CONVOLUTIONS OF GENERALIZED STIRLING NUMBERS AND DEGENERATE BERNOULLI POLYNOMIALS

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Abstract. We present a general convolution formula involving the generalized Stirling numbers of Hsu and Shiue and the degenerate Bernoulli polynomials of Carlitz. As special cases, this formula yields new convolution formulas for weighted Stirling numbers of both kinds, Bernoulli polynomials of both kinds, binomial coefficients, and harmonic numbers. The formula also encompasses several familiar recurrences for these combinatorial sequences.

1. Introduction

Hsu and Shiue [5] defined generalized Stirling number pairs by the generating function

$$k!\sum_{n=k}^{\infty} S(n,k;\alpha,\beta,r)\frac{t^n}{n!} = (1+\alpha t)^{r/\alpha}\left(\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right)^k \tag{1.1}$$

where $(\alpha,\beta)\neq(0,0)$. The usual Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$ are given by the parameters $(\alpha,\beta,r)=(1,0,0)$ and $(\alpha,\beta,r)=(0,1,0)$, respectively. (When $\alpha=0$ or $\beta=0$, the equation is understood to mean the limit as $\alpha\to 0$ or $\beta\to 0$.) The parameters $(1,0,-x)$ and $(0,1,x)$ give Carlitz' weighted Stirling numbers of the first and second kinds, and the parameters $(1,\lambda,0)$ give the degenerate Stirling numbers of Carlitz. Hsu and Shiue demonstrated that there is, in general, a duality between the generalized Stirling numbers with parameters (α,β,r) and $(\beta,\alpha,-r)$. These sequences are of great importance in combinatorics and number theory. See [6] for further algebraic and combinatorial viewpoints on these sequences.

Carlitz [2] defined the *degenerate Bernoulli polynomials* $\beta_n^{(w)}(\lambda,x)$ for $\lambda\neq 0$ by means of the generating function

$$\left(\frac{t}{(1+\lambda t)^{\mu}-1}\right)^w (1+\lambda t)^{\mu x} = \sum_{n=0}^{\infty}\beta_n^{(w)}(\lambda,x)\frac{t^n}{n!}, \tag{1.2}$$

where $\lambda\mu=1$. When $\lambda\to 0$, we obtain the usual order w Bernoulli polynomials $B_n^{(w)}(x)=\beta_n^{(w)}(0,x)$, which are generated by

$$\left(\frac{t}{e^t-1}\right)^w e^{xt} = \sum_{n=0}^{\infty} B_n^{(w)}(x)\frac{t^n}{n!}, \tag{1.3}$$

and for the case $\mu=0$, we have $\lim_{\lambda\to\infty}\lambda^{-n}\beta_n^{(w)}(\lambda,\lambda x)=n!b_n^{(w)}(x)$, where the order w Bernoulli polynomials of the second kind $b_n^{(w)}(x)$ are defined by

$$\left(\frac{t}{\log(1+t)}\right)^w (1+t)^x = \sum_{n=0}^{\infty} b_n^{(w)}(x)t^n. \tag{1.4}$$

In all cases involving Bernoulli numbers, when the order $w = 1$ or the argument $x = 0$, we will often suppress that part of the notation. These polynomials are also important in combinatorics and number theory, being related to power sums, generalized factorial sums [2, 8], divided differences of binomial coefficients [1], game theory probabilities [3], and coset products in factor rings [7].

In this paper, we demonstrate a different kind of relation between these sequences, which may be expressed as a general convolution identity. Several special cases will be observed such as

$$\sum_{j=0}^{n} \binom{n}{j} (H_n - H_j) b_j = n \tag{1.5}$$

and

$$\sum_{j=0}^{n} \binom{n}{j} (H_n - H_j) \frac{B_j^{(j)}}{j!} = 1, \tag{1.6}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number.

2. Main Result

Theorem 2.1. *If $k \geqslant w$, we have*

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j, k; \alpha, \beta, r) \beta_j^{(w)}(\lambda, x) \beta^j = \frac{\binom{n}{w}}{\binom{k}{w}} S(n-w, k-w; \alpha, \beta, r+\beta x),$$

where $\lambda\beta = \alpha$; and for $k \leqslant w$, we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j, k; \alpha, \beta, r) \beta_j^{(w)}(\lambda, x) \beta^j = \binom{n}{k} \beta_{n-k}^{(w-k)}(\lambda, x + \frac{r}{\beta}) \beta^{n-k}.$$

Proof. If $\lambda\beta = \alpha$, then replacing t with βt yields

$$\sum_{n=0}^{\infty} \beta_n^{(w)}(\lambda, x) \beta^n \frac{t^n}{n!} = \left(\frac{\beta t}{(1+\alpha t)^{\beta/\alpha} - 1} \right)^w (1+\alpha t)^{\beta x/\alpha}. \tag{2.1}$$

Therefore, if $k \geqslant w$, we have

$$\left[\sum_{n=k}^{\infty} S(n, k; \alpha, \beta, r) \frac{t^n}{n!} \right] \left[\sum_{n=0}^{\infty} \beta_n^{(w)}(\lambda, x) \beta^n \frac{t^n}{n!} \right] = \frac{t^w}{k!} (1+\alpha t)^{(r+\beta x)/\alpha} \left(\frac{(1+\alpha t)^{\beta/\alpha} - 1}{\beta} \right)^{k-w}, \tag{2.2}$$

whereas, if $k \leqslant w$, we may write

$$\left[\sum_{n=k}^{\infty} S(n, k; \alpha, \beta, r) \frac{t^n}{n!} \right] \left[\sum_{n=0}^{\infty} \beta_n^{(w)}(\lambda, x) \beta^n \frac{t^n}{n!} \right] = \frac{t^k}{k!} (1+\lambda\beta t)^{\mu(x+r/\beta)} \left(\frac{\beta t}{(1+\lambda\beta t)^{\mu} - 1} \right)^{w-k}. \tag{2.3}$$

Equating coefficients of $t^n/n!$ gives the result. $\square$

3. Limiting Cases

When $\lambda = 0$, our convolution involves the order w Bernoulli polynomials and weighted Stirling numbers of the second kind, and the result is either a weighted Stirling number of the second kind or a Bernoulli polynomial, depending on whether $k \geqslant w$.

Corollary 3.1. *($\lambda = 0$ case) If $k \geqslant w$, we have*

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;0,1,r) B_j^{(w)}(x) = \frac{\binom{n}{w}}{\binom{k}{w}} S(n-w,k-w;0,1,r+x);$$

and for $k \leqslant w$, we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;0,1,r) B_j^{(w)}(x) = \binom{n}{k} B_{n-k}^{(w-k)}(x+r).$$

When $\mu = 0$, our convolution involves the order w Bernoulli polynomials of the second kind and weighted Stirling numbers of the first kind, and the result is either a weighted Stirling number of the first kind or a Bernoulli polynomial of the second kind, depending on whether $k \geqslant w$.

Corollary 3.2. *($\mu = 0$ case) If $k \geqslant w$, we have*

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;1,0,r) j! b_j^{(w)}(x) = \frac{\binom{n}{w}}{\binom{k}{w}} S(n-w,k-w;1,0,r+x);$$

and for $k \leqslant w$, we have

$$\sum_{j=0}^{n-k} \frac{S(n-j,k;1,0,r)}{(n-j)!} b_j^{(w)}(x) = \frac{b_{n-k}^{(w-k)}(x+r)}{k!}.$$

We remark that in [9, eq. (4.3)], there is a convolution formula for Bernoulli polynomials of the second kind with weighted Stirling numbers of the second kind. That formula seems to share a kind of duality with the above corollaries.

4. Zero-order Cases

In this section, we consider the specializations of the main result when $k = 0$, $w = 0$, or $k - w = 0$. When $k = w$, the sum reduces to a single falling factorial or power; this occurs because $S(n,0;\alpha,\beta,r) = (r|\alpha)_n$, where

$$(r|\alpha)_n = r(r-\alpha)\cdots(r-(n-1)\alpha) \tag{4.1}$$

denotes the generalized falling factorial with increment α, with convention $(r|\alpha)_0 = 1$ [5, eq. (8)].

Corollary 4.1. *($k = w$ case) We have*

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;\alpha,\beta,r) \beta_j^{(k)}(\lambda,x) \beta^j = \binom{n}{k} (r+\beta x|\alpha)_{n-k},$$

where $\lambda\beta = \alpha$; in particular for $\lambda = 0$, we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;0,1,r) B_j^{(k)}(x) = \binom{n}{k} (r+x)^{n-k};$$

and for $\mu = 0$, we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;1,0,r) j! b_j^{(k)}(x) = \binom{n}{k} (r+x|1)_{n-k}.$$

When $r = x = 0$ in the above corollary, we obtain the orthogonality relations

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;\alpha,\beta,0) \beta_j^{(k)}(\lambda) \beta^j = \delta_{n,k}, \tag{4.2}$$

where $\delta_{n,k}$ is the Kronecker delta; in particular, we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k) B_j^{(k)} = \delta_{n,k} \tag{4.3}$$

and

$$\sum_{j=0}^{n-k} \binom{n}{j} s(n-j,k) j! b_j^{(k)} = \delta_{n,k} \tag{4.4}$$

in terms of the usual Stirling numbers $s(n,k)$ and $S(n,k)$ of the first and second kinds.

In the case $k = 0$, the generalized Stirling number disappears from the convolution, and we obtain a recurrence involving Bernoulli polynomials only.

Corollary 4.2. *($k = 0$ case) We have*

$$\sum_{j=0}^{n} \binom{n}{j} (r|\alpha)_{n-j} \beta_j^{(w)}(\lambda, x) \beta^j = \beta_n^{(w)}(\lambda, x + (r/\beta)) \beta^n,$$

where $\lambda\beta = \alpha$; in particular for $\lambda = 0$, we have

$$\sum_{j=0}^{n} \binom{n}{j} r^{n-j} B_j^{(w)}(x) = B_n^{(w)}(x+r)$$

and for $\mu = 0$, we have

$$\sum_{j=0}^{n} \binom{r}{n-j} b_j^{(w)}(x) = b_n^{(w)}(x+r).$$

Note that the second equation ($\lambda = 0$) of this corollary is a well-known recurrence for Bernoulli polynomials, particularly in the case $x = 0$. The third equation ($\mu = 0$) does not appear to be so well known.

In the case $w = 0$, the Bernoulli polynomial disappears from the convolution, and we obtain a recurrence involving Stirling numbers only.

Corollary 4.3. *($w = 0$ case) We have*

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;\alpha,\beta,r)(x|\lambda)_j \beta^j = S(n,k;\alpha,\beta,r+\beta x),$$

where $\lambda\beta = \alpha$; in particular for $\lambda = 0$, we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;0,1,r) x^j = S(n,k;0,1,r+x)$$

and for $\mu = 0$, we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;1,0,r) j! \binom{x}{j} = S(n,k;1,0,r+x).$$

These two special cases ($\lambda = 0$ and $\mu = 0$) are well-known recurrences for weighted Stirling numbers, particularly in the case $r = 0$.

5. First-order Cases

When $k = 1$ or $w = 1$, the generalized Stirling number may be simplified to

$$S(n,1;\alpha,\beta,r) = \beta^{-1}\left((r+\beta|\alpha)_n - (r|\alpha)_n\right) \tag{5.1}$$

in terms of generalized falling factorials. This may be proven by induction from the recurrence

$$S(n+1,k;\alpha,\beta,r) = S(n,k-1;\alpha,\beta,r) + (k\beta - n\alpha + r)S(n,k;\alpha,\beta,r) \tag{5.2}$$

[5, eq. (7)]. Taking the limit as $\beta \to 0$ yields

$$S(n,1;1,0,r) = (r|1)_n \left[\frac{1}{r} + \frac{1}{r-1} + \cdots + \frac{1}{r-n+1}\right]. \tag{5.3}$$

Therefore, we have the following corollary.

Corollary 5.1. *($k = w = 1$ case) We have*

$$\sum_{j=0}^{n-1} \binom{n}{j} \left((r+\beta|\alpha)_{n-j} - (r|\alpha)_{n-j}\right) \beta_j(\lambda,x)\beta^{j-1} = n(r+\beta x|\alpha)_{n-1},$$

where $\lambda\beta = \alpha$; in particular for $\lambda = 0$, we have

$$\sum_{j=0}^{n-1} \binom{n}{j} \left((r+1)^{n-j} - r^{n-j}\right) B_j(x) = n(r+x)^{n-1}$$

and for $\mu = 0$, we have

$$\sum_{j=0}^{n-1} \binom{r}{n-j} \left[\frac{1}{r} + \frac{1}{r-1} + \cdots + \frac{1}{r-(n-j-1)}\right] b_j(x) = \binom{r+x}{n-1}.$$

In the case $r = 0$, the $\lambda = 0$ case of the above corollary reflects the usual recurrence and difference equation for the Bernoulli polynomials. In the $\mu = 0$ case, the weighted Stirling numbers of the first kind reduce to generalized harmonic numbers; in particular taking $r = n$, we obtain

$$\sum_{j=0}^{n} \binom{n}{j} (H_n - H_j) b_j(x) = \binom{n+x}{n-1}, \tag{5.4}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number, and more specifically for $x = 0$, we get

$$\sum_{j=0}^{n} \binom{n}{j} (H_n - H_j) b_j = n. \tag{5.5}$$

Taking $x = -1$ and using the identity $B_n^{(n)} = n! b_n(-1)$ [4, eq. (2.10)] yields the identity

$$\sum_{j=0}^{n} \binom{n}{j} (H_n - H_j) \frac{B_j^{(j)}}{j!} = 1 \tag{5.6}$$

for the Nörlund numbers $B_n^{(n)}$.

These identities are special cases of a general class as illustrated in the previous corollary. We conclude with a general statement of the result for $k = 1$.

Corollary 5.2. *($k = 1$ case) We have*

$$\sum_{j=0}^{n-1} \binom{n}{j} \left((r+\beta|\alpha)_{n-j} - (r|\alpha)_{n-j}\right) \beta_j^{(w)}(\lambda, x)\beta^{j-1} = n\beta_{n-1}^{(w-1)}(x + (r/\beta))\beta^{n-1},$$

where $\lambda\beta = \alpha$; in particular for $\lambda = 0$, we have

$$\sum_{j=0}^{n-1} \binom{n}{j} \left((r+1)^{n-j} - r^{n-j}\right) B_j^{(w)}(x) = nB_{n-1}^{(w-1)}(x + r)$$

and for $\mu = 0$, we have

$$\sum_{j=0}^{n-1} \binom{r}{n-j} \left[\frac{1}{r} + \frac{1}{r-1} + \cdots + \frac{1}{r-(n-j-1)}\right] b_j^{(w)}(x) = b_{n-1}^{(w-1)}(x + r).$$

In the case $k = 2$, the Stirling numbers may also in some cases be expressed in terms of generalized harmonic numbers of higher order. We leave the details to the reader.

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