REDUCTION FORMULAS FOR FIBONACCI SUMMATIONS

L. CARLITZ*
Duke University, Durham, North Carolina

1. INTRODUCTION

In a recent paper [1], Brother Alfred Brousseau has obtained a chain of formulas of the following kind.

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+2}} \quad ,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+2}} = \frac{5}{12} - \frac{3}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} \quad ,$$

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} \cdots F_{n+4}} = \frac{97}{2640} + \frac{40}{11} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} \cdots F_{n+6}} \quad .$$

As an application he has computed the value of the sum

$$S = \sum_{n=1}^{\infty} \frac{1}{F_n}$$

to twenty-five decimal places. It does not seem to be known whether the sum S is a rational number.

If we define

$$S_k = \sum_{n=1}^{\infty} \frac{(-1)^{k(n-1)}}{F_n F_{n+1} \cdots F_{n+2k}} \qquad (k = 0, 1, 2, \cdots) \quad ,$$

the above special results suggest that generally

(1.1) $$S_{k+1} = a_k + b_k S_k,$$

where a_k, b_k are rational numbers. We shall show below that this is indeed true and moreover we shall obtain explicit formulas for a_k, b_k. Also we obtain explicit formulas for the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{k(n-1)}}{F_n F_{n+1} \cdots F_{n+2k-1}} \qquad (k = 1, 2, 3, \cdots) .$$

Indeed we shall prove these results in a somewhat more general setting. In place of the Fibonacci numbers F_n we take the numbers u_n defined by

$$u_0 = 0, \qquad u_1 = 1, \qquad u_{n+1} = (\alpha + \beta)u_n - \alpha\beta u_{n-1} \qquad (n = 1, 2, 3, \cdots),$$

where α, β are distinct, and consider the sums

$$U_k = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k}}$$

and

$$T_k = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k-1}} .$$

We show that

(1.2) $$U_{k+1} = c_k + d_k U_k,$$

where c_k, d_k are rational functions of α, β that are determined explicitly. As for T_k, we show that

$$T_k = c_k' + \frac{d_k'}{\alpha},$$

where c_k', d_k' are rational functions of α, β that are determined explicitly. Also it is assumed, in order to assure convergence, that

$$|\alpha| > |\beta|, \qquad |\alpha| > 1 .$$

2. SOME PRELIMINARY RESULTS

To begin with, let α, β denote indeterminates and put

$$(2.1) \qquad u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad , \qquad v_n = \alpha^n + \beta^n .$$

Then, of course,

$$(2.2) \qquad \begin{cases} u_{n+1} = (\alpha + \beta)u_n - \alpha\beta u_{n-1} \\ v_{n+1} = (\alpha + \beta)v_n - \alpha\beta v_n \end{cases} .$$

Next define

$$(2.3) \qquad (u)_0 = 1, \qquad (u)_n = u_1 u_2 \cdots u_n$$

and

$$(2.4) \qquad \begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{(u)_n}{(u)_k (u)_{n-k}} = \begin{Bmatrix} n \\ n - k \end{Bmatrix} .$$

It follows from the definition that

$$(2.5) \qquad \begin{aligned} \begin{Bmatrix} n + 1 \\ k \end{Bmatrix} &= \alpha^k \begin{Bmatrix} n \\ k \end{Bmatrix} + \beta^{n-k+1} \begin{Bmatrix} n \\ k - 1 \end{Bmatrix} \\ &= \beta^k \begin{Bmatrix} n \\ k \end{Bmatrix} + \alpha^{n-k+1} \begin{Bmatrix} n \\ k - 1 \end{Bmatrix} \end{aligned} .$$

Clearly $u_n v_n$, $(u)_n$, $\begin{Bmatrix} n \\ k \end{Bmatrix}$ are symmetric polynomials in α, β; the last assertion is a consequence of (2.5).

Let

$$(2.6) \qquad R_{2k}(x) = \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x^j \quad,$$

$$(2.7) \qquad R_{2k-1}(x) = \sum_{j=0}^{2k-1} (-1)^j \left\{ {2k - 1 \atop j} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \alpha^{j+1} x^j \quad.$$

Then, by (2.5),

$$R_{2k}(x) = \sum_{j=0}^{2k} (-1)^j (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x^j \left[\alpha^j \left\{ {2k - 1 \atop j} \right\} + \beta^{2k-j} \left\{ {2k - 1 \atop j - 1} \right\} \right]$$

$$= \sum_{j=0}^{2k-1} (-1)^j \left\{ {2k - 1 \atop j} \right\} x^j \left[(\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \alpha^j \right.$$

$$\left. - (\alpha\beta)^{\frac{1}{2}j(j+1)(j+2)-(j+1)k} \beta^{2k-j+1} x \right]$$

$$= \sum_{j=0}^{2k-1} (-1)^j \left\{ {2k - 1 \atop j} \right\} x^j \left[\alpha^{\frac{1}{2}j(j+1)(j+2)-jk-1} \beta^{\frac{1}{2}j(j+1)-jk} \right.$$

$$\left. - \alpha^{\frac{1}{2}(j+1)(j+2)-jk-j} \beta^{\frac{1}{2}j(j+1)-jk+k} x \right]$$

$$= (\alpha^{-1} - \alpha^k \beta^k x) \sum_{j=0}^{2k-1} (-1)^j \left\{ {2k - 1 \atop j .} \right\} \alpha^{\frac{1}{2}(j+1)(j+2)-jk} \beta^{\frac{1}{2}j(j+1)-jk} x^j \quad,$$

so that

$$(2.8) \qquad R_{2k}(x) = (\alpha^{-1} - \alpha^{-k} \beta^k x) R_{2k-1}(x) \quad.$$

Similarly,

$$R_{2k+1}(x) = \sum_{j=0}^{2k+1} (-1)^j \left\{ {2k+1 \atop j} \right\} \alpha^{\frac{1}{2}(j+1)(j+2)-j(k+1)} \beta^{\frac{1}{2}j(j+1)-j(k+1)} x^j$$

$$= \sum_{j=0}^{2k+1} (-1)^j \alpha^{\frac{1}{2}(j+1)(j+2)-j(k+1)} \beta^{\frac{1}{2}j(j+1)-j(k+1)} x^j$$

$$\cdot \left[\beta^j \left\{ {2k \atop j} \right\} + \alpha^{2k-j+1} \left\{ {2k \atop j-1} \right\} \right]$$

$$= \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} x^j \left[\alpha^{\frac{1}{2}(j+1)(j+2)-jk-j} \beta^{\frac{1}{2}j(j+1)-jk} \right.$$

$$- \alpha^{\frac{1}{2}(j+2)(j+3)-(j+1)(k+1)+2k-j}$$

$$\left. \cdot \beta^{\frac{1}{2}(j+1)(j+2)-(j+1)(k+1)} x \right]$$

$$= \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} x^j \left[\alpha^{\frac{1}{2}j(j+1)-jk+1} \beta^{\frac{1}{2}j(j+1)-jk} \right.$$

$$\left. - \alpha^{\frac{1}{2}j(j+1)-jk+k+2} \beta^{\frac{1}{2}j(j+1)-jk-k} x \right]$$

$$= (\alpha - \alpha^{k+1}\beta^{-k}x) \sum_{s=0}^{2k} (-1)^k \left\{ {2k \atop j} \right\} \alpha^{\frac{1}{2}j(j+1)-jk} \beta^{\frac{1}{2}j(j+1)-jk} x^j$$

and so

$$(2.9) \qquad\qquad R_{2k+1}(x) = (\alpha - \alpha^{k+1}\beta^{-k}x) R_{2k}(x) .$$

Combining (2.8) and (2.9) we get

$$(2.10) \quad R_{2k}(x) = (\alpha\beta)^{-k+1}(\alpha^{k-1} - \beta^k x)(\beta^{k-1} - \alpha^k x)R_{2k-2}(x) \qquad\qquad (k \geq 1)$$

and therefore

$$(2.11) \quad R_{2k}(x) = (\alpha\beta)^{-\frac{1}{2}k(k-1)} \prod_{j=1}^{k} (\alpha^{j-1} - \beta^j x)(\beta^{j-1} - \alpha^j x)$$

$$= (\alpha\beta)^{-\frac{1}{2}k(k-1)} \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} - v_{2j-1}x + (\alpha\beta)^j x^2 \right] ,$$

with v_{2j-1} defined by (2.1).

The recurrence (2.10) can be generalized in the following way. Let

$$\xi = (x_0, \ x_1, \ x_2, \ \cdots)$$

denote an arbitrary sequence and define

$$R_{2k}(\xi) = \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \, x_j \ .$$

Then, exactly as above, we have

(2.12) $$R_{2k}(\xi) = (\alpha\beta)^{-k+1} \sum_{j=0}^{2k-2} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-j(k-1)}$$

$$\cdot \left[(\alpha\beta)^{k-1} x_j - v_{2k-1} x_{j+1} + (\alpha\beta)^k x_{j+2} \right] \ .$$

It follows from (2.12) that

(2.13) $$R_{2k+2}(\xi) - R_{2k}(\xi)$$

$$= \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} \alpha\beta^{\frac{1}{2}j(j+1)-jk} \left[-(\alpha\beta)^{-k} v_{2k+1} x_{j+1} + \alpha\beta \, x_{j+2} \right]$$

3. A SECOND PROOF OF EQ. (2.11)

It may be of interest to show that (2.11) can be obtained from a known result. We recall that

(3.1)
$$\prod_{j=0}^{k-1} (1 - q^j x) = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{1}{2} j (j-1)} x^j \quad ,$$

where

$$\begin{bmatrix} k \\ j \end{bmatrix} = \frac{(1 - q^k)(1 - q^{k-1}) \cdots (1 - q^{k-j+1})}{(1 - q)(1 - q^2) \cdots (1 - q^j)} \quad .$$

Replacing q by α/β ,

$$\begin{bmatrix} k \\ j \end{bmatrix} \longrightarrow \frac{(\beta^k - \alpha^k)(\beta^{k-1} - \alpha^{k-1}) \cdots (\beta^{k-j+1} - \alpha^{k-j+1})}{(\beta - \alpha)(\beta^2 - \alpha^2) \cdots (\beta^j - \alpha^j)} \beta^{j^2 - jk}$$

$$= \begin{Bmatrix} k \\ j \end{Bmatrix} \beta^{j^2 - jk}$$

Thus (2.1) becomes

$$\beta^{-\frac{1}{2} k (k-1)} \prod_{j=0}^{k-1} (\beta^j - \alpha^j x) = \sum_{j=0}^{k} (-1) \begin{Bmatrix} k \\ j \end{Bmatrix} \alpha^{\frac{1}{2} j (j-1)} \beta^{\frac{1}{2} j (j+1) - jk} x^j \quad .$$

In particular, if k is replaced by 2k, we get

(3. 2) $$\beta^{-k(2k-1)} \prod_{j=0}^{2k-1} (\beta^j - \alpha^j x) = \sum_{j=0}^{2k} (-1)^j \begin{Bmatrix} 2k \\ j \end{Bmatrix} \alpha^{\frac{1}{2} j (j-1)} \beta^{j(j+1) - 2jk} x_j \quad .$$

Now replace x by $\alpha^{1-k} \beta^k x$ and (3.2) becomes

$$\beta^{-k(2k-1)} \prod_{j=0}^{2k-1} (\beta^j - \alpha^{j-k+1} \beta^k x)$$

$$= \sum_{j=0}^{2k} (-1)^j \begin{Bmatrix} 2k \\ j \end{Bmatrix} (\alpha\beta)^{\frac{1}{2} j (j+1) - jk} x^j \quad ,$$

so that

$$R_{2k}(x) = \beta^{-k(2k-1)} \prod_{j=0}^{2k-1} (\beta^j - \alpha^{j-k+1}\beta^k x)$$

(3.3)

$$= \prod_{j=0}^{2k-1} (1 - \alpha^{j-k+1}\beta^{k-j}x) = \prod_{j=1}^{2k} (1 - \alpha^{k-j+1}\beta^{j-k}x) \ ;$$

at the last step we have replaced j by $2k - j$.

Now on the other hand,

$$\prod_{j=1}^{k} (\alpha^{j-1} - \beta^j x)(\beta^{j-1} - \alpha^j x)$$

$$= (\alpha\beta)^{\frac{1}{2}k(k-1)} \prod_{k=1}^{k} (1 - \alpha^{-j+1}\beta^j x)(1 - \alpha^j\beta^{-j+1}x)$$

$$= (\alpha\beta)^{\frac{1}{2}k(k-1)} \prod_{j=0}^{k-1} (1 - \alpha^{j-k+1}\beta^{k-j}x)(1 - \alpha^{k-j}\beta^{j-k+1}x)$$

$$= (\alpha\beta)^{\frac{1}{2}k(k-1)} \prod_{j=0}^{k-1} (1 - \alpha^{j-k+1}\beta^{k-j}x) \cdot \prod_{j=k}^{2k-1} (1 - \alpha^{j-k+1}\beta^{k-j}x)$$

$$= (\alpha\beta)^{\frac{1}{2}k(k-1)} \prod_{j=0}^{2k-1} (1 - \alpha^{j-k+1}\beta^{k-j}x) \ .$$

Substitution in (3.3) gives

$$R_{2k}(x) = (\alpha\beta)^{-\frac{1}{2}k(k-1)} \prod_{j=1}^{k} (\alpha^{j-1} - \beta^j x)(\beta^{j-1} - \alpha^j x) \ ,$$

which is the first of (2.11).

4. THE MAIN RESULTS

We consider next the expansion into partial fractions of

$$(4.1) \qquad \frac{x^k}{(1 - x)(\alpha - \beta x)(\alpha^2 - \beta^2 x) \cdots (\alpha^{2k} - \beta^{2k} x)} = \sum_{j=0}^{2k} \frac{A_j}{\alpha^j - \beta^j x} \quad,$$

where A_j is independent of x. We find that

$$(4.2) \qquad (\alpha - \beta)^{2k} A_j = (-1)^j \frac{(\alpha\beta)^{\frac{1}{2}j(j+1)-jk}}{(u)_j (u)_{2k-j}} \quad,$$

where, as above,

$$(u)_n = \prod_{j=1}^{n} \frac{\alpha^j - \beta^j}{\alpha - \beta} \;.$$

Thus we have the identity

$$(4.3) \qquad \frac{(\alpha - \beta)^{2k} x^k}{(1 - x)(\alpha - \beta x) \cdots (\alpha^{2k} - \beta^{2k} x)}$$

$$= \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} \frac{(\alpha\beta)^{\frac{1}{2}j(j+1)-jk}}{\alpha^j - \beta^j x} \qquad \cdot$$

For $x = \alpha^{-n}\beta^n$, the left member of (4.3) becomes

$$\frac{\alpha - \beta}{u_n u_{n+1} \cdots u_{n+2k}} \alpha^{n(k+1)} \beta^{nk} \quad,$$

while the right member becomes

$$\frac{\alpha^n}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} \frac{(\alpha\beta)^{\frac{1}{2}j(j+1)-jk}}{\alpha^{n+j} - \beta^{n+j}} \qquad \cdot$$

We have therefore the identity

$$(4.4) \qquad \frac{(\alpha\beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k}} = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2j \\ j \end{matrix} \right\} \frac{(\alpha\beta)^{\frac{1}{2}j(j+1)-jk}}{\alpha^{n+j} - \beta^{n+j}} \quad .$$

Now put

$$(4.5) \qquad U_k = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k}} \qquad (k = 0, 1, 2, \cdots)$$

and in particular, for $k = 0$,

$$(4.6) \qquad U = U_0 = \sum_{n=1}^{\infty} \frac{1}{u_n} \quad .$$

To assure convergence, we assume that

$$|\alpha| > |\beta|, \qquad |\alpha| > 1 \ .$$

Then, by (4.4) and (4.5),

$$U = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \sum_{n=1}^{\infty} \frac{1}{u_{n+j}}$$

$$= \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \cdot U$$

$$- \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \sum_{n=1}^{j} \frac{1}{u_n} \quad .$$

The coefficient on the right is equal to

$$\frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{array}{c} 2k \\ j \end{array} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk}$$

$$= \frac{1}{(u)_{2k}} S_{2k}(1)$$

$$= \frac{(\alpha\beta)^{-\frac{1}{2}k(k-1)}}{(u)_{2k}} \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} - v_{2j+1} + (\alpha\beta)^j \right] \quad .$$

We have therefore

$$\begin{aligned}
U_k &= \frac{(\alpha\beta)^{-\frac{1}{2}k(k-1)}}{(u)_{2k}} \; U \cdot \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} - v_{2j+1} + (\alpha\beta)^j \right] \\
&\quad - \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{array}{c} 2k \\ j \end{array} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \sum_{n=1}^{j} \frac{1}{u_n} \quad .
\end{aligned}$$

(4.7)

More generally, if we put

$$(4.8) \qquad U_k(x) = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{nk} x^{n+2k}}{u_n u_{n+1} \cdots u_{n+2k}}$$

and in particular, for $k = 0$,

$$(4.9) \qquad U(x) = U_0(x) = \sum_{n=1}^{\infty} \frac{x^n}{u_n} \quad ,$$

then as above,

$$U_k(x) = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \sum_{n=1}^{\infty} \frac{x^{n+2k}}{u_{n+j}}$$

$$= \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x^{2k-j} U(x)$$

$$- \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x^{2k-j} \sum_{n=1}^{j} \frac{x^n}{u_n} \quad .$$

Since

$$\sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x^{2k-j} = x^{2k} S_{2k}(x^{-1})$$

$$= (\alpha\beta)^{-\frac{1}{2}k(k-1)} x^{2k} \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} - v_{2j-1} x^{-1} + (\alpha\beta)^j x^{-2} \right]$$

$$= (\alpha\beta)^{-\frac{1}{2}k(k-1)} \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} x^2 - v_{2j-1} x + (\alpha\beta)^j \right] \quad ,$$

it is clear that

$$U_k(x) = \frac{(\alpha\beta)^{-\frac{1}{2}k(k-1)}}{(u)_{2k}} U(x) \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} x^2 - v_{2j-1} x + (\alpha\beta)^j \right]$$

(4.10)

$$- \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{j(j+1)-jk} x^{2k-j} \sum_{n=1}^{j} \frac{x^n}{u_n} \quad .$$

It follows from (4.10) that

(4.11) $U_{k+1}(x) - \dfrac{x^2 - (\alpha\beta)^k v_{2k+1} x + \alpha\beta}{u_{2k+1} u_{2k+2}} U_k(x)$

$$= - \frac{x^{2k+2}}{(u)_{2k+2}} \left\{ \sigma_{2k+2}(x) - \left[1 - (\alpha\beta)^{-k} v_{2k+1} x^{-1} + \alpha\beta x^{-2} \right] \sigma_{2k}(x) \right\},$$

where

$$\sigma_{2k}(x) = \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{j(j+1)-jk} x^{-j} \sum_{n=1}^{j} \frac{x^n}{u_n} \quad .$$

If we now apply (2.13) to $\sigma_{2k}(x)$ with

(4.12) $$x_j = x^{-j} \sum_{n=1}^{j} \frac{x^n}{u_n} \quad ,$$

we get

$$\sigma_{2k+2}(x) - \sigma_{2k}(x) = \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk}$$

$$\cdot \left[-(\alpha\beta)^{-k} v_{2k+1} x_{j+1} + \alpha\beta x_{j+2} \right] .$$

Thus, by (4.12), (4.11) reduces to

$$U_{k+1}(x) - \frac{x^2 - (\alpha\beta)^{-k} v_{2k+1} x + \alpha\beta}{u_{2k+1} u_{2k+2}} U_k(x)$$

(4.13) $$= \frac{x^{2k+2}}{(u)_{2k+2}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk}$$

$$\cdot \left[(\alpha\beta)^{-k} \frac{v_{2k+1}}{u_{j+1}} - \alpha\beta \left(\frac{x^{-1}}{u_{j+1}} + \frac{1}{u_{j+2}} \right) \right] .$$

In particular, for $x = 1$, (4.13) becomes

(4.14)
$$U_{k+1} - \frac{1 - (\alpha\beta)^{-k} v_{2k+1} x + \alpha\beta}{u_{2k+1} u_{2k+2}} U_k$$

$$= \frac{1}{(u)_{2k+2}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk}$$

$$\cdot \left[(\alpha\beta)^{-k} \frac{v_{2k+1}}{u_{j+1}} - \alpha\beta \left(\frac{1}{u_{j+1}} + \frac{1}{u_{j+2}} \right) \right] .$$

5. APPLICATION TO FIBONACCI SUMMATIONS

We now consider the special case

(5.1)
$$\alpha + \beta = 1, \qquad \alpha\beta = -1 .$$

Then

(5.2)
$$u_n = F_n, \qquad v = L_n ,$$

the Fibonacci and Lucas numbers, respectively. Also $U_k(x)$ becomes

(5.3)
$$\sum_{n=1}^{\infty} \frac{(-1)^{nk} x^{n+2k}}{F_n F_{n+1} \cdots F_{n+2k}}$$

and in particular U_k becomes

(5.4)
$$\sum_{n=1}^{\infty} \frac{(-1)^{nk}}{F_n F_{n+1} \cdots F_{n+2k}} = (-1)^k S_k .$$

Formula (4.10) reduces to

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk} x^{n+2k}}{F_n F_{n+1} \cdots F_{n+2k}} = \frac{1}{(F)_{2k}} \prod_{j=1}^{k} (x^2 + (-1)^j L_{2j-1} x - 1) \cdot \sum_{n=1}^{\infty} \frac{x^n}{F_n}$$

(5.5)

$$- \frac{1}{(F)_{2k}} \sum_{j=0}^{2k} (-1)^{\frac{1}{2}j(j-1)-jk} \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} x^{2k-j} \sum_{n=1}^{j} \frac{x^n}{F_n} \; ,$$

where now

$$\left\{ \begin{matrix} n \\ j \end{matrix} \right\} = \frac{F_n F_{n-1} \cdots F_{n-j+1}}{F_1 F_2 \cdots F_j}$$

and

$$(F)_{2k} = F_1 F_2 \cdots F_{2k} \; .$$

In particular, for $x = 1, -1$, (5.5) reduces to

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk}}{F_n F_{n+1} \cdots F_{n+2k}} = \frac{(-1)^{\frac{1}{2}k(k+1)}}{(F)_{2k}} \prod_{j=1}^{k} L_{2j-1} \cdot \sum_{n=1}^{\infty} \frac{1}{F_n}$$

(5.6)

$$- \frac{1}{(F)_{2k}} \sum_{j=0}^{2k} (-1)^{\frac{1}{2}j(j-1)-jk} \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} \sum_{n=1}^{j} \frac{1}{F_n} \; ,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n(k+1)}}{F_n F_{n+1} \cdots F_{n+2k}} = \frac{(-1)^{\frac{1}{2}k(k-1)}}{(F)_{2k}} \prod_{j=1}^{k} L_{2j-1} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n}$$

(5.7)

$$- \frac{1}{(F)_{2k}} \sum_{j=0}^{2k} (-1)^{\frac{1}{2}j(j+1)-jk} \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} \sum_{n=1}^{j} \frac{(-1)^n}{F_n} \; .$$

For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} F_{n+2}} = -S + 3 \ ,$$

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} = -\frac{2}{3} S + \frac{41}{18} \ ,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} \cdots F_{n+6}} = \frac{11}{60} S + \frac{17749}{28800} \ ,$$

where

$$S = \sum_{n=1}^{\infty} \frac{1}{F_n} \ .$$

We note also that (4.14) yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{n(k+1)}}{F_n F_{n+1} \cdots F_{n+2k+2}} + \frac{(-1)^k L_{2k+1}}{F_{2k+1} F_{2k+2}} \sum_{n=1}^{\infty} \frac{(-1)^{nk}}{F_n F_{n+1} \cdots F_{n+2k}}$$

(5.8)

$$= \frac{1}{(F)_{2k+2}} \sum_{j=0}^{2k} (-1)^{\frac{1}{2}j(j-1)-jk} \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} \left[(-1)^k \frac{L_{2k+1}}{F_{j+1}} + \frac{1}{F_{j+1}} + \frac{1}{F_{j+2}} \right] .$$

For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} F_{n+2}} + \sum_{n=1}^{\infty} \frac{1}{F_n} = 3 \ ,$$

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} \cdots F_{n+4}} - \frac{2}{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} F_{n+2}} = \frac{5}{18} \ ,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} \cdots F_{n+6}} + \frac{11}{40} \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} \cdots F_{n+4}} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 8} \frac{97}{40} = \frac{97}{9600} \quad ,$$

in agreement with the special results obtained in [1].

It should be observed that the formulas of this section depend essentially on $\alpha\beta = -1$. Very similar results can be stated for $\alpha\beta = 1$. Thus, in particular we can obtain results like the above for such sums as

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n} F_{2n+2} \cdots F_{2n+4k}}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk}}{F_{3n} F_{3n+3} \cdots F_{3n+6k}} \quad .$$

6. SOME ADDITIONAL RESULTS

Returning to the general case, we shall now evaluate the sum

$$(6.1) \qquad T_k = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k-1}} \qquad (k = 1, 2, 3, \cdots) \quad .$$

Multiplying (4.4) by $(\alpha\beta)^n / u_{n+2k+1}$, we get

$$\frac{(\alpha\beta)^{n(k+1)}}{u_n u_{n+1} \cdots u_{n+2k+1}} = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \begin{Bmatrix} 2k \\ j \end{Bmatrix} \frac{(\alpha\beta)^{\frac{1}{2}j(j+1)-jk+n}}{u_{n+j} u_{n+2k+1}}$$

so that

$$\sum_{n=1}^{\infty} \frac{(\alpha\beta)^{n(k+1)}}{u_n u_{n+1} \cdots u_{n+2k+1}} = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \begin{Bmatrix} 2k \\ j \end{Bmatrix} (\alpha\beta)^{\frac{1}{2}j(j+1)-j(k+1)}$$

(6.2)

$$\cdot \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{n+j}}{u_{n+j} u_{n+2k+1}} \quad .$$

Now consider the sum

(6.3)
$$A_r = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^n}{u_n u_{n+r}} \quad .$$

Since

$$u_{n+r} u_{n-1} - u_n u_{n+r-1} = (\alpha\beta)^n u_r \quad ,$$

we have

$$\frac{u_{n-1}}{u_n} - \frac{u_{n+r-1}}{u_{n+r}} = \frac{(\alpha\beta)^n u_r}{u_n u_{n+r}} \quad .$$

In this identity, take $n = 1, 2, \cdots, N$ and sum. Then

$$u_r \sum_{n=1}^{N} \frac{(\alpha\beta)^n}{u_n u_{n+r}} = \sum_{n=1}^{N} \frac{u_{n-1}}{u_n} - \sum_{n=1}^{N} \frac{u_{n+r-1}}{u_{n+r}}$$

(6.4)

$$= \sum_{n=1}^{r} \frac{u_{n-1}}{u_n} - \sum_{n=1}^{r} \frac{u_{N+n-1}}{u_{N+n}} \quad .$$

Since we have assumed that

ON THE COEFFICIENTS OF A GENERATING SERIES

M. BERESIN, E. LEVINE, and D. LUBELL
Adelphi University, Garden City, New York

1. INTRODUCTION

Our object of study is the generating series

(1)
$$\prod_{n=1} \left(1 + \rho x^{u_n} \right) = \sum^{\infty} \epsilon(n) x^n ,$$

where the coefficients $\epsilon(n)$ are polynomials in ρ, and where $\{ u_n \}$ is the sequence defined by

(2)
$$u_1 = 1, \quad u_2 = 2, \quad u_n = u_{n-1} + u_{n-2} \quad \text{for} \quad n > 2 .$$

Theorem 1. The values assumed by the coefficients $\epsilon(n)$ as $n = 0$, 1, 2, $\cdots$ range over a finite set if and only if ρ is one of the numbers 0, $-1, \omega$, or ω^2, where ω and ω^2 are the complex cube roots of unity.

The theorem has applications to partition theory. It implies the existence of certain symmetries, which we illustrate in Section 5, among the partitions of integers into terms of the sequence $\{u_n\}$. Sections 3 and 4 are devoted to the proof of Theorem 1. In Section 2, some preliminary recursion formulas are obtained, which find application in Sections 3 and 4.

For an added comment, see note at conclusion of this article.

2. RECURSION FORMULAS FOR $\epsilon(n)$

For each natural number n, let $\nu(n)$ denote the largest index k for which $u_k \le n$. Thus $\nu(n)$ is defined by the condition that

(3)
$$u_{\nu(n)} \le n < u_{\nu(n)+1} .$$

Writing $\epsilon(m) = 0$ for negative m, we prove that

Lemma 1. For $n > 1$, $\epsilon(n)$ satisfies the recursion

467

(4) $\qquad \epsilon(n) = \rho\epsilon(n - u_\nu) + \rho\epsilon(n - u_{\nu-1}) - \rho^2\epsilon(n - 2u_{\nu-1})$,

where we have written ν for $\nu(n)$.

For a fixed natural number n, write $f(x) \equiv g(x)$ if $f(x)$ and $g(x)$ are formal power series whose difference contains only terms of degree greater than n. Then (1) and (3) imply that

$$\sum_{m=0}^{n} \epsilon(m)x^m \equiv \prod_{m=1}^{\nu(n)} \left(1 + \rho x^{u_m}\right) .$$

From (2) and (3) it follows that

$$\left(1 + \rho x^{u_\nu}\right)^{-1}\left(1 + \rho x^{u_{\nu-1}}\right)^{-1} \equiv 1 - \rho x^{u_\nu} - \rho x^{u_{\nu-1}} + \rho^2 x^{2u_{\nu-1}} .$$

so that

$$\left(1 - \rho x^{u_\nu} - \rho x^{u_{\nu-1}} + \rho^2 x^{2u_{\nu-1}}\right)\sum_{m=0}^{n} \epsilon(m)x^m \equiv \prod_{m=1}^{\nu-2} \left(1 + \rho x^{u_m}\right) .$$

Equating coefficients of x^n, we find that

(5) $\qquad \epsilon(n) - \rho\epsilon(n - u_\nu) - \rho\epsilon(n - u_{\nu-1}) + \rho^2\epsilon(n - 2u_{\nu-1})$

is the coefficient of x^n in

$$\prod_{m=1}^{\nu-2} \left(1 + \rho x^{u_m}\right) .$$

Now from the identity

$$\sum_{m=1}^{\nu-2} u_m = u_\nu - 2 ,$$

(an immediate consequence of (2)), it is clear that

$$\deg \prod_{m=1}^{\nu-2} \left(1 + \rho x^{u_m} \right) = u_\nu - 2 \leq n - 2 ,$$

so that (5) vanishes, proving the lemma.

In sequel, a shall denote a natural number and we shall write σ for $\nu(a)$. From the inequalities

$$u_{\sigma+1} \leq 2u_\sigma \leq a + u_\sigma < u_{\sigma+1} + u_\sigma = u_{\sigma+2} ,$$

$$u_{\sigma+2} = u_\sigma + u_{\sigma+1} \leq a + u_{\sigma+1} < 2u_{\sigma+1} \leq u_{\sigma+3} ,$$

$$u_{\sigma+n} \leq a + u_{\sigma+n} < u_{\sigma+1} + u_{\sigma+n} \leq u_{\sigma+n+1} \quad \text{for} \quad n \geq 2 ,$$

we obtain

$$(6) \qquad \nu(a + u_{\sigma+n}) = \begin{cases} \sigma + n + 1 & \text{if } 0 \leq n < 2 \\ \sigma + n & \text{if } 2 \leq n \end{cases} .$$

Applying the fundamental recursion (4),

$$(7) \qquad \epsilon(a + u_\sigma) = \rho\epsilon(a - u_{\sigma-1}) + \rho\epsilon(a) - \rho^2\epsilon(a - u_\sigma) ,$$

$$(8) \qquad \epsilon(a + u_{\sigma+1}) = \rho\epsilon(a - u_\sigma) + \rho\epsilon(a) ,$$

$$\epsilon(a + u_{\sigma+2}) = \rho\epsilon(a) + \rho\epsilon(a + u_\sigma) - \rho^2\epsilon(a - u_{\sigma-1})$$

from which it follows that

$$(9) \qquad \epsilon(a + u_{\sigma+2}) = \rho(1 + \rho)\epsilon(a) - \rho^3\epsilon(a - u_\sigma) .$$

Lemma 2. For $h \geq 1$ and $\rho \neq 1$ we have

$$(10) \qquad \epsilon(a + u_{\sigma+2h}) = \frac{\rho(1 - \rho^{h+1})}{1 - \rho} \epsilon(a) - \rho^{h+2}\epsilon(a - u_\sigma) .$$

For $k > 1$, Eq. (6) and Lemma 1 imply that

(11)
$$\epsilon(a + u_{\sigma+2k}) = \rho\epsilon(a) + \rho\epsilon(a + u_{\sigma+2k-2})$$

since the term $\rho^2 \epsilon(a - u_{\sigma+2k-3})$ vanishes. Multiplying both sides of (11) by ρ^{-k} and summing,

$$\sum_{k=2}^{h} \rho^{-k}\epsilon(a + u_{\sigma+2k}) = \frac{\rho^{-1} - \rho^{-h}}{1 - \rho^{-1}}\epsilon(a) + \sum_{k=1}^{h-1} \rho^{-k}\epsilon(a + u_{\sigma+2k}) ,$$

so that, for $h \geq 2$,

$$\epsilon(a + u_{\sigma+2h}) = \frac{\rho(1 - \rho^{h-1})}{1 - \rho}\epsilon(a) + \rho^{h-1}\epsilon(a + u_{\sigma+2}) .$$

An appeal to (9) proves the lemma.

Lemma 3. For $h \geq 1$ and $\rho \neq 1$ we have

(12)
$$\epsilon(a + u_{\sigma+2h+1}) = \frac{\rho(1 - \rho^{h+1})}{1 - \rho}\epsilon(a) .$$

For $k > 1$, Eq. (16) and Lemma 1 imply that

$$\epsilon(a + u_{\sigma+2k+1}) = \rho\epsilon(a) + \rho\epsilon(a + u_{\sigma+2k+1}) .$$

Treating this in the same manner as (11), we get

(13)
$$\epsilon(a + u_{\sigma+2h+1}) = \frac{\rho(1 - \rho^{h-1})}{1 - \rho}\epsilon(a) + \rho^{h-1}\epsilon(a + u_{\sigma+3})$$

for $h \geq 2$. But (6), (8) and Lemma 1 imply that

$$\epsilon(a + u_{\sigma+3}) = \rho\epsilon(a) + \rho\epsilon(a + u_{\sigma+1}) - \rho^2\epsilon(a - u_{\sigma}) = \rho(1 + \rho)\epsilon(a) .$$

Inserting this identity in (13), we arrive at (12), which is seen to hold for $h = 1$ as well.

3. NECESSITY THAT $\rho = 0, -1, \omega,$ or ω^2

We can now prove that if the coefficients $\epsilon(1), \epsilon(2), \epsilon(3), \cdots$ range over a finite set of values, then ρ must be one of the numbers $0, -1, \omega,$ or ω^2.

From (1) and (2), it is clear that $\epsilon(1) = \rho$ and $\nu(1) = 1$. Taking $a = 1$ and $\sigma = \nu(a) = 1$ in (12),

$$\epsilon(1 + u_{2h+2}) = \frac{\rho(1 - \rho^{h+1})}{1 - \rho}$$

for $h \geq 1$. If these values all lie in a finite set, then ρ must be either zero or a root of unity.

Taking $h = 1$ in (12), we get for $a \geq 0$,

$$(14) \qquad \epsilon(a + u_{\sigma+3}) = \rho(1 + \rho)\epsilon(a) .$$

Letting $a', a'', a''', \cdots,$ and $\sigma', \sigma'', \sigma''', \cdots$ be defined by

$$a' = a + u_{\sigma+3}, \quad \sigma' = \nu(a') ,$$
$$a'' = a' + u_{\sigma'+3}, \quad \sigma'' = \nu(a'') ,$$
$$a''' = a'' + u_{\sigma''+3}, \quad \sigma''' = \nu(a''') ,$$

etc., we obtain by iterating (14),

$$\epsilon(a^{(t)}) = \rho^t(1 + \rho)^t \epsilon(a) ;$$

since these values all lie in a finite set, $\rho(1 + \rho)$ must either be zero or a root of unity. Thus, either $\rho = 0$, $\rho = -1$, or both ρ and $1 + \rho$ are roots of unity, in which case it is a simple deduction that $\rho = \omega$ or $\rho = \omega^2$.

4. SUFFICIENCY OF $\rho = 0, -1, \omega,$ or ω^2; THE METHOD OF DESCENT

If $\rho = 0$, it follows directly from (1) that $\epsilon(0) = 1$ and $\epsilon(n) = 0$ for $n \neq 0$. For the case $\rho = -1, \omega,$ or ω^2, we shall employ a method of descent.

The next lemma is needed only for $\rho = \omega$ or ω^2. It is valid, however, for all ρ.

<u>Lemma 4.</u> For each natural number n, $\epsilon(n) - \rho\,\epsilon(n - u_\nu)$ either vanishes or is of the form $\rho^h \epsilon(m)$ for some $h \geq 0$ and some $m < n$.

We define a finite descending chain of natural numbers $n^{(0)} > n^{(1)} > n^{(2)} > \cdots$ as follows:

$$n^{(0)} = n, \qquad \nu^{(0)} = \nu = \nu(n) \, .$$

If

$$n^{(k)} \leq 2u_{\nu^{(k)}-1} \, ,$$

the chain terminates at $n^{(k)}$; if, on the other hand,

$$n^{(k)} > 2u_{\nu^{(k)}-1} \, ,$$

define $n^{(k+1)}$ and $\nu^{(k+1)}$ by

$$n^{(k+1)} = n^{(k)} - u_{\nu^{(k)}-1} \, , \qquad \nu^{(k+1)} = \nu^{(k)} - 1 \, .$$

First, we show by induction on k that $\nu^{(k)} = \nu(n^{(k)})$, for if the chain extends to $n^{(k+1)}$, then

$$u_{\nu^{(k+1)}} = u_{\nu^{(k)}-1} < n^{(k)} - u_{\nu^{(k)}-1} = n^{(k+1)}$$

and

$$n^{(k+1)} = n^{(k)} - u_{\nu^{(k)}-1} < u_{\nu^{(k)}+1} - u_{\nu^{(k)}-1} = u_{\nu^{(k)}} = u_{\nu^{(k+1)}+1} \, .$$

Next, applying (4) to $n^{(k)}$, we arrive at

$$\epsilon(n^{(k)}) - \rho\epsilon(n^{(k)} - u_{\nu^{(k)}}) = \rho\left\{\epsilon(n^{(k+1)}) - \rho\epsilon(n^{(k+1)} - u_{\nu^{(k+1)}})\right\} ;$$

it follows that

$$\epsilon(n) - \rho\epsilon(n - u_\nu) = \rho^k\left\{\epsilon(n^{(k)}) - \rho\epsilon(n^{(k)} - u_{\nu^{(k)}})\right\} .$$

If $n^{(k)}$ is the last term in the chain, then (4) applied to $n^{(k)}$ yields

$$\epsilon(n^{(k)}) - \rho\epsilon(n^{(k)} - u_{\nu^{(k)}}) = \begin{cases} \rho\epsilon(m^{(k)} - u_{\nu^{(k)}-1}) & \text{if } n^{(k)} < 2u_{\nu^{(k)}-1} \\ \rho\left\{\epsilon(u_{\nu^{(k)}-1}) - \rho\right\} & \text{if } n^{(k)} = 2u_{\nu^{(k)}-1} \end{cases} .$$

Hence, in the first case,

$$\epsilon(n) - \rho\epsilon(n - u_\nu) = \rho^{k+1}\epsilon(n^{(k)} - u_{\nu^{(k)}-1}) .$$

Finally, (4) applied to u_t yields

$$\epsilon(u_t) = \rho + \rho\epsilon(u_{t-2}) ,$$

so that

$$(15) \qquad \epsilon(u_t) - \rho = \begin{cases} 0 \text{ if } t = 1 \text{ or } t = 2 \\ \rho\epsilon(u_{t-2}) \quad \text{otherwise} \end{cases} .$$

Therefore, the second case results in

$$\epsilon(n) - \rho\epsilon(n - u) = \begin{cases} 0 \text{ if } \nu^{(k)} \leq 3 \\ \rho^{k+2}\epsilon(u_{\nu^{(k)}-2}) \text{ otherwise} \end{cases} ,$$

and the lemma is proved.

Lemma 5. If $k \geq 2$ and $\rho = -1, \omega$, or ω^2, then $\epsilon(a + u_{\sigma+k})$ either vanishes or is of the form $\pm \rho^t \epsilon(m)$ for some $t \geq 0$ and some $m < a + u_{\sigma+k}$.

If k is odd, the result is a direct consequence of Lemma 3.

If k is even and $\rho = -1$, then Lemma 2 implies that $\epsilon(a + u_{\sigma+k})$ equals either $\epsilon(a - u_\sigma)$ or $-\epsilon(a) - \epsilon(a - u_\sigma)$ which, according to (8), in turn equals $\epsilon(a + u_{\sigma+1})$.

If is is even and $\rho = \omega$ or $\rho = \omega^2$, then Lemma 2 implies that

$$\epsilon(a + u_{\sigma+k}) = \begin{cases} \rho\left\{\epsilon(a) - \rho\epsilon(a - u_\sigma)\right\} & \text{if } k \equiv 0 \pmod 3 \\ -\epsilon(a) - \epsilon(a - u_\sigma) & \text{if } k \equiv 2 \pmod 3 \\ -\rho\epsilon(a - u_\sigma) & \text{if } k \equiv 1 \pmod 3 \end{cases} \quad .$$

In the first case, Lemma 4 yields the desired form; in the third case, the result is manifest. Finally, in the second case, Eq. (8) gives

$$-\epsilon(a) - \epsilon(a - u_\sigma) = -\rho^2\epsilon(a + u_{\sigma+1}) \quad .$$

To complete the proof of the theorem, we show by a method of descent that if $\rho = -1, \omega$, or ω^2, then for every n, either

$$\epsilon(n) = \pm\rho^t$$

for some $t \geq 0$, or

$$\epsilon(n) = 0 \quad .$$

Suppose this were false. Then choosing the smallest positive n for which the theorem fails, we need only apply Lemma 5 to arrive at a contradiction. Hence, it suffices to show that n admits a representation

$$n = a + u_{\sigma+k}$$

with $k \geq 2$. We may assume that $n \neq u_t$, since (15) easily implies that

$$\epsilon(u_t) = \frac{\rho\left(1 - \rho^{\left[\frac{t+1}{2}\right]}\right)}{1 - \rho} \ .$$

which is of the required form for $\rho = -1$, ω, or ω^2. Taking

$$a = n - u_\nu \ ,$$

we therefore have $a > 0$. Now

$$a = n - u_\nu < u_{\nu+1} - u_\nu = u_{\nu-1} \ ,$$

so that

$$\sigma = \nu(a) \leq \nu - 2 \ .$$

Therefore,

$$n = a + u_\nu = a + u_{\sigma+k} \ ,$$

where $k \geq 2$.

5. APPLICATIONS AND GENERALIZATION

Theorem 1 can be interpreted as a statement about partitions of natural numbers as sums of distinct terms of the sequence $\{u_n\}$ defined by (2).

Letting $A_{k,d}(N)$ denote the number of ways N can be written as a sum

$$N = u_{n_1} + u_{n_2} + \cdots + u_{n_h} \ ,$$

where $h \equiv d \pmod{k}$ and

$$n_1 < n_2 < \cdots < n_h \ ,$$

Theorem 1 asserts that

$$A_{2,0}(N) - A_{2,1}(N) \ ,$$

$$A_{3,0}(N) - A_{3,1}(N) \ ,$$

$$A_{3,0}(N) - A_{3,2}(N)$$

are all bounded as N varies over the natural numbers; moreover, if $k > 3$, then there exists d such that the difference

$$A_{k,0}(N) - A_{k,d}(N)$$

is not bounded.

Theorem 1 can be proven in the same way for any sequence $\{v_n\}$ such that

$$v_n = v_{n-1} + v_{n-2} \ ,$$

and can be interpreted as an analogous assertion about partitions of the form

$$N = v_{n_1} + v_{n_2} + \cdots + v_{n_h} \ .$$

Lemma 5, however, has more precise consequences for the sequence $\{u_n\}$ defined by (2). It is easy to see that $\epsilon(N) = 0$ or ± 1 if $\rho = -1$, and that $\epsilon(N) = 0, \pm 1, \pm \omega,$ or $\pm \omega^2$ if $\rho = \omega^2$. The partition-theoretic consequence of this observation is that for each N,

$$\left| A_{2,0}(N) - A_{2,1}(N) \right| \leq 1$$

and

$$\left| A_{3,0}(N) - A_{3,1}(N) \right| + \left| A_{3,1}(N) - A_{3,2}(N) \right| + \left| A_{3,2}(N) - A_{3,0}(N) \right| \leq 1 .$$

NOTE: The truth of Theorem 1 for the special case $\rho = 1$ is a consequence of results found in [4]. The special case $\rho = 1$ is also a consequence of results found in later papers (see [5] and [1]). The interest in series (1) for

ON GENERALIZED BASES FOR REAL NUMBERS

J. L. BROWN, JR.
Ordnance Research Laboratory
The Pennsylvania State University, State College, Pennsylvania

1. INTRODUCTION

The purpose of this paper is to give an exposition of certain results due to J. A. Fridy [1], [2], using a somewhat different approach. In [2], Fridy considers a non-increasing sequence

$$\{r_i\}_1^\infty$$

of real numbers with

$$\lim_{i \to \infty} r_i = 0$$

and defines, for two given non-negative integer sequences

$$\{k_i\}_1^\infty$$

and

$$\{m_i\}_1^\infty \ ,$$

the sequence $\{r_i\}$ to be a $\{k,m\}$ base for the interval $(-S^*,S)$ if for each $x \in (-S^*,S)$, there is an integer sequence

$$\{a_i\}_1^\infty$$

such that

$$x = \sum_1^\infty a_i \, r_i$$

477

with $-m_i \leq a_i \leq k_i$ for each $i \geq 1$, where

$$S = \sum_1^\infty k_i r_i$$

and

$$S^* = \sum_1^\infty m_i r_i \ .$$

When the $\{k_i\}$ and $\{m_i\}$ sequences are specialized to $k_i = n - 1$ for all $i \geq 1$ and $m_i = 0$ for all $i \geq 1$, Fridy [1] has termed the resulting $\{k,m\}$ base an "n-base" and developed a necessary and sufficient condition for a sequence $\{r_i\}$ to be an n-base. He also notes in a subsequent paper [2] that a necessary and sufficient condition for a 2-base had been given by Kakeya [3] much earlier. The main result of Fridy's second paper derives from a Lemma which gives a necessary and sufficient condition for $\{r_i\}$ to be a $\{k,0\}$ base ([2], pp. 194-196). Since an n-base is a specialization of a $\{k,0\}$ base, this latter condition for a $\{k,0\}$ base subsumes the earlier result for an n-base in [1]. Moreover, the derivation of the necessary and sufficient condition for a $\{k,m\}$ base follows directly ([2], Theorem 1, pp. 196-197) once the condition for a $\{k,0\}$ base is established.

Our point of departure here is to show that the characterizing condition for a $\{k,0\}$ base is itself almost immediate from Kakeya's condition for a 2-base. This follows from the observation that $\{r_i\}$ is a $\{k,0\}$ base if and only if a certain augmented sequence (obtained by repeating each r_i, in order k_i times) is a 2-base; the details are given below in Theorem 1. (cf. the development in [4].)

In order to keep the presentation self-contained, a proof of Kakeya's result is also given as Lemma 1, where we have emphasized the possibility of obtaining expansions of the required form with an infinite number of the expansion coefficients being equal to zero. This particular constraint will be seen to be important in Section 3, which deals with uniqueness of the expansions.

As illustrations of some of the results, we show in Section 4 that the Cantor expansion is a special case in which unique expansions are obtained. A Lemma is then established which gives a useful sufficient condition for the existence of expansions (non-unique, in general), and this Lemma is applied to show that an arbitrary positive number may be expressed (non-uniquely) as a sum of distinct reciprocal primes. A similar result holds for the Fibonacci numbers

$$\{F_i\}_1^\infty$$

where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$; that is, any real number

$$x \in \left(0, \sum_1^\infty \frac{1}{F_i} \right)$$

may be represented (again, non-uniquely) as a distinct sum of reciprocal Fibonacci numbers. Along the same lines, we show that any real number

$$x \in \left(-\sum_1^\infty \frac{1}{F_i}, \ \sum_1^\infty \frac{1}{F_i} \right)$$

has an expansion of the form

$$x = \sum_1^\infty \frac{\epsilon_i}{F_i} \ ,$$

where each $\epsilon_i = \epsilon_i(x)$ is either a $+1$ or -1.

2. EXISTENCE OF REPRESENTATIONS

Lemma 1: (KAKEYA): Let

$$\{r_i\}_1^\infty$$

be a non-increasing sequence of real numbers such that

$$\lim_{i \to \infty} r_i = 0$$

and

$$(1) \qquad r_p \le \sum_{p+1}^\infty r_i \qquad \text{for} \qquad p = 1, 2, 3, \cdots \, .$$

If

$$\sum_1^\infty r_i = S \, ,$$

finite or infinite, then for each x is [0,S), there exist binary coefficients $\alpha_i = \alpha_i(x)$ such that

$$(2) \qquad x = \sum_1^\infty \alpha_i \, r_i$$

and $\alpha_i = 0$ for infinitely many values of i.

   Proof. The case $S = +\infty$ is straightforward and left to the reader. It is also apparent that the Lemma holds for $x = 0$.

   Now, for S finite, let x be given in (0,S). Choose n_1 as the smallest positive integer such that $r_{n_1} \le x$. If equality holds, the lemma is proved for x; if not, choose n_2 as the smallest integer $>n_1$ for which

$$r_{n_2} \le x - r_{n_1} \, .$$

Again, equality at this stage implies the result. Otherwise, we continue the process, and in general, n_k is the smallest integer $>n_{k-1}$ for which

$$r_{n_k} \leq x - \sum_1^{k-1} r_{n_i} .$$

The process either terminates with an equality sign after a finite number of steps, or else we obtain an infinite series

$$\sum_1^\infty r_{n_i} ;$$

we focus our attention on the latter case. Clearly,

$$\sum_1^\infty r_{n_i}$$

converges since

$$\sum_1^p r_{n_i} \leq x$$

for any choice of p. Let

$$\beta = \sum_1^\infty r_{n_i} .$$

First, we show $n_i > n_{i-1} + 1$ for infinitely many values of i. If not, there exists a smallest integer k such that $n_{k+j} = n_k + j$ for $j = 1, 2, \cdots$. Then $n_k > 1$, since

$$\beta \le x < \sum_{1}^{\infty} r_i = S .$$

If $k = 1$,

$$x \ge \beta = \sum_{n_1}^{\infty} r_i \ge r_{n_1-1} ,$$

thereby contradicting our choice of n_1. Hence, $k > 1$, and we write

$$\beta = \sum_{1}^{k-1} r_{n_i} + \sum_{n_k}^{\infty} r_i$$

with $n_k > n_{k-1} + 1$ from our definition of k. Then

$$x - \sum_{1}^{k-1} r_{n_i} \ge \beta - \sum_{1}^{k-1} r_{n_i} = \sum_{n_k}^{\infty} r_i \ge r_{n_k-1} ,$$

which implies $n_k = n_{k-1} + 1$, a contradiction. We conclude $n_i > n_{i-1} + 1$ for infinitely many i.

Lastly, we show $\beta = x$. For, if not, $\beta < x$ and there exists N such that $p \ge N$ implies

$$r_{n_p} < x - \beta = x - \sum_{1}^{\infty} r_{n_i} \le x - \sum_{1}^{p} r_{n_i} ,$$

which in turn implies $n_{p+1} = n_p + 1$ for each $p \ge N$, a contradiction to our previous assertion.   q. e. d.

The principal Lemma in Fridy's paper ([2], pp. 194-196) may now be derived quite simply from Lemma 1:

Theorem 1. Let

$$\{r_i\}_1^\infty$$

be a non-increasing sequence of real numbers with $\lim\limits_{i \to \infty} r_i = 0$ and let

$$\{k_i\}_1^\infty$$

be an arbitrary sequence of positive integers. Then every real number x in

$$\left[0, \ \sum_1^\infty k_i\, r_i \right)$$

can be expanded in the form

$$(3) \qquad\qquad\qquad x = \sum_1^\infty \beta_i\, r_i \ ,$$

with β_i integers satisfying $0 \le \beta_i \le k_i$ for $i = 1, 2, \cdots$ if and only if

$$(4) \qquad\qquad\qquad r_p \le \sum_{p+1}^\infty k_i\, r_i \qquad \text{for} \quad p = 1, 2, 3, \cdots \ .$$

Further, the expansion in (3) can be accomplished such that $\beta_i < k_i$ for infinitely many values of i.

Proof. To show necessity of (4), assume there exists $m > 0$ such that

$$r_m > \sum_{m+1}^\infty k_i\, r_i$$

and choose x such that

$$\sum_{m+1}^{\infty} k_i r_i < x < r_m \ .$$

If x has an expansion of the form (3), we must have $\beta_1 = \beta_2 = \cdots = \beta_m = 0$ since $x < r_m$, but then

$$x = \sum_{m+1}^{\infty} \beta_i r_i \leq \sum_{m+1}^{\infty} k_i r_i < x \ ,$$

a contradiction.

Conversely, assume (4) holds and consider the sequence

$$\{g_i\}_1^{\infty} \ ,$$

defined to consist of each term r_i, in order, repeated k_i times; that is

$$\{g_i\}_1^{\infty} = \underbrace{r_1, \ r_1, \ r_1,}_{k_1 \text{ times}} \quad \underbrace{r_2, \ r_2, \ r_2, \ r_2, \cdots,}_{k_2 \text{ times}} \underbrace{r_n, \ r_n, \ r_n,}_{k_n \text{ times}} \cdots \ .$$

Using (4), we observe

$$g_p \leq \sum_{p+1}^{\infty} g_i$$

for $p = 1, 2, 3, \cdots$. Thus, Lemma 1 guarantees binary coefficients α_i such that any x in

$$\left[0, \ \sum_1^{\infty} g_i \right)$$

has an expansion of the form

$$(5) \qquad x = \sum_{1}^{\infty} \alpha_i \, g_i$$

with $\alpha_i = 0$ for infinitely many i. Replacing (5) in terms of the r_i, and noting

$$\sum_{1}^{\infty} g_i = \sum_{1}^{\infty} k_i \, r_i \; ,$$

we have that any x in

$$\left[0, \; \sum_{1}^{\infty} k_i \, r_i \right)$$

can be written in the form

$$x = \sum_{1}^{\infty} \beta_i \, r_i$$

with $0 \le \beta_i \le k_i$ and $\beta_i < k_i$ for infinitely many i.   q. e. d.

3. UNIQUENESS OF REPRESENTATIONS

Thus, condition (4) is both necessary and sufficient for the existence of expansions in the form (3). We give a result next in Lemma 2 concerning the uniqueness of such expansions independently of the existence question.

Definition.   Let

$$\{ r_i \}_1^{\infty}$$

be a non-increasing sequence of real numbers with $\lim_{i \to 0} r_i = 0$ and let

$$\{k_i\}_1^\infty$$

be an arbitrary but fixed sequence of positive integers.   Let

$$\{\beta_i\}_1^\infty \quad \text{and} \quad \{\gamma_i\}_1^\infty$$

be two sequences of integers which satisfy $0 \leq \beta_i \leq k_i$ and $0 \leq \gamma_i \leq k_i$ for $i = 1, 2, 3, \cdots$. Further, let $\beta_i < k_i$ for infinitely many i and $\gamma_i < k_i$ for infinitely many i. Then

$$\{r_i\}_1^\infty$$

will be said to possess the <u>uniqueness property [Property U]</u> if and only if the equality

$$\sum_1^\infty \beta_i \, r_i = \sum_1^\infty \gamma_i \, r_i$$

implies $\beta_i = \gamma_i$ for each $i \geq 1$.

  <u>Lemma 2.</u> Let

$$\{r_i\}_1^\infty \quad \text{and} \quad \{k_i\}_1^\infty$$

be given as in the preceding definition.   Then

$$\{r_i\}_1^\infty$$

possesses Property U if

$$(6) \qquad r_p \geq \sum_{p+1}^\infty k_i \, r_i \qquad \text{for} \quad p = 1, 2, 3, \cdots \ .$$

<u>Proof.</u>   Assume (6) holds and that

$$\sum_1^\infty \beta_i \, r_i \;=\; \sum_1^\infty \gamma_i \, r_i$$

with $\{\beta_i\}$ and $\{\gamma_i\}$ as in the definition.   If the two representatives are not identical, let m be the smallest positive integer i such that $\beta_i \neq \gamma_i$. Then

$$\beta_m \, r_m + \sum_{m+1}^\infty \beta_i \, r_i \;=\; \gamma_m \, r_m + \sum_{m+1}^\infty \gamma_i \, r_i \;,$$

or assuming $\beta_m > \gamma_m$ without loss of generality,

$$(7) \qquad\qquad (\beta_m - \gamma_m) \;=\; \sum_{m+1}^\infty (\gamma_i - \beta_i) \, r_i \quad .$$

Now, $\gamma_i - \beta_i < k_i$ for some $i \geq m + 1$ (otherwise $\gamma_i = k_i$ for all $i \geq m + 1$, contrary to choice of $\{\gamma_i\}$), and therefore, from (7),

$$r_m \leq (\beta_m - \gamma_m) \, r_m \;<\; \sum_{m+1}^\infty k_i \, r_i \;\;,$$

contradicting condition (6) for $p = m$. We conclude $\gamma_i = \beta_i$ for all $i \geq 1$, giving Property U.   q. e. d.

   <u>Lemma 3.</u>   Take

$$\{r_i\}_1^\infty \qquad \text{and} \qquad \{k_i\}_1^\infty$$

as before.   If

$$r_p \leq \sum_{p+1}^{\infty} k_i r_i$$

for $p = 1, 2, 3, \cdots$, then

$$(8) \qquad r_p = \sum_{p+1}^{\infty} k_i r_i \qquad (p = 1, 2, 3, \cdots)$$

is a necessary and sufficient condition for $\{r_i\}$ to possess Property U.

Proof. Sufficiency follows from Lemma 2. To show necessity, assume that there exists an integer $m > 0$ such that

$$r_m < \sum_{m+1}^{\infty} k_i r_i \; ,$$

and choose x to satisfy

$$r_m < x < \sum_{m+1}^{\infty} k_i r_i \; .$$

By Theorem 1, x has an expansion of the form

$$x = \sum_{1}^{\infty} \beta_i r_i$$

with $0 \leq \beta_i \leq k_i$ for $i \geq 1$ and $\beta_i < k_i$ for many i. Further, at least one of the coefficients $\beta_1, \beta_2, \cdots, \beta_m$ must be different from zero.

Since the sequence

$$\{r_i\}_{m+1}^{\infty}$$

also satisfies the conditions of Theorem 1 and

$$x < \sum_{m+1}^{\infty} k_i \, r_i \quad,$$

the number x has an expansion of the form

$$x = \sum_{m+1}^{\infty} \gamma_i \, r_i$$

with $0 \le \gamma_i \le k_i$ for $i \ge m+1$ and $\gamma_i < k_i$ for infinitely many i. Thus

$$x = \sum_{m+1}^{\infty} \gamma_i \, r_i = \sum_{1}^{\infty} \beta_i \, r_i$$

and $\beta_i = \gamma_i$ does not hold for all $i \ge 1$, showing Property U does not hold. q.e.d.

Theorem 2.   Let

$$\{r_i\}_1^{\infty} \qquad \text{and} \qquad \{k_i\}_1^{\infty}$$

be sequences as in Theorem 1. Then every real number x in

$$\left[0, \; \sum_{1}^{\infty} k_i \, r_i \right)$$

has one and only one expansion

$$(8) \qquad x = \sum_{1}^{\infty} \beta_i \, r_i$$

with $0 \le \beta_i \le k_i$ for $i \ge 1$ and $\beta_i < k_i$ for infinitely many i, if and only if

$$(9) \qquad r_p = \sum_{p+1}^{\infty} k_i \, r_i$$

for $p = 1, 2, 3, \cdots$, or equivalently,

$$(10) \qquad r_p = S \cdot \prod_{i=1}^{p} \frac{1}{1 + k_i}$$

for all $p \ge 1$, where

$$S = \sum_{1}^{\infty} k_i \, r_i \ .$$

Proof. From Theorem 1, we must have

$$r_p \le \sum_{p+1}^{\infty} k_i \, r_i$$

for $p \ge 1$, while from Lemma 3 and the uniqueness requirement,

$$r_p = \sum_{p+1}^{\infty} k_i \, r_i$$

for $p \geq 1$. Equation (10) follows on noting

$$r_{p+1} = \sum_{p+2}^{\infty} k_i r_i = r_p - k_{p+1} r_{p+1} \, ,$$

or

$$(11) \qquad\qquad r_{p+1} = \frac{r_p}{1 + k_{p+1}}$$

for $p \geq 1$. Since

$$r_1 = \sum_{2}^{\infty} k_i r_i = S - r_1 k_1,$$

we have

$$r_1 = \frac{S}{1 + k_1} \, ,$$

and iteration using (11) leads to (10).   q. e. d.

4. APPLICATIONS

CANTOR EXPANSION ([5], Theorem 1.6, p. 7): "Let a_1, a_2, a_3, $\cdots$ be a sequence of positive integers, all greater than 1. Then any real number α is uniquely expressible in the form

$$(12) \qquad\qquad \alpha = c_0 + \sum_{i=1}^{\infty} \frac{c_i}{a_1 a_2 \cdots a_i}$$

with integers c_i satisfying the inequalities $0 \leq c_i \leq a_i - 1$ for all $i \geq 1$ and $c_i < a_i - 1$ for infinitely many i."

Proof.   In Theorem 2, identify

$$r_i = \frac{1}{a_1 a_2 \cdots a_i}$$

and $k_i = a_i - 1$ for $i \geq 1$. Then condition (11) is clearly satisfied. Now, for given α, let $c_0 = [\alpha]$, the greatest integer contained in α, so that

$$0 \leq \alpha - [\alpha] < 1 = \sum_1^\infty k_i r_i = \sum_1^\infty \frac{a_i - 1}{a_1 a_2 \cdots a_i} \quad .$$

Then Theorem 2 implies a unique expansion in the form (12) as required. q. e. d.

Next, we give a useful sufficient condition for the existence of expansions as specified in Theorem 1.

Lemma 4.   A sufficient condition for

$$r_p \leq \sum_{p+1}^\infty k_i r_i \qquad (p \geq 1)$$

is

(13) $$r_p \leq (k_{p+1} + 1)r_{p+1} \quad .$$

for all $p \geq 1$.

Proof. Assume (13) is satisfied. Then

$$\sum_{p+1}^\infty r_i \leq \sum_{p+1}^\infty (k_{i+1} + 1)r_{i+1} = \sum_{p+1}^\infty k_{i+1} r_{i+1} + \sum_{p+1}^\infty r_{i+1} \quad .$$

Thus,

$$r_{p+1} = \sum_{p+1}^{\infty} r_i - \sum_{p+1}^{\infty} r_{i+1} \le \sum_{p+1}^{\infty} k_{i+1} \, r_{i+1} = \sum_{p+1}^{\infty} k_i \, r_i - k_{p+1} \, r_{p+1}$$

or

$$(1 + k_{p+1}) r_{p+1} \le \sum_{p+1}^{\infty} k_i \, r_i \ .$$

Since $r_p \le (1 + k_{p+1}) r_{p+1}$, we have

$$r_p \le \sum_{p+1}^{\infty} k_i \, r_i$$

for all $p \ge 1$ as required.

Example 1. Let x be an arbitrary real number satisfying

$$0 \le x < \sum_{1}^{\infty} \frac{1}{F_i} \quad ,$$

where $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \ge 2$ specify the Fibonacci numbers. Then

$$x = \sum_{1}^{\infty} \frac{\alpha_i}{F_i} \quad ,$$

with $\alpha_i = \alpha_i(x)$ a binary coefficient for each $i \ge 1$. Further, $\alpha_i = 0$ for infinitely many i.

Proof. Here $k_i = 1$ for all $i \ge 1$. Clearly

$$\left\{ \frac{1}{F_i} \right\}_1^\infty$$

is non-increasing and

$$\lim_{i \to \infty} \frac{1}{F_i} = 0 \ .$$

By condition (13) of Lemma 4, a sufficient condition for Theorem 1 to be applicable is $r_p \leq 2r_{p+1}$, or equivalently,

$$\frac{1}{F_p} \leq \frac{2}{F_{p+1}} \qquad (p \geq 1) \ ,$$

where

$$\{r_i\}_1^\infty = \left\{ \frac{1}{F_i} \right\}_1^\infty \ .$$

But this is merely the condition $F_{p+1} \leq 2F_p$, which is obvious for $p \geq 1$ and the result follows from Theorem 1.

Example 2. Let x be an arbitrary real number satisfying $0 \leq x < \infty$. Then

$$x = \sum_1^\infty \frac{\alpha_i}{p_i} \ ,$$

where

$$\{p_i\}_1^\infty = \{2, \ 3, \ 5, \ 7, \ 11, \ \cdots \}$$

is the sequence of primes and $\alpha_i = \alpha_i(x)$ is a binary coefficient for each $i \geq 1$. Further $\alpha_i = 0$ for infinitely many i.

Proof. Again, we apply Theorem 1 with

$$r_i = \frac{1}{p_i}$$

for $i \geq 1$ and $k_i = 1$ for all $i \geq 1$. Condition (13) reduces to $p_{i+1} \leq 2p_i$, and this latter inequality holds for all $i \geq 1$ by Betrand's postulate ([6], p. 171). Since

$$\left\{ \frac{1}{p_i} \right\}_1^\infty$$

is non-increasing and

$$\lim_{i \to \infty} \frac{1}{p_i} = 0,$$

the result follows from Theorem 1 and the well-known divergence of the series

$$\sum_1^\infty \frac{1}{p_i}$$

([6], Theorem 8.3, p. 168).

Example 3. Let x be an arbitrary real number with

$$-\sum_1^\infty \frac{1}{F_i^2} \leq x \leq \sum_1^\infty \frac{1}{F_i} .$$

Then x possesses an expansion of the form

$$(14) \qquad x = \sum_1^\infty \frac{\epsilon_i}{F_i} ,$$

where each $\epsilon_i = \epsilon_i(x)$ is either $+1$ or -1.

Proof. For

$$x \in \left(-\sum_1^\infty \frac{1}{F_i}, \quad \sum_1^\infty \frac{1}{F_i} \right),$$

we have

$$0 < \frac{1}{2}\left(x + \sum_1^\infty \frac{1}{F_i} \right) < \sum_1^\infty \frac{1}{F_i},$$

so that by Example 1,

$$\frac{1}{2}\left(x + \sum_1^\infty \frac{1}{F_i} \right) = \sum_1^\infty \frac{\alpha_i}{F_i},$$

where each α_i is a binary digit. Equivalently,

$$x = \sum_1^\infty \frac{2\alpha_i - 1}{F_i},$$

and we note that $2\alpha_i - 1$ is either $+1$ or -1 depending on whether $\alpha_i = 1$ or $\alpha_i = 0$, respectively; this establishes the expansion in the stated form.

<div align="center">REFERENCES</div>

1. J. A. Fridy, "A Generalization of n–Scale Number Representation," American Mathematical Montly, Vol. 72, No. 8, October, 1965, pp. 851–855.

2. J. A. Fridy, "Generalized Bases for the Real Numbers," The Fibonacci Quarterly, Vol. 4, No. 3, October, 1966, pp. 193–201.

ON MODULI FOR WHICH THE FIBONACCI SEQUENCE
CONTAINS A COMPLETE SYSTEM OF RESIDUES

S. A. BURR
Bell Telephone Laboratories, Inc., Whippany, New Jersey

Shah [1] and Bruckner [2] have considered the problem of determining which moduli m have the property that the Fibonacci sequence $\{u_n\}$, defined in the usual way, contains a complete system of residues modulo m. Following Shah we say that m is defective if m does not have this property.

The results proved in [1] include: (I) If m is defective, so is any multiple of m; in particular, $8n$ is always defective. (II) if p is a prime not 2 or 5, p is defective unless $p \equiv 3$ or $7 \pmod{20}$. (III) If p is a prime $\equiv 3$ or $7 \pmod{20}$ and is not defective, then the set $\{0, \pm 1, \pm u_3, \pm u_4, \pm u_5, \cdots, \pm u_n\}$, where $h = (p+1)/2$, is a complete system of residues modulo p. In [2], Bruckner settles the case of prime moduli by showing that all primes are defective except 2, 3, 5, and 7.

In this paper we complete the work of Shah and Bruckner by proving the following result, which completely characterizes all defective and nondefective moduli.

Theorem. A number m is not defective if and only if m has one of the following forms:

$$5^k, \quad 2 \cdot 5^k, \quad 4 \cdot 5^k,$$
$$3^j \cdot 5^k, \quad 6 \cdot 5^k,$$
$$7 \cdot 5^k, \quad 14 \cdot 5^k,$$

where $k \geq 0$, $j \geq 1$.

Thus almost all numbers are defective. We will prove a series of lemmas, from which the theorem will follow directly. We first make some useful definitions.

We say a finite sequence of integers $(a_1, a_2, \cdots, a_r)$ is a <u>Fibonacci cycle modulo m</u> if it satisfies $a_i + a_{i+1} \equiv a_{i+2} \pmod{m}$, $i = 1, \cdots, r-2$, as well as $a_{r-1} + a_r \equiv a_1 \pmod{m}$ and $a_r + a_1 \equiv a_2 \pmod{m}$, and furthermore $(a_1, a_2, \cdots, a_q)$ does not have these properties for any $q < r$. (As

the name implies, it is convenient to regard the cycles as circular.) We say r is the length of the cycle. For any m, we also call (km) a Fibonacci cycle modulo m of length 1. We call two Fibonacci cycles _equivalent_ if one is congruent termwise modulo m to a cyclic permutation of the other. Finally, we define a _complete Fibonacci system modulo m_ to be a maximal set of pairwise inequivalent Fibonacci cycles modulo m. Note that the total number of terms appearing in such a system is m^2.

The idea behind this definition is simple; it is a compact way of representing all possible Fibonacci sequences modulo m. For example, the following are complete Fibonacci systems modulo 2, 3, 4, and 5, respectively:

$\{(0, 1, 1), (0)\}$,

$\{(0, 1, 1, 2, 0, 2, 2, 1), (0)\}$,

$\{0, 1, 1, 2, 3, 1), (0, 3, 3, 2, 1, 3), (0, 2, 2), (0)\}$,

$\{(0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1), (1, 3, 4, 2), (0)\}$.

For larger m the structure of these systems can become quite intricate and is worthy of study in itself. We will not undertake such a study here. Instead, we will proceed to the lemmas. The first lemma gives another proof of the result of Bruckner; it is included to illustrate the above ideas..

Lemma 1. If p is a prime which is not defective, then p = 2, 3, 4, or 7.

Proof. Assume the contrary, and let $p > 7$ be a nondefective prime. Then $p \equiv 3$ or 7 (mod 20), and (III) holds. From this it is easily seen either directly or from (5.5) and (5.6) of [1] that

$$C_1 = (0, 1, 1, \cdots, u_{h-2}, u_{n-1}, u_h, -u_{h-1}, u_{h-2}, \cdots, 1, -1,$$
$$0, -1, -1, \cdots, -u_{h-2}, -u_{h-1}, -u_h, u_{h-1}, -u_{h-2}, \cdots, -1, 1)$$

is a Fibonacci cycle of length 2p + 2 modulo p.

Let C_k, k = 1, $\cdots$, (p - 1)/2, be the finite sequence formed by multiplying the terms of C_1 by k. Clearly each C_k is a Fibonacci cycle modulo p. But they are all inequivalent, since C_j equivalent to C_k implies

$j \equiv \pm k \pmod{p}$, which implies $j = k$. Since all the $(p-1)/2$ sequences C_k are inequivalent, the set

$$\{C_1, \cdots, C_{(p-1)/2}, (0)\}$$

is a complete Fibonacci system (modulo p) because the total number of terms appearing is

$$\frac{p-1}{2} \cdot (2p + 2) + 1 = p^2 .$$

Consider the finite sequence of integers 5, -2, 3, 1, 4, 5. This satisfies the Fibonacci difference equation, and hence must be congruent term-by-term to a portion of some C_k (possibly wrapped end around). Thus some C_k has two congruent terms five steps apart. Therefore, multiplying each term by the inverse of k, we see that C_1 has two congruent terms five steps apart. But examination of the definition of C_1 shows that this implies that for some $3 \leq j \leq h$ either $u_j \equiv \pm 1 \pmod{p}$ or $u_j \equiv \pm u_k \pmod{p}$ for some $k \neq j$, $3 \leq k \leq h$. (Note that here we have used $p > 7$.) But this contradicts (III), so the lemma is proved.

By property (I) it suffices to consider moduli divisible only by 2, 3, 5, and 7. We first deal with the powers of 3.

Lemma 2. No power of three is deficient.

Proof. We begin by determining a complete Fibonacci system modulo 3^n. It is well known that the rank and period of 3^n are $4 \cdot 3^{n-1}$ and $8 \cdot 3^{n-1}$ respectively. That is, the smallest $m > 0$ for which $3^n \mid u_m$ is $4 \cdot 3^{n-1}$, and for all m,

$$u_m \equiv u_{m+8 \cdot 3^{n-1}} \pmod{3^n} .$$

Thus

$$C = (0, 1, 1, 2, \cdots, u_{8 \cdot 3^{n-1}})$$

is a Fibonacci cycle modulo 3^n. But it is easily from the above facts that

$$u_{4 \cdot 3^{n-1}+1} \equiv -1 \pmod{3^n},$$

so that

$$C_1 = \left(0, \ 1, \ 1, \ 2, \ \cdots, \ 0, \ -1, \ -1, \ -2, \ \cdots, \ u_{8 \cdot 3^{n-1}-1} \right)$$

is an equivalent Fibonacci cycle.

For each integer k prime to 3 in the range $0 < k < \frac{1}{2} \cdot 3^n$, let C_k be the sequence formed by multiplying each term of C_1 by k. As in the previous lemma, the C_k are all inequivalent Fibonacci cycles. The total number of such C_k is $\frac{1}{2}\phi(3^n) = 3^{n-1}$, where ϕ is the Euler function. Hence, the total number of terms appearing in the C_k is $8 \cdot 3^{m-2}$. Consider also the sequences formed by multiplying by 3 every term of a complete Fibonacci system modulo 3^{n-1}. This clearly forms a set of inequivalent Fibonacci cycles modulo 3^n, and the total number of terms appearing in the cycles is 3^{2n-2}. Furthermore, none of these cycles is equivalent to any C_k. Therefore, these cycles, together with the C_k, form a complete Fibonacci system modulo 3^n, since the total number of terms is then

$$8 \cdot 3^{2n-2} + 3^{2n-2} = 3^{2n} \ .$$

It is well known that the expression $\left| a^2 + ab - b^2 \right|$, where a and b are two consecutive terms of a sequence satisfying the Fibonacci difference equation, is an invariant of the sequence. Consequently, an invariant of any such sequence modulo m is the pair of residue classes corresponding to $\pm(a^2 + ab - b^2)$, and the same applies to Fibonacci cycles.

We now show that any Fibonacci cycle modulo 3^n with invariant corresponding to ± 1 is equivalent to C_1. Certainly such a cycle must be equivalent to some C_k, since the invariants of the other cycle are divisible by 3. Such a C_k must satisfy $k^2 \equiv \pm 1 \pmod{3^n}$. But

$$k^2 \equiv -1 \pmod{3^n}$$

is impossible, so $(k + 1)(k - 1) \equiv 0 \pmod{3^n}$, so that $k = 1$ and the cycle is equivalent to C_1.

From this, we see that the lemma will be proved if it can be shown that for any a there is a b such that

$$a^2 + ab - b^2 \equiv \pm 1 \pmod{3^n} .$$

In fact, we will even show this for

$$a^2 + ab - b^2 \equiv -1 .$$

This is obvious for $n = 1$. Now suppose the above to have been proved for some value $n \geq 1$, and let b be such that

$$a^2 + ab - b^2 \equiv -1 \pmod{3^n} ,$$

let

$$a^2 + ab - b^2 = A \cdot 3^n - 1 .$$

We will determine an $x = 3^n t + b$ such that

$$a^2 + ax - x^2 \equiv -1 \pmod{3^{n+1}} .$$

We have

$$
\begin{aligned}
a^2 + ax - x^2 &\equiv a^2 + 3^n at + ab + 2 \cdot 3^n bt + b^2 \\
&\equiv 3^n(a + 2b)t + (a^2 + ab - b^2) \\
&\equiv 3^n(a + 2b)t + 3^n A - 1 \pmod{3^{n+1}} .
\end{aligned}
$$

Thus x will have the desired property if

$$(a + 2b)t + A \equiv 0 \pmod{3} .$$

But $3 \nmid a + 2b$, for otherwise $a \equiv b$, and

$$a^2 \equiv a^2 + ab - b^2 \equiv -1 \pmod{3} \ ,$$

which is impossible. Therefore, the above congruence has a solution and the lemma is proved.

We now consider the effect of the prime 5. We will prove a general lemma which is of some interest in itself.

<u>Lemma 3.</u> Suppose that the Fibonacci sequence $\{u_n\}$ has period k modulo m, and that it has period $5k$ modulo $5m$. For some n and a let $u_n \equiv a \pmod{m}$. Then $u_n, u_{k+n}, \cdots, u_{4k+n}$ are congruent to $a, m + a, \cdots, 4m + a \pmod{5m}$ in some order.

<u>Proof.</u> We consider two cases, depending on whether or not $5 | m$. We first assume $5 \nmid m$. Then the period of $5m$ is the g. c. d. of k and the period of 5, which is 20. Since this period is to equal $5k$, we have $k \equiv 4, 8, 12, 16 \pmod{20}$. Now, a cycle modulo 5 which corresponds to the standard Fibonacci sequence is

$$(0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1).$$

From this it may be verified that $u_n, u_{k+n}, \cdots, u_{4k+n}$ are congruent modulo 5 to 0, 1, 2, 3, 4 in some order. For instance, if $n \equiv 0 \pmod{20}$ they are congruent respectively to 9, 3, 1, 4, 2. Since each of these is congruent to a modulo m, they are congruent in some order to $a, m + a, \cdots, 4m + a$. This completes the first case.

We now assume $5 | m$. Since the Fibonacci sequence has period k modulo m, $u_n, u_{k+n}, \cdots, u_{4k+n}$ are all congruent to a modulo m and hence are each congruent to $im + a$ modulo $5m$ for some choice of $0 \le i \le 4$. Our object is to show that the value of i is different for each of the five terms. Set $u_{n+1} \equiv b \pmod{m}$. Then $u_{n+1}, u_{m+n+1}, \cdots, u_{4m+n+1}$ are each congruent to $jm + b$ for some $0 \le j \le 4$. Speaking in terms of the concept we have defined, there are 25 pairs congruent modulo $5m$ to $(im + a, jm + b)$ appearing within a complete Fibonacci system modulo $5m$, of which 5 appear in the cycle corresponding to the standard Fibonacci sequence. Our object is to show that each of these 5 gives a different value of i.

Since

$$a^2 + ab - b^2 \equiv \pm 1 \pmod{m} \ ,$$

we may set

$$a^2 + ab - b^2 = mA \pm 1 \ .$$

Applying this same invariant to the pair $(im + n, \ jm + b)$, we have

$$(im + a)^2 + (im + a)(jm + b) - (jm + b)^2$$

$$= i^2 m^2 + ijm^2 - j^2 m^2 + ((2a + b)i + (a - 2b)j)m + a^2 + ab - b^2$$

$$= m^2(i^2 + ij - j^2) + m((2a + b)i + (a - 2b)j) + mA \pm 1 \ .$$

This last expression will be congruent to ± 1 (modulo $5m$) if and only if

$$(2a + b)i + (a - 2b)j + A \equiv 0 \pmod{5} \ .$$

However, $2a + b \not\equiv 0 \pmod{5}$ since otherwise

$$\pm 1 \equiv a^2 - ab - b^2 \equiv a^2 - 2a^2 - 4a^2 \equiv 0 \pmod{5};$$

similarly $a - 2b \not\equiv 0 \pmod{5}$.

Consequently, for each of the 5 possible choices of i, there is exactly one j satisfying the above congruence. Hence only these 5 pairs could appear as consecutive pairs in the Fibonacci sequence. Since i is different in each case, the lemma is proved.

We now deal with the other primes, and combinations thereof.

Lemma 4. The numbers 8, 12, 18, 21, 28, and 49 are deficient; the numbers 4, 6, 14, and 20 are nondeficient.

Proof. The arithmetic involved in verifying these facts is left to the reader.

We now can easily prove the main result.

Proof of Theorem. Lemmas 1 and 4, along with (I), show that the numbers of the theorem are the only possible nondeficient numbers. All numbers 3^j are nondeficient by Lemma 2. Furthermore, the periods of 6, 14, 20,

and 30 are 24, 48, 60, and $8 \cdot 3^{j-1}$, respectively, so that by Lemma 3, all numbers $6 \cdot 5^k$, $14 \cdot 5^k$, $20 \cdot 5^k$, $3^j \cdot 5^k$ are all nondeficient. Applying (I) again we see that all numbers of the theorem are nondeficient. Thus, the theorem is proved.

It would be interesting to extend this work by considering more general-ly the problem of characterizing, at least partially, the residue classes that appear in the Fibonacci sequence with respect to a general modulus, as well as their multiplicities. A small start on this large problem has been made by [1], [2], and the present work, especially Lemma 3. Also of interest, both as an aid to the above and for itself, would be a systematic study of com-plete Fibonacci systems, whose structure can be quite complicated. In par-ticular, it would be useful to know the set of lengths and multiplicities of the cycles. Considerable information, especially for prime moduli, bearing on this problem exists in various places; see for instance [3], [4]. Of course, these problems can be generalized to sequences satisfying other recurrence relations.

REFERENCES

1. A. P. Shah, "Fibonacci Sequence Modulo m," Fibonacci Quarterly, Vol. 6, 1968, pp. 139-141.

2. G. Bruckner, "Fibonacci Sequence Modulo A Prime $p \equiv 3 \pmod 4$," Fibonacci Quarterly, Vol. 8, 1970, pp. 217-220.

3. D. D. Wall, "Fibonacci Series Modulo m," Amer. Math Monthly, Vol. 67, 1960, pp. 525-532.

4. D. M. Bloom, "On Periodicity in Generalized Fibonacci Sequences," Amer. Math. Monthly, Vol. 72, 1965, pp. 856-861.

◆◇◆◇◆

NINTH ANNUAL FALL CONFERENCE OF THE FIBONACCI ASSOCIATION
Nov. 13, 1971   COLLEGE OF THE HOLY NAMES, Oakland, California

Morning Session

A Triangle for the Fibonacci Powers
    Charles Pasma, San Jose State College, San Jose, California
On the Number of Primitive Solutions of $x^2 - xy - y^2 = a$ in Positive Relative-ly Prime Integers, Professor V. E. Hoggatt, Jr., San Jose State College
    Free Discussion Period

COMBINATIONS AND THEIR DUALS

C. A. CHURCH, JR.
University of North Carolina, Greensboro, North Carolina

In [3] this author gave derivations of certain results for restricted combinations by simple extensions of the first problem in Riordan [4, p. 14]. In these derivations k-combinations of the first n natural numbers were obtained by one-one correspondence with arrangements of plus signs and minus signs on a line. In what follows "dual" results are obtained by the symmetric interchange of the pluses and minuses.

For notation, terminology, and basic combinatorial results we follow Riordan [4]. By k-combinations will be meant k-combinations of the first n natural numbers.

To establish the correspondence, consider the arrangements of p pluses and q minuses on a line. If $p = k$ and $q = n - k$, each arrangement corresponds in a one-one way with a k-combination of the first n natural numbers as follows. Arrange the first n natural numbers on a line in their natural (rising) order; place a plus sign under each integer selected and a minus sign under each integer not selected.

It is well known that there are

$$\binom{p + q}{p}$$

arrangements of p pluses and q minuses on a line. With $p = k$ and $q = n - k$ we get the familiar

$$C(n,k) = \binom{n}{k}$$

k-combinations. The dual in this case gives nothing new since

$$C(n,k) = C(n,n - k) \ .$$

Starting with the first problem in Riordan [4, p. 14], with pluses and minuses interchanged, there are

(1)
$$\binom{q + 1}{p}$$

arrangements of p pluses and q minuses on a line with no two pluses together [3]. With p = k and q = n - k we get Kaplansky's result [4, p. 198] that there are

(2)
$$\binom{n - k + 1}{k}$$

k-combinations with no two consecutive integers in the same combination.

To get the dual in this case, interchange p and q in (1). Then with p = k and q = n - k we have that there are

(3)
$$\binom{k + 1}{n - k}$$

k-combinations with no two consecutive integers omitted from the same combination $(n - 1)/2 \le k \le n$.

In [3] we also rederived the circular case of Kaplansky's lemma [4, p. 198]. That is, there are

(4)
$$\frac{p + q}{q} \binom{q}{p}$$

arrangements of p pluses and q minuses on a circle with no two consecutive pluses, and

$$\frac{n}{n - k} \binom{n - k}{k}$$

circular k-combinations with no two consecutive integers, where n and 1 are taken to be consecutive. The dual in this case is that there are

$$\frac{n}{k} \binom{k}{n-k}$$

circular k-combinations with no two consecutive integers omitted, $n/2 \leq k \leq n$.

In rederiving (5) below, a result of Abramson and Moser [2], which generalizes (2), we got that there are

$$\binom{p-1}{r-1} \binom{q+1}{r}$$

arrangements of p pluses and q minuses on a line with exactly r blocks of consecutive pluses. With $p = k$ and $q = n - k$ there are

(5) $$\binom{k-1}{r-1} \binom{n-k+1}{r}$$

k-combinations with exactly r blocks of consecutive integers. This reduces to (2) when $r = k$. The dual in this case is

$$\binom{n-k-1}{r-1} \binom{k+1}{r}$$

k-combinations with exactly r blocks of consecutive integers omitted. This reduces to (3) when $r = n - k$.

There are circular k-combinations corresponding to (5), see [2] or [3], and the appropriate dual.

Another generalization of (2) is that there are

$$\binom{q+p-bp+b}{p}$$

arrangements of p pluses and q minuses on a line with at least b minuses between any two pluses [3], and

(6) $$\binom{n-bk+b}{k}$$

k-combinations such that if i occurs in a combination, none of i + 1, i + 2, $\cdots$, i + b can [4, p. 222]. Here the dual is

$$\binom{n \ - \ b(n \ - \ k) \ + \ b}{n \ - \ k}$$

k-combinations such that if i is omitted, none of the i + 1, i + 2, $\cdots$, i + b are, $b(n - 1)/(b + 1) \leq k \leq n.$

For the circular k-combinations corresponding to (6) see [3, (5b)] or [4, p. 222]. The dual follows readily from [3, (9b)].

Combining the restrictions in (5) and (6), we have

$$\binom{p \ - \ 1}{r \ - \ 1}\binom{q \ - \ (b \ - \ 1)(r \ - \ 1) \ + \ 1}{r}$$

arrangements of p pluses and q minuses on a line with exactly r blocks of pluses, each block separated by at least b minuses. Thus there are

$$\binom{k \ - \ 1}{r \ - \ 1}\binom{n \ - \ k \ - \ (b \ - \ 1)(r \ - \ 1) \ + \ 1}{r}$$

k-combinations with r blocks of consecutive integers with atlease b consecutive integers omitted between each block [3, (4b)]. The dual is

$$\binom{n \ - \ k \ - \ 1}{r \ - \ 1}\binom{k \ - \ (b \ - \ 1)(r \ - \ 1) \ + \ 1}{r}$$

k-combinations with r + 1 blocks of at least b consecutive integers in each, since there are only r gaps.

Clearly, additional results of the type we have considered above can be obtained from similar enumerations in the literature. Additional enumerations for which the duals are immediate appear in [3].

In closing, one enumeration and its dual should be mentioned. Expansion of the enumerating generating function

$$(1 \ + \ t \ + \ t^2 \ + \ \cdots \ + \ t^j)^{q+1}$$

gives

$$f(p,q; j + 1) = \sum_{r=0}^{\left[\frac{p}{j+1}\right]} (-1)^r \binom{q + 1}{r} \binom{q + p - r(j + 1)}{q} \quad ,$$

the number of arrangements of p pluses and q minuses on a line with at most j pluses between two minuses, before the first minus, and after the last. With p = k and q = n - k we get Abramson's [1]

$$A_{j+1}(n,k) = \sum_{r=0}^{\left[\frac{k}{j+1}\right]} (-1)^r \binom{n - k + 1}{r} \binom{n - r(j + 1)}{n - k} ,$$

the number of k-combinations with blocks of at most j consecutive integers. Its dual is

$$\sum_{r=0}^{\left[\frac{n-k}{j+1}\right]} (-1)^r \binom{k + 1}{r} \binom{n - r(j + 1)}{k} \quad ,$$

the number of k-combinations with blocks of at most j consecutive integers omitted. See also (5).

REFERENCES

1. M. Abramson, "Explicit Expressions for a Class of Permutation Problems," Canad. Math. Bull., 7(1964), pp. 345-350.

2. M. Abramson and W. Moser, "Combinations, Successions, and the n-Kings Problem," Mathematics Magazine, 39 (1966), pp. 269-273.

3. C. A. Church, Jr., "Combinations and Successions," Mathematics Magazine, 41 (1968), pp. 123-128.

4. J. Riordan, An Introduction to Combinatorial Analysis, Wiley, New York, 1958.

5. V. E. Hoggatt, Jr., "Combinatorial Problems for Generalized Fibonacci Numbers," Fibonacci Quarterly, Vol. 9, Dec., 1970, pp. 456-462.

$$|\alpha| \; > \; |\beta| \; , \qquad |\alpha| \; > \; 1 \; ,$$

it is evident that

$$\lim_{N=\infty} \; \frac{u_{N+n-1}}{u_{N+n}} \; = \; \frac{1}{\alpha} \; .$$

Thus (6.4) implies

$$(6.5) \qquad u_r A_r \; = \; u_r \sum_{n=1}^{\infty} \frac{(\alpha\beta)^n}{u_n u_{n+r}} \; = \; \sum_{n=1}^{r} \frac{u_{n-1}}{u_n} \; - \; \frac{r}{\alpha} \; .$$

Returning to (6.2), we have

$$T_{k+1} \; = \; \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j-1)-jk} \sum_{n=1} \frac{(\alpha\beta)^n}{u_n u_{n+2k-j+1}}$$

$$- \; \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j-1)-jk} \sum_{n=1}^{j} \frac{(\alpha\beta)^n}{u_n u_{n+2k-j+1}} \; .$$

Therefore we have

$$T_{k+1} \; = \; \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j-1)-jk} A_{2k-j+1}$$

$$= \; \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j-1)-jk} \sum_{n=1}^{j} \frac{(\alpha\beta)^n}{u_n u_{n+2k-j+1}} \; .$$

In particular, when $\alpha + \beta = 1$, $\alpha\beta = -1$, (6.6) reduces to

$$\sum_{n=1}^{\infty} \frac{(-1)^{n(k+1)}}{F_n F_{n+1} \cdots F_{n+2k+1}} = \frac{1}{(F)_{2k}} \sum_{j=0}^{2k} (-1)^{\frac{1}{2}j(j+1)-jk} \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} A_{2k-j+1}$$

$$(6.7) \qquad\qquad - \frac{1}{(F)_{2k}} \sum_{j=0}^{2k} (-1)^{\frac{1}{2}j(j+1)-jk} \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\}$$

$$\cdot \sum_{n=1}^{j} \frac{(-1)^n}{F_n F_{n+2k-j+1}} \qquad ,$$

where now $\left\{ \begin{matrix} 2k \\ j \end{matrix} \right\}$ and A_{2k-j+1} are expressed in terms of Fibonacci numbers.

REFERENCE

1. Brother Alfred Brousseau, "Summation of Infinite Fibonacci Series," Fibonacci Quarterly, Vol. 7 (1969), pp. 143-168.

◆◇◆◇◆

[Continued from page 476.]

$\rho = 1$ stems from its application to the partitioning of integers into distinct Fibonacci numbers. These applications are investigated in the papers listed in References. When ρ is a root of unity, series (1) again has partition — theoretic congruence which we exploited to some extent in Section 5.

REFERENCES

1. L. Carlitz, "Fibonacci Representations," Fibonacci Quarterly, Vol. 6, 1968, pp. 193-220.

2. L. Carlitz, "Fibonacci Representations — II," Fibonacci Quarterly, Vol. 8, 1970, pp. 113-134.

3. H. H. Ferns, "On Representations of Integers as Sums of Distinct Fibonacci Numbers," Fibonacci Quarterly, Vol. 3, 1965, pp. 21-30.

4. V. E. Hoggatt, Jr., and S. L. Basin, "Representations by Complete Sequences," Fibonacci Quarterly, Vol. 1, No. 3, pp. 1-14.

5. D. A. Klarner, "Representations of N as a Sum of Distinct Elements from Special Sequences," Fibonacci Quarterly, Vol. 4, 1966, pp. 289-306.

◆◇◆◇◆

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-186   Proposed by James Desmond, Florida State University, Tallahasse, Florida.

The generalized Fibonacci sequence is defined by the recurrence relation

$$U_{n-1} + U_n = U_{n+1} ,$$

where n is an integer and U_0 and U_1 are arbitrary fixed integers.

For a prime p and integers n, r, s and t show that

$$U_{np+r} \equiv U_{sp+t} \pmod{p}$$

if $p \equiv \pm 1 \pmod 5$ and $n + r = s + t$, and that

$$U_{np+r} \equiv (-1)^{r+t} U_{sp+t} \pmod{p}$$

if $p \equiv \pm 2 \pmod 5$ and $n - r = s - t$.

H-187   Proposed by Ira Gessel, Harvard University, Cambridge, Massachusetts.

Problem: Show that a positive integer n is a Fibonacci number if and only if either $5n^2 + 4$ or $5n^2 - 4$ is a square.

512

H-188   Proposed by Raymond E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Prove that there are no even perfect Fibonacci numbers.

SOLUTIONS
A NORMAL DETERMINANT

H-168   Proposed by David A. Klarner, University of Alberta, Edmonton, Alberta, Canada.

If

$$a_{ij} = \binom{i + j - 2}{i - 1}$$

for $i, j = 1, 2, \cdots, n$, show that $\det a_{ij} = 1$.

Solution by F. D. Parker, St. Lawrence University, Canton, New York.

It will be convenient to denote the given matrix by M_n, and its deter-
minant by $d(M_n)$, and then to prove the result by mathematical induction.

Since

$$a_{ij} = \binom{i + j - 2}{i - 1} ,$$

we have the two identities

$$a_{ij} - a_{i-1,j} = a_{i,j-1} ,$$

and

$$a_{ij} - a_{i,j-1} = a_{i-1,j} .$$

If we subtract from each column (except the first) of M_n the preceding
column, the second identity shows that

$$d(M_n) = d(C_{11}, C_{i-1,2}, C_{i-1,3}, \cdots, C_{i-1,n}) ,$$

where c_{ij} represents a <u>column</u> whose elements are given by a_{ij}. We notice that the first row of this new matrix is $(1, 0, 0, \cdots)$. Now if we subtract from each row (except the first) of the new matrix the preceding row, the first identity produces the matrix

$$Mn'' = \begin{pmatrix} 1 & \overline{0} \\ I & M_{n-1} \end{pmatrix}$$

where $\overline{0}$ is a row vector of <u>zeros,</u> I is a column vector of <u>ones.</u> The determinant has not been changed by these operations so that we have

$$d(Mn) = d(Mn'') = d(Mn - 1) .$$

Thus $d(Mn)$ is a constant and, since $d(m1) = 1$, then $d(Mn) = 1.$

Also solved by C. B. A. Peck and M. Yoder.

PRIME TARGET

H-169   Proposed by Francis DeKoven, Highland Park, Illinois. (Correction).

Show $n^2 + 1$ is a prime if and only if $n \neq ab + cd$ with $ad - bc = \pm 1$ for integers $a, b, c, d > 0.$

Solution by Robert Guili, San Jose State College, San Jose, California. (Partial)

Note: Z denotes the set of positive integers.
Solution by contradiction: If

$$n = ab + cd; \qquad ad - bc = \pm 1 ,$$

then

$$n^2 = a^2b^2 + 2abcd + c^2d^2 ; \qquad 1 = a^2d^2 - 2abcd + b^2c^2 .$$

So

$$n^2 + 1 = a^2b^2 + a^2d^2 = c^2d^2 + b^2c^2$$
$$= a^2(b^2 + d^2) + c^2(d^2 + b^2)$$
$$= (a^2 + c^2)(b^2 + d^2)$$

which is not true.

EDITORIAL COMMENT

The second part of this proof intended here was not complete. The late proposer made the same logical oversight. However, the second proof he submitted was more complete and can appear at a later date.

Editor V. E. H.

Also solved by the Proposer.

NON-EXISTENT

H-171   Proposed by Douglas Lind, Stanford University, Stanford, California.

Does there exist a continuous real-valued function f defined on a compact interval I of the real line such that

$$\int_I f(x)^n dx = F_n .$$

What if we require f only be measurable?

Solution by the Proposer.

We claim that such a measurable function f does not exist. By the Binet formula,

$$F_n = (a^n - b^n)/\sqrt{5} \ ,$$

where

$$a = (1 + \sqrt{5})/2, \qquad b = (1 - \sqrt{5})/2 \ .$$

For any measurable real-valued function g defined on I and any $p \geq 1$ we define

$$\|g\|_{p,I} \equiv \|g\|_p = \left(\int_I |g(x)|^p dx \right)^{1/p}$$

which is taken to be $+\infty$ if $|g|^p$ is not Lebesgue integrable. Also, let

$$\|g\|_{\infty,I} \equiv \|g\|_\infty = \text{ess sup} \{|g(x)|; \ x \in I\} = \inf \{t: \mu(g^{-1}(t,\infty)) = 0\} \ ,$$

where μ denotes Lebesgue measure on the real line. It is well known that since $\mu(I) < \infty$,

$$\lim_{p\to\infty} \|g\|_p = \|g\|_\infty \ ,$$

where $\|g\|_\infty$ is possibly ∞.

Now suppose that f is a real-valued function on I such that

$$F_n = \int_I f^n(x) \, dx$$

for $n = 1, 2, \cdots$. Then

$$\|f\|_\infty = \lim_{n\to\infty} \|f\|_n = \lim_{n\to\infty} F_n^{1/n} = a \ .$$

Let

$$A = \{x \in I: f(x) = a\} \ ,$$

$$B = \{x \in I: f(x) = -a\} \ .$$

Then for n = 2k we have

$$\frac{a^{2k} - b^{2k}}{\sqrt{5}} = \int_I f^{2k}(x)dx = \{\mu(A) + \mu(B)\}a^{2k} + \int_{I-(A\cup B)} f^{2k}(x)dx \ ,$$

so that

$$(*) \qquad \frac{1}{\sqrt{5}} - \mu(A) - \mu(B) = \frac{1}{\sqrt{5}}\left(\frac{b}{a}\right)^{2k} + \int_{I-(A\cup B)} \left[\frac{f(x)}{a}\right]^{2k} dx \ .$$

Since $|f(x)/a| < 1$ for almost all $x \in I - (A\cup B)$,

$$\{f(x)/a\}^{2k} \to 0$$

a.e. on I - (A∪B) as k → ∞, so by Lebesgue's Dominated Convergence Theorem, the right-hand integral approaches 0 as k → ∞. Since

$$|b/a| < 1, \qquad (b/a)^{2k} \to 0$$

as k → ∞, so letting k → ∞ in (*) shows

$$\mu(A) + \mu(B) = 1/\sqrt{5} \ .$$

Now if we put n = 2k + 1, we have

$$\frac{a^{2k+1} - b^{2k+1}}{\sqrt{5}} = \{\mu(A) - \mu(B)\}a^{2k+1} + \int_{I-(A\cup B)} f^{2k+1}(x)dx \ ,$$

and the same reasoning as before shows

$$\mu(A) - \mu(B) = 1/\sqrt{5} \ ,$$

Hence $\mu(B) = 0$ and $\mu(A) = 1/\sqrt{5}$. Letting K = I - A, we thus have

$$\frac{-b^n}{\sqrt{5}} = \int_K f^n(x)dx \ .$$

Now

$$|b| = \lim_{n\to\infty} \|f\|_{n,K} = \|f\|_{\infty,K} \ ,$$

so

$$\|f(x)\| \le |b|$$

for almost all $x \in K$. Let

$$C = \{x \in K: f(x) = b\} \ ,$$

$$D = \{x \in K: f(x) = -b\} \ .$$

Then

$$\frac{-b^{2k}}{\sqrt{5}} = \{\mu(C) + \mu(D)\}b^{2k} + \int_{K-(C\cup D)} f^{2k}(x)dx \ ,$$

so that

$$\frac{1}{\sqrt{5}} + \mu(C) + \mu(D) = - \int_{K-(C\cup D)} \left[\frac{f(x)}{b}\right]^{2k} dx \ .$$

Reasoning as before, we see by dominated convergence that the right-hand integral approaches 0 as $k \to \infty$. But this contradicts the fact that the left side is strictly positive. This contradiction shows that such an f does not exist.

We remark that the situation is different for Lucas numbers. For let $1 = [0,2]$, $f(x) = a$ if $0 \le x < 1$, $f(x) = b$ if $1 \le x \le 2$. Then

$$\int_I f^n(x)dx = a^n + b^n = L_n \; .$$

However, one can show using the methods above that f cannot be replaced by a continuous function.

Editorial Note: Robert Giuli noted that

$$\int_b^a \frac{nx^{n-1}}{\sqrt{5}} \, dx = F_n \; ,$$

although this does not satisfy the proposal. It might be interesting to reconsider the proposal with restrictions on f, such as boundedness, etc.

HISTORY REPEATS

H-172   Proposed by David Englund, Rockford College, Rockford, Illinois.

Prove or disprove the "identity,"

$$F_{kn} = F_n \sum_{t=1}^{\left[\frac{k+1}{2}\right]} (-1)^{(n+1)(t+1)} \binom{k - t}{t - 1} L_n^{k-2t+1} \; .$$

where F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively, and $[x]$ denotes the greatest integer function.

Solution by Douglas Lind, Stanford University

This is Problem H-135 (this Quarterly, Vol. 6, 1968, pp. 143-144: solution, Vol. 7, 1969, pp. 518-519), and appears as Eq. (3.15) in "Compositions and Fibonacci Numbers" by V. E. Hoggatt, Jr. , and D. A. Lind (this Quarterly, Vol. 7, 1969, pp. 253-266).

Also solved by Wray Brady and L. Carlitz.

FIBONACCI VERSUS DIOPHANTUS

H-173   Proposed by George Ledin, Jr., Institute of Chemical Biology, University of San Francisco, San Francisco, California

Solve the Diophantine equation,

$$x^2 + y^2 + 1 = 3xy \; .$$

Solution by L. Carlitz, Duke University, Durham, North Carolina.

The equation

$$(*) \qquad\qquad x^2 + y^2 + 1 = 3xy$$

can be written in the form

$$(ax - 3y)^2 - 5y^2 = -4 \; ,$$

where $a = 2$. We recall that the general (positive) solution of

$$x^2 - 5y^2 = -4$$

is given by

$$\left(\frac{1 + \sqrt{5}}{2}\right)^{2n+1} = \frac{u_n + v_n\sqrt{5}}{2} \qquad (n = 0, 1, 2, \cdots) \; ,$$

so that

$$
\begin{cases}
u_n = \dfrac{1}{2^{2n}} \displaystyle\sum_{r=0}^{n} \binom{2n + 1}{2r} 5^r \\[2em]
v_n = \dfrac{1}{2^{2n}} \displaystyle\sum_{r=0}^{n} \binom{2n + 1}{2r + 1} 5^r \; .
\end{cases}
$$

On the other hand, the Fibonacci number F_{n+1} satisfies

$$F_{n+1} = \frac{1}{2^n} \sum_{2r \leq n} \binom{n+1}{2r+1} 5^r \ ,$$

so that $v_n = F_{2n+1}$. Moreover,

$$u_n + v_n = 2F_{2n+2} \ ,$$

which gives

$$u_n = 2F_{2n+2} - F_{2n+1} \ .$$

Since

$$y = v_n, \quad 2x - 3y = u_n \ ,$$

it follows that

$$2x = u_n + 3v_n = 2F_{2n+2} + 2F_{2n+1} = 2F_{2n+3} \ ,$$

so that $x = F_{2n+3}$. Hence we have the general solution of (*) with $x > y$:

$$x = F_{2n+3}, \quad y = F_{2n+1} \qquad (n = 0, 1, 2, \cdots) \ .$$

The solution $x = y = 1$ is evidently obtained by taking $n = -1$.

Also solved by W. Barley, M. Herdy, C. B. A. Peck, C. Bridger, J. A. H. Hunter, and the Proposer.

SUM PROJECT

H-175 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$(1 + z + \tfrac{1}{3}z^2)^{-n-1} = \sum_{k=0}^{\infty} a(n,k)z^k \quad .$$

Show that

(1) $$a(n,n) = \frac{2 \cdot 5 \cdot 8 \cdots (2n - 1)}{n!}$$

(II) $$\sum_{s=0}^{n} \binom{n - s}{s} \binom{2n - s}{n} \left(-\frac{1}{3}\right)^s = \frac{2 \cdot 5 \cdot 8 \cdots (3n - 1)}{n!}$$

(III) $$\sum_{r=0}^{\infty} \binom{n + r}{r} \binom{2n - r}{n} (-\omega)^r = (\omega^2\sqrt{-3})^n \frac{2 \cdot 5 \cdot 8 \cdots (3n - 1)}{n!} \quad ,$$

where

$$\omega = \frac{1}{2}(-1 - \sqrt{-3}) \quad .$$

Solution by the Proposer.

(I) If $z = wf(z)$, $f(0) \neq 0$, where $f(z)$ is analytic about the origin, then (Polya-Szegö, <u>Aufgaben und Lehrsatze aus der Analysis</u>, Vol. 1, p. 125)

$$z = \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} (f(x))^n \right]_{x=0}$$

$$= \sum_{n=0}^{\infty} \frac{w^{n+1}}{(n+1)!} \left[\frac{d^n}{dx^n} (f(x))^{n+1} \right]_{x=0} \quad .$$

Take

$$f(z) = (1 - z + \tfrac{1}{3} z^2)^{-1} \ ,$$

so that

$$(*) \qquad \left[\frac{d^n}{dx^n} (f(x))^{n+1}\right]_{x=0} = n!\, a(n,n) \ .$$

On the other hand, $z = wf(z)$ becomes

$$z(1 - z + \tfrac{1}{3} z^2) = w \ ,$$

which reduces to

$$(1 - z)^3 = 1 - 3w \ .$$

It follows that

$$z = 1 - (1 - 3w)^{\frac{1}{3}}$$

$$= \sum_{n=1}^{\infty} (-1)^n \binom{-\tfrac{1}{3}}{n} 3^n w^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \binom{-\tfrac{2}{3}}{n} \frac{3^n w^n}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n - 1)}{(n + 1)!} w^n \ .$$

Comparison with $(*)$ gives

$$a(n,n) = \frac{2 \cdot 5 \cdot 8 \cdots (3n - 1)}{n!} \qquad \bullet$$

(II).   Since

$$(1 - z + \tfrac{1}{3}z^2)^{-n-1} = \sum_{r=0}^{\infty} \binom{n + r}{r} z^r (1 - \tfrac{1}{3}z)^r$$

$$= \sum_{r=0}^{\infty} \binom{n + r}{r} z^r \sum_{s=0}^{r} \binom{r}{s}\left(-\tfrac{1}{3}\right)^s z^s$$

$$= \sum_{k=0}^{\infty} z^k \sum_{r+s=k} \binom{n + r}{r}\binom{r}{s}\left(-\tfrac{1}{3}\right)^s ,$$

it follows that

$$a(n,n) = \sum_{s=0}^{n} \binom{n - s}{s}\binom{2n - s}{n}\left(-\tfrac{1}{3}\right)^s .$$

(III).   Put

$$1 - z + \tfrac{1}{3} z^2 = (1 - \alpha z)(1 - \beta z) .$$

It is easily verified that

$$\alpha = -\frac{\omega^2}{\sqrt{-3}} , \qquad \beta = \frac{\omega}{\sqrt{-3}} .$$

Then

$$(1 - z + \tfrac{1}{3} z^2)^{-n-1} = (1 - \alpha z)^{-n-1}(1 - \beta z)^{-n-1}$$

$$= \sum_{r=0}^{\infty} \binom{n + r}{r} \alpha^r z^r \sum_{s=0}^{\infty} \binom{n + s}{s} \beta^s z^s ,$$

so that

$$a(n,n) = \sum_{r+s=n} \binom{n+r}{r}\binom{n+s}{s}\alpha^r \beta^s$$

$$= \sum_{r=0}^{\infty} \binom{n+r}{r}\binom{2n-r}{n}\left(-\frac{\omega^2}{\sqrt{-3}}\right)^r \left(\frac{\omega}{\sqrt{-3}}\right)^{n-r}$$

$$= \frac{\omega^n}{(\sqrt{-3})^n} \sum_{r=0}^{n} \binom{n+r}{r}\binom{2n-r}{n}(-\omega)^r .$$

◆◇◆◇◆

[Continued from page 496.]

GENERALIZED BASES FOR REAL NUMBERS

3. S. Kakeya, "On the Partial Sums of an Infinite Series," Sci. Reports Tohoku Imp. U. (1), 3 (1914), pp. 159-163.

4. J. L. Brown, Jr., "On the Equivalence of Completeness and Semi-Completeness for Integer Sequences," Mathematics Magazine, Vol. 36, No. 4, Sept.-Oct., 1963, pp. 224-226.

5. I. Niven, "Irrational Numbers," Carus Mathematical Monograph No. 11, John Wiley and Sons, Inc., 1956.

6. I. Niven and H. S. Zuckerman, An Introduction to the Theory of Numbers, John Wiley and Sons, Inc., 1960.

◆◇◆◇◆

CHALLENGE

"In what way does the cubic congruence

$$x^3 - 15x + 25 \equiv 0 \pmod{p}, p \text{ a prime}$$

relate to the Fibonacci numbers?

Generalize to other recurring series."

John Brillhart and Emma Lehmer

THE SUM OF THE FIRST n POSITIVE INTEGERS—GEOMETRICALLY

FREDERICK STERN
San Jose State College, San Jose, California

The familiar formula $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$ follows from counting in two ways, the number of intersections of $(n + 1)$ lines in the plane, assuming that no two of these lines are parallel and no three intersect at the same point. On the one hand, since any two of the lines intersect at a point distinct from the point of intersection of any other pair, the number of points of intersection is the same as the number of distinct pairs of lines:

$$\binom{n + 1}{2} = \frac{1}{2}n(n + 1) \ .$$

On the other hand, suppose the lines are numbered, completely arbitrarily, from 1 to $(n + 1)$. Counting the number of intersections sequentially, the second line intersects the first at one point. The third line intersects each of the first two at two distinct points — giving a partial total of $1 + 2$ intersections. The fourth line intersects each of the first 3 at three points — giving a partial total of $1 + 2 + 3$ intersections. Thus, the $(k + 1)^{st}$ line intersects the first through the kth lines at k distinct points so that the lines numbered 1 through $(k + 1)$ intersect at $1 + 2 + \cdots + k$ distinct points. Finally, we see in this way that the $n + 1$ lines intersect at $1 + 2 + \cdots + n$ distinct points. Thus, we have counted the same number of points in two ways and have arrived at the familiar formula.

Afternoon Session

Phyllotaxis: The Facts and the Theory
    Dr. Irving Adler, North Bennington, Vermont
Telephone Grammars: An Elementary Example in the Mathematical Theory of Context-Free Languages
    George Ledin, Jr. , Institute of Chemical Biology, University of S. F.
The Periodic Properties of a Linear Recurrent Sequence over a Ring
    Prof. Donald W. Robinson, Brigham Young University, Provo, Utah

Free Discussion Period

SOME STRIKING PROPORTIONS IN THE MUSIC OF BELA BARTÓK

EDWARD A. LOWMAN
200 Santa Clara Avenue, Oakland, California

Unlike the techniques discussed in a previous article*, relatively simple Fibonacci proportions can be used in the organization of larger units of musical time. In common diatonic practices the lengths of phrases and sections, expressed in measures, are generally some power of two: four, eight, sixteen, and thirty-two. Fibonacci numbers, as numerical expressions of the golden mean, offer other ways of creating proportion which largely avoid these divisions. Naturally, just as the older phrases could be sometimes extended, shortened, or grouped in unusual ways without destroying the overall sense of balance, Fibonacci proportions need not always be exact or consistent to achieve their intended effect.

Many contemporary composers are using Fibonacci proportions in this way, but some of the most striking examples are found in the music of an earlier master: Bartók. Bartók's use of Fibonacci proportions evidently springs from an interest in the golden mean. Ernő Lendvai, in his book Bartók: sa vie et son oeuvre (Budapest, 1957), has pointed out many examples from Bartók's music where the golden mean is the major dividing point of a piece.

If a unity is divided into two parts according to the golden mean, the larger part will be 0.618 and the smaller will be 0.382. The first movement for the Sonata for Two Pianos and Percussion has 443 measures, and its golden mean is therefore 443 x 0.618 = 274. The recapitulation (the return to material from the beginning) begins in measure 274. In the first movement of the Divertimento for String Orchestra, the recapitulation begins at the golden mean (measured in ternary units instead of measures to compensate for meter changes), as it does also in the first movement of Contrasts. Three examples are cited from the sixth volume of Mikrokosmos. In "Free Variations," the golden mean comes at the molto più calmo; in "From the Diary of a Fly," it falls at the climax (which is marked with a double

---

* "An Example of Fibonacci Numbers Used to Generate Rhythmic Values in Modern Music," this Quarterly, Vol. 9, No. 4, pp. 423-426.

sforzando), and in "Divided Arpeggios," the recapitulation begins at the gold-
en mean.

It is only one step further to casting subdivisions in Fibonacci propor-
tions. The first movement of Music for Strings, Percussion, and Celeste is
88 measures long. If we allow a measure's silence at the end, we have 89.
The fff climax of the movement arrives after 55 measures, of which the
strings play the first 34 with mutes, removing them for the last 21. The first
34 measures are subdivided further, as the exposition (the movement is a
fugue) is 21 bars long. The 34 measures following the climax are divided in-
to 13 and 21 by the replacement of the mutes at measure 69, and the final 21
measures are divided again by a change of texture into groups of thirteen and
eight. The following diagram illustrates these divisions. It will be noted al-
so that before the climax longer units are followed by shorter ones, while the
reverse tends to be true after the climax. Thus pace becomes a major factor
in shaping the movement.
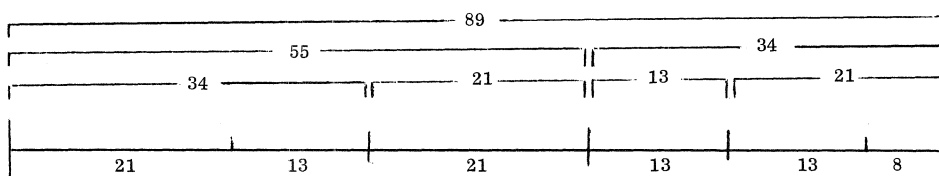
First Movement of Music for Strings, Percussion, and Celeste

A diagram of the third movement of Music shows considerable, but not
exclusive use of Fibonacci porportions. Here the smaller units are cast
mostly in the familiar fives, eights, and thirteens, but the overall balance of
the movement is less obvious, and highly individual.
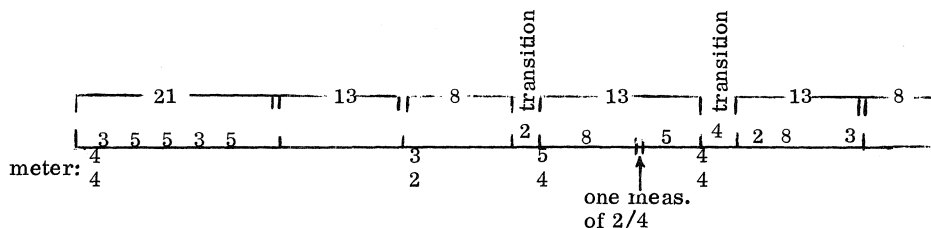
Third Movement of Music for Strings, Percussion, and Celeste

A PRIMER FOR THE FIBONACCI NUMBERS: PART IX

MARJORIE BICKNELL
A. C. Wilcox High School, Santa Clara, California
and
VERNER E. HOGGATT, JR.
San Jose State College, San Jose, California

TO PROVE: F_n DIVIDES F_{nk}

For many years, it has been known that the nth Fibonacci number F_n divides F_m if and only if n divides m, $n > 2$. Many different proofs have been given; it will be instructive and entertaining to examine some of them.

Some special cases are very easy. It is obvious that F_k divides F_{2k}, for $F_{2k} = F_k L_k$. If we wish only to prove that F_n divides F_{nk} when k is a power of 2, the identity

$$F_{2^j n} = F_n L_n L_{2n} L_{4n} \cdots L_{2^{j-1}n}$$

suffices.

1. PROOFS USING THE BINET FORM

Perhaps the simplest proof to understand is one which depends upon simple algebra and the Binet form (see [1]),

$$(1) \qquad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad ,$$

where

$$\alpha = (1 + \sqrt{5})/2, \qquad \beta = (1 - \sqrt{5})/2$$

are the roots of $x^2 - x - 1 = 0$. Then

$$F_n = \frac{\alpha^{nk} - \beta^{nk}}{\alpha - \beta} = \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right)(M) = F_k M \ ,$$

529

where

$$M = \alpha^{(n-1)k} + \alpha^{(n-2)k}\beta^{k} + \alpha^{(n-3)k}\beta^{2k} + \cdots$$
$$+ \alpha^{k}\beta^{(n-2)k} + \beta^{(n-1)k} \quad .$$

If M is an integer, F_k divides F_{nk}, $k \neq 0$.

Since $\alpha\beta = -1$, if $(n-1)k$ is odd, pairing the first and last terms, second and next to last terms, and so on,

$$M = (\alpha^{(n-1)k} + \beta^{(n-1)k}) + (-1)^{k}(\alpha^{(n-3)k} + \beta^{(n-3)k})$$
$$+ (-1)^{2k}(\alpha^{(n-5)k} + \beta^{(n-5)k}) + \cdots$$
$$= L_{(n-1)k} + (-1)^{k}L_{(n-3)k} + (-1)^{2k}L_{(n-5)k} + \cdots \quad ,$$

where the nth Lucas number is given by

(2) $$L_n = \alpha^n + \beta^n \quad .$$

Thus, M is the sum of integers, and hence an integer. If $(n-1)k$ is even, the symmetric pairs can again be formed except for the middle term which is

$$(\alpha\beta)^{(n-1)k/2} = (-1)^{(n-1)k/2} \quad ,$$

again making M an integer. Thus, F_k divides F_{nk}, or, F_n divides F_m if n divides m. See also H-172, this issue.

2. PROOFS BY MATHEMATICAL INDUCTION

Other proofs can be derived, starting with a known identity and using mathematical induction. For example, use the known identity (see [2])

(3) $$F_{m+n} = F_m F_{n+1} + F_{m-1}F_n \quad .$$

Let $m = nk$:

(4) $$F_{nk+n} = F_{n(k+1)} = F_{nk}F_{n+1} + F_{nk-1}F_n \quad .$$

Obviously, F_n divides F_n and F_n divides F_{2n}, for $F_{2n} = F_n L_n$, so that F_n divides F_{kn} for $k = 1, 2$. Assume that F_n divides F_{in} for $i = 1, 2, \cdots, k$. Then, since F_n divides F_n and F_n divides F_{kn}, identity (4) forces F_n also to divide $F_{n(k+1)}$, so that F_n divides F_{kn} for all positive integers k.

Another identity, easily proved using (2) and (2), which leads to an easy proof by mathematical induction is

$$(5) \qquad\qquad L_n F_{m-n} + F_n L_{m-n} = 2F_m \ .$$

Let $m = nk$, yielding

$$(6) \qquad\qquad L_n F_{n(k-1)} + F_n L_{n(k-1)} = 2F_{nk} \ .$$

If F_n divides F_n and F_n divides $F_{n(k-1)}$, then F_n must divide F_{nk}, $|F_n| > 2$.

A less obvious identity given by Siler [3] also yields a proof by mathematical induction:

$$(7) \qquad ((-1)^n + 1 - L_n)\left(\sum_{i=1}^{k} F_{in} \right) = (-1)^n F_{kn} - F_{n(k+1)} + F_n \ .$$

If F_n divides F_{in} for $i = 1, 2, 3, \cdots, k$, then F_n is a factor of the left-hand member of (7). Since F_n divides F_n and F_n divides F_{kn}, F_n must also divide $F_{n(k+1)}$, so that F_n divides F_{kn} for all positive integers k.

3. PROOFS FROM GENERATING FUNCTIONS AND POLYNOMIALS

Now let us look for elegance. Suppose that we have proved the generating function identity given in [4],

$$\frac{F_n x}{1 - L_n x + (-1)^n x^2} = \sum_{k=0}^{\infty} F_{nk} x^k \ .$$

Then, since the leading coefficient of the divisor is one and the resulting operations of division are multiplying, adding, and subtracting integers, the quotient coefficients F_{nk}/F_n of powers of x are integers, and F_n divides F_{nk} for all integers $k \geq 0$.

Let us develop a generating function for a related proof that L_n divides L_{kn} whenever k is odd. Applying (2) and the formula for summing an infinite geometric progression,

$$\sum_{i=0}^{\infty} L_{(2i+1)n} x^i = \sum_{i=0}^{\infty} \alpha^{n(2i+1)} x^i + \sum_{i=0}^{\infty} \beta^{n(2i+1)} x^i$$

$$= \frac{\alpha^n}{1 - \alpha^{2n} x} + \frac{\beta^n}{1 - \beta^{2n} x}$$

$$= \frac{(\alpha^n + \beta^n)(1 - (-1)^n x)}{1 - (\alpha^{2n} + \beta^{2n})x + (\alpha\beta)^{2n} x^2}$$

$$= \frac{L_n (1 - (-1)^n x)}{1 - L_{2n} x + x^2} \quad .$$

Then

$$\sum_{i=0}^{\infty} \frac{L_{(2i+1)n}}{L_n} x^i = \frac{1 - (-1)^n x}{1 - L_{2n} x + x^2} \quad ,$$

so that by the same reasoning given for the Fibonacci generating function above, $L_{(2i+1)n}/L_n$ is an integer.

Next, we prove that $L_{(2k+1)n}/L_n$ is an integer another way. Now it is true that

$$L_{(2k+1)n} = L_n L_{2kn} - (-1)^{n+1} L_{(2k-1)n}$$

so that

$$\frac{L_{(2k+1)n}}{L_n} = L_{2n} - (-1)^{n+1}\frac{L_{(2k-1)n}}{L_n}$$

Thus, we are set up to use mathematical induction since when $k = 1$, it is clear that L_n divides L_n. Thus, if $L_{(2k-1)n}/L_n$ is an integer, then $L_{(2k+1)n}/L_n$ is also an integer. The proof is complete by mathematical induction.

We can carry this one step further, and prove that L_m is not divisible by L_n if $m \neq (2k+1)n$, $n \geq 2$.

$$L_{(2k+1)n+j} = L_n L_{2kn+j} + (-1)^n L_{(2k-1)n+j}, \qquad j = 1, 2, 3, \cdots, 2n - 1 .$$

Thus, given that some $j = 1, 2, 3, \cdots$, or $2n - 1$ exists so that $L_{(2k+1)n+j}$ is divisible by L_n, then by the method of infinite descent, $L_{(2k-1)n+j}$ is divisible by L_n for this same $j = 1, 2, 3, \cdots$, or $2n - 1$. This will ultimately yield the inequality

$$-|L_n| < L_{-n+j} < L_n ,$$

which is clearly a contradiction since the L_s in that range are all smaller than L_n, $n \geq 2$. The same technique can be used on F_{nk} and F_n to prove that F_n divides F_m only if n divides m, $n > 2$. (Since $F_2 = 1$ divides all F_n, we must make the qualification $n > 2$.)

If the theory of Fibonacci polynomials is at our disposal, the theorem that F_n divides F_m if and only if n divides m, $n > 2$, becomes a special case. (See [5].)

If the following identity is accepted (proved in [5]),

$$F_m = F_n\left(\sum_{i=0}^{p-1} (-1)^{in}L_{m-(2i+1)n}\right) + (-1)^{pn}F_{m-2pn}, \qquad p \geq 1 ,$$

when $|n| < |m|$, $n \neq 0$, the identity can be interpreted in terms of quotients and remainders; the quotient being a sum of Lucas numbers and the remainder

of least absolute value being a Fibonacci number or its negative. The remainder is zero if and only if either $F_{m-2pn} = 0$ or $F_{m-2pn} = \pm F_n$, in which case the quotient is changed by ± 1. In the first case, $m - 2pn = 0$, so that m is an even multiple of n; and in the second, $m - 2pn = \pm n$, with m an odd multiple of n. So, F_n divides F_m if and only if n divides m, $n > 2$.

That F_n divides F_m only if n divides m can also be proved through use of the Euclidean Algorithm [2] or as the solution to a Diophantine equation [6] to establish that

$$(F_m, F_n) = F_{(m,n)} \qquad (m \geq n > 2) ,$$

or, that the greatest common divisor of two Fibonacci numbers is a Fibonacci number whose subscript is the greatest common divisor of the subscripts of the other two Fibonacci numbers.

4. THE GENERAL CASE

A second proof that L_n divides L_m if and only if $m = (2k + 1)n$, $n \geq 2$, provides a springboard for studying the general case. The identity

$$(8) \qquad L_{m+n} = F_{m+1}L_n + F_m L_{n-1}$$

indicates that L_n divides L_{m+n} if L_n divides F_m. Since

$$F_{2p} = L_p F_p ,$$

L_p divides F_{2p}. But since

$$F_{2(k+1)p} = F_{2kp+2p} = F_{2kp} F_{2p+1} + F_{2kp-1} F_{2p} ,$$

whenever L_p divides F_{2kp}, it must divide $F_{2(k+1)p}$, and we have proved by mathematical induction that L_p divides F_{2kp} for all positive integers k. Then, returning to (8), if $m = 2kn$, L_n divides L_{m+n}, or,

$$L_{2kn+n} = L_{(2k+1)n} = F_{2kn+1}L_n + F_{2kn}L_{n-1} ,$$

so that L_n divides $L_{(2k+1)n}$.

To prove that L_n divides L_m only if $m = (2k + 1)n$, $n \geq 2$, we prove that L_n divides F_m only if $m = 2kn$, $n \geq 2$. We use the identity

$$F_{2n-j} = L_n F_{n-j} + (-1)^{n+1} F_{-j}, \qquad j = 1, 2, \cdots, n - 1$$

to show that L_n cannot divide F_{2n-j}. If L_n divides F_{2n-j}, then L_n must divide F_{-j}, but $L_n > F_n > |F_{-j}|$, clearly a contradiction. Thus, L_n divides L_m if and only if $m = (2k + 1)n$. A proof of this same theorem using algebraic numbers is given by Carlitz in [7].

Now we consider the general case. Given a Fibonacci sequence defined by

$$H_1 = p, \qquad H_2 = q, \qquad H_{n+2} = H_{n+1} + H_n ,$$

under what circumstances does H_n divide H_m?

Studying a sequence such as

$$1, \ 4, \ 5, \ 9, \ 14, \ 23, \ 37, \ 60, \ 97, \ 157, \ 254, \ 411, \ 665, \ 1076, \ \cdots$$

quickly convinces one that each member divides other members of the sequence in a regular fashion. For example, 5 divides itself and every fifth member thereafter, while 4 divides itself and every sixth member thereafter.

The mystery is resolved by the identity

$$H_{m+n} = F_{m+1} H_n + F_m H_{n-1} .$$

If H_n divides F_m, then H_n divides every m^{th} term of the sequence thereafter. Further, divisibility of terms of $\{H_n\}$ by an arbitrary integer p can be predicted using tables of Fibonacci entry points. If H_k is divisible by p, then H_{k+e} is the next member of the sequence divisible by p, where e is the entry point of p for the Fibonacci sequence. For example, if 41 divides H_n, then 41 divides H_{n+20} and 41 divides H_{n+20k} since 20 is the sub-

script of the first Fibonacci number divisible by 41, but 41 will divide no member of the sequence between H_n and H_{n+20}.

REFERENCES

1. Verner E. Hoggatt, Jr., Fibonacci and Lucas Numbers (Boston; Houghton Mifflin Co. , 1969), pp. 37-39.

2. N. N. Vorobyov, Fibonacci Numbers (Boston: D. C. Heath and Co., 1963), pp. 22-24.

3. Ken Siler, "Fibonacci Summations," Fibonacci Quarterly, Vol. 1, No. 3, October, 1963, pp. 67-69.

4. V. E. Hoggatt, Jr. , and D. A. Lind, "A Primer for the Fibonacci Numbers: Part VI," Fibonacci Quarterly, Vol. 5, No. 5, Dec., 1967, pp. 445-460.

5. Marjorie Bicknell, "A Primer for the Fibonacci Numbers: Part VII," Fibonacci Quarterly, Vol. 8, No. 4, October, 1970, pp. 407-420.

6. Glen Michael, "A New Proof for an Old Property," Fibonacci Quarterly, Vol. 2, No. 1, February, 1964, pp. 57-58.

7. Leonard Carlitz, "A Note on Fibonacci Numbers," Fibonacci Quarterly, Vol. 2, No. 1, February, 1964, pp. 15-28.

[Continued from page 528.]

Perhaps the most carefully wrought example is the introduction to the first movement of the Sonata for Two Pianos and Percussion. Here the divisions, while not conforming to numbers of the Fibonacci series (0,1), are all determined by the golden mean. Measures 2–17 (the first measure is simply a roll on the timpani) contain 46 ternary (3/8) units, the most appropriate for study in a passage which contains both 6/8 and 9/8 measures. The golden mean of 46 is 28, which is the dividing line between the second and the third statements of the theme, and the place where the theme becomes inverted. The golden mean of 28 is 17.3, the juncture of the first and second statements of the theme. The two cymbal notes further subdivide the first and

second statements according to the golden mean, while the third statement is so partitioned by the entrance of the tam-tam. There are other passages of the Sonata in which the phrase structure is organized according to the golden mean, but most amazing is the fact that the entire piece is so proportioned. It contains 6,432 eighth notes, and the division between the first and second movements falls but one eighth note from 3,975, the golden mean.

Will a listener be aware of the "Fibonacci proportions" as they go by? Probably not, yet they will do their job just the same. What the listener will perceive is a sense of balance, a feeling that the musical events he hears occur at the "right" places, that they form intriguing patterns in time. Composers have always played with our perception of time, causing a moment to seem interminable or a whole passage to foreshorten or "telescope" into a single recollection. To ears accustomed to fours, eights, and sixteens, these new proportions will undoubtedly seem curious in their effectiveness, but so may the phrases of a Renaissance motet or a passage of Gregorian chant.

When we analyze music, the result is a number of graphs, charts, and explanations showing how a piece achieves its effect. We must remember, however, that composers seldom create their pieces in this manner. Bartók may have had a mathematical interest in the golden mean, or he may have hit upon the Fibonacci series while consciously searching for shapes other than powers of two for his musical ideans. More likely, however, the techniques grew out of the shapes of the musical ideas themselves, just as have most new techniques throughout music history. One can imagine his realizing at some point that these proportions were what his ideas had been approaching all along. The technique was thus a means of focusing and clarifying the effect. Whatever the procedure Bartók used, we know that performers and listeners recognized the exceptional balance and proportion of this music long before anyone discovered its "astonishing" use of Fibonacci numbers and the golden mean.

ACKNOWLEDGEMENT

A FIBONACCI CROSTIC

MARJORIE BICKNELL
A. C. Wilcox High School, Santa Clara, California

Use the definitions below to write the words to which they refer; then enter the appropriate letters in the diagram to complete a quotation from a mathematical work. The first letters of the defined words give the author and title from which the quotation is taken. The end of each word is indicated by a black square following it.

A. The first, fourth, and fifth clue numbers for word A; also, the sum of its five clue numbers.

$$\overline{76}\ \ \overline{40}\ \ \overline{36}\ \ \overline{29}\ \ \overline{18}$$

B. A Fibonacci count-down.

$$\overline{23}\ \ \overline{17}\ \ \overline{65}\ \ \overline{88}\ \ \overline{66}\ ,\quad \overline{43}\ \ \overline{156}\ \ \overline{124}\ \ \overline{42}\ ,$$
$$\overline{66}\ \ \overline{88}\ \ \overline{115}\ \ \overline{32}\ \ \overline{32}\ ,\ \overline{175}\ \ \overline{52}\ \ \overline{38}$$

C. An undiscovered number.

$$\overline{169}\ \ \overline{24}\ \ \overline{46}\ \quad \overline{138}\ \ \overline{105}\ \ \overline{10}\ \ \overline{184}\ \ \overline{105}\ \ \overline{90}\ \ \overline{161}$$

D. Compass point.

$$\overline{174}\ \ \overline{96}\ \ \overline{7}\ \ \overline{130}\ \ \overline{150}\ \ \overline{185}\ \ \overline{134}\ \ \overline{39}\ \ \overline{130}$$
$$\overline{12}\ \ \overline{99}\ \quad \overline{174}\ \ \overline{47}\ \ \overline{7}\ \ \overline{130}\ \ \overline{67}$$

E. Rank given by subscript of first positive Fibonacci number divisible by a number.

$$\overline{5}\ \ \overline{19}\ \ \overline{61}\ \ \overline{11}\ \ \overline{93}\ \ \overline{119}\ \ \overline{149}\ \ \overline{25}\ \ \overline{51}\ \ \overline{170}$$

F. See G.

$$\overline{129}\ \ \overline{27}\ \ \overline{163}\ \ \overline{183}\ \ \overline{91}$$

G. Platonic solid in which the (word F) of a diagonal of a face to an edge is the Golden Section.

$$\overline{81}\ \ \overline{113}\ \ \overline{81}\ \ \overline{125}\ \ \overline{107}\ \ \overline{110}\ \ \overline{131}\ \ \overline{125}\ \ \overline{171}\ \ \overline{141}\ \ \overline{177}\ \ \overline{157}$$

H. Phyllotaxis finds the numbers of A and J here and in V.

$$\overline{57}\ \ \overline{26}\ \quad \overline{9}\ \ \overline{20}\ \ \overline{57}\ \ \overline{28}\ \ \overline{84}\ \ \overline{104}\ \ \overline{16}$$

I. $E = IR$.

$$\overline{147}\ \ \overline{164}\ \ \overline{53}\ \ \overline{181}\ \quad \overline{182}\ \ \overline{3}\ \ \overline{74}$$

J. See H and first eight clue numbers of J.

$$\overline{2}\ \ \overline{1}\ \ \overline{13}\ \ \overline{8}\ \ \overline{55}\ \ \overline{21}\ \ \overline{144}\ \ \overline{6}$$

K. Secret emblem
of the Pythagoreans.

$\overline{109}$ $\overline{100}$ $\overline{35}$ $\overline{120}$ $\overline{71}$ $\overline{122}$ $\overline{62}$ $\overline{71}$ $\overline{146}$ $\overline{146}$ $\overline{153}$

$\overline{146}$ $\overline{56}$ $\overline{94}$ $\overline{49}$ $\overline{44}$ $\overline{22}$ $\overline{82}$ $\overline{146}$

L. Boundless.

$\overline{128}$ $\overline{86}$ $\overline{59}$ $\overline{123}$ $\overline{86}$ $\overline{140}$ $\overline{155}$ $\overline{167}$

M. Music of the   ? .

$\overline{69}$ $\overline{78}$ $\overline{145}$ $\overline{136}$ $\overline{112}$ $\overline{136}$ $\overline{121}$

N. Type of number
in which the sum of
the aliquot divisors
exceeds the number
itself.

$\overline{54}$ $\overline{118}$ $\overline{48}$ $\overline{135}$ $\overline{33}$ $\overline{143}$ $\overline{135}$ $\overline{152}$

O. Exponents used
in hand computations.

$\overline{37}$ $\overline{133}$ $\overline{158}$ $\overline{116}$ $\overline{79}$ $\overline{85}$ $\overline{87}$ $\overline{108}$ $\overline{172}$ $\overline{60}$

P. Necessary
and sufficient.

$\overline{14}$ $\overline{58}$ $\overline{104}$

Q. Form
$F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$.

$\overline{117}$ $\overline{179}$ $\overline{64}$ $\overline{142}$ $\overline{15}$

R. He proved
$e^{i\pi} = -1$

$\overline{165}$ $\overline{92}$ $\overline{30}$ $\overline{165}$ $\overline{41}$

S. Horizontal
    arrays.

$\overline{154}$ $\overline{173}$ $\overline{159}$ $\overline{166}$

T. The Golden
    Section.

$\overline{139}$ $\overline{31}$ $\overline{70}$ $\overline{176}$ $\overline{101}$ or $\overline{70}$ $\overline{50}$ $\overline{63}$

U. Asked secretly
by the mathema-
tically uniniatiated
about phyllotaxis,
the Golden Section,
and the ubiquity of
Fibonacci numbers.

$\overline{127}$ $\overline{89}$ $\overline{132}$ $\overline{162}$ $\overline{75}$ $\overline{151}$

$\overline{137}$ $\overline{103}$ $\overline{132}$ $\overline{83}$ $\overline{162}$ $\overline{83}$ $\overline{73}$ $\overline{106}$ $\overline{114}$ $\overline{132}$

V. See H.

$\overline{98}$ $\overline{4}$ $\overline{160}$ $\overline{45}$ $\overline{34}$ $\overline{98}$ $\overline{4}$ $\overline{4}$ $\overline{77}$ $\overline{34}$

W. What $5x^2$,
Pascal's triangle,
and friction have
in common.

$\overline{168}$ $\overline{80}$ $\overline{95}$ $\overline{178}$ $\overline{178}$ $\overline{111}$ $\overline{168}$ $\overline{68}$ $\overline{95}$ $\overline{148}$ $\overline{180}$ $\overline{126}$

X. Hypothesis

$\overline{72}$ $\overline{97}$

A Fibonacci Crostic puzzle grid.

A POLYNOMIAL REPRESENTATION OF FIBONACCI NUMBERS

DAVID G. BEVERAGE
San Diego State College, San Diego, California

Douglas Lind has proposed the problem, "Does there exist a polynomial $P(x)$ for which $P(F_k) = F_{nk}$?". In particular, it is known that, if $L_n(x)$ is the nth Lucas Polynomial, then

$$L_n(L_{2k+1}) = L_{(2k+1)n} \ .$$

The answer to the Fibonacci formulation of the problem is in the affirmative also. The following theorem expresses an infinite number of polynomials, defined recursively, satisfying the properties for Fibonacci numbers analogous to those of the Lucas numbers and the Lucas polynomials.

Theorem.

$$F_{(2n+1)k} = 5^n F_k^{2n+1} - \sum_{n=1}^{n} \binom{2n+1}{n+1-s} \left[(-1)^{k+1} \right]^{n+1-s} F_{(2s-1)k}$$

Proof. Using the Binet form,

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} \ ,$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}$$

and

$$\beta = \frac{1 - \sqrt{5}}{2} \ ,$$

then

$$F_k^{2n+1} = \left[\frac{\alpha^k - \beta^k}{\alpha - \beta}\right]^{2n+1} = \sum_{s=0}^{2n+1} \binom{2n+1}{s} \frac{(\alpha^k)^{2n+1-s}(-\beta^k)^s}{(\alpha - \beta)^{2n+1}}$$

$$= \left\{\sum_{s=0}^{n} + \sum_{s=n}^{2n+1}\right\} \binom{2n+1}{s} \frac{(\alpha^k)^{2n+1-s}(-\beta^k)^s}{(\alpha - \beta)^{2n+1}} \qquad .$$

Noting that

$$\binom{2n+1}{s} = \binom{2n+1}{2n+1-s}$$

and substituting $t = 2n + 1 - s$ in the second of the summations from $s = n$ to $s = 2n + 1$, then

$$F_k^{2n+1} = \sum_{s=0}^{n} \frac{\binom{2n+1}{s}(\alpha\beta)^{ks}(-1)^s}{(\alpha - \beta)^{2n}} \left[\frac{(\alpha^k)^{2n+1-2s} - (\beta^k)^{2n+1-2s}}{(\alpha - \beta)}\right] .$$

Substituting $(\alpha - \beta) = \sqrt{5}$ and $\alpha\beta = -1$, noting the Binet form in the last expression of each term of the summation, and solving for the term in which $s = 0$, the theorem can be obtained.

Examples. For $n = 1$,

$$F_{3k} = 5F_k^3 + (-1)^k 3F_k .$$

Hence, the polynomial $P(x) = 5x^2 + 3x$ satisfies $P(F_{2k}) = F_{6k}$, and the polynomial $P(x) = 5x^3 - 3x$ satisfies $P(F_{2k+1}) = F_{6k+3}$. To determine F_{5k} in terms of F_k by the theorem above, it is necessary to substitute the F_{3k} expression above into the theorem with $n = 2$; one obtains

$$F_{5k} = 25F_k^5 + 25(-1)^k F_k^3 + 5F_k ,$$

so that the polynomials

$$P_1(x) = 25x^5 + 25x^3 + 5x$$

and

$$P_2(x) = 25x^5 - 25x^3 + 5x$$

satisfy

$$P_1(F_{2k}) = F_{10k}$$

and

$$P_2(F_{2k+1}) = F_{10k+5} .$$

Similarly, one may obtain

$$F_{7k} = 5^3 F_k^7 + 7 \cdot 5^2 (-1)^k F_k^5 + 70 F_k^3 + 7(-1)^k F_k$$

with polynomials

$$P_3(x) = 125 x^7 + 175 x^5 + 70 x^3 + 7x$$

or

$$P_4(x) = 125 x^7 - 175 x^5 + 70 x^3 - 7x ,$$

where

$$P_3(F_{2k}) = F_{14k}$$

and

$$P_4(F_{2k+1}) = F_{14k+7} .$$

An interesting congruence result may be obtained by taking $2n + 1 =$ odd prime p in the theorem.

$$F_{pk} \equiv 5^{\frac{p-1}{2}} F_k \pmod{p} ,$$

for $(p, 5) = 1$. Here,

$$5^{\frac{p-1}{2}} \equiv \left(\frac{5}{p} \right) \pmod{p} ,$$

using the Legendre symbol, and hence

$$F_{p^s k} \equiv F_k \pmod{p}$$

or

$$F_{p^s k} \equiv (-1)^s F_k \pmod{p}$$

according as $p \equiv \pm 1 \pmod{5}$ or $p \equiv \pm 2 \pmod{5}$.

I would like to take this opportunity to express my sincere appreciation to V. C. Harris for many years of kindness, and years of encouragement and assistance in mathematics. See also [1].

REFERENCE

1. Ellen King, "Some Fibonacci Inverse Trigonometry," unpublished paper, San Jose State College Master's Thesis, 1969.

◆◇◆◇◆

ELEMENTARY PROBLEMS AND SOLUTIONS
Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

NOTATION: $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$;
$L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$.

PROPOSED PROBLEMS

B-220 Proposed by Guy A. R. Guillotte, Montreal, P. Q., Canada.

Let p_m be the mth prime. Prove that p_m and p_{m+1} are twin primes (i.e., $p_{m+1} = p_m + 2$) if and only if

$$\sum_{n=1}^{m} (p_{n+1} - p_n) = p_m .$$

B-221 Proposed by R. Garfield, College of Insurance, New York, N. Y.

Prove that

$$\sum_{n=1}^{\infty} (1/F_n L_n) = \sum_{n=1}^{\infty} (1/F_{2n}) .$$

B-222   Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Find a formula for K_n where $K_1 = 1$ and

$$K_{n+1} = (K_1 + K_2 + \cdots + K_n) + F_{2n+1} .$$

B-223   Proposed by Edgar Karst, University of Arizona, Tuscon, Arizona.

Find a solution of

$$x^y + (x + 3)^y - (x + 4)^y = u^v + (u + 3)^v - (u + 4)^v$$

in the form

$$x = F_m, \quad y = F_n, \quad u = L_v, \quad \text{and} \quad v = L_s .$$

B-224   Proposed by Lawrence Somer, Champaign, Illinois.

Let m be a fixed positive integer. Prove that no term in the sequence F_1, F_3, F_5, F_7, $\cdots$ is divisible by $4m - 1$.

B-225   Proposed by John Ivie, Berkeley, California.

Let a_0, $\cdots$, a_{j-1} be constants and let $\{f_n\}$ be a sequence of integers satisfying

$$f_{n+j} = a_{j-1} f_{n+j-1} + a_{j-2} f_{n+j-2} + \cdots + a_0 f_n; \quad n = 0, 1, 2, \cdots .$$

Find a necessary and sufficient condition for $\{f_n\}$ to have the property that every integer m is an exact divisor of some f_k.

SOLUTIONS
A SEQUENCE OF MULTIPLES OF 12

B-202   Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let F_1, F_2, $\cdots$ be the Fibonacci Sequence 1, 1, 2, 3, 5, 8, $\cdots$ with $F_{n+2} = F_{n+1} + F_n$. Let

$$G_n = F_{4n-2} + F_{4n} + F_{4n+2} \cdot$$

(i) Find a recursion formula for the sequence G_1, G_2, $\cdots$.

(ii) Show that each G_n is a multiple of 12.

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

(i) The sequence $\{G_n\}$ satisfies $G_{n+2} = 7G_{n+1} - G_n$ since each of the sequences $\{F_{4n-2}\}$, $\{F_{4n}\}$, and $\{F_{4n+2}\}$ has this recursion relation.

(ii) Since $G_0 = 0$ and $G_1 = 12$, mathematical induction using Part (i) proves that $12 | G_n$ for $n \geq 0$.

Also solved by T. E. Stanley, Gregory Wulczyn, and the Proposer.

A SEQUENCE OF MULTIPLES OF 168

B-203   Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Show that $F_{8n-4} + F_{8n} + F_{8n+4}$ is always a multiple of 168.

Solution by T. E. Stanley, City University, London, England.

The following generalizes on B-202 and B-203.

Let

$$E(n,k,r) = F_{kn-r} + F_{kn} + F_{kn+r} \cdot$$

The formulas

$$F_{kn+r} = F_{r-1}F_{kn} + F_r F_{kn+1}$$
$$F_{kn-r} = (-1)^r (F_{r-1} F_{kn} - F_r F_{kn-1})$$

are well known. Thus, if r is even, we have

$$E(n,k,r) = (F_r +)F_{r-1} + 1)F_{kn} \cdot$$

Now F_k divides F_{kn} for each n and so $E(n,k,r)$ is a multiple of

$$(F_r + 2F_{r-1} + 1)F_k$$

for even r. Then $E(n, 8, 4)$ is a multiple of $(3 + 4 + 1)21 = 168$, which establishes B-203.

Also solved by Gregory Wulczyn and the Proposer.

Editor's Note: Combining thoughts from the solutions of B-202 and B-203, one can show that $F_{kn-2s} + F_{kn} + F_{kn+2s}$ is a multiple of $(L_{2s}+1)F_{kn}$ for $n = 1, 2, 3, \cdots$.

GENERATING FUNCTION FOR F_{2n-1}

B-204   Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Show that

(i) $F_1 x + F_2 x^2 + F_5 x^3 + F_7 x^4 + \cdots = (x - x^2)/(1 - 3x + x^2)$ for $|x| < (3 - \sqrt{5})/2$.

(ii) $1 + 2x + 3x^2 + 4x^3 + \cdots = 1/(1 - x)^2$ for $|x| < 1$.

(iii) $nF_1 + (n - 1)F_3 + (n - 2)F_5 + \cdots + 2F_{3n-2} + F_{2n-1} = F_{2n+1} - 1$.

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

(i) Let

$$f(x) = (1 - x)/(1 - 3x + x^2)$$

and let its Maclaurin expansion be

(1) $$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots .$$

Then (1) converges for $|x| < |r|$, where r is the root of $1 - 3x + x^2 = 0$ of least absolute value, i.e., $r = (3 - \sqrt{5})/2$. Multiplying both sides of (1) by $1 - 3x + x^2$ gives us

(2) $$1 - x = (1 - 3x + x^2)c_0 + c_1 x + c_2 x^2 + \cdots) .$$

Expanding the right side of (2) and equating coefficients of x^m on both sides of (2), leads to

(3) $c_0 = 1,$ $c_1 = 2,$ $c_{n+2} - 3c_{n+1} + c_n = 0$ for $n \geq 0$.

This implies that $c_n = F_{2n+1}$ and Part (i) is proved.

(ii) This follows by term-by-term differentiation of

$$1 + x + x^2 + \cdots = 1/(1 - x), \quad |x| < 1.$$

(iii) Let $G_n = nF_1 + (n - 1)F_3 + 2F_{2n-2} + F_{2n-1}$. Then the generating function for the G_n is found by multiplying the series of Parts (i) and (ii) to be

$$1/[(1 - x)(1 - 3x + x^2)] = G_1 + G_2 x + G_3 x^2 + \cdots.$$

This implies that $G_1 = 1,$ $G_2 = 4,$ $G_3 = 12,$ and

(4) $G_{n+3} - 4G_{n+2} + 4G_{n+1} - G_n = 0$.

Since $F_{2n+1} - 1$ satisfies the same initial conditions and the same recurrence relation (4) as G_n, Part (iii) is established.

Also solved by the Proposer.

ANOTHER CONVOLUTION FOR F_{2n-1}

B-205   *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.*

Show that

$$(2n - 1)F_1 + (2n - 3)F_3 + (2n - 5)F_5 + \cdots + 3F_{2n-3} + F_{2n-1} = L_{2n} - 2.$$

where L_n is the nth Lucas number (i.e., $L_1 = 1,$ $L_2 = 3,$ $L_{n+2} = L_{n+1} + L_n$).

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

The solution is similar to that of B-204. Instead of Part (ii) of B-204, one uses

$$1 + 3x + 5x^2 + \cdots = (1 + x)/(1 - x)^2, \quad |x| < 1,$$

which may be obtained by differentiating term-by-term in

$$y + y^3 + y^5 + \cdots = y/(1 - y^2), \quad |x| < 1,$$

and then substituting $y^2 = x$.

A GEOMETRIC SERIES

B-206   Proposed by Guy A. Guillotte, Montreal, Quebec, Canada.

Let $a = (1 + \sqrt{5})/2$ and sum

$$\sum_{n=1}^{\infty} \frac{1}{aF_{n+1} + F_n} \quad .$$

Solution by C. B. A. Peck, State College, Pennsylvania.

From the Fibonacci Quarterly, Vol. 1, No. 3, p. 54,

$$a^{n+1} = aF_{n+1} + F_n .$$

Hence the sum is

$$(1/a^2)[1 - (1/a)] = 1/(a^2 - a) = 1,$$

since $a^2 - a - 1 = 0$.

Also solved by Gregory Wulczyn and the Proposer.

ANOTHER GEOMETRIC SERIES

B-207   Proposed by Guy A. Guillotte, Montreal, Quebec, Canada.

Sum

$$\sum_{n=1}^{\infty} \frac{1}{F_n + \sqrt{5}\, F_{n+1} + F_{n+2}}\quad .$$

Solution by C. B. A. Peck, State College, Pennsylvania.

The equation

$$F_n + \sqrt{5}\, F_{n+1} + F_{n+2} = L_{n+1} + \sqrt{5}\, F_{n+1} = 2a^{n+1}\ ,$$

along with B-206, show that the sum desired here is 1/2.

Also solved by Gregory Wulczyn and the Proposer.

◆◇◆◇◆

FIBONACCI NOTE SERVICE

The Fibonacci Quarterly is offering a service in which it will be possible for its readers to secure background notes for articles. This will apply to the following:

(1) Short abstracts of extensive results, derivations, and numerical data.
(2) Brief articles summarizing a large amount of research.
(3) Articles of standard size for which additional background material may be obtained.

Articles in the Quarterly for which this note service is available will indicate the fact together with the number of pages in question. Requests for these notes should be made to:

Brother Alfred Brousseau
St. Mary's College
Moraga, Calif. 94575

The notes will be Xeroxed.

The price for this service is four cents a page (including postage, materials, and labor).

VOLUME INDEX

GOULD, H. W. "Equal Products of Generalized Binomial Coefficients,"
Vol. 9, No. 4, pp. 337-346.

GRASSL, RICHARD M. Problems Proposed: B-202, Vol. 9, No. 1, p. 106;
B-203, Vol. 9, No. 1, p. 106; B-214, Vol. 9, No. 4, p. 438; B-196,
Vol. 9, No. 4, p. 440; B-202, Vol. 9, No. 5, p. 545; B-203, Vol. 9,
No. 5, p. 546. Problems Solved: B-196, Vol. 9, No. 4, p. 441; B-
202, Vol. 9, No. 5, p. 546; B-203, Vol. 9, No. 5, p. 547.

GRONEMEIJER, CARL. Problem Proposed: B-187, Vol. 9, No. 1, p. 110.
Problem Solved: B-187, Vol. 9, No. 1, p. 111.

GIULI, ROBERT. Problems Solved: H-166, Vol. 9, No. 4, p. 397; H-169,
Vol. 9, No. 5, p. 514.

GUILLOTTE, GUY A. Problems Proposed: B-206, Vol. 9, No. 1, p. 107;
B-207, Vol. 9, No. 1, p. 107; B-210, Vol. 9, No. 2, p. 217; B-191,
Vol. 9, No. 2, p. 219; B-218, Vol. 9, No. 4, p. 439; B-220, Vol. 9,
No. 5, p. 544; B-206, Vol. 9, No. 5, p. 549; B-207, Vol. 9, No. 5,
p. 550. Problems Solved: B-191, Vol. 9, No. 2, p. 220; B-206, Vol.
9, No. 5, p. 549; B-207, Vol. 9, No. 5, p. 550.

GUPTA, A. K. Problems Solved: B-184, Vol. 9, No. 1, p. 108; B-195, Vol.
9, No. 2, p. 224.

HANSELL, WALTER. "The Hidden Hexagon Squares," Vol. 9, No. 2, p.
120 (co-author, V. E. Hoggatt, Jr.).

HERDY, M. Problem Solved: H-173, Vol. 9, No. 5, p. 521.

HILLMAN, A. P. "A Symmetric Substitute for Stirling Numbers," Vol. 9,
No. 1, pp. 51-60 (co-authors, P. L. Mana and C. T. McAbee). Editor
of Elementary Problems and Solutions: Vol. 9. No. 1, pp. 106-112;
Vol. 9, No. 2, pp. 217-224; Vol. 9, No. 4, pp. 438-448; Vol. 9, No.
5, pp. 544-550.

HITCHCOCK, JOHN A. Problems Solved: B-190, Vol. 9, No. 2, p. 219;
B-192, Vol. 9, No. 2, p. 221.

HOBBS, STU. Problem Solved: B-192, Vol. 9, No. 2, p. 221.

HOGGATT, V. E., JR. "Twenty-Four Master Identities," Vol. 9, No. 1,
pp. 1-17 (co-authors, John W. Phillips and H. T. Leonard). "The
Hidden Hexagon Squares," Vol. 9, No. 2, pp. 120, 133 (co-author,
Walter Hansell). "Some Special Fibonacci and Lucas Generating Func-
tions," Vol. 9, No. 2, pp. 121-133. "A Property of Multinomial Co-
efficients," Vol. 9, No. 4, pp. 351-356 (co-author, G. L. Alexander-
son). "Some More Fibonacci Diophantine Equations," Vol. 9, No. 4,
p. 437. "A Primer for the Fibonacci Numbers: Part IX," Vol. 9, No.
5, pp. 529-536 (co-author, Marjorie Bicknell). Problems Proposed:
B-204, Vol. 9, No. 1, p. 106; B-205, Vol. 9, No. 1, p. 107; B-208,
Vol. 9, No. 2, p. 217; B-209, Vol. 9, No. 2, p. 217; B-211, Vol. 9,
No. 2, p. 218; B-193, Vol. 9, No. 2, p. 221; H-183, Vol. 9, No. 4, p.
389; B-216, Vol. 9, No. 4, p. 439; B-222, Vol. 9, No. 5, p. 545; B-
204, Vol. 9, No. 5, p. 547; B-205, Vol. 9, No. 5, p. 548. Problems
Solved: B-193, Vol. 9, No. 2, p. 223; B-204, Vol. 9, No. 5, p. 548.

HORADAM, A. F. "Pell Identities," Vol. 9, No. 3, pp. 245-252. "Some Properties of Certain Generalized Fibonacci Matrices," Vol. 9, No. 3, pp. 264-276 (co-author, J. E. Walton). "Reciprocals of Generalized Fibonacci Numbers," Vol. 9, No. 3, pp. 299-306 (co-author, A. G. Shannon). "A Generalized Pythagorean Theorem," Vol. 9, No. 3, pp. 307-312 (co-author, A. G. Shannon).

HUNTER, J. A. H. Problems Solved: B-187, Vol. 9, No. 1, p. 111; B-189, Vol. 9, No. 1, p. 112; B-191, Vol. 9, No. 2, p. 220; H-173, Vol. 9, No. 5, p. 521.

IVIE, JOHN. Problem Proposed: B-225, Vol. 9, No. 5, p. 545.

JAISWAL, D. V. Problems Solved: H-157, Vol. 9, No. 1, p. 67; H-158, Vol. 9, No. 1, p. 69; H-163, Vol. 9, No. 2, p. 145; H-164, Vol. 9, No. 4, p. 396.

JENTSCH, W. "On an Initial Value Problem for Linear Partial Difference Equations," Vol. 9, No. 3, pp. 313-323.

KARST, EDGAR. Problem Proposed: B-223, Vol. 9, No. 5, p. 545. Problems Solved: B-185, Vol. 9, No. 1, p. 109; B-190, Vol. 9, No. 2, p. 219; B-191, Vol. 9, No. 2, p. 220; B-192, Vol. 9, No. 2, p. 221.

KEGEL, JOHN. Problems Solved: B-184, Vol. 9, No. 1, p. 108; B-186, Vol. 9, No. 1, p. 110; B-192, Vol. 9, No. 2, p. 221; B-193, Vol. 9, No. 2, p. 221; B-194, Vol. 9, No. 2, p. 223 (corrected); B-195, Vol. 9, No. 2, p. 224.

KENNEY, CHARLES. Problems Solved: B-185, Vol. 9, No. 1, p. 109; B-187, Vol. 9, No. 1, p. 111.

KING, BRUCE W. Problem Proposed: B-184, Vol. 9, No. 1, p. 107. Problems Solved: H-160, Vol. 9, No. 1, p. 72; B-184, Vol. 9, No. 1, p. 108.

KING, SCOTT. Problem Solved: B-192, Vol. 9, No. 2, p. 221.

KLAMKIN, MURRAY S. Problem Proposed: H-164, Vol. 9, No. 4, p. 390. Problems Solved: H-164, Vol. 9, No. 4, p. 393; B-196, Vol. 9, No. 4, p. 441; B-197, Vol. 9, No. 4, p. 442; B-198, Vol. 9, No. 4, p. 443.

KLARNER, DAVID S. Problems Proposed: H-162, Vol. 9, No. 1, p. 72; H-161, Vol. 9, No. 2, p. 137; H-168, Vol. 9, No. 5, p. 513.

KOHNHAUSER, JOSEPH. Problem Solved: B-188, Vol. 9, No. 1, p. 111.

KRISHNAR, S. Problem Proposed: H-182, Vol. 9, No. 2, p. 134.

LA BARBERA, SAL. "Fibonacci, Lucas and the Egyptians," Vol. 9, No. 2, pp. 177-187.

LAFER, PHIL. "Discovering the Square-triangular Numbers," Vol. 9, No. 1, pp. 93-105.

LAW, A. G. Problem Solved: H-157, Vol. 9, No. 1, p. 65.

LEDIN, GEORGE, JR. Problem Proposed: H-173, Vol. 9, No. 5, p. 520. Problem Solved: H-173, Vol. 9, No. 5, p. 521.

LEHMER, D. Problem Proposed: H-160, Vol. 9, No. 1, p. 71. Problem Solved: H-160, Vol. 9, No. 1, p. 72.

LEHMER, E. Problem Proposed: H-160, Vol. 9, No. 1, p. 71. Problem Solved: H-160, Vol. 9, No. 1, p. 72.

LEONARD, H. T. "Twenty-Four Master Identities," Vol. 9, No. 1, pp. 1-17 (co-authors, V. E. Hoggatt, Jr., and John W. Phillips).

LEVINE, E. "On the Coefficients of a Generating Series," Vol. 9, No. 5, pp. 467-476 (co-authors, M. Beresin and D. Lubell).

LIND, DOUGLAS A. "A Determinant Involving Generalized Binomial Co-efficients," Vol. 9, No. 2, pp. 113-119. Problem Proposed: H-171, Vol. 9, No. 5, p. 515. Problems Solved: H-171, Vol. 9, No. 5, p. 515; H-172, Vol. 9, No. 5, p. 519.

LORD, GRAHAM. Problem Solved: B-198, Vol. 9, No. 4, p. 444.

LOWMAN, EDWARD A. "An Example of Fibonacci Numbers Used to Generate Rhythmic Values in Modern Music," Vol. 9, No. 4, pp. 423-426. "Some Striking Proportions in the Music of Bela Bartok," Vol. 9, No. 5, pp. 527-528.

LUBELL, D. "On the Coefficients of a Generating Series," Vol. 9, No. 5, pp. 567-576 (co-authors, E. Levine and M. Beresin).

MC ABEE, C. T. "A Symmetric Substitute for Stirling Numbers," Vol. 9, No. 1, pp. 51-60 (co-authors, P. L. Mana and A. P. Hillman).

MADACHY, JOSEPH S. Editor of: "Recreational Mathematics, Asymptotic Euclidean Type Constructions Without Euclidean Tools," Vol. 9, No. 2, pp. 199-216 (author, Jean J. Pedersen).

MANA, PHIL. "A Symmetric Substitute for Stirling Numbers," Vol. 9, No. 1, pp. 51-60 (co-authors, A. P. Hillman and C. T. McAbee). Problems Proposed: B-189, Vol. 9, No. 1, p. 111; B-194, Vol. 9, No. 2, p. 223; B-215, Vol. 9, No. 4, p. 438; B-197, Vol. 9, No. 4, p. 441; B-198, Vol. 9, No. 4, p. 442. Problems Solved: B-189, Vol. 9, No. 1, p. 112; B-190, Vol. 9, No. 2, p. 218; B-194, Vol. 9, No. 2, p. 223; B-197, Vol. 9, No. 4, p. 442; B-198, Vol. 9, No. 4, p. 444; B-200, Vol. 9, No. 4, p. 446; B-202, Vol. 9, No. 5, p. 546; B-204, Vol. 9, No. 5, p. 547; B-205, Vol. 9, No. 5, p. 549.

MANGERON, D. "On the Generation of Fibonacci Numbers and the 'Poly-vibrating' Extension of These Numbers," Vol. 9, No. 3, pp. 324-328 (co-authors, M. N. Oguztorelli and V. E. Poterasu).

MILSOM, JOHN W. Problems Solved: B-184, Vol. 9, No. 1, p. 108; B-185, Vol. 9, No. 1, p. 109; B-187, Vol. 9, No. 1, p. 111; H-163, Vol. 9, No. 2, p. 145; B-191, Vol. 9, No. 2, p. 220; B-192, Vol. 9, No. 2, p. 221; B-195, Vol. 9, No. 2, p. 224.

MOST, MEL. Problem Proposed: B-201, Vol. 9, No. 4, p. 446. Problem Solved: B-201, Vol. 9, No. 4, p. 448.

MUNZENRIDER, JOSEPH P. "A New Anthesis," Vol. 9, No. 2, pp. 163-176.

NEWMAN, HENRY. Problems Solved: B-184, Vol. 9, No. 1, p. 108; B-186, Vol. 9, No. 1, p. 110; B-189, Vol. 9, No. 1, p. 112.

OGUZTORELLI, M. N. "On the Generation of Fibonacci Numbers and the 'Polyvibrating' Extension of These Numbers," Vol. 9, No. 3, pp. 324-328 (co-authors, D. Mangeron and V. E. Poterasu).

PARKER, F. D. Problems Solved: B-201, Vol. 9, No. 4, p. 447; H-168, Vol. 9, No. 5, p. 513.

PECK, C. B. A. Problems Solved: H-162, Vol. 9, No. 1, p. 72; B-184, Vol. 9, No. 1, p. 108; B-185, Vol. 9, No. 1, p. 109; B-186, Vol. 9, No. 1, p. 110; B-187, Vol. 9, No. 1, p. 111; B-188, Vol. 9, No. 1, p. 111; B-189, Vol. 9, No. 1, p. 112; H-163, Vol. 9, No. 2, p. 145; B-191, Vol. 9, No. 2, p. 220; B-192, Vol. 9, No. 2, p. 221; B-194, Vol. 9, No. 2, p. 223; H-164, Vol. 9, No. 4, p. 392; H-165, Vol. 9, No. 4, p. 396; H-168, Vol. 9, No. 5, p. 514; H-173, Vol. 9, No. 5, p. 521; B-206, Vol. 9, No. 5, p. 549; B-207, Vol. 9, No. 5, p. 550.

PEDERSON, JEAN J. "Recreational Mathematics, Asymptotic Euclidean Type Constructions without Euclidean Tools," Vol. 9, No. 2, pp. 199-216 (edited by Joseph S. Madachy).

PERRY, DARLA. Problems Solved: B-191, Vol. 9, No. 2, p. 220; B-192, Vol. 9, No. 2, p. 221.

PHILLIPS, JOHN W. "Twenty-Four Master Identities," Vol. 9, No. 1, pp. 1-17 (co-authors, V. E. Hoggatt, Jr., and H. T. Leonard).

POTERASU, V. E. "On the Generation of Fibonacci Numbers and the 'Polyvibrating' Extension of These Numbers," Vol. 9, No. 3, pp. 324-328 (co-authors, D. Mangeron and M. N. Oguztorelli).

PRIMROSE, E. J. F. "Relations Between a Sequence of Fibonacci Type and the Sequence of its Partial Sums," Vol. 9, No. 3, pp. 296-298 (co-author, Max Rumney).

RATCHFORD, M. Problem Solved: H-163, Vol. 9, No. 2, p. 145.

ROSENFELD, AZRIEL. Problem Solved: B-191, Vol. 9, No. 2, p. 220.

WALTON, J. E. "Some Properties of Certain Generalized Fibonacci Matrices," Vol. 9, No. 3, pp. 264-276 (co-author, A. F. Horadam).

WESSNER, JOHN. Problems Solved: B-185, Vol. 9, No. 1, p. 109; B-186, Vol. 9, No. 1, p. 109; B-189, Vol. 9, No. 1, p. 112.

WHITNEY, RAYMOND E. Editor of "Advanced Problems and Solutions," Vol. 9, No. 1, pp. 61-73; Vol. 9, No. 2, pp. 134-145; Vol. 9, No. 4, pp. 389-401; Vol. 9, No. 5, pp. 512-525. Problems Proposed: H-184, Vol. 9, No. 4, p. 389; H-188, Vol. 9, No. 5, p. 513.

WLODARSKI, J. "Fibonacci and Lucas Numbers Tend to Obey Benford's Law," Vol. 9, No. 1, pp. 87-88. "Regular Polyhedrons and Pascal's Triangle," Vol. 9, No. 2, p. 146. "A Number Problem," Vol. 9, No. 2, p. 195. "The Possible End of the Periodic Table of Elements and the 'Golden Ratio'," Vol. 9, No. 1, p. 82. "The Golden Ratio in an Electrical Network," Vol. 9, No. 2, p. 188.

WULCZYN, GREGORY. Problems Solved: B-202, Vol. 9, No. 5, p. 546; B-203, Vol. 9, No. 5, p. 547; B-206, Vol. 9, No. 5, p. 549; B-207, Vol. 9, No. 5, p. 550.

WYATT, ANDREW. Problems Solved: B-190, Vol. 9, No. 2, p. 219; B-192, Vol. 9, No. 2, p. 221.

YODER, M. Problems Solved: H-157, Vol. 9, No. 1, p. 67; H-158, Vol. 9, No. 1, p. 69; H-159, Vol. 9, No. 1, p. 70; H-160, Vol. 9, No. 1, p. 72; H-163, Vol. 9, No. 2, p. 145; H-164, Vol. 9, No. 4, p. 392; H-165, Vol. 9, No. 4, p. 396; H-166, Vol. 9, No. 4, p. 397; H-167, Vol. 9, No. 4, p. 401; H-168, Vol. 9, No. 5, p. 514.

ZEITLIN, D. "General Identities for Recurrent Sequences of Order Two," Vol. 9, No. 4, pp. 357-388. Problem Proposed: B-195, Vol. 9, No. 2, p. 223. Problems Solved: H-157, Vol. 9, No. 1, p. 67; H-160, Vol. 9, No. 1, p. 72; B-185, Vol. 9, No. 1, p. 109; B-187, Vol. 9, No. 1, p. 111; B-189, Vol. 9, No. 1, p. 112; B-191, Vol. 9, No. 2, p. 219; B-192, Vol. 9, No. 2, p. 221; B-193, Vol. 9, No. 2, p. 223; B-194, Vol. 9, No. 2, p. 223; B-195, Vol. 9, No. 2, p. 224; B-198, Vol. 9, No. 4, p. 444; B-199, Vol. 9, No. 4, p. 444.

ZUPARKO, REV. ROBERT. Problems Solved: B-190, Vol. 9, No. 2, p. 219; B-192, Vol. 9, No. 2, p. 221.