}}}\right| \\
& X_{3}=\lim _{\mathrm{n}}\left|\frac{\mathrm{~F} \mathrm{n}_{\mathrm{m}}}{\mathrm{~F}_{\mathrm{n}}^{\mathrm{m}}}\right| \\
& X_{4}=\lim _{\mathrm{n}}\left|\frac{{ }^{\mathrm{F}}{ }_{\mathrm{n}} \mathrm{~m}}{}\right| \\
& \mathrm{X}_{5}=\lim _{\mathrm{n} \rightarrow 0}\left|\frac{\mathrm{~L}_{\mathrm{n}}-2}{\mathrm{n}}\right|
\end{aligned}
$$

Solution by David Zeitlin, Minneapolis, Minnesota.
EDITORIAL NOTE: We have assumed Binét Extensions,

$$
\mathrm{F}_{\mathrm{x}}=\frac{\alpha^{\mathrm{x}}-\beta^{\mathrm{X}}}{\alpha-\beta}, \mathrm{L}_{\mathrm{x}}=\alpha^{\mathrm{X}}+\beta^{\mathrm{x}}
$$

in the calculations of $x_{2}, x_{3}, \cdots, x_{5}$ since we are concerned with neighborhoods of zero!
(1) As $\mathrm{n} \rightarrow \infty, \mathrm{F}_{\mathrm{n}} / \alpha^{\mathrm{n}} \rightarrow(\alpha-\beta)^{-1}$. Let $\mathrm{p}_{\mathrm{n}}=\mathrm{F}_{\mathrm{n}+1}$ and $\mathrm{q}_{\mathrm{n}}=\mathrm{F}_{\mathrm{n}}$. Then, as $\mathrm{n} \rightarrow \infty$,

$$
\mathrm{F}_{\mathrm{p}_{\mathrm{n}}} \int^{\mathrm{p}_{\mathrm{n}}} \rightarrow(\alpha-\beta)^{-1} \quad \text { and } \quad-\mathrm{F}_{\mathrm{q}_{\mathrm{n}}}^{\mathrm{d}} / \alpha^{2 q_{\mathrm{n}}} \rightarrow(\alpha-\beta)^{-\alpha}
$$

Since $\alpha q_{n}-p_{n} \rightarrow 0$, we have $x_{1}=(\alpha-\beta)^{\alpha-1} \cong 5^{(\alpha-1) / 2} \cong 5^{.309 \cong 1.644}$.
For x real, we define $\mathrm{L}_{\mathrm{x}}=\alpha^{\mathrm{X}}+\beta^{\mathrm{X}}$ and $\mathrm{F}_{\mathrm{x}}=\left(\alpha^{\mathrm{X}}-\beta^{\mathrm{x}}\right) /(\alpha-\beta)$. Let $Y_{i}, i=2,3,4,5$, denote the limits without absolute value signs; then $X_{i}=\left|Y_{i}\right|$
(2) Using L'Hospital's rule, we have (since $\alpha \beta=-1$ ),

$$
\mathrm{Y}_{2}=\lim _{\mathrm{x} \rightarrow 0}\left(\mathrm{~F}_{\mathrm{x}}^{\mathrm{m}} / \mathrm{x}^{\mathrm{m}}\right)=\frac{\log \alpha-\log \beta}{\alpha-\beta}=\frac{2 \log \alpha-\mathrm{i} \pi}{\alpha-\beta}
$$

where $\mathrm{i}^{2}=-1$, and $\log (-1)=\mathbf{i} \pi$, using principal values. Thus,

$$
\mathrm{X}_{2}=\left|\mathrm{Y}_{2}\right|=\sqrt{\left(4 \log ^{2} \alpha+\pi^{2}\right) / 5}
$$

(3) Using L'Hospital's rule, we have

$$
Z_{3}=\lim _{x \rightarrow 0} \frac{x}{F_{x}}=\frac{\alpha-\beta}{\log \alpha-\log \beta}=\frac{\alpha-\beta}{2 \log \alpha-i \pi}
$$

Thus,

$$
Y_{3}=\lim _{x \rightarrow 0}\left(F_{x}^{m} / F_{x}^{m}\right)=Y_{2} \cdot Z_{3}^{m}
$$

and so

$$
\mathrm{X}_{3}=\left|\mathrm{Y}_{3}\right|=\left|\mathrm{Y}_{2}\right| \cdot\left|\mathrm{Z}_{3}\right|^{\mathrm{m}}=\mathrm{X}_{2}\left|\mathrm{Z}_{3}\right|^{\mathrm{m}}=\left(\frac{4 \log ^{2} \alpha+\pi^{2}}{5}\right)^{-(\mathrm{m}-1) / 2}
$$

(4) We readily find that

$$
Y_{4}=\lim _{x \rightarrow 0}\left(\underset{x}{F} / x^{m-1} F_{x}\right)=Y_{2} \cdot Z_{3}=1
$$

and so $X_{4}=1$.
(5) Using L'Hospital's rule, we have

$$
Y_{5}=\lim _{x \rightarrow 0}\left(L_{x}-2\right) / x=\log \alpha+\log \beta=\mathrm{i} \pi
$$

and so $\mathrm{X}_{5}=\left|\mathrm{Y}_{5}\right|=\pi$.

## SHADES OF EULER

H-149 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tennessee. For $s=\sigma+i t$, let

$$
P(s)=\Sigma p^{-s}
$$

where the summation is over the primes. Set

$$
\begin{aligned}
& \sum_{n=1}^{\infty} a(n) n^{-s}=[1+P(s)]^{-1} \\
& \sum_{n=1}^{\infty} b(n) n^{-s}=[1-P(s)]^{-1} .
\end{aligned}
$$

Determine the coefficients $a(n)$ and $b(n)$.

Solution by the Proposer.
For $n=p_{1}{ }^{a_{1}} \cdots p_{m}{ }^{a_{m}}$ let $\rho(\mathrm{n})=a_{1}+\cdots+a_{m}$ and $\lambda(n)=(-1)^{\rho(n)}$. We claim that

$$
a(n)=a\left(p_{1}^{a_{1}} \cdots p_{m}^{a_{m}}\right)=\frac{\lambda(n)\left(a_{1}+\cdots+a_{m}\right)!}{a_{1}!\cdots a_{m}^{!}}
$$

and that $b(n)=|a(n)|$.
The proof is by induction on $\rho(\mathrm{n})$. If $\rho(\mathrm{n})=1, \mathrm{n}$ is prime and we have $a(n)+a(1)=0$ and the validity of the assertion is obvious. Since in general, we have

$$
a(n)+a\left(n / p_{1}\right)+\cdots+a\left(n / p_{m}\right)=0
$$

the result follows by induction. A similar method works for $b(n)$, except that here we have

$$
b(n)-b\left(n / p_{1}\right)-\cdots-b\left(n / p_{m}\right)=0 .
$$

Also solved by L. Carlitz, D. Lind, D. Klarner, and M. Yoder.

## TRIPLE THREAT

H-150 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

Show that

$$
25 \sum_{p=1}^{n-1} \sum_{q=1}^{p} \sum_{r=1}^{q} F_{2 r-1}^{2}=F_{4 n}+(n / 3)\left(5 n^{2}-14\right)
$$

where $F_{n}$ is the $n^{\text {th }}$ Fibonacci number.

Solution by the Proposer.
To establish this result, we need the following identities which have already been established earlier (Fibonacci Quarterly, December, 1966, pp. 369-372):

$$
\begin{gathered}
5\left(F_{1}^{2}+F_{3}^{2}+\cdots+F_{2 n-1}^{2}\right)=F_{4 n}+2 n \\
F_{4}+F_{8}+\cdots+F_{4 n}=F_{2 n} F_{2 n+2} \\
5\left(F_{2} F_{4}+F_{4} F_{6}+\cdots+F_{2 n-2} F_{2 n}=F_{4 n}-3 n\right.
\end{gathered}
$$

Hence,

$$
5 \sum_{1}^{q} F_{2 r-1}^{2}=F_{4 q}+2 q
$$

Or,

$$
5 \sum_{q=1}^{p} \sum_{r=1}^{q} F_{2 r-1}^{2}=\sum_{1}^{p} F_{4 q}+2 \sum_{1}^{p} q=F_{2 p} F_{2 p+2}+(p+1) p
$$

Hence,
$25 \sum_{p=1}^{n-1} \sum_{q=1}^{p} F_{2 r=1}^{2}=5 \sum_{1}^{n-1} F_{2 p} F_{2 p+2}+5 \sum_{1}^{n-1} p^{2}+5 \sum_{1}^{n-1} p=$

$$
\begin{aligned}
& =F_{4 n}-3 n+(5 / 6) n(n-1)(2 n-1)+(5 / 2) n(n-1)= \\
& =F_{4 n}+(n / 3)\left(5 n^{2}-14\right)
\end{aligned}
$$

Also solved by C. Peck, M. Yoder, A. Shannon, S. Hamelin, and D. Jaiswal. EDITORIAL NOTE. C. B. A. Peck, in his solution, obtained the identity

$$
25 \sum_{\mathrm{q}=1}^{\mathrm{n}} \sum_{r=1}^{\mathrm{q}} \mathrm{~F}_{2 \mathrm{r}-1}^{2}=\mathrm{L}_{4 \mathrm{n}+2}+5 \mathrm{n}(\mathrm{n}+1)-3
$$

[Continued from page 371.]
and also $\operatorname{ctn} \operatorname{arc} \cos \varphi=\sin \operatorname{arc} \cos \varphi=\sqrt{\varphi}$. The results are summarized below.


# RECREATIONAL MATHEMATICS 

# Edited by <br> JOSEPH S. MADACHY <br> 4761 Bigger Road, Kettering, Ohio <br> ANGLE MULTISECTION BY PARALLEL STRAIGHTEDGES 

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Trisection of an arbitrary angle is, of course, impossible by means of compass and unmarked straightedge alone, but the attempt to do this by other means has fascinated mathematicians since the time of the ancient Greeks.

The simplest trisection is probably Archimedes' famous paper strip construction. This method involves using a straightedge with two marks, known as an application of the "insertion principle." It is illustrated in Fig. 1.

The angle to be trisected here is $\theta$. A circle of arbitrary radius $r$ is drawn whose center $O$ is the vertex of the given angle. The sides of the angle intersect the circle at points A and $\mathrm{B} . \overrightarrow{\mathrm{BO}}$ is extended. A segment of length $r$ is marked on a straightedge or paper strip. The edge is placed so that it passes through $A$ and so that one endpoint of the marked segment intersects the circle at $C$ and the other endpoint falls at $D$ on $\overrightarrow{\mathrm{BO}}$, outside the circle. Then $m \angle C D O$, here marked $a$, is one-third $m \theta$. The proof is easily seen: $\triangle$ OCD, having two sides of length $r$, is isosceles, so that $\mathrm{m} / \mathrm{COD}=\mathrm{m} \angle \mathrm{CDO}=\mathrm{a}$. By the exterior angle theorem, $\mathrm{m} / \mathrm{ACO}=\mathrm{m} / \mathrm{CAO}$ $=2 \mathrm{a}$, since $\angle \mathrm{OAC}$ is also isosceles. The given angle $\theta$ forms an exterior angle of $\triangle$ OAD. Thus $\mathrm{m} \theta=2 \mathrm{a}+\mathrm{a}=3 \mathrm{a}$.

A solution of the problem using parallel straightedges, and a generalization are given here.

In Fig. 2, let the angle to be trisected be $\theta$; a circle of arbitrary radius $r$ is drawn with center at vertex $O$. $A$ and $B$ are the intersection points of the sides of the angle and the circle. $K$ and $F$ represent the two parallel straightedges; $K$ passes through point $A$ and $F$ passes through center O. C and E are points where each straightedge intersects the circle. Mark point $D$ on straightedge $K$ such that $C D=r$. Adjust the $\overrightarrow{\mathrm{BO}}$, for the reason that $\square \mathrm{OCDE}$ is a rhombus. To prove it, let $\mathrm{D}^{\prime}$ be the point of intersection of K and $\overrightarrow{\mathrm{BO}} . \Delta \mathrm{OPC}$ and $\triangle \mathrm{OEP}$ are $\cong$ by $\mathrm{H}-\mathrm{L}$; $\angle \mathrm{CD}^{\prime} \mathrm{P} \cong \angle \mathrm{EOP}$ by alt. int. angles; $\overline{\mathrm{CP}} \cong \overline{\mathrm{EP}}$ by cpctc. Therefore, $\Delta \mathrm{OEP}$ $=\Delta D^{\prime} \mathrm{CP}$ by SAA. It then follows that $\mathrm{OE}=\mathrm{OC}=\mathrm{CD}^{\prime}, \overline{\mathrm{CD}^{\prime}}$ parallel to $\overline{\mathrm{OE}}$, diagonals are $\perp$, therefore $\square O C D^{\prime} E$ is a rhombus, and $\mathrm{D}=\mathrm{D}^{\prime}$. Thus $\mathrm{m} \angle \mathrm{ODA}=\mathrm{a}=1 / 3 \mathrm{~m} \theta . \quad \angle \mathrm{DOE}$ and the angle vertical to it have measure a .

It is possible to broaden the scope of the previous method to certain problems of multisection - that of dividing an arbitrary angle into a given number of equal parts - combining the ideas used in the Archimedes trisection with the properties of what might be thought of as a set of "collapsing rhombuses." Specifically, it makes possible division of a given arbitrary angle into $2^{n}+1$ equal parts, where $n$ may be any positive integer. In fact, the angle may be divided into any number of parts which is a divisor of a number of the form $2^{n}+1$ - for example, if $n=5$, it is possible to divide the angle into 33 parts or 11 parts by taking three of the parts each equal to $1 / 33$ of the angle. The method shown above for trisection represents the case where $n=1$; that is, $2^{n}=3$. When $n=2,3,4, \cdots$, it may be seen that any given angle may be divided into $5,9,17, \ldots$ equal parts, respectively.

If $n=2$ so that $2^{n}+1=5$, we have a 5 -section as shown in Fig. 3. In each case, incidentally, there appear $n$ rhombuses - we see two here. As before the angle to be 5 -sected is represented by $\theta$, the circle is drawn with radius $r$ and center $O$. An inserted length equal to $r$ on straightedge K has endpoints C and D , with C on the circle. Straightedge F is parallel to $K$. $K$ and $F$ pass through points $A$ and $O$, respectively. $E$ is the point of intersection of $F$ and the circle. This time $\square$ OEDC is a rhombus with diagonals $\overline{\mathrm{OD}}$ and $\overline{\mathrm{CE}}$ (sides $\overline{\mathrm{CD}}$ and $\overline{\mathrm{OE}}$ are equal inlength and parallel). H is the point of intersection of diagonal $\overline{\mathrm{OD}}$ and the circle. K is adjusted so that $\overline{\mathrm{HE}} \perp \overrightarrow{\mathrm{BO}}$. Now using OD as radius, O as center, draw a circle concentric to the original. The intersection of $F$ and this circle is point M. Draw $\overline{\mathrm{DM}}$. It can be seen that $\overline{\mathrm{DM}} \perp \overrightarrow{\mathrm{BO}}$. पODGM is a rhombus so that $O D=D G$. If $m / D G O=a$, then $m \angle D O G=a$. By the
exterior angle theorem $m \angle C D O=m \angle C O D=2 a$. Now $\angle A C O$ is an exterior angle to $\triangle C O G$, so that $m \angle A C O=m \angle C O G+m \angle C G O=3 a+a=4 a=$ $\mathrm{m} / \mathrm{OAC}$. The given angle, $\theta$, is an exterior angle to $\triangle \mathrm{AOG}$, so that $\mathrm{m}=4 \mathrm{a}+\mathrm{a}=5 \mathrm{a}$. By alt. interior angles, $\mathrm{m} \angle \mathrm{GOE}=\mathrm{m} \angle \mathrm{DGO}=\mathrm{a}$, which is the measure also of the angle vertical to $\angle$ GOE. Thus the angle $\theta$ is 5-sected.

In each case, it may be seen, as in Fig. 4, which shows a 9 -section with 3 rhombuses, that a diagonal of each small rhombus becomes a side of the next larger rhombus. Each succeeding case is similar to that outlined in the 5 -section, in principle. The properties of the rhombus, in particular its equal sides and perpendicular diagonals, provide the means for this interesting method whereby the chain of rhombuses could be extended to infinity.


Fig. 1 Archimedes' Trisection


# A NOTE ON FIBONACCI FUNCTIONS 

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Recently, a number of authors [1, 2, 3] have considered Fibonacci functions - continuous functions possessing properties related to Fibonacci sequences. In this note, some Fibonacci functions are derived and their properties verified. The derivation is based on the following definition.

Definition: If $f$ is an infinitely differentiable function and $f$ satisfies the recursion relation:

$$
\begin{equation*}
f(x+2)=f(x)+f(x+1) \tag{1}
\end{equation*}
$$

then f is a Fibonacci function.
An immediate consequence of the definition is:
Theorem 1. If $f(x)$ is a Fibonacci function, then $f^{\prime}(x)$ and $\int f(x) d x$ are also.

The theorem is established by elementary calculus.

$$
\begin{aligned}
\mathrm{f}^{\prime}(\mathrm{x}+2) & =[\mathrm{f}(\mathrm{x})+\mathrm{f}(\mathrm{x}+1)]^{\prime}= \\
& =\mathrm{f}^{\prime}(\mathrm{x})+\mathrm{f}^{\prime}(\mathrm{x}+1) \\
\int \mathrm{f}(\mathrm{x}+2) \mathrm{dx} & =\int[\mathrm{f}(\mathrm{x})+\mathrm{f}(\mathrm{x}+1)] \mathrm{dx}= \\
& =\int \mathrm{f}(\mathrm{x}) \mathrm{dx}+\int \mathrm{f}(\mathrm{x}+1) \mathrm{dx}
\end{aligned}
$$

Theorem 2. If $\mathrm{f}(\mathrm{x})$ and $\mathrm{g}(\mathrm{x})$ are Fibonacci functions, then their sum is also.

Proof. Let $F(x)=f(x)+g(x)$. Then

$$
\begin{aligned}
F(x+2)=f(x+2)+g(x+2) & =[f(x+1)+g(x+1)]+[f(x)+g(x)]= \\
& =F(x+1)+F(x) .
\end{aligned}
$$

Theorem 3. If $\mathrm{f}(\mathrm{x})$ is a Fibonacci function and c is a real constant, then $\mathrm{cf}(\mathrm{x})$ is a Fibonacci function.

Proof. Let $F(x)=c f(x)$. Then

$$
F(x+2)=c f(x+2)=c[f(x+1)+f(x)]=c f(x+1)+c f(x)=
$$

$$
=F(x+1)+F(x)
$$

Since the function $e^{(p+k \pi i) x}$ where $p$ is a real constant, $k$ an integer, and $i=\sqrt{-1}$ is real for integer values of $x$, we look for Fibonacci functions of the form $y=e^{d x}$ where $d$ is complex. Substitution into the recursion relation (1) yields

$$
\begin{equation*}
e^{d(x+2)}-e^{d(x+1)}-e^{d x}=0 \tag{3}
\end{equation*}
$$

or,

$$
\begin{equation*}
e^{d x}\left(d^{2 d}-e^{d}-1\right)=0 \tag{4}
\end{equation*}
$$

Since 0 is omitted by the first factor of (4),

$$
\begin{equation*}
e^{2 d}-e^{d}-1=0 \tag{5}
\end{equation*}
$$

Solving (5) for $e^{d}$ :

$$
e^{d_{1}}=\frac{1}{2}(1+\sqrt{5})=\alpha
$$

and

$$
e^{d_{2}}=\frac{1}{2}(1-\sqrt{5})=\beta
$$

Let $d_{1}=a_{1}+b_{1} i$, then

$$
a^{a_{1}}\left(\cos b_{1}+i \sin b_{1}\right)=\alpha
$$

Since $\alpha>0, a_{1}=1 n \alpha=0.48$ and $b_{1}=2 \mathrm{k} \pi$ for $k$ an integer. Similarly, if $d_{2}=a_{2}+b_{2} i$, then

$$
e^{a_{2}}\left(\cos b_{2}+i \sin b_{2}\right)=\beta
$$

Since $\beta<0, \mathrm{a}_{2}=\ln |\beta|$ and $\mathrm{b}_{2}=(2 \mathrm{k}+\mathrm{i})$ for k an integer. Furthermore,

$$
1=\mid\left(e^{a_{1}} \cos 2 k\right)\left(e^{a_{2}} \cos (2 n+1)\left|=\left|e^{a_{1}}\right|\right| e^{a_{2}} \mid\right.
$$

and so $a_{2}=-1 n \alpha=-0.48$, or $a_{2}=-a_{1}$. Thus, the subscript on $a$ is not necessary and two solutions of (1) are:

$$
\mathrm{y}(\mathrm{x})=\mathrm{e}^{\mathrm{ax}} \cos 2 \mathrm{k} \pi \mathrm{x}
$$

and

$$
\mathrm{y}(\mathrm{x})=\mathrm{e}^{-\mathrm{ax}} \cos (2 \mathrm{k}+1) \pi \mathrm{x}
$$

Applying Theorems 2 and 3, we have:

$$
\begin{equation*}
\mathrm{y}(\mathrm{x})=\mathrm{c}_{1} \mathrm{e}^{\mathrm{ax}} \cos 2 \mathrm{k} \pi \mathrm{x}+\mathrm{c}_{2} \mathrm{e}^{-\mathrm{ax}} \cos (2 \mathrm{n}+1) \pi \mathrm{x} \tag{6}
\end{equation*}
$$

where $\mathrm{a}=1 \mathrm{n} \alpha ; \mathrm{k}$ and n integers. Equation (6) may be written:

$$
\begin{equation*}
y(x)=c_{1} e^{(a+2 k \pi i) x}+c_{2} e^{(-a+(2 n+1) \pi) x} \tag{7}
\end{equation*}
$$

Some interesting and useful relations between $e^{a}$ and $e^{-a}$ can be derived by substituting the values of $d_{1}$ and $d_{2}$ into Eq. (5).

$$
\begin{gathered}
e^{(a+2 k \pi i) 2}-e^{a+2 k \pi i}-1=0 \\
e^{2 a} e^{4 k \pi i}-e^{a} e^{2 k \pi i}-1=0 \\
e^{2 a}-e^{a}-1=0
\end{gathered}
$$

or

$$
\begin{equation*}
e^{2 a}=1+e^{a} \tag{8}
\end{equation*}
$$

Also,

$$
\begin{gathered}
e^{(-a+(2 \mathrm{k}+1) \pi \mathrm{i}) 2}-e^{(-\mathrm{a}+(2 \mathrm{k}+1) \pi \mathrm{i})}-1=0 \\
\mathrm{e}^{-2 \mathrm{a}} \mathrm{e}^{2(2 \mathrm{k}+1) \pi \mathrm{i}}-\mathrm{e}^{-\mathrm{a}} \mathrm{e}^{2 \mathrm{k} \pi \mathrm{i}} \mathrm{e}^{\pi i}-1=0 \\
\mathrm{e}^{-2 \mathrm{a}}+\mathrm{e}^{-\mathrm{a}}-1-0
\end{gathered}
$$

or
(9)

$$
\mathrm{e}^{-2 \mathrm{a}}=1-\mathrm{e}^{-\mathrm{a}}
$$

Furthermore,

$$
\begin{equation*}
\mathrm{e}^{\mathrm{a}}+\mathrm{e}^{-\mathrm{a}}=|\alpha|+|\beta|=\sqrt{5} . \tag{10}
\end{equation*}
$$

The trigonometric identity $\cos \mathrm{k} \pi(\mathrm{x}+2)=\cos \mathrm{k} \pi \mathrm{x}=-\cos \mathrm{k} \pi(\mathrm{x}+1)$, relations (8) and (9), and some algebra verify that (6) is a solution to (1).

Since (6) is a differentiable function satisfying relation (1) in view of Theorem 1,

$$
\begin{equation*}
\mathrm{y}^{\prime}(\mathrm{x})=\left(\mathrm{c}_{1} \mathrm{e}^{a \mathrm{x}} \cos 2 \mathrm{k} \pi \mathrm{x}+\mathrm{c}_{2} \mathrm{e}^{-\mathrm{ax}} \cos (2 \mathrm{n}+1) \pi \mathrm{x}\right)^{\prime} \tag{11}
\end{equation*}
$$

and

$$
\begin{equation*}
\int y(x) d x=\int\left[c_{1} e^{\mathrm{ax}} \cos 2 \mathrm{k} \pi \mathrm{x}+\mathrm{c}_{2} \mathrm{e}^{-\mathrm{ax}} \cos (2 \mathrm{n}+1 ; \pi \mathrm{x}] \mathrm{dx}\right. \tag{12}
\end{equation*}
$$

are also Fibonacci functions.
The values of $c_{1}$ and $c_{2}$ for which Eq. (6) assumes the Fibonacci numbers for integer x can be computed by applying the conditions $\mathrm{y}(0)=0$ and $y(1)=1$. That is,

$$
\begin{equation*}
c_{1}+c_{2}=0 \tag{13}
\end{equation*}
$$

$$
\mathrm{c}_{1} \mathrm{e}^{\mathrm{a}}-\mathrm{c}_{2} \mathrm{e}^{-\mathrm{a}}=1
$$

The solutions to the system (13) are $c_{1}=1 / \sqrt{5}$ and $c_{2}=-1 \sqrt{5}$. Thus, the Fibonacci functions that agree with the Fibonacci numbers for integer x are

$$
\mathrm{y}(\mathrm{x})=\left(\mathrm{e}^{\mathrm{ax}} \cos 2 \mathrm{k} \pi \mathrm{x}-\mathrm{e}^{-\mathrm{ax}} \cos (2 \mathrm{n}+1) \pi \mathrm{x}\right) / \sqrt{5}
$$

The function $f(x)=\left(a^{x}-b^{x} \cos \pi x\right) / \sqrt{5} \quad$ [2] is a special case of (14), where $\mathrm{k}=0, \mathrm{n}=0$, and $\mathrm{a}^{\mathrm{x}}$ and $\mathrm{b}^{\mathrm{x}}$ are not identified as exponentials base e.

The usual extension of the Fibonacci sequence to the negative integers satisfies the relation $F_{-n}=(-1)^{n+1} F_{n}$. For integer values of $x$, the Fibonacci functions (14) have the same property.

Since

$$
\cos 2 \mathrm{kn} \pi=(-1)^{2 \mathrm{kn}}=1
$$

and

$$
\cos (2 \mathrm{kn}+\mathrm{n}) \pi=(-1)^{2 \mathrm{kn}}(-1)^{\mathrm{n}}=(-1)^{\mathrm{n}}
$$

we have

$$
\begin{aligned}
\sqrt{5} y(-n)=e^{-a n} \cos 2 k \pi(-n) & -e^{a n} \cos (2 n+1) \pi(-n)= \\
e^{-a n}-(-1)^{n} e^{a n} & =(-1)^{n+1}\left(e^{a n}-e^{-a n}\right)= \\
& =(-1)^{n+1} y(n) \sqrt{5}
\end{aligned}
$$

## REFERENCES

1. M. Elmore, "Fibonacci Functions," Fibonacci Quarterly, 5 (1967), pp. 371-382.
2. F. Parker, "A Fibonacci Function," Fibonacci Quarterly, 6 (1968), pp. 1-2.
3. A. Scott, "Continuous Extensions of Fibonacci Identities," Fibonacci Quarterly, 6 (1968), pp. 245-250.

# COMPLETE DIOPHANTINE SOLUTION OF THE PYTHAGOREAN TRIPLE 

( $\mathbf{a}, \mathbf{b}=\mathbf{a}+\mathbf{1 , c}$ )
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In connection with problem B-123 (Fibonacci Quarterly 5 (1967), p. 288) the question was raised whether Pell numbers provide the only possible Diophantine solutions for the Pythagorean triple ( $a, b=a+1, c$ ). To prove that this is indeed so, it is necessary and sufficient to show that the general solution for the Pythagorean triple, when modified for this special case, acquires a form identical with relationships that are characteristic for Pell numbers, $\mathrm{P}_{\mathrm{n}}$ 。

The proof is based on a property of a class of sequences, of which both the Fibonacci and the Pell sequences are particular cases. By applying the recursion formula that is specific to the Pell sequence, an identity for the general sequence is transformed into one that is valid only for the Pell sequence. It is precisely this identity that must be satisfied by the special Pythagorean triples.

We start with the single-membered, purely periodic, infinite continued fraction

$$
\frac{1}{g^{+}} \cdots=-g+\frac{\sqrt{g^{2}+4}}{2}
$$

where $g$ is a positive integer. The limiting value shown is the positive root, $x_{1}$, of $x^{2}+g x-1=0$ 。 The numerators and denominators of the convergents are obtained by the well-known recursion formula

$$
\begin{equation*}
G_{n+2}=g G_{n+1}+G_{n} \quad\left(G_{0}=0, G_{1}=1\right) \tag{1}
\end{equation*}
$$

These numbers appear in the powers of, e. g.,

$$
-\mathrm{x}_{2}=\frac{\mathrm{g}}{2}+\frac{\sqrt{\mathrm{g}^{2}+4}}{2}
$$

Oct. 1970

$$
\begin{equation*}
\left(-x_{2}\right)^{n+3}=G_{n+2}+G_{n+3} x_{2}=\left(\frac{g}{2} G_{n+3}+G_{n+2}\right)+G_{n+3} \frac{\sqrt{g^{2}+4}}{2} \tag{2}
\end{equation*}
$$

Now
(3) $\left(\frac{g}{2} G_{n+3}+G_{n+2}\right)^{2}=\frac{g^{2}}{4} G_{n+3}^{2}+g G_{n+3} G_{n+2}+G_{n+2}^{2}$,
and

$$
\begin{equation*}
\left(G_{n+3} \frac{\sqrt{g^{2}+4}}{2}\right)^{2}=\frac{g^{2}}{4} G_{n+3}^{2}+g G_{n+3} G_{n+2}+G_{n+3} G_{n+1} \tag{4}
\end{equation*}
$$

By subtracting (4) from (3) and using (1), we obtain

$$
\begin{equation*}
-\left(G_{n+1}^{2}-G_{n+2} G_{n}\right)=G_{n+2}^{2}-G_{n+3} G_{n+1} \tag{5}
\end{equation*}
$$

(the left side being simply a transformation of the right).
Since (5) holds for all values of $n$, each side must equal the same constant. Upon substituting $G_{0}=0$ and $G_{1}=1$ on the left (note that the value of $G_{2}=g$ is not really needed), we find $-\left(1^{2}-g \cdot 0\right)=-1$, and hence

$$
\begin{equation*}
G_{n+2} G_{n}-G_{n+1}^{2}=(-1)^{n+1}=G_{n}^{2}+g G_{n+1} G_{n}-G_{n+1}^{2} \tag{6}
\end{equation*}
$$

When $\mathrm{g}=1$, the left side reduces to the well-known Fibonacci-number identity

$$
F_{n+2} F_{n}-F_{n+1}^{2}=(-1)^{n+1}
$$

When $\mathrm{g}=2$, the right side of (6) can be rewritten as the Pell-number identity

$$
\left(P_{n}+P_{n+1}\right)^{2}-2 P_{n+1}^{2}=(-1)^{n+1}
$$

whence

$$
\begin{equation*}
P_{n}+P_{n+1}=\sqrt{2 P_{n+1}^{2}+(-1)^{n+1}} \tag{7}
\end{equation*}
$$

Also, since the recursion formula for Pell numbers is

$$
P_{n+2}=2 P_{n+1}+P_{n}\left(P_{0}=0, \quad P_{1}=1\right)
$$

from (1) above, we have

$$
\begin{equation*}
P_{n+2}-P_{n+1}=P_{n+1}+P_{n} \tag{8}
\end{equation*}
$$

These equations have thus been shown to be characteristic for Pell numbers.
The Diophantine solution for the Pythagorean triple, with legs a and b and hypotenuse c , is $2 \mathrm{pq}=\mathrm{a}$ or $\mathrm{b}, \mathrm{p}^{2}-\mathrm{q}^{2}=\mathrm{b}$ or a , and $\mathrm{p}^{2}+\mathrm{q}^{2}=$ $c$, where $p$ and $q$ are positive integers of different parity and $p>q$. When $b=a+1, p^{2}-q^{2}-2 p q= \pm 1$. Solving for $p$ and $q$, and rearranging terms,

$$
\begin{equation*}
p+q=\sqrt{2 p^{2} \pm 1} \tag{9}
\end{equation*}
$$

$$
\begin{equation*}
p-q=\sqrt{2 q^{2} \pm 1} . \tag{10}
\end{equation*}
$$

Since (7) and (8) are obviously equivalent to (9) and (10), p and $q$ must be Pell numbers. In fact, when $q=P_{n}$ then $p=P_{n+1}$. The even leg of the triangle is

$$
2 P_{n+1} P_{n}=a_{n} \text { or } b_{n}
$$

the odd leg,

$$
P_{n+1}^{2}-P_{n}^{2}=b_{n} \text { or } a_{n}
$$

and the hypotenuse,

$$
\text { OF THE PYTHAGOREAN TRIPLE }(a, b=a+1, c)
$$

$$
P_{n+1}^{2}+P_{n}^{2}=P_{2 n+1}=c_{n}
$$

Also, the smaller leg is

$$
\sum_{m=1}^{2 n} P_{m}=a_{n} \text { or } b_{n}
$$

Except for the lowest nontrivial value 3, the values for both legs are obviously composite numbers.
[Continued from p. 379.]
TABLE 2

$$
\begin{aligned}
& \mathrm{f}_{1}=\mathrm{e}_{1} \quad \mathrm{f}_{2}=\mathrm{e}_{2} \quad \mathrm{f}_{3}=\mathrm{e}_{1} \mathrm{e}_{2}-\mathrm{e}_{3} \\
& f_{4}=e_{3}-e_{1} e_{2}-e_{1} e_{3}+e_{4}+e_{2}\binom{-e_{1}}{2} f_{5}=-e_{1} e_{2}+e_{3}-e_{2} e_{3}+e_{5}+e_{1}\binom{-e_{2}}{2} \\
& f_{6}=e_{1} e_{2}-e_{3}+2 e_{1} e_{3}-2 e_{4}-2 e_{2}\binom{-e_{1}}{2}+e_{1} e_{4}-e_{6}-e_{2}\binom{-e_{1}}{3}-e_{3}\binom{-e_{1}}{2} \\
& f_{7}=e_{1} e_{2}-e_{3}+e_{1} e_{3}-e_{4}-e_{2}\binom{-e_{1}}{2}+e_{2} e_{3}-e_{5}-e_{1}\binom{-e_{2}}{2} \\
& +e_{1} e_{5}-e_{7}+e_{2} e_{4}-e_{1} e_{2} e_{3}+\binom{-e_{1}}{2}\binom{-e_{2}}{2} \\
& f_{8}=e_{1} e_{2}-e_{3}+2 e_{2} e_{3}-2 e_{5}-2 e_{1}\binom{-e_{2}}{2}+e_{2} e_{5}-e_{8}-e_{3}\binom{-e_{2}}{2}-e_{1}\binom{-e_{2}}{3}
\end{aligned}
$$

TABLE 3

$$
\begin{array}{lll}
h_{1}=e_{2} & h_{2}=e_{1} & h_{3}=e_{1} e_{2}-e_{3} \\
h_{4}=-e_{5}+e_{1}\binom{e_{2}}{2} & h_{5}=-e_{4}+e_{2}\binom{e_{1}}{2} & h_{6}=-e_{8}+e_{1}\binom{e_{2}}{3} \\
h_{7}=-e_{7}+\binom{e_{1}}{2}\binom{e_{2}}{2} & h_{8}=-e_{6}+e_{2}\binom{e_{1}}{3} &
\end{array}
$$

## FIBONACCI TO THE RESCUE

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Every year my PUZZLER problem in the December issue of the magazine Canadian Consulting Engineer is made the subject of a contest, the prize being won by the first acceptable theoretical solution that is opened on a specified date. This last December, in 1969, the problem was as follows:

Charlie was through with the paper. He'd read the comics, studied the sports pages, and even glanced at the headlines of world news. And now he was doodling on the margin of the front page.
"It's funny about all our ages," he said suddenly. "Yours is in the same proportion to mine as mine is to one less than our two ages combined."

Mary had been watching the two children playing with the boy's new scooter on the sidewalk outside. "What of it?" she asked. "A lot of fractions. "
"I mean the complete years." Her husband smiled. "But the funny thing is that the same applies to the ages of the two kids."

Do you know Mary's age?
Many acceptable solutions were received from readers. But one of these, not the first opened however, was most ingenious in its use of the Fibonacci concept for dealing with the relevant diophantine equation. For this reason, I give it in full, as received from the originator, Michael R. Buckley of Toronto, Canada.

We notice Fibonacci lurking between the lines. Let Mary's and Charlie's ages be x and y respectively. Then:

$$
\frac{y}{x}=\frac{x+y-1}{y}
$$

Obviously, the larger x and y are, the more golden becomes their ratio. The simplest solution is $\mathrm{x}=\mathrm{y}=1$ : here x and y being [Continued on p. 420.]

# A PRIMER FOR THE FIBONACCI NUMBERS: PART VII 

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AN INTRODUCTION TO FIBONACCI POLYNOMIALS AND THEIR DIVISIBILITY PROPERTIES

An elementary study of the Fibonacci polynomials yields some general divisibility theorems, not only for the Fibonacci polynomials, but also for Fibonacci numbers and generalized Fibonacci numbers. This paper is intended also to be an introduction to the Fibonacci polynomials.

Fibonacci and Lucas polynomials are special cases of Chebyshev polynomials, and have been studied on a more advanced level by many mathematicians. For our purposes, we define only Fibonacci and Lucas polynomials.

## 1. THE FIBONACCI POLYNOMIALS

The Fibonacci polynomials $\left\{\mathrm{F}_{\mathrm{n}}(\mathrm{x})\right\}$ are defined by

$$
\begin{equation*}
F_{1}(x)=1, \quad F_{2}(x)=x, \quad \text { and } F_{n+1}(x)=x F_{n}(x)+F_{n-1}(x) . \tag{1.1}
\end{equation*}
$$

Notice that, when $\mathrm{x}=1, \mathrm{~F}_{\mathrm{n}}(1)=\mathrm{F}_{\mathrm{n}}$, the $\mathrm{n}^{\text {th }}$ Fibonacci number. It is easy to verify that the relation

$$
\begin{equation*}
\mathrm{F}_{-\mathrm{n}}(\mathrm{x})=(-1)^{\mathrm{n}+1} \mathrm{~F}_{\mathrm{n}}(\mathrm{x}) \tag{1.2}
\end{equation*}
$$

extends the definition of Fibonacci polynomials to all integral subscripts. The first ten Fibonacci polynomials are given below:

$$
\begin{aligned}
& \mathrm{F}_{1}(\mathrm{x})=1 \\
& \mathrm{~F}_{2}(\mathrm{x})=\mathrm{x} \\
& \mathrm{~F}_{3}(\mathrm{x})=\mathrm{x}^{2}+1 \\
& \mathrm{~F}_{4}(\mathrm{x})=\mathrm{x}^{3}+2 \mathrm{x} \\
& \mathrm{~F}_{5}(\mathrm{x})=\mathrm{x}^{4}+3 \mathrm{x}^{2}+1 \\
& \mathrm{~F}_{6}(\mathrm{x})=\mathrm{x}^{5}+4 \mathrm{x}^{3}+3 \mathrm{x} \\
& \mathrm{~F}_{7}(\mathrm{x})=\mathrm{x}^{6}+5 \mathrm{x}^{4}+6 \mathrm{x}^{2}+1 \\
& \\
& 407
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{F}_{8}(\mathrm{x}) & =\mathrm{x}^{7}+6 \mathrm{x}^{5}+10 \mathrm{x}^{3}+4 \mathrm{x} \\
\mathrm{~F}_{9}(\mathrm{x}) & =\mathrm{x}^{8}+7 \mathrm{x}^{6}+15 \mathrm{x}^{4}+10 \mathrm{x}^{2}+1 \\
\mathrm{~F}_{10}(\mathrm{x}) & =\mathrm{x}^{9}+8 \mathrm{x}^{7}+21 \mathrm{x}^{5}+20 \mathrm{x}^{3}+5 \mathrm{x}
\end{aligned}
$$

It is important for Section 4, to notice that the degree of $F_{n}(x)$ is $|n|-1$ for $\mathrm{n} \neq 0$. Also, $\mathrm{F}_{0}(\mathrm{x})=0$.

In Table 1, the coefficients of the Fibonacci polynomials are arranged in ascending order. The sum of the $n^{\text {th }}$ row is $F_{n}$, and the sum of the $n^{\text {th }}$ diagonal of slope 1 , formed by beginning on the $n$th row, left column, and going one up and one right to get the next term, is given by

$$
2^{(\mathrm{n}-1) / 2}=2 \cdot 2^{(\mathrm{n}-3) / 2}
$$

when n is odd.

Table 1
Fibonacci Polynomial Coefficients Arranged in Ascending Order

| n | $\mathrm{x}^{0}$ | $\mathrm{x}^{1}$ | $\mathrm{x}^{2}$ | $\mathrm{x}^{3}$ | $\mathrm{x}^{4}$ | $\mathrm{x}^{5}$ | $\mathrm{x}^{6}$ | $\mathrm{x}^{7}$ | $\mathrm{x}^{8}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 1 |  |  |  |  |  |  |  |  |
| 2 | 0 | 1 |  |  |  |  |  |  |  |
| 3 | 1 | 0 | 1 |  |  |  |  |  |  |
| 4 | 0 | 2 | 0 | 1 |  |  |  |  |  |
| 5 | 1 | 0 | 3 | 0 | 1 |  |  |  |  |
| 6 | 0 | 3 | 0 | 4 | 0 | 1 |  |  |  |
| 7 | 1 | 0 | 6 | 0 | 5 | 0 | 1 |  |  |
| 8 | 0 | 4 | 0 | 10 | 0 | 6 | 0 | 1 |  |
| 9 | 1 | 0 | 10 | 0 | 15 | 0 | 7 | 0 | 1 |

To compare with Pascal's triangle, the sum of the $\mathrm{n}^{\text {th }}$ row there is $2^{\mathrm{n}}$, and the sum of the $n^{\text {th }}$ diagonal of slope one is $F_{n}$. In fact, the (alternate) diagonals of slope 1 in Table 1 produce Pascal's triangle.

If the successive binomial expansions of $(x+1)^{n}$ are written in descending order,
$\mathrm{n}=0: 1$
$\mathrm{n}=1: \mathrm{x}+1$
$\mathrm{n}=2: \mathrm{x}^{2}+2 \mathrm{x}+1$
$n=3: x^{3}+3 x^{2}+3 x+1$
$n=4: x^{4}+4 x^{3}+6 x^{2}+4 x+1$
the sum of the 4 th diagonal of slope 1 is $F_{4}(x)=x^{4}+3 x^{2}+1$, and the sum of the $n$th diagonal of slope 1 is $F_{n}(x)$, or,

$$
\begin{equation*}
F_{n}(x)=\sum_{j=0}^{[(n-1) / 2]}(n-j-1) x^{n-2 j-1} \tag{1.3}
\end{equation*}
$$

for $[x]$ the greatest integer contained in $x$, and binomial coefficient $\binom{n}{j}$, as given by Swamy [1] and others.

## 2. LUCAS POLYNOMIALS AND GENERAL FIBONACCI POLYNOMIALS

The Lucas polynomials $\left\{L_{n}(x)\right\}$ are defined by

$$
\begin{equation*}
L_{0}(x)=2, \quad L_{1}(x)=x, \quad L_{n+1}(x)=x L_{n}(x)+L_{n-1}(x) \tag{2.1}
\end{equation*}
$$

Again, when $x=1, L_{n}(1)=L_{n}$, the $n^{\text {th }}$ Lucas number. Lucas polynomials have the properties that

$$
\begin{align*}
& L_{n}(x)=F_{n+1}(x)+F_{n-1}(x)=x F_{n}(x)+2 F_{n-1}(x)  \tag{2.2}\\
& x L_{n}(x)=F_{n+2}(x)-F_{n-2}(x)
\end{align*}
$$

and can be extended to negative subscripts by

$$
\begin{equation*}
\mathrm{L}_{-\mathrm{n}}(\mathrm{x})=(-1)^{\mathrm{n}_{\mathrm{L}}} \mathrm{~L}_{\mathrm{n}}(\mathrm{x}) \tag{2.3}
\end{equation*}
$$

The degree of $L_{n}(x)$ is $\left.\mid n\right\rfloor$, as can be observed in the following list of the first ten Lucas polynomials:

$$
\begin{aligned}
& \mathrm{L}_{1}(\mathrm{x})=\mathrm{x} \\
& \mathrm{~L}_{2}(\mathrm{x})=\mathrm{x}^{2}+2 \\
& \mathrm{~L}_{3}(\mathrm{x})=\mathrm{x}^{2}+3 \mathrm{x} \\
& \mathrm{~L}_{4}(\mathrm{x})=\mathrm{x}^{4}+4 \mathrm{x}^{2}+2 \\
& \mathrm{~L}_{5}(\mathrm{x})=\mathrm{x}^{5}+5 \mathrm{x}^{3}+5 \mathrm{x} \\
& \mathrm{~L}_{6}(\mathrm{x})=\mathrm{x}^{6}+6 \mathrm{x}^{4}+9 \mathrm{x}^{2}+2 \\
& \mathrm{~L}_{7}(\mathrm{x})=\mathrm{x}^{7}+7 \mathrm{x}^{5}+14 \mathrm{x}^{3}+7 \mathrm{x} \\
& \mathrm{~L}_{8}(\mathrm{x})=\mathrm{x}^{8}+8 \mathrm{x}^{6}+20 \mathrm{x}^{4}+16 \mathrm{x}^{2}+2 \\
& \mathrm{~L}_{9}(\mathrm{x})=\mathrm{x}^{9}+9 \mathrm{x}^{7}+27 \mathrm{x}^{5}+30 \mathrm{x}^{3}+9 \mathrm{x} \\
& \mathrm{~L}_{10}(\mathrm{x})=\mathrm{x}^{10}+10 \mathrm{x}^{8}+35 \mathrm{x}^{6}+50 \mathrm{x}^{4}+25 \mathrm{x}^{2}+2
\end{aligned}
$$

If the Lucas polynomial coefficients are arranged in ascending order in a leftjustified triangle similar to that of Table 1 , the sum of the $n^{\text {th }}$ row is $L_{n}$, and the sum of the $n^{\text {th }}$ diagonal of slope 1 is given by $3 \cdot 2^{(n-2) / 2}$ for even $\mathrm{n}, \mathrm{n} \geq 2$.

When general Fibonacci polynomials are defined by

$$
\begin{equation*}
\mathrm{H}_{1}(\mathrm{x})=\mathrm{a}, \quad \mathrm{H}_{2}(\mathrm{x})=\mathrm{bx}, \quad \mathrm{H}_{\mathrm{n}}(\mathrm{x})=\mathrm{xH}_{\mathrm{n}-1}(\mathrm{x})+\mathrm{H}_{\mathrm{n}-2}(\mathrm{x}) \tag{2.4}
\end{equation*}
$$

then

$$
\begin{equation*}
\mathrm{H}_{\mathrm{n}}(\mathrm{x})=\mathrm{bx} \mathrm{~F}_{\mathrm{n}-1}(\mathrm{x})+a \mathrm{~F}_{\mathrm{n}-2}(\mathrm{x}) \tag{2.5}
\end{equation*}
$$

If the coefficients of the $\left\{H_{n}(x)\right\}$, written in ascending order, are placed in a left-justified triangle such as Table 1 , then the sum of the $n^{\text {th }}$ diagonal of slope 1 is

$$
(\mathrm{a}+\mathrm{b}) \cdot 2^{(\mathrm{n}-3) / 2}=(\mathrm{a}+\mathrm{b}) \cdot 2^{[(\mathrm{n}-2) / 2]}
$$

for odd $\mathrm{n}, \mathrm{n} \geq 3$. (Notice that, if $\mathrm{a}=2, \mathrm{~b}=1$, then $\mathrm{H}_{\mathrm{n}+1}(\mathrm{x})=\mathrm{L}_{\mathrm{n}}(\mathrm{x})$, and if $\mathrm{a}=\mathrm{b}=1, \mathrm{H}_{\mathrm{n}}(\mathrm{x})=\mathrm{F}_{\mathrm{n}}(\mathrm{x})$.)

## 3. A MATRIX GENERATOR FOR FIBONACCI POLYNOMIALS

Since Fibonacci polynomials appear as the elements of the matrix defined below, many identities can be derived for Fibonacci polynomials using matrix theory, as done by Hayes [2] and others, and as done for Fibonacci numbers by Basin and Hoggatt [3].

It is easily established by mathematical induction that the matrix

$$
\mathrm{Q}=\left(\begin{array}{ll}
\mathrm{x} & 1 \\
1 & 0
\end{array}\right)
$$

when raised to the $\mathrm{k}^{\text {th }}$ power, is given by

$$
Q^{k}=\left(\begin{array}{ll}
\mathrm{F}_{\mathrm{k}+1}(\mathrm{x}) & \mathrm{F}_{\mathrm{k}}(\mathrm{x})  \tag{3.1}\\
\mathrm{F}_{\mathrm{k}}(\mathrm{x}) & \mathrm{F}_{\mathrm{k}-1}(\mathrm{x})
\end{array}\right)
$$

for any integer $k$, where $Q^{0}$ is the identity matrix and $Q^{-k}$ is the matrix inverse of $Q^{k}$. Since $\operatorname{det} Q=-1, \operatorname{det} Q^{k}=(\operatorname{det} Q)^{k}=(-1)^{k}$ gives us

$$
\begin{equation*}
\mathrm{F}_{\mathrm{k}+1}(\mathrm{x}) \mathrm{F}_{\mathrm{k}-1}(\mathrm{x})-\mathrm{F}_{\mathrm{k}}^{2}(\mathrm{x})=(-1)^{\mathrm{k}} \tag{3.2}
\end{equation*}
$$

Since $Q^{m} Q^{n}=Q^{m+n}$ for all integers $m$ and $n$, matrix multiplication of $\mathrm{Q}^{\mathrm{m}}$ and $\mathrm{Q}^{\mathrm{n}}$ gives
$Q^{m} Q^{n}=\left(\begin{array}{ll}F_{m+1}(x) F_{n+1}(x)+F_{m}(x) F_{n}(x) & F_{m+1}(x) F_{n}(x)+F_{m}(x) F_{n-1}(x) \\ F_{m}(x) F_{n+1}(x)+F_{m-1}(x) F_{n}(x) & F_{m}(x) F_{n}(x)+F_{m-1}(x) F_{n-1}(x)\end{array}\right)$
while

$$
Q^{m+n}=\left(\begin{array}{ll}
F_{m+n+1}(x) & F_{m+n}(x) \\
F_{m+n}(x) & F_{m+n-1(x)}
\end{array}\right)
$$

Equating elements in the upper right corner gives

$$
\begin{equation*}
\mathrm{F}_{\mathrm{m}+\mathrm{n}}(\mathrm{x})=\mathrm{F}_{\mathrm{m}+1}(\mathrm{x}) \mathrm{F}_{\mathrm{n}}(\mathrm{x})+\mathrm{F}_{\mathrm{m}}(\mathrm{x}) \mathrm{F}_{\mathrm{n}-1}(\mathrm{x}) \tag{3.3}
\end{equation*}
$$

Replacing n by ( -n ) and using the identity

$$
\mathrm{F}_{-\mathrm{n}}(\mathrm{x})=(-1)^{\mathrm{n}+1} \mathrm{~F}_{\mathrm{n}}(\mathrm{x})
$$

gives

$$
\left.\mathrm{F}_{\mathrm{m}-\mathrm{n}}(\mathrm{x})=(-1)^{\mathrm{n}_{( }} \mathrm{F}_{\mathrm{m}+1}(\mathrm{x}) \mathrm{F}_{\mathrm{n}}(\mathrm{x})+\mathrm{F}_{\mathrm{m}}(\mathrm{x}) \mathrm{F}_{\mathrm{n}+1}(\mathrm{x})\right)
$$

Then,

$$
\begin{aligned}
\mathrm{F}_{\mathrm{m}+\mathrm{n}}(\mathrm{x})+(-1)^{\mathrm{n}_{\mathrm{F}_{\mathrm{m}-\mathrm{n}}}(\mathrm{x})} & =\mathrm{F}_{\mathrm{m}}(\mathrm{x}) \mathrm{F}_{\mathrm{n}-1}(\mathrm{x})+\mathrm{F}_{\mathrm{m}}(\mathrm{x}) \mathrm{F}_{\mathrm{n}+1}(\mathrm{x}) \\
& =\mathrm{F}_{\mathrm{m}}(\mathrm{x}) \mathrm{L}_{\mathrm{n}}(\mathrm{x})
\end{aligned}
$$

If we replace $n$ by $k$ and $m$ by $m-k$ above, we can obtain

$$
\begin{equation*}
\mathrm{F}_{\mathrm{m}}(\mathrm{x})=\mathrm{L}_{\mathrm{k}}(\mathrm{x}) \mathrm{F}_{\mathrm{m}-\mathrm{k}}(\mathrm{x})+(-1)^{\mathrm{k}+1_{\mathrm{F}}} \mathrm{~F}_{\mathrm{m}-2 \mathrm{k}}(\mathrm{x}) \tag{3.4}
\end{equation*}
$$

which results in the divisibility theorems of the next section
4. DIVISIBחITY PROPERTIES OF FIBONACCI AND LUCAS POLYNOMIALS

Lemma. The Fibonacci polynomials $\mathrm{F}_{\mathrm{m}}(\mathrm{x})$ satisfy

$$
\begin{aligned}
& F_{m}(x)=F_{m-k}(x)\left(\sum_{i=0}^{p-1}(-1)^{i(m-k)} L_{(2 i}+1\right) k-2 i m \\
&(x)) \\
&+(-1)^{p(m-k)+m+1} F_{(2 p-1) m-2 p k}(x)
\end{aligned}
$$

for all integers $m$ and $k$, and for $p \geq 1$.
Proof: If $\mathrm{p}=1$, the Lemma is just Equation (3.4). For convenience, call $\mathrm{Q}_{\mathrm{p}}(\mathrm{x})$ the sum of Lucas polynomials in the Lemma. Then, assume that the Lemma holds when $p=j$, or that

1970]
(A)

$$
\mathrm{F}_{\mathrm{m}}(\mathrm{x})=\mathrm{F}_{\mathrm{m}-\mathrm{k}}(\mathrm{x}) \mathrm{Q}_{\mathrm{j}}(\mathrm{x})+(-1)^{\mathrm{j}(\mathrm{~m}-\mathrm{k})+\mathrm{m}+1} \mathrm{~F}_{(2 \mathrm{j}-1) \mathrm{m}-2 \mathrm{jk}}(\mathrm{x})
$$

Substitute ( $2 \mathrm{jk}-(2 \mathrm{j}-1) \mathrm{m}$ ) for m in Equation (3.4), giving
$F_{2 j k-(2 j-1) m}(x)=L_{k^{\prime}}(x) F_{2 j k-(2 j-1) m-k^{\prime}}(x)+(-1)^{k^{\prime}+1} F_{2 j k-(2 j-1) m-2 k^{\prime}}(x)$.

Since we want to express $\mathrm{F}_{2 \mathrm{jk}-(2 \mathrm{j}-1) \mathrm{m}}(\mathrm{x})$ in terms of $\mathrm{F}_{\mathrm{m}-\mathrm{k}}(\mathrm{x})$, set

$$
2 \mathrm{jk}-(2 \mathrm{j}-1) \mathrm{m}-\mathrm{k}^{\prime}=\mathrm{m}-\mathrm{k}
$$

and solve for $\mathrm{k}^{\prime}$, yielding $\mathrm{k}^{\prime}=(2 \mathrm{j}+1) \mathrm{k}-2 \mathrm{jm}$, so that
$\mathrm{F}_{2 \mathrm{jk}-(2 \mathrm{j}-1) \mathrm{m}}(\mathrm{x})=\mathrm{L}_{(2 \mathrm{j}+1) \mathrm{k}-2 \mathrm{jm}}(\mathrm{x}) \mathrm{F}_{\mathrm{m}-\mathrm{k}}(\mathrm{x})+(-1)^{\mathrm{k}+1} \mathrm{~F}_{(2 \mathrm{j}+1) \mathrm{m}-(2 \mathrm{j}+2) \mathrm{k}}(\mathrm{x})$.

Substituting into (A) and using

$$
\mathrm{F}_{-\mathrm{n}}(\mathrm{x})=(-1)^{\mathrm{n}+1} \mathrm{~F}_{\mathrm{n}}(\mathrm{x})
$$

to simplify gives
$\begin{aligned} & F_{m}(x)=\left(Q_{j}(x)+(-1)^{j(m-k)} L_{(2 j+1) k-2 j m}{ }^{(x))} F_{m-k}(x)\right. \\ &+(-1)^{(j+1)(m-k)+m+1} F_{(2 j+1) m-(2 j+2) k}(x),\end{aligned}$
which is the Lemma when $p=j+1$, completing a proof by mathematical induction.

Notice that the Lemma yields an interesting identity for Fibonacci numbers, given below:
(4.1) $F_{m}=F_{m-k}\left(\sum_{i=0}^{p-1}(-1)^{i(m-k)} L_{m-(2 i+1)(m-k)}\right)+(-1)^{p(m-k)} F_{m-2 p(m-k)}$,

To establish (4.1), use algebra on the subscripts of the Lemma and then take $\mathrm{x}=1$.

Theorem 1: Whenever a Fibonacci polynomial $\mathrm{F}_{\mathrm{m}}(\mathrm{x})$ is divided by a Fibonacci polynomial $\mathrm{F}_{\mathrm{m}-\mathrm{k}}(\mathrm{x}), \mathrm{m} \neq \mathrm{k}$, of lesser or equal degree, the remainder is always a Fibonacci polynomial or the negative of a Fibonacci polynomial, and the quotient is a sum of Lucas polynomials whenever the division is not exact. Explicitly, for $p \geq 1$,
(i) the remainder is $\pm \mathrm{F}_{(2 \mathrm{p}-1) \mathrm{m}-2 \mathrm{kp}}(\mathrm{x})$ when

$$
\frac{2 p|m|}{2 p+1}>|k|>\frac{(2 p-2)|m|}{2 p-1}
$$

or, equivalently, if $\pm F_{m-2 p(m-k)}(x)$ for

$$
\frac{|m|}{2 p+1}<|m-k|<\frac{|m|}{2 p-1}
$$

(ii) the quotient is $\pm \mathrm{L}_{\mathrm{k}}(\mathrm{x})$ when $|\mathrm{k}|<2|\mathrm{~m}| / 3$;
(iii) the quotient is given by

$$
Q_{p}(x)=\sum_{i=0}^{p-1}(-1)^{i(m-k)} L_{(2 i+1) k-2 i m}(x)=\sum_{i=0}^{p-1}(-1)^{i(m-k)} L_{m-(2 i+1)(m-k)}(x)
$$

for $m, k$, and $p$ related as in (i), and by $Q_{p}(x)+(-1)^{p(m-k)}$ if

$$
\mathrm{k}=\frac{2 \mathrm{pm}}{2 \mathrm{p}+1}
$$

(iv) the division is exact when

$$
k=\frac{2 p m}{2 p+1} \quad \text { or } \quad k=\frac{(2 p-1) m}{2 p}
$$

Proof: When

$$
\frac{2 p|m|}{2 p+1}>|k|>\frac{(2 p-2)|m|}{2 p-1}
$$

and the degree of $F_{m}(x)$ is greater than that of $F_{m-k}(x)$, we can show that $|m|>|m-k|>|(2 p-1) m-2 p k|$. Since the degree of $F_{n}$ is $|n|-1$, we can interpret the Lemma in terms of quotients and remainders for the restrictions on $\mathrm{m}, \mathrm{k}$, and p above, establishing (i), (ii), and (iii). As for (iv), the division is exact if

$$
\mathrm{k}=\frac{(2 \mathrm{p}-1) \mathrm{m}}{2 \mathrm{p}}
$$

for then

$$
F_{(2 \mathrm{p}-1) \mathrm{m}-2 \mathrm{pk}}(\mathrm{x})=\mathrm{F}_{0}(\mathrm{x})=0
$$

When $\mathrm{k}=\frac{2 \mathrm{pm}}{2 \mathrm{p}+1}$,

$$
\begin{aligned}
\mathrm{F}_{(2 \mathrm{p}-1) \mathrm{m}-2 \mathrm{pk}}(\mathrm{x}) & =\mathrm{F}_{\mathrm{k}-\mathrm{m}}(\mathrm{x})=(-1)^{\mathrm{m}-\mathrm{k}+1} \mathrm{~F}_{\mathrm{m}-\mathrm{k}}(\mathrm{x}) \\
& =(-1)^{\mathrm{m}+1} \mathrm{~F}_{\mathrm{m}-\mathrm{k}}(\mathrm{x})
\end{aligned}
$$

because k is even. Referring to the Lemma, increasing the quotient by

$$
(-1)^{\mathrm{p}(\mathrm{~m}-\mathrm{k})+\mathrm{m}+1+\mathrm{m}+1}=(-1)^{\mathrm{p}(\mathrm{~m}-\mathrm{k})}
$$

will make the division exact.
Corollary 1.1: $F_{q}(x)$ divides $F_{m}(x)$ if and only if $q$ divides $m$.
Proof: If $q$ divides $m$, then either $m / 2 p=q$ or $m /(2 p+1)=q$. Let $q=m-k$ and apply Theorem 1 .

If $\mathrm{F}_{\mathrm{q}}(\mathrm{x})$ divides $\mathrm{F}_{\mathrm{m}}(\mathrm{x})$, then let $\mathrm{q}=\mathrm{m}-\mathrm{k}$ and consider the remainder of Theorem 1. Either

$$
\mathrm{F}_{(2 \mathrm{p}-1) \mathrm{m}-2 \mathrm{pk}}(\mathrm{x})=\mathrm{F}_{0}(\mathrm{x})
$$

or

$$
\mathrm{F}_{(2 \mathrm{p}-1) \mathrm{m}-2 \mathrm{pk}}(\mathrm{x})= \pm \mathrm{F}_{\mathrm{m}-\mathrm{k}}(\mathrm{x})
$$

giving

$$
k=\frac{(2 p-1) m}{2 p}, \quad k=\frac{(2 p-2) m}{2 p-1}, \quad \text { or } \quad k=\frac{2 p m}{2 p+1}
$$

by equating subscripts. The possibilities give $q=m-k=m / 2 p$ or $q=$ $m /(2 p-1)$ or $q=m /(2 p+1)$, so that $q$ divides $m$.

Corollary 1.2: If the Fibonacci number $\mathrm{F}_{\mathrm{m}}$ is divided by $\mathrm{F}_{\mathrm{m}-\mathrm{k}}, \mathrm{m} \neq$ k , then the remainder of least absolute value is always a Fibonacci number or its negative. Further,
(i) the remainder is $\pm F_{m-2 p(m-k)}$ when

$$
\frac{|m|}{2 p+1}<|m-k|<\frac{|m|}{2 p-1}, \quad m-k \neq 2
$$

and the quotient is the sum of Lucas numbers;
(ii) the quotient is $\pm L_{k}$ when $|\mathrm{k}|<2|\mathrm{~m}| / 3$, for Lucas number $L_{k^{\circ}}$

Proof: Let $\mathrm{x}=1$ throughout Theorem 1. Since the magnitudes of Fibonacci numbers are ordered by their subscripts, $\pm F_{m-2 p(m-k)}$ represents a remainder (unless $m-k=2$ since $F_{2}=F_{1}=1$ ).

To illustrate Corollary 1.2, divide $F_{13}$ by $F_{7}$ :

$$
233=17 \cdot 13+12=18 \cdot 13+(-1)
$$

Now, 12 is the remainder in usual division, but we consider the positive and negative remainders with absolute value less than that of the divisor, so that $(-1)=-\mathrm{F}_{1}$ is the remainder of least absolute value. Here, $\mathrm{m}=13, \mathrm{k}=6$ $<2 \mathrm{~m} / 3, \mathrm{p}=1$, and the quotient is $\mathrm{L}_{6}=18$. The remainders found upon dividing one Fibonacci number by another have been discussed by Taylor [4], and Halton [5].

Corollary 1.3: The Fibonacci number $\mathrm{F}_{\mathrm{q}}$ divides $\mathrm{F}_{\mathrm{m}}$ if and only if q divides $m,|q| \neq 2$.

Proof: If $q$ divides $m$, let $x=1$ in Corollary 1.1. If $F_{q}$ divides $F_{m}$, let $q=m-k$. The remainder of Corollary 1.2 becomes $F_{m-2 p(m-k)}$ $=\mathrm{F}_{0}=0$ or $\mathrm{F}_{\mathrm{m}-2 \mathrm{p}(\mathrm{m}-\mathrm{k})}= \pm \mathrm{F}_{\mathrm{m}-\mathrm{k}^{*}}$. The algebra on the subscripts follows the proof of Corollary 1.1, which will prove that $q$ divides $m$, provided that there are no cases of mistaken identity, such as $\mathrm{F}_{\mathrm{s}}=\mathrm{F}_{\mathrm{q}},|\mathrm{s}| \neq|\mathrm{q}|$, and
such that $s$ does not divide $m$. Thus, the restriction $|q| \neq 2$ since

$$
\mathrm{F}_{2}=\mathrm{F}_{1}=1
$$

Unfortunately, as pointed out by E. A. Parberry, Corollary 1.3 cannot be proved immediately from Corollary 1.1 by simply taking $x=1$. That $F_{q}$ divides $F_{m}$ does not imply that $F_{q}(x)$ divides $F_{m}(x)$, just as that $f(1)$ divides $\mathrm{g}(1)$ does not imply that $\mathrm{f}(\mathrm{x})$ divides $\mathrm{g}(\mathrm{x})$ for arbitrarypolynomials $f(x)$ and $g(x)$. Also, Webb and Parberry [8] have proved that a Fibonacci polynomial $\mathrm{F}_{\mathrm{m}}(\mathrm{x})$ is irreducible over the integers if and only if m is prime. But, if $m$ is prime, while $F_{m}$ is not divisible by any other Fibonacci number $F_{q},|q| \geq 3, F_{m}$ is not necessarily a prime. How to determine all values of $m$ for which $F_{m}$ is prime when $m$ is prime, is an unsolved problem.

Corollary 1.4: There exist an infinite number of sequences $\left\{S_{n}\right\}$ having the division property that, when $S_{m}$ is divided by $S_{m-k}, m \neq k$, the remainder of least absolute value is always a member of the sequence or the negative of a member of the sequence.

Proof: We can let x be any integer in the Lemma and throughout Theorem 1. If $x=2$, one such sequence is $\ldots, 0,1,2,5,12,29,70,169, \cdots$ 。

Theorem 2: Whenever a Lucas polynomial $L_{m}(x)$ is divided by a Lucas polynomial $L_{m-k}(x), m \neq k$, of lesser degree, a non-zero remainder is always a Lucas polynomial or the negative of a Lucas polynomial. Explicitly,
(i) non-zero remainders have the form $\pm \mathrm{L}_{(2 \mathrm{p}-1) \mathrm{m}-2 \mathrm{pk}}(\mathrm{x})$ when

$$
\frac{2 p|m|}{2 p+1}>|k|>\frac{(2 p-2)[m \mid}{2 p-1}
$$

or, equivalently, $\pm \mathrm{L}_{2 \mathrm{p}(\mathrm{m}-\mathrm{k})-\mathrm{m}}{ }^{(\mathrm{x})}$ for

$$
\frac{|m|}{2 p+1}<|m-k|<\frac{|m|}{2 p-1}
$$

(ii) if $|\mathrm{k}|<2|\mathrm{~m}| / 3$, the quotient is $\pm \mathrm{L}_{\mathrm{k}}(\mathrm{x})$;
(iii) the division is exact when $\mathrm{k}=2 \mathrm{pm} /(2 \mathrm{p}+1), \mathrm{p} \neq 0$.

Proof: Since the proof parallels that of the Lemma and Theorem 1, details are omitted. Identity (3.生) is used to establish

$$
\begin{equation*}
\mathrm{L}_{\mathrm{m}}(\mathrm{x})=\mathrm{L}_{\mathrm{k}}(\mathrm{x}) \mathrm{L}_{\mathrm{m}-\mathrm{k}}(\mathrm{x})+(-1)^{\mathrm{k}+1} \mathrm{~L}_{\mathrm{m}-2 \mathrm{k}}(\mathrm{x}) \tag{4.2}
\end{equation*}
$$

Since $L_{-n}(x)=(-1)^{n_{1}} L_{n}(x)$, it can be proved that

$$
\mathrm{L}_{\mathrm{m}}(\mathrm{x})=\mathrm{Q}_{\mathrm{p}}(\mathrm{x}) \mathrm{L}_{\mathrm{m}-\mathrm{k}}(\mathrm{x}) \pm \mathrm{L}_{(2 \mathrm{p}-1) \mathrm{m}-2 \mathrm{pk}}(\mathrm{x})
$$

for $|m| \geq|m-k| \geq|(2 p-1) m-2 p k|$. Since the degree of $L_{n}(x)$ is $|n|$, the rest of the proof is similar to that of Theorem 1. However, notice that it is necessary to both proofs that $F_{-n}(x)= \pm F_{n}(x)$ and $L_{-n}(x)= \pm L_{n}(x)$.

Corollary 2.1: The Lucas polynomial $\mathrm{L}_{\mathrm{q}}(\mathrm{x})$ divides $\mathrm{L}_{\mathrm{m}}(\mathrm{x})$ if and only if m is an odd multiple of q .

Proof: If $m=(2 p+1) q$, let $q=m-k$ and Theorem 2 guarantees that $L_{q}(x)$ divides $L_{m}(x)$.

If $L_{q}(x)$ divides $L_{m}(x)$, then let $q=m-k$. For the division to be exact, the term $\pm \mathrm{L}_{(2 \mathrm{p}-1) \mathrm{m}-2 \mathrm{pk}}(\mathrm{x})$ must equal $\mathrm{L}_{\mathrm{m}-\mathrm{k}}(\mathrm{x})$ since it cannot be the zero polynomial. Then, either $k=2 p m /(2 p+1)$ or $k=2 p m /(2 p-1)$, so $q=m-k=m /(2 p+1)$ or $q=m /(2 p-1)$. In either case, $m$ is an odd multiple of $q$.

Corollary 2.2: If a Lucas number $\mathrm{L}_{\mathrm{m}}$ is divided by $\mathrm{L}_{\mathrm{m}-\mathrm{k}}$, then the non-zero remainder of least absolute value is always a Lucas number or its negative with the form $\pm L_{2 p(m-k)-m}$ for

$$
\frac{|m|}{2 p+1}<|m-k|<\frac{|m|}{2 p-1}
$$

and the quotient is $\pm \mathrm{L}_{\mathrm{k}}$ when $|\mathrm{k}|<2|\mathrm{~m}| / 3$.
Proof: Let $\mathrm{x}=1$ throughout the development of Theorem 2.
Corollary 2.3: The Lucas number $L_{q}$ divides $L_{m}$ if and only if $m=$ $(2 s+1) q$ for some integer $s$. (This result is due to Carlitz [6]).

Proof: If $m=(2 s+1) q$, let $x=1$ in Corollary 2.1. If $L_{q}$ divides $L_{m}$, take $q=m-k$ and examine the remainder $L_{2 p(m-k)-m}$ of Corollary 2.2 which must equal $\mathrm{L}_{\mathrm{m}-\mathrm{k}}$ or $\mathrm{L}_{\mathrm{k}-\mathrm{m}}$ since it cannot be zero. The algebra follows that given in Corollary 2.1. Since there are no Lucas numbers such that $L_{q}=L_{s}$ where $|q| \neq|s|$, and since $L_{q} \neq 0$ for any $q$, there are no restrictions.

Since the generalized Fibonacci polynomials $H_{m}(x)$ satisfy $H_{m}(x)=$ $\mathrm{bxF}_{\mathrm{m}-1}(\mathrm{x})+\mathrm{aF}_{\mathrm{m}-2}(\mathrm{x})$, we can show that

$$
\begin{equation*}
\mathrm{H}_{\mathrm{m}}(\mathrm{x})=\mathrm{L}_{\mathrm{k}}(\mathrm{x}) \mathrm{H}_{\mathrm{m}-\mathrm{k}}(\mathrm{x})+(-1)^{\mathrm{k}+1_{1}} \mathrm{H}_{\mathrm{m}-2 \mathrm{k}}(\mathrm{x}) \tag{4.3}
\end{equation*}
$$

but since $H_{m}(\mathrm{x}) \neq \pm \mathrm{H}_{-\mathrm{m}}(\mathrm{x})$, we have a more limited theorem.
Theorem 3: Whenever a generalized Fibonacci polynomial $H_{m}(x)$ is divided by $\mathrm{H}_{\mathrm{m}-\mathrm{k}}(\mathrm{x}), 2 \mathrm{~m} / 3>\mathrm{k}>0$, any non-zero remainder is always another generalized Fibonacci polynomial or its negative, and the quotient is $L_{k}(\mathrm{x})$.

As a consequence of Theorem 3, when a generalized Fibonacci number $\mathrm{H}_{\mathrm{m}}$ is divided by $\mathrm{H}_{\mathrm{q}}$, a non-zero remainder of least absolute value is guaranteed to be another generalized Fibonacci number only when $|m-q| \leqslant 2 m / 3$. Taylor [4] has proved that, of all the generalized Fibonacci sequences $\left\{\mathrm{H}_{\mathrm{m}}\right\}$ satisfying the recurrence $H_{m}=H_{m-1}+H_{m-2}$, the only sequences with the division property that the non-zero remainders of least absolute value are always a member of the sequence or the negative of a member of the sequence, are the Fibonacci and Lucas sequences. For your further reading, Hoggatt [7] gives a lucid description of divisibility properties of Fibonacci and Lucas numbers.

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[Continued from p. 406.]
the first two terms in the Fibonacci series. Who could resist the temptation to test the conjecture that $y / x=F_{n+1} / F_{n}$ ?

Now let $x=k F_{n}, y=k F_{n+1^{\circ}}$ Then,

$$
F_{n+1} / F_{n}=\left[k\left(F_{n}+F_{n+1}\right)-1\right] / k F_{n+1}
$$

so

$$
\mathrm{k}\left(\mathrm{~F}_{\mathrm{n}+1}^{2}-\mathrm{F}_{\mathrm{n}} \mathrm{~F}_{\mathrm{n}+2}\right)=-\mathrm{F}_{\mathrm{n}}
$$

but

$$
\mathrm{F}_{\mathrm{n}+1}^{2}-\mathrm{F}_{\mathrm{n}} \mathrm{~F}_{\mathrm{n}+2}=(-1)^{\mathrm{n}}
$$

hence n is odd, and we have $\mathrm{k}=\mathrm{F}_{\mathrm{n}}$. So,

$$
\begin{aligned}
& \mathrm{x}=\mathrm{F}_{2 \mathrm{~m}-1}^{2}=1,4,25,169, \text { etc. } \\
& \mathrm{y}=\mathrm{F}_{2 \mathrm{~m}-1} \mathrm{~F}_{2 \mathrm{~m}}=1,6,40,273, \text { etc }
\end{aligned}
$$

Hence, the children were 4 and 6 years old, Charlie 40, and Mary 25.


# AN EXTENSION OF A THEOREM OF EULER 

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Leonhard Euler, the great mathematician of the 18th Century, wrote some of the greatest works ever read by man. Among the numerous mathematical interests of this genius was the study of the problem of partitions.

A partition of an arbitrary positive integer, say $\underline{n}$, is a representation of $n$ as the sum of any number of integral parts. For example, the number 6 has 11 partitions, since
$6=5+1=4+2=4+1+1=3+3=3+2+1=3+1+1+1$ $=2+2+2=2+2+1+1=2+1+1+1+1=1+1+1+1+1+1$.
(Problem. Show that the number of partitions of 7 is equal to 15. )
It is apparent that to find the partition of some positive integer $n$ using the same methods we used to find the solution for $n=6$ above is a clumsy and difficult procedure. To overcome this difficulty, Euler combined the partition with a generating function which led to his discovering a powerful recurrence formula with which to attack the study of partitions.
(For those readers who wish to discover how Euler found his recurrence formula, the author recommends they read a chapter on partitions in any good book on number theory. One such book is An Introduction to the Theory of Numbers by Niven and Zuckerman, published by John Wiley and Sons, Inc. )

EULER'S FAMOUS RECURRENCE FORMULA is usually written as follows:

$$
\begin{align*}
p(n)= & p(n-1)+p(n-2)-p(n-5)-p(n-7)+\cdots \\
& +(-1)^{k+1} p\left(n-\frac{1}{2} k(3 k-1)\right)+(-1)^{k+1} p\left(n-\frac{1}{2} k(3 k+1)\right)+\cdots, \tag{1}
\end{align*}
$$

where $\mathrm{p}(0)=1$ and $\mathrm{n}, \mathrm{k}=1,2,3, \ldots$.
To understand the above theorem (1), let us go back to the beginning of this paper where we have shown that the number $\mathrm{n}=6$ has 11 partitions.

Then, what the formula says in a very precise way is that $p(6)=11$, or more generally, $p(n)$ is the number of ways an arbitrary positive integer, say $n$, can be partitioned into equal or distinct parts.

Now that we know what $p(n)$ means, let us find out how to use the formula (1) and in so doing, it will become evident how greatly the genius of Euler reduced the work required in order to find a solution for each $p(n)$ ( $\mathrm{n}=0,1,2, \cdots$ ).

We define $p(0)=1$ so that in (1) when $n=1$, we have

$$
p(1)=p(1-1)=p(0)=1
$$

and now that we have found a value for $p(1)$, we are in a position to find a value (or the number of partitions) for $p(2)$ since for $n=2$, we have

$$
\mathrm{p}(2)=\mathrm{p}(2-1)+\mathrm{p}(2-2)=\mathrm{p}(1)+\mathrm{p}(0)=1+1=2 .
$$

We continue in this exact way, using the information that $\mathrm{p}(0)=1, \mathrm{p}(1)=1$, and $p(2)=2$ to find a value for $p(3)$ and then step-by-step values for $p(4)$, $\mathrm{p}(5), \mathrm{p}(6)$, and so on.

In the following are examples of how to find values for the $p(n)$ when $\mathrm{n}=1,2,3, \cdots, 7$ by using Leonhard Euler's very important recurrence formula (in (1)): We define $p(0)=1$, then

$$
\begin{aligned}
& \mathrm{p}(1)=\mathrm{p}(1-1)=\mathrm{p}(0)=1 \\
& \mathrm{p}(2)=\mathrm{p}(2-1)+\mathrm{p}(2-2)=\mathrm{p}(1)+\mathrm{p}(0)=1+1=2 \\
& \mathrm{p}(3)=\mathrm{p}(3-1)+\mathrm{p}(3-2)=\mathrm{p}(2)+\mathrm{p}(1)=2+1=3 \\
& \mathrm{p}(4)=\mathrm{p}(4-1)+\mathrm{p}(4-2)=\mathrm{p}(3)+\mathrm{p}(2)=3+2=5 \\
& \mathrm{p}(5)=\mathrm{p}(5-1)+\mathrm{p}(5-2)-\mathrm{p}(5-5)=\mathrm{p}(4)+\mathrm{p}(3)-\mathrm{p}(0)=5+3-1=7 \\
& \mathrm{p}(6)=\mathrm{p}(6-1)+\mathrm{p}(6-2)-\mathrm{p}(6-5)=\mathrm{p}(5)+\mathrm{p}(4)-\mathrm{p}(1)=7+5-1=11
\end{aligned}
$$

(Compare this way of finding $p(6)=11$ with the way we showed that the number $n=6$ has 11 partitions at the beginning of the paper, and you will
realize the magnificence of Euler's formula as a systematized labor-saving device, especially for large $n$ where his formula is really needed.)

For our last example, we find $\mathrm{p}(7)$ :

$$
\begin{aligned}
\mathrm{p}(7) & =\mathrm{p}(7-1)+\mathrm{p}(7-2)-\mathrm{p}(7-5)-\mathrm{p}(7-7)= \\
& =\mathrm{p}(6)+\mathrm{p}(5)-\mathrm{p}(2)-\mathrm{p}(0)=11+7-2-1=15 .
\end{aligned}
$$

Continuing in this exact way, we may, of course, step-by-step find values for $\mathrm{p}(8), \mathrm{p}(9), \mathrm{p}(10), \cdots, \mathrm{p}(\mathrm{n}), \cdots$, where n runs through the positive integers $\mathrm{n}=8,9,10, \cdots$, to infinity.

It is evident that Euler's great recurrence formula (1) systematized the study of partitions. However, to determine the values of $p(n)$ for still large n required an enormous amount of work. (For example, to show that p (243) $=133978259344888$, we must first find a value for each $p(n)(n=0,1,2$, $\cdots, 242$ ) from $\mathrm{p}(0)$ through $\mathrm{p}(242)$ inclusive.) To this end, in what follows of this paper, we show how to greatly reduce the work required in finding values of the $p(n)$ by applying a new theorem from a paper entitled "Recurrence Formulas," by Joseph Arkin and Richard Pollack (The Fibonacci Quarterly, Vol. 8, No. 1, February, 1970, pp. 4-5).

In fact, using formula (1) of "Recurrence Formulas" and applying the method that has been found by this author to formula (1) so greatly reduces the work involved that to find the value of, say, $p(243)$, it would only be necessary to know the value of each $p(0)$ through $p(122)$. The reduction in work is evident, since in Euler's recurrence formula, to find a value for $p(243)$, we must first find a value for each $p(0)$ through $p(242)$.

To explain the new method of determining the value of any partition $(p(n))$, we shall, as examples, find the values of $p(16)$ and $p(17)$.

Then, to find the value of $p(16)$, we set up the table on the next page.

## EXPLANATION OF HOW TABLE WAS MADE

1. On row $A$ we have placed the values of $(-1)^{k+1} \frac{1}{2} k(3 k \mp 1)$ for $k=1,2,3$.
2. We then take half of 16 to get 8 and so we write under column a the consecutive numbers from 8 through 16.
3. Now, next to the number 8 (under column $\underline{a}$ ) we place under column $\underline{b}$ the value $p(16-8)$, next to the number 9 (under $\underline{a}$ ) we place under column $\underline{b}$ the value $p(16-9)$ and so on to complete column $\underline{b}$ with $p(16-16)=p(0)$.

| A | 1 | 2 | -5 | -7 | 12 | 15 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{\mathrm{a}}{8} \frac{\mathrm{~b}}{\mathrm{p}(8)}$ | $\mathrm{p}(7)$ | p (6) | -p(3) | -p(1) |  |  |
| $9 \mathrm{p}(7)$ |  | $\mathrm{p}(7)$ | -p(4) | -p(2) |  |  |
| $10 \mathrm{p}(6)$ |  |  | -p(5) | -p(3) |  |  |
| $11 \mathrm{p}(5)$ |  |  | -p(6) | -p(4) |  |  |
| $12 \mathrm{p}(4)$ |  |  | -p(7) | -p(5) | $\mathrm{p}(0)$ |  |
| $13 \mathrm{p}(3)$ |  |  |  | -p(6) | $\mathrm{p}(1)$ |  |
| $14 \mathrm{p}(2)$ |  |  |  | -p(7) | $\mathrm{p}(2)$ |  |
| $15 \mathrm{p}(1)$ |  |  |  |  | $\mathrm{p}(3)$ | $\mathrm{p}(0)$ |
| $16 \mathrm{p}(0)$ |  |  |  |  | p(4) | $\mathrm{p}(1)$ |

4. We fill in the rest of the table in the same way we plot a graph. For example, the $\mathrm{p}(7)$ under the column where $\mathrm{A}=1$ and on the row where $\underline{a}=8$ is said to be in box $(8,1)$ or more generally this $p(7)$ is found in box $(\underline{a}, A)$ where $\underline{a}=8$ and $A=1$. Now, it will be observed that in each box ( $\underline{a}, A$ ) we find the term $p(a-A)$, or in box $(a,-A)$ we find the term $-p(a-A)$ except that there are no terms $\mp p(a-A)$ entered at all when $\frac{1}{2} \cdot 16=8 \leqq \mathrm{a}-\mathrm{A}<0$ 。

Let us consider a few numerical examples of what was just said. Written into the five boxes $(8,1)$, $(8,2),(13,-7),(13,15)$, and $(16,7)$ we find respectively the following: $p(8-1)=p(7), p(8-2)=p(6),-p(13-7)=$ $-\mathrm{p}(6)$, no entry (since $13-15=-2<0$ ) andno entry (since $16-7=9>8$ ).

Now that we have filled in the table, we then multiply each partition under the column $\underline{b}$ in the table together with the sum of the partitions directly to its right and on the same row. We then have the following products (row-by-row):

$$
\begin{aligned}
& \mathrm{p}(8)[\mathrm{p}(7)+\mathrm{p}(6)-\mathrm{p}(3)-\mathrm{p}(1)] \\
& \mathrm{p}(7)[\mathrm{p}(7)-\mathrm{p}(4)-\mathrm{p}(2)] \\
& \mathrm{p}(6)[-\mathrm{p}(5)-\mathrm{p}(3)] \\
& \mathrm{p}(5)[-\mathrm{p}(6)-\mathrm{p}(4)] \\
& \mathrm{p}(4)[-\mathrm{p}(7)-\mathrm{p}(5)+\mathrm{p}(0)] \\
& \mathrm{p}(3)[-\mathrm{p}(6)+\mathrm{p}(1)] \\
& \mathrm{p}(2)[-\mathrm{p}(7)+\mathrm{p}(2)]
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{p}(1)[\mathrm{p}(2)+\mathrm{p}(0)] \\
& \mathrm{p}(0)[\mathrm{p}(4)+\mathrm{p}(1)]
\end{aligned}
$$

and after replacing the $p()$ with their numerical values, we find $p(16)$ as follows:

$$
\begin{aligned}
& 22(15+11-3-1)=22 \cdot 22=484 \\
& 15(15-5-2)=15 \cdot 8=120 \\
& 11(-7-3)=-11 \cdot 10=-110 \\
& 7(-11-5)=-7 \cdot 16=-112 \\
& 5(-15-7+1)=-5 \cdot 21=-105 \\
& 3(-11+1)=-3 \cdot 10=-30 \\
& 2(-15+2)=-2 \cdot 13=-26 \\
& 1(3+1)= 1 \cdot 4=4 \\
& 1(5+1)= 1 \cdot 6=\frac{6}{231}=\mathrm{p}(16) .
\end{aligned}
$$

To find a numerical value for $p(17)$, we use the exact methods that were used to find a numerical value for $\mathrm{p}(16)$. The important difference is that since 17 is an odd number, we must then take half of $17-1$ and then complete the following table using the same methods that we used to complete a table for $\mathrm{p}(16)$ (that is, we shall begin by writing under column a the consecutive numbers from 8 through 17, etc.).

To find the value of $\mathrm{p}(17)$, we erect the following table:
A

| 1 | 2 | -5 | -7 | 12 | 15 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  $\underline{b}$ <br> 8 $\mathrm{p}(9)$ | $\mathrm{p}(7)$ | $\mathrm{p}(6)$ | $-\mathrm{p}(3)$ | $-\mathrm{p}(1)$ |  |  |
| 9 | $\mathrm{p}(8)$ |  | $\mathrm{p}(7)$ | $-\mathrm{p}(4)$ | $-\mathrm{p}(2)$ |  |
| 10 | $\mathrm{p}(7)$ |  |  | $-\mathrm{p}(5)$ | $-\mathrm{p}(3)$ |  |
| 11 | $\mathrm{p}(6)$ |  |  | $-\mathrm{p}(6)$ | $-\mathrm{p}(4)$ |  |
| 12 | $\mathrm{p}(5)$ |  |  | $-\mathrm{p}(7)$ | $-\mathrm{p}(5)$ | $\mathrm{p}(0)$ |
| 13 | $\mathrm{p}(4)$ |  |  |  | $-\mathrm{p}(6)$ | $\mathrm{p}(1)$ |
| 14 | $\mathrm{p}(3)$ |  |  |  | $-\mathrm{p}(7)$ | $\mathrm{p}(2)$ |
| 15 | $\mathrm{p}(2)$ |  |  |  |  |  |
| 16 | $\mathrm{p}(1)$ |  |  |  |  | $\mathrm{p}(3)$ |
| 17 | $\mathrm{p}(0)$ |  |  |  | $\mathrm{p}(0)$ |  |

We now multiply each partition under column $\underline{b}$ in the table together with the sum of the partitions directly to its right and on the same row to get the following products:

$$
\begin{aligned}
& \mathrm{p}(9)[\mathrm{p}(7)+\mathrm{p}(6)-\mathrm{p}(3)-\mathrm{p}(1)] \\
& \mathrm{p}(8)[\mathrm{p}(7)-\mathrm{p}(4)-\mathrm{p}(2)] \\
& \mathrm{p}(7)[-\mathrm{p}(5)-\mathrm{p}(3)] \\
& \mathrm{p}(6)[-\mathrm{p}(6)-\mathrm{p}(4)] \\
& \mathrm{p}(5)[-\mathrm{p}(7)-\mathrm{p}(5)+\mathrm{p}(0)] \\
& \mathrm{p}(4)[-\mathrm{p}(6)+\mathrm{p}(1)] \\
& \mathrm{p}(3)[-\mathrm{p}(7)+\mathrm{p}(2)] \\
& \mathrm{p}(2)[\mathrm{p}(3)+\mathrm{p}(0)] \\
& \mathrm{p}(1)[\mathrm{p}(4)+\mathrm{p}(1)] \\
& \mathrm{p}(0)[\mathrm{p}(5)+\mathrm{p}(2)]
\end{aligned}
$$

and after replacing the $p()$ with their numerical values, we find $p(17)$ as follows:

$$
\begin{aligned}
30(15+11-3-1) & =30 \cdot 22=660 \\
22(15-5-2) & =22 \cdot 8=176 \\
15(-7-3) & =-15 \cdot 10=-150 \\
11(-11-5) & =-11 \cdot 16=-176 \\
7(-15-7+1) & =-7 \cdot 21=-147 \\
5(-11+1) & =-5 \cdot 10=-50 \\
3(-15+2) & =-3 \cdot 13=-39 \\
2(3+1) & =2: 4=8 \\
1(5+1) & =1 \cdot 6=6 \\
1(7+2) & =1 \cdot 9=\frac{9}{297}=\mathrm{p}(17) .
\end{aligned}
$$

In conclusion, it may be interesting to mention that we could have used smaller $p()$ to find $p(16)$ and $p(17)$. For example: Since $p(0)$ through $p(8)$ will determine $p(16)$ and by the methods used in this paper, it is evident that the numerical values of $p(0)$ through $p(4)$ will enable us to find values for $p(0)$ through $p(8)$ then we need only have used the values of the
partitions $p(0)$ through $p(4)$ to find $p(16)$. This reduction rule is applicable in finding the value of any $p(n)$, however it defeats the purpose of the method to reduce too much.

Of course, applying the method of this paper to find small partitions like $p(16)$ or $p(17)$ does not show the method to its fullest - but when used to find a value for large partitions, like say, $p(243)=133978259344888$, the method shown in this paper very greatly reduces the work involved.
[Continued from p. 364.]
and the induction is complete. Thus the C-array is precisely the B-array. Thus, $\mathrm{B}_{\mathrm{m}, \mathrm{p}} \equiv \mathrm{C}_{\mathrm{m}, \mathrm{p}}$, and further, the pattern observed by Umansky and Karst persists for all $\mathrm{n} \geq 1, \mathrm{~m} \geq 2$. The case $\mathrm{m}=1$ was earlier verified.

Theorem 2 (Independent). If one ignores the signs of the coefficients in Array C, then the sum across the $m^{\text {th }}$ row is $L_{m}$ 。

Proof. Interchange the firstcolumn on the right with the column on the left and set $n=1$. The left column is now $-L_{m}$ and all of the terms on the right are negative. Equality still holds since Theorem 1 is true. Thus

$$
1+\sum_{j=1}^{[\mathrm{m} / 2]} C_{m, j}=\sum_{j=0}^{[m / 2]} C_{m, j}=L_{m}
$$

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# FIBONACCI SERIES IN THE SOLAR SYSTEM 

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## ABSTRACT

The Fibonacci Series is shown to predict the distances of the moons of Jupiter, Saturn and Uranus from their respective primary. The planets are shown to have a trend which follows the Fibonacci Series with individual offsets attributed to planetary densities.

## 1. INTRODUCTION

Many series exist where successive terms are a function of previous terms. When this function is a linear combination, each term can be expressed as

$$
z_{n}=\sum_{j=1}^{n} a_{j} z_{n-j}+C
$$

where $\left\{a_{j}\right\}$ is a weighting function acting upon each successive $\left\{z_{n-j}\right\}$, and C is a constant.

The Fibonacci series $\left\{f_{n}\right\}=\{0,1,1,2,3,5,8,13, \cdots\}$ which is of the form

$$
\begin{equation*}
f_{n}=f_{n-1}+f_{n-2} \tag{1}
\end{equation*}
$$

is an excellent example of this with $\left\{\mathrm{a}_{\mathrm{j}}\right\}=\{1,1,0, \cdots, 0\}$ and $\mathrm{C}=0$.
Subsequently, however, a modified Fibonacci Series will be employed having the form $\{1,2,3,5, \ldots\}$ but with again Eq. (1) holding.

## 2. PLANET MOONS

The distances of moons from their parent planet have not been known to follow any sequence. The mean distances $z_{n}$ of the moons of Jupiter, Saturn,
and Uranus, as shown in Figs. 1 through 5, respectively, can be assigned to terms in the Fibonacci Series $f_{n}$ in the form

$$
\begin{equation*}
z_{n}=m f_{n}+b \tag{2}
\end{equation*}
$$

The following table shows the "best fit" values of $m$ and $b$ for Jupiter, Saturn, and Uranus and the possible values for Mars and Neptune which, because they have only two moons each, cannot be fitted to a linear relationship.

Table: Values of $m$ and $b$

| Primary |  | $\left(10^{3} \mathrm{~km}\right)$ |  |
| :---: | :---: | :---: | :---: |
|  | No. of Moons | $\underline{\mathrm{m}}$ | b |
| Mars | 2 | $\left\{\begin{array}{c}14 \\ 7 \\ 3.5\end{array}\right.$ | $\left\{\begin{array}{r}-4.8 \\ 2.3 \\ 5.8\end{array}\right.$ |
| Jupiter | 12 | 240 | -50 |
| Saturn | 9 | 50 | 140 |
| Uranus | 5 | 70 | 60 |
| Neptune | 2 | $\left\{\begin{array}{l}420 \\ 250\end{array}\right.$ | $\left\{\begin{array}{l}-66 \\ 104\end{array}\right.$ |

Equation (2) is expressible in the form

$$
z_{n}=z_{n-1}+z_{n-2}-b
$$

comparable to Eq. (1).

## 3. PLANETS

Figure 6 shows the mean distance of the planets from the sun, $z_{n}$, plotted against the Fibonacci Series $f_{n}$. As can be seen, the trend of the distances follow the series, although the individual values seem to be offset in a sinusoidal manner from it. Figure 7, however, shows the quantity $z_{n} / f_{n}$ for


Fig. 1 First Moons of Jupiter


Fig. 2 Moons of Jupiter


Fig. 3 First Moons of Saturn


Fig. 4 Moons of Saturn
[Oct.


Fig. 5 Moons of Uranus


Fig. 6 The Planets
[Oct.


Fig. $7 \mathrm{z}_{\mathrm{n}} / \mathrm{f}_{\mathrm{n}}$ and $\delta_{\mathrm{n}}$ for Planet Number n
each planet. Superimposed upon this figure are the respective planet densities $\delta_{n}$ (in terms of earth densities). Figure 8 is a plot of $z_{n} / f_{n}$ against $\delta_{n-1}$ which yields

$$
z_{n} / f_{n}=0.56-0.28 \delta_{n-1}
$$

(in astronomical units). The actual form of the density dependence of the offset is not readily determinable. It is conjectured from further study that the offset of a particular planet is either due to the average density of that planet and the planet previous, or is due to some weighting of previous densities with terms of the Fibonacci Series.

## 4. CONCLUSIONS

The significance of the values of $m$ and $b$ for the moons of the planets is left for subsequent interpretation. The quantity m may be regarded as a planetary scale length although neither $m$, $b$, or the ratio $b / m$ seems to bear any relationship to such planetary parameters as density, mass, or volume.

The fact that the distance of the moons follows the Fibonacci Series means that a particular moon's position is dependent upon the positions of the previous two moons closer to the primary. Also, the moons seem to reside and, in the case of Jupiter, even congregate at "potential levels" predicted by the series.

## 5. ADDENDUM

The offset of a planet from the Fibonacci Series $f_{n}$ being a function of the density of the previous planet is found to hold also for the offset of the moons of Jupiter and Saturn. The "normalized" offset for the moons of these two planets and for the planets of the sun is of the form,

$$
\frac{z_{n}+C}{\bar{m} f_{n}}=1.0-0.09 \delta_{n-1}
$$

although the reliability of the density data is poor.


Fig. 8 "Offset" versus Previous Planet Density [Continued on p. 448.]

## A LUCAS ANALOGUE

## BROTHER ALFRED BROUSSEAU <br> St. Mary's College, California

The Lucas sequence is defined in terms of the roots $r_{1}$ and $r_{2}$ of the equation $x^{2}=x+1$ by the formula:

$$
L_{\mathrm{n}}=\mathrm{r}_{1}^{\mathrm{n}}+\mathrm{r}_{2}^{\mathrm{n}}
$$

For this simple quadratic equation, the roots can be calculated as:

$$
r_{1}=\frac{1+\sqrt{5}}{2}, \quad r_{2}=\frac{1-\sqrt{5}}{2}
$$

It can be ascertained directly that $r_{1}+r_{2}=1$ and $r_{1}^{2}+r_{2}^{2}=3$. Then, since

$$
\mathrm{r}_{1}^{\mathrm{n}+1}=\mathrm{r}_{1}^{\mathrm{n}}+\mathrm{r}_{1}^{\mathrm{n}-1}
$$

and

$$
\mathrm{r}_{2}^{\mathrm{n}+1}=\mathrm{r}_{2}^{\mathrm{n}}+\mathrm{r}_{2}^{\mathrm{n}-1}
$$

it follows by mathematical induction that if:

$$
L_{n-1}=r_{1}^{n-1}+r_{2}^{n-1}
$$

and

$$
\mathrm{L}_{\mathrm{n}}=\mathrm{r}_{1}^{\mathrm{n}}+\mathrm{r}_{2}^{\mathrm{n}}
$$

then

$$
L_{n+1}=r_{1}^{n+1}+r_{2}^{n+1}
$$

by direct addition of the two equations.

If we seek to extend this idea to a sequence in which the last three consecutive terms are added to get the next term, the corresponding equation to be used is: $x^{3}=x^{2}+x+1$, which has three roots $r_{1}, r_{2}, r_{3}$. These need not be calculated. It suffices to know that:

$$
\sum_{i=1}^{3} r_{i}=1, \quad \sum_{i, j=1}^{3}{ }^{\prime} r_{i} r_{j}=-1 \quad \text { and } \quad r_{1} r_{2} r_{3}=1
$$

using the standard relations between roots and coefficients of an equation. The analogous definition of a sequence in terms of the roots would be:

$$
\mathrm{T}_{\mathrm{n}}=\mathrm{r}_{1}^{\mathrm{n}}+\mathrm{r}_{2}^{\mathrm{n}}+\mathrm{r}_{3}^{\mathrm{n}}
$$

The first three values calculated on the basis of symmetric function formulas would be:

$$
\begin{gathered}
T_{1}=\sum_{i=1}^{3} r_{i}=1 \\
T_{2}=\sum_{i=1}^{3} r_{i}^{2}=\left(\sum r_{i}\right)^{2}-2 \Sigma r_{i} r_{j}=3 \\
T_{3}=\sum r_{i}^{3}=\left(\sum r_{i}\right)^{3}-3\left(\sum r_{i}\right)\left(\sum r_{i} r_{j}\right)+3 r_{1} r_{2} r_{3} \\
T_{3}=1+3+3=7
\end{gathered}
$$

With these values defined, $\mathrm{T}_{4}$ would be equal to

$$
\Sigma r_{i}^{4}
$$

on the basis of the relation

$$
r_{i}^{4}=r_{i}^{3}+r_{i}^{2}+r_{i}
$$

Carrying the procedure one more step, if the last four quantities in a sequence are added to obtain the next quantity, the roots of the equation

$$
x^{4}=x^{3}+x^{2}+x+1
$$

would be employed with the sequence defined by:

$$
Q_{n}=\sum_{i=1}^{4} r_{i}^{n}
$$

With the aid of symmetric functions, it can be shown that:

$$
Q_{1}=1, \quad Q_{2}=3, \quad Q_{3}=7, \quad Q_{4}=15
$$

From these preliminary investigations, two points emerge: (1) As we extend the Lucas analogue, the basic starting quantities carry over from one stage to the next; (2) All the initial quantities are of the form $2^{k}-1$. That these relations remain generally true is not difficult to prove.

Consider the $\mathrm{n}^{\text {th }}$ order recursion relation:

$$
x^{n}=x^{n-1}+x^{n-2}+x^{n-3}+\cdots+x+1
$$

which corresponds to adding the last n quantities to obtain the next quantity in a sequence. Expressed as a polynomial equated to zero, this becomes:

$$
x^{n}-x^{n-1}-x^{n-2}-x^{n-3}-\cdots-x-1=0
$$

so that regardless of the degree of the equation:

$$
\Sigma r_{i}=1, \quad \Sigma r_{i} r_{j}=-1, \quad \Sigma r_{i} r_{j} r_{k}=1, \text { etc. }
$$

Then, since

$$
\Sigma r_{i}^{j}
$$

is expressible in terms of those summations which have $j$ or less components, it follows that the sum must be the samefor any two equations having a degree greater than or equal to $j$. This accounts for the persistence of the basic starting quantities from one stage to the next.

Now assume that for the equation of degree $n-1$

$$
\Sigma r_{i}^{j}=2^{j}-1 \quad(j=1,2, \cdots, n-1)
$$

By the equation for the $n^{\text {th }}$ degree:

$$
\begin{aligned}
\Sigma r_{i}^{n} & =\Sigma r_{i}^{n-1}+\Sigma r_{i}^{n-2}+\Sigma r_{i}^{n-3}+\cdots+\Sigma r_{i}+\Sigma 1 \\
& =\left(2^{n-1}-1\right)+\left(2^{n-2}-1\right)+\left(2^{n-3}-1\right) \cdots(2-1)+n \\
& =\sum_{i=0}^{n-1} 2^{k}=2^{n}-1
\end{aligned}
$$

This proves the second contention.

## CONCLUSION

For an $n^{\text {th }}$ order recursion relation of the form

$$
\mathrm{T}_{\mathrm{m}+\mathrm{n}+1}=\mathrm{T}_{\mathrm{m}+\mathrm{n}}+\mathrm{T}_{\mathrm{m}+\mathrm{n}+1}+\mathrm{T}_{\mathrm{m}+\mathrm{n}-2}+\cdots+\mathrm{T}_{\mathrm{m}+1}
$$

it is possible to define a Lucas-type sequence using the roots of the equation $\mathrm{x}^{\mathrm{n}}=\mathrm{x}^{\mathrm{n}-1}+\mathrm{x}^{\mathrm{n}-2}+\mathrm{x}^{\mathrm{n}-2}+\cdots+\mathrm{x}+1$ with

$$
\mathrm{T}_{\mathrm{m}}=\Sigma \mathrm{r}_{\mathrm{i}}^{\mathrm{m}}
$$

With such a definition, the first n starting values would be given by:

$$
\mathrm{T}_{\mathrm{k}}=2^{\mathrm{k}}-1 \quad(\mathrm{k}=1,2, \cdots, \mathrm{n})
$$

# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by
A. P. HILLMAN

University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheetor sheets in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-190 A repeat of B-186 with a typographical error corrected.
Let $L_{n}$ be the $\mathrm{n}^{\text {th }}$ Lucas number and show that

$$
L_{5 n} / L_{n}=\left[L_{2 n}-3(-1)^{n}\right]^{2}+(-1)^{n_{2}} 25 \mathrm{~F}_{\mathrm{n}}^{2}
$$

B-191 Proposed by Guy A. Guillottee, Montreal, Quebec, Canada.
In this alphametic, each letter represents a particular but different digit, all ten digits being represented here. It must only be that well-known mathematical teaser from Toronto, J, A. H. Hunter, but what is the value of HUNTER?

M R
HUNTER
MADE
A
TEASER

B-192 Proposed by Warren Cheves, Littleton, North Carolina.
Prove that $F_{3 n}=L_{n} F_{2 n}-(-1)^{n} F_{n}$ 。

B-193 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that $L_{n+p} \pm L_{n-p}$ is $5 F_{p} F_{n}$ or $L_{p} L_{n}$ depending on the choice of sign on whether $p$ is even or odd.

B-194 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.
Show that $L_{n+4 k}-L_{n}=5 F_{k}\left[F_{n+3 k}+(-1)^{n} F_{n+k}\right]$.
B-195 Proposed by David Zeitlin, Minneapolis, Minnesota.
Let $\left\{\begin{array}{l}n \\ r\end{array}\right\}$ denote $L_{n} L_{n-1} \cdots L_{n-r+1} / L_{1} L_{2} \cdots L_{r}$. Show that

$$
L_{n}^{3} / 6=\left\{\begin{array}{c}
n+2 \\
3
\end{array}\right\}-2\left\{\begin{array}{c}
n+1 \\
3
\end{array}\right\}-\left\{\begin{array}{l}
n \\
3
\end{array}\right\} .
$$

Following is a list of solvers whose names were inadvertently omitted from lists in recent issues:

B-144 Don Allen
B-148, B-149, B-150, B-151, B-153 - D. V. Jaiswal
B-160, B-161, B-163 - H. V. Krishna
B-166 Michael Yoder
B-167 T. J. Cullen, Bruce W. King, R. W. Sielaff, Michael Yoder
B-168 Michael Yoder
B-169 Wray G. Brady, Michael Yoder
B-170, B-171 - Michael Yoder

## SOLUTIONS

A CUBIC IDENTITY
B-172 Proposed by Gloria C. Padilla, Albuquerque High School, Albuquerque, New Mexico.

Let $F_{0}=0, F_{1}=1$, and $F_{n+2}=F_{n}+F_{n+1}$ for $n=0,1, \cdots$. Show that

$$
\mathrm{F}_{\mathrm{n}+2}^{3}=\mathrm{F}_{\mathrm{n}}^{3}+\mathrm{F}_{\mathrm{n}+1}^{3}+3 \mathrm{~F}_{\mathrm{n}} \mathrm{~F}_{\mathrm{n}+1} \mathrm{~F}_{\mathrm{n}+2}
$$

Solution by C. B. A. Peck, State College, Pennsylvania.
We obtain the desired result from $(x+y)^{3}=x^{3}+y^{3}+3 x y(x+y)$ on substituting $F_{n}=x$ and $F_{n+1}=y$ and hence $F_{n+2}=x+y$. Note that the result does not depend on the initial values

Also solved by W. C. Barley, Wray G. Brady, T. J. Cullen, Andrew Dias, David Englund, Herta T. Freitag, Bernard G. Hoerbelt, John E. Homer, Jr., John Ivie, Bruce W. King, Peter A. Lindstrom, Bruce Lynn, John W. Milsom, Klaus-Günther Recke, Michael Rennie, Gerald Satlow, A. G. Shannon, Richard W. Sielaff, Charles W. Trigg, John Wessner, Gregory Wulczyn, Michael Yoder, and the Proposer.

## ANOTHER CUBIC IDENTITY

B-173 Proposed by Gloria C. Padilla, Albuquerque High School, Albuquerque, New Mexico.

Show that

$$
F_{3 n}=F_{n+2}^{3}-F_{n-1}^{3}-3 F_{n} F_{n+1} F_{n+2}
$$

Solution by T. J. Cullen, California State Polytechnic College, Pomona, California.
From Formula XXI on p. 68 of Vol. 1, Fibonacci Quarterly:

$$
F_{3 n}=F_{n}^{3}+F_{n+1}^{3}-F_{n-1}^{3}
$$

Hence, by B-172,

$$
F_{3 n}=F_{n+2}^{3}-F_{n-1}^{3}-3 F_{n} F_{n+1} F_{n+2}
$$

Also solved by W. C. Barley, Wray G. Brady, Herta T. Freitag, Bernard G. Hoerbelt, John E. Homer, Jr., John Ivie, Bruce W. King, Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, Klaus-Günther Recke, A. G. Shannon, Charles W. Trigg, Gregory Wulczyn, Michael Yoder, David Zeitlin, and the Proposer.

MODULO 10
B-174 Proposed by Mel Most, Ridgefield Park, New Jersey.

Let a be a non-negative integer. Show that in the sequence

$$
2 \mathrm{~F}_{\mathrm{a}+1}, \quad 2^{2} \mathrm{~F}_{\mathrm{a}+2}, \quad 2^{3} \mathrm{~F}_{\mathrm{a}+3}, \cdots
$$

all differences between successive terms must end in the same digit.

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico.

The sequence satisfies the recurrence

$$
S_{n+2}=2 S_{n+1}+4 S_{n} \quad\left(S_{n}=2^{n} F_{a+n}\right)
$$

Thus

$$
s_{n+2}-s_{n+1}=\left(S_{n+1}-s_{n}\right)+5 S_{n} \equiv s_{n+1}-s_{n}(\bmod 10)
$$

since all $\mathrm{S}_{\mathrm{n}}$ are even.
Also solved by T. J. Cullen, David Englund, Hertä T. Freitag, C. B. A. Peck, Klaus-Günther Recke, A. G. Shannon, Charles W. Trigg, John Wessner, David Zeitlin, and the proposer.

## A GENERALIZED 2-BY-2 DETERMINANT

B-175 Composed from the Solution by David Zeitlin to B-155.
Let $r$ and $q$ be constants and let $U_{0}=0, U_{1}=1, U_{n+2}=r U_{n+1}-$ $q U_{n}$. Show that

$$
\mathrm{U}_{\mathrm{n}+\mathrm{a}} \mathrm{U}_{\mathrm{n}+\mathrm{b}}-\mathrm{U}_{\mathrm{n}+\mathrm{a}+\mathrm{b}} \mathrm{U}_{\mathrm{n}}=q^{\mathrm{n}} \mathrm{U}_{\mathrm{a}} \mathrm{U}_{\mathrm{b}}
$$

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico.

For $a=0$, the identity is obviously true; and if it is true for $a=1$, it will be true by induction for all a. Similarly, the identity need only be proved for $b=1$, and the problem reduces to that of proving $U_{n+1}^{2}-U_{n+2} U_{n}$ $=q^{n}$. For $n=0$, this is true; and since

$$
\begin{aligned}
U_{n+2}^{2}-U_{n+3} U_{n+1} & =U_{n+2}\left(r U_{n+1}-q U_{n}\right)-\left(r U_{n+2}-q U_{n+1}\right) U_{n+1} \\
& =q\left(U_{n+1}^{2}-U_{n+2} U_{n}\right),
\end{aligned}
$$

the identity is verified for all $a, b$, and $n$.

Also solved by Wray G. Brady, T. J. Cullen, Herta T. Freitag, C. B. A. Peck, Klaus-Günther Recke, A. G. Shannon, Gregory Wulczyn, and David Zeitlin.

## CUBES IN TERMS OF FIBONOMIALS ON DIAGONAL 3

B-176 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.
Let $\left[\begin{array}{l}n \\ r\end{array}\right]$ denote the Fibonomial Coefficient

$$
F_{n} F_{n-1} \cdots F_{n-r+1} / F_{1} F_{2} \cdots F_{r}
$$

Show that

$$
\mathrm{F}_{\mathrm{n}}^{3}=\left[\begin{array}{c}
\mathrm{n}+2 \\
3
\end{array}\right]-2\left[\begin{array}{c}
\mathrm{n}+1 \\
3
\end{array}\right]-\left[\begin{array}{l}
\mathrm{n} \\
3
\end{array}\right] .
$$

Solution by David Zeitlin, Minneapolis, Minnesota.
Let $H_{n}$ satisfy $H_{n+2}=H_{n+1}+H_{n}$, and define

$$
\left\{\begin{array}{l}
n \\
r
\end{array}\right\}=H_{n} H_{n-1} \cdots H_{n-r+1} / H_{1} H_{2} \cdots H_{r}
$$

where $H_{i}>0, i=1,2, \ldots$. Since
(1)

$$
2 H_{n}^{2}=H_{n+2} H_{n+1}-2 H_{n+1} H_{n-1}-H_{n-1} H_{n-2}
$$

multiplication of (1) by $\mathrm{H}_{\mathrm{n}}$ gives

$$
2 \mathrm{H}_{\mathrm{n}}^{3}=\mathrm{H}_{1} \mathrm{H}_{2} \mathrm{H}_{3}\left(\left\{\begin{array}{c}
\mathrm{n}+2  \tag{2}\\
3
\end{array}\right\}-2\left\{\begin{array}{c}
\mathrm{n}+1 \\
3
\end{array}\right\}-\left\{\begin{array}{l}
\mathrm{n} \\
3
\end{array}\right\}\right)
$$

The desired result is obtained from (2) for

$$
\mathrm{H}_{\mathrm{n}}=\mathrm{F}_{\mathrm{n}}, \text { where }\left\{\begin{array}{l}
\mathrm{n} \\
\mathrm{r}
\end{array}\right\}=\left[\begin{array}{l}
\mathrm{n} \\
\mathrm{r}
\end{array}\right] .
$$

Also solved by L. Carlitz, T. J. Cullen, Herta T. Freitag, John E. Homer, Jr., Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, Klaus-Günther Recke, A. G. Shannon, Charles W. Trigg, Gregory Wulczyn, Michael Yoder, and the Proposer.

FOURTH POWERS IN TERMS OF FIBONOMIALS
B-177 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Using the notation of $\mathrm{B}-176$, show that

$$
\mathrm{F}_{\mathrm{n}}^{4}=\left[\begin{array}{c}
\mathrm{n}+3 \\
4
\end{array}\right]-\mathrm{a}\left[\begin{array}{c}
n+2 \\
4
\end{array}\right]-a\left[\begin{array}{c}
n+1 \\
4
\end{array}\right]+\left[\begin{array}{l}
\mathrm{n} \\
4
\end{array}\right]
$$

for some integer a and find a .

Solution by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.
Letting $\mathrm{n}=2$, we find that a would have to be 4. Then letting $\mathrm{a}=$ 4, both sides satisfy the same fourth-order (i. $\mathrm{e}_{0}$, five-term) recurrence relation. Hence it suffices to verify the formula for $n=0,1,2,3$ and it follows for all values of $n$ by induction.

Also solved by L. Carlitz, T. J. Cullen, Herta T. Freitag, John E. Homer, Jr., C. B. A. Peck, Klaus-Günther Recke, Charles W. Trigg, Gregory Wulczyn, Michael Yoder, David Zeitlin, and the Proposer.
[Continued from p. 438.]

It is found, also, that the slope $m$ of the distances $z_{n}$ versus the Fibonacci Series $f_{n}$ for each planet system is a power law function of the mass $M$ and radius $R$ of the planet in the form

$$
\mathrm{m} \propto M^{3} \mathrm{R}^{-7}
$$

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".... The binder is made of heavy weight virgin vinyl, electronically sealed over rigid board equipped with a clear label holder extending $2-3 / 4^{\prime \prime}$ high from the bottom of the backbone, round cornered, fitted with a $11 / 2^{" m}$ multiple mechanism and 4 heavy wires."

The name, FIBONACCI QUARTERLY, is printed in gold on the front of the binder and the spine. The color of the binder is dark green. There is a small pocket on the spine for holding a tab giving year and volume. The se latter will be supplied with each order if the volume or volumes to be bound are indicated.

The price per binder is $\$ 3.50$ which includes postage (ranging from $50 \hat{\zeta}$ to $80 \hat{\zeta}$ for one binder). The tabs will be sent with the receipt or invoice.

All orders should be sent to: Brother Alfred Brousseau, Managing Editor, St. Mary's College, Calif. 94575

