A DETERMINANT INVOLVING GENERALIZED BINOMIAL COEFFICIENTS

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1. INTRODUCTION

Define the Fibonacci numbers F_n by $F_1 = F_2 = 1$,

$$F_{n+2} - F_{n+1} - F_n = 0 . \tag{1.1}$$

This difference equation may be extended in both directions, yielding

$$F_{-n} = (-1)^{n+1}F_n .$$

Lucas [2] has shown that the $n \times n$ determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & \cdots \\ -1 & 1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & 1 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix}_n = F_{n+1} \tag{1.2}$$

This is also a consequence of Problems B-13 [5] and B-16 [6] in this Quarterly. Note that the rows of (1.2) are the negatives of the coefficients of the difference equation (1.1) obeyed by the Fibonacci numbers. The squares of the Fibonacci numbers obey

$$F^2_{n+3} - 2F^2_{n+2} - 2F^2_{n+1} + F^2_n = 0 . \tag{1.3}$$

If we take the negatives of the coefficients of (1.3) and place them in a determinant analogous to (1.2), we find

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$$\text{(1.4)} \qquad \begin{vmatrix} 2 & 2 & -1 & 0 & \cdots \\ -1 & 2 & 2 & -1 & \cdots \\ 0 & -1 & 2 & 2 & \cdots \\ 0 & 0 & -1 & 2 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix}_n = F_{n+1}F_{n+2}$$

Equation (1.4), which appears to be new, may be proved by expanding along the last column and using induction on n. It is our aim to generalize (1.2) and (1.4), first for the Fibonacci sequence, and then for arbitrary second-order recurring sequences.

2. THE FIBONACCI CASE

We define the Fibonacci generalized binomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}$ by

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{F_m F_{m-1} \cdots F_{m-r+1}}{F_1 F_2 \cdots F_r} \quad (r > 0); \qquad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1 .$$

Note that $\begin{bmatrix} m \\ r \end{bmatrix}$ is defined for all integers and all non-negative integers r, and that

$$\text{(2.1)} \qquad \begin{bmatrix} m \\ r \end{bmatrix} = 0 \quad \text{for} \quad m = 0, 1, \cdots, r-1 .$$

It is convenient to set

$$\text{(2.2)} \qquad \begin{bmatrix} m \\ r \end{bmatrix} = 0 \quad \text{for} \quad r < 0 .$$

Jarden [1] showed that the term-by-term product P_n of $k - 1$ sequences each of which obeys (1.1) satisfies

$$\text{(2.3)} \qquad \sum_{j=0}^{k} (-1)^{j(j+1)/2} \begin{bmatrix} k \\ j \end{bmatrix} P_{n-j} = 0 .$$

In particular, if each is the Fibonacci sequence we have

$$\sum_{j=0}^{k} (-1)^{j(j+1)/2} \begin{bmatrix} k \\ j \end{bmatrix} F_{n-j}^{k-1} = 0 \ . \tag{2.4}$$

This becomes (1.1) for $k = 2$, and (1.3) for $k = 3$.

Determinants of the form

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \tag{2.5}$$

are known as recurrents. We shall put the coefficients of (2.4) into an $n \times n$ recurrent and show its value is yet another generalized binomial coefficient. We remark that a general method for evaluating recurrents, from which the results here would follow, appears to date back to H. Faure (see [3], Vol. 2, p. 212). However, our approach seems somewhat more direct, and the specific results novel enough to warrant separate attention.

Put

$$D_{n,k} = \det(a_{rs}) \ ,$$

where

$$a_{rs} = -(-1)^{(s-r+1)(s-r+2)/2} \begin{bmatrix} k+1 \\ s-r+1 \end{bmatrix} \quad (r, s = 1, 2, \cdots, n) \ .$$

Recalling (2.1) and (2.2), we see that $D_{n,1}$ is simply (1.2), and that $D_{n,2}$ is (1.4).

For $n > k$, expansion of $\det(a_{rs})$ along the last column and simplification gives

$$D_{n,k} = -\sum_{j=1}^{k+1} (-1)^{j(j+1)/2} \begin{bmatrix} k+1 \\ j \end{bmatrix} D_{n-j,k} \ . \tag{2.6}$$

If we define

$$D_{0,k} = 1; \quad D_{-n,k} = 0 \quad \text{for} \quad n = 1, 2, \cdots, k-1, \tag{2.7}$$

then (2.6) remains valid for $n \geq 1$. Now for fixed k, $\begin{bmatrix} n \\ k \end{bmatrix}$ is the product of k sequences each obeying (1.1), so that using Jarden's result (2.3) we see

$$\sum_{j=0}^{k+1} (-1)^{j(j+1)/2} \begin{bmatrix} k+1 \\ j \end{bmatrix} \begin{bmatrix} n-j \\ k \end{bmatrix} = 0 . \tag{2.8}$$

By (2.1) and (2.7),

$$D_{n,k} = \begin{bmatrix} n+1 \\ k \end{bmatrix} \quad (n = -k+1, -k+2, \cdots, 0) ,$$

and by (2.6) and (2.8) both $D_{n,k}$ and $\begin{bmatrix} n+k \\ k \end{bmatrix}$ obey the same $(k+1)^{st}$-order recurrence relation. Hence,

$$D_{n,k} = \begin{bmatrix} n+k \\ k \end{bmatrix} . \tag{2.9}$$

Note that this reduces to (1.2) and (1.4) for $k = 1, 2$, respectively.

3. EXTENSION TO SECOND-ORDER RECURRING SEQUENCES

Let the sequence $\{U_n\}$ be defined by $U_0 = 0$, $U_1 = 1$,

$$U_{n+2} - pU_{n+1} + qU_n = 0 \quad (q \neq 0) . \tag{3.1}$$

Let a and b be the roots of the auxiliary polynomial $x^2 - px + q$ of (3.1). We deal only with the case in which (3.1) is ordinary in the sense of R. F. Torretto and J. A. Fuchs [4], i.e., we assume that either $a = b$ or $a^n \neq b^n$ for $n > 0$. It follows that

$$U_n = \begin{cases} \dfrac{a^n - b^n}{a - b} & \text{if} \quad a \neq b\,, \\ na^{n-1} & \text{if} \quad a = b\,. \end{cases}$$

We define the U-generalized binomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}_u$ by

$$\begin{bmatrix} m \\ r \end{bmatrix}_u = \frac{U_m U_{m-1} \cdots U_{m-r+1}}{U_1 U_2 \cdots U_r} \quad (r > 0); \qquad \begin{bmatrix} m \\ 0 \end{bmatrix}_u = 1\,.$$

Note that

$$\begin{bmatrix} m \\ r \end{bmatrix}_u = 0 \qquad (m = 0, 1, \cdots, r - 1)\,. \tag{3.2}$$

As with the usual binomial coefficients, we define

$$\begin{bmatrix} m \\ r \end{bmatrix}_u = 0 \qquad (r < 0)\,. \tag{3.3}$$

In a generalization of (2.3), Jarden has shown that the term-by-term product Q_n of any $k - 1$ sequences, each obeying (3.1), satisfies

$$\sum_{j=0}^{k} (-1)^j q^{j(j-1)/2} \begin{bmatrix} k \\ j \end{bmatrix}_u Q_{n-j} = 0\,. \tag{3.4}$$

Equation (3.4) indeed reduces to (2.3) when $p = -q = 1$. We shall use the negatives of the coefficients of (3.4) to form a recurrent as before.

Let

$$D_{n,k}(U) = \det\,(b_{rs})\,,$$

where

$$b_{rs} = -(-1)^{s-r+1} q^{(s-r)(s-r+1)} \begin{bmatrix} k+1 \\ s-r+1 \end{bmatrix}_u \qquad (r,s = 1,2,\cdots,n) .$$

We find it convenient to set

$$D_{0,k}(U) = 1; \quad D_{-n,k}(U) = 0 \qquad (n = 1, 2, \cdots, k-1) . \tag{3.5}$$

Then expansion of det (b_{rs}) along the last column gives

$$D_{n,k}(U) = -\sum_{j=1}^{k+1} (-1)^j q^{j(j-1)/2} \begin{bmatrix} k+1 \\ j \end{bmatrix}_u D_{n-j,k} \tag{3.6}$$

for all $n \geq 1$.

Noticing that $\begin{bmatrix} n \\ k \end{bmatrix}_u$ is the product of k sequences eacy obeying (3.1), we see from (3.4) that

$$\sum_{j=0}^{k+1} (-1)^j q^{j(j-1)/2} \begin{bmatrix} k+1 \\ j \end{bmatrix}_u \begin{bmatrix} n-j \\ k \end{bmatrix}_u = 0 . \tag{3.7}$$

Then

$$D_{n,k}(U) = \begin{bmatrix} n+k \\ k \end{bmatrix}_u \qquad (n = -k+1, -k+2, \cdots, 0) ,$$

and by (3.6) and (3.7), $D_{n,k}(U)$ and $\begin{bmatrix} n+k \\ k \end{bmatrix}_u$ obey the same $(k+1)^{st}$-order recurrence relation. Hence,

$$D_{n,k}(U) = \begin{bmatrix} n+k \\ k \end{bmatrix}_u . \tag{3.8}$$

We conclude by investigating some particular cases of (3.8). First note that it reduces to (2.9) for $p = -q = 1$. If

$$p = L_s = F_{s-1} + F_{s+1}, \quad q = (-1)^s ,$$

then $U_n = F_{sn}$, so that for $k = 2$, (3.8) yields

$$\begin{vmatrix} L_s & (-1)^{s+1} & 0 & 0 & \cdots \\ -1 & L_s & (-1)^{s+1} & 0 & \cdots \\ 0 & -1 & L_s & (-1)^{s+1} & \cdots \\ 0 & 0 & -1 & L_s & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n = F_{s(n+1)} .$$

Putting $s = 1$ proves (1.2).

If we let $p = 2$, $q = 1$, then $a = b = 1$ and $U_n = n$. In this case,

$$\begin{bmatrix} m \\ r \end{bmatrix}_u = \binom{m}{r},$$

the usual binomial coefficient. Equation (3.8) then yields

$$\det\left[-(-1)^{s-r+1}\binom{k+1}{s-r+1}\right] = \binom{n+k}{k} \qquad (r,s = 1, 2, \cdots, n) .$$

In particular, for $k = 2$, we find

$$\begin{vmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & -1 & 2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n = n + 1 ,$$

which first seems to have been noted by Welstenholme (see [3], Vol. 3, p. 394). Letting $k = 3$, we obtain

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THE HIDDEN HEXAGON SQUARES

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INTRODUCTION

Pascal's arithmetic triangle has been much studied. Further study continues to produce evidence of the great fertility of this array of numbers. Here we divulge a very surprising result.

<u>Theorem</u>. Let $\binom{m}{n}$ be such that $0 < n < m$, $m \geq 2$, then the product of the six binomial coefficients surrounding $\binom{m}{n}$ is a perfect integer square.

<u>Proof</u>. The six binomial coefficients are:

$$\binom{m-1}{n-1}, \binom{m-1}{n}, \binom{m}{n+1}, \binom{m+1}{n+1}, \binom{m+1}{n}, \text{ and } \binom{m}{n-1}.$$

The product is

$$\frac{(m-1)!}{(n-1)!(m-n)!} \times \frac{(m-1)!}{n!(m-1-n)!} \times \frac{m!}{(n+1)!(m-n+1)!} \times \frac{(m+1)!}{(n+1)!(m-n)!} \times$$

$$\times \frac{m!}{(n-1)!(m-n+1)} = \left[\frac{(m+1)!\,m!\,(m-1)!}{(n-1)!n!(n+1)!(m-n-1)!(m-n)!(m-n+1)!}\right]^2$$

Since each binomial coefficient is an integer, the product is an integer, and since the square of a rational number is an integer if and only if the rational number is an integer, it follows that the product is an integer square.

<u>Corollary</u>. Each alternate triad of the six binomial coefficients have equal products.
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SOME SPECIAL FIBONACCI AND LUCAS GENERATING FUNCTIONS

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In [1], Hoggatt and Bicknell derived by matrix methods that

$$\sum_{i=0}^{2n+2} \binom{2n+2}{i} F_i^2 = 5^n L_{2n+2}$$

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_i^2 = 5^n F_{2n+1}$$

We next list three more similar sums.

(a) $$\sum_{k=0}^{n} \binom{n}{k} F_k = 1^n F_{2n}$$

(b) $$\sum_{k=0}^{n} \binom{n}{k} F_{3k} = 2^n F_{2n}$$

(c) $$\sum_{k=0}^{n} \binom{n}{k} F_{4k} = 3^n F_{2n} \quad .$$

Identity (a) is well known, while (b) was in a private communication from D. Lind, and (c) is a special case of Problem B-88 in the Fibonacci Quarterly, April, 1966, p. 149.

In [2], various special related results are also derived by matrix methods. Here, we derive a new class of generating functions by following the suggestion given in [3]. The column generators for Pascal's left-adjusted triangle are

$$g_n(x) = \frac{x^n}{(1-x)^{n+1}}, \qquad n = 0, 1, 2, \cdots ,$$

while the generating function for the Fibonacci numbers is

$$G(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n .$$

If we now sum

$$\sum_{n=0}^{\infty} F_n g_n(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} F_k x^n$$

$$= \frac{1}{1-x} \sum_{n=0}^{\infty} F_n \left(\frac{x}{1-x}\right)^n$$

$$= \frac{1}{1-x} \frac{\frac{x}{1-x}}{1 - \left(\frac{x}{1-x}\right) - \left(\frac{x}{1-x}\right)^2}$$

$$= \frac{x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n} x^n .$$

Thus

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n} .$$

Now, if we sum

$$\sum_{n=0}^{\infty} F_{2n} g_n(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} F_{2k} \right) x^n = \frac{x}{1 - 5x + 5x^2} .$$

This is a special case of the general class of identities

$$\frac{L_m x}{1 - 5F_m x + (-1)^{m+1} 5x^2} = L_m x + 5F_{2m}x^2 + 5L_{3m}x^3 + 5^2 F_{4m}x^4$$
$$+ 5^2 L_{5m}x^5 + \cdots + 5^k F_{2km}x^{2k}$$
$$+ 5^k L_{(2k+1)m}x^{2k+1} + \cdots .$$

We discuss first a related special case. To see this requires a few identities and a neat trick in algebra. It is easy to establish that

$$\frac{3 - 2x}{1 - 3x + x^2} = \sum_{k=0}^{\infty} L_{2k+2}\, x^k$$

$$\frac{x - x^2}{1 - 3x + x^2} = \sum_{k=0}^{\infty} F_{2k+1}\, x^k .$$

Now,

$$\frac{3x^2 - 10x^4}{1 - 15x^2 + 25x^4} = \sum_{k=0}^{\infty} L_{2k+2}\, x^{2k+2}\, 5^k$$

$$\frac{x(1 - 5x^2)}{1 - 15x^2 + 25x^4} = \sum_{k=0}^{\infty} F_{2k+1}\, x^{2k+1}\, 5^k .$$

Notice that

$$\frac{x + 3x^2 - 5x^3 - 10x^4}{1 - 15x^2 + 25x^4} = \frac{(x - 2x^2)(1 + 5x + 5x^2)}{(1 - 5x + 5x^2)(1 + 5x + 5x^2)} = \frac{x - 2x^2}{1 - 5x + 5x^2} .$$

Next, we need

$$\frac{x(1-x)}{1-2x-2x^2+x^3} = \sum_{n=0}^{\infty} F_n^2 x^n .$$

Summing as before,

$$\sum_{n=0}^{\infty} F_n^2 g_n(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_k^2\right)x^n$$

$$= \sum_{n=0}^{\infty} F_n^2 \frac{x^n}{(1-x)^{n+1}} = \frac{1}{1-x}\sum_{n=0}^{\infty} F_n^2\left(\frac{x}{1-x}\right)^n$$

$$= \frac{1}{1-x}\;\frac{\frac{x}{1-x}\left(1-\frac{x}{1-x}\right)}{1-2\left(\frac{x}{1-x}\right)-2\left(\frac{x}{1-x}\right)^2+\left(\frac{x}{1-x}\right)^3}$$

$$= \frac{x-2x^2}{1-5x+5x^2} .$$

Thus

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}F_k^2\right]x^n = \frac{x-2x^2}{1-5x+5x^2} = \sum_{k=0}^{\infty} 5^k\left(F_{2k+1}+xL_{2k+2}\right)x^{2k+1}$$

and

$$\sum_{k=0}^{2n+2}\binom{2n+2}{k}F_k^2 = 5^n L_{2n+2}$$

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}F_k^2 = 5^n F_{2n+1} ,$$

which are given in the first paragraph of the paper. Clearly, then, the

$$\sum_{k=0}^{n} \binom{n}{k} F_k^2 \qquad \text{and} \qquad \sum_{k=0}^{n} \binom{n}{k} F_{2k}$$

are related. We return now to the special case

$$\frac{x}{1 - 5x + 5x^2} = L_1 x + F_2 5x^2 + L_3 5x^3 + \cdots \quad .$$

To see this, we write

$$\frac{x + x^2}{1 - 3x + x^2} = \sum_{k=0}^{\infty} L_{2k+1} x^{k+1}$$

$$\frac{x}{1 - 3x + x^2} = \sum_{k=0}^{\infty} F_{2k} x^k \quad .$$

Next,

$$\frac{x(1 + 5x^2)}{1 - 15x^2 + 25x^4} = \sum_{k=0}^{\infty} L_{2k+1} 5^k x^{2k+1}$$

and

$$\frac{5x^2}{1 - 15x^2 + 25x^4} = \sum_{k=0}^{\infty} F_{2k} 5^k x^{2k} \quad .$$

Thus,

$$\frac{x(1 + 5x + 5x^2)}{1 - 15x^2 + 25x^4} = \sum_{k=0}^{\infty} 5^k (F_{2k} + xL_{2k+1})x^{2k}.$$

But,

$$\begin{aligned} 1 - 15x^2 + 25x^4 &= 1 + 10x^2 + 25x^4 - 25x^2 = (1 + 5x^2)^2 - (5x)^2 \\ &= (1 + 5x + 5x^2)(1 - 5x + 5x^2). \end{aligned}$$

Thus,

$$\frac{x}{1 - 5x + 5x^2} = \sum_{k=0}^{\infty} 5^k (F_{2k} + xL_{2k+1})x^{2k}$$

and

$$\sum_{k=0}^{2n} \binom{2n}{k} F_{2k} = 5^n F_{2n}$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{2k} = 5^n L_{2n+1} .$$

We now return to our general class of identities

$$\frac{L_m x}{1 - 5F_m x + (-1)^{m+1} 5x^2} = \sum_{k=0}^{\infty} 5^k (F_{2km} + xL_{(2k+1)m})x^{2k} .$$

We begin by writing

$$\frac{xL_m(1 + (-1)^{m+1}x^2)}{1 - L_{2m}x + x^2} = \sum_{k=0}^{\infty} L_{(2k+1)m}\, x^{k+1}$$

while

$$\frac{xL_m(1 + (-1)^{m+1}5x^2)}{1 - 5L_{2m}x^2 + 25x^4} = \sum_{k=0}^{\infty} L_{(2k+1)m} 5^k x^{2k+1} .$$

Next,

$$\frac{F_{2m}5x^2}{1 - 5L_{2m}x + 25x^2} = \sum_{k=0}^{\infty} F_{2km} 5^k x^{2k} ,$$

$$\frac{xL_m(1 + (-1)^{m+1}5x^2) + xL_m(F_m 5x)}{1 - 5L_{2m}x^2 + 25x^4} = \sum_{k=0}^{\infty} 5^k (F_{2km} + xL_{(2k+1)m})x^{2k} ;$$

$$\frac{xL_m(1 + 5xF_m + (-1)^{m+1}5x^2)}{1 - 5L_{2m}x^2 + 25x^4} = \frac{xL_m}{1 - 5F_m x + (-1)^{m+1}5x^2}$$

since

$$\begin{aligned} 5L_{2m} = 5L_m^2 + 10(-1)^{m+1} &= 5(5F_m^2 + 4(-1)^m + 2(-1)^{m+1}) \\ &= 25F_m^2 - 10(-1)^{m+1} . \end{aligned}$$

Thus

$$1 - 5L_{2m}x^2 + 25x^4 = 1 + 10(-1)^{m+1}x^2 + 25x^4 - 25F_m^2x^2$$

or

$$\begin{aligned} (1 - 5L_{2m}x^2 + 25x^4) &= (1 + 5(-1)^{m+1}x^2)^2 - 25F_m^2x^2 \\ &= (1 - 5F_m x + (-1)^{m+1}5x^2)(1 + 5F_m x + 5(-1)^{m+1}x^2). \end{aligned}$$

We now return to the general problem.

Remember,

$$\frac{L_m x}{1 - 5F_m x + (-1)^{m+1} 5x^2} = \sum_{k=0}^{\infty} 5^k (F_{2km} + x L_{(2k+1)m}) x^{2k} .$$

We start with the general problem. The generating function for every m^{th} Fibonacci number is

$$\frac{F_m x}{1 - L_m x + (-1)^m x^2} = \sum_{k=0}^{\infty} F_{km} x^k .$$

Consider the sum

$$\sum_{n=0}^{\infty} F_{mn} g_n(x) = \frac{1}{1-x} \sum_{n=0}^{\infty} F_{mn} \left(\frac{x}{1-x}\right)^n$$

$$= \frac{1}{1-x} \frac{F_m \frac{x}{1-x}}{1 - L_m \left(\frac{x}{1-x}\right) + (-1)^m \left(\frac{x}{1-x}\right)^2}$$

$$= \frac{F_m x}{(1-x)^2 - L_m x(1-x) + (-1)^m x^2}$$

$$= \frac{F_m x}{1 - (L_m + 2)x + (L_m + 1 + (-1)^m) x^2}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} F_{km} \right] x^n .$$

Now, from

$$L_m^2 = L_{2m} + 2(-1)^m$$

and

$$L_n^2 - 5F_n^2 = 4(-1)^n$$

one can obtain four useful identities:

$$L_{4m} + 2 = L_{2m}^2$$

$$L_{4m} - 2 = L_{2m}^2 - 4 = 5F_{2m}^2$$

$$L_{4m+2} + 2 = L_{2m+1}^2 + 4 = 5F_{2m+1}^2$$

$$L_{4m+2} - 2 = L_{2m+1}^2 .$$

Thus, for $m = 2a$ (even)

$$\frac{F_{2s}x}{1 - (L_{2s} + 2)x + (L_{2s} + 2)x^2} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}F_{2sk}\right]x^n .$$

We now discuss two special cases.

(A)
$$\frac{F_{4m}x}{1 - L_{2m}^2 x + L_{2m}^2 x^2} = \frac{F_{2m}(L_{2m}x)}{1 - L_{2m}(L_{2m}x) + (L_{2m}x)^2}$$

$$= \sum_{n=0}^{\infty} F_{2mn}(L_{2m}x)^n$$

$$= \sum_{n=0}^{\infty} F_{2mn}L_{2m}^n x^n .$$

Thus,

$$\sum_{k=0}^{n} \binom{n}{k} F_{4mk} = L_{2m}^{n} F_{2mn} .$$

This is Problem H-88, Fibonacci Quarterly, April 1966, p. 149.

(B)
$$\frac{F_{4m+2}\,x}{1 - (L_{4m+2} + 2)x + (L_{4m+2} + 2)x^2}$$

$$= \frac{L_{2m+1}(F_{2m+1}\,x)}{1 - 5F_{2m+1}(F_{2m+1}\,x) + 5(F_{2m+1}\,x)^2}$$

$$= \frac{L_{2m+1}\,y}{1 - 5F_{2m+1}y + 5y^2}$$

$$= \sum_{k=0}^{\infty} 5^k (F_{(4m+2)k} + y\,L_{(2k+1)(2m+1)})y^{2k}$$

$$= \sum_{k=0}^{\infty} 5^k (F_{(4m+2)k} + F_{2m+1}L_{(2k+1)(2m+1)})F_{2m+1}^{2k}\,x^{2k} .$$

Thus,

$$\sum_{k=0}^{2n} \binom{2n}{k} F_{(4m+2)k} = 5^n F_{(4m+2)n} F_{2m+1}^{2n}$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{(4m+2)k} = 5^n L_{(2n+1)(2m+1)} F_{2m+1}^{2n+1} .$$

Suppose, on the other hand, that we wish to alternate the signs in the above sums. Consider the sums

$$\sum_{m=0}^{\infty} F_{km}(-1)^m g_m(x) = \frac{1}{1-x} \frac{F_m \frac{-x}{1-x}}{1 - L_m \frac{-x}{1-x} + (-1)^m \left(\frac{-x}{1-x}\right)^2}$$

$$= -\frac{F_m x}{(1-x)^2 + L_m x(1-x) + (-1)^m x^2}$$

$$= \frac{-F_m x}{1 + (L_m - 2)x - (L_m - 1 - (-1)^m)x^2}$$

again for even m. Thus,

$$\frac{-F_{4m} x}{1 + 5F_{2m}^2 x + 5F_{2m}^2 x^2} = \frac{L_{2m}(-F_{2m} x)}{1 - 5F_{2m}(-F_{2m} x) + 5(-F_{2m} x)^2}$$

$$= \frac{L_{2m} y}{1 - 5F_{2m} y + 5y^2}$$

$$= \sum_{k=0}^{\infty} 5^k (F_{2mk} + y L_{(2k+1)2m}) y^{2k}$$

$$= \sum_{k=0}^{\infty} 5^k (F_{2mk} - F_{2m} L_{(2k+1)2m}) F_{2m}^{2k} x^{2k} .$$

Thus,

$$\sum_{k=0}^{2n} (-1)^{2n+k} \binom{2n}{k} F_{4mk} = F_{2m}^{2n} F_{2mn}$$

$$\sum_{k=0}^{2n+1} (-1)^{2n+1+k} \binom{2n+1}{k} F_{4mk} = F_{2m}^{2n+1} L_{(2k+1)(2n+1)} .$$

Proceeding similarly with

$$\frac{F_{2m+1}(-L_{2m+1}x)}{1 - L_{2m+1}(-L_{2m+1}x) - (-L_{2m+1}x)^2} = \frac{F_{2m+1}y}{1 - L_{2m+1}y - y^2}$$

$$= \sum_{k=0}^{\infty} F_{(2m+1)k}\, y^k$$

$$= \sum_{k=0}^{\infty} (-1)^k L_{2m+1}^k F_{(2m+1)k}\, x^k .$$

Thus

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} F_{(4m+2)k} = L_{2m+1}^n F_{(2m+1)n} .$$

There remains unsolved

$$\sum_{k=0}^{n} \binom{n}{k} F_k^m \qquad \text{and} \qquad \sum_{k=0}^{n} \binom{n}{k} F_{mk}$$

for m odd and greater than 3. Corresponding formulas are given also in [4] as follows:

$$\sum_{k=0}^{2n} \binom{2n}{k} L_{(4m+2)k} = 5^n L_{(2m+1)2n} F_{2m+1}^{2n}$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} L_{(4m+2)k} = 5^n F_{(2m+1)(2n+1)} F_{2m+1}^{2n+1}$$

$$\sum_{k=0}^{2n} \binom{n}{k} (-1)^{n+k} L_{(4m+2)k} = L_{(2m+1)n} L_{2m+1}^n$$

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k L_{4mk} = 5^n L_{4mn} F_{2m}^{2n}$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^{k+1} L_{4mk} = 5^n F_{2m(2n+1)} F_{2m}^{2n+1} .$$

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[Continued from page 120.]

SOME FURTHER RESULTS

There are several other configurations which yield products of binomial coefficients which are squares. For instance, if two hexagons H_1 and H_2 have a common entry, then the ten terms obtained by omitting the common entry have a product which is an integral square. Thus, one can build up a long serpentine configuration, or in fact build up snowflake curves.

Secondly, it should be noted in passing that all results above hold for generalized binomial coefficient arrays, in particular for the FIBONOMIAL COEFFICIENTS.

ADVANCED PROBLEMS AND SOLUTIONS

Edited By
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-181 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Prove the identity

$$\sum_{m,n=0}^{\infty} (am + cn)^m (bm + dn)^n \frac{u^m v^n}{m!n!} = \frac{1}{(1 - ax)(1 - dy) - bcxy}$$

where

$$u = xe^{-(ax+by)}, \qquad v = ye^{-(cx+dy)} .$$

H-182 Proposed by S. Krishnar, Berthampur, India.

Prove or disprove

(i) $$\sum_{k=1}^{m} \frac{1}{k^2} \equiv 0 \pmod{2m + 1} ,$$

and

(ii) $$\sum_{k=1}^{m} \frac{1}{(2k-1)^2} \equiv 0 \pmod{2m+1},$$

when $2m + 1$ is prime and larger than 3.
[See Special Problem on page 216.]

SOLUTIONS

GONE BUT NOT FORGOTTEN

H-102 Proposed by J. Arkin, Suffern, New York. (For convenience, the problem is restated, using $B_n = A_m$.)

Find a closed expression for B_n in the following recurrence relation.

(H) $$\left[\frac{n}{2}\right] + 1 = B_n - B_{n-3} - B_{n-4} - B_{n-5} + B_{n-7} + B_{n-8} + B_{n-9} - B_{n-12},$$

where $n = 0, 1, 2, \cdots$ and the first thirteen values of B_0 through B_{12} are 1, 1, 2, 3, 5, 7, 10, 13, 18, 23, 30, 37, and 47, and $[x]$ is the greatest integer contained in x.

Solution by the Proposer.

In a recent paper* this author introduced a new notation, and because of the new method in the paper, we are, for the first time, able to find explicit formulas in such recurrence relations as H-102.

We denote by $p_m(n)$ the number of partitions of n into parts not exceeding m, where

(1) $$F_m(x) = 1/(1-x)(1-x^2)\cdots(1-x^m) = \sum_{n=0}^{\infty} p_m(n)x^n,$$

and $p_m(0) = 1$.

The new notation we mentioned above is defined as follows:

---

*Joseph Arkin, "Researches on Partitions," Duke Mathematical Journal, Vol. 38, No. 3 (1970), pp. 304-409.

$$A(m,n) = 1 \quad \text{if} \quad m \text{ divides } n \tag{2}$$

$$A(m,n) = 0 \quad \text{if} \quad m \text{ does not divide } n\ ,$$

where

$$m = 1, 2, 3, \cdots, \qquad n = 0, 1, 2, \cdots,$$

and

$$A(m,0) = 1\ .$$

Now, in (1), it is plain that

$$F_2(x)/(1 - x^3)(1 - x^4)(1 - x^5) = \sum_{n=0}^{\infty} p_5(n)x^n\ ,$$

and we have

$$F_2(x) = (1 - x^3)(1 - x^4)(1 - x^5) \sum_{n=0}^{\infty} p_5(n)x^n\ . \tag{3}$$

Then, combining the coefficients in (3) leads to

$$\begin{aligned} p_2(n) = {} & p_5(n) - p_5(n-3) - p_5(n-4) - p_5(n-5) + p_5(n-7) \\ & + p_5(n-8) + p_5(n-9) - p_5(n-12)\ , \end{aligned} \tag{4}$$

and it is evident that the right side of (4) is identical to the right side of (H).

Now* it was shown that

*Joseph Arkin, "Researches on Partitions," Duke Mathematical Journal, Vol. 38, No. 3 (1970), Eq. (6), p. 404.

$$p_2(2u) = u + 1$$

and

$$p_2(2u + 1) = u + 1 \qquad (u = 0, 1, 2, \cdots),$$

so that

$$p_2(n) = [n/2], \tag{5}$$

where $n = 0, 1, 2, \cdots$, and $[x]$ is the greatest integer contained in x.

Then, combining (5) with the left side of (4) and since

$$B_n = p_5(n) \qquad (n = 0, 1, 2, \cdots),$$

it remains to find an explicit formula for the $p_5(n)$.

To this end*, we see that

$$p_5(n) = \frac{1}{17280}\begin{bmatrix} 6n^4 + 180n^3 + 1860n^2 + 7650n + 7719 \\ (270n + 2025)(-1)^n \\ 1920A(3,n) \\ 2160(A(4,n) + A(4,n+3)) \\ 3456A(5,n) \end{bmatrix}$$

A LARGE ORDER

H-161 Proposed by David Klarner, University of Alberta, Edmonton, Alberta, Canada.

Let

$$b(n) = \sum_{a_1+a_2+\cdots+a_i=n} \binom{a_1 + a_2}{a_2}\binom{a_2 + a_3}{a_3}\cdots\binom{a_{i-1} + a_i}{a_i},$$

*Joseph Arkin, "Researches on Partitions," <u>Duke Mathematical Journal</u>, Vol. 38, No. 3 (1970), Eq. (19), p. 406.

where the sum is extended over all compositions of n and the contribution to the sum is 1 when there is only one part in the composition. Find an asymptotic estimate for $b(n)$.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$b_k(n) = \sum_{a_1+\cdots+a_k=n} \binom{a_1+a_2}{a_2}\binom{a_2+a_3}{a_3}\cdots\binom{a_{k-1}+a_k}{a_k} ,$$

$$\frac{1}{f_k(x)} = \sum_{n=0}^{\infty} b_k(n)\, x^n .$$

It is known (see "A Binomial Identity Arising from a Sorting Problem," SIAM Review, Vol. 6 (1964), pp. 20-30), that $f_k(x)$ is equal to the following determinant of order $k+1$:

$$\begin{vmatrix} 1 & x & & & & \\ 1 & 1 & x & & & \\ & 1 & 1 & x & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & 1 & x \\ & & & & 1 & 1 \end{vmatrix}$$

It follows that

$$f_{n+1}(x) = f_n(x) - xf_{n-1}(x) .$$

Since $f_0(x) = 1$, $f_1(x) = 1 - x$, we find that

$$F(z) = \sum_{k=0}^{\infty} f_k(x)\, z^k = \frac{1 - xz}{1 - z + xz^2} ,$$

In the next place,

$$\frac{1 - xz}{1 - z + xz^2} = \frac{1}{\alpha - \beta}\left(\frac{\alpha^2}{1 - \alpha z} - \frac{\beta^2}{1 - \beta z}\right) ,$$

where

$$\alpha + \beta = 1, \qquad \alpha\beta = x .$$

It follows that

$$f_k(x) = \frac{\alpha^{k+2} - \beta^{k+2}}{\alpha - \beta} ,$$

so that

$$\sum_{n=0}^{\infty} b_k(n)\, x^n = \frac{\alpha - \beta}{\alpha^{k+2} - \beta^{k+2}} . \tag{1}$$

Now, if $k = 2r + 1$,

$$\begin{aligned}\frac{\alpha^k - \beta^k}{\alpha - \beta} &= \prod_{s=1}^{k-1} (\alpha - \beta e^{2\pi is/k}) \\ &= \prod_{s=1}^{r} (\alpha - \beta e^{2\pi is/k})(\alpha - \beta e^{-2\pi is/k}) \\ &= \prod_{s=1}^{r}\left(\alpha^2 - 2\alpha\beta \cos \frac{2\pi s}{k} + \beta^2\right) \\ &= \prod_{s=1}^{r}\left(1 - 4x \cos^2 \frac{\pi s}{k}\right) .\end{aligned}$$

If we put

(3)
$$\prod_{s=1}^{r}\left(1 - 4x\cos^2\frac{\pi s}{k}\right)^{-1} = \sum_{s=1}^{r}\frac{A_s}{1 - 4x\cos^2\frac{\pi s}{k}},$$

we find that

$$A_s = \frac{\cos^{2(r-1)}\frac{\pi s}{k}}{\prod\limits_{\substack{t=1\\t\neq s}}^{r}\left(\cos^2\frac{\pi s}{k} - \cos^2\frac{\pi t}{k}\right)} = \frac{2^{r-1}\cos^{2(r-1)}\frac{\pi s}{k}}{\prod\limits_{\substack{t=1\\t\neq s}}^{r}\left(\cos\frac{2\pi s}{k} - \cos\frac{2\pi t}{k}\right)}$$

$$= \frac{\cos^{2(r-1)}\frac{\pi s}{k}}{\prod\limits_{\substack{t=1\\t\neq s}}^{r}\sin\frac{\pi(t+s)}{k}\sin\frac{\pi(t-s)}{k}}.$$

But

$$\prod_{\substack{t=1\\t\neq s}}^{r}\sin\frac{\pi(t+s)}{k}\sin\frac{\pi(t-s)}{k} = (-1)^{s-1}\frac{\prod\limits_{t=1}^{2r}\sin\frac{\pi t}{k}}{\sin\frac{\pi s}{k}\sin\frac{2\pi s}{k}}$$

$$= \frac{(-1)^{s-1}k}{2^k\sin^2\frac{\pi s}{k}\cos\frac{\pi s}{k}},$$

so that

(4)
$$A_s = (-1)^{s-1}\frac{2^k\sin^2\frac{\pi s}{k}\cos^{2r-1}\frac{\pi s}{k}}{k}.$$

Then, by (2), and (3) and (4),

$$\frac{\alpha - \beta}{\alpha^{k+2} - \beta^{k+2}} = \frac{2^{k+2}}{k+2} \cdot \sum_{s=1}^{\frac{1}{2}(k+1)} (-1)^{s-1} \frac{\sin^2 \frac{\pi s}{k+2} \cos^k \frac{\pi s}{k+2}}{1 - 4x\cos^2 \frac{\pi s}{k+2}}$$

$$= \frac{2^{k+2}}{k+2} \sum_{s=1}^{\frac{1}{2}(k+1)} (-1)^{s-1} \sin^2 \frac{\pi s}{k+2} \cos^k \frac{\pi s}{k+2} \sum_{n=0}^{\infty} (4x)^n \cos^{2n} \frac{\pi s}{k+2}$$

$$= \frac{2^{k+2}}{k+2} \sum_{n=0}^{\infty} (4x)^n \sum_{s=1}^{\frac{1}{2}(k+1)} (-1)^{s-1} \sin^2 \frac{\pi s}{k+2} \cos^{k+2n} \frac{\pi s}{k+2} .$$

Therefore, by (1),

$$b_k(n) = \frac{2^{k+2n+2}}{k+2} \sum_{s=1}^{\frac{1}{2}(k+1)} (-1)^{s-1} \sin^2 \frac{\pi s}{k+2} \cos^{k+2n} \frac{\pi s}{k+2} \qquad (k \text{ odd}) . \tag{5}$$

This implies the asymptotic formula

$$b_k(n) \sim \frac{2^{k+2n+2}}{k+2} \sin^2 \frac{\pi}{k+2} \cos^{k+2n} \frac{\pi}{k+2} \qquad (k \text{ odd}) . \tag{6}$$

Next, if $k = 2r$,

$$\frac{\alpha^k - \beta^k}{\alpha - \beta} = \frac{\alpha^k - \beta^k}{\alpha^2 - \beta^2} = \prod_{s=1}^{r-1} (\alpha - \beta e^{2\pi i s/k})(\alpha - \beta e^{-2\pi i s/k})$$

$$= \prod_{s=1}^{r-1} \left(1 - 4x\cos^2 \frac{\pi s}{k}\right) .$$

If we put

$$\prod_{s=1}^{r-1} \left(1 - 4x\cos^2 \frac{\pi s}{k}\right) = \sum_{s=1}^{r-1} \frac{A_s}{1 - 4x\cos^2 \frac{\pi s}{k}} ,$$

we get

$$A_s = \frac{\cos^{2(r-2)} \frac{\pi s}{k}}{\prod_{\substack{t=1 \\ t \neq s}}^{r-1} \left(\cos \frac{2\pi s}{k} - \cos \frac{2\pi t}{k} \right)}$$

$$= \frac{2^{r-2} \cos^{2(r-2)} \frac{\pi s}{k}}{\prod_{\substack{t=1 \\ t \neq s}}^{r-1} \left(\cos \frac{2\pi s}{k} - \cos \frac{2\pi t}{k} \right)} = \frac{\cos^{2(r-2)} \frac{\pi s}{k}}{\prod_{\substack{t=1 \\ t \neq s}}^{r-1} \sin \frac{\pi(t+s)}{k} \sin \frac{\pi(t-s)}{k}} .$$

Since

$$\prod_{\substack{t=1 \\ t \neq s}}^{r-1} \sin \frac{\pi(t-s)}{k} \sin \frac{\pi(t-s)}{k} = (-1)^{s-1} \frac{\prod_{t=1}^{2r-1} \sin \frac{\pi t}{k}}{\sin \frac{\pi s}{k} \sin \frac{2\pi s}{k} \sin \frac{\pi(r+s)}{k}}$$

$$= (-1)^{s-1} \frac{k}{2^k \sin^2 \frac{\pi s}{k} \cos^2 \frac{\pi s}{k}} ,$$

it follows that

$$A_s = (-1)^{s-1} \frac{2^k \sin^2 \frac{\pi s}{k} \cos^{k-2} \frac{\pi s}{k}}{k} .$$

Then

$$\frac{\alpha - \beta}{\alpha^{k+2} - \beta^{k+2}} = \frac{2^{k+2}}{k+2} \sum_{s=1}^{\frac{1}{2}k} (-1)^{s-1} \frac{\sin^2 \frac{\pi s}{k+2} \cos^k \frac{\pi s}{k+2}}{1 - 4x \cos^2 \frac{\pi s}{k+2}}$$

$$= \frac{2^{k+2}}{k+2} \sum_{n=0}^{\infty} (4x)^n \sum_{s=1}^{\frac{1}{2}k} (-1)^{s-1} \sin^2 \frac{\pi s}{k+2} \cos^{k+2n} \frac{\pi s}{k+2} ,$$

so that

$$(7)\qquad b_k(n) = \frac{2^{k+2n+2}}{k+2}\sum_{s=1}^{\frac{1}{2}k}(-1)^{s-1}\sin^2\frac{\pi s}{k+2}\cos^{k+2n}\frac{\pi s}{k+2} \qquad (k \text{ even}) .$$

This implies the asymptotic result

$$(8)\qquad b_k(n) \sim \frac{2^{k+2n+2}}{k+2}\sin^2\frac{\pi s}{k+2}\cos^{k+2n}\frac{\pi}{k+2} \qquad (k \text{ even}) .$$

We may combine (5) and (7) in the single formula

$$(9)\qquad b_k(n) = \frac{2^{k+2n+2}}{k+2}\sum_{s=1}^{[\frac{1}{2}(k+1)]}(-1)^{s-1}\sin^2\frac{\pi s}{k+2}\cos^{k+2n}\frac{\pi s}{k+2}$$

and (6) and (8) in

$$(10)\qquad b_k(n) \sim \frac{2^{k+2n+2}}{k+2}\sin^2\frac{\pi}{k+2}\cos^{k+2n}\frac{\pi}{k+2} .$$

LUCA-NACCI

H-163 Proposed by H. H. Ferns, Victoria, B. C., Canada.

Prove the following identities:

$$(1)\qquad \sum_{k=1}^{n} 2^{2k-2}L_kF_{k+3} = 2^{2n}F_{n+1}^2 - 1$$

$$(2)\qquad 5\sum_{k=1}^{n} 2^{2k-2}F_kL_{k+3} = 2^{2n}L_{n+1}^2 - 1 ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively.

Solution by A. G. Shannon, Mathematics Department, University of Papua and New Guinea, Boroko, T.P.N.G.

1. <u>n = 1</u>; $$\sum_{k=1}^{n} 2^{2k-2} L_k F_{k+3} = L_1 F_4 = 3 ,$$

and

$$2^n F_{n+1}^2 - 1 = 2^2 F_2^2 - 1 = 3 .$$

Assume identity true for n. Then,

$$\sum_{k=1}^{n} 2^{2k-2} L_k F_{k+3} + 2^{2n} L_{n+1} F_{n+4} = \sum_{k=1}^{n+1} 2^{2k-2} L_k F_{k+3}$$

$$\begin{aligned} 2^{2n} F_{n+1}^2 &- 1 + 2^{2n} L_{n+1} F_{n+4} \\ &= 2^{2n} (F_{n+1}^2 + (F_n + F_{n+2})(F_{n+3} + F_{n+2})) - 1 \\ &= 2^{2n} (F_{n+1}^2 + 2F_{n+2}^2 + 2F_n F_{n+2} + F_n F_{n+1} + F_{n+1} F_{n+2}) - 1 \\ &= 2^{2n} (2F_{n+2}^2 + F_{n+2}(2F_n + 2F_{n+1})) - 1 \\ &= 2^{2n+2} F_{n+2}^2 - 1 , \end{aligned}$$

which proves the result.

2. It can be readily shown that

$$L_k F_{k+3} = F_k L_{k+3} + 4(-1)^k , \tag{3}$$

by using

$$L_k = \alpha^k + \beta^k$$

and

$$F_k = (\alpha^k - \beta^k)(\alpha - \beta)^{-1} .$$

From (1) above, it follows that

$$(4) \qquad 5\sum_{k=1}^{n} 2^{k-2} L_k F_{k+3} = 2^{2n}(\alpha^{n+1} - \beta^{n+1})^2 - 5 .$$

With (3), the left-hand side of (4) becomes

$$5\sum_{k=1}^{n} 2^{2k-2} F_k L_{k+3} + 20\sum_{k=1}^{n} 2^{2k-2}(-1)^k$$

$$= 5\sum_{k=1}^{n} 2^{2k-2} F_k L_{k+3} + (2^{2n+2}(-1)^n - 4) .$$

The right-hand side of (4) reduces to

$$2^{2n}(\alpha^{2n+2} + \beta^{2n+2} + 2(-1)^n) - 5$$

$$= (2^{2n} L_{n+1} - 1) + (2^{2n+2}(-1)^n - 4) ,$$

and result (2) follows.

Also solved by M. Yoder, C. B. A. Peck, J. Milsom, M. Ratchford, D. V. Jaiswal, and the Proposer.

REGULAR POLYHEDRONS AND PASCAL'S TRIANGLE

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It is known that any convex polyhedron has three parameters. Numerical values of parameters of all regular polyhedrons are shown below.

Polyhedron	F	V	E
1. Tetrahedron	4	4	6
2. Hexahedron	6	8	12
3. Octahedron	8	6	12
4. Dodecahedron	12	20	30
5. Icosahedron	20	12	30 ,

where F represents the number of faces, V the number of vertices, and E the number of edges.

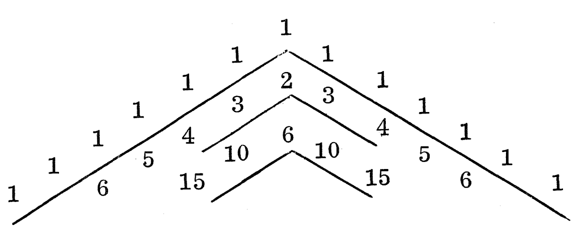
Numerical values of these parameters form a sequence:

$$4,\ 6,\ 8,\ 12,\ 20,\ 30\ .$$

It is remarkable that the half-values of all members of this sequence form two apexes of Pascal's triangle.

The first apex is situated just below the edge-series of ones and the second one below the first apex.

Both apexes look like this:



SEQUENCES WITH A CHARACTERISTIC NUMBER

IRVING ADLER
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1. A Fibonacci sequence $a_0, a_1, a_2, \cdots, a_n, \cdots$ is called a Fibonacci sequence if it satisfies the recursion relation

$$a_{n+2} = a_{n+1} + a_n \; . \tag{1}$$

A well-known property of such a sequence is that there exists a number α such that

$$a_n a_{n+2} - a_{n+1}^2 = (-1)^n \alpha \tag{2}$$

for all $n = 0, 1, 2, \cdots$. The number α is called the characteristic number of the sequence [1]. The purpose of this paper is to explore the significance of the characteristic number [2] and to identify all sequences that have a characteristic number. We shall consider only sequences of rational numbers.

2. We call a sequence geometric if there exist numbers a and r such that

$$a_n = ar^n, \qquad n = 0, 1, 2, \cdots \; . \tag{3}$$

If a sequence is geometric, then

$$a_n a_{n+2} - a_{n+1}^2 = 0, \qquad n = 0, 1, 2, \cdots \; . \tag{4}$$

Conversely, suppose Eq. (4) holds. If $a_n \neq 0$ for all n, then

$$\frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1}}{a_n}, \qquad n = 0, 1, 2, \cdots \; . \tag{5}$$

Then the sequence satisfies (3) with $a = a_0$, and

$$r = \frac{a_1}{a_0} .$$

If $a_n = 0$ for some n, then by Eq. (4), $a_{n+1} = 0$. If $n \geq 2$, then by Eq. (4),

$$a_{n-2}a_n - a_{n-1}^2 = 0,$$

and $a_{n-1} = 0$. Hence, if $a_n = 0$ for some n, then $a_n = 0$ for all $n = 1, 2, 3, \cdots$. That is, either every term of the sequence is 0, or only a_0 is not 0. In the first case, the sequence satisfies Eq. (3) with $a = 0$, and r arbitrary. In the second case, it satisfies Eq. (3) with $a = a_0$, and $r = 0$. Therefore, a sequence is geometric if and only if it satisfies Eq. (4). Equation (4) is a special case of Eq. (2) with $d = 0$. Since Eq. (2), with $d \neq 0$ represents a minor deviation from the typical behavior of a geometric sequence, we shall call any sequence satisfying Eq. (2) with $d \neq 0$ a <u>parageometric sequence.</u>

3. We shall call a sequence <u>almost geometric</u> if it is not geometric, but there exist numbers r_n such that $a_{n+1} = a_n r_n$ for $n = 0, 1, 2, \cdots$, and the sequence (r_n) approaches a limit as n becomes infinite. For example, in the Fibonacci sequence defined by

$$F_0 = 1, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n = 0, 1, 2, \cdots, \tag{6}$$

the terms of the sequence are given by the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2} \tag{7}$$

Then

$$r_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} = \frac{\alpha - \beta\left(\frac{\beta}{\alpha}\right)^n}{1 - \left(\frac{\beta}{\alpha}\right)^n}$$

$$= \frac{\alpha + \beta\left(\frac{1}{\alpha^2}\right)^n (-1)^{n+1}}{1 + \left(\frac{1}{\alpha^2}\right)^n (-1)^{n+1}} .$$

But

$$\frac{1}{\alpha^2} = \frac{4}{6 + 2\sqrt{5}} < 1 .$$

Therefore,

$$\lim_{n \to \infty} \left(\frac{1}{\alpha^2}\right)^n = 0 ,$$

and $\lim_{n \to \infty} r_n = \alpha$. So the Fibonacci sequence, defined by (6), which is parageometric with $d = 1$, is also almost geometric.

4. We shall call a sequence alternating if $a_{2k} = a$, $a_{2k+1} = b$, $a \neq b$, for all $n = 0, 1, 2, \cdots$. An alternating sequence satisfies Eq. (2) with $d = a^2 - b^2$. Then $d = 0$ if and only if $b = -a$. So, an alternating sequence is geometric if and only if $b = -a$, and it is parageometric in all other cases. However, a parageometric alternating sequence is not almost geometric. In fact, if $a = 0$ and $b \neq 0$, then r_n cannot be defined for even n. If $a \neq 0$ and $b = 0$, then r_n cannot be defined for odd n. If neither a nor b is zero, then

$$r_n = \frac{b}{a}$$

for even n, and

$$r_n = \frac{a}{b}$$

for odd n, and

$$\frac{a}{b} \neq \frac{b}{a} ,$$

so, while r_n is defined for all n, it does not approach a limit as n becomes infinite. Hence, every alternating sequence is not almost geometric.

5. We shall call a sequence eventually almost geometric if it is not almost geometric, but the sequence obtained by deleting the first k terms, for some positive integer k, is almost geometric. For example, the sequence $0, 1, 0, 1, 0, a_5, a_6, a_7, \cdots$, where $a_{m+5} = F_m$ for $m = 0, 1, 2, \cdots$, is parageometric and is not almost geometric, but it is eventually almost geometric. Similarly, the sequence $8, 5, 3, 2, 1, 1, 0, 1, 0, 1, 0, a_{11}, a_{12}, \cdots$, where $a_{m+11} = F_m$ for $m = 0, 1, 2, \cdots$, is parageometric, is not almost geometric, but it is eventually almost geometric.

We shall call a sequence eventually alternating if it is not alternating, but the sequence obtained by deleting the first k terms, for some positive integer k, is alternating. For example, the sequence $8, 5, 3, 2, 1, 1, a_7, a_8, \cdots$, where a_{6+n} is 0 for odd n, and is 1 for even n, is parageometric, is not alternating, but is eventually alternating.

6. We can now state our principal result.

Theorem. If a sequence is not geometric, and no term of the sequence is 0, it is parageometric if and only if it satisfies the recursion relation

$$a_{n+2} = k a_{n+1} + a_n \tag{8}$$

for some rational number k. If $k = 0$, the sequence is alternating, and if $k \neq 0$, the sequence is almost geometric.

A zero term may occur in the sequence only if the absolute value of its characteristic number is a perfect square. If there is a zero term in the sequence, then either the sequence is alternating, or the sequence is eventually alternating, or the sequence is eventually almost geometric. In the first case, the sequence satisfies the recursion relation (8) with $k = 0$. In the second case, for some index $i > 0$, $a_0, a_1, \cdots, a_i$ is a fragment of an almost geometric sequence satisfying the recursion relation (8) for some $k \neq 0$, and

$a_{i+1}, a_{i+2}, \cdots, a_{i+n}, \cdots$ is an alternating sequence satisfying (8) with $k = 0$. In the third case, there are two possibilities: (1) For some index $j > 0$, $a_0, a_1, \cdots, a_j$ is a fragment of an alternating sequence satisfying the recursion relation (8) with $k = 0$, and $a_{i+1}, a_{i+2}, \cdots, a_{j+n}, \cdots$ is an almost geometric sequence satisfying (8) for some $k \neq 0$. (2) For some non-negative index i, $a_0, a_1, \cdots, a_i$ is a fragment of an almost geometric sequence satisfying the recursion relation (8) for some $k \neq 0$; for some positive index $j > i$, $a_i, a_{i+1}, \cdots, a_j$ is a fragment of an alternating sequence satisfying the recursion relation (8) with $k = 0$; and $a_{j+1}, a_{j+2}, \cdots, a_{j+n}, \cdots$ is an almost geometric sequence satisfying (8) for some $k \neq 0$. Consequently, a parageometric sequence consists of at most three consecutive segments each of which satisfies the recursion relation (8) for some value of k.

Proof. (1) Let (a_n) be a sequence that is not geometric and with $a_n \neq 0$ for all $n = 0, 1, 2, \cdots$. If it is parageometric, we have

$$a_n a_{n+2} - a_{n+1}^2 = (-1)^n d \; .$$

Then

$$a_{n+1} a_{n+3} - a_{n+2}^2 = (-1)^{n+1} d \; .$$

Therefore,

$$a_n a_{n+2} - a_{n+1}^2 + a_{n+1} a_{n+3} - a_{n+2}^2 = 0 \; .$$

Hence

$$a_{n+1}(a_{n+3} - a_{n+1}) = a_{n+2}(a_{n+2} - a_n) \; .$$

Then, since $a_n \neq 0$ for all n,

$$\frac{a_{n+3} - a_{n+1}}{a_{n+2}} = \frac{a_{n+2} - a_n}{a_{n+1}} \; .$$

Thus

$$\frac{a_{n+2} - a_n}{a_{n+1}} = k$$

for some rational constant k and all values of $n = 0, 1, 2, \cdots$. Then (a_n) satisfies Eq. (8).

Conversely, suppose the sequence satisfies (8). Then

$$\frac{a_{n+3} - a_{n+1}}{a_{n+2}} = \frac{a_{n+2} - a_n}{a_{n+1}} = k \; .$$

Consequently,

$$a_{n+1}a_{n+3} - a_{n+2}^2 = -(a_n a_{n+2} - a_{n+1}^2)$$

for all $n = 0, 1, 2, \cdots$. If we let $d = a_0 a_2 - a_1^2$, then we have

$$a_n a_{n+2} - a_{n+1}^2 = (-1)^n d \; .$$

Since the sequence is not geometric, $d \neq 0$, and the sequence is parageometric. If $k = 0$, then $a_{n+2} = a_n$. Since the sequence is not geometric, $a_n \neq a_{n+1}$. Hence it is alternating. The characteristic equation associated with (8) is

$$x^2 - kx - 1 = 0 \; , \tag{9}$$

whose roots are

$$r = \frac{k + \sqrt{k^2 + 4}}{2} \quad , \quad s = \frac{k - \sqrt{k^2 + 4}}{2} \; . \tag{10}$$

Then, by the theory of linear recurrence relations [3],

(11) $$a_n = ar^n + bs^n ,$$

where a and b have the values

(12) $$a = \frac{a_1 - a_0 s}{\sqrt{k^2 + 4}} , \qquad b = \frac{a_0 r - a_1}{\sqrt{k^2 + 4}} .$$

$$d = a_0 a_2 - a_1^2 = (a + b)(ar^2 + bs^2) - (ar + bs)^2 = ab(r - s)^2 = ab(k^2 + 4) .$$

Since $d \neq 0$, it follows that $a \neq 0$ and $b \neq 0$. If $k > 0$,

$$\left|\frac{s}{r}\right| < 1 .$$

If $k < 0$,

$$\left|\frac{r}{s}\right| < 1 .$$

$$r_n = \frac{a_{n+1}}{a_n} = \frac{ar^{n+1} + bs^{n+1}}{ar^n + bs^n} = \frac{r + \frac{b}{a} s \left(\frac{s}{r}\right)^n}{1 + \frac{b}{a}\left(\frac{s}{r}\right)^n} = \frac{\frac{a}{b} r \left(\frac{r}{s}\right)^n + s}{\frac{a}{b}\left(\frac{r}{s}\right)^n + 1} .$$

If $k > 0$,

$$\left|\frac{s}{r}\right| < 1 ,$$

and $\lim_{n \to \infty} r_n = r$. If $k < 0$,

$$\left|\frac{r}{s}\right| < 1 ,$$

and $\lim_{n \to \infty} r_n = s$. Consequently, if $k \neq 0$, the sequence is almost geometric.

(2) If some term $a_k = 0$, then

$$a_k a_{k+2} = a_{k+1}^2 = (-1)^n d ,$$

and hence

$$-a_{k+1}^2 \cdot = (-1)^k d .$$

If k is odd, d is a perfect square. If k is even, -d is a perfect square. Since $d \neq 0$, $a_{n+1} \neq 0$. If $k \neq 0$, we have

$$a_{k-1} a_{k+1} - a_k^2 = (-1)^{n-1} d ,$$

or

$$a_{k-1} a_{k+1} = (-1)^{k-1} d .$$

Then

$$a_{k-1} a_{k+1} = a_{k+1}^2 ,$$

and

$$a_{k+1} (a_{k-1} - a_{k+1}) = 0 .$$

Then, since $a_{k+1} \neq 0$, $a_{k-1} = a_{k+1}$. That is, every zero term in the sequence is flanked by a pair of equal non-zero terms. Consequently, if $a_n = a_m = 0$, with $k < m$, then $m - k > 1$. If $a_k = 0$, it is possible that $a_{k+2} = 0$, and $a_{n-2} = 0$ if it exists. Then a_k belongs to a sequence of alternate zero terms

$$a_{k-2\ell} = a_{k-2+2\ell} = \cdots = a_{k-2} = a_k = a_{k+2} = \cdots = a_{k+2m} = 0 ,$$

where $\ell \geq 0$, $2\ell \leq k$, and $m \geq 0$.

$$(a_{k-2\ell-1}) = a_{k-2\ell+1} = \cdots = a_{k-1} = a_{k+1} = \cdots = a_{k+2m+1} \neq 0 ,$$

where the parentheses around the term $a_{k-2\ell-1}$ indicate that it is included only if it exists. (That is, if $k - 2\ell \neq 0$.) Then $a_{k-2\ell-1}$, $a_{k-2\ell}$, $\cdots$, a_{k+2m+1}, which is a segment of the sequence (a_n), is an alternating sequence with zero terms alternating with non-zero terms. Let us extend this alternating sequence as far as we can to both lower and higher indices by including a_{k+2m+2} and a_{k+2m+3} if $a_{k+2m+2} = 0$, and by including $a_{k-2\ell-2}$ and $a_{k-2\ell-3}$ if they exist and $a_{k-2\ell-2} = 0$. Then the following four possibilities arise, depending on whether or not the alternating sequence begins with a_0 on the left and whether or not it terminates on the right:

I. The alternating sequence begins with a_0, and does not terminate.

II. The alternating sequence begins with a_i, $i > 0$, and does not terminate.

III. The alternating sequence begins with a_0, and terminates with a_j, $j > 0$.

IV. The alternating sequence begins with a_i, $i > 0$, and terminates with a_j, $j > i$.

In case I, the sequence (a_n) is an alternating sequence, with either the odd-numbered terms or the even-numbered terms equal to zero. That is, it has the form $0, a, 0, a, 0, a, \cdots$ or $a, 0, a, 0, a, 0, \cdots$, where $a \neq 0$. Such a sequence satisfies the recursion relation (8) with $k = 0$.

In case II, $a_i \neq 0$, $a_{i+1} = 0$, and $a_{i-1} \neq 0$. The infinite sequence $a_i, a_{i+1}, \cdots$ is an alternating sequence of the form $a, 0, a, 0, \cdots$. We shall show that for every $n < i$, $a_n \neq 0$.

In case III, $a_j \neq 0$, $a_{j-1} = 0$, and $a_{j+1} \neq 0$. The finite sequence a_0, $a_1, \cdots, a_j$ has the form $0, a, 0, a, \cdots, 0, a$ or $a, 0, a, 0, \cdots, 0, a$. We shall show that for every $n > j$, $a_n \neq 0$.

In case IV, $a_i \neq 0$, $a_{i+1} = 0$, $a_{i-1} \neq 0$, $a_j \neq 0$, $a_{j-1} = 0$, $a_{j+1} \neq 0$. The finite sequence $a_i, \cdots, a_j$ has the form $a, 0, a, 0, \cdots, 0, a$. We shall show that for every $n < i$ and every $n > j$, $a_n \neq 0$.

Suppose $a_j \neq 0$, $a_{j-1} = 0$, $a_{j+1} \neq 0$ (cases III and IV). We shall call these assumptions Assumptions A. We show that for every $n > j$, $a_n \neq 0$. From

$$a_{j-1}a_{j+1} - a_j^2 = (-1)^{j-1}d$$

we get $a_j^2 = (-1)^j d$.

(13) $$a_j a_{j+2} - a_{j+1}^2 = (-1)^j d = a_j^2 \; .$$

Therefore,

$$a_j(a_{j+2} - a_j) = a_{j+1}^2 \; .$$

We consider first the case where $a_j > 0$. Then, since $a_{j+1} \neq 0$, $a_{j+2} - a_j > 0$, and $a_{j+2} > a_j > 0$.

(14) $$a_{j+1}a_{j+3} - a_{j+2}^2 = (-1)^{j+1}d = -a_j^2 \; .$$

Therefore,

$$a_{j+1}a_{j+3} = a_{j+2}^2 - a_j^2 > 0 \; .$$

Then a_{j+3} is not zero, and has the same sign as a_{j+1}. From (13) and (14),

$$a_j a_{j+2} - a_{j+1}^2 + a_{j+1}a_{j+3} - a_{j+2}^2 = 0 \; .$$

Therefore,

$$a_{j+2}(a_{j+2} - a_j) = a_{j+1}(a_{j+3} - a_{j+1}) \; .$$

Hence $a_{j+3} - a_{j+1}$ has the same sign as a_{j+1} and a_{j+3}. This, if $a_{j+1} > 0$, $a_{j+3} > a_{j+1}$, and if $a_{j+1} < 0$, $a_{j+3} < a_{j+1}$. In either case, $|a_{j+3}| > |a_{j+1}| > 0$. Now we proceed by induction. Assume that $a_{j+2k} > a_{j+2k-2} > \cdots > a_j > 0$, that $a_{j+2n+1}, a_{j+2k+1}, a_{j+2k-3}, \cdots, a_{j+1}$ have the same sign, and that

$$|a_{j+2k+1}| > |a_{j+2k-1}| > \cdots > |a_{j+1}| > 0 \; .$$

(15) $$a_{j+2k}a_{j+2k+2} - a_{j+2k+1}^2 = (-1)^{j+2k}d = (-1)^j d = a_j^2 .$$

(16) $$a_{j+2k-1}a_{j+2k+1} - a_{j+2k}^2 = (-1)^{j+2k-1}d = -a_j^2 .$$

$$a_{j+2k}a_{j+2k+2} - a_{j+2k+1}^2 + a_{j+2k-1}a_{j+2j+1} - a_{j+2k}^2 = 0 .$$

$$a_{j+2k}(a_{j+2k+2} - a_{j+2k}) = a_{j+2k+1}(a_{j+2k+1} - a_{j+2k-1}) .$$

Then, since a_{j+2k+1}, a_{j+2k-1}, and $a_{j+2k+1} = a_{j+2k-1}$ have the same sign, and $a_{j+2k} > 0$, $a_{j+2k+2} - a_{j+2k} > 0$, and

$$a_{j+2k+2} > a_{j+2k} > \cdots > a_j > 0 .$$

(17) $$a_{j+2k+1}a_{j+2k+3} - a_{j+2k+2}^2 = (-1)^{j+2k+1}d = (-1)^{j+1}d = -a_j^2 .$$

Therefore,

$$a_{j+2k+1}a_{j+2k+3} = a_{j+2k+2}^2 - a_j^2 > 0 .$$

Therefore, a_{j+2k+1} and a_{j+2k+3} have the same sign. From (15) and (17), we get

$$a_{j+2k}a_{j+2k+2} - a_{j+2k+1}^2 + a_{j+2k+1}a_{j+2k+3} - a_{j+2k+2}^2 = 0.$$

Then

$$a_{j+2k+1}(a_{j+2k+3} - a_{j+2k+1}) = a_{j+2k+2}(a_{j+2k+2} - a_{j+2k}) .$$

Therefore, $a_{j+2n+3} - a_{j+2n+1}$ has the same sign as a_{j+2k+1}. Hence,

$$a_{j+2k+3},\ a_{j+2k+1},\ \cdots,\ a_{j+1}$$

have the same sign, and

$$|a_{j+2k+3}| > |a_{j+2k+1}| > \cdots > |a_{j+1}| > 0 .$$

If $a_j < 0$, a similar argument shows that

$$a_{j+2k} < a_{j+2k-2} < \cdots < a_j < 0, \quad a_{j+2k+1}, \quad a_{j+2k-1}, \quad \cdots, \quad a_{j+1}$$

have the same sign, and

$$|a_{j+2k+1}| > |a_{j+2k-1}| > \cdots > |a_{j+1}| > 0 .$$

Hence, for every $n > j$, $a_n \neq 0$.

Suppose $i > 0$, $a_i \neq 0$, $a_{i+1} = 0$, $a_{i-1} \neq 0$ (cases II and IV). We shall call these assumptions Assumptions B. Because of the symmetry with respect to i of the indices in the equation

$$a_{i-1}a_{i+1} - a_i^2 = (-1)^{i-1}d = (-1)^{i+1}d ,$$

and because Assumptions A are symmetrical to Assumptions B with respect to i if we write i instead of j in Assumptions A, the argument above proceeds just as well in the direction of decreasing indices. Hence, for every $n < i$, $a_n \neq 0$. Then by (1), in cases II and IV, the sequence $a_0, \cdots, a_{i-1}$ satisfies Eq. (8) for some $k \neq 0$, and is a finite segment of an almost geometric sequence; and in cases III and IV, the sequence $a_{j+1}, a_{j+2}, \cdots, a_{j+n}, \cdots$ satisfies Eq. (8) for some $k \neq 0$, and is an almost geometric sequence. This completes the proof of the theorem.

An example of case IV is given in Section 5. Another example is the sequence

$$58, \; 24, \; 10, \; 4, \; 2, \; 0, \; 2, \; 0, \; 2, \; 0, \; 2, \; 8, \; 34, \; 144, \; \cdots .$$

In this sequence, the characteristic number $\alpha = 4$. The sequence is made up of three consecutive segments:

I. 58, 24, 10, 4;

II. 2, 0, 2, 0, 2, 0, 2;

III. 8, 34, 144, $\cdots$;

where Segment I is a fragment of an almost geometric sequence satisfying the recurrence relation $a_{n+2} = -2a_{n+1} + a_n$. Segment II is a fragment of an alternating sequence satisfying the recurrence relation $a_{n+2} = a_n$; Segment III is an almost geometric sequence satisfying the recurrence relation $a_{n+2} = 4a_{n+1} + a_n$.

7. Consider the set of all almost geometric sequences satisfying the recurrence relation (8) with given $k \neq 0$. The associated characteristic equation is (9), where roots are r and s given in (10). If r and s are irrational, the theory of these sequences is analogous to that of rational Fibonacci sequences. For example, just as the set of all rational Fibonacci sequences can be given a field structure isomorphic to the field extension $R(\alpha)$ (see [4]), the set of all rational sequences satisfying the recurrence relation (8) with given $k \neq 0$ such that r is irrational can be given a field structure isomorphic to the field extension $R(r)$. In fact, we may represent each such sequence $a_0, a_1, \cdots$ by the ordered pair (a_0, a_1), since the sequence is fully determined by its first two terms and the recurrence relation (8). Then $(a_0, a_1) \mapsto a_0 + a_1 r$ is an isomorphism if we define addition and multiplication of sequences by

$$(a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1) .$$

$$(a_0, a_1)(b_0, b_1) = (a_0b_0 + a_1b_1, a_0b_1 + a_1b_0 + ka_1b_1) .$$

8. If

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}} \; \ddots$$

is a continued fraction, the convergents

$$c_n = \frac{p_n}{q_n}$$

for $n = 1, 2, 3, \cdots$ are given by $p_0 = 1$, $q_0 = 0$, $p_1 = a_1$, $q_1 = 1$, and $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n > 1$ [5]. If we let $a_1 = a_2 = \cdots = k \neq 0$, where k is rational, then the equations take the form $p_0 = 1$, $q_0 = 0$, $p_1 = k$, $q_1 = 1$, and $p_n = kp_{n-1} + p_{n-2}$, $q_n = kq_{n-1} + q_{n-2}$ for $n > 1$. Moreover, $q_1 = p_0$, and $q_2 = kq_1 + a_0 = k = p_1$. Hence, for all $n > 0$, $q_n = p_{n-1}$. Then

$$c_n = \frac{p_n}{q_n} = \frac{p_n}{p_{n-1}}$$

for $n > 0$. In this case, $\lim_{n\to\infty} C_n = r$, where r is a root of $x^2 - kx - 1 = 0$. Moreover, the relation

$$p_{i+2}\, q_{i+1} - p_{i+1}\, q_{i+2} = (-1)^i$$

in this case takes the form

$$p_i\, p_{i+2} - p_{i+1}^2 = (-1)^i = (-1)^i d ,$$

where $d = 1$. Hence the sequence $p_1, p_2, \cdots, p_n, \cdots$ is a parageometric sequence with characteristic number 1, and is also an almost geometric sequence satisfying the recursion relation $p_{n+2} = kp_{n+1} + p_n$. If k is a positive integer, the sequence is related to the golden-type rectangle [6].

9. Every sequence that has a characteristic number d is either geometric (with $d = 0$) or parageometric (with $d \neq 0$). If it is parageometric, it consists of at most three consecutive segments, each of which satisfies the recursion relation (8) for some value of k. If it is a geometric sequence (ar^n), and $r \neq 0$, it satisfies the recursion relation (8) with $k = r = 1/r$. If $r = 0$, the sequence is $a, 0, 0, \cdots$, and is composed of two consecutive segments a and $0, 0, \cdots$, each of which trivially satisfies Eq. (8). Hence, every sequence that satisfies Eq. (2) and therefore has a characteristic number

α consists of at most three consecutive segments each of which satisfies Eq. (8) for some value of k.

Let us now consider any sequence satisfying Eq. (8), to see if it also satisfies Eq. (2) and hence has a characteristic number. If the sequence is geometric, it satisfies Eq. (2) with $d = 0$. If the sequence is not geometric, and no term of the sequence is 0, we have already shown in Section 6 that it satisfies Eq. (2) with $d \neq 0$. Suppose now that the sequence is not geometric and contains a term $a_j = 0$. Then the method of proof used in Section 6 breaks down. However, this case can be covered by a general proof that does not require that all terms of the sequence be different from 0.

Let $a_0, a_1, \cdots, a_{n-1}, \cdots$ be a sequence satisfying Eq. (8) for some value of K. Let $d = a_0a_2 - a_1^2$. Then, for $n = 0$, the sequence satisfies Eq. (2). We now proceed by induction. Assume

$$a_n a_{n+2} - a_{n+1}^2 = (-1)^n d$$

for some fixed n.

$$\begin{aligned} a_{n+1}a_{n+3} - a_{n+2}^2 &= a_{n+1}(ka_{n+2} + a_{n+1}) - a_{n+2}^2 \\ &= ka_{n+1}a_{n+2} - a_{n+2}^2 + a_{n+1}^2 \\ &= a_{n+2}(ka_{n+1} - a_{n+2}) + a_{n+1}^2 \\ &= a_{n+2}(-a_n) + a_{n+1} = -(a_n a_{n+2} - a_{n+1}^2) \\ &= (-1)^{n+1}d \,. \end{aligned}$$

Hence, every sequence satisfying Eq. (8) also satisfies Eq. (2), and therefore has a characteristic number.

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[Continued from page 119.]

$$\begin{vmatrix} 3 & -3 & 1 & 0 & \cdots \\ -1 & 3 & -3 & 1 & \cdots \\ 0 & -1 & 3 & -3 & \cdots \\ 0 & 0 & -1 & 3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n = \frac{1}{2}(n+1)(n+2) .$$

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A NEW ANTHESIS

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Louis Pasteur pursued and assessed his studies in the light of his belief that:

> "...there is a cosmic dissymmetric influence which presides constantly and naturally over the molecular organization of principles [sic] immediately essential to life; and that, in consequence of this, the species of the three kingdoms, by their structure, by their form, by the disposition of their tissues, have a definite relation to the movements of the universe." [1]

Fathoming the occurrence of a specific angle of dynamic orientation throughout a range of pehnomena involving gravitation and electromagnetism: stellar and atomic systems and living molecules, may fulfill such belief that there is unitary, fundamental interdependency (<u>relativity</u>) of each such system one upon the others, mutually generated of cosmic necessity.

In 1783, Herschel found in the constellation Hercules the point among Earth's neighborhood of "fixed" stars toward which the Solar System moves, the Solar Apex. The pole of the ecliptic, known from ancient times, is in the constellation Draco. Though the individual members of the Solar System are variously oriented, the system as a whole <u>spirals</u> toward Hercules along a trajectory inclined about 37° from the pole of the ecliptic [2].

Galactocentrically, that is of great interest. The plane of the ecliptic intersects the plane of the Milky Way, our Galaxy, at points in Sagittarius and Gemini. Beyond Sagittarius lie the mass center and dynamic foci of the Galaxy. Thus, the Solar System plane is about perpendicular to the galectic center, while the axis of the Solar System is inclined to the plane of the Galaxy. As the Solar System spirals along its galactic orbit, toward Cephaus, it would seem that we would rise out of the Galaxy, but, the fact is that the Galaxy itself moves overall at an inclination to its plane, at an angle that may be determinable with respect to our neighbors in the local cluster of galaxies. It may also be determinable how galactic clusters in general are oriented: their members among themselves; one cluster to others [3].

The axis of the overall motion of the Solar System is of additional interest with respect to the "abandoned" theory of the Æther. Adolph Grünbaum [4] argues that acceptance or rejection of the Æther depends substantially upon one's philosophical and historical comprehension of the issue, as well as upon scientific criteria. Arthur Moestler [5] reviews the history of the rejection of the Æther and concludes that contemporary scientists have glossed the issue. On theoretical grounds, P. A. M. Dirac [6] has reconsidered and found a reconceived Æther necessary.

Somewhat like Laplace not needing the "hypothesis" of God, one may say, Einstein [7] early argued that Relativity Theory eliminated the need for Æther postulates, experiments, and interpretations. Firm in his belief that God does not cast dice, however, by 1952, Einstein wrote:

> "...the foundation of electromagnetic <u>theory</u> taught that a particular inertial system must be given preference, namely, that of the luminiferous aether at rest...
>
> "Since the special theory of relativity revealed the physical equivalence of all inertial systems, it proved the untenability of the hypothesis of an aether at rest...
>
> "It appears therefore more natural to think of physical reality as a four-dimensional existence, instead of, as hitherto, the <u>evolution</u> of a three-dimensional existence.
>
> "This rigid four-dimensional space of the special theory of relativity is to some extent a four-dimensional analogue of H. A. Lorentz's rigid three-dimensional aether." [8]

Inquiry as to the results of the Michaelson-Morely experimental program would, then, seem as legitimate as it is interesting. As summarized by Robert W. Wood:

> "The most exhaustive series of observations extending over a period of thirty years have been made by D. C. Miller..." He "com-

puted the velocity and direction of the earth's absolute motion in space, on the assumption that the observed effects were real. Astronomical observations indicate that the solar system is moving with a velocity of 19 kms/sec with respect to the brighter stars toward the constellation of Hercules. Miller's results showed an absolute motion in the opposite direction of 208 kms/sec.... A very full and convincing account of these laborious observations and calculations will be found in Review of Modern Physics, Vol. 5, No. 3, July, 1933." [9]

Herbert Dingle, a member of the British solar eclipse expeditions of 1927, 1932, and 1940, to test Einstein's prediction of the bending of light rays passing through the gravitational field of the Sun, has long questioned the Special Theory of Relativity and all its consequents, arguing that there is a serious error at the root of Einstein's mathematical reasoning. The history of science is largely a human story of accurate results obtained in terms of inadequate theory. Dingle insists that, some time, however inconvenient it may be, the inadequacies of Special Relativity must be faced [10].

The Æther is still considerable. But a concept is best judged by its fruits. Beginning in 1925, in terms of his revamping of the Æther, Carl F. Krafft [11] discovered what is fully described by the title of his first, privately published monograph of 1927, Spiral Molecular Structures the Basis of Life, which is replete with diagrams. In that and subsequent writings, which Mendelianly remain unrecognized, Krafft fully developed a theory of helical molecular structure for proteins, with full understanding of the genetic import of his discovery, and much more.

The years 1925—1927 were those when Schroedinger, Meisenberg, Born, Jordan, Wigner, Pauli, Fermi, Dirac, de Broglie, Base, Einstein, et al, were developing the fundaments of quantum mechanics.

In 1948, Linus Pauling [12] discovered the base of contemporary knowledge of helical molecular structure of proteins, in terms of which F. H. C. Crick and J. D. Watson discovered the helical molecular structure of Deoxyribose Nucleic Acid (DNA). Presenting their theory and structure, Crick and Watson wrote:

"We have assumed an angle of 36° between adjacent residues in the same chain.... The structure is an open one, and its water content is rather high. At lower water contents we would expect the bases to tilt so that the structure could become more compact." [13]

That assumed 36° molecule of the DNA double helix, articulated with respect to the molecular axis, to which the purine and pyrimidine bases of the genetic code are perpendicular and planar.

In 1927, Krafft theorized that the spiral structure of proteins provided an explanation of optical activity. This is as yet unsettled. Discussing the "Origin and Role of Optical Isomery in Life," A. S. Garay states:

"Living organisms possess only one of two possible optical isomers. There is no generally accepted theory for the origin of this asymmetry." [14]

It is, however, generally thought that polarized light is necessary to the origin, development, and maintenance of life. Garey asks: "What is the source of circularly polarized light in nature?" A. Dauvillier notes:

"Rectilinear polarized light exists in the solar light diffused by the sky and is produced in nature by reflexion, at an incidence of 37°, from the surface of water or on flat crystalline facets. The light from the sky is not polarized elliptically. Circularly polarized light, which is obtained by causing rectilinear polarized light to fall on a quarter-wave plate — or a Fresnel parallelepiped, may also be produced naturally by polarized light falling on a birefringent quartz or apar crystal." [15]

In the molecule of water, the hydrogens are bonded at an angle of 104.6° across the pole of the oxygen. Regarded in an upsidedown, Alice sort of way, one may say that the hydrogens are oriented to oxygen at 180 — 2 x 37.7° [16].

Investigating "Rotary Brownian Movement. The Shape of Protein Molecules as Determined from Viscosity and Double Refraction of Flow," John T. Edsall wrote:

> "All these measurements involve the rotation and partial orientation of protein molecules in an external field of force... the orientation achieved is only partial, since it is opposed by the disorienting action of the Brownian movement of the molecules... involving rotation of the molecules about their axes, arising from thermal agitation. Its effect is to produce a purely random distribution of molecular orientations, in the absence of external orienting forces. In the presence of such orienting forces, a steady state is gradually achieved, a state intermediate between the two limiting conditions of complete orientation and of complete disorder. The exact character of this intermediate state depends on the magnitude of the orienting forces relative to that of the rotary Brownian movement." [17]

In terms of the rationale then presented, the experiment performed on various proteins and other substances involved having a fixed core within a concentric tube rotatable to impart motion to a solution contained in the tube, through which polarized light is passed, its behavior being measured. For at least myosin, the protein of muscle, the angular parameters emerged as 53° and/or 37°.

(Edsall, in another discussion, makes the only reference to the work of Carl F. Krafft which I have yet found [18].)

Gunther S. Stant, both historian and practitioner of molecular biology, discussing the seemingly startling permanence of the genetic code over geologic time, ventures the possible explanation that:

> "... there exists some as yet unfathomed geometrical or stereochemical relation between the anticodon nucleotids triplet and the amino acid which it represents. Indeed, if such a relation exists, it would be bound to hold one of the keys to understanding the origin of life." [19]

In "Toward a Definition of Mind," citing the work of D. L. Reiser [20], Harold Kelman wrote this beautiful passage:

"The forming process is metaphorically a spiral, constituted of an intimately connected sequence of levels or a continuum of transformations with movement possible from depth to surface and vice versa. The helix is of crucial import in Indian cosmology. 'Nature moves in a helical pattern in time, so that spiral forms get ingrained at many levels.... This is all part of a galactic rotation in which a Cosmic field plays an important part in transmitting spin (angular momentum) to matter.' This is one facet of Reiser's concept of cosmic imagination which moves in similar directions to my ideas on cosmic minding.

"X-ray crystellography reveals DNA as a double stranded alpha helix [sic].... It directs protein synthesis and heredity. What was intuited thousands of years ago regarding life and living is being confirmed by science or science confronts us with ancient truth." [21]

In terms of the evidence marshaled for the universal occurrence of an angle of about 37°–38°, it seems that now and henceforth, it shall be less metaphorical to make such assertions.

A. N. Whitehead wrote:

"...the search for a reason is always the search for an actual fact which is the vehicle of reason. The antological principle..., constitutes the first step in the description of the universe as a solidarity of many actual entities." [22]

and:

"The task of reason is to fathom the deeper depths of the many-sidedness of things. We must not expect simple answers to far-reaching questions." [23]

A fact is not an answer, but should be of service in approaching an answer, however Zenoic the process of approach may ultimately be.

Musing in Autumn on "The Secret of Life," Loran Eiseley says:

"...I have come to suspect that the mystery may just as well be solved in a carved and intricate seed case out of which the life has flown, as in the seed itself." [24]

Natural Philosophy, as contemporary as it is ancient and honorable, is especially pertinent in this two-cultured era, when I am "credentially" a "litterateur." As such, however, I am particularly a student of the Philosophy of Owen Barfield, whose preface compelled my attention to E. Grant-Watson's exposition [25] of many beautifully amazing aspects of structure and behavior of living creatures. This was my introduction to the botanical principle of Phyllotaxis; the aesthetic principle of the Golden Section; and to the significance of the Fibonacci Series.

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \cdots,$$

or

$$1/1,\ 1/2,\ 2/3,\ 3/5,\ 5/8,\ 8/13,\ 13/21,\ 21/34,\ 34/55,\ 55/89,\ 89/144,\cdots.$$

These fractions, $1/2 = 0.5$; $2/3 = 0.66$; $3/5 = 0.6$; successively vacillate until $a/b = 0.6180\cdots$.

This Fibonacci Number, $0.6180\cdots$, is the value of the Sine of $38.166°$.

"Elected Silence, sing to me
And beat upon my whorlèd ear....
Be shellèd, eyes, with double dark
And find the uncreated light.... ."
G. M. Hopkins, S. J., "The Habit of Perfection."

Having discovered the equiangular, logarithmic spiral, which is the shape of the shell of the Chambered Nautilus; whose equation is satisfied by the Fibonacci Number, Jacob Bernoulli (1654-1705) had this figure inscribed on his tombstone, with the inscription:

"Eadem mutata resurgo (Though changed I shall rise the same)."
[26]

The French poet, Paul Vallery, having caught Einstein's allusion to a "mollusc" of reference [27], essaying on "Man and the Sea Shell," wrote:

"Without the slightest effort life creates a very 'generalized' relativity....

"It does not separate its geometry from its physics but endows each species with all the axioms and more or less 'differential' invariants it needs to maintain a satisfactory harmony between the individual and the world around it....

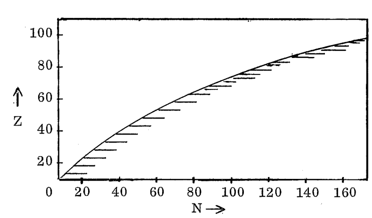
"The pattern of the colored furrows or bands that curve around the shell, and of the bands that intersect them, reminds us of 'geodesic lines' and suggests the existence of some sort of 'field of force' which we are unable to discern, but whose action would give the growth of the shell the irresistible torsion and rhythmic progress we observe in the finished product." [28]

A line segment is a Golden Section if it may be split A . . B . . . . . C such that AB/BC = BC/AC.

Turning the segment AB perpendicular to BC and completing the rectangle produces the Golden Rectangle.

A Golden Rectangle may be constructed geometrically by taking any line as the base of a square; drawing the square; bisecting the square; drawing a diagonal in one of the created interior rectangles; using that diagonal as a radius; swinging that radius until it intersects a line extended from the original base; from which point, completing a new rectangle containing the original square.

The Golden Section, and forms based thereon, occur throughout artistic endeavors from the Greeks through the Renaissance through Bartok's music [29]. A beautiful and extraordinarily meaningful example of a Golden Rectangle is this graph by Linus Pauling:



"Fig. 8. A curve of proton number Z as a function of neutron number N, calculated as described in the text. The horizontal lines show the ranges of stable isotopes for alternate Z-even elements (for large Z the four most stable isotopes)." [30]

The value of the Tangent of 38.1+° is 0.7854, which is the value of (pi)/4, which is generated by the Gregory/Leibniz Series: [31]

$$1 - 1/3 + 1/5 - 1/7 + 1/9 \cdots .$$

Assessing the mass ratios of electron, positron, proton, neutron, the mesons and hyperons, etc., John J. Grebe found "A Periodic Table for Fundamental Particles," as:

> "The existence of unique relations among the fundamental particles based on exponentials of (pi)/4 has been discovered in the search for symmetry, unity, and simple structure." [32]

Grebe has also employed the logarithmic spiral ingeniously in plotting a three-dimensional Chambered Nautilus of a graph ordering all things in terms of frequencies of cycles per nanosecond through cycles per eons [33].

Hermann Weyl, whose discussion [34] of the significance of the Fibonacci Series eases one's mind in daring such sweeping claims as are being presented herein, has stated:

> " "Perhaps the philosophically most relevant feature of modern science is the emergence of abstract symbolic structures as the hard core of objectivity behind — as Eddington put it — the colorful tale of the subjective storyteller mind.... In the progress of science such elementary structures as roughly correspond to obvious facts are often later recognized as founded on structures of a deeper level, and in this reduction the limits of their validity are revealed." [35]

The objection that the Fibonacci Number and its associated Golden Angle at best only approximate to established parameters in astronomy, from which inference about gravitation may be made; in the interaction of electromagnetic radiation with matter, from which inference about electromagnetism and about chemical structure may be made; in microphysics; and in biology; may be met by stating that appropriate investigation of the deviations from equality may be profoundly significant. The radius of curvature in non-Euclidian Space Time may be involved. The variation of gravitation with time may be involved.

Considering such things in "Gravitation — An Enigma," R. H. Dicke says:

> "The chief conclusion... is that it is a serious lack of observational data that keeps one from drawing a clear portrait or gravitation. Each tiny fragment of information appears as a star shining through a murky haze. Conclusions regarding the most fundamental of physical concepts are based on numbers which may be off by a factor of 100." [36]

Rectification may be achieved by recognition and pursuit of the unification that seems possible in terms of the Golden Angle.

Before finding the details presented here, in an unpublished paper written in January 1968, on the significance of Carl F. Krafft's work and neglect, I perhaps gave myself as litterateur too free a rein when I wrote:

> "Krafft unambiguously formulated, and, over several years, amplified and exploited his idea of spiral molecular structure as the basis of life. In terms of his development of Descartes' vortices and his unique 'panpsychism,'... one can visualize Life as an inherent function of energized matter: From electrons spiraling about nuclei, which consist of spiraling mason clouds, yielding atoms which aggregate into stars and planets which spiral about each other and about a focus of a spiral galaxy; which galaxies perhaps ultimately spiral about each other as their overall form of motion in the universe, which may not be expanding if Krafft's ((and my independent)) interpretation of the red shift in the spectra of galaxies prove more accurate than the Doppler effect interpretation of that red shift. As an inherent product of universal spiral motion, under suitable conditions, matter is energized to spiral into molecular structures which live, evolve, and finally are energized to such density of redundant interaction as to resonate

"self-consciousness, yielding Teilhard de Chardin's Noosphere in a way he doubtless would have rejoiced in understanding.

"It may be flamboyant, but I think that the model of DNA 'sculpted' by Crick and Watson, and the models of the alpha-helix by Pauling and Corey; of hemoglobin by Perutz; of myoglobin by Kendrew; and of insulin by Frederick Sanger; should be enshrined along with if not above the works of Praxiteles and Michelangelo, as the greatest testimony to the truest humanistic value of science (man's cosmically emplaced impulse to know) in fathoming the secret of the ultimate, or at least penultimate, formative principle of life... ."

Perceiving the European philosophical tradition as a series of footnotes to Plato, A. N. Whitehead wrote:

"...if we had to render Plato's general point of view with the least changes made necessary by the intervening two thousand years of human experience in social organization, in aesthetic attainments, in science, and in religion, we should have to set about the construction of a philosophy of organism." [37]

Ludwig von Bertalanffy, independent cofounder with Whitehead and others of contemporary Organismic Philosophy, recently concluded an excellent critique of psychology in the modern world:

"...science is more than an accumulation of facts and technological exploitation of knowledge in the service of the Establishment; it may still be able to present a grand view and to become deeply humanistic in its endeavor. If we achieve as much as contributing a bit toward humanization of science, we have done our share in the service of society and civilization." [38]

G. D. Birkhoff, whose gravitational theory needs review of itself if not in the context of this anthesis; who perceptively explored Aesthetics mathematically [39], wrote:

"The prophetic conjecture that Nature is mathematical is one which which goes back to Pythagoras and the ancient Greeks. The scientific "development of the intervening 25 centuries has only served to establish this conjecture to a remarkable degree. The complementary fact that mathematics is natural is, however, just beginning to be grasped...

"The essential genetic foundation here is obvious. The mental codification of the facts of Nature in logical and mathematical terms has its origin in the uniformity of Nature and of Mind." [40]

Many thinkers, addressing themselves to problems engendered by the growth of detailed knowledge, advocate radically new departures. This Anthesis, however, is rooted in Tradition embracing Goethe, Leonardo de Vinci, Leonard of Pisa/Fibonacci, Plato, Pythagoras, and so many more. This Anthesis affirms Man and his capacity to know, and, what though, not thinking, at the foot of the cross men cast dice for the seamless raiment, this Anthesis affirms Einstein's belief that God does not cast dice.

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FIBONACCI, LUCAS, AND THE EGYPTIANS

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1. INTRODUCTION

One of the obvious distinctions between Egyptian mathematics and the mathematics of other cultures is its additive character of the dependent arithmetic. A typical example is recognized when we examine the algorithm used by the Egyptians in doing multiplication in comparison to other algorithms.

Multiplication (Egyptian Style) is done by a doubling-summing process similar to the one shown in the following example. Let us solve the following problem: 19 x 65. The Egyptians noted that the number 19 was equal to $1 + 2 + 16$ (the sum of powers of two), hence, by the addition of appropriate multiples of 65 the Egyptians arrived at the desired result. We may arrange the problem in the following way:

doubling	1*	*65	
	2*	*130	doubling
doubling	4	260	doubling
doubling	8	520	doubling
doubling	16*	*1040	doubling
	19	1235	

Upon careful examination of the processes used in this algorithm, we find that there are two basic concepts that contribute to its efficiency. Namely, they are the concepts of distributivity and completeness. The latter conceived by Professor Verner E. Hoggatt, Jr. [1].

We can easily identify the role which is played by the distributive law in the algorithm, for example, in the preceding problem $65 \times 19 = 65\ (1 + 2 + 16)$. However, the contribution made by the concept of completeness is not self-evident. Let us turn to the definition of completeness before we examine its role in the Egyptian algorithm.

Definition. A sequence S of positive integers is said to be complete if and only if every element n, where n is an element of the positive integers can be represented as a sum of distinct elements of S.

The sequence used in the Egyptian method of multiplication the author shall describe as T, where $T_n = 2^n$ $(n \geq 0)$. In order to show that T is complete, we must first prove the following lemma.

Lemma 1. $T_0 + T_1 + T_2 + T_3 + \cdots + T_{n-1} = T_n - 1$.

Proof. We shall prove this lemma by mathematical induction. Here, we have

$$P(n) : T_0 + T_1 + T_2 + T_3 + \cdots + T_{n-1} = T_n - 1 .$$

Then $P(1) : T_0 = T_1 - 1$ is easily seen to be true since $1 = 2 - 1$. Thus, we have accomplished our inductive basis.

Now, suppose that

$$P(K) : T_0 + T_1 + T_2 + T_3 + \cdots + T_{k-1} = T_k - 1$$

is true (the inductive assumption), and we must then prove:

$$P(K + 1) : T_0 + T_1 + T_2 + T_3 + \cdots + T_k = T_{k+1} - 1.$$

By our inductive assumption, we know that

$$T_0 + T_1 + T_2 + T_3 + \cdots + T_{k-1} = T_k - 1 .$$

Hence, by substitution into $P(k + 1)$, we have that

$$T_k - 1 + T_k = T_{k+1} - 1.$$

It follows that

$$2T_k - 1 = T_{k+1} - 1 ,$$

hence, $2T_k = T_{k+1}$. Since $T_k = 2^k$, we have that $2T_k = T_{k+1}$. Therefore, we have shown that if $P(K)$ is true, then $P(K + 1)$ is true, and we have completed the inductive transition.

Employing Lemma 1, we may prove the following theorem.

Theorem 1. The sequence T, where $T_n = 2^n$ $(n \geq 0)$ is a complete sequence.

Proof. As an inductive basis, we know that

$$\begin{aligned} 1 &= 1 \\ 2 &= 2 \\ 3 &= 1 + 2 \\ 4 &= 4 \\ 5 &= 1 + 4 \\ 6 &= 2 + 4 \\ 7 &= 1 + 2 + 4, \quad \text{etc.} \end{aligned}$$

Hence, we must assume that there are representations for all the positive integers N:

$$1 < N < 2^{n+1} - 1.$$

Therefore, we must show that there are representations for all positive integers M:

$$2^{n+1} - 1 < M < 2^{n+2} - 1 .$$

By subtracting 2^{n+1} from the above inequality, we have that

$$-1 < M - 2^{n+1} < 2^{n+2} - 2^{n+1} - 1 .$$

Let $Q = M - 2^{n+1}$; hence, $-1 < Q < 2^{n+1} - 1$. This leads us to the conclusion that Q is representable as a sum of powers of 2 by our inductive assumption. And, from this, we can conclude that M is representable as a sum of powers of 2 since $M = Q + 2^{n+1}$ and

$$2^{n+1} - 1 = 1 + 2 + 2^2 + 2^3 + \cdots + 2^n .$$

Hence, we have completed our inductive transition.

2. FIBONACCI-EGYPTIAN METHOD

As we noted in the introduction, the necessary and sufficient conditions for the Egyptian algorithm to "work" are completeness and distributivity.

The author, upon reaching this conclusion, went in search of other sequences that would satisfy the above conditions. The first sequence examined proved to be fruitful. It was the Fibonacci sequence. It is obvious that the distributive law is satisfied, since we are working solely with positive integers; however, it is not so obvious that the Fibonacci sequence is complete. Let us then prove this fact.

As before, we must prove a lemma before proving the main theorem. It is the following:

Lemma 2.

$$F_{n+2} - 1 = F_1 + F_2 + F_3 + F_4 + \cdots + F_n .$$

Proof. We shall prove the lemma by mathematical induction.

$$P(n) : F_{n+2} - 1 = F_1 + F_2 + F_3 + F_4 + \cdots + F_n .$$

Then $P(1) : F_3 - 1 = F_1$ which is true, since $2 - 1 = 1$. Thus, we have accomplished our inductive basis. Now we must suppose that

$$P(K) : F_{k+2} - 1 = F_1 + F_2 + F_3 + F_4 + \cdots + F_k$$

is true (the inductive assumption), and we must then prove:

$$P(K + 1) : F_{k+3} - 1 = F_1 + F_2 + F_3 + F_4 + \cdots + F_{k+1} .$$

By the addition of F_{k+1} to both sides of the equation $P(K)$, we have

$$F_{k+2} + F_{k+1} - 1 = F_1 + F_2 + F_3 + \cdots + F_k + F_{k+1},$$

which lieads us to

$$F_{k+3} - 1 = F_1 + F_2 + F_3 + \cdots + F_{k+1}$$

by the recursion relation for Fibonacci numbers, namely

$$F_{n+3} = F_{n+2} + F_{n+1} .$$

Using this lemma, we may prove the following theorem.

Theorem 2. The Fibonacci numbers form a complete sequence.

Proof. The inductive proof will be considered in the following way. We observe that

$$\begin{aligned}
1 &= F_1 = F_2 \\
2 &= F_3 = F_2 + F_1 \\
3 &= F_4 = F_3 + F_2 \\
4 &= F_4 + F_2 = F_3 + F_2 + F_1, \quad \text{etc.}
\end{aligned}$$

We shall use this as our inductive basis. Next, we must assume that there are representations for all positive integers N, such that

$$1 < N < F_{n+2} - 1$$

is true. We must therefore show that there are representations for all positive integers M, such that

$$F_{n+2} - 1 < M < F_{n+3} - 1.$$

By subtracting an F_{n+2} from the above inequality, we have that

$$-1 < M - F_{n+2} < F_{n+3} - F_{n+2} - 1 .$$

Let $Q = M - F_{n+2}$; hence,

$$-1 < Q < F_{n+1} - 1 .$$

This leads us to the conclusion that Q is representable as a sum of Fibonacci numbers by our inductive assumption. And from this, we may conclude that M is representable as a sum of Fibonacci numbers, since

$$M = Q + F_{n+2}$$

and

$$F_{n+2} - 1 = F_1 + F_2 + F_3 + \cdots + F_n .$$

Hence, we have completed our inductive argument.

Let us examine the Fibonacci-Egyptian method for multiplication. For example, consider the problem 19 x 65. We note that

$$19 = 1 + 5 + 13 ,$$

all of which are Fibonacci numbers. Together with the Fibonacci recursion relation, and the following set-up, we may approach the problem in the following way:

$$\begin{array}{rrrr}
 & 1* & *\ 65 & \\
+ & 2 & 130 & + \\
+ & 3 & 195 & + \\
+ & 5* & *325 & + \\
+ & 8 & 520 & + \\
+ & 13* & *845 & + \\
\hline
 & 19 & 1235 &
\end{array}$$

One may observe that in the preceding example, the entire Fibonacci sequence was not used. Upon examination, one will find that the first number of the sequence has been truncated. This does not, however, effect either the completeness of the sequence nor the distributivity. The author shall refer to the Fibonacci sequence with one element omitted as <u>the Deleted F Sequence</u>. Hence, let us prove the following theorem.

<u>Theorem 3.</u> The deleted F sequence, where $f_n = F_n$ $(n \geq 1)$ with arbitrary F_n not used, is complete.

Proof. From the previously proven theorem, it was noted that we may represent any positive integer n, where $1 \leq n \leq F_{n+1} - 1$ by using only the Fibonacci numbers F_1 through F_{n-1}, without using F_n. Hence, we shall consider F_n as the arbitrary Fibonacci number to be omitted. We may observe that F_{n+1} can represent itself. Since this is true, it is noted that we now have representations for $1 \leq n \leq 2F_{n+1} - 1$. Since we have increased our upper bound from what it was formerly, we may use this particular technique so that we may have representations for any positive integer without using F_n. For example, if $F_n = 1$, which is proposed to be the deleted number, then the sequence would remain complete.

Therefore, we have another method for multiplication which may be employed by those who have not mastered the traditional algorithm.

3. LUCAS-EGYPTIAN METHOD

Another sequence which proves fruitful in using our algorithm is the Lucas sequence. The Lucas sequence is composed of the numbers

$$(1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ \cdots)$$

and can be used effectively for the base sequence in an Egyptian multiplication problem. However, there is one acute difficulty in the consideration of this sequence for our algorithm; it does not have any representation for the positive integer 2. Therefore, something must be done to the sequence before we can apply it to our algorithm, since without a representation for the number 2 it is not complete.

The author chose to augment the sequence in the following way and define his Augmented Lucas Sequence as $A_n = L_{n-1}$, where $A_1 = 2$, $A_2 = 1$, $A_3 = 3$, and so on.

The reader will observe that this augmented sequence has a representation for 2 and also observe the recursion relation for the Lucas Sequence, namely $A_{n+1} = A_n + A_{n-1}$. Hence, we may use it for our base sequence in the Egyptian algorithm. The problem 18 x 54 may be set up in the following fashion.

$$
\begin{array}{cccc}
 & 2 & 108 & \\
+ & 1 & 54 & + \\
+ & 3 & 162 & + \\
+ & 4 & 216 & + \\
+ & 7 & 378 & + \\
+ & 11 & 594 & + \\
+ & 18 & 972 & + \\
\hline
 & 18 & 972 &
\end{array}
$$

The augmented Lucas sequence is complete and may be proved to be in a similar fashion to Theorem 2, by use of Lemma 3, which states

Lemma 3.

$$L_0 + L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 1.$$

Proof. Using an inductive proof, we have as our basis

$$P(1) : L_0 + L_1 = L_3 - 1$$

which is true, since $2 + 1 = 4 - 1$. Our inductive assumption is

$$P(K) : L_0 + L_1 + L_2 + L_3 + \cdots + L_k = L_{k+2} - 1 .$$

We must then prove that

$$P(K + 1) : L_0 + L_1 + L_2 + L_3 + \cdots + L_{k+1} = L_{k+3} - 1$$

is true. This may be accomplished by adding a L_{k+1} to both sides of $P(K)$. Hence, we have that

$$L_0 + L_1 + L_2 + \cdots + L_k + L_{k+1} = L_{k+2} + L_{k+1} - 1,$$

which leads us to the fact that

$$L_0 + L_1 + L_2 + L_3 + L_4 + \cdots + L_{k+1} = L_{k+3} - 1 .$$

Hence, our induction transition is complete.

Invoking this lemma, we may prove the following theorem.

Theorem 4. The augmented Lucas sequence is complete.

Proof. As our inductive basis, we have that

$$\begin{aligned} 1 &= L_1 \\ 2 &= L_0 \\ 3 &= L_2 \\ 4 &= L_3 \\ 5 &= L_3 + L_1, \quad \text{etc.} \end{aligned}$$

As our inductive assumption, we assume that for N, a positive integer, there are representations for N in terms of Lucas numbers so that

$$1 < N < L_{n+2} - 1 .$$

Hence, we must prove that for M, a positive integer, M is representable as a sum of Lucas numbers between the intervals of

$$L_{n+2} - 1 < M < L_{n+3} - 1 .$$

Using the same idea as described in the previously proven theorems, we shall subtract an L_{n+2} from the above inequality. Hence, we have that

$$-1 < M - L_{n+2} < L_{n+3} - L_{n+2} - 1 .$$

Let $Q = M - L_{n+2}$. Therefore,

$$-1 < Q < L_{n+3} - L_{n+2} - 1.$$

This leads us to the conclusion that

$$-1 < Q < L_{n+1} - 1 .$$

We may conclude that Q is representable as a sum of augmented Lucas numbers. And from this, we can conclude that M is representable as a sum of augmented Lucas numbers, since $M = Q + L_{n+2}$.

Other sequences may be investigated and tested for completeness; however, no others with starting values other than (1,1), (1,2), and (2,1) will be found which satisfy the generalized Fibonacci recursion relation. In general, other sequences that are complete will follow the following generalized recursion relation

$$G_n = \sum_{q=n-j}^{n-1} G_q \qquad j = (2, 3, 4, \cdots),$$

and where the starting values for the above sequences are taken from either the augmented Lucas sequence or the deleted F sequence. For example, let us examine the Tribonacci sequence, where three numbers are added. The generalized recursion relation would look like the following:

$$G_n = \sum_{q=n-3}^{n-1} G_q .$$

Hence, the sequence would be

$$(1, 2, 3, 6, 11, 20, \cdots) .$$

In general j determines the number of terms to be added together and also the number of starting values to be taken from either the deleted F sequence or the augmented Lucas sequence.

The author at this point feels that it would be valuable for the reader to have a simple method for determining whether a sequence is or is not complete. It was observed and proven by John L. Brown, Jr. [2] that the necessary and sufficient conditions for a sequence to be complete is that the sequence satisfy the following general summation formula

$$A_{n+1} \leq 1 + \sum_{i=1}^{n} A_i$$

where $A_1 = 1$. Hence, we now have a convenient way in which to determine a sequence complete.

The material submitted in this paper is not completely theoretical and does have very definite practical application. The author used both the deleted F sequence and the augmented Lucas sequence in conjunction with the Egyptian method in a class of "slow learners." The results were phenomenal. Those students who could not multiply by traditional means were then given a method even they could handle. You see, all one needs to be proficient in the methods given above is an adequate understanding of simple addition. The author found that most slow learners could add correctly, however, they could not multiply. Therefore, this algorithm best fit the needs of those students.

The concepts mentioned throughout the paper may also be used in advanced mathematics classes. Hence, as one can see, the utility of these topics and their applications is boundless.

It is the author's intent that the reader search for other complete sequences and establish those concepts revealed in this paper, so that he may transfer the concepts to others and hence, give many an algorithm for multiplication which they may not already have.

The author would also like the reader to be aware of the fact that it is sometimes advantageous to use one complete sequence over another. For example, it is better to use the Lucas sequence when multiplying the numbers 18 x 432, than it is to use the Fibonacci sequence or the powers of two sequence, since 18 is an element of the Lucas sequence. Therefore, this was the primary reason the author went in search of other complete sequences.

The author hopes that the methods for multiplication developed in this paper will be tried, and hopes that the success of those using them will be as rich as his own.

[Continued on page 194.]

THE GOLDEN RATIO IN AN ELECTRICAL NETWORK

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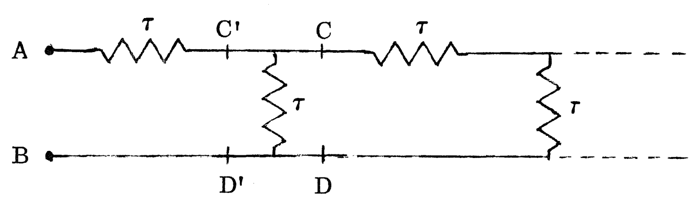
At the end of June 1967, Poland called together an international physics olympiad for grammar-school students in Warsaw. Five countries participated: Bulgaria, Poland, Romania, Czechoslovakia, and Hungary.

During this competition, the following problem was presented, among others:

An infinite network consists of the resistors r. Calculate the resistance between points A and B.



The solution of this problem can be presented in different ways. One quite brief version is possible as follows:

Suppose the resistance of the infinite network on the right-hand side of points C and D is equal to r_n.

If we go one step to the left from points C and D to C' and D', the resistance of the network would be

$$r_{C'D'} = \frac{r \cdot r_n}{r + r_n}$$

in accordance with relation:

$$\frac{1}{r_{C'D'}} = \frac{1}{r} + \frac{1}{r_n} \quad .$$

The next resistor r is added on the left behind the resistance $r_{C'D'}$: therefore, the resistance between A and B is

[Continued on page 194.]

KAPREKAR'S ROUTINE WITH TWO-DIGIT INTEGERS

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Kaprekar's routine consists of rearranging the digits (not all alike) of an integer, N_0, to form the largest and the smallest possible integers, finding their difference, N_1, and repeating the operation on N_1 and on the subsequent differences until a terminal situation is reached. He found [1] that when the routine is applied to any four-digit (not all alike) integer in the decimal system, the self-producing 6174 is eventually reached. The routine has been expanded to other number bases [2], [3], and to three-digit [4] and five-digit [5] integers.

When applied to two-digit integers, an integer and its reverse are involved, the smaller being subtracted from the larger in each step of the routine. In the system with base r, all differences are multiples of $r - 1$. Each step may be called a reversal-subtraction-operation (RSO).

There are three possible terminal situations which may result when the routine is applied to a two-digit integer, namely:

A. If at any step of the repetitious routine, an integer with two like digits is produced, its N_1 will be 00.

B. A self-producing integer is formed. That is, the integer is reproduced when subjected to an RSO. For example: 37 in the scale of eleven, where $73 - 37 = 37$.

C. A regenerative loop is formed, in which an RSO on one member produces the next member. For example: in the scale of nine, $53 - 35 = 17$, $71 - 17 = 53$, and so on.

In each of these categories, if all two-digit integers in a particular system lead to the same result, it is said to be unanimous.

The two-digit ordered integers ab and $\overline{a+k\ b+k}$ have the same N_1. Hence, to investigate the entire field in the scale of r, it will be sufficient to examine only the $r - 1$ integers with the form $\overline{r-1\ b}$, where $b < r - 1$. Each of these is the representative (rep) of all two-digit integers from which an RSO produces the same N_1. Any two-digit integer can be converted into its rep by addition of an appropriate multiple of 11. Thus, $52 + 44 = 96$ in the decimal system, so 52 - 25 = 27 = 96 - 69. All integers with the same rep have the same value of a - b.

THE EXAMINATION PROCEDURE

An RSO is performed on each of the reps. Each difference, N_1, is converted into its own rep. These results are assembled into flow charts such as those given below. All N_1's with the same rep are placed below a common subtraction line. Their common rep is placed below them on the left. The category letter of each terminal situation is placed below it.

Thus, only r - 1 RSO's are necessary to examine the entire field in the base r. The number of steps necessary to convert any integer into the terminal result can be read directly from the chart after locating its rep. In the scale of five, three RSO's convert 12 (which has the rep 43) into the self-producing 13.

In the charts for bases eleven and twelve, the symbols X and E stand for the digits ten and eleven, respectively.

OPERATIONAL FLOW CHARTS

Base Two	Base Three	Base Four		Base Five	
10	21	30	31	43	41
01	12	03	13	34	14
01	02	21	12	04	22
B					A
	20	32		40	
	02	23		04	
	11	03		31	
	A	C			
				42	
				24	
				13	
				B	

Base Six

50	53	
05	35	
41	14	
52		51
25		15
23		32
54		
45		
05		
C		

Base Seven

65	63	61
56	36	16
06	24	42
60	64	
06	46	
51	15	
62		
26		
33		
A		

Base Eight

70	75		71
07	57		17
61	16		52
72		73	74
27		37	47
43		34	25
			B
76			
67			
07			
C			

Base Nine

85	81		83
58	18		38
26	62		44
			A
84		87	
48		78	
35		08	
86		80	
68		08	
17		71	
82			
28			
53			
C			

Base Ten

90	97	
09	79	
81	18	
92		95
29		59
63		36
96	91	
69	19	
27	72	
94		93
49		39
45		54
98		
89		
09		
C		

Base Eleven

X9	X5	X3	X7	X1
9X	5X	3X	7X	1X
0X	46	64	28	82
X0	X8		X4	
0X	8X		4X	
91	19		55	
			A	
X2				
2X				
73				
X6				
6X				
37				
B				

Base Twelve

E0	E9	
0E	9E	
X1	1X	
E2		E7
2E		7E
83		38
E6	E3	
6E	3E	
47	74	
E8		E1
8E		1E
29		92
E4	E5	
4E	5E	
65	56	
EX		
XE		
0E		
C		

SUMMARY AND GENERALIZATIONS

1. Every system with an odd base has a sequence leading to 00, since

$$\overline{r-1}\ \overline{(r-3)/2} - \overline{(r-3)/2}\ \overline{r-1} = \overline{(r-1)/2}\ \overline{(r-1)/2}\,.$$

In bases three and seven, 00 is unanimous.

2. If a self-producing integer, kx with $k < x$, exists in a system with base r, then

$$(rx + k) - (rk + x) = rk + x$$

whereupon

$$r = (2x - k)/(x - 2k) = 2 + 3k/(x - 2k) .$$

Then, since $x < r$, self-producing integers, which will have the form $k\,\overline{2k + 1}$, exist in and only in systems with bases of the form $3k + 2$.

Such bases are two (in which 01 is unanimous), five, eight, and eleven.

3. Both $\overline{r - 1}\,c$ and $\overline{r - 1}\,\overline{r - 3 - c}$ have N_1's which are the reverse of each other, since

$$\overline{r - 1}\,c - c\,\overline{r - 1} = \overline{r - 2 - c}\;\overline{c + 1}$$

and

$$\overline{r - 1}\;\overline{r - 3 - c} - \overline{r - 3 - c}\;\overline{r - 1} = \overline{c + 1}\;\overline{r - 2 - c} .$$

Hence, the N_1's have the same rep.

4. If r is even and not of the form $3k + 2$, the result of application of RSO's to the reps in that system is a unanimous regenerative loop of $r/2$ elements.

REFERENCES

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2. J. H. Jordan, "Self-Producing Sequences of Digits," American Mathematical Monthly, 71 (January, 1964), pp. 61-64.
3. Charles W. Trigg, "Predictive Indices for Kaprekar's Routine," Journal of Recreational Mathematics, scheduled for publication in October, 1970.
4. Charles W. Trigg, "All Integers Lead to $\cdots$," submitted to The Arithmetic Teacher.
5. Charles W. Trigg, "Kaprekar's Routine with Five-Digit Integers," submitted to Mathematics Magazine.

[Continued from page 188.]

$$r_{AB} = r + \frac{r \cdot r_n}{r + r_n}$$

Since the network is infinite, we can disregard the addition of one section of each sequence. This allows to determine the resistance between points A and B as equal to the resistance between C and D.

Consequently,

$$r_n = r + \frac{r \cdot r_n}{r + r_n} \quad .$$

After solving this equation, we have ,

$$r_n = r \cdot \frac{1 + \sqrt{5}}{2} = r \cdot \phi \qquad ,$$

where ϕ is the Golden Ratio.

See also, S. L. Basin, "The Fibonacci Sequence as it Appears in Nature," Fibonacci Quarterly, Vol. 1, No. 1, p. 53.

[Continued from page 187.]

Editorial Note: The question remains how the students are to find the Fibonacci or Lucas representation for the first factor. To find the Fibonacci representation for 28, we subtract the largest Fibonacci number not exceeding 28, namely 21. This leaves 28 - 21 = 7; so our next choice is 5; 28 - 21 - 5 = 2, a Fibonacci number. Thus, 28 = 21 + 5 + 2. This will always yield the representation with the least number of summands.

REFERENCES

1. V. E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton-Mifflin Company, Boston (1969), pp. 69-72.
2. John L. Brown, Jr., "Note on Complete Sequence of Integers," American Mathematical Monthly, Vol. 67 (1960), pp. 557-560.

A NUMBER PROBLEM

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The last digit of one number is 6. Take away this digit and put it in front of a given number.

Thus a new number can be formed which will be 6 times bigger than the original one.

What smallest of all possible numbers satisfies this condition?

Simple Solution

$$
\begin{array}{r}
6 \\
36 \\
216 \\
1296 \\
7776 \\
46656 \\
\cdots\cdots \\
\hline
1016949 \cdots\cdots 677966
\end{array}
$$

a period with 58 digits

The number to be found is:

1016949152542372881355932203389830508474576271186440677966 .

Find this number in another way using the Fibonacci terms.

Remark. 2, 8, 34, 144, ⋯, etc. are third, sixth, ninth, twelfth, ⋯ (general: F_{3n}, where $n = 1, 2, 3, 4, \cdots$), etc. terms of Fibonacci sequence 1, 1, 2, 3, 5, ⋯ .

The solution by using the Fibonacci numbers is given on the following page.

[Continued on page 198.]

NUMBERS THAT ARE BOTH TRIANGULAR AND SQUARE THEIR TRIANGULAR ROOTS AND SQUARE ROOTS

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There is an infinite series of numbers, N, which for integral T and S:

(1) $$\tfrac{1}{2}T(T+1) = N = S^2 .$$

The first nine members of the series are tabulated below, together with their triangular roots, square roots, and index numbers, n.

n	T	N	S
0	0	0	0
1	1	1	1
2	8	36	6
3	49	1225	35
4	288	41616	204
5	1681	1413721	1189
6	9800	48024900	6930
7	57121	1631432881	40391
8	332928	55420693056	235416

By inspection of the tabulation, we note the recursive formula for N:

(2) $$N_n = 34\,N_{n-1} - N_{n-2} + 2 ,$$

from which we can develop a generalized formula for N:

(3) $$N_n = \frac{1}{32}\left[(17 + 12\sqrt{2})^n + (17 - 12\sqrt{2})^n - 2\right] .$$

Similarly,

(4) $$T_n = 7T_{n-1} - 7T_{n-2} + T_{n-3} ,$$

and

(5) $$T_n = \frac{1}{4}[(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n - 2].^*$$

Also:

(6) $$S_n = 6S_{n-1} - S_{n-2} ,$$

and

$$S_n = \frac{1}{8}\sqrt{2}\,[(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n] .$$

Other recursive formulas and relations were found by inspection of the tabulation:

(7) $$S_{2n} = S_n(S_{n+1} - S_{n-1})$$

(8) $$T_{2n-1} = (T_n - T_{n-1})^2$$

(9) $$S_{2n-1} = N_n - N_{n-1}$$

(10) $$T_{2n} = 8N_n$$

(11) $$T_n - T_{n-1} = S_n + S_{n-1}$$

(12) $$T_{2n-1} = (S_n + S_{n-1})^2$$

(13) $$S_{2n-1} = (S_n - S_{n-1})(T_n - T_{n-1})$$

(14) $$N_n - N_{n-1} = (S_n - S_{n-1})(T_n - T_{n-1})$$

*This simplification of the author's more complicated formula was furnished by Hoggatt

(15) $$T_{2n} = 8S_n^2$$

(16) $$S_{2n-1} = (S_n - S_{n-1})T_{2n-1}^{\frac{1}{2}}$$

(17) $$N_n - N_{n-1} = (S_n - S_{n-1})(S_n + S_{n-1}) \, .$$

By the use of the recursive formulas, the tabulation was extrapolated for negative index numbers. It was found to be perfectly reflexive about 0 except that the values of S became negative for negative index numbers, while the values of N and T remained positive. All generalized formulas and recursive formulas and relations held for the reflected series.

[Continued from page 195.]

Solution by Using the Fibonacci Terms

$$\begin{array}{l}
2 \\
\;8 \\
\;34 \\
\;144 \\
\;\;610 \\
\;\;2584 \\
\;\;10946 \\
\;\;\;46368 \\
\;\;\;196418 \\
\;\;\;\;832040 \\
\;\;\;\;\cdots\cdots \\
\hline
3389\cdots\cdots
\end{array}$$

$$3 \times 3389\cdots = 1016949\cdots \, .$$

Edited by
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ASYMPTOTIC EUCLIDEAN TYPE CONSTRUCTIONS WITHOUT EUCLIDEAN TOOLS [1]

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INTRODUCTION

> "..., Gauss made the remarkable discovery that those, and only those, regular polygons having a prime number of sides p can be constructed with straight edge and compasses if and only if p is of the form $2^{2^n} + 1$. Now the ancient Greeks had found how to construct with straight edge and compasses regular polygons of 3, 4, 5, 6, 8, 10 and 15 sides. If in the formula $p = 2^{2^n} + 1$ we set $n = 0$ and 1, we obtain the primes 3 and 5 respectively — cases already known to the Greeks. For $n = 2$, we find $p = 17$, which is a prime number. Therefore Gauss showed that a regular polygon of 17 sides is constructible with straight edge and compasses, which was unknown to the Greeks. Gauss was vastly proud of this discovery, and he said that it induced him to choose mathematics instead of philology as his life work." [2]

This quote from Howard W. Eves' recent two-volume set, In Mathematical Circles, suggests that the construction of regular polygons having a prime number of sides is not easy, even when possible, with a straight edge and compass. Note that Gauss showed it is impossible to construct with a ruler and compass the regular seven-sided polygon. Furthermore, one method for showing that a general angle θ cannot be trisected with Euclidean tools involves showing that it is impossible to trisect the angle whose

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[2]Howard W. Eves, In Mathematical Circles, Quadrants III and IV, Prindle, Weber and Schmidt, Inc., Boston, 1969, p. 113.

measure is $\pi/3$ — hence, the nine-sided regular polygon is not constructible with a ruler and compass either.*

The first part of this article deals with a way to approximate, by folding a paper strip, any regular polygon whose number of sides is of the form $2^n \pm 1$, for some natural number n. Note that when $n = 3$, the expression $2^n \pm 1$ yields 7 and 9.

A modification of the iterative folding sequences used on paper strips is presented. It suggests a method for approximating an angle having measure $\theta/(2^n + 1)$, where n is any natural number and θ is any given angle whose measure is between 0 and π Particularly interesting is the case when $n = 1$, which produces a trisection approximation process.

Finally, as an illustration, instructions are given describing how paper strips may be used to construct models of regular convex dodecahedra. The constructions suggest, as will be seen, that a "parallel strip" classification of certain polyhedra might provide an interesting point of view from which to study their properties.

FOLDING SEQUENCES INVOLVING ONE ITERATIVE EQUATION

As an elementary example, take a roll of ordinary adding machine tape and make a fold on any straight line, t_0, near the end of the tape so that t_0 crosses one of the parallel edges of the tape at a point, A_0. Fold again through A_0 to bisect one of the angles formed by t_0 and an edge of the tape. Do this so that the newly created transversal, t_1, goes towards the roll of paper. One endpoint of t_1 is A_0, the other endpoint is named "A_1." Now fold the tape through A_1, bisecting the obtuse angle created by t_1 and the edge of the tape. This fold yields yet another transversal, t_2, whose endpoints are A_1, A_2. To continue this folding process always bisect, by folding through A_n, the obtuse angle, having sides t_n and an edge of the tape; thereby obtaining a new transversal, t_{n+1}, having endpoints A_n, A_{n+1} (for $n = 1, 2, 3, \cdots$). The acute angle formed by t_n and an edge of the tape is denoted x_{n-1}.

*Howard W. Eves, An Introduction to the History of Mathematics, Rinehart and Company, Inc., New York, 1953, pp. 96-98, p. 107.

For the most accurate results, both in this case and all other examples which follow, fold the tape so that whenever transversals are formed, the tape remains folded on these creases and the next fold always occurs on the portion of the paper strip which comes from the top of the existing configuration. Thus, the triangles which are formed will either stack up or form a zig-zag type pattern in the folding plane, but the configuration formed will never need to be turned over during the folding process. One quickly discovers, however, that certain rotations of the configuration in the folding plane facilitates the folding process. Figure 1 illustrates one case of how the unfolded tape appears after the folding process has taken place.

When the above folding process is accurately carried out, an accordian-like stack of triangles results. And, it soon becomes visually apparent that successive triangles are getting more and more alike — consequently, the measure of x_n must approach $\pi/3$ as n gets large.

For skeptics, the proof can be ascertained. First, note that since the edges of the tape are parallel, the measures of successive acute angles always satisfy the equation

$$2x_n + x_{n-1} = \pi \quad ,$$

where $n = 1, 2, 3, \cdots$. Successive computations of x_1, x_2, x_3, etc., yields

$$x_n = \frac{\pi}{2}\,[1 + (-1/2)^1 + (-1/2)^2 + \cdots + (-1/2)^{n-1}] + (-1/2)^n x_0 \quad ,$$

which can be verified by mathematical induction. Then, using the formula for the sum of a geometric sequence, it follows that

$$x_n = \frac{\pi}{3}\,[1 - (-1/2)^n] + (-1/2)^n x_0 \ .$$

Consequently,

$$\lim_{n\to\infty} x_n = \pi/3 \ .$$

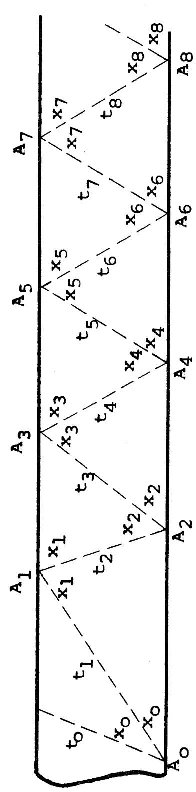

Figure 1

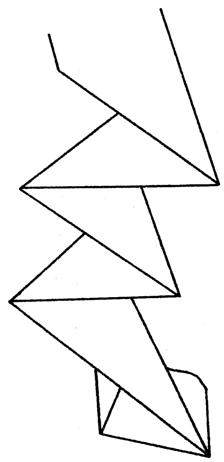

Figure 2(a)

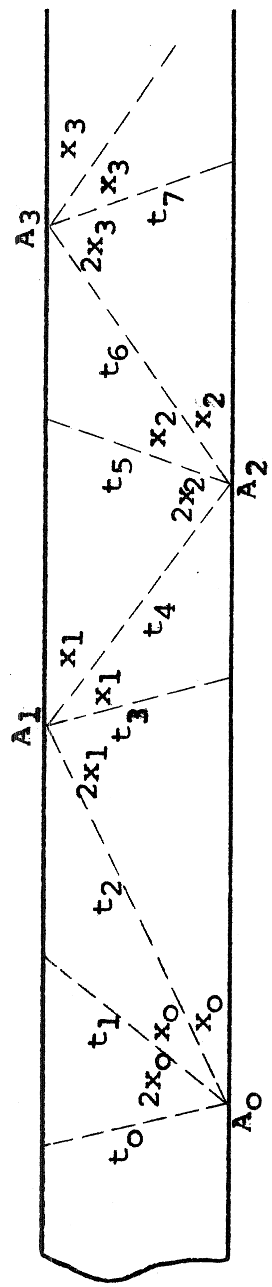

Figure 2(b)

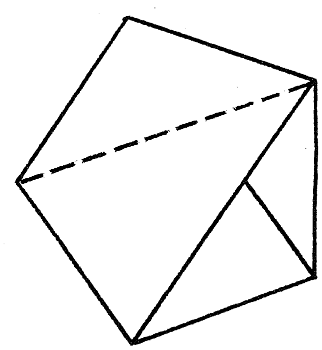

Figure 2(c)

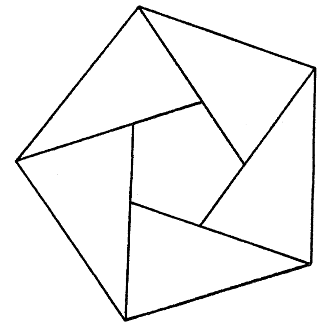

Figure 2(d)

Notice that the difference, in radians, between x_n (which is formed by the $(n + 2)^{nd}$ fold of the tape) and $\pi/3$ is

$$(-1/2)^n [x_0 - \pi/3] .$$

This means each new accurate fold on the tape produces an angle whose measure is twice as close to $\pi/3$ as its predecessor. In fact, the maximum value for the actual error (which occurs when x_0 approaches zero) indicates that one can always expect an approximation of $\pi/3$ with accuracy better than one minute after 14 folds. But, as one is not likely to choose x_0 close to zero, this degree of accuracy will occur, in most cases, when $n < 14$.

It turns out to be practical, in the paper tape construction of models, to have the following:

Visual Criterion. When the consecutive longest transversals formed on a tape by an iterative folding process, appear to be of the same length, then the tape is called usable.

Suppose the length of successive transversals obtained from some iterative folding process approaches some fixed value, in the limit sense. Then there must exist some number $k \neq 0$ such that consecutive acute angles formed by those transversals and an edge of the tape converge to an angle having measure π/k.

Definition. On a usable tape, whose successive smallest acute angles converge to π/k; when the portion not satisfying the Visual Criterion is cut off, the remaining tape is denoted "$T(\pi/k)$."

Accordingly, the usable tape produced in the above example is denoted "$T(\pi/3)$" and called a "pi thirds tape."

The method of obtaining $T(\pi/k)$ implies that there will always be some natural number, p, such that the transversals t_n, where $n < p$, will not appear on that tape. But, it is not necessary to identify p. Thus, in describing constructions, reference to a transversal t_n on $T(\pi/k)$ will mean any transversal on $T(\pi/k)$. However, once t_n has been identified for use in a particular construction, then t_{n+q} (where q is any natural number) will mean the q^{th} transversal following t_n.

Since $T(\pi/3)$ contains approximations of equilateral triangles, it may be used to construct models of hexagons and deltahedra. As an example, cut $T(\pi/3)$ on t_n and t_{n+10}, then fold the ten triangle strip on t_{n+3} and t_{n+6}. Now, because straight lines are easier to fold than to cut, the t_{n+10} end of the tape is wrapped around t_n when the tape is folded on t_{n+9} to complete the model of a hexagon. Note that the definitive edges do not include either of the cut edges t_n, t_{n+10}.

The above folding process generalizes in the following way.

Theorem 1. If

(1) n is some fixed natural number.

(2) A paper tape of width w is folded on some transversal, t_0, which crosses one of the parallel edges of the tape at A_0.

(3) One angle formed by t_0 and an edge of the tape is then divided into 2^n parts, by folding through A_0; creating, in order, the new set of transversals, $t_1, t_2, t_3, \cdots, t_n$, where $t_1 < t_2 < t_3 < \cdots < t_n$. The measure of the acute angle formed by t_n and the edge of the tape is denoted x_0. The endpoint of t_n which lies on the opposite edge of the tape from A_0 is called A_1.

(4) In general, folds are made so as to divide into 2^n parts the obtuse angle having vertex A_k and an interior with no transversals. The new transversals, $t_{kn+1}, t_{kn+2}, \cdots, t_{kn+n}$, are such that $t_{kn+1} < t_{kn+2} < \cdots < t_{kn+n}$. The endpoint of t_{kn+n}, called A_{k+1}, lies on the opposite edge of the tape from A_k (for $k = 1, 2, 3, \cdots$). The measure of the acute angle formed between t_{kn+n} and an edge of the tape is denoted x_k.

Then $\lim_{k \to \infty} x_k = \pi/(2^n + 1)$ and consequently, this folding process produces $T(\pi/(2^n + 1))$.

Proof. From the description of the folding process, it follows that the measures of successive acute angles satisfy the equation

$$2^n x_k + x_{k-1} = \pi \quad , \tag{1}$$

where $k = 1, 2, 3, \cdots$. Then, using mathematical induction, it can be shown that

$$x_k = \frac{\pi}{2^n + 1}\left[1 - (-1/2^n)^k\right] + (-1/2^n)^k x_0 ,$$

for $k = 1, 2, 3, \cdots$. But, since $\left|-1/2^n\right| < 1$, it follows immediately that

$$\lim_{k\to\infty} x = \pi/(2^n + 1) .$$

The theorem is surprisingly fruitful. For example, Figure 2(a) illustrates how the folded tape appears just after the folding process has taken place with $n = 2$. Figure 2(b) shows how this same tape appears when it is unfolded. This folding process produces the usable tape, $T(\pi/5)$. If $T(\pi/5)$ is cut on t_{2n} and t_{2n+6}, and folded on t_{2n+1}, t_{2n+3}, t_{2n+5}, a model of the regular pentagon shown in Figure 2(c) is formed. The sides of this pentagon approximate $w/\sin(2\pi/5)$. But, a regular pentagon whose sides approximate $w/\sin(\pi/5)$ may also be formed from $T(\pi/5)$. To see this, cut $T(\pi/5)$ on t_{2n+1} and t_{2n+13}; then fold in a winding fashion on the transversals t_{2n+2}, t_{2n+4}, t_{2n+6}, t_{2n+8}, t_{2n+10}, t_{2n+12}. The result, a model of a regular pentagon with a pentagonal hole in the center, is shown in Figure 2(d).

As another example, consider the results of the theorem when $n = 3$. Figure 3(a) shows how the beginning of the tape which produces $T(\pi/9)$ might appear. Once $T(\pi/9)$ has been obtained, it may be used to construct models of regular 9-gons whose sides approximate either $w/\sin(\pi/9)$, $w/\sin(2\pi/9)$, or $w/\sin(4\pi/9)$. This is done by folding $T(\pi/9)$ on consecutive transversals whose labels are equal to $0 \pmod 3$, $2 \pmod 3$, and $1 \pmod 3$, respectively. Figure 3(b) illustrates the regular 9-gon which is formed by folding on t_{3n+1}, t_{3n+4}, t_{3n+7}, $\cdots$, t_{3n+28}; and whose sides approximate $w/\sin(4\pi/9)$.

In general, $T(\pi/(2^n + 1))$ will produce models of n non-congruent regular $(2^n + 1)$-gons whose sides approximate $w/\sin(2^k\pi/(2^n + 1))$, $k = 1, 2, \cdots, (n - 1)$. The actual construction involves folding $T(\pi/(2^n + 1))$ on successive transversals whose labels are equal to $0 \pmod n$, $(n-1) \pmod n$, $(n - 2) \pmod n$, $\cdots$, $1 \pmod n$, respectively.

A BONUS

Suppose the folding process described in the theorem takes place on a piece of paper whose straight edges are <u>not</u> parallel. Thus, suppose angle

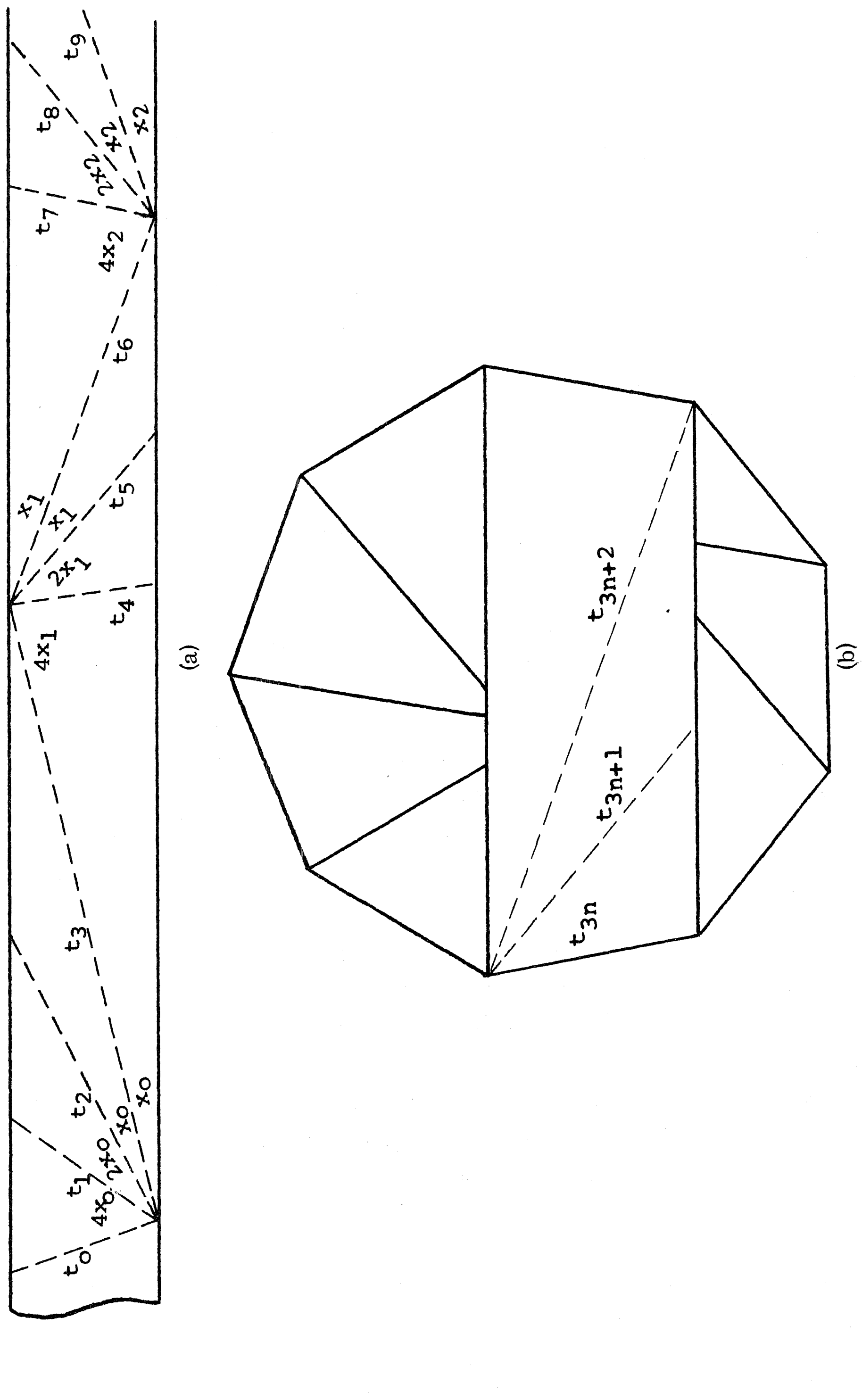

Figure 3

ABC, having measure θ (between 0 and π), and supplementary angle ABD, occur so that DBC lies on the edge of a piece of paper. Then the paper is cut along the line AB (see Figure 4). A point, A_0, is selected between D and B and the paper is folded, through A_0 on some line, t_0, which is not parallel to AB. The transversals t_k (where $k = 1, 2, \cdots$) are formed by folding so that t_1 bisects the angle formed by t_0 and A_0B, determining a point, A_1, on the line containing AB. And, in general, t_k bisects the angle $A_{k-2}A_{k-1}B$, determining a point A_k on the line containing $A_{k-2}B$ (when $k \geq 2$). The measure of the angle A_1A_0B is denoted x_0 and half the measure of angle $A_{k-2}A_{k-1}B$ is denoted x_{k-1} for $k \geq 2$.

Then, since the sum of the measures of the interior angles in any triangle is always equal to π it follows that

$$2x_k + x_{k-1} + (\pi - \theta) = \pi \quad ,$$

when $k = 1, 2, 3, \cdots$. Thus,

$$2x_k + x_{k-1} = \theta \,, \tag{2}$$

when $k = 1, 2, 3, \cdots$.

But this is similar to Equation 1, where $n = 1$. In fact, a review of the proof for Theorem 1 reveals that it would not have been any more difficult if "π" were replaced with "θ" and that Equation 2 would lead to the result

$$\lim_{k \to \infty} x_k = \theta/3 \,.$$

Thus the method illustrated in Figure 4 really represents a trisection approximation method for angles whose measure is between 0 and π. As a practical matter it is not, in this case, possible to fold accurately indefinitely, as was the case with parallel lines. Nevertheless, the method is effective — especially when judicious choices of A_0 and t_0 are made — i.e., choose A_0 as far away from B as the paper will allow and make a visual guess when folding t_0 so that when x_0 is formed, it will be as close to $\theta/3$ as possible.

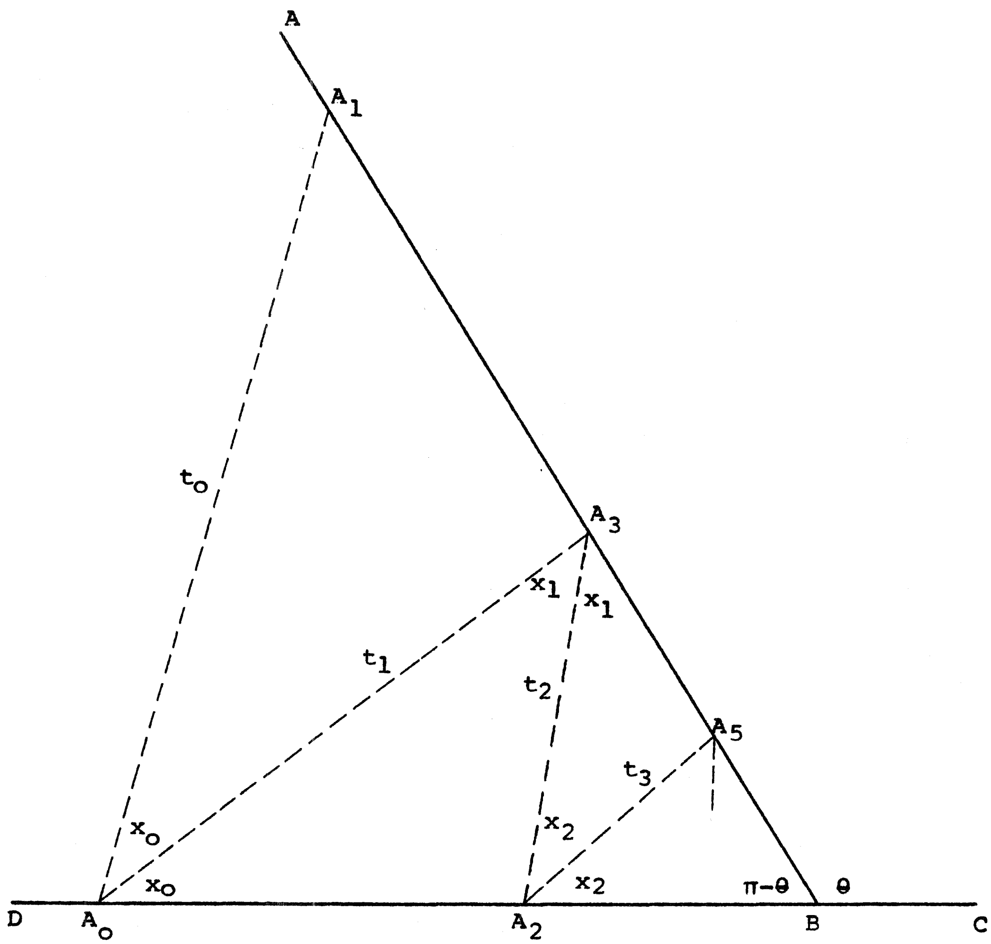

Figure 4

OTHER FOLDING SEQUENCES

The folding sequences considered thus far have involved just one iterative equation. But, as the next theorem shows, other folding sequences do exist.

Theorem 2. If

(1) n is some fixed natural number greater than 1.

(2) A paper tape, of width w, is folded on a transversal, t_0, which crosses an edge of the tape at some point, A_0.

(3) The angle formed by t_0 and the edge of the tape having vertex A_0 is divided, by folding, into 2^n parts producing transversals t_1,

$t_2, t_3, \cdots, t_n$ so that $t_1 < t_2 < t_3 < \cdots < t_n$; and t_n has endpoints A_0, A_1. The measure of the acute angle which t_n makes with the edge of the tape is denoted x_0.

(4) The obtuse angle at A_1 is bisected, creating a new transversal t_{n+1}. It has endpoints A_1, A_2 and forms an acute angle with an edge of the tape, denoted x_1.

(5) In general, either (i) the obtuse angle at A_k is divided into 2^n parts, when k is even, so that each new transversal is longer than its predecessor and the last transversal folded creates the point A_{k+1} on the opposite edge of the tape from A_k; or (ii) the obtuse angle at A_k is bisected, when k is odd. In either case, the measure of the acute angle between the transversal joining A_k, A_{k+1} and an edge of the tape is denoted x_k.

Then $\lim_{k \to \infty} x_{2k} = \pi/(2^{n+1} - 1)$ and, consequently, the folding sequence produces $T(\pi/(2^{n+1} - 1))$.

Proof. By the description of the folding process, it follows that the measures of consecutive acute angles satisfy

$$\left.\begin{aligned} 2x_{2k-1} + x_{2k-2} &= \pi \\ 2^n x_{2k} + x_{2k-1} &= \pi \end{aligned}\right\} \text{ for } k = 1, 2, 3, \cdots . \tag{3}$$

Solving for x_{2k-1} in the first iterative equation, then for x_{2k} in the second yields

$$x_{2k} = (\pi + x_{2k-2})/2^{n+1} .$$

It can then be shown, by mathematical induction, that

$$x_{2k} = \frac{\pi}{2^{n+1} - 1} \left[1 - (1/2^{n+1})^k\right] + (1/2^{n+1})^k x_0 ,$$

for $k = 1, 2, 3, \cdots$. Thus

$$\lim_{k \to \infty} x_{2k} = \pi/(2^{n+1} - 1) .$$

In general, if $T(\pi/(2^{n+1} - 1))$ is folded on all t_{k-1}, t_{k-2}, where $k = 0 \pmod{n + 1}$, a regular $(2^{n+1} - 1)$-gon will be formed.

As an example, suppose $n = 2$ in Theorem 2. Figure 5(a) illustrates how the beginning of this tape, which produces $T(\pi/7)$, might appear.

If $T(\pi/7)$ is folded, in a winding fashion, on all t_{k-1}, t_{k-2}, where $k = 0 \pmod 3$, the model formed is a regular seven-sided polygon (Figure 5(b)), whose sides approximate $w/\sin(\pi/7)$.

Likewise, if $T(\pi/7)$ is folded on all t_k where $k \neq 1 \pmod 3$, the result is a seven-sided polygon whose sides approximate $w/\sin(3\pi/7)$. If this is done so that the folds on t_k, when $k = 0 \pmod 3$, wrap the tape around the polygon being formed; then the result appears as shown in Figure 5(c).

Note, however, that as illustrated in Figure 5(d), if $T(\pi/7)$ is folded on all t_k where $k \neq 2 \pmod 3$, a regular seven-sided <u>star</u> polygon is formed whose sides approximate $w/\sin(\pi/7)$. It can be shown that the shortest distance between consecutive vertices approximates $w/\sin(2\pi/7)$.

CONSTRUCTING DODECAHEDRA WITH $T(\pi/5)$

When cash register tape (which is more porous than adding machine tape) is used with white glue, surprisingly sturdy models of polyhedra may be made.

To construct a dodecahedron, for example, fold the cash register tape to obtain $T(\pi/5)$ containing at least 90 usable triangles. Cut $T(\pi/5)$ on t_{2n} and t_{2n+56}, then fold the resulting strip, glueing the overlapping portions in position as shown in Figure 6. Label the edges of the pentagons as shown. The polyhedron is completed by first forming a ring-like figure and glueing one of the shaded parallelograms on top of the other. Then join the remaining 18 pairs of edges so that edges labeled with like numbers correspond with each other. Tabs for joining the edges may be conveniently obtained by cutting on nineteen successive long transversals of $T(\pi/5)$.

If the tabs are labeled so that when they are glued in place it preserves the numbers shown on each of the edges, it is then possible, upon completion of the dodecahedron, to observe that

<u>The dodecahedron, formed from $T(\pi/5)$ of width w, and whose edge approximates $w/\sin(2\pi/5)$ may be constructed with no fewer than six</u>

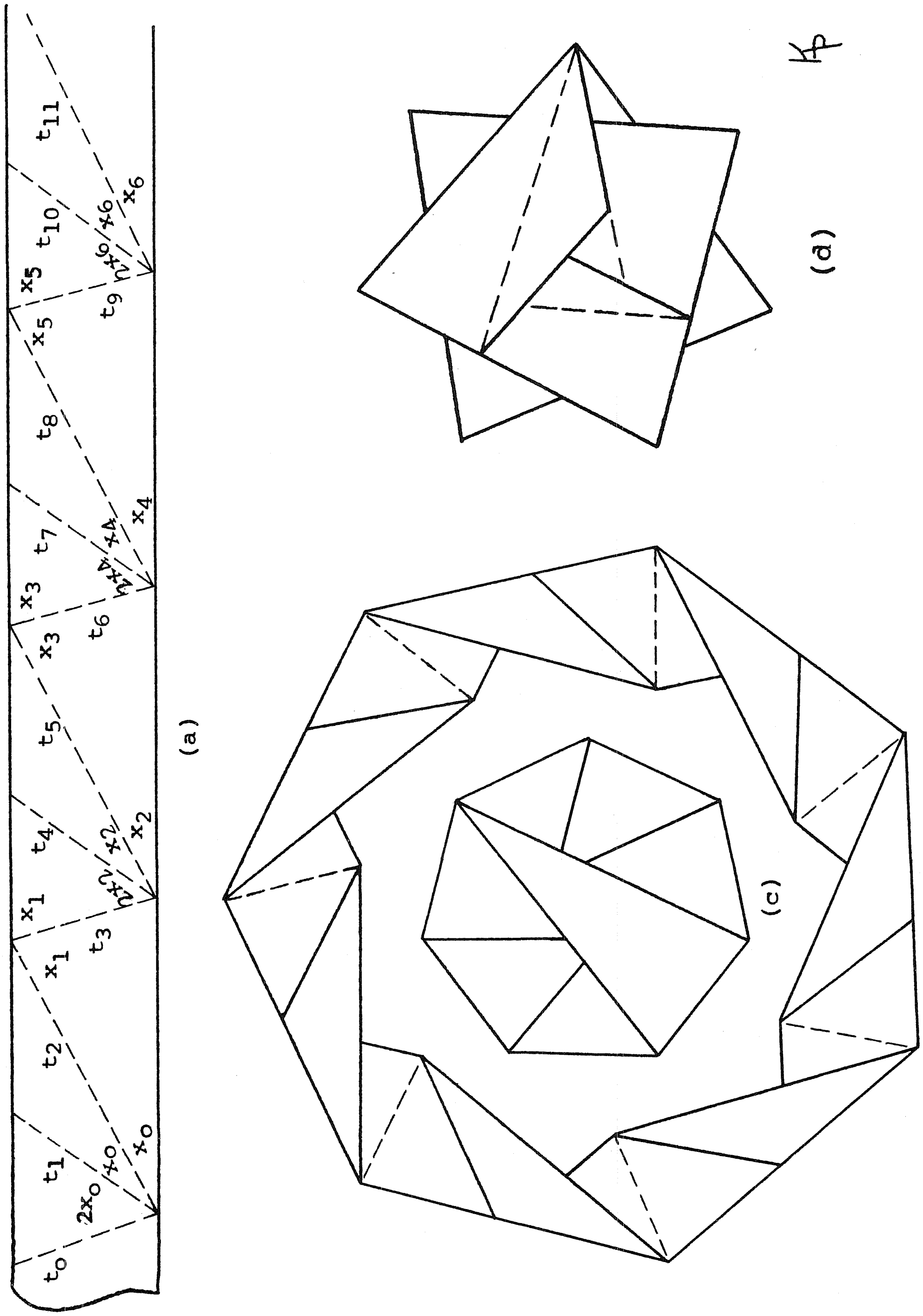

Figure 5

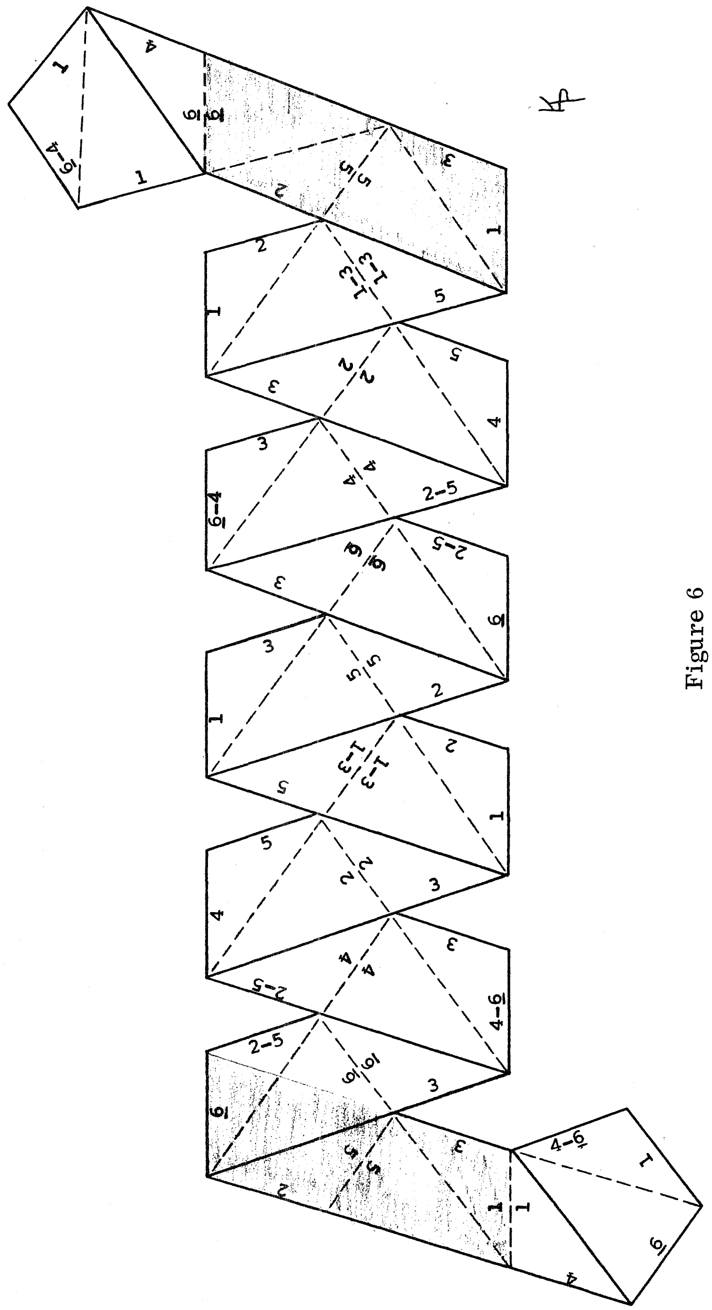

Figure 6

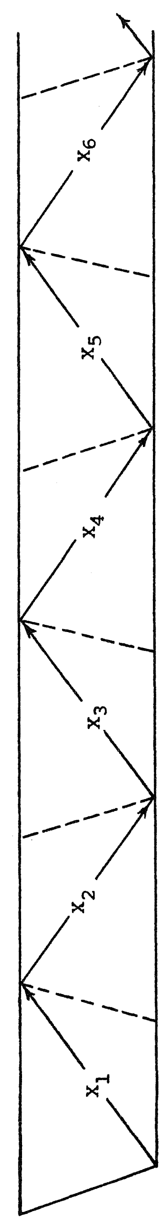

Figure 7

bands, each of which contains 12 consecutive triangles from $T(\pi/5)$. (In practice, an extra triangle would be required on some bands — but, since it serves only as a tab, it is not counted.)

To see that this is true, take a strip of $T(\pi/5)$ which contains 12 triangles and observe that it is possible to position it on the completed dodecahedron so that its short transversals all coincide with edges whose label includes the symbol "1." But, it may also be positioned in five other ways so that its short transversals coincide with the edges each of whose labels include the symbols "2," "3," "4," "5," and "6," respectively. Because the label on every edge contains at least one number, six bands are sufficient for this particular construction of the dodecahedron. Note that if any one number were removed from the labels on this dodecahedron, there would be some edges with no label. Therefore, at least six bands are necessary for the construction of this dodecahedron.

This model may be used to show that if a dodecahedron were constructed from six bands, each containing 12 consecutive triangles from $T(\pi/5)$, there would be six edges crossed by exactly two bands and those edges would be oriented so that (a) their midpoints are the vertices of an inscribed octahedron; (b) the collection of pentagonal diagonals parallel to those six edges form the edges of an inscribed cube; and, since alternative vertices of a cube define vertices of a tetrahedron, (c) the vertices of two distinct inscribed tetrahedra may be identified on this model.

A second, somewhat different, dodecahedron may be constructed using $T(\pi/5)$. This model is particularly easy to make from gummed tape. Cash register tape and white glue produce a better looking model but, having one side gummed makes the description of the construction easier. Accordingly, the following instructions are given for gummed tape.

First, cut from $T(\pi/5)$ six strips of 22 triangles each. The first portion of a typical strip is shown, with the gummed side down, in Figure 7. Label the ungummed side of each of the strips by replacing the letter "X" shown in Figure 7 with the letters "A," "B," "C," "D," "E," "F," on the first, second, third, fourth, fifth, and sixth strips, respectively. As an example, the first strip, called "strip A," will have its eleven long transversals labeled "A_1," "A_2," $\cdots$, "A_{11}," consecutively, and all transversals will be labeled with an arrow which points to the endpoint of the next long transversal.

The following notational device is convenient: If X, Y, Z represent members of $\{A, B, C, D, E, F\}$ and if m, n, k are natural numbers, then "$X_m \rightarrow Y_n$" means that: the gummed side of strip X is glued onto the ungummed side of strip Y so that the transversal marked "$-X_m\rightarrow$" coincides with the transversal marked "$-Y_n\rightarrow$" and the arrows point in the <u>same</u> direction.

"$X_m \leftrightarrow Y_n$" means that: the gummed side of strip X is glued onto the ungummed side of strip Y so that the transversal marked "$-X_m\rightarrow$" coincides with the transversal marked "$-Y_n\rightarrow$" and the arrows point in <u>opposite</u> directions.

"$X_m \rightarrow Y_n \rightarrow Z_k$" means that: $X_m \rightarrow Y_n$ and $Y_n \rightarrow Z_k$.

Using this notational device, the dodecahedron is assembled as follows:

I.	$E_7 \rightarrow A_6$	II.	$A_5 \leftrightarrow D_8$
	$D_7 \rightarrow E_6$		$B_5 \leftrightarrow E_8$
	$C_7 \rightarrow D_6$		$C_5 \leftrightarrow A_8$
	$B_7 \rightarrow C_6$		$D_5 \leftrightarrow B_8$
	$A_7 \rightarrow B_6$		$E_5 \leftrightarrow C_8$

III. The F strip may now be woven in and out so that

$$D_9 \rightarrow F_2$$
$$F_3 \rightarrow B_4$$
$$E_9 \rightarrow F_4$$
$$F_5 \rightarrow C_4$$
$$A_9 \rightarrow F_6$$
$$F_7 \rightarrow D_4$$
$$B_9 \rightarrow F_8$$
$$F_9 \rightarrow E_4$$
$$C_9 \rightarrow F_{10}$$
$$F_{11} \rightarrow F_1 \rightarrow A_4$$

IV.	$A_3 \longleftrightarrow C_{10}$	V.	$A_{11} \longrightarrow A_1 \longrightarrow E_2$
	$B_3 \longleftrightarrow D_{10}$		$B_{11} \longrightarrow B_1 \longrightarrow A_2$
	$C_3 \longleftrightarrow E_{10}$		$C_{11} \longrightarrow C_1 \longrightarrow B_2$
	$D_3 \longleftrightarrow A_{10}$		$D_{11} \longrightarrow D_1 \longrightarrow C_2$
	$E_3 \longleftrightarrow B_{10}$		$E_{11} \longrightarrow E_1 \longrightarrow D_2$

This dodecahedron is also formed from exactly six bands, but each band contains 20 triangles (not counting the overlapping tabs) from $T(\pi/5)$. Comparing the two completed polyhedra, one will note many similarities and differences. The first and most obvious difference is that the one has some holes in it and that it appears to be "woven together." A most effective model of the second dodecahedron may be made if six different colored strips are used in its construction. In fact, it is not even necessary to use glue, for one can hold the various strips together as indicated by the instructions, with 30 paper clips. Then, when the dodecahedron is finished, all of the paper clips, except those six which hold three thicknesses of tape together, may be removed.

If the places where bands overlap themselves are discounted, all of the edges of the second dodecahedron are crossed by exactly two bands. If one imagines the arrows on this dodecahedron to be roads on which travel is permitted only in the direction of the arrows, it can be seen that, if one leaves the pentagonal cycle $A_{11}B_{11}C_{11}D_{11}E_{11}$, all roads lead to the cycle

$$D_9 F_3 E_9 F_5 A_9 F_7 B_9 F_9 C_9 F_{11}$$

and, leaving that cycle, all roads lead to the cycle $A_7B_7C_7D_7E_7$, from which there is no escape.

Many other polyhedra may be constructed with paper strips. If the reader wishes to try devising some paper tape constructions for other polyhedra, the following references may be useful.

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[Continued from page 135.]

SPECIAL ADVANCED PROBLEM

H-182S Proposed by Paul Erdös, University of Colorado, Boulder, Colorado.

Prove that there is a sequence of integers $n_1 < n_2 < \cdots$ satisfying

$$\frac{\sigma(n_k)}{n_k} \to \infty \quad \text{and} \quad \frac{\sigma(\sigma(n_k))}{\sigma(n_k)} \to 1 ,$$

where

$$(n) = \sum_{d|n} d$$

(the sum of the integer divisors of n.)

[From Conference on NUMBER THEORY, March 24-27, Washington State University, Pullman, Washington.]

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-208 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n.$$

Prove both of the following and generalize:

(a) $$F^2_{n+2} = 3F^2_{n+1} - F^2_n - 2(-1)^n$$

(b) $$L^2_{n+2} = 3L^2_{n+1} - L^2_n + 10(-1)^n \ .$$

B-209 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Do the analogue of B-208 for the Pell sequence defined by

$$P_0 = 0, \quad P_1 = 1, \quad P_{n+2} = 2P_{n+1} + P_n, \quad \text{and} \quad Q_n = P_n + P_{n-1} \, .$$

B-210 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Prove that $S > 803/240$, where

$$S = \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \cdots \quad .$$

B-211 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let F_n be the n^{th} term in the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$. Solve the recurrence

$$D_{n+1} = 2D_n + F_{2n+1}$$

subject to the initial conditions $D_1 = 1$ and $D_2 = 3$.

B-212 Proposed by Tomas Djerverson, Albrook College, Tigertown on the Rio.

Give examples of interesting functions f and g such that

$$f(m,n) = g(m + n) - g(m) - g(n) .$$

(One example is $f(m,n) = mn$ and $g(n) = \binom{n}{2} = n(n - 1)/2$.)

B-213 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Given n points on a straight line, find the number of subsets (including the empty set) of the n points in which consecutive points are not allowed. Also find the corresponding number when the points are on a circle.

SOLUTIONS

A SIXTY-ORDER FIBONACCI-LUCAS IDENTITY

B-190 A repeat of B-186 with a typographical error corrected.

Let L_n be the n^{th} Lucas number and show that

$$L_{5n}/L_n = [L_{2n} - 3(-1)^n]^2 + (-1)^n 25F_n^2 \quad .$$

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Since L_{5n} and

$$\{[L_{2n} - 3(-1)^n]^2 + (-1)^n 25 F_n^2\} L_n$$

satisfy the same sixth-order linear homogeneous recurrence, the result is proved by verifying it for $n = -2, -1, 0, 1, 2,$ and 3 (and then relying on mathematical induction).

Also solved by W. C. Barley, Wray G. Brady, Warren Cheves, Herta T. Freitag, Jo Carol Gordon, John A. Hitchcock, Edgar Karst, Bob Topley, Andrew Wyatt, Rev. Robert Zuparko, and the Proposer.

THE HUNTER UNVEILED

B-191 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.

In this alphametic, each letter represents a particular but different digit, all ten digits being represented here. It must only be that well-known mathematical teaser from Toronto, J. A. H. Hunter, but what is the value of HUNTER?

```
    MR
HUNTER
  MADE
     A
------
TEASER
```

Solution by David Zeitlin, Minneapolis, Minnesota.

The value of HUNTER is 198207, where the unique solution is given by

```
    57
198207
  5340
     3
------
203607
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We note that

(1) $$2R + E + A = 10C_1 + R,$$

(2) $$C_1 + M + E + D = 10C_2 + E,$$

(3) $$C_2 + T + A = 10C_3 + S,$$

(4) $$C_3 + N + M = 10C_4 + A,$$

(5) $$C_4 + U = 10C_5 + E,$$

and

(6) $$C_5 + H = T,$$

where C_i, $i = 1, 2, \cdots, 5$, are carry-overs from the i^{th} column. Since $C_5 = 1$, we find from (5) that $C_4 = 1$, with $U = 9$ and $E = 0$; and thus, from (1), that $C_1 = 1$. From (2), $C_2 = 1$, and from (3), $C_3 = 0$ or 1. All cases for $C_3 = 1$ are non-solutions. For $C_3 = 0$, the single solution is obtained when

$$(A,D,E,H,M,N,R,S,T,U) = (3,4,0,1,5,8,7,6,2,9).$$

Also solved by W. C. Barley, Wray G. Brady, Albert Gommel, Jo Carol Gordon, J. A. H. Hunter, Edgar Karst, John W. Milsom, C. B. A. Peck, Darla Perry, Azriel Rosenfeld, and the Proposer.

A FOURTH-ORDER F-L IDENTITY

B-192 Proposed by Warren Cheves, Littleton, North Carolina.

Prove that $F_{3n} = L_nF_{2n} - (-1)^nF_n$.

Solution by Herta T. Freitag, Hollins, Virginia.

One needs to show that

$$\alpha^{3n} - \beta^{3n} = (\alpha^n + \beta^n)(\alpha^{2n} - \beta^{2n}) - (-1)^n(\alpha^n - \beta^n),$$

where α and β are $(1 \pm \sqrt{5})/2$.

This, however, is immediately seen by using the relationship $\alpha\beta = -1$, and simplifying.

Also solved by W. C. Barley, Wray G. Brady, Mike Franusich, Jo Carol Gordon, John A. Hitchcock, Stu Hobbs, Edgar Karst, John Kegel, Scott King, John W. Milsom, C. B. A. Peck, Darla Perry, Patricia Shay, Don C. Stevens, Bob Tepley, Andrew Wyatt, David Zeitlin, Rev. Robert Zuparko, and the Proposer.

ANOTHER F-L IDENTITY

B-193 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that $L_{n+p} \pm L_{n-p}$ is $5F_pF_n$ or L_pL_n, depending on the choice of sign on whether p is even or odd.

Solution by John Kegel, Fort Lauderdale, Florida.

It is a well-known fact that

(1) $$F_n = \frac{a^n - b^n}{a - b}$$

and

(2) $$L_n = a^n + b^n ,$$

where

$$a = \frac{1}{2}(1 + \sqrt{5})$$

and

$$b = \frac{1}{2}(1 - \sqrt{5}) ,$$

which also gives

(3) $$ab = -1$$

and

(3') $$(a - b)^2 = 5 .$$

Now

$$\begin{aligned} L_p L_n &= (a^p + b^p)(a^n + b^n) \\ &= a^{n+p} + b^{n+p} + a^p b^n + b^p a^n \\ &= L_{n+p} + (ab)^p (a^{n-p} + b^{n-p}) ; \end{aligned}$$

Thus

(4) $$L_p L_n = L_{n+p} + (-1)^p L_{n-p} .$$

Likewise,

$$\begin{aligned} 5 F_p F_n &= 5\left(\frac{a^p - b^p}{a - b}\right)\left(\frac{a^n - b^n}{a - b}\right) \\ &= \frac{5}{(a - b)^2} (a^{n+p} + b^{n+p} - a^p b^n - a^n b^p) \\ &= \frac{5}{5} [L_{n+p} - (a^p b^p)(a^{n-p} + b^{n-p})] ; \end{aligned}$$

Thus

(5) $$5 F_p F_n = L_{n+p} - (-1)^p L_{n-p} ,$$

Hence (4) and (5) give

(6) $$L_{n+p} \pm L_{n-p} = 5 F_p F_n \text{ or } L_p L_n \quad (p \text{ odd or even})$$

and the proof is complete.

Also solved by W. C. Barley, Wray G. Brady, Herta T. Freitag, Jo Carol Gordon, David Zeitlin, and the Proposer.

SECOND ORDER IN n, FIFTH IN k

B-194 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico. (Corrected Statement).

Show that

$$L_{n+4k} - L_n = 5F_k[F_{n+3k} + (-1)^k F_{n+k}] .$$

Solution by C. B. A. Peck, State College, Pennsylvania.

From results of Brother Alfred (Fibonacci Quarterly, Vol. 1, No. 4, p. 55), or H. H. Ferns (Fibonacci Quarterly, Vol. VII, No. 1, p. 1), we have

$$5F_u F_v = [L_{u+v} - (-1)^v L_{u-v}] .$$

Replace v by k and u successively by $n + 3k$ and $n + k$. Multiply the second of these identities by $(-1)^k$ and add to the first; this gives the (corrected) desired result.

Also solved and corrected by Wray G. Brady, Herta T. Freitag, John Kegel, David Zeitlin, and the Proposer. The error in the statement was also noted by W. C. Barley.

GENERALIZED FIBONOMIALS

B-195 Proposed by David Zeitlin, Minneapolis, Minnesota.

Let $\left\{ {n \atop r} \right\}$ denote $L_n L_{n-1} \cdots L_{n-r+1}/L_1 L_2 \cdots L_r$. Show that

$$L_n^3/6 = \left\{ {n+2 \atop 3} \right\} - 2\left\{ {n+1 \atop 3} \right\} - \left\{ {n \atop 3} \right\} .$$

Solution by A. K. Gupta, University of Arizona, Tuscon, Arizona.

From formula (2) (on p. 447 of Fibonacci Quarterly, Vol. 8, No. 4) of the Proposer's solution to B-176, we have

$$\text{(A)} \qquad 2H^3 = H_1H_2H_3\left\{\begin{bmatrix} n+2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} n+1 \\ 3 \end{bmatrix} + \begin{bmatrix} n \\ 3 \end{bmatrix}\right\},$$

where H_n satisfies

$$H_{n+2} = H_{n+1} + H_n$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} = H_n H_{n-1} \cdots H_{n-r+1} / H_1 H_2 \cdots H_r .$$

The desired result is obtained from (A) for $H_n + L$

$$H_n + L_n, \quad \begin{Bmatrix} n \\ r \end{Bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix},$$

and

$$H_1H_2H_3 = L_1L_2L_3 = 2\cdot 1\cdot 3 = 6 \quad .$$

Also solved by W. C. Barley, Wray G. Brady, Warren Cheves, Herta T. Freitag, John Kegel, John W. Milsom, and the Proposer.