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# THE FIBONACCI QUARTERLY 

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# A FIBONACCI CIRCULANT 

D. A. LIND

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1. Put

$$
D_{n, r}=\left|\begin{array}{ccccc}
F_{r} & F_{r+1} & F_{r+2} & \cdots & F_{r+n-1} \\
F_{r+n-1} & F_{r} & F_{r+1} & \cdots & F_{r+n-2} \\
F_{r+n-2} & F_{r+n-1} & F_{r} & \cdots & F_{r+n-3} \\
\vdots & & & & \vdots \\
F_{r+1} & F_{r+2} & F_{r+3} & \cdots & F_{r}
\end{array}\right|,
$$

where $F_{n}$ denotes the Fibonacci numbers defined by

$$
F_{1}=F_{2}=1, \quad F_{n+2}=F_{n+1}+F_{n}
$$

We show that

$$
\begin{equation*}
D_{n, r}=\frac{\left(F_{r}-F_{n+r}\right)^{n}-\left(F_{n+r-1}-F_{r-1}\right)^{n}}{1-L_{n}+(-1)^{n}} \tag{1}
\end{equation*}
$$

where $L_{n}=F_{n-1}+F_{n+1}$ is the $n^{\text {th }}$ Lucas number.
A circulant is a determinant of the form
(2)

$$
C\left(a_{0}, \cdots, a_{n-1}\right)=\left|\begin{array}{ccccc}
a_{0} & a_{1} & a_{2} & \cdots & a_{n-1} \\
a_{n-1} & a_{0} & a_{1} & \cdots & a_{n-2} \\
a_{n-2} & a_{n-1} & a_{0} & \cdots & a_{n-3} \\
\vdots & & & & \vdots \\
a_{1} & a_{2} & a_{3} & \cdots & a_{0}
\end{array}\right|
$$

It is known (see [1, Vol. 3, pp. 374-375] and [3, p. 39]) that

$$
\begin{equation*}
C\left(a_{0}, \cdots, a_{n-1}\right)=\sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} a_{j} \omega_{k}^{j}\right), \tag{3}
\end{equation*}
$$

where the

$$
\omega_{\mathrm{k}}=\cos \frac{2 \mathrm{k} \pi}{\mathrm{n}}+\mathrm{i} \sin \frac{2 \mathrm{k} \pi}{\mathrm{n}}
$$

are the $\mathrm{n}^{\text {th }}$ roots of unity. To establish (3) rapidly, multiply

$$
\mathrm{C} \equiv \mathrm{C}\left(\mathrm{a}_{0}, \cdots, a_{\mathrm{n}-1}\right)
$$

by the Vandermonde determinant $V=\left|\omega_{i}^{j}\right| \quad(i, j=0,1, \cdots, n-1)$. Denoting the right side of (3) by $P$, by factoring out common factors, one finds $\mathrm{CV}=$ PV, and since $V \neq 0$, (3) follows.

Now $D_{n, r}$ is a special case of (2) with

$$
a_{j}=F_{j+r}=\frac{\alpha^{j+r}-\beta^{j+r}}{\alpha-\beta}
$$

in which $\alpha=(1+\sqrt{5}) / 2, \quad \beta=(1-\sqrt{5}) / 2$. Thus by (3),

$$
\begin{aligned}
D_{n, r} & =\prod_{\mathrm{k}=0}^{\mathrm{n}-1}\left(\sum_{j=0}^{\mathrm{n}-1} \frac{\alpha^{\mathrm{r}}\left(\alpha \omega_{k}\right)^{\mathrm{j}}-\beta^{\mathrm{r}}\left(\beta \omega_{\mathrm{k}}\right)^{\mathrm{j}}}{\alpha-\beta}\right) \\
& =(\alpha-\beta)^{-\mathrm{n}} \prod_{\mathrm{k}=0}^{\mathrm{n}-1}\left(\frac{\alpha^{\mathrm{r}}\left[1-\left(\alpha \omega_{\mathrm{k}}\right)^{\mathrm{n}}\right]}{1-\alpha \omega_{\mathrm{k}}}-\frac{\beta^{\mathrm{r}}\left[1-\left(\beta \omega_{\mathrm{k}}\right)^{\mathrm{n}}\right]}{1-\beta \omega_{\mathrm{k}}}\right) \\
& =\prod_{\mathrm{k}=0}^{\mathrm{n}-1} \frac{\alpha^{\mathrm{r}}-\beta^{\mathrm{r}}-\alpha^{\mathrm{n}+\mathrm{r}}+\beta^{\mathrm{n}+\mathrm{r}}+\left[\alpha^{\mathrm{r}-1}-\beta^{\mathrm{r}-1}-\alpha^{\mathrm{n}+\mathrm{r}-1}+\beta^{\mathrm{n}+\mathrm{r}-1}\right] \omega_{\mathrm{k}}}{(\alpha-\beta)\left(1-\alpha \omega_{\mathrm{k}}\right)\left(1-\beta \omega_{\mathrm{k}}\right)} \\
& =\prod_{\mathrm{k}=0}^{\mathrm{n}-1} \frac{\mathrm{~F}_{\mathrm{r}}-\mathrm{F}_{\mathrm{n}+\mathrm{r}}-\left(\mathrm{F}_{\mathrm{n}+\mathrm{r}-1}-\mathrm{F}_{\mathrm{r}-1}\right) \omega_{\mathrm{k}}}{\left(1-\alpha \omega_{\mathrm{k}}\right)\left(1-\beta \omega_{\mathrm{k}}\right)}
\end{aligned}
$$

Now for any x and y ,
(4) $\prod_{k=0}^{n-1}\left(x-y \omega_{k}\right)=y^{n} \prod_{k=0}^{n-1}\left(\frac{x}{y}-\omega_{k}\right)=y^{n}\left[\left(\frac{x}{y}\right)^{n}-1\right]=x^{n}-y^{n}$.

Therefore
$\prod_{k=0}^{n-1}\left[F_{r}-F_{n+r}-\left(F_{n+r-1}-F_{r-1}\right) \omega_{k}\right]=\left(F_{r}-F_{n+r}\right)^{n}-\left(F_{n+r-1}-F_{r-1}\right)^{n}$,
and
$\prod_{\mathrm{k}=0}^{\mathrm{n}-1}\left(1-\alpha \omega_{\mathrm{k}}\right)\left(1-\beta \omega_{\mathrm{k}}\right)=\prod_{\mathrm{k}=0}^{\mathrm{n}-1}\left(1-\alpha \omega_{\mathrm{k}}\right) \prod_{\mathrm{k}=0}^{\mathrm{n}-1}\left(1-\beta \omega_{\mathrm{k}}\right)$

$$
=\left(1-\alpha^{\mathrm{n}}\right)\left(1-\beta^{\mathrm{n}}\right)=1-\mathrm{L}_{\mathrm{n}}+(-1)^{\mathrm{n}}
$$

where we have used $L_{n}=\alpha^{n}+\beta^{n}$. This establishes (1).
We note that this evaluation of $D_{n, k}$ simplifies if $n$ is even. Ruggles [2] has shown that

$$
F_{n+p}-F_{n-p}=\left\{\begin{array}{ll}
L_{n} F_{p}, & p \text { even } \\
F_{n} L_{p}, & p \text { odd }
\end{array} .\right.
$$

It follows that if $n \equiv 0(\bmod 4)$,

$$
D_{n, r}=\frac{F_{\frac{1}{2} n}^{n}\left[L_{r+\frac{1}{2} n}^{n}-L_{r-1+\frac{1}{2} n}^{n}\right]}{2-L_{n}}
$$

and if $n \equiv 2(\bmod 4)$,

$$
\mathrm{D}_{\mathrm{n}, \mathrm{r}}=\frac{\mathrm{L}_{\frac{1}{2} \mathrm{n}}^{\mathrm{n}}\left[\mathrm{~F}_{\mathrm{r}+\frac{1}{2} \mathrm{n}}^{\mathrm{n}}-\mathrm{F}_{\mathrm{r}-1+\frac{1}{2} \mathrm{n}}^{\mathrm{n}}\right]}{2-\mathrm{L}_{\mathrm{n}}}
$$

2. The generalization of (1) to second-order recurring sequences uses the same techniques. Consider the sequence $\left\{W_{n}\right\}$ defined by

$$
\mathrm{W}_{\mathrm{n}+2}=\mathrm{p} \mathrm{~W}_{\mathrm{n}+1}-\mathrm{qW} \mathrm{~W}_{\mathrm{n}},
$$

$W_{0}$ and $W_{1}$ arbitrary, where $p^{2}-4 q \neq 0$. Let $a$ and $b$ be the roots of the auxiliary polynomial, so that $a \neq b$ and $a b=q$. We shall assume that neither $a$ nor $b$ is an $n^{\text {th }}$ root of unity. Since the roots are distinct, there are constants $A$ and $B$ such that $W_{n}=A a^{n}+B b^{n}$. Define the sequence $\left\{\mathrm{V}_{\mathrm{n}}\right\}$ by $\mathrm{V}_{\mathrm{n}}=\mathrm{a}^{\mathrm{n}}+\mathrm{b}^{\mathrm{n}}$.

Put

$$
\mathrm{D}_{\mathrm{n}, \mathrm{r}}(\mathrm{~W})=\mathrm{C}\left(\mathrm{~W}_{\mathrm{r}}, \mathrm{~W}_{\mathrm{r}+1}, \cdots, \mathrm{~W}_{\mathrm{n}+\mathrm{n}-1}\right)
$$

Setting $a_{j}=W_{j+r}=A a^{j+r}+B b^{j+r}$ in (3) gives

$$
\begin{aligned}
D_{n, r}(W) & =\prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} A a^{r}\left(a \omega_{k}\right)^{j}+B b^{r}\left(b \omega_{k}\right)^{j}\right) \\
& =\prod_{k=0}^{n-1}\left(\frac{A a^{r}\left(1-a^{n}\right)}{1-a \omega_{k}}+\frac{B b^{r}\left(1-b^{n}\right)}{1-b \omega_{k}}\right) \\
& =\prod_{k=0}^{n-1} \frac{W_{r}-W_{n+r}-q\left(W_{r-1}-W_{n+r-1}\right) \omega_{k}}{\left(1-a \omega_{k}\right)\left(1-b \omega_{k}\right)} \\
& =\frac{\left(W_{r}-W_{n+r}\right)^{n}-q^{n}\left(W_{r-1}-W_{n+r-1}\right)^{n}}{1-V_{n}+q^{n}}
\end{aligned}
$$

which agrees with (1) by taking $p=1, q=-1, W_{n}=F_{n}$, and $V_{n}=L_{n}$.
3. We now consider a slight variant of the above. Put

$$
\mathrm{E}_{\mathrm{n}, \mathrm{r}}=\left|\begin{array}{ccccc}
\mathrm{F}_{\mathrm{r}} & \mathrm{~F}_{\mathrm{r}+1} & \mathrm{~F}_{\mathrm{r}+2} & \cdots & \mathrm{~F}_{\mathrm{r}+\mathrm{n}-1} \\
-\mathrm{F}_{\mathrm{r}+\mathrm{n}-1} & \mathrm{~F}_{\mathrm{r}} & \mathrm{~F}_{\mathrm{r}+1} & \cdots & \mathrm{~F}_{\mathrm{r}+\mathrm{n}-2} \\
-\mathrm{F}_{\mathrm{r}+\mathrm{n}-2} & -\mathrm{F}_{\mathrm{r}+\mathrm{n}-1} & \mathrm{~F}_{\mathrm{r}} & \cdots & \mathrm{~F}_{\mathrm{r}+\mathrm{n}-3} \\
\vdots & & & & \vdots \\
-\mathrm{F}_{\mathrm{r}+1} & -\mathrm{F}_{\mathrm{r}+2} & -\mathrm{F}_{\mathrm{r}+3} & \cdots & \mathrm{~F}_{\mathrm{r}}
\end{array}\right|
$$

We shall prove

$$
\begin{equation*}
\mathrm{E}_{\mathrm{n}, \mathrm{r}}=\frac{\left(\mathrm{F}_{\mathrm{r}}+\mathrm{F}_{\mathrm{n}+\mathrm{r}}\right)^{\mathrm{n}}+(-1)^{\mathrm{n}}\left(\mathrm{~F}_{\mathrm{n}+\mathrm{r}-1}+\mathrm{F}_{\mathrm{r}-1}\right)^{\mathrm{n}}}{1+\mathrm{L}_{\mathrm{n}}+(-1)^{\mathrm{n}}} \tag{5}
\end{equation*}
$$

A determinant of the form

$$
S\left(a_{0}, \cdots, a_{n-1}\right)=\left|\begin{array}{ccccc}
a_{0} & a_{1} & a_{2} & \cdots & a_{n-1} \\
-a_{n-1} & a_{0} & a_{1} & \cdots & a_{n-2} \\
-a_{n-2} & -a_{n-1} & a_{0} & \cdots & a_{n-3} \\
\vdots & & & & \vdots \\
-a_{1} & -a_{2} & -a_{3} & \cdots & a_{0}
\end{array}\right|
$$

is termed a skew circulant. Scott [1, Vol. 4, p. 356] has shown that

$$
\begin{equation*}
S\left(a_{0}, \cdots, a_{n-1}\right)=\prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} a_{j} \epsilon_{k}^{j}\right), \tag{6}
\end{equation*}
$$

where the

$$
\epsilon_{\mathrm{k}}=\cos \frac{(2 \mathrm{k}+1) \pi}{2}+\mathrm{i} \sin \frac{(2 \mathrm{k}+1) \pi}{2}
$$

are the $n^{\text {th }}$ roots of -1 . To prove (6) quickly, multiply $S\left(a_{0}, \cdots, a_{n-1}\right)$ by the Vandermonde determinant $\left|\epsilon_{i}^{j}\right|(i, j=0,1, \cdots, n-1)$, and treat as in the proof of (3).

To evaluate $E_{n, r}$ let $a_{j}=F_{j+r^{*}}$ A development similar to Section 1 shows that

$$
\mathrm{E}_{\mathrm{n}, \mathrm{r}}=\operatorname{II}_{\mathrm{k}=0}^{\mathrm{n}-1}\left(\sum_{\mathrm{j}=0}^{\mathrm{n}-1} \frac{\alpha^{\mathrm{r}}\left(\alpha \boldsymbol{\epsilon}_{\mathrm{k}}\right)^{\mathrm{j}}-\beta^{\mathrm{r}}\left(\beta \boldsymbol{\epsilon}_{\mathrm{k}}\right)^{\mathrm{j}}}{\alpha-\beta}\right)
$$

(7)

$$
=\prod_{\mathrm{k}=0}^{\mathrm{n}-1} \frac{\mathrm{~F}_{\mathrm{n}+\mathrm{r}}+\mathrm{F}_{\mathrm{r}}+\left[\mathrm{F}_{\mathrm{n}+\mathrm{r}-1}+\mathrm{F}_{\mathrm{r}-1}\right]_{\mathrm{k}}}{\left(1-\alpha \epsilon_{\mathrm{k}}\right)\left(1-\beta \epsilon_{\mathrm{k}}\right)} .
$$

For arbitrary x and y ,
(8) $\prod_{\mathrm{k}=0}^{\mathrm{n}-1}\left(\mathrm{x}-\mathrm{y} \epsilon_{\mathrm{k}}\right)=\mathrm{y} \prod_{\mathrm{k}=0}^{\mathrm{n}-1}\left(\frac{\mathrm{x}}{\mathrm{y}}-\epsilon_{\mathrm{k}}\right)=\mathrm{y}^{\mathrm{n}}\left[\left(\frac{\mathrm{x}}{\mathrm{y}}\right)^{\mathrm{n}}+1\right]=\mathrm{x}^{\mathrm{n}}+\mathrm{y}^{\mathrm{n}}$.

Application of this to (7) yields the desired result (5).
We remark that as before (5) simplifies for even n. Ruggles [2] has shown that

$$
F_{n+p}+F_{n-p}= \begin{cases}F_{n} L_{p}, & p \text { even } \\ L_{n} F_{p}, & p \text { odd }\end{cases}
$$

Then if $n \equiv 0(\bmod 4)$,

$$
\mathrm{E}_{\mathrm{n}, \mathrm{r}}=\frac{\mathrm{F}_{\frac{1}{2} \mathrm{n}}^{\mathrm{n}}\left(\mathrm{~F}_{\mathrm{r}+\frac{1}{2} \mathrm{n}}^{\mathrm{n}}+\mathrm{F}_{\mathrm{r}-1+\frac{1}{2} \mathrm{n}}^{\mathrm{n}}\right)}{2+\mathrm{L}_{\mathrm{n}}}
$$

while if $n \equiv 2(\bmod 4)$,

$$
\mathrm{E}_{\mathrm{n}, \mathrm{r}}=\frac{\mathrm{F}_{\frac{1}{2} \mathrm{n}}^{\mathrm{n}}\left(\mathrm{~L}_{\mathrm{r}+\frac{1}{2} \mathrm{n}}^{\mathrm{n}}+\mathrm{L}_{\mathrm{r}-1+\frac{1}{2} \mathrm{n}}^{\mathrm{n}}\right)}{2+\mathrm{L}_{\mathrm{n}}}
$$

Note that the latter yields on comparison with the determinant the identity

$$
5\left(\mathrm{~F}_{\mathrm{r}+1}^{2}+\mathrm{F}_{\mathrm{r}}^{2}\right)=\mathrm{L}_{\mathrm{r}+1}^{2}+\mathrm{L}_{\mathrm{r}}^{2}=5 \mathrm{~F}_{2 \mathrm{r}+1}
$$

4. The extension of this to second-order recurring sequences involves no new ideas, and the details are therefore omitted. Let $W_{n}$ and $V_{n}$ be as before, with the exception that we require $a$ and $b$ not be $n^{n}$ th $\operatorname{roots}^{n}$ of -1 rather than +1 to avoid division by zero. Put

$$
\mathrm{E}_{\mathrm{n}, \mathrm{r}}(\mathrm{~W})=\mathrm{S}\left(\mathrm{~W}_{\mathrm{r}}, \mathrm{~W}_{\mathrm{r}+1}, \cdots, \mathrm{~W}_{\mathrm{r}+\mathrm{n}-1}\right)
$$

Using (6) and (8), we find

$$
E_{n, r}(W)=\frac{\left(W_{n+r}+W_{r}\right)+q^{n}\left(W_{n+r-1}+W_{r-1}\right)^{n}}{1+V_{n}+q^{n}}
$$

which reduces to (5) when $\mathrm{q}=-\mathrm{q}=1, \mathrm{~W}_{\mathrm{n}}=\mathrm{F}_{\mathrm{n}}$, and $\mathrm{V}_{\mathrm{n}}=\mathrm{L}_{\mathrm{n}}$.

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1. Thomas Muir, The Theory of Determinants (4 Vols.), Dover, New York, 1960.
2. I. D. Ruggles, "Some Fibonacci Results Using Fibonacci-Type Sequences," Fibonacci Quarterly, 1 (1963), No. 2, pp. 75-80.
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# COMBINATORIAL PROBLEMS FOR GENERALIZED FIBONACCI NUMBERS 

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Theorem 1. The number of subsets of $\{1,2,3, \cdots, n\}$ which have k elements and satisfy the constraint that $i$ and $i+j(j=1,2,3, \cdots, a)$ do not appear in the same subsetis

$$
\mathrm{f}_{\mathrm{a}}(\mathrm{n}, \mathrm{k})=(\mathrm{n}-\underset{\mathrm{k}}{\mathrm{ka}+\mathrm{a}})
$$

where $\binom{n}{k}$ is the binomial coefficient. We count $\phi$, the empty set, as a subset.

Comments. Before proceeding with the proof, we note with Riordan [1], that for $a=1$, the result is due to Kaplansky. If, for fixed $n$, one sums over all k-part subsets, he gets Fibonacci numbers,

$$
F_{n+1}=\sum_{k=0}^{[(n+1) / 2]}\binom{n-k+1}{k} \quad, \quad(n \geq 0)
$$

where [x] is the greatest integer function. The theorem above is a problem given in Riordan [2].

Proof. Let $\mathrm{g}_{\mathrm{a}}(\mathrm{n}, \mathrm{k})$ be the number of admissible subsets selected from the set $\{1,2,3, \cdots, n\}$. Then

$$
g_{a}(n+1, k)=g_{a}(n, k)+g_{a}(n-a, k-1)
$$

since $g_{a}(n, k)$ counts all admissible subsets without element $n+1$ while $g_{a}(n-a, k-1)$ counts all the admissible subsets which contain element $\mathrm{n}+1$. If element $\mathrm{n}+1$ is in any such subset, then the elements $\mathrm{n}, \mathrm{n}-1$, $\mathrm{n}-2, \mathrm{n}-3, \cdots, \mathrm{n}-\mathrm{a}+1$ cannot be in the subset. We select $\mathrm{k}-1$ elements from the $\mathrm{n}-\mathrm{a}$ elements $1,2,3, \cdots, \mathrm{n}-\mathrm{a}$ to make admissible subsets and add $n+1$ to each subset. The count is precisely $g_{a}(n-a, k-1)$.

$$
f_{a}(n, k)=\left(n-k_{k}^{k}+a\right), \quad k \geq 0
$$

But, since the $f_{a}(n, k)$ are binomial coefficients,

$$
\left.\begin{array}{rl}
f_{a}(n+1, k)=(n+1-k a+a \\
k
\end{array}\right)=\binom{n-k a+a}{k}+\binom{n-a-(k-1) a+a}{k-1} .
$$

Thus, $f_{a}(n, k)$ and $g_{a}(n, k)$ satisfy the same recurrence relation. Since the boundary conditions are

$$
\mathrm{g}_{\mathrm{a}}(\mathrm{n}, 1)=\mathrm{f}_{\mathrm{a}}(\mathrm{n}, 1)=\mathrm{n},
$$

and

$$
\mathrm{g}_{\mathrm{a}}(1, \mathrm{n})=\mathrm{g}_{\mathrm{n}}(1, \mathrm{n})=0, \quad \mathrm{n}>1,
$$

the arrays are identical. This concludes the proof of Theorem 1.
We note that, for fixed $k \geq 0$, the number of $k$-part subsets of $\{1,2,3, \cdots, n\}$ for $n=0,1,2, \cdots$, are aligned in the $k^{\text {th }}$ column of Pascal's left-adjusted triangle. If one sums for fixed $n$ the number of $k$ part subsets, one obtains

$$
\mathrm{v}_{\mathrm{a}}(\mathrm{n}, \mathrm{a})=\sum_{k=0}^{\left[\frac{n+a}{a+1}\right]} f_{\mathrm{n}}(\mathrm{n}, \mathrm{k})=\sum_{k=0}^{\left[\frac{n+a}{a+1}\right]}\binom{n-k a+a}{k},
$$

where [x] is the greatest integer function. These are precisely the generalized Fibonacci numbers of Harris and Styles [3]. There,

$$
u(n ; p, 1)=\sum_{k=0}^{[n /(p+1)]}\binom{n-k p)}{k}
$$

so that

$$
V_{a}(n, a)=u(n+a ; n, 1)
$$

Clearly, if we select only certain $k$-part subsets ( $b \geq 1$ )

$$
\mathrm{V}_{\mathrm{a}}(\mathrm{n}, \mathrm{a}, \mathrm{~b})=\sum_{\mathrm{k}=0}^{\left[\frac{n+a}{\mathrm{a}+\mathrm{b}}\right]}\binom{m-k a+\mathrm{a}}{\mathrm{~kb}}
$$

then

$$
V_{a}(n, a, b)=u(n+a ; a, b)
$$

Thus, one has a nice combinatorial problem in restricted subsets whose solution sequences are the generalized Fibonacci numbers defined in [3] and studied in [4], [5], [6], [7], [11], and [12].

## GENERALIZATION

We extend Theorem 1 to all generalized Pascal triangles.
Theorem 2. The number of subsets of $\{1,2,3, \cdots, n\}$ with $k$ elements in which $i, i+j(j=1,2, \ldots$, a) are not in the same subset nor are simultaneously all of the integers $i+j a+1(j=0,1,2, \cdots, r-1)$, in the same subset, is

$$
\mathrm{f}_{\mathrm{a}}(\mathrm{n}, \mathrm{k}, \mathrm{r})=\left\{\begin{array}{c}
\mathrm{n}-\mathrm{ka}+\mathrm{a} \\
\mathrm{k}
\end{array}\right\}_{\mathrm{r}}
$$

where

$$
\left(1+x+x^{2}+\cdots+x^{r-1}\right)^{n}=\sum_{i=0}^{n(r-1)}\left\{\begin{array}{c}
n \\
i
\end{array}\right\}_{r} x^{i}
$$

We call

$$
\left\{\begin{array}{l}
n \\
i
\end{array}\right\}_{r}
$$

the r -nomial coefficients, and n designates the row and i designates the column in the generalized Pascal triangle induced by the expansion of

$$
\left(1+x+x^{2}+\cdots+x^{r-1}\right)^{n}, n=0,1,2, \cdots
$$

Proof. Let $g_{a}(n, k, r)$ be the number of admissible subsets selected with elements from $\{1,2,3, \cdots, n\}$. Then

$$
\begin{aligned}
g_{a}(n+1, k, r)=g_{a}(n, k, r) & +g_{a}(n-a, k-1, r)+g_{a}(n-2 a, k-2, r) \\
& +\cdots+g_{a}(n-(r-1) a, k-r+1, r)
\end{aligned}
$$

Consider the set of numbers $n+1, n-a+1, n-2 a+1, n-3 a+1, \cdots$, $n-(r-1) a+1$. The general term $g_{a}(n-s a, k-s, r)$ gives the number of admissible subsets which require the use of $n+1, \mathrm{n}-\mathrm{a}+1, \mathrm{n}-2 \mathrm{a}+1$, $\cdots, n-(s-1) a+1$, disallows the integer $n-s a+1$, but permits the use of the integers $\mathrm{n}=1,2,3, \cdots, \mathrm{n}-\mathrm{sa}$ in the subsets subject to the constraints that integers $i, i+j(j=1,2,3, \cdots, a)$ do not appear in the same subset. This concludes the derivation of the recurrence relation.

Next, consider

$$
f_{a}(\mathrm{n}, \mathrm{k}, \mathrm{r})=\left\{\begin{array}{c}
\mathrm{n}-\mathrm{ka}+\mathrm{a} \\
\mathrm{k}
\end{array}\right\}_{\mathrm{r}}
$$

Since $f_{a}(n, k, r)$ is an $r$-nomial coefficient, then

$$
\begin{aligned}
f_{a}(n+1, k, r)=f_{a}(n, k, r) & +f_{a}(n-a, k-1, r)+\cdots+f_{a}(n-s a, k-s, r) \\
& +\cdots+f_{a}(n-(r-1) a, k-r+1, r)
\end{aligned}
$$

Thus, $f_{a}(n, k, r)$ and $g_{a}(n, k, r)$ both obey the same recurrence relation, and

$$
\begin{aligned}
& \mathrm{f}_{\mathrm{a}}(\mathrm{n}, 1, \mathrm{r})=\mathrm{g}_{\mathrm{a}}(\mathrm{n}, 1, \mathrm{r})=\mathrm{n} \\
& \mathrm{f}_{\mathrm{a}}(1, \mathrm{n}, \mathrm{r})=\mathrm{g}_{\mathrm{a}}(1, \mathrm{n}, \mathrm{r})=0, \quad \mathrm{n}>1
\end{aligned}
$$

for all $\mathrm{n} \geq 0$, so that the arrays are identical for all $\mathrm{k} \geq 0$.
Summing, for fixed $\mathrm{n} \geq 0$, over all numbers of all k-part subsets yields

$$
\mathrm{v}_{\mathrm{a}}(\mathrm{n}, \mathrm{a}, \mathrm{r})=\sum_{\mathrm{k}=0}^{\left[\frac{(\mathrm{n}+\mathrm{a})(\mathrm{r}-1)}{1+\mathrm{a}(\mathrm{r}-1)}\right]}\{\mathrm{n}-\underset{\mathrm{k}}{\mathrm{ka}}+\mathrm{a}\}_{\mathrm{r}}
$$

If we now generalize the "generalized Fibonacci numbers, $u(n ; p, q)$, of Harris and Styles [3]" to the generalized Pascal triangles obtained from the expansions $\left(1+x+x^{2}+\cdots+x^{r-1}\right)^{n}, n=0,1,2,3, \cdots$,

$$
u(n ; p, q, r)=\sum_{k=0}^{\left[\frac{n(r-1)}{q+p(r-1)}\right]}\left\{\begin{array}{c}
n-k p \\
k q
\end{array}\right\}_{r}
$$

there are precisely

$$
p+\left[\frac{q}{r-1}\right]+1
$$

ones at the beginning of each $u(n ; p, q, r)$ sequence. Our application starts with just one 1. Let

$$
m=\left[\frac{q}{r-1}\right]
$$

the greatest integer in $q /(r-1)$. Then,

$$
u(n+a+m ; a, b, r)=\sum_{k=0}^{\left[\frac{(n+a+m)(r-1)}{b+a(r-1)}\right]}\left\{\begin{array}{c}
n+a+m-k a \\
k b
\end{array}\right\}_{r}
$$

Thus the solution set to the number of subsets of $\{1,2,3, \cdots, n\}$ subject to the constraints that no pairs $i, i+j(j=1,2,3, \cdots, a)$ are to be allowed in the same subset, nor are all of $i+j a+1(j=0,1,2,3, \cdots, r-1)$ to be allowed in the same subset, are the generalized Fibonacci numbers of Harris and Styles generalized to Pascal triangles induced from the expansions of

$$
\left(1+x+x^{2}+\cdots+x^{r-1}\right)^{n}, \quad n=0,1,2,3, \cdots .
$$

One notes that the r-nacci generalized Fibonacci numbers

$$
u(n ; 1,1, r)=\sum_{k=0}^{\left[\frac{n(r-1)}{r}\right]}\left\{\begin{array}{c}
n-k \\
k
\end{array}\right\}_{r}
$$

are not generally obtained by setting $a=0$ in the above formulation. However, the generalized Fibonacci sequences for the binomial triangle are obtained if $r=2$. The other $r$-nacci number sequences are obtained if the subsets are simply restricted from containing simultaneously r consecutive integers from the set $\{1,2,3, \cdots, n\}$ but there is no restriction of $r>2$ about pairs of consecutive integers. Thus, for these $r$-nacci sequences ( $r>2$ ), we cannot simply set $a=1$. However, the formulas look identical. Let

$$
\mathrm{V}(\mathrm{n} ; 1,1, \mathrm{r})=\mathrm{u}(\mathrm{n}+1 ; 1,1, \mathrm{r}) ;
$$

then

$$
\mathrm{V}(\mathrm{n} ; 1,1, \mathrm{r})=\sum_{\mathrm{k}=0}^{\left[\frac{(\mathrm{n}+1)(\mathrm{r}-1)}{\mathrm{r}}\right]}\left\{n-\frac{k}{k}+1\right\}_{r}
$$

which is seen to be the generalization of Kaplansky's lemma to generalized Pascal triangles.

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# APPLICATION OF RECURSIVE SEQUENCES TO DIOPHANTINE EQUATIONS 

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ABSTRACT
In a former version of this paper ("Iteration Algorithms for Certain Sums of Squares"), Karst, by composition of simple sums of squares, found six iteration algorithms of which he could prove the first by means of the generalized Pell equation and the second by the permanence of formal laws. For the remaining four, equivalent to the solution of $x^{2}-\left(k^{2}+1\right) y^{2}=k^{2}$, with $k$ $=1,2$, and 3, Finkelstein and London were able to furnish a unifying proof by the use of class numbers and quadratic fields. This justifies the new title.

The following three-step iteration algorithm to generate x in $2 \mathrm{x}+1=$ $\mathrm{a}^{2}$ and $3 \mathrm{x}+1=\mathrm{b}^{2}$, simultaneously, was mentioned in [6, p .211 ]:

$$
\begin{aligned}
1 \cdot 10-1 & =9 & 9^{2} & =81 \\
9 \cdot 10-1 & =89 & 89^{2} & =7921
\end{aligned} r(81-1) / 2=40=x_{1} .
$$

Proof. From $2 x+1=a^{2}$ and $3 x+1=b^{2}$ comes $3 a^{2}-2 b^{2}=1$. If $a_{n}, b_{n}$ is any solution of this generalized Pell equation, then

$$
a_{n+1}=5 a_{n}+4 b_{n}, \quad b_{n+1}=6 a_{n}+5 b_{n}
$$

is the next larger one. From these, we can obtain immediately

$$
a_{n+1}+a_{n-1}=10 a_{n}, \quad b_{n+1}+b_{n-1}=10 b_{n},
$$

which is equivalent to the algorithm.

For the $\mathrm{n}^{\text {th }}$ formula, we use the usual approach by linear substitution (for example, [1, p. 181]) and obtain

$$
x_{n}=\left[(\sqrt{6}+2)(5+2 \sqrt{6})^{n}+(\sqrt{6}-2)(5-2 \sqrt{6})^{\mathrm{n}}\right]^{2} / 48-1 / 2
$$

This formula has three shortcomings: (1) it uses fractions, (2) it employs roots, and (3) it has n in the exponent. The algorithm above has none of them.

Similar arguments are valid for a four-step iteration algorithm [4] to generate x in $\mathrm{x}^{2}+(\mathrm{x}+1)^{2}=\mathrm{y}^{2}$.

Sometimes, the $n^{\text {th }}$ term formula may be simple, as for $a^{2}+b^{2}+(a b)^{2}$. $=c^{2}$, a and b consecutive positive integers [2]. Here, we have

$$
(n-1)^{2}+n^{2}+(n-1) n^{2}=\left(n^{2}-n+1\right)^{2}
$$

and hence we need no algorithm. But for $\mathrm{a}=1$, an algorithm would be helpful. Let us first find some clues to such an algorithm. We have by hand and by a table of squares:

$$
\begin{aligned}
& 1^{2}+0^{2}+0^{2}=1^{2}=\left(0^{2}+1\right)^{2} \\
& 1^{2}+2^{2}+2^{2}=3^{2}=\left(2^{2}-1\right)^{2} \\
& 1^{2}+12^{2}+12^{2}=17^{2}=\left(4^{2}+1\right)^{2} \\
& 1^{2}+70^{2}+70^{2}=99^{2}=\left(10^{2}-1\right)^{2} .
\end{aligned}
$$

The alternating +1 and -1 in the last column, which shows a constant pattern, suggests the possibility of an algorithm. If we can find all $b$, say from $b_{3}=12$ on, we will also have all c. After some trials and errors we obtain

## Iteration Algorithm I

$$
\begin{aligned}
6 \cdot 2-0 & =12 \\
6 \cdot 12-2 & =70 \\
6 \cdot 70-12 & =408 \\
6 \cdot 408-70 & =2378 \\
6 \cdot 2378-408 & =13860 \\
6 \cdot 13860-2378 & =80782
\end{aligned}
$$

which yields easily the next four results:

$$
\begin{aligned}
& 1^{2}+408^{2}+408^{2}=577^{2}=\left(24^{2}+1\right)^{2} \\
& 1^{2}+2378^{2}+2378^{2}=3363^{2}=\left(58^{2}-1\right)^{2} \\
& 1^{2}+13860^{2}+13860^{2}=19601^{2}=\left(140^{2}+1\right)^{2} \\
& 1^{2}+80782^{2}+80782^{2}=114243^{2}=\left(338^{2}-1\right)^{2}
\end{aligned}
$$

Similarly, we approach the case $\mathrm{a}=2$, We have by hand and by a table of squares:

$$
\begin{aligned}
& 2^{2}+1^{2}+2^{2}=3^{2}=\left(1^{2}+2\right)^{2} \\
& 2^{2}+3^{2}+6^{2}=7^{2}=\left(3^{2}-2\right)^{2} \\
& 2^{2}+8^{2}+16^{2}=18^{2}=\left(4^{2}+2\right)^{2} \\
& 2^{2}+21^{2}+42^{2}=47^{2}=\left(7^{2}-2\right)^{2}
\end{aligned}
$$

The alternating +2 and -2 in the last column, which shows a constant pattern, suggest the possibility of an algorithm. If we can find all $b$, say from $b_{3}=8$ on, we will also have all c. After some trials and errors we obtain

| Iteration Algorithm III |  |
| ---: | :--- |
| $3 \cdot 3-1$ | $=8$ |
| $3 \cdot 8-3$ | $=21$ |
| $3 \cdot 21-8$ | $=55$ |
| $3 \cdot 55-21$ | $=144$ |
| $3 \cdot 144-55$ | $=377$ |
| $3 \cdot 377-144$ | $=987$ |

which yields easily the next four results:

$$
\begin{aligned}
& 2^{2}+55^{2}+110^{2}=123^{2}=\left(11^{2}+2\right)^{2} \\
& 2^{2}+144^{2}+288^{2}=322^{2}=\left(18^{2}-2\right)^{2} \\
& 2^{2}+377^{2}+754^{2}=843^{2}=\left(29^{2}+2\right)^{2} \\
& 2^{2}+987^{2}+1974^{2}=2207^{2}=\left(47^{2}-2\right)^{2}
\end{aligned}
$$

Slightly differently behaves the case of $a=3$. We have by hand and by a table of squares

$$
\begin{aligned}
& 3^{2}+0^{2}+0^{2}=3^{2}=\left(0^{2}+3\right)^{2} \\
& 3^{2}+2^{2}+6^{2}=7^{2}=\left(2^{2}+3\right)^{2} \\
& 3^{2}+4^{2}+12^{2}=13^{2}=\left(4^{2}-3\right)^{2} \\
& 3^{2}+18^{2}+54^{2}=57^{2} \\
& 3^{2}+80^{2}+240^{2}=253^{2}=\left(16^{2}-3\right)^{2} \\
& 3^{2}+154^{2}+462^{2}=487^{2}=\left(22^{2}+3\right)^{2} \\
& 3^{2}+684^{2}+2052^{2}=2163^{2}
\end{aligned}
$$

Here the doubly alternating +3 and -3 in the last column would show a constant pattern, if the exceptional values $57^{2}$ and $2163^{2}$ could be eliminated. This suggests obviously the possibility of two algorithms. To obtain further results, we write an Integer-FORTRAN program for the IBM 1130 which yields:

$$
\begin{array}{lr}
3^{2}+3038^{2}+9114^{2}= & 9607^{2}=\left(98^{2}+3\right)^{2} \\
3^{2}+5848^{2}+17544^{2}=18493^{2}=\left(136^{2}-3\right)^{2} \\
3^{2}+25974^{2}+77922^{2}=82137^{2} \\
3^{2}+115364^{2}+346092^{2}=364813^{2}=\left(604^{2}-3\right)^{2} \\
3^{2}+222070^{2}+666210^{2}=702247^{2}=\left(838^{2}+3\right)^{2} \\
3^{2}+986328^{2}+2958984^{2}=3119043^{2} \\
3^{2}+4380794^{2}+13142382^{2}=13853287^{2}=\left(3722^{2}+3\right)^{2}
\end{array}
$$

Now we want to find an algorithm which should generate the sequence 80,154 , $3038,5848,115364,222070,4380794, \cdots$. Let the terms $b_{1}=0, b_{2}=2$, and $b_{3}=4$ be given; then $b_{0}=-4$ is the left neighbor of $b_{1}=0$, since

$$
3^{2}+(-4)^{2}+(-12)^{2}=13^{2}=\left(4^{2}-3\right)^{2}
$$

is the logical extension to the left. With this new initializing and some trials and errors, we obtain the Iteration Algorithm III on the following page. Now there remains only to find an algorithm which should generate 25974,986328 , ... . Here, we have not far to go, since such an algorithm is already contained in the former one, and we obtain Iteration Algorithm IV on the following page.

Iteration Algorithm III

$$
\begin{aligned}
38 \cdot 2-(-4) & =80 \\
2 \cdot 80-2: 4+2 & =154 \\
38 \cdot 80-2 & =3038 \\
2 \cdot 3038-2 \cdot 154+80 & =5848 \\
38 \cdot 3038-80 & =115364 \\
2 \cdot 115364-2: 5848+3038 & =222070 \\
38 \cdot 115364-3038 & =4380794
\end{aligned}
$$

Iteration Algorithm IV

$$
\begin{aligned}
38 \cdot 684-18 & =25974 \\
38 \cdot 25974-684 & =986328
\end{aligned}
$$

Finally, one could ask: Does there exist a general formula for solving $\mathrm{x}^{2}+$ $\mathrm{y}^{2}+\mathrm{z}^{2}=\mathrm{w}^{2}$ ? The answer is yes. Let
$\mathrm{x}=\mathrm{p}^{2}+\mathrm{q}^{2}-\mathrm{r}^{2}, \quad \mathrm{y}=2 \mathrm{pr}, \quad \mathrm{z}=2 \mathrm{qr}, \quad$ and $\quad \mathrm{w}=\mathrm{p}^{2}+\mathrm{q}^{2}+\mathrm{r}^{2} ;$
then $x^{2}+y^{2}+z^{2}=w^{2}$ becomes $0=0$. But this formula has two shortcomings: (a) it uses fractions, and (b) it employs roots, since, for example, the solution $3^{2}+2^{2}+6^{2}=7^{2}$ requires $\mathrm{p}=\sqrt{2} / 2, \quad \mathrm{q}=3 \sqrt{2} / 2$, and $\mathrm{r}=$ $\sqrt{2}$.

Now we shall prove how the integer solutions of certain Diophantine equations of the second degree, equivalent to Iteration Algorithms I-IV, can be found by recursive sequences. We will consider the equation

$$
\begin{equation*}
x^{2}-\left(k^{2}+1\right) y^{2}=k^{2} \tag{1}
\end{equation*}
$$

with $\mathrm{k}=1,2$, and 3. Further, $\mathrm{x}_{\mathrm{n}}$ and $\mathrm{y}_{\mathrm{n}}$ will denote integer solutions of (1).

If $k=1$, Eq. (1) becomes $x^{2}-2 y^{2}=1$. By Theorem 3 of [3], the recurrence formula for this equation is given by

$$
\mathrm{y}_{\mathrm{n}}=6 \mathrm{y}_{\mathrm{n}+1}-\mathrm{y}_{\mathrm{n}-1}, \quad \mathrm{n}>2
$$

with $\mathrm{y}_{1} \neq 2$ and $\mathrm{y}_{2}=12$.

If $\mathrm{k}=2$, Eq. (1) becomes

$$
\begin{equation*}
x^{2}-5 y^{2}=4 \tag{2}
\end{equation*}
$$

This equation belongs to the quadratic field $Q(\theta), \theta=\sqrt{5}$, which has $(1,(1+\theta) / 2)$ as an integral basis, and its fundamental unit is $\boldsymbol{\epsilon}_{0}=(1+\theta) / 2$. Since the class number of $Q(\theta)$ is 1 and the discriminant $D \equiv 5(\bmod 8)$, the ideal (2) is prime [5, p. 66]. Hence, all the algebraic integers of $\mathrm{Q}(\theta)$ of norm 4 are associates of 2. Thus, if $x_{n}+y_{n} \theta$ is an algebraic integer of norm 4, we get

$$
\mathrm{x}_{\mathrm{n}}+\mathrm{y}_{\mathrm{n}} \theta=2 \epsilon_{0}^{2 \mathrm{n}}=2 \epsilon_{1}^{\mathrm{n}}
$$

where $\boldsymbol{\epsilon}_{1}=\boldsymbol{\epsilon}_{0}^{2}=(3+\theta) / 2$.
Remark. Since we want all the algebraic integers of norm 4, we have only considered the even powers of $\epsilon_{0}$. Noting that

$$
\epsilon_{1}^{\mathrm{n}+1}=3 \epsilon_{1}^{\mathrm{n}}-\epsilon_{1}^{\mathrm{n}-1}
$$

we obtain

$$
\mathrm{y}_{\mathrm{n}+1}=3 \mathrm{y}_{\mathrm{n}}-\mathrm{y}_{\mathrm{n}-1}, \quad \mathrm{n}>2
$$

with $\mathrm{y}_{1}=1$ and $\mathrm{y}_{2}=3$. It can easily be shown, by using the well-known identity

$$
\mathrm{L}_{\mathrm{n}}^{2}-5 \mathrm{~F}_{\mathrm{n}}^{2}=4(-1)^{\mathrm{n}}
$$

of the Lucas and Fibonacci numbers, that $y_{n}=F_{2 n}$ and $x_{n}$ $=\mathrm{L}_{2 \mathrm{n}}$. If $\mathrm{k}=3$, Eq. (1) becomes

$$
\begin{equation*}
x^{2}-10 y^{2}=9 \tag{3}
\end{equation*}
$$

This equation belongs to the field $\mathrm{Q}(\theta), \theta=\sqrt{10}$, which has $(1, \theta)$ as an integral basis, $\epsilon_{0}=3+\theta$ as its fundamental unit, and class number 2.

Since the discriminant $D \equiv 1(\bmod 3)$, the ideal (3) becomes $P_{1} P_{2}$, where $P_{1}$ and $P_{2}$ are distinct prime ideals of norm 3. Thus there are 3 distinct ideals of norm 9. Since $3,7-2 \theta, 7+2 \theta$ are non-associated integers of norm 9 , all the integers of norm 9 are associates of one of these 3 integers. It follows that
(4) $\quad\left\{\begin{array}{c}\mathrm{x}_{3 \mathrm{n}}+\mathrm{y}_{3 \mathrm{n}} \theta=3 \epsilon_{0}^{2 \mathrm{n}}=3 \epsilon_{1}^{\mathrm{n}}, \\ \mathrm{x}_{3 \mathrm{n}+1}+\mathrm{y}_{3 \mathrm{n}+1} \theta=(7-2 \theta) \epsilon_{0}^{2 \mathrm{n}}=(7-2 \theta) \epsilon_{1}^{\mathrm{n}}, \\ \mathrm{x}_{3 \mathrm{n}+2}+\mathrm{y}_{3 \mathrm{n}+2} \theta=(7+2 \theta) \epsilon_{0}^{2 \mathrm{n}}=(7+2 \theta) \epsilon_{1}^{\mathrm{n}}\end{array}\right.$

By applying Theorem 3 of [3], we find that $\epsilon_{1}^{n}$ satisfies the recurrence formula

$$
u_{n+2}=38 u_{n+1}-u_{n}
$$

where $u_{n}$ is either the constant term of the coefficient of $\theta$ for $\epsilon_{1}^{n}$. Thus the recurrence formulas for Eqs. (4) are, for $n>2$,

$$
\left\{\begin{array}{lll}
b_{3 n}=38 b_{3 n-3}-b_{3 n-6}, & b_{1}=57, & b_{6}=684 \\
b_{3 n+1}=38 b_{3 n-2}-b_{3 n-5}, & b_{1}=2, & b_{4}=80 \\
b_{3 n+2}=38 b_{3 n-1}-b_{3 n-4}, & b_{2}=4, & b_{5}=154
\end{array}\right.
$$

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# ON A CLASS OF DIFFERENCE EQUATIONS 

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The purpose of this article is to examine sequences generated by a certain class of difference equations and to encourage further investigations into their properties. We shall be interested in sequences satisfying the recurrence relation,
where k is a positive integer.
It may be shown by a simple inductive argument that
(2)

$$
\mathrm{v}_{\mathrm{n}}=\frac{(\mathrm{k}+1)^{\mathrm{F}_{\mathrm{n}}}-1}{\mathrm{k}} \quad(\mathrm{n} \geq 1)
$$

where $F_{n}$ denotes the $n^{\text {th }}$ Fibonacci number.
When we wish to emphasize the dependence on the parameter, $k$, we shall write $\mathrm{v}_{\mathrm{n}} \equiv \mathrm{v}_{\mathrm{n}}(\mathrm{k})$.

$$
\text { A MODEL FOR }\left\{v_{n}\right\}_{n=1}^{\infty}
$$

Let $b$ denote an integer $(b \geq 2)$. Consider the sequence defined as follows:

$$
\begin{equation*}
\theta_{\mathrm{n}}=\frac{\mathrm{F}_{\mathrm{n}}}{11 \cdots 1} \text { (b) } \quad(\mathrm{n} \geq 1) . \tag{3}
\end{equation*}
$$

where (b) denotes base b. Obviously,

$$
\begin{equation*}
\theta_{n}=\sum_{i=0}^{F_{n}-1} b^{i}=\frac{b^{F_{n}}-1}{b-1} \quad(n \geq 1) \tag{4}
\end{equation*}
$$

As above, we shall write $\theta_{\mathrm{n}} \equiv \theta_{\mathrm{n}}$ (b). From Eqs. (2) and (4), we see that

$$
\mathrm{v}_{\mathrm{n}}(\mathrm{k}+1)=\theta_{\mathrm{n}}(\mathrm{~b})
$$

$b^{n}-1$ has been called the $n^{\text {th }}$ Fermatian function of $b$ and

$$
B_{n} \equiv \frac{b^{n}-1}{b-1}
$$

has been called a reduced Fermatian of index b. (See [1].) We note that $B_{F_{n}}=\theta_{\mathrm{n}}$.

If we are willing to abuse the language, we may extend the allowed values of b . Formally, if $\mathrm{k}=0$, Eq. (1) becomes the usual Fibonacci recurrence relation. Then $b=k+1=1$, and if we interpret the $1^{\prime} s$ in (3) as tally marks,

$$
\begin{aligned}
\theta_{\mathrm{n}} & =1(1)^{\mathrm{F}_{\mathrm{n}}-1}+\cdots+1(1)^{0} \\
& =\frac{\mathrm{F}_{\mathrm{n}}}{11 \cdots 1}
\end{aligned}
$$

Similarly, if $k=-1$, then $b=0$. With the agreement that $0^{0}=1$,

$$
\begin{align*}
\theta_{\mathrm{n}} & =1(0)^{\mathrm{F}_{\mathrm{n}}-1}+\cdots+1(0)^{0} \\
& =\frac{\mathrm{F}_{\mathrm{n}}}{11 \cdots 1}(0) \tag{0}
\end{align*}
$$

Thus $\theta_{\mathrm{n}} \equiv 1$. But the solution of (1) in this case is

$$
\mathrm{v}_{\mathrm{n}}(-1) \equiv 1 \quad(\mathrm{n} \geq 1)
$$

Using similar interpretations for negative bases, we can extend (1) and (3) to negative integers.

$$
\text { DIVISIBILITY PROPERTIES OF }\left\{v_{n}\right\}_{n=1}^{\infty}
$$

It is interesting to note that if

$$
\left\{\mathrm{v}_{\mathrm{n}}(1)\right\}_{\mathrm{n}=1}^{\infty}
$$

contains an infinite number of primes, then there would be an infinite number of Fibonacci and Mersenne primes.

In this section, we shall assume $k=9 \quad(b=10)$ unless otherwise specified.

Theorem 1.
(a)
(b)

$$
\left.\begin{array}{ll}
\left(\theta_{\mathrm{n}}, \mathrm{n}+1\right.
\end{array}\right)=1 \quad(\mathrm{n} \geq 1) ;
$$

Proof. a) Deny! Then there is a pair such that $\left(\theta_{m}, \theta_{m+1}\right)=d>1$. But $\mathrm{d}\left|\mathrm{v}_{\mathrm{n}+2}, \mathrm{~d}\right| \mathrm{v}_{\mathrm{n}+1}$ implies $\mathrm{d} \mid \mathrm{v}_{\mathrm{n}}$. Thus, after repeated use of the above, we would have $\left(\theta_{1}, \theta_{2}\right) \geq d \geq 1$. Contradiction.
b) Similar to part a).

Theorem 2. None of the $\theta_{\mathrm{n}}$ are perfect.
Proof. Any odd perfect number is congruent to 1 modulo 4 (see [2]). But

$$
\theta_{\mathrm{n}} \equiv 3(\bmod 4) \quad \text { for } \mathrm{n} \geq 3
$$

Theorem 3. $3 \mid \theta_{\mathrm{n}}$ if and only if $4 \mid \mathrm{n}$.
Proof. Clearly, $3 \mid \theta_{n}$ if and only if $3 \mid F_{n}$. Thus $F_{4} \mid F_{n}$ and the result follows.

Theorem 4. $11 \mid \theta_{\mathrm{n}}$ if and only if $3 \mid \mathrm{n}$.
Proof. $11 \mid \theta_{n}$ if and only if $2=F_{3} \mid F_{n}$ and the result follows.
Theorem 5. a) $7 \mid \theta_{\mathrm{n}}$ if and only if 12 n ;
b) $13 \mid \theta_{\mathrm{n}}$ if and only if 12 n .

Proof. a) Consider the congruences,

$$
\begin{array}{rlrl}
1 & \equiv 1(\bmod 7), & 10 & \equiv 3(\bmod 7), \\
100 & \equiv 2(\bmod 7) \\
1,000 & \equiv-1(\bmod 7), & 10,000 & \equiv-3(\bmod 7), \\
100,000 & \equiv-2(\bmod 7)
\end{array}
$$

Clearly $7 \mid \theta_{n}$ if and only if $6 \mid F_{n}$ ．But $6 \mid F_{n}$ is equivalent to $2 \mid F_{n}$ and $3 \mid F_{n}$ of $3 \mid n$ and $4 \mid n$ and the result follows．
b）Similar to a），considering the congruences modulo 13.
In light of the above，we have the unusual result that $3 \mid \theta_{n}$ and $11 \mid \theta_{n}$ implies $7 \mid \theta_{\mathrm{n}}$ and $13 \mid \theta_{\mathrm{n}}$ 。

We mention some other results which the reader might like to establish．
Assertion 1：$\quad 18 \mid F_{n}$ implies $19 \mid \theta_{n}$ 。
Assertion 2：$\quad 41 \mid \theta_{n}$ if and only if $5 \mid n$ ．
Assertion 3：$\quad 271 \mid \theta_{\mathrm{n}}$ if and only if $5 \mid \mathrm{n}$ 。
Assertion 4：$\quad 73\left|\theta_{n}, 101\right| \theta_{n}, \quad 137 \mid \theta_{n}$ if and only if $6 \mid n$ 。

$$
\text { GENERATING FUNCTIONS FOR }\left\{\mathrm{v}_{\mathrm{n}}(\mathrm{k})\right\}_{\mathrm{n}=1}^{\infty}
$$

One area which might be worth investigating is that of obtaining gener－ ating functions for the sequences．Of course，since

$$
\begin{equation*}
\frac{1}{1-x-x^{2}}=\sum_{i=1}^{\infty} F_{i} x^{i-1} \tag{6}
\end{equation*}
$$

we have

$$
\begin{equation*}
\frac{1}{1-x-x^{2}}=\sum_{i=1}^{\infty} \frac{\log \left[1+k v_{i}(k)\right]}{\log (k+1)} x^{i-1} \tag{7}
\end{equation*}
$$

but one should be able to do better than this．

## ALTERNATE RELATIONSHIPS

We present two results along these lines．
Theorem 6．

$$
\theta_{n+2}(2)=2 \prod_{i=1}^{n}\left[1+\theta_{i}(2)\right]-1 \quad(n \geq 1)
$$

Proof．Since

$$
2^{\mathrm{F}_{\mathrm{n}}}=1+\theta_{\mathrm{n}}(2)
$$

and

$$
\sum_{i=1}^{n} F_{i}=F_{n+2}-1 \quad(n \geq 1)
$$

the result easily follows.
Theorem 7。

$$
1+\theta_{2 n}(2)=\prod_{i=1}^{n}\left[1+\theta_{2 i-1}(2)\right] \quad(n \geq 1)
$$

Proof. The result is readily obtained from

$$
\sum_{i=1}^{n} F_{2 i-1}=F_{2 n}
$$

GENERALIZATION TO OTHER RECURSIVELY DEFINED SEQUENCES
We conclude our discussion with one result in this area.
Theorem 8. If

$$
\left\{u_{n}\right\}_{n=1}^{\infty}
$$

is a recursively defined positive integer sequence satisfying the linear difference equation

$$
\begin{equation*}
\sum_{i=0}^{m} \alpha_{i} u_{n+i}=\beta \quad(n \geq 1) \quad(\text { order } m) \tag{8}
\end{equation*}
$$

and boundary conditions $\left\{u_{1}, u_{2}, \cdots, u_{m-1}\right\}$, where $\beta$ and $\alpha_{i}$ for $i \in\{0$, $1, \cdots, m\}$ are constants, and if

$$
\beta_{\mathrm{n}}=\frac{u_{\mathrm{n}}}{11 \cdots 1} \text { (b) } \quad(\mathrm{n} \geq 1) ;
$$

then
(9)

$$
\min _{i=0}^{m}\left[1+(b-1) \beta_{n+i}\right]^{\alpha}=b^{\beta} \quad(n \geq 1)
$$

Proof. Since

$$
\beta_{n}=\frac{b^{u_{n}}-1}{b-1} \quad(n \geq 1)
$$

we have

$$
\mathrm{b}^{\mathrm{u}_{\mathrm{k}}}=1+(\mathrm{b}-1) \beta_{\mathrm{k}}
$$

for $\mathrm{k} \geq 1$ and the result readily follows.

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1. L. E. Dickson, History of the Theory of Numbers, Vol. 1, p. 385.
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## ON A CONJECTURE OF DMITRI THORO*

## DAVID G. BEVERAGE

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Denoting the $\mathrm{n}^{\text {th }}$ term of the Fibonacci sequence $1,1,2,3,5, \cdots$, by $F_{n}$, where $F_{n+2}=F_{n+1}+F_{n}$, it is well known that

$$
F_{n}^{2}-F_{n-1} F_{n+1}=(-1)^{n+1}
$$

If odd prime p divides $\mathrm{F}_{\mathrm{n}-1}$, then

$$
\mathrm{F}_{\mathrm{n}}^{2} \equiv(-1)^{\mathrm{n}+1} \quad(\bmod \mathrm{p})
$$

so that $(-1)^{\mathrm{n}+1}$ is a quadratic residue modulo p . Clearly, for $\mathrm{n}=2 \mathrm{k}$, this implies -1 is a quadratic residue modulo $p$, and accordingly, $p \equiv 1(\bmod$ [Continued on page 537.]

# THE FIBONACCI NUMBERS CONSIDERED AS A PISOT SEQUENCE* 

MORRIS JACK DeLEON

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Charles Pisot [1] was the first to consider the sequence, $\left\{a_{n}\right\}_{n=0}^{\infty}$, of natural numbers determined from two natural numbers $a_{0}$ and $a_{1}$ such that

$$
2 \leq a_{0}<a_{1}
$$

(1) and

$$
-\frac{1}{2}<a_{n+2}-\frac{a_{n+1}^{2}}{a_{n}} \leq \frac{1}{2}
$$

for all $n \geq 0$. The Fibonacci numbers with the first two terms deleted satisfy Eq. (1).

Peter Flor [2] called the sequences which satisfy (1) Pisot sequences of the second kind. Flor also considered the sequence of natural numbers determined from two natural numbers $a_{0}$ and $a_{1}$ such that

$$
2 \leq a_{0}<a_{1}
$$

(2) and

$$
-\frac{1}{2} \leq a_{n+2}-\frac{a_{n+1}^{2}}{a_{n}}<\frac{1}{2}
$$

for all $\mathrm{n} \geq 0$. He called these sequences Pisot sequences of the first kind. For a Pisot sequence of the first (second) kind $a_{n+2}$ is simply the nearest integer to $a_{n+1}^{2} / a_{n}$, where in case of ambiguity we choose the smaller

[^0]Dec. 1970
THE FIBONACCI NUMBERS
(larger) integer. By Pisot sequence we shall mean a sequence that satisfies (1) and (2).

By

$$
\left\{\mathrm{F}_{\mathrm{n}}+\mathrm{k}\right\}_{\mathrm{n}=\mathrm{n}_{0}}^{\infty}
$$

we mean the sequence formed by adding $k$ to each term of the sequence

$$
\left\{\mathrm{F}_{\mathrm{n}}\right\}_{\mathrm{n}=\mathrm{n}_{0}}^{\infty}
$$

where $F_{n}$ is the $n^{\text {th }}$ Fibonacci number. In this paper necessary and sufficient conditions for

$$
\left\{\mathrm{F}_{\mathrm{n}}+\mathrm{k}\right\}_{\mathrm{n}=\mathrm{n}_{0}}^{\infty}
$$

to be a Pisot sequence are given.
The main result is
Theorem. Let

$$
\left\{F_{n}\right\}_{n=1}^{\infty}
$$

be the Fibonacci sequence. The sequence

$$
\left\{\mathrm{F}_{\mathrm{n}}+1\right\}_{\mathrm{n}=\mathrm{n}_{0}}^{\infty}
$$

is a Pisot sequence of the first kind (second kind) iff $n_{0} \geq 6\left(n_{0} \geq 4\right)$. The sequence

$$
\left\{F_{n}-1\right\}_{n=n_{0}}^{\infty}
$$

is a Pisot sequence iff $n_{0} \geq 7$. The sequence $\left\{F_{n}\right\}$ is a Pisot sequence of the first kind (second kind) iff $n_{0} \geq 4\left(n_{0} \geq 3\right)$. If $|k|>1$ then there exists no integer $\mathrm{n}_{0}$ such that

$$
\left\{\mathrm{F}_{\mathrm{n}}+\mathrm{k}\right\}_{\mathrm{n}=\mathrm{n}_{0}}^{\infty}
$$

is a Pisot sequence.
We shall need two lemmas in order to prove the theorem.

$$
\begin{aligned}
& \text { Lemma 1。 } \quad F_{n+2}-2 F_{n+1}+F_{n}=F_{n-2} \text {. } \\
& \text { Proof. } F_{n+2}-2 F_{n+1}+F_{n}=\left(F_{n+1}+F_{n}\right)-2 F_{n+1}+F_{n} \\
& =-F_{n+1}+2 F_{n}=-\left(F_{n}+F_{n-1}\right)+2 F_{n} \\
& =F_{n}-F_{n-1}=F_{n-2} \quad:: \\
& \text { Lemma 2. } \quad\left(F_{n+2}+k\right)\left(F_{n}+k\right)-\left(F_{n+1}+k\right)^{2} \\
& =(-1)^{\mathrm{n}+1}+\mathrm{kF}_{\mathrm{n}-2} \text {. } \\
& \text { Proof. } \quad\left(\mathrm{F}_{\mathrm{n}+2}+\mathrm{k}\right)\left(\mathrm{F}_{\mathrm{n}}+\mathrm{k}\right)-\left(\mathrm{F}_{\mathrm{n}+1}+\mathrm{k}\right)^{2} \\
& =\left(F_{n+2} F_{n}-F_{n+1}^{2}\right)+k\left(F_{n+2}-2 F_{n+1}+F_{n}\right) \\
& =(-1)^{\mathrm{n}+1}+\mathrm{kF}_{\mathrm{n}-2} \text {. }
\end{aligned}
$$

The last equality is true since

$$
F_{n+2} F_{n}-F_{n+1}^{2}=(-1)^{n+1}::
$$

We are now able to begin the proof of the theorem. From the definition of a Pisot sequence and Lemma 2, we have that

$$
\left\{\mathrm{F}_{\mathrm{n}}+\mathrm{k}\right\}_{\mathrm{n}=\mathrm{n}_{0}}^{\infty}
$$

is a Pisot sequence of the first kind iff

$$
\begin{equation*}
2 \leq \mathrm{F}_{\mathrm{n}_{0}}+\mathrm{k}<\mathrm{F}_{\mathrm{n}_{0}+1}+\mathrm{k} \tag{i}
\end{equation*}
$$

and
(iia)

$$
-\left(\mathrm{F}_{\mathrm{n}}+\mathrm{k}\right) \leq 2\left[(-1)^{\mathrm{n}+1}+\mathrm{kF}_{\mathrm{n}-2}\right] \quad \text { for all } \mathrm{n} \geq \mathrm{n}_{0}
$$

$$
\begin{equation*}
2\left[(-1)^{\mathrm{n}+1}+\mathrm{kF}{ }_{\mathrm{n}-2}\right]<\mathrm{F}_{\mathrm{n}}+\mathrm{k} \quad \text { for all } \mathrm{n} \geq \mathrm{n}_{0} \tag{iib}
\end{equation*}
$$

are satisfied. Also $\left\{\mathrm{F}_{\mathrm{n}}+\mathrm{k}\right\}_{\mathrm{n}=\mathrm{n}_{0}}^{\infty}$ is a Pisot sequence of the second kind iff (i) and
(iiia) $\quad-\left(\mathrm{F}_{\mathrm{n}}+\mathrm{k}\right)<2\left[(-1)^{\mathrm{n}+1}+\mathrm{kF}_{\mathrm{n}-2}\right] \quad$ for all $\mathrm{n} \geq \mathrm{n}_{0}$
(iiib) $\quad 2\left[(-1)^{\mathrm{n}+1}+\mathrm{kF}_{\mathrm{n}-2}\right] \leq \mathrm{F}_{\mathrm{n}}+\mathrm{k} \quad$ for all $\mathrm{n} \geq \mathrm{n}_{0}$.

We shall first consider the case $k=1$.

$$
\mathrm{F}_{\mathrm{n}}+1=2 \mathrm{~F}_{\mathrm{n}-2}+\mathrm{F}_{\mathrm{n}-3}+1>2\left[\mathrm{~F}_{\mathrm{n}-2}+1\right] \geq 2\left[\mathrm{~F}_{\mathrm{n}-2}+(-1)^{\mathrm{n}+1}\right]
$$

iff $\mathrm{n} \geq 6$. Thus (iib) is satisfied iff $\mathrm{n} \geq 6$. Also,

$$
\mathrm{F}_{\mathrm{n}}+1=2 \mathrm{~F}_{\mathrm{n}-2}+\mathrm{F}_{\mathrm{n}-3}+1 \geq 2\left[\mathrm{~F}_{\mathrm{n}-2}+1\right] \geq 2\left[\mathrm{~F}_{\mathrm{n}-2}+(-1)^{\mathrm{n}+1}\right]
$$

iff $n \geq 4$. Thus (iiib) is satisfied iff $n \geq 4$. Since

$$
2\left[(-1)^{n+1}+F_{n-2}\right] \geq 0>-\left(F_{n}+1\right) \quad \text { for all } n \geq 3
$$

(iia) and (iiia) are satisfied for $n \geq 3$. It is clear that (i) is satisfied if $n_{0} \geq$ 2. Thus

$$
\left\{\mathrm{F}_{\mathrm{n}}+1\right\}_{\mathrm{n}=\mathrm{n}_{0}}^{\infty}
$$

is a Pisot sequence of the first kind iff $n_{0} \geq 6$ and it is a Pisot sequence of the second kind iff $\mathrm{n}_{0} \geq 4$ 。

Next, we consider the case $k=-1$. If $n=6$, both (iia) and (iiia) are not satisfied. If $\mathrm{n}=7$, both (iia) and (iiia) are satisfied. Now

$$
\mathrm{F}_{\mathrm{n}}-1=2 \mathrm{~F}_{\mathrm{n}-2}+\mathrm{F}_{\mathrm{n}-3}-1>2\left[\mathrm{~F}_{\mathrm{n}-2}+1\right] \quad \text { if } \mathrm{n} \geq 8
$$

Thus,

$$
-\left(\mathrm{F}_{\mathrm{n}}-1\right)<2\left[-1-\mathrm{F}_{\mathrm{n}-2}\right] \leq 2\left[(-1)^{\mathrm{n}+1}-\mathrm{F}_{\mathrm{n}-2}\right]
$$

if $n \geq 8$. Therefore, (iia) and (iiia) are satisfied iff $n \geq 7$. Since

$$
2\left[(-1)^{\mathrm{n}+1}-\mathrm{F}_{\mathrm{n}-2}\right] \leq 0<\mathrm{F}_{\mathrm{n}}-1
$$

if $n \geq 3$, both (iib) and (iiib) are satisfied for $n \geq 3$. It is clear that (i) is satisfied for $n \geq 4$. Thus,

$$
\left\{F_{n}-1\right\}_{n=n_{0}}^{\infty}
$$

is a Pisot sequence iff $n_{0} \geq 7$.
Now we consider the case $k=0$. It is clear that (i) is satisfied iff $\mathrm{n} \geq 3$. Both (iia) and (iiia) are satisfied for $\mathrm{n} \geq 3$. Also (iiib) is satisfied for $\mathrm{n} \geq 3$, but (iib) is satisfied iff $\mathrm{n} \geq 4$. Thus

$$
\left\{F_{n}\right\}_{n=n_{0}}^{\infty}
$$

is a Pisot sequence of the first kind iff $n_{0} \geq 4$, and

$$
\left\{\mathrm{F}_{\mathrm{n}}\right\}_{\mathrm{n}=\mathrm{n}_{0}}^{\infty}
$$

is a Pisot sequence of the second kind iff $n_{0} \geq 3$.
We shall show that if $|k|>1$, then there exists no integer $n_{0}$ such that

$$
\left\{\mathrm{F}_{\mathrm{n}} \neq \mathrm{k}\right\}_{\mathrm{n}=\mathrm{n}_{0}}^{\infty}
$$

is a Pisot sequence. This will be accomplished by showing that (iia) or (iiia) implies that $k>-2$ and that (iib) or (iiib) implies that $2<k$.

Dividing (iia) by $\mathrm{F}_{\mathrm{n}-2}$ yields that

$$
-\frac{F_{n}}{F_{n-1}} \cdot \frac{F_{n-1}}{F_{n-2}}-\frac{k}{F_{n-2}} \leq \frac{2(-1)^{n+1}}{F_{n-2}}+2 k
$$

for $\mathrm{n} \geq \mathrm{n}_{0}$. After taking the limit of both sides as $\mathrm{n} \rightarrow \infty$ and remembering that

$$
\lim _{\mathrm{n} \rightarrow \infty} \frac{\mathrm{~F}_{\mathrm{n}+1}}{\mathrm{~F}_{\mathrm{n}}}=\frac{1+\sqrt{5}}{2}<2
$$

we have that

$$
-4<-\lim \left(\frac{F_{n+1}}{F_{n}}\right)^{2} \leq 2 k
$$

Thus

$$
-2<\mathrm{k}
$$

In a similar manner, one can show that (iib) or (iiib) implies that $\mathrm{k}<$ 2. : :

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FALL RESEARCH CONFERENCE
St. Mary's College, Saturday, October 17, 1970
9:15 Registration
10:00-10:50 - Combinatorial Problems Leading to Generalized Fibonacci Numbers. Verner E. Hoggatt, Jr., San Jose State College
11:00-11:50 - How Fibonacci Numbers Helped Solve Hilbert's Tenth Problem Professor Julia Robinson, University of California, Berkeley
1:30-2:20 - Explicit Determination of Perron Matrices. Professor Helmut Hasse, Visiting Lecturer, San Diego State College
2:30-3:20 - Asymptotic Fibonacci Ratios. Brother Alfred Brousseau, St. Mary's College
3:30-4:00 - Fibonacci Correlations in Bishop Pine. Brother Alfred Brousseau, St. Mary's College.

# MODULO ONE UNIFORM DISTRIBUTION OF THE SEQUENCE OF LOGARITHMS OF CERTAIN RECURSIVE SEQUENCES 

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Let $\left\{x_{j}\right\}_{1}^{\infty}$ be a sequence of real numbers with corresponding fractional parts $\left\{\beta_{j}\right\}_{1}^{\infty}$, where $\beta_{j}=x_{j}-\left[x_{j}\right]$ and the bracket denotes the greatest integer function. For each $\mathrm{n} \geq 1$, we define the function $\mathrm{F}_{\mathrm{n}}$ on $[0,1]$ so that $\mathrm{F}_{\mathrm{n}}(\mathrm{x})$ is the number of those terms among $\beta_{1}, \cdots, \beta_{\mathrm{n}}$ whichlie in the interval $[0, x)$, divided by $n$. Then $\left\{x_{j}\right\}_{1}^{\infty}$ is said to be uniformly distributed modulo one iff $\mathrm{lim}_{\infty} \mathrm{F}_{\mathrm{n}}(\mathrm{x})=\mathrm{x}$ for all $\mathrm{x} \in[0,1]$. In other words, each interval of the form $[0, \mathrm{x})$ with $\mathrm{x} \in[0,1]$, contains asymptotically a proportion of the $\beta_{n}{ }^{\prime} \mathrm{s}$ equal to the length of the interval, and clearly the same will be true for any subinterval $(\alpha, \beta)$ of $[0,1]$. The classical Weyl criterion ( $[1], p .76$ ) states that $\left\{x_{j}\right\}_{1}^{\infty}$ is uniformly distributed mod 1 iff

$$
\begin{equation*}
\lim _{\mathrm{n} \rightarrow \infty} \frac{1}{\mathrm{n}} \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{e}^{2 \pi \mathrm{i} \nu \mathrm{x}_{\mathrm{j}}}=0 \quad \nu \geq 1 \tag{1}
\end{equation*}
$$

An example of a sequence which is uniformly distributed $\bmod 1$ is $\{n \xi\}_{n=0}^{\infty}$ where $\boldsymbol{\xi}$ is an arbitrary irrational number (see [1], p. 81 for a proof using Weyl's criterion).

The purpose of this paper is to show that the sequence $\left\{\ln \mathrm{V}_{\mathrm{n}}\right\}_{1}^{\infty}$ is uniformly distributed $\bmod 1$, where $\left\{\mathrm{V}_{\mathrm{n}}\right\}_{1}^{\infty}$ is defined by a linear recurrence of the form
(2)

$$
\mathrm{V}_{\mathrm{n}+\mathrm{k}}=\mathrm{a}_{\mathrm{k}-1} \mathrm{~V}_{\mathrm{n}+\mathrm{k}-1}+\cdots+\mathrm{a}_{0} \mathrm{~V}_{\mathrm{n}} \quad \mathrm{n} \geq 1
$$

the initial terms $V_{1}, V_{2}, \cdots, V_{k}$ being given positive numbers. In (2), we assume that the coefficients are non-negative rational numbers with $a_{0} \neq 0$, and that the associated polynomial $x^{k}-a_{k-1} x^{k-1}-\cdots-a_{1} x-a_{0}$, has roots $\beta_{1}, \beta_{2}, \cdots, \beta_{\mathrm{k}}$ which satisfy the inequality $0<\left|\beta_{1}\right|<\ldots<\left|\beta_{\mathrm{k}}\right|$. Additionally, we require that $\left|\beta_{j}\right| \neq 1$ for $j=1,2, \cdots, k$.

In particular, our result implies that any sequence $\left\{U_{n}\right\}_{1}^{\infty}$ which satisfies the Fibonacci recurrence $U_{n+2}=U_{n+1}+U_{n}$ for $n \geq 1$ with $U_{1}=k_{1}$ and $\mathrm{U}_{2}=\mathrm{k}_{2}$ arbitrary positive initial terms (not necessarily integers) will have the property that $\left\{\ln \mathrm{U}_{\mathrm{n}}\right\}_{1}^{\infty}$ is uniformly distributed mod 1. With $\mathrm{k}_{1}=$ 1 , $\mathrm{k}_{2}=1$, we obtain this conclusion for the classical Fibonacci sequence (see [2], Theorem 1), while for $k_{1}=1, k_{2}=3$, an analogous result is seen to hold for the Lucas sequence.

Before proving the main theorem, we prove two lemmas:
Lemma 1. If $\left\{x_{j}\right\}_{1}^{\infty}$ is uniformly distributed $\bmod 1$ and $\left\{y_{j}\right\}_{1}^{\infty}$ is such that $\lim _{j \rightarrow \infty}\left(x_{j}-y_{j}\right)=0$, then $\left\{y_{j}\right\}_{i}^{\infty}$ is uniformly distributed mod ${ }_{1}^{1}$.

Proof. From the hypothesis and the continuity of the exponential function, it follows that

$$
\lim _{j \rightarrow \infty}\left(e^{2 \pi i v x_{j}}-e^{2 \pi i \nu y_{j}}\right)=0
$$

But it is well known ([3], Theorem B, p. 202) that if $\left\{\gamma_{n}\right\}$ is a sequence of real numbers converging to a finite limit $L$, then

$$
\lim _{\mathrm{n} \rightarrow \infty} \frac{1}{\mathrm{n}} \sum_{1}^{\mathrm{n}} \gamma_{j}=\mathrm{L}
$$

Taking $\gamma_{j}=e^{2 \pi i \nu x_{j}}-e^{2 \pi i \nu y_{j}}$, we have

$$
\lim _{\mathrm{n} \rightarrow \infty} \frac{1}{\mathrm{n}} \sum_{\mathrm{j}=1}^{\mathrm{n}}\left(\mathrm{e}^{2 \pi \mathrm{i} \nu \mathrm{x}_{\mathrm{j}}}-\mathrm{e}^{\left.2 \pi \mathrm{i} \nu \mathrm{y}_{j}\right)=0}\right.
$$

Since

$$
\lim _{\mathrm{n} \rightarrow \infty} \frac{1}{\mathrm{n}} \sum_{1}^{\mathrm{n}} \mathrm{e}^{2 \pi \mathrm{i} \nu \mathrm{x}_{\mathrm{j}}}=0
$$

by Weyl's criterion, we also have

$$
\lim _{\mathrm{n} \rightarrow \infty} \frac{1}{\mathrm{n}} \sum_{1}^{\mathrm{n}} \mathrm{e}^{2 \pi \mathrm{i} \nu y_{j}}=0
$$

and the sufficiency of Weyl's criterion proves the sequence $\left\{y_{j}\right\}_{1}^{\infty}$ to be uniformly distributed mod 1.

Lemma 2. If $\alpha$ is a positive algebraic number not equal to one, then $\ln \alpha$ is irrational.

Proof. Assume, to the contrary, $\ln \alpha=(p / q)$, where $p$ and $q$ are non-zero integers. Then $e^{p / q}=\alpha$, so that $e^{p}=\alpha^{q}$. But $\alpha^{p}$ is algebraic, since the algebraic numbers are closed under multiplication ([1], p. 84). Thus $e^{p}$ is algebraic, in turn implying $e$ is algebraic. But $e$ is known to be transcendental ([1], p. 25) so that a contradiction is obtained.

Theorem. Let $\left\{\mathrm{V}_{\mathrm{n}}\right\}_{1}^{\infty}$ be a sequence generated by the recursion relation,
(2) $\quad \mathrm{V}_{\mathrm{n}+\mathrm{k}}=\mathrm{a}_{\mathrm{k}-1} \mathrm{~V}_{\mathrm{n}+\mathrm{k}-1}+\cdots+\mathrm{a}_{1} \mathrm{~V}_{\mathrm{n}+1}+\mathrm{a}_{0} \mathrm{~V}_{\mathrm{n}} \quad(\mathrm{n} \geq 1)$,
where $a_{0}, a_{1}, \cdots, a_{k-1}$ are non-negative rational coefficients with $a_{0} \neq 0$, $k$ is a fixed integer, and

$$
\begin{equation*}
\mathrm{V}_{1}=\gamma_{1}, \quad \mathrm{~V}_{2}=\gamma_{2}, \cdots, \quad \mathrm{~V}_{\mathrm{k}}=\gamma_{\mathrm{k}} \tag{3}
\end{equation*}
$$

are given positive values for the initial terms. Further, we assume that the polynomial $\mathrm{x}^{\mathrm{k}}-\mathrm{a}_{\mathrm{k}-1} \mathrm{x}^{\mathrm{k}-1}-\cdots-\mathrm{a}_{1} \mathrm{x}-\mathrm{a}_{0}$ has k distinct roots $\beta_{1}, \beta_{2}, \cdots$, $\beta_{\mathrm{k}}$ satisfying $0<\left|\beta_{1}\right|<\cdots<\left|\beta_{\mathrm{k}}\right|$ and such that none of the roots has magnitude equal to 1 . Then $\left\{\ln V_{n}\right\}_{1}^{\infty}$ is uniformly distributed mod 1.

Proof. The general solution of the recurrence (2) is

$$
\begin{equation*}
\mathrm{V}_{\mathrm{n}}=\sum_{\mathrm{j}=1}^{\mathrm{k}} \alpha_{\mathrm{j}} \beta_{\mathrm{j}}^{\mathrm{n}} \quad(\mathrm{n} \geq 1) \tag{4}
\end{equation*}
$$

where the arbitrary constants $\alpha_{1}, \alpha_{2}, \cdots, \alpha_{\mathrm{k}}$ are determined by the specification of the initial terms in (3). [It is easily checked that the determinant of the kxk matrix $\left(\beta_{\mathrm{j}}^{\mathrm{i}}\right)$ does not vanish, so that determination of the $\alpha_{j}^{\prime} \mathrm{s}$ is
unique.] Since the initial terms were not all zero, at least one of the $\alpha_{j}^{\prime}$ s is non-zero. Let $p$ be the largest value of $j$ for which $\alpha_{j} \neq 0$, so that $p \geq 1$. Then

$$
\mathrm{V}_{\mathrm{n}}=\sum_{1}^{\mathrm{p}} \alpha_{\mathrm{j}} \beta_{\mathrm{j}}^{\mathrm{n}}
$$

and

$$
\left|1-\frac{v_{n}}{\alpha_{p} \beta_{p}^{n}}\right|=\left|\sum_{1}^{p-1} \frac{\alpha_{j} \beta_{j}^{n}}{\alpha_{p} \beta_{p}^{n}}\right| \leq \sum_{1}^{p-1}\left|\frac{\alpha_{j}}{\alpha_{p}}\right|\left|\frac{\beta_{j}}{\beta_{p}}\right|^{n}
$$

But

$$
\left|\frac{\beta_{\mathrm{j}}}{\beta_{\mathrm{p}}}\right|<1
$$

for $j=1,2, \cdots, p-1$, and hence,

$$
\lim _{\mathrm{n}}\left(\frac{\mathrm{v}_{\mathrm{n}}}{\left|\alpha_{\mathrm{p}} \beta_{\mathrm{p}}\right|}\right)=1
$$

or equivalently,

$$
\begin{equation*}
\lim _{\mathrm{n} \rightarrow \infty}\left[\ln V_{\mathrm{n}}-\ln \left|\alpha_{\mathrm{p}} \beta_{\mathrm{p}}\right|^{\mathrm{n}}\right]=0 \tag{5}
\end{equation*}
$$

Since $\beta_{p}$ is algebraic, it is easily verified that $\left|\beta_{p}\right|$ is also algebraic. Moreover, $\left|\beta_{p}\right| \neq 1$ by hypothesis so that $\ln \left|\beta_{p}\right|$ is irrational by application of Lemma 2. But the sequence $\{n \xi\}_{1}^{\infty}$ is uniformly distributed mod 1 whenever $\xi$ is irrational; therefore, the sequence

$$
\left\{n \ln \left|\beta_{p}\right|\right\}_{1}^{\infty}=\left\{\ln \left|\beta_{p}\right|^{n}\right\}_{1}^{\infty}
$$

is uniformly distributed mod 1 and the same is true for the sequence

$$
\left\{\ln \left|\alpha_{p}\right|\left|\beta_{p}\right|^{n}\right\}_{i}^{\infty}
$$

From (5) and Lemma 1, it is then clear that $\left\{\ln \mathrm{V}_{\mathrm{n}}\right\}_{1}^{\infty}$ is uniformly distributed $\bmod 1$ as asserted. q.e.d.

The specialization to sequences satisfying the Fibonacci recurrence, $\mathrm{U}_{\mathrm{n}+2}=\mathrm{U}_{\mathrm{n}+1}+\mathrm{U}_{\mathrm{n}}(\mathrm{n} \geq 1)$, is immediate since the relevant polynomial in this case is $x^{2}-x-1$, and there are two distinct roots of unequal magnitude, namely

$$
\frac{1 \pm \sqrt{5}}{2} .
$$

From the theorem, we conclude $\left\{\ln \mathrm{U}_{\mathrm{n}}\right\}_{1}^{\infty}$ is uniformly distributed mod 1 independently of the (non-zero) values specified for $U_{1}$ and $U_{2}$.

Lastly, we give an example to show that our assumption on the roots of the associated polynomial cannot be relaxed. Consider the recurrence $\mathrm{V}_{\mathrm{n}+2}$ $=\mathrm{V}_{\mathrm{n}}$ for $\mathrm{n} \geq 1$ with $\mathrm{V}_{1}=1, \mathrm{~V}_{2}=1$. Then clearly $\mathrm{V}_{\mathrm{n}}=1$ for all $\mathrm{n} \geq 1$ so that $\left\{\ln V_{n}\right\}_{1}^{\infty}$ is a sequence of zeroes and hence not uniformly distributed mod 1. The associated polynomial in this case is $x^{2}-1$ which has two distinct roots, $\pm 1$; however, the roots have magnitude unity, and therefore, the conditions of our theorem are not met.

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# ADVANCED PROBLEMS AND SOLUTIONS 

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-175 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
Put

$$
\left(1+z+\frac{1}{3} z^{2}\right)^{-n-1}=\sum_{k=0}^{\infty} a(n, k) z^{k}
$$

Show that
(I)

$$
a(n, n)=\frac{2 \cdot 5 \cdot 8 \cdots(3 n-1)}{n!}
$$

(II) $\quad \sum_{s=0}^{n}\binom{n-s}{s}\binom{2 n-s}{n}\left(-\frac{1}{3}\right)^{s}=\frac{2 \cdot 5 \cdot 8 \ldots(3 n-1)}{n!}$
(III) $\sum_{r=0}^{\infty}\binom{n+r}{r}\binom{2 n-r}{n}(-\omega)^{r}=\left(\omega^{2} \sqrt{-3}\right)^{n} \frac{2 \cdot 5 \cdot 8 \cdots(3 n-1)}{n!}$,
where

$$
\omega=\frac{1}{2}(-1+\sqrt{-3})
$$

In the "Collected Papers of Srinivas Ramanujan," edited by G. H. Hardy, P. V. Sheshu Aiyer, and B. M. Wilson, Cambridge University Press, 1927, on p. 326, Q. 427 reads as follows:

Show that

$$
\frac{1}{e^{2 \pi}-1}+\frac{2}{e^{4 \pi}-1}+\frac{3}{e^{6 \pi}-1}+\cdots=\frac{1}{24}+\frac{1}{8 \pi}
$$

Provide a proof.

H-177 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
Let $R(N)$ denote the number of solutions of

$$
\mathrm{N}=\mathrm{F}_{\mathrm{k}_{1}}+\mathrm{F}_{\mathrm{k}_{2}}+\cdots+\mathrm{F}_{\mathrm{k}_{\mathrm{r}}} \quad(\mathrm{r}=1,2,3, \cdots)
$$

where

$$
\mathrm{k}_{1}>\mathrm{k}_{2}>\ldots>\mathrm{k}_{\mathrm{r}}>1
$$

Show that
(1) $\left.R\left(\mathrm{~F}_{2 \mathrm{n}} \mathrm{F}_{2 \mathrm{~m}}\right)=\mathrm{R}_{2 \mathrm{n}+1} \mathrm{~F}_{2 \mathrm{~m}}\right)=(\mathrm{n}-\mathrm{m}) \mathrm{F}_{2 \mathrm{~m}}+\mathrm{F}_{2 \mathrm{~m}-1} \quad(\mathrm{n} \geq \mathrm{m})$,
(2) $\quad \mathrm{R}\left(\mathrm{F}_{2 \mathrm{n}} \mathrm{F}_{2 \mathrm{~m}+1}\right)=(\mathrm{n}-\mathrm{m}) \mathrm{F}_{2 \mathrm{~m}+1} \quad(\mathrm{n}>\mathrm{m})$,
(3) $\quad R\left(F_{2 n+1} F_{2 m+1}\right)=(n-m) F_{2 m+1}$
$(\mathrm{n}>\mathrm{m})$,
(4) $R\left(F_{2 n+1}^{2}\right)=R\left(F_{2 n}^{2}\right)=F_{2 n-1}$
( $\mathrm{n} \geq 1$ ) 。

SOLUTIONS
SUM INVERSION
H-151 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
a. Put

$$
\left(1-a x^{2}-b x y-c y^{2}\right)^{-1}=\sum_{m, n=0}^{\infty} A_{m, n} x^{m} y^{n}
$$

Show that

$$
\sum_{n=0}^{\infty} A_{n, n} x^{n}=\left\{1-2 b x+\left(b^{2}-4 a c\right) x^{2}\right\}^{-\frac{1}{2}}
$$

B. Put

$$
(1-a x-b x y-c y)^{-1}=\sum_{m, n=0}^{\infty} B_{m, n} x^{m} y^{n}
$$

Show that

$$
\sum_{n=0}^{\infty} B_{n, n} x^{n}=\left\{(1-b x)^{2}-4 a c x\right\}^{-\frac{1}{2}}
$$

Solution by M. L. J. Hautus and D. A. Klarner, Technological University, Eindhouen, the Netherlands.

In a paper submitted to the Duke Mathematical Journal (the diagonal of a double power series), we have proved the following result:

Theorem. Suppose

$$
F(x, y)=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n) x^{m} y^{n}
$$

converges for all $x$ and $y$ such that $|x|<A,|y|<B$, then for all $z$ such that $|z|<A B$, we have

$$
\frac{1}{2 \pi i} \int_{C} F(s, z / s) \frac{d s}{s}=\sum_{n=0}^{\infty} f(n, n) z^{n}
$$

where $C$ is the circle $\{s:|s|=(A+|z| / B) / 2\}$. Furthermore, if $F(s, z / s) / s$ has only isolated singularities inside $C$, then the integral can be evaluated by summing the residues of $\mathrm{F}(\mathrm{s}, \mathrm{z} / \mathrm{s}) / \mathrm{s}$ at these singularities. Coincidentally, we gave Carlitz' Problem B as an example in our paper. Problem A can be treated in just the same way. According to the theorem cited,

$$
\sum_{n=0}^{\infty} A_{n, n} x^{n}=\frac{1}{2 \pi i} \int_{C} \frac{-s d s}{a s^{4}-(1-b x) s^{2}+c x^{2}}
$$

The singularities of the integrand are

$$
\pm \theta_{1}=\left(\frac{1-\mathrm{bx}-\left(1-2 \mathrm{bx}+\mathrm{b}^{2} \mathrm{x}^{2}-4 a c x^{2}\right)^{\frac{1}{2}}}{2 \mathrm{a}}\right)^{\frac{1}{2}}
$$

and

$$
\pm \theta_{2}=\left(\frac{1-b x+\left(1-2 b x+b^{2} x^{2}-4 a c x^{2}\right)^{\frac{1}{2}}}{2 a}\right)^{\frac{1}{2}}
$$

and the singularities $\pm \theta_{1}$ tend to 0 with $x$ while the singularities $\pm \theta_{2}$ do not. Thus, the contour $C$ must include $\pm \theta_{1}$ and exclude $\pm \theta_{2}$; using the residue theorem, we easily calculate

$$
\frac{1}{2 \pi \mathrm{i}} \int_{\mathrm{C}} \frac{-\mathrm{sds}}{\mathrm{a}\left(\mathrm{~s}-\theta_{1}\right)\left(\mathrm{s}+\theta_{1}\right)\left(\mathrm{s}-\theta_{2}\right)\left(\mathrm{s}+\theta_{2}\right)}=\frac{1}{\mathrm{a}\left(\theta_{2}^{2}-\theta_{1}^{2}\right)}
$$

Substituting the values of $\theta_{1}$ and $\theta_{2}$ given above yields the desired result. A generalization of Problems $A$ and $B$ can be given as follows:

Let

$$
F(x, y)=\sum_{m, n} f(m, n) x^{m} y^{n}=\left(1-a x^{k}-b x y-c y^{k}\right)^{-1}
$$

and let

$$
F(x)=\sum_{n} f(n, n) x^{n}
$$

Then according to the theorem cited above, we have

$$
F(x)=\frac{1}{2 \pi i} \int_{C} \frac{-s^{k-1} d s}{a s^{2 k}-(1-b x) s^{k}+c x^{k}}
$$

Set $\omega=e^{2 \pi i / k}$, then the singularities of the integrand are $\omega^{j} \theta_{1}, \omega^{j_{\theta_{2}}}$ for $j=1, \ldots, k$, where

$$
\begin{aligned}
& \theta_{1}=\left(\frac{1-b x-\left(1-2 b x+b^{2} x^{2}-4 a c x^{k}\right)^{\frac{1}{2}}}{2 a}\right)^{1 / k}, \\
& \theta_{2}=\left(\frac{1-b x+\left(1-2 b x+b^{2} x^{2}-4 a c x^{k}\right)^{\frac{1}{2}}}{2 a}\right)^{1 / k}
\end{aligned}
$$

Since $\theta_{1}$ tends to 0 with x and $\theta_{2}$ does not, C includes the singularities $\omega^{j} \theta_{1}$ for $j=1, \cdots, k$, but excludes the singularities $\omega^{j} \theta_{2}$ for $j=1, \cdots$, k. Now we have the residue theorem to find that

$$
\begin{aligned}
\mathrm{F}(\mathrm{x}) & =\frac{1}{2 \pi \mathrm{i}} \int_{\mathrm{C}} \frac{-s^{\mathrm{k}-1} \mathrm{ds}}{\left(\mathrm{~s}^{\mathrm{k}}-\theta_{1}^{\mathrm{k}}\right)\left(\mathrm{s}^{\mathrm{k}}-\theta_{2}^{\mathrm{k}}\right) \mathrm{a}}=\frac{1}{2 \pi i \mathrm{k}} \int_{\mathrm{C}} \frac{-d \mathrm{t}}{\left(\mathrm{t}-\theta_{1}^{\mathrm{k}}\right)\left(\mathrm{t}-\theta_{2}^{\mathrm{k}}\right) \mathrm{a}}=\frac{1}{\mathrm{a}\left(\theta_{2}^{\mathrm{k}}-\theta_{1}^{\mathrm{k}}\right)} \\
& =\left(1-2 \mathrm{bx}+\mathrm{b}^{2} \mathrm{x}^{2}-4 a c \mathrm{x}^{-\frac{1}{2}}\right)^{2},
\end{aligned}
$$

where $C^{\prime}$ is a contour in the $t$-plane which encircles the singularity $\theta_{1}^{k}, k$ times but excludes $\theta_{2}^{\mathrm{k}}$.

Also solved by D. V. Jaiswal and the Proposer.

## HIDDEN IDENTITY

H-153 Proposed by J. Ramanna, Government College, Mercara, India.
Show that
(i) $\quad 4 \sum_{0}^{n} \mathrm{~F}_{3 \mathrm{k}+1} \mathrm{~F}_{3 \mathrm{k}+2}\left(2 \mathrm{~F}_{3 \mathrm{k}+1}^{2}+\mathrm{F}_{6 \mathrm{k}+3}\right)\left(2 \mathrm{~F}_{3 \mathrm{k}+2}^{2}+\mathrm{F}_{6 \mathrm{k}+3}\right)=\mathrm{F}_{3 \mathrm{n}+3}^{6}$
(ii) $16 \sum_{0}^{n} \mathrm{~F}_{3 \mathrm{k}+1} \mathrm{~F}_{3 \mathrm{k}+2} \mathrm{~F}_{6 \mathrm{k}+3}\left(2 \mathrm{~F}_{6 \mathrm{k}+3}^{2}-\mathrm{F}_{3 \mathrm{k}}^{2} \mathrm{~F}_{3 \mathrm{k}+3}^{2}\right)=\mathrm{F}_{3 \mathrm{n}+3}^{3}$.

Hence, generalize (i) and (ii) for $\mathrm{F}_{3 \mathrm{n}+3}^{2 \mathrm{r}}$.

## Solution by the Proposer.

We note that (i) and (ii) are easily verified for $k=0$ and $k=1$ and assume the results for $k=r$ and prove them for $k=r+1$. Thus we need show, on subtracting (i) and (ii) for $n=r$ from (i) and (ii) for $n=r+1$, respectively, that
(2)
(i) $\begin{gathered}\left.4 \mathrm{~F}_{3(\mathrm{r}+1)+1} \mathrm{~F}_{3(\mathrm{r}+1)+2}{ }^{\left(2 \mathrm{~F}_{3(\mathrm{r}+1)+1}^{2}\right.}+\mathrm{F}_{6(\mathrm{r}+1)+3}^{6}\right)\left(2 \mathrm{~F}_{3(\mathrm{r}+1)+2}^{2}+\mathrm{F}_{6(\mathrm{r}+1)+3}\right) \\ =\mathrm{F}_{3(1)+3}-\mathrm{F}_{3 \mathrm{r}+3}\end{gathered}$
(ii) $16 \mathrm{~F}_{3(\mathrm{r}+1)+1} \mathrm{~F}_{3(\mathrm{r}+1)+2} \mathrm{~F}_{6(\mathrm{r}+1)+3}\left(2 \mathrm{~F}_{6(\mathrm{r}+1)+3}^{2}-\mathrm{F}_{3(\mathrm{r}+1)}^{2} \mathrm{~F}_{3(\mathrm{r}+1)+3}^{2}\right)$ $=\mathrm{F}_{3(\mathrm{r}+1)+3}^{8}-\mathrm{F}_{3 \mathrm{r}+3}^{8}$.

Equations (1) are true since
(i) $4 \mathrm{~F}_{3(\mathrm{r}+1)+1} \mathrm{~F}_{3(\mathrm{r}+1)+2}\left(2 \mathrm{~F}_{3(\mathrm{r}+1)+1}^{2}+\mathrm{F}_{6(\mathrm{r}+1)+3}\right)\left(2 \mathrm{~F}_{3(\mathrm{r}+1)+2}^{2}+\mathrm{F}_{6(\mathrm{r}+1)+3}\right)$
(3)

$$
=\left(\mathrm{F}_{3(\mathrm{r}+1)+2}+\mathrm{F}_{3(\mathrm{r}+1)+1}\right)^{6}-\left(\mathrm{F}_{3(\mathrm{r}+1)+2}-\mathrm{F}_{3(\mathrm{r}+1)+1}\right)
$$

$$
\begin{aligned}
& \text { (ii) } \begin{aligned}
16 \mathrm{~F}_{3(\mathrm{r}+1)+1} \mathrm{~F}_{3(\mathrm{r}+1)+2} \mathrm{~F}_{6(\mathrm{r}+1)+3}\left(\mathrm{~F}_{3(\mathrm{r}+1)+2}^{4}\right. & +6 \mathrm{~F}_{3(\mathrm{r}+1)+2}^{2} \mathrm{~F}_{3(\mathrm{r}+1)+1}^{2} \\
& \left.+\mathrm{F}_{3(\mathrm{r}+1)+1}^{4}\right)
\end{aligned} \\
& =\mathrm{F}_{3(\mathrm{r}+1)+3}^{8}-\mathrm{F}_{3 \mathrm{r}+3}^{8}=16 \mathrm{~F}_{3(\mathrm{r}+1)+1} \mathrm{~F}_{3(\mathrm{r}+1)+2} \mathrm{~F}_{6(\mathrm{r}+1)+3}\left(2 \mathrm{~F}_{6(\mathrm{r}+1)+3}^{2}\right. \\
& \left.-\mathrm{F}_{3(\mathrm{r}+1)}^{2} \mathrm{~F}_{3(\mathrm{r}+1)+3}^{2}\right)
\end{aligned}
$$

Hence, the desired results follow.

Also solved by D. V. Jaiswal and C. C. Yalavigi.

## TRIPLE THREAT

H-154 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
Show that for $m, n, p$ integers $\geq 0$,

$$
\begin{aligned}
& \sum_{i, j, k \geq 0}\binom{m+1}{j+k+1}\binom{n+1}{i+k+1}\binom{p+1}{i+j+1} \\
& \quad=\sum_{a=0}^{m} \sum_{b=0}^{n} \sum_{c=0}^{p}\binom{m-a+b}{b}\binom{a-b+c}{c}\binom{p-c+a}{a}
\end{aligned}
$$

and generalize.
Solution by the Proposer.
Put

$$
\begin{gathered}
S_{m, n, p}=\sum_{i, j, k \geq 0}\binom{m+1}{j+k+1}\binom{n+1}{i+k+1}\binom{p+1}{i+j+1} \\
T_{m, n, p}=\sum_{a=0}^{m} \sum_{b=0}^{n} \sum_{c=0}^{p}\binom{m-a+b}{b}\binom{n-b+c}{c}\binom{p-c+a}{a}
\end{gathered}
$$

Then
$\sum_{m, n, p=0}^{\infty} S_{m, n, p} x^{m} y^{n} z^{p}$

$$
\begin{aligned}
& =\sum_{i, j, k=0} x^{j+k} y^{i+k} z^{i+j} \sum_{m=0}^{\infty}\binom{m+j+k+1}{j+k+1} x^{m} \sum_{n=0}^{\infty}\binom{n+i+k+1}{i+k+1} y^{n} \\
& \qquad \sum_{p=0}^{\infty}\binom{p+i+j+1}{i+j+1} z^{p} \\
& =\sum_{i, j, k=0}^{\infty} x^{j+k y^{i+k} z^{i+j}(1-x)^{-j-k-2}(1-y)^{-i-k-2}(1-z)^{-i-j-2}} \\
& =(1-x)^{-2}(1-x)^{-2}(1-z)^{-2}\left(1-\frac{y z}{(1-y)(1-z)}\right)^{-1}\left(1-\frac{x z}{(1-x)(1-z)}\right)^{-1} \\
& =(1-y-z)^{-1}(1-x-z)^{-1}(1-x-y)^{-1} \cdot
\end{aligned}
$$

In the next place,

$$
\begin{aligned}
\sum_{m, n, p=0}^{\infty} & T_{m, n, p} x^{m} y^{n} z^{p} \\
& =\sum_{a, b, c=0}^{\infty} x^{a} y^{b} z^{c} \sum_{m=0}^{\infty}\binom{m+b}{b} x^{m} \sum_{n=0}^{\infty}\binom{n+c}{c} y^{n} \sum_{p=0}^{\infty}\binom{p+a}{a} z^{p} \\
= & \sum_{a, b, c=0}^{\infty} x^{a} y^{b} z^{c}(1-x)^{-b-1}(1-y)^{-c-1}(1-z)^{-a-1} \\
= & (1-x)^{-1}(1-y)^{-1}(1-z)^{-1}\left(1-\frac{x}{1-1}\right)^{-1}\left(1-\frac{y}{1-x}\right)^{-1}\left(1-\frac{z}{1-y}\right)^{-1} \\
= & (1-x-z)^{-1}(1-x-y)^{-1}(1-y-z)^{-1},
\end{aligned}
$$

and the result follows at once.

## GENERALIZED VERSION

Let $\mathrm{k} \geq 2$ and $\mathrm{n}_{1}, \mathrm{n}_{2}, \cdots, \mathrm{n}_{\mathrm{k}}$ non-negative integers. Show that

$$
\begin{gathered}
\sum\binom{n_{1}+1}{a_{k}+a_{1}+1}\binom{n_{2}+1}{a_{1}+a_{2}+1} \cdots\binom{n_{k}+1}{a_{k-1}+a_{k}+1} \\
=\sum\binom{a_{1}+a_{2}}{a_{1}}\binom{a_{3}+a_{4}}{a_{3}} \cdots\binom{a_{2 k-1}+a_{2 k}}{a_{2 k-1}}
\end{gathered}
$$

where the first summation is over all non-negative $a_{1}, \cdots, a_{k}$ while in the second sum
$a_{2 k}+a_{1}=n_{1}, \quad a_{2}+a_{3}=n_{2}, \quad a_{4}+a_{5}=n_{3}, \cdots, a_{2 k-2}+a_{2 k-1}=n_{k}$.

Solution. Let $S\left(n_{1}, \cdots, n_{k}\right)$ denote the first sum and $T\left(n_{1}, \cdots, n_{k}\right)$ denote the second sum. Consider the expansion of
$\phi=\phi\left(x_{1}, \cdots, x_{k}\right)=\left(1-x_{1}-x_{2}\right)^{-1}\left(1-x_{2}-x_{3}\right)^{-1} \cdots\left(1-x_{k}-x_{1}\right)^{-1}$.

Since

$$
(1-x-y)^{-1}=((1-x)(1-y)-x y)^{-1}=\sum_{a=0}^{\infty} \frac{x^{a}}{(1-x)^{a+1}(1-y)^{b+1}}
$$

we have

$$
\dot{\phi}=\sum_{a_{1}, \cdots, a_{k}=0}^{\infty} \frac{x_{1} k^{a_{k}+a_{1}} x_{x_{1}} a^{+a} 2 \ldots x_{k-1}{ }_{\left(1-x_{1}\right)^{a_{k}+a_{1}+2}}^{\left(1-x_{2}\right)^{a_{1}+a_{2}+2}} \ldots\left(1-x_{k}\right)^{a_{k-1}+a_{k}+2}}{\ldots}
$$

$$
\begin{aligned}
& \phi=\sum_{a_{1}, \cdots, a_{k}=0}^{\infty}{x_{1}}_{a_{k}+a_{1}}^{x_{1}}{ }_{1}+a_{2} \ldots x_{k}^{a_{k-1}+a_{k}} \\
& \sum_{b_{1}, \cdots, b_{k}=0}^{\infty}\binom{a_{k}+a_{1}+b_{1}+1}{a_{k}+a_{1}+1}\binom{a_{1}+a_{2}+b_{2}+1}{a_{1}+a_{2}+1} \ldots \\
& \cdots\binom{a_{k-1}+a_{k}+b_{k}+1}{a_{k-1}+a_{k}+1} x_{1}{ }_{1}{ }_{x_{2}}{ }_{2}{ }_{2} \ldots x_{k}{ }_{k} \\
& =\sum_{n_{1}, \cdots, n_{k}=0}^{\infty}{x_{1}{ }^{n_{1}}{ }_{x_{2}}{ }^{n_{2}} \ldots x_{k}^{n_{k}} S\left(n_{1}, n_{2}, \cdots, n_{k}\right) .}^{\infty}
\end{aligned}
$$

On the other hand, since

$$
\begin{aligned}
& (1-x-y)^{-1}=\sum_{a, b=0}^{\infty}\binom{a+b}{a} x^{a} y^{b}, \\
& \phi=\sum_{a_{1}, a_{2}=0}^{\infty}\binom{a_{1}+a_{2}}{a_{1}}{ }_{x_{1}}^{a_{1}}{ }_{x_{2}}^{a_{2}} \sum_{a_{3}, a_{4}=0}^{\infty}\binom{a_{3}+a_{4}}{a_{3}}{ }_{x_{2}}^{a_{3}}{ }_{x_{3}}^{a_{4}} \ldots \\
& \sum_{a_{2 k-1}, a_{2 k}=0}\binom{a_{2 k-1}+a_{2 k}}{a_{2 k-1}}{ }_{x_{k}}^{a_{2 k-1}}{ }_{x_{1}}^{a_{2 k}} \\
& =\sum_{n_{1}, \cdots, n_{k}=0}^{\infty}{ }_{x_{1}}^{n_{1}}{ }_{x_{2}}^{n_{2}} \cdots x_{k}^{n_{k}} T\left(n_{1}, n_{2}, \cdots, n_{k}\right) \quad .
\end{aligned}
$$

The stated result now follows at once.

## RECURRING THEME

H-155 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

The Fibonacci polynomials are defined by

$$
f_{n+1}(x)=x_{n}(x)+f_{n-1}(x)
$$

with $f_{1}(x)=1$ and $f_{2}(x)=x_{0}$ Let $z_{r, S}=f_{r}(x) f_{S}(y)$. If $z_{r, S}$ satisfies the relation

$$
z_{r+4, s+4}+a z_{r+3, S+3}+b z_{r+2, S+2}+c z_{r+1, S+1}+d z_{r, S}=0
$$

show that

$$
a=c=-x y, \quad b=-\left(x^{2}+y^{2}+2\right) \quad \text { and } \quad d=1
$$

## Solution by the Proposer.

Let $u_{r}=f_{r}(x)$ and $v_{r}=f_{r}(y)$. Then,

$$
\begin{aligned}
\mathrm{z}_{\mathrm{r}+4, \mathrm{~s}+4} & =\mathrm{u}_{\mathrm{r}+4} \mathrm{v}_{\mathrm{s}+4}=\left(\mathrm{xu}_{\mathrm{r}+3}+\mathrm{u}_{\mathrm{r}+2}\right)\left(\mathrm{yv}_{\mathrm{s}+3}+\mathrm{v}_{\mathrm{S}+2}\right) \\
& =\mathrm{xyz}_{r+3, s+3}+\mathrm{z}_{\mathrm{r}+2, \mathrm{~s}+2}+\left(\mathrm{x} u_{\mathrm{r}+3} \mathrm{v}_{\mathrm{s}+2}+y u_{\mathrm{r}+2} \mathrm{v}_{\mathrm{s}+3}\right)
\end{aligned}
$$

Now,

$$
\begin{aligned}
& \left(x u_{r+3} v_{s+2}+y u_{r+2} v_{s+3}\right) \\
& =x\left(x u_{r+2}+u_{r+1}\right) v_{s+2}+y\left(y v_{s+2}+v_{s+1}\right) u_{r+2} \\
& =\left(x^{2}+y^{2}\right) z_{r+2, s+2}+\left(x u_{r+1} v_{S+2}+y v_{s+1} u_{r+2}\right) \\
& =\left(x^{2}+y^{2}\right) z_{r+2, s+2}+x u_{r+1}\left(y v_{s+1}+v_{s}\right)+y v_{s+1} u_{r+2} \\
& =\left(x^{2}+y^{2}\right) z_{r+2, s+2}+x y z_{r+1, s+1}+x u_{r+1} v_{s}+u_{r+2}\left(v_{S+2}-v_{S}\right) \\
& =\left(x^{2}+y^{2}\right) z_{r+2, s+2}+x y z_{r+1, s+1}+v_{S}\left(x u_{r+1}-u_{r+2}\right)+z_{r+2, s+2} \\
& =\left(x^{2}+y^{2}+1\right) z_{r+2, s+2}+x y z_{r+1, s+1}+v_{S}\left(-u_{r}\right)=
\end{aligned}
$$

$$
=\left(x^{2}+y^{2}+1\right) z_{r+2, s+2}+\mathrm{xy} \mathrm{z}_{\mathrm{r}+1, \mathrm{~s}+1}+\mathrm{z}_{\mathrm{r}, \mathrm{~s}} .
$$

Hence,

$$
\left.\begin{array}{rl}
\mathrm{z}_{\mathrm{r}+4, \mathrm{~s}+4}= & \mathrm{xy} \mathrm{z} \\
\mathrm{r}+3, \mathrm{~s}+3 \\
& +\left(\mathrm{x}^{2}+\mathrm{y}^{2}+2\right) \mathrm{z}_{\mathrm{r}+2, \mathrm{~s}+2}
\end{array}+\mathrm{xy} \mathrm{z}_{\mathrm{r}+1, \mathrm{~s}+1}\right)
$$

Thus,

$$
\begin{array}{ll}
\mathrm{a}=-\mathrm{xy}, & \mathrm{~b}=-\left(\mathrm{x}^{2}+\mathrm{y}^{2}+2\right) \\
\mathrm{c}=-\mathrm{xy}, & \mathrm{~d}=1
\end{array}
$$

Also solved by W. Brady, D. Zeitlin, and D. V. Jaiswal.
Late Acknowledgement: D. V. Jaiswal solved H-126, H-127, H-129, H-131.


## LETTER TO THE EDITOR

DAVID G. BEVERAGE

## San Diego State College, San Diego, California

In regard to the two articles, "A Shorter Proof," by Irving Adler (December, 1969, Fibonacci Quarterly), and "1967 as the Sum of ThreeSquares," by Brother Alfred Brousseau (April, 1967, Fibonacci Quarterly), the general result is as follows:
$\mathrm{x}^{2}+\mathrm{y}^{2}+\mathrm{z}^{2}=\mathrm{n}$ is solvable if and only if n is not of the form $4^{\mathrm{t}}(8 \mathrm{k}+$ 7), for $\mathrm{t}=0,1,2, \cdots, \mathrm{k}=0,1,2, \cdots$.

Since $1967=8(245)+7, \quad 1967 \neq \mathrm{x}^{2}+\mathrm{y}^{2}+\mathrm{z}^{2}$. A lesser result known to Fermat and proven by Descartes is that no integer $8 \mathrm{k}+7$ is the sum of three rational squares.**

[^1]
## DETERMINATION OF HERONIAN TRIANGLES

JOHN R. CARLSON

San Diego State College, San Diego, California

1. A Pythagorean triangle is defined as any right-triangle having integral sides. Using the well-known relationship $a^{2}+b^{2}=c^{2}$ where $a, b$, and $c$ are the two sides and the hypotenuse respectively, it is obvious that one of the sides must be even. Hence, the area of such a triangle is also an integer. In his book*, Ore introduces the generalization of this situation: a triangle is called Heronian if it has integral sides and area. He further comments that, "although we know a considerable number of Heronian triangles, we have no general formula giving them all." In this paper, we propose to find all such triangles and prove a few basic properties concerning them.
2. Since every Pythagorean triangle is Heronian, and since Pythagorean triangles are completely described by the well-known formulas for the sides $u^{2}+v^{2}, u^{2}-v^{2}$, and $2 u v$, the real problem is to characterize all non-right-angled Heronian triangles. We first give an obvious property.

Lemma 1. Let $a, b, c$, and $n$ all be integers. Then the triangle with sides of na, nb, and nc is Heronian if and only if the "reduced" triangle with sides $a, b, c$ is Heronian.

Proof. We shall use Heron's formula for the area of a triangle

$$
\begin{equation*}
A=\sqrt{s(s-a)(s-b)(s-c)}, \tag{1}
\end{equation*}
$$

where

$$
s=\frac{1}{2}(a+b+c)
$$

Let $A$ be the area of triangle $a, b, c$ and let $A^{\prime}$ be the area of $n a, n b, n c$. Then Eq。 (1) shows us immediately that

[^2]$$
\mathrm{A}^{\prime}=\mathrm{n}^{2} \mathrm{~A}
$$

Hence, if $A$ is an integer, so is $A^{\prime}$. For the converse, suppose that $A^{\prime}$ is integral. Then Eq. (2) insures that $A$ is at least rational. On the other hand, Eq. (1) implies that $A$ is the square root of an integer, which is well known to be either integral or irrational. Thus we conclude that A must be an integer and the lemma is proven.

Before we proceed to our first theorem, we illustrate with two examples. Suppose we juxtapose (or "adjoin") the two Pythagorean triangles 5, 12, 13 and $9,12,15$ so that their common-length sides coincide。 Clearly a (non-right-angled) Heronian triangle results with sides of $13,14,15$ and area equalling 84 .

As a second example, we adjoin the triangles $12,16,20$ and $16,63,65$ (one of which is primative) to obtain the Heronian triangle 20, 65, 75 which may be reduced to $4,13,15$, a primitive Heronian triangle with area equalling 24. That these two examples illustrate all possible events is the content of our first theorem.

Theorem 1. A triangle is Heronian if and only if it is the adjunction of two Pythagorean triangles along a common side, or a reduction of such an adjunction.

Proof. One direction is clear: since every Pythagorean triangle is Heronian, so is every adjunction of two. The previous lemma guarantees that every reduction is also Heronian.

Thus, let us suppose that the triangle $a, b, c$ is Heronian and try to prove that it is either the adjunction or a reduction of an adjunction of two Pythagorean triangles. First we assume the obvious: that from some vertex we may draw a perpendicular to the opposite side, thus dividing the given triangle into two "adjoined" right triangles. That such is possible is easily shown. Let the length of the altitude so constructed be x and let the base c be thus divided into segments $c_{1}$ and $c_{2}$, so that $c=c_{1}+c_{2}$. Since the triangle is Heronian, it is clear that

$$
x=\frac{2 A}{c}
$$

is rational. Let

$$
x=\frac{m}{n}
$$

be reduced to lowest terms.
By the law of cosines, $b^{2}=a^{2}+c^{2}-2 a c \cos w_{1}$ and thus

$$
\cos \mathrm{w}_{1}=\frac{\mathrm{a}^{2}+\mathrm{c}^{2}-\mathrm{b}^{2}}{2 \mathrm{ac}}
$$

is rational. Hence both $c_{1}=b \cos w_{1}$ and $c_{2}=c-c_{1}$ are also rational. If the numbers $x, c_{1}$, and $c_{2}$ are in fact all integers, we are done. Otherwise we look at the triangle having sides na, nb, and nc. From elementary geometry, this triangle is similar to the original one and thus the new altitude is equal to $n x=m$, an integer. But $n c_{1}$, still rational, is given by

$$
\sqrt{\mathrm{n}^{2} \mathrm{a}^{2}-\mathrm{m}^{2}}
$$

which is, as before, either integral or irrational, and thus must be integral. Likewise $\mathrm{nc}_{2}$ is integral, and the new enlarged triangle is the adjunction of two Pythagorean triangles. Thus the original is a reduction of an adjunction and we are through.

Since the sides of any Pythagorean triangle are given by $u^{2}+v^{2}, u^{2}-$ $\mathrm{v}^{2}$, and 2uv, we now have a method for finding all Heronian triangles.

Corollary 1. A triangle is Heronian if and only if its sides are given by either (3) $u^{2}+v^{2}, r^{2}+s^{2}$, and $u^{2}-v^{2}+r^{2}-s^{2}$; where $r s=u v$; or (4) $u^{2}+v^{2}, r^{2}+s^{2}$, and $2(u v+r s)$; where $r^{2}-s^{2}=u^{2}-v^{2}$; or (5) a reduction by any constant factor in either case (3) or (4).
3. Although the preceding theorem and its corollary give formulations for finding all Heronian triangles, there are many properties of Heronian triangles that are not obvious from examination of the special subset of rightangled triangles. Some of these properties will be given here.

Lemma 2. A primitive Heronian triangle is isosceles if and only if it has sides given by (3), (4), or (5) with $r=u$ and $s=v_{0}$

Proof. Since a triangle is primitive only when one side is even, the equal sides of the isosceles triangle must be odd, say $2 m+1$. Let the even
side be 2 n . Then the semiperimeter is given by $2 \mathrm{~m}+\mathrm{n}+1$ and the area, given by Eq. (1) becomes

$$
\sqrt{(2 m+n+1)(2 m-n+1)(n)(n)},
$$

so that

$$
A=n \sqrt{(2 m+1)^{2}-n^{2}}
$$

If this is to be an integer, there must be an integer $Q$ such that

$$
(2 \mathrm{~m}+1)^{2}-\mathrm{n}^{2}=\mathrm{Q}^{2}
$$

Thus the number $2 m+1$ is the hypotenuse of a Pythagorean triangle, which means, of course, that $2 m+1$ is as given in Corollary 1. Conversely, every triangle described by those formulae will be isosceles if $r=u$ and $\mathrm{s}=\mathrm{v}$ 。

We note in particular that any number of the form $4 \mathrm{n}+2$ may be used as the even side of a primitive isosceles Heronian triangle, by using sides $2 n^{2}+2 n+1, \quad 2 n^{2}+2 n+1$, and $4 n+2$. We will show, in fact, that any integer greater than two may be used as the side of a primitive non-right-angled Heronian triangle. Before we do, we shall establish the following.

Lemma 3. No Heronian triangle has a side of either 1 or 2.
Proof. Since the sides of the triangle must be integers, the difference between two sides is either 0 or an integer $\geq 1$. This latter case would preclude the use of 1 as a side. But an isosceles triangle with side one is also impossible, since in that event, we must have two sides equal to 1 , and hence the third side either 0 or 2 . Thus we have only to show that 2 cannot be used as the side of an Heronian triangle. Suppose, to the contrary, that we do have a triangle with sides $2, a$, and $b$. Then the area as given by (1) is

$$
A=\sqrt{s(s-2)(s-a)(s-b)}
$$

where $a+b+2=2 \mathrm{~s}$. The only values of a and b which satisfy this last equation are $a=b=s-1$. Thus the area becomes

$$
A=\sqrt{s(s-2)(1)(1)}
$$

and we must have $s(s-2)=Q^{2}$ for some Q. But this is impossible, so we are done。

Using the formulae for the sides of a Pythagorean triangle, it is easy to show that every integer greater than two can be used as a side in a finite number of Pythagorean triangles. This observation has the following remarkable generalization.

Theorem 2. Let a be an integer greater than two. Then there exists an infinitude of primitive Heronian triangles with one side of length a.

Proof. If a is odd, we may use sides given by

$$
\begin{equation*}
\text { a, } \frac{1}{2}(a t-1), \quad \text { and } \quad \frac{1}{2}(a t+1) \tag{6}
\end{equation*}
$$

where $t$ is a solution of the Pellian equation

$$
\begin{equation*}
t^{2}-\left(a^{2}-1\right) y^{2}=1 \tag{7}
\end{equation*}
$$

Since $a^{2}-1$ is even and never a perfect square, Eq. (7) has an infinitude of solutions for $t$, each of them odd, so that (6) lists only integers. Since

$$
\frac{1}{2}(a t-1)+1=\frac{1}{2}(a t+1)
$$

the triangle is obviously primitive. We compute the area of the triangle by (1) and find

$$
A^{2}=\frac{1}{2}(t+1) a \frac{1}{2}(t-1) a \frac{1}{2}(a+1) \frac{1}{2}(a-1)=\left(\frac{1}{4}\right)^{2} a^{2}\left(a^{2}-1\right)\left(t^{2}-1\right)
$$

But, by Eq. (7), we have $\mathrm{t}^{2}-1=\left(\mathrm{a}^{2}-1\right) \mathrm{y}^{2}$ so that

$$
A^{2}=\left(\frac{1}{4}\right)^{2} a^{2}\left(a^{2}-1\right)^{2} y^{2}
$$

and thus

$$
A=\frac{1}{2}(a-1) \frac{1}{2}(a+1) a y
$$

an integer, and hence the triangle is Heronian.
Next, suppose that $a$ is even, say $a=2 n$, where $n$ is odd. Then we may use

$$
\begin{equation*}
\text { a, } \operatorname{tn}-2, \text { and } \operatorname{tn}+2, \tag{8}
\end{equation*}
$$

where $t$ is any odd solution of

$$
\begin{equation*}
\mathrm{t}^{2}-\left(\mathrm{n}^{2}-4\right) \mathrm{y}^{2}=1 \tag{9}
\end{equation*}
$$

Since $n$ is odd, $n^{2}-4$ is also odd and thus an infinitude of odd values of $t$ is available. That (8) forms a primitive triangle is clear; we prove it is Heronian by computing
$A^{2}=(t+1) n(t-1) n(n+2)(n-2)=n^{2}\left(n^{2}-4\right)\left(t^{2}-1\right)=n^{2}\left(n^{2}-4\right)^{2}(y)^{2}$
so that $A$ is an integer.
Lastly, suppose that $a=2 n$, where $n$ is even. Then we may use
(10)

$$
a, \operatorname{tn}-1, \text { and } \operatorname{tn}+1
$$

where $t$ is any solution of

$$
\begin{equation*}
\mathrm{t}^{2}-\left(\mathrm{n}^{2}-1\right) \mathrm{y}^{2}=1 \tag{11}
\end{equation*}
$$

The proof follows the same lines as that just given. Thus our theorem is proven for all cases.

It should be obvious that the formulations given in the above proof are simply chosen from an infinitude of possibilities. Thus we could also have shown, for example, that the triangle having sides

$$
a, \frac{1}{2} a(x-1)+1,
$$

and

$$
\frac{1}{2} a(x+1)-1
$$

where x is found from

$$
x^{2}-(a-1) y^{2}-1
$$

which will have an infinitude of solutions as long as a-1 is not a perfect square.
4. We conclude with a few observations and a short list of examples. It is easy to show that every primitive Heronian triangle has exactly one even side. A simple check of all the possibilities obtained from the adjunction of two Pythagorean triangles (which are known to have either one or three even sides) will suffice to prove this. From this, we conclude that the area of any Heronian triangle is divisible by 2 , since, if $s$ is even, a factor of 2 divides $A^{2}$ (and thus also A) while, if $s$ is odd, then $s-a$ will be even where a is an odd side of the triangle, and so again 2 divides the area.

Since the area of any Pythagorean triangle is given by

$$
A=u v(u-v)(u+v)
$$

it is easy to show that such a triangle has area divisible by three. A simple analysis of adjunctions and possible reductions will then show that every Heronian triangle has area divisible by three.

Following is alist of the "first" fewprimitive (non-right-angled) Heronian triangles:

| $\mathrm{a}=3$ | $\mathrm{~b}=25$ | $\mathrm{c}=26$ |
| ---: | ---: | ---: |
| 4 | 13 | 15 |
| 5 | 5 | 6 |
| 5 | 5 | 8 |
| 5 | 29 | 30 |
| 6 | 25 | 29 |
| 7 | 15 | 20 |
| 8 | 29 | 35 |


| $\mathrm{a}=9$ | $\mathrm{b}=.10$ | $\mathrm{c}=17$ |
| :---: | :---: | :---: |
| 9 | 65 | 70 |
| 10 | 13 | 13 |
| 10 | 17 | 21 |
| 11 | 13 | 20 |
| 12 | 17 | 25 |
| 13 | 13 | 24 |
| 13 | 14 | 15 |
| 13 | 20 | 21 |
| 13 | 37 | 40 |
| 14 | 25 | 25 |
| 15 | 28 | 41 |
| 15 | 37 | 44 |
| 15 | 41 | 52 |
| 16 | 17 | 17 |
| 17 | 17 | 30 |
| 17 | 25 | 26 |
| 17 | 25 | 28 |
| 18 | 41 | 41 |
| 19 | 20 | 37 |
| 20 | 37 | 51 |
| 21 | 85 | 104 |
| 22 | 61 | 61 |
| 23 | 212 | 225 |
| 24 | 37 | 37 |
| 25 | 25 | 48 |
| 25 | 29 | 36 |
| 25 | 39 | 40 |
| 25 | 51 | 74 |
| 26 | 85 | 85 |
| 27 | 676 | 701 |
| 28 | 85 | 111 |
| 29 | 29 | 40 |
| 29 | 29 | 42 |

[Continued on page 551.]

# PINEAPPLES AND FIBONACCI NUMBERS 

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In the book, Fibonacci and Lucas Numbers, by V. E. Hoggatt, Jr., there is a paragraph mentioning the spiral curves on pineapples which makes reference to Mathematical Diversions by Hunter and Madachy (Princeton, N. J., published by D. Van Nostrand Co., Inc., in 1963). Thus, parastichies on pineapples were known, then, sometime prior to 1963.

The author first became aware of the spiral curves and the agreement with Fibonacci numbers about 1951 upon reading an article in a magazine put out by the American Association for the Advancement of Science which pertained to Fibonacci Numbers. At that time, the author was working for the Maiu Pineapple Company, Kahului, Maui, Hawaii, and after reading this article, started checking pineapples. The vast majority of pineapples checked had 8-13-21 rows of fruitlets (eyes). A few runts were 5-8-13.

Since a pineapple with more fruitlets for a given size would likely have a finer texture and would be better for eating, the author was interested in finding a pineapple with $13-21-34$ rows. No such pineapples were ever found, however.

Giving evidence to the fact that numbers of spiral rows on pineapples were studied even earlier, the Experiment Station of the Association of Hawaiian Pineapple Canners published an article on such a study as early as 1933. The article by M. B. Linford was published in the Pineapple Quarterly, Vol. III, No. 4, December 1933, pp. 185-195. It was entitled "Fruit Quality Studies II, Eye Number and Eye Weight," and mentioned the number of rows of fruitlets on pineapples as basically 8-13 spirals, ranging from 5-8-13 -21 for the several types of spirals. The article did not, however, draw any connection between these numbers and the Fibonacci Numbers.

Following is a quotation from a section of this article which had the heading, "A Method of Estimating Eye Numbers."
"A Cayenne pineapple fruit of usable size consists of from 100 to 200 fruitlets. This makes counting eyes in anylarge numbers of fruits both tedious and expensive. Where great precision is required there is no short cut, and when counts are made of very young inflorescences long before flowering the use of a low-power microscope is essential.
"For use where such precision is not required, however, the regular arrangement of eyes (fruitlets) suggested a method of estimating which is much more rapid. As may be seen in Figure 1, the fruitlets are aligned in two series of spirals, one rising to the right, the other to the left. One series of spirals is steep and is found, in Cayenne and at least some other varieties, to be composed of 13 separate rows of eyes. The other series, less steep, is composed of 8 longer spirals. Various irregularities occur, but in examining many individual fruits over a period of two years, no specimen has yet been found in which the basic pattern was other than 8 and 13 spirals. It follows from this uniformity that a count of eyes in one spiral, multiplied by the number of similar spirals should yield a figure close to the actual number of eyes. Errors are introduced by two factors: Some spirals have more eyes than others of the same fruit. In Figure 1b some spirals contain 11 eyes, some contain 12. Chance will determine which spiral is counted. Then as shown in Figure 2, irregularities may result in there being more or less than the regular number of spirals through part of the length of the fruit. By actual test, it has been determined that smaller errors result from counting the longer spiral and multiplying by 8 , than from counting the short spiral and multiplying by 13. For this reason, the standard procedure adopted is to count the long spiral. On some fruits, this ascends to the right of the observer, on some to the left. After a little practice, it is recognized readily as the less steep of the two spirals which bound the four sides of an eye, considering the eye as a square standing on one corner. In case of doubt, it is safest to verify the number of spirals before counting eyes.
"Confusion may arise with extraordinarily small or large fruits from the fact that two other series of spirals may sometimes be recognized. These include steep, nearly vertical spirals, of which there are 21, and very flat spirals of which there are only 5 . Thus the numbers of spirals of the several types are $5,8,13$ and 21 . The number of eyes counted in any spiral must be multiplied by the number of similar spirals.
"In other plant parts where spiral patterns occur, chance alone usually determines whether the spiral winds to the right or to the left. This was tested for pineapples by recording direction of spirals along with eye number for a large number of fruits during seasin 1932. Results are shown in Table 1. If chance determines we should expect to find equal numbers of right and left patterns. Actually among 9,008 fruits, right spirals were found 49.4 percent of the time, left spirals 50.6. Only one of the lots shown in this table deviated markedly from the expectation, and the 24 separate plots in that group showed no agreement among themselves; some had more left spirals and some more right. Chance alone seems to determine the direction of these spirals in the pineapple."

## THE POWERS OF THREE

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Any number may be expressed in powers of three by addition or subtraction of the numbers those powers represent.

$$
\begin{aligned}
& 13=3^{2}+3^{1}+3^{0} \\
& 14=3^{3}-3^{2}-3^{1}-3^{0}
\end{aligned}
$$

The powers used in such expressions are whole integers, no fractional powers being involved.

The number of terms required to express a number approximates twice the number of digits in the number; the greater the number of digits required the more closely this limit is approached.

Any such expression of a number need contain no repetition of any given power.

Such expressions are easily handled in arithmetic processes by observation of algebraic rules regarding exponents.

Discussion of this digital system follows.
It will be noted from Table 1 that the powers of three follow a routine sequence in the expressions for the numbers, appearing first behind the positive sigh, then changing to the negative sign, and then disappearing from the statement.

The appearance of powers in the statements follows a fixed sequence:

1. The power appears for the first time at a number equal to one-half of the value of that power with 1 added.
2. It appears behind the positive sign.
3. It remains in the statement, and behind the positive sign, for a series of statements equal in number to the value of that power of three.
4. It then becomes negative in the following statement.
5. It remains in the statement, and behind the negative sign, for a series of statements equal in number to the value of that power of three.
6. It then disappears in the following statement.

Table 1

| Number expressed in | Powers of Three |
| :---: | ---: |
| 1 | $+3^{0}$ |
| 2 | $+3^{1}-3^{0}$ |
| 3 | $+3^{1}$ |
| 4 | $+3^{1}+3^{0}$ |
| 5 | $+3^{2}-3^{1}-3^{0}$ |
| 6 | $+3^{2}-3^{1}$ |
| 7 | $+3^{2}-3^{1}+3^{0}$ |
| 8 | $+3^{2}-3^{0}$ |
| 9 | $+3^{2}+3^{2}+3^{0}$ |
| 10 | $+3^{2}+3^{1}-3^{0}$ |
| 11 | $+3^{2}+3^{1}+3^{0}$ |
| 12 | $+3^{3}-3^{2}-3^{1}-3^{0}$ |
| 13 | $+3^{3}-3^{2}-3^{1}$ |
| 14 | $+3^{3}-3^{2}-3^{1}+3^{0}$ |
| 15 | $+3^{3}-3^{2}-3^{0}$ |
| 16 | $+3^{3}-3^{2}$ |

7. It remains out of the statement for a series of statements equal in number to the value of that power of three.
8. It then reappears in the statement, and behind the positive sign, and repeats the sequence outlined above without limit.

Determination of the proper statement for any given number is outlined below.

## Determination of Statement

1. Subtract 1 from the given number.
2. Divide the remainder by 3 , setting the quotient below the dividend and the remainder from the division to the right, whether this remainder be 2 , or 1 , or 0 .
3. If the remainder after division is 1 or 2 proceed as directed below, but whenever the remainder after division is 0 it is necessary to subtract 1 from the quotient before proceeding to treat it as a dividend, as below.
4. Divide again by 3 as directed in step 2, and continue this process until the dividend is 0 with 0 remainder, watching throughout the process outlined in Step 3 。
5. The column of remainders 2 , or 1 , or 0 , which have been set to the right is now numbered, beginning at the top with 0 and proceeding with $1,2,3$, etc., and ending with the highest number in the sequence opposite the final 0 remainder. The numbers in this sequence are the powers of three.
6. Fixation of signs for the various powers, or exclusion from the statement, is determined by the remainders:
When the remainder is 0 the sign is positive.
When the remainder is 1 the sign is negative.
When the remainder is 2 the power is excluded from the statement.

Demonstration of this process follows:


Consider the sequence thus, bottom to top:

| Remainders: | 0 | 2 | 1 | 0 | 2 | 1 | 0 | 2 | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Signs: | + |  | - | + |  | - | + |  | - |
| Powers: | 8 |  | 6 | 5 |  | 3 | 2 |  | 0 |

Statement for the given number:

$$
6056=3^{8}-3^{6}+3^{5}-3^{3}+3^{2}-3^{0}
$$

Proof of statement:
Positive powers: $\quad 6561+243+9=6813$
Negative powers: - $729-27-1=\underline{-757}$
Given number: 6056

For simplicity, in further discussion, the digit 3 will not be used in power statements except when the exponent 3 is required. Statements will use only the digits designating the powers. Treated thus the statement for 6056 would be written thus: 6056 equivalent, $+8-6+5-3+2-0$.

## ADDITION OF STATEMENTS

In handling two or more such statements arithmetically, it is almost inevitable that there will be duplication of one or more of the powers. Consider the addition of the statements for the numbers two and three:

$$
\begin{aligned}
& 2 \text { equivalent, }+1-0 \\
& 3 \text { equivalent, }+1
\end{aligned}
$$

Addition of +1 to the statement gives $+1+1-0$. This is re-written thus: $+2-1-0$. The next higher power is given the sign of the duplicated power, following it by the duplicated power with reversed sign; obviously $3+3=$ 9-3.

When a power is triplicated, the next higher power is given the sign of the triplicated power, which is then dropped from the statement. Consider addition of the following statements:

| 2 | equivalent, | $+1-0$ |
| :---: | :---: | :---: |
| 3 | $"$ | +1 |
| $\frac{4}{9}$ | $"$ | $+1+0$ |
| 9 |  | +2 |

Note that the unlike signs calcelled and removed 0 power from the statement for the sum.

When the number of repetitions exceeds three, they can be eliminated step-wise by application of the processes outlined above.

SUBTRACTION OF STATEMENTS
Subtraction is performed by changing the signs on all of the powers in the statement being subtracted and then performing addition as abovel.

## MULTIPLICATION OF STATEMENTS

In performing multiplication, the digits representing the powers are added irrespective of sign, and the signs follow algebraic rule:

Multiplication of like signs yields the positive sign.
Multiplication of unlike signs yields the negative sign.
Consider multiplication of statements for 13 and 14:

$$
\begin{array}{lr}
14: & +3-2-1-0 \\
13 & \frac{+2+1+0}{+3-2-1-0} \\
& +4-3-2-1 \\
& +5-4-3-2 \\
\hline+5-4+3-2+1-0
\end{array}
$$

Consider the totalling of those products step-wise:

1. The -0 comes down unchanged.
2. The duplicated -1 becomes $-2+1$
3. The triplicated -2 becomes -3 , with the -2 carried forward from Step 2 brought down unchanged.
4. The duplicated -3 has been triplicated by the -3 carried forward from Step 3 so it becomes -4 and the +3 is brought down unchanged.
5. The unlike signs for 4 cancel so the -4 carried forward from Step 4 is brought down unchanged.
6. The +5 is brought down unchanged.

Proof of statement:

| Positive powers: $\quad$$243+27+3$ $=273$ <br> Negative powers: $\quad-81$ $-9-1$$=\frac{-91}{182}$ |  |
| ---: | :--- |
| $13 \times 14$ | $=18$ |

## DIVISION OF STATEMENTS

This process brings another rule into play. Before stating the rule, consider these two terms, $+4-3$ and +3 , the statements for 54 and 27 . If these terms are added, the cancellation of unlike signs will leave +4 , the statement for 81 . If they are subtracted, the sign representing 27 would be changed. It would then be duplicated. After treatment for duplication, it would cancel the +4 and leave +3 , the statement for 27 .

The rule in dividing power statements once more involves like and unlike signs, but with an extra specification added: like or unlike signs of adjacent powers as in the case of the statement for 54 . The rule: when adjacent powers in the dividend have unlike signs, the two powers must be considered as a duplication of the lower power, both with the sign of the higher power. This is a reversal of the rule for the additive process and is not unreasonable since division is a subtractive process.

Demonstration of division will be the reverse of the example given for multiplication. The statement for 182 is divided by the statement for below. The notes referenced are found on the following page.


## Notes:

1. In this case, all the powers are both adjacent and unlike. The rule will be applied only to the leading pair.
2. Rewritten giving the duplicated lower power the sign of the higher power.
3. The +2 in the divisor, when subtracted from the +4 in the dividend, yields +2 for the quotient. When the sign is changed for subtraction, cancellation will result.
4. The duplicated -2 became $-3+2$ and the +2 came down. Unlike signs for 3 cancelled and the -3 brought forward came down. The unaffected +4 came down, making this another case of unlike signs for adjacent powers.
5. Rewritten as in Note 2.
6. The +2 in the divisor, when subtracted from the +3 in the remainder, yields +1 for the quotient. When the sign is changed for subtraction, cancellation will result.
7. Unlike signs cancel and +3 comes down with -0 .
8. The +2 in the divisor, when subtracted from the +3 in the remainder, yields +1 for the quotient. When the sign is changed for subtraction, cancellation will result. The duplication of +1 in the quotient can be cared for after completion of the division.
9. Unlike signs cancel, and unaffected terms come down.
10. The +2 in the divisor now encounters -2 in the remainder. Disregarding signs, the 2 from 2 yields 0 for the quotient. Adding -0 to +2 will yield -2 ; when the sign is changed for subtraction, cancellation will result.

The quotient reads $+2+1+1=0$. There is duplication of +1 . This is changed to $+2-1$. This change duplicates +2 . This is changed to $+3-2$. Now the corrected statement reads $+3-2-1-0$, the power statement for 14 。

This is as far as I have investigated this curiosity with any success. Perhaps someone else can find a way to extract roots and raise to higher powers without simple multiplication. Decimal fractions can, of course, be handled by appropriate multiplication by powers of 10 , as can uneven divisions that result in significant remainders.

I apologize for using the expression "negative powers" when values involved are not reciprocals. This misuse was a short-cut through verbose explanation.

# SOME REMARKS ON THE ORDERING OF GENERAL FIBONACCI SEQUENCES 

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In The Fibonacci Quarterly, Vol. 1, No. 4, December, 1963, Brother U. Alfred suggested a method whereby the general Fibonacci sequences could be ordered. The remarks of this paper are intended to supplement, rather than supplant, those of Brother Alfred. Another system of ordering is proposed herein.

We shall obtain ageneral Fibonacci sequence by taking any two integers, $a$ and $b$, and employing the relationship $a+b=c$. Utilizing the set of integers for indexing, with $\mathrm{a}=\mathrm{T}_{\mathrm{n}}$ and $\mathrm{b}=\mathrm{T}_{\mathrm{n}+1}$, and requiring that $\mathrm{T}_{\mathrm{n}}<$ $T_{n+1}$, we may define the general sequence by using the recursive form:

$$
T_{n}+T_{n+1}=T_{n+2}
$$

To eliminate possible confusion, we adjust the indexing such that $T_{0}$ is the smallest non-negative term of the sequence.

It has been shown [1], and is easily verified, that when $T_{0}$ is the smallest non-negative term of any general Fibonacci sequence,

$$
2 \mathrm{~T}_{0}<\mathrm{T}_{1} ; \quad 2 \mathrm{~T}_{\mathrm{n}+1}>\mathrm{T}_{\mathrm{n}+2}, \quad(\mathrm{n}=0,1,2,3, \cdots)
$$

Thus, while any two successive terms of a general Fibonacci sequence are sufficient to define the entire sequence, we should like to employ $T_{0}$ and $T_{1}$. Hence, we have a unique representation of each general Fibonacci sequence, i.e.,

$$
\left(\mathrm{T}_{0}, \mathrm{~T}_{1}\right)=\mathrm{T}_{0}, \mathrm{~T}_{1}, \mathrm{~T}_{2}, \mathrm{~T}_{3}, \mathrm{~T}_{4}, \cdots
$$

Attention is called to the fact that we do not require any two successive terms to be relatively prime and employ the single restriction that $\mathrm{T}_{0}$ be the smallest non-negative term of the sequence.

Since we now have a unique representation of each general Fibonacci sequence, one of the next logical steps would be to devise a method of ordering
the sequences. Such a method should be easy to apply, virtually by visual inspection. Given any representation of a general Fibonacci sequence, we immediately have the first two terms, the third term is a simple summation, and we can easily calculate the characteristic number, D, from the wellknown relation [2]

$$
\mathrm{T}_{1}^{2}-\mathrm{T}_{1} \mathrm{~T}_{0}-\mathrm{T}_{0}^{2}=\mathrm{D}
$$

From these few properties we should like to accomplish the desired ordering of all general Fibonacci Sequences.

Utilizing the property that each sequence has a unique characteristic number, D, Brother U. Alfred has suggested a method of arranging the general Fibonacci sequences with respect to the value of $D$. Where several sequences have the same $D$, the size of $T_{0}$ becomes the second criterion. With the restriction that successive terms of the sequence be relatively prime, no doubt to eliminate multiples of a sequence, he suggests the following convention:

Given: | D | $\left(\mathrm{T}_{0}, \mathrm{~T}_{1}\right)$ |  |
| ---: | :--- | :--- |
|  |  | $(0,1)$ |
| 5 | $(1,3)$ |  |
| 11 | $(1,4),(2,5)$ |  |
| 19 | $(1,5),(3,7)$ |  |
| 29 | $(1,6),(4,9)$ |  |
| 31 | $(2,7),(3,8)$ |  |
| 41 | $(1,7),(5,11)$ |  |
| 55 | $(1,8),(6,13)$ |  |
|  |  |  |

Let $S_{n}^{\prime},(n=1,2,3,4, \cdots)$, denote the $n^{\text {th }}$ sequence of the ordering, and we have

$$
\begin{aligned}
& S_{1}^{p}=(0,1), \quad S_{2}^{\prime}=(1,3), \quad S_{3}^{1}=(1,4), \quad S_{4}^{\prime}=(2,5), \quad S_{5}^{\prime}=(1,5), \\
& S_{6}^{p}=(3,7), \quad S_{7}^{\prime}=(1,6), \quad S_{8}^{p}=(4,9), \quad S_{9}^{p}=(2,7), \quad S_{10}^{\prime}=(3,8), \cdots,
\end{aligned}
$$

By dropping the restriction that any two successive terms be relatively prime, but retaining the same principle, we must then modify the above method as follows:

Given: | D | $\left(\mathrm{T}_{0}, \mathrm{~T}_{1}\right)$ |
| ---: | :--- | :--- |
| 1 | $(0,1)$ |
| 4 | $(0,2)$ |
| 5 | $(1,3)$ |
| 9 | $(0,3)$ |
| 11 | $(1,4),(2,5)$ |
| 16 | $(0,4)$ |
| 19 | $(1,5),(3,7)$ |
| 20 | $(2,6)$ |
| 25 | $(0,5)$ |
| 29 | $(1,6),(4,9)$ |
| 31 | $(2,7),(3,8)$ |
| 36 | $(0,6)$ |
| 41 | $(1,7),(5,11)$ |
| 44 | $(2,8),(4,10)$ |
| 45 | $(3,9)$ |
| $\vdots$ |  |

Hence, we now have, with $S_{n}^{\prime \prime}(n=1,2,3,4, \cdots)$ denoting the $n^{\text {th }}$ sequence of the modified ordering
$S_{1}^{\prime \prime}=(0,1), \quad S_{2}^{\prime \prime}=(0,2), \quad S_{3}^{\prime \prime}=(1,3), \quad S_{4}^{\prime \prime}=(0,3), \quad S_{5}^{\prime \prime}=(1,4)$,
$S_{6}^{\prime \prime}=(2,5), \quad S_{7}^{\prime \prime}=(0,4), \quad S_{8}^{\prime \prime}=(1,5), \quad S_{9}^{\prime \prime}=(3,7), \quad S_{10}^{\prime \prime}=(2,6), \cdots$.

Let us examine three of the sequences, observing the above systems of ordering.

$$
\begin{aligned}
& S_{8}^{\prime}=S_{13}^{\prime \prime}=(4,9)=4,9,13,22,35,57,92, \quad \cdots D=29 \\
& S_{11}^{\prime}=S_{17}^{\prime \prime}=(1,7)=1,7,8,15,23,38,61, \quad \cdots D=41 \\
& S_{12}^{\prime}=S_{18}^{\prime \prime}=(5,11)=5,11,16,27,43,69,112, \cdots D D=41 .
\end{aligned}
$$

Here note that there is no simple relationship between the $k^{\text {th }}$ term of $S_{1}^{\prime}$ and $S_{j}^{\prime}$ (or $S_{i}^{\prime \prime}$ and $S_{j}^{\prime \prime}$ ) as compared to the relationship of $i$ and $j$, thus
eliminating one of the most desirable results we should like to have from a system of ordering. For example: Given $S_{23}^{9}$ and $S_{25}^{9}$, is the $\mathrm{k}^{\text {th }}$ term of $\mathrm{S}_{23}^{\boldsymbol{\phi}}$ smaller than the $\mathrm{k}^{\text {th }}$ term of $\mathrm{S}_{25}^{\mathrm{p}}$ ?

Furthermore, additional investigation has not revealed any simple relation for comparing $k^{\text {th }}$ terms of individual sequences using the readily available information we should like to use for an ordering. Thus it appears, at least for the present, we shall have to be content with a method of arranging the general Fibonacci sequences so that they may be designated, or counted, for example:

$$
S_{1070}^{p}=(?, ?) \quad \text { or } \quad(237,475)=S_{7}^{?}
$$

Utilizing a table of $D^{\prime} \mathrm{S}$ up to a value of 1000 , one finds, for $D$ greater than 5, at least two sequences associated with each D. Occasionally, but without a recognizable pattern* four sequences are found to be associated with a particular D. Therefore, without extensive calculation or the aid of a lengthy table, we would not know how many sequences precede a sequence having a D in, for example, the $1,000,000$ region. Hence, an ordering, using $D$ as an index, becomes unwieldy with larger values of $D$, the characteristic number of the sequence.

With the above limitation in mind, an alternate proposal for ordering is presented herein. There is no noteworthy advantage claimed, other than the convenience of obtaining $S_{n}$ or $\left(T_{0}, T_{1}\right)$ where large indices and/or initial terms are involved. Again, there is no simple relationship between the $k^{\text {th }}$ term of $S_{i}$ as compared to the $k^{\text {th }}$ term of $S_{j}$.

Let us arrange the unique representation of each general Fibonacci sequence, $\left(\mathrm{T}_{0}, \mathrm{~T}_{1}\right)$, in an infinite matrix array in the following manner:

| $(0,1)$ | $(0,2)$ | $(0,3)$ | $(0,4)$ | $(0,5)$ | $\cdots 0$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $(1,3)$ | $(1,4)$ | $(1,5)$ | $(1,6)$ | $(1,7)$ | -.. |
| $(2,5)$ | $(2,6)$ | $(2,7)$ | $(2,8)$ | $(2,9)$ | $\cdots$ |
| $\stackrel{:}{(\mathrm{j}, 2 \mathrm{j}+1)}$ | $(\mathrm{j}, 2 \mathrm{j}+2)$ | $(\mathrm{j}, 2 \mathrm{j}+3)$ | $(\mathrm{j}, 2 \mathrm{j}+4)$ | $(\mathrm{j}, 2 \mathrm{j}+5)$ | -• |
| : | ( j = | , 1, 2, 3, | ...) |  |  |

[^3]Given the above display of general Fibonacci sequence representations, the remaining problem is that of choosing the system of ordering to be employed. Note that the representation is in the $\mathrm{T}_{0}+1$ row and the $\mathrm{T}_{1}-2 \mathrm{~T}_{0}$ column. Two methods that might be considered are shown below, together with certain comments. Observe that the numbers in the following displays are those of the indexing set and only reflect the position of, or number assigned to, each general Fibonacci sequence.

## I. Diagonal Method

| 1 | 2 | 4 | 7 | 11 | 16 | 22 | $\ldots$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 3 | 5 | 8 | 12 | 17 | 23 |  |  |
| 6 | 9 | 13 | 18 | 24 |  |  |  |
| 10 | 14 | 19 | 25 |  |  |  |  |
| 15 | 20 | 26 |  |  |  |  |  |
| 21 | 27 |  |  |  |  |  |  |
| 28 |  |  |  |  |  |  |  |
| $\vdots$ |  |  |  |  |  |  |  |

There are several methods of associating the proper row and column of any given integer in the above display of positive integers, thus each reader is free to choose his own favorite scheme. Using the diagonal method of ordering, with $S_{n}$ denoting the $\mathrm{n}^{\text {th }}$ sequence, we obtain

$$
\begin{array}{ll}
\mathrm{S}_{1070}=(?, ?)=(34,80) & \begin{array}{l}
1070 \text { appears in the } 35 \text { th row and the 12th } \\
\text { column. } \mathrm{T}_{0}+1=35 \text { and } \mathrm{T}_{1}-2 \mathrm{~T}_{0}=12 .
\end{array} \\
(237,475)=\mathrm{S}_{?}=\mathrm{S}_{28411} & \begin{array}{l}
28411 \text { is found in the } 237+1 \text { row and the } \\
475-2 \times 237 \text { column. }
\end{array}
\end{array}
$$

Only a short calculation is required to obtain the desired information, $n$ or $S_{n}$, or $\left(T_{0}, T_{1}\right)$, when given the corresponding data. To retain consistency, the indexing set is the set of positive integers, $(\mathrm{n}=1,2,3,4, \ldots)$ 。
II. Modified Sides of Squares Method

| 1 | 2 | 5 | 10 | 17 | 26 | 37 | $\cdots$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 3 | 4 | 6 | 11 | 18 | 27 | 38 |  |
| 7 | 8 | 9 | 12 | 19 | 28 | 39 |  |
| 13 | 14 | 15 | 16 | 20 | 29 | 40 |  |
| 21 | 22 | 23 | 24 | 25 | 30 | 41 |  |
| 31 | 32 | 33 | 34 | 35 | 36 | 42 |  |
| 43 | 44 | 45 | 46 | 47 | 48 | 49 |  |
| $\vdots$ |  |  |  |  |  |  |  |

Again, the reader is free to use any one of several well-known methods of obtaining the row and column of a given integer, and we observe, with $\mathrm{S}_{\mathrm{n}}$ denoting the $\mathrm{n}^{\text {th }}$ sequence of the ordering ( $\mathrm{n}=1,2,3,4, \cdots$ )

$$
\begin{array}{ll}
\mathrm{S}_{1070}^{*}=(?, ?)=(32,78) & \begin{array}{l}
1070 \text { is in the } 33^{\text {rd }} \text { row and } \\
\text { the } 14^{\text {th }} \text { column. }
\end{array} \\
(237,475)=\mathrm{S}_{?}^{*}=\mathrm{S}_{56407}^{*} & \begin{array}{l}
\text { The } 238^{\text {th }} \text { row and the } 1^{\text {st }} \\
\text { column is the position of } \\
56407 .
\end{array}
\end{array}
$$

With regard to the two demonstrated methods of ordering, the diagonal is possibly a more elegant attack for it parallels that used to count other infinite matrix arrays.

In conclusion, we have proposed another method of arranging or ordering the general Fibonacci sequences - the first being that as suggested by Brother U. Alfred. Undoubtedly, there are still others. As implied earlier, this property of a unique representation of each general Fibonacci sequence demands the ultimate adoption of some system of ordering to assist the growing number of Fibonacci devotees. References 3, 4, 5, and 6 were added by the Editor.

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1. Brother U. Alfred, "On the Ordering of Fibonacci Sequences," Fibonacci Quarterly, Vol. 1, No. 4, December, 1963, pp. 41-46.
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4. Charles H. King, "Conjugate Generalized Fibonacci Sequences," Fibonacci Quarterly, Vol. 6, No. 1 (1968), pp. 46-49.
5. Eugene Levine, "Fibonacci Sequences with Identical Characteristic Values," Fibonacci Quarterly, Vol. 6, No. 3 (1968), pp. 75-80.
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# THE EDUCATIONAL VALUE IN MATHEMATICS 

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The educational value in mathematics is higher to the perceptive individual than the manufacture of things and the mere solving of problems.

I would like to offer a different approach to the significance of the Golden Ratio. It is for the purpose of making clear to the layman that beyond the useful pursuit of mathematics for its own sake (which is not an end in itself) lies the deep philosophic content, and a content that constitutes a very important ingredient of philosophy.

Plato's Divided Line and Euclid's Golden Section bear identical ratios. From Plato's exposition, we derive much of the philosophic content.

Let me begin by noting that the Creative Right Triangle of Pythagoras $(3,4,5)$ is itself created by the encompassing environment of three Right Triangles whose perpendiculars bear the relation of 2 to 1 . This type of right triangle will be termed, "the Celestial Right Triangle," occupying threetenths of a Square. See Figure 1.

From this locating a cosmic position for the creative triangle, it becomes immediately inviting to search for properties of the celestial triangle which may lead to many analogies.

We will have more to say about this illustration, which is replete with Fibonacci and Lucas numbers, as we unfold the picture.

The location of the right angular point of the triangle illustrated in Fig1 lies exactly in the following distances from the boundaries of the square: South, 1; East, 2; West, 3; and North, 4 (which is $5 \times 5$ ).

When the area of this square is "one," the area of the creative triangle is three-tenths, and as the cardinal numbers increase, the area of the creative triangle increases in the arithmetical progression of $0.3,0.6,0.9$, etc.

Therefore, the area of the creative triangle reaches identity with cardinal numbers at its intervals in the arithmetical progression of 10 when its area becomes $3,6,9,12$, etc.

When the area of the square is expressed by cardinal numbers squared, the areas of the creative triangles increase by an increasing progression of $4,9,16,25$, etc. The right triangle has sides $3,4,5$ multiplied by $\sqrt{5} / 2$.


Fig. 1 The Celestial Right Triangle
When the areas of the creative triangles reach our squared numbers, the encompassing square is ten-thirds of them; that is, if the area of the creative triangle is 36 , that square is 120. This indicates a series of Cosmic squares in the progression of

| 10 | , | 20 | , | 30 | $53-1 / 3$, | $83-1 / 3$, |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 120 | , | $163-1 / 3$, | $213-1 / 3$, | 270 | $333-1 / 3$, |  |
| $403-1 / 3$, | 480 | , | $563-1 / 3$, | $653-1 / 3$, | 750, | etc. |

From Figure 1, this means that we have another Cosmic series of squares, formed with progressive side lengths of $\sqrt{5}, 2 \sqrt{5}, 3 \sqrt{5}, 4 \sqrt{5}$, etc., when the square area becomes
5,20 , 45 , 80 , 125 , 180 , 245 , 320 , 405 , $500,605,720$, etc. The areas of the creative triangle are three-tenths of them, or, in the series of $6,24,54,96,150,216$, etc., and therefore in the increasing progression of $4,9,16,25$, etc.

In Figure 2, the point of Golden Section is determined at once by taking the difference between the length of the hypotenuse and the length of the greater perpendicular, and adding this difference to the length of one-half of this perpendicular.

Dr. Verner E. Hoggatt, Jr., in his book, Fibonacci and Lucas Numbers, ably explains the same situation by a different method of construction. The square thus divided into these ratio segments, yields in three dimensions, 27 prisms to form the cube. Among them, only four have variant volumes.


[^4]Fig. 2 The point of Golden Section located by Simple Subtraction and Addition

In Figure 3, Plato had a term for the lines EH and GF, which he called, "the Intelligible."

Line Lengths

$$
\begin{aligned}
& \mathrm{AC}=\frac{1}{2 \sqrt{5}+4} \\
& \mathrm{GH}=\frac{1}{\sqrt{5}+2} \\
& \mathrm{AB}=1 \quad \mathrm{EC}=0.5 \\
& \mathrm{CF}=\frac{\sqrt{5}}{2} \\
& \mathrm{EG}=\frac{2}{\sqrt{5}+3} \quad \\
& \mathrm{EH}=\frac{2}{\sqrt{5}+1} \begin{array}{c}
\text { The Golden } \\
\text { Ratio }
\end{array}
\end{aligned}
$$

Areas of Right Triangles CEF and CDF are each 0.25 .
Areas of Golden Rectangles

$$
\begin{array}{ll}
(\mathrm{EG})^{2}=\frac{2}{3 \sqrt{5}+7} & (\mathrm{EH})^{2}=\frac{2}{\sqrt{5}+3} \quad \mathrm{ABFE}=\frac{2}{\sqrt{5}+1} \quad \mathrm{GHHG}=\frac{2}{5 \sqrt{5}+11} \\
\text { GEG }^{\prime} \mathrm{G}=\frac{2}{3 \sqrt{5}+7} & \mathrm{GHH}^{\prime} \mathrm{G}^{\prime}=\frac{1}{\sqrt{5}+2} \quad \text { EEGG }=\frac{2}{\sqrt{5}+3} \quad \mathrm{EEHH}=\frac{2}{\sqrt{5}+1}
\end{array}
$$

Cube of Intelligence:

$$
(\mathrm{EH})^{3}=\frac{1}{\sqrt{5}+1}
$$

Cube of Mathematics

$$
(\mathrm{GH})^{3}=\frac{1}{17 \sqrt{5}+38} \quad(\mathrm{GH})^{2}=\frac{1}{4 \sqrt{5}+9} \quad(\mathrm{EG})^{3}=\frac{1}{4 \sqrt{5}+9}
$$

Fig. 3 Illustrating the Segmented Areas and Volumes

It may be observed in Fig. 3 that the cube of the "intelligible" is composed of eight prisms, four only with variant volumes. Their volumes are:

$$
\begin{array}{rlrl}
(\mathrm{EG})^{3} & =\frac{1}{4 \sqrt{5}+9} & \text { One } \\
(\mathrm{EG})^{2} \times \mathrm{GH} & =\frac{2}{12 \sqrt{5}+29} & & \text { Three } \\
(\mathrm{GH})^{2} \times \mathrm{EG} & =\frac{2}{21 \sqrt{5}+47} & & \text { Three } \\
(\mathrm{GH})^{3} & & =\frac{1}{17 \sqrt{5}+38} &
\end{array}
$$

Plato termed the line GH , "Mathematics."
This cube of the "intelligible" has eight positions upon the cube, with one important feature - that they all share in their construction the center cube, the cube of "mathematics."

These denominators each share numbers appearing in both the Fibonacci and Lucas series.

The sum of these eight prisms, that is, the volume of the cube of the "intelligible," is

$$
\alpha^{-3}=\frac{2}{2 \sqrt{5}+4}=\frac{1}{\sqrt{5}+2}
$$

the difference between ratios.
Editorial Note: If

$$
\alpha=\frac{1+\sqrt{5}}{2}
$$

then

$$
\alpha^{n}=\frac{L_{n}+F_{n} \sqrt{5}}{2}
$$

thus,

$$
\begin{aligned}
&(\mathrm{EG})^{3}=\frac{2}{8 \sqrt{5}+18}=\alpha^{-6}, \frac{2}{13 \sqrt{5}+29}=\alpha^{-7}, \frac{2}{21 \sqrt{5}+47}=\alpha^{-8}, \text { and } \\
&(\mathrm{GH})^{3}=\frac{2}{34 \sqrt{5}+76}=\alpha^{-9}
\end{aligned}
$$

Glancing again at Fig。1, an illustration of the "birth" of the creative triangle, you may have noted that the number 1234 is not divisible by "eleven." See Fig. 4 and accompanying text.

But, suppose we observe these distances in order of rotation. We find they run as follows: 1243, 1342, 2134, 2431, 4213, and 4312, 3124, 3421.

These quaternaries are divisible by 11, and the same holds good for the remaining possible forty. Figure 4 illustrates the potential of 48 out of 144 combinations.

The relative proportional areas in Fig. 5 are: $-1,4,4,6,5$, or 1, 4, 4, 11. By rotating the triangle into its eight possible positions within the square, we obtain 24 points which coincide exactly with the points of intersection of perpendicular and horizontal lines within the square of $60 \times 60$.

By plotting these points, we are provided with the center of the inscribed circle, at $x=0, y=0$; and by bisecting the triangle, we have the point, $x=0, y=15$ shown in Fig. 5.

Upon making a plotted graph for each of the other triangular positions, the plotted values of the triangle are merely a matter of sign and number interchange. The inscribed circle will roll around the circumference of the circumscribed circle $5 \times 0.5$ times. The area of the circumscribed circle contains the area of the smaller circle $2.5 \times 2.5$ times. Area of triangle BDP is 90. The area of triangle EOP is 20 and 5/11ths. Multiplying all areas by 11 to clear denominator, we have a total area of $\mathrm{xx} \mathrm{c} 1080=11,880$. Therefore, the 3,600 square units each enjoy an area of eleven.

Such circumferences increase in the arithmetical progression of $3.6 \sqrt{5} \times \pi$ 。

| Coefficients of $\sqrt{5}$ |  | Times Increased |
| :---: | :---: | :---: |
| 11.30976 | 0.5 |  |
| 22.61952 |  | 1.0 |
| 33.92928 |  | 1.5 |
| 45.23904 |  | 2.0 |
| 56.5488 |  | 2.5 |
| 67.85856 |  | 3.0 |



This illustration gives the numerical evidence at a glance. Any four numbers taken in their rotary position of sequence within the same circle are divisible by eleven. These 48 numbers, out of a total of 144 possible combinations are unique in their divisibility by eleven. The remaining 96 are not exactly divisible by eleven.

This chart is not intended to bear any idea of magic, but it does reveal analogies to known Law. The causation of this law is shown in the previous pages. Philosophic research is just as rewarding as scientific research. "Ominia numeris sita sunt." (All things lie veiled in numbers.)

Fig. 4 The Number "Eleven" Chart


Fig. 5 The Number "Eleven" Chart

| Coefficients of $\sqrt{5}$ | Inscribed Circumference |
| :---: | :---: |
| 37.6992 | $12 \times \sqrt{5} \times \pi$ |
|  | Increase |
| 56.5488 | $\overline{18 \times \sqrt{5} \times \pi}$ |
|  | Circumscribed Circumference |
| 94.248 | $30 \times \sqrt{5} \times \pi$ |
| $\text { tors: } \begin{array}{rlllll}  & 4 & \times & 6 & x & 11 \\ & = & 94.248 \end{array}$ | $\text { x } 21 \times 0.001$ |

The sums and products of these quantities yield two quantities in both numerators and denominators.

Both Fibonacci and Lucas numbers are represented in these numerators and denominators.

This indicates the home of spirals making the dynamic symmetry within the Golden Rectangles.


ERRATA
Please make the following corrections in "Sums Involving Fibonacci Numbers," Vol. 7, No. 1, pp. 92-98:

The first half of Eq. (3), line 3, page 95, should read as follows:

$$
\sum_{r=0}^{n} v_{r}(p, q)=2+\frac{p T_{n}-2 q T_{n-1}}{1-p+q}
$$

Please make the following corrections in "Identities Involving Generalized Fibonacci Numbers," Vol. 7, No. 1, pp. 66-72:
Page 67 - Please correct line 5 to read:
It is also easy to see that $\mathrm{H}_{\mathrm{n}}=\mathrm{pF}_{\mathrm{n}}+\mathrm{qF}_{\mathrm{n}-1}$ where $\mathrm{F}_{\mathrm{n}}$ is the $\mathrm{n}^{\text {th }}$ Fibonacci Page 69 - Please change the last part of the last sentence of page to read:. . for the Fibonacci numbers we get in the generalized Fibonacci numbers the identity: ...
Page 71 - Please correct Eq. (26) to read as follows:

$$
\sum_{\mathrm{r}=1}^{\mathrm{n}} \mathrm{H}_{2 \mathrm{r}-1}^{3}=\frac{1}{4}\left[\left(\mathrm{H}_{2 \mathrm{n}}^{3}-\mathrm{q}^{3}\right)+3 \mathrm{e}\left(\mathrm{H}_{2 \mathrm{n}}-\mathrm{q}\right)\right] .
$$

# CERTAIN ARITHMETICAL PROPERTIES OF $1 / 2 k(a k \pm 1)$ 

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Define

$$
A_{u}=\frac{1}{2} u(a u-1) \quad \text { and } \quad B_{u}=\frac{1}{2} u(a u+1)
$$

where $a \neq 0$ is a positive rational integer.
In this paper, we discuss

$$
A_{u}+A_{v}=A_{k}, \quad B_{u}+B_{v}=B_{k}, \quad A_{u} A_{v}=A_{k} \quad \text { and } \quad B_{u} B_{v}=B_{k}
$$

where the suffixes $(u, v, k)$ are positive rational integers. In particular, we shall, for the first time, finally settle the question, and prove that if one solution of

$$
\frac{1}{2} \mathrm{u}(\mathrm{au} \pm 1) \frac{1}{2} \mathrm{v}(\mathrm{v} \pm 1)=\frac{1}{2} \mathrm{k}(\mathrm{k} \pm 1)
$$

exists for integral $u, v, k$, then an infinite number of other such solutions also exist.

Theorem 1. If a is an odd integer, then the suffixes are integers in

$$
\begin{equation*}
A_{\frac{1}{2}}\left(a^{2} q^{2}+(2 a-1) q+2\right)=A_{a q+1}+A_{\frac{1}{2}}\left(a^{2} q^{2}+2 a q-q\right) \tag{2}
\end{equation*}
$$

and

$$
\mathrm{B}_{\frac{1}{2}}\left(\mathrm{a}^{2} \mathrm{q}^{2}+(2 \mathrm{a}+1) \mathrm{q}+2\right)=\mathrm{B}_{\frac{1}{2}}\left(\mathrm{a}^{2} \mathrm{q}^{2}+2 a q+q\right)+\mathrm{B}_{a q+1}
$$

where $\mathrm{q}=0,1,2, \cdots$ 。
Proof. The proof is immediate, using elementary algebra to show identities.

Theorem 2. If

$$
\begin{equation*}
\mathrm{n}=\frac{1}{2}\left(\mathrm{a}^{4} \mathrm{q}^{2}+(2 a-1) a q+2\right), \quad m=a^{2} q+1 \tag{3}
\end{equation*}
$$

and

$$
\mathrm{w}=\left(\mathrm{a}\left(\mathrm{n}^{2}+1\right)-(\mathrm{n}-1)\right) / 2 \mathrm{a},
$$

then

$$
A_{n}=A_{n-1}+A_{m}=A_{w}-A_{w-1}, \quad \text { (with } q=0,1,2, \cdots \text { ) }
$$

where the suffixes are integers when a is an odd integer.
Proof. In (2), we replace $q$ with $a q$ and then solve for $\underline{w}$ in

$$
A_{n}=\frac{1}{2} n(n-1)=\frac{1}{2}(2 a w-a-1)=A_{w}-A_{w-1} .
$$

We complete the proof by observing that $\underline{w}$ and $\underline{n}$ are integers when $\underline{a}$ is odd.

In the same way we got (3), we get the following:
Corollary. If
$\mathrm{n}=\frac{1}{2}\left(\mathrm{a}^{4} \mathrm{q}^{2}+(2 \mathrm{a}+1) \mathrm{aq}+2\right), \quad \mathrm{m}=\mathrm{a}^{2} \mathrm{q}+1, \quad$ and $\mathrm{w}=\left(\mathrm{a}\left(\mathrm{n}^{2}+1\right)+\mathrm{n}-1\right) / 2 \mathrm{a}$, then

$$
\mathrm{B}_{\mathrm{n}}=\mathrm{B}_{\mathrm{n}-1}+\mathrm{B}_{\mathrm{m}}=\mathrm{B}_{\mathrm{w}}-\mathrm{B}_{\mathrm{w}-1} \text {, (with } \mathrm{q}=0,1,2, \cdots \text { ) }
$$

where the suffixes are integers when $\underline{a}$ is an odd integer.
Remark. It should be noted that R. T. Hansen, in a recent paper [1], found solutions for the special case when $\mathrm{a}=3$ in (2), for the A sum, and in (3).

We now discuss the paired products in the following:

$$
A_{u} A_{v}=A_{k} \quad \text { and } \quad B_{u} B_{v}=B_{k}
$$

for integer suffixes ( $u, v, k$ ).
Theorem 3. If $\underline{a}$ is an odd integer, the Pell equation,

$$
\begin{equation*}
K^{2}=8 a p^{2}+8 a+1 \tag{4}
\end{equation*}
$$

is solvable in rational integers, and $K+1$ and $2 \mathrm{p}^{2}+1 \equiv 0(\bmod a)$, then in $A_{v} A_{u}=A_{k}$, the suffixes ( $u, v, k$ ) are the following integers

$$
\begin{gather*}
\mathrm{k}=\left(2 \mathrm{p}^{2}+1\right)\left(2 \mathrm{p}^{3}+2 \mathrm{p}^{2}+2 \mathrm{p}+1\right) / \mathrm{a} \\
\mathrm{u}=\left(2 \mathrm{p}^{2}+1\right)\left(2 \mathrm{p}^{2}+2 \mathrm{p}+1\right) / \mathrm{a} \tag{4.1}
\end{gather*}
$$

and

$$
\mathrm{v}=(\mathrm{K}+1) / 2 \mathrm{a}=\left(1+\left(8 \mathrm{ap}^{2}+8 \mathrm{a}+1\right)^{\frac{1}{2}}\right) / 2 \mathrm{a}
$$

Proof. It is evident (by elementary means) that the identities in (4.1) balance the equation $A_{v} A_{u}=A_{k}$. We complete the proof by noting that the congruences are self-evident in (4.1).

Theorem 4. If $\underline{a}$ is an odd integer, the Pell equation

$$
\begin{equation*}
K=8 a p^{2}+8 a+1 \tag{5}
\end{equation*}
$$

is solvable in rational integers, and $K-1$ and $4 p^{2}+3 \equiv 0(\bmod a)$, then in $B_{u} B_{v}=B_{k}$, the suffixes ( $u, v, k$ ) are the following integers

$$
\begin{gather*}
\mathrm{k}=\left(4 \mathrm{p}^{2}+3\right)\left(4 \mathrm{p}^{5}+4 \mathrm{p}^{4}+5 \mathrm{p}^{3}+3 \mathrm{p}^{2}+\mathrm{p}\right) / \mathrm{a} \\
\mathrm{u}=\left(4 \mathrm{p}^{2}+3\right)\left(4 \mathrm{p}^{4}+4 \mathrm{p}^{3}+3 \mathrm{p}^{2}+\mathrm{p}\right) / \mathrm{a} \tag{5.1}
\end{gather*}
$$

and

$$
\mathrm{v}=(\mathrm{K}-1) / 2 \mathrm{a}=\left(1+\left(8 \mathrm{ap}^{2}+8 \mathrm{a}+1\right)^{\frac{1}{2}}\right) / 2 \mathrm{a}
$$

Proof. It is evident (by elementary means) that the identities in (5.1) balance the equation $B_{u} B_{v}=B_{k}$. The congruences in (5.1) of a are selfevident.

Euler [2] proved that if

$$
\mathrm{y}^{2}-\mathrm{Ax}^{2}=\mathrm{B}
$$

is solvable in integers, its solution reduces to the integration of the equation

$$
y_{t+2}-2 m y_{t+1}=0
$$

in finite differences, the integral being

$$
\mathrm{y}=(\mathrm{r}+\mathrm{s}) / 2, \quad \mathrm{x}=(\mathrm{r}-\mathrm{s}) /\left(2(\mathrm{~A})^{\frac{1}{2}}\right)
$$

where

$$
\mathrm{r}=\left(\mathrm{Y}+\mathrm{X}(\mathrm{~A})^{\frac{1}{2}}\right)\left(\mathrm{m}+\mathrm{n}(\mathrm{~A})^{\frac{1}{2}}\right)^{\mathrm{z}-1}, \quad \mathrm{~s}=\left(\mathrm{Y}-\mathrm{X}(\mathrm{~A})^{\frac{1}{2}}\right)\left(\mathrm{m}-\mathrm{n}(\mathrm{~A})^{\frac{1}{2}}\right)^{\mathrm{z}-1}
$$

$\mathrm{Y}, \mathrm{X}$ being the least integral solutions of $\mathrm{Y}^{2}-\mathrm{AX} X^{2}=B$, and $m, n$ being the least integral solutions of $\mathrm{m}^{2}-\mathrm{An}^{2}=1$. This is Euler's theorem in changed notation.

Theorem 5. If a is an odd prime,

$$
\begin{equation*}
\mathrm{K}_{\mathrm{t}}, \mathrm{P}_{\mathrm{t}} \quad(\mathrm{t}=1,2,3, \cdots) \tag{7}
\end{equation*}
$$

are integer solutions (where $K_{1}, P_{1}$ are the least integer solutions) of

$$
\phi(t)=K_{t}^{2}=8 a P_{t}^{2}+8 a+1
$$

and if there exists a $K_{j}$ and a $P_{j}$ which are the least integer solutions of $\phi(t)$ such that $K_{j}+1 \equiv 0(\bmod a)$ and $2 P_{j}^{2} \equiv-1(\bmod a)$, then the number of solutions of $A_{u} A_{v}=A_{k}$ are infinite for the following integer suffixes ( $u, v, k$ ):

$$
\begin{gather*}
k=\left(2 P_{i}^{2}+1\right)\left(2 P_{i}^{3}+2 P_{i}^{2}+2 P_{i}+1\right) / a \\
u=\left(2 P_{i}^{2}+1\right)\left(2 P_{i}^{2}+2 P_{i}+1\right) / a \tag{7.1}
\end{gather*}
$$

and

$$
\mathrm{v}=\left(\mathrm{K}_{\mathrm{i}}+1\right) / 2 \mathrm{a}=\left(1+\left(8 \mathrm{aP}_{\mathrm{i}}^{2}+8 \mathrm{a}+1\right)^{\frac{1}{2}}\right) / 2 \mathrm{a}
$$

where $i=j+w(a-1) a \quad(w=0,1,2, \cdots)$.
Proof. Since $K_{j}, P_{j}$ are integers, $K_{j}+1 \equiv 0(\bmod a)$ and $2 P_{j}^{2}+1=$ $0(\bmod a)$, then combining (7.1) with (4.1), it is evident that the $u, v, k$ in (7.1) are integers.

Now, combining (6) with the equation $\phi(\mathrm{j})$ in (7), we write

$$
\left(K_{j}+P_{j}(8 a)^{\frac{1}{2}}\right)\left(m+n(8 a)^{\frac{1}{2}}\right)^{w a(a-1)}=K_{w a(a-1)+j}+P_{w a(a-1)+j}(8 a)^{\frac{1}{2}}
$$

$$
\begin{equation*}
\left(K_{j}-P_{j}(8 a)^{\frac{1}{2}}\right)\left(m-n(8 a)^{\frac{1}{2}}\right)^{w a(a-1)}=K_{w a(a-1)+j}-P_{w a(a-1)+j}(8 a)^{\frac{1}{2}} \tag{8}
\end{equation*}
$$

where $\underline{\mathrm{a}}$ is an odd prime and $\mathrm{w}=0,1,2, \cdots$.
In (6), it is evident that $(\mathrm{m}, \mathrm{a})=1$, and since $\underline{a}$ is an odd prime, we have, by Fermat's familiar theorem ( m , a are integers with $\underline{a}$ an odd prime, $(\mathrm{m}, \mathrm{n})=1$, then $\left.\mathrm{m}^{\mathrm{a}-1} \equiv 1(\bmod \mathrm{a})\right)$

$$
\left(\mathrm{m} \pm \mathrm{n}(8 \mathrm{a})^{\frac{1}{2}}\right)^{\mathrm{wa}(\mathrm{a}-1)} \equiv 1(\bmod \mathrm{a})
$$

which leads to (in (8)),

$$
K_{j} \equiv K_{w a(a-1)+j}(\bmod a) \quad \text { and } \quad P_{j} \equiv P_{w a(a-1)+j}
$$

and we complete the proof by noting that these congruences satisfy the conditions of Theorem 5.

Corollary 1. In (7), it is almost immediate that

$$
\begin{equation*}
1 \leq j \leq a(a-1) \tag{9}
\end{equation*}
$$

Since, if $j=s a(a-1)+d$ (where $1=d=a(a-1)$ and $a=0,1,2, \cdots)$, it is evident that

$$
\mathrm{K}_{\mathrm{wa}(\mathrm{a}-1)+\mathrm{j}}=\mathrm{K}_{\mathrm{a}(\mathrm{a}-1)(\mathrm{w}+\mathrm{s})+\mathrm{d}} \equiv \mathrm{~K}_{\mathrm{d}}(\bmod \mathrm{a})
$$

and

$$
P_{\mathrm{wa}(\mathrm{a}-1)+\mathrm{j}}=\mathrm{P}_{\mathrm{a}(\mathrm{a}-1)(\mathrm{w}+\mathrm{s})+\mathrm{d}} \equiv \mathrm{P}_{\mathrm{d}}(\bmod \mathrm{a})
$$

Corollary 2. If $\underline{a}$ is an odd prime, and

$$
K_{j}-1 \equiv 0 \quad(\bmod a) \quad \text { and } \quad 4 P_{j}^{2}+3 \equiv 0(\bmod a)
$$

then the number of solutions of $B_{u} B_{v}=B_{k}$ are infinite for the following integer suffixes ( $u, v, k$ ):

$$
\begin{gathered}
\mathrm{k}=\left(4 \mathrm{P}_{1}^{2}+3\right)\left(4 \mathrm{P}_{1}^{5}+4 \mathrm{P}_{1}^{4}+5 \mathrm{P}_{1}^{3}+3 \mathrm{P}_{1}^{2}+\mathrm{P}_{1}\right) / \mathrm{a} \\
\mathrm{u}=\left(4 \mathrm{P}_{1}^{2}+3\right)\left(4 \mathrm{P}_{1}^{4}+4 \mathrm{P}_{1}^{3}+3 \mathrm{P}_{1}^{2}+\mathrm{P}_{1}\right) / \mathrm{a}
\end{gathered}
$$

and

$$
\mathrm{v}=\left(\mathrm{K}_{\mathrm{j}}-1\right) / 2 \mathrm{a}=\left(1+\left(8 \mathrm{aP} P_{1}^{2}+8 \mathrm{a}+1\right)^{\frac{1}{2}}\right) / 2 \mathrm{a}
$$

where

$$
i=j+w a(a-1) \quad(w=0,1,2, \cdots)
$$

and

$$
1 \leq j \leq a(a-1)
$$

We shall give one application in pentagonal numbers for infinite paired products in (7-7.1).

In (7-7.1), let $a=3$, then

$$
\mathrm{K}^{2}=24 \mathrm{P}^{2}+25 \quad \text { and } \quad \mathrm{m}^{2}=24 \mathrm{n}^{2}+1
$$

where the first solutions are

$$
\mathrm{K}_{1}=7, \quad \mathrm{P}_{1}=1, \quad \text { and } \quad \mathrm{m}=5, \quad \mathrm{n}=1
$$

We then find that $j=4$ and 6 , so that $i=6 w+4$ and $6 \mathrm{w}+6$, and we write $\left(7+(24)^{\frac{1}{2}}\right)\left(5+(24)^{\frac{1}{2}}\right)^{6 \mathrm{w}+3}=\left(\mathrm{K}_{4}+\mathrm{P}_{4}(24)^{\frac{1}{2}}\right)\left(5+(24)^{\frac{1}{2}}\right)^{6 \mathrm{w}}=\mathrm{K}_{6 \mathrm{w}+4}+\mathrm{P}_{6 \mathrm{w}+4}(24)^{\frac{1}{2}}=\mathrm{r}$,
and
$\left(7-(24)^{\frac{1}{2}}\right)\left(5-(24)^{\frac{1}{2}}\right)^{6 \mathrm{w}+3}=\left(\mathrm{K}_{4}-\mathrm{P}_{4}(24)^{\frac{1}{2}}\right)\left(5-(24)^{\frac{1}{2}}\right)^{6 \mathrm{w}}=\mathrm{K}_{6 \mathrm{w}+4}-\mathrm{P}_{6 \mathrm{w}+4}(24)^{\frac{1}{2}}=\mathrm{s}$,
so that $(\mathrm{r}+\mathrm{s}) / 2=\mathrm{K}_{6 \mathrm{w}+4}$ and $(\mathrm{r}-\mathrm{s}) / 2=\mathrm{P}_{6 \mathrm{w}+4}$, In the same way, we find $(r+s) / 2=K_{6 w+6}$ and $(r-s) / 2=P_{6 w+6}$. Then combining these results with (6) and (7.1), we conclude our application.

## REFERENCES

1. R. T. Hansen, "Arithmetic of Pentagonal Numbers," Fibonacci Quarterly, Vol. 8, No. 2 (1970), pp. 83-87.
2. L. Euler, Comm. Arith. Coll., I, pp. 316-336.

[Continued from page 475.]
4). Hence, if odd prime $p$ divides $F_{2 k-1}$, then $p$ is not of the form $4 s+$ 3, thus proving Conjecture 2 of Dmitri Thoro.* The proof by Leonard Weinstein** came to my attention at a later time and is distinct from the above proof.
*Dmitri Thoro, "Two Fibonacci Conjectures," Fibonacci Quarterly, Oct.
1965, pp. 184-186.
** Leonard Weinstein, "Letter to the Editor," Fibonacci Quarterly, Feb.
1966, p. 88.

## ERRATA

Please make the following corrections in 'Some Results on Fibonacci Quaternions," Vol. 7, No. 2, pp. 201-210.
Page 201 - The first displayed equation on the page should read:

$$
\mathrm{i}^{2}=\mathrm{j}^{2}=\mathrm{k}^{2}=-1, \quad \mathrm{ij}=-\mathrm{ji}=\mathrm{k} ; \quad \mathrm{jk}=-\mathrm{kj}=\mathrm{i} ; \quad \mathrm{ki}=-\mathrm{ik}=\mathrm{j} .
$$

Page 205 - Change the bracketed part of Eq. (27) to read:

$$
\left[F_{r}^{2} T_{0}+F_{2 r}\left(Q_{0}-3 k\right)\right]
$$

Page 208 - Change the first terms of Eq. (74) to read:

$$
\mathrm{T}_{\mathrm{n}+\mathrm{t}^{\mathrm{F}} \mathrm{n}+\mathrm{r}}=\cdots
$$

# FIBONACCI NUMBERS <br> AS PATHS OF A ROOK ON A CHESSBOARD <br> EDWARD T. FRANKEL <br> Schenectady, New York 

The purpose of this article is to show that Fibonacci numbers can be derived by enumerating the number of different routes of a rook from one corner of a chessboard to the opposite corner when the moves of the rook are limited by restrictive fences.

Consider the chessboard array of binomial coefficients or figurate numbers in Fig. 1. It is well known that the number in any square or cell represents the number of different routes of a rook from the upper left corner to that cell, provided that the rook moves are either horizontal to the right or vertically downward。*

| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 |
| 1 | 4 | 10 | 20 | 35 | 56 | 84 | 120 |
| 1 | 5 | 15 | 35 | 70 | 126 | 210 | 330 |
| 1 | 6 | 21 | 56 | 126 | 252 | 462 | 792 |
| 1 | 7 | 28 | 84 | 210 | 462 | 924 | 1716 |
| 1 | 8 | 36 | 120 | 330 | 792 | 1716 | 3432 |

Fig. 1. Number of Rook Paths from Corner of Chessboard

Figure 2 shows the same chessboard array in standard combinatorial notation:

$$
\binom{h}{k}=h!/ k!(h-k)!=\binom{h}{h-k} .
$$

[^5]| $\binom{0}{0}$ | $\binom{1}{0}$ | $\binom{2}{0}$ | $\binom{3}{0}$ | $\binom{4}{0}$ | $\binom{5}{0}$ | $\binom{6}{0}$ | $\binom{7}{0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\binom{1}{1}$ | $\binom{2}{1}$ | $\binom{3}{1}$ | $\binom{4}{1}$ | $\binom{5}{1}$ | $\binom{6}{1}$ | $\binom{7}{1}$ | $\binom{8}{1}$ |
| $\binom{2}{2}$ | $\binom{3}{2}$ | $\binom{4}{2}$ | $\binom{5}{2}$ | $\binom{6}{2}$ | $\binom{7}{2}$ | $\binom{8}{2}$ | $\binom{9}{2}$ |
| $\binom{3}{3}$ | $\binom{4}{3}$ | $\binom{5}{3}$ | $\binom{6}{3}$ | $\binom{7}{3}$ | $\binom{8}{3}$ | $\binom{9}{3}$ | $\binom{10}{3}$ |
| $\binom{4}{4}$ | $\binom{5}{4}$ | $\binom{6}{4}$ | $\binom{7}{4}$ | $\binom{8}{4}$ | $\binom{9}{4}$ | $\binom{10}{4}$ | $\binom{11}{4}$ |
| $\binom{5}{5}$ | $\binom{6}{5}$ | $\binom{7}{5}$ | $\binom{8}{5}$ | $\binom{9}{5}$ | $\binom{10}{5}$ | $\binom{11}{5}$ | $\binom{12}{5}$ |
| $\binom{6}{6}$ | $\binom{7}{6}$ | $\binom{8}{6}$ | $\binom{9}{6}$ | $\binom{10}{6}$ | $\binom{11}{6}$ | $\binom{12}{6}$ | $\binom{13}{7}$ |
| $\binom{7}{7}$ | $\binom{8}{7}$ | $\binom{9}{7}$ | $\binom{10}{7}$ | $\binom{11}{7}$ | $\binom{12}{7}$ | $\binom{13}{7}$ | $\binom{14}{7}$ |

Fig. 2. Rook Paths in Combinatorial Notation
Figure 3 shows a chessboard array where the moves of a rook are limited by the indicated pattern of horizontal and vertical restrictive fences.

| 1 | 1 | 1 |  |  |  |  |  |
| ---: | ---: | :--- | ---: | ---: | ---: | ---: | ---: |
| 1 | 2 | 3 | 3 |  |  |  |  |
|  | 2 | 5 | 8 | 8 |  |  |  |
|  |  | 5 | 13 | 21 | 21 |  |  |
|  |  |  | 13 | 34 | 55 | 55 |  |
|  |  |  |  | 34 | 89 | 144 | 144 |
|  |  |  |  |  | 89 | 233 | 377 |
|  |  |  |  |  |  | 233 | 610 |

Fig. 3 Rook Paths Limited by Restrictive Fences
The array begins with number one in the top left corner. Inasmuch as the number in any cell is the sum of the numbers immediately above it and to the left of it, the pattern of restrictive fences results in the entire array being composed of Fibonacci numbers.

Figure 4 shows the chessboard with the same pattern of fences as in Fig. 3, but with the numbers in the Fibonacci notation where $F_{0}=F_{1}=1$; $\mathrm{F}_{2}=2 ; \mathrm{F}_{3}=3$; and, in general, $\mathrm{F}_{\mathrm{n}+2}=\mathrm{F}_{\mathrm{n}+1}+\mathrm{F}_{\mathrm{n}}$ 。

| $\mathrm{F}_{0}$ | $\mathrm{~F}_{1}$ | $\mathrm{~F}_{1}$ |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{~F}_{0}$ | $\mathrm{~F}_{2}$ | $\mathrm{~F}_{3}$ | $\mathrm{~F}_{3}$ |  |  |  |  |
|  | $\mathrm{~F}_{2}$ | $\mathrm{~F}_{4}$ | $\mathrm{~F}_{5}$ | $\mathrm{~F}_{5}$ |  |  |  |
|  |  | $\mathrm{~F}_{4}$ | $\mathrm{~F}_{6}$ | $\mathrm{~F}_{7}$ | $\mathrm{~F}_{7}$ |  |  |
|  |  |  | $\mathrm{~F}_{6}$ | $\mathrm{~F}_{8}$ | $\mathrm{~F}_{9}$ | $\mathrm{~F}_{9}$ |  |
|  |  |  |  | $\mathrm{~F}_{8}$ | $\mathrm{~F}_{10}$ | $\mathrm{~F}_{11}$ | $\mathrm{~F}_{11}$ |
|  |  |  |  |  | $\mathrm{~F}_{10}$ | $\mathrm{~F}_{12}$ | $\mathrm{~F}_{13}$ |
|  |  |  |  |  |  | $\mathrm{~F}_{12}$ | $\mathrm{~F}_{14}$ |

Fig. 4. Limited Rook Paths in Fibonacci Notation
Comparing Fig. 4 with Fig. 2, it is noted that $F_{0}$ corresponds to $\binom{0}{0}$ and $\mathrm{F}_{14}$ corresponds to $\binom{14}{7}$. Comparing Fig. 3 with Fig. 1, we see that the number of Fibonacci rook paths from corner to corner is 610, whereas the number of unrestricted paths is 3432. The difference of 2822 must be the number of routes which are eliminated because of the restrictive fences. This can be verified by tabulating the effect of each restrictive fence as in the analysis on the following page.

To generalize, in a chessboard of $(\mathrm{n}+1)^{2}$ cells, the number of unrestricted rook paths from corner to corner is $\binom{2 n}{n}$, the number of Fibonacci rook paths is $F_{2 n}$, and the number of paths that are eliminated by the pattern of horizontal and vertical fences is

$$
\begin{aligned}
\binom{2 n}{n}-F_{2 n} & =F_{0}\binom{2 n-2}{n}+F_{1}\binom{2 n-3}{n} \\
& +F_{2}\binom{2 n-4}{n-1}+F_{3}\binom{2 n-5}{n-1} \\
& +\cdots+\cdots \\
& +F_{2 n-6}\binom{4}{3}+F_{2 n-5}\binom{3}{3} \\
& +F_{2 n-4}\binom{2}{2} .
\end{aligned}
$$

## ANALYSIS OF ELIMINATED ROOK PATHS

A
Number of Fibonacci paths
from origin to cells with
fences

Cells with horizontal
fences

$$
\begin{aligned}
& \mathrm{F}_{0}=1 \\
& \mathrm{~F}_{2}=2
\end{aligned}
$$

$\left.\begin{array}{ll}\binom{12}{7} & =792 \\ \binom{10}{6} & =210\end{array}\right) 792$

Subtotal

B
Number of unrestricted rook paths from fence to lower right corner

Cells with horizontal

Subtotal

Cells with vertical fences
$A \times B$
Number of paths eliminated by fences

1912

$\mathrm{F}_{1}=1$
$\mathrm{~F}_{3}=3$
$\mathrm{~F}_{5}=8$
$\mathrm{~F}_{7}=21$
$\mathrm{~F}_{9}=55$
$\binom{11}{7}=792 \quad 330$
$\binom{9}{6}=84 \quad 262$
$\binom{7}{5}=21 \quad 168$
${ }_{9}=55$

| $55 \quad\binom{3}{3}=$ |  | 55 |
| :---: | :---: | :---: |
|  | Subtotal | 910 |
| Total number of eliminated paths |  | 2822 |
| Total number of Fibonacci paths, |  | 610 |
| Total number of unrestricted path | $\binom{14}{7}$ | 3432 |

# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

## B-196 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let $a_{0}, a_{1}, a_{2}, \cdots$, and $b_{0}, b_{1}, b_{2}, \cdots$ be two sequences such that

$$
b_{n}=\binom{n}{0} a_{n}+\binom{n}{1} a_{n-1}+\binom{n}{2} a_{n-2}+\cdots+\binom{n}{n} a_{0} \quad n=0,1,2, \cdots
$$

Give the formula for $a_{n}$ in terms of $b_{n}, \cdots, b_{0} \cdot$

B-197 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let the Pell Sequence be defined by $P_{0}=0, P_{1}=1$, and $P_{n+2}=$ $2 P_{n+1}+P_{n}$. Show that there is a sequence $Q_{n}$ such that

$$
P_{n+2 k}=Q_{k} P_{n+k}-(-1)^{k} P_{n}
$$

and give initial conditions and the recursion formula for $Q_{n}$.

Let $c_{n}$ be the coefficient of $x_{1} x_{2} \cdots x_{n}$ in the expansion of

$$
\begin{array}{r}
\left(-x_{1}+x_{2}+x_{3}+\cdots+x_{n}\right)\left(x_{1}-x_{2}+x_{3}+\cdots+x_{n}\right)\left(x_{1}+x_{2}-x_{3}+\cdots+x_{n}\right) \cdots \\
\\
\left(x_{1}+x_{2}+x_{3}+\cdots+x_{n-1}-x_{n}\right) .
\end{array}
$$

For example, $c_{1}=-1, c_{2}=2, c_{3}=-2, c_{4}=8$, and $c_{5}=8$. Show that

$$
c_{n+2}=n c_{n+1}+2(n+1) c_{n}, \quad c_{n}=n c_{n-1}+(-2)^{n}
$$

and

$$
\lim _{n \rightarrow \infty}\left(c_{n} / n!\right)=e^{-2}
$$

B-199 Proposed by M. J. DeLeon, Florida Atlantic University, Boca Raton, Florida.

Define the Fibonacci and Pell numbers by

$$
\begin{array}{lll}
F_{1}=1, & F_{2}=1, \quad F_{n+2}=F_{n+1}+F_{n} & n \geq 1 \\
P_{1}=1, & P_{2}=2, \quad P_{n+2}=2 P_{n+1}+P_{n} & n \geq 1
\end{array}
$$

Prove or disprove that $\mathrm{P}_{6 \mathrm{k}}<\mathrm{F}_{11 \mathrm{k}}$ for $\mathrm{k} \geq 1$.

B-200 Proposed by M. J. DeLeon, Florida Atlantic University, Boca Raton, Florida.

With the notation of $\mathrm{B}-199$, prove or disprove that

$$
\mathrm{F}_{11 \mathrm{k}}<\mathrm{P}_{6 \mathrm{k}+1} \quad \text { for } \mathrm{k} \geq 1
$$

B-201 Proposed by Mel Most, Ridgefield Park, New Jersey.
Given that a very large positive integer $k$ is a term $F_{n}$ in the Fibonacci Sequence, describe an operation on $k$ that will indicate whether $n$ is even or odd.

## SOLUTIONS

## DOUBLING NEED NOT BE TROUBLING

For all positive integers $n$ show that

$$
\mathrm{F}_{2 \mathrm{n}+2}=\sum_{\mathrm{i}=1}^{\mathrm{n}} 2^{\mathrm{n}-\mathrm{i}} \mathrm{~F}_{2 \mathrm{i}-1}+\mathrm{z}^{\mathrm{n}},
$$

and

$$
F_{2 n+3}=\sum_{i=1}^{n} 2^{n-i} F_{2 i}+2^{n+1}
$$

Generalize.

Solution by Herta T. Freitag, Hollins, Virginia.

Our generalization states that for all positive integers n and for all positive integers a
(1)

$$
\sum_{i=1}^{n} 2^{n-1} F_{2 i+(a-3)}+2^{n} F_{n}=F_{2 n+a}
$$

The proof is by mathematical induction on $n$.
Relationship (1), for $n=1$, claims that

$$
\mathrm{F}_{\mathrm{a}-1}+2 \mathrm{~F}_{\mathrm{a}}=\mathrm{F}_{\mathrm{a}+2}
$$

which is, indeed, the case
Assuming that (1) holds for some positive integer, say $k$, we have:
(2)

$$
\sum_{i=1}^{k} 2^{k-i} F_{2 i+(a-3)}+2^{k} F_{a}=F_{2 k+a}
$$

To see if

$$
\begin{equation*}
\sum_{i=1}^{k+1} 2^{k+1-i} F_{2 i+(a-3)}+2^{k+1} F_{n}=F_{2 k+a+2} \tag{3}
\end{equation*}
$$

we recognize-on the basis of our assumption (2)-that the left side of (3) equals

$$
2 \mathrm{~F}_{2 \mathrm{k}+\mathrm{a}}+\mathrm{F}_{2 \mathrm{k}+\mathrm{a}-1}
$$

which, however, is seen to be the number $\mathrm{F}_{2 \mathrm{k}+\mathrm{a}}+\mathrm{F}_{2 \mathrm{k}+\mathrm{a}+1}$ and, hence, $\mathrm{F}_{2 \mathrm{k}+\mathrm{a}+2}{ }^{\text {. }}$

This completes our proof by the principle of mathematical induction.
The relationships stated in the problem now become special ases of our generalization (1), whereby $\mathrm{a}=2$ establishes the first, and $\mathrm{a}=3$ the second, of the two given formulas.

Also solved by C. B. A. Peck, A. G. Shannon (T.P.N.G.), and the Proposer.

## A SURJECTION (NOT MONOTONIC)

B-179 Based on Douglas Lind's Problem B-165.
Let $Z^{+}$consist of the positive integers and let the function $b$ from $\mathrm{Z}^{+}$to $\mathrm{Z}^{+}$be defined by $\mathrm{b}(1)=\mathrm{b}(2)=1, \quad \mathrm{~b}(2 \mathrm{k})=\mathrm{b}(\mathrm{k})$, and $\mathrm{b}(2 \mathrm{k}+1)=$ $b(k+1)+b(k)$ for $k=1,2, \cdots$. Show that every positive integer $m$ is a value of $b(n)$ and that $b(n+1) \geq b(n)$ for all positive integers $n$.

## Solution

Put

$$
B(x)=\sum_{k=1}^{\infty} \mathrm{b}(\mathrm{k}) \mathrm{x}^{\mathrm{k}-1}
$$

Then

$$
\begin{aligned}
B(x) & =\sum_{k=1}^{\infty} b(2 k) x^{2 k-1}+\sum_{k=1}^{\infty} b(2 k-1) x^{2 k-2} \\
& =\sum_{k=1}^{\infty} b(k) x^{2 k-1}+1+\sum_{k=1}^{\infty}[b(k)+b(k+1)] x^{2 k} \\
& =x B\left(x^{2}\right)+x^{2} B\left(x^{2}\right)+\sum_{k=0}^{\infty} b(k+1) x^{2 k}
\end{aligned}
$$

so that

$$
\mathrm{B}(\mathrm{x})=\left(1+\mathrm{x}+\mathrm{x}^{2}\right) \mathrm{B}\left(\mathrm{x}^{2}\right)
$$

It follows that

$$
B(x)=\prod_{n=0}^{\infty}\left(1+x^{2^{n}}+x^{2^{n+1}}\right)
$$

It is evident from this generating function that every positive integer m is a value of $b(n)$. However, the statement $b(n+1) \geq b(n)$ is false:

$$
b(2 k+2)=b(k+1)<b(k+1)+b(k)=b(2 k+1)
$$

For additional properties of $b(k)$, see: "A Problem in Partitions Related to the Stirling Numbers," Bull. Amer. Math. Soc., Vol. 70 (1964), pp. 275-278; also: D. A. Lind, "An Extension of Stern's Diatomic Series," Duke Math. Journal, Vol. 36 (1969), pp. 53-60.

Outline
Every positive integer $m$ is a value of $b(n)$ since

$$
\mathrm{m}=\mathrm{b}\left(2^{\mathrm{m}-1}+1\right)
$$

This is easily established by mathematical induction using the definition of $b(n)$ and the fact that $b\left(2^{n}\right)=1$ 。

## BUNNY PATHS?

B-180 Proposed by Reuben C. Drake, North Carolina A \& T University, Greensboro, North Carolina.

Enumerate the paths in the Cartesian plane from $(0,0)$ to $(\mathrm{n}, 0)$ that consist of directed line segments of the four following types:

| Type | I | II | III | IV |
| :---: | :---: | :---: | :---: | :---: |
| Initial Point | $(k, 0)$ | $(k, 0)$ | $(k, 1)$ | $(k, 1)$ |
| Terminal Point | $(k, 1)$ | $(k+1,0)$ | $(k+1,1)$ | $(k+1,0)$ |

Solution by L. Carlitz, Duke University, Durham, North Carolina.
Let $f(n)$ denote the total number of paths from $(0,0)$ to $(n, 0)$. Let $f_{0}(n)$ denote the number of paths ending with segment of Type II, and $f_{1}(n)$ the number ending with a segment of Type IV. Then we have

$$
\begin{gathered}
\mathrm{f}_{0}(\mathrm{n}+1)=\mathrm{f}_{0}(\mathrm{n})+\mathrm{f}_{1}(\mathrm{n})=\mathrm{f}(\mathrm{n}) \\
\mathrm{f}_{1}(\mathrm{n}+1)=\mathrm{f}(0)+\mathrm{f}(1)+\cdots+\mathrm{f}(\mathrm{n})
\end{gathered}
$$

It follows that

$$
\mathrm{f}(\mathrm{n}+1)=\mathrm{f}(\mathrm{n})+\sum_{\mathrm{k}=0}^{\mathrm{n}} \mathrm{f}(\mathrm{k})
$$

Put

$$
F(x)=\sum_{n=0}^{\infty} f(n) x^{n}
$$

Then

$$
\begin{aligned}
F(x) & =1+\sum_{n=0}^{\infty}\left\{f(n)+\sum_{k=0}^{n} f(k)\right\} x^{n} \\
& =1+x F(x)+\frac{x}{1-x} F(x),
\end{aligned}
$$

so that

$$
F(x)=\frac{1-x}{1-3 x+x^{2}}
$$

Since

$$
\frac{x}{1-3 x+x^{2}}=\sum_{n=0}^{\infty} F_{2 n} x^{n}
$$

in the usual notation for Fibonacci numbers, it follows that

$$
f(n)=F_{2 n+1}
$$

## Moreover,

$$
f_{0}(n)=F_{2 n-1}
$$

and

$$
\mathrm{f}_{1}(\mathrm{n})=\mathrm{F}_{1}+\mathrm{F}_{3}+\cdots+\mathrm{F}_{2 \mathrm{n}-1}=\mathrm{F}_{2 \mathrm{n}} .
$$

B-181 Proposed by J. B. Roberts, Reed College, Portland, Oregon.
Let $m$ be a fixed integer and let $G_{-1}=0, G_{1}=1, G_{n}=G_{n-1}+G_{n-2}$ for $n \geq 1$. Show that $G_{0}, G_{m}, G_{2 m}, G_{3 m}, \cdots$ is the sequence of upper left principal minors of the infinite matrix
$\left(\begin{array}{ccccc}1 & 1 & 0 & 0 & \cdots \\ G_{m-2} & G_{m-2}+G_{m} & 1 & 0 & \cdots \\ 0 & (-1)^{m} & G_{m-2}+G_{m} & 1 & \cdots \\ 0 & 0 & (-1)^{m} & G_{m-2}+G_{m} & \cdots \\ 0 & 0 & 0 & (-1)^{m} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots\end{array}\right)$

Solution by the Proposer.
Expansion of the typical minor $\mathrm{M}_{\mathrm{k}}, \mathrm{k}>2$, by means of the elements of its last row, yields the recurrence relation

$$
M_{k}=\left(G_{m-2}+G_{m}\right) M_{k-1}-(-1)^{m_{M-2}} M_{k}
$$

Induction, making use of the identity

$$
G_{\mathrm{n}}=\left(\mathrm{G}_{\mathrm{k}-2}+\mathrm{G}_{\mathrm{k}}\right) \mathrm{G}_{\mathrm{n}-\mathrm{k}}-(-1)^{\mathrm{k}} \mathrm{G}_{\mathrm{n}-2 \mathrm{k}}
$$

(itself easily proved by induction), yields the conclusion.

## CONGRUENCES

B-182 Proposed by James E. Desmond, Florida State University, Tallahassee, Florida.

Show that for any prime $p$ and any integer $n$,

$$
F_{n p} \equiv F_{n} F_{p}(\bmod p) \quad \text { and } \quad L_{n p} \equiv L_{n} L_{p} \equiv L_{n}(\bmod p)
$$

## Solution by the Proposer.

We have, from Hardy and Wright, Theory of Numbers, Oxford University Press, London, 1954, p. 150, that $F_{p-1} \equiv 0(\bmod p)$ and $F_{p} \equiv 1(\bmod$ p) if $\mathrm{p} \equiv \pm 1(\bmod 5)$ and that $\mathrm{F}_{\mathrm{p}+1} \equiv 0(\bmod \mathrm{p})$ and $\mathrm{F}_{\mathrm{p}} \equiv-1(\bmod \mathrm{p})$ if $p \equiv \pm 2(\bmod 5)$. Therefore, $L_{p} \equiv 1(\bmod p)$ for all primes $p$, the case $p=5$ being clear.

From I. D. Ruggles, "Some Fibonacci Results Using Fibonacci-Type Sequences," Fibonacci Quarterly, Vol. 1, No. 2 (1963), p. 77, we have

$$
\mathrm{F}_{\mathrm{r}+\mathrm{S}}=\mathrm{L}_{\mathrm{s}} \mathrm{~F}_{\mathrm{r}}+(-1)^{\mathrm{S}+1} \mathrm{~F}_{\mathrm{r}-\mathrm{S}}
$$

for all integers $r$ and $s$. Let $s=p$, a prime, and let $r=n p$ for $a n$ arbitrary integer $n$. Then

$$
F_{(n+1) p}=L_{p} F_{n p}+(-1)^{p+1} F_{(n-1) p}
$$

so that

$$
F_{(n+1) p} \equiv F_{n p}+F_{(n-1) p}(\bmod p)
$$

for any integer $n$ and any prime $p$, the case $p=2$ being clear. Similarly, we obtain

$$
L_{(n+1) p} \equiv L_{n p}+L_{(n-1) p}(\bmod p)
$$

for any integer $n$ and any prime $p$, since $L_{m}=F_{m+1}+F_{m-1}$ for all integers m. Now, using induction on $n$, the proposition is true for $n=0,1$. Suppose $\mathrm{F}_{\mathrm{np}} \equiv \mathrm{F}_{\mathrm{n}} \mathrm{F}_{\mathrm{p}}(\bmod \mathrm{p})$ for $\mathrm{n}=0,1, \ldots, \mathrm{k}$ with $\mathrm{k}>0$. Then

$$
\mathrm{F}_{(\mathrm{k}+1) \mathrm{p}} \equiv \mathrm{~F}_{\mathrm{kp}}+\mathrm{F}_{(\mathrm{k}-1) \mathrm{p}} \equiv \mathrm{~F}_{\mathrm{k}} \mathrm{~F}_{\mathrm{p}}+\mathrm{F}_{\mathrm{k}-1} \mathrm{~F}_{\mathrm{p}}(\bmod \mathrm{p})
$$

Suppose $\mathrm{F}_{\mathrm{np}} \equiv \mathrm{F}_{\mathrm{n}} \mathrm{F}_{\mathrm{p}}(\bmod \mathrm{p})$ for $\mathrm{n}=1,0, \cdots, \mathrm{k}$ with $\mathrm{k}>1$. Then

$$
\mathrm{F}_{(\mathrm{k}-1) \mathrm{p}} \equiv \mathrm{~F}_{(\mathrm{k}+1) \mathrm{p}}-\mathrm{F}_{\mathrm{kp}} \equiv \mathrm{~F}_{\mathrm{k}+1} \mathrm{~F}_{\mathrm{p}}-\mathrm{F}_{\mathrm{k}} \mathrm{~F}_{\mathrm{p}} \equiv \mathrm{~F}_{\mathrm{k}-1} \mathrm{~F}_{\mathrm{p}}(\bmod \mathrm{p})
$$

Therefore, the Fibonacci congruence relation is true for any prime $p$ and any integer $n$. The Lucas congruence relation can be proved by an argument similar to that given above.

## PALINDROME CUBES

B-183 Proposed by Gustavus J. Simmons, Sandia Corporation, Albuquerque, New Mexico.

A positive integer is a palindrome if its digits read the same forward or backward. The least positive integer $n$, such that $n^{2}$ is a palindrome but $n$ is not, is 26. Let $S$ be the set of $n$ such that $n^{3}$ is a palindrome but n is not. Is S empty, finite, or infinite?

Comment by the Proposer.
Since $2201^{3}$ is the palindrome 10662526601, $S$ is not empty. This is all that is known about the set S .

[Continued from page 506.]

| $\mathrm{a}=$29 <br> 30 | $\mathrm{~b}=$ | 35 |
| :---: | ---: | ---: |
| 31 | 113 | $\mathrm{c}=$ |
| 32 | 97 | 113 |
| 33 | 65 | 120 |
| 34 | 34 | 65 |
| 35 | 145 | 65 |
| 36 | 73 | 145 |
| 37 | 61 | 102 |
| 38 | 37 | 65 |
| 39 | 181 | 70 |
| 40 | 41 | 181 |
|  | 101 | 50 |
|  |  | 101 |

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[^0]:    * This paper is taken from Chapter Four of the author's doctoral dissertation at the Pennsylvania State University. The author would like to express his gratitude to Research Professor S. Chowla, his advisor.

[^1]:    *William H. Leveque, Topics in Number Theory, Vol. 1, p. 133.
    **Leonard E. Dickson, History of the Theory of Numbers, Vol. II, Chap. VII, p. 259.

[^2]:    *Oystein Ore, Invitation to Number Theory, pp. 59-60, Random House, New York, 1967.

[^3]:    *See [5] and [6].

[^4]:    * Because the Hypotenuse AC minus the greater perpendicular DC equals XC. **:DE then becomes the length of the Golden Section, equal to DY and XC.

[^5]:    *Edouard Lucas, Théorie des Nombres, Paris, 1891, page 83.

