REPRESENTATION OF $\frac{1}{2}(\mathbf{F_n}-\mathbf{1})(\mathbf{F_{n+1}}-\mathbf{1})$ AND $\frac{1}{2}(\mathbf{F_n}-\mathbf{1})(\mathbf{F_{n+2}}-\mathbf{1})$

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Abstract. Let $a, b \in \mathbb{N}$ be relatively prime. We consider $(a-1)(b-1)/2$, which arises in the study of the pqth cyclotomic polynomial, where p, q are distinct primes. We prove two possible representations of $(a-1)(b-1)/2$ as nonnegative integral linear combinations of a and b. Surprisingly, for each pair (a, b), only one of the two representations exists and the representation is also unique. We then investigate the representations of $(F_n-1)(F_{n+1}-1)/2$ and $(F_n-1)(F_{n+2}-1)/2$, where F_i is the ith Fibonacci number, and observe several nice patterns.

1. Motivation and Main Results

The nth cyclotomic polynomial is defined as

$$\Phi_n(x) \;=\; \prod_{m=1,\ (m,n)=1}^{n} (x - e^{\frac{2\pi i m}{n}}).$$

Naturally, much work has been done on the values of the coefficients of $\Phi_n(x)$. Numbers of the form $(a-1)(b-1)/2$ with $(a, b) = 1$ arise in the study of the midterm coefficient of the pqth cyclotomic polynomial, where p, q are distinct primes. (Note that the degree of $\Phi_{pq}(x)$ is $\phi(pq) = (p-1)(q-1)$, where ϕ is the Euler totient function, so its midterm coefficient has degree $(p-1)(q-1)/2$.) In particular, these polynomials have been fully characterized by the work of Beiter [1], Carlitz [2], and Lam and Leung [6]. These authors used different and clever approaches.

In computing the midterm coefficient of $\Phi_{pq}(x)$, Beiter sketched a proof that $(p-1)(q-1)/2$ can be uniquely written as $\alpha q + \beta p + \delta$, where $0 \le \alpha \le p-1$, $\beta \ge 0$, and $\delta \in \{0, 1\}$. In this article, we provide an alternate proof of the result applied to any relatively prime numbers.

Theorem 1.1. *Let $a, b \in \mathbb{N}$ be relatively prime. Consider the following two equations.*

$$xa + yb \;=\; \frac{(a-1)(b-1)}{2}. \tag{1.1}$$

$$xa + yb + 1 \;=\; \frac{(a-1)(b-1)}{2}. \tag{1.2}$$

Exactly one of the two equations has nonnegative integral solution(s) and the solution is unique.

Example 1.2. We observe that both representations of $(a-1)(b-1)/2$ can happen. If $a = 3$ and $b = 5$, we have $1 \cdot 3 + 0 \cdot 5 + 1 = (3-1)(5-1)/2$. If $a = 11$ and $b = 31$, we have $8 \cdot 11 + 2 \cdot 31 = (11-1)(31-1)/2$. Our theorem is also related to Problem E1637 [5], which states that for $k \ge (a-1)(b-1)$, there exist nonnegative solution(s) to $xa + yb = k$. Our theorem gives examples of k smaller than $(a-1)(b-1)$, which still makes $xa + yb = k$ have a unique nonnegative solution.

Corollary 1.3. *Let p, q be distinct primes. Then $(p-1)(q-1)/2$ can be uniquely written as $\alpha q + \beta p + \delta$ for some $0 \le \alpha \le p-1$, $\beta \ge 0$, and $\delta \in \{0, 1\}$.*

This corollary is what Beiter used in computing the midterm coefficient of $\Phi_{pq}(x)$. We now present another proof, which is shorter with the use of a strong theorem of Dresden, that was not available when [1] first appeared.

Alternate proof of Corollary 1.3. By [4, Theorem 1], the midterm coefficient of $\Phi_{pq}(x)$ is odd. By [1, Theorem 1], $(p-1)(q-1)/2 = \alpha q + \beta p + \delta$ in exactly one way. □

Next, given a pair of consecutive Fibonacci numbers (F_n, F_{n+1}), we investigate the nonnegative integral solutions to

$$F_n x + F_{n+1} y = (F_n - 1)(F_{n+1} - 1)/2, \tag{1.3}$$
$$1 + F_n x + F_{n+1} y = (F_n - 1)(F_{n+1} - 1)/2. \tag{1.4}$$

From Theorem 1.1 and $(F_n, F_{n+1}) = 1$[1], we know that exactly one of these equations has a unique nonnegative integral solution. By convention, we index the Fibonacci sequence as follows:

$$F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5, \ldots.$$

Table 1 provides the first several cases.

n	F_n	F_{n+1}	Equation	x	y
3	2	3	(1.4)	0	0
4	3	5	(1.4)	1	0
5	5	8	(1.4)	1	1
6	8	13	(1.3)	2	2
7	13	21	(1.3)	6	2
8	21	34	(1.3)	6	6
9	34	55	(1.4)	10	10
10	55	89	(1.4)	27	10
11	89	144	(1.4)	27	27
12	144	233	(1.3)	44	44
13	233	377	(1.3)	116	44
14	377	610	(1.3)	116	116

Table 1

We observe two patterns here. First, equation (1.3) and equation (1.4) are used alternatively with period 3. Second, the first and the third row of each period give $x = y$. The two patterns are summarized by the following theorem.

[1] from Cassini's identity: $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

Theorem 1.4. *For $k \geq 1$, the following formulas are correct.*

$$\frac{1}{2}(F_{6k-1}-1)F_{6k}+\frac{1}{2}(F_{6k-1}-1)F_{6k+1} = \frac{(F_{6k}-1)(F_{6k+1}-1)}{2}. \tag{1.5}$$
$$\frac{1}{2}(F_{6k+1}-1)F_{6k+1}+\frac{1}{2}(F_{6k-1}-1)F_{6k+2} = \frac{(F_{6k+1}-1)(F_{6k+2}-1)}{2}. \tag{1.6}$$
$$\frac{1}{2}(F_{6k+1}-1)F_{6k+2}+\frac{1}{2}(F_{6k+1}-1)F_{6k+3} = \frac{(F_{6k+2}-1)(F_{6k+3}-1)}{2}. \tag{1.7}$$
$$1+\frac{1}{2}(F_{6k+2}-1)F_{6k+3}+\frac{1}{2}(F_{6k+2}-1)F_{6k+4} = \frac{(F_{6k+3}-1)(F_{6k+4}-1)}{2}. \tag{1.8}$$
$$1+\frac{1}{2}(F_{6k+4}-1)F_{6k+4}+\frac{1}{2}(F_{6k+2}-1)F_{6k+5} = \frac{(F_{6k+4}-1)(F_{6k+5}-1)}{2}. \tag{1.9}$$
$$1+\frac{1}{2}(F_{6k+4}-1)F_{6k+5}+\frac{1}{2}(F_{6k+4}-1)F_{6k+6} = \frac{(F_{6k+5}-1)(F_{6k+6}-1)}{2}. \tag{1.10}$$

Remark 1.5. If n is a multiple of 3, then F_n is even [3]. Let $k \in \mathbb{N}$. Because $(F_{3k}, F_{3k+1}) = (F_{3k+2}, F_{3k+3}) = 1$, F_{3k+1} and F_{3k+2} are odd. Hence, F_n is even if and only if n is a multiple of 3. Therefore, if $n \not\equiv 0 \bmod 3$, $\frac{1}{2}(F_n - 1)$ is a nonnegative integer.

Similarly, given (F_n, F_{n+2}), we can consider two following equations.

$$F_n x + F_{n+2} y = (F_n - 1)(F_{n+2} - 1)/2. \tag{1.11}$$
$$1 + F_n x + F_{n+2} y = (F_n - 1)(F_{n+2} - 1)/2. \tag{1.12}$$

From Theorem 1.1 and $(F_n, F_{n+2}) = 1$[2], we know that exactly one of these equations has a unique nonnegative integral solution. Table 2 provides the first several cases.

n	F_n	F_{n+2}	Equation	x	y
1	1	2	(1.11)	0	0
2	1	3	(1.11)	0	0
3	2	5	(1.11)	1	0
4	3	8	(1.12)	2	0
5	5	13	(1.12)	2	1
6	8	21	(1.12)	6	1
7	13	34	(1.11)	10	2
8	21	55	(1.11)	10	6
9	34	89	(1.11)	27	6
10	55	144	(1.12)	44	10
11	89	233	(1.12)	44	27
12	144	377	(1.12)	116	27

Table 2

Again, equation (1.11) and equation (1.12) appear alternately with period 3. The following theorem is compatible with Theorem 1.4 and the proof is similar, so we omit it.

[2]from the identity: $F_n^2 - F_{n-2}F_{n+2} = (-1)^n$ [7].

Theorem 1.6. *For $k \geq 0$, the following formulas are correct.*

$$\frac{F_{6k+2}-1}{2}F_{6k+1} + \frac{F_{6k-1}-1}{2}F_{6k+3} = \frac{(F_{6k+1}-1)(F_{6k+3}-1)}{2}. \tag{1.13}$$

$$\frac{F_{6k+2}-1}{2}F_{6k+2} + \frac{F_{6k+1}-1}{2}F_{6k+4} = \frac{(F_{6k+2}-1)(F_{6k+4}-1)}{2}. \tag{1.14}$$

$$\frac{F_{6k+4}-1}{2}F_{6k+3} + \frac{F_{6k+1}-1}{2}F_{6k+5} = \frac{(F_{6k+3}-1)(F_{6k+5}-1)}{2}. \tag{1.15}$$

$$1 + \frac{F_{6k+5}-1}{2}F_{6k+4} + \frac{F_{6k+2}-1}{2}F_{6k+6} = \frac{(F_{6k+4}-1)(F_{6k+6}-1)}{2}. \tag{1.16}$$

$$1 + \frac{F_{6k+5}-1}{2}F_{6k+5} + \frac{F_{6k+4}-1}{2}F_{6k+7} = \frac{(F_{6k+5}-1)(F_{6k+7}-1)}{2}. \tag{1.17}$$

$$1 + \frac{F_{6k+1}-1}{2}F_{6k} + \frac{F_{6k-2}-1}{2}F_{6k+2} = \frac{(F_{6k}-1)(F_{6k+2}-1)}{2}. \tag{1.18}$$

2. Proofs

We first prove the following lemma.

Lemma 2.1. *For any integers n, x, y, a, b with a, b positive and $(a, b) = 1$, we consider the equation $xa + yb = n$. If there is a solution r, s with $r < b$ and $s < 0$, then there are no solutions with x, y nonnegative.*

Proof. All solutions are of the form $(x, y) = (r + tb, s - ta)$ for some $t \in \mathbb{Z}$. To get $y \geq 0$, we must have $t < 0$, but then $x < 0$. □

Proof of Theorem 1.1. Let $k = (a - 1)(b - 1)/2$.

Let $0 \leq r_1 \leq b - 1$ be chosen such that $ar_1 \equiv k \bmod b$ and $s_1 = (k - ar_1)/b$.

Let $0 \leq r_2 \leq b - 1$ be chosen such that $ar_2 \equiv (k - 1) \bmod b$ and $s_2 = (k - 1 - ar_2)/b$.

Observe that

$$a(r_1 + r_2) = ar_1 + ar_2 \equiv 2k - 1 = a(b - 1) - b \equiv a(b - 1) \bmod b.$$

So, $b \mid (r_1 + r_2 - (b - 1))$ and so, $r_1 + r_2 = b - 1$. We compute

$$s_1 + s_2 = \frac{k - ar_1 + k - ar_2 - 1}{b} = \frac{2k - a(b - 1) - 1}{b} = -1.$$

Hence, exactly one of s_1, s_2 is nonnegative. By definition, $r_1a + s_1b = k$ and $r_2a + s_2b + 1 = k$. Therefore, we have shown that equation (1.1) or equation (1.2) has a nonnegative solution (x, y), whereas the other equation has no nonnegative solutions because of Lemma 2.1.

It remains to prove that equation (1.1) has at most one nonnegative solution. (A similar claim and proof hold for equation (1.2).) Let (x_1, y_1) and (x_2, y_2) be two nonnegative solutions of equation (1.1). Clearly, $0 \leq x_1, x_2 \leq (b-1)$, so $|x_1 - x_2| \leq b-1$. Because $x_1a + y_1b = x_2a + y_2b$, $(x_1 - x_2)a = -(y_1 - y_2)b$. Because $(a, b) = 1$, b divides $x_1 - x_2$, which, with $|x_1 - x_2| \leq b - 1$, implies that $x_1 = x_2$. It follows that $y_1 = y_2$. Hence, equation (1.1) has at most one nonnegative solution. □

Proof of Theorem 1.4. We prove formulas (1.5), (1.6), and (1.8). Proofs for the remaining formulas are similar. We use identity $(d7)$ [7] repeatedly

$$F_nF_{n+3} = F_{n+1}F_{n+2} + (-1)^{n-1}. \tag{2.1}$$

Write

$$\begin{aligned}
&\frac{1}{2}(F_{6k-1}-1)F_{6k}+\frac{1}{2}(F_{6k-1}-1)F_{6k+1}\\
&=\frac{1}{2}(F_{6k-1}-1)(F_{6k}+F_{6k+1})\\
&=\frac{1}{2}F_{6k-1}F_{6k+2}-\frac{1}{2}(F_{6k}+F_{6k+1})\\
&=\frac{F_{6k}F_{6k+1}+1}{2}-\frac{1}{2}(F_{6k}+F_{6k+1}) \text{ due to (2.1)}\\
&=\frac{(F_{6k}-1)(F_{6k+1}-1)}{2}.
\end{aligned}$$

This proves formula (1.5).

Next, we have

$$\begin{aligned}
&\frac{1}{2}(F_{6k+1}-1)F_{6k+1}+\frac{1}{2}(F_{6k-1}-1)F_{6k+2}\\
&=\frac{1}{2}F_{6k+1}^2-\frac{1}{2}F_{6k+1}+\frac{1}{2}F_{6k-1}F_{6k+2}-\frac{1}{2}F_{6k+2}\\
&=\frac{1}{2}F_{6k+1}^2-\frac{1}{2}F_{6k+1}+\frac{1}{2}(F_{6k}F_{6k+1}+1)-\frac{1}{2}F_{6k+2} \text{ due to (2.1)}\\
&=\frac{1}{2}F_{6k+1}^2-\frac{1}{2}F_{6k+1}+\frac{1}{2}((F_{6k+2}-F_{6k+1})F_{6k+1}+1)-\frac{1}{2}F_{6k+2}\\
&=\frac{1}{2}F_{6k+1}F_{6k+2}-\frac{1}{2}(F_{6k+1}+F_{6k+2})+\frac{1}{2} \;=\; \frac{(F_{6k+1}-1)(F_{6k+2}-1)}{2}.
\end{aligned}$$

This proves formula (1.6)

Finally, we prove formula (1.8). The left side is

$$\begin{aligned}
&1+\frac{1}{2}(F_{6k+2}-1)F_{6k+3}+\frac{1}{2}(F_{6k+2}-1)F_{6k+4}\\
&=1+\frac{1}{2}F_{6k+2}F_{6k+5}-\frac{1}{2}(F_{6k+3}+F_{6k+4})\\
&=1+\frac{1}{2}(F_{6k+3}F_{6k+4}-1)-\frac{1}{2}(F_{6k+3}+F_{6k+4})\\
&=\frac{(F_{6k+3}-1)(F_{6k+4}-1)}{2},
\end{aligned}$$

which is the right side. □

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