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# THE FIBONACCI QUARTERLY 

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## TWENTY-FOUR MASTER IDENTITIES

## V. E. HOGGATT, JR., JOHN W. PHILLIPS, and H. T. LEONARD, JR. San Jose State College, San Jose, California

## 1. INTRODUCTION

The area of Fibonacci research is expanding and generalized, and a large number of known identities have been listed in many articles in these pages and in the booklet [1]. Many new results and old will be summarized in the forthcoming Concordance, edited by George Ledin, Jr., to appear in 1971. Here, we generalize the results of John Halton [2]. Leonard in his thesis [3] also expanded upon this in several directions. David Zeitlin has promised an all-encompassing paper to follow upon this generalization theme.

## 2. THE HILBERT TENTH PROBLEM

In [4] Matijasevic proves Lemma 17: $\mathrm{F}_{\mathrm{m}}^{2} \mid \mathrm{F}_{\mathrm{mr}}$ iff $\mathrm{F}_{\mathrm{m}} \mid \mathrm{r}$. At the end of the English translation, the translators suggest a sequence of lemmas leading to a simplified derivation. We now prove it in an even simpler way.

Let

$$
\alpha=\frac{1+\sqrt{5}}{2}, \quad \text { and } \quad \beta=\frac{1-\sqrt{5}}{2},
$$

then

$$
\alpha^{\mathrm{m}}=\alpha \mathrm{F}_{\mathrm{m}}+\mathrm{F}_{\mathrm{m}-1} \quad \text { and } \quad \beta^{\mathrm{m}}=\beta \mathrm{F}_{\mathrm{m}}+\mathrm{F}_{\mathrm{m}-1}
$$

Recall

$$
\mathrm{F}_{\mathrm{n}}=\frac{\alpha^{\mathrm{n}}-\beta^{\mathrm{n}}}{\alpha-\beta}
$$

then

$$
\begin{aligned}
\mathrm{F}_{\mathrm{mr}} & =\frac{\alpha^{\mathrm{mr}}-\beta^{\mathrm{mr}}}{\alpha-\beta}=\sum_{\mathrm{k}=0}^{\mathrm{r}}\binom{\mathrm{r}}{\mathrm{k}} \mathrm{~F}_{\mathrm{m}}^{\mathrm{k}} \mathrm{~F}_{\mathrm{m}-1}^{\mathrm{r}-\mathrm{k}} \frac{\left(\alpha^{\mathrm{k}}-\beta^{\mathrm{k}}\right)}{\alpha-\beta} \\
& =\sum_{\mathrm{k}=0}^{\mathrm{r}}\binom{\mathrm{r}}{\mathrm{k}} \mathrm{~F}_{\mathrm{m}}^{\mathrm{k}} \mathrm{~F}_{\mathrm{m}-1}^{\mathrm{r}-\mathrm{k}} \mathrm{~F}_{\mathrm{k}}
\end{aligned}
$$

Next, $\quad F_{0}=0$, and $F_{m}^{2}$ divides all terms for $k \geq 2$. Thus,

$$
\mathrm{F}_{\mathrm{mr}} \equiv\binom{\mathrm{r}}{1} \mathrm{~F}_{\mathrm{m}} \mathrm{~F}_{\mathrm{m}-1}^{\mathrm{r}-1} \mathrm{~F}_{1} \equiv \mathrm{rF}_{\mathrm{m}} \mathrm{~F}_{\mathrm{m}-1}^{\mathrm{r}-1} \quad\left(\bmod \mathrm{~F}_{\mathrm{m}}^{2}\right)
$$

Since $\left(\mathrm{F}_{\mathrm{m}}, \mathrm{F}_{\mathrm{m}-1}\right)=1$, then the result follows easily. A shilar result could have been derived from

$$
\alpha^{\mathrm{m}}=\mathrm{F}_{\mathrm{m}+1}-\beta \mathrm{F}_{\mathrm{m}} \quad \text { and } \quad \beta^{\mathrm{m}}=\mathrm{F}_{\mathrm{m}+1}-\alpha \mathrm{F}_{\mathrm{m}}
$$

## 3. THE DERIVATIONS

Let $\alpha^{k}=A F_{k+t}+B F_{k^{*}}$ Then,

$$
\begin{aligned}
\sqrt{5} \alpha^{\mathrm{k}} & =\mathrm{A}\left(\alpha^{\mathrm{k}+\mathrm{t}}-\beta^{\mathrm{k}+\mathrm{t}}\right)+\mathrm{B}\left(\alpha^{\mathrm{k}}-\beta^{\mathrm{k}}\right) \\
& =\alpha^{\mathrm{k}}\left(\mathrm{~A} \alpha^{\mathrm{t}}+\mathrm{B}\right)-\beta^{\mathrm{k}}\left(\mathrm{~A} \beta^{\mathrm{t}}+\mathrm{B}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sqrt{5} & =\mathrm{A} \alpha^{\mathrm{t}}+\mathrm{B} \\
0 & =\mathrm{A} \beta^{\mathrm{t}}+\mathrm{B}
\end{aligned}
$$

and

$$
\mathrm{A}=\sqrt{5} /\left(\alpha^{\mathrm{t}}-\beta^{\mathrm{t}}\right)=1 / \mathrm{F}_{\mathrm{t}} ; \quad \mathrm{B}=-\beta^{\mathrm{t}} \mathrm{~A}=-\beta^{\mathrm{t}} / \mathrm{F}_{\mathrm{t}}
$$

and thus

$$
\begin{equation*}
\mathrm{F}_{\mathrm{k}+\mathrm{t}}=\alpha^{\mathrm{k}} \mathrm{~F}_{\mathrm{t}}+\beta^{\mathrm{t}} \mathrm{~F}_{\mathrm{k}} \tag{1}
\end{equation*}
$$

Since $k$ and $t$ are arbitrary integers, we may interchange them:

$$
\begin{equation*}
\mathrm{F}_{\mathrm{k}+\mathrm{t}}=\beta^{\mathrm{k}} \mathrm{~F}_{\mathrm{t}}+\alpha^{\mathrm{t}} \mathrm{~F}_{\mathrm{k}} \tag{2}
\end{equation*}
$$

Equation (1) yields
(3) $\quad \alpha^{\mathrm{j}} \mathrm{F}_{\mathrm{k}+\mathrm{t}}^{\mathrm{n}}=\alpha^{\mathrm{j}} \sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}} \mathrm{F}_{\mathrm{t}}^{\mathrm{n}-\mathrm{i}} \mathrm{F}_{\mathrm{k}}^{\mathrm{i}} \alpha^{\mathrm{k}(\mathrm{n}-\mathrm{i})} \beta^{\mathrm{ti}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{\mathrm{ti}} \mathrm{F}_{\mathrm{t}}^{\mathrm{n}-\mathrm{i}} \mathrm{F}_{\mathrm{k}}^{\mathrm{i}} \alpha^{\mathrm{k}(\mathrm{n}-\mathrm{i})-\mathrm{ti}+\mathrm{j}}$,
and, in a similar manner, Eq. (2) gives us:

$$
\begin{equation*}
\beta^{\mathrm{j}} \mathrm{~F}_{\mathrm{k}+\mathrm{t}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{\mathrm{ti}} \mathrm{~F}_{\mathrm{t}}^{\mathrm{n}-\mathrm{i}} \mathrm{~F}_{\mathrm{k}}^{\mathrm{i}} \beta^{\mathrm{k}(\mathrm{n}-\mathrm{i})-\mathrm{ti}+\mathrm{j}} \tag{4}
\end{equation*}
$$

Substituting (4) for (3), and dividing by $\sqrt{5}$ gives:
(A)

$$
\mathrm{F}_{\mathrm{j}} \mathrm{~F}_{\mathrm{k}+\mathrm{t}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{\mathrm{ti}} \mathrm{~F}_{\mathrm{t}}^{\mathrm{n}-\mathrm{i}} \mathrm{~F}_{\mathrm{k}}^{\mathrm{i}} \mathrm{~F}_{\mathrm{k}(\mathrm{n}-\mathrm{i})-\mathrm{ti}+\mathrm{j}}
$$

while adding (3) and (4) results in
(B)

$$
L_{j} F_{k+t}^{n}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{t i} F_{t}^{n-i} F_{k}^{i} L_{k(n-i)-t i+j}
$$

We note that

$$
\begin{aligned}
F_{k(n-i)-t i+j}^{2} & =\frac{1}{5}\left(L_{2 k(n-i)-2 t i+2 j}-2(-1)^{k(n-i)-t i+j}\right) \\
L_{k(n-i)-t i+j}^{2} & =L_{2 k(n-i)-2 t i+2 j}+2(-1)^{k(n-i)-t i+j}
\end{aligned}
$$

and that

$$
\begin{equation*}
2(-1)^{j}\left[\mathrm{~F}_{2 \mathrm{k}}(-1)^{\mathrm{t}}+\mathrm{F}_{2 \mathrm{t}}(-1)^{\mathrm{k}}\right]^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}} \mathrm{~F}_{2 \mathrm{t}}^{\mathrm{n}-\mathrm{i}} \mathrm{~F}_{2 \mathrm{k}}^{\mathrm{i}}\left[2(-1)^{\mathrm{k}(\mathrm{n}-\mathrm{i})-\mathrm{ti}+\mathrm{j}}\right] \tag{5}
\end{equation*}
$$

Substitute $2 \mathrm{j}, 2 \mathrm{k}$, and 2 t for $\mathrm{j}, \mathrm{k}$, and t in (B), and subtract (5) to get:

We add the same equations to conclude that:

$$
L_{2 j} F_{2(k+t)}^{n}+2(-1)^{j}\left[F_{2 k}(-1)^{t}+F_{2 t}(-1)^{k}\right]^{n}=\sum_{i=0}^{n}\binom{n}{i} F_{2 t}^{n-i} F_{2 k}^{i} L_{k(n-i)-t i+j}^{2}
$$

These expressions may be simplified by observing that

$$
\left[F_{2 t}(-1)^{t}+F_{2 t}(-1)^{\mathrm{k}}\right]^{\mathrm{n}}=(-1)^{\mathrm{tn}}\left[\mathrm{~F}_{2 \mathrm{k}}+(-1)^{\mathrm{k}-\mathrm{t}} \mathrm{~F}_{2 \mathrm{t}}\right]^{\mathrm{n}}
$$

and that from the well-known identity

$$
L_{h} F_{g}=F_{g+h}+(-1)^{h^{2}} F_{g-h}
$$

it follows (by letting $\mathrm{g}=\mathrm{k}+\mathrm{t}$ and $\mathrm{h}=\mathrm{k}-\mathrm{t}$ ) that

$$
\mathrm{F}_{2 \mathrm{k}}+(-1)^{\mathrm{k}-\mathrm{t}} \mathrm{~F}_{2 \mathrm{t}}=\mathrm{L}_{\mathrm{k}-\mathrm{t}} \mathrm{~F}_{\mathrm{k}+\mathrm{t}}
$$

Thus
(C) $\quad \mathrm{L}_{2 \mathrm{j}} \mathrm{F}_{2(\mathrm{k}+\mathrm{t})}^{\mathrm{n}}-2(-1)^{\mathrm{j}+\operatorname{tn}} \mathrm{F}_{\mathrm{k}+\mathrm{t}}^{\mathrm{n}} \mathrm{L}_{\mathrm{k}-\mathrm{t}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}} \mathrm{F}_{2 \mathrm{t}}^{\mathrm{n}-\mathrm{i}} \mathrm{F}_{2 \mathrm{k}}^{\mathrm{i}} \mathrm{F}_{\mathrm{k}(\mathrm{n}-\mathrm{i})-\mathrm{ti}+\mathrm{j}}^{2}$,
and
(D) $\quad L_{2 j} \mathrm{~F}_{2(\mathrm{k}+\mathrm{t})}^{\mathrm{n}}+2(-1)^{\mathrm{j}+\operatorname{tn}} \mathrm{F}_{\mathrm{k}+\mathrm{t}}^{\mathrm{n}} \mathrm{L}_{\mathrm{k}-\mathrm{t}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}} \mathrm{F}_{2 \mathrm{t}}^{\mathrm{n}-\mathrm{i}} \mathrm{F}_{2 \mathrm{k}}^{\mathrm{i}} \mathrm{L}_{\mathrm{k}(\mathrm{n}-\mathrm{i})-\mathrm{ti}+\mathrm{j}}^{2}$.

We rewrite (1) and (2), using $m$ in place of $k$ :

$$
\alpha^{\mathrm{t}} \mathrm{~F}_{\mathrm{m}}=\mathrm{F}_{\mathrm{m}+\mathrm{t}}-\beta^{\mathrm{m}} \mathrm{~F}_{\mathrm{t}} \quad \text { and } \quad \beta^{\mathrm{t}} \mathrm{~F}_{\mathrm{m}}=\mathrm{F}_{\mathrm{m}+\mathrm{t}}-a^{\mathrm{m}} \mathrm{~F}_{\mathrm{t}}
$$

Therefore,

$$
\alpha^{\mathrm{kt}} \mathrm{~F}_{\mathrm{m}}^{\mathrm{k}}=\sum_{\mathrm{h}=0}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{~h}}(-1)^{\mathrm{h}} \mathrm{~F}_{\mathrm{m}+\mathrm{t}}^{\mathrm{k}-\mathrm{h}} \mathrm{~F}_{\mathrm{t}}^{\mathrm{h}} \beta^{\mathrm{mh}}
$$

and

$$
\beta^{\mathrm{kt}} \mathrm{~F}_{\mathrm{m}}^{\mathrm{k}}=\sum_{\mathrm{h}=0}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{~h}}(-1)^{\mathrm{h}} \mathrm{~F}_{\mathrm{m}+\mathrm{t}}^{\mathrm{k}-\mathrm{h}} \mathrm{~F}_{\mathrm{t}}^{\mathrm{h}} \mathrm{a}^{\mathrm{mh}}
$$

Multiplying the first equation by $\alpha^{\mathrm{n}-\mathrm{kt}}$ and the second by $\beta^{\mathrm{n}-\mathrm{kt}}$, we get:

$$
\begin{equation*}
\alpha^{\mathrm{n}} \mathrm{~F}_{\mathrm{m}}^{\mathrm{k}}=\sum_{\mathrm{h}=0}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{~h}}(-1)^{\mathrm{h}+\mathrm{n}-\mathrm{kt}} \mathrm{~F}_{\mathrm{m}+\mathrm{t}}^{\mathrm{k}-\mathrm{h}} \mathrm{~F}_{\mathrm{t}}^{\mathrm{h}} \beta^{\mathrm{mh}-\mathrm{n}+\mathrm{kt}} \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta^{\mathrm{n}} \mathrm{~F}_{\mathrm{m}}^{\mathrm{k}}=\sum_{\mathrm{h}=0}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{~h}}(-1)^{\mathrm{h}+\mathrm{n}-\mathrm{kt}} \mathrm{~F}_{\mathrm{m}+\mathrm{t}}^{\mathrm{k}-\mathrm{h}} \mathrm{~F}_{\mathrm{t}}^{\mathrm{h}} \alpha^{\mathrm{mh}-\mathrm{n}+\mathrm{kt}} \tag{7}
\end{equation*}
$$

We subtract (7) from (6) to get

$$
\mathrm{F}_{\mathrm{n}} \mathrm{~F}_{\mathrm{m}}^{\mathrm{k}}=\sum_{\mathrm{h}=0}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{~h}}(-1)^{\mathrm{h}+\mathrm{n}-\mathrm{kt}+1} \mathrm{~F}_{\mathrm{m}+\mathrm{t}}^{\mathrm{k}-\mathrm{h}} \mathrm{~F}_{\mathrm{t}}^{\mathrm{h}} \mathrm{~F}_{\mathrm{mn}-\mathrm{n}+\mathrm{kt}}
$$

or equivalently,

$$
(-1)^{\mathrm{n}+1} \mathrm{~F}_{\mathrm{n}} \mathrm{~F}_{\mathrm{m}}^{\mathrm{k}}=(-1)^{\mathrm{kt}} \sum_{\mathrm{h}=0}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{~h}}(-1)^{\mathrm{h}} \mathrm{~F}_{\mathrm{m}+\mathrm{t}}^{\mathrm{k}-\mathrm{h}} \mathrm{~F}_{\mathrm{t}}^{\mathrm{h}} \mathrm{~F}_{\mathrm{mh}-\mathrm{n}+\mathrm{kt}}
$$

Adding (6) and (7), we get:

$$
L_{n} F_{m}^{k}=\sum_{h=0}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{~h}}(-1)^{\mathrm{h}+\mathrm{n}-\mathrm{kt}} \mathrm{~F}_{\mathrm{m}+\mathrm{t}}^{\mathrm{k}-\mathrm{h}} \mathrm{~F}_{\mathrm{t}}^{\mathrm{h}} \mathrm{~L}_{\mathrm{mh}-\mathrm{n}+\mathrm{kt}}
$$

or

$$
(-1)^{n} L_{n} F_{m}^{k}=(-1)^{k t} \sum_{h=0}^{k}\binom{\mathrm{k}}{\mathrm{~h}}(-1)^{\mathrm{h}} \mathrm{~F}_{\mathrm{m}+\mathrm{t}}^{\mathrm{k}-\mathrm{h}} \mathrm{~F}_{\mathrm{t}}^{\mathrm{h}} \mathrm{~L}_{\mathrm{mh}-\mathrm{n}+\mathrm{kt}}
$$

Finally, we replace $(-1)^{n+1} F_{n}$ with $F_{-n} ;(-1)^{n} L_{n}$ with $L_{-n}$; and $-n$ with n to obtain:
(E)

$$
F_{n} F_{m}^{k}=(-1)^{k t} \sum_{h=0}^{k}\binom{k}{h}(-1)^{h} F_{m+t}^{k-h} \mathrm{~F}_{\mathrm{t}}^{\mathrm{h}} \mathrm{~F}_{m h+n+k t}
$$

(see [3]), and

$$
\begin{equation*}
L_{n} F_{m}^{k}=(-1)^{k t} \sum_{h=0}^{k}\binom{k}{h}(-1)^{h} F_{m+t}^{k-h} F_{t}^{h} L_{m h+n+k t} \tag{F}
\end{equation*}
$$

As before, we observe that

$$
\mathrm{F}_{\mathrm{mh}+\mathrm{n}+\mathrm{kt}}^{2}=\frac{1}{5}\left(\mathrm{~L}_{2 \mathrm{mn}+2 \mathrm{n}+2 \mathrm{kt}}-2(-1)^{\mathrm{mh}+\mathrm{n}+\mathrm{kt}}\right),
$$

that

$$
\mathrm{L}_{\mathrm{mh}+\mathrm{n}+\mathrm{kt}}^{2}=\mathrm{L}_{2 \mathrm{mh}+2 \mathrm{n}+2 \mathrm{kt}}+2(-1)^{\mathrm{mh}+\mathrm{n}+\mathrm{kt}}
$$

that
$2(-1)^{\mathrm{n}+\mathrm{kt}}\left[\mathrm{F}_{2(\mathrm{~m}+\mathrm{t})}+(-1)^{\mathrm{m}+1} \mathrm{~F}_{2 \mathrm{t}}\right]^{\mathrm{k}}=\sum_{\mathrm{h}=0}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{h}}(-1)^{\mathrm{h}} \mathrm{F}_{2(\mathrm{~m}+\mathrm{t})}^{\mathrm{k}-\mathrm{F}} \mathrm{F}_{2 \mathrm{t}}^{\mathrm{h}}\left[2(-1)^{\mathrm{mh}+\mathrm{n}+\mathrm{kt}}\right]$
and that
$L_{h} F_{g}=F_{g+h}+(-1)^{h} F_{g-h} \Rightarrow($ with $g=m+2 t ; h=m): F_{2(m+t)}+(-1)^{m+1} F_{2 t}=$ $L_{m+2 t^{F}}{ }_{m}$.
We replace $\mathrm{m}, \mathrm{n}$ and t in ( F ) with $2 \mathrm{~m}, 2 \mathrm{n}$ and 2 t and perform the obvious subtraction and addition to obtain:
(G) $\mathrm{L}_{2 \mathrm{n}} \mathrm{F}_{2 \mathrm{~m}}^{\mathrm{k}}-2(-1)^{\mathrm{n}+\mathrm{kt}} \mathrm{L}_{\mathrm{m}+2 \mathrm{t}}^{\mathrm{k}} \mathrm{F}_{\mathrm{m}}^{\mathrm{k}}=5 \sum_{\mathrm{h}=0}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{h}}(-1)^{\mathrm{h}} \mathrm{F}_{2(\mathrm{~m}+\mathrm{t})}^{\mathrm{k}-\mathrm{h}} \mathrm{F}_{2 t^{\mathrm{h}}}^{\mathrm{F}_{\mathrm{mh}+\mathrm{n}+\mathrm{kt}}^{2}}$,
and
(H) $\quad L_{2 n} F_{2 m}^{k}+2(-1)^{n+k t_{L}}{ }_{m+2 t^{k}}^{F_{m}^{k}}=\sum_{h=0}^{k}\binom{\mathrm{k}}{\mathrm{h}}(-1)^{\mathrm{h}} \mathrm{F}_{2(\mathrm{~m}+\mathrm{t})}^{\mathrm{k}-\mathrm{h}} \mathrm{F}_{2 t^{\mathrm{L}}}^{\mathrm{L}_{m h+n+k t}^{2}}$.

Starting with $\alpha^{\mathrm{m}}=\mathrm{AF}_{\mathrm{m}+\mathrm{k}}+\mathrm{BL}_{\mathrm{m}}$.
By a procedure identical with that used to obtain (1) and (2), we get:

$$
\begin{equation*}
\alpha^{\mathrm{m}} \mathrm{~L}_{\mathrm{k}}=\sqrt{5} \mathrm{~F}_{\mathrm{m}+\mathrm{k}}+\beta^{\mathrm{k}} \mathrm{~L}_{\mathrm{m}}, \tag{8}
\end{equation*}
$$

and
(9)

$$
\beta^{\mathrm{m}_{L_{k}}}=-\sqrt{5} \mathrm{~F}_{\mathrm{m}+\mathrm{k}}+\alpha^{\mathrm{k}} \mathrm{~L}_{\mathrm{m}},
$$

which lead to

$$
\begin{equation*}
\alpha^{m n+j_{2}} L_{k}^{n}=\sum_{i=0}^{n}\binom{n}{i} \sqrt{5}^{\mathrm{i}_{F_{m+k}}^{i}} L_{m}^{n-i} \beta^{k(n-i)} \alpha^{j} \tag{10}
\end{equation*}
$$

and
(11)

$$
\beta^{\mathrm{mn}+\mathrm{j}} \mathrm{~L}_{\mathrm{k}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{\mathrm{i}} \sqrt{5}^{\mathrm{i} F_{m+k^{i}}^{i} \mathrm{~L}_{\mathrm{m}}^{\mathrm{n}-\mathrm{i}} \alpha^{\mathrm{k}(\mathrm{n}-\mathrm{i})}{ }_{\beta}^{\mathrm{j}} . . . . . .}
$$

Subtracting (11) from (10) and dividing by $\sqrt{5}$ gives

$$
F_{m n+j} L_{k}^{n}=(-1)^{j} \sum_{i=0}^{n}\binom{n}{i} \sqrt{5}^{i-1} F_{m+k^{i}} L_{m}^{n-i}\left[\beta^{k(n-i)-j}-(-1)^{i} \alpha^{k(n-i)-j}\right]
$$

or

$$
F_{m n+j} L_{k}^{n}=(-1)^{j+1} \sum_{i=0}^{[n / 2]}\binom{n}{2 i} 5^{i} F_{m+k}^{2 i} L_{m}^{n-2 i} F_{k(n-2 i)-j}
$$

(I)

$$
+(-1)^{j} \sum_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2 i+1} 5^{i} F_{m+k}^{2 i+1} L_{m}^{n-2 i-1} L_{k(n-2 i-1)-j}
$$

and adding (10) and (11) yields:

1971]

$$
L_{m n+j} L_{k}^{n}=(-1)^{j} \sum_{i=0}^{n}\binom{n}{i} \sqrt{5}^{i} F_{m+k}^{i} L_{m}^{n-i}\left[\beta^{k(n-i)-j}+(-1)^{i} \alpha^{k(n-i)-j}\right]
$$

or

$$
L_{m n+j} L_{k}^{n}=(-1)^{j} \sum_{i=0}^{[n / 2]}\binom{n}{2 i} 5^{i} F_{m+k}^{2 i} L_{m}^{n-2 i} L_{k(n-2 i)-j}
$$

(J)

$$
+(-1)^{j+1} \sum_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2 i+1} 5^{i} F_{m+k}^{2 i+1} L_{m}^{n-2 i-1} F_{k(n-2 i-1)-j}
$$

Equations (8) and (9) may be rewritten:

$$
\sqrt{5} \mathrm{~F}_{\mathrm{m}+\mathrm{k}}=\alpha^{\mathrm{m}} \mathrm{~L}_{\mathrm{k}}-\beta^{\mathrm{k}} \mathrm{~L}_{\mathrm{m}}
$$

and

$$
\sqrt{5} \mathrm{~F}_{\mathrm{m}+\mathrm{k}}=-\beta^{\mathrm{m}} \mathrm{~L}_{\mathrm{k}}+\alpha^{\mathrm{k}} \mathrm{~L}_{\mathrm{m}}
$$

which give

$$
\begin{equation*}
\alpha^{\mathrm{j}} \sqrt{5}^{\mathrm{n}} \mathrm{~F}_{\mathrm{m}+\mathrm{k}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{\mathrm{i}} \mathrm{~L}_{\mathrm{k}}^{\mathrm{n}-\mathrm{i}} \mathrm{~L}_{\mathrm{m}}^{\mathrm{i}} \alpha^{\mathrm{m}(\mathrm{n}-\mathrm{i})+\mathrm{j}_{\beta} \mathrm{ki}} \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta^{\mathrm{j}} \sqrt{5}^{\mathrm{n}} \mathrm{~F}_{\mathrm{m}+\mathrm{k}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{\mathrm{n}-\mathrm{i}} \mathrm{~L}_{\mathrm{k}}^{\mathrm{n}-\mathrm{i}} \mathrm{~L}_{\mathrm{m}}^{\mathrm{i}} \alpha^{\mathrm{ki}} \beta^{\mathrm{m}(\mathrm{n}-\mathrm{i})+\mathrm{j}} \tag{13}
\end{equation*}
$$

Adding (12) and (13), we get:
$\sqrt{5}^{n} L_{j} F_{m+k}^{n}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{(k+1) i^{n}} L_{k}^{n-i} L_{m}^{i}\left[\alpha^{m(n-i)-k i+j}+(-1)^{n} \beta^{m(n-i)-k i+j}\right]$,
which, in turn, provides
(K)

$$
5^{n} L_{j} F_{m+k}^{2 n}=\sum_{i=0}^{2 n}\binom{2 n}{i}(-1)^{(k+1) i_{L}}{ }_{k}^{2 n-i} L_{m}^{i} L_{m(2 n-i)-k i+j}
$$

and
(L) $\quad 5^{n} L_{j} F_{m+k}^{2 n+1}=\sum_{i=0}^{2 n+1}\binom{2 n+1}{i}(-1)^{(k+1) i_{L}} L_{k}^{2 n+1-i_{1}} L_{m}^{i} F_{m(2 n+1-i)-k i+j}$

We subtract (13) from (12) to get:
$\sqrt{5}^{n+1} F_{j} F_{m+k}^{n}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{(k+1) i_{1}} L_{k}^{n-i} L_{m}^{i}\left[\alpha^{m(n-i)-k i+j}-(-1)^{n} \beta^{m(n-i)-k i+j}\right]$
from which we get
(M)

$$
5^{n} F_{j} F_{m+k}^{2 n}=\sum_{i=0}^{2 n}\binom{2 n}{i}(-1)^{(k+1) i_{1}} L_{k}^{2 n-i_{1}} L_{m}^{i} F_{m(2 n-i)-k i+j}
$$

and
(N) $\quad 5^{n+1} F_{j} F_{m+k}^{2 n+1}=\sum_{i=0}^{2 n+1}\binom{2 n+1}{i}(-1)^{(k+1) i_{L}} L_{k}^{2 n+1-i_{1}} L_{m}^{i} L_{m(2 n+1-i)-k i+j}$.

Once again, we note that

$$
\begin{aligned}
& 2(-1)^{j}\left[L_{2 k}-(-1)^{k-m} L_{2 m}\right]^{2 n} \\
&=\sum_{i=0}^{2 n}\binom{2 n}{i}(-1)^{i} L_{2 k}^{2 n-i} L_{2 m}^{i}\left[2(-1)^{m(2 n-i)-k i+j}\right]
\end{aligned}
$$

In (K), we let $j, k$, and $m$ be replaced by $2 \mathrm{j}, 2 \mathrm{k}$, and 2 m and subtract (M):

$$
\begin{align*}
5^{n} L_{2 j} F_{2(m+k)}^{2 n} & -2(-1)^{j}\left[L_{2 k}-(-1)^{k-m} L_{2 m}\right]^{2 n} \\
& =5 \sum_{i=0}^{2 n}\binom{2 n}{i}(-1)^{i} L_{2 k}^{2 n-i} L_{2 m}^{i} F_{m(2 n-i)-k i+j}^{2} \tag{15}
\end{align*}
$$

The corresponding addition provides

$$
\begin{aligned}
5^{n} L_{2 j} F_{2(m+k)}^{2 n}+2(-1)^{j} & {\left[L_{2 k}-(-1)^{k-m_{1}} L_{2 m}\right]^{2 n} } \\
& =\sum_{i=0}^{2 n}\binom{2 n}{i}(-1)^{i} L_{2 k}^{2 n-i} L_{2 m}^{i} L_{m(m-i)-k i+j}^{2}
\end{aligned}
$$

Since

$$
\begin{aligned}
5 \mathrm{~F}_{\mathrm{k}+\mathrm{m}} \mathrm{~F}_{\mathrm{k}=\mathrm{m}} & =\left(\alpha^{\mathrm{k}+\mathrm{m}}-\beta^{\mathrm{k}+\mathrm{m}}\right)\left(\alpha^{\mathrm{k}-\mathrm{m}}-\beta^{\mathrm{k}-\mathrm{m}}\right) \\
& =\alpha^{2 \mathrm{k}}-(\alpha \beta)^{\mathrm{k}-\mathrm{m}}\left(\alpha^{2 \mathrm{n}}+\beta^{2 \mathrm{~m}}\right)+\beta^{2 \mathrm{k}}
\end{aligned}
$$

or

$$
\begin{equation*}
5 \mathrm{~F}_{\mathrm{k}+\mathrm{m}} \mathrm{~F}_{\mathrm{k}-\mathrm{m}}=\mathrm{L}_{2 \mathrm{k}}-(-1)^{\mathrm{k}-\mathrm{m}_{\mathrm{L}_{2 \mathrm{~m}}}, ~} \tag{17}
\end{equation*}
$$

we can rewrite (15) and (16):

$$
\begin{align*}
5^{\mathrm{n}-1} \mathrm{~L}_{2 j} \mathrm{~F}_{2(\mathrm{~m}+\mathrm{k})}^{2 \mathrm{n}} & -2 \cdot 5^{2 \mathrm{n}-1}(-1)^{j_{F}} \mathrm{~F}_{\mathrm{k}+\mathrm{m}}^{2 \mathrm{n}} \mathrm{~F}_{\mathrm{k}-\mathrm{m}}^{2 \mathrm{n}} \\
= & \sum_{\mathrm{i}=0}^{2 \mathrm{n}}\binom{2 \mathrm{n}}{i}(-1)^{\mathrm{i}} \mathrm{~L}_{2 k}^{2 n-\mathrm{i}_{\mathrm{k}}} \mathrm{~L}_{2 m}^{\mathrm{i}} \mathrm{~F}_{\mathrm{m}(2 n-\mathrm{i})-\mathrm{ki}+j}^{2} \tag{P}
\end{align*}
$$

and
(Q)

$$
\begin{aligned}
5^{n} L_{2 j} F_{2(m+k)}^{2 n} & +2 \cdot 5^{2 n}(-1)^{j} F_{k+m}^{2 n} F_{k-m}^{2 n} \\
& =\sum_{i=0}^{2 n}\binom{2 n}{i}(-1)^{i} L_{2 k}^{2 n-i} L_{2 m}^{i} L_{m(2 k-i)-k i+j}^{2}
\end{aligned}
$$

We next observe that

$$
\begin{aligned}
2(-1)^{\mathrm{m}+j}\left[\mathrm{~L}_{2 k}\right. & -(-1)^{\left.\mathrm{k}-\mathrm{m}_{L_{2 m}}\right]^{2 n+1}} \\
& =\sum_{i=0}^{2 n+1}\binom{2 n+1}{i}(-1)^{i} L_{2 k}^{2 n+i-j} L_{2 m}^{i}\left[2(-1)^{m(2 n+1-i)-k i+j}\right]
\end{aligned}
$$

and again we employ (17) and treat (N) as we did (K) to conclude
(R)

$$
5^{n+1} F_{2 j} F_{2(m+k)}^{2 n+1}+2 \cdot 5^{2 n+1}(-1)^{m+j} F_{k+m}^{2 n+1} F_{k-m}^{2 n+1}
$$

$$
=\sum_{i=0}^{2 n+1}\binom{2 n+1}{i}(-1)^{i} L_{2 k}^{2 n+1-i} L_{2 m}^{i} L_{m(2 n+1-i)-k i+j}^{2}
$$

and

$$
5^{n} F_{2 j} F_{2(m+k)}^{2 n+1}-2 \cdot 5^{2 n}(-1)^{m+j} F_{k+m}^{2 n+1} F_{k=m}^{2 n+1}
$$

(S)

$$
=\sum_{i=0}^{2 n+1}\binom{2 n+1}{i}(-1)^{i} L_{2 k}^{2 n+1-i} L_{2 m}^{i} F_{m(2 n+1-i)-k i+j}^{2} .
$$

Starting with

$$
\begin{equation*}
\alpha^{\mathrm{m}}=\mathrm{AL}_{\mathrm{m}+\mathrm{k}}+\mathrm{BL}_{\mathrm{m}} \tag{18}
\end{equation*}
$$

we get

$$
\mathrm{L}_{\mathrm{m}+\mathrm{k}}=\sqrt{5} \alpha^{\mathrm{m}} \mathrm{~F}_{\mathrm{k}}+\beta^{\mathrm{k}} \mathrm{~L}_{\mathrm{m}}
$$

Interchanging variables does not produce a second useful equation. However,

$$
\begin{equation*}
\beta^{m}=A^{\prime} L_{m+k}+B^{\prime} L_{m} \tag{19}
\end{equation*}
$$

yields

$$
\mathrm{L}_{\mathrm{m}+\mathrm{k}}=-\sqrt{5} \beta^{\mathrm{m}} \mathrm{~F}_{\mathrm{k}}+\alpha^{\mathrm{k}} \mathrm{~L}_{\mathrm{m}}
$$

Proceeding as usual, we get

$$
\alpha^{\mathrm{j}} \mathrm{~L}_{\mathrm{m}+\mathrm{k}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{(\mathrm{m}+1) \mathrm{i}} \sqrt{5}^{\mathrm{i}} \mathrm{~L}_{\mathrm{m}}^{\mathrm{n}-\mathrm{i}} \mathrm{~F}_{\mathrm{k}}^{\mathrm{i}} \alpha^{\mathrm{k}(\mathrm{n}-\mathrm{i})-\mathrm{mi}+\mathrm{j}}
$$

and

$$
\beta^{\mathrm{j}_{\mathrm{L}}^{\mathrm{n}}}{ }_{\mathrm{m}+\mathrm{k}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{\mathrm{mi}} \sqrt{5}^{\mathrm{i}} \mathrm{~L}_{\mathrm{m}}^{\mathrm{n}-\mathrm{i}} F_{\mathrm{k}}^{\mathrm{i}} \beta^{\mathrm{k}(\mathrm{n}-\mathrm{i})-\mathrm{mi}+\mathrm{j}}
$$

Adding,

$$
L_{j} L_{m+k}^{n}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{m i} \sqrt{5}^{i_{i}} L_{m}^{n-i} F_{k}^{i}\left[(-1)^{i} \alpha^{k(n-i)-m i+j}+\beta^{k(n-i)-m i+j}\right]
$$

or equivalently,

$$
L_{j} L_{m+k}^{n}=\sum_{i=0}^{[n / 2]}\binom{n}{2 i} 5^{i} L_{m}^{n-2 i_{m}} F_{k}^{2 i_{k}} L_{k(n-2 i)-2 m i+j}
$$

(T)

$$
+(-1)^{m} \sum_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2 i+1} 5^{i+1} L_{m}^{n-2 i-1} F_{k}^{2 i+1} F_{k(n-2 i-1)-m(2 i+1)+j}
$$

and subtracting,

$$
\sqrt{5} \mathrm{~F}_{\mathrm{j}} \mathrm{~L}_{\mathrm{m}+\mathrm{k}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{\mathrm{mi}} \sqrt{5}^{\mathrm{i}_{\mathrm{L}}} \mathrm{~m}_{\mathrm{m}}^{\mathrm{n}-\mathrm{i}} \mathrm{~F}_{\mathrm{k}}^{\mathrm{i}}\left[(-1)^{\mathrm{i}} \alpha^{\mathrm{k}(\mathrm{n}-\mathrm{i})-\mathrm{mi}+\mathrm{j}}-\beta^{\mathrm{k}(\mathrm{n}-\mathrm{i})-\mathrm{mi}+\mathrm{j}}\right]
$$

or

$$
\mathrm{F}_{\mathrm{j}} \mathrm{~L}_{\mathrm{m}+\mathrm{k}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{[\mathrm{n} / 2]}\binom{\mathrm{n}}{\mathrm{i}} 5^{\mathrm{i}} \mathrm{~L}_{\mathrm{m}}^{\mathrm{n}-2 \mathrm{i}_{\mathrm{F}} \mathrm{~F}_{\mathrm{k}}^{2 \mathrm{i}_{\mathrm{k}}}{ }_{k(\mathrm{n}-2 \mathrm{i})-\mathrm{m}(2 \mathrm{i})+\mathrm{j}}{ }^{2} .}
$$

(U)

$$
+(-1)^{m} \sum_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2 i+1} 5^{i} L_{m}^{n-2 i-1} F_{k}^{2 i+1} L_{k(n-2 i-1)-m(2 i+1)+j}
$$

We rewrite (18) and (19) and proceed as before:

$$
\mathrm{L}_{\mathrm{m}} \alpha^{\mathrm{k}}=\mathrm{L}_{\mathrm{m}+\mathrm{k}}+\sqrt{5} \mathrm{~B}^{\mathrm{m}} \mathrm{~F}_{\mathrm{k}}
$$

and

$$
\mathrm{L}_{\mathrm{m}}^{\beta^{\mathrm{k}}}=\mathrm{L}_{\mathrm{m}+\mathrm{k}}-\sqrt{5} \alpha^{\mathrm{m}} \mathrm{~F}_{\mathrm{k}}
$$

yield

$$
\alpha^{\mathrm{kn}+\mathrm{j}_{1}} \mathrm{~L}_{\mathrm{m}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{\mathrm{j}} \sqrt{5}^{\mathrm{i}} L_{m+\mathrm{k}}^{\mathrm{n}-\mathrm{i}} \mathrm{~F}_{\mathrm{k}}^{\mathrm{i}} \beta^{\mathrm{mi}-\mathrm{j}}
$$

and

$$
\beta^{\mathrm{kn}+\mathrm{j}_{2}} \mathrm{~L}_{\mathrm{m}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{\mathrm{i}+\mathrm{j}} \sqrt{5}^{\mathrm{i}} \mathrm{~L}_{\mathrm{m}+\mathrm{k}}^{\mathrm{n}-\mathrm{i}} \mathrm{~F}_{\mathrm{k}}^{\mathrm{i}} \alpha^{m \mathrm{i}-\mathrm{j}}
$$

We add, to give

$$
\mathrm{L}_{\mathrm{kn}+\mathrm{j}} \mathrm{~L}_{\mathrm{m}}^{\mathrm{n}}=(-1)^{\mathrm{j}} \sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}} \sqrt{5}{ }^{\mathrm{i}} \mathrm{~L}_{\mathrm{m}+\mathrm{k}}^{\mathrm{n}-\mathrm{i}} \mathrm{~F}_{\mathrm{k}}^{\mathrm{i}}\left[\beta^{\mathrm{mi}-\mathrm{j}}+(-1)^{\mathrm{i}} \alpha^{\mathrm{mi}-\mathrm{j}}\right]
$$

or

$$
L_{k n+j} L_{m}^{n}=(-1)^{j} \sum_{i=0}^{[n / 2]}\binom{n}{2 i} 5^{i} L_{m+k}^{n-2 i_{k}} F_{k}^{2 i^{n}} L_{2 m i-j}
$$

(V)

$$
+(-1)^{j+i}\left[\begin{array}{c}
\frac{n-1}{2} \\
\left.\sum_{i=0}\right] \\
2 i+1
\end{array}\right) 5^{n+1} L_{m+k}^{n-2 i-1} F_{k}^{2 i+1} F_{m(2 i+1)-j}
$$

and subtract, for

$$
\sqrt{5} \mathrm{~F}_{\mathrm{kn}+\mathrm{j}} \mathrm{~L}_{\mathrm{m}}^{\mathrm{n}}=\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}}(-1)^{\mathrm{j}} \sqrt{5}^{\mathrm{i}} \mathrm{~L}_{\mathrm{m}+\mathrm{k}}^{\mathrm{n}-\mathrm{i}} \mathrm{~F}_{\mathrm{k}}^{\mathrm{i}}\left[\beta^{\mathrm{mi}-\mathrm{j}}-(-1)^{\mathrm{i}} \alpha^{\mathrm{mi}-\mathrm{j}}\right]
$$

or

$$
\mathrm{F}_{\mathrm{kn}+\mathrm{j}} \mathrm{~L}_{\mathrm{m}}^{\mathrm{n}}=(-1)^{\mathrm{j}+1} \sum_{\mathrm{i}=0}^{[\mathrm{n} / 2]}\binom{\mathrm{n}}{2 \mathrm{i}} 5^{\mathrm{i}} \mathrm{~L}_{\mathrm{m}+\mathrm{k}}^{\mathrm{n}-2 \mathrm{i}_{\mathrm{k}}^{2 \mathrm{i}} \mathrm{~F}_{2 \mathrm{mi}-\mathrm{j}}}
$$

(W)

$$
+(-1)^{j} \sum_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2 i+1} 5^{i} L_{m+k}^{n-2 i-1} F_{k}^{2 i+1} L_{m(2 i+1)-j}
$$

## 3. EXTENSION TO FIBONACCI AND LUCAS POLYNOMIALS

The Fibonacci polynomials $\left\{f_{n}(x)\right\}$ are defined by:

$$
\mathrm{f}_{1}(\mathrm{x})=1 ; \quad \mathrm{f}_{2}(\mathrm{x})=\mathrm{x} ; \quad \mathrm{f}_{\mathrm{n}+2}(\mathrm{x})=\mathrm{xf}_{\mathrm{n}+1}(\mathrm{x})+\mathrm{f}_{\mathrm{n}}(\mathrm{x})
$$

The Lucas polynomials are similarly defined:

$$
\ell_{1}(x)=x ; \quad \ell_{2}(x)=x^{2}+2 ; \quad \ell_{n+2}(x)=x \ell_{n+1}(x)+\ell_{n}(x)
$$

Let $\lambda_{1}$ and $\lambda_{2}$ be the roots of $\lambda^{2}=x \lambda+1$;

$$
\lambda_{1}(x)=\frac{1}{2}\left(x+\sqrt{x^{2}+4}\right) ; \quad \lambda_{2}(x)=\frac{1}{2}\left(x-\sqrt{x^{2}+4}\right) .
$$

It is easily verified that:

$$
\mathrm{f}_{\mathrm{n}}(\mathrm{x})=\left(\lambda_{1}^{\mathrm{n}}(\mathrm{x})-\lambda_{2}^{\mathrm{n}}(\mathrm{x})\right) /\left(\lambda_{1}(\mathrm{x})-\lambda_{2}(\mathrm{x})\right)
$$

and

$$
\ell_{\mathrm{n}}(\mathrm{x})=\lambda_{1}^{\mathrm{n}}(\mathrm{x})+\lambda_{2}^{\mathrm{n}}(\mathrm{x})
$$

In view of the striking similarities between the Binet forms of the Fibonacci and Lucas polynomials, and the corresponding forms for the Fibonacci and Lucas sequences, it is nardly surprising that there exists an identity involving $\lambda_{1}(x), \quad \lambda_{2}(x), f_{n}(x)$ and $\ell_{n}(x)$ paralleling each identity involving $\alpha, \beta, \mathrm{F}_{\mathrm{n}}$, and $\mathrm{L}_{\mathrm{n}}$. For example, corresponding to (A), we get:

$$
\begin{equation*}
f_{i}(x) f_{k+t}^{n}(x)=\sum_{i=0}^{n}\binom{n}{i}(-1)^{t i_{f}} f_{t}^{n-i}(x) f_{k}^{i}(x) f_{k n+j-(k+t) i}(x), \tag{19'}
\end{equation*}
$$

and, corresponding to ( E ), we have:

$$
f_{n}(x) f_{m}^{k}(x)=(-1)^{k t} \sum_{h=0}^{k}\binom{k}{h}(-1)^{h} f_{m+t^{k}-h_{t}^{h}}(x) f_{m n+n+k t}(x)
$$

In fact, the identities (A) through (W) are special cases of the Fibonacci-Lucas polynomial identities, obtained by setting $\mathrm{x}=1$.

One observes that $f_{n}(2)$ obeys: $C_{n+2}=2 C_{n+1}+C_{n} ; C_{0}=0, C_{1}=1$. This sequence is the Pell sequence. Since

$$
\ell_{\mathrm{n}}(\mathrm{x})=\mathrm{f}_{\mathrm{n}+1}(\mathrm{x})+\mathrm{f}_{\mathrm{n}-1}(\mathrm{x})
$$

one can define

$$
\ell_{\mathrm{n}}(2)=\mathrm{C}_{\mathrm{n}}^{*}=\mathrm{C}_{\mathrm{n}+1}+\mathrm{C}_{\mathrm{n}-1}
$$

to make complete substitutions in identities (A)-(W).

## 4. A FURTHER EXTENSION

Let $g_{n}(x)$ obey $g_{n+2}(x)=x g_{n+1}(x)-g_{n}(x) ; g_{0}(x)=0 ; g_{1}(x)=1$.
Then

$$
\begin{aligned}
\mathrm{g}_{\mathrm{n}}(\mathrm{x}) & =1 /\left(\sqrt{\mathrm{x}^{2}+4}\right)\left\{\left[\left(\mathrm{x}+\sqrt{\left.\mathrm{x}^{2}+4\right)} / 2\right]^{\mathrm{n}}-\left[\left(\mathrm{x}-\sqrt{\left.\mathrm{x}^{2}+4\right)} / 2\right]^{\mathrm{n}}\right\}\right.\right. \\
& =\left(\lambda_{1}^{\mathrm{n}}-\lambda_{2}^{\mathrm{n}}\right) /\left(\lambda_{1}-\lambda_{2}\right)
\end{aligned}
$$

where $\lambda_{1}$ and $\lambda_{2}$ are roots of $\lambda_{2}-x \lambda+1=0$. Also, let

$$
h_{n}(x)=\lambda_{1}^{n}+\lambda_{2}^{n}=g_{n+1}(x)-g_{n-1}(x)
$$

These sequences of polynomials are simply related to the Chebychev polynomials of the first and second kind.

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## FIBONACCI NUMBERS AND EULERIAN POLYNOMIALS

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Given a sequence of numbers $\left\{\lambda_{0}, \lambda_{1}, \cdots\right\}$ one can define a linear operator on $\Pi$, the set of all polynomials, by means of the symbolic relation

$$
\begin{equation*}
\Lambda x^{n}=(x+\lambda)^{n} \quad n=0,1,2, \cdots, \tag{1}
\end{equation*}
$$

where it is understood that after expanding $(x+\lambda)^{n}$, we replace $\lambda^{k}$ by $\lambda_{k}$. This operator can also be represented on $\Pi$ by a differential operator of infinite order. Indeed, one can show that if

$$
\begin{equation*}
\Lambda=\sum_{n=0}^{\infty} \frac{\lambda_{n}}{n!} D^{n} \quad D=d / d x \tag{2}
\end{equation*}
$$

then

$$
\Lambda f(x)=f(x+\lambda) \quad f \in \Pi
$$

A third representation of this operator can be obtained using a wellknown theorem of Boas [2]. Given any sequence of numbers $\lambda_{0}, \lambda_{1}, \lambda_{2}, \ldots$ we can find a function of $\alpha(t)$ of bounded variation on $(0, \infty)$ such that

$$
\lambda_{\mathrm{n}}=\int_{0}^{\infty} \mathrm{t}^{\mathrm{n}} \mathrm{~d} \alpha(\mathrm{t})
$$

so that

$$
\begin{equation*}
\Lambda f(x)=\int_{0}^{\infty} f(x+t) d \alpha(t) \quad f \in \Pi \tag{3}
\end{equation*}
$$

We shall refer to the sequence $\lambda_{0}, \lambda_{1}, \lambda_{2}, \cdots$ as the sequence of moments corresponding to the operator $\Lambda_{\text {. }}$

Naturally, all the representations (1), (2), and (3) are valid (and define the same operator) on П. However, we can extend the definition formally to functions with power series expansion.

In this note, we are interested in a class of "mean operators" defined by

$$
\begin{equation*}
\operatorname{Mf}(\mathrm{x})=\mu \mathrm{f}\left(\mathrm{x}+\mathrm{c}_{1}\right)+(1-\mu) \mathrm{f}\left(\mathrm{x}+\mathrm{c}_{2}\right) \tag{4}
\end{equation*}
$$

where $\mu, c_{1}, c_{2}$ are given numbers. Obviously, $M$ takes polynomials into polynomials of the same degree.

To determine the corresponding moments, we note that

$$
M x^{n}=\sum_{k=0}^{n}\binom{n}{k} x^{n-k}\left[\mu c_{1}^{k}+(1-\mu) c_{2}^{k}\right]=\sum_{k=0}^{n}\binom{n}{k} x^{n-k_{m}} m_{k}
$$

so that

$$
\mathrm{m}_{\mathrm{n}}=\mu \mathrm{c}_{1}^{\mathrm{n}}+(1-\mu) \mathrm{c}_{2}^{\mathrm{n}} \quad \mathrm{n}=0,1,2, \cdots
$$

It is easy to verify that

$$
\begin{gather*}
m_{n+1}=\left(c_{1}+c_{2}\right) m_{n}-c_{1} c_{2} m_{n-1} \quad n=1,2, \cdots  \tag{5}\\
m_{0}=1, \quad m_{1}=\mu c_{1}+(1-\mu) c_{2}
\end{gather*}
$$

To find the inverse operator $M^{-1}$, put

$$
\begin{equation*}
M^{-1}=\sum_{k=0}^{\infty} \frac{m_{k}^{*}}{k!} D^{k} \tag{6}
\end{equation*}
$$

Then $M^{-1} M x^{n}=x^{n}$ for all $n$ imply that

$$
\left.\sum_{k=0}^{j}\binom{j}{k} m_{k} m_{j-k}^{*}=0 \quad \text { (if } j>0\right) \quad \text { and } \quad 1 \quad(\text { if } j=0)
$$

Thus, if we multiply (7) by $\mathrm{t}^{\mathrm{n}} / \mathrm{n}$ ! and sum over all integers $\mathrm{n} \geq 0$, we get

$$
\left[\sum_{\mathrm{k}=0}^{\infty} \frac{\mathrm{m}_{\mathrm{k}}}{\mathrm{k!}} \mathrm{t}^{\mathrm{k}}\right]\left[\sum_{\mathrm{k}=0}^{\infty} \frac{\mathrm{m}_{\mathrm{k}}^{*}}{\mathrm{k!}} \mathrm{t}^{\mathrm{k}}\right]=1
$$

so that

$$
\sum_{k=0}^{\infty} \frac{m_{k}^{*}}{k!} t^{k}=\frac{1}{\mu e^{c_{1} t}+(1-\mu) e^{c_{2} t}}
$$

If we recall the Eulerian polynomials [1] defined by means of

$$
\frac{1-\lambda}{e^{t}-\lambda} e^{x t}=\sum_{n=0}^{\infty} H_{n}(x \mid \lambda) t^{n} / n!
$$

then we see that

$$
m_{n}^{*}=\left(c_{2}-c_{1}\right)^{n_{H}} H_{n}\left(\left.\frac{c_{1}}{c_{1}-c_{2}} \right\rvert\, \frac{\mu}{\mu-1}\right)
$$

Thus the operator inverse to the operator (1) is given by

$$
\begin{equation*}
M^{-1}=\sum_{n=0}^{\infty} \frac{\left(c_{2}-c_{1}\right)^{n}}{n!} H_{n}\left(\left.\frac{c_{1}}{c_{1}-c_{2}} \right\rvert\, \frac{\mu}{\mu-1}\right) D^{n} . \tag{8}
\end{equation*}
$$

In particular, if we take $\mu=1 / 2, c_{1}=(1+\sqrt{5}) / 2, c_{2}=(1-\sqrt{5}) / 2$ in the above, we see that $m_{n}=F_{n+1}$. Thus the Fibonacci numbers $F_{n+1}$,
$\mathrm{n}=0,1,2, \cdots$ are the moments corresponding to the mean operator
(9)

$$
\delta \mathrm{f}(\mathrm{x})=\frac{1}{2} \quad \mathrm{f}\left\{\left(\mathrm{x}+\frac{1}{2}+\frac{\sqrt{5}}{2}\right)+\mathrm{f}\left(\mathrm{x}+\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\right\}
$$

$$
=\sum_{n=0}^{\infty} \frac{F_{n+1}}{n!} D^{n} f(x)
$$

If we note that $H_{n}(x \mid-1)=E_{n}(x)$, the Euler polynomials generated by

$$
\frac{2 e^{x t}}{e^{t}+1}
$$

we find that the operator inverse to $\delta$ is

$$
\delta^{-1} f(x)=\sum_{n=0}^{\infty}(-1)^{n} \frac{5^{n / 2}}{n!} E_{n}\left(\frac{1+\sqrt{5}}{2 \sqrt{5}}\right) D^{n} f(x)
$$

The moments corresponding to $\delta^{-1}$ are the numbers

$$
(-1)^{\mathrm{n}} 5^{\mathrm{n} / 2} \mathrm{E}_{\mathrm{n}}\left(\frac{1+\sqrt{5}}{2 \sqrt{5}}\right) \quad \mathrm{n}=0,1,2, \cdots
$$

Another special case is when $\mu=1 / 2, c_{1}=0, c_{2}=1$. We get that $\lambda_{0}=1, \quad \lambda_{\mathrm{k}}=1 / 2(\mathrm{k}>0)$ are the moments of the operator

$$
\begin{equation*}
\delta \mathrm{f}(\mathrm{x})=\frac{1}{2}\{\mathrm{f}(\mathrm{x})+\mathrm{f}(\mathrm{x}+1)\} \tag{10}
\end{equation*}
$$

The moments of $\delta^{-1}$ have, therefore, the generating relation

$$
\frac{2}{1+e^{t}}=\sum_{n=0}^{\infty} \lambda_{n}^{*} t^{n} / n!
$$

and thus

$$
\lambda_{\theta}^{*}=1, \quad \lambda_{\mathrm{n}}^{*}=\left(1-2^{\mathrm{n}}\right) \frac{\mathrm{B}_{\mathrm{n}}}{\mathrm{n}} \quad(\mathrm{n} \geq 1),
$$

where $B_{n}$ are the Bernoulli numbers.
Similarly, if we consider another mean operator, namely,
(11)

$$
\operatorname{Lf}(\mathrm{x})=\frac{1}{\mathrm{n}} \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{f}(\mathrm{x}+\mathrm{kh}),
$$

we see that the moments corresponding to L are the numbers

$$
\ell_{0}=1, \quad \ell_{\mathrm{m}}=\frac{\mathrm{h}^{\mathrm{m}}}{\mathrm{n}} \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{k}^{\mathrm{m}}=\frac{\mathrm{h}^{\mathrm{m}}}{\mathrm{n}}\left\{\frac{\mathrm{~B}_{\mathrm{m}+1}(\mathrm{n})-\mathrm{B}_{\mathrm{m}+1}}{\mathrm{~m}+1}\right\},
$$

where $B_{m}{ }^{(x)}$ are the Bernoulli polynomials and $B_{m}=B_{m}{ }^{(0)}$.
The moments corresponding to the inverse operator $L^{-1}$ are

$$
\ell_{\mathrm{m}}^{*}=\mathrm{h}^{\mathrm{m}} \sum_{\mathrm{k}=0}^{\mathrm{m}}\binom{\mathrm{~m}}{\mathrm{k}} \frac{\mathrm{~B}_{\mathrm{k}}}{\mathrm{~m}-\mathrm{k}+1} \mathrm{n}^{\mathrm{k}} \quad(\mathrm{~m}=0,1,2, \cdots) .
$$

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# COMPOSITION OF $\Phi_{3}(X)$ MODULO $m$ <br> SISTER CHRISTELLE THEUSCH <br> Dominican College, Racine, Wisconsin 

## 1. INTRODUCTION

In an earlier issue of this quarterly, Cohn* investigated the value of the residues modulo $n$ of $X^{m}$ when $0 \leq X \leq(n-1)$. The object of this paper is to study the value set modulo $m$ of another function - the cyclotomic polynomial $\Phi_{3}(\mathrm{X})=\mathrm{X}^{2}+\mathrm{X}+1$, and further to consider some properties of the composition of this function with itself $n$ times. We will denote this $n-$ fold composition by

$$
\mathrm{n}: \Phi_{3}(\mathrm{X})=\Phi_{3}\left(\Phi_{3}\left(\cdots\left(\Phi_{3}(\mathrm{X})\right) \cdots\right)\right)
$$

We define

$$
\Psi(\mathrm{m}, \mathrm{n})=\left\{\mathrm{n} \Phi_{3}(\mathrm{X}) \quad(\bmod \mathrm{m}) \mid 0 \leq \mathrm{X}<\mathrm{m}\right\},
$$

and such that if $\alpha$ is in $(m, n)$, then $0 \leq \alpha<m$. Further, we let $r(m)$ be the minimum $n$ for which $\Psi(m, n)=\Psi(m, n+1)$ and refer to $\Psi(m, r(m))$ simply as $\Psi(m)$. The cardinality of $\Psi(m, n)$ will be denoted by $N(\Psi(m, n))$.

## 2. PROPERTIES

Definition. We say that $f(X)$ is modulo m-symmetric if $f(X) \equiv$ $\mathrm{f}(-\mathrm{X}-1)(\bmod \mathrm{m})$ and that $\mathrm{f}(\mathrm{X})$ is modulo m -doubly symmetric if $\mathrm{f}(\mathrm{X}) \equiv$ $\mathrm{f}(\mathrm{m} / 2-\mathrm{X}-1) \equiv \mathrm{f}(\mathrm{m} / 2+\mathrm{X}) \equiv \mathrm{f}(-\mathrm{X}-1)(\bmod \mathrm{m})$ for $0 \leq \mathrm{X}<\mathrm{m}$.

Property 1. $\mathrm{n}: \Phi_{3}(\mathrm{X})$ is modulo m -symmetric.
Proof. We have

$$
\Phi_{3}(\mathrm{X})=\mathrm{X}^{2}+\mathrm{X}+1=\mathrm{X}^{2}+2 \mathrm{X}+1-\mathrm{X}-1+1=\Phi_{3}(-\mathrm{X}-1)
$$

and hence also
*John H. E. Cohn, "On m-tic Residues Modulo n," Fibonacci Quarterly, 5 (1967), pp. 305-318.

$$
\mathrm{n}: \Phi_{3}(\mathrm{X})=\mathrm{n}: \Phi_{3}(-\mathrm{X}-1) .
$$

We note that X and $-\mathrm{X}-1$ cannot simultaneously be elements of $\Psi(\mathrm{m})$ since $\quad \mathrm{r}: \Phi_{3}(\mathrm{X})$ is modulo m -symmetric.

Property 2. $\mathrm{n}: \Phi_{3}(\mathrm{X})$ is modulo 2 p -doubly symmetric.
Proof. Elementary calculations yield

$$
\Phi_{3}(\mathrm{p}-\mathrm{X}-1) \equiv \Phi_{3}(\mathrm{p}+\mathrm{X}) \equiv \mathrm{p}^{2}+\mathrm{p}+\mathrm{X}^{2}+\mathrm{X}+1(\bmod 2 \mathrm{p})
$$

Now

$$
\mathrm{p}^{2}+\mathrm{p}=2 \mathrm{p}[(\mathrm{p}+1) / 2] \equiv 0(\bmod 2 \mathrm{p})
$$

and hence

$$
\Phi_{3}(\mathrm{X}) \equiv \Phi_{3}(\mathrm{p}+\mathrm{X})(\bmod 2 \mathrm{p})
$$

These congruences together with Property 1 yield the result.
Property 3. $N(\Psi(p, 1))$ is $(p+1) / 2$.
Proof. Since $\Phi_{3}(\mathrm{X})$ is modulo p-symmetric $N(\Psi(p, 1))$ is at most $(p+1) / 2$. Suppose

$$
\Phi_{3}(\mathrm{X}) \equiv \Phi_{3}(\mathrm{X}+\mathrm{a}) \quad(\bmod \mathrm{p})
$$

with $a \not \equiv 0(\bmod p)$. Then, simple calculations yield

$$
a(2 X+a+1) \equiv 0 \quad(\bmod p)
$$

Since $a \not \equiv 0(\bmod p)$, we must have $X+a \equiv-X-1(\bmod p)$.
Property 4. $N(\Psi(m)) \neq 1$ for $m>2$.
Proof. Clearly a necessary condition that $N(\Psi(\mathrm{~m}))=1$ is that $\Phi_{3}(\mathrm{X}) \equiv$ $\mathrm{X}(\bmod \mathrm{m})$ for exactly one X where $0 \leq \mathrm{X}<\mathrm{m}$. In order for the above congruence to hold, we need $X^{2} \equiv-1(\bmod m)$. However, for $m>2$, this congruence has either two distinct solutions or no solutions.

Property 5. $N\left(\Psi\left(2^{\mathrm{n}}\right)\right)=2^{\mathrm{n}-1} ; \quad \mathrm{r}\left(2^{\mathrm{n}}\right)=1$.
Proof. First, we note that for any $\alpha$ in $\Psi\left(2^{n}, 1\right)$ we have $\alpha \equiv 1(\bmod$ 2). Thus since $X$ and $-X-1$ are of opposite parity modulo $2^{n}$ and $\Phi_{3}(X)$ is modulo $2^{\mathrm{n}}$-symmetric $\Psi\left(2^{\mathrm{n}}, 1\right)$ is completely determined by $\Phi_{3}(2 \mathrm{k}$ $\mathrm{k}=1, \cdots, 2^{\mathrm{n}-1}$. Suppose that

$$
\Phi_{3}\left(2 \mathrm{k}_{1}-1\right) \equiv \Phi_{3}\left(2 \mathrm{k}_{2}-1\right) \quad\left(\bmod 2^{\mathrm{n}}\right)
$$

with $1 \leq \mathrm{k}_{1}, \mathrm{k}_{2} \leq 2^{\mathrm{n}-1}$. It can readily be verified that this supposition yields

$$
4\left(\mathrm{k}_{1}^{2}-\mathrm{k}_{2}^{2}\right)-2\left(\mathrm{k}_{1}-\mathrm{k}_{2}\right) \equiv 0\left(\bmod 2^{\mathrm{n}}\right)
$$

and hence

$$
\left(\mathrm{k}_{1}-\mathrm{k}_{2}\right)\left(2 \mathrm{k}_{1}+2 \mathrm{k}_{2}-1\right) \equiv 0\left(\bmod 2^{\mathrm{n}-1}\right)
$$

from which it follows immediately that $\mathrm{k}_{1}=\mathrm{k}_{2}$. Hence, $\mathrm{N}\left(\Psi\left(2^{\mathrm{n}}, 1\right)\right)=2^{\mathrm{n}-1}$ and since $\alpha \equiv 1(\bmod 2)$ we must have $\mathrm{r}\left(2^{\mathrm{n}}\right)=1$.

Property 6. If $\mathrm{p} \equiv=11,13,17,19$ modulo 20 then $\mathrm{r}(\mathrm{p})>1$.
Proof. Let

$$
\Phi_{3}((\mathrm{p}-1) / 2) \equiv \beta(\bmod \mathrm{p})
$$

First we note that if

$$
\Phi_{3}(\mathrm{X}) \not \equiv(\mathrm{p}-1) / 2(\bmod \mathrm{p})
$$

for all X , then properties 1 and 3 imply that $\beta$ is an element of $\Psi(p, 1)$ while it is not in $\Psi(p)$ and hence $r(p)>1$. Now from

$$
X^{2}+X+1 \equiv(p-1) / 2(\bmod p)
$$

it follows that

$$
2 X^{2}+2 X+3 \equiv 0(\bmod p)
$$

The quadratic formula indicates that -5 must be a quadratic residue modulo $p$ if this congruence has a solution. However -5 is a quadratic non-residue for the p in the hypothesis.

Property 7. $\mathrm{N}(\Psi(\mathrm{m}))$ is multiplicative.
Proof. Let

$$
m=p_{1}^{e_{1}} \cdots p_{t}^{e_{t}}
$$

For each $\gamma$ in $\Psi(\mathrm{m})$ there exists an X such that

$$
\mathrm{r}: \Phi_{3}(\mathrm{X})-\gamma \equiv 0(\bmod \mathrm{~m})
$$

Thus

$$
\mathrm{r}: \Phi_{3}(\mathrm{X})-\gamma \equiv 0\left(\bmod \mathrm{p}_{\mathrm{i}}^{\mathrm{e}^{\mathrm{i}}}\right), \quad 1 \leq \mathrm{i} \leq \mathrm{t}
$$

and hence

$$
\gamma \equiv \alpha_{i}\left(\bmod \mathrm{p}_{\mathrm{i}}^{\mathrm{i}}\right)
$$

for some $\alpha_{i}$ in $\Psi\left(p_{i}{ }_{i}\right)$. The Chinese Remainder Theorem assures a unique $\gamma, \quad 0 \leq \gamma<\mathrm{m}$, as a solution to this system of congruences, and hence

$$
\left.\left.\left.N(\Psi(\mathrm{~m})) \leq{\underset{1}{\mathrm{t}}}_{\mathrm{M}_{1}}^{\mathrm{N}\left(\Psi \left(\mathrm{p}_{\mathrm{i}}\right.\right.}{ }_{\mathrm{i}}\right)\right)\right]
$$

To see that equality actually holds, we suppose

$$
\gamma \equiv \alpha_{i}\left(\bmod \mathrm{p}_{\mathrm{i}}^{\mathrm{i}}\right), \quad 1 \leq \mathrm{i} \leq \mathrm{t}
$$

Since

$$
\mathrm{r}: \Phi_{3}(\mathrm{X})-\gamma \equiv \Phi_{3}(\mathrm{X})-\alpha_{\mathrm{i}} \equiv 0\left(\bmod \mathrm{p}_{\mathrm{i}}^{\mathrm{i}}\right)
$$

has a solution for each i we are guaranteed a solution to the congruence

$$
\mathrm{r}: \Phi_{3}(\mathrm{X})-\gamma \equiv 0 \quad(\bmod \mathrm{~m})
$$

Thus $\gamma$ is in $\Psi(m)$.
Property 8. $\quad \mathrm{r}(\mathrm{m})=\max \mathrm{r}\left(\mathrm{p}_{\mathrm{i}}{ }^{\mathrm{i}}\right), \quad 1 \leq \mathrm{i} \leq \mathrm{t}$.
Proof. We denote

$$
\max r\left(p_{i}{ }_{i}\right)
$$

by $\mathrm{r}^{\prime}$ and consider

$$
\mathrm{r}^{\prime}: \Phi_{3}(\mathrm{X}) \equiv \gamma(\bmod \mathrm{m})
$$

Since for

$$
\mathrm{r}^{\prime}: \Phi_{3}(\mathrm{X}) \equiv \gamma \equiv \alpha_{\mathrm{i}}\left(\bmod \mathrm{p}_{\mathrm{i}}^{\mathrm{e}_{\mathrm{i}}}\right)
$$

we have $\alpha_{i}$ in

$$
\Psi\left(\mathrm{p}_{\mathrm{i}}^{\mathrm{e}_{\mathrm{i}}}, \mathrm{r}^{\prime}\right)=\Psi\left(\mathrm{p}_{\mathrm{i}}^{\mathrm{i}^{\prime}}\right)
$$

$\gamma$ must be in $\Psi(\mathrm{m})$. On the other hand, for $\mathrm{n}<\mathrm{r}^{\prime}$, there exists at least one p such that for $\mathrm{n}: \Phi_{3}(\mathrm{X}) \equiv \gamma(\bmod \mathrm{m})$,

$$
\mathrm{n}: \Phi_{3}(\mathrm{X}) \equiv \gamma \equiv \alpha_{\mathrm{i}}\left(\bmod \mathrm{p}_{\mathrm{i}}{ }^{\mathrm{i}}\right)
$$

with $\alpha_{i}$ not in $\Psi\left(p_{i}{ }^{i_{i}}\right)$ and hence $\gamma$ cannot be in $\Psi(m)$.

## 3. EXTENSION

We note that Properties 7 and 8 can easily be extended to the composition of other cyclotomic polynomials $n: \Phi_{p}(X)$ modulo $m$. However, the other properties given are not generally valid for $n: \Phi_{p}(X)$. In particular, for $\Phi_{5}(X)$ we have $\mathrm{r}\left(2^{\mathrm{n}}\right)=\mathrm{n}$ and $\mathrm{N}\left(\Psi\left(2^{\mathrm{n}}\right)\right)=1$ with

$$
\Psi\left(2^{\mathrm{n}}\right)=2+2^{2}+\cdots+2^{\mathrm{n}}-1 \text { for } \mathrm{n}=1, \cdots, 6
$$

# SOME SUMMATION FORMULAS* 

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1. Multiple summation formulas of a rather unusual kind can be obtained in the following way. Let

$$
\begin{equation*}
f(x)=1-a_{1} x-a_{2} x^{2}-\cdots \tag{1.1}
\end{equation*}
$$

denote a series that converges for small x . Put

$$
\begin{equation*}
\frac{1}{f(x)}=1+b_{1} x+b_{2} x^{2}+\cdots \tag{1.2}
\end{equation*}
$$

so that

$$
\begin{equation*}
\frac{1}{f(x)(1-y)}=\sum_{m, n=0}^{\infty} b_{m} x^{m} y^{n} \quad\left(b_{0}=1\right) \tag{1.3}
\end{equation*}
$$

Replacing y by $\mathrm{x}^{-1} \mathrm{y}$, Eq. (1.3) becomes

$$
\begin{equation*}
\frac{1}{f(x)\left(1-x^{-1} y\right)}=\sum_{m, n=0}^{\infty} b_{m} x^{m-n} y^{n} \tag{1.4}
\end{equation*}
$$

Let k denote a fixed non-negative integer. Then that part of the righthand side of (1.4) that contains terms in $\mathrm{x}^{-\mathrm{k}}$ is evidently

$$
\begin{equation*}
\sum_{m=0}^{\infty} b_{m} y^{m+k}=y^{k} / f(y) \tag{1.5}
\end{equation*}
$$

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On the other hand, since
$\left(1-a_{1} x-a_{2} x^{2}-\cdots\right)\left(1-x^{-1} y\right)=\left(1+a_{1} y\right)-x^{-1} y-\left(a_{1}-a_{2} y\right) x-\left(a_{2}-a_{3} y\right) x^{2}-\cdots$

It follows that

$$
\begin{aligned}
& \frac{1}{f(x)\left(1-x^{-1} y\right)}=\sum_{n=0}^{\infty} \frac{\left[x^{-1} y+\left(a_{1}-a_{2} y\right) x+\left(a_{2}-a_{3} y\right) x^{2}+\cdots\right]^{n+1}}{\left(1+a_{1} y\right)^{n+1}} \\
& =\sum_{r=0}^{\infty} \sum_{s_{j}=0}^{\infty} \frac{\left(r+s_{1}+s_{2}+\cdots\right)!}{r!s_{1}!s_{2}!\cdots} \cdot \frac{y^{r}\left(a_{1}-a_{2} y\right)^{s_{1}}\left(a_{2}-a_{3} y\right)^{s_{2}} \cdots}{\left(1+a_{1} y\right)^{r+s_{1}+s_{2}+\cdots+1}} \\
& \cdot x^{-r+s_{1}+2 s_{2}+3 s_{3}+\cdots} .
\end{aligned}
$$

The part of the multiple summation on the right that contains terms in $x^{-k}$ is obtained by taking

$$
\mathrm{r}=\mathrm{k}+\mathrm{s}_{1}+2 \mathrm{~s}_{2}+3 \mathrm{~s}_{3}+\cdots
$$

Comparison with (1.5) therefore yields the following identity:
(1.6)

$$
\sum_{s_{j}=0}^{\infty} \frac{\left(k+2 s_{1}+3 s_{2}+4 s_{3}+\cdots\right)!}{s_{1}!s_{2}!\cdots\left(k+s_{1}+2 s_{2}+3 s_{3}+\cdots\right)!}
$$

$$
\frac{y^{s_{1}+2 s_{2}+3 s_{3}+\cdots}\left(a_{1}-a_{2} y\right)^{s_{1}}\left(a_{2}-a_{3} y\right)^{s_{2}}}{\left(1+a_{1} y\right)^{2 s_{1}+3 s_{2}+\cdots}}
$$

$$
=\frac{\left(1+a_{1} y\right)^{k+1}}{1-a_{1} y-a_{2} y^{2}-\cdots}
$$

If we take

$$
z_{j}=a_{j} y^{j} \quad(j=1,2,3, \cdots)
$$

(1.6) becomes
(1.7) $\sum_{s_{j}=0}^{\infty} \frac{\left(k+2 s_{1}+3 s_{2}+4 s_{3}+\cdots\right)!}{s_{1}!s_{2}!\cdots\left(k+s_{1}+2 s_{2}+\cdots\right)!} \cdot \frac{\left(z_{1}-z_{2}\right)^{s_{1}}\left(z_{2}-z_{3}\right)^{s_{2}} \cdots}{\left(1+z_{1}\right)^{2 s_{1}+3 s_{2}+\cdots}}=\frac{(1+z)^{k+1}}{1-z_{1}-z_{2}-\cdots}$.

If we now put

$$
z_{j}-z_{j+1}=u_{j} \quad(j=1,2,3, \cdots)
$$

so that

$$
z_{j}=u_{j}+u_{j+1}+u_{j+2}+\cdots \quad(j=1,2,3, \cdots)
$$

we get
(1.7)

$$
\begin{gathered}
\sum_{s_{j}=0}^{\infty} \frac{\left(k+2 s_{1}+3 s_{2}+4 s_{3}+\cdots\right)!}{s_{1}!s_{2}!\cdots\left(k+s_{1}+2 s_{2}+\cdots\right)!} \cdot \frac{u_{1}^{s_{1}} u_{2}^{s_{2}} \cdots}{\left(1+u_{1}+u_{2}+u_{3}+\cdots\right)^{2 s_{1}+3 s_{2}+\cdots}} \\
=\frac{\left(1+u_{1}+u_{2}+\cdots\right)^{k+1}}{1-u_{1}-2 u_{2}-3 u_{3}-\cdots}
\end{gathered}
$$

where

$$
u_{1}+u_{2}+u_{3}+\cdots
$$

is absolutely convergent.
2. There are numerous special cases of the above identities that may be noted. To begin with, we take

$$
u_{3}=u_{4}=\cdots=0
$$

Changing the notation slightly, Eq. (1.7) gives

$$
\begin{equation*}
\sum_{r, s=0}^{\infty} \frac{(k+2 r+3 s)!}{r!s!(k+r+2 s)!} \cdot \frac{u^{r} v^{s}}{(1+u+v)^{2 r+3 s}}=\frac{(1+u+v)^{k+1}}{1-u-2 v} \tag{2.1}
\end{equation*}
$$

In particular, for $v=0$, Eq. (2.1) reduces to

$$
\begin{equation*}
\sum_{r=0}^{\infty} \frac{(k+2 r)!}{r!(k+r)!} \frac{u^{r}}{(1+u)^{2 r}}=\frac{(1+u)^{k+1}}{1-u} \tag{2.2}
\end{equation*}
$$

This is easily verified for $\mathrm{k}=0$. Indeed,

$$
\sum_{r=0}^{\infty}\binom{2 r}{r} \frac{u^{r}}{(1+u)^{2 r}}=\left\{1-\frac{4 u}{(1+u)^{2}}\right\}^{-\frac{1}{2}}=\frac{1+u}{1-u}
$$

in agreement with the special case of (2.2).
If we take all $u_{j}=0$ except $u_{p-1}$, we get

$$
\begin{equation*}
\sum_{s=0}^{\infty}\binom{k+p s}{s} \frac{u^{s}}{(1+u)^{p s}}=\frac{(1+u)^{k+1}}{1-(p-1) u} \tag{2.3}
\end{equation*}
$$

Summations like (2.3) are usually obtained by means of the Lagrange-Burmann expansion formula. For example, it is proved [1, p. 126, No. 216] that

$$
\begin{equation*}
\sum_{n=0}^{\infty}\binom{\alpha+\beta n}{n} w^{n}=\frac{(1+z)^{\alpha}}{1-\beta w(1+z)^{\beta+1}} \tag{2.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{w}=\frac{\mathrm{z}}{(1+\mathrm{z})^{\beta}} \tag{2.5}
\end{equation*}
$$

Making use of (2.5), the right member of (2.4) is seen to be equal to

$$
\frac{(1+z)^{\alpha+1}}{1-(\beta-1) z}
$$

so that (2.4) is in agreement with (2.3).
It should be observed that (2.3) has been proved above only for integral $\mathrm{k} \geq 0, \mathrm{p} \geq 1$. However, since

$$
\begin{aligned}
\sum_{s=0}^{\infty}\binom{k+p s}{s} \frac{u^{s}}{(1+u)^{k+p s+1}} & =\sum_{s=0}^{\infty}\binom{k+p s}{s} u^{s} \sum_{r=0}^{\infty}(-1)^{r}\binom{k+r+p s}{r} u^{r} \\
& =\sum_{n=0}^{\infty} u^{n} \sum_{s=0}^{n}(-1)^{n-s}\binom{k+p s}{s}\binom{k+n-s+p s}{n-s}
\end{aligned}
$$

it follows that (2.3) is equivalent to

$$
\begin{equation*}
\sum_{s=0}^{n}(-1)^{n-s}\binom{k+p s}{s}\binom{k+n-s+p s}{n-s}=(p-1)^{n} \tag{2.6}
\end{equation*}
$$

Since (2.6) is a polynomial identity that holds for

$$
\mathrm{k}=0,1,2, \cdots ; \quad \mathrm{p}=1,2,3, \cdots,
$$

it therefore holds for arbitrary $k, p$.
3. The proof that (2.3) holds for arbitrary $k, p$ suggests that (1.7) also holds for arbitrary k. We divide both sides of (1.7) by

$$
\left(1+u+u_{2}+\cdots\right)^{\mathrm{k}+1}
$$

Then since

$$
\begin{aligned}
\left(1+u_{1}\right. & \left.+u_{2}+\cdots\right)^{-k-2 s_{1}-3 s_{2}-\cdots-1} \\
& =\sum_{r=0}^{\infty}(-1)^{r}\left(k+r+2 s_{1}+3 s_{2}+\cdots\right)\left(u_{1}+u_{2}+\cdots\right)^{r} \\
& =\sum_{r_{j}=0}^{\infty}\binom{\left.k+r+2 s_{1}+3 s_{2}+\cdots\right)}{r} \frac{r!}{r_{1}!r_{2}!\cdots} u_{1}^{r_{1}} u_{2}^{s_{2}} \cdots
\end{aligned}
$$

where $r=r_{1}+r_{2}+\ldots$, it follows that the left member of (1.7) is equal to

$$
\begin{gathered}
\sum_{r_{j}=0}^{\infty}(-1)^{r} \sum_{s_{j}=0}^{\infty} \frac{\left(k+2 s_{1}+3 s_{2}+\cdots\right)!}{s_{1}!s_{2}!\cdots\left(k+s_{1}+2 s_{2}+\cdots\right)!}\binom{k+r+2 s_{1}+3 s_{2}+\cdots}{r} \frac{r!}{r_{1}!r_{2}!} \\
\cdot u_{1}^{r_{1}+s_{1}} u_{2}^{r_{2}+s_{2}+\cdots} .
\end{gathered}
$$

Hence (1.7) is equivalent to

$$
\begin{gather*}
\sum_{r_{j}+s_{j}=n_{j}}^{\infty}(-1)^{r}\binom{\left.k+2 s_{1}+3 s_{2}+\cdots\right)}{s}\left(k+r+2 s_{1}+3 s_{2}+\cdots\right) \\
\cdot \frac{s!}{s_{1}!s_{2}!\cdots} \cdot \frac{r!}{r_{1}!r_{2}!\cdots}  \tag{3.1}\\
=\frac{\left(n_{1}+n_{2}+\cdots\right)!}{n_{1}!n_{2}!\cdots} 1^{n_{1} n_{2} n_{2} n_{3}}
\end{gather*}
$$

where

$$
\mathrm{r}=\mathrm{r}_{1}+\mathrm{r}_{2}+\cdots, \quad \mathrm{s}=\mathrm{s}_{1}+\mathrm{s}_{2}+\cdots
$$

Since (3.1) is a polynomial identity in $k$, it is valid for arbitrary k. Therefore (1.7) is proved for arbitrary $k$.
4. Another special case of (1.7) that is of some interest is obtained by taking all $u_{j}=0$ except $u_{p-1}$ and $u_{q-1}$. We evidently get
(4.1) $\sum_{r, s=0}^{\infty}\binom{k+p r+q s}{r+s}\binom{r+s}{r} \frac{u^{r} v^{s}}{(1+u+v)^{p r+q s}}=\frac{(1+u+v)^{k+1}}{1-(p-1) u-(q-1) v} \quad(q \neq p)$.

As above, we can assert that (4.1) holds for all $k$, $p$, q. This can evidently be extended in an obvious way, thus furnishing extensions of (2.3) involving an arbitrary number of parameters.

We remark that (4.1) is equivalent to

$$
\begin{array}{r}
\sum_{\substack{r+i=m \\
s+j=n}}(-1)^{i+j}\binom{k+p r+q s}{r+s}\binom{k+p r+q s+i+j}{i+j}\binom{r+s}{r}\binom{i+j}{i} \\
=\binom{m+n}{m}(p-1)^{m}(q-1)^{n}, \tag{4.2}
\end{array}
$$

which is itself a special case of (3.1).

## REFERENCE

1. G. Pólya and G. Szegö, Aufaben und Lehrsätze aus der Analysis, 1, Berlin, 1925.

## LETTER TO THE EDITOR

DAVID ZEITLIN
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In the note by W. R. Spickerman, "A Note on Fibonacci Functions," Fibonacci Quarterly, October, 1970, pp. 397-401, his Theorem 1, p. 397, states that if $f(x)$ is a Fibonacci function, i. e.,

$$
\begin{equation*}
\mathrm{f}(\mathrm{x}+2)=\mathrm{f}(\mathrm{x}+1)+\mathrm{f}(\mathrm{x}) \tag{1}
\end{equation*}
$$

then $\int f(x) d x$ is also a Fibonacci function. Since $\int f(x) d x=h(x)+C$, where $C$ is the arbitrary constant of integration, the above result assumes that $\mathbf{C}=0$. Thus, a formulation of this result in terms of a definite integral seems apropos.
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# ON ITERATIVE FIBONACCI SUBSCRIPTS 

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The Fibonacci sequence is defined by the recurrence relation $\mathrm{F}_{\mathrm{n}}+$ $F_{n+1}=F_{n+2}$ and the initial values $F_{1}=F_{2}=1$.

The main result of this paper is
Theorem 4. For positive integers $a, k, m$ and $n$ such that $k \geq m$,


The proof of Theorem 4 will depend on all results preceding it in this paper.

Let N be the set of natural numbers.
Definition 1. For any $a, b$ in $N$ the symbol $f_{n}(a, b)$ is defined for each $n$ in $N$ as follows:

$$
\begin{gather*}
\mathrm{f}_{1}(\mathrm{a}, \mathrm{~b})=\mathrm{F}_{\mathrm{ab}} \\
\mathrm{f}_{\mathrm{n}+1}(\mathrm{a}, \mathrm{~b})=\mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}(\mathrm{a}, \mathrm{~b})\right)
\end{gather*}
$$

By induction, we observe that

$$
\mathrm{f}_{\mathrm{n}}(\mathrm{a}, \mathrm{~b})=\mathrm{F}_{\mathrm{aF}}
$$

Definition 2. For any $a$ in $N$ the symbol $f_{n}(a)$ is defined for each $n$ in N as follows:
ii)

$$
\begin{gather*}
f_{1}(a)=F \\
f_{n+1}(a)=f_{1}\left(a, f_{n}(a)\right)
\end{gather*}
$$

By induction, we observe that $f_{n}(a)=f_{n}(a, 1)$.
If $a, b$ are in $N$, we write $a \mid b$ if and only if there exists some $c$ in $N$ such that $b=a c$.

In the sequel we shall let $a, b$ and $c$ denote arbitrary elements of $N$.
Lemma 1. If $b \mid c$, then $f_{1}(a, b) \mid f_{1}(a, c)$ for all $a$ in $N$.
Proof. If $b \mid c$, then $a b \mid a c$ for all $a$ in N. From Hardy and Wright [1, p. 148] we have, if $n>0$, then $F_{n} \mid F_{r n}$ for every $r>0$. So in the present notation $f_{1}(a, b)=F_{a b} \mid F_{a c}=f_{1}(a, c)$ for all $a$ in $N$.

Lemma 2. If $b \mid f_{1}(a, c)$, then $b f_{1}(a, c) \mid f_{1}(a, b c)$.
Proof. From Vinson [2] we have in the present notation,

$$
F_{a c b}=\sum_{j=1}^{b}\binom{b}{j} F_{a c}^{j} F_{a c-1}^{b-j} F_{j}
$$

For $\mathrm{j}=1$, we have

$$
\mathrm{bF}_{\mathrm{ac}} \left\lvert\,\binom{\mathrm{b}}{1} \mathrm{~F}_{\mathrm{ac} \mathrm{~F}_{\mathrm{ac}-1}^{\mathrm{b}-1} \mathrm{~F}_{1} .}\right.
$$

For $j>1$, we have, since $b \mid f_{1}(a, c)=F a c$, that

$$
\mathrm{bF}_{a c}\left|F_{a c}^{2}\right| \sum_{j=2}^{b}\binom{b}{j} F_{a c}^{j} F_{a c-1}^{b-j} F_{j}
$$

Thus $\mathrm{bf}_{1}(\mathrm{a}, \mathrm{c})=\mathrm{bF} \mathrm{ac} \mid \mathrm{F}_{\mathrm{acb}}=\mathrm{f}_{1}(\mathrm{a}, \mathrm{bc})$.
Corollary 1. If $b \mid f_{n+1}(a)$, then $b f_{n+1}(a) \mid f_{1}\left(a, b f_{n}(a)\right)$.
Corollary 2. If $b \mid f_{1}(a)$, then $\mathrm{bf}_{1}(a) \mid f_{1}(a, b)$.
Theorem 1. If $b \mid c$, then $f_{n}(a, b) \mid f_{n}(a, c)$.
Proof. We use induction on $n$. The case $n=1$ is true by Lemma 1, Suppose $f_{q}(a, b) \mid f_{q}(a, c)$. Then by Lemma 1 and Definition 1 ,

$$
f_{1}\left(a, f_{q}(a, b)\right)=f_{q+1}(a, b) \mid f_{q+1}(a, c)=f_{1}\left(a, f_{q}(a, c)\right)
$$

Corollary 3. $f_{n}(a) \mid f_{n}(a, c)$.
Theorem 2. $f_{m}\left(a, f_{n}(a, b)\right)=f_{m+n}(a, b)$.
Proof. We use induction on m . The case $\mathrm{m}=1$ is true by Definition 1. Suppose $f_{q}\left(a, f_{n}(a, b)\right)=f_{q+n}(a, b)$. Then by Definition 1 ,

$$
\mathrm{f}_{\mathrm{q}+1}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}(\mathrm{a}, \mathrm{~b})\right)=\mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{\mathrm{q}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}(\mathrm{a}, \mathrm{~b})\right)\right)=\mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{\mathrm{q}+\mathrm{n}}(\mathrm{a}, \mathrm{~b})\right)=\mathrm{f}_{\mathrm{q}+1+\mathrm{n}}(\mathrm{a}, \mathrm{~b})
$$

Corollary 4: $\mathrm{f}_{\mathrm{m}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}(\mathrm{a})\right)=\mathrm{f}_{\mathrm{m}+\mathrm{n}}(\mathrm{a})$.
Lemma 3. $f_{n}(a) f_{m+n}(a)$ for $m \geq 0$.
Proof. The case $m=0$ is clear. Suppose $m>0$. Then by corollaries 3 and 4,

$$
\mathrm{f}_{\mathrm{n}}(\mathrm{a}) \mid \mathrm{f}_{\mathrm{n}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{m}}(\mathrm{a})\right)=\mathrm{f}_{\mathrm{m}+\mathrm{n}}(\mathrm{a}) .
$$

Lemma 4. $f_{n}(a) f_{n}(a) \mid f_{2 n}(a)$.
Proof. We use induction on n. By corollary 2 and definition 2, $f_{1}(a) \mid f_{1}(a)$ implies

$$
\mathrm{f}_{1}(\mathrm{a}) \mathrm{f}_{1}(\mathrm{a}) \mid \mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{1}(\mathrm{a})\right)=\mathrm{f}_{2}(\mathrm{a})
$$

so the case $n=1$ is true. Suppose $f_{q}(a) f_{q}(a) \mid f_{2 q}(a)$. Then by Lemma 1,

$$
f_{1}\left(a, f_{q}(a) f_{q}(a)\right) \mid f_{1}\left(a, f_{2 q}(a)\right)=f_{2 q+1}(a)
$$

and by Lemma 1 again,

$$
\begin{equation*}
\mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{\mathrm{q}}(\mathrm{a}) \mathrm{f}_{\mathrm{q}}(\mathrm{a})\right)\right) \mid \mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{2 \mathrm{q}+1}(\mathrm{a})\right)=\mathrm{f}_{2(\mathrm{q}+1)}(\mathrm{a}) \tag{1}
\end{equation*}
$$

Since $f_{q+1}(a) \mid f_{q+1}(a)$ we have, by Corollary 1 ,

$$
\begin{equation*}
f_{q+1}(a) f_{q+1}(a) \mid f_{1}\left(a, f_{q+1}(a) f_{q}(a)\right) \tag{2}
\end{equation*}
$$

By Lemma 3, $f_{q}(a) \mid f_{q+1}(a)$ so by Corollary 1, $f_{q}(a) f_{q+1}(a) \mid f_{1}\left(a, f_{q}(a) f_{q}(a)\right)$. Therefore, by Lemma 1,

$$
\mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{\mathrm{q}}(\mathrm{a}) \mathrm{f}_{\mathrm{q}+1}(\mathrm{a})\right) \mid \mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{\mathrm{q}}(\mathrm{a}) \mathrm{f}_{\mathrm{q}}(\mathrm{a})\right)\right)
$$

By Equations (1) and (2), the proof is complete.
Theorem 3. $f_{m}(a) f_{n}(a) \mid f_{m+n}(a)$.
Proof. It is sufficient to prove the theorem for all $n \geq m$. Let $n=$ $\mathrm{m}+\mathrm{r}$ where $\mathrm{r} \geq 0$. We use induction on r . The case $\mathrm{r}=0$ is true by Lemma 4. Suppose $f_{m}(a) f_{m+q}(a) \mid f_{2 m+q}(a)$ for $q \geq 0$. Then, by Lemma 1 ,

$$
\begin{equation*}
\mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{\mathrm{m}}(\mathrm{a}) \mathrm{f}_{\mathrm{m}+\mathrm{q}}(\mathrm{a})\right) \mid \mathrm{f}_{1}\left(\mathrm{a}, \mathrm{f}_{2 \mathrm{~m}+\mathrm{q}}(\mathrm{a})\right)=\mathrm{f}_{2 \mathrm{m+q+1}}(\mathrm{a}) \tag{3}
\end{equation*}
$$

By Lemma 3, $f_{m}(a) \mid f_{m+q+1}(a)$, so by Corollary 1,

$$
f_{m}(a) f_{m+q+1}(a) \mid f_{1}\left(a, f_{m}(a) f_{m+q}(a)\right)
$$

By Equation (3), the proof is complete.
Lemma 5. $f_{m+n}(a) \mid f_{m}\left(a, f_{n}^{k}(a)\right)$ for $k>0$ 。
Proof. By Theorem 1, and Corollary 4, $f_{n}(a) \mid f_{n}^{k}(a)$ implies

$$
\mathrm{f}_{\mathrm{m}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}(\mathrm{a})\right)=\left.\mathrm{f}_{\mathrm{m}+\mathrm{n}}(\mathrm{a})\right|_{\mathrm{f}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}^{\mathrm{k}}(\mathrm{a})\right)
$$

Lemma 6. $f_{n}(a) f_{m}\left(a, f_{n}^{k}(a)\right) \mid f_{m}\left(a, f_{n}^{k+1}(a)\right)$ for $k \geq 0$.
Proof. The case $\mathrm{k}=0$ is true by Theorem 3 and Corollary 4. Suppose $\mathrm{k}>0$. We now use induction on m . By Lemmas 3 and 5,

$$
f_{n}(a) \mid f_{n+1}(a) f_{1}\left(a, f_{n}^{k}(a)\right)
$$

for $\mathrm{k}>0$. So by Lemma 2,

$$
f_{n}(a) f_{1}\left(a, f_{n}^{k}(a)\right) \mid f_{1}\left(a, f_{n}(a) f_{n}^{k}(a)\right)=f_{1}\left(a, f_{n}^{k+1}(a)\right)
$$

So the case $m=1$ is true. Suppose

$$
\mathrm{f}_{\mathrm{n}}(\mathrm{a}) \mathrm{f}_{\mathrm{q}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}^{\mathrm{k}}(\mathrm{a})\right) \mid \mathrm{f}_{\mathrm{q}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}^{\mathrm{k}+1}(\mathrm{a})\right)
$$

for $k>0$. Then by Lemma 1 ,

$$
\begin{equation*}
f_{1}\left(a, f_{n}(a) f_{q}\left(a, f_{n}^{k}(a)\right)\right) \mid f_{1}\left(a, f_{q}\left(a, f_{n}^{k+1}(a)\right)\right)=f_{q+1}\left(a, f_{n}^{k+1}(a)\right) \tag{4}
\end{equation*}
$$

by Definition 1. By Lemmas 3 and 5,

$$
f_{n}(a)\left|f_{q+1+n}(a)\right| f_{q+1}\left(a, f_{n}^{k}(a)\right)=f_{1}\left(a, f_{q}\left(a, f_{n}^{k}(a)\right)\right)
$$

for $\mathrm{k}>0$, which implies by Lemma 2 that

$$
f_{n}(a) f_{q+1}\left(a, f_{n}^{k}(a)\right) \mid f_{1}\left(a, f_{n}(a) f_{q}\left(a, f_{n}^{k}(a)\right)\right)
$$

So by Eq. (4), the proof is complete.
Lemma 7. $f_{n}^{k}(a) f_{n}\left(a, f_{n}^{k-1}(a)\right)$ for $k>0$.
Proof. We use induction on $k$. The case $k=1$ is clear. Suppose

$$
\mathrm{f}_{\mathrm{n}}^{\mathrm{q}}(\mathrm{a}) \mid \mathrm{f}_{\mathrm{n}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}^{\mathrm{q}-1}(\mathrm{a})\right)
$$

for $q>0$. Then

$$
\mathrm{f}_{\mathrm{n}}^{\mathrm{q}+1}(\mathrm{a})\left|\mathrm{f}_{\mathrm{n}}(\mathrm{a}) \mathrm{f}_{\mathrm{n}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}^{\mathrm{q}-1}(\mathrm{a})\right)\right| \mathrm{f}_{\mathrm{n}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}^{\mathrm{q}}(\mathrm{a})\right)
$$

for $q-1 \geq 0$, by Lemma 6 .
Theorem 4. $f_{n}^{k}(a) \mid f_{m n}\left(a, f_{n}^{k-m}(a)\right)$ for $k \geq m>0$.
Proof. We use induction on m . The case $\mathrm{m}=1$ is true by Lemma 7 .
Suppose

$$
\mathrm{f}_{\mathrm{n}}^{\mathrm{k}}(\mathrm{a}) \mid \mathrm{f}_{\mathrm{qn}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}^{\mathrm{k}-\mathrm{q}}(\mathrm{a})\right)
$$

for $k \geq q>0$. Then by Theorems 1 and 2,

$$
\begin{equation*}
f_{n}\left(a, f_{n}^{k}(a)\right) \mid f_{n}\left(a, f_{q n}\left(a, f_{n}^{k-q}(a)\right)\right)=f_{(q+1) n}\left(a, f_{n}^{k+1-(q+1)}(a)\right) \tag{5}
\end{equation*}
$$

where $k+1 \geq q+1>0$. By Lemma 7,

$$
\mathrm{f}_{\mathrm{n}}^{\mathrm{k}+1}(\mathrm{a}) \mid \mathrm{f}_{\mathrm{n}}\left(\mathrm{a}, \mathrm{f}_{\mathrm{n}}^{\mathrm{k}}(\mathrm{a})\right)
$$

for $k+1>0$. Therefore, by Eq. (5),
for $k+1 \geq q+1>0$, and the proof is complete.

## ACKNOWLEDGEMENT

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## REFERENCES

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2. John Vinson, "The Relation of the Period Modulo $m$ to the Rank of Apparition of $m$ in the Fibonacci Sequence," Fibonacci Quarterly, Vol. 1, No. 2, April 1963, p. 38.
[Continued from page 34.]
Theorem. Let $f(x)$ be a Fibonacci function (see [1]). Then,
(2)

$$
\int_{i}^{2} f(t) d t=A \quad(A \text { is a constant })
$$

is a necessary and sufficient condition that

$$
\begin{equation*}
g(x)=\int_{0}^{x} f(t) d t+A, \quad g(0)=A \tag{3}
\end{equation*}
$$

also be a Fibonacci function.
Proof. Necessity. If $\mathrm{g}(\mathrm{x})$ is a Fibonacci function, then $\mathrm{g}(\mathrm{x}+2)=$ $\mathrm{g}(\mathrm{x}+1)+\mathrm{g}(\mathrm{x})$. For $\mathrm{x}=0, \mathrm{~g}(2)=\mathrm{g}(1)+\mathrm{g}(0)$, which simplifies to (2).

Sufficiency. By integration, we have

$$
\int_{0}^{x} f(t+2) d t=\int_{0}^{x} f(t+1) d t+\int_{0}^{x} f(t) d t
$$

Let $\mathrm{t}+2=\mathrm{u}$ and $\mathrm{t}+1=\mathrm{v}$ to obtain

$$
\begin{equation*}
\int_{8}^{x+2} f(u) d u=\int_{1}^{x+1} f(v) d v+\int_{0}^{x} f(t) d t \tag{4}
\end{equation*}
$$

Using (3), we obtain from (4), $g(x+2)=g(x+1)+g(x)$, by using (2).

# BINET FORMS BY LAPLACE TRANSFORM 

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In past articles of the Fibonacci Quarterly, several methods have been suggested for solutions to $\mathrm{n}^{\text {th }}$-order difference equations.

In a series of articles entitled "Linear Recursive Relations," J. A. Jeske attacks and solves this problem by use of generating functions [1, p. 69], [2, p. 35], [3, p. 197].

In another series of articles also entitled, "Linear Recursive Relations," Brother Alfred Brousseau, one of the founders of the Fibonacci Quarterly, outlines a method of finding Binet forms using matrices [4, p. 99], [5p p. 194], [6, p. 295], [7, p. 533].

What I propose to do here is to find a general solution to the linear homogenous difference equation with distinct roots to the characteristic. The method of solution will be Laplace Transform.

Unfortunately, the Laplace Transform does not deal with discrete functions. So, to make the problem applicable, define the continuous function $y(t)$ such that $y(t)=a_{n} \quad n \leq t<n+1 n=0,1,2, \cdots$, where $a_{n}, n \in Z$, is the sequence of the difference equation. This changes the discrete sequence to a continuous and integrable function.

The following is the Laplace Transform pair:

$$
\begin{gathered}
Y(s)=\int_{0}^{\infty} e^{-s t} y(t) d t \\
y(t)=\frac{1}{2 \pi i} \int_{c-i_{\infty}}^{c+i \infty} e^{t s} Y(s) d s .
\end{gathered}
$$

The inversion formula is messy. It is a contour integral, and requires a knowledge of complex variables. In our case, we will "recognize" the resultant inverse. The following Lemma illustrates the integration of our step function $y(t)$, and will be used in a subsequent theorem.
[Feb.
Lemma 1. If $\mathrm{y}(\mathrm{t})=\mathrm{a}_{\mathrm{n}}, \mathrm{n} \leq \mathrm{t}<\mathrm{n}+1, \mathrm{n}=0,1,2, \cdots$,

$$
L\{y(t+j)\}=e^{s j} Y(s)-\left(\frac{1-e^{-s}}{s}\right) \sum_{n=0}^{j-1} a_{n} e^{s(j-n)}
$$

Proof. By definition,

$$
\mathrm{L}\{\mathrm{y}(\mathrm{t}+\mathrm{j})\}=\int_{Q}^{\infty} \mathrm{y}(\beta+\mathrm{j}) \mathrm{e}^{-\mathrm{s} \beta} \mathrm{~d} \beta
$$

Let $\beta+\mathrm{j} \rightarrow \mathrm{t}$. Then

$$
\begin{aligned}
L\{y(t+j)\} & =\int_{j}^{\infty} y(t) e^{-s(t-j)} d t \\
& =e^{s j} \int_{0}^{\infty} y(t) e^{-s t} d t . \\
& =e^{s j} \int_{0}^{\infty} y(t) e^{-s t} d t-e^{s j} \int_{0}^{j} y(t) e^{-s t} d t \\
& =e^{s j} Y(s)-e^{s j} \sum_{n=0}^{j-1} a_{n} \int_{n}^{n+1} e^{-s t} d t
\end{aligned}
$$

since $\mathrm{y}(\mathrm{t})=\mathrm{a}_{\mathrm{n}} \mathrm{n} \leq \mathrm{t}<\mathrm{n}+1$,

$$
\begin{aligned}
& =e^{s j} Y(s)-e^{s j} \sum_{n=0}^{j-1} a_{n}\left(e^{-s n}\left(\frac{1-e^{-s}}{s}\right)\right) \\
& =e^{s j} Y(s)-\left(\frac{1-e^{-s}}{s}\right) \sum_{n=0}^{j-1} a_{n} e^{s(j-n)}
\end{aligned}
$$

The next Lemma will provide the inverse that we will later "recognize."

Lemma 2. If $\mathrm{y}(\mathrm{t})=\alpha^{\mathrm{n}}, \mathrm{n} \leq \mathrm{t}<\mathrm{n}+1, \mathrm{n}=0,1,2, \cdots, \quad$ where $\alpha$ is a constant, then

$$
\mathrm{Y}(\mathrm{~s})=\left(\frac{1-\mathrm{e}^{-\mathrm{s}}}{\mathrm{~s}}\right)\left(\frac{1}{1-\alpha \mathrm{e}^{-s}}\right)
$$

Proof. By definition,

$$
\begin{aligned}
Y(s) & =\int_{0}^{\infty} y(t) e^{-s t} d t \\
& =\sum_{n=0}^{\infty} \int_{n}^{n+1} \alpha^{n} e^{-s t} d t \\
& =\sum_{n=0}^{\infty} \alpha^{n} e^{-s n}\left(\frac{1-e^{-s}}{s}\right) \\
& =\left(\frac{1-e^{-s}}{s}\right) \sum_{n=0}^{\infty}\left(\left(\alpha e^{-s}\right)^{n}\right. \\
& =\left(\frac{1-e^{-s}}{s}\right)\left(\frac{1}{1-\alpha e^{-s}}\right)
\end{aligned}
$$

The third and last Lemma is a very slight modification to the Partial Fractions Theorem to fit our particular needs. Here $Q(x)$ has distinct roots $\alpha_{i}$

Lemma 3. Let $Q(x)$ be a polynomial, degree $N$. Let $P(x)$ be a polynomial, degree $\leq N$. Then if

$$
\frac{P(x)}{Q(x)}=\sum_{i=1}^{N} \frac{\gamma_{i}}{\left(1-\alpha_{i} x^{-1}\right)} \quad, \gamma_{i}=\frac{P\left(\alpha_{i}\right)}{\alpha_{i} Q^{\prime}\left(\alpha_{i}\right)}
$$

Proof. Let

$$
\frac{P(x)}{Q(x)}=\sum_{i=1}^{N} \frac{\gamma_{i}}{\left(1-\alpha_{i} x^{-1}\right)}
$$

Then

$$
\begin{gathered}
P(x)=\sum_{i=1}^{N} \frac{\gamma_{i} Q(x)}{\left(1-\alpha_{i} x^{-1}\right)} \\
P(x)=\sum_{i=1}^{N} \frac{x \gamma_{i} Q(x)}{x-\alpha_{i}} \\
\lim _{x \rightarrow \alpha_{j}} P(x)=\sum_{i=1}^{M}\left[\left[x \lim _{i \rightarrow \alpha_{j}} x\right]\left[\lim _{i \rightarrow \alpha_{j}} \frac{Q(x)}{\left(x-\alpha_{i}\right)}\right]\right]
\end{gathered}
$$

The limit on the right is $Q^{\prime}\left(\alpha_{j}\right)$ when $i=j$ and 0 otherwise. Therefore,

$$
\begin{aligned}
& P\left(\alpha_{j}\right)=\alpha_{j} \gamma_{j} Q^{\prime}\left(\alpha_{j}\right) \\
& \Rightarrow \gamma_{j}=\frac{P\left(\alpha_{j}\right)}{\alpha_{j} Q^{\prime}\left(\alpha_{j}\right)}
\end{aligned}
$$

We now have sufficient information to solve the problem. First, we find the transform of the difference equation producing $\left\{a_{n} \mid n=Z^{+}\right\}$.

Theorem 1. If $\mathrm{y}(\mathrm{t})=\mathrm{a}_{\mathrm{n}}, \mathrm{n} \leq \mathrm{t}<\mathrm{n}+1, \mathrm{n}=0,1,2, \cdots, \quad$ and

$$
\sum_{j=0}^{N} A_{j} y(t+j)=0:
$$

$A_{j}$ are coefficients: $N$ is the degree, then the transform

$$
Y(s)=\left(\frac{1-e^{-s}}{s}\right) \frac{\sum_{i=1}^{N} A_{j} \sum_{n=0}^{j-1} a_{n} e^{s(j-n)}}{\sum_{j=0}^{N} A_{j} e^{s j}}
$$

Proof.

$$
\begin{gathered}
L\left\{\sum_{j=0}^{N} A_{j} y(t+j)\right\}=\sum_{j=0}^{N} A_{j} L\{y(t+j)\}=0 \\
A_{0} Y(s)+\sum_{j=1}^{N} A_{j} L\{y(t+j)\}=0
\end{gathered}
$$

From Lemma 1,

$$
\begin{gathered}
A_{0} Y(s)+\sum_{j=1}^{N} A_{j}\left[e^{s j} Y(s)-\sum_{n=0}^{j-1} a_{n} e^{s(j-n)}\left(\frac{1-e^{-s}}{s}\right)\right]=0 \\
A_{0} Y(s)+\sum_{j=1}^{N} A_{j} e^{s j} Y(s)=\sum_{j=1}^{N} A_{j} \sum_{n=0}^{j-1} a_{n} e^{s(j-n)}\left(\frac{1-e^{-s}}{s}\right) \\
Y(s) \\
Y(s)=\left(\frac{1-e^{-s}}{s}\right) \frac{\sum_{j=1}^{N} A_{j} \sum_{n=0}^{j-1} a_{n} e^{s(j-n)}}{\sum_{j=0}^{N} A_{j} e^{s j}}
\end{gathered}
$$

The transform is actually a quotient of polynomials in $e^{s}$. The following is a corollary based on the previous theorem and Lemma 3. We get

Corollary. If $\mathrm{y}(\mathrm{t})=\mathrm{a}_{\mathrm{n}}, \mathrm{n} \leq \mathrm{t}<\mathrm{n}+1, \mathrm{n}=0,1,2, \cdots, \quad$ and

$$
\sum_{j=0}^{N} A_{j} y(t+j)=0
$$

and the roots of

$$
\sum_{j=0}^{N} A_{j} x^{j}
$$

distinct $\left(\alpha_{i}\right)$, then

$$
\mathrm{Y}(\mathrm{~s})=\left(\frac{1-\mathrm{e}^{-\mathrm{s}}}{\mathrm{~s}}\right) \sum_{\mathrm{i}=1}^{\mathrm{N}} \frac{\gamma_{\mathrm{i}}}{\left(1-\alpha_{\mathrm{i}} \mathrm{e}^{-\mathrm{s}}\right)}
$$

where

$$
\gamma_{i}=\frac{\sum_{j=1}^{N} A_{j} \sum_{n=0}^{j-1} a_{n} \alpha_{i}^{j-n}}{\sum_{j=1}^{N} j A_{j} \alpha_{i}^{j}}
$$

Proof. Let $x=e^{s}$. Then

$$
\begin{gathered}
P(x)=\sum_{j=1}^{N} A_{j} \sum_{n=0}^{j-1} a_{n} x^{j-n} \\
Q(x)=\sum_{j=0}^{N} A_{j} x^{j} \\
Q^{\prime}(x)=\sum_{j=1}^{N} j A_{j} x^{j-1}
\end{gathered}
$$

Then if the roots of $\mathrm{Q}(\mathrm{x}), \alpha_{\mathrm{i}}$, are distinct

$$
\frac{P(x)}{Q(x)}=\sum_{i=1}^{N} \frac{\gamma_{i}}{1-\alpha_{i} e^{-\dot{s}}}
$$

where

$$
\gamma_{i}=\frac{P\left(\alpha_{i}\right)}{\alpha_{i} Q^{\prime}\left(\alpha_{i}\right)}=\frac{\sum_{j=1}^{N} A_{j} \sum_{n=0}^{j-1} a_{n} \alpha_{i}^{j-n}}{\alpha_{i} \sum_{j=1}^{N} j A_{j} \alpha_{i}^{j-1}}
$$

Therefore,

$$
\mathrm{Y}(\mathrm{~s})=\left(\frac{1-\mathrm{e}^{-\mathrm{s}}}{\mathrm{~s}}\right) \sum_{\mathrm{i}=1}^{\mathrm{N}} \frac{\gamma_{\mathrm{i}}}{\left(1-\alpha_{\mathrm{i}} \mathrm{e}^{-\mathrm{s}}\right)}
$$

where

$$
\gamma_{i}=\frac{\sum_{j=1}^{j} A_{j} \sum_{n=0}^{j-1} a_{n} a_{i}^{j-n}}{\sum_{j=1}^{N} j A_{j} \alpha_{i}^{j}}
$$

The Corollary gives a very nice little package to unravel. Finding the inverse is a direct result of Lemma 2.

Theorem 2. If $\mathrm{y}(\mathrm{t})=\mathrm{a}_{\mathrm{n}}, \quad \mathrm{n} \leq \mathrm{t}<\mathrm{n}+1, \mathrm{n}=0,1,2, \cdots$, and

$$
\sum_{j=0}^{N} A_{j} y(t+j)=0
$$

then

$$
\mathrm{y}(\mathrm{t})=\sum_{\mathrm{i}=1}^{\mathrm{N}} \gamma_{\mathrm{i}} \alpha_{\mathrm{i}}^{\mathrm{n}} \quad \mathrm{n}=0,1,2, \cdots
$$

where

$$
\gamma_{i}=\frac{\sum_{j=1}^{N} A_{j} \sum_{n=0}^{j-1} a_{n} \alpha_{i}^{j-n}}{\sum_{j=1}^{N} j A_{j} \alpha_{i}^{j}}
$$

The proof is implicit from the Corollary and Lemma 2. Consider the following problem of Pell:

$$
P_{n+2}=2 P_{n+1}+P_{n} \quad \begin{aligned}
& P_{0}=0 \\
& P_{1}=1
\end{aligned}
$$

Translating this into our terminology yields:

$$
\begin{array}{rlrl}
\mathrm{y}(\mathrm{t}+2)-2 \mathrm{y}(\mathrm{t}+1)-\mathrm{y}(\mathrm{t}) & =0 & & \mathrm{n} \leq \mathrm{t}<\mathrm{n}+1 \quad \mathrm{n}=0,1,2, \cdots \\
\mathrm{~A}_{0} & =-1 \\
\mathrm{~A}_{1} & =-2 \\
\mathrm{~A}_{2} & =1 & & a_{0}=0 \\
a_{1}=1
\end{array}
$$

Since $\alpha^{2}-2 \alpha-1=0, \alpha=1 \pm \sqrt{2}$. Let

$$
\begin{aligned}
& \alpha_{1}=1+\sqrt{2} \\
& \alpha_{2}=1-\sqrt{2}
\end{aligned}
$$

Now from Theorem 2:

$$
\begin{gathered}
y(t)=\gamma_{1} \alpha_{1}^{n}+\gamma_{2} \alpha_{2}^{n} \\
\gamma_{i}=\frac{A_{1} a_{0} \alpha_{i}+A_{2}\left(a_{0} \alpha_{i}^{2}+a_{1} \alpha_{i}\right)}{A_{1} \alpha_{i}+2 A_{2} \alpha_{i}^{2}}
\end{gathered}
$$

After reducing with $\mathrm{a}_{0}=0$,

$$
\gamma_{i}=\frac{A_{2} a_{1}}{2 A_{2} \alpha_{i}+A_{1}} \quad \text { or } \quad \gamma_{i}=\frac{1}{2 \alpha_{i}-2}
$$

Since $\alpha_{1}=1+\sqrt{2}$,

$$
\gamma_{2}=\frac{1}{2(1+\sqrt{2})-2}=-\frac{1}{2 \sqrt{2}}
$$

Since $\alpha_{2}=1-\sqrt{2}$,

$$
\gamma_{2}=\frac{1}{2(1-\sqrt{2})-2}=-\frac{1}{2 \sqrt{2}}
$$

Therefore,

$$
\mathrm{y}(\mathrm{t})=\mathrm{a}_{\mathrm{n}}=\frac{\alpha_{1}^{\mathrm{n}}-\alpha_{2}^{\mathrm{n}}}{2 \sqrt{2}}
$$

which of course we recognize is the Binet form for the Pell sequence. In fact, similarly we can find Binet forms for Fibonacci, Lucas, or any other Homogencus Linear Difference Equations where roots to $\Sigma_{i} A_{i} x^{i}$, the characteristic, are distinct.

One more logical extension of Fibonacci sequence is the Tribonacci. This problem is the Fibonacci equation extended to the next degree.

$$
T_{n+2}=T_{n+2}+T_{n+1}+T_{n}
$$

In this instance, the most difficult part lies in solving the characteristic equation,

$$
m^{3}-m^{2}-m-1=0
$$

for its roots using Cardan formulae. This involves a little algebra and a little time. The procedure yields the roots,

$$
\begin{aligned}
& \alpha_{1}= \frac{1}{3}+\left(\frac{19}{27}+\frac{1}{3}\left(\frac{11}{3}\right)^{1 / 2}\right)^{1 / 3}+\left(\frac{19}{27}-\frac{1}{3}\left(\frac{11}{3}\right)^{1 / 2}\right)^{1 / 3} \\
& \alpha_{2}= \frac{1}{3}-\frac{1}{2}\left(\frac{19}{27}+\frac{1}{3}\left(\frac{11}{3}\right)^{1 / 2}\right)^{1 / 3}-\frac{1}{2}\left(\frac{19}{27}-\frac{1}{3}\left(\frac{11}{3}\right)^{1 / 2}\right)^{1 / 3} \\
&+\frac{i \sqrt{3}}{2}\left[\left(\frac{19}{27}+\frac{1}{3}\left(\frac{11}{3}\right)^{1 / 2}\right)^{1 / 3}-\left(\frac{19}{27}-\frac{1}{3}\left(\frac{11}{3}\right)^{1 / 2}\right)^{1 / 3}\right] \\
& \alpha_{3}=\bar{\alpha}_{2}
\end{aligned}
$$

Now for those of us more furtunate fellows, we can simplify some of this by means of a computer, which yields:

$$
\begin{aligned}
& \alpha_{1}=1.84 \\
& \alpha_{2}=-0.42+0.61 \mathrm{i} \\
& \alpha_{3}=-0.42-0.61 \mathrm{i}
\end{aligned}
$$

From Theorem 2,

$$
\left.\begin{array}{rlrl}
A_{3}=1 & a_{0} & =1 \\
A_{2}= & A_{1}=A_{0}=-1 & a_{1} & =0 \\
a_{2} & =0
\end{array}\right] \begin{array}{ll} 
\\
\gamma_{i}=\frac{A_{1} a_{0} \alpha_{i}+A_{2}\left(a_{0} \alpha_{i}^{2}+a_{i} \alpha_{i}\right)+A_{3}\left(a_{0} \alpha_{i}^{3}+a_{1} \alpha_{i}^{2}+a_{2} \alpha_{i}\right)}{A_{1} \alpha_{i}+2 A_{2} \alpha_{i}^{2}+3 A_{3} \alpha_{i}^{3}}
\end{array}
$$

Reduced, $\quad\left(\alpha^{3}=\alpha^{2}+\alpha+1\right)$

$$
\gamma_{i}=\frac{1}{\alpha_{i}^{2}+2 \alpha_{i}+3}
$$

Therefore,
$\mathrm{y}(\mathrm{t})=\mathrm{a}_{\mathrm{n}}=\mathrm{T}_{\mathrm{n}}=\left(\frac{1}{\alpha_{1}^{2}+2 \alpha_{1}+3}\right) \alpha_{1}^{\mathrm{n}}+\left(\frac{1}{\alpha_{2}^{2}+2 \alpha_{2}+3}\right) \alpha_{2}^{\mathrm{n}}+\left(\frac{1}{\alpha_{3}^{2}+2 \alpha_{3}+3}\right) \alpha_{3}^{\mathrm{n}}$.

Now, you, too, can find your own Binet forms.

## FOOD FOR THOUGHT

Brother Alfred Brousseau says, for $\mathrm{N}=2$,

$$
\boldsymbol{\gamma}_{1}=\frac{a_{0} \alpha_{2}-a_{1}}{\alpha_{2}-\alpha_{1}} \quad \gamma_{2}=\frac{a_{0} \alpha_{1}-a_{1}}{\alpha_{1}-\alpha_{2}}
$$

[Continued on page 112.]

## A SYMMETRIC SUBSTITUTE FOR STIRLING NUMBERS

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Let

$$
\binom{x}{0}=1
$$

and

$$
\binom{x}{r}=\frac{x(x-1)(x-2) \cdots(x-r+1)}{1: 2 \cdot 3 \cdots r}
$$

for all complex numbers x and all positive integers r . It is well known that

$$
\begin{equation*}
\binom{0}{r}+\binom{1}{r}+\binom{2}{r}+\cdots+\binom{n}{r}=\binom{n+1}{r+1}, \tag{1}
\end{equation*}
$$

and that for every non-negative integer $d$ there exist integers $s_{d 0}, s_{d 1}, \cdots$, $s_{\text {dd }}$ such that

$$
\begin{equation*}
\mathrm{x}^{\mathrm{d}}=\mathrm{s}_{\mathrm{d} 0}\binom{\mathrm{x}}{0}+\mathrm{s}_{\mathrm{d} 1}\binom{\mathrm{x}}{1}+\mathrm{s}_{\mathrm{d} 2}\binom{\mathrm{x}}{2}+\cdots+\mathrm{s}_{\mathrm{dd}}\binom{\mathrm{x}}{\mathrm{~d}} \tag{2}
\end{equation*}
$$

holds for all x . (The $\mathrm{s}_{\mathrm{dj}}$ are related to the Stirling numbers of the second kind.) Using (1) and (2), one obtains the summation formulas

$$
\text { (3) } \quad 0^{\mathrm{d}}+1^{\mathrm{d}}+2^{\mathrm{d}}+\cdots+\mathrm{n}^{\mathrm{d}}=\mathrm{s}_{\mathrm{d} 0}\binom{\mathrm{n}+1}{1}+\mathrm{s}_{\mathrm{d} 1}\binom{\mathrm{n}+1}{2}+\cdots+\mathrm{s}_{\mathrm{dd}}\binom{\mathrm{n}+1}{\mathrm{~d}+1}
$$

This paper presents alternates for (2) and (3) in which the $s_{d j}$ are replaced by coefficients having symmetry properties and other advantages. Part of the work generalizes with the help of Dov Jarden's results from the $\binom{n}{r}$
to generalized binomial coefficients.

Using the well known

$$
\binom{x}{r}+\binom{x}{r+1}=\binom{x+1}{r+1}
$$

one easily proves
(4) $\binom{x}{r-s}=\binom{s}{0}\binom{x+s}{r}-\binom{s}{1}\binom{x+s-1}{r}+\cdots+(-1)^{s}\binom{s}{s}\binom{x}{r}$
by mathematical induction. Then (2) and (4) imply that for every non-negative integer $d$ there exist integers $a_{d j}$ such that

$$
\begin{equation*}
x^{d}=s_{d 0}\binom{x}{d}+a_{d 1}\binom{x+1}{d}+\cdots+a_{d d}\binom{x+d}{d} \tag{5}
\end{equation*}
$$

From (1) and (5), one now obtains
(6) $0^{d}+1^{d}+\cdots+n^{d}=a_{d 0}\binom{x+1}{d+1}+a_{d 1}\binom{x+2}{d+1}+\cdots+a_{d d}\binom{x+d+1}{d+1}$.

For example,

$$
\begin{gathered}
x^{2}=\binom{x}{2}+\binom{x+1}{2}, \quad x^{3}=\left(\begin{array}{l}
x \\
3 \\
3
\end{array}\right)+4\binom{x+1}{3}+\binom{x+2}{3}, \\
x^{4}=\binom{x}{4}+11\binom{x+1}{4}+11\binom{x+2}{4}+\binom{x+3}{4}, \\
x^{5}=\binom{x}{5}+26\binom{x+1}{5}+66\binom{x+2}{5}+26\binom{x+3}{5}+\binom{x+4}{5}, \\
0^{2}+1^{2}+2^{2}+\cdots+n^{2}=\binom{n+1}{3}+\binom{n+2}{3} .
\end{gathered}
$$

and

$$
\sum_{k=0}^{n} k^{3}=\binom{n+1}{4}+4\binom{n+2}{4}+\binom{n+3}{4}
$$

The listed cases of (5) suggest that the following may be true:

$$
\begin{gather*}
a_{d d}=0  \tag{7}\\
a_{d 0}=1=a_{d, d-1}  \tag{8}\\
a_{d j}=a_{d, d-1-j}
\end{gather*}
$$

(9)

$$
\begin{equation*}
a_{d 0}+a_{d 1}+\cdots+a_{d, d-1}=1 \cdot 2 \cdot 3 \ldots d=d! \tag{10}
\end{equation*}
$$

Successively letting x be $0,-1,1,-2,2, \cdots$ in (5) establishes (7), (8), and (with the help of mathematical induction) the symmetry formula (9). These substitutions also prove that the $a_{d j}$ are unique. One obtains (10) from

$$
\begin{aligned}
\frac{1}{d+1}=\int_{0}^{1} x^{d} d x & =\lim _{n \rightarrow \infty}\left(\left[\left(\frac{1}{n}\right)^{d}+\left(\frac{2}{n}\right)^{d}+\cdots+\left(\frac{n}{n}\right)^{d}\right] / n\right) \\
& =\lim _{n \rightarrow \infty}\left[\left(1^{d}+2^{d}+\cdots+n^{d}\right) / n^{d+1}\right] \\
& =\left[a_{d 0}+a_{d 1}+\cdots+a_{d, d-1}\right] /(d+1)!
\end{aligned}
$$

A recursion formula for the $a_{d j}$ is derived as follows:
$\sum_{j=0}^{d} a_{d+1, j}\binom{x+j}{d+1}=x^{d+1}$

$$
\begin{aligned}
& =x \sum_{j=0}^{d-1} a_{d j}\binom{x+j}{d} \\
& =\sum_{j=0}^{d-1} a_{d j}[(x-d+j)+(d-j)]\binom{x+j}{d}
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{j=0}^{d-1} a_{d j}\left[(d+1)\binom{x+j}{d+1}+(d-j)\binom{x+j}{d}\right] \\
& =\sum_{j=0}^{d-1} a_{d j}\left[(d+1)\binom{x+j}{d+1}+(d-j)\binom{x+j+1}{d+1}-(d-j)\binom{x+j}{d+1}\right] \\
& =\sum_{j=0}^{d-1} a_{d j}(j+1)\binom{x+j}{d+1}+\sum_{j=0}^{d-1} a_{d j}(d-j)\binom{x+j+1}{d+1} \\
& =\sum_{j=0}^{d-1} a_{d j}(j+1)\binom{x+j}{d+1}+\sum_{k=1}^{d} a_{d, k-1}(d-k+1)\binom{x+k}{d+1} \\
& =a_{d 0}\binom{x}{d+1}+\sum_{j=1}^{d-1}\left[(j+1) a_{d j}+(d-j+1) a_{d, j-1}\right]\binom{x+j}{d+1} \\
& +a_{d, d-1}\binom{x+d}{d+1} .
\end{aligned}
$$

This and uniqueness of the $a_{d j}$ imply that for $j=1,2, \cdots, d-1$ onehas

$$
\begin{equation*}
a_{d+1, j}=(j+1) a_{d j}+(d-j+1) a_{d, j-1} \tag{11}
\end{equation*}
$$

Using $a_{d 0}=1$ and (11) gives us $a_{d+1,1}=2 a_{d 1}+d$. Let $E$ be the operator on functions of $d$ such that $E y_{d}=y_{d+1}$. Then $(E-2) a_{d 1}=d$ and

$$
\left(E^{2}-2 E+1\right)(E-2) a_{d 1}=(d+2)-2(d+1)+d=0
$$

It follows from the theory of linear homogeneous difference equations with constant coefficients that there are constants $e_{0}, e_{1}$, and $e_{2}$ such that

$$
a_{d 1}=e_{0}+e_{1} d+e_{2} \cdot 2^{d} \quad \text { for } \quad d=1,2,3, \cdots
$$

Using the known values of $a_{11}, a_{21}$, and $a_{31}$ one solves for $e_{0}, e_{1}$, and $e_{2}$ and thus shows that $a_{d 1}=2^{d}-d-1$. Similarly, one sees that

$$
(E-1)^{3}(E-2)^{2}(E-3) a_{d 2}=0
$$

and hence that there are constants $f_{i}$ such that

$$
a_{d 2}=\left(f_{0}+f_{1} d+f_{2} d^{2}\right)+\left(f_{3}+f_{4} d\right) 2^{d}+f_{5} \cdot 3^{d}
$$

Determining the $f_{i}$, one finds that

$$
a_{d 2}=3^{d}-\binom{d+1}{1} 2^{d}+\binom{d+1}{2}
$$

Now (or after additional cases) one conjectures that

$$
\begin{equation*}
a_{d j}=\sum_{k=0}^{j}(-1)^{k}(j+1-k)^{d}\binom{d+1}{k} \tag{12}
\end{equation*}
$$

Because of the symmetry formula (9), we know that (12) is equivalent to

$$
\begin{equation*}
a_{d j}=\sum_{k=0}^{d-j-1}(-1)^{k}(d-j-k)^{d}\binom{d+1}{k} \tag{13}
\end{equation*}
$$

Substituting (13) into (5) gives us

$$
\begin{equation*}
x^{d}=\sum_{j=0}^{d-1}\left\{\left[\sum_{k=0}^{d-j-1}(-1)^{k}(d-j-k)^{d}\binom{d+1}{k}\right]\binom{x+j}{d}\right\} \tag{14}
\end{equation*}
$$

Since the $\mathrm{a}_{\mathrm{dj}}$ that satisfy (5) are unique, one can prove (13) and (12) by showing that (14) is an identity in x . Since both sides of (14) are polynomials in
$x$ of degree $d$, it suffices to verify (14) for the $d+1$ values $x=0,1$, $\cdots$, d. For such an $x$, (14) becomes

$$
\begin{equation*}
x^{d}=x^{d}+\sum_{r=1}^{x-1}\left\{r^{d} \sum_{j=0}^{x-r}(-1)^{j}\binom{d+1}{j}\binom{x+d-r-j}{d}\right\} \tag{15}
\end{equation*}
$$

Since $\binom{h}{d}=0$ for $h=0,1, \cdots, d-1$, one has

$$
\begin{equation*}
\sum_{j=0}^{x-r}(-1)^{j}\binom{d+1}{j}\binom{x+d-r-j}{d}=\sum_{j=0}^{d+1}(-1)^{j}\binom{d+1}{j}\binom{x+d-r-j}{d} \tag{16}
\end{equation*}
$$

The right side sum in (16) is zero since it is a $(d+1)^{\text {st }}$ difference of a polynomial of degree $d$. Hence (15) becomes the tautology $\mathrm{x}^{\mathrm{d}}=\mathrm{x}$. This establishes (13) and (12).

We next apply some of the above material to convolution formulas. It is well known (and easily shown by Maclaurin's expansion or Newton's binomial expansion) that

$$
\begin{equation*}
(1-x)^{-d-1}=\binom{d}{d}+\binom{d+1}{d} x+\binom{d+2}{d} x^{2}+\cdots \text { for }-1<x<1 \tag{17}
\end{equation*}
$$

Using (5) and (17), we obtain

$$
\begin{array}{r}
\left(a_{d, d-1}+a_{d, d-2} x+\cdots+a_{d, 0} x^{d-1}\right)(1-x)^{-d-1}=1^{d}+2^{d} x+3^{d} x^{2}+\cdots  \tag{18}\\
|x|<1
\end{array}
$$

Now let
(19) $p(d, x)=a_{d, 0}+a_{d, 1} x+\cdots+a_{d, d-1} x^{d-1}=a_{d, d-1}+a_{d, d-2}+\cdots+a_{d, 0} x^{d-1}$.

Then (18) can be rewritten as
(20)

$$
p(d, x) \cdot(1-x)^{-d-1}=\sum_{j=0}^{\infty}(j+1)^{d} x^{j}, \quad|x|<1 .
$$

Also let
(21) $p(d, x) p(e, x)=q(d, e, x)=c_{d, e, 0}+c_{d, e, 1} x+\cdots+c_{d, e, d+e-2} x^{d+e-2}$.

Then

$$
\begin{equation*}
\sum_{k=0}^{n} k^{d}(n-k)^{e} \tag{22}
\end{equation*}
$$

is the coefficient of $x^{n}$ in the Maclaurin expansion of

$$
q(d, e, x)(1-x)^{-d-e-2}
$$

i. e., (22) is equal to

$$
\begin{equation*}
\sum_{j=0}^{d+e-2} c_{d, e, j}\binom{n+1+j}{d+e+1} \tag{23}
\end{equation*}
$$

For example, since $p(3, x)=1+4 x+x^{2}$, and $p(2, x)=1+x$, we have

$$
\mathrm{q}(3,2, \mathrm{x})=\left(1+4 \mathrm{x}+\mathrm{x}^{2}\right)(1+\mathrm{x})=1+5 \mathrm{x}+5 \mathrm{x}^{2}+\mathrm{x}^{3}
$$

and it follows from the equality of (22) and (23) that

$$
\sum_{k=0}^{n} k^{3}(n-k)^{2}=\binom{n+1}{6}+5\binom{n+2}{6}+5\binom{n+3}{6}+\binom{n+4}{6}
$$

We note that the recursion formula (11) for the $a_{d j}$ can also be derived from (20) using

$$
\begin{equation*}
\mathrm{d}\left[\operatorname{xp}(d, x)(1-x)^{-d-1}\right] / d x=p(d+1, x)(1-x)^{-d-2} \tag{24}
\end{equation*}
$$

Next we turn to generalizations of (5) and (12) in which the sequence $0,1,2,3, \cdots$ is replaced by any sequence $U_{0}, U_{1}, U_{2}, U_{3}, \cdots$ satisfying

$$
\begin{gather*}
\mathrm{U}_{0}=0, \mathrm{U}_{1}=1, \quad \mathrm{U}_{\mathrm{n}+2}=\mathrm{g} \mathrm{U}_{\mathrm{n}+1}-\mathrm{h} \mathrm{U}_{\mathrm{n}} \text { for } \mathrm{n}=0,1,2, \cdots,  \tag{25}\\
\text { and } \mathrm{h}^{2}=1 .
\end{gather*}
$$

The following table indicates some of the well-known sequences that are included for special values of $g$ and $h$ :

| $g$ | h | Sequence |  |
| :--- | :---: | :--- | :--- |
| 2 | 1 | Natural Numbers: | $\mathrm{U}_{\mathrm{n}}=\mathrm{n}$ |
| 1 | -1 | Fibonacci Numbers: $\mathrm{U}_{\mathrm{n}}=\mathrm{F}_{\mathrm{n}}$ |  |
| 2 | -1 | Pell Numbers: | $\mathrm{U}_{\mathrm{n}}=\mathrm{P}_{\mathrm{n}}$ |
| $\mathrm{L}_{\mathrm{k}}$ | $(-1)^{\mathrm{k}}$ | $\mathrm{U}_{\mathrm{n}}=\mathrm{F}_{\mathrm{kn}} / \mathrm{F}_{\mathrm{k}}$ |  |

A key formula for the generalized sequence $U_{n}$ is the addition formula

$$
\mathrm{U}_{2} \mathrm{U}_{\mathrm{m}+\mathrm{n}+2}=\mathrm{U}_{\mathrm{m}+2} \mathrm{U}_{\mathrm{n}+2}-\mathrm{U}_{\mathrm{m}} \mathrm{U}_{\mathrm{n}}
$$

which is established by double induction using (25) and verification for the four cases in which $(\mathrm{m}, \mathrm{n})$ is $(0,0),(0,1),(1,0)$, and $(1,1)$.

We now assume that ( $\mathrm{g}, \mathrm{h}$ ) is not $(1,1)$ in (25); then (25) is ordinary in the sense of Torretto-Fuchs (see [1]) and so $\mathrm{U}_{\mathrm{n}} \neq 0$ for $\mathrm{n}>0$. Then we use the Torretto-Fuchs notation

$$
\left[\begin{array}{l}
\mathrm{n} \\
0
\end{array}\right]=1, \quad\left[\begin{array}{l}
\mathrm{n} \\
\mathrm{r}
\end{array}\right]=\frac{\mathrm{U}_{\mathrm{m}} \mathrm{U}_{\mathrm{n}-1} \cdots \mathrm{U}_{\mathrm{n}-\mathrm{r}+1}}{\mathrm{U}_{1} \mathrm{U}_{2} \cdots \mathrm{U}_{\mathrm{r}}} \quad \text { for } \quad \mathrm{r}=1,2, \cdots
$$

for Dov Jarden's generalized binomials. Jarden showed 2 that

$$
\sum_{j=0}^{d+1}(-1)^{j_{h}}{ }^{j(j-1) / 2}\left[\begin{array}{c}
d+1  \tag{26}\\
j
\end{array}\right] Z_{n-j}=0
$$

if $Z_{n}$ is the term-by-term product of the $n{ }^{\text {th }}$ terms of $d$ sequences each of which satisfies the recursion formula (25). The sequence $Z_{n}=\left[\begin{array}{l}n \\ d\end{array}\right]$ is such a product, hence

$$
\sum_{j=0}^{d+1}(-1)^{j} h^{j(j-1) / 2}\left[\begin{array}{c}
d+1  \tag{27}\\
j
\end{array}\right]\left[\begin{array}{c}
n-j \\
d
\end{array}\right]=0
$$

We are now in a position to give the following generalizations of (5) and (12):
(28) $\quad U_{n}^{d}=B_{d 0}\left[\begin{array}{c}n+d-1 \\ d\end{array}\right]+B_{d 1}\left[\begin{array}{c}n+d-2 \\ d\end{array}\right]+\cdots+B_{d, d-1}\left[\begin{array}{l}n \\ d\end{array}\right]$,
where

$$
B_{d j}=\sum_{k=0}^{d-j-1}(-1)^{k} h^{k(k-1) / 2} U_{j+1-k}^{d}\left[\begin{array}{c}
d+1  \tag{29}\\
k
\end{array}\right]
$$

Formula (29) is established in the same fashion as for formula (12), with the vanishing of the sums of (16) replaced by (27).

We do not generalize the summation formula (6) since we are not able to give a generalization of formula (1). However, we do present the following summation formulas involving the generalized sequence $U_{n}$ :

$$
\begin{gather*}
\mathrm{U}_{2}+\mathrm{U}_{4}+\cdots+\mathrm{U}_{2 \mathrm{n}}=\left(\mathrm{U}_{\mathrm{n}+1}^{2}+\mathrm{U}_{\mathrm{n}}^{2}-\mathrm{U}_{1}^{2}\right) / \mathrm{U}_{2}  \tag{30}\\
\mathrm{U}_{1} \mathrm{U}_{3}+\mathrm{U}_{2} \mathrm{U}_{6}+\cdots+\mathrm{U}_{\mathrm{n}} \mathrm{U}_{3 \mathrm{n}}=\mathrm{U}_{\mathrm{n}} \mathrm{U}_{\mathrm{n}+1} \mathrm{U}_{2 \mathrm{n}+1} / \mathrm{U}_{2} \tag{31}
\end{gather*}
$$

$$
\begin{equation*}
\mathrm{U}_{1}^{2} \mathrm{U}_{2}+\mathrm{U}_{2}^{2} \mathrm{U}_{4}+\cdots+\mathrm{U}_{\mathrm{n}}^{2} \mathrm{U}_{2 \mathrm{n}}=\mathrm{U}_{\mathrm{n}}^{2} \mathrm{U}_{\mathrm{n}+1}^{2} / \mathrm{U}_{2} \tag{32}
\end{equation*}
$$

These formulas are easily probed by mathematical induction using the following special cases of the above addition formula:

$$
\begin{gather*}
\mathrm{U}_{\mathrm{n}+2}^{2}-\mathrm{U}_{\mathrm{n}}^{2}=\mathrm{U}_{2} \mathrm{U}_{2 \mathrm{n}+2}  \tag{33}\\
\mathrm{U}_{\mathrm{n}+2} \mathrm{U}_{2 \mathrm{n}+3}-\mathrm{U}_{\mathrm{n}} \mathrm{U}_{2 \mathrm{n}+1}=\mathrm{U}_{2} \mathrm{U}_{3 \mathrm{n}+3} \tag{34}
\end{gather*}
$$

The special case of (31) in which $U_{n}=F_{n}$ is Recke's problem [3]which brought to mind the well-known formula

$$
\begin{equation*}
1^{2}+2^{2}+3^{2}+\cdots+n^{2}=n(n+1)(2 n+1) / 6 \tag{35}
\end{equation*}
$$

These two special cases inspired the generalization (31). Then (32) was obtained as a generalization of the well-known

$$
\begin{equation*}
1^{3}+2^{3}+3^{3}+\cdots+\mathrm{n}^{3}=\mathrm{n}^{2}(\mathrm{n}+1)^{2} / 2 \tag{36}
\end{equation*}
$$

The proofs of (31) and (32) produced (33) as a byproduct; then (30) follows readily using the telescoping sum

$$
\sum_{\mathrm{k}=1}^{\mathrm{n}}\left[\mathrm{U}_{2} \mathrm{U}_{2 \mathrm{k}}\right]=\sum_{\mathrm{k}=0}^{\mathrm{n}}\left[\mathrm{U}_{\mathrm{k}+1}^{2}-\mathrm{U}_{\mathrm{k}-1}^{2}\right]=\mathrm{U}_{\mathrm{n}+1}^{2}+\mathrm{U}_{\mathrm{n}}^{2}-\mathrm{U}_{1}^{2}
$$

Some special cases of (28) and a special case of (33) above were proposed by one of the authors [4].

Formulas (5), (11), and (13) go back to J. Worpitzky and G. Frobenius (see [5] and [6]). These have been generalized in a different manner from our formulas (28) and (29) by L. Carlitz [7].
[Continued on page 73.]

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-178 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
Put

$$
a_{m, n}=\binom{m+n}{m}^{2}
$$

Show that $a_{m, n}$ satisfies no recurrence of the type

$$
\sum_{j=0}^{r} \sum_{k=0}^{s} c_{j, k} a_{m-j, n-k}=0 \quad(m \geq r, n \geq s)
$$

where the $c_{j, k}$ and $r, s$ are all independent of $m, n$.
Show also that $a_{m, n}$ satisfies no recurrence of the type

$$
\sum_{j=0}^{r} \sum_{k=0}^{n} c_{j, k} a_{m-j, n-k}=0 \quad(m \geq r, n \geq 0)
$$

where the $c_{j, k}$ and $r$ are independent of $m, n$.

H-179 Proposed by D. Singmaster, Bedford College, University of London, London, England.

Let k numbers $\mathrm{p}_{1}, \mathrm{p}_{2}, \cdots, \mathrm{p}_{\mathrm{k}}$ be given. Set $\alpha_{\mathrm{n}}=0$ for $\mathrm{n}<0$; $\alpha_{0}=1$ and define $\alpha_{\mathrm{n}}$ by the recursion

$$
\alpha_{n}=\sum_{i=1}^{n} p_{i} \alpha_{n-i} \quad \text { for } n>0
$$

1. Find simple necessary and sufficient conditions on the $p_{i}$ for

$$
\lim _{\mathrm{n} \rightarrow \infty} \alpha_{\mathrm{n}}
$$

to exist and be: (a) finite and nonzero; (b) zero; (c) infinite.
2. Are the conditions: $p_{i} \geq 0$ for $i=1,2, \cdots, p_{i}>0$ and

$$
\sum_{i=1}^{n} p_{i}=1
$$

sufficient for $\lim _{\mathrm{n}}{ }^{\infty} \alpha_{\mathrm{n}}$ to exist, be finite, and be nonzero?
H-180 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
Show that

$$
\begin{aligned}
& \sum_{k=0}^{n}\binom{n}{k}^{3} F_{k}=\sum_{2 k \leq n} \frac{(n+k)!}{(k!)^{3}(n-2 k)!} F_{(2 n-3 k)} \\
& \sum_{k=0}^{n}\binom{n}{k}^{3} L_{k}=\sum_{2 k \leq n} \frac{(n+k)!}{(k!)^{3}(n-2 k)!} L_{(2 n-3 k)}
\end{aligned}
$$

where $\mathrm{F}_{\mathrm{k}}$ and $\mathrm{L}_{\mathrm{k}}$ denote the $\mathrm{k}^{\text {th }}$ Fibonacciand Lucas numbers, respectively.

H-156 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Prove the identity

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{n^{2}} z^{n}}{(q)_{n}} \prod_{k=1}^{\infty}\left(1-q^{k}\right)= \sum_{n=-\infty}^{\infty} q^{n^{2}} z^{n} \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q)} 2 k \\
& z^{-k} \\
&-\sum_{n=-\infty}^{\infty} q^{n(n+1)} z^{n} \sum_{k=0}^{\infty} \frac{q^{(k+1)^{2}} \frac{(q)}{(q k+1} z^{-k}}{}
\end{aligned}
$$

where

$$
(q)_{n}=(1-q)\left(1-q^{2}\right) \cdots\left(1-q^{n}\right)
$$

## Solution by the Proposer.

We shall make use of the Euler identity

$$
\prod_{n=0}^{\infty}\left(1-q^{n} z\right)=\sum_{n=0}^{\infty}(-1)^{n} q^{\frac{1}{2}} n(n-1) z^{n} /(q)_{n}
$$

and the Jacobi identity

$$
\prod_{n=1}^{\infty}\left(1-q^{2 n}\right)\left(1-q^{2 n-1} z\right)\left(1-q^{2 n-1} z^{-1}\right)=\sum_{n=-\infty}^{\infty} q^{n^{2}} z^{n}
$$

Now we have

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{n^{2}} z^{n}}{q^{n}} \prod_{k=1}^{\infty}\left(1-q^{k}\right)=\sum_{n=0}^{\infty} q^{n^{2}} z^{n} \prod_{k=1}^{\infty}\left(1-q^{n+k}\right) \\
& =\sum_{n=-\infty}^{\infty} q^{n^{2}} z^{n} \prod_{k=1}^{\infty}\left(1-q^{n+k}\right) \\
& =\sum_{n=-\infty}^{\infty} q^{n^{2}} z^{n} \sum_{k=0}^{\infty}(-1)^{k} q^{\frac{1}{2} k(k+1)+n k} /(q)_{k} \\
& =\sum_{k=0}^{\infty}(-1)^{k} q^{\frac{1}{2} k(k+1)} /(q){ }_{k} \sum_{n=-\infty}^{\infty} q^{n^{2}}\left(q^{k} z\right)^{n} \\
& =\sum_{n=0}^{\infty}(-1)^{k} q^{\frac{1}{2} k(k+1)} /(q) k \cdot \prod_{n=1}^{\infty}\left(1-q^{2 n}\right)\left(1+q^{2 n+k-1} z\right)\left(1+q^{2 n-k-1} z^{-1}\right) \\
& =\prod_{n=1}^{\infty}\left(1-q^{2 n}\right) \cdot \sum_{k=0}^{\infty} \frac{q^{k(2 k+1)}}{(q)} \prod_{n=1}^{\infty}\left(1+q^{2 n+2 k-1} z\right)\left(1+q^{2 n-2 k-1} z\right) \\
& -\prod_{n=1}^{\infty}\left(1-q^{2 n}\right) \cdot \sum_{k=0}^{\infty} \frac{q^{(k+1)(2 k+1)}}{(q)} \prod_{n=1}^{\infty}\left(1+q^{2 n+2 k}\right)\left(1+q^{2 n-2 k-2} z^{-1}\right) \\
& =\sum_{n=-\infty}^{\infty} q^{n^{2}} z^{n} \sum_{k=0}^{\infty} \frac{q^{k(2 k+1)}}{(q)_{2 k}} \cdot \frac{\left(1+q^{-2 k+1} z^{-1}\right) \cdots\left(1+q^{-1} z^{-1}\right)}{(1+q z) \cdots\left(1+q^{2 k-1} z\right)} \\
& -\sum_{n=-\infty}^{\infty} q^{n(n+1)} z^{n} \sum_{k=0}^{\infty} \frac{q^{(k+1)(2 k+1)}}{(q)} 2 k+1 \quad \frac{\left(1+q^{-2 k} z^{-1}\right) \cdots\left(1+q^{-2} z^{-1}\right)}{\left(1+q^{2} z\right) \cdots\left(1+q^{2 k} z\right)} \\
& =\sum_{n=-\infty}^{\infty} q^{n^{2}} z^{n} \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q)} 2 k \text { } z^{-k}-\sum_{n=-\infty}^{\infty} q^{n(n+1)} z^{n} \sum_{k=0}^{\infty} \frac{q^{(k+1)^{2}}}{(q)} 2 k+1-
\end{aligned}
$$

H-157 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada (corrected)

A set of polynomials $c_{n}(x)$, which appears in network theory is defined by

$$
c_{n+1}(x)=(x+2) c_{n}(x)-c_{n-1}(x) \quad(n \geq 1)
$$

with

$$
c_{0}(x)=1 \text { and } c_{1}(x)=(x+2) / 2
$$

(a) Find a polynomial expression for $\mathrm{c}_{\mathrm{n}}(\mathrm{x})$.
(b) Show that

$$
2 c_{n}(x)=b_{n}(x)+b_{n-1}(x)=B_{n}(x)-B_{n-2}(x)
$$

where $B_{n}(x)$ and $b_{n}(x)$ are the Morgan-Voyce polynomials as defined in the Fibonacci Quarterly, Vol. 5, No. 2, p. 167.
(c) Show that $2 \mathrm{c}_{\mathrm{n}}^{2}(\mathrm{x})-\mathrm{c}_{2 \mathrm{n}}(\mathrm{x})=1$.
(d) If

$$
\mathrm{Q}=\left[\begin{array}{cr}
(x+2) & -1 \\
1 & 0
\end{array}\right]
$$

show that

$$
\left[\begin{array}{ll}
c_{n} & -c_{n-1} \\
c_{n-1} & -c_{n-2}
\end{array}\right]=\frac{1}{2}\left(Q^{n}-Q^{n-2}\right) \text { for }(n \geq 2)
$$

Hence deduce that $c_{n+1} c_{n-1}-c_{n}^{2}=x(x+4) / 4$.
Solution by A. G. Law, University of Saskatchewan, Regina, Saskatchewan, Canada.

Let $\left\{\mathrm{c}_{\mathrm{n}}(\mathrm{x})\right\}$ be the family of polynomials prescribed by the recurrence

$$
\begin{equation*}
\mathrm{y}_{\mathrm{n}+1}=(\mathrm{x}+2) \mathrm{y}_{\mathrm{n}}-\mathrm{y}_{\mathrm{n}-1}, \quad \mathrm{n} \geq 1, \tag{*}
\end{equation*}
$$

with $y_{0}=1$ and $y_{1}=1+x / 2$. It can be derived, with the aid of [1], that

$$
c_{n}(x)=\frac{4^{n}(n!)^{2}}{(2 n)!} P_{n}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1+x / 2), \quad n \geq 1
$$

where $P_{n}^{(-1 / 2,-1 / 2)}$ is the $n^{\text {th }}$-degree Jacobi polynomial. Consequently [3], $\mathrm{c}_{\mathrm{n}}(\mathrm{x})=\cos \mathrm{n} \theta$, where $\cos \theta=1+\mathrm{x} / 2$, for $\mathrm{n} \geq 1$.

A half-angle formula gives immediately that $2 \mathrm{c}_{\mathrm{n}}^{2}-\mathrm{c}_{2 \mathrm{n}} \equiv 1, \mathrm{n} \geq 1$. Similarly, each relation

$$
c_{n+1}(x) c_{n-1}(x)-c_{n}^{2}(x)=x(x+4) / 4
$$

is also just a trigonometric identity.
The coupled recurrence

$$
b_{n}=x B_{n-1}+b_{n-1} ; \quad B_{n}=(x+1) B_{n-1}+b_{n-1} \quad(n \geq 1),
$$

where $\mathrm{b}_{0} \equiv \mathrm{~B}_{0} \equiv 1$ shows that

$$
b_{n+1}=(x+2) b_{n}-b_{n-1}
$$

for $\mathrm{n} \geq 1$. Hence,

$$
b_{n+1}=(x+1)\left(b_{n}+b_{n-1}\right)-b_{n-2} ;
$$

that is, $y_{n}=\left(b_{n}+b_{n-1}\right) / 2$ satisfies recurrence (*) and, so,

$$
\left(b_{n}+b_{n-1}\right) / 2 \equiv c_{n}
$$

for $\mathrm{n} \geq 1$. Similarly, $2 \mathrm{c}_{\mathrm{n}} \equiv \mathrm{B}_{\mathrm{n}}-\mathrm{B}_{\mathrm{n}-1}$ for $\mathrm{n} \geq 1$.
Finally, since each $\mathrm{b}_{\mathrm{n}}(\mathrm{x})$ is a known sum (see [2]), $2 \mathrm{c}_{\mathrm{n}}=\mathrm{b}_{\mathrm{n}}+\mathrm{b}_{\mathrm{n}-1}$ yields the explicit formula:

$$
c_{n}(x)=x^{n} / 2+\sum_{k=0}^{n-1} \frac{n}{n-k}\binom{n+k-1}{n-k-1} x^{k}
$$

for $\mathrm{n} \geq 1$.

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3. G. Szego, "Orthogonal Polynomials," American Mathematical Society Colloquium Publications, Vol. XXIII (1939).

Also solved by D. Zeitlin, D. V. Jaiswal, M. Yoder, and the Proposer.

## IN THEIR PRIME

H-158 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

If $f_{n}(x)$ be the Fibonacci polynomial as defined in $H-127$, show that
(a) For integral values of $x, f_{n}(x)$ and $f_{n+1}(x)$ are prime to each other.
(b) $\quad\left\{1+\sum_{1}^{n}\left(1 / \mathrm{f}_{2 \mathrm{n}-1} \mathrm{~F}_{2 \mathrm{n}+1}\right)\right\}\left\{1-\mathrm{x}^{2} \sum_{1}^{\mathrm{n}}\left(1 / \mathrm{f}_{2 \mathrm{n}} \mathrm{f}_{2 \mathrm{n}+2}\right)\right\}=1$.

Solution by the Proposer.
(a) It may easily be established by induction that

$$
\mathrm{f}_{\mathrm{n}+1}(\mathrm{x}) \mathrm{f}_{\mathrm{n}-1}(\mathrm{x})-\mathrm{f}_{\mathrm{n}}^{2}(\mathrm{x})=(-1)^{\mathrm{n}}
$$

Hence, for integral values of $x, f_{n}(x)$ and $f_{n+1}(x)$ are prime to each other.
(b) It may also be established by induction that

$$
f_{n+1}(x) f_{n-2}(x)-f_{n}(x) f_{n-1}(x)+(-1)^{n} x=0
$$

Hence,

$$
x \frac{1}{f_{2 n+1} f_{2 n-1}}=\frac{f_{2 n+2}}{f_{2 n+1}}-\frac{f_{2 n}}{f_{2 n-1}}
$$

Thus,

$$
x \sum_{1}^{n} \frac{1}{f_{2 n+1} f_{2 n-1}}=\frac{f_{2 n+2}}{f_{2 n+1}}-\frac{f_{2}}{f_{1}}=\frac{f_{2 n+2}}{f_{2 n+1}}-x
$$

Or,
(2)

$$
1+\sum_{1}^{n} \frac{1}{f_{2 n+1} f_{2 n-1}}=\frac{1}{x} \frac{f_{2 n+2}}{f_{2 n}}
$$

Also, from (1), we have

$$
-x \frac{1}{f_{2 n} f_{2 n+2}}=\frac{f_{2 n+1}}{f_{2 n+2}}-\frac{f_{2 n+1}}{f_{2 n}}
$$

Hence,

$$
\begin{aligned}
-x \sum_{1}^{n} \frac{1}{f_{2 n} f_{2 n+2}} & =\frac{f_{2 n+1}}{f_{2 n+2}}-\frac{f_{3}}{f_{2}} \\
& =\frac{x_{2 n+2}+f_{2 n+1}}{f_{2 n+2}}-\frac{f_{3}}{f_{2}} \\
& =\frac{f_{2 n+1}}{f_{2 n+2}}-\frac{x^{2}+1}{x}+x=\frac{f_{2 n+1}}{f_{2 n+2}}-\frac{1}{x}
\end{aligned}
$$

Thus,

$$
\begin{equation*}
1-x^{2} \sum_{1}^{n} \frac{1}{f_{2 n} f_{2 n+2}}=x \frac{f_{2 n+1}}{f_{2 n+2}} \tag{3}
\end{equation*}
$$

Hence from (2) and (3), we have

$$
\left\{1+\sum_{1}^{n} \frac{1}{f_{2 n+1} f_{2 n-1}}\right\}\left\{1-x^{2} \sum_{1}^{n} \frac{1}{f_{2 n} f_{2 n+2}}\right\}=1 .
$$

Also solved by A. Shannon, M. Yoder, and D. V. Jaiswal.

## HARMONY

H-159 Proposed by Clyde Bridger, Springfield College, Springfield, Illinois.
Let

$$
D_{k}=\frac{c^{k}-d^{k}}{c-d}
$$

and

$$
\mathrm{E}_{\mathrm{k}}=\mathrm{c}^{\mathrm{k}}+\mathrm{d}^{\mathrm{k}}
$$

where $c$ and $d$ are the roots of $z^{2}=a z+b$. Consider the four numbers $e$, $f, x, y$, where $e=c^{k}$ and $f=d^{k}$ are the roots of

$$
z^{2}-z E_{k}+(-b)^{k}=0
$$

and y is the harmonic conjugate of x with respect to e and f . Find y when

$$
\mathrm{x}=\frac{\mathrm{D}_{\mathrm{nk}+\mathrm{k}}}{\mathrm{D}_{\mathrm{nk}}} \quad(\mathrm{k} \neq 0)
$$

Solution by the Proposer.
The condition to be met is

$$
\frac{x-e}{x-f} \cdot \frac{y-f}{y-e}=-1
$$

(See page 69, R. M. Winger, Projective Geometry, Heath, 1923.) This leads directly to

$$
2 x y-E_{k}(x+y)+2(-b)^{k}=0
$$

For the given value of $x$,

$$
y=\frac{E_{k} D_{n k+k}-2(-b)^{k} D_{n k}}{2 D_{n k+k}-E_{k} D_{n k}}
$$

It is easy to verify from the definitions of $D_{k}$ and $E_{k}$ that the numerator reduces to $E_{n k+k} D_{k}$ and that the denominator reduces to $E_{n k} D_{k}$. Hence,

$$
\mathrm{y}=\frac{\mathrm{E}_{\mathrm{nk}+\mathrm{k}}}{\mathrm{E}_{\mathrm{nk}}}
$$

Note that when $\mathrm{a}=\mathrm{b}=1$, and $\mathrm{k}=1$,

$$
\frac{\mathrm{F}_{\mathrm{n}+1}}{\mathrm{~F}_{\mathrm{n}}} \quad \text { and } \quad \frac{\mathrm{L}_{\mathrm{n}+1}}{\mathrm{~L}_{\mathrm{n}}}
$$

are harmonic conjugates with respect to the roots of $z^{2}=z+1$.

Find the roots and the discriminant of

$$
x^{3}-(-1)^{k_{3}} 3-L_{3 k}=0
$$

Solution by L. Carlitz, Duke University, Durham, North Carolina.
Somewhat more generally, we may consider the equation

$$
\begin{equation*}
\mathrm{x}^{3}-3(\alpha \beta)^{\mathrm{k}} \mathrm{x}-\left(\alpha^{3 \mathrm{k}}+\beta^{3 \mathrm{k}}\right)=0 \tag{*}
\end{equation*}
$$

where $\alpha, \beta$ are arbitrary. This equation evidently reduces to

$$
x^{3}-3(-1)^{k} x-L_{3 k}=0
$$

where $\alpha, \beta$ are the roots of

$$
z^{2}-z-1=0
$$

Let $\omega, \omega^{2}$ denote the complex cube roots of 1 and put

$$
\mathrm{x}_{1}=\alpha^{\mathrm{k}}+\beta^{\mathrm{k}}, \quad \mathrm{x}_{2}=\omega \alpha^{\mathrm{k}}+\omega^{2} \beta^{\mathrm{k}}, \quad \mathrm{x}_{3}=\omega^{2} \alpha^{\mathrm{k}}+\omega \beta^{\mathrm{k}}
$$

Then it is easily verified that $x_{1}, x_{2}, x_{3}$ are the roots of $(*)$.
By the familiar formula for the discriminant of a cubic, or directly by computing $\left(x_{1},-x_{2}\right)^{2}\left(x_{2}-x_{3}\right)^{2}$, we find that the discriminant is given by

$$
\mathrm{D}=-27\left(\alpha^{3 \mathrm{k}}-\beta^{3 \mathrm{k}}\right)^{2}
$$

For the special case

$$
x^{3}-3(-1)^{k} x-L_{3 k}=0
$$

the roots are $\mathrm{x}_{1}=\mathrm{L}_{\mathrm{k}}$ and $\mathrm{x}_{2}, \mathrm{x}_{3}$, where

$$
\mathrm{x}_{2}+\mathrm{x}_{3}=-\mathrm{L}_{\mathrm{k}}, \quad \mathrm{x}_{2} \mathrm{x}_{3}=\mathrm{L}_{2 \mathrm{k}}-(-1)^{\mathrm{k}}
$$

The discriminant reduces to

$$
-135 \mathrm{~F}_{3 \mathrm{k}}^{2}
$$

Also solved by M. Yoder, D. Zeitlin, B. King, A. Shannon, and the Proposers.

## BE NEGATIVE

H-162 Proposed by David A. Klarner, University of Alberta, Edmonton, Alberta, Canada.

Suppose $a_{i j} \geq 1$ for $i, j=1,2, \cdots$. Show there exists an $x \geq 1$ such that

$$
(-1)^{n}\left|\begin{array}{llll}
a_{11}-x & a_{12} & \cdots & a_{1 n} \\
a_{21} & a_{22}-x^{2} & \cdots & a_{2 n} \\
\vdots & \vdots & & \vdots \\
a_{n 1} & a_{n 2} & \cdots & a_{n n}-x
\end{array}\right| \leq 0
$$

for all n .

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsy/vania.
Let $\mathrm{D}(\mathrm{n})$ be the determinant.

$$
D(1)=(-1)^{1}\left|a_{11}-x\right|=x-a_{11} \leq 0
$$

if $\mathrm{x} \leq \mathrm{a}_{11}$. Since $\mathrm{x}, \mathrm{a}_{11} \geq 1$, any x satisfying $\mathrm{a}_{11} \geq \mathrm{x} \geq 1$ will do. Suppose $\mathrm{a}_{11}=1$; then $\mathrm{x}=1$ is the only answer for $\mathrm{n}=1$. The statement requires an $x$ for all $n$. Can we reach a contradiction in the case $a_{11}=1$ ? While

$$
\mathrm{D}(2)=-\mathrm{a}_{12} \mathrm{a}_{21} \leq-1<0,
$$

$D(3)=-\left|\begin{array}{lll}0 & a_{12} & a_{13} \\ a_{21} & a_{22}-1 & a_{23} \\ a_{31} & a_{32} & a_{33}-1\end{array}\right| \xlongequal{2 a_{23}-a_{21}-a_{23} a_{31}-a_{13} a_{21} a_{32}} \begin{array}{r} \\ +a_{13} a_{22} a_{31}-a_{13} a_{31} .\end{array}$

Each term here has the sign preceding it, as all factors are positive. Given $a_{i j}$ with $i \neq j$, we can take $a_{22}$ and/or $a_{33}$ so large that the positive terms dominate, since these factors occur only in positive terms. Thus we reach a contradiction of the inequality for $n=3, a_{11}=1$.
[Continued from page 60.]

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# A PRIMER FOR THE FIBONACCI NUMBERS: PART VIII 

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## SEQUENCES OF SUMS FROM PASCAL'S TRIANGLE

There are many ways to generalize Fibonacci numbers, one way being to consider them as a sequence of sums found from diagonals in Pascal's triangle [1], [2]. Since Pascal's triangle and computations with generating functions are so interrelated with the Fibonacci sequence, we introduce a way to find such sums in this section of the Primer.

## 1. INTRODUCTION

Some elementary but elegant mathematics solves the problem of finding the sums of integers appearing on diagonals of Pascal's triangle. Writing Pascal's triangle in a left-justified manner, the problem is to find the infinite sequence of sums $p / q$ of binomial coefficients appearing on diagonals $p / q$ for integers $p$ and $q, p=q \geq 1, q>0$, where we find entries on a diagonal $p / q$ by counting up $p$ and right $q$, starting in the left-most column. (Notice that, while the intuitive idea of "slope" is useful in locating the diagonals, the diagonal $1 / 2$, for example, is not the same as $2 / 4$ or $3 / 6$.) As an example, the sums $2 / 1$ on diagonals formed by going up 2 and right 1 are $1,1,1,2,3,4,6,9,13,19,28, \cdots$, as illustrated below:


Some sequences of sums are simple to find. For example, the sums $0 / 1$ formed by going up 0 and right 1 are the sums of integers appearing in each row, the powers of 2 . The sums $0 / 2$ are formed by alternate integers in a row, also powers of 2 . The sums $1 / 1$ give the famous Fibonacci sequence $1,1,2,3,5,8,13,21, \cdots$, defined by $F_{1}=F_{2}=1, F_{n}=$ $F_{n-1}+F_{n-2}$. The sums $-1 / 2$, found by counting down 1 and right 2, give the Fibonacci numbers with odd subscripts, $1,2,5,13,34,89, \cdots, F_{2 n+1}$, ... While the problem is not defined for negative "slope" less than or equal to -1 nor for summing columns, the diagonals $-1 / 1$ are the same as the columns of the array, and the sum of the first $j$ integers in the $n^{\text {th }}$ column is the same as the $j^{\text {th }}$ array in the $(n+1)^{s t}$ column.

To solve the problem in general, we develop some generating functions.

## 2. GENERATING GUNCTIONS FOR THE COLUMNS OF PASCAL'S TRIANGLE

Here, a generating function is an algebraic expression which lists terms in a sequence as coefficients in an infinite series. For example, by the formula for summing an infinite geometric progression,

$$
\begin{equation*}
\frac{a}{1-r}=a+a r+a r^{2}+a r^{3}+\cdots, \quad|r|<1, \tag{1}
\end{equation*}
$$

we can write a generating function for the powers of 2 as
(2) $\frac{1}{1-2 \mathrm{x}}=1+2 \mathrm{x}+4 \mathrm{x}^{2}+8 \mathrm{x}^{3}+\cdots+2^{n} \mathrm{x}^{n}+\cdots, \quad|\mathrm{x}|<1 / 2$.

Long division gives a second verification that $1 /(1-2 x)$ generates powers of 2 , and long division can be used to compute successive coefficients of powers of x for any generating function which follows.

We need some other generating functions to proceed. By summing the geometric progression,
(3)

$$
\frac{1}{1-x}=1+x+x^{2}+x^{3}+\cdots=\sum_{k=0}^{\infty}\binom{k}{0} x^{k}, \quad|x|<1
$$

By multiplying series or by taking successive derivatives of (3), one finds
(4)

$$
\frac{1}{(1-x)^{2}}=1+2 x+3 x^{2}+\cdots+k x^{k-1}+\cdots=\sum_{k=0}^{\infty}\binom{k}{1} x^{k}, \quad|x|<1
$$

(5)

$$
\frac{1}{(1-x)^{3}}=1+3 x+6 x^{2}+10 x^{3}+15 x^{4}+\cdots=\sum_{k=0}^{\infty}\binom{k}{2} x^{k}, \quad|x|<1
$$

Computation of the $n^{\text {th }}$ derivative of (3) shows that

$$
\frac{1}{(1-x)^{n+1}}=\sum_{k=0}^{\infty}\binom{k}{n} x^{k}, \quad n=0,1,2,3, \cdots,
$$

is a generating function for the integers appearing in the $n^{\text {th }}$ column of Pascal's triangle, or equivalently, the column generator for the $n{ }^{\text {th }}$ column, where we call the left-most column the zero ${ }^{\text {th }}$ column. As a restatement, the columns of Pascal's triangle give the coefficients of the binomial expansion of $(1-x)^{-n-1}, n=0,1,2, \cdots,|x|<1$, or of $(1+x)^{-n-1}$ if taken with alternating signs.

## 3. SOME PARTICULAR SUMS DERIVED USING COLUMN GENERATORS

It is easy to prove that the rows in Pascal's triangle have powers of 2 as their sums: merely let $\mathrm{x}=1$ in $(\mathrm{x}+1)^{\mathrm{n}}, \mathrm{n}=0,1,2, \cdots$. But, to demonstrate the method, we work out the sums $0 / 1$ of successive rows using column generators.

First write Pascal's triangle to show the terms in the expansions of $(x+1)^{n}$. Because we want the exponents of $x$ to be identical in each row so that we will add the coefficients in each row by adding the column generators, multiply the columns successively by $1, x, x^{2}, x^{3}, \cdots$, making

|  | 1x | 1x |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $1 \mathrm{x}^{2}$ | $2 \mathrm{x}^{2}$ | $1 \mathrm{x}^{2}$ |  |  |
|  | $1 \mathrm{x}^{3}$ | $3 \mathrm{x}^{3}$ | $3 x^{3}$ | $1 x^{3}$ |  |
|  | $1 \mathrm{x}^{4}$ | $4 \mathrm{x}^{4}$ | $6 \mathrm{x}^{4}$ | $4 \mathrm{x}^{4}$ | $1 \mathrm{x}^{4}$ |
|  | -•• | $\cdots$ | -• | -.. | -•• |
| generators: | $\frac{1}{1-x}$ | $\frac{x}{(1-x)^{2}}$ | $\frac{x^{2}}{(1-x)^{3}}$ | $\frac{x^{3}}{-x)^{4}}$ | $\frac{x^{4}}{-x)^{5}}$ |

Then the sum $S$ of column generators will have the sums $0 / 1$ of the rows appearing as coefficients of successive powers of X 。 But, S is ageometric progression with ratio $\mathrm{x} /(1-\mathrm{x})$, so by (1),

$$
S=\frac{\frac{1}{1-x}}{1-\frac{x}{1-x}}=\frac{1}{1-2 x}
$$

for

$$
\left|\frac{x}{1-x}\right|<1 \quad \text { or } \quad|x|<1 / 2
$$

the generating function for powers of 2 given earlier in (2).
If we want the sums $0 / 2$, we sum every other generating function.

$$
S^{*}=\frac{1}{1-x}+\frac{x^{2}}{(1-x)^{3}}+\frac{x^{4}}{(1-x)^{5}}+\cdots
$$

and again sum the geometric progression to find

$$
\begin{aligned}
S=\frac{1-x}{1-2 x}= & \frac{1}{1-2 x}-\frac{x}{1-2 x} \\
= & \left(1+2 x+4 x^{2}+8 x^{3}+\cdots+2^{n} x^{n}+\cdots\right) \\
& -\left(x+2 x^{2}+4 x^{3}+\cdots+2^{n-1} x^{n}+\cdots\right) \\
= & 1+\left(x+2 x^{2}+4 x^{3}+\cdots+2^{n-1} x^{n}+\cdots\right)
\end{aligned}
$$

which again generates powers of 2 as verified above.
We have already noted that the sums $1 / 1$ give the Fibonacci numbers. To use column generators, we must multiply the columns successively by $1, x^{2}, x^{4}, x^{6}, \ldots$, so that the exponents of $x$ will be the same along each diagonal $1 / 1$. The sum $\mathbb{S}^{* *}$ of column generators becomes

$$
S^{* *}=\frac{1}{1-x}+\frac{x^{2}}{(1-x)^{2}}+\frac{x^{4}}{(1-x)^{3}}+\frac{x^{6}}{(1-x)^{4}}+\cdots
$$

again a geometric progression, so that

$$
\mathrm{S}=\frac{\frac{1}{1-\mathrm{x}}}{1-\frac{x^{2}}{1-x}}=\frac{1}{1-\mathrm{x}-\mathrm{x}^{2}}
$$

for

$$
\left|\frac{x^{2}}{1-x}\right|<1 \quad \text { or } \quad|x|<\left(\frac{1+\sqrt{5}}{2}\right)^{-1} .
$$

This means that, for $x$ less than the positive root of $x^{2}+x-1=0$,
(6) $\frac{1}{1-x-x^{2}}=1+1 x+2 x^{2}+3 x^{3}+5 x^{4}+8 x^{5}+\cdots+F_{n} x^{n-1}+\cdots \quad$.

For generating functions, we are concerned primarily with the coefficients of $x$ rather than values of $x$, but a particular example is interesting at this point. Both series (4) and (6) converge when $\mathrm{x}=1 / 2$; let $\mathrm{x}=$ $1 / 2$ in those series to form

$$
\begin{aligned}
& 4=1+2 \cdot \frac{1}{2}+3 \cdot \frac{1}{4}+4 \cdot \frac{1}{8}+5 \cdot \frac{1}{16}+6 \cdot \frac{1}{32}+\cdots+\mathrm{n} \cdot \frac{1}{2^{\mathrm{n}-1}}+\cdots, \\
& 4=1+1 \cdot \frac{1}{2}+2 \cdot \frac{1}{4}+3 \cdot \frac{1}{8}+5 \cdot \frac{1}{16}+8 \cdot \frac{1}{32}+\cdots+\mathrm{F}_{\mathrm{n}} \cdot \frac{1}{2^{\mathrm{n}-1}}+\cdots,
\end{aligned}
$$

the same result whether we use the natural numbers or the Fibonacci numbers as coefficients of the powers of $1 / 2$ :

Also, $x=0.1$ in (6) gives, upon division by 100,

$$
\frac{1}{89}=0.0112358
$$

21
34
55
89
the reciprocal of a Fibonacci number with successive Fibonacci numbers making up its decimal expansion.

We are now in a position to solve the general problem of finding the sums $p / q$.
4. SEQUENCES OF SUMS p/q APPEARING ALONG ANY DIAGONAL

To find the sequence of sums appearing along the diagonals $p / 1$, multiply the columns of Pascal's triangle successively by $1, x^{p+1}, x^{2(p+1)}$, $x^{3(p+1)}, \cdots$, so that the exponents of $x$ appearing on each diagonal $p / 1$ will be the same, giving

$$
\begin{array}{lcccc}
1 & & & & \\
1 x^{p} & 1 x^{p+1} & & & \\
1 x^{2} & 2 x^{p+2} & 1 x^{2 p+2} & & \\
1 x^{3} & 3 x^{p+3} & 3 x^{2 p+3} & 1 x^{3 p+3} & \\
1 x^{4} & 4 x^{p+4} & 6 x^{2 p+4} & 4 x^{3 p+4} & 1 x^{4 p+4}
\end{array}
$$

generators: $\frac{1}{1-x} \frac{x^{p+1}}{(1-x)^{2}} \quad \frac{x^{2 p+2}}{(1-x)^{3}} \quad \frac{x^{3 p+3}}{(1-x)^{4}} \quad \frac{x^{4 p+4}}{(1-x)^{5}}$.

The sum S of column generators is a geometric progression, so that
(7) $\quad S=\frac{\frac{1}{1-x}}{1-\frac{x^{p+1}}{1-x}}=\frac{1}{1-x-x^{p+1}}, \quad p \geq 1, \quad\left|\frac{x^{p+1}}{1-x}\right|<1$,
with $S$ convergent for $|x|$ less than the positive root of $x^{p+1}+x-1=0$. Then, the generating function (7) gives the sums $\mathrm{p} / 1$ as coefficients of successive powers of $x$. [Reader: Show $|x|<1 / 2$ is sufficient. Editor.]

In conclusion, the sequence of sums $p / q$ are found by multiplying successive $q^{\text {th }}$ columns by $1, x^{p+q}, x^{2(p+q)}, x^{3(p+q)}, \cdots$, making the sum of column generators be

$$
S^{*}=\frac{1}{1-x}+\frac{x^{p+q}}{(1-x)^{q+1}}+\frac{x^{2 p+2 q}}{(1-x)^{2 q+1}}+\frac{x^{3 p+3 q}}{(1-x)^{3 q+1}}+\cdots .
$$

Summing that geometric progression yields the generating function

$$
S^{*}=\frac{(1-x)^{q-1}}{(1-x)^{q}-x^{p+q}}, \quad p+q \geq 1, \quad q>0
$$

which converges for $|x|$ less than the absolute value of the root of smallest absolute value of $x^{p+q}-(1-x)^{q}=0$ and which gives the sums of the binomial coefficients found along the diagonals $p / q$ as coefficients of successive powers of x . [Reader: Show $|\mathrm{x}|<1 / 2$ is sufficient. Editor.]

Some references for readings related to the problem of this paper follow but the list is by no means exhaustive. We leave the reader with the problem of determining the properties of particular sequences of sums arising in this paper.

## REFERENCES

1. V. C. Harris and Carolyn C. Styles, "A Generalization of Fibonacci Numbers," Fibonacci Quarterly, Vol. 2, No. 4, Dec., 1964, pp. 277-289.
2. V. C. Harris and Carolyn C. Styles, "Generalized Fibonacci Sequences Associated with a Generalized Pascal Triangle," Fibonacci Quarterly, Vol. 4, No. 3, October, 1966, pp. 241-248.
3. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Diagonal Sums of Generalized Pascal Triangles," Fibonacci Quarterly, Vol. 7, No. 4, Nov., 1969, pp. 341-358.
4. V. E. Hoggatt, Jr., and D. A. Lind, "A Primer for the Fibonacci Numbers: Part VI: Generating Functions," Fibonacci Quarterly, Vol. 5, No. 5, Dec., 1967, pp. 445-460.
5. Mark Feinberg, "New Slants," Fibonacci Quarterly, Vol. 2, No. 3, Oct., 1964, pp. 223-227.
6. V. E. Hoggatt, Jr., "A New Angle on Pascal's Triangle," Fibonacci Quarterly, Vol. 6, No. 4, Oct. , 1968, pp. 221-234.

## ERRATA

In Volume 8, No. 5, December, 1970, issue of the Fibonacci Quarterly, please make the following changes:

Page 457: Please change the equation on line 10 to read as follows:

$$
\mathrm{f}_{\mathrm{a}}(1, \mathrm{n})=\mathrm{g}_{\mathrm{a}}(1, \mathrm{n}) \quad \mathrm{n}>1
$$

Page 472: Please change Eqs. (a) and (b) of Theorem 1 to read:
(a) $\left(\theta_{n}, \theta_{n+1}\right)=1 \quad(n \geq 1)$;
(b) $\left(\theta_{\mathrm{n}}, \theta_{\mathrm{n}+2}\right)=1 \quad(\mathrm{n} \geq 1)$.

Page 488: Please change Eq. (1) to read:

$$
R\left(\mathrm{~F}_{2 \mathrm{n}} \mathrm{~F}_{2 \mathrm{~m}}\right)=\mathrm{R}\left(\mathrm{~F}_{2 \mathrm{n}+1} \mathrm{~F}_{2 \mathrm{~m}}\right)=(\mathrm{n}-\mathrm{m}) \mathrm{F}_{2 \mathrm{~m}}+\mathrm{F}_{2 \mathrm{~m}-1} \quad(\mathrm{n} \geq \mathrm{m})
$$

# THE POSSIBLE END OF THE PERIODIC TABLE OF ELEMENTS AND THE "GOLDEN RATIO" 

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Atomic nuclei consist of protons (P) and neutrons (N). The number of protons in the nucleus is equal to the position number, or atomic number ( Z ), of the elements in Mendeleev's Periodic System of Elements.

The heaviest element - found in nature long ago - the uranius (U), has for a long time occupied the last place ( $Z=92$ ) in the Periodic Table.

During the last 30 years, the Periodic Table has become bigger on account of production of the artificial "transuranium elements" (with $Z>92$ ). At present, the list of known elements already contains 104 names.

The atomic physicists of the USA and USSR are now making great efforts to create some super-heavy elements in their laboratories.

It is known that all elements with $\mathrm{Z}>90$ are spontaneously fissionable. The spontaneous fission half-life of nuclei decreases rapidly with increasing Z and makes the creation of super-heavy elements more and more difficult.

Only in the region of two possible "islands of stability" some atomic nuclei have more chances of being relatively stabilized. According to some theoretical consideration, the best probability for a comparatively stable super-heavy element is at atomic number 114 (nucleus ${ }_{114}[\mathrm{x}]{ }^{298}$ ), but there are some theoretical indications that suggest this would occur also at atomic number 126 (nucleus ${ }_{126}[\mathrm{y}]^{310}$ ).

It should be noted that a probability for the creation of these two hypothetical elements, besides fulfilling all remaining theoretical conditions, still depends on the value of fission parameter $Z^{2} / A$ (where $A$ is mass number equal to number of $P+$ number of $N$ ) for every nucleus.

This parameter is a criterion of instability of nucleus against spontaneous fission and has a general trend of increasing with increasing $Z$. So for ${ }_{92} \mathrm{U}^{238}$ the value of $\mathrm{Z}^{2} / \mathrm{A}=35.6$, for ${ }_{94} \mathrm{Pu}^{239}=37.0$, for ${ }_{98} \mathrm{Cf}^{246}=39.0$, etc.

On the basis of the liquid-drop model of atomic nuclei, the fission parameter has a limiting value equal to 44.0. (See the illustration on page 92.)
[Continued on page 92.]

# PYTHAGORAS REVISITED 

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A Pythagorean triplet consists of three numbers ( $a, b, c$ ) in which $a^{2}+b^{2}=c^{2}$. Such triplets are generated by $m$ and $n$ where $m^{2}-n^{2}=a$, $2 \mathrm{mn}=\mathrm{b}$, and $\mathrm{m}^{2}+\mathrm{n}^{2}=\mathrm{c}$.

What are the conditions that in such triplets $a+b=L^{2}$, as in (9, 40, 41)?

Set $\mathrm{m}+\mathrm{n}=\mathrm{K}$ and substitute $\mathrm{K}-\mathrm{n}=\mathrm{m}$ in $\mathrm{a}+\mathrm{b}=\mathrm{L}^{2}$, letting $\mathrm{a}=$ $\mathrm{m}^{2}-\mathrm{n}^{2}$ and $\mathrm{b}=2 \mathrm{mn}$. Then, $\mathrm{K}^{2}-2 \mathrm{n}=\mathrm{L}^{2}$ or $\mathrm{K}^{2}-\mathrm{L}^{2}=2 \mathrm{n}$.

This last equation is of the form $A^{2}-B^{2}=2 C^{2}$, whose general solution is $\mathrm{A}=\mathrm{t}^{2}+2 \mathrm{u}^{2}, \quad \mathrm{~B}=\mathrm{t}-2 \mathrm{u}^{2}$, and $\mathrm{C}=2 \mathrm{tu}$ [1].

Hence, $K=t^{2}+2 u^{2}, L=t^{2}-2 u^{2}$, and $n=2 t u$. Since $m=K-n$, then by substitution, $m=t^{2}+2 u^{2}-2 t u$ 。

We desire to choose $m>n$. This condition will obtain when $t^{2}+2 u^{2}$ : $\mathrm{tu}>4$.

Several other Pythagorean triplets of this type are (133, 156, 205), ( $2461,5460,5989$ ), and $(12,549,34,540,36,749)$.
I. What are the conditions that in Pythagorean triplets $a+b=L^{2}$ and $\mathrm{m}+\mathrm{n}=\mathrm{K}^{2}$, and in $(1,690,128 ; 9,412,096 ; 9,562,640)$ in which $\mathrm{m}=2372$ and $\mathrm{n}=1984$ ?

Since $m=K^{2}-n$ and the conditions for $a+b=L^{2}$ have been found above, we can set $t^{2}+2 u^{2}-2 t u=K^{2}-2 t u$. Then $K^{2}=t^{2}+2 u^{2}$ or $K^{2}-t^{2}$ $=2 u^{2}$, an equation of the form $A^{2}-B^{2}=2 C^{2}$. Whence

$$
\begin{aligned}
\mathrm{K} & =\mathrm{x}^{2}+2 \mathrm{y}^{2} \\
\mathrm{t} & =\mathrm{x}^{2}-2 \mathrm{y}^{2} \\
\mathrm{u} & =2 \mathrm{xy}
\end{aligned}
$$

Therefore,

$$
m=\left(x^{2}-2 y^{2}\right)^{2}+2(2 x y)^{2}-2(2 x y)\left(x^{2}-2 y^{2}\right)
$$

or

$$
m=\left(x^{2}+2 y^{2}\right)^{2}-4 x y\left(x^{2}-2 y^{2}\right)
$$

and

$$
n=4 x y\left(x^{2}-2 y^{2}\right) .
$$

We desire to choose $\mathrm{m}>\mathrm{n}$. This condition will obtain when $\left(\mathrm{x}^{2}+2 \mathrm{y}^{2}\right)^{2}$ : $x y\left(x^{2}-2 y^{2}\right)>8$.

In the example above, $x=8, y=1$. Another such triplet is one in which $\mathrm{x}=15$ and $\mathrm{y}=2, \mathrm{~m}=28,249$ and $\mathrm{n}=26,040$.
II. What are the conditions that in Pythagorean triplets $a+b+c=M^{2}$, as in $(63,16,65) ?$

Since $a=m^{2}-n^{2}, b=2 m n$, and $c=m^{2}+n^{2}$, we can set

$$
\mathrm{m}^{2}-\mathrm{n}^{2}+2 \mathrm{mn}+\mathrm{m}^{2}+\mathrm{n}^{2}=\mathrm{M}^{2}
$$

by substitution. Then

$$
2 m^{2}+2 m n-M^{2}=0
$$

Use the quadratic equation formula to solve for $m$. Then

$$
m=-2 n \pm \sqrt{4 \mathrm{n}^{2}+8 \mathrm{M}^{2}}: 4
$$

or

$$
m=-n \pm \sqrt{\mathrm{n}^{2}+2 \mathrm{M}^{2}}: 2
$$

We will show that $n^{2}+2 M^{2}$ is a perfect square when $n=d^{2}-2 e^{2}$ and $\mathrm{M}=2 \mathrm{de}$.

Set $n^{2}+2 M^{2}=P^{2}$. Then

$$
\mathrm{P}^{2}-\mathrm{n}^{2}=2 \mathrm{M}^{2}
$$

which is an equation of the type $A^{2}-B^{2}=C^{2}$. Whence

$$
\begin{aligned}
\mathrm{P} & =\mathrm{d}^{2}+2 \mathrm{e}^{2} \\
\mathrm{n} & =\mathrm{d}^{2}-2 \mathrm{e}^{2} \\
\mathrm{M} & =2 \mathrm{de}
\end{aligned}
$$

Then, by substitution,

$$
\mathrm{m}=-\mathrm{d}^{2}+2 \mathrm{e}^{2} \pm \sqrt{\left(\mathrm{d}^{2}-2 \mathrm{e}^{2}\right)^{2}}+\sqrt{2(2 \mathrm{de})^{2}: 2}
$$

or $m=2 e^{2},-e^{2}$. Discard the negative result.
We desire to choose $m>n$. This condition will obtain when $d<2 e$ and $d^{2}>2 \mathrm{e}^{2}$.

In triplets of this type, there is the bonus that $m+n$ is also a square, namely, $d^{2}$.
III. What are the conditions that in Pythagorean triplets $a+b=L^{2}$ and $\mathrm{a}^{2}=\mathrm{b}+\mathrm{c}$, as in $(57,1624,1625)$ ?

Since $\mathrm{a}^{2}=\mathrm{b}+\mathrm{c}$, then by substitution,

$$
\left(\mathrm{m}^{2}-\mathrm{n}^{2}\right)^{2}=2 \mathrm{mn}+\mathrm{m}^{2}+\mathrm{n}^{2}
$$

or

$$
\left(\mathrm{m}^{2}-\mathrm{n}^{2}\right)^{2}=(\mathrm{m}+\mathrm{n})^{2}
$$

whence, $m=n+1$.
We have shown earlier that if $a+b=L^{2}$, then $m=t^{2}+2 u^{2}-2 t u$ and $n=2$ tu. Since $m=n+1$ if $a^{2}=b+c$, then set

$$
\mathrm{t}^{2}-2 \mathrm{tu}+2 \mathrm{u}^{2}=2 \mathrm{tu}+1
$$

or

$$
t^{2}-4 t u+2 u^{2}-1=0
$$

Solve for $t$ using the quadratic equation formula. Then

$$
t=4 u \pm \sqrt{16 u^{2}-8 u^{2}+4}: 2
$$

or

$$
t=2 u \pm \sqrt{2 u^{2}+1}
$$

Now $2 u^{2}+1$ will be a perfect square when $u=0,2,12,70,408, \cdots$, a recurrent series in which $q_{1}=0, q_{2}=2$, and $q_{n}=6 q_{n-1}-q_{n-2}$.

As $u=0,2,12,70,408, \cdots$, $t$ correspondingly equals $\pm 1,4 \pm 3$. $24 \pm 17, \quad 140 \pm 99, \quad 816 \pm 577, \cdots$.

The first six Pythagorean triplets in which $a+b=L^{2}$ and $a^{2}=b+c$ are listed below in abbreviated form, since in these triplets, $n=m-1$ and $\mathrm{c}=\mathrm{B}+1$.

| M | A | B |
| ---: | ---: | ---: |
|  | 9 | 40 |
| 29 | 57 | 1,624 |
| 169 | 337 | 56,784 |
| 985 | 1,969 | $1,938,480$ |
| 5,741 | 11,481 | $65,906,680$ |
| 33,461 | 66,921 | $2,239,210,120$ |

## REFERENCE

1. Albert H. Beiler, Recreations in the Theory of Numbers, Dover Publications, Inc., New York, 1964, p. 129

# FIBONACCI AND LUCAS NUMBERS <br> TEND TO OBEY BENFORD'S LAW 

## J. WLODARSKI

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In numbers that appear in tables of physical and chemical constants and similar tabulations, the digit 1 appears as first digit almost three times more often, as one would expect.

This phenomenal distribution of first digit is known at least for over 30 years when the physicist Frank Benford published a paper on this subject [1].

In his paper, Benford offered a general "law of anomalous numbers. " The probability that a random decimal begins with digit $p$ is

$$
\log (p+1)-\log p
$$

where logarithms are based on 10 .
Now the mathematician Ralph A. Raimi has, in [2], recently concerned himself with analysis of this number phenomenon, but did not refer to Fibonacci and Lucas numbers.

It seems that the first digits of Fibonacci and Lucas numbers tend to obey very closely the formula of probability offered by Benford. The chart on the following page shows the relation of frequencies for the first 100 Fibonacci Numbers and the first 100 Lucas numbers with frequencies of Benford's law. It would be interesting to make use of much more than 100 Fibonacci and Lucas numbers for the purpose of further analyzing Benford's law.

## REFERENCES

1. Frank Benford, Proceedings of the American Philosophical Society, Vol. 78, No. 4, pp. 551-572 (March 31, 1938).
2. Ralph A. Raimi, Scientific American, Vol. 221, No. 6, pp. 109-120 (December 1969).

| $\infty$ | $\begin{aligned} & \text { Rog } \\ & \\ & \hline \end{aligned}$ |
| :---: | :---: |
| $\infty$ |  |
| － |  |
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$\%$ \％용 The frequencies according to Benford＇s law are given by $\square$ bars and numbers under them．The frequencies of first
 under them．


# GREATEST COMMON DIVISORS IN ALTERED FIBONACCI SEQUENCES 

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Let the Fibonacci and Lucas sequence be defined as usual:

$$
\begin{gathered}
\mathrm{F}_{\mathrm{n}+1}=\mathrm{F}_{\mathrm{n}}+\mathrm{F}_{\mathrm{n}-1}, \quad \mathrm{~L}_{\mathrm{n}+1}=\mathrm{L}_{\mathrm{n}}+\mathrm{L}_{\mathrm{n}-1}, \quad \mathrm{n}=2,3, \cdots, \\
\mathrm{~F}_{1}=\mathrm{F}_{2}=\mathrm{L}_{1}=1, \quad \mathrm{~L}_{2}=3
\end{gathered}
$$

It is well known that successive members of the Fibonacci sequence are relatively prime, but if we alter the sequence slightly by letting

$$
\mathrm{G}_{\mathrm{n}}=\mathrm{F}_{\mathrm{n}}+(-1)^{\mathrm{n}}, \quad \mathrm{n}=1,2, \cdots,
$$

then we have very different behavior, as can be seen in the following table:

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\mathrm{G}_{\mathrm{n}}$ | 0 | 2 | 1 | 4 | 4 | 9 | 12 | 22 | 33 | 56 |
| $\left(\mathrm{G}_{\mathrm{n}}, \mathrm{G}_{\mathrm{n}+1}\right)$ |  | 1 |  | 4 |  | 3 |  | 11 |  | 8 |
| n | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |  |
| $\mathrm{G}_{\mathrm{n}}$ | 88 | 145 | 232 | 378 | 609 | 988 | 1596 | 2585 | 4180 |  |
| $\left(\mathrm{G}_{\mathrm{n}}, \mathrm{G}_{\mathrm{n}+1}\right)$ |  | 29 |  | 21 |  | 76 |  | 55 |  |  |

Inspection of the table shows that the first, third, fifth, $\cdots$ entries in the $\left(G_{n}, G_{n+1}\right)$ line are the second, fourth, sixth, $\cdots$ Fibonacci numbers, and the second, fourth, sixth, ... entries are the third, fifth, seventh,... Lucas numbers. It is the purpose of this note to prove this and some related results which are corollaries of Theorem 1 below.

Theorem 1:

$$
\begin{align*}
& \mathrm{F}_{4 \mathrm{n}}+1=\mathrm{F}_{2 \mathrm{n}-1} \mathrm{~L}_{2 \mathrm{n}+1}, \quad \mathrm{~F}_{4 \mathrm{n}}-1:=\mathrm{F}_{2 \mathrm{n}+1} \mathrm{~L}_{2 \mathrm{n}-1},  \tag{1}\\
& \mathrm{~F}_{4 \mathrm{n}+1}+1=\mathrm{F}_{2 \mathrm{n}+1} \mathrm{~L}_{2 \mathrm{n}}, \quad \mathrm{~F}_{4 \mathrm{n}+1}-1=\mathrm{F}_{2 \mathrm{n}} \mathrm{~L}_{2 \mathrm{n}+1} \text {, } \\
& \mathrm{F}_{4 \mathrm{n}+2}+1=\mathrm{F}_{2 \mathrm{n}+2} \mathrm{~L}_{2 \mathrm{n}}, \quad \mathrm{~F}_{4 \mathrm{n}+2}-1=\mathrm{F}_{2 \mathrm{n}} \mathrm{~L}_{2 \mathrm{n}+2} \text {, } \\
& F_{4 n+3}+1=F_{2 n+1} L_{2 n+2}, F_{4 n+3}-1=F_{2 n+2} L_{2 n+1} \text {, }
\end{align*}
$$

$\mathrm{n}=1,2, \ldots$.

Proof. From [1, p. 59], we have

$$
\begin{array}{ll}
F_{n+p}+F_{n-p}=F_{n} L_{p}, & p \text { even } \\
F_{n+p}+F_{n-p}=F_{p} L_{n}, & p \text { odd } \\
F_{n+p}-F_{n-p}=F_{p} L_{n}, & p \text { even }, \\
F_{n+p}-F_{n-p}=F_{n} L_{p}, & p \text { odd } . \tag{8}
\end{array}
$$

Using (6), we get

$$
\begin{aligned}
F_{4 n}+1=F_{4 n}+F_{2} & =F_{(2 n+1)+(2 n-1)}+F_{(2 n+1)-(2 n-1)} \\
& =F_{2 n-1} L_{2 n+1}
\end{aligned}
$$

Using (5), we get

$$
\begin{aligned}
F_{4 n+1}+1=F_{4 n+1}+F_{1} & =F_{(2 n+1)+2 n}+F_{(2 n+1)-2 n} \\
& =F_{2 n+1} L_{2 n} .
\end{aligned}
$$

Similar applications of (5)-(8) give the remaining six identities in (1)-(4).
Although it is not known whether or not the Fibonacci sequence contains infinitely many primes, Theorem 1 shows that the sequences $\left\{F_{n}+1\right\}$ and $\left\{F_{n}-1\right\}$ contain only finitely many primes.

Corollary 1. $F_{n}+1$ is composite for $n \geq 4$ and $F_{n}-1$ is composite for $\mathrm{n} \geq 7$.

Proof. From Theorem 1, $\mathrm{F}_{8} \pm 1, \mathrm{~F}_{9} \pm 1, \mathrm{~F}_{10} \pm 1, \cdots$ are all composite because all of the factors on the right-hand sides of the equations in (1)-(4) are greater than one. Inspection of early values of $F_{n}$ then completes the proof.

The property of greatest common divisors noted at the beginning of this note is proved in

Corollary 2.

$$
\begin{aligned}
& \left(G_{4 n}, G_{4 n+1}\right)=L_{2 n+1}, \quad\left(G_{4 n+2}, G_{4 n+3}\right)=F_{2 n+2}, \\
& \left(G_{4 n+1}, G_{4 n+3}\right)=L_{2 n+1},
\end{aligned}
$$

$\mathrm{n}=1,2, \cdots$.

Proof. From Theorem 1, we have

$$
\begin{aligned}
\left(G_{4 n}, G_{4 n+1}\right) & =\left(F_{4 n}+1, F_{4 n+1}-1\right)=\left(L_{2 n+1} F_{2 n-1}, L_{2 n+1} F_{2 n}\right) \\
& =L_{2 n+1}\left(F_{2 n-1}, F_{2 n}\right)=L_{2 n+1}
\end{aligned}
$$

The proofs of other equations are similar, the last one needing the fact that $\left(F_{2 n}, F_{2 n+2}\right)=1, \quad n=1,2, \cdots$.

Using Theorem 1 in a similar way, we can prove
Corollary 3. If $H_{n}=F_{n}-(-1)^{n}, n=1,2, \cdots$, then

$$
\begin{array}{cc}
\left(\mathrm{H}_{4 \mathrm{n}}, \mathrm{H}_{4 \mathrm{n}+1}\right)= & \mathrm{F}_{2 \mathrm{n}+1}, \\
& \left(\mathrm{H}_{4 \mathrm{n}+2}, \mathrm{H}_{4 \mathrm{n}+3}\right)=\mathrm{L}_{2 \mathrm{n}+2}, \\
\left(\mathrm{H}_{4 \mathrm{n}+1}, \mathrm{H}_{4 \mathrm{n}+3}\right)= & \mathrm{F}_{2 \mathrm{n}+1},
\end{array}
$$

$\mathrm{n}=1,2, \cdots$.
It would be natural to now consider the sequences $\left\{L_{n}+(-1)^{n}\right\}$ and $\left\{\mathrm{L}_{\mathrm{n}}-(-1)^{\mathrm{n}}\right\}$, but different methods are needed.

The authors wish to thank the Editor for valuable suggestions.
Note. The readers may wish to prove the additional ones listed below. Editor.
A.
$\left(\mathrm{F}_{4 \mathrm{n}+1}+1, \mathrm{~F}_{4 \mathrm{n}+2}+1\right)=\mathrm{L}_{2 \mathrm{n}}$,
B. $\quad\left(\mathrm{F}_{4 \mathrm{n}+1}+1, \mathrm{~F}_{4 \mathrm{n}+3}+1\right)=\mathrm{F}_{2 \mathrm{n}+1}$,
C. $\quad\left(F_{4 n+1}-1, F_{4 n+2}-1\right)=F_{2 n}$,
D. $\quad\left(F_{4 n+1}-1, F_{4 n+3}-1\right)=L_{2 n+1}$,
E. $\quad\left(F_{4 n-1}-1, F_{4 n+1}-1\right)=F_{2 n}$,
F. $\quad\left(\mathrm{F}_{4 \mathrm{n}-1}+1, \mathrm{~F}_{4 \mathrm{n}+1}+1\right)=\mathrm{L}_{2 \mathrm{n}}$,
G. $\quad\left(\mathrm{F}_{4 \mathrm{n}+3}+1, \mathrm{~F}_{4 \mathrm{n}}-1\right)=\mathrm{F}_{2 \mathrm{n}+1}$,
H.
$\left(\mathrm{F}_{4 \mathrm{n}+3}+1, \mathrm{~F}_{4 \mathrm{n}+2}-1\right)=\mathrm{F}_{2 \mathrm{n}}$,
I.

$$
\left(F_{4 n+4}-1, F_{4 n+3}-1\right)=L_{2 n+1}
$$

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The Fission Parameter $Z^{2} / A$ for Some Representative Nuclides
The limiting value $Z^{2} / A=2 a_{S} / a_{c}=44$, from the incompressible lieuid-drop model is shown dotted. All the nuclides shown above $Z=90$ exhibit spontaneous fission but not as their major mode of decay. (See [1].)

All nuclei with parameter $Z^{2} / A>44$ could not practically exist, because they would decay already in "statu Nascendi." (See [1], [2].)

For the aforementioned possible comparatively stable candidate in the region of the first "island stability" the nucleus ${ }_{114}[\mathrm{x}]^{298}$ has the value of $Z^{2} / A=43.6$, which is very close to the limiting value of the fission parameter $=44$.

It seems that the element with $Z=114$ would be practically the last one in the Periodic Table of Elements. The most stable candidate at the element with $Z=114$ is the nucleus $114[\mathrm{x}]^{298}$. His proton-neutron ratio $\mathrm{Z} / \mathrm{N}=$ $0.6195 \cdots$ and this value is one of the best approximations to the "Golden Ratio" in the world of atoms. (See [3].)

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# DISCOVERING THE SQUARE-TRIANGULAR NUMBERS 

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Among the many mathematical gems which fascinated the ancient Greeks, the Polygonal Numbers were a favorite. They offered a variety of exciting problems of a wide range of difficulty and one can find numerous articles about them in the mathematical literature even up to the present time.

To the uninitiated, the polygonal numbers are those positive integers which can be represented as an array of points in a polygonal design. For example, the Triangular Numbers are the numbers $1,3,6,10, \cdots$ associated with the arrays


The square numbers are just the perfect squares $1,4,9,16, \cdots$ associated with the arrays:

$$
\begin{aligned}
& \mathrm{x} \times \mathrm{x} \mathrm{x} \\
& \text { x } x \text { x } x \text { x } x
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{x}, \mathrm{x} \mathrm{x}, \mathrm{x} \mathrm{x} \mathrm{x}, \mathrm{XX} \mathrm{x} \mathrm{x} \text {, }
\end{aligned}
$$

Similar considerations lead to pentagonal numbers, hexagonal numbers and so on.

One of the nicer problems which occurs in this topic is to determine which of the triangular numbers are also square numbers, i.e., which of the numbers

$$
1,3,6,10, \cdots, \frac{\mathrm{n}(\mathrm{n}+1)}{2}, \ldots
$$

are perfect squares. There are several ways of approaching this problem [1]. I would like to direct your attention to a very elementary method using the discovery approach advocated so well by Polya [2].

A few very natural questions arise, such as, "Are there any squaretriangular numbers?". This is easily answered since

$$
1^{2}=\frac{1 \cdot 2}{2}=1
$$

is such a number. To show that more than this trivial case occurs, we find that

$$
36=6^{2}=\frac{8 \cdot 9}{2}
$$

is also a square triangular number. One would then naturally ask, "Are there infinitely many square-triangular numbers?". This is considerably more difficult to answer since a careful check reveals the next one to be

$$
35^{2}=\frac{49 \cdot 50}{2}=1225
$$

and we see that they do not appear to be very dense. In seeking to answer the last question, one quite naturally asks, "Is there a formula which always yields such a number, or better yet, is there a formula which yields all such numbers?". This, in turn, leads us to ask, "Is there a pattern in these numbers which would help us guess a formula?".

To find a pattern from the three cases $1,36,1225$, seems rather futile, so we apply a little (!) more arithmetic to find that the next two cases are

$$
204^{2}=\frac{288 \cdot 289}{2}=41,616
$$

and

$$
1189^{2}=\frac{1681 \cdot 1682}{2}=1,413,721
$$

We now seek a pattern from the five cases: $1 ; 36 ; 1225 ; 41,616 ; 1,413,721$. One is immediately discouragingly impressed by the relative scarcity of square-triangular numbers and the possibility of a nice easy-to-guess pattern seems quite remote; but, having gone this far, it does not hurt to atleast pursue this course a little further. Let us introduce some notation to facilitate the work by calling $\mathrm{S}_{\mathrm{n}}, \mathrm{T}_{\mathrm{n}}$, and $(\mathrm{ST})_{\mathrm{n}}$ the $\mathrm{n}^{\text {th }}$ square, triangular, and square-triangular numbers, respectively. Organizing our data to date, then, we have:

$$
\begin{aligned}
& 1=(\mathrm{ST})_{1}=\mathrm{S}_{1} \quad=\quad 1^{2}=\frac{1 \cdot 2}{2}=\mathrm{T}_{1} \\
& 36=(\mathrm{ST})_{2}=\mathrm{S}_{2}=6^{2}=\frac{8 \cdot 9}{2}=\mathrm{T}_{8} \\
& 1,225=(\mathrm{ST})_{3}=\mathrm{S}_{35}=35^{2}=\frac{49 \cdot 50}{2}=\mathrm{T}_{49} \\
& 41,616=(\mathrm{ST})_{4}=\mathrm{S}_{204}=204^{2}=\frac{288 \cdot 289}{2}=\mathrm{T}_{288} \\
& 1,413,721=(\mathrm{ST})_{5}=\mathrm{S}_{1189}=1189^{2}=\frac{1681 \cdot 1682}{2}=\mathrm{T}_{1681}
\end{aligned}
$$

The adventurous reader is encouraged at this point to look for a pattern and formula on his own before reading any further.

For the unsuccessful guessers or those wishing to compare results, let us carry on by writing the numbers in various ways; in particular, we might look at them in prime factored form.

$$
\begin{aligned}
1 & =(\mathrm{ST})_{1}=\mathrm{S}_{1}=1^{2}=(1 \cdot 1)^{2}=\frac{1 \cdot 2}{2}=\frac{2}{2}=\mathrm{T}_{1} \\
36 & =(\mathrm{ST})_{2}=\mathrm{S}_{6}=6^{2}=(2 \cdot 3)^{2}=\frac{8 \cdot 9}{2}=\frac{2^{3} \cdot 3^{2}}{2}=\mathrm{T}_{8} \\
1,225 & =(\mathrm{ST})_{3}=\mathrm{S}_{35}=35^{2}=(5 \cdot 7)^{2}=\frac{49 \cdot 50}{2}=\frac{2 \cdot 5^{2} \cdot 7^{2}}{2}=\mathrm{T}_{49}
\end{aligned}
$$

$$
\begin{aligned}
41,616 & =(\mathrm{ST})_{4}=\mathrm{S}_{204}=204^{2}=\left(2^{2} \cdot 3 \cdot 17\right)^{2}=\frac{288 \cdot 289}{2}=\frac{2^{5} \cdot 3^{2} \cdot 17^{2}}{2}=\mathrm{T}_{288} \\
1,413,721 & =(\mathrm{ST})_{5}=\mathrm{S}_{1189}=1189^{2}=(29 \cdot 41)^{2}=\frac{1681 \cdot 1682}{2}=\frac{2 \cdot 29^{2} \cdot 41^{2}}{2}=\mathrm{T}_{1681}
\end{aligned}
$$

Is there a pattern now? We note that as far as patterns are concerned, the form of the $S_{n}{ }^{\prime} s$ is a little nicer than that of the $T_{n}{ }^{\prime} s$, but essentially they are the same, so we shall concentrate on the $S_{n}{ }^{\prime} s$.

Since we only have five cases at hand, and the sixth case is likely to be a bit far off, we must make the most of what we have. We might note that three of the cases are the square of exactly two factors whereas the trivial case $s_{1}=1^{2}$ could be written with any number of $1^{\prime} s$ and

$$
S_{204}=\left(2^{2} \cdot 3 \cdot 17\right)^{2}
$$

could be reduced to the square of two factors if we dropped the requirement of prime factors. It might be worthwhile to write each $S_{n}$ as the square of two factors. This allows no options except for $S_{204}$ which could then be written in five non-trivial ways, namely,

$$
S_{204}=(2 \cdot 102)^{2}=(3 \cdot 68)^{2}=(4 \cdot 51)^{2}=(6 \cdot 34)^{2}=(12.17)^{2}
$$

Do any of these fit into a pattern with the other four? If we looked only for the monotone increasing pattern of the factors we would choose $S_{204}=(12 \cdot 17)^{2}$. Now, looking at the data so arranged, we have:

$$
\begin{aligned}
& \mathrm{S}_{1}=(1 \cdot 1)^{2} \\
& \mathrm{~S}_{6}=(2 \cdot 3)^{2} \\
& \mathrm{~S}_{35}=(5 \cdot 7)^{2} \\
& \mathrm{~S}_{204}=(12 \cdot 17)^{2} \\
& \mathrm{~S}_{1189}=(29 \cdot 41)^{2}
\end{aligned}
$$

Look hard, now, for there is a very nice pattern here; and in fact, it is recursive of a Fibonacci type. Do you see that $1+1=2,1+2=3,2+3=$
$5,2+5=7, \quad 5+7=12, \quad 5+12=17, \quad 12+17=29, \quad$ and $12+29=41 ?$ Let us write this into our data as:

$$
\begin{aligned}
& (\mathrm{ST})_{1}=(1 \cdot 1)^{2} \\
& (\mathrm{ST})_{2}=(2 \cdot 3)^{2}=(1+1)^{2} \cdot(1+1+1)^{2}=(1+1)^{2} \cdot(2 \cdot 1+1)^{2} \\
& (\mathrm{ST})_{3}=(5 \cdot 7)^{2}=(2+3)^{2} \cdot(2+2+3)^{2}=(2+3)^{2} \cdot(2 \cdot 2+3)^{2} \\
& (\mathrm{ST})_{4}=(12 \cdot 17)^{2}=(5+7)^{2} \cdot(5+5+7)^{2}=(5+7)^{2} \cdot(2 \cdot 5+7)^{2} \\
& (\mathrm{ST})_{5}=(29 \cdot 41)^{2}=(12+17)^{2} \cdot(12+12+17)^{2}=(12+17)^{2} \cdot(2 \cdot 12+17)^{2}
\end{aligned}
$$

Before formalizing and trying to prove this guess, it would be well to test it as much as possible to see if it works at all. Our first test will be to see if $(29+41)^{2}(29+29+41)^{2}$ is a triangular number.

$$
(29+41)^{2}(29+29+41)^{2}=70^{2} \cdot 99^{2}=4900 \cdot 9801=\frac{9800 \cdot 9801}{2}
$$

is triangular and our confidence in our guess is considerably strengthened. Our next test will be to see if this new square-triangular number is, in fact, the next one; i. e., is it $(S T)_{6}$ ? This involves checking to see if there are any squares between $\mathrm{T}_{1681}$ and $\mathrm{T}_{9800}$ which is hardly an inviting exercise in arithmetic. Therefore, let us use the sometimes wise advice that "If you can't prove it, generalize it."

In order to proceed on with a proof we introduce a bit more notation. Let $a_{n}$ be defined by the recursive relation $a_{0}=0, a_{1}=1$, and $a_{n}=$ $2 \mathrm{a}_{\mathrm{n}-1}+\mathrm{a}_{\mathrm{n}-2}$ for $\mathrm{n} \geq 2$. For $\mathrm{n}=1,2,3,4,5$, this gives us the sequence $1,2,5,12,29$ which we recognize as the first factors for $(S T)_{1},(S T)_{2}$, $(\mathrm{ST})_{2},(\mathrm{ST})_{4},(\mathrm{ST})_{5}$, respectively. We also note that the second factors 1 , $3,7,17,41$ are $a_{1}+a_{0}, a_{2}+a_{1}, a_{3}+a_{2}, a_{4}+a_{3}, a_{5}+a_{4}$, respectively.

Finally, before proceeding with our proof, we notice that in order to prove a positive integer m is a triangular number, it suffices to show that there exists a positive integer $n$ such that

$$
\mathrm{m}=\frac{\mathrm{n}(\mathrm{n}+1)}{2}
$$

or equivalently that there exists positive integers $a$ and $b$ such that

$$
m=\frac{a b}{2}
$$

with $|a-b|=1$.
We now attempt to prove the following conjecture which is a formalized generalization from our data.

Conjecture A: $(S T)_{n}=a_{n}^{2}\left(a_{n}+a_{n-1}\right)^{2}$ for $n=1,2,3, \cdots$
Proof: We will attempt the proof in two parts.
(1) The sequence of numbers $a_{n}^{2}\left(a_{n}+a_{n-1}\right)^{2}$ for $n=1,2,3, \ldots$ are square-triangular numbers.
(2) This sequence is in fact all of the square-triangular numbers.

Clearly, $a_{n}^{2}\left(a_{n}+a_{n-1}\right)^{2}$ is a square number for all $n \geq 1$ so we concentrate on showing these numbers are also triangular for $n \geq 1$. Using mathematical induction, we first dispense with the case for $n=1$ as $1^{2}(1+0)^{2}=$ $1=(2 \cdot 1) / 2$ with $|2-1|=1$.

Now assume $\mathrm{a}_{\mathrm{n}}^{2}\left(\mathrm{a}_{\mathrm{n}}+\mathrm{a}_{\mathrm{n}-1}\right)^{2}$ is triangular with

$$
\mathrm{a}_{\mathrm{n}}^{2}\left(\mathrm{a}_{\mathrm{n}}+\mathrm{a}_{\mathrm{n}-1}\right)^{2}=\frac{2 \mathrm{a}_{\mathrm{n}}^{2}\left(\mathrm{a}_{\mathrm{n}}+\mathrm{a}_{\mathrm{n}-1}\right)^{2}}{2}
$$

and

$$
\left|2 a_{n}^{2}-\left(a_{n}+a_{n-1}\right)^{2}\right|=1
$$

for some $n \geq 1$. Then

$$
a_{n+1}^{2}\left(a_{n+1}+a_{n}\right)^{2}=\frac{2\left(2 a_{n}+a_{n-1}\right)^{2}\left(2 a_{n}+a_{n-1}+a_{n}\right)^{2}}{2}
$$

where

$$
\begin{aligned}
& \left|2\left(2 a_{n}+a_{n-1}\right)^{2}-\left(3 a_{n}+a_{n-1}\right)^{2}\right| \\
& \quad=\left|\left(8 a_{n}^{2}+8 a_{n} a_{n-1}+2 a_{n-1}^{2}\right)-\left(9 a_{n}^{2}+6 a_{n} a_{n-1}+a_{n-1}^{2}\right)\right| \\
& \quad=\left|-a_{n}^{2}+2 a_{n} a_{n-1}+a_{n-1}^{2}\right| \\
& \quad=\left|a_{n}^{2}-2 a_{n} a_{n-1}-a_{n-1}^{2}\right|=\left|2 a_{n}^{2}-\left(a_{n}+a_{n-1}\right)^{2}\right|=1 .
\end{aligned}
$$

Therefore $a_{n+1}^{2}\left(a_{n+1}+a_{n}\right)^{2}$ is also triangular and (1) is proved. Now let

$$
\mathrm{m}_{1}^{2}=\frac{\mathrm{k}_{1}\left(\mathrm{k}_{1}+1\right)}{2}
$$

be an arbitrary square-triangular number. There are two cases which can be considered, namely $k_{1}$ even or $k_{1}$ odd. It is immaterial which we consider first, as we will be alternating back and forth from one to the other in a descending, sequence of square-triangular numbers which will terminate finally at $(S T)_{1}=1$. To be definite, let $k_{1}$ be odd which implies

$$
\frac{\mathrm{k}_{1}+1}{2}
$$

is an integer and

$$
\left(\mathrm{k}_{1}, \frac{\mathrm{k}_{1}+1}{2}\right)=1
$$

(we are using the common notation of letting ( $a, b$ ) denote the greatest common divisor of a and b). Therefore

$$
\mathrm{m}_{1}^{2}=\mathrm{k}_{1} \frac{\left(\mathrm{k}_{1}+1\right)}{2}
$$

a square implies that both $\mathrm{k}_{1}$ and $\left(\mathrm{k}_{1}+1\right) / 2$ are squares. Let

$$
\frac{\mathrm{k}_{1}+1}{2}=\mathrm{b}_{1}^{2}
$$

and $k_{1}=c_{1}^{2}$ with $b_{1}, c_{1} \geq 1$. Now $k_{1} \geq 1$ implies $b_{1} \leq c_{1}$ and, in fact, $b_{1}=c_{1}$, if and only if

$$
\mathrm{k}_{1}=\frac{\mathrm{k}_{1}+1}{2}
$$

if and only if $\mathrm{k}_{1}=1$ if and only if $\mathrm{m}_{1}^{2}=1$, in which case

$$
\mathrm{m}_{1}^{2}=\mathrm{a}_{1}^{2}\left(\mathrm{a}_{1}+\mathrm{a}_{0}\right)^{2}=(\mathrm{ST})_{1}
$$

and we are done. Consider then $b_{1}<c_{1}$ and define $b_{2}=c_{1}-b_{1}, c_{2}=$ $2 \mathrm{~b}_{1}-\mathrm{c}_{1}$, and $\mathrm{m}_{2}^{2}=\mathrm{b}_{2}^{2} \mathrm{c}_{2}^{2}$.

Since

$$
2 \mathrm{~b}_{1}^{2}-\mathrm{c}_{1}^{2}=\mathrm{k}_{1}+1-\mathrm{k}_{1}=1
$$

we factor and get

$$
\left(\sqrt{2} b_{1}-c_{1}\right)\left(\sqrt{2} b_{1}+c_{1}\right)=1
$$

where $b_{1}, c_{1} \geq 1$ implies $\sqrt{2} b_{1}+c_{1}>0$, which implies that $\sqrt{2} b_{1}-c_{1}$ $>0$, so $\sqrt{2} b_{1}>c_{1}$. Now $3 / 2>\sqrt{2}$ so it also follows that $3 b_{1} / 2>c_{1}$ which is equivalent to $3 b_{1}>2 c_{1}$ and thus $2 b_{1}-c_{1}>c_{1}-b_{1}$. Also, $c_{1}>b_{1}$ implies both that $c_{1}-b_{1}>0$ and that $b_{1}>2 b_{1}-c_{1}$, which then gives us the inequality $\mathrm{c}_{1}>\mathrm{b}_{1}>2 \mathrm{~b}_{1}-\mathrm{c}_{1}>\mathrm{c}_{1}-\mathrm{b}_{1}>0$, or equivalently, $\mathrm{c}_{1}>\mathrm{b}_{1}>\mathrm{c}_{2}>$ $\mathrm{b}_{2}>0$.

Furthermore,

$$
\mathrm{m}_{2}^{2}=\mathrm{b}_{2}^{2} \mathrm{c}_{2}^{2}=\frac{2 \mathrm{~b}_{2}^{2} \mathrm{c}_{2}^{2}}{2}
$$

where

$$
\left|2 \mathrm{~b}_{2}^{2}-\mathrm{c}_{2}^{2}\right|=\left|2\left(\mathrm{c}_{1}-\mathrm{b}_{1}\right)^{2}-\left(2 \mathrm{~b}_{1}-\mathrm{c}_{1}\right)^{2}\right|=\left|\mathrm{c}_{1}^{2}-2 \mathrm{~b}_{1}^{2}\right|=|-1|=1
$$

so $\mathrm{m}_{2}^{2}$ is also a square-triangular number and is necessarily smaller than $m_{1}^{2}$ from the last inequality. Now let

$$
\mathrm{m}_{2}^{2}=\frac{2 \mathrm{~b}_{2}^{2} \mathrm{c}_{2}^{2}}{2}=\frac{\mathrm{k}_{2}\left(\mathrm{k}_{2}+1\right)}{2}
$$

with $2 \mathrm{~b}^{2}=\mathrm{k}_{2}$ (since $2 \mathrm{~b}_{2}^{2}-\mathrm{c}_{2}^{2}=-1$ gives $c_{2}^{2}=2 \mathrm{~b}_{2}^{2}+1$ ) and we get $\mathrm{k}_{2}$ even as predicted earlier. It might be observed that in this case $m_{2}^{2} \neq 1$ which is equivalent to our fact that $c_{2}>b_{2}$.

Now continue in the same manner by defining $b_{3}=c_{2}-b_{2}, c_{3}=2 b_{2}-$ $c_{2}$, and $m_{3}^{2}=b_{3}^{2} c_{3}^{2}$. In this case, $b_{3} \leq c_{3}$ since if $b_{3}>c_{3}$, then by substitution $c_{2}-b_{2}>2 b_{2}-c_{2}$ which implies $c_{2}>3 b_{2} / 2$. Recalling that

$$
2 b_{2}^{2}-c_{2}^{2}=c_{1}^{2}-2 b_{1}^{2}=-1
$$

which is equivalent to $c_{2}^{2}-2 b_{2}^{2}=1$, we get by using $c_{2}>3 b_{2} / 2$ that

$$
\left(3 \mathrm{~b}_{2} / 2\right)^{2}-2 \mathrm{~b}_{2}^{2}<1
$$

This implies

$$
\frac{b_{2}^{2}}{4}<1
$$

which implies $b_{2}$ is a positive integer with square less than 4 , or that $b_{2}=$ 1. This, however, yields $\mathrm{c}_{2}^{2}-2 \cdot 1^{2}=1$ or $\mathrm{c}_{2}^{2}=3$ in which case $\mathrm{c}_{2}=\sqrt{3}$ must be a positive integer, which is false. Thus the hypothesis that $b_{3}>c_{3}$ is false and $b_{3} \leq c_{3}$ as claimed. We might note that $b_{3}=c_{3}$ is equivalent to $\mathrm{c}_{2}=3 \mathrm{~b}_{2} / 2$ which implies

$$
1=c_{2}^{2}-2 b_{2}^{2}=\left(3 b_{2} / 2\right)^{2}-2 b_{2}^{2}=\frac{b^{2}}{4}
$$

which implies that $b_{2}=2$ and also that

$$
\mathrm{c}_{2}=\frac{3}{2} \cdot 2=3
$$

This, in turn, gives us $b_{3}=c_{3}=1$ as well as $b_{1}=5, c_{1}=7$, so

$$
\mathrm{m}_{1}^{2}=5^{2} \cdot 7^{2}=\mathrm{a}_{3}^{2}\left(\mathrm{a}_{3}+\mathrm{a}_{2}\right)^{2}=(\mathrm{ST})_{3}
$$

and we are done.
In general, with $b_{3} \leq c_{3}$,

$$
\mathrm{m}_{3}^{2}=\mathrm{b}_{3}^{2} \mathrm{c}_{3}^{2}=\frac{2 \mathrm{~b}_{3}^{2} \mathrm{c}_{3}^{2}}{2}
$$

with

$$
\left|2 \mathrm{~b}_{3}^{2}-\mathrm{c}_{3}^{2}\right|=\left|2\left(\mathrm{c}_{2}-\mathrm{b}_{2}\right)^{2}-\left(2 \mathrm{~b}_{2}-\mathrm{c}_{2}\right)^{2}\right|=\left|\mathrm{c}_{2}^{2}-2 \mathrm{~b}_{2}^{2}\right|=1
$$

so $\mathrm{m}_{3}^{2}$ is again a square-triangular number. Since

$$
\mathrm{c}_{2}>\mathrm{b}_{2}>2 \mathrm{~b}_{2}-\mathrm{c}_{2} \geq \mathrm{c}_{2}-\mathrm{b}_{2}>0
$$

or equivalently

$$
\mathrm{c}_{2}>\mathrm{b}_{2}>\mathrm{c}_{3} \geq \mathrm{b}_{3}>0
$$

$\mathrm{m}_{3}^{2}$ is again smaller than $\mathrm{m}_{2}^{2}$. If we let

$$
\mathrm{m}_{3}^{2}=\frac{2 \mathrm{~b}_{3}^{2} \mathrm{c}_{3}^{2}}{2}=\frac{\left(\mathrm{k}_{3}+1\right) \mathrm{k}_{3}}{2}
$$

with $k_{3}=c_{3}^{2}$ (since from above $2 b_{3}^{2}-c_{3}^{2}=1$ gives $2 b_{3}^{2}=c_{3}^{2}+1$ ) we have $2 b_{3}^{2}=k_{3}+1$. Thus $k_{3}$ is now odd as in the first case and one can proceed in exactly the same manner generating new and smaller square-triangular numbers until we finally arrive at $\mathrm{m}_{\mathrm{n}}^{2}=\mathrm{b}_{\mathrm{n}}^{2} \mathrm{c}_{\mathrm{n}}^{2}=1$ with $\mathrm{b}_{\mathrm{n}}=\mathrm{c}_{\mathrm{n}}=1$.

This gives us

$$
1=b_{n}=c_{n-1}-b_{n-1}
$$

and

$$
1=b_{n}=c_{n-1}-b_{n-1}
$$

which gives $b_{n-1}=2$ and $c_{n-1}=3$ when solved. It follows that

$$
2=b_{n-2}=c_{n-2}-b_{n-2}
$$

and

$$
3=c_{n-1}=2 b_{n-2}-c_{n-2}
$$

which yields $b_{n-2}=5$ and $c_{n-2}=7, \cdots$. In general, for $j \geq 2, b_{j+1}=$ $c_{j}-b_{j}$,

$$
c_{j+1}=2 b_{j}-c_{j}
$$

and

$$
b_{j}=c_{j-1}-b_{j-1}, \quad c_{j}=2 b_{j-1}-c_{j-1}
$$

Therefore,

$$
\begin{aligned}
2 b_{j}+b_{j+1}=2\left(c_{j-1}-b_{j-1}\right) & +\left(c_{j}-b_{j}\right)=2 c_{j-1}-2 b_{j-1}+\left(2 b_{j-1}-c_{j-1}\right) \\
& -\left(c_{j-1}-b_{j-1}\right)=b_{j-1}
\end{aligned}
$$

and

$$
\begin{aligned}
b_{j}+b_{j-1}=b_{j} & +\left(2 b_{j}+b_{j+1}\right)=3 b_{j}+b_{j+1}=3 c_{j-1}-3 b_{j-1} \\
& +\left(2 b_{j-1}-c_{j+1}\right)-\left(c_{j-1}-b_{j-1}\right)=c_{j-1}
\end{aligned}
$$

We have just done the computation for an induction proof that $b_{j}=a_{n-j+1}$ and $c_{j}=a_{n-j+1}+a_{n-j}$ for $j=1,2, \cdots, n$. In particular, for $j=1$, it follows that

$$
m_{1}^{2}=b_{1}^{2} c_{1}^{2}=a_{n}^{2}\left(a_{n}+a_{n-1}\right)^{2}
$$

and $m_{1}^{2}$ is in our sequence as claimed, and (2) is proved.
Since the sequence is monotonically increasing, we have that

$$
(\mathrm{ST})_{\mathrm{n}}=\mathrm{a}_{\mathrm{n}}^{2} \cdot\left(\mathrm{a}_{\mathrm{n}}+\mathrm{a}_{\mathrm{n}-1}\right)^{2}
$$

as claimed, and our Conjecture $A$ is true.
Thus the empirical data of five casesled usto guess a very nice recursion formula which turned out to be valid. So even though the square-triangular numbers are very sparse, not only in relation to the positive integers, but also in relation to either the square or triangular numbers themselves, there are still infinitely many of them and they behave very well. In fact, they behave beautifully.

There are many other very nice relationships in these numbers which are left for the reader to derive and/or prove. A few of these are listed here to whet the appetite.
(i) $(S T)_{1}=a_{1}^{2}\left(a_{1}+a_{0}\right)^{2}$
$(S T)_{2}=a_{2}^{2}\left(a_{2}+a_{1}\right)^{2}=\left(2 a_{1}+a_{0}\right)^{2}\left(3 a_{1}+a_{0}\right)^{2}$
:

$$
(\mathrm{ST})_{\mathrm{n}}=\mathrm{a}_{\mathrm{n}}^{2}\left(\mathrm{a}_{\mathrm{n}}+\mathrm{a}_{\mathrm{n}-1}\right)^{2}=\left(2 \mathrm{a}_{\mathrm{n}-1}+\mathrm{a}_{\mathrm{n}-2}\right)^{2}\left(3 \mathrm{a}_{\mathrm{n}-1}+\mathrm{a}_{\mathrm{n}-2}\right)^{2} \text { for } \mathrm{n} \geq 2
$$

(ii) $(\mathrm{ST})_{\mathrm{n}}$ is odd if and only if n is odd if and only if $\mathrm{a}_{\mathrm{n}}$ is odd.
(iii) $a_{n}=\frac{(1+\sqrt{2})^{n}-(1-\sqrt{2})^{n}}{2 \sqrt{2}}$ and $(\mathrm{ST})_{\mathrm{n}}=\left(\frac{(1+\sqrt{2})^{2 \mathrm{n}}-(1-\sqrt{2})^{2 \mathrm{n}}}{4 \sqrt{2}}\right)^{2}$.
(iv) $\quad 2 a_{n}^{2}-\left(a_{n}+a_{n-1}\right)^{2}=(-1)^{n-1} \quad$ for $\quad n \geq 1$.
(v) If $\quad(\mathrm{ST})_{\mathrm{n}}=\mathrm{a}_{\mathrm{n}}^{2}\left(\mathrm{a}_{\mathrm{n}}+\mathrm{a}_{\mathrm{n}-1}\right)^{2}=\mathrm{S}_{\mathrm{u}_{\mathrm{n}}}=\mathrm{u}_{\mathrm{n}}^{2}=\mathrm{T}_{\mathrm{v}_{\mathrm{n}}}=\frac{\mathrm{v}_{\mathrm{n}}\left(\mathrm{v}_{\mathrm{n}}+1\right)}{2}$,
then $v_{n+1}=u_{n+1}+v_{n}+u_{n}$ for $n \geq 1$. This may be proven with or without (vi) below.
(vi) The sequences of $u_{n} ' s$ and $v_{n}$ 's are generated by the recursive formulae:

$$
\begin{aligned}
& u_{0}=0, u_{1}=1, \text { and } u_{n}=6 u_{n-1}-u_{n-2} \text { for } n \geq 2, \\
& v_{0}=0, v_{1}=1, \text { and } v_{n}=6 v_{n-1}-v_{n-2}+2 \text { for } n \geq 2
\end{aligned}
$$

(vii)

$$
\begin{aligned}
& \mathrm{u}_{\mathrm{n}}=\frac{(3+2 \sqrt{2})^{\mathrm{n}}-(3-2 \sqrt{2})^{\mathrm{n}}}{4 \sqrt{2}} \\
& \mathrm{v}_{\mathrm{n}}=\frac{(4+3 \sqrt{2})(3+2 \sqrt{2})^{\mathrm{n}-1}-(4-3 \sqrt{2})(3-2 \sqrt{2})^{\mathrm{n}-1}}{4 \sqrt{2}}-\frac{1}{2}
\end{aligned}
$$

(viii) The square-triangular numbers are precisely the numbers $x^{2} y^{2}$ such that $x^{2}-2 y^{2}=1$ or $x^{2}-2 y^{2}=-1$ with $x$ and $y$ positive
integers. These types of Diophantine equations are commonly known as Pell's Equations.

Having seen these very nice results, the mathematician naturally asks, "What about the triangular-pentagonal numbers, square-pentagonal numbers, and so on?". This is not at present completely answered, butmany in-roads have been made by some outstanding mathematicians. In particular, W. Sierpinski devoted some time to this problem [3], but perhaps the nicest result so far obtained is one derived by Diane (Smith) Lucas as an undergraduate at Washington State University. In a paper (not yet published) she obtained the very beautiful result that for $3 \leq m<n$, there exist infinitely many numbers which are both $n$-gonal and $m$-gonal if and only if
(i) $\mathrm{m}=3$ and $\mathrm{n}=6$
or
(ii) $(m-2)(n-2)$ is not a perfect square.

With the machinery she developed, it is quite easy to derive for example, that the $\mathrm{n}^{\text {th }}$ pentagonal-triangular number

$$
\left(\mathrm{P}_{5,3}\right)_{\mathrm{n}}=\frac{(2-\sqrt{3})(97+56 \sqrt{3})^{\mathrm{n}}+(2+\sqrt{3})(97-56 \sqrt{3})^{\mathrm{n}}-4}{48}
$$

which is a result obtained by Sierpinski.

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2. George Polya, Mathematics and Plausible Reasoning, Princeton University Press, Vol. 1, 1954.
3. W. Sierpinski, "Sur les Nombres Pentagonaux," Bull. Soc. Royale Sciences Leige, 33 (1964), pp. 513-517.

# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by<br>A. P. HILLMAN<br>University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-202 Proposed by Richard M. GrassI, University of New Mexico, Albuquerque, New Mexico.

Let $F_{1}, F_{2}, \cdots$ be the Fibonacci Sequence $1,1,2,3,5,8, \cdots$ with $F_{n+2}=F_{n+1}+F_{n}$. Let

$$
\mathrm{G}_{\mathrm{n}}=\mathrm{F}_{4 \mathrm{n}-2}+\mathrm{F}_{4 \mathrm{n}}+\mathrm{F}_{4 \mathrm{n}+2}
$$

(i) Find a recursion formula for the sequence $G_{1}, G_{2}, \cdots$.
(ii) Show that each $G_{n}$ is a multiple of 12 .

B-203 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Show that $F_{8 n-4}+F_{8 n}+F_{8 n+4}$ is always a multiple of 168 .

B-204 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let $\mathrm{F}_{1}=\mathrm{F}_{2}=1$ and $\mathrm{F}_{\mathrm{n}+2}=\mathrm{F}_{\mathrm{n}+1}+\mathrm{F}_{\mathrm{n}}$. Show that
(i) $F_{1} x+F_{3} x^{2}+F_{5} x^{3}+F_{7} x^{4}+\cdots=\left(x-x^{2}\right) /\left(1-j x+x^{2}\right)$
for $|x|<(3-\sqrt{5}) / 2$.
(ii) $1+2 \mathrm{x}+3 \mathrm{x}^{2}+4 \mathrm{x}^{3}+\cdots=1 /(1-\mathrm{x})^{2}$ for $|\mathrm{x}|<1$.
(iii) $n F_{1}+(n-1) F_{3}+(n-2) F_{5}+\cdots+2 F_{3 n-2}+F_{2 n-1}=F_{2 n+1}-1$.

B-205 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that
$(2 n-1) F_{1}+(2 n-3) F_{3}+(2 n-5) F_{5}+\cdots+3 F_{2 n-3}+F_{2 n-1}=L_{2 n}-2$,
where $L_{n}$ is the $n^{\text {th }}$ Lucas number (i.e., $L_{1}=1, L_{2}=3, L_{n+2}=L_{n+1}$ $+L_{n}$ ).

B-206 Proposed by Guy A. Guillotte, Montreal, Quebec, Canada.
Let $\mathrm{a}=(1+\sqrt{5}) / 2$ and sum

$$
\sum_{n=1}^{\infty} \frac{1}{a F_{n+1}+F_{n}}
$$

B-207 Proposed by Guy A. Guillotte, Montreal, Quebec, Canada.
Sum

$$
\sum_{n=1}^{\infty} \frac{1}{F_{n}+\sqrt{5} F_{n+1}+F_{n+2}}
$$

## SOLUTIONS

CONTRACTING INTO A SQUARE
B-184 Proposed by Bruce W. King, Adirondack Community College, Glen Falls, New York.

Let the sequence $\left\{T_{n}\right\}$ satisfy $T_{n+2}=T_{n+1}+T_{n}$ with arbitrary initial conditions. Let

$$
\mathrm{g}(\mathrm{n})=\mathrm{T}_{\mathrm{n}}^{2} \mathrm{~T}_{\mathrm{n}+3}^{2}+4 \mathrm{~T}_{\mathrm{n}+1}^{2} \mathrm{~T}_{\mathrm{n}+2}^{2}
$$

Show the following:
(i)

$$
\mathrm{g}(\mathrm{n})=\left(\mathrm{T}_{\mathrm{n}+1}^{2}+\mathrm{T}_{\mathrm{n}+2}^{2}\right)^{2}
$$

(ii) If $\mathrm{T}_{\mathrm{n}}$ is the Lucas number $\mathrm{L}_{\mathrm{n}}$,

$$
\mathrm{g}(\mathrm{n})=25 \mathrm{~F}_{2 \mathrm{n}+3}^{2}
$$

(See Fibonacci Quarterly, Problems H-101, October, 1968, and B-160, April, 1968.)

Solution by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Substituting $T_{n}=T_{n+2}-T_{n+1}$ and $T_{n+3}=T_{n+2}+T_{n+1}$ into $g(n)$ we have

$$
\begin{aligned}
g(n) & =\left(T_{n+2}-T_{n+1}\right)^{2}\left(T_{n+2}+T_{n+1}\right)^{2}+4 T_{n+1}^{2} \cdot T_{n+2}^{2} \\
& =\left(T_{n+2}^{2}-T_{n+1}^{2}\right)^{2}+4 T_{n+1}^{2} T_{n+2}^{2} \\
& =T_{n+1}^{4}+2 T_{n+1}^{2} T_{n+2}^{2}+T_{n+2}^{4} \\
& =\left(T_{n+1}^{2}+T_{n+2}^{2}\right)^{2} .
\end{aligned}
$$

Thus (i) is established.
By substituting in terms of $r$ and $s$ in the usual way, (ii) is established.

$$
\begin{aligned}
\left(\mathrm{L}_{\mathrm{n}+1}^{2}+\mathrm{L}_{\mathrm{n}+2}^{2}\right)^{2} & =\left[\left(\mathrm{r}^{\mathrm{n}+1}+\mathrm{s}^{\mathrm{n}+1}\right)^{2}+\left(\mathrm{r}^{\mathrm{n}+2}+\mathrm{s}^{\mathrm{n}+2}\right)^{2}\right]^{2} \\
& =\left[\mathrm{r}^{2 \mathrm{n}+3}\left(\mathrm{r}^{-1}+\mathrm{r}\right)+\mathrm{s}^{2 \mathrm{n}+3}\left(\mathrm{~s}^{-1}+\mathrm{s}\right)\right]^{2} \\
& =\left[(\mathrm{r}-\mathrm{s})\left(\mathrm{r}^{2 \mathrm{n}+3}-\mathrm{s}^{2 \mathrm{n}+3}\right)\right]^{2} \\
& =25 \mathrm{~F}_{2 \mathrm{n}+3}^{2}
\end{aligned}
$$

where $r$ and $s$ are the roots of $x^{2}-x-1=0$.
Also solved by W. C. Barley, A. K. Gupta, John Kegel, John W. Milsom, Henry Newmon, C. B. A. Peck, A. G. Shannon (Australia), and the Proposer.

## LUCAS RATIO I

B-185 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
Show that

$$
\mathrm{L}_{5 \mathrm{n}} / \mathrm{L}_{\mathrm{n}}=\mathrm{L}_{2 \mathrm{n}}^{2}-(-1)^{\mathrm{n}} \mathrm{~L}_{2 \mathrm{n}}-1
$$

Solution by C. B. A. Peck, State College, Pennsy/vania.
Substitute in the r.h.s. $L_{n}=a^{n}+b^{n}$ where $a b=-1$, multiply by $a^{n}+b^{n} \neq 0$ afterward to get $a^{5 n^{n}}+b^{5 n}$.

Also solved by W. C. Barley, Wray G. Brady, Warren Chaves, Herta T. Freitag, Edgar Karst, Charles Kenney, John W. Milsom, John Wessner, David Zeitlin, and the Proposer.

## LUCAS RATIO II

B-186 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
Show that

$$
\mathrm{L}_{5 \mathrm{n}} / \mathrm{L}_{\mathrm{n}}=\left[\mathrm{L}_{2 \mathrm{n}}-(-1)^{\mathrm{n}_{3}} 3\right]^{2}+(-1)^{\mathrm{n}_{2}} 2 \mathrm{~F}_{\mathrm{n}}^{2}
$$

(For n even, this result has been given by D. Jarden in the Fibonacci Quarterly, Vol. 5 (1967), p. 346.)

Solution by John Wessner, Montana State University, Bozeman, Montana.
Using the well-known identity,

$$
\mathrm{L}_{2 \mathrm{n}}=5 \mathrm{~F}_{\mathrm{n}}^{2}+2(-1)^{\mathrm{n}}
$$

and the result of Problem B-185, we find

$$
\begin{aligned}
\mathrm{L}_{5 \mathrm{n}} / \mathrm{L}_{\mathrm{n}} & =\mathrm{L}_{2 \mathrm{n}}^{2}-(-1)^{\mathrm{n}} \mathrm{~L}_{2 \mathrm{n}}-1 \\
& =\left[\mathrm{L}_{2 \mathrm{n}}-2(-1)^{\mathrm{n}}\right]^{2}+5(-1)^{\mathrm{n}} \mathrm{~L}_{2 \mathrm{n}}-10 \\
& =\left[\mathrm{L}_{2 \mathrm{n}}-3(-1)^{\mathrm{n}}\right]^{2}+5(-1)^{\mathrm{n}}\left[5 \mathrm{~F}_{\mathrm{n}}^{2}+2(-1)^{\mathrm{n}}\right]-10 \\
& =\left[\mathrm{L}_{2 \mathrm{n}}-3(-1)^{\mathrm{n}}\right]^{2}+25(-1)^{\mathrm{n}} \mathrm{~F}_{\mathrm{n}}^{2}
\end{aligned}
$$

This is the result given by Jarden in the reference. The " 5 " in the problem statement was a misprint.

The following solved the corrected problem or pointed out the misprint: W. C. Barley, Wray G. Brady, Herta T. Freitag, John Kegel, Henry Newmon, C. B. A. Peck, and the Proposer.

## A DIOPHANTINE EQUATION

B-187 Proposed by Carl Gronemeijer, Saramoc Lake, New York.
Find positive integers x and y , with x even, such that

$$
\left(x^{2}+y^{2}\right)\left(x^{2}+x+y^{2}\right)\left(x^{2}+\frac{3}{2} x+y^{2}\right)=1,608,404
$$

Solution by Richard L. Breisch, Pennsy/vania State University, University Park, Pennsy/vania.

Since

$$
\left(x^{2}+y^{2}\right)<\left(x^{2}+x+y^{2}\right)<\left(x^{2}+\frac{3}{2} x+y^{2}\right)
$$

$\left(x^{2}+y^{2}\right)<\sqrt[3]{1,608,404}$. Hence, it is sufficient to consider $x$ and $y$ such that $\left(x^{2}+y^{2}\right)<117$; that requires $0<\mathrm{x} \leq 10$ and $0<\mathrm{y} \leq 10$. Since $1,608,414$ factors into $2^{2} \cdot 7 \cdot 17 \cdot 31 \cdot 109,\left(x^{2}+y^{2}\right)$ must equal either

$$
68=4+64=2^{2} \cdot 17
$$

or

$$
24=9+25=2 \cdot 17
$$

or

$$
109=100+9
$$

Only this last value works, and thus with $\mathrm{x}=10$ and $\mathrm{y}=3$, we get

$$
109 \cdot 119 \cdot 124=1,608,404
$$

Also solved by W. C. Barley, Wray G. Brady, Herta T. Freitag, J. A. H. Hunter (Canada), Charles Kenney, John W. Milsom, C. B. A. Peck, David Zeitlin, and the Proposer.

## INSCRIBED CIRCUMSCRIBED QUADRILATERAL

B-188 Proposed by A. G. Shannon, University of Papua and New Guinea, Boroko, Papua.
Two circles are related so that there is a trapezoid $A B C D$ inscribed in one and circumscribed in the other. $A B$ is the diameter of the larger circle which has center $O$, and $A B$ is parallel to $C D$. $\theta$ is half of angle AOD. Prove that $\sin \theta=(-1+\sqrt{5}) / 2$.

Solution by Joseph Konhauser, Macalester College, St. Paul, Minnesota.
In a circumscribed quadrilateral, sums of opposite sides are equal, so

$$
A B+D C=A D+B C
$$

Substituting $A B=2 r$,

$$
\mathrm{DC}=2 \mathrm{r} \sin \theta(\pi / 2)-2 \theta, \quad \mathrm{AD}=2 \mathrm{r} \sin \theta,
$$

where $r$ is the radius of the larger circle, we obtain, after simplifying,

```
sin}0=1-\mp@subsup{\operatorname{sin}}{}{2}0
```

It follows that $\sin \theta=(-1+\sqrt{5}) / 2$.

Also solved by Richard L. Breisch, Herta T. Freitag, C. B. A. Peck, John Wessner, and the Proposer.

## FIBONACCI EXPONENTS

B-189 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $a_{0}=1, a_{1}=7$, and $a_{n+2}=a_{n+1} a_{n}$ for $n \geq 0$. Find the last digit (i. e., units digit) of agga。

## Solution by David Zeitlin, Minneapolis, Minnesota.

The units digit has a repetitive cycle of six digits: $1,7,7,9,3,7$. Since agg9 is the $1,000^{\text {th }}$ term, and $1000=6(166)+4$, the required units digit is 9.

Also solved by W. C. Barley, Wray G. Brady, Richard L. Breisch, Warren Chaves, Herta T. Freitag, J. A. H. Hunter (Cañada), Henry Newmon, C. B. A. Peck, Richard W. Sielaff, John Wessner, and the Proposer.
[Continued from page 50.]
show that Theorem 2 yields an equivalent formula.

## REFERENCES

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2. James A. Jeske, "Linear Recursive Relations, Part II,"Fibonacci Quarterly, Vol. 1, No. 4, p. 35.
3. James A. Jeske, "Linear Recursive Relations, Part III," Fibonacci Quarterly, Vol. 2, No. 2, p. 197.
4. Brother Alfred Brousseau, "Linear Recursive Relations, Lesson III," Fibonacci Quarterly, Vol. 7, No. 1, p. 99.
5. Brother Alfred Brousseau, "Linear Recursive Relations, Lesson IV," Fibonacci Quarterly, Vol. 7, No. 2, p. 194.
6. Brother Alfred Brousseau, "Linear Recursive Relations, Lesson V," Fibonacci Quarterly, Vol. 7, No. 3, p. 295.
7. Brother Alfred Brousseau, "Linear Recursive Relations, Lesson VI," Fibonacci Quarterly, Vol. 7, No. 5, p. 533.
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