HOW TO SUM THE SQUARES OF THE TETRANACCI NUMBERS AND THE FIBONACCI m-STEP NUMBERS

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Abstract. In this paper, we present explicit formulas for the sum of the first n Tetranacci numbers and for the sum of the squares of the first n Tetranacci numbers. In the second half of the paper, we will prove an explicit formula for the sum of the first n Fibonacci m-step numbers. This formula will give us explicit expressions for the sum of the first n Pentanacci numbers, the first n Hexanacci numbers, the first n Heptanacci numbers, and so on.

1. Introduction

The two explicit expressions for the sum of the first n Fibonacci numbers and for the sum of the squares of the first n Fibonacci numbers given by

$$\sum_{k=1}^{n} F_k = F_{n+1} + F_n - 1 = F_{n+2} - 1 \quad \text{and} \quad \sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$$

are well-known [2, 4].

In the past 10 years, several explicit closed form expressions for sums of Tribonacci numbers, such as

$$\sum_{k=1}^{n} T_k = \frac{1}{2}(T_{n+2} + T_n - 1) \quad \text{and} \quad \sum_{k=1}^{n} T_k^2 = \frac{4T_n T_{n+1} - (T_{n+1} - T_{n-1})^2 + 1}{4}$$

were discovered and published [3, 1]. The study of these papers has led us to find the equivalents for the Tetranacci numbers.

The sequence of Tetranacci numbers $\{\mathrm{Te}_k\}_{k=-\infty}^{\infty}$ is defined by the initial conditions $\mathrm{Te}_0 = 0$, $\mathrm{Te}_1 = \mathrm{Te}_2 = 1$, $\mathrm{Te}_3 = 2$ and the Tetranacci recurrence relation [8, 6]

$$\mathrm{Te}_{k+4} = \mathrm{Te}_k + \mathrm{Te}_{k+1} + \mathrm{Te}_{k+2} + \mathrm{Te}_{k+3} \quad \text{for all integers } k.$$

The Tetranacci numbers are therefore higher analogues of the Fibonacci numbers $\{F_k\}_{k=-\infty}^{\infty}$ and the Tribonacci numbers $\{T_k\}_{k=-\infty}^{\infty}$. The initial conditions for the Tetranacci numbers are not uniform in the literature and we have chosen them in the above way because then its generating function takes the simplest form. Moreover, this choice is similar to that in [1] for the Tribonacci numbers.

In this paper, we give the corresponding explicit formulas for the sum $\sum_{k=1}^{n} \mathrm{Te}_k$ of the first n Tetranacci numbers and for the sum $\sum_{k=1}^{n} \mathrm{Te}_k^2$ of the squares of the first n Tetranacci numbers. In the second half of the paper, we will present an explicit formula for the sum $\sum_{k=1}^{n} F_k^{(m)}$ of the first n Fibonacci m-step numbers. To prove these three formulas, we will use induction on n.

We have searched our formulas in the literature and on the internet, but we could find none of them. We believe that these formulas are original.

2. The Explicit Formula for the Sum of the First n Tetranacci Numbers

In this section, we prove

Theorem 2.1. *(Explicit formula for the sum of the first n Tetranacci numbers) For all nonnegative integers n,*

$$\sum_{k=1}^{n} \mathrm{Te}_k = \frac{1}{3}\left(\mathrm{Te}_{n+3} - \mathrm{Te}_{n+1} + \mathrm{Te}_n - 1\right).$$

Proof. We will prove this formula by induction.
Because we have $\mathrm{Te}_0 = 0$, $\mathrm{Te}_1 = \mathrm{Te}_2 = 1$, $\mathrm{Te}_3 = 2$, and $\mathrm{Te}_4 = 4$, the above formula holds for $n = 0$ and $n = 1$ by the calculations

$$0 = \sum_{k=1}^{0} \mathrm{Te}_k = \frac{1}{3}\left(\mathrm{Te}_3 - \mathrm{Te}_1 + \mathrm{Te}_0 - 1\right) = \frac{1}{3}(2 - 1 + 0 - 1) = 0,$$
$$1 = \mathrm{Te}_1 = \sum_{k=1}^{1} \mathrm{Te}_k = \frac{1}{3}\left(\mathrm{Te}_4 - \mathrm{Te}_2 + \mathrm{Te}_1 - 1\right) = \frac{1}{3}(4 - 1 + 1 - 1) = 1.$$

Assuming that the formula holds for a nonnegative integer n, we will show that it also holds for $n + 1$.
Because of the above Tetranacci recurrence relation, we have

$$\begin{aligned}
&\mathrm{Te}_{n+4} = \mathrm{Te}_n + \mathrm{Te}_{n+1} + \mathrm{Te}_{n+2} + \mathrm{Te}_{n+3}\\
\iff\ &\mathrm{Te}_{n+1} = \mathrm{Te}_{n+4} - \mathrm{Te}_{n+3} - \mathrm{Te}_{n+2} - \mathrm{Te}_n\\
\iff\ &3\,\mathrm{Te}_{n+1} = \mathrm{Te}_{n+4} - \mathrm{Te}_{n+3} - \mathrm{Te}_{n+2} + 2\,\mathrm{Te}_{n+1} - \mathrm{Te}_n\\
\iff\ &\mathrm{Te}_{n+1} = \frac{1}{3}\left(\mathrm{Te}_{n+4} - \mathrm{Te}_{n+3} - \mathrm{Te}_{n+2} + 2\,\mathrm{Te}_{n+1} - \mathrm{Te}_n\right),
\end{aligned}$$

which implies

$$\begin{aligned}
\sum_{k=1}^{n+1} \mathrm{Te}_k &= \sum_{k=1}^{n} \mathrm{Te}_k + \mathrm{Te}_{n+1}\\
&= \frac{1}{3}\left(\mathrm{Te}_{n+3} - \mathrm{Te}_{n+1} + \mathrm{Te}_n - 1\right) + \frac{1}{3}\left(\mathrm{Te}_{n+4} - \mathrm{Te}_{n+3} - \mathrm{Te}_{n+2} + 2\,\mathrm{Te}_{n+1} - \mathrm{Te}_n\right)\\
&= \frac{1}{3}\left(\mathrm{Te}_{n+4} - \mathrm{Te}_{n+2} + \mathrm{Te}_{n+1} - 1\right).
\end{aligned}$$

These calculations prove the claimed formula is also true for $n + 1$. □

The above formula is equivalent to the formula for $\sum_{k=1}^{n} \mathrm{Te}_k$ given in [8]. We have presented it in this form because the formula for all Fibonacci m-step numbers given below suggests this is its most natural form.

3. The Explicit Formula for the Sum of the Squares of the First n Tetranacci Numbers

In this section, we prove

Theorem 3.1. *(Explicit formula for the sum of the squares of the first n Tetranacci numbers) For all nonnegative integers n,*

$$\sum_{k=1}^{n} \mathrm{Te}_k^2 = \frac{3\,\mathrm{Te}_n\,\mathrm{Te}_{n+1} - (\mathrm{Te}_{n+1} - \mathrm{Te}_{n-1})^2 + (\mathrm{Te}_n + \mathrm{Te}_{n-3})\,\mathrm{Te}_{n-2} + 1}{3}.$$

Proof. We will prove this formula also by induction. From the recurrence relation of the Tetranacci numbers, we have that $\mathrm{Te}_{-3} = 1$, $\mathrm{Te}_{-2} = 0$, $\mathrm{Te}_{-1} = 0$, $\mathrm{Te}_0 = 0$, $\mathrm{Te}_1 = \mathrm{Te}_2 = 1$, and $\mathrm{Te}_3 = 2$. The above formula is satisfied for $n = 0$ and $n = 1$ because

$$0 = \sum_{k=1}^{0} \mathrm{Te}_k^2 = \frac{3\,\mathrm{Te}_0\,\mathrm{Te}_1 - (\mathrm{Te}_1 - \mathrm{Te}_{-1})^2 + (\mathrm{Te}_0 + \mathrm{Te}_{-3})\,\mathrm{Te}_{-2} + 1}{3} = \frac{3 \cdot 0 \cdot 1 - 1^2 + 1 \cdot 0 + 1}{3} = 0,$$

$$1 = \mathrm{Te}_1^2 = \sum_{k=1}^{1} \mathrm{Te}_k^2 = \frac{3\,\mathrm{Te}_1\,\mathrm{Te}_2 - (\mathrm{Te}_2 - \mathrm{Te}_0)^2 + (\mathrm{Te}_1 + \mathrm{Te}_{-2})\,\mathrm{Te}_{-1} + 1}{3}$$
$$= \frac{3 \cdot 1 \cdot 1 - 1^2 + 1 \cdot 0 + 1}{3} = 1.$$

We assume that the above formula is true for a positive integer n and show that the correctness of the formula for n implies the correctness of the formula for $n+1$.
We have the following identity, namely

$$\mathrm{Te}_{n+1}^2 = \frac{3\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n+2} - (\mathrm{Te}_{n+2} - \mathrm{Te}_n)^2 + (\mathrm{Te}_{n+1} + \mathrm{Te}_{n-2})\,\mathrm{Te}_{n-1} + 1}{3}$$
$$- \frac{3\,\mathrm{Te}_n\,\mathrm{Te}_{n+1} - (\mathrm{Te}_{n+1} - \mathrm{Te}_{n-1})^2 + (\mathrm{Te}_n + \mathrm{Te}_{n-3})\,\mathrm{Te}_{n-2} + 1}{3},$$

which is true, because using the Tetranacci recurrence relation several times, the above identity is equivalent to

$$\begin{aligned}
3\,\mathrm{Te}_{n+1}^2 &= 3\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n+2} - (\mathrm{Te}_{n+2} - \mathrm{Te}_n)^2 + (\mathrm{Te}_{n+1} + \mathrm{Te}_{n-2})\,\mathrm{Te}_{n-1} + 1 \\
&\quad - \left[3\,\mathrm{Te}_n\,\mathrm{Te}_{n+1} - (\mathrm{Te}_{n+1} - \mathrm{Te}_{n-1})^2 + (\mathrm{Te}_n + \mathrm{Te}_{n-3})\,\mathrm{Te}_{n-2} + 1\right] \\
&= 3\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n+2} - (\mathrm{Te}_{n+2} - \mathrm{Te}_n)^2 + (\mathrm{Te}_{n+1} + \mathrm{Te}_{n-2})\,\mathrm{Te}_{n-1} + 1 - 3\,\mathrm{Te}_n\,\mathrm{Te}_{n+1} \\
&\quad + (\mathrm{Te}_{n+1} - \mathrm{Te}_{n-1})^2 - (\mathrm{Te}_n + \mathrm{Te}_{n-3})\,\mathrm{Te}_{n-2} - 1 \\
&= (3\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n+2} - 3\,\mathrm{Te}_n\,\mathrm{Te}_{n+1}) - (\mathrm{Te}_{n+2} - \mathrm{Te}_n)^2 + \mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n-2}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n+1}^2 \\
&\quad - 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n-1}^2 - \mathrm{Te}_n\,\mathrm{Te}_{n-2} - \mathrm{Te}_{n-3}\,\mathrm{Te}_{n-2} \\
&= 3\,\mathrm{Te}_{n+1}(\mathrm{Te}_{n+2} - \mathrm{Te}_n) - (\mathrm{Te}_{n+2} - \mathrm{Te}_n)^2 + \mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n-2}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n+1}^2 \\
&\quad - 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n-1}^2 - \mathrm{Te}_n\,\mathrm{Te}_{n-2} - \mathrm{Te}_{n-3}\,\mathrm{Te}_{n-2} \\
&= 3\,\mathrm{Te}_{n+1}(\mathrm{Te}_{n+1} + \mathrm{Te}_{n-1} + \mathrm{Te}_{n-2}) - (\mathrm{Te}_{n+2} - \mathrm{Te}_n)^2 + \mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n-2}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n+1}^2 \\
&\quad - 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n-1}^2 - \mathrm{Te}_n\,\mathrm{Te}_{n-2} - \mathrm{Te}_{n-3}\,\mathrm{Te}_{n-2} \\
&= 3\,\mathrm{Te}_{n+1}^2 + 3\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + 3\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-2} - (\mathrm{Te}_{n+2} - \mathrm{Te}_n)^2 + \mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n-2}\,\mathrm{Te}_{n-1} \\
&\quad + \mathrm{Te}_{n+1}^2 - 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n-1}^2 - \mathrm{Te}_n\,\mathrm{Te}_{n-2} - \mathrm{Te}_{n-3}\,\mathrm{Te}_{n-2}
\end{aligned}$$

$$
\begin{aligned}
&= 4\,\mathrm{Te}_{n+1}^2 + 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + 3\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-2} - (\mathrm{Te}_{n+2} - \mathrm{Te}_n)^2 + \mathrm{Te}_{n-2}\,\mathrm{Te}_{n-1} + \mathrm{Te}_{n-1}^2\\
&\quad - \mathrm{Te}_n\,\mathrm{Te}_{n-2} - \mathrm{Te}_{n-3}\,\mathrm{Te}_{n-2}\\
&= 4\,\mathrm{Te}_{n+1}^2 + 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-2} - (\mathrm{Te}_{n+2} - \mathrm{Te}_n)^2 + (\mathrm{Te}_{n+1} - \mathrm{Te}_n - \mathrm{Te}_{n-3})\,\mathrm{Te}_{n-2}\\
&\quad + \mathrm{Te}_{n-1}\,\mathrm{Te}_{n-2} + \mathrm{Te}_{n-1}^2\\
&= 4\,\mathrm{Te}_{n+1}^2 + 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-2} - (\mathrm{Te}_{n+1} + \mathrm{Te}_{n-1} + \mathrm{Te}_{n-2})^2\\
&\quad + (\mathrm{Te}_{n-1} + \mathrm{Te}_{n-2})\,\mathrm{Te}_{n-2} + \mathrm{Te}_{n-1}\,\mathrm{Te}_{n-2} + \mathrm{Te}_{n-1}^2\\
&= 4\,\mathrm{Te}_{n+1}^2 + 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1} + 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-2} - \mathrm{Te}_{n+1}^2 - \mathrm{Te}_{n-1}^2 - \mathrm{Te}_{n-2}^2 - 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-1}\\
&\quad - 2\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n-2} - 2\,\mathrm{Te}_{n-1}\,\mathrm{Te}_{n-2} + \mathrm{Te}_{n-1}\,\mathrm{Te}_{n-2} + \mathrm{Te}_{n-2}^2 + \mathrm{Te}_{n-1}\,\mathrm{Te}_{n-2} + \mathrm{Te}_{n-1}^2\\
&= 3\,\mathrm{Te}_{n+1}^2 .
\end{aligned}
$$

This implies

$$
\begin{aligned}
\sum_{k=1}^{n+1} \mathrm{Te}_k^2 &= \sum_{k=1}^{n} \mathrm{Te}_k^2 + \mathrm{Te}_{n+1}^2\\
&= \frac{3\,\mathrm{Te}_n\,\mathrm{Te}_{n+1} - (\mathrm{Te}_{n+1} - \mathrm{Te}_{n-1})^2 + (\mathrm{Te}_n + \mathrm{Te}_{n-3})\,\mathrm{Te}_{n-2} + 1}{3}\\
&\quad + \left(\frac{3\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n+2} - (\mathrm{Te}_{n+2} - \mathrm{Te}_n)^2 + (\mathrm{Te}_{n+1} + \mathrm{Te}_{n-2})\,\mathrm{Te}_{n-1} + 1}{3} \right.\\
&\quad \left. - \frac{3\,\mathrm{Te}_n\,\mathrm{Te}_{n+1} - (\mathrm{Te}_{n+1} - \mathrm{Te}_{n-1})^2 + (\mathrm{Te}_n + \mathrm{Te}_{n-3})\,\mathrm{Te}_{n-2} + 1}{3} \right)\\
&= \frac{3\,\mathrm{Te}_{n+1}\,\mathrm{Te}_{n+2} - (\mathrm{Te}_{n+2} - \mathrm{Te}_n)^2 + (\mathrm{Te}_{n+1} + \mathrm{Te}_{n-2})\,\mathrm{Te}_{n-1} + 1}{3}
\end{aligned}
$$

and finishes the proof. □

To see how the above formula works, we compute the sum $\sum_{k=1}^{n} \mathrm{Te}_k^2$ for $n = 7$ and $n = 10$. We have that

$$
\begin{aligned}
1152 = \sum_{k=1}^{7} \mathrm{Te}_k^2 &= \frac{3\,\mathrm{Te}_7\,\mathrm{Te}_8 - (\mathrm{Te}_8 - \mathrm{Te}_6)^2 + (\mathrm{Te}_7 + \mathrm{Te}_4)\,\mathrm{Te}_5 + 1}{3}\\
&= \frac{3 \cdot 29 \cdot 56 - (56 - 15)^2 + (29 + 4) \cdot 8 + 1}{3}\\
&= \frac{4872 - 1681 + 264 + 1}{3}\\
&= 1152
\end{aligned}
$$

and

$$
\begin{aligned}
59216 = \sum_{k=1}^{10} \mathrm{Te}_k^2 &= \frac{3\,\mathrm{Te}_{10}\,\mathrm{Te}_{11} - (\mathrm{Te}_{11} - \mathrm{Te}_9)^2 + (\mathrm{Te}_{10} + \mathrm{Te}_7)\,\mathrm{Te}_8 + 1}{3}\\
&= \frac{3 \cdot 208 \cdot 401 - (401 - 108)^2 + (208 + 29) \cdot 56 + 1}{3}\\
&= \frac{250224 - 85849 + 13272 + 1}{3}\\
&= 59216.
\end{aligned}
$$

4. The Explicit Formula for the Sum of the First n Fibonacci m-Step Numbers

Let m be a positive integer. The sequence of Fibonacci m-step numbers $\left\{F_k^{(m)}\right\}_{k=-\infty}^{\infty}$ is defined by the initial conditions $F_k^{(m)} = 0$ for all $(2-m) \le k \le 0$, $F_1^{(m)} = F_2^{(m)} = 1$ and the Fibonacci m-step recurrence relation [5]

$$F_{k+m}^{(m)} = \sum_{i=0}^{m-1} F_{k+i}^{(m)} \text{ for all integers } k.$$

In this section, we prove

Theorem 4.1. *(Explicit formula for the sum of the first n Fibonacci m-step numbers)* [7] *For all integers $m \ge 2$ and all nonnegative integers n,*

$$\sum_{k=1}^{n} F_k^{(m)} = \frac{1}{m-1}\left(F_{n+m-1}^{(m)} - \sum_{k=1}^{m-3} kF_{n+m-k-2}^{(m)} + F_n^{(m)} - 1\right).$$

Proof. We will prove this formula again by induction on n.
Let $m \ge 3$. From the above recurrence relation for the Fibonacci m-step numbers, we get that $F_0^{(m)} = 0$, $F_1^{(m)} = F_2^{(m)} = 1$, $F_3^{(m)} = 2$, $F_4^{(m)} = 4$, …, $F_l^{(m)} = 2^{l-2}$ for all $2 \le l \le m+1$, …, $F_m^{(m)} = 2^{m-2}$, $F_{m+1}^{(m)} = 2^{m-1}$.
The above formula is true for $n = 0$ and $n = 1$ because

$$\begin{aligned}
0 = \sum_{k=1}^{0} F_k^{(m)} &= \frac{1}{m-1}\left(F_{m-1}^{(m)} - \sum_{k=1}^{m-3} kF_{m-k-2}^{(m)} + F_0^{(m)} - 1\right)\\
&= \frac{1}{m-1}\left(2^{m-3} - \sum_{k=1}^{m-4} k2^{m-k-4} - m + 2\right) = 0,\\
1 = F_1^{(m)} = \sum_{k=1}^{1} F_k^{(m)} &= \frac{1}{m-1}\left(F_m^{(m)} - \sum_{k=1}^{m-3} kF_{m-k-1}^{(m)} + F_1^{(m)} - 1\right)\\
&= \frac{1}{m-1}\left(2^{m-2} - \sum_{k=1}^{m-3} k2^{m-k-3}\right) = 1.
\end{aligned}$$

Assuming the formula holds for a nonnegative integer n, we will show that it is also true for $n+1$.
Because of the above Fibonacci m-step recurrence relation, we have

$$\begin{aligned}
F_{n+m}^{(m)} &= \sum_{k=0}^{m-1} F_{n+k}^{(m)}\\
\iff F_{n+m}^{(m)} &= F_n^{(m)} + F_{n+1}^{(m)} + \sum_{k=2}^{m-1} F_{n+k}^{(m)}\\
\iff F_{n+1}^{(m)} &= F_{n+m}^{(m)} - \sum_{k=2}^{m-1} F_{n+k}^{(m)} - F_n^{(m)}
\end{aligned}$$

$$\Longleftrightarrow (m-1)F_{n+1}^{(m)} = F_{n+m}^{(m)} - \sum_{k=2}^{m-1} F_{n+k}^{(m)} + (m-2)F_{n+1}^{(m)} - F_n^{(m)}$$

$$\Longleftrightarrow F_{n+1}^{(m)} = \frac{1}{m-1}\left(F_{n+m}^{(m)} - \sum_{k=2}^{m-1} F_{n+k}^{(m)} + (m-2)F_{n+1}^{(m)} - F_n^{(m)}\right),$$

which implies

$$\begin{aligned}
\sum_{k=1}^{n+1} F_k^{(m)} &= \sum_{k=1}^{n} F_k^{(m)} + F_{n+1}^{(m)} \\
&= \frac{1}{m-1}\left(F_{n+m-1}^{(m)} - \sum_{k=1}^{m-3} kF_{n+m-k-2}^{(m)} + F_n^{(m)} - 1\right) \\
&+ \frac{1}{m-1}\left(F_{n+m}^{(m)} - \sum_{k=2}^{m-1} F_{n+k}^{(m)} + (m-2)F_{n+1}^{(m)} - F_n^{(m)}\right) \\
&= \frac{1}{m-1}\left(F_{n+m}^{(m)} + F_{n+m-1}^{(m)} - \sum_{k=1}^{m-3} kF_{n+m-k-2}^{(m)} - \sum_{k=2}^{m-1} F_{n+k}^{(m)} + (m-2)F_{n+1}^{(m)} - 1\right) \\
&= \frac{1}{m-1}\left(F_{n+m}^{(m)} + F_{n+m-1}^{(m)} - \sum_{k=1}^{m-3} kF_{n+m-k-2}^{(m)} - \sum_{k=2}^{m-2} F_{n+k}^{(m)} - F_{n+m-1}^{(m)} + (m-2)F_{n+1}^{(m)} - 1\right) \\
&= \frac{1}{m-1}\left(F_{n+m}^{(m)} - \sum_{k=1}^{m-3} kF_{n+m-k-2}^{(m)} - \sum_{k=2}^{m-2} F_{n+k}^{(m)} + (m-2)F_{n+1}^{(m)} - 1\right) \\
&= \frac{1}{m-1}\left(F_{n+m}^{(m)} - \sum_{k=1}^{m-4} kF_{n+m-k-2}^{(m)} - (m-3)F_{n+1}^{(m)} - \sum_{k=2}^{m-2} F_{n+k}^{(m)} + (m-2)F_{n+1}^{(m)} - 1\right) \\
&= \frac{1}{m-1}\left(F_{n+m}^{(m)} - \sum_{k=1}^{m-4} kF_{n+m-k-2}^{(m)} - \sum_{k=2}^{m-2} F_{n+k}^{(m)} + F_{n+1}^{(m)} - 1\right) \\
&= \frac{1}{m-1}\left(F_{n+m}^{(m)} - \sum_{k=1}^{m-4} kF_{n+m-k-2}^{(m)} - \sum_{k=0}^{m-4} F_{n+m-k-2}^{(m)} + F_{n+1}^{(m)} - 1\right) \\
&= \frac{1}{m-1}\left(F_{n+m}^{(m)} - \sum_{k=2}^{m-3} (k-1)F_{n+m-k-1}^{(m)} - \sum_{k=1}^{m-3} F_{n+m-k-1}^{(m)} + F_{n+1}^{(m)} - 1\right) \\
&= \frac{1}{m-1}\left(F_{n+m}^{(m)} - \sum_{k=1}^{m-3} (k-1)F_{n+m-k-1}^{(m)} - \sum_{k=1}^{m-3} F_{n+m-k-1}^{(m)} + F_{n+1}^{(m)} - 1\right) \\
&= \frac{1}{m-1}\left(F_{n+m}^{(m)} - \sum_{k=1}^{m-3} kF_{n+m-k-1}^{(m)} + \sum_{k=1}^{m-3} F_{n+m-k-1}^{(m)} - \sum_{k=1}^{m-3} F_{n+m-k-1}^{(m)} + F_{n+1}^{(m)} - 1\right) \\
&= \frac{1}{m-1}\left(F_{n+m}^{(m)} - \sum_{k=1}^{m-3} kF_{n+m-k-1}^{(m)} + F_{n+1}^{(m)} - 1\right).
\end{aligned}$$

Therefore, the claimed formula also holds for $n+1$, which finishes the proof, because for $m = 2$, the formula becomes the identity $\sum_{k=1}^{n} F_k = F_{n+1} + F_n - 1 = F_{n+2} - 1$ for the Fibonacci numbers already mentioned in the introduction and that is true for all nonnegative integers n. □

As a corollary, we obtain the following summation formulas for the Pentanacci numbers $P_k = F_k^{(5)}$, the Hexanacci numbers $H_k = F_k^{(6)}$, the Heptanacci numbers $\mathrm{He}_k = F_k^{(7)}$ and the Octanacci numbers $O_k = F_k^{(8)}$.

Corollary 4.2. *(Explicit formulas for the sum of the first n Pentanacci, Hexanacci, Heptanacci and Octanacci numbers)* [7]
For all nonnegative integers n,

$$\sum_{k=1}^{n} P_k = \frac{1}{4}\left(P_{n+4} - P_{n+2} - 2P_{n+1} + P_n - 1\right),$$

$$\sum_{k=1}^{n} H_k = \frac{1}{5}\left(H_{n+5} - H_{n+3} - 2H_{n+2} - 3H_{n+1} + H_n - 1\right),$$

$$\sum_{k=1}^{n} \mathrm{He}_k = \frac{1}{6}\left(\mathrm{He}_{n+6} - \mathrm{He}_{n+4} - 2\,\mathrm{He}_{n+3} - 3\,\mathrm{He}_{n+2} - 4\,\mathrm{He}_{n+1} + \mathrm{He}_n - 1\right),$$

$$\sum_{k=1}^{n} O_k = \frac{1}{7}\left(O_{n+7} - O_{n+5} - 2O_{n+4} - 3O_{n+3} - 4O_{n+2} - 5O_{n+1} + O_n - 1\right).$$

5. Conclusion

We have presented and proved closed form expressions for the sum $\sum_{k=1}^{n} \mathrm{Te}_k$ and for the sum $\sum_{k=1}^{n} \mathrm{Te}_k^2$ involving the Tetranacci numbers. In addition, we proved a summation formula for the sum $\sum_{k=1}^{n} F_k^{(m)}$ of the first n Fibonacci m-step numbers.

The structure of all the summation formulas in this paper is similar and clearly visible. There exist many other closed form expressions for Tetranacci sums of the same structure, which are new and previously unpublished, for example:

$$\sum_{k=1}^{n} k\,\mathrm{Te}_k = \frac{1}{3}\left[\left(n+\frac{2}{3}\right)\mathrm{Te}_n - \left(n+\frac{2}{3}\right)\mathrm{Te}_{n+1} - \mathrm{Te}_{n+2} + \left(n-\frac{1}{3}\right)\mathrm{Te}_{n+3}\right] + \frac{7}{9}$$

as well as

$$\sum_{k=1}^{n} k^2\,\mathrm{Te}_k = \frac{1}{3}\left[\left(n^2+\frac{4}{3}n+\frac{14}{9}\right)\mathrm{Te}_n - \left(n^2+\frac{4}{3}n-\frac{22}{9}\right)\mathrm{Te}_{n+1} - \left(2n-\frac{5}{3}\right)\mathrm{Te}_{n+2} + \left(n^2-\frac{2}{3}n+\frac{11}{9}\right)\mathrm{Te}_{n+3}\right] - \frac{59}{27}$$

and

$$\sum_{k=1}^{n} k \operatorname{Te}_k^2 = \left(\frac{1}{3}n + \frac{10}{9}\right) \operatorname{Te}_n \operatorname{Te}_{n+3} + \left(\frac{2}{3}n + \frac{43}{18}\right) \operatorname{Te}_{n+1} \operatorname{Te}_{n+3} + \left(n + \frac{10}{3}\right) \operatorname{Te}_{n+2} \operatorname{Te}_{n+3}$$
$$- \left(\frac{1}{3}n + \frac{16}{9}\right) \operatorname{Te}_n \operatorname{Te}_{n+1} - \frac{1}{6} \operatorname{Te}_n \operatorname{Te}_{n+2} - \frac{5}{6} \operatorname{Te}_{n+1} \operatorname{Te}_{n+2} - \left(\frac{1}{3}n + \frac{29}{18}\right) \operatorname{Te}_n^2$$
$$- \left(\frac{4}{3}n + \frac{28}{9}\right) \operatorname{Te}_{n+1}^2 - (n+3) \operatorname{Te}_{n+2}^2 - \left(\frac{1}{3}n + \frac{23}{18}\right) \operatorname{Te}_{n+3}^2 + \frac{11}{18}$$

and

$$\sum_{k=1}^{n} k^2 \operatorname{Te}_k^2 = \left(\frac{1}{3}n^2 + \frac{20}{9}n + \frac{277}{54}\right) \operatorname{Te}_n \operatorname{Te}_{n+3} + \left(\frac{2}{3}n^2 + \frac{43}{9}n + \frac{343}{27}\right) \operatorname{Te}_{n+1} \operatorname{Te}_{n+3}$$
$$+ \left(n^2 + \frac{20}{3}n + \frac{277}{18}\right) \operatorname{Te}_{n+2} \operatorname{Te}_{n+3} - \left(\frac{1}{3}n^2 + \frac{32}{9}n + \frac{553}{54}\right) \operatorname{Te}_n \operatorname{Te}_{n+1}$$
$$- \left(\frac{1}{3}n - \frac{2}{9}\right) \operatorname{Te}_n \operatorname{Te}_{n+2} - \left(\frac{5}{3}n + \frac{56}{9}\right) \operatorname{Te}_{n+1} \operatorname{Te}_{n+2} - \left(\frac{1}{3}n^2 + \frac{29}{9}n + \frac{242}{27}\right) \operatorname{Te}_n^2$$
$$- \left(\frac{4}{3}n^2 + \frac{56}{9}n + \frac{799}{54}\right) \operatorname{Te}_{n+1}^2 - \left(n^2 + 6n + \frac{79}{6}\right) \operatorname{Te}_{n+2}^2 - \left(\frac{1}{3}n^2 + \frac{23}{9}n + \frac{164}{27}\right) \operatorname{Te}_{n+3}^2 + \frac{62}{27}.$$

The proofs of these four formulas are left to the reader.

6. Acknowledgment

Many thanks to the referee for valuable suggestions that improved this paper.

This work was supported by SNSF (Swiss National Science Foundation) under grant 169247.

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MSC2010: Primary 11B39, Secondary 65B10.

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