EQUAL PRODUCTS OF GENERALIZED BINOMIAL COEFFICIENTS

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In a letter dated 24 March 1970, Professor V. E. Hoggatt, Jr., has communicated to me the following interesting result: "Choose a binomial coefficient $\binom{n}{k}$ inside Pascal's triangle. There are six bordering terms of Pascal's triangle surrounding $\binom{n}{k}$. The product of all six is a perfect square." As he notes, the theorem is also true for the generalized binomial coefficients $\left\{{n \atop k}\right\}$ discussed in [1]. In a later communication (22 April 1970), Hoggatt has noted that a corresponding extension to multinomial coefficients holds true. (See [2].)

We may arrange the six binomial coefficients as follows:

$$(1)\qquad \begin{array}{ccccc} & \left\{{n-1 \atop k-1}\right\} & & \left\{{n-1 \atop k}\right\} & \\ \left\{{n \atop k-1}\right\} & & \left\{{n \atop k}\right\} & & \left\{{n \atop k+1}\right\} \\ & \left\{{n+1 \atop k}\right\} & & \left\{{n+1 \atop k+1}\right\} & \end{array}$$

Here the braces denote the generalized binomial coefficients studied in [1] and defined by

$$(2)\qquad \left\{{n \atop k}\right\} = \frac{[n]!}{[k]!\,[n-k]!} \quad ,$$

with the generalized factorials given by

$$[n]! = A_n A_{n-1} \cdots A_2 A_1, \qquad [0]! = 1 \;,$$

where $\{A_1, A_2, \cdots\}$ is an arbitrary sequence except that $A_i \neq 0$. In the present paper, we shall abbreviate the factorial notation and agree to write (n) instead of $[n]!$.

Now it is easily seen that

$$
\begin{aligned}
(3) \qquad & \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}\begin{Bmatrix} n-1 \\ k \end{Bmatrix}\begin{Bmatrix} n \\ k-1 \end{Bmatrix}\begin{Bmatrix} n \\ k+1 \end{Bmatrix}\begin{Bmatrix} n+1 \\ k \end{Bmatrix}\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \\
& = \left(\frac{(n-1)(n)(n+1)}{(k-1)(k)(k+1)(n-1-k)(n-k)(n+1-k)}\right)^2
\end{aligned}
$$

so that the hexagon theorem is indeed true in general.

Moreover, this is true because in fact two products are equal:

$$
(4) \qquad \begin{Bmatrix} n-1 \\ k \end{Bmatrix}\begin{Bmatrix} n \\ k-1 \end{Bmatrix}\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}\begin{Bmatrix} n \\ k+1 \end{Bmatrix}\begin{Bmatrix} n+1 \\ k \end{Bmatrix} .
$$

The arrangement of these terms in the original hexagon suggests a Star of David, and we will refer to this form of the theorem as the Star of David property. This property motivates the following paper.

Instead of searching for squares in the general Pascal triangle, we will look for equal products of generalized coefficients. The first such problem which we solve is to find equal products of five binomial coefficients, just as Hoggatt's Star of David property gives such a result for equal products of three binomial coefficients.

First of all, however, we ought to examine into the question of whether there are any other equal products of three. To keep the problem within reasonable bounds we will consider only what happens when we make all six possible permutations of the lower indices $k-1$, k, $k+1$ in a product such as that in (4). The six possible products of three binomial coefficients yield the relation (4) and the remaining set of four products are in general unequal. For example,

$$
\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}\begin{Bmatrix} n \\ k \end{Bmatrix}\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = \frac{(n-1)(n)(n+1)}{(k-1)(n-k)(k)(n-k)(k+1)(n-k)}
$$

and

$$
\begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix}\begin{Bmatrix} n \\ k-1 \end{Bmatrix}\begin{Bmatrix} n+1 \\ k \end{Bmatrix} = \frac{(n-1)(n)(n+1)}{(k+1)(n-2-k)(k-1)(n+1-k)(k)(n+1-k)},
$$

which are two different things. We should remark that the simplest possible case of equal products

$$\left\{ \begin{matrix} n+a \\ k+c \end{matrix} \right\} \left\{ \begin{matrix} n+b \\ k+d \end{matrix} \right\} = \left\{ \begin{matrix} n+a \\ k+d \end{matrix} \right\} \left\{ \begin{matrix} n+b \\ k+c \end{matrix} \right\} \tag{5}$$

has only the trivial solutions $a = b$ or $c = d$ or both.

Demanding relation (5) is something rather different from the knowledge that

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} = \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \left\{ \begin{matrix} n-j \\ k-j \end{matrix} \right\},$$

a true identity, because we are concerned solely with permutations of the lower or upper indices.

To go ahead with the situation for a product of five coefficients, we note first that it is not necessary to enumerate all possible products which can be written. It will be sufficient for our purposes to see first of all in how many ways the numbers $k-2$, $k-1$, k, $k+1$, $k+2$ may be added to the numbers $n-k-2$, $n-k-1$, $n-k$, $n-k+1$, $n-k+2$ so as to yield some or all of the numbers $n-2$, $n-1$, n, $n+1$, $n+2$. Now, $k-2$ may be paired with $n-k$, $n-k+1$, or $n-k+2$ only, unless we wish to admit elements such as $n-3$ or $n+3$. Our paper will exclude consideration of any numbers in the upper index position other than $n-2, \cdots, n+2$.

A list of possible pairings can be written as follows:

$$\begin{array}{l|l} k-2 & n-k,\ n-k+1,\ n-k+2 \\ k-1 & n-k-1,\ n-k,\ n-k+1,\ n-k+2 \\ k & n-k-2,\ n-k-1,\ n-k,\ n-k+1,\ n-k+2 \\ k+1 & n-k-2,\ n-k-1,\ n-k,\ n-k+1 \\ k+2 & n-k-2,\ n-k-1,\ n-k \end{array} \tag{6}$$

If we denote the five numbers $n-k-2$, $n-k-1$, $n-k$, $n-k+1$, $n-k+2$ by, respectively, A, B, C, D, E then we may set up the chart more conveniently as follows:

$$
(7)\qquad
\begin{array}{l|l}
k-2 & C,\ D,\ E \\
k-1 & B,\ C,\ D,\ E \\
k & A,\ B,\ C,\ D,\ E \\
k+1 & A,\ B,\ C,\ D \\
k+2 & A,\ B,\ C
\end{array}
$$

and all arrangements necessary to consider then may be found by choosing arrangements of the distinct letters in columns, where one letter only may be chosen from a given row in (7). There appear to be just 31 possible combinations:

(8)

C	C	C	C	C	C	C	D	D	D	D	D	D
B	D	D	E	E	E	E	B	B	C	C	E	E
E	E	E	A	B	D	D	E	E	E	E	A	A
D	B	A	D	D	A	B	A	C	A	B	B	C
A	A	B	B	A	B	A	C	A	B	A	C	B

D	D	D	D	E	E	E	E	E	E	E	E	E
E	E	E	E	D	D	D	D	D	D	C	C	C
B	B	C	C	A	A	B	B	C	C	D	D	B
A	C	A	B	B	C	A	C	A	B	A	B	D
C	A	B	A	C	B	C	A	B	A	B	A	A

E	E	E	E	E
C	B	B	B	B
A	D	D	A	C
D	A	C	D	D
B	C	A	C	A

They give a remarkable collection of identities. First of all, there are six combinations that yield the desired $n-2,\ n-1,\ n,\ n+1,\ n+2$: CEBDA, CDEAB, DBECA, DEABC, EBDAC, and ECABD. The resulting generalized binomial coefficient product identities are:

$$
(12)\qquad
\left\{
\begin{aligned}
& \left\{\begin{matrix} n-2 \\ k-2 \end{matrix}\right\} \left\{\begin{matrix} n+1 \\ k-1 \end{matrix}\right\} \left\{\begin{matrix} n-1 \\ k \end{matrix}\right\} \left\{\begin{matrix} n+2 \\ k+1 \end{matrix}\right\} \left\{\begin{matrix} n \\ k+2 \end{matrix}\right\} \\
&= \left\{\begin{matrix} n-2 \\ k-2 \end{matrix}\right\} \left\{\begin{matrix} n \\ k-1 \end{matrix}\right\} \left\{\begin{matrix} n+2 \\ k \end{matrix}\right\} \left\{\begin{matrix} n-1 \\ k+1 \end{matrix}\right\} \left\{\begin{matrix} n+1 \\ k+2 \end{matrix}\right\} \\
&= \left\{\begin{matrix} n-1 \\ k-2 \end{matrix}\right\} \left\{\begin{matrix} n-2 \\ k-1 \end{matrix}\right\} \left\{\begin{matrix} n+2 \\ k \end{matrix}\right\} \left\{\begin{matrix} n+1 \\ k+1 \end{matrix}\right\} \left\{\begin{matrix} n \\ k+2 \end{matrix}\right\}
\end{aligned}
\right.
$$

$$
\begin{aligned}
&= \begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix} \\
&= \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix} \\
&= \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix} .
\end{aligned}
$$

If we next equate these products in pairs, we find that a common factor cancels out in a number of cases, so that we obtain three different pairs of equal products of five coefficients:

$$
\begin{aligned}
&\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} \\
&= \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix}
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix} \\
&= \begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix}
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} \\
&= \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix}
\end{aligned}
\tag{12}
$$

These identities are the natural extension of the Star of David property (4).

Of those cases in (9) where a common factor cancels out, we appear to get twelve equal products of four binomial coefficients:

$$
\begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix}
\tag{13}
$$

$$
\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}
\tag{14}
$$

(15)

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix}$$

(16)

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix}$$

(17)

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix}$$

(18)

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix}$$

(19

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix}$$

(20)

$$\begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix}$$

(21)

$$\begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix}$$

(22)

$$\begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix}$$

(23)

$$\begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix}$$

(24)

$$\begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix}$$

But these twelve are not all distinct. In relation (13) replace k by k - 1 and n by n - 1. This shows that (13) is equivalent to (22). Similarly, (14) and (24) are equivalent and (17) and (23) are equivalent. Thus we obtain nine distinct relations. Of these, only the first, relation (13), has consecutive integers in both the upper and lower index positions, and is thereby an elegant companion to (4). It is an octagonal equivalent of the original Star of David property:

(25)

$$\begin{array}{ccccc}
 & \left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\} & \left\{\begin{matrix} n-1 \\ k \end{matrix}\right\} & \left\{\begin{matrix} n-1 \\ k+1 \end{matrix}\right\} & \\
\left\{\begin{matrix} n \\ k-1 \end{matrix}\right\} & \left\{\begin{matrix} n \\ k \end{matrix}\right\} & \left\{\begin{matrix} n \\ k+1 \end{matrix}\right\} & & \left\{\begin{matrix} n \\ k+2 \end{matrix}\right\} \\
 & \left\{\begin{matrix} n+1 \\ k \end{matrix}\right\} & \left\{\begin{matrix} n+1 \\ k+1 \end{matrix}\right\} & \left\{\begin{matrix} n+1 \\ k+2 \end{matrix}\right\} & \\
\left\{\begin{matrix} n+1 \\ k-1 \end{matrix}\right\} & & & & \\
 & \left\{\begin{matrix} n+2 \\ k \end{matrix}\right\} & \left\{\begin{matrix} n+2 \\ k+1 \end{matrix}\right\} & \left\{\begin{matrix} n+2 \\ k+2 \end{matrix}\right\} &
\end{array}$$

We return next to the 31 permutations in (8). There are 25 of these which yield products having some repetitions among the numbers n - 2, n - 1, n, n + 1, n + 2. It is worthwhile to explore these. Three of these stand alone: CEADB, DBEAC, and EDCBA. Three pairs give equal products of five coefficients: CBEDA and EBADC; DCEAB and DEBAC; CEDAB and DEACB. Four trios give inequalities: CDEBA, EBCDA, EDABC; DCEBA, ECBDA, EDBAC; DEBCA, DECAB, ECDAB; CEDBA, EBDCA, EDACB. Finally, there is a set of four equalities of products: DECBA, ECDBA, EDCAB, EDBCA. Exploring all the possible pairings, case-by-case, we find first of all three sets of equal products of five coefficients, as follows:

(26)

$$\left\{\begin{matrix} n-1 \\ k-2 \end{matrix}\right\}\left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n+2 \\ k \end{matrix}\right\}\left\{\begin{matrix} n \\ k+1 \end{matrix}\right\}\left\{\begin{matrix} n \\ k+2 \end{matrix}\right\} = \left\{\begin{matrix} n \\ k-2 \end{matrix}\right\}\left\{\begin{matrix} n \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n-1 \\ k \end{matrix}\right\}\left\{\begin{matrix} n-1 \\ k+1 \end{matrix}\right\}\left\{\begin{matrix} n+2 \\ k+2 \end{matrix}\right\}$$

(27)

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix},$$

(28)

$$\begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix},$$

and these form interesting geometric patterns when marked in the Pascal triangle. The left and right members in each identity are symmetrical with respect to $\begin{Bmatrix} n \\ k \end{Bmatrix}$ as a central point.

Next, we obtain five sets of equal products of four coefficients:

(29)

$$\begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix}.$$

(30)

$$\begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix},$$

(31)

$$\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix},$$

(32)

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix},$$

(33)

$$\begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix}.$$

The remainder of the relations found are equal products of three coefficients. The most interesting of these results from equating the permutations CBEDA and EBADC:

(34)

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix}.$$

This is an extension of Hoggatt's original Star of David, and within the Pascal triangle it forms a Star of David with each point moved out one unit further in each direction. What is more, it is easily verified that we have a quite general Star of David formula:

(35)
$$\begin{Bmatrix} n-a \\ k-a \end{Bmatrix} \begin{Bmatrix} n+a \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+a \end{Bmatrix} = \begin{Bmatrix} n \\ k-a \end{Bmatrix} \begin{Bmatrix} n-a \\ k \end{Bmatrix} \begin{Bmatrix} n+a \\ k+a \end{Bmatrix},$$

where a is an arbitrary integer. Some similar extensions of other relations developed in this paper are possible. It should also be possible to find multinomial extensions.

Relation (34) also follows upon equating permutations CDEBA and EDABC.

Relations equivalent to Hoggatt's original formula are obtained in five cases. Finally, there are six remaining cases:

(36)
$$\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix}$$

(37)
$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}$$

(38)
$$\begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix}$$

(39)
$$\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix}$$

(40)
$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}$$

(41)
$$\begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix}.$$

These offer various geometric patterns because they do not involve consecutive integers for the upper and lower indices. As a matter of fact, they all represent Star of David patterns, rotated differently than the original pattern.

Each of the formulas (36) — (41) represents a Star with two points in common with the original Star. From these relations, by means of the substitutions $k+1$ for k, or $a+1$ for n, etc., it is easy to see that relations (36) and (40) are the same, and relations (37) and (39) are the same. The others are distinct from each other and from these. The result is that relations (36), (37), (38), and (41) are the four distinct relations given. One can easily find, as we did in the case for products of five coefficients, whether there are any other distinct such relations.

It would seem to be possible to program the entire procedure for a modern digital computer, which could tirelessly check out all possible cases, and this would make it very easy to tabulate all possible equal products of binomial coefficients within any specified range of parameters. A program could evidently be written along the lines of the procedure used here. Some results, such as formula (35), would not be immediately evident to a computer program, but even here a computer can be programmed to look for certain patterns.

Finally, it would be interesting to find out whether any of the products of the type studied here could be studied in the context of generating functions, as coefficients in power series.

REFERENCES

1. H. W. Gould, "The Bracket Function and Fontene-Ward Generalized Binomial Coefficients with Application to Fibonomial Coefficients," Fibonacci Quarterly, Vol. 7 (1969), Feb., No. 1, pp. 23-40, 35.
2. V. E. Hoggatt, Jr., and G. L. Alexanderson, "A Property of Multinomial Coefficients," Fibonacci Quarterly, Vol. 9, No. 4, pp. 351-356.

THE LEAST REMAINDER ALGORITHM

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Lamé's theorem [1] asserts that the number of divisions n required to find the greatest common divisor (a, b) of a and b $(a \geq b)$ using the Euclidean algorithm does not exceed five times the number of digits p in b. More precisely,

$$n < \frac{p}{\log \xi} + 1, \qquad \text{where} \qquad \xi = \frac{1 + \sqrt{5}}{2} .$$

It is also known [2], [3] that the number of divisions required to find (μ_{n+1}, μ_n) is n and that

$$\left[\frac{p_n - 1}{\log \xi}\right] \leq n - 1 \leq \left[\frac{p_n}{\log \xi}\right] \qquad , \tag{1}$$

where p_n is the number of digits in μ_n and $\mu_1 = 1$, $\mu_2 = 2$ and $\mu_n = \mu_{n-1} + \mu_{n-2}$ $(n > 2)$ are the Fibonacci numbers. Thus the upper bound given by Lamé's theorem is about the best possible and it has been shown [3], [4] that the upper and lower bounds in (1) are attained for infinitely many n.

We recall that the remainders in the ordinary Euclidean algorithm are always positive but that shorter algorithms may be obtained by allowing negative remainders. A well known result of Kronecker [1] asserts that the least-remainder algorithm (L. R. A.) is never longer than any other Euclidean algorithm. The purpose of this note is to derive results analogous to (1) for the L. R. A. To do this, we define v_1 1, $v_2 = 2$ and $v_m = 2v_{m-1} + v_{m-2}$ $(m > 2)$. This sequence has been applied to a similar problem by Shea [5].

Let

$$\begin{aligned}
a &= bq_1 + e_1 b_1 \\
b &= b_1 q_2 + e_2 b_2 \\
&\;\;\vdots \qquad\quad \vdots \\
b_{m-2} &= b_{m-1} q_m + e_m b_m \\
b_{m-1} &= b_m q_{m+1}
\end{aligned}$$

be the L. R. A. for (a,b), where $a_k = \pm 1$ $(k = 1, \cdots, m)$ and $a > b \geq 2b_1 \geq 4b_2 \geq \cdots \geq 2^m b_m > 0$. Then the required number of divisions is $m + 1$ and [1]

$$b_m \geq 1 = v_1, \quad b_{m-1} \geq 2b_m \geq 2 = v_2,$$
$$b_{m-2} \geq 2b_{m-1} + b_m \geq 2v_2 + v_1 = v_3, \cdots .$$

Hence

$$b_{m-k} \geq v_k + 1 \quad \text{and} \quad b \geq v_{m+1} .$$

Now let $N = 1 + \sqrt{2}$. Then

$$N < \frac{1}{2} \cdot 5 = \frac{1}{2} v_3 ,$$

$$N^2 = 2N + 1 < \frac{1}{2}(2v_3 + v_2) = \frac{1}{2} v_4, \cdots .$$

Hence,

$$N^{m-1} < \frac{1}{2} v_{m+1} \leq \frac{1}{2} b .$$

If p is the number of digits in b, then $b < 10^p$ and

$$m - 1 < \frac{\log b - \log 2}{\log N} < \frac{p - \log 2}{\log N} \quad \text{or} \quad m + 1 \leq 2 + \left[\frac{p - \log 2}{\log N}\right] .$$

Also,

$$N > 2 = v_2, \quad N^2 = 2N + 1 > 2v_2 + v_1 = v_3, \quad \cdots .$$

Hence $N^{n-1} > v_n$. If q_n is the number of digits in v_n, then

$$v_n \geq 10^{q_n - 1}$$

and

$$n - 1 > \frac{q_n - 1}{\log N} \quad .$$

The L. R. A. for (v_{n+1}, v_n) is

$$\begin{aligned} v_{n+1} &= 2v_n + v_{n-1} \\ v_n &= 2v_{n-1} + v_{n-2} \\ &\vdots \\ v_3 &= 2v_2 + v_1 \\ v_2 &= 2v_1 \end{aligned}$$

and the required number of divisions is n. Thus

$$\left[\frac{q_n - 1}{\log N}\right] \le n - 2 \le \left[\frac{q_n - \log 2}{\log N}\right] \tag{2}$$

and the upper bound for the required number of divisions in the L. R. A. is about the best possible.

We now show that both the upper and lower bounds in (2) are attained for infinitely many n. Using standard difference equation techniques, it is easily shown that

$$v_n = \frac{1}{2\sqrt{2}} \left[(1 + \sqrt{2})^n - (1 - \sqrt{2})^n\right]$$

and it follows that

$$v_n \sim \frac{N^n}{2\sqrt{2}} \quad .$$

Let ϕ_n be the fractional part (mantissa) of $\log v_n$. Then, since

$$q_n = 1 + [\log v_n],$$

we have

$$q_n = 1 + \log v_n - \phi_n .$$

Hence

(3) $$q_n = 1 + n \log N - \log 2\sqrt{2} - \phi_n + o(1) .$$

But (3) implies that

$$n > \frac{q_n - \log 2}{\log N} + \frac{\phi_n - \frac{1}{4}}{\log N}$$

for all sufficiently large n. Thus

$$n - 2 \geq \left[\frac{q_n - \log 2}{\log N}\right]$$

If $\phi_n \geq 1/4 + \log N$ and n is sufficiently large. Also, (3) implies that

$$n < \frac{q_n - 1}{\log N} + \frac{\phi_n + \frac{1}{2}}{\log N}$$

for all sufficiently large n. Thus

$$n - 2 \leq \left[\frac{q_n - 1}{\log N}\right]$$

if $\phi_n \leq 2 \log N - 1/2$ and n is sufficiently large.

The desired results will follow when it is shown that the sequence $\{\log v_n\}$ is uniformly distributed modulo one [6]. The proof is almost identical to that of a similar result [3] and is therefore omitted. Also, further discussion of such results occurs elsewhere [7].

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A PROPERTY OF MULTINOMIAL COEFFICIENTS

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ABSTRACT

The multinomial coefficients "surrounding" a given multinomial coefficient in a generalized Pascal pyramid are partitioned into subsets such that the product of the coefficients in each subset is a constant N and such that the product of all the coefficients "surrounding" a given m-nomial coefficient is N^m. The result is then generalized to other numerical triangles or pyramids.

1. INTRODUCTION

In the paper by Hansell and Hoggatt [1] the following is proved:

Theorem. The product of the six binomial coefficients surrounding each binomial coefficient $\binom{n}{k}$, $(n \geq 2;\ 0 < k < n)$, in Pascal's triangle is a perfect integer square, N^2. Further, each triad formed by taking alternate binomial coefficients has product N.

Further results in the plane are obtained by Gould in [4].

In this paper, we generalize this theorem to generalized Pascal pyramids in m-space.

2. SELECTING THE MULTINOMIAL COEFFICIENTS

Let us expand $(x_1 + x_2 + x_3 + \cdots + x_m)^n$, $(m \geq 2;\ n = 0, 1, 2, \cdots)$:

$$(x_1 + \cdots + x_m)^n = \sum_{\substack{k_1 + \cdots + k_m = n \\ (k_j \geq 0)}} \binom{k_1 + k_2 + \cdots + k_m}{k_1,\ k_2,\ \cdots\ k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} .$$

Here

$$\binom{k_1 + k_2 + \cdots + k_m}{k_1, \quad k_2, \quad \cdots, \quad k_m} = \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1!\,k_2!\,k_3!\,\cdots\,k_m!} \quad .$$

The recurrence relation is

$$\text{(R)} \quad \binom{k_1 + k_2 + \cdots + k_m}{k_1, \quad k_2, \quad \cdots, \quad k_m} = \sum_{j=1}^{m} \binom{k_1 + k_2 + \cdots + k_m - 1}{k_1 - \delta_{1j},\ k_2 - \delta_{2j},\ \cdots,\ k_m - \delta_{mj}} .$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

Given a multinomial coefficient (inside the pyramid, i.e., $k_s \geq 1$, $s = 1, 2, \cdots, m$)

$$A = \binom{k_1 + k_2 + \cdots + k_m}{k_1, \quad k_2, \quad \cdots, \quad k_m}$$

there are m multinomial coefficients

$$\binom{k_1 + k_2 + k_3 + \cdots + k_m - 1}{k_1 - \delta_{1j},\ k_2 - \delta_{2j},\ \cdots,\ k_m - \delta_{mj}}, \qquad (j = 1, 2, \cdots, m)$$

which "contribute to A" by means of recurrence relation (R); that is, lie directly above A in the pyramid. These same m multinomial coefficients contribute to $m(m - 1)$ multinomial coefficients

$$\binom{k_1 + k_2 + \cdots + k_{m-1} + k_m}{k_1 - \delta_{1j} + \delta_{1k},\ k_2 - \delta_{2j} + \delta_{2k},\ \cdots,\ k_m - \delta_{mj} + \delta_{mk}}, \quad (j, k = 1, 2, \cdots, m; j \neq k)$$

which are all on the same level as A. There are also m multinomial coefficients which are contributed to by A, namely those of the form

$$\binom{k_1 + k_2 + \cdots + k_m + 1}{k_1 + \delta_{1j},\ k_2 + \delta_{2j},\ \cdots,\ k_m + \delta_{mj}}, \quad (j = 1, 2, \cdots, m) \ .$$

Thus there are m above A, $m(m-1)$ on the same level as A and m below A. These $m(m+1)$ multinomial coefficients we say are adjacent to A, and geometrically surround A.

3. THE PRINCIPAL RESULT

Theorem. The product of the $m(m+1)$ multinomial coefficients adjacent to A is a perfect integer m^{th} power.

Proof. In the following, $s = 1, 2, \cdots, m$. On the level above A, the number $k_s - 1$ appears once; on the level with A the number $k_s - 1$ appears $m - 1$ times; $k_s - 1$ does not appear in the level below A. (In the level with A, k_s appears $(m-1)(m-2)$ times; and on the level below A, k_s appears $m - 1$ times. On the level above A, $k_s + 1$ does not appear; on the level with A, $k_s + 1$ appears $(m-1)$ times; and on the level below A, $k_s + 1$ appears once. Thus, in the denominator of the product, $(k_s - 1)!$ appears m times, $(k_s)!$ appears $m(m-1)$ times, and $(k_s + 1)!$ appears m times. The product, therefore, of all $m(m+1)$ multinomial coefficients adjacent to A is:

$$P = \frac{\left[\left(\sum_{i=1}^{m} k_i - 1\right)!\right]^m \left[\left(\sum_{i=1}^{m} k_i\right)!\right]^{m(m-1)} \left[\left(\sum_{i=1}^{m} k_i + 1\right)!\right]^m}{\prod_{i=1}^{m} [k_i - 1!]^m [k_i!]^{m(m-1)} [(k_i + 1)!]^m}$$

$$= \left[\frac{\left(\sum_{i=1}^{m} k_i - 1\right)! \left[\left(\sum_{i=1}^{m} k_i\right)!\right]^{m-1} \left(\sum_{i=1}^{m} k_i + 1\right)!}{\prod_{i=1}^{m} (k_i - 1)!\,(k_i!)^{m-1}(k_i + 1)!}\right]^m = N^m$$

N. B. N is an integer, since (p/q) reduced to lowest terms with $q \neq 1$ is not an integer when raised to the m^{th} power. But the product is an integer, since each multinomial coefficient factor is an integer.

We next prove the following rather surprising result.

Theorem: The $m(m+1)$ multinomial coefficients adjacent to A with product N^m, can be decomposed into m sets of $(m+1)$ multinomial coefficients such that the product over each set is N. Furthermore, the construction yields sets of $(m+1)$ multinomial coefficients such that permuting the subscripts cyclically on any one set $m-1$ times produces all the other sets. Thus the m sets are congruent by rotation.

Proof. We now describe a construction for the sets. Recall that the product within each set must be

$$N = \frac{\left(\sum_{i=1}^{m} k_i - 1\right)!\left(\sum_{i=1}^{m} k_i\right)!^{m-1}\left(\sum_{i=1}^{m} k_i + 1\right)!}{\prod_{i=1}^{m} (k_i - 1)!(k_i!)^{m-1}(k_i + 1)!}$$

For convenience, we introduce the following notation for the multinomial coefficient

$$\binom{k_1 + k_2 + \cdots + k_m}{k_1, \quad k_2, \quad \cdots, \quad k_m} = (0, 0, 0, \cdots, 0)$$

so that by introducing -1 or $+1$ as entries in the m-tuple, we can raise or lower one of the k_i and thus represent adjacent coefficients. For example:

$$\binom{k_1 + k_2 + \cdots + k_m - 1}{k_1 - 1, \; k_2, k_3, \cdots, k_m} = (-1, 0, 0, \cdots, 0)$$

and

$$\binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, k_3 + 1, k_4, \cdots, k_{m-1}, k_m - 1} = (0, 0, 1, 0, \cdots, 0, -1) \quad .$$

Thus a subset of multinomial coefficients of the type desired could be represented as an $(m+1) \times m$ matrix, where each row is a vector as described above, each row representing one of the adjacent multinomial coefficients. Each subset, in order to have the proper numerator in the product, must have one coefficient from above, one from below, and $m - 1$ from the same level as the given coefficient. We shall adopt the convention that the first row represents the coefficient above and the $(m + 1)^{st}$ row the coefficient below. Let the $(m + 1) \times m$ matrix have entries a_{ij}. It is necessary to consider two separate cases.

For m odd, let

$$\left.\begin{array}{l} a_{jj} = -1 \\ a_{m+2-j,j} = +1 \\ a_{ij} = 0 \text{ otherwise} \end{array}\right\} \begin{array}{l} i = 1, 2, \cdots, m + 1 \\ j = 1, 2, \cdots, m \end{array} .$$

We illustrate with $m = 5$:

$$C = \begin{array}{c} \left(\begin{array}{ccccc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ 3 \\ 1 \\ 4 \\ 2 \\ \\ \end{array} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \text{Spacing between } -1 \text{ and } +1 \text{ on the middle } m - 1 \text{ rows.}$$

This corresponds to six multinomial coefficients whose 5 lower arguments are given in rows of this matrix. The other four sets are obtained by rotating cyclically the column vectors of matrix C. We note that $(k_s - 1)!$ appears once, $k_s!$ appears $m - 1$ times, and $(k_s + 1)!$ appears once in each of the five sets, $s = 1, 2, \cdots, m$.

For m even, let

$$\begin{array}{ll} a_{jj} = -1 & j = 1, 2, \cdots, m \\ a_{k+1,m+1-k} = +1 & k = 1, 2, \cdots, (m/2) - 1 \\ a_{m+1-k,k} = +1 & k = 1, 2, \cdots, (m/2) + 1 \\ a_{m+1,(m/2)+1} = 1 & \\ a_{ij} = 0 \text{ otherwise} & \left\{\begin{array}{l} i = 1, 2, \cdots, m + 1 \\ j = 1, 2, \cdots, m \end{array}\right. \end{array}$$

We illustrate for $m = 6$:

$$C' = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 & 0 & 0 \end{pmatrix} \begin{array}{c} \\ \left.\begin{array}{c} 4 \\ 2 \\ 5 \\ 3 \\ 1 \end{array}\right\} \begin{array}{l} \text{Spacing between } -1 \text{ and} \\ +1 \text{ on the middle} \\ m - 1 \text{ rows .} \end{array} \\ \\ \end{array}$$

In both matrices C and C' a cyclic permutation of the column vectors does not produce a duplication before m steps. Thus each set of $m + 1$ elements are distinct and the m sets exhaust the collection of $m(m + 1)$ multinomial coefficients adjacent to A.

It should be noted that the above construction does not yield the only possible partitioning. There exist other partitionings into subsets with the desired property in both the even and odd cases. For example, for $m = 3$, the above construction yields

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

but

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

also have the desired property.

For $m = 4$, the following is a partitioning different from that yielded by the above process:

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GENERAL IDENTITIES FOR RECURRENT SEQUENCES OF ORDER TWO

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1. INTRODUCTION

Let W_0, W_1, $a \neq 0$, and $b \neq 0$ be arbitrary real numbers, and define

$$(1.1) \qquad W_{n+2} = aW_{n+1} - bW_n, \quad a^2 - 4b \neq 0, \qquad (n = 0, 1, \cdots),$$

$$(1.2) \qquad U_n = (\alpha^n - \beta^n)/(\alpha - \beta) \qquad (n = 0, 1, \cdots),$$

$$(1.3) \qquad V_n = \alpha^n + \beta^n \qquad (n = 0, 1, \cdots),$$

$$(1.4) \qquad W_{-n} = (W_0 V_n - W_n)/b^n \qquad (n = 0, 1, \cdots),$$

where $\alpha \neq \beta$ are roots of $x^2 - ax + b = 0$. If $W_0 = 0$ and $W_1 = 1$, then $W_n \equiv U_n$, $n = 0, 1, \cdots$; and if $W_0 = 2$ and $W_1 = a$, then $W_n \equiv V_n$, $n = 0, 1, \cdots$. Our first result is

Theorem 1. Let W_n and W_n^* be solutions of (1.1). Let r, m, and n be integers (+, –, or 0). Then, for $k = 0, 1, \cdots$,

$$(1.5) \qquad \sum_{i=0}^{k} (-1)^i \binom{k}{i} W_{r+m}^{k-i} W_r^i W_{n+rk+im}^*$$

$$= b^{rk} U_m^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j W_{n+j}^* \quad .$$

Special cases of (1.5) are given by

Corollary 1.

$$(1.6) \qquad \sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+m}^{k-i} U_r^i U_{n+rk+im} = b^{rk} U_m^k U_n \quad ,$$

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+m}^{k-i} U_r^i W_{n+rk+im} = b^{rk} U_m^k W_n , \tag{1.7}$$

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+m}^{k-i} U_r^i V_{n+rk+im} = b^{rk} U_m^k V_n , \tag{1.8}$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+m}^{2k-i} V_r^i U_{n+2kr+im} = (a^2 - 4b)^k b^{2kr} U_m^{2k} U_n , \tag{1.9}$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+m}^{2k-i} V_r^i W_{n+2kr+im} = (a^2 - 4b)^k b^{2kr} U_m^{2k} W_n , \tag{1.10}$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+m}^{2k-i} V_r^i V_{n+2kr+im} = (a^2 - 4b)^k b^{2kr} U_m^{2k} V_n , \tag{1.11}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{r+m}^{2k+1-i} V_r^i V_{n+(2k+1)r+im} = -(a^2 - 4b)^{k+1} b^{(2k+1)r} U_m^{2k+1} U_n , \tag{1.12}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{r+m}^{2k+1-i} V_r^i U_{n+(2k+1)r+im} = -(a^2 - 4b)^k b^{(2k+1)r} U_m^{2k+1} V_n . \tag{1.13}$$

Our next result related to Theorem 1 is

Theorem 2. Let W_n be a solution of (1.1). Let r, m, and n be integers (+, -, or 0). Then, for $k = 0, 1, \cdots,$ we have

$$V_m^k W_{kr+n}$$

$$= \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} (W_1 U_{n-2jm} - bW_0 U_{n-2jm-1})$$

(1.14)
$$+ \sum_{j=0}^{[(k-1)/2]} \binom{k}{2j+1} (a^2 - 4b)^j b^{(2j+1)m} V_{r+m}^{k-2j-1} U_r^{2j+1} (W_1 V_{n-(2j+1)m} - bW_0 V_{n-(2j+1)m-1})$$

(1.15)
$$V_m^k U_{kr+n} = \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} U_{n-2jm}$$

$$+ \sum_{j=0}^{[(k-1)/2]} \binom{k}{2j+1} (a^2 - 4b)^j b^{(2j+1)m} V_{r+m}^{k-2j-1} U_r^{2j+1} V_{n-(2j+1)m} ,$$

(1.16)

$$V_m^k V_{kr+n} = \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} V_{n-2jm}$$

$$+ \sum_{j=0}^{[(k-1)/2]} \binom{k}{2j+1} (a^2 - 4b)^{j+1} b^{(2j+1)m} V_{r+m}^{k-2j-1} U_r^{2j+1} U_{n-(2j+1)m} .$$

2. PROOF OF THEOREM 1

Let $W_n^* = S_1\alpha^n + S_2\beta^n$ and $W_n = C_1\alpha^n + C_2\beta^n$, $n = 0, 1, \cdots,$ where S_i and C_i, $i = 1, 2$, are arbitrary constants. Since $W_0 = C_1 + C_2$ and $W_1 = C_1\alpha + C_2\beta$, we readily find that

$$(2.1) \qquad (\alpha - \beta)^k C_1^k = (W_1 - \beta W_0)^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j \beta^j ,$$

$$(2.2) \qquad (\alpha - \beta)^k C_2^k = (\alpha W_0 - W_1)^k = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j \alpha^j .$$

Let L denote the left-hand side of (1.5). Then, using the binomial theorem and representation of W_n^*, we have

$$(2.3) \qquad L = S_1\alpha^{n+rk}(W_{r+m} - \alpha^m W_r)^k + S_2\beta^{n+rk}(W_{r+m} - \beta^m W_r)^k .$$

Since

$$W_{r+m} - \alpha^m W_r = (\beta^m - \alpha^m)\beta^r C_2$$

and

$$W_{r+m} - \beta^m W_r = (\alpha^m - \beta^m)\alpha^r C_1 ,$$

we obtain, using $\alpha\beta = b$, (2.1), (2.2), and (1.2),

$$L = S_1 b^{rk}\alpha^n(\beta^m - \alpha^m)^k C_2^k + S_2 b^{rk}\beta^n(\alpha^m - \beta^m)^k C_1^k$$

$$(2.4) \qquad = b^{rk}U_m^k\left\{S_1\alpha^n(-1)^k(\alpha - \beta)^k C_2^k + S_2\beta^n(\alpha - \beta)^k C_1^k\right\}$$

$$= b^{rk}U_m^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j (S_1\alpha^{n+j} + S_2\beta^{n+j}) = R ,$$

where R denotes the right-hand side of (1.5).

If $W_r \equiv U_r$ and $W_n^* \equiv U_n$, then $W_0 = 0$ and (1.5) gives the special case (1.6), noting that all terms in the right-hand sum of (1.5) vanish except for $j = 0$.

Since

$$(2.5) \qquad W_n = W_0 U_{n+1} + (W_1 - aW_0)U_n ,$$

we obtain (1.7) from (1.6); and (1.8) from (1.7) when $W_n \equiv V_n$.

If $W_n \equiv V_n$ (i.e., $C_1 = C_2 = 1$), then (2.4), with $k = p$, gives

$$L = b^{rp} U_m^p (\alpha - \beta)^p (S_1\alpha^n(-1)^p + S_2\beta^n) \ . \tag{2.6}$$

Noting that $(\alpha - \beta)^2 = a^2 - 4b$, then (2.6), for $W_n^* \equiv U_n$ (i.e., $S_1 = -S_2 = 1/(\alpha - \beta)$), gives (1.9) for $p = 2k$ and (1.13) for $p = 2k + 1$. Using (2.5), we get (1.10) from (1.9); and (1.11) from (1.10) when $W_n \equiv V_n$.

If $W_n^* \equiv V_n$ (i.e., $S_1 = S_2 = 1$), then (2.6) gives (1.12) for $p = 2k + 1$.

If $a = -b = 1$, then $U_n \equiv F_n$, and (1.6) gives the identity of Halton [1, p. 34] as a special case.

3. PROOF OF THEOREM 2

Our method is a generalization of a proof used in the unpublished Master's thesis of Vinson [2, pp. 14-16]. If we treat α^r and β^r as the unknowns in the system $(\alpha - \beta)U_r = \alpha^r - \beta^r$ and $V_{r+m} = \alpha^m\alpha^r + \beta^m\beta^r$, we obtain

$$V_m\alpha^r = V_{r+m} + (\alpha - \beta)\beta^m U_r \quad\text{and}\quad V_m\beta^r = V_{r+m} - (\alpha - \beta)\alpha^m U_r \ .$$

Since $W_{kr+n} = C_1\alpha^n(\alpha^r)^k + C_2\beta^n(\beta^r)^k$, we obtain

$$V_m^k W_{kr+n} = C_1\alpha^n(V_{r+m} + (\alpha - \beta)\beta^m U_r)^k + C_2\beta^n(V_{r+m} - (\alpha - \beta)\alpha^m U_r)^k$$

$$= \sum_{i=0}^{k} \binom{k}{i} (\alpha - \beta)^i V_{r+m}^{k-i} U_r^i (\alpha\beta)^{mi} (C_1\alpha^{n-mi} + (-1)^i C_2\beta^{n-mi}) \ . \tag{3.1}$$

Now

$$C_1\alpha^{n-mi} + (-1)^i C_2\beta^{n-mi}$$

$$= \left[(W_1 - \beta W_0)\alpha^{n-mi} + (-1)^i(\alpha W_0 - W_1)\beta^{n-mi}\right] / (\alpha - \beta)$$

$$= W_1 \frac{(\alpha^{n-mi} - (-1)^i\beta^{n-mi})}{\alpha - \beta} - bW_0 \frac{(\alpha^{n-mi-1} - (-1)^i\beta^{n-mi-1})}{\alpha - \beta} \ .$$

Since $(\alpha - \beta)^2 = a^2 - 4b$, we obtain (1.14) from (3.1) for $i = 2j$ and $i = 2j + 1$.

If $W_n \equiv U_n$, then $W_0 = 0$, $W_1 = 1$, and thus (1.14) gives (1.15). If $W_n \equiv V_n$, then $W_0 = 2$, $W_1 = a$, and thus (1.14) gives (1.16), noting that $V_n = aU_n - 2bU_{n-1}$ and that

$$aV_n - 2bV_{n-1} = 2V_{n+1} - aV_n = (a^2 - 4b)U_n \ .$$

4. EXTENDED RESULTS

Our next class of results are of a higher level order than Theorem 1, since we now essentially replace W_n^* in (1.5) by its cross-product with itself.

Theorem 3. Let W_n and W_n^* be solutions of (1.1). Let r, m, p, and n be integers (+, -, or 0). Then, for $k = 0, 1, \cdots$,

$$\begin{aligned} &\left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} W_{r+2m}^{k-i} W_r^i W_{p+im}^* W_{n+im}^*\right)(a^2 - 4b) \\ (4.1) \quad &= b^{rk} U_{2m}^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j (W_1^{2*} V_{p+n-rk+j} - 2bW_0^* W_1^* V_{p+n-rk+j-1} \\ &\qquad + b^2 W_0^{2*} V_{p+n-rk+j-2}) \\ &\quad - (W_1^{2*} - aW_0^* W_1^* + bW_0^{2*})\, b^n V_{p-n} U_m^k (W_1 V_{r+m} - bW_0 V_{r+m-1})^k \ . \end{aligned}$$

Corollary 3. In special cases of (4.1), we have

$$\begin{aligned} &\sum_{i=0}^{k} (-1)^i \binom{k}{i} W_{r+2m}^{k-i} W_r^i V_{p+im} V_{n+im} \\ (4.2) \quad &= b^{rk} U_{2m}^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j V_{p+n-rk+j} \\ &\quad + b^n V_{p-n} U_m^k (W_1 V_{r+m} - bW_0 V_{r+m-1})^k \ , \end{aligned}$$

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+2m}^{k-i} U_r^i V_{p+im} V_{n+im}$$

(4.3)

$$= b^{rk} U_{2m}^k V_{p+n-rk} + b^n V_{p-n} U_m^k V_{r+m}^k ,$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+2m}^{2k-i} V_r^i V_{p+im} V_{n+im}$$

(4.4)

$$= (a^2 - 4b)^k b^{2rk} U_{2m}^{2k} V_{p+n-2rk} + (a^2 - 4b)^{2k} b^n V_{p-n} U_m^{2k} U_{r+m}^{2k} ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{r+2m}^{2k+1-i} V_r^i V_{p+im} V_{n+im}$$

(4.5)

$$= -(a^2 - 4b)^{k+1} b^{r(2k+1)} U_{2m}^{2k+1} U_{p+n-r(2k+1)}$$

$$+ (a^2 - 4b)^{2k+1} b^n V_{p-n} U_m^{2k+1} U_{r+m}^{2k+1} ,$$

$$(a^2 - 4b) \sum_{i=0}^{k} (-1)^i \binom{k}{i} W_{r+2m}^{k-i} W_r^i U_{p+im} U_{n+im}$$

(4.6)

$$= b^{rk} U_{2m}^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j V_{p+n-rk+j}$$

$$- b^n V_{p-n} U_m^k (W_1 V_{r+m} - bW_0 V_{r+m-1})^k ,$$

$$(a^2 - 4b) \sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+2m}^{k-i} U_r^i U_{p+im} U_{n+im}$$

(4.7)

$$= b^{rk} U_{2m}^k V_{p+n-rk} - b^n V_{p-n} U_m^k V_{r+m}^k ,$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+2m}^{2k-i} V_r^i U_{p+im} U_{n+im} \tag{4.8}$$

$$= (a^2 - 4b)^{k-1} b^{2rk} U_{2m}^{2k} V_{p+n-2rk} - (a^2 - 4b)^{2k-1} b^n V_{p-n} (U_m U_{r+m})^{2k} ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{r+2m}^{2k+1-i} V_r^i U_{p+im} U_{n+im}$$

$$= -(a^2 - 4b)^k b^{r(2k+1)} U_{2m}^{2k+1} U_{p+n-r(2k+1)} \tag{4.9}$$

$$- (a^2 - 4b)^{2k} b^n V_{p-n} U_m^{2k+1} U_{r+m}^{2k+1} .$$

Closely associated with Theorem 3 is

Theorem 4. Let W_n be a solution of (1.1). Let r, m, p, and n be integers (+, -, or 0). Then, for k = 0, 1, ⋯,

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} W_{r+2m}^{k-i} W_r^i U_{p+im} V_{n+im}$$

$$= b^{rk} U_{2m}^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j U_{p+n-rk+j} \tag{4.10}$$

$$+ b^n U_{p-n} U_m^k (W_1 V_{r+m} - bW_0 V_{r+m-1})^k .$$

Corollary 4. As special cases of (4.10), we have

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+2m}^{k-i} U_r^i U_{p+im} V_{n+im} \tag{4.11}$$

$$= b^{rk} U_{2m}^k U_{p+n-rk} + b^n U_{p-n} U_m^k V_{r+m}^k ,$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+2m}^{2k-i} V_r^i U_{p+im} V_{n+im}$$

(4.12)

$$= (a^2 - 4b)^k b^{2rk} U_{2m}^{2k} U_{p+n-2rk} + (a^2 - 4b)^{2k} b^n U_{p-n} U_m^{2k} U_{r+m}^{2k} ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{r+2m}^{2k+1-i} V_r^i U_{p+im} V_{n+im}$$

(4.13)

$$= -(a^2 - 4b)^k b^{r(2k+1)} U_{2m}^{2k+1} V_{p+n-r(2k+1)}$$

$$+ (a^2 - 4b)^{2k+1} b^n U_{p-n} U_m^{2k+1} U_{r+m}^{2k+1} .$$

Remarks. Since $U_{2n} = U_n V_n$, we note that for $p = n$, (4.10), (4.11), (4.12) and (4.13) reduce to special cases, respectively, of (1.5), (1.6), (1.9), and (1.13).

5. PROOF OF THEOREM 3

We readily find that

$$W^*_{p+im} W^*_{n+im} = S_1^2 \alpha^{p+n} \alpha^{2mi} + S_2^2 \beta^{p+n} \beta^{2mi} + S_1 S_2 b^n V_{p-n} b^{mi} . \tag{5.1}$$

Let $(a^2 - 4b) \cdot L$ denote the left-hand side of (4.1). Then, the binomial theorem, using (5.1), gives

$$\begin{aligned} L = {} & S_1^2 \alpha^{p+n} (W_{r+2m} - \alpha^{2m} W_r)^k + S_2^2 \beta^{p+n} (W_{r+2m} - \beta^{2m} W_r)^k \\ & + S_1 S_2 b^n V_{p-n} (W_{r+2m} - b^m W_r)^k . \end{aligned} \tag{5.2}$$

Since $W_n = C_1 \alpha^n + C_2 \beta^n$, we have, using (2.1) and (2.2) for $k = 1$,

$$(W_{r+2m} - b^m W_r)^k = U_m^k (W_1 V_{r+m} - b W_0 V_{r+m-1})^k \equiv Y . \tag{5.3}$$

Noting the relations cited after (2.3), we have

$$
\begin{aligned}
L &= S_1^2 b^{rk} \alpha^{p+n-rk} C_2^k (\beta^{2m} - \alpha^{2m})^k \\
(5.4) \qquad &\quad + S_2^2 b^{rk} \beta^{p+n-rk} C_1^k (\alpha^{2m} - \beta^{2m})^k + S_1 S_2 b^n V_{p-n} Y \\
&= b^{rk} U_{2m}^k \left\{ S_1^2 \alpha^{p+n-rk} (-1)^k (\alpha - \beta)^k C_2^k + S_2^2 \beta^{p+n-rk} (\alpha - \beta)^k C_1^k \right\} \\
&\quad + S_1 S_2 b^n V_{p-n} Y \quad .
\end{aligned}
$$

Recalling (2.1) and (2.2), we now have

$$
\begin{aligned}
L &= b^{rk} U_{2m}^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j (S_1^2 \alpha^{p+n-rk+j} + S_2^2 \beta^{p+n-rk+j}) \\
(5.5) \qquad &\quad + S_1 S_2 b^n V_{p-n} Y \quad .
\end{aligned}
$$

Since $(\alpha - \beta)S_1 = W_1^* - \beta W_0^*$ and $(\alpha - \beta)S_2 = \alpha W_0^* - W_1^*$, additional simplification of (5.5), using $\alpha\beta = b$, $\alpha + \beta = a$, and $(\alpha - \beta)^2 = a^2 - 4b$, yields (4.1).

If $W_n^* \equiv V_n$, then $W_0^* = 2$ and $W_1^* = a$, and thus (4.1) gives (4.2), noting that

$$a^2 V_c - 4ab V_{c-1} + 4b^2 V_{c-2} = (a^2 - 4b) V_c \quad .$$

We get (4.3) from (4.2) when $W_n \equiv U_n$.

If $W_n \equiv V_n$, then (4.2) gives (4.4) and (4.5), which are also obtained from (5.4), where $S_1 = S_2 = C_1 = C_2 = 1$.

If $W_n^* \equiv U_n$, then (4.1) gives (4.6), which gives (4.7) for $W_n \equiv U_n$. If $W_n \equiv V_n$, then (4.6) gives (4.8) and (4.9), which are also obtained from (5.4), where now $C_1 = C_2 = 1$ and $S_1 = -S_2 = (\alpha - \beta)^{-1}$.

6. PROOF OF THEOREM 4

We readily find that

$$(6.1) \qquad U_{p+im} V_{n+im} = \frac{\alpha^{p+n}}{\alpha - \beta} \alpha^{2mi} - \frac{\beta^{p+n}}{\alpha - \beta} \beta^{2mi} + b^n U_{p-n} b^{mi} \quad .$$

Let L denote the left-hand side of (4.10). Then, the binomial theorem, using (6.1), gives

$$\begin{aligned} L &= \frac{\alpha^{p+n}}{\alpha-\beta}(W_{r+2m} - \alpha^{2m}W_r)^k - \frac{\beta^{p+n}}{\alpha-\beta}(W_{r+2m} - \beta^{2m}W_r)^k \\ &\quad + b^n U_{p-n}(W_{r+2m} - b^m W_r)^k \\ &= b^{rk}U_{2m}^k\left(\alpha^{p+n-rk}(-1)^k(\alpha-\beta)^k C_2^k - \beta^{p+n-rk}(\alpha-\beta)^k C_1^k\right)/(\alpha-\beta) \\ &\quad + b^n U_{p-n} Y . \end{aligned} \tag{6.2}$$

Using (2.1) and (2.2) in (6.2) gives the desired result (4.10).

We obtain (4.11) from (4.10), where $W_n \equiv U_n$. If $W_n \equiv V_n$, then (4.10) gives (4.12) and (4.13), which are also obtained from (6.2), where $C_1 = C_2 = 1$.

7. ADDITIONAL SUMS

Closely related in proof to the above theorems are the following results:

Theorem 5. Let W_n and W_n^* be solutions of (1.1). Let m, p, and n be integers (+, -, or 0). Then

$$\begin{aligned} &\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} W_{p+im} W^*_{n-im} \qquad (k > 0) \\ &\quad = b^p U_m^{2k}(a^2 - 4b)^{k-1} Z_1(m,k) , \end{aligned} \tag{7.1}$$

where

$$\begin{aligned} &W_0 W_1^* V_{n-2jm-p+1} - (b W_0 W_0^* + W_1 W_1^*)V_{n-2jm-p} \\ &\quad + bW_0^* W_1 V_{n-2jm-p-1} \equiv Z_1(m,j) , \end{aligned} \tag{7.2}$$

$$\begin{aligned} &\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} W_{p+im} W^*_{n-im} \qquad (k = 0, 1, \cdots) \\ &\quad = b^p U_m^{2k+1}(a^2 - 4b)^k \cdot Z_2(m,k), \end{aligned} \tag{7.3}$$

where

$$(7.4)\quad \begin{aligned} &W_0 W_1^* U_{n-(2j+1)m-p+1} - (bW_0 W_0^* + W_1 W_1^*) U_{n-(2j+1)m-p} \\ &\quad + b W_0^* W_1 U_{n-(2j+1)m-p-1} \equiv Z_2(m,j) \quad . \end{aligned}$$

Corollary 5. As special cases of (7.1) and (7.3), we have

$$(7.5)\quad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} U_{p+im} U_{n-im} = -b^p U_m^{2k} (a^2 - 4b)^{k-1} V_{n-2km-p} \text{ ,}$$

$$(7.6)\quad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{p+im} V_{n-im} = b^p U_m^{2k} (a^2 - 4b)^{k} V_{n-2km-p} \text{ ,}$$

$$(7.7)\quad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} U_{p+im} V_{n-im} = -b^p U_m^{2k} (a^2 - 4b)^{k} U_{n-2km-p} \text{ ,}$$

$$(7.8)\quad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{p+im} U_{n-im} = b^p U_m^{2k} (a^2 - 4b)^{k} U_{n-2km-p} \text{ ,}$$

$$(7.9)\quad \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} U_{p+im} U_{n-im} = -b^p U_m^{2k+1} (a^2 - 4b)^{k} U_{n-(2k+1)m-p} \text{ ,}$$

$$(7.10)\quad \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{p+im} V_{n-im} = b^p U_m^{2k+1} (a^2 - 4b)^{k+1} U_{n-(2k+1)m-p} \text{ ,}$$

$$(7.11)\quad \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} U_{p+im} V_{n-im} = -b^p U_m^{2k+1} (a^2 - 4b)^{k} V_{n-(2k+1)m-p} .$$

Theorem 6. Let W_n and W_n^* be solutions of (1.1). Let m, p, and n be integers (+, -, or 0). Then, for $k = 0, 1, \cdots$,

$$\left(\sum_{i=0}^{k} \binom{k}{i} W_{p+im} W^*_{n-im}\right)(a^2 - 4b) = 2^k Z_3 + b^p V_m^k Z_4 \quad , \tag{7.12}$$

where

$$\begin{aligned} Z_3 &= W_1 W_1^* V_{p+n} - b(W_0^* W_1 + W_0 W_1^*) V_{p+n-1} + b^2 W_0 W_0^* V_{p+n-2} \\ &\equiv Z_3(p,n) \end{aligned} \tag{7.13}$$

$$\begin{aligned} Z_4 = W_0 W_1^* V_{n-km-p+1} &- (bW_0 W_0^* + W_1 W_1^*) V_{n-km-p} \\ &+ bW_0^* W_1 V_{n-km-p-1} \quad . \end{aligned} \tag{7.14}$$

Corollary 6. As special cases of (7.12), we have

$$\begin{aligned} &\left(\sum_{i=0}^{k} \binom{k}{i} U_{p+im} U_{n-im}\right)(a^2 - 4b) \\ &\qquad = 2^k V_{p+n} - b^p V_m^k V_{n-km-p} \quad , \end{aligned} \tag{7.15}$$

$$\sum_{i=0}^{k} \binom{k}{i} V_{p+im} V_{n-im} = 2^k V_{p+n} + b^p V_m^k V_{n-mk-p} \quad , \tag{7.16}$$

$$\sum_{i=0}^{k} \binom{k}{i} U_{p+im} V_{n-im} = 2^k U_{p+n} - b^p V_m^k U_{n-mk-p} \quad . \tag{7.17}$$

Remarks. Special cases of (7.5, (7.6), (7.9), and (7.10) for $U_n \equiv F_n$ and $V_n \equiv L_n$ were given, using matrix methods, in the paper by Hoggatt and Bicknell [3].

8. PROOF OF THEOREMS 5 AND 6

We readily find that

$$W_{p+im}W^*_{n-im} = S_1C_2\alpha^{n-im}\beta^{p+im} + C_1S_2\alpha^{p+im}\beta^{n-im} + C_1S_1\alpha^{p+n} + C_2S_2\beta^{p+n} . \tag{8.1}$$

For $r > 0$, we obtain, using the binomial theorem,

$$\begin{aligned}\sum_{i=0}^{r}(-1)^i\binom{r}{i}W_{p+im}W^*_{n-im} &= S_1C_2\alpha^n\beta^p(1-\alpha^{-m}\beta^m)^r + C_1S_2\alpha^p\beta^n(1-\alpha^m\beta^{-m})^r \\ &= b^pU_m^r(\alpha-\beta)^r(S_1C_2\alpha^{n-mr-p} + (-1)^rC_1S_2\beta^{n-mr-p}) .\end{aligned} \tag{8.2}$$

Using (2.1) and (2.2), (8.2) gives (7.1) for $r = 2k$ and (7.3) for $r = 2k + 1$.

Special cases (7.5), $\cdots$, and (7.11) are readily obtained from (7.1) and (7.3) for the choices indicated.

Using (8.1), we readily find that

$$\sum_{i=0}^{k}\binom{k}{i}W_{p+im}W^*_{n-im} = 2^k(C_1S_1\alpha^{p+n} + C_2S_2\beta^{p+n}) + b^pV_m^k(S_1C_2\alpha^{n-mk-p} + C_1S_2\beta^{n-mk-p}) . \tag{8.3}$$

Using (2.1) and (2.2), (8.3) reduces to (7.12). Special cases (7.15), $\cdots$, (7.17), are readily obtained from (7.12).

9. MORE SUMS

Introduction of new integer parameters requires that we redefine certain identities by notationally including parameters previously suppressed for simplicity. Thus, we define $Z_1(m, j, p, n)$ by (7.2); $Z_2(m,j,p,n)$ by

(7.4), and $Z_3(p,n)$ by (7.13). Using (8.1), we can obtain the following results, whose lengthy details are omitted.

Theorem 7. Let W_n, W_n^* and W_n^{**} be solutions of (1.1). Let m, p, n, and r be integers (+, -, or 0). Then, we have

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} b^{-mi} W^{**}_{r+2im} W_{p+im} W^*_{n-im} \qquad (k > 0)$$

$$\begin{aligned} (9.1) \qquad &= (a^2 - 4b)^{k-1} b^{-2km} U_m^{2k} W^{**}_{r+2km} Z_3(p,n) \\ &+ b^{p+r} (a^2 - 4b)^{k-1} U_{2m}^{2k} [W_0^{**} Z_2(4mk, 0, p + r - 1, n) \\ &- W_1^{**} Z_2(4mk, 0, p + r, n)], \end{aligned}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} b^{-mi} W^{**}_{r+2im} W_{p+im} W^*_{n-im} \qquad (k \geq 0)$$

$$\begin{aligned} (9.2) \qquad &= (a^2 - 4b)^{k-1} b^{-m(2k+1)} U_m^{k+1} [bW_0^{**} V_{r+m(2k+1)-1} \\ &- W_1^{**} V_{r+m(2k+1)}] Z_3(p,n) \\ &+ b^{p+r} (a^2 - 4b)^{k-1} U_{2m}^{2k+1} [W_0^{**} Z_1(m, 2k + 1, p + r - 1, n) \\ &- W_1^{**} Z_1(m, 2k + 1, p + r, n)], \end{aligned}$$

$$(a^2 - 4b) \sum_{i=0}^{k} \binom{k}{i} b^{-mi} W^{**}_{r+2im} W_{p+im} W^*_{n-im} \qquad (k \geq 0)$$

$$\begin{aligned} (9.3) \qquad &= 2^k b^p [W_1^{**} Z_2(0, k, p - r, n) - bW_0^{**} Z_2(0, k, p - r + 1, n)] \\ &+ b^{p+r} V_{2m}^k [W_0^{**} Z_2(2mk, 0, p + r - 1, n) - W_1^{**} Z_2(2mk, 0, p + r, n)] \\ &+ b^{-mk} V_m^k W^{**}_{r+mk} Z_3(p,n) . \end{aligned}$$

Remarks. As a typical special case, we get from (9.1),

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} b^{-mi} U_{r+2im} U_{p+im} U_{n-im} \qquad (k > 0)$$

(9.4)
$$= (a^2 - 4b)^{k-1} b^{-2km} U_m^{2k} U_{r+2km} V_{p+n}$$
$$+ b^{p+r} (a^2 - 4b)^{k-1} U_{2m}^{2k} U_{n-4mk-p-r} \;.$$

Theorem 8. Let W_n, W_n^*, and W_n^{**} be solutions of (1.1). Let m, p, n, q, and r be integers (+, –, or 0). Then

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} W_{p+im} W_{q+im} W^*_{n-im} W^*_{r-im} \qquad (k > 0)$$

(9.5)
$$= b^p U_m^{2k} (a^2 - 4b)^{k-2} Z_1(m, k, p, n) Z_3(q, r)$$
$$+ b^q U_m^{2k} (a^2 - 4b)^{k-2} Z_1(m, k, q, r) Z_3(p, n)$$
$$+ b^{p+q} (a^2 - 4b)^{k-2} U_{2m}^{2k} A(V_{n+r-p-q-4mk}) \;,$$

where

(9.6)
$$\begin{aligned} A(W_i^{**}) = {} & (W_0 W_1^*)^2 W^{**}_{i+2} - 2W_0 W_1^* (bW_0 W_0^* + W_1 W_1^*) W^{**}_{i+1} \\ & + [b^2 (W_0 W_0^*)^2 + 4bW_0 W_1 W_0^* W_1^* + (W_1 W_1^*)^2] W_i^{**} \\ & - 2b W_0^* W_1 (bW_0 W_0^* + W_1 W_1^*) W^{**}_{i-1} + b^2 (W_0^* W_1)^2 W^{**}_{i-2} \;. \end{aligned}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} W_{p+im} W_{q+im} W^*_{n-im} W^*_{r-im} \qquad (k \geq 0)$$

(9.7)
$$= b^p U_m^{2k+1} (a^2 - 4b)^{k-1} Z_2(m, k, p, n) Z_3(q, r)$$
$$+ b^q U_m^{2k+1} (a^2 - 4b)^{k-1} Z_2(m, k, q, r) Z_3(p, n)$$
$$+ b^{p+q} (a^2 - 4b)^{k-1} U_{2m}^{2k+1} A(U_{n+r-p-q-2m(2k+1)}) \;.$$

Remarks. As a special case of (9.5), we have

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} U_{p+im} U_{q+im} U_{n-im} U_{r-im} \qquad (k > 0)$$

(9.8)
$$= -b^p U_m^{2k} (a^2 - 4b)^{k-2} V_{n-2km-p} V_{q+r} - b^q U_m^{2k} (a^2 - 4b)^{k-2} V_{r-2km-q} V_{p+n} + b^{p+q} (a^2 - 4b)^{k-2} U_{2m}^{2k} V_{n+r-p-q-4mk} .$$

For Fibonacci, F_n, and Lucas, L_n, sequences, we obtain from (9.8), with $a = -b = 1$,

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} F_{p+im} F_{q+im} F_{n+im} F_{r+im} \qquad (k > 0)$$

(9.9)
$$= (-1)^{r+1} 5^{k-2} F_m^{2k} L_{n+2km+p} L_{q-r} + (-1)^{n+1} 5^{k-2} F_m^{2k} L_{r+2km+q} L_{p-n} + 5^{k-2} F_{2m}^{2k} L_{n+r+p+q+4mk} .$$

Theorem 9. Let W_n, W_n^*, and W_n^{**} be solutions of (1.1). Let m, p, n, q, r, and t be integers (+, -, or 0). Then

(9.10)
$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} b^{-mi} W_{t+2im}^{**} W_{p+im} W_{q+im} W_{n-im}^* W_{r-im}^* \qquad (k > 0)$$

$$= b^{p+t} (a^2 - 4b)^{k-2} U_{2m}^{2k} [W_0^{**} Z_2(4mk, 0, p+t-1, n) - W_1^{**} Z_2(4mk, 0, p+t, n)] Z_3(q,r)$$
$$+ b^{q+t} (a^2 - 4b)^{k-2} U_{2m}^{2k} [W_0^{**} Z_2(4mk, 0, q+t-1, r) - W_1^{**} Z_2(4mk, 0, q+t, r)] Z_3(p,n)$$
$$+ b^{p+q} (a^2 - 4b)^{k-2} U_m^{2k} [W_1^{**} A(U_{n+r+t-p-q-2mk}) - bW_0^{**} A(U_{n+r+t-p-q-2mk-1})]$$
$$+ b^{p+q+t} (a^2 - 4b)^{k-2} U_{3m}^{2k} [W_0^{**} A(U_{n+r-p-q-t-6mk+1}) - W_1^{**} A(U_{n+r-p-q-t-6mk})]$$
$$+ b^{-2mk} (a^2 - 4b)^{k-2} U_m^{2k} W_{t+2mk}^{**} Z_3(p,n) Z_3(q,r)$$
$$+ b^{-2mk+n+q} (a^2 - 4b)^{k-2} U_m^{2k} W_{t+2mk}^{**} V_{p+r-n-q} D(W_0, W_1) D(W_0^*, W_1^*) ,$$

where

$$D(W_0, W_1) = -(W_1^2 - aW_0W_1 + bW_0^2) \; . \tag{9.11}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} b^{-mi} W^{**}_{t+2im} W_{p+im} W_{q+im} W^*_{n-im} W^*_{r-im} \qquad (k \geq 0)$$

$$\begin{aligned}
&= b^{p+t}(a^2 - 4b)^{k-2} U_{2m}^{2k+1} [W_0^{**} Z_1(m, 2k+1, p+t-1, n) \\
&\qquad - W_1^{**} Z_1(m, 2k+1, p+t, n)] Z_3(q, r) \\
&+ b^{q+t}(a^2 - 4b)^{k-2} U_{2m}^{2k+1} [W_0^{**} Z_1(m, 2k+1, q+t-1, r) \\
&\qquad - W_1^{**} Z_1(m, 2k+1, q+t, r)] Z_3(p, n) \\
&+ b^{p+q}(a^2 - 4b)^{k-2} U_m^{2k+1} [W_1^{**} A(V_{n+r+t-p-q-m(2k+1)}) \\
&\qquad - bW_0^{**} A(V_{n+r+t-p-q-m(2k+1)-1})] \\
&+ b^{p+q+t}(a^2 - 4b)^{k-2} U_{3m}^{2k+1} [W_0^{**} A(V_{n+r-p-q-t-3m(2k+1)+1}) \\
&\qquad - W_1^{**} A(V_{n+r-p-q-t-3m(2k+1)})] \\
&+ b^{-m(2k+1)}(a^2 - 4b)^{k-2} U_m^{2k+1} [bW_0^{**} V_{t+m(2k+1)-1} \\
&\qquad - W_1^{**} V_{t+m(2k+1)}] Z_3(p, n) Z_3(q, r) \\
&+ b^{-m(2k+1)+n+q}(a^2 - 4b)^{k-2} U_m^{2k+1} (bW_0^{**} V_{t+m(2k+1)-1} \\
&\qquad - W_1^{**} V_{t+m(2k+1)}) D(W_0^*, W_1^*) D(W_0, W_1) V_{p+r-n-q} .
\end{aligned} \tag{9.12}$$

Theorem 10. Let W_n and W_n^* be solutions of (1.1). Let m, p, n, q, r, t, and s be integers (+, -, or 0). Then, for $k > 0$,

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} W_{p+im} W_{q+im} W_{s+im} W^*_{n-im} W^*_{r-im} W^*_{t-im}$$

$$\begin{aligned}
&= b^s U_m^{2k} (a^2 - 4b)^{k-3} Z_1(m, k, s, t) Z_3(p, n) Z_3(q, r) \\
&+ b^p U_m^{2k} (a^2 - 4b)^{k-3} Z_1(m, k, p, n) Z_3(q, r) Z_3(s, t) \\
&+ b^q U_m^{2k} (a^2 - 4b)^{k-3} Z_1(m, k, q, r) Z_3(p, n) Z_3(s, t) \\
&+ b^{p+q} U_{2m}^{2k} (a^2 - 4b)^{k-3} Z_3(s, t) A(V_{n+r-p-q-4mk}) \\
&+ b^{p+s} U_{2m}^{2k} (a^2 - 4b)^{k-3} Z_3(q, r) A(V_{n+t-p-s-4mk}) \\
&+ b^{q+s} U_{2m}^{2k} (a^2 - 4b)^{k-3} Z_3(p, n) A(V_{r+t-q-s-4mk})
\end{aligned} \tag{9.13}$$

(Continued, next page.)

$$+ b^{p+q+t}U_m^{2k}(a^2 - 4b)^{k-3}D(W_0, W_1)D(W_0^*, W_1^*)Z_1(m,k,p+q+t-r-s,n)$$
$$+ b^{n+q+s}U_m^{2k}(a^2 - 4b)^{k-3}D(W_0, W_1)D(W_0^*, W_1^*)Z_1(m,k,s,t)V_{p+r-n-q}$$
$$+ W_0W_1^*b^{p+q+s}U_{3m}^{2k}(a^2 - 4b)^{k-3}A(V_{n+r+t-p-q-s-6mk+1})$$
$$- (bW_0W_0^* + W_1W_1^*)b^{p+q+s}U_{3m}^{2k}(a^2 - 4b)^{k-3}A(V_{n+r+t-p-q-s-6mk})$$
$$+ W_0^*W_1b^{p+q+s+1}U_{3m}^{2k}(a^2 - 4b)^{k-3}A(V_{n+r+t-p-q-s-6mk-1}),$$

(9.14)
$$\sum_{i=0}^{2k+1}(-1)^i\binom{2k+1}{i}W_{p+im}W_{q+im}W_{s+im}W_{n-im}^*W_{r-im}^*W_{t-im}^* \qquad (k \geq 0)$$
$$= b^sU_m^{2k+1}(a^2 - 4b)^{k-2}Z_2(m,k,s,t)Z_3(p,n)Z_3(q,r)$$
$$+ b^pU_m^{2k+1}(a^2 - 4b)^{k-2}Z_2(m,k,p,n)Z_3(q,r)Z_3(s,t)$$
$$+ b^qU_m^{2k+1}(a^2 - 4b)^{k-2}Z_2(m,k,q,r)Z_3(p,n)Z_3(s,t)$$
$$+ b^{p+q}U_{2m}^{2k+1}(a^2 - 4b)^{k-2}Z_3(s,t)A(U_{n+r-p-q-2m(2k+1)})$$
$$+ b^{p+s}U_{2m}^{2k+1}(a^2 - 4b)^{k-2}Z_3(q,r)A(U_{n+t-p-s-2m(2k+1)})$$
$$+ b^{q+s}U_{2m}^{2k+1}(a^2 - 4b)^{k-2}Z_3(p,n)A(U_{r+t-q-s-2m(2k+1)})$$
$$+ b^{p+q+t}U_m^{2k+1}(a^2 - 4b)^{k-2}D(W_0, W_1)D(W_0^*, W_1^*)Z_2(m,k,p+q+t-r-s,n)$$
$$+ b^{n+q+s}U_m^{2k+1}(a^2 - 4b)^{k-2}D(W_0, W_1)D(W_0^*, W_1^*)Z_2(m,k,s,t)V_{p+r-n-q}$$
$$+ W_0W_1^*b^{p+q+s}U_{3m}^{2k+1}(a^2 - 4b)^{k-2}A(U_{n+r+t-p-q-s+1-3m(2k+1)})$$
$$- (bW_0W_0^* + W_1W_1^*)b^{p+q+s}U_{3m}^{2k+1}(a^2 - 4b)^{k-2}A(U_{n+r+t-p-q-s-3m(2k+1)})$$
$$+ W_0^*W_1b^{p+q+s+1}U_{3m}^{2k+1}(a^2 - 4b)^{k-2}A(U_{n+r+t-p-q-s-1-3m(2k+1)}).$$

<u>Remarks.</u> As a special case of (9.13), we have

(9.15)
$$\sum_{i=0}^{2k}(-1)^i\binom{2k}{i}U_{p+im}U_{q+im}U_{s+im}U_{n-im}U_{r-im}U_{t-im} \qquad (k > 0)$$
$$= -b^sU_m^{2k}(a^2 - 4b)^{k-3}V_{t-2km-s}V_{p+n}V_{q+r}$$
$$- b^pU_m^{2k}(a^2 - 4b)^{k-3}V_{n-2km-p}V_{q+r}V_{s+t}$$
$$- b^qU_m^{2k}(a^2 - 4b)^{k-3}V_{r-2km-q}V_{p+n}V_{s+t}$$

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$$+ b^{p+q} U_{2m}^{2k} (a^2 - 4b)^{k-3} V_{s+t} V_{n+r-p-q-4mk}$$

$$+ b^{p+s} U_{2m}^{2k} (a^2 - 4b)^{k-3} V_{q+r} V_{n+t-p-s-4mk}$$

$$+ b^{q+s} U_{2m}^{2k} (a^2 - 4b)^{k-3} V_{p+n} V_{r+t-q-s-4mk}$$

$$- b^{p+q+t} U_{m}^{2k} (a^2 - 4b)^{k-3} V_{n-2km-p-q-t+r+s}$$

$$- b^{n+q+s} U_{m}^{2k} (a^2 - 4b)^{k-3} V_{t-2km-s} V_{p+r-n-q}$$

$$- b^{p+q+s} U_{3m}^{2k} (a^2 - 4b)^{k-3} V_{n+r+t-p-q-s-6mk} \quad .$$

10. BINOMIAL SUMS WITH TRIPLE CROSS-PRODUCTS

The following results are an extension of Theorem 3.

Theorem 11. Let W_n be a solution of (1.1). Let m, p_1, p_2, and p_3 be integers (+, -, or 0). Let $p_1 + p_2 + p_3 = p$. Then

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i \left(\prod_{j=1}^{3} W_{p_j+im} \right) \qquad (k > 0)$$

$$(10.1) \qquad = U_{3m}^{2k} (a^2 - 4b)^{k-1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i U_{p+2mk-i}$$

$$+ D(W_0, W_1) b^{4mk} U_m^{2k} (a^2 - 4b)^{k-1} \sum_{j=1}^{3} b^{p_j} W_{p-2p_j-2mk} \quad ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i \left(\prod_{j=1}^{3} W_{p_j+im} \right) \qquad (k \geq 0)$$

$$(10.2) \qquad = -U_{3m}^{2k+1} (a^2 - 4b)^{k-1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i V_{p+m(2k+1)-i}$$

$$- D(W_0, W_1) b^{2m(2k+1)} U_m^{2k+1} (a^2 - 4b)^{k-1} Y_1 \quad ,$$

where

$$(10.3)\qquad Y_1 = \sum_{j=1}^{3} b^{p_j}(W_1 V_{p-2p_j-m(2k+1)} - bW_0 V_{p-2p_j-m(2k+1)-1}) .$$

Special cases of Theorem 11, with $p_1 = p_2 = p_3 = n$, are given by

$$(10.4)\qquad \begin{aligned} &\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i U_{n+im}^3 \qquad (k > 0) \\ &= (a^2 - 4b)^{k-1}(U_{3m}^{2k} U_{3n+2mk} - 3b^{n+4mk} U_m^{2k} U_{n-2mk}) , \end{aligned}$$

$$(10.5)\qquad \begin{aligned} &\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i V_{n+im}^3 \qquad (k > 0) \\ &= (a^2 - 4b)^{k}(U_{3m}^{2k} V_{3n+2mk} + 3b^{n+4mk} U_m^{2k} V_{n-2mk}) , \end{aligned}$$

$$(10.6)\qquad \begin{aligned} &\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i U_{n+im}^3 \qquad (k \ge 0) \\ &= (a^2 - 4b)^{k-1}(-U_{3m}^{2k+1} V_{3n+m(2k+1)} + 3b^{n+2m(2k+1)} U_m^{2k+1} V_{n-m(2k+1)}) , \end{aligned}$$

$$(10.7)\qquad \begin{aligned} &\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i V_{n+im}^3 \qquad (k \ge 0) \\ &= (a^2 - 4b)^{k+1}(-U_{3m}^{2k+1} U_{3n+m(2k+1)} - 3b^{n+2m(2k+1)} U_m^{2k+1} U_{n-m(2k+1)}) . \end{aligned}$$

11. BINOMIAL SUMS WITH FOUR CROSS-PRODUCTS

The following results are an extension of Theorem 3.

Theorem 12. Let W_n be a solution of (1.1). Let m, p_1, p_2, p_3, and p_4 be integers (+, -, or 0). Let $p_1 + p_2 + p_3 + p_4 = p$. Then

$$(11.1) \qquad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i \left(\prod_{j=1}^{4} W_{p_j+im} \right) \qquad (k > 0)$$

$$= U_{4m}^{2k} (a^2 - 4b)^{k-2} \sum_{i=0}^{4} (-1)^i \binom{4}{i} W_1^{4-i} (b W_0)^i V_{p+2mk-i}$$

$$+ U_m^{2k} U_{2m}^{2k} D^2(W_0, W_1)(a^2 - 4b)^{2k-2} \sum_{i=1}^{3} b^{2mk+p_1+p_{1+i}} V_{p-2p_1-2p_{1+i}}$$

$$+ U_{2m}^{2k} D(W_0, W_1)(a^2 - 4b)^{k-2} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i \sum_{j=1}^{4} b^{4mk+p_j} V_{p-2p_j-2mk-i},$$

$$(11.2) \qquad \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i \left(\prod_{j=1}^{4} W_{p_j+im} \right) \qquad (k \geq 0)$$

$$= -U_{4m}^{2k+1} (a^2 - 4b)^{k-1} \sum_{i=0}^{4} (-1)^i \binom{4}{i} W_1^{4-i} (bW_0)^i U_{p+m(2k+1)-i}$$

$$+ (U_m U_{2m})^{2k+1} D^2(W_0, W_1)(a^2 - 4b)^{2k-1} \sum_{i=1}^{3} b^{m(2k+1)+p_1+p_{1+i}} V_{p-2p_1-2p_{1+i}}$$

$$- U_{2m}^{2k+1} D(W_0, W_1)(a^2 - 4b)^{k-1}$$

$$\cdot \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i \sum_{j=1}^{4} b^{2m(2k+1)+p_j} U_{p-2p_j-m(2k+1)-i} .$$

Special cases of Theorem 12, with $p_1 = p_2 = p_3 = p_4 = n$, are given by

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i U_{n+im}^4 \qquad (k > 0)$$

$$(11.3) \qquad \begin{aligned} &= U_{4m}^{2k} (a^2 - 4b)^{k-2} V_{4n+2mk} \\ &+ 6\, U_m^{2k} U_{2m}^{2k} (a^2 - 4b)^{2k-2}\, b^{2mk+2n} \\ &- 4\, U_{2m}^{2k} (a^2 - 4b)^{k-2}\, b^{4mk+n} V_{2n-2mk}, \end{aligned}$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i V_{n+im}^4 \qquad (k > 0)$$

$$(11.4) \qquad \begin{aligned} &= U_{4m}^{2k} (a^2 - 4b)^{k} V_{4n+2mk} \\ &+ 6\, U_{2m}^{2k} (a^2 - 4b)^{2k}\, b^{2mk+2n} U_m^{2k} \\ &+ 4\, U_{2m}^{2k} (a^2 - 4b)^{k}\, b^{4mk+n} V_{2n-2mk}, \end{aligned}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i U_{n+im}^4 \qquad (k \geq 0)$$

$$(11.5) \qquad \begin{aligned} &= -U_{4m}^{2k+1} (a^2 - 4b)^{k-1} U_{4n+m(2k+1)} \\ &+ 6(U_m U_{2m})^{2k+1} (a^2 - 4b)^{2k-1}\, b^{m(2k+1)+2n} \\ &+ 4\, U_{2m}^{2k+1} (a^2 - 4b)^{k-1}\, b^{2m(2k+1)+n} U_{2n-m(2k+1)}, \end{aligned}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i V_{n+im}^4 \qquad (k \geq 0)$$

$$(11.6) \qquad \begin{aligned} &= -U_{4m}^{2k+1} (a^2 - 4b)^{k+1} U_{4n+m(2k+1)} \\ &+ 6(U_m U_{2m})^{2k+1} (a^2 - 4b)^{2k+1}\, b^{m(2k+1)+2n} \\ &- 4\, U_{2m}^{2k+1} (a^2 - 4b)^{k+1}\, b^{2m(2k+1)+n} U_{2n-m(2k+1)}. \end{aligned}$$

12. BINOMIAL SUMS WITH MIXED CROSS-PRODUCTS

The following results are companion results for Theorems 11 and 12.

Theorem 13. Let W_n be a solution of (1.1). Let m, p_1, p_2, and p_3 be integers (+, -, or 0). Let $p_1 + p_2 + p_3 = p$. Then

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i \left(V_{p_1+im} \prod_{j=2}^{3} W_{p_j+im} \right) \qquad (k > 0)$$

(12.1)

$$= U_{3m}^{2k} (a^2 - 4b)^{k-1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p+2mk-i}$$

$$+ D(W_0, W_1) b^{4mk} U_m^{2k} (a^2 - 4b)^{k-1} \sum_{j=2}^{3} b^{p_j} V_{p-2p_j-2mk}$$

$$+ U_m^{2k} (a^2 - 4b)^{k-1} b^{4mk+p_1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p-2p_1-2mk-i} ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i \left(V_{p_1+im} \prod_{j=2}^{3} W_{p_j+im} \right) \qquad (k \ge 0)$$

(12.2)

$$= -U_{3m}^{2k+1} (a^2 - 4b)^k \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i U_{p+m(2k+1)-i}$$

$$- D(W_0, W_1) b^{2m(2k+1)} U_m^{2k+1} (a^2 - 4b)^k \sum_{j=2}^{3} b^{p_j} U_{p-2p_j-m(2k+1)}$$

$$- U_m^{2k+1} (a^2 - 4b)^k b^{2m(2k+1)+p_1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i U_{p-2p_1-m(2k+1)-i}$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{3} W_{p_j+im} \right) \qquad (k > 0)$$

(12.3)

$$= U_{3m}^{2k} (a^2 - 4b)^{k-1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i U_{p+2mk-i}$$

$$+ U_m^{2k} D(W_0, W_1)(a^2 - 4b)^{k-1} \sum_{j=2}^{3} b^{4mk+p_j} U_{p-2p_j-2mk}$$

$$- (a^2 - 4b)^{k-1} U_m^{2k} b^{4mk+p_1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i U_{p-2p_1-2mk-i} ,$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{3} V_{p_j+im} \right) \qquad (k > 0)$$

(12.4)

$$= (a^2 - 4b)^k U_{3m}^{2k} U_{p+2mk}$$

$$+ U_m^{2k} (a^2 - 4b)^k \sum_{j=2}^{3} b^{4mk+p_j} U_{p-2p_j-2mk}$$

$$- (a^2 - 4b)^k U_m^{2k} b^{4mk+p_1} U_{p-2p_1-2mk} ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{3} W_{p_j+im} \right) \qquad (k \geq 0)$$

(12.5)

$$= -U_{3m}^{2k+1} (a^2 - 4b)^{k-1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^i (bW_0)^i V_{p+m(2k+1)-i}$$

$$- U_m^{2k+1} D(W_0, W_1)(a^2 - 4b)^{k-1} \sum_{j=2}^{3} b^{2m(2k+1)+p_j} V_{p-2p_j-m(2k+1)}$$

$$+ (a^2 - 4b)^{k-1} U_m^{2k+1} b^{2m(2k+1)+p_1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p-2p_1-m(2k+1)-i} ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{3} V_{p_j+im} \right) \qquad (k \geq 0)$$

$$= -U_{3m}^{2k+1} (a^2 - 4b)^k V_{p+m(2k+1)}$$

(12.6)

$$- U_m^{2k+1} (a^2 - 4b)^k \sum_{j=2}^{3} b^{2m(2k+1)+p_j} V_{p-2p_j-m(2k+1)}$$

$$+ (a^2 - 4b)^k U_m^{2k+1} b^{2m(2k+1)+p_1} V_{p-2p_1-m(2k+1)} .$$

Theorem 14. Let W_n be a solution of (1.1). Let m, p_1, p_2, p_3, and p_4 be integers (+, -, or 0).

Let $p_1 + p_2 + p_3 + p_4 = p$. Then

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i \left(V_{p_1+im} \prod_{j=2}^{4} W_{p_j+im} \right) \qquad (k > 0)$$

$$= (a^2 - 4b)^{k-1} U_{4m}^{2k} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i U_{p+2mk-i}$$

(12.7)

$$+ D(W_0, W_1)(a^2 - 4b)^{2k-1} (U_m U_{2m})^{2k} \sum_{i=1}^{3} b^{2mk+p_1+p_{1+i}} W_{p-2p_1-2p_{1+i}}$$

$$+ D(W_0, W_1)(a^2 - 4b)^{k-1} U_{2m}^{2k} \sum_{j=2}^{4} b^{4mk+p_j} W_{p-2p_j-2mk}$$

$$+ (a^2 - 4b)^{k-1} U_{2m}^{2k} b^{4mk+p_1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i U_{p-2p_1-2mk-i} ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i \left(V_{p_1+im} \prod_{j=2}^{4} W_{p_j+im} \right) \qquad (k \geq 0)$$

$$= -(a^2 - 4b)^{k-1} U_{4m}^{2k+1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i V_{p+m(2k+1)-i}$$

(12.8)

$$+ D(W_0, W_1)(a^2 - 4b)^{2k} (U_m U_{2m})^{2k+1} \sum_{i=1}^{3} b^{m(2k+1)+p_1+p_{1+i}} W_{p-2p_1-2p_{1+i}}$$

$$- (a^2 - 4b)^{k-1} U_{2m}^{2k+1} b^{2m(2k+1)+p_1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i V_{p-2p_1-m(2k+1)-i}$$

$$- (a^2 - 4b)^{k-1} D(W_0, W_1) U_{2m}^{2k+1} b^{2m(2k+1)} Y_2(2k+1) \quad ,$$

where

(12.9) $$Y_2(k) = \sum_{j=2}^{4} b^{p_j} (W_1 V_{p-2p_j-mk} - bW_0 V_{p-2p_j-mk-1}) \quad ,$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i \left(\prod_{j=1}^{2} V_{p_j+im} \right) \left(\prod_{j=3}^{4} W_{p_j+im} \right) \qquad (k > 0)$$

(12.10)

$$= (a^2 - 4b)^{k-1} U_{4m}^{2k} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p+2mk-i}$$

$$+ (a^2 + 4b)^{2k-1} (U_m U_{2m})^{2k} b^{2mk+p_1+p_2} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p-2p_1-2p_2-i}$$

$$+ D(W_0, W_1)(a^2 - 4b)^{2k-1} (U_m U_{2m})^{2k} \sum_{j=1}^{2} b^{2mk+p_1+p_{2+j}} V_{p-2p_1-2p_{2+j}}$$

(Continued, next page.)

$$+ (a^2 - 4b)^{k-1}U_{2m}^{2k}\sum_{j=1}^{2} b^{4mk+p_j}\sum_{i=0}^{2}(-1)^i\binom{2}{i}W_1^{2-i}(bW_0)^iV_{p-2p_j-2mk-i}$$

$$+ (a^2 - 4b)^{k-1}D(W_0, W_1)U_{2m}^{2k}\sum_{j=3}^{4} b^{4mk+p_j}V_{p-2p_j-2mk} \quad ,$$

(12.11)
$$\sum_{i=0}^{2k+1}(-1)^i\binom{2k+1}{i}(b^mV_{3m})^{2k+1-i}V_m^i\left(\prod_{j=1}^{2}V_{p_j+im}\right)\left(\prod_{j=3}^{4}W_{p_j+im}\right) \qquad (k \ge 0)$$

$$= -(a^2 - 4b)^kU_{4m}^{2k+1}\sum_{i=0}^{2}(-1)^i\binom{2}{i}W_1^{2-i}(bW_0)^iU_{p+m(2k+1)-i}$$

$$+ (a^2 - 4b)^{2k}(U_mU_{2m})^{2k+1}b^{m(2k+1)+p_1+p_2}\sum_{i=0}^{2}(-1)^i\binom{2}{i}W_1^{2-i}(bW_0)^iV_{p-2p_1-2p_2-i}$$

$$+ D(W_0, W_1)(a^2 - 4b)^{2k}(U_mU_{2m})^{2k+1}\sum_{j=1}^{2} b^{m(2k+1)+p_1+p_{2+j}}V_{p-2p_1-2p_{2+j}}$$

$$- (a^2 - 4b)^kU_{2m}^{2k+1}\sum_{j=1}^{2} b^{2m(2k+1)+p_j}\sum_{i=0}^{2}(-1)^i\binom{2}{i}W_1^{2-i}(bW_0)^iU_{p-2p_j-m(2k+1)-i}$$

$$- (a^2 - 4b)^kU_{2m}^{2k+1}D(W_0, W_1)\sum_{j=3}^{4} b^{2m(2k+1)+p_j}U_{p-2p_j-m(2k+1)} \quad ,$$

(12.12)
$$\sum_{i=0}^{2k}(-1)^i\binom{2k}{i}(b^mV_{3m})^{2k-i}V_m^i\left(U_{p_1+im}\prod_{j=2}^{4}W_{p_j+im}\right) \qquad (k > 0)$$

$$= (a^2 - 4b)^{k-2}U_{4m}^{2k}\sum_{i=0}^{3}(-1)^i\binom{3}{i}W_1^{3-i}(bW_0)^iV_{p+2mk-i}$$

(Continued, next page.)

$$- (a^2 - 4b)^{2k-2} D(W_0, W_1)(U_m U_{2m})^{2k} \sum_{j=1}^{3} b^{2mk+p_1+p_{1+j}} (W_1 V_{p-2p_1-2p_{1+j}} - bW_0 V_{p-2p_1-2p_{1+j}-1})$$

$$- (a^2 - 4b)^{k-2} U_{2m}^{2k} b^{4mk+p_1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i V_{p-2p_1-2mk-i}$$

$$+ (a^2 - 4b)^{k-2} U_{2m}^{2k} D(W_0, W_1) \sum_{j=2}^{4} b^{4mk+p_j} (W_1 V_{p-2p_j-2mk} - bW_0 V_{p-2p_j-2mk-1}) ,$$

$$(12.13) \quad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{4} V_{p_j+im} \right) \qquad (k > 0)$$

$$= (a^2 - 4b)^k U_{4m}^{2k} U_{p+2mk}$$

$$- (a^2 - 4b)^{2k} U_m^{2k} U_{2m}^{2k} \sum_{j=1}^{3} b^{2mk+p_1+p_{1+j}} U_{p-2p_1-2p_{1+j}}$$

$$- (a^2 - 4b)^k U_m^{2k} b^{4mk+p_1} U_{p-2p_1-2mk}$$

$$+ (a^2 - 4b)^k U_{2m}^{2k} \sum_{j=2}^{4} b^{4mk+p_j} U_{p-2p_j-2mk} ,$$

$$(12.14) \quad \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{4} W_{p_j+im} \right) \qquad (k \geq 0)$$

$$= - (a^2 - 4b)^{k-1} U_{4m}^{2k+1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i U_{p+m(2k+1)-i}$$

$$- (a^2 - 4b)^{2k-1} D(W_0, W_1)(U_m U_{2m})^{2k+1} \sum_{j=1}^{3} b^{m(2k+1)+p_1+p_{1+j}} (W_1 V_{p-2p_1-2p_{1+j}} - bW_0 V_{p-2p_1-2p_{1+j}-1})$$

(Continued, next page.)

$$-(a^2-4b)^{k-1}D(W_0,W_1)U_{2m}^{2k+1}\sum_{j=2}^{4} b^{2m(2k+1)+p_j}W_{p-2p_j-m(2k+1)}$$

$$+(a^2-4b)^{k-1}U_{2m}^{2k+1}b^{2m(2k+1)+p_1}\sum_{i=0}^{3}(-1)^i\binom{3}{i}W_1^{3-i}(bW_0)^iU_{p-2p_1-m(2k+1)-i}\ ,$$

$$\text{(12.15)}\quad \sum_{i=0}^{2k+1}(-1)^i\binom{2k+1}{i}(b^mV_{3m})^{2k+1-i}V_m^i\left(U_{p_1+im}\prod_{j=2}^{4}V_{p_j+im}\right)\qquad (k>0)$$

$$= -(a^2-4b)^kU_{4m}^{2k+1}V_{p+m(2k+1)}$$

$$-(a^2-4b)^{2k+1}(U_mU_{2m})^{2k+1}\sum_{j=1}^{3}b^{m(2k+1)+p_1+p_{1+j}}U_{p-2p_1-2p_{1+j}}$$

$$-(a^2-4b)^kU_{2m}^{2k+1}\sum_{j=2}^{4}b^{2m(2k+1)+p_j}V_{p-2p_j-m(2k+1)}$$

$$+(a^2-4b)^kU_{2m}^{2k+1}b^{2m(2k+1)+p_1}V_{p-2p_1-m(2k+1)}\ .$$

13. REMARKS ON THE PAPER BY CARLITZ AND FERNS [4]

All the important identities of Sections 1 and 2 of the above paper are special cases of our general results. Indeed, for the proper choices of parameters, our result, (1.7), contains as special cases, identities (1.6), (1.8), (1.10), (1.11), and (1.12) of [4, pp. 62-64]. In [4, pp. 65-66], I noted misprints and omissions in (2.9) (for n odd), (2.10) (for n even and odd), and (2.11) (for n even and odd). If these errors are corrected, we can then say, for the proper choices of parameters, our (1.9) gives their (2.8) and (2.10) for even n; our (1.13) gives their (2.8) and (2.10) for odd n. Also, our (1.12) gives their (2.9) and (2.11) for odd n. The odd and even n refers to their identity usage, not ours. Our (1.11) gives their (2.9) and (2.11) for even n.

The remaining portion of [4] obtained transformation identities for the Fibonacci and Lucas sequences as an application of Legendre and Jacobi polynomials. Under proper linear substitutions, these same polynomials could give transformation identities for U_n and V_n of (1.1). We will illustrate these ideas with a pair of identities suitable for our purposes.

The following pair of polynomial identities,

$$x^{2n+1} - (x-1)^{2n+1} = \sum_{i=0}^{n} \frac{2n+1}{2i+1}\binom{n+i}{2i}(x^2-x)^{n-i}, \tag{13.1}$$

$$x^{2n+2} - (x-1)^{2n+2} = (2x-1)\sum_{i=0}^{n} \binom{n+1+i}{2i+1}(x^2-x)^{n-i}, \tag{13.2}$$

appeared as a proposed problem 4356, p. 479, in the American Mathematical Monthly, 56 (1949), and their solution, in the same journal, 58 (1951), appears on pp. 268-269. We now proceed to apply (13.1) and (13.2) to obtain identities for U_n and V_n of (1.1).

Recalling that α and β are roots of $x^2 = ax - b$ (see (1.1)), set $x = (a/b)y$ in (13.1) to obtain

$$a^{2n+1}y^{2n+1} - (ay-b)^{2n+1} = \sum_{i=0}^{n} \frac{2n+1}{2i+1}\binom{n+i}{2i} b^{2i+1}[ay(ay-b)]^{n-i} . \tag{13.3}$$

Thus, (13.3) for $y = \alpha$ and $y = \beta$ gives the identities

$$a^{2n+1}V_{2n+1} - V_{4n+2} = \sum_{i=0}^{n} \frac{2n+1}{2i+1}\binom{n+i}{2i} b^{2i+1}a^{n-i}V_{3n-3i} , \tag{13.4}$$

$$a^{2n+1}U_{2n+1} - U_{4n+2} = \sum_{i=0}^{n} \frac{2n+1}{2i+1}\binom{n+i}{2i} b^{2i+1}a^{n-i}U_{3n-3i} . \tag{13.5}$$

In (13.2), set $x = (ay)/(2b)$ to obtain for even $n = 2k$, noting that $a\alpha - 2b = \alpha^2 - b = \alpha(\alpha - \beta)$, $\alpha\beta - 2b = \beta(\beta - \alpha)$,

$$\left\{a^{4k+2} - (a^2 - 4b)^{2k+1} - (a^2 - 4b)^k a^{2k}(4b)(2k + 1)\right\}V_{4k+2}$$

$$(13.6) \qquad = 2\sum_{j=0}^{k-1} (a^2 - 4b)^j a^{2j}(2b)^{4(k-j)+1}\binom{4k + 1 - 2j}{2j}V_{4j+2}$$

$$+ 2\sum_{i=1}^{k} (a^2 - 4b)^i a^{2i-1}(2b)^{4(k-i)+3}\binom{4k + 2 - 2i}{2i - 1}U_{4i} \quad .$$

An identity similar to (13.6) is obtained for $n = 2k + 1$. We note that the factor $(2x - 1)$ in (13.2) is troublesome for obtaining identities in U_n and V_n for (1.1), but is not so for the Fibonacci sequences.

Additional identities for U_n and V_n are readily obtained from (13.1). For complete generality, we note that α and β satisfy $x^m = U_m x - bU_{m-1}$. Thus, from (13.1), we obtain, having set $x = (U_m y)/(bU_{m-1})$, the general identity

$$(13.7) \qquad U_m^{2n+1} W_{2n+1+p} - W_{2mn+m+p}$$

$$= \sum_{i=0}^{n} \frac{2n + 1}{2i + 1}\binom{n + i}{2i} U_m^{n-i}(bU_{m-1})^{2i+1} W_{(m+1)(n-i)+p} \quad ,$$

where U_n and W_n are solutions of (1.1).

It should be noted that (13.1) gives Fibonacci identities that are not special cases of (13.7). As a partial listing, we have

$$(13.8) \qquad L_{2n+1} = \sum_{i=0}^{n} \frac{2n + 1}{2i + 1}\binom{n + i}{2i} \quad ,$$

$$(13.9) \qquad H_{12n+6+p} - 4^{2n+1}H_{6n+3+p} = \sum_{i=0}^{n} \frac{2n + 1}{2i + 1}\binom{n + i}{2i}4^{n-i}H_{9n-9i+p} \quad ,$$

[Continued on page 421.]

ADVANCED PROBLEMS AND SOLUTIONS

Edited By
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-183 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California.

Consider the display indicated below.

1						
1	1					
2	2	1				
5	4	3	1			
13	9	7	4	1		
34	22	16	11	5	1	
89	56	38	27	16	6	1
			...			

(i) Find an expression for the row sums.
(ii) Find a generating function for the row sums.
(iii) Find a generating function for the rising diagonal sums.

H-184 Proposed by Raymond E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Define the cycle α_n $(n = 1, 2, \cdots)$ as follows:

(1) $\alpha_n = (1234 + \cdots + F_n)$, where F_n denotes the n^{th} Fibonacci number.

Now construct a sequence of permutations

$$\left\{\alpha_n^{F_i}\right\}_{i=1}^{\infty}, \qquad (n = 1, 2)$$

where

$$\text{(ii)} \qquad \alpha_n^{F_{i+2}} = \alpha_n^{F_i} \cdot \alpha_n^{F_{i+1}} \qquad (i \geq 1) .$$

Finally, define a sequence $\{u_n\}_{n=1}^{\infty}$ as follows:
u_n is the period of (ii); i.e., u_n is the smallest positive integer such that

$$\text{(iii)} \qquad \alpha_n^{F_{i+u_n}} = \alpha_n^{F_i} \qquad (i \geq N) .$$

a. Find a closed-form expression for u_n.

b. If possible, show $N = 1$ is the minimum positive integer for which (iii) holds for all n.

H-185 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$(1 - 2x)^n = \sum_{k=0}^{n} (-1)^{n-k}\binom{n+k}{2k}\binom{2k}{k}(1 - v)^{n-k}\,{}_2F_1[-k,\ n + k + 1;\ k + 1;\ x] ,$$

where ${}_2F_1[a, b;\ c;\ x]$ denotes the hypergeometric function.

SOLUTIONS

H-127 REVISITED

H-164 Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.

Generalize H-127 and find a recurrence relation for the product

$$C_n = A_n(x)\,B_n(y),$$

where A_n and B_n satisfy the general second-order recurrence equations:

(1) $$A_{n+1}(x) = R(x)\,A_n(x) + S(x)\,A_{n-1}(x)$$

(2) $$B_{n+1}(y) = P(y)\,B_n(y) + Q(y)\,B_{n-1}(y),$$

$a \geq 1$ and A_0, A_1, B_0, B_1 arbitrary.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

We consider the following more general situation. Let E denote the operator defined by $Ef(n) = f(n+1)$. Let $\alpha_1, \cdots, \alpha_r$, $\beta_1, \cdots, \beta_s$ denote $r + s$ arbitrary constants and assume that

(1) $$(E - \alpha_1) \cdots (E - \alpha_r)\,A_n = 0$$

(2) $$(E - \beta_1) \cdots (E - \beta_s)\,B_n = 0\,.$$

If $C_n = A_n B_n$, we shall show that

(3) $$\prod_{i=1}^{r}\prod_{j=1}^{s}(E - \alpha_i\beta_j)\cdot C_n = 0\,.$$

If the α's are distinct and the β's are distinct, the proof of this assertion is easy. In this case, the general solution of (1) is given by

$$A_n = c_1\alpha_1^n + \cdots + c_r\alpha_r^n\ ,$$

where $c_1, \cdots, c_r$ are independent of n; the general solution of (2) is

$$B_n = d_1\beta_1^n + \cdots + d_s\beta_s^n\ ,$$

where $d_1, \cdots, d_s$ are independent of n. Then

$$C_n = c_1 d_1 (\alpha_1 \beta_1)^n + \cdots + c_r d_s (\alpha_r \beta_s)^n$$

and (3) follows at once.

For the general case we require the following lemma. Let

$$(E - \alpha)^r A_n = 0, \qquad (E - \beta)^s B_n = 0 .$$

Then $C_n = A_n B_n$ satisfies

$$(E - \alpha\beta)^{r+s-1} C_n = 0.$$

To prove this, note that

$$A_n = P_{r-1}(n) \alpha^n ,$$

where $P_{r-1}(n)$ is a polynomial in n of degree $r - 1$ with arbitrary constant coefficients:

$$B_n = Q_{a-1}(n) \beta^n ,$$

where $Q_{s-1}(n)$ is a polynomial in n of degree $s - 1$ with arbitrary constant coefficients. Then

$$C_n = P_{r-1}(n) Q_{s-1}(n) (\alpha\beta)^n$$

and the assertion follows at once.

Now let

$$(E - \alpha_1)^{e_1} \cdots (E - \alpha_r)^{e_r} A_n = 0$$

$$(E - \beta_1)^{f_1} \cdots (E - \beta_s)^{f_s} B_n = 0,$$

where the α's and β's are distinct. Then, by the lemma,

$$\prod_{i=1}^{r}\prod_{j=1}^{s}(E - \alpha_i\beta_j)^{e_i+f_j-1} \cdot C_n = 0 . \tag{4}$$

This result is somewhat stronger than (3). The degree of the operator in the left member of (4) is equal to

$$\sum_{i=1}^{r}\sum_{j=1}^{s}(e_i + f_j - 1) = s\sum_{i=1}^{r} e_i + r\sum_{j=1}^{s} f_j - rs .$$

When some of the α's and β's are equal, c_n may satisfy a recurrence of even lower degree. For example, if

$$(E - \alpha_1)\cdots(E - \alpha_r)A_n = 0,$$

$$(E - \alpha_1)\cdots(E - \alpha_r)B_n = 0 ,$$

then C_n satisfies

$$(E - \alpha_1^2)(E - \alpha_1\alpha_2)\cdots(E - \alpha_r^2)C_n = 0 ,$$

a recurrence of order $n(n + 1)/2$.

Also solved by C. B. A. Peck, M. Yoder, and the Proposer.

SHORT-TERM INDUCTION

H-165 Proposed by H. H. Ferns, Victoria, B.C., Canada.

Prove the identity

$$\sum_{i=1}^{n}\binom{n}{i}\frac{F_{ki}}{F_{k-2}^{i}} = \left(\frac{F_k}{F_{k-2}}\right)^n F_{2n} \qquad (k \neq 2) ,$$

where F_i denotes the i^{th} Fibonacci number.

Solution by the Proposer.

The proof of the given identities is based on the two identities:

(1) $$F_{n-2} + \alpha^n = \alpha^2 F_n$$

(2) $$F_{n-2} + \beta^n = \beta^2 F_n ,$$

in which $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. These are readily proved by induction on n. Thus, if in (1), we put $n = 1$, we get

$$F_{-1} + \alpha = \alpha^2 F_i$$

or

$$1 + \alpha = \alpha^2 ,$$

which is true. Assuming that (1) is true for $n = 1, 2, \cdots, r = 1, r$, we have

$$F_{r-3} + \alpha^{r-1} = \alpha^2 F_{r-1}$$

and

$$F_{r-2} + \alpha^r = \alpha^2 F_r .$$

Adding corresponding members of these two equations, we get

$$\begin{aligned} F_{r-3} + F_{r-2} + \alpha^{r-1} + \alpha^r &= \alpha^2 (F_{r-1} + F_r) \\ F_{r-1} + \alpha^{r-1}(1 + \alpha) &= \alpha^2 F_{r+1} \\ F_{r-1} + \alpha^{r+1} &= \alpha^2 F_{r+1} . \end{aligned}$$

Hence the induction is complete for the proof of (1). The proof of (2) is similar.

Continuing with the proof of the given identity, we have from (1)

$$\left(1 + \frac{\alpha^k}{F_{k-2}}\right)^n = \left(\frac{F_k}{F_{k-2}}\alpha^2\right)^n \qquad (k \neq 2) \quad .$$

Hence

$$\text{(3)} \qquad \sum_{i=0}^{n} \binom{n}{i}\left(\frac{\alpha^k}{F_{k-2}}\right)^i = \left(\frac{F_k}{F_{k-2}}\right)^n \alpha^{2n} \quad .$$

In a similar manner (2) yields

$$\text{(4)} \qquad \sum_{i=0}^{n} \binom{n}{i}\left(\frac{\beta^k}{F_{k-2}}\right)^i = \left(\frac{F_k}{F_{k-2}}\right)^n \beta^{2n} \quad .$$

Subtracting members of (4) from the corresponding members of (3) we have

$$\sum_{i=1}^{n} \binom{n}{i} \frac{F_{ki}}{F_{k-2}^i} = \left(\frac{F_k}{F_{k-2}}\right)^n F_{2n} \qquad (k \neq 2) \quad .$$

This completes the proof of the given identity.

Note that addition of (3) and (4) yields

$$\sum_{i=1}^{n} \binom{n}{i} \frac{L_{ki}}{F_{k-2}^i} = \left(\frac{F_k}{F_{k-2}}\right)^n L_{2n} - 2 \qquad (k \neq 2) \quad .$$

Some special cases are of interest. Putting $k = 1$ and $k = 3$ in these two identities, we get the following.

$$\sum_{i=1}^{n} \binom{n}{i} F_i = F_{2n}, \qquad \sum_{i=1}^{n} \binom{n}{i} F_{3i} = 2^n F_{2n}$$

$$\sum_{i=1}^{n} \binom{n}{i} L_i = L_{2n} - 2, \qquad \sum_{i=1}^{n} \binom{n}{i} L_{3i} = 2^n L_{2n} - 2 .$$

Also solved by A. Shannon, M. Yoder, C. B. A. Peck, L. Carlitz, and D. V. Jaiswal.

SUM EVEN INDEX

H-166 Proposed by H. H. Ferns, Victoria, B.C., Canada (Corrected).

Prove the identity

$$F_{2mn} = \begin{cases} \displaystyle\sum_{i=1}^{n} \binom{n}{i} L_m^i F_{mi}, & \text{if } m \text{ is odd} \\ \displaystyle\sum_{i=1}^{n} (-1)^{n+i} \binom{n}{i} L_m^i F_{mi}, & \text{if } m \text{ is even} . \end{cases}$$

Solution by the Proposer.

In the identity (this Journal, Vol. 7, No. 2, p. 174),

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{F_k}{F_{m-k}}\right)^i F_{mi+\lambda} = \left(\frac{F_m}{F_{m-k}}\right)^n F_{nk+\lambda} - F_\lambda, \qquad (m \neq k) ,$$

put $\lambda = 0$ and $k = 2m$. We get

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{F_{2m}}{F_{-m}}\right)^i F_{mi} = \left(\frac{F_m}{F_{-m}}\right)^n F_{2mn}$$

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{F_{2m}}{(-1)^{m+1}F_m}\right)^i F_{mi} = \left(\frac{F_m}{(-1)^{m+1}F_m}\right)^n F_{2m}$$

$$\sum_{i=1}^{n} (-1)^{(m+1)i} \binom{n}{i} \left(\frac{F_m L_m}{F_m}\right)^i F_{mi} = (-1)^{(m+1)n} F_{2mn} \quad .$$

Hence

$$F_{2mn} = \begin{cases} \displaystyle\sum_{i=1}^{n} \binom{n}{i} L_m^i F_{mi}, & \text{if } m \text{ is odd} \\ \displaystyle\sum_{i=1}^{n} (-1)^{n+i} \binom{n}{i} L_m^i F_{mi}, & \text{if } m \text{ is even} \end{cases}$$

Also solved by M. Shannon, B. Giuli, and M. Yoder.

HIGHER BRACKET

H-167 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$S_k = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k}} \quad .$$

Show that, for $k \geq 0$,

$$\text{(A)} \qquad F_{2k+2} S_{2k+2} = k + 1 - \sum_{n=1}^{2k} \frac{k - [\frac{1}{2}(n-1)]}{F_n F_{n+2}} \quad ,$$

(B) $$F_{2k+1}S_{2k+1} = S_1 - k + \sum_{n=0}^{2k+1} \frac{k - \left[\frac{n}{2}\right]}{F_n F_{n+2}} ,$$

where $[a]$ denotes the greatest integer function.

Special cases of (A) and (B) have been proved by Brother Alfred Brousseau, "Summation of Infinite Fibonacci Series," Fibonacci Quarterly, Vol. 7, No. 2, April, 1969, pp. 143-168.

Solution by the Proposer.

1. Proof of (A). It follows from the identity

$$F_{n+2k}F_{2k+2} - F_{n+2k+2}F_{2k} = F_n$$

that

$$\begin{aligned} F_{2k+2}S_{2k+2} - F_{2k}S_{2k} &= F_{2k+2}\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2k+2}} - F_{2k}\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2k}} \\ &= \sum_{n=1}^{\infty} \frac{F_{n+2k}F_{2k+2} - F_{n+2k+2}F_{2k}}{F_n F_{n+2k} F_{n+2k+2}} \\ &= \sum_{n=1}^{\infty} \frac{1}{F_{n+2k}F_{n+2k+2}} \\ &= \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} - \sum_{n=1}^{2k} \frac{1}{F_n F_{n+2}} . \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1 ,$$

we get

$$F_{2k+2}S_{2k+2} - F_{2k}S_{2k} = 1 - \sum_{n=1}^{2k} \frac{1}{F_n F_{n+2}} \qquad (k \geq 0) .$$

Then, by addition,

$$F_{2k+2}S_{2k+2} = k + 1 - \sum_{j=1}^{k} \sum_{n=1}^{2j} \frac{1}{F_n F_{n+2}}$$

$$= k + 1 - \sum_{n=1}^{2k} \frac{1}{F_n F_{n+2}} \sum_{n \leq 2j \leq 2k} 1 .$$

The inner sum is equal to

$$\sum_{\frac{n}{2} \leq j \leq k} 1 = k - \sum_{1 \leq j < \frac{n}{2}} 1 = k - \left[\tfrac{1}{2}(n - 1)\right] .$$

Therefore

$$F_{2k+2}S_{2k+2} = k + 1 - \sum_{n=1}^{2k} \frac{k - \left[\frac{1}{2}(n - 1)\right]}{F_n F_{n+2}} .$$

This evidently proves (A).

2. Proof of (B). It follows from the identity

$$F_{n+2k+1}F_{2k-1} - F_{n+2k-1}F_{2k+1} = F_n$$

that

$$F_{2k+1}S_{2k+1} - F_{2k-1}S_{2k-1} = F_{2k+1}\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+2k+1}} - F_{2k-1}\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+2k-1}}$$

$$= \sum_{n=1}^{\infty}\frac{F_{2k+1}F_{n+2k-1} - F_{2k-1}F_{n+2k+1}}{F_nF_{n+2k-1}F_{n+2k+1}}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{F_{n+2k-1}F_{n+2k+1}}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+2}} + \sum_{n=1}^{2k-1}\frac{1}{F_nF_{n+2}} ,$$

so that

$$F_{2k+1}S_{2k+1} - F_{2k-1}S_{2k-1} = -1 + \sum_{n=1}^{2k-1}\frac{1}{F_nF_{n+2}} .$$

Then, by addition,

$$F_{2k+1}S_{2k+1} - S_1 = -k + \sum_{j=1}^{k}\sum_{n=1}^{2j-1}\frac{1}{F_nF_{n+2}}$$

$$= -k + \sum_{n=1}^{2k-1}\frac{1}{F_nF_{n+2}}\sum_{n<2j\le 2k} 1 .$$

The inner sum is equal to

$$\sum_{\frac{n}{2}<j\le k} 1 = k - \sum_{j\le\frac{n}{2}} 1 = k - \left[\frac{n}{2}\right] .$$

Therefore,

$$F_{2k+1}S_{2k+1} = S_1 - k + \sum_{n=1}^{2k-1} \frac{k - \left[\frac{n}{2}\right]}{F_n F_{n+2}} .$$

This proves (B).

Also solved by M. Yoder.

[Continued from page 350.]

REFERENCES

1. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill, 1939, pp. 43-52.
2. R. L. Duncan, "Note on the Euclidean Algorithm," The Fibonacci Quarterly, Vol. 4, No. 4, pp. 367-368.
3. R. L. Duncan, "An Application of Uniform Distributions to the Fibonacci Numbers," The Fibonacci Quarterly, Vol. 5, No. 2, pp. 137-140.
4. J. L. Brown, Jr., "On Lamé's Theorem," The Fibonacci Quarterly, Vol. 5, No. 2, pp. 153-160.
5. Dale D. Shea, "On the Number of Divisions Needed in Finding the Greatest Common Divisor," The Fibonacci Quarterly, Vol. 7, No. 4, pp. 337-
6. Ivan Niven, "Irrational Numbers," Carus Monograph No. 11, M. A. A., 1956, Chapter 6.
7. L. Kuipers, "Remark on a Paper by R. L. Duncan Concerning the Uniform Distribution mod 1 of the Sequence of the Logarithms of the Fibonacci Numbers," The Fibonacci Quarterly, Vol. 7, No. 5, pp. 465-466 and 473.

ADDITIONS TO THE SUMMATION OF RECIPROCAL FIBONACCI AND LUCAS SERIES

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1. In two recent papers [1], [2], Brother U. Alfred Brousseau surveyed the status of the summation of infinite reciprocal Fibonacci series. In this paper, we will add a few summations to those of Brother Brousseau.

We will use the notations Ln and Fn for the n^{th} Lucas and Fibonacci numbers.

2. We have from Bromwich [3]

$$(1)\qquad \frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \cdots + \frac{x^{2^n}}{1-x^{2^{(n+1)}}} = \frac{1}{1-x} - \frac{1}{1-x^{2^{(n+1)}}} .$$

The left-hand expression of (1) can be converted to

$$\frac{\sqrt{5}}{r^{(2^j m)} - r^{(-2^j m)}}$$

if one substitutes $x = r^{2m}$ with r an integer and multiplies by $\sqrt{5}$. We then have

$$(2)\qquad \sum_{j=1}^{n} \frac{1}{F(2^j m)} = \sqrt{5}\left(\frac{1}{r^{(2m)} - 1} - \frac{1}{r^{(m2^{n+1})} - 1}\right) .$$

Clearly (2) gives rise to the infinite formula

$$(3)\qquad \sum_{j=1}^{\infty} \frac{1}{F(2^j m)} = \frac{\sqrt{5}}{r^{2m} - 1} .$$

3. One can easily show

$$\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \cdots + \frac{2^n}{x^{2^n}+1} \tag{4}$$

$$= \frac{1}{x-1} - \frac{2^{n+1}}{x^{2^{(n+1)}} - 1}$$

(Jolley [4] gives this formula for the infinite case.) Equation (4) can be converted by the substitution $x = r^{4m}$ into

$$\sum_{j=0}^{n} \frac{2^j s(2^{j+1}m)}{L(2^{j+1}m)} = \frac{1}{r^{4m}-1} - \frac{2^{n+1}}{r^{(2^n m)} - 1} \tag{5}$$

Since the final term of (5) goes to zero as $n \to \infty$, Eq. (5) gives rise to

$$\sum_{j=0}^{\infty} \frac{2^j s(2^{j+1}m)}{L(2^{j+1}m)} = \frac{1}{r^{4m}-1} \tag{6}$$

4. The author has not found the following summation formula in the literature:

$$\sum \frac{x^{(3^n)} + x^{2(3^n)}}{x^{(3^{n+1})} - 1} = \frac{1}{x^3 - 1} - \frac{1}{x^{(3^{n+1})} - 1} \tag{7}$$

If in (7) we set $x = r^{(4m+2)}$, we obtain

$$\sum_{j=0}^{n} \frac{F(2m+1)3^j}{L(2m+1)3^{j+1}} = \frac{1}{5}\,\frac{1}{r^{(12m+6)} - 1} - \frac{1}{r^{(4m+2)3^{n+1}} - 1} \tag{8}$$

while if in (7) we set $x = r^{(4m)}$,

$$\sum_{j=0}^{n} \frac{F(2m3^j)}{L(2m3^j)} = 5\left(\frac{1}{r^{12m} - 1} - \frac{1}{r^{(12m\,3^{n+1})} - 1}\right) . \tag{9}$$

4. Formula (7) suggests the following generalization:

$$\sum_{i=1}^{n} \left\{ \frac{\sum_{j=1}^{k-1} x^{(jk^{j-1})}}{1 - x^{(k^i)}} \right\} = \sum_{i=1}^{n} \frac{x^{(k^i-1)} - x^{(k^i)}}{\left[1 - x^{(k^i)}\right]\left[1 - x^{(k^i-1)}\right]} = \frac{1}{1 - x} - \frac{1}{1 - x^{(k^m)}} \tag{10}$$

In (10), if $x = r^{(4n)}$,

$$\sum_{i=1}^{n} \frac{F\left(2n(k^i - k^{i-1})\right)}{F(2nk^i)F(2nk^{i-1})} = \sqrt{5}\left[\frac{1}{r^{4n} - 1} - \frac{1}{x^{4nk^m} - 1}\right] \tag{11}$$

In (10) with k odd and $x = r^{(4n+2)}$, we have

$$\sum_{i=1}^{m} \frac{L\left((2n+1)(k^i - k^{i-1})\right)}{L\left\{(2n+1)k^i\right\}L\left\{(2n+1)k^{i-1}\right\}} = \frac{1}{r^{(4n+2)} - 1} - \frac{1}{r^{(4n+2)k^m} - 1} \tag{12}$$

Both (11) and (12) become infinite in an obvious way.

REFERENCES

1. Brother Alfred Brousseau, "Summation of Infinite Fibonacci Series," The Fibonacci Quarterly, Vol. 7, No. 2, pp. 143-168.

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A SERIES FORM FOR THE FIBONACCI NUMBERS F_{12n}

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ABSTRACT

The Fibonacci Numbers, F_{12n}, have been found to be expressible as a series:

$$F_{12(2m-1)} = (2m - 1)F_{12}\sum_{j=1}^{m}\left[\frac{C(m - 1 + j, 2j - 1)}{(m - 1 + j)}(5 \times 12^4)^{j-1}\right]$$

and

$$F_{12(2m)} = F_{24}\sum_{j=1}^{m}\left[C(m - 1 + j, 2j - 1)(5 \times 12^4)^{j-1}\right],$$

where m is the running index and $C(p,q)$ is the combination of p and q. The rationale by which these series were derived is given: namely, by writing F_k in the basic twelve system, recognizing groupings among the digits, and writing a summation series for the corresponding groups within sequential numbers.

1. INTRODUCTION

A list of 571 Fibonacci numbers, F_k, given by Basin and Hoggatt [1] shows that F_1 is 1 or 1^2 and F_{12} is 144 or 12^2. No other such coincidences were found, at least for the second power. Other relations will be shown below for F_{12n}, where n is an integer.

If d_i is the digit in the 10^{i-1} place, then there are cyclic relations among the d_i. That is, d_1 of $F_{k+60} = d_1$ of F_k, and d_2 of $F_{k+300} = d_2$ of F_k. The cycling period for d_3 is not to be found from the numbers in the above table. However, see Hoggatt [2].

To further the study of F_{12n} let us examine Table 1 in which F_k is written in the base twelve. The first 72 Fibonacci numbers are shown with X and e representing the tenth and eleventh digits, respectively. If D_i is the digit in the 12^{i-1} place, it is evident that D_1 of $F_{k+24} = D_1$ of F_k, and D_2 of $F_{k+24} = D_2$ of F_k. By examining an expansion of Table 1, one finds that D_3 of $F_{k+288} = D_3$ of F_k. These cyclic relations suggest that the digits change in shorter cycles in the base twelve than in the base ten. Therefore, a sequence of digits might be more readily recognized in the base twelve than in the base ten. This paper presents the rationale by which the particular series were found.

Table 1

THE FIRST 61 FIBONACCI NUMBERS IN THE BASE TWELVE

k	F_k	k	F_k	k	F_k
1	1	25	37,501	49	1,611,102,X01
2	1	26	5X,301	50	2,533,148,601
3	2	27	95,802	51	3,e44,24e,402
4	3	28	133,e03	52	6,477,397,X03
5	5	29	209,705	53	X,3ee,627,205
6	8	30	341,608	54	14,876,X03,008
7	11	31	54e,111	55	23,076,42X,211
8	19	32	890,719	56	37,931,231,219
9	2X	33	1,21e,82X	57	5X,9X7,65e,42X
10	47	34	1,Xe0,347	58	96,718,890,647
11	75	35	3,10e,e75	59	135,504,32e,X75
12	100	36	5,000,300	60	210,021,000,500
13	175	37	8,110,275	61	345,525,330,375
14	275	38	11,110,575	62	555,546,330,875
15	42X	39	19,220,82X	63	89X,X6e,661,02X
16	6X3	40	2X,331,1X3	64	1,234,3e5,991,8X3
17	e11	41	47,551,X11	65	1,e13,265,432,911
18	1,5e4	42	75,882,ee4	66	3,147,65e,204,5e4
19	2,505	43	101,214,X05	67	5,05X,904,637,305
20	2,Xe9	44	176,X97,9e9	68	8,1X6,363,83e,8e9
21	6,402	45	278,0e0,802	69	11,245,068,277,002
22	X,2ee	46	432,e88,5ee	70	19,42e,40e,Xe6,8ee
23	14,701	47	6Xe,079,201	71	2X,674,478,171,901
24	22,X00	48	e22,045,800	72	47,XX3,888,068,600

2. GROUPS OF FOUR DIGITS

Certain F_k's have been extracted from Table 1 and its logical extension to form Tables 2(a) and 3(a); F_{12n} are shown where n is an odd integer in Table 2(a) and an even integer in Table 3(a).

The smaller F_k's contain a surprising number of zeros so that they almost naturally fall into groups of four digits. For example, in Table 2(a),

$$F_{60} = 210021000500 \qquad \text{or} \qquad 21 - 0021 - 0005 - 00$$

in which the four groups contain the small integers 21, 21, 5, and 0. When written in the base ten,

$$F_{60} = 25 \times 12^{10} + 25 \times 12^6 + 5 \times 12^2 .$$

Table 2(a)

FIBONACCI NUMBERS IN BASE TWELVE

n	F_{12n}	j=9	8	7	6	5	4	3	2	1		Row R
1	F_{12}									1	00	1
3	F_{36}								5	0003	00	2
5	F_{60}							21	0021	0005	00	3
7	F_{84}						X5	0127	005X	0007	00	4
9	F_{108}					441	0799	0483	0106	0009	00	5
11	F_{132}				1985	3e83	3224	1145	01Xe	000e	00	6
13	F_{156}			9062	e616	e615	e350	2772	031e	0011	00	7
15	F_{180}	3	9274	3784	6922	3571	8676	5576	04X4	0013	00	8

Table 2(b)

n	F_{12n}	j=9	8	7	6	5	4	3	2	1		Row R
13	F_{156}			9061	$\underline{1}$e615	$\underline{1}$e615	e350	2772	031e	0011	00	7
15	F_{180}		$\underline{3}$9265	$\underline{e}$3773	$\underline{11}$6916	$\underline{8}$356e	$\underline{2}$8676	5576	04X4	0013	00	8

"Borrowed" integers

Table 3(a)
FIBONACCI NUMBERS IN BASE TWELVE

n	F_{12n}	j=7	6	5	4	3	2	1		Row R
2	F_{24}									1
4	F_{48}						e22	22X	00	2
6	F_{72}					47XX	3888	0686	00	3
8	F_{96}			1	e364	3e50	9398	08e4	00	4
10	F_{120}		9	8581	6420	19e7	6774	0e22	00	5
12	F_{144}	40	6463	0821	X679	8X86	873X	1150	00	6

Table 3(b)

n	F_{12n}	j=7	6	5	4	3	2	1		Row R
8	F_{96}				1e362	23e50	9398	08e4	00	4
10	F_{120}			9856X	136414	819e6	16774	0e22	00	5
12	F_{144}		4063X2	810784	59X660	198X84	2873X	1150	00	6

_ "Borrowed" integers

Form Table 4(a) from Table 2(a) by writing the integers within each group in the base ten. In writing this table, certain numbers were "borrowed" from one group to expand a following group. (The "borrowed" digits are indicated in Table 2(b) by underlining.)

Likewise, form Table 5(a) from Table 3(a) using "borrowed" integers as shown in Table 3(b) as necessary. Further, since F_{24n}/F_{24} is an integer, form Table 5(b) by dividing all groups by 322 which is the only group in F_{24} ($F_{24} = 322 \times 12^2$). Tables 4(a) and 5(b) are similar, especially the heads of the columns — 1, 5, 25, 125, etc., which are powers of 5. Of course, the integer five plays a large role in the expression for Fibonacci numbers

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

so one should not be surprised to find the digit 5 prominent in any expression for F_k.

By dividing each column by the number at its head, we obtain Table 4(b) from Table 4(a) and Table 5(c) from Table 5(b). Again, this process requires that digits be "borrowed" as noted above.

Table 4(a)

FIBONACCI NUMBERS IN BASE TEN WITH SPECIAL NOTATION

n	F_n	j= 8	7	6	5	4	3	2	1	Row R
		$\times 12^{30}$	$\times 12^{26}$	$\times 12^{22}$	$\times 12^{18}$	$\times 12^{14}$	$\times 12^{10}$	$\times 12^{6}$	$\times 12^{2}$	
1	F_{12}								1	1
3	F_{36}							5	3	2
5	F_{60}						25	25	5	3
7	F_{84}					125	175	70	7	4
9	F_{108}				625	1,125	675	150	9	5
11	F_{132}			3,125	6,875	5,500	1,925	275	11	6
13	F_{156}		15,625	40,625	40,625	19,500	4,550	455	13	7
15	F_{180}	78,125	234,375	281,250	171,875	66,250	9,450	700	15	8

Table 4(b)

n	F_n	j= 8	7	6	5	4	3	2	1	Row R
		$\times 12^{30}$ $\times 5^{7}$	$\times 12^{26}$ $\times 5^{6}$	$\times 12^{22}$ $\times 5^{5}$	$\times 12^{18}$ $\times 5^{4}$	$\times 12^{14}$ $\times 5^{3}$	$\times 12^{10}$ $\times 5^{2}$	$\times 12^{6}$ $\times 5^{1}$	$\times 12^{2}$ $\times 5^{0}$	
1	F_{12}								1	1
3	F_{36}							1	3	2
5	F_{60}						1	5	5	3
7	F_{84}					1	7	14	7	4
9	F_{108}				1	9	27	30	9	5
11	F_{132}			1	11	44	77	55	11	6
13	F_{156}		1	13	65	156	182	91	13	7
15	F_{180}	1	15	90	275	450	378	140	15	8

3. SERIES FORMS

The rows in Tables 4 and 5 will be designated by R and the columns by j counting from right to left. For Table 5(c) the integers in each group are the binomial coefficients or the combinations $C(R - 1 + j,\ 2j - 1)$. For Table 4(b), the integers in each group are not the binomial coefficient or the combinations, but can be so expressed when a multiplier and a divisor are included: the expression is

$$\frac{(2R - 1)C(R - 1 + j, 2j - 1)}{(R - 1 + j)} .$$

Table 5(a)

FIBONACCI NUMBERS IN BASE TEN WITH SPECIAL NOTATION

n	F_n	j= 6	5	4	3	2	1	Row R
		$x12^{22}$	$x12^{18}$	$x12^{14}$	$x12^{10}$	$x12^{6}$	$x12^{2}$	
2	F_{24}						322	1
4	F_{48}					1,610	644	2
6	F_{72}				8,050	6,440	966	3
8	F_{96}			40,250	48,300	16,100	1,288	4
10	F_{120}		201,250	322,000	169,050	32,200	1,610	5
12	F_{144}	1,006,250	2,012,500	1,449,000	450,800	56,350	1,932	6

Table 5(b)

n	F_n							
		$x12^{22}$	$x12^{18}$	$x12^{14}$	$x12^{10}$	$x12^{6}$	$x12^{2}$	
		x322	x322	x322	x322	x322	x322	
2	F_{24}						1	1
4	F_{48}					5	2	2
6	F_{72}				25	20	3	3
8	F_{96}			125	150	50	4	4
10	F_{120}		625	1,000	525	100	5	5
12	F_{144}	3,125	6,250	4,500	1,400	175	6	6

Table 5(c)

n	F_n							
		$x12^{22}$	$x12^{18}$	$x12^{14}$	$x12^{10}$	$x12^{6}$	$x12^{2}$	
		x322	x322	x322	x322	x322	x322	
		$x5^5$	$x5^4$	$x5^3$	$x5^2$	$x5^1$	$x5^0$	
2	F_{24}						1	1
4	F_{48}					1	2	2
6	F_{72}				1	4	3	3
8	F_{96}			1	6	10	4	4
10	F_{120}		1	8	21	20	5	5
12	F_{144}	1	10	36	56	35	6	6

Of course, F_k is the sum of the numbers in a row multiplied by the proper factors for each column. It is convenient to write 12^2 as F_{12} and $322\ F_{12}$ as F_{24}. One notes that in summing, j runs from 1 to R in every case. In Table 2(a), n = 2R - 1, and in Table 3(a), n = 2R. Therefore, for the odd multiples of 12, one has

$$F_{12(2R-1)} = (2R - 1)F_{12}\sum_{j=1}^{R}\left[\frac{C(R - 1 + j,\ 2j - 1)}{(R - 1 + j)}\ 5^{j-1} \times 12^{4(j-1)}\right]$$

and for the even multiples of 12, one has

$$F_{12(2R)} = F_{24}\sum_{j=1}^{R}\left[C(R - 1 + j,\ 2j - 1) \times 5^{j-1} \times 12^{4(j-1)}\right].$$

5. SUMMARY

(A) d_i for F_k occur in cycles. The cycles of d_i are shorter when F_k is written in the base twelve than when F_k is written in the base ten.

(B) F_{12n} written in the base twelve may be split into groups of four digits. Some borrowing among groups may be needed for the larger numbers to retain integers in the groups.

(C) When the groups are rewritten in the base ten, certain features stand out: (1) For n odd, F_{12n}/F_{12} is the integer shown in Table 4(a). (2) For n even, F_{12n}/F_{24} is the integer shown in Table 5(b). (3) Each column of integers is divisible by a power of five given by 5^{j-1} where j is the number of the column counting from right to left. (4) The quotients left after dividing by 5^{j-1} are expressible as combinations and factors using the row and column designators. (5) F_k is formed by summing the numbers in its row multiplied by the proper factors for each column.

(D) F_{12n} may be expressed by the summation series that are given in the Abstract.

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REFERENCES

1. S. L. Basin and V. E. Hoggatt, Jr., "The First 571 Fibonacci Numbers," Recreational Mathematics Magazine, No. 11, October 1962, pp. 19-30.
2. V. E. Hoggatt, Jr., "A Type of Periodicity for the Fibonacci Numbers," Math. Magazine, Jan.-Feb. 1955, pp. 139-142.

[Continued from page 404.]

2. Brother Alfred Brousseau, "Fibonacci Infinite Series — Research Topic," The Fibonacci Quarterly, Vol. 7, No. 2, pp. 211-217.
3. T. J. L. Bromwich, An Introduction to the Theory of Infinite Series, Macmillan, London, 1959. Page 24, Problem 15.
4. L. B. W. Jolley, Summation of Infinite Series, Dover, New York, 1961. Page 18, Formula 81.

THE FIBONACCI ASSOCIATION

PROGRAM OF THE EIGHTH ANNIVERSARY MEETING

Saturday April 24, 1971

HARNEY SCIENCE CENTER — UNIVERSITY OF SAN FRANCISCO
Sponsored by the Institute of Chemical Biology

Welcome: George Ledin, Jr., Institute of Chemical Biology, University of San Francisco

"Divisibility Properties of the Fibonomial Triangle,"
A. P. Hillman, University of New Mexico, Albuquerque, New Mexico

"Broadening Your Fibonacci Horizons,"
Brother Alfred Brousseau, St. Mary's College, California

"Golden and Silver Rectangles,"
Marjorie Bicknell, A. C. Wilcox High School, Santa Clara, Calif.

[Continued on page 436.]

TRIANGLES DE FIBONACCI

W. C. BARLEY
Los Gatos High School, Los Gatos, California

Consider a triangle whose sides have lengths represented by Fibonacci numbers and whose area is non-zero. In fact, while you are at it, consider several.

Also consider the possibility that the area, perimeter, and altitude to the base might be expressible in terms of Fibonacci numbers.

It doesn't take long to discover that all the triangles under consideration are isosceles. Further, it is obvious that the perimeter is already expressed in terms of Fibonacci numbers. But what about the area and the altitude to the base?

To aid and accomplish this end, it is suggested that Hero's formula for finding the area of a triangle be used. That is:

$$A = \sqrt{S(S - a)(S - b)(S - c)} \ ,$$

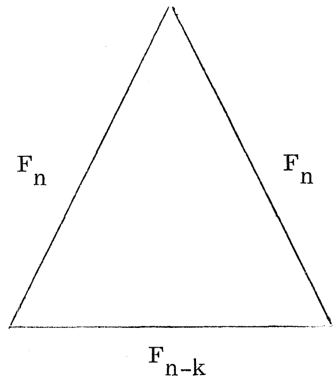
where

$$S = \frac{1}{2}(a + b + c)$$

if a, b, and c are the sides of the triangle.

Before continuing, it might be helpful to classify the triangles into some general categories and thus avoid random samples. The aim here, should it be attempted before reading on, would be to classify ALL the Fibonacci triangles.

The form selected here will be where the sides have length F_n, F_n, F_{n-k} with k an integer. Thus the groups of triangles might be represented as follows:



$k = -1$	$k = 0$	$k = 1$
2, 2, 3	1, 1, 1	2, 2, 1
3, 3, 5	2, 2, 2	3, 3, 1
5, 5, 8	3, 3, 3	5, 5, 3
etc.	etc.	etc.

$k = 2$	$k = 3$	$k = 4$
3, 3, 1	5, 5, 1	8, 8, 1
5, 5, 2	8, 8, 2	13, 13, 2
8, 8, 3	13, 13, 3	21, 21, 3
etc.	etc.	etc.

and so on for $k \geq -1$.

Now for an intuitive idea of what the area of a Fibonacci triangle might be for any k.

For $k = -1$,

SIDES		AREA (using Hero's formula)	
2, 2, 3	F_3, F_3, F_4	$\frac{3}{4}\sqrt{1 \cdot 7}$	$\frac{F_4}{L_3}\sqrt{F_1 L_4}$
3, 3, 5	F_4, F_4, F_5	$\frac{5}{4}\sqrt{1 \cdot 11}$	$\frac{F_5}{L_3}\sqrt{F_2 L_5}$
5, 5, 8	F_5, F_5, F_6	$\frac{8}{4}\sqrt{2 \cdot 18}$	$\frac{F_6}{L_3}\sqrt{F_3 L_6}$
8, 8, 13	F_6, F_6, F_7	$\frac{13}{4}\sqrt{3 \cdot 29}$	$\frac{F_7}{L_3}\sqrt{F_4 L_7}$
13, 13, 21	F_7, F_7, F_8	$\frac{21}{4}\sqrt{5 \cdot 47}$	$\frac{F_8}{L_3}\sqrt{F_5 L_8}$
Without using F_n, F_n, and F_{n-k} in the formula, what is the area in general?	$\cdot$ $\cdot$ $\cdot$		$\cdot$ $\cdot$ $\cdot$
	F_n, F_n, F_{n+1}		$\frac{F_{n+1}}{L_3}\sqrt{F_{n-2}L_{n+1}}$

Once again, the generalization is found by looking at these two columns. What APPEARS to be the relationship between the specific example and its answer — then generalize. It will be seen later how this can be proved, or really how it can be verified as one expression for the area.

The reader may now wish to complete the following:

$k = 0$

	$F_n,\ F_n,\ F_n$		$\dfrac{F_n^2}{L_3}\sqrt{F_4}$

$k = 1$

	$F_n,\ F_n,\ F_{n-1}$		$\dfrac{F_{n-1}}{L_3}\sqrt{L_{n-1}F_{n+2}}$

k = 2

	$F_n,\ F_n,\ F_{n-2}$		$\frac{F_{n-2}}{L_3}\sqrt{F_{n+1}(F_{n-4} + L_{n+1})}$

k = 3

	$F_n,\ F_n,\ F_{n-3}$		$\frac{F_{n-3}}{L_3}\sqrt{L_n(F_{n-4} + F_{n+1})}$

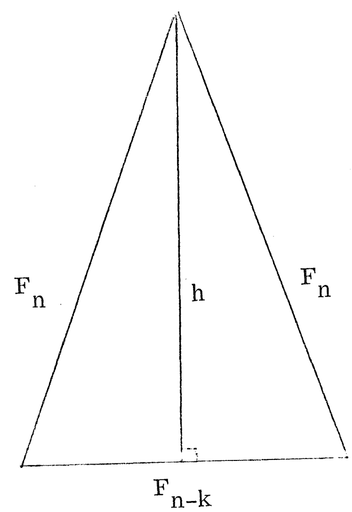
There is no need to stop at $k = 3$ and the interested reader may later with to continue. However, temporarily put aside the information found so far.

The Pythagorean Theorem and the fact that the base of an isosceles triangle is bisected by the altitude to that base leads to:

$$h^2 + \left(\frac{1}{2} F_{n-k}\right)^2 = F_n^2$$

or

$$h^2 = \frac{4F_n^2 - F_{n-k}}{4}$$



and

$$h = \frac{1}{2}\sqrt{4F_n^2 - F_{n-k}^2}$$

or

$$h = \frac{\sqrt{F_3^2 F_n^2 - F_{n-k}^2}}{F_3}$$

Then since $A = 1/2\ bh$,

$$A = \frac{F_{n-k}}{L_3}\sqrt{F_3^2 F_n^2 - F_{n-k}^2}$$

It is unlikely that this was what was arrived at using Hero's formula — and it might prove interesting to equate the two at this time. That is, for $k = -1$,

$$A = \frac{F_{n+1}}{L_3}\sqrt{F_{n-2}\, L_{n+1}}$$

using Hero's formula

$$A = \frac{F_{n+1}}{L_3}\sqrt{F_3^2 F_n^2 - F_{n+1}^2}$$

using $A = 1/2\ bh$. Therefore,

$$\frac{F_{n+1}}{L_3}\sqrt{F_{n-2}L_{n+1}} = \frac{F_{n+1}}{L_3}\sqrt{F_3^2 F_n^2 - F_{n+1}^2}$$

or

$$F_{n-2}L_{n+1} = F_3^2 F_n^2 - F_{n+1}^2$$

and

$$L_{n+1} = \frac{F_3^2 F_n^2 - F_{n+1}^2}{F_{n-2}} ,$$

which is the same as

$$\boxed{L_n = \frac{F_3^2 F_{n-1}^2 - F_n^2}{F_{n-3}}}$$

Thus, a new method for obtaining identities.

Although $k = 0$ is rather uninteresting, the reader may now wish to check and see what identities are produced by other k's. The results can be proved by conventional methods.

Anyone for Lucas triangles?

[Continued from page 356.]

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

4. EXTENSION TO OTHER NUMERICAL TRIANGLES

Although we have established the equality of products over the selected m sets of $(m + 1)$ elements where the elements are multinomial coefficients, the results remain valid when the sequence $1, 2, 3, \cdots, n, \cdots$ in the multinomial coefficients is replaced by the Fibonacci Sequence $F_1, F_2, F_3, \cdots, F_n, \cdots$ ($F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.) The generalized binomial coefficients in [3] are integers and the generalized multinomial coefficients are integers in [2]. This is enough to guarantee the validity of both the theorems. This occurs because the identification of the various factors was independent of the particular function. In the case of the original theorem, $f(n) = n!$. In the case of the extension, $f(n) = F_1F_2 \cdots F_n$. Thus

$$\binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, k_3, \cdots, k_m} = \frac{f(k_1 + \cdots + k_m)}{f(k_1)f(k_2) \cdots f(k_m)} = \frac{\prod_{i=1}^{k_1+\cdots+k_m} F_i}{\prod_{i=1}^{k_1} F_i \prod_{i=1}^{k_2} F_i \cdots \prod_{i=m}^{k_m} F_i}$$

(as in [2]), instead of

$$\binom{k_1 + \cdots + k_m}{k_1, k_2, \cdots, k_m} = \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1!k_2! \cdots k_m!} .$$

The corresponding N is

$$N = \frac{\prod_{i=1}^{n-1} F_i \left(\prod_{i=1}^{n} F_i\right)^{m-1} \prod_{i=1}^{n+1} F_i}{\prod_{i=1}^{m}\left(\prod_{j=1}^{k_i-1} F_j \left(\prod_{j=1}^{k_i} F_j\right)^{m-1} \prod_{j=1}^{k_i+1} F_j\right)} ,$$

where $n = k_1 + k_2 + \cdots + k_m$.

REFERENCES

1. Walter Hansell and V. E. Hoggatt, Jr., "The Hidden Hexagon Squares," Fibonacci Quarterly, Vol. 9, No. 2, p. 120.
2. Eugene Kohlbecker, "On a Generalization of Multinomial Coefficients for the Fibonacci Sequence," Fibonacci Quarterly, Vol. 4, No. 1, pp. 307-312.
3. V. E. Hoggatt, Jr., "Generalized Binomial Coefficients and the Fibonacci Numbers," Fibonacci Quarterly, Vol. 5, No. 4, pp. 383-400.
4. Henry W. Gould, "Equal Products of Generalized Binomial Coefficients," Fibonacci Quarterly, Vol. 9, No. 4, pp. 337-346.

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where

$$H_{n+2} = H_{n+1} + H_n .$$

The following identities were obtained from (13.2):

$$(13.10) \qquad H_{4n+4+p} - H_{2n+2+p} = \sum_{i=0}^{n} \binom{2n+1-i}{i} H_{3i+3+p} ,$$

$$H_{8n+8+p} - 5^{n+1} H_{4n+4+p}$$

$$(13.11) \qquad = 3 \sum_{i=0}^{[n/2]} \binom{2n+1-2i}{2i} 5^i H_{12i+3+p}$$

$$+ 3 \sum_{j=0}^{[(n-1)/2]} \binom{2n-2j}{2j+1} 5^j (H_0 L_{12j+8+p} + H_1 L_{12j+9+p}) .$$

Many more Fibonacci identities are readily obtainable from (13.1) and (13.2).

14. REMARKS ON THE PAPER BY HOGGATT, PHILLIPS, AND LEONARD [5]

All the 22 identities in [5] are special cases of our general results. The 22 identities appear in the Master's thesis of Leonard [6]. The notation (A, 1.6) means that identity A of [5] is a special case of our identity (1.6). Thus, we have the remaining identity pairings for special cases of our results: (B, 1.8), (C, 4.7), (D, 4.3), (E, 1.6), (F, 1.8), (G, 4.7), (H, 4.3), (I, 1.15), (J, 1.16), (K, 1.11), (L, 1.13), (M, 1.9), (N, 1.12), (P, 4.8), (Q, 4.4), (R, 4.5), (S, 4.9), (T, 1.16), (U, 1.15), (V, 1.16), and (W, 1.15).

Since A and E are obtained as special cases of our (1.6), A and E are therefore not independent, i.e., by a change of parameters, A can be transformed to E and vice versa. Thus, a perusal of the above pairings gives us the following dependent identity grouping: (A,E; 1.6), (B,F; 1.8), (I,U,W; 1.15), (J,T,V; 1.16), (D,H; 4.3), (C,G; 4.7). Since K, L. M,N, P, Q, R, and S are independent, the 22 identities A, B, $\cdots$, W, contains now only 14 independent identities.

REFERENCES

1. John H. Halton, "On a General Fibonacci Identity," Fibonacci Quarterly, Vol. 3 (1965), pp. 31-43.
2. John Vinson, Modulos m properties of the Fibonacci Numbers, Master's Thesis, 1961, Oregon State University, pp. 14-16.
3. Verner E. Hoggatt, Jr., and Marjorie Bicknell, "Some New Fibonacci Identities," Fibonacci Quarterly, Vol. 2, 1964, pp. 29-32.
4. L. Carlitz and H. H. Ferns, "Some Fibonacci and Lucas Identities," Fibonacci Quarterly, 8 (1970), pp. 61-73.
5. V. E. Hoggatt, John W. Phillips, and H. T. Leonard, Jr., "Twenty-four Master Identities," Fibonacci Quarterly, 9 (1971), pp. 1-17.
6. H. T. Leonard, Jr., Fibonacci and Lucas Number Identities and Generating Functions, Master's Thesis, January, 1969, San Jose State College.
7. John Wessner, "Binomial Sums of Fibonacci Powers," Fibonacci Quarterly, 4 (1966), pp. 355-358.

AN EXAMPLE OF FIBONACCI NUMBERS USED TO GENERATE RHYTHMIC VALUES IN MODERN MUSIC

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It has been said often enough that mathematics and music are somehow related. Whether or not such a statement can be supported in detail, proportion is certainly a major structural and expressive element in music. Time, in particular, seems to want clear-cut divisions and organization, reflecting perhaps our uneasiness in dealing with it.

In music of the twentieth century, the occurrence of two devices of temporal organization involving Fibonacci numbers stands out. One of these is the structuring of the lengths of phrases and sections in Fibonacci proportions. The other is the use of Fibonacci numbers, as well as other mathematical series and functions (even random number tables), to generate what are known as "irrational" rhythmic values.

Rhythm in Western music of the past was based on durational patterns which could be reconciled easily to some short repeating unit of time. This unit is the "beat." However long or short the beat, the rhythmic values of a passage would be multiples or divisions of it simple enough that the beat itself could be perceived.

Some composers, however, have felt that uniform beats grouped in twos, threes, and fours produced rigidity and "squareness," the so-called "tyranny of the barline." Claude Debussy, for example, deliberately obscured the beat by using long tones which did not always move at the beginning of a beat.

Not until more recent times, however, have composers tried to write irrational rhythms, rhythms which suggested no beat at all. From the outset, composers found that generating such rhythms from little numerical "games" stimulated their imaginations, assured a measure of consistency, and taught them to free their minds from old and ingrained habits. There was, after all, the present danger of falling back into beats without realizing it.

New techniques are often like this. In diatonic music the key signature appears at the beginning of every line of music, while the time signature appears only at the beginning of the piece or at points of change. We are told that in the seventeenth century, when diatonic practice was new, musicians needed to be reminded of the key signature, while they were familiar with the older times signature. It has been suggested that the strictness of early "twelve-tone" music springs in part from a similar problem: the composers' ears were too well trained in diatonic harmony to be completely consistent in the new "atonal" medium.

Although many kinds of manipulations can produce irrational rhythms, composers have been most interested in numbers which do things. A random number table is just a bunch of numbers. A chart made from various permutations of the Fibonacci series (0,1), a great favorite with many composers, constantly reveals surprising and provocative relationships. In the composer's mind, these are often transformed immediately into musical ideas.

The following diagram is a simple example given as an exercise by Jean-Claude Eloy (el'wah'), a prominent pupil of Pierre Boulez, when he was teaching at the Berkeley campus of the University of California. He begins with six members of the Fibonacci series (0,1) and multiplies them by the numbers one through six so as to produce six rows of differing lengths (Figure 1).

He then scrambles the numbers one through six, or "permutes" them as he used to say, according to an arbitrarily chosen law, in this case starting with the middle two numbers and working outward by pairing the next larger with the next smaller and the largest with the smallest. The numbers 123456 now read 435261. Beside each of these numbers he now places a row from Figure 1, while the four-number row beside the number four of the new column, the three-number row beside the number three, etc. (Figure 2). These rows, too, are permuted, using the same principles, but alternating between starting at the middle and working outward and starting at the extremes and working inward. He also alternates between placing the larger number of each pair first or second. The Roman numerals represent groupings based on the position of the number six, the only one which appears three times in the array. This grouping is used at this point only to continue the alternation of permutation methods on the basis of odd- and even-numbered rows.

1	2	3	5	8	13
2	4	6	10	16	
3	6	9	15		
4	8	12			
5	10				
6					

Figure 1

4	15	3	9	6			I
3	4	12	8				
5	16	2	10	4	6		II
2	5	10					
6	13	1	8	2	5	3	III
1	6						

Figure 2

The rows are separated in Figure 3, and an integer is placed beneath each member. For the row containing four numbers, the numbers one through four are permuted and distributed. When all the rows have been treated in this manner, these new numbers are used to determine the number of integers to be placed at each point in yet a third row. (The permutation game continues.) When the number one appears in the second row, it simply carries down to the third to produce some long values.

Now the musical problem is posed. In each vertical group of three, the uppermost figure is to represent an amount of time, measured in seconds. The second figure represents the number of segments into which this period of time is to be divided, and the figures of the third level give the relative lengths of these segments. Thus 15/3/324 is fifteen seconds divided into three parts whose lengths can be expressed by the ratio 3:2:4. These proportions are to be written in traditional notation, with a quarter note representing one second (♩ = 60).

Row one is written out in Figure 4. As can be seen, seconds must be divided into fifths and tenths to express the three-second and nine-second units. The notation ⌐5(𝅘𝅥𝅯):4¬ means five sixteenth notes in the time of four. (Remember that a quarter note represents a second.)

I
15" 3" 9" 6"
3 2 4 1 (1234 → 3241)
324 41 1234

II
1. 4" 12" 8"
2 1 3 (123 213)
23 123

2. 16" 2" 10" 4" 6"
3 4 2 5 1 (12345 → 34251)
514 2315 24 35142

III
1. 5" 10"
2 1 (12 → 21)
21

2. 13" 1" 8" 2" 5" 3"
3 4 2 5 1 6 (123456 → 342516)
516 5243 36 61524 436152

3. 6"
1

Figure 3

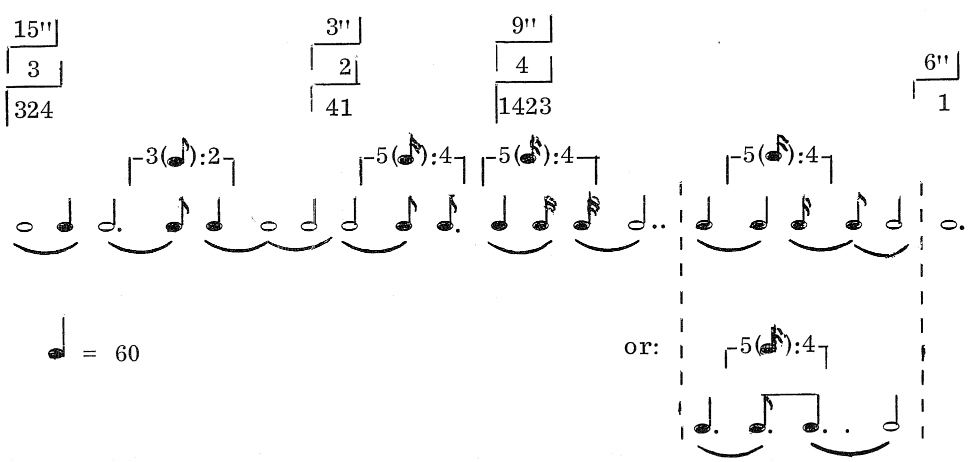

Figure 4

[Continued on page 436.]

CONTINUED FRACTIONS OF QUADRATIC FIBONACCI RATIOS

BROTHER ALFRED BROUSSEAU
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In a previous article [1] the author investigated the continued fraction representation of linear Fibonacci ratios. As a sequence of this work a study has been made of certain quadratic ratios and their representation in continued fractions. The program as carried out was twofold: (1) Ascertaining the pattern or patterns; (2) Proving that these patterns hold in general. We shall begin with a couple of elementary examples and then report more fully a case of greater difficulty. Other patterns discovered and proved will then be listed.

THE RATIO F^2_{n+1}/F^2_n

The pattern in this case can be devined readily from a few examples.

$$F_5^2/F_4^2 = 25/9 = (2,\ 1,\ 3,\ 1,\ 1)$$
$$F_6^2/F_5^2 = 64/25 = (2,\ 1,\ 1,\ 3,\ 1,\ 1,\ 1)$$
$$F_7^2/F_6^2 = 169/64 = (2,\ 1,\ 1,\ 1,\ 3,\ 1,\ 1,\ 1,\ 1)$$

It appears that in general

$$F^2_{n+1}/F^2_n = (2,\ 1_{n-3},\ 3,\ 1_{n-2})$$

where the subscripts of the 1's indicate the number of times the quotient 1 occurs at the point in question.

We first examine the initial portion of the expansion represented by $2,\ 1_{n-3}$. Forming a table of convergents:

		2	1	1	1	1	1	1
0	1	2	3	5	8	13	21	34
1	0	1	1	2	3	5	8	13

we can conclude that

$$(2, 1_{n-3}) = F_n / F_{n-2} \quad .$$

If we now adjoin the 3 to the above table we have

1	1	3
F_{n-1}	F_n	$3F_n + F_{n-1}$
F_{n-3}	F_{n-2}	$3F_{n-2} + F_{n-3}$.

Additional 1's simply mean that the last two convergents are being treated as the first two terms of a Fibonacci sequence. Now if we start a sequence with a and b, the n^{th} term is

$$T_n = F_{n-2}a + F_{n-1}b \quad .$$

In the present instance, therefore, we have for the numerator

$$F_{n-2}F_n + F_{n-1}(3F_n + F_{n-1}) \ ,$$

which can be shown to be equal to F^2_{n+1}. Similarly the denominator comes out F^2_n.

THE RATIO L^2_n / F^2_n

In this case, the pattern can be derived directly from two formulas, namely:

$$L^2_{2n} = 3F^2_{2n} + 4$$
$$L^2_{2n+1} = 3F^2_{2n+1} - 4 \quad .$$

From the first relation it follows that

$$L^2_{2n} / F^2_{2n} = 5 + 4/F^2_{2n} \quad .$$

Then if $F_{2n} \equiv 0 \pmod 2$,

$$L_{2n}^2/F_{2n}^2 = (5,\ F_{2n}^2/4)\ .$$

If $F_{2n} \equiv 1 \pmod 2$,

$$L_{2n}^2/F_{2n}^2 = (5,\ [F_{2n}^2/4],\ 4)$$

where the square brackets indicate the greatest integer function.

From the second relation,

$$L_{2n+1}^2/F_{2n+1}^2 = 4 + (F_{2n+1}^2 - 4)/F_{2n+1}^2\ .$$

Then

$$F_{2n+1}^2/(F_{2n+1}^2 - 4) = 1 + 4/(F_{2n+1}^2 - 4)\ .$$

If $F_{2n+1} \equiv 0 \pmod 2$, the final outcome is

$$L_{2n+1}^2/F_{2n+1}^2 = (4,\ 1,\ (F_{2n+1}^2 - 4)/4)\ .$$

If $F_{2n+1} \equiv 1 \pmod 2$,

$$L_{2n+1}^2/F_{2n+1}^2 = (4,\ 1,\ [(F_{2n+1}^2 - 4)/4],\ 4).$$

THE RATIO F_n^2/F_{n-3}^2

The case we shall consider in some detail is the ratio F_n^2/F_{n-3}^2 as it is sufficiently complex to bring out the techniques required in finding and priving the patterns. We list first the continued fraction expansions for $n = 4$ to $n = 35$. (See Table 1.)

From this table, it appears that for $n > 12$, the patterns arrange themselves modulo 6 as follows:

$n = 6k$. $(17,\ (1,\ 16)_{k-2},\ 1,\ 15,\ 17,\ (1,\ 16)_{k-3},\ 1,\ 17)$

$n = 6k + 1$. $(17,\ (1,\ 16)_{k-2},\ 1,\ 17,\ 1,\ 3,\ (1,\ 2,\ 1,\ 3)_{k-1},\ 2)$

$n = 6k + 2$. $(17,\ (1,\ 16)_{k-1},\ 1_3,\ 3,\ (1,\ 2,\ 1,\ 3)_{k-1},\ 2)$

$n = 6k + 4$. $(17,\ (1,\ 16)_{k-1},\ 1,\ 8,\ (1,\ 16)_k)$

$n = 6k + 5$. $(17,\ (1,\ 16)_{k-1},\ 1,\ 24,\ (1,\ 16)_k)$

Table 1

CONTINUED FRACTION EXPANSION OF F_n^2/F_{n-3}^2

n	
4	(9)
5	(25)
6	(16)
7	(18,1,3,2)
8	(17,1,1,1,3,2)
9	(18,16)
10	(17,1,8,1,16)
11	(17,1,24,1,16)
12	(17,1,15,18)
13	(17,1,17,1,3,1,2,1,3,2)
14	(17,1,16,1,1,1,3,1,2,1,3,2)
15	(17,1,17,15,1,17)
16	(17,1,16,1,8,1,16,1,16)
17	(17,1,16,1,24,1,16,1,16)
18	(17,1,16,1,15,17,1,17)
19	(17,1,16,1,17,1,3,1,2,1,3,1,2,1,3,2)
20	(17,1,16,1,16,1,1,1,3,1,2,1,3,1,2,1,3,2)
21	(17,1,16,1,17,15,1,16,1,17)
22	(17,1,16,1,16,1,8,1,16,1,16,1,16)
23	(17,1,16,1,16,1,24,1,16,1,16,1,16)
24	(17,1,16,1,16,1,15,17,1,16,1,17)
25	(17,1,16,1,16,1,17,1,3,1,2,1,3,1,2,1,3,1,2,1,3,2)
26	(17,1,16,1,16,1,16,1,1,1,3,1,2,1,3,1,2,1,3,1,2,1,3,2)
27	(17,1,16,1,16,1,17,15,1,16,1,16,1,17)
28	(17,1,16,1,16,1,16,1,8,1,16,1,16,1,16,1,16)
29	(17,1,16,1,16,1,16,1,24,1,16,1,16,1,16,1,16)
30	(17,1,16,1,16,1,16,1,15,17,1,16,1,16,1,17)
31	(17,1,16,1,16,1,16,1,17,1,3,1,2,1,3,1,2,1,3,1,2,1,3,1,2,1,3,2)
32	(17,1,16,1,16,1,16,1,16,1,1,1,3,1,2,1,3,1,2,1,3,1,2,1,3,1,2,1,3,2)
33	(17,1,16,1,16,1,16,1,17,15,1,16,1,16,1,16,1,17)
34	(17,1,16,1,16,1,16,1,16,1,8,1,16,1,16,1,16,1,16,1,16)
35	(17,1,16,1,16,1,16,1,16,1,24,1,16,1,16,1,16,1,16,1,16)

To establish these patterns, it is first necessary to examine various portions of the expansion and prove that their forms continue to hold for all values of k. For the first portion, we have Table 2.

Table 2

FIRST PORTION OF THE EXPANSION

Quotient	Numerator	Denominator	Numerator	Denominator
17	17	1	$F_9/2$	$F_3/2$
1	18	1	$F_{12}/8$	$F_6/8$
16	305	17	$F_{15}/2$	$F_9/2$
1	323	18	$F_{18}/8$	$F_{12}/8$
16	5473	305	$F_{21}/2$	$F_{15}/2$
1	5796	323	$F_{24}/8$	$F_{18}/8$
16	98209	5473	$F_{27}/2$	$F_{21}/2$
1	104005	5796	$F_{30}/8$	$F_{24}/8$

Let p_n/q_n be the partial quotient for the n^{th} step in Table 2. Then assuming that for n odd

$$\begin{aligned} p_n &= (F_{3n+6})/2 , \\ q_n &= F_{3n}/2 , \\ p_{n+1} &= (F_{3n+9})/8 , \\ q_{n+1} &= (F_{3n+3})/8 , \end{aligned}$$

it would follow that

$$\begin{aligned} p_{n+2} &= 16(F_{3n+9})/8 + (F_{3n+6})/2 \\ &= (4F_{3n+9} + F_{3n+6})/2 = (F_{3n+12})/2 . \end{aligned}$$

Similarly, for q_{n+2},

$$p_{n+3} = (F_{3n+12})/2 + (F_{3n+9})/8 = (F_{3n+15})/8 .$$

Similarly for q_{n+3}.

Thus, the pattern is seen to hold by mathematical induction. Consider next the portion $(1,16)_k$.

Table 3

PORTION $(1,16)_k$

Quotient	Numerator	Denominator
1	1	1
16	17	1
1	18	17
16	305	288
1	323	305
16	5473	5168
1	5796	5473
16	98209	92736

It appears that for n odd,

$$p_n = (F_{3n+3})/8 ,$$
$$q_n = F_{3n}/2 ,$$
$$p_{n+1} = (F_{3n+6})/2 ,$$
$$q_{n+1} = (F_{3n+6})/2 - F_{3n}/2 .$$

Again, this pattern can be shown to hold by mathematical induction. Another part of some of the patterns is $(1, 2, 1, 3)_k$. (See Table 4).

This pattern continues leading to assumed values for $n \equiv 0 \pmod 4$ as follows.

$$p_n = (L_{3+6k})/4 - F_{6k}/2 ,$$
$$p_{n+1} = (L_{3+6k})/4$$

From this assumption,

Table 4

PORTION $(1, 2, 1, 3)_k$

Quotient	Numerator	Denominator	Numerator	Denominator
1	1	1	$L_3/4$	$F_1 + F_0/8$
2	3	2	$F_6/2 - L_3/4$	$(3L_3 + 4F_2)/8$
1	4	3	$F_6/2$	$F_6/2 - L_3/4$
3	15	11	$L_9/4 - F_6/2$	$11\,F_6/8$
1	19	14	$L_9/4$	$F_7 + F_6/8$
2	53	39	$F_{12}/2 - L_9/4$	$(3L_9 + 4F_8)/8$
1	72	53	$F_{12}/2$	$F_{12}/2 - L_9/4$
3	269	198	$L_{15}/4 - F_{12}/2$	$11\,F_{12}/8$

$$\begin{aligned} p_{n+2} &= 2(L_{3+6k})/4 + (L_{3+6k})/4 - F_{6k}/2 \\ &= L_{3+6k} - F_{6k}/2 - (L_{3+6k})/4 \\ &= F_{4+6k} + F_{2+6k} - F_{6k}/2 - (L_{3+6k})/4 \\ &= F_{4+6k} + (F_{3+6k})/2 - (L_{3+6k})/4 \\ &= (F_{6+6k})/2 - (L_{3+6k})/4 , \end{aligned}$$

which agrees with the observed pattern.

From this,

$$p_{n+3} = (F_{6+6k})/2 - (L_{3+6k})/4 + (L_{3+6k})/4 = (F_{6+6k})/2 .$$

Then

$$\begin{aligned} p_{n+4} &= 3(F_{6+6k})/2 + (F_{6+6k})/2 - (L_{3+6k})/4 \\ &= 10\,(F_{6+6k})/4 - (L_{3+6k})/4 - (F_{6+6k})/2 \\ &= (F_{9+6k})/4 + 2\,(F_{8+6k})/4 - (F_{6+6k})/2 \\ &= (L_{9+6k})/4 - (F_{6+6k})/2 . \end{aligned}$$

Finally,

$$p_{n+5} = (L_{9+6k})/4 - (F_{6+6k})/2 + F_{6+6k})/2 = (L_{9+6k})/4 .$$

Similar considerations show that the q's follow the observed pattern.

The next step is to put the pieces together for the six given cases. For $n = 6k$, the first part is given by the partial quotients $(17, (1,16)_{k-2}, 1)$. The last part can be remodeled to this same form by changing the final 17 to 16, 1. Between these two sets of quantities is 15. Thus, the numerator and denominator can be evaluated from Table 5.

Table 5

Quotients	16	1	$15 + (F_{6k-6})/F_{6k}$
Numerator	$(F_{6k-3})/2$	$F_{6k}/8$	
Denominator	F_{6k-9}	$(F_{6k-6})/8$	

The numerator would therefore be

$$\begin{aligned} &15\, F_{6k}/8 + (F_{6k-6})F_{6k}/8F_{6k} + (F_{6k-3})/2 \\ &= 16\, F^2_{6k}/8F_{6k} \end{aligned} .$$

The denominator evaluates to

$$15\,(F_{6k-6})/8 + F^2_{6k-6} + (F_{6k-9})/2 ,$$

which after some calculation gives $16\, F^2_{6k-3}/8F_{6k}$. Thus, the ratio represented is F^2_{6k}/F^2_{6k-3}. Similar considerations apply to the other five cases.

SUMMARY OF RESULTS

1. F^2_{n+1}/F^2_n. Pattern already given.
2. Patterns of F^2_n / F^2_{n-2}.

$$F^2_{4k+1}/F^2_{4k-1} = (6,\ (1,5)_{k-1},\ 3,\ (1,5)_{k-2},\ 1,\ 6)$$

$$F^2_{4k+2}/F^2_{4k} = (6,\ (1,5)_{k-2},\ 1,\ 6,\ 8,\ (1,5)_{k-2}, 1, 6)$$

$$F^2_{4k+3}/F^2_{4k+1} = (6, (1,5)_{k-1},\ 1,3,5,(1,5)_{k-2},\ 1,6)$$

$$F^2_{4k+4}/F^2_{4k+2} = (6,\ (1,5)_{k-1},\ 1,8,6,\ (1,5)_{k-2},\ 1,6) .$$

All these results hold down to $k = 2$.

3. Patterns of F_n^2/F_{n-3}^2. Already given.

4. Patterns of L_n^2/L_{n-1}^2.

$$L_n^2/L_{n-1}^2 = (2, L_{n-5}, 2, 9 \text{ expansion of } L_{n-3}/L_{n-8}) \text{ for } n \quad 9.$$

5. Patterns of L_n^2/L_{n-2}^2.

$$L_{4k}^2/L_{4k-2}^2 = \left(6, (1,5)_{k-2}, 1, 4, 2, 33, \text{ expansion of } \frac{(L_{4k-6})/3}{L_{4k-1}}\right)$$

$$L_{4k+1}^2/L_{4k-1}^2 = (6, (1,5)_{k-2}, 1, 6, 1, 1, 2, 1, 3, 33, \text{ expansion of } (L_{4k-10}/3)/L_{4k-15})$$

$$L_{4k+2}^2/L_{4k}^2 = (6, (1,5)_{k-1}, 1, 1, 1, 1, 2, 3, 2, 1, 3, 33, \text{ expansion of } (L_{4k-14}/3)/L_{4k-19})$$

$$L_{4k+3}^2/L_{4k+1}^2 = (6, (1,5)_{k-1}, 1, 19, \text{ expansion of } (L_{4k+2}/3)/L_{4k-3})$$

6. Pattern of L_n^2/F_n. Previously given.

7. Patterns of L_n^2/F_{n-1}^2.

$$L_{5k+5}^2/F_{5k+4}^2 = (13, 11_k, 2, 4, 11_k) \text{ down to } k = 0.$$

$$L_{5k+6}^2/F_{5k+5}^2 = (13, 11_{k-1}, 10, 1, 24, 11_k) \text{ down to } k = 1.$$

$$L_{5k+7}^2/L_{5k+6}^2 = (13, 11_k, 7, 9, 11_k) \text{ down to } k = 0.$$

$$L_{5k+8}^2/F_{5k+7}^2 = (13, 11_k, 14, 12, 11_k) \text{ down to } k = 0.$$

$$L_{5k+9}^2/F_{5k+8}^2 = (13, 11_k, 10, 3, 1, 10, 11_k) \text{ down to } k = 0.$$

REFERENCE

1. Brother Alfred Brousseau, "Continued Fractions of Fibonacci and Lucas Ratios," Fibonacci Quarterly, Dec. 1964, pp. 269-276.

The question always arises: "Is it music?" Well no, not yet, no more than a few geometric shapes sketched on canvas constitute a painting. The composer can use his numbers to build up larger structural units as well as using them to "control" other elements such as texture, timbre, dynamics, and even pitch. In this example, Eloy used his groupings designated by Roman numerals to suggest further development. If the patterns suggested by the numerical scheme do not produce the kinds of sounds and structures the composer desires, however, he must either depart from them or try a new scheme. It is not surprising that many composers use the results of such an exercise simply to stimulate their imaginations, without resorting to thorough-going applications.

Regardless of the techniques employed, composers, having already passed through a period of re-evaluation concerning pitch structures, have launched into a far-reaching reconsideration of time and its musical organization. In this endeavor, Fibonacci proportions have been among the most favored and most useful tools, providing an alternative both to the old techniques and to randomness.

[Continued from page 412.]

"A Symmetric Substitute for Stirling Numbers,"
Professor A. P. Hillman, University of New Mexico, Albuquerque

"A Bouquet of Convolutions,"
Professor V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

"On a Generalized Catalan Sequence,"
Richard Jow, Graduate Student, San Jose State College

"On a Theorem of Suryanarayana,"
Professor Hugh Edgar, San Jose State College

SOME MORE FIBONACCI DIOPHANTINE EQUATIONS

V. E. HOGGATT, JR.
San Jose State College, San Jose, California

It is well known that the Quadratic Diophantine equation $y^2 - 5x^2 = \pm 4$ has solutions in integers <u>if</u> and <u>only if</u> $y = L_n$ and $x = F_n$, n an integer. For a proof by infinite descent see [2]. The underlying identity is

$$L_n^2 - 5F_n^2 = 4(-1)^n.$$

There are other quadratic Diophantine equations which are Fibonacci-related. In "Fibonacci to the Rescue" [1], there occurs

$$x^2 + x(y - 1) - y^2 = 0. \tag{1}$$

The proof that solutions in positive integers are possible if and only if $x = F_{2p+1}^2$ and $y = F_{2p+1}F_{2p+2}$ appears novel.

Solve quadratic equation (1) for x. In order for x to be an integer, the quadratic discriminant

$$(y - 1)^2 + 4y^2 = k^2.$$

Set $y - 1 = m^2 - n^2$, $2y = 2mn$, and $k = m^2 + n^2$ so that

$$m^2 - mn - n^2 = -1,$$

which, when solved for m yields

$$m = \frac{n \pm \sqrt{5n^2 - 4}}{2} .$$

Thus m is an integer if and only if $5n^2 - 4 = s^2$. It follows that $n = F_{2p+1}$ and $s = L_{2p+1}$ for some integer p.

It follows that $m = F_{2p+2}$ or $-F_{2p}$ since $L_{2p+1} = F_{2p+1} + 2F_{2p}$.

Thus $y = mn = F_{2p+2}F_{2p+1}$ or $-F_{2p+1}F_{2p}$. Since $k = m^2 + n^2$, it follows that, for

$$m = F_{2p+2} \quad \text{and} \quad n = F_{2p+1},$$
$$x = -F_{2p+2}^2 \quad \text{or} \quad F_{2p+1}^2 \quad \text{and} \quad y = F_{2p+2}F_{2p+1},$$

while for $m = -F_{2p}$, $n = F_{2p+1}$,

$$x = -F_{2p}^2 \quad \text{or} \quad F_{2p+1}^2 \quad \text{and} \quad y = -F_{2p+1}F_{2p} .$$

These are the only integral solutions to $x^2 + x(y - 1) - y^2 = 0.$

[Continued on page 448.]

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

DEFINITIONS

$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n;\ L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n.$

PROBLEMS PROPOSED IN THIS ISSUE

B-214 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let n be a random positive integer. What is the probability that L_n has a remainder of 11 on division by 13? [Hint: Look at the remainders for $n = 1, 2, 3, 4, 5, 6, \cdots$.]

B-215 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Prove that for all positive integers n the quadratic $q(x) = x^2 - x - 1$ is an exact divisor of the polynomial

$$p_n(x) = x^{2n} - L_n x^n + (-1)^n$$

and establish the nature of $p_n(x)/q(x)$. [Hint: Evaluate $p_n(x)/q(x)$ for $n = 1, 2, 3, 4, 5.$]

B-216 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Solve the recurrence $D_{n+1} = D_n + L_{2n} - 1$ for D_n, subject to the initial condition $D_1 = 1$.

B-217 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

A triangular array of numbers $A(n,k)$ $(n = 0, 1, 2, \cdots;\ 0 \le k \le n)$ is defined by the recurrence

$$A(n + 1, k) = A(n, k - 1) + (n + k + 1)A(n,k) \qquad (1 \le k \le n)$$

together with the boundary conditions

$$A(n,0) = n!\ , \qquad A(n,n) = 1\ .$$

Find an explicit formula for $A(n,k)$.

B-218 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.

Let $a = (1 + \sqrt{5})/2$ and show that

$$\mathrm{Arctan} \sum_{n=1}^{\infty} [1/(aF_{n+1} + F_n)] = \sum_{n=1}^{\infty} \mathrm{Arctan}\ (1/F_{2n+1})\ .$$

B-219 Proposed by Tomas Djerverzon, Albrook College, Tigertown, New Mexico.

Let k be a fixed positive integer and let $a_0, a_1, \cdots, a_k$ be fixed real numbers such that, for all positive integers n,

$$\frac{a_0}{n} + \frac{a_1}{n+1} + \cdots + \frac{a_k}{n+k} = 0\ .$$

Prove that $a_0 = a_1 = \cdots = a_k = 0$.

SOLUTIONS

INVERTING A CONVOLUTION

B-196 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let $a_0, a_1, a_2, \cdots,$ and $b_0, b_1, b_2, \cdots$ be two sequences such that

$$b_n = \binom{n}{0}a_n + \binom{n}{1}a_{n-1} + \binom{n}{2}a_{n-2} + \cdots + \binom{n}{n}a_0 \qquad a = 0, 1, 2, \cdots .$$

Give the formula for a_n in terms of $b_n, \cdots, b_0$.

Solution by A. C. Shannon, New South Wales, I. T., N.S.W., Australia.

We are given

$$b_n = \sum_{r=0}^{n} \binom{n}{r} a_r ,$$

and so

$$\sum_{n=0}^{\infty} b_n x^n/n! = \sum_{n=0}^{\infty} \sum_{r=0}^{n} a_r x^n/r!\,(n-r)!$$

$$= e^x \sum_{n=0}^{\infty} a_n x^n/n! ,$$

Thus

$$\sum_{n=0}^{\infty} a_n x^n/n! = e^{-x} \sum_{n=0}^{\infty} b_n x^n/n!$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} b_r(-x)^{n-r} x^r/r!\,(n-r)!$$

which gives

$$a_n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} b_r$$

$$= \binom{n}{0} b_n - \binom{n}{1} b_{n-1} + \binom{n}{2} b_{n-2} + \cdots + (-1)^n \binom{n}{n} b_0 .$$

Also solved by J. L. Brown, Jr., T. J. Cullen, Herta T. Freitag, M. S. Klamkin, and the Proposer.

AN IM-PELL-ING FORMULA

B-197 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let the Pell Sequence be defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. Show that there is a sequence Q_n such that

$$P_{n+2k} = Q_k P_{n+k} - (-1)^k P_n ,$$

and give initial conditions and the recursion formula for Q_n.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

We have

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} ,$$

where α, β are the roots of

$$\alpha^2 = 2\alpha + 1 .$$

Since $\alpha\beta = -1$,

$$P_{n+2k} + (-1)^k P_n = \frac{\alpha^{n+2k} - \beta^{n+2k}}{\alpha - \beta} + \alpha^k\beta^k \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$$= \frac{(\alpha^k + \beta^k)(\alpha^{n+k} - \beta^{n+k})}{\alpha - \beta} .$$

Thus if we put

$$Q_k = \alpha^k + \beta^k ,$$

we have

$$P_{n+2k} = Q_k P_{n+k} - (-1)^k P_n .$$

Clearly,

$$Q_{k+2} = 2Q_{k+1} + Q_k, \qquad Q_0 = Q_1 = 2 .$$

Also,

$$\sum_{k=0}^{\infty} Q_k x^k = \frac{1}{1 - \alpha x} + \frac{1}{1 - \beta x} = \frac{2 - 2x}{1 - 2x - x^2} .$$

Also solved by Clyde A. Bridger, T. J. Cullen, Herta T. Freitag, M. S. Klamkin, and the Proposer.

PERMUTATIONS, DERANGEMENTS, AND THESE THINGS

B-198 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let c_n be the coefficient of $x_1x_2 \cdots x_n$ in the expansion of

$$(-x_1 + x_2 + x_3 + \cdots + x_n)(x_1 - x_2 + x_3 + \cdots + x_n)(x_1 + x_2 - x_3 + \cdots + x_n)$$
$$\cdots (x_1 + x_2 + x_3 + \cdots + x_{n-1} - x_n) .$$

For example,

$$c_1 = -1, \quad c_2 = 2, \quad c_3 = -2, \quad c_4 = 8 \quad \text{and} \quad c_5 = 8 .$$

Show that

$$c_{n+2} = nc_{n+1} + 2(n+1)c_n, \qquad c_n = nc_{n-1} + (-2)^n ,$$

and

$$\lim_{n \to \infty} (c_n/n!) = e^{-2} .$$

Solution by M. S. Klamkin, Ford Motor Company, Dearborn, Michigan.

Letting $x = \Sigma x_1$, the given produce is a special case of (i.e., for $a = 2$)

$$(x - ax_1)(x - ax_2) \cdots (x - ax_n) =$$
$$x^n - ax^{n-1}\Sigma x_i + a^2 x^{n-2}\Sigma x_i x_j - a^3 x^{n-3}\Sigma x_i x_j x_k + \cdots .$$

Then by the multinomial theorem, the coefficient of $x_1 x_2 \cdots x_n$ is given by the sum

$$n! - a\binom{n}{1}[(n-1)!] + a^2\binom{n}{2}[(n-2)!] - \cdots$$

or

$$c_n(a) = n!\left\{1 - a + \frac{a^2}{2!} - \frac{a^3}{3!} + \cdots + \frac{(-a)^n}{n!}\right\} .$$

It now immediately follows that

$$c_n - nc_{n-1} = (-a)^n ,$$
$$c_{n+2} - (n+2)c_{n+1} = -a[c_{n+1} - (n+1)c_n] .$$

Also, since

$$e^{-a} = 1 - a + \frac{a^2}{2!} - \frac{a^3}{3!} + \cdots ,$$

$$\lim_{n \to \infty} c_n/n! = e^{-a} .$$

Now let $a = 2$ to give the desired special case.

Also solved by L. Carlitz, Herta T. Freitag, Graham Lord, David Zeitlin, and the Proposer.

A FIBONACCI-PELL INEQUALITY

B-199 Proposed by M. J. DeLeon, Florida Atlantic University, Boca Raton, Florida.

Define the Fibonacci and Pell numbers by

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad n \ge 1;$$

$$P_1 = 1, \quad P_2 = 2, \quad P_{n+2} = 2P_{n+1} + P_n \quad n \ge 1.$$

Prove or disprove that $P_{6k} < F_{11k}$ for $k \ge 1$.

Solution by David Zeitlin, Minneapolis, Minnesota.

Let α, β be the roots of $x^2 = x + 1$, and A, B the roots of $x^2 = 2x + 1$. Now,

$$Y_k \equiv F_{11k} = (\alpha^{11k} - \beta^{11k})/(\alpha - \beta)$$

satisfies

$$(E - \alpha^{11})(E - \beta^{11})Y_k = 0, \qquad \text{or} \qquad Y_{k+2} - 199\,Y_{k+1} - Y_k = 0,$$

and

$$Z_k \equiv P_{6k} = (A^{6k} - B^{6k})/(A - B)$$

satisfies

$$(E - A^6)(E - B^6)Z_k = 0 ,$$

or

$$Z_{k+2} - 198 Z_{k+1} + Z_k = 0,$$

where $E^n R_k = R_{k+n}$.

Let

$$W_k \equiv Z_k - Y_k \equiv P_{6k} - F_{11k} .$$

Then

$$(1) \quad W_{k+2} = Z_{k+2} - Y_{k+2} = 198(Z_{k+1} - Y_{k+1}) - Y_{k+1} - Y_k - Z_k ,$$

or

$$W_{k+2} - 198 W_{k+1} \quad 0, \qquad k = 0, 1, \cdots ,$$

and thus

$$W_k < (198)^{k-1} W_1 \qquad k = 2, 3, \cdots .$$

Since

$$W_1 = P_6 - F_{11} = 70 - 89 = -19 < 0 ,$$

$W_k \equiv P_{6k} - F_{11k} < 0$ for $k = 1, 2, \cdots$. Thus, the stated inequality is true.

Also solved by Wayne Vucenic and the Proposer.

A CLOSE CALL

B-200 Proposed by M. J. DeLeon, Florida Atlantic University, Boca Raton, Florida.

With the notation of B-199, prove or disprove that

$$F_{11k} < P_{6k+1} \quad \text{for} \quad k \geq 1 .$$

Solution by Phil Mana and Wayne Vucenic, University of New Mexico, Albuquerque, New Mexico.

Let

$$a = (1 + \sqrt{5})/2, \quad b = (1 - \sqrt{5})/2, \quad c = 1 + \sqrt{2}, \quad \text{and} \quad d = 1 - \sqrt{2} .$$

Then

$$F_{11k} = (a^{11k} - b^{11k})/\sqrt{5}, \qquad P_{6k+1} = (c^{6k+1} - d^{6k+1})/2\sqrt{2} .$$

Since $|a| > |b|$ and $|c| > |d|$, it can easily be seen that as $k \to \infty$ the limit of F_{11k}/P_{6k+1} is a positive constant times the limit of $(a^{11}/c^6)^k$. Since

$$a^{11} = (L_{11} + F_{11}\sqrt{5})/2 = (199 + 89\sqrt{5})/2 \qquad 99 + 70\sqrt{2} = c^6 ,$$

$(a^{11}/c^6)^k \to +\infty$ as $k \to \infty$ and so ultimately $F_{11k} > P_{6k+1}$. Computer calculation shows that when $k = 128$, $F_{11k} > 8 \times 10^{293} > P_{6k+1}$.

Also solved by the Proposer.

PARITY OF n

B-201 Proposed by Mel Most, Ridgefield Park, New Jersey.

Given that a very large positive integer k is a term F_n in the Fibonacci Sequence, describe an operation on k that will indicate whether n is even or odd.

I. Solution by F. D. Parker, St. Lawrence University, Canton, New York.

Undoubtedly, there are many possible tests. Here is only one test. From the identities

$$F_n^2 = F_{n-1}F_{n+1} - (-1)^n$$
$$F_{n+1} = F_n + F_{n-1} \quad ,$$

we get

$$F_n + 2F_{n-1} = \sqrt{5F_n^2 + 4(-1)^n}$$

Therefore n is even if $5F_n^2 + 4$ is a perfect square; otherwise, n is odd, with the single exception of $n = 1$ or $n = 2$. In this case, no test prevails since $F_1 = F_2 = 1$.

II. Solution by Wayne Vucenic, Student, University of New Mexico, Albuquerque, New Mexico.

The ratio between consecutive terms of the Fibonacci sequence, F_{n+1}/F_n, approaches α by oscillation as n approaches infinity, where α is the Greek golden ratio, or $\frac{1}{2}(1 + \sqrt{5})$, which is $1.61803\cdots$. Thus, if n is even,

(1) $$\frac{F_{n+1}}{F_n} = \alpha + A_n, \quad \text{and } A_n \text{ decreases as } n \text{ increases;}$$

if n is odd,

(2) $$\frac{F_{n+1}}{F_n} = \alpha - B_n, \quad \text{and } B_n \text{ decreases as } n \text{ increases.}$$

If n is even, from Equation (1),

$$F_{n+1} = F_n(\alpha + A_n)$$

$$F_{n+1} = \alpha F_n + A_n F_n$$
$$\alpha F_n = F_{n+1} - A_n F_n ;$$

if n is odd, from Equation (2),

$$\alpha F_n = F_{n+1} + B_n F_n .$$

These equations show that αF_n will be less than F_{n+1} if n is even, and will be greater than F_{n+1} if n is odd.

As n increases, A_n and B_n decrease fast enough that, if $n \geq 2$, $A_n F_n < 0.5$ and $B_n F_n < 0.5$.

Thus, if $n \geq 2$, it is possible to determine whether n is even or odd by multiplying F_n by α, then seeing if the product is greater than or less than the nearest integer which will be F_{n+1}. For example, given that $F_n = 21$, $21 \times 1.618 = 33.978$. This is less than the nearest integer, 34, thus a is even.

Also solved by the Proposer.

[Continued from page 437.] When X and Y are -ve integers,

$$X = (2 - L_{4k})/5, \quad Y = (X - F_{4k})/2 , \quad k = 1, 2, 3, \ldots.$$

And the general solution in +ve integers is:

$$X = (2 + L_{4k-2})/5 = F^2_{2k-1}; \quad Y = (X + F_{4k-2})/2 = F_{2k-1}F_{2k}$$

The author found the first set of integral solutions while others were found by Guy Guillotte

REFERENCES

1. J. A. H. Hunter, "Fibonacci to the Rescue," Fibonacci Quarterly, Oct. 1970, p. 406.
2. David Ferguson, "Letters to the Editor," Fibonacci Quarterly, Feb. 1970, p. 88.